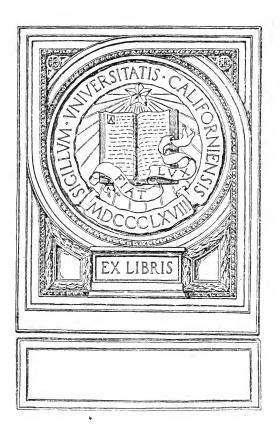
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 $\mathbf{B}\mathbf{Y}$

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Containing a Large Number of Practical Exercises, with Answers, and 330 Illustrations

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PREFACE.

This book has been designed primarily to meet the requirements of technical students and artisans as a practical text-book on Mathematics. Teachers who do not combine instruction in Practical Mathematics with Practical Geometry, and who have thus been unable to adopt the Authors' existing text-books* in which the two subjects are combined, will, it is hoped, find this book adapted to their requirements.

It is believed that the book will be found to contain all the mathematics that the average technical student or practical man requires to enable him to follow intelligently technical or scientific subjects which he may have to study, and also to deal with practical problems which he may encounter in connection with his trade or profession. No previous knowledge of the subject is assumed, and the work is intended to cover a two, or three, years' course of instruction according to the capacity of the student.

The syllabus of the Board of Education "Lower Stage" Examination in Practical Mathematics has been taken as a guide, but in many sections the book will be found to exceed the requirements of the syllabus. Students studying for this examination should consult the note facing p. 1 as to which parts of the book may be omitted as regards the examination. The small section of the syllabus dealing with Descriptive Geometry has been purposely omitted, as it is more properly included in the companion volume by the Authors on Practical Geometry and Graphics. Its exclusion from this volume has, moreover, enabled the Authors to deal more fully with Integration and applications to practical problems, thus rendering the work more complete as a text-book and guide to the student of Practical Mathematics.

^{* &}quot;Practical Mathematics and Geometry," 3 parts. London, 1911-12.

The practical application of principles discussed to problems met with daily by engineers, builders, architects, surveyors, draughtsmen, and others, has been made a prominent feature of the book. The sections dealing with Rates of Increase, Differentiation, and Integration are intended to give the student an acquaintance with these subjects sufficient to enable him to read advanced text-books on sciences allied to his trade or profession, and to apply the calculus to practical problems.

The needs of the student who has to work unaided by a teacher have been carefully kept in view, and he should have no difficulty in proceeding intelligently, if he will master each section before proceeding to the next, and also work the exercises, which have been carefully chosen and compiled so as to be of a graduated and practical nature. The Exercises marked B. E. are taken from the Board of Education Examination Papers in Practical Mathematics and in Practical Geometry: those marked B. E. (2) being selected from the old Stage 2 Examination.

The Authors cordially thank Mr. F. P. Johns who has read the MSS, and proofs, and made many valuable suggestions, and also the Publisher who has rendered every possible assistance in the preparation of the work.

E. L. B. F. C.

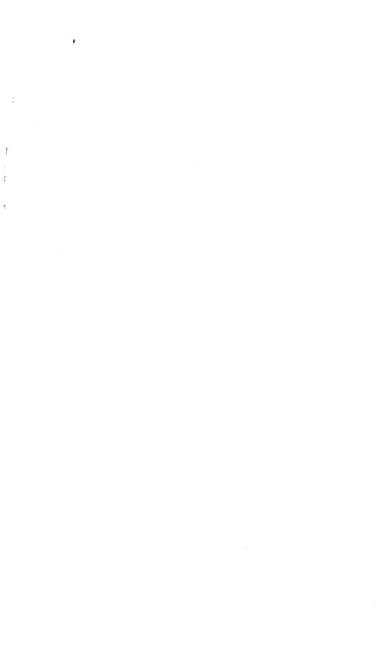
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Sept., 1912.

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BOARD OF EDUCATION "LOWER STAGE" EXAMINATION.

The student reading for the above examination may omit the following Articles: 291, 295 to 301; 304; 306; 326 to 328; 336 to 351; 367 to 375; 385 to 404; and also Appendix I.

NOTE.

The following contractions are sometimes used :-

 \perp^{r} = perpendicular.

 \triangle = triangle.

: = therefore.

> = greater than.

∠ = angle.

| = parallel to.

 \therefore = because.

< = less than.



PRACTICAL MATHEMATICS.

CHAPTER I.

INTRODUCTION TO ALGEBRA.

1. In arithmetic we deal principally with the numbers of things—for example, we speak of 50 bricks, or 30 miles, or 25 horses; in each case, the number denotes the quantity of the particular thing we are speaking about, and the particular thing is specified by writing its name in full.

In algebra we adopt a similar plan as regards quantity, but we choose some **symbol** to represent the thing, and hence we specify the thing by its representative symbol.

Thus, if we let the symbol x represent a brick, then 50x represents 50 bricks; similarly, if y represents a mile, then 30y represents 30 miles; and if z represents a horse, then 25z represents 25 horses. The number, which in reality multiplies the symbol, is called a **coefficient**, and the number and symbol taken together is called a **term**.

2. When we speak of one article, we usually omit the coefficient 1. Thus, instead of 1x we should write x, if we

wished to represent, say, 1 brick.

The student must understand clearly that a symbol may be used to represent anything we please. The same letter may denote a certain thing in one problem and a different thing in a different problem, but in any particular problem one symbol is always retained for one thing and a different symbol for a different thing. For example, in one problem x may stand for the number of bricks per foot run in a wall, whereas in a second problem x may stand for the pressure in lbs. per square inch of the steam in a locomotive boiler. The use of symbols is a sort of shorthand method of expressing facts.

3. Signs of Operation. To indicate that one quantity

P. M.

is equal to another we use the sign = for "equals" or "is

equal to." Thus, 3 multiplied by 2 = 6.

The operation of addition is denoted by the sign +, called the *plus* sign. Thus, 3 plus 5 is equal to 8, or 3 added to 5 is equal to 8, is written 3 + 5 = 8.

Similarly, if we use a symbol, say x, to denote a thing,

then 3x + 5x = 8x.

The operation of subtraction is denoted by the sign –, called the *minus* sign. Thus, 5 minus 3 equals 2, is written 5-3=2. Using symbols again we have 5x-3x=2x.

The operation of multiplication may be indicated in several ways. The sign commonly used is \times . Thus, 4 multiplied by 5 equals 20 is written $4 \times 5 = 20$. Four times x multiplied by 3 equals 12x is written $4x \times 3 = 12x$.

Another method of indicating the operation of multiplication, used with symbols, is to place a dot between the symbols. Thus, a multiplied by x may be written $a \cdot x$. This should not be used with figures, for a dot is used to indicate the decimal point. See Art. 20.

Two or more letters written together with no sign between them indicates that they are to be multiplied together. Thus, a multiplied by b is written ab; x multiplied by a is written ab; b is a multiplied by a is written as

plied by y multiplied by z is written xyz.

The quantity to be multiplied is called the *multiplicand*, the quantity we multiply by is called the *multiplier*, and the

result is called the product.

The operation of division is indicated by the sign \div , or by placing the quantity to be divided, called the *dividend*, above the quantity by which it is to be divided, called the *divisor*. Thus, 4 divided by 2 equals 2 is written $4 \div 2 = 2$

or $\frac{4}{2} = 2$; 8x divided by 2 equals 4x is written $8x \div 2 =$

4x, or $\frac{8x}{2} = 4x$. The result of dividing one quantity by another is called the *quotient*. In the above examples, 2 and 4x are quotients.

4. If any arithmetical quantity, or an algebraic term, is multiplied by an equal quantity or term it is said to be

squared, or raised to the second power.

Thus, $3 \times 3 = 3$ squared. $x \times x = x$ squared.

Arithmetically we can represent the value of 3 squared by a single number, which is the product of 3×3 , i.e., 9. In the case of an algebraic symbol we cannot do this, so instead of writing x squared as the product x..x, we write it as x^2 , which means that two equal quantities x have been multiplied together. The number 2, which denotes that the symbol x has been multiplied by x, or squared, is written above and slightly to the right of the symbol x, and it is called an **index**.

This process is extended to eases in which more than two equal arithmetical or algebraic quantities are multiplied together.

Thus, $3 \times 3 \times 3$ is called 3 **cubed** or 3 raised to the **third power**. 3 cubed may be written 3^3 or 27.

Similarly $x \times x \times x = x$ cubed, or x raised to the third power. It is written x^3 , the index 3 showing that three equal quantities x are multiplied together.

Also $3 \times 3 \times 3 \times 3 = 3$ raised to the **fourth power**, which is written 3^4 or 81. $x \times x \times x \times x = x$ raised to the fourth power, written x^4 ; and so on.

An algebraic term consisting of a number and a symbol, such as 3x, can be raised to any power.

Thus, 3x squared means 3 squared multiplied by x squared. This $= 3 \times 3 \times x \times x$, and it may be written $3^2 \cdot x^2$. We should, however, multiply out the square of 3 and we then get $3 \times 3 \times x^2 = 9x^2$, which is read as "nine x squared."

Similarly 3x raised to the third power, or cubed, is $3 \times 3 \times 3 \times x \times x \times x = 27x^3$.

Also 3x raised to the fourth power = $3 \times 3 \times 3 \times 3 \times x \times x \times x \times x \times x = 81x^4$.

The index corresponds to the power to which the symbol is raised and also to the number of such symbols multiplied together. Thus, the square or second power has an index 2, the cube or third power an index 3, and so on. There is no limit to the power to which a symbol may be raised.

The operation of raising quantities or terms to given powers is called **involution**.

Any quantity or term raised to the *first* power is represented by the quantity or term itself; thus, x raised to the first power written fully would be x^1 , but this has the same value as x, hence x and x^1 represent the same thing.

5. If a given number or algebraic term can be produced

by multiplying together two or more equal numbers or algebraic terms, then each of the equal numbers or terms is said to be a **root** of the given number or term. Thus, $9 = 3 \times 3$; hence 3 is a root of 9. Again, $x^3 = x \times x \times x$; hence x is a root of x^3 .

The process of extracting roots of given quantities or terms is denoted by the sign $\sqrt{\ }$ placed in front of the quantity or term to be operated upon. Thus, $\sqrt{\ }$ means the root of 9. $\sqrt{\ }x$ means the root of x. Now we may require the square root of, say, 9—that is, a number which multiplied by an equal number will give 9 as product; or we may require the cube root, say, of 27—that is, a number which multiplied by two equal numbers will give as product 27; or we may require the fourth root, say, of 81—that is, a number which multiplied by three equal numbers will give as product 81; and so on. In order to indicate which root is required, a small index figure corresponding to the required root is written before the root sign, thus:—

 $\sqrt[3]{9}$ means the square or second root of 9, and since $3 \times 3 = 9$, this root is 3.

 $\sqrt[3]{27}$ means the cube or third root of 27, and since $3 \times 3 \times 3 = 27$, this root is 3.

 $\sqrt[4]{81}$ means the fourth root of 81, and since $3 \times 3 \times 3 \times 3 = 81$, this root is 3.

It is usual to omit the index figure 2 for square root, and to use the root sign only, thus, $\sqrt{9}$ and $\sqrt[2]{9}$ each mean the square root of 9.

The operation of extracting roots of quantities or terms

is called evolution.

6. Algebraic addition, multiplication, and division. The chief difference between an algebraic quantity and an arithmetical quantity is that whereas in arithmetic we always deal with positive quantities, in algebra a quantity may be positive or negative, and we must always think of a quantity as having both sign and magnitude.

For example, suppose that a man has 50 pounds in gold and 50 shillings in silver, and, further, that he owes one man a debt of 20 pounds, and a second man a debt of 15 shillings. This states his position arithmetically, and we could easily calculate how much he

would have left if he paid his debts. Now, if we denote each pound by x, and each shilling by y, then the man has 50x and 50y. He owes 20x to one man, and 15y to the other. We will now denote money which the man possesses by a plus sign, and money which he owes by a minus sign. +50x and +50y then represents algebraically the money he possesses; -20x and -15y represents the money he owes. Suppose now that the man pays the debts; he then has left 30 pounds and 35 shillings, and he owes nothing. We represent this result as follows: +30x + 35y (since the plus sign denotes money possessed), and this result is called the **algebraic sum** of the quantities given, that is, the algebraic sum of +50x + 50y - 20x and -15y is +30x + 35y.

Now take a case in which the debts exceed the money possessed. A man has 20 pounds and he owes 30 pounds. If, as before, we call each pound possessed +x and each pound owing -x, then an

algebraic statement of his position is +20x and -30x.

If he pays the debt as far as he is able to do so, it will be seen that he pays away the whole 20 pounds and still owes 10 pounds. This is represented algebraically as -10x, the minus sign indicating that the result is 10 pounds still owing and not 10 pounds possessed. Thus, the algebraic sum of +20x and -30x is seen to be -10x.

While the above illustration shows the difference between plus and minus quantities, it must be borne in mind that it is only an

illustration.

7. An algebraic sum is by no means confined to similar examples; the chief point to be observed is that an algebraic sum may result in a positive quantity, or a negative quantity.

We can arrive at these results without reference to the illustration given above.

Ex. 1. Find the algebraic sum of +50x, +50y, -20x, -15y. Arrange the x terms in one column and the y terms in a separate column, and in each case take the sign with the term.

We thus get
$$+\frac{50x}{-20x} + \frac{50y}{-15y}$$

Now subtract the coefficient of the smaller term containing x (that is, 20) from that of the greater term (that is, 50), the difference is 30; we give to this result the same sign as that of the greater term, in this case a + sign, and the result is thus written + 30x.

Similarly, for the quantity containing y, we get +35y, and hence the algebraic sum is +30x + 35y.

Ex. 2. Find the algebraic sum of -30x and +20x.

Following the above rule we have
$$\frac{-30x}{+20x}$$

$$-10x$$

The smaller term 20x subtracted from the greater term 30x gives

as difference 10x, and since the greater term has the minus sign we attach the minus sign to the result.

The student must bear in mind that in adding and subtracting algebraic terms, the symbols must not be left out of the result. Thus 50x - 30x = 20x, not 20, as is commonly written by beginners; for, suppose x = 2, then 50x = 100 and 30x = 60, and 50x - 30x = 100 - 60 which = 40. Hence, to write 50x - 30x = 20 would be incorrect.

Ex. Find the algebraic sum of the following terms:
$$+12x$$
, $-15y$, $+4z$, $-15x$, $+13y$, $-7z$, $+14y$, $-18x$, $+11z$, $+8x$, $-3z$, $-4y$.

First arrange the terms with their signs in columns as below, placing the x terms, y terms, and z terms in separate columns thus:—

We will find the result, first, by an illustration. Suppose each letter x represents a shilling, each letter y a frane, and each letter z a dollar, and the plus sign represents money possessed, and the minus sign money owing. Then, taking the shillings, we find that a man has 12 shillings and 8 shillings, total 20 shillings (+20x), while he owes 15 shillings and also 18 shillings, total debt 33 shillings (-33x). The nett result is that he still owes 13 shillings, and this we represent by -13x.

Treating the francs similarly, we find that the man has 13 francs and 14 francs, total 27 francs (+27y), and that he owes 15 francs and 4 francs, total debt 19 francs (-19y). The nett result is that he has 8 francs left after paying the debt, and this we represent by +8y.

Treating the dollars similarly, he possesses 4 dollars and 11 dollars, total 15 dollars (+ 15z), while he owes 7 dollars and 3 dollars, total debt 10 dollars (- 10z). The nett result is that he has 5 dollars left after paying the debt, and this we represent by + 5z.

The total result is now seen to be

$$-13x + 8y + 5z$$
.

or, 13 shillings owing, and 8 francs and 5 dollars in hand.

If we now place this result beneath the columns, we get

 $\begin{array}{ccccc}
+ 12x & - 15y & + 4z \\
- 15x & + 13y & - 7z \\
- 18x & + 14y & + 11z \\
+ 8x & - 4y & - 3z
\end{array}$

Applying the algebraic method without the illustration, we arrive at the same result.

Thus, adding together the x terms which have a plus sign we get

Subtract and attach sign of the larger total and we get .
$$-13x$$

Algebraic sum as before -13x + 8y + 5z.

The second method given above is the one to be adopted in practice.

8. It must be remembered that to any term you can only add other like terms, if the result is to be given as a single term. As an illustration we could say that the sum of 6 bricks, 7 bricks, and 2 bricks is 15 bricks; and similarly, we can represent the sum of 6x, 7x, and 2x as 15x; but we cannot represent the sum of 6x, 7y, 2z as a single term, any more than we can represent the sum of 6 bricks, 7 screws, and 2 bicycles as a single total.

The sum of 6x, 7y, and 2z, is written +6x + 7y + 2z, and is called an **algebraic expression**, that is, a series of algebraic terms connected by signs.

Two or more terms can be regarded as like terms when they contain like symbols with like indices, and hence are of equal value.

Thus, x and x^2 are not like terms, for one is simply x whereas the other is $x \times x$. If x = 2, then one term = 2 and the other 4, and hence they are not alike in value. Similarly a^2 and a^3 are not alike, neither are such terms as x^2y and xy^2 , a^2b^3 and a^3b^2 , and so on.

The order in which the symbols occur does not matter—for example,

abe has the same value as acb; if a=2, b=3, c=4, then $abc=2\times 3\times 4$, and $acb=2\times 4\times 3$, and in each case the product is 24. Further examples of like terms are a^3b^2c and b^2ca^3 ,

also $x^2y^2z^3$ and $y^2x^2z^3$.

Further examples of unlike terms are

 $a^{3}b^{2}c$ and $a^{2}b^{3}c$, also $x^{2}y^{2}z^{3}$ and $x^{3}y^{2}z$.

A simple test as to whether two terms are, or are not, alike in value, is to write each term out fully without indices; thus, by writing a^2b^3 as $a \times a \times b \times b \times b$ it is seen to be unlike a^3b^2 , which becomes $a \times a \times a \times b \times b$. Again, writing $a^2b^2c^3$ as $a \times a \times b \times b \times c \times c \times c$ it is seen to be like or equal to $a^2c^3b^2$, which becomes $a \times a \times c \times c \times c \times b \times b$.

9. Every algebraic term is made up of a sign, a coefficient, a symbol, and an index; for example, in the term $-3x^2$ we have a sign -, a coefficient 3, a symbol x, and an index 2. This term is spoken of as "minus three x squared."

If the coefficient is 1, it is omitted; thus, $-x^2$ is really

 $-1x^2$, "minus one x squared."

If the index is 1, it is omitted; thus, -3x is really $-3x^1$, "minus three x to the first power."

If the sign is plus, it is omitted when the term stands alone, or when at the beginning of an expression; thus,

 $3x^2$ is really $+3x^2$, "plus three x squared."

It will thus be seen that when we write simply a symbol x we really mean $+1x^1$, "plus one x to the first power," but in accordance with the above statements the sign, coefficient, and index are omitted.

When we write -x, we really mean $-1x^1$, "minus one x to the first power." Here we omit the coefficient and index, but not the sign.

Similarly, $-x^2$ means $-1x^2$, the coefficient being omitted; also -3x means $-3x^1$, the index being omitted. Thus, an expression which when written fully is

$$+1x^{1}-3x^{2}+1x^{3}-2y^{2}$$

would in practice be written

$$x = 3x^2 + x^3 - 2y^2$$

In the following examples the algebraic sum of each column of terms is shown separately on the right.

Ex. 1. Find the algebraic sum of

Ex. 2. Find the algebraic sum of 7x - 10y + 12z, -4x + 6y - 8z, -3x + 2y - 4z.

Ex. 3. Find the algebraic sum of $2a^3b^2 - 4a^2b^3$, $-6a^3l^2 + 7a^2b^3$ and $-11b^3a^2$.

Exercises.

Find the algebraic sum of—

- (1) 3a 5b + 8x 15y 9a + 11b 12y 15b + 17x.
- (2) 4x 7y 12z + 11x + 15y + 13z 14x + 10y.
- (3) a + 4b c 6d 14a + 11b + 7d 7b + 11c 12a. (4) $x^2 + 3y^2 - 10x + 11y - 4x^2 + 2y^2 - 15y + 7x^2 - 3y^2$
- $\begin{array}{c} (3) \ 3a^2 + 2b^2 3a^2 + 11a + 12b 3a^2 + 3b^2 3a + 1b + 3b^2 \\ + 4a 10b + 11c^2 12b^2 + 4a^2 7a + 8b. \end{array}$
- (7) $a^2b + ab^2 4ab + 4a^2b 6ab + 4ab^2 8a^2b + 9ab^2 + 11ab$.
- $\begin{array}{l} + (8) x^2 + y^2 + a^2x + by^2 + 3x^2 5a^2x + 6y^2 + 7by^2 \\ + 8a^2x 10y^2. \end{array}$
- (9) 5abc + 8ab + 11ac + 12bc 4abc + 11ab 5ac + 8bc 5abc + 6ac 8bc + 4ab 5ac 10bc + abc.
- (10) $12a^2x^3 + 7a^2x^2 + 8ax 5a^2x^2 + 8a^2x^8 20a^2x^3 2x^2x^2 8ax$.

Answers.

$$\begin{array}{llll} (1) & -6a - 9b + 25x - 27y. & (7) - 3a^2b + 14ab^2 + ab. \\ (2) & x + 18y + z. & (8) & 2x^2 - 3y^2 + 4a^2x + \\ (3) & -25a + 8b + 10c + d. & 8by^2. \\ (4) & 4x^2 + 2y^2 - 24x + 8y. & (9) & 23ab + 7ac + 2bc - \\ (5) & 3a^2 - 7b^2 + 13c^2 + 3a & 3abc. \\ & + 17b. & (10) & 0. \end{array}$$

(6) 9xy + 10a - 6x. 10. **Multiplication**. Any al

10. **Multiplication.** Any algebraic term may be multiplied by a number; for example, 3x multiplied by 3 will be equal to 3x + 3x + 3x or = 9x.

As an illustration, if x represents 1 pound, then 3x represents 3 pounds, and the product, when 3 pounds is multiplied by 3, is 9 pounds or 9x. Similarly, $2x^2 \times 4 = 2x^2 + 2x^2 + 2x^2 + 2x^2 = 8x^2$; $4x^3 \times 5 = 20x^3$; and so on,

An algebraic expression may be multiplied by a number as illustrated by the following example: Multiply 3x - 5y + 10z by 5.

If we think of +3x as 3 pounds in pocket, -5y as a debt of 5 francs, and +10z as 10 dollars in pocket, and also that there are five persons whose financial positions are so represented, then the total number of pounds possessed by the five people is $5 \times 3 = 15$ pounds, or +15x, the total debt in francs is $5 \times 5 = 25$ francs, or -25y, and the total number of dollars is $10 \times 5 = 50$ dollars, or +50z; hence the product is 15x - 25y + 50z.

It will be seen that this product is obtained by multiplying the coefficient of each term in the expression by the multiplier 5.

When a term consisting of a symbol with an index, such as x^3 , is multiplied by a similar term, such as x^2 , the product is obtained as follows:—

$$x^3 = x \times x \times x$$
, and $x^2 = x \times x$;
hence $x^3 \times x^2 = x \times x \times x \times x \times x \times x$ which $= x^5$ by Art. 4.

If we call x^3 the multiplicand and x^2 the multiplier, we see that the sum of the indices of the multiplicand and multiplier (viz., 3 + 2 = 5) gives the index of the product.

Thus,
$$a^2 \times a^4 = a^{4+2} = a^6$$

 $y^3 \times y^4 = y^{3+4} = y^7$
 $y^4 \times y^4 = y^4 + y^5$ and so on.

The product of a term consisting of a coefficient, symbol, and index, such as $5x^2$, and a similar term, such as $3x^3$, is equal to $5 \times x \times x$ multiplied by $3 \times x \times x \times x$, which is equal to $15 \times x \times x \times x \times x \times x \times x$ or $15x^5$. Thus, the coefficient 15 of the product is the product of the coefficients 5 and 3 of the multiplicand and multiplier, and the symbol x^5 of the product is the product of the symbols x^2 and x^3 of the multiplicand and multiplier.

Similarly,
$$2x^2 \times 3x^4 = 6x^{2+4} = 6x^6$$

 $5a^4 \times 6a = 30a^{4+1} = 30a^5$
 $3y^3 \times 4y^2 = 12y^{3+2} = 12y^5$,

We have still to investigate the rules for the sign of the product. The following cases will occur:—

(1.) Signs of multiplier and multiplicand both + (plus).

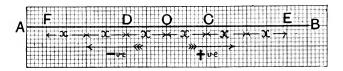


Fig. 1.

- (2.) Signs of multiplier and multiplicand both (minus).
- (3.) Signs of multiplier and multiplicand different, one + and the other -.

A graphical illustration will serve for this investigation. Suppose we are at a point O in a road AB, Fig. 1. Consider distances measured to the right (towards B from O) as positive, and distances measured to the left (towards A from O) as negative. Let us denote a distance of two miles by a length x. Then if C and D are each two miles from O but in opposite directions, OC = +x, and OD = -x. If we now walk towards B a distance equal to six miles, or three times the distance x, we arrive at E; hence OE = x + x + x or 3x. Similarly, a distance equal to three times the distance x towards A is OF, a distance of three times -x, or -3x. Thus, the product of +3 and +x is OE = +3x, and the product of +3 and -x is OF = -3x.

Now if we wish a person to walk six miles towards B, we can indicate this distance as a distance = three times +x from O or +3 times a distance x from O; if we wish the person to walk a similar

distance towards A from O, we can indicate this distance as a distance = three times -x from O, or we could say walk a distance -3 times the distance x from O. In the latter case, a distance +x when multiplied by -3 gives a distance -3x; thus multiplying by -3 instead of +3 changes the direction only, and does not change the distance from O. Hence a distance -3 times -x is opposite in direction to +3 times -x. We already know that +3 times -x is a distance oOF from O; hence -3 times -x is a distance equal to OF from O, but opposite in direction, i.e., it is equal to OE, which is +3x. Thus, $-3 \times -x = +3x$.

If we collect these results we find that

(1.)
$$+ 3$$
 multiplied by $+ x = + 3x$.
(2.) $+ 3$,, $-x = - 3x$.
(3.) $- 3$,, $+ x = - 3x$.
(4.) $- 3$,, $-x = + 3x$.

And from these results we deduce the following rules for the sign of a product:—

Rule 1. Like signs, whether + or -, when multiplied together, give + as the sign of the product, as in (1.) and (4.) above.

Rule 2. Unlike signs give — as the sign of the product, as in (2.) and (3.) above.

 $Ex. 1. + 5x^2$ multiplied by $-3x^3 = -15x^5$ (as signs are unlike, therefore product is minus; coefficients are 5 and 3, therefore coefficient of product is $5 \times 3 = 15$; indices are 2 and 3, therefore index of product is 2 + 3 = 5).

Ex. 2.
$$-3x \times 2x^2 = -6x^3$$
 (written fully $-3x^1 \times 2x^2 = -6x^{1+2} = -6x^3$), Ex. 3.
$$-x \times 2x^5 = -2x^6$$
 (written fully $-1x^1 \times +2x^5 = -2x^{1+5} = -2x^6$), Ex. 4.
$$x^3 \times x^3 = x^6$$
 (written fully $+1x^3 \times +1x^3 = x^{3+3} = +x^6$). Ex. 5.
$$-x^3 \times -x^3 = x^6$$
 (written fully $-1x^3 \times -1x^3 = x^3 + x = +x^6$). Ex. 6.
$$+x^3 \times -x^3 = -x^6$$

11. If the terms to be multiplied together do not contain the same symbols, then we cannot add together the index figures for the product.

We proceed as follows:-

The product of +a and +b is written ab, which, as we have already seen, denotes a multiplied by b.

The product of +a and -b is written -ab, following the rule for the sign of a product.

If there are coefficients with different symbols in the two terms, we multiply together the coefficients and write down all the symbols with their indices, thus:—

$$\begin{array}{rcl}
2a \times 3b &= 6ab \\
-2a \times 3b &= -6ab \\
-3a^2 \times 4b^2 &= -12a^2b^2
\end{array}$$

If there are one or more of the same symbols in each term, and other different symbols, we multiply together the coefficients, add the indices of the similar symbols, and write down all other letters with their indices, thus:—

which example written out fully is
$$-2 \times 3 = -6 \text{ for coefficient of product,}$$

$$-2 \times 3 = -6 \text{ for coefficient of product,}$$

$$a^2 \times a^3 = a^5 \text{ for product of } a^2 \text{ and } a^3.$$

Then $-6 a^5$ must be multiplied by b^3 , and we get $-6a^5 b^3$, and this in turn is multiplied by e^4 , which gives $-6a^5 b^3 e^4$.

In the following examples, each term in the product is shown worked out separately.

Ex. 1. Find the product of
$$3a^2 + 5ab - 6c^2$$
 and $5a^3$.

Ex. 2. Find the product of $a^2b^3 - 2b^4c^5 + 3c^3d^2 - 2a^2bc^2d$ and $-3a^2b^2c^3$.

$$\begin{array}{c} a^2b^3 - 2b^4c^5 + 3c^3d^2 - 2a^2bc^2d \\ -3a^2b^2c^3 \\ \hline -3a^4b^5c^3 + 6a^2b^6c^8 - 9a^2b^2c^6d^2 + 6a^4b^3c^5d \\ \hline +1a^2b^3 & \times -3a^2b^2c^3 = -3x^4b^5c^3 \\ -2b^4c^5 & \times -3a^2b^2c^3 = +6a^2b^6c^8 \\ +3x^3d^2 & \times -3a^2b^2c^3 = -9a^2b^2c^6d^2 \\ -2a^2bc^2d & \times -3a^2b^2c^3 = +6a^4b^8c^5d \end{array}$$

Ex. 3. Find the product of
$$x^2 + 2xy + y^2$$
 and -1 .

Exercises.

Find the product of—
(1) a + b - c and 3a.
(2) x + y - z and 2x.
(3) x - y + a - b and -3x.
(4) a + b - c + x - y and 2ab.
(5) $a^2x + b^2y - c^2z$ and ax^2 .
(6) $a^2b^2x + c^2d^2y - a^2c^2z$ and a^3b .
(7) 8a + 9b - 11c and $-7a^2$.
(8) $5a^2b + 6b^2c^2 - 11a^2c^2$ and $-6a^3b^2c$.
(9) $6x^3y^2z + 5x^2y^2z^3 - 11xyz$ and $4x^2y^3z^4$.
(10) -a - b - c - x - y - z and $-4a^3x^2z^3$.

(11) $-x + y^2 - z^2 + abc$ and $-4x^2yza^2b$. (12) x squared plus 3y squared is multiplied by three times the product of a squared, y cubed, and z. Write this statement down

algebraically and find the complete product.

(13) Write down algebraically and find the product of: The sum of x squared and the cube of y multiplied by five times the product of the square of a, the fourth power of b, the square of x, and the square of y.

Answers.

12. **Division.** The process of dividing an algebraic expression by a number can be illustrated by again thinking of money possessed and owing.

Suppose a man receives 504 pounds each year, but that he has one debt of 60 francs and a second debt of 84 dollars each year to pay, and he wishes to spread the debt and the income evenly over the year. We can represent the 504 pounds as +504x, the 60 francs debt as -60y, and

the 84 dollars debt as -84z. His yearly account is then +504x-60y-84z. Each month he should receive one-twelfth of the 504 pounds, and should pay one-twelfth of each debt, thus:—

504x - 60y - 84z divided by 12 will be 42x - 5y - 7z, and this represents his account each month.

In this example it will be seen that the coefficient of each term in the dividend is divided by the divisor to obtain the coefficients of the terms in the quotient. The sign and the symbol of each term remains the same in dividend and quotient.

To divide a symbol having an index, such as x^3 , by a like symbol having a different index, such as x^2 , let us consider x = 2. Then $x^3 = x \times x \times x = 2 \times 2 \times 2 = 8$.

Also $x^2 = x \times x = 2 \times 2 = 4$, and thus $x^3 \div x^2$ is $= \frac{x \times x \times x}{x \times x}$ or $= \frac{2 \times 2 \times 2}{2 \times 2}$ or $= \frac{8}{4}$ which = 2 or x.

Now if we subtract the index of the divisor from that of the dividend to get the index of the quotient we get $\frac{x^3}{x^2} = x^{3-2} = x^1$, which agrees with our former result. Hence, to divide one symbol by a like symbol we subtract the indices.

Thus,
$$a^5 \div a^2 = a^{5-2} = a^3$$

 $x^7 \div x^3 = x^{7-3} = x^4$
 $y^4 \div y = y^{4-1} = y^3$

The rules for dividing a term, such as $15a^5x^3$, by a term having the same sign but a different coefficient and indices, such as $3a^3x$, are (1) for the coefficient of the quotient divide the coefficient of the dividend by that of the divisor; (2) for the indices of the quotient subtract the index of each symbol in the divisor from the index of the *like* symbol in the dividend, thus:—

$$15a^5x^3 + 3a^3x$$
 gives $\frac{15}{3} = 5$ as coefficient; $\frac{a^5}{a^3} = a^{5-3} = a^2$,

and
$$\frac{x^3}{x} = x^{3-1} = x^2$$
 as the symbols; hence $\frac{15a^5x^3}{3a^3x} = 5a^2x^2$.

Rule for Signs in Division.

When the signs of the dividend and divisor are atike, the sign of the quotient is +; when they are unlike, the sign of the quotient is -.

Thus,
$$\frac{+15x}{+3} = +5x$$
; $\frac{-15x}{-3} = +5x$
 $\frac{+12x}{-2} = -6x$; $\frac{-12x}{2} = -6x$.

This rule can be proved by multiplying quotient and divisor together, and comparing the result with the dividend. The result should be equal to the dividend.

The result of dividing any symbol by a different symbol for example, dividing b by x, we can only represent as $\frac{b}{r}$.

If there are symbols in the dividend which have no corresponding symbol in the divisor, then such symbols appear in the quotient unaltered.

Thus,
$$\frac{15x^3yz}{5x} = 3x^2yz$$
$$\frac{12a^2b^3e^2}{3ac} = 4ab^3c.$$

If we have any symbol with an index in the dividend, and a corresponding symbol and index in the divisor, then these symbols, when divided, give as quotient 1; thus, $\frac{c^2}{c^2} = 1$, just as $\frac{3}{3} = 1$.

Hence
$$\frac{12a^3b^2c}{4abc} = 3a^2b$$
 since $\frac{c}{c} = 1$.

13. By the rule for division of one term by another, we can show that any quantity raised to the power zero is equal to 1.

Since any quantity, say x^2 , divided by an equal quantity is equal to 1, therefore $\frac{x^2}{x^2} = 1$.

Now by our rule for division we subtract the indices of

dividend and divisor and take the difference as the index of the quotient; thus, $\frac{x^2}{x^2} = x^{2-2} = x^0$. But we know that $\frac{x^2}{x^2} = 1$, hence x^0 must = 1.

The following examples illustrate the rules for division:—

$$\frac{6a^{2}x}{3^{-}} = 2a^{2}x. \qquad \frac{-6a^{3}x^{2}}{2ax} = -3a^{2}x.
\frac{-6a^{2}x}{-3a} = 2ax. \qquad \frac{12a^{5}b^{4}r^{3}d}{-4abcd} = -3a^{4}b^{3}c^{2}.
\frac{6a^{2}x}{3ax} = 2a. \qquad \frac{12a^{3}b^{2}x^{2}}{4ax} = 3a^{2}b^{2}x.$$

Exercises.

(1) Divide $4x^2y^2$ by 2xy; and $15a^3b^2c$ by $3a^2c$.

(4) Simplify
$$\frac{16a^3x^3z^2}{8a^2xz^2}$$
; and $\frac{8a^3x^2b^3z}{2a^2xb}$

(1) Divide
$$4x^2y^2$$
 by $2xy$; and $15a^2b^2c$ by $3a^2c$.
(2) Divide $-6a^2b^2c$ by $2ab$; and $12x^3y^3z^3$ by $-4xy^2z$.
(3) Divide $12x^3y^2a^2b$ by $-3xya^2z$; and $-21a^4x^2y^2$ by $-3xa^3y$.
(4) Simplify $\frac{16a^3x^3z^2}{8a^2xz^2}$; and $\frac{8a^3x^2b^3z}{2a^2xb}$.
(5) Simplify $\frac{x^3y^2z}{y^2zx^3}$; and $\frac{15a^4b^3x^2}{3a^2x^2 \times 5a^2b^3}$.

(6) Find the result of dividing the product of eight, the square of x, the cube of y, and the fourth power of z, by minus four times the product of x, the square of y, and the cube of z.

(7) What is the quotient when the dividend is $-12a^3b^2x^3y^2z$ and

the divisor is $-3a^2bx^2yz$.

(8) Divide $15a^2b^2xy^2$ by 3abxy and divide $-20a^3b^4x^2y^3$ by $4a^2b^3x^2y^2$, and find the algebraic sum of the quotients.

Answers.

(1) 2 xy; $5ab^2$.

- (5) 1; 1.
- $\begin{array}{l} (2) 3abc; 3x^2y^3z^2. \\ (3) 4x^2yb; 7axy. \end{array}$
- (6) -2xyz. (7) 4absy.

(4) 2ax2; 4axb2z.

(8) 0.

14. Numerical value of a term or expression. Since each of the symbols in an algebraic term or expression can be represented by a number, the numerical value of the term or expression can be found by substituting for each symbol its numerical value, carrying out the various operations indicated by the signs, and finding the algebraic sum of the numbers thus obtained. The following examples illustrate the method:—

Ex. 1. Find the value of
$$3xyz$$
 when $x = 2$, $y = 3$, $z = 4$, $3xyz = 3 \times 2 \times 3 \times 4$ = 72.

Ex. 2. Find the rather of $4ax \rightarrow 3ay + 5xy - 3x^2 + y^2$ when a = 2, x = 4, y = 5.

$$4ax = 4 \times 2 \times 4 = 32$$

 $3ay = 3 \times 2 \times 5 = 30$
 $5xy = 5 \times 4 \times 5 = 100$
 $3x^2 = 3 \times 4 \times 4 = 48$
 $y^2 = 5 \times 5 = 25$

Hence the given expression becomes 32 - 30 + 100 - 48 + 25= 157 - 78= 79.

Ex. 3. If a = 2, b = 3, c = 4, x = 5, y = 9, z = 16, find the numerical value of $2ab + 3a^2 + 4bc - 2xy + \sqrt{y} - \sqrt[4]{z}$.

The given expression then becomes 12 + 12 + 48 - 90 + 3 = 2 and this is equal to +75 - 92, which again = -17; hence the numerical value of the given expression is = 17.

If the expression can be simplified before the numerical values are substituted, this should be done.

For example, if we have an expression such as

$$\frac{3a^2x^2}{2ax} + \frac{5x^3y}{x^2} - 4x^3y - 3xy$$

we can simplify the expression to

$$\frac{3ax}{2} + 5xy - 4x^8y - 3xy$$

and further to $\frac{3}{5}ax + 2xy - 4x^3y$, by bringing like terms together.

If now a = 2, x = 3, and y = 4, we get

$$\frac{3 \times 2 \times 3}{2} + (2 \times 3 \times 4) - (4 \times 3^{3} \times 4)$$

$$0 + 24 - 432$$

$$33 - 432$$

$$- 399$$

Exercises ::

Find the numerical value of

- (1) 3a + 5b 8c + 11d when a = 2, b = 3, c = 4, d = 5.
- (2) $5a^2 + 3b^3 + 5c 4d^2$ when a = 1, b = 4, c = 6, d = 3.
- (3) $12x^2y 5y^3z^2 + 6a^2x^2y^3$ when a = 2, x = 4, y = 3, z = 1
- (4) $16a^3b^2x 12a^2b^3x^4 + 12a^3b^4$ when a = 1, b = 3, x = 0.
- (5) $-6a^2c 11a^3b^2 + 12ab^2c^3$ when a = -1, b = -2, c = 4.
- (6) $-12a^2x^2y^3 + 5axy^2 + 12a^3xy$ when a = 4, x = -2, y = 3.
- (7) $5a^2b^3x^2 12abx + 4ax^2 11bx^3$ when a = -2, b = 4, x = 10.
- (8) Find the numerical value of the product of x^2 and y^3 divided by the product of x and y when x = -4 and y = 6.
- (9) Find the numerical value of $\frac{4x^3y^2z}{2xy}$ when x = 2, y = 4, and z = 10.
- (10) Multiply $3a^3x^2y$ by $-8ax^3y^2z$ and divide the product by $-6x^4a^2y^2z$ and find the numerical value if a=3, x=-2, and y=4.
- (11) Find the numerical value of 7a 8b + 10x 15y 5a + 7x 9b when a = 3, b = 4, x = 6, y = 9.
- (12) Find the numerical value of $a^2b^3 bc^2 + cd a^3b^2c + \sqrt{b^2a}$ when a = 4, b = 3, c = 7, d = 0.
- (13) What is the value of $4a^2b^3 3a^3x^2 + 6abx$, when a = 2, b = -3, x = -1.
- (14) Find the product of 2ab and $-3x^2ab^2$, and divide the result by 6abx, and find the numerical value if a = -2, b = -1, x = 3,
- (15) If x = 2a and y = -3b, find the value of $5x^2y^3$, and the numerical value when a = 2, b = -5.
- (16) Express algebraically five times the square of x multiplied by four times the cube of y, the product divided by the cube of x multiplied by minus twice the square of y.
- (17) Write out an expression for the sum of fice times the product of x squared and minus y, three times the cube of x, seven times the product of minus a and b, and nine times the quotient of x squared and minus x.

Answers.

- (1) 44.
- (2) 191. (3) 10809.
- (4) 972.
- (5) 3052.
- (6) 25704. (7) 84160.
- (8) 144.
- (9) 320.

- . (10) 288.
 - (11) 95.
 - (12) 3741.
 - (13) 420.(14) - 6.
 - (15) 270,000.
 - $(16) \ \frac{20x^2y^3}{-2x^3y^2} = -\frac{10y}{x}.$
 - $(17) 5x^2y + 3x^3 7ab 9x$

CHAPTER II.

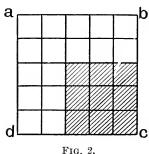
TO VULGAR AND DECIMAL INTRODUCTION FRACTIONS.

15. A numeral which expresses an exact number of units is called a whole number or an integral number.

Thus, if the unit is 1 mile, then 5 miles and 10 miles are

respectively 5 times and 10 times the unit.

When the unit is divided into any number of equal parts, and one or more parts (less than the number into which



it is divided) are taken, the expression which represents the number of parts taken is called a fraction. we take a square a, b, c, d, Fig. 2, as unit and divide it into 25 squares as shown, then each square represents one-twenty-fifth of a unit, and the shaded part will represent nine-twenty-fifths of the unit. These results are written respectively $\frac{1}{2.5}$ and $\frac{9}{2.5}$, and each is called a fraction.

be seen that a fraction is represented by two numbers written one above the other and separated by a horizontal line. The number below the line is that which states the number of parts into which the unit is divided, and is called the denominator. The number above the line states the number of such parts taken, and is called the numerator.

16. Fractions written in this way, that is, with a numerator and denominator, are called vulgar fractions. the denominator exceeds the numerator in magnitude, the fraction is called a proper fraction. If the numerator exceeds the denominator in magnitude, the fraction is called an improper fraction, and it may be reduced to an integral or whole number and a proper fraction by dividing the numerator by the denominator, the quotient giving the integral number, and the remainder divided by the denominator giving the proper fraction.

Thus $\frac{3}{5}$, $\frac{7}{8}$, $\frac{5}{6}$, $\frac{1}{4}$, are proper fractions,

and the latter, when reduced to integral numbers and proper fractions, become respectively $5\frac{1}{7}$, $7\frac{1}{4}$, $4\frac{3}{5}$, $9\frac{1}{11}$, which are called mixed numbers.

Instead of using a horizontal line to separate the numerator and denominator, fractions may be written as follows: $1/_2$, $2/_3$, $4/_5$.

Algebraic fractions are written in the same manner as arithmetic fractions, and have precisely the same meaning. Thus, if we divide the unit into x parts and take three of such parts, the fraction representing this result is written $\frac{3}{x}$ or 3/x. Again, if we divide the unit, say, into 25 parts and take a number of such parts represented by y, then the fraction representing this result is written $\frac{y}{25}$ or $\frac{y}{25}$. Again, we may divide the unit into x parts, and select a number of these parts represented by y; the fraction then becomes $\frac{y}{x}$ or $\frac{y}{x}$

Exercises.

(2) A square is divided into m parts, and 3, 5, x, and y parts respectively are taken. Write out each as a fraction.

17. A fraction is not altered in magnitude if we multiply or divide both the numerator and denominator by any number. Thus, $\frac{2}{3}$ is the same magnitude as $\frac{4}{6}$ or $\frac{6}{9}$, which

⁽¹⁾ Write down as fractions—two-fifteenths, seven-eighths, elevenfifteenths.

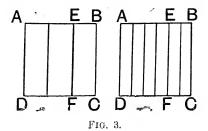
^{(3) £50} is divided amongst x people. What fraction indicates the amount that each receives? If x = 10, find the amount received by each.

fractions are obtained by multiplying the numerator and denominator of the fraction $\frac{2}{3}$ by 2 or 3.

This result may be illustrated by taking a square ABCD, Fig. 3, as unit. First divide it into three equal strips, then the area AEFD represents the fraction $\frac{2}{3}$.

Now divide it into six equal strips, the area AEFD now represents $\frac{4}{6}$. But the areas AEFD are equal in the two figures; thus we may conclude that $\frac{2}{3}$ and $\frac{4}{6}$ are equal in magnitude.

18. A fraction in which the numerator and denominator cannot both be divided by a number greater than 1 is said to be in its **lowest terms**. Thus, $\frac{2}{3}$ is in its lowest terms,



whereas $\frac{a}{9}$ is not, since 3 will divide both numerator and denominator.

One number is said to be a **factor** or measure of another number if it will divide the other number without remainder. Thus, 3 is a factor of 6 or 9, but not of 10.

A number which is exactly divisible by another number is called a **multiple** of the latter number. Thus, 6 is a multiple of 3, or 2, but not of 4 or 5.

A prime number is one which is only divisible, without remainder, by itself and by unity. Thus, 11, 17, 13, 3, 2, are prime numbers.

Two or more numbers may have a **common factor** or measure. 12, 16, and 20 have a common factor or measure 2, and another common factor or measure 4.

19. A fraction cannot be said to be in its lowest terms if its numerator and denominator have a common factor

greater than 1. To reduce a fraction to its lowest terms, divide the numerator and denominator by any common factor. and repeat the process until the only common factor left is 1.

To reduce $\frac{255}{450}$ to its lowest terms, we note that a common factor is 3; hence, dividing, we get $\frac{8.5}{1.5.3}$. common factor of 85 and 153 is 17; hence, dividing, we get $\frac{5}{9}$. The only common factor of 5 and 9 is 1; hence, $\frac{5}{9}$ is the fraction $\frac{2}{4}\frac{5}{5}\frac{5}{9}$ reduced to its lowest terms.

Ex. 1. Reduce $\frac{15}{24}$ to its lowest terms. A common factor is 3, hence $\frac{15}{24} \Rightarrow \frac{5}{8}$. Ex. 2. Reduce $\frac{51}{221}$ to its lowest terms. A common factor is 17, hence $\frac{51}{221} = \frac{3}{131}$

Exercises.

(1) Write down the following fractions:—one-ninth, two-fifteenths, five-thirty-seconds, twelve-fifty-sevenths, and seven-hundredths,

(2) Write the following fractions as equivalent fractions with 32 as denominator :-

 $\frac{1}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{16}$, $\frac{7}{8}$, $\frac{1}{4}$, $\frac{11}{16}$, and $\frac{9}{24}$.

(3) Reduce the following fractions to their lowest terms:

 $\frac{357}{408}, \frac{3811}{816}, \frac{1824}{881}, \frac{306}{1881}, \frac{276}{27408}.$ (4) Reduce the following improper fractions to mixed numbers:—

1 $\frac{1}{1}$ 83, $\frac{2}{2}$ 18, $\frac{4}{1}$ 132, $\frac{5}{1}$ 736, $\frac{2}{3}$ 7378. (5) A quantity represented by $\frac{6}{a^3}$ a^2 is divided into a number of parts represented by $2a^2x$. What is the expression representing each part?

Answers.

 $\begin{array}{c} (3) \ \ \frac{7}{8}, \ \frac{11}{16}, \ \frac{2}{3}, \ \frac{51}{314}, \ \frac{2}{571}, \\ (4) \ \ \frac{167}{4}, \ 43\frac{1}{5}, \ 375\frac{7}{11}, \ 452, \ 72\frac{18}{43}. \end{array}$

(5) 3ax.

20. Decimal fractions. A fraction in which the denominator is some power of 10 is called a decimal fraction; thus, $\frac{1}{10}$, $\frac{3}{100}$, $\frac{7}{1000}$, are decimal fractions. If we take a number such as 333, the 3 on the right represents 3 units, the middle 3 represents 30 units, and the 3 on the left represents 300 units; thus each figure has ten times the value of the figure on the right of it. If we extend this notation beyond the unit figure and use some sign to indicate definitely the position of the unit figure, then each figure to the right of the sign will represent a fraction of the unit, and the first figure will be ten times the value of the second, the second ten times the third, and so on.

To indicate the division between the integral and fractional figures a dot, (·), called the **decimal point**, is used and is placed immediately to the right of the unit figure; thus $333\cdot333$ represents $300 + 30 + 3 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$. By comparing these values, it is seen that each 3 has ten times the value of the 3 to the right of it, whether the 3 represents an integral number or a fractional one.

In a similar way, a quantity such as 364.749 represents $300 + 60 + 4 + \frac{7}{10} + \frac{4}{100} + \frac{9}{1000}$, and it is read as three hundred and sixty-four *point* (or *decimal*) seven, four, nine, not three hundred and sixty-four point seven hundred

and forty-nine.

21. A vulgar fraction or an improper fraction can be converted into a decimal fraction if it is reduced to an equivalent fraction with 10 or some power of 10 as denominator, provided we have an integral number as numerator.

Thus
$$\frac{3}{5} = \frac{\frac{1}{10}}{\frac{6}{10}} = 0.1$$

$$\frac{31}{50} = \frac{62}{100} = \frac{60}{100} + \frac{2}{100}$$

$$= \frac{6}{10} + \frac{2}{100}$$

$$= 0.62$$

$$\frac{1}{40} = \frac{2\frac{1}{2}}{100} = \frac{25}{1000}$$

$$= \frac{20}{1000} + \frac{5}{1000}$$

$$= \frac{2}{100} + \frac{5}{1000}$$

$$= 0.025.$$

Since we have 0 tenths, 2 hundredths, and 5 thousandths.

For the tenths follow the decimal point immediately.

For the first figure after the decimal point represents tenths, and the second represents hundredths.

Here we make denominator into 100 if we multiply numerator and denominator by $\frac{100}{40}$. We then get $2\frac{1}{2}$ as numerator. As we require an integral number as numerator, we again multiply numerator and denominator by 10, getting $\frac{25}{100}$.

22. The above process of conversion is somewhat lengthy. A much quicker method is adopted in practice. Consider the fraction $\frac{3}{5}$ which is shown above to be 0.6. If we add a cipher to the numerator and then divide the numerator by the denominator, we get $\frac{3.0}{5} = 6$. If when we add

a cipher to the numerator we write down the decimal point

in the quotient, we get 0.6.

This process will be found to hold good for any fraction whatever; hence, to convert any vulyar fraction into a decimal fraction, divide the numerator as far as possible by the denominator; when we can no longer divide place the decimal point in the quotient and add ciphers to the numerator. Continue the division as far as necessary.

Ex. 1. Convert 33 into a decimal fraction.

Divide 33 by 25. Quotient is 1 and remainder is 8. Add a cipher to remainder, and put decimal point in quotient, then 80 divided by 25 gives quotient 3 and remainder 5. Add another cipher to remainder, then 50 divided by 25 gives quotient 2 and no remainder, hence $\frac{33}{2} = 1.32$.

Ex. 2. Convert $\frac{1}{40}$ into a decimal fraction.

Divide 1 by 40. As we cannot do this, put decimal point in quotient and add a cipher to dividend, which then becomes 10. Divide 10 by 40. This we cannot do. Put a cipher in quotient, and add another cipher to dividend, which now becomes 100. Divide 100 by 40. Quotient is 2 and remainder 20. Add cipher to remainder, then $\frac{200}{40}$ is 5 and we have no remainder, hence $\frac{1}{10} = 0.025$.

25)33(1·32 25
80
75
50
50

40)100(·025 <u>80</u>, <u>2</u>00 200

Exercise.

Show that $\frac{1}{8} = 0.125$, $\frac{1}{4} = 0.25$, $\frac{2}{8} = 0.375$, $\frac{1}{2} = 0.5$, $\frac{5}{8} = 0.625$, $\frac{3}{4} = 0.75$, $\frac{7}{8} = 0.875$. Commit these results to memory.

23. In converting vulgar fractions into decimal fractions, the following cases will occur:—

(1) The numerator will divide by the denominator without remainder, if sufficient ciphers are added—

for example,
$$\frac{1}{4} = 0.125$$
, $\frac{3138}{12} = 209.2$.

(2) The decimal will not divide out so as to give no remainder—

for example, $\frac{29}{17} = 1.70588235294117$, etc.

These decimals are called recurring decimals.

(3) The decimal will not divide out without remainder,

but a particular series of figures, commencing with the first figure after the decimal point, will be repeated—

for example, $\frac{1}{7} = 0.142857$, 142857, 142857, etc. $\frac{1}{6} = 0.333333$, etc.

These decimals are called **repeating decimals**. The figures which repeat are only written down once, with a dot placed above the first and last of the repeating figures.

Thus, $\frac{1}{7}$ is written = 0.142857; $\frac{1}{3} = 0.3$, instead of writing out the repeating figures several times as above.

(4) The decimal will not divide out, but certain figures repeat, as in (3) above, although one or more figures immediately following the decimal point do not repeat—

for example, $\frac{31.9}{15} = 209.733333$, etc., written 209.73, $\frac{824}{2475} = 0.3329292929$, etc., written 0.3329.

These decimals are called mixed repeating decimals. 24. A result similar to that shown at (2) above, viz., $\frac{2.9}{1.7} = 1.70588235294117$, etc., is of no value for practical purposes. We know that each figure to the right of the decimal point represents a smaller fraction than the one preceding it, hence all figures beyond the fourth or fifth figures represent fractions so small that they may be discarded. In the above case, the fifth figure 8 represents $\frac{8}{1000000}$, the sixth figure 2 represents $\frac{2}{10000000}$, the seventh figure 3 represents $\frac{3}{10.000,000}$, and so on. To the nearest five decimal places, the above value is 1.70588, for the sixth figure is 2, and hence 1.705882 is nearer to 1.70588 than to 1.70589 in value. To the nearest four decimal places, the value is 1.7059, for the fifth figure is 8, and 1.70588 is nearer in value to 1.7059 than to 1.7058. In approximating these values, we increase the last figure retained by 1 if the first figure discarded is equal to, or exceeds, 5.

In many cases in practice, decimal results are only of any value to the first or second place; we must be guided

by the accuracy required in the problem in hand.

Exercises.

(1) Express 374:2968 in hundreds, tens, units, and fractions with 10, 100, etc., as denominator, as in Art. 20.

(2) Reduce $\frac{3}{8}$, $\frac{7}{16}$, $\frac{5}{32}$, and $\frac{11}{64}$ to decimal fractions.

(3) Reduce $\frac{13}{33}, \frac{19}{18}, \frac{27}{35}, \frac{10}{910}$ and $\frac{1238}{709915}$ to decimal fractions correct to four significant figures.

(4) Reduce \$\frac{514}{7}\$, \$\frac{216}{11}\$, and \$\frac{329}{27}\$ to decimal form.
 (5) Reduce \$\frac{5}{5}\$, \$\frac{43}{8}\$, \$\frac{63}{8}\$, to decimal fractions.

(6) Find the decimal which represents $\frac{1}{75}$, $\frac{2}{3}$, $\frac{1}{9}$, and $\frac{5}{4}$.

(7) What fractional part of £1 is sixpence. Express this as a decimal fraction.

(8) Express one inch as a decimal part of a foot, and also as a decimal of a yard.

(9) A sum of £50 is divided into 300 parts. What decimal parts

of £1 is each part?

(10) There are ten centimetres in one decimetre and ten decimetres in one metre. What fractional part of a metre is a centimetre and a decimetre?

(11) Express £18 17s. 3d, in pounds. (B. E. (2), 1906.)

(12) Express £0 17s, 9d. as the decimal of a pound. (B. E., 1906.) (13) Express £45 7s, 8d, as pounds and a decimal of a pound. (B. E., 1907.)

Answers.

- (2) 0·375, 0·4375, 0·15625, 0·171875.
- (3) 0.2407, 0.1538, 0.1111,
- 0.3804, 0.0016.
 - (4) 73·428571, 19·63, 12·148. (5) 3·2, 4·6, 6·375.
 - (6) 0.013, 0.6, 0.1, 0.714285.
- $(7) \frac{1}{40}$, 0 025.
- (8) 0.083, 0.027. (9) 0.16.
- (10) 0.01, 0.1.
- (11) 18·8625. (12) 0·8875.
- (12) 0 8875.
- 25. The forms of decimals set out at (1) and (2), Art. 23, are converted into vulgar fractions by writing as numerator the whole of the decimal part, and as denominator writing the numeral 1 for the decimal point and a cipher for each decimal figure.

Thus
$$\begin{array}{ll} 0.125 = & \frac{1055}{1005} = \frac{1}{8} \\ 209 \cdot 2 = & 209 \cdot \frac{2}{10} = 209 \cdot \frac{1}{5}, \\ 1.70588 = & 1 \frac{70588}{100,000} = 1 \frac{1}{17} \text{ approximately.} \end{array}$$

Note that when we convert a vulgar fraction, such as $1\frac{1}{1}\frac{2}{7}$, into a recurring decimal fraction, and then select, say, five decimal figures to represent this result, we do not get exactly back to $1\frac{1}{1}\frac{2}{7}$ when we convert the decimal fraction into a vulgar fraction. This is due to the fact that the five figures selected only represent approximately the decimal equivalent of the vulgar fraction, hence when this approximate decimal fraction is converted to a vulgar fraction we only get approximately the original value,

26. Repeating decimals, as in (3), Art. 23, are converted into vulgar fractions by putting the repeating figures in the numerator, and by putting a numeral 9 for each repeating figure in the denominator.

Thus,
$$0.3 = \frac{3}{9} = \frac{1}{3}$$
.
 $0.142857 = \frac{142857}{999999} = \frac{1}{7}$.

27. Mixed repeating decimals, as in (4), Art. 23, are converted into vulgar fractions, as follows:—

The numerator is = (whole of decimal figures - non-

repeating figures).

The denominator has a numeral 9 for each repeating figure, and a cipher for each non-repeating figure.

Thus
$$0.33\dot{2}\dot{9} = \frac{(3329 - 33)}{9900}$$

$$= \frac{3296}{9900} = \frac{824}{2475}.$$
Also
$$209.7\dot{3} = 209 \frac{(73 - 7)}{90} = 209 \frac{66}{90} = 209 \frac{11}{15}.$$

$$= \frac{3146}{15}.$$

Exercises.

- (1) Convert 46:25, 3:786, 0:0297, 0:435, 0:0002, 0:025, and 0:75 to vulgar fractions or mixed numbers.
 - (2) Convert 0.0628, 0.375, 0.285714, 0.6, 0.3694, to vulgar fractions.

(3) Find the value of 0.142857 of a guinea.

- (4) Find the value of 0.3 of 12, and 0.6 of twelve shillings.
- (5) Write 0.367294158 correct to four, five, six, seven, and eight decimal figures.

Answers.

- (1) $46\frac{1}{4}$, $3\frac{399}{500}$, $\frac{297}{10000}$, $\frac{87}{200}$, $\frac{1}{5000}$, $\frac{1}{40}$, $\frac{3}{4}$.
- (2) $\frac{311}{4950}$, $\frac{125}{335}$, $\frac{2}{7}$, $\frac{2}{3}$, $\frac{1829}{4950}$. (3) 3s.
- (4) 4; 8s.

2

(5) 0.3673, 0.36729, 0.367294, 0.3672942, 0.36729416.

CHAPTER III.

MEASUREMENT OF LENGTH; SUPERFICIAL AND VOLUMETRIC MEASURE.

28. In order to measure the length of any line, straight or curved, we must first select some *definite length* as a *standard* to which we can compare the line we wish to measure. Any definite length can be used as a standard of measure, and such length, whether it be comparatively great or small, must of necessity, so long as it is being used as a standard of measure, be *unity* (or *one*).

29. When we have a standard of linear measure, we can apply it to any line, and see whether such line is less than our measure, or, if greater, how many times our measure is contained in the line; we can express by whole or fractional numbers the length of the line in terms of the

standard unit.

The standard unit of linear measure in Britain is the yard. The yard is divided into three equal parts, each of which is called a foot. The foot is divided into twelve equal parts called inches. The inch may be subdivided into any number of equal parts. Particular names are also given to certain magnitudes, each of which contains a certain number of yards. In Table A, the linear measures most commonly used in Britain are given.

Linear Measure.

Table A.	Table B.
12 inches = 1 foot.	12 inches = 1 foot.
3 feet = 1 yard.	3 feet = 1 yard.
$5\frac{1}{2}$ yards = 1 pole.	22 yards = 1 chain.
40 poles = 1 furlong.	10 chains = 1 furlong.
8 furlongs = 1 mile.	8 furlongs = 1 mile.
3 miles = 1 league.	3 miles = 1 league.

30. There are other measures in use. In surveying, for example, lengths are usually measured with a "Gunter's

chain." This is a chain 22 yards long which consists of 100 equal links. A chain is thus equal to 4 poles, and 10 chains = 1 furlong. Substituting chain for pole in Table A above, we obtain Table B.

31. In the Metric or French system, the standard unit of length is the Metre. By dividing, and multiplying the metre by 10, particular lengths are obtained, as shown in the table below. Latin prefixes denote division of the metre, Greek denote multiplication.

Metric Table of Linear Measure.

```
10 millimetres (min.) = 1 centimetre (cm.)
10 centimetres (cm.) = 1 decimetre (dcm.)
10 decimetres (dem.) = 1 metre (m.)
10 metres (m.) = 1 dekametre (dkm.)
10 dekametres (dkm.) = 1 hektometre (hkm.)
10 hektometres (hkm.) = 1 kilometre (km.).
```

The following **Table of Equivalents** will be found useful:—

Linear Measures.

French.		British.		British.		French.
Millimetre	=	0.0394 inch.		Inch	=	2.5399 centimetres
Centimetre	=	0.3937 "		Foot	≈	30.479 ,,
Decimetre	=	3.937 ,,		Yard	=	0.914 metre.
Metre	=	39.3708 ,,	- 1	Chain	42	20.116 ,,
Hektometre	=	0 0621 mile.		Furlong	⋍	201.164 ,,
Kilometre	æ	0.6214 ,,	ļ	Mile	==	1.609 kilometre.

32. Each of the magnitudes given in the preceding tables may be used as a unit in measuring lengths. In selecting a unit to measure any particular distance, we must be guided by the magnitude it is desired to measure. It would be useless to state the distance between London and Edinburgh in inches; such a distance is most conveniently expressed in miles, hence the mile would be the unit selected for such distances. On the other hand, it would be absurd to give the length of a room in poles, or the width of a sheet of drawing-paper in feet. The length of a room would be given in feet and the width of a sheet of drawing-paper in inches.

The following examples show how a number which expresses a particular magnitude in terms of one unit

length may be converted into a number which will express the same magnitude in terms of another unit.

Ex. 1. Find the number of yards, and the number of feet; in one mile.

1 mile = 8 furlongs₁
=
$$(8 \times 40)$$
 poles₁
= $(8 \times 40 \times 5\frac{1}{2})$ yards = 1760 yards;
= $(8 \times 40 \times 5\frac{1}{2} \times 3)$ feet = 5280 feet;

Commit these two results to memory,

Ex. 2. Convert 8 miles, 50 yards, 2 feet, into feet.

8 miles =
$$(8 \times 5,280)$$
 feet = $42,240$ feet.
50 yards = 150 feet.

Hence 8 miles 50 yards 2 feet = (42,240 + 150 + 2) feet. = 42,392 feet.

Ex. 3. Express in miles, yards, feet, and inches a length of 860,732 inches.

$$860,732 \text{ inches} = \frac{860,732}{12} \text{ feet} = 71,727 \text{ ft. 8 in.}$$

$$71,727 \text{ feet} = \frac{71,727}{3} \text{ yards} = 23,909 \text{ yards.}$$

$$23,909 \text{ yards} = \frac{23,909}{1,760} \text{ miles} = 13 \text{ miles } 1,029 \text{ yards.}$$

Hence 860,732 inches = 13 miles 1,029 yards 0 feet 8 inches. This could be done as follows:—

$$\frac{12)860,732}{3)71,727}$$
 feet + 8 inches,

1,760)23,909 yards + 0 feet, 13 miles + 1,029 yards,

i.e., 860.732 inches = 13 miles 1,029 yards 0 feet 8 inches.

Ex. 4. What is the length in inches of each link in a Gunter's chain?

1 chain = 22 yards = (22 × 36 inches) = 100 links

1 chain = 22 yards =
$$(22 \times 36 \text{ inches})$$
 = 100 links.
Each link = $\frac{22 \times 36}{100}$ inches = $\frac{792}{100}$ inches,
= $7\frac{23}{25}$ inches.

Exercises.

(1) Find the number of feet in 5 furlongs 7 poles 3 yards, and convert 763,947 inches into miles.

(2) How many chains are there in 5 miles? Convert 75,000 links into miles,

(3) A plot of land is 200 ft. long and 80 ft. wide. What is the cost of fencing the plot on each side at 3s, 6d, per linear yard?

(4) A motor car travels 15 miles in an hour. What is its velocity in kilometres per hour?

(5) An aeroplane attains a height of 1,500 metres, and a velocity of 55 kilometres per hour. Convert these values into British measures.

(6) A train travels at 40 miles per hour. What velocity is this in feet per second? How long will the train take to travel 65 miles?

(7) The winding-rope of a colliery weighs 25 lb. per yard, and the eage weighs 6 tons. The shaft is 1,000 yards deep. Find the weight on the rope at the winding-drum when the cage is at the bottom and when it is 400 yards down.

(8) A train 300 ft. long, travelling at 60 miles per hour, passes another train 350 ft. long travelling at 40 miles per hour. How long do they take to clear each other (a) when travelling in the same direction; (b) in the opposite directions?

(9) Convert 37,854 inches to yards, and 11,639 yards to miles.

(10) Convert 7 miles 37 yards 2 feet and 10 inches into inches.

(11) How many miles are there in 7,634 chains.

(12) Convert 3,769,418 inches to miles, yards, feet, and inches.

Answers.

- (1) 3,424\frac{1}{2}; 12 mls. 100 vds. 2 ft. 3 ins. (7) 38,440; 23,440 lb.
- (2) 400; 9 $\frac{3}{2}$.
- (3) £32 13s. 4d.
- (4) 24·135.
- (5) 1,641 yds.; 34·2.
- (6) $58\frac{2}{3}$; $1\frac{5}{8}$ hrs.

- (8) 22·16 secs.; 4·43 secs.
- (9) $1,051\frac{1}{2}$ yds.; 6.61 mls.
- (10) 444,886 in.
- (11) 95.4.
- (12) 59 mls. 866 yds. 0 ft. 2 in.

33. Superficial Measure. If we multiply together two equal units of length, we get as product a unit of area which

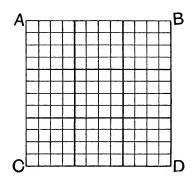


FIG. 4.

is a square, the side of which is unit length. The area of the square ABCD, Fig. 4, and its side AB are corresponding

units of area and length respectively. If the unit of length is 1 in., the unit of area is called a *square inch*; similarly, if the unit of length is a foot, a metre, a chain, or a mile, the corresponding unit of area is called a square foot, a square metre, a square chain, or a square mile.

We frequently wish to compare two or more areas which may be given in terms of different units, and to do this we require to know the number of times one unit of area is contained in another unit. Draw on squared paper a square ABCD, Fig. 4, and make each side of the square 12 divisions in length. By counting the squares in the area ABCD, we ascertain that there are 144 squares, each

having a side 1 division in length, in the square which has a side 12 divisions in length. If we regard each division as 1 in., then each small square is 1 sq. in.; also the side AB is 12 in. or 1 ft., and the area ABCD is 1 sq. ft. We thus see that 1 sq. ft. contains 144 sq. in. Similarly, we can show by means of a square, Fig. 5, that a square of side 1 yd. long, and hence 1 sq. yd. in area, contains 9

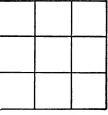


Fig. 5.

squares each of side 1 ft. long; hence 1 sq. yd. contains 9 sq. ft.

If we compare the results obtained above with our Table A of linear measures (Art. 29), we find that the number of square inches in a square foot is obtained by squaring the number of inches in the linear foot; and the number of square feet in a square yard is obtained by squaring the number of feet in the linear yard. And if we square each of the numbers in the first column of the linear table, a corresponding table of square measure is obtained, thus:—

Linear Measure.

12 inches = 1 foot. 3 feet = 1 yard. $5\frac{1}{2}$ yards = 1 pole. 320 poles = 1 mile.

Square Measure.

 $(12)^2$ or 144 sq. inches = 1 sq. foot. $(3)^2$ or 9 sq. feet = 1 sq. yard. $(5\frac{1}{2})^2$ or $30\frac{1}{4}$ sq. yards = 1 sq. pole. $(320)^2$ or 10^2 , 400 sq. poles = 1 s₁, mile. 34. In square measure, we usually employ two units not included in the above Table, viz., the *rood* and the *acre*.

Now 40 sq. poles = 1 rood, and 4 roods = 1 acre, hence by calculation we see that 640 acres = 1 sq. mile.

Ex. (1) Find the number of square yards in an acre, and (2) show that there are 640 acres in a square mile.

(1) 1 acre = 4 roods. = $(4 \times 40) = 160$ sq. poles. = $(160 \times 30\frac{1}{4}) = 4,840$ sq. yards.

(2) From above Table, 1 sq. mile = 102,400 sq. poles, hence as there are 160 sq. poles in 1 acre, there are $\frac{102,400}{160} = 640$ acres in 1 sq. mile.

The relation between an acre and a square chain is found as follows:—

22 yds. \equiv 1 chain (linear), hence (22)² or 484 sq. yds. \equiv 1 sq. chain. But an acre \equiv 4,840 sq. yds. and 1 sq. chain \approx 484 sq. yds. hence 1 acre \equiv 10 sq. chains. Again, 100 links \approx 1 chain (linear), hence 1 sq. chain \approx (100)² or 10,000 sq. links, and 1 acre \approx 100,000 sq. links. The above results are tabulated below.

British Square Measure.

Table A.

144 sq. inches = 1 sq. foot.

9 sq. feet = 1 sq. yard.

30\frac{1}{4} sq. yards = 1 sq. pole.

40 sq. poles = 1 rood.

4 roods = 1 acre.

640 acres = 1 sq. mile.

Table B,
484 sq. yards = 1 sq. chain.
10 sq. chains = 1 acre,
1 sq. chain = 10,000 sq.links.
100,000 sq. links = 1 acre.

35. In the metric system, units of area corresponding to units of length are obtained by squaring the numbers in the first column of the Table in Art. 31, as shown below.

Metric Square Measure.

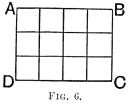
The following table of equivalents will be found useful:-

Square Measures.

British. French.	French. British.
1 sq. inch = 6.451 sq. cms.	1 sq. metre = 10.764 sq. feet or
1 sq. foot = 9.29 sq. dcms.	1·196 sq. yards.
1 sq. mile = 2.599 sq. kilos.	1 sq. kilo. = 0.3861 sq. miles.

36. Area of a rectangle. We already know that to find the area of a square we multiply together the lengths of two sides, or the length and breadth of the square. Draw on squared paper a rectangle ABCD, Fig. 6, making the length AB = 4 divisions, and the breadth BC = 3 divisions. There are twelve squares each having side

of one division in length in the area ABCD, hence the area of the rectangle contains 12 unit areas. The length (4 units) of the rectangle multiplied by the breadth (3 units) is also equal to 12 units; hence to find the area of a rectangle, multiply the length by the breadth.



If a denote the length and b the breadth of a rectangle, then the area A expressed in symbols is A = ab.

If each linear division in Fig. 6 represents 1 in., the area is given in square inches; if each division represents 1 ft., the area is in square feet, and so on. In calculating areas, the student must not multiply one dimension, say length, in feet, by a second dimension, say breadth, in inches. Both dimensions must be in the same units, e.g., length in feet and breadth in feet, or length in inches and breadth in inches.

37. To find the area of a parallelogram. Draw on squared paper a parallelogram ABCD, Fig. 7, making the length AB = 20 divisions, and the perpendicular distance h between the parallel sides AB and CD (called the height) = 10 divisions. Draw BE and AF perpendicular to AB meeting CD in E and meeting CD produced in E. Then by counting the divisions, we can ascertain that

BE = AF and EC = FD, and thus the triangle BEC is equal in area to the triangle AFD. If we remove the triangle BEC and place it in the position AFD, we see that the area of the parallelogram is equal to that of the rectangle ABEF, which area is $20 \times 10 = 200$ units. The length of the rectangle is the same as that of the parallelogram, and the height h of the parallelogram is equal to the breadth of the rectangle; hence the area of a

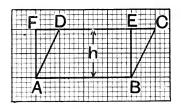


FIG. 7.

paratelogram is equal to the product of the length and the height.

Symbolically, if A = area, l = length, and h = height of a parallelogram,

A = lh.

38. To find the area of a triangle when the base and height are given. We may have three forms of triangle, as shown in Fig. 8. In each form let AC be the base and h the height, i.e., the perpendicular distance between the base (produced if necessary) and the vertex. Draw three such triangles on squared paper, making AC = 15 divisions and taking the vertex B on a line 10 divisions from AC. Draw BD and DC parallel respectively to the two sides CA, AB of each triangle. We thus get three parallelograms ABDC, each formed of two equal triangles ACB, BDC; hence the area of each triangle ACB is half the area of the corresponding parallelogram ABDC, which we know to be 15×10 or 150 units. Each triangle is thus 75 units in area. Now the base AC of each triangle is 15 units long, and the height h is 10 units, and the area

75 units is thus equal to half the product of the base and the height. Hence the rule

Area of triangle = $\frac{1}{2}$ base \times height, or in symbols $A = \frac{1}{2}bh$ where A = area, b = base, and h = height.

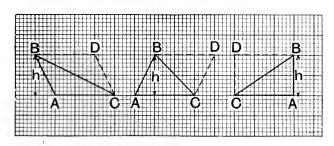


Fig. 8.

Ex. A triangular plate has a 16-ft. base and is 12 ft. high. Find its area,

Area =
$$\frac{1}{2}$$
 base × height,
= $\frac{1}{2}$ × 16 × 12.
= 96 sq. ft,

Exercises.

- (1) Find the number of square inches in a plate 3 yds, long and 2 yds, wide,
- (2) How many square yards and square feet are there in 156,738 sq. in.?
 - (3) Reduce 12 sq. yds., 2 sq. ft., 75 sq. in. to square inches.
- (4) How many stone slabs 3 ft, by 2 ft, will be required for a courtyard 15 yds, long and 12 yds, wide?
- (5) A rectangular plot of land is measured with a Gunter's chain and found to be 52 chains 50 links long, and 25 chains wide. Find its area in acres and square yards.
 - (6) Convert 10,000 sq. metres into square kilometres.
- (7) A parallelogram is 25 ft. long and 15 ft. in height. Find its area.
 - (8) A triangle is 20 ft. long and 12 ft. in height. Find its area.
- (9) A room 15 ft. by 12 ft. is to be floored with boards $4\frac{1}{2}$ in, wide. How many feet run will be required?
- (10) How many square yards of carpet are required for a room 25 ft. by 12 ft. if a border 1 ft. wide all round is not carpeted?
- (11) A rectangular plot of land $\frac{1}{2}$ mile long and 400 ft. wide is to be cut up into building plots, each having 40 ft. frontage and 200 ft.

depth. How many such plots can be obtained and what is the value

of the plot of land at £4 per foot frontage?

(12) An acre of land is to be divided in the following portions:— $\frac{5}{6}$, $\frac{3}{10}$, $\frac{3}{60}$, and $\frac{3}{6}$. How many square yards does each portion contain, and what fraction of an acre is left?

Answers.

39. Volumetric or cubic measure. In measuring the volume of a solid, whether it is bounded by plane or curved surfaces, we have to determine the number of times it contains a particular volume, which we call a *unit of volume*. All units of volume are cubes the faces of which are squares of unit area, and the edges corresponding units of length. If each edge of a cube is 1 in. long, the volume is a cubic inch; similarly, a cube having an edge 1 ft. long is a cubic foot, and one having an edge 1 yd. long is a cubic vard.

Now suppose the cube ABCDEGH shown in Fig. 9 has each edge 1 ft. long. It is thus a cubic foot. We can divide the edge DH into 12 equal parts each 1 in. long and cut the cube into 12 slabs each 1 in. thick. Now if we draw two sets of lines on the face ABCD of the top slab 1 in. apart, one set being perpendicular to the other, as shown, we divide the face into 144 squares each of area 1 sq. in., and by cutting the slab along these lines we should divide it into 144 cubes each having an edge 1 in. long, and hence = 1 cub. in.; each slab thus contains 144 cub. in., and as there are 12 slabs in the large cube, there are $144 \times 12 = 1,728$ cub. in. in 1 cub. ft.

Similarly, we can show there are 27 cub. ft. in a cubic yard.

Isometric paper* may be used to make sketches of cubes and to verify the above statements.

 $^{{}^*}$ Isometric paper is ruled with three sets of parallel lines so arranged as to divide the surface into equilateral triangles,

We thus have for our British table of cubic measure :-

1,728 cubic inches = 1 cubic foot. 27 cubic feet = 1 cubic yard.

Each quantity in the first column is seen to be the cube of the corresponding quantity in the linear lab'e. Thus,

12 inches = 1 foot and $(12)^3$ or 1,728 cubic inches = 1 cubic foot. 3 feet = 1 yard and $(3)^3$ or 27 cubic feet = 1 cubic yard.

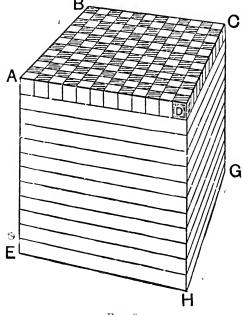


FIG. 9.

Similarly, in the metric system, units of cubic measure are obtained by *cubing* the numbers in column one of the linear measure table in Art. 31.

Metric System Cubic Measure.

1,000 cubic millimetres = 1 cubic centimetre, 1,000 cubic centimetres = 1 cubic decimetre, 1,000 cubic decimetres = 1 cubic metre. 1,000 cubic metres = 1 cubic dekametre. 1,000 cubic dekametres = 1 cubic hektometre. 1,000 cubic hektometres = 1 cubic kilometre.

The following equivalents may be useful:-

1 cubic inch = 16.387 cubic centimetres,

or

1 cubic centimetre = 0.061 cubic inch.

40. Volume of a rectangular block. Consider a slab of material 3 in. long, 2 in. wide, and 1 in. deep,

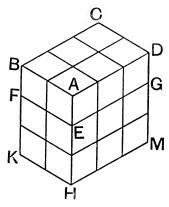


Fig. 10.

such as ABCDEFG, Fig. 10. This slab can be divided, as shown, into six cubes each having edges 1 in. long. Suppose now that three of the slabs are placed vertically one upon the other, forming the block ABCDHKM. Each of the three slabs contains 6 cub. in., hence the complete block contains 3×6 or 18 cub. in. If we take the area ABCD of one end of the block and multiply this by the height AH (the area being in square inches and the height in inches) we get:—

Area ABCD
$$\times$$
 height AH = $(3 \times 2 \text{ sq. in.}) \times (3 \text{ in.})$
= 18 cub. in.

This agrees with the result previously obtained; hence we conclude that the volume of any rectangular block is equal to the product of the area of the end face and the height.

Symbolically, if V = volume, A = area of end face, and h = height, then V = Ah.

Ex. 1. Find the weight of a rectangular block of stone, 12 ft. long 5 ft. wide, and 3 ft. deep, if each cubic foot weighs 145 lb.

Volume =
$$5 \times 3 \times 12$$

= 180 cub. ft.
Weight = 180×145
= $26,100$ lb.

Ex. 2. A storage room is 25 ft. long, 15 ft. wide, and 12 ft. high. How many cubic feet does it contain?

Capacity or volume = $15 \times 12 \times 25$ = 4,500 cub. ft.

Exercises.

(1) Find the number of cubic inches in 12 cub. yds. 23 cub. ft.

(2) A brickwork pier is 3 ft. square and 12 ft. high. Find the

number of cubic feet of brickwork it contains.

(3) A tank is 25 ft. long, 12 ft. wide, and 8 ft. deep. How many cubic feet of water does it contain? If 1 cub. ft. of water $= 62\frac{1}{2}$ lb., find the weight of water in the tank when full.

(4) A block of stone is 15 ft. by 12 ft. by 5 ft. What is its weight

if each cubic foot weighs 145 lb.?

(5) Compute approximately the number of bricks in a wall 200 ft.

long, 6 ft. high, and 11 bricks wide.

(6) A storage room is x ft. long, y ft. wide, and z ft. high. Represent

its capacity symbolically.

(7) A reservoir is 50 ft. long, 30 ft. wide, and 8 ft. deep. Find what weight of water it contains, and how many gallons it contains. Take 1 cub. ft. of water = $62\frac{1}{2}$ lbs.

Answers.

(1) 599616. (5) 19,200. (6) xyz cub. ft.

(3) 2,400 cub. ft.; 150,00 lb. (7) 750,000 lb.; 75,000 galls.

(4) 130,500 lb.

CHAPTER IV.

USE OF SQUARED PAPER.

41. If we wish to fix definitely a point in space so that at any future time we can again find the exact point, we must make certain measurements. Consider a small electric lamp in a room, as shown in the sketch, Fig. 11; suppose that this lamp is to be removed, and that the ceiling from which the lamp hangs is to be renewed, so that the point from which the wires are suspended will be obliterated. After the renewal of the ceiling we wish to replace the lamp in exactly the same position in the room.

Before removing the lamp we must measure (1) its distance from the ceiling, say 6 ft.; (2) its perpendicular distance from the back wall, say 7 ft.; (3) its perpendicular distance from the side wall, say 5 ft. The lamp may now be removed. and when we have to replace it, we first find the point on the ceiling from which the lamp hangs by measuring 7 ft. from the back wall and 5 ft. from the side wall. If we now attach the lamp to this point by a wire 6 ft. long, the

lamp is back in its old position.

42. The student will see from this illustration that in order to fix a point in space we require to know its distance from each of three fixed planes (the two walls and the ceiling). we knew only two of these distances, say the two distances from the walls, we could not again find the exact position of the lamp, although we could readily find the point on the ceiling from which the lamp hangs. Thus, to fix the position of a point in a plane (the ceiling) we require its distance from each of two fixed lines in the plane (the two lines in which the walls meet the ceiling).

43. Now consider a sheet of drawing-paper. the position of any point thereon when we know its distance from two edges of the paper, say the left-hand edge and the bottom. Again, consider a sheet of squared paper ruled with squares of $\frac{1}{10}$ -in. side. If we know that a point P is 30 divisions from the left OY, Fig. 12, and 40 divisions above the bottom OX, we can at once locate the position of the point by counting 30 divisions horizontally along OX,

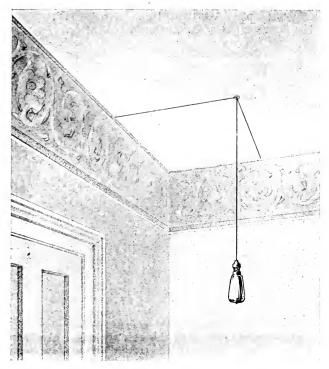


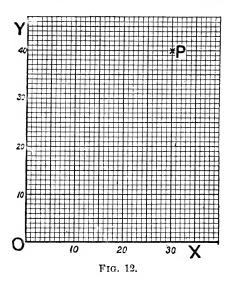
Fig. 11.

and then counting 40 divisions vertically to the point P. This method of locating the point P is called **plotting a point**.

The two edges or fixed lines OX and OY, from which the measurements are made, are called the Axes of Reference. Their point of intersection O is called the Origin, and the

two measurements 30 and 40 which locate the point P are called **co-crdinates**. The point P is referred to as the point whose co-ordinates are 30, 40, and this is denoted by $P_{(30,40)}$, the co-ordinate which is given first is always the distance measured horizontally or parallel to the axis of reference OX, and that given second is the vertical measurement parallel to the axis of reference OY.

The horizontal axis OX is often called the axis of x, and



the measurements parallel to this axis are called **abscissæ** or x co-ordinates. Similarly, the vertical axis is often called the axis of y, and the measurements parallel to this axis are called **ordinates** or y co-ordinates.

44. In the above example, we plotted a point P whose co-ordinates are 30, 40, by taking each division on the squared paper as unity, and measuring 30 divisions horizontally and 40 divisions vertically. In this case, we were simply locating a definite point on the paper. Now suppose the point P to be a point on a lawn which is 30 ft. from one

edge and 40 ft. from another edge at right angles. If we take each division on the squared paper as 1 ft., and the two lines OX, OY to represent two edges of the lawn, we have thus a scale plan of the lawn and the position of the point P. Our scale is $\frac{1}{10}$ in. to 1 ft., and horizontal and vertical measurements are both made to the same scale.*

Suppose that a drain passes under the lawn and we wish to make a record of its position so that we can find any point on the drain by measurement from two edges of the lawn. We will take as an example a lawn 100 ft. long and 60 ft. wide; the rectangle ABCD, Fig. 13, drawn on squared paper to a scale of $\frac{1}{10}$ in. to 1 ft. represents a plan of the lawn. Measurements for locating the drain are made from the two edges AB and BC of the lawn—for example, one point of the drain is 30 ft. from BC and 22 ft. from AB. A series of corresponding measurements are given in the following table:—

Distance from AB	10	14	18	22	26	30	34	38	42	46	50
Distance from BC	0	10	20	30	40	50	60	70	80	90	100

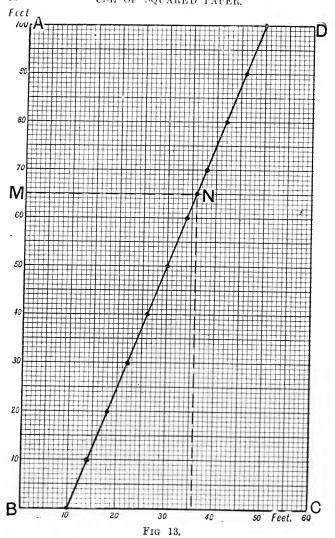
and each pair of measurements are the co-ordinates of points on the plan of the drain. Plot the points $P(_{10-0})$, $Q(_{14,10})$, $R(_{18,20})$, $S(_{22,30})$, etc., on squared paper and join all these points by a line; this line will be the required plan of the drain, as shown in Fig. 13, and from this plan we can locate any point of the drain—e.g., if we require to make a connection to the drain at a point which is 65 ft. from BC, we should find this point by drawing through the division 65 a line MN parallel to BC to intersect the plan of the drain; the distance MN is now seen to be 36 ft., hence the required point is 65 ft. from BC and 36 ft. from AB.

Exercise.

A tennis court is 78 ft. long and 36 ft. wide. The net passes across the middle of the court. A line is drawn along each side of the court 4.5 ft. from each long edge, and two lines across the court each 21 ft. from the net. A line is also drawn down the centre of the court to meet each cross line. Draw on squared paper to scale a plan of a tennis court, and calculate the area of each of the ten divisions into which the court is divided.

45. In the example in Art. 44, we are recording two sets of measurements which are not very different in magnitude,

^{*} In all the squared-paper diagrams in this book, 10 squares are to be reckoned as 1 in, and lines $\frac{1}{2}$ in, apart are thickened.



and hence we use the same scale for both sets of measurements in making our plan. It often happens in practice that we require to make a record of two corresponding sets of measurements which differ very considerably in magnitude. For example, suppose at equal horizontal distances of a mile we measure the height above sea level of a high road which passes over some hills. These heights probably will not exceed 400 or 500 ft.; hence to plot these points on squared paper to scale we should require a very long sheet if we were plotting horizontally distances to represent miles (each of which is 5,280 ft.), and vertically proportional distances to represent, say, only 150 ft.

Now this difficulty is overcome by the use of different scales along the two axes of reference. To take an actual case, the distances of certain points on a road measured horizontally from O, and the heights of these points above

sea level are as shown in the following table:-

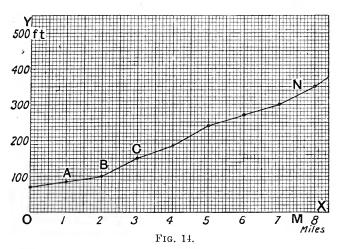
Distance from O in miles	0	1	2	3	4	5	6	7	8
Height of road above sea level in feet	70	85	100	150	185	240	270	300	350

To make a record on squared paper we must plot a series of points A, B, C, etc., whose co-ordinates are A (1 mile, 85 ft.), B (2 miles, 100 ft.), C (3 miles, 150 ft.), etc. We take as our scale for distances from O, each inch to represent 1 mile, and for the heights above sea level, each inch to represent 100 ft. The horizontal scale is thus 1 in. to 1 mile, and the vertical scale is $\frac{1}{100}$ in. to 1 ft.

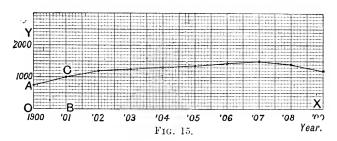
Referring to Fig. 14, the point A is 10 divisions (representing 1 mile) from OY, and 8.5 divisions (representing 85 ft.) from OX, and points B, C, etc., are similarly plotted. By joining up all the points we get a representation of the inclines of the road, and such a representation is used in conjunction with a contour map. From this Fig. we can find the height at any point of the road; thus, at 7½ miles from O the height is given by MN to scale which is 32.5 divisions on map and = 325 ft. actually.

46. In Fig. 14, the line joining the plotted points represents how the inclination varies from point to point along the road, and from it we get an idea of the steepness of the road at various points. It can also be said that it represents the way in which the height of

the road above the sea varies from point to point. Thus, two magnitudes are represented, one of which varies in



value according to the value of the other, but these magnitudes are of the same kind, that is, both are distances, one



measured from a point O and the other measured above sea level.

In a similar manner, magnitudes of related quantities which are different in kind are often met with in practice, and records of such magnitudes can also be made on

squared paper. Examples of such related quantities are the number of bicycles or other articles made each year by any particular firm, or the highest shade temperature each day, or the height of the barometer at different times, or the maximum wind pressure each day, or the number of hours of sunshine per day, and so on,

Each of these sets of magnitudes may be recorded on squared paper. As an example, we will suppose that a firm makes the number of bicycles each year shown in the

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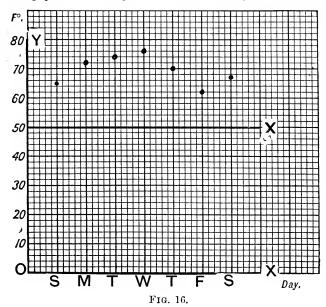
Year.	1900.	1901.	1902.	1903.	1904.	1905.	1906.	1907.	1908.	1909.
Number of bicycles	750	1000	1200	1250	1300	1350	1420	1500	1400	1200

To record these statisties on squared paper we take points equidistant, say by 1 inch along OX, Fig. 15, to represent years; thus we have a scale of 1 in. to represent 1 year; along OY we must have a scale for number of bicycles, say each inch represents 1,000 bicycles. Our vertical scale is thus $\frac{1}{10}$ in. to 100 bicycles. If we now mark off OA = 7.5 divisions, OA will represent (7.5 × 100) or 750 bicycles for the year 1900, BC is 10 divisions to represent 1,000 bicycles in 1901, and so on for the ten years. By joining the plotted points, we get a line which shows graphically how the number of bicycles made each year has varied.

47. The total number of bicycles made in the 10 years is 12,370. Now, if the firm had made an equal number of bicycles each year, and their total output for the 10 years was, as above, 12,370, then they would have made \(\frac{12370}{60}\) or 1237 bicycles each year. This number 1237 is called an average, and we should say that a firm which made bicycles as shown in the above table had an average output of 1237 bicycles per year. An ordinate in Fig. 15 which represents 1237 bicycles, is called an average ordinate of the ten ordinates OA, BC, etc. In general, an average of a number of magnitudes is that magnitude which, if multiplied by the number, gives a product equal to the sum of the magnitudes.

48. In many cases it is unnecessary to show on the squared paper the full length of the line which represents to scale some particular quantity. We can obtain all the information we require if we show only the upper parts of each line, provided we suitably mark our scale; we thus

require less space for the record, or, if necessary, we can use a larger scale than when we have to show the full lines. An example of this kind, which is to be found in many newspapers, is a temperature chart, showing the maximum



shade temperature each day, say for a week. The table below gives the observed temperatures for a week:—

Day.	S in.	Mon.	Tue.	Wed.	Thur.	Fri.	Sat.
Temperature in degrees Fahrenheit	65	72	74	76	70	62	67

If we chose a vertical scale for temperature so that each $\frac{1}{2}$ inch represents a temperature of 10°, and plot the given values on squared

paper, as shown in Fig. 16, we find that all the points lie more than 3 in., and less than 4 in., above OX, and the space required for the record is thus 4 in. wide. By taking the line 50-X as the horizontal axis of reference, and considering that each point on the horizontal axis represents a temperature of 50° instead of 0° , we can plot all the points in a space of $1\frac{1}{2}$ in. wide, and still derive all the information required from the chart. Under these conditions, we can dispense with that part of Fig. 16 which lies below the line 50-X. In plotting the points, e.g., the point representing 76° , we have 76-50=26 as the length to be set off above 50-X: each

					Α,	м.		_			Р.М.		
Time of Observa- tion.	10 p.m.	Mid- night.	2	4	6	8	10	Noon.	2	4	6	8	10
Inches of glycerine,	321.7	321.65	321.52	321.5	321.6	321.7	321.7	321.6	321.7	321.6	321.7	322	321.9

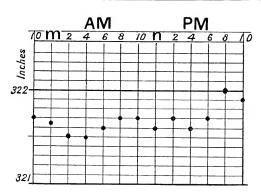


Fig. 17.

division represents 2°, hence the point representing 76° is 13 divisions above 50—X. The vertical scale of such a chart would begin at 50

instead of 0, as shown in the upper part of Fig. 16.

An actual example, which illustrates during twenty-four hours the height of the barometer in inches of glycerine, is taken from *The Times* of May 5th, 1910, and shown in Fig. 17. As an exercise, the student should note that the readings on the chart agree with those shown in the table above; also note that the scale used is a fairly large one, and further that the horizontal axis of reference corresponds to a reading of 321 in, instead of 0 in.

49. The line or curve which joins up a series of plotted points is called a graph. Graphs are now extensively used as they furnish a very ready and easy means of information as compared with many columns of figures. All kinds of statistics can be and are represented graphically. Among others may be mentioned the variation in price of metals or other raw material or manufactured articles, fluctuation in trade, railway receipts and expenditure, and so on. Whitaker's Almanack or any annual handbook of statistics will give the necessary data for practice in plotting graphs. In the following exercises, which should be carefully worked by the student, statistics are given from Whitaker's Almanack of 1908. In those examples in which a probable value for some later date is asked for, this value is obtained by producing the curve or graph beyond the last plotted point, care being taken to keep the direction of the graph in accordance with that for the preceding two or three points. Values obtained in this way are fairly reliable where the plotted magnitudes are steadily increasing or steadily decreasing, but they must not be used to predict probable future values of quantities which fluctuate rapidly. In Exercise 2 (f), (h), (i) and Exercises 3 and 4, will be found values of the former kind, in which the predicted value may be relied upon, whereas in Exercise 1 a predicted value would be unreliable.

Exercises.

The following readings, excepting those in Ex. 5, are taken from Whitaker's Almanack. Make records of each set on squared paper.

(1) Data for May 12th to 18th inclusive, 1907.

Day.	May.	12th.	13th.	14th.	15th.	16th.	17th.	18th.
degrees F. Mean baromet inches of m	ercury	\$1.5 29.609 2.0 11.7	66°9 29°702 1°0 0°6	56·2 29·669 2·2 0	61°S 29°669 1°4 6°3	58·5 29·923 2·9 3·7	56·9 30·028 3·8 S·1	53·0 30·016 9·0 9·1

				ÜSI	E :	ΟF	SQ	UAR	ΕĐ	Ρ.	APER						53	
1:006.	30.375	9 057	3,954	10,101	886	1.83	23,063	1240.3	108.58	S1.01	1		26.11	634.0	1,744	1,557	460.68	
1905.	97 ·S125	3 116	0.11.6		792	1.65	21 22 1- 21 21	1199.0	105.13	20-06	00.07		84 85	2,939	1,519	1.118	9.201	
1904.	578.96	100	727.7	1011	717	1-51	22,034	1198.8	103.78	51.69	19.01		83.38	637.6	1,461	166	371.05	
1903.	57.45	2 5	3,101	nio,no	150	1.19	22,435	1195-3	103.08	92.89	98.07		× 1.82	0.049	1,450	1,026	360.37	
1902	94.1 195	1. 0.	3,108	6.5.6.9	694	1.43	22,152	1188.5	102.00	18.19	49.09		89.08	9.609	1,382	506	349-54	
1901.	27.1875	0007 8	3,099	010(22	633	1.25	870,22	1172.4	62-66	61.19	41.45		18.40	551-2	1,488	823	347.86	
1900.	89 64 70	5	3,441	, 200 () .	695	1.44	21,855	1142.3	28.86	£4.19	40.52		80.08	552.6	1,394	850	354.37	
1899.	100	193 0	3,607	010,41	256	1.43	91,700	1106.7	68-26	60.09	30-19		69.94	583.3	1,227	769	320.53	
1898.	26:9375	0000	3,503	185,55	192	1.37	91,659	1062-9	50.16	55.96	37.36		73.53	585.8	1,231	616	294.01	
1897.	27.5695	9900	2,986	24,031	591	0.62	21,433	1030-4	88.38	53.08	92.28		69·S3	I	1,051	202	504-17	
Year,	(a) Average yearly price of silver per standard Troy ounce in London in nearce	(b) Number of ships passing each	(c) Number of people emigrating	(d) Number of ships (excluding warships) launched in the	United Kingdom		(f) Afties of rankasy open in the United Kingdom	in United Kingdom on rail- ways, in million	(h) Total receipts Kingdom	(i) Total expenditure millions of	(j) Amount deposited in Post Office Savings Bank in millions of nounds	(k) Amount of tobacco consumed in the United Kingdom in	millions of Ibs.	of pounds	(a) Institute of United States in (b) Institute of United States in	millions of dollars.	willions of pounds.	I are liveraged to the limit of the control of

(3) The expectation of life at certain ages is shown in the appended table. Plot on squared paper and find the probable expectation of life of a man at the age of 45 and of a woman at the age of 44.

Age	32.	33.	34.	35.	36.	37.	38.	39.	40.	41.	42.	43.
Male years. Female years.	30·71 33·0	30·01 32·3	29·33 31·6							24.65 26.78		23·35 25·42

(4) The population of England and Wales, Scotland, and Ireland in millions are shown in the table below. Plot a graph and find the probable population of each country in 1911.

Year.	1801.	1811.	1821.	1831.	1841.	1851.	1861.	1871.	1881.	1891.	1901
England											
and Wales	8.85	10.16	12.0	14.16	15.91	17:93	20.01	22.71	25.97	29.00	32.5
Scotland Ireland	1.61	1.81	2.09 6.80	2:36 7:77	2.62 8.18	2·89 6·55	3.06 5.80	3·36 5·41	3·74 5·17	4·03 4·70	4.4

(5) The following values of imports and exports are for the month of April from 1900 to 1910. Plot two curves showing respectively the fluctuation in imports and exports.

Year.	1900.	1901.	1902.	1903.	1904.	1905.	1906.	1907.	1908.	1909.	1910.
Imports in millions £. Exports in millions £.											

Board of Trade Returns.

(6) The revenue derived by the British Government from the Suez Canal is shown in the table below. Plot on squared paper and estimate the probable revenue in 1907–8.

Year.	1896-7.	1897-8.	1898-9.	1899-00.	1900-01.	1901-2.	1902-3.	1903-4.	1904-5.	1905-6.	1906-7
Revenue in mil- lions £		0.698	0.679	0.802	0.812	0.848	0.934	0.936	0.990	1.023	1.054

(7) The	Civil	Service	Estimates	for	expenditure	\mathbf{for}	the	years
from 1894-	5 to 19	907-8 are	as follows	:	•			•

Year,	1894-5.	1895-6.	1896-7.	1897-8.	18º8-9.	1899-00.	1900-01
Millions £ .	20.02	20.65	21.21	21.59	23:19	23.68	24.40
Year.	1901-2.	1902-3,	1:03-4.	1904-5.	1905–6.	1906–7.	1907-8.
Millions £	. 25.17	28.03	28.21	29.70	30.34	31.0	31.78

Plot on squared paper and find the probable expenditure in 1908-9. (8) The imports and exports of wholly or mainly manufactured articles into the United Kingdom for the years 1902 to 1906 are shown in the table below. Plot on squared paper.

Year.	1902.	1903.	1904.	1905.	1906.
Imports in millions & Exports in millions &	132:48	134·56	135·17	142·54	155·81
	227:54	234·79	243·83	269·77	305·53

50. In many cases, corresponding values are so related that if the values are plotted on squared paper the points so found all lie on a straight line. Now we only require to find two points to fix any straight line; hence if we know that the graph when plotted will be a straight line, and further if we have data to plot two points on the line, we can at once determine the graph.

For example, corresponding values of inches and centimetres give a straight line graph.

We know that 1 in. is equal to 2.54 cm., hence 10 in. = 25.4 cm., and we also know that 0 in. = 0 cm.; thus, in order to draw a graph showing the relationship between inches and centimetres, we plot the point P. Fig. 18, the co-ordinates of which are 10 and 25.4, and join this point to the origin, for the co-ordinates of the origin are 0, 0. From this graph, we can readily convert inches into centimetres and vice versã.

For example, 5 in. =
$$12.7$$
 cm. also 20 cm. = 7.9 in.

The graph showing the relationship between degrees of heat on a Fahrenheit scale and on a Centigrade scale is also a straight line.

To obtain this graph, we know that for water 0° C. = 32° F. (freezing points)

and 100° C. = 212° F. (boiling points),

hence by plotting the points P (0, 32) and Q (100, 212), Fig. 19, we get the required graph by joining PQ.

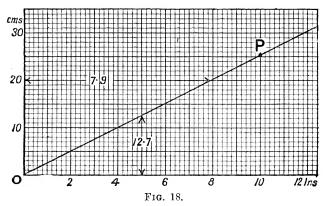
Ex. From this graph convert 50° C. into degrees F. 50° C. = 122° F.

reading length RS.

Ex. Convert 200° F. into degrees C. 200° F. = 93° C., reading length TU.

Exercises.

The student should draw the following graphs and keep them for reference in his work. They are all straight lines, and can be found



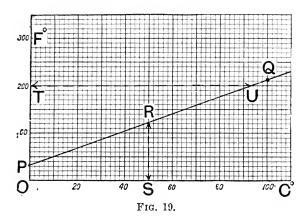
by plotting one point on each and joining to the origin, as in Art. 50.

(1) 100 lb. = 45·36 kilos. Plot a graph from which corresponding values of lbs. and kilograms may be read off, and find from the curve the number of kilograms equivalent to 65 lb.

Answer, 29.5.

(2) A pint of water at 4° C. is equal to 0.57 litres. Plot a graph from which corresponding values of pints, quarts, or gallons and litres of water can be read off.

- (3) 1 litre of water = 61.025 cub, in. Plot a graph showing corresponding values of litres and cubic inches.
- (4) 1 radian = 57.3 degrees. Plot a graph showing corresponding values of radians and degrees.
 - (5) The circumference of a circle = 3.1416 times the diameter.



Show this relationship by a graph which will read for all circles up to 25 in. diameter.

(6) 33,000 ft.-lbs. (mechanical units of work) = 746 watts (electrical units). Draw a graph for watts and ft.-lbs.

```
1 cub. ft. of water weighs
                                                62:3 lb.
(7)
           1 cub. in. of cast iron weighs
                                                  0.26 lb.
                         wrought iron weighs 0.28 lb.
                         steel
                                                  0.29 lb.
                          brass
                                                  0.301 lb.
                         aluminium
                                                  0.092 lb.
                                           11
                                                  0.319 lb.
                         copper
                                           33
                         lead
                                                  0.414 lb.
                                           ,,
                          gold
                                                  0:69 lb.
                                                  0.38 lb.
                         silver
                                           ,,
                                                  0.27 lb.
                         tin
```

In each of the above eases, draw a graph showing corresponding weights and volumes.

(8) A knot (nautical mile) is 6,080 ft. Plot a graph showing the relation between miles and knots. What velocity in miles per hour corresponds to 27.5 knots per hour?

(9) I mile = 80 chains, and 1 chain = 22 yds. Draw graphs for miles and chains, and yards and chains. Convert 175 chains into miles, and 85 yds. into chains.

- (10) 1 cub. ft. of air at 0° C. and 1 atmosphere pressure weighs 0.0807 lb. Draw a graph for conversion of weights and volumes of air.
- (11) Atmospheric pressure averages about 14·7 lb. per square inch. Draw a graph to convert lbs. per square inch into atmospheres, and vice rerså.
- (12) A head of 2.3 ft. of water, or a head of 2.03 in. of mercury, gives a pressure of 1 lb. per square inch. Plot a graph to convert

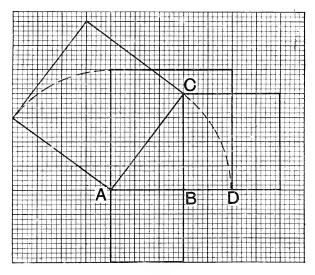


FIG. 20.

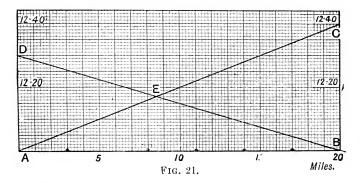
pressure readings in lbs. per square inch into corresponding readings in feet head of water or inches of mercury.

51. Squared paper may be used to show that in a right-angled triangle the square constructed on the hypotenuse (the side opposite to the right-angle) is equal in area to the two squares on the two sides which contain the right-angle. Thus, draw the triangle ABC, Fig. 20, and take $\frac{1}{2}$ indivisions as units. Make AB = 3 units, BC = 4 units. Draw a square on AB; it contains 9 unit squares. Draw a square on BC; it contains 16 unit squares. Now set off AD = AC, and on AD construct a square. This square is

equal in area to the square on AC, and it is found to contain 25 unit squares. Thus, square on AC (25 units) = square on AB (9 units) + square on BC (16 units).

52. Some problems are more readily solved by using

52. Some problems are more readily solved by using squared paper than in any other way. An example is as follows: On a single-line railway, 20 miles long, two trains start at the same time, say noon, one from each end. The train from the end A travels at 30 miles per hour, and that from end B travels at 40 miles per hour. Each train is to make one stop only, and the stations are at distances 3, 8,



11, 14, and 17 miles from the end A. At what station must the two trains pass if each is to complete the journey in as short a time as possible?

We set out on squared paper a length AB to represent the railway, and mark off the stations to scale as shown in Fig. 21, in which $\frac{1}{2}$ in represents 1 mile. Now take a vertical scale in which $\frac{1}{2}$ in, represents 5 minutes, and take the line AB as representing 12 o'clock. The train from A would arrive at B in 40 minutes if it did not stop at all. Mark off C to represent the time 12.40 (time of arrival at B) and join AC. Now notice that the ordinate at each point representing a station gives the time the train passes the station; thus, at 12.6 the train from A passes station 3 miles from A, at 12.16 it arrives at the station 8 miles from A, and so on. The train from B would arrive at A at 12.30 if it did not stop. Mark off D to represent 12.30 (to the same scale as that used for C) and join BD. Similarly, from BD we can find the time the train p .sses each station; thus it passes the second station

from B at 12.9, and so on. The point E in which AC and BD intersect gives the place and time at which the trains would meet, and this is 8.5 miles from A and the time is 12.17. Now as there is a station 8 miles from A, and the next is 11 miles from A, the train from A should stop at the station 8 miles from A to allow the train from B to pass. If there were two sets of rails, the trains would pass at the point 8.5 miles from A. The accuracy of this result can be tested as follows:—

Both trains leave at noon.

Train from A travels 1 mile in 2 min., and hence takes 8.5×2 or 17.0 min. to reach the meeting place.

Train from B travels 1 mile in $1\frac{1}{2}$ min., and has to travel (20 - 8.5) or 11.5 miles. This takes (11.5×1.5) or 17.25 min. Thus both trains have been travelling for 17 min. approximately

when they reach the meeting place.

In examples in which the two trains or other vehicles do not leave at the same time, the points C and D (representing the times of arrival) would be joined to two points found in a similar manner but representing the times of departure. The remainder of the problem is solved as above.

Exercises.

(1) A single-line railway is 30 miles long, and two trains start, one from each end. One travels at 30 miles per hour and starts from the end A, and the other at 45 miles per hour and starts from B. Each train is to stop once only. How far from the end A would they meet? If the stations are at distances of 3, 8, 14, 18, and 24 miles from A, at which station should the trains pass if each is to complete its journey in as short a time as possible.

(2) Two motorists, A and B, start from London and Coventry respectively. The distance is 91 miles. A can travel 25 miles per hour and B at 20 miles per hour. They wish to dine together at some town en route and to send an advance order to that effect. How far must the town be from London, and in what time after

starting will they meet?

(3) From Crewe to Euston is 158 miles. A non-stop train leaves Crewe at 6.20 p.m. and arrives at Euston at 9.15 p.m. A non-stop train leaves Euston at 5.30 p.m. and arrives at Crewe at 8.23 p.m. How far from Crewe and at what time do these trains pass each other, assuming that each train travels at a constant rate throughout the journey? What is the average speed of each train in miles per hour?

Answers.

(1) 12 mls. from A; the station (3) 56 mls. from Crewe; time of passing 7.22 p m.; 54.2 mls. per hour; 54.8 mls. per hour.

(2) $\frac{1}{2}$ mis.; 2 nrs. $\frac{1}{3}$ min. after nour; $\frac{1}{2}$ mis. per nour

starting.

CHAPTER V.

OPERATIONS WITH VULGAR AND DECIMAL FRACTIONS; CONTRACTED METHODS.

53. In Chapter II. we have discussed certain rules relating to arithmetic and algebraic fractions, and we have also discussed decimal fractions and the methods of converting vulgar into decimal fractions and *vice versâ*.

In the present chapter we shall deal with the operations of multiplication, division, addition, and subtraction of frac-

tions, and with contracted methods of multiplication and division of decimals.

54. **Multiplication.** A fraction may be multiplied by a whole number, or a whole number may be multiplied by a fraction, or a fraction may be multiplied by a second fraction. The method of multiplying together whole numbers and fractions is illustrated by the following simple case:—



FIG. 22.

Suppose a square abed, Fig. 22, is

divided into five parts as shown, and one of these parts is selected; we then have an area (shown shaded) which is $\frac{1}{5}$ of the whole square; if we take three of these parts we have $\frac{3}{5}$ of the whole square; hence three times a fraction of $\frac{1}{5}$ is equal to a fraction $\frac{\pi}{6}$.

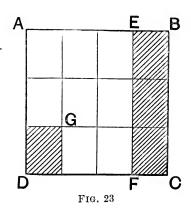
This result would have been obtained if we had multiplied the numerator of the fraction $\frac{1}{5}$ by 3, and used the product

as the numerator of the product of $\frac{1}{5}$ and 3.

The product 2×3 is equal to the product 3×2 . In the same way the product $\frac{1}{5} \times 3$ is equal to the product $3 \times \frac{1}{5}$. In each case the result is $\frac{3}{5}$. That this is true can also be seen by drawing three squares each equal to *abcd*

and dividing each into five strips. If we now take one strip from each square we have in all three strips, each of which is $\frac{1}{2}$ of the unit. The total is $\frac{3}{2}$ of the unit; hence we conclude that the product of $3 \times \frac{1}{2}$ is equal to $\frac{3}{2}$.

If we divide a unit square ABCD, Fig. 23, into four strips, each equal to the strip EBCF, shown shaded, then each strip is $\frac{1}{4}$ of the unit. Now divide each strip into three parts, such as DG; each one of these parts is $\frac{1}{3}$ of a strip, or $\frac{1}{3}$ of $\frac{1}{4}$ of the unit. But the whole square is now divided into 12 of these smaller parts; thus each small part represents $\frac{1}{12}$ of a unit, and hence the product of $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$. The numerator of the product



is seen to be the product of the numerators of the separate fractions, and the denominator of the product is the product of the denominators of the separate fractions.

This process may be extended to any number of fractions.

For example, the product of

 $\frac{1}{4} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{10} = \frac{1}{240}, \text{ since product of numerators} = 1 \text{ and product of denominators} = 240$ $\text{Also} \qquad \frac{3}{8} \times \frac{1}{5} = \frac{3}{40}$

Also
$$-\frac{5}{8} \times \frac{3}{5} = \frac{1}{40}$$

and $\frac{5}{7} \times \frac{3}{4} \times \frac{9}{11} = \frac{135}{308}$

Any integral or whole number may be regarded as a fraction in which the number itself forms the numerator, and the number 1 forms the denominator.

Thus, 3 may be regarded as
$$\frac{3}{1}$$

5 may be regarded as $\frac{5}{1}$.
The product of 5×3 then $= \frac{5}{1} \times \frac{3}{1} = \frac{15}{1}$.

This method of regarding whole numbers is useful in multiplication, division, addition, and subtraction, when we are dealing with whole numbers and fractions.

Ex. Find the product of
$$\frac{1}{5}$$
, 3 and $\frac{7}{8}$.
$$\frac{1}{5} \times \frac{3}{1} \times \frac{7}{8} = \frac{21}{40}.$$

55. In multiplying together a series of fractions, it is usual to divide any numerator and any denominator by any number which can be seen to be a common factor, and to replace the divided numbers by the quotients.

Thus,
$$\frac{2}{3} \times \frac{15}{24} \times \frac{12}{30}$$
 may be treated as follows:—
Divide 2 and 24 by 2 and we get $\frac{1}{3} \times \frac{15}{12} \times \frac{12}{30}$.

Now divide each number 12 by 12, and the numbers 15 and 30 by 15 and we get $\dot{}$

$$\frac{1}{3} \times \frac{1}{1} \times \frac{1}{2}$$
 which is $= \frac{1}{6}$.

In practice this would be done by cancelling out common factors; thus—

$$\frac{\frac{1}{8}}{\frac{3}{3}} \times \underbrace{\frac{1}{1}}_{\frac{54}{3}} \times \underbrace{\frac{1}{34}}_{\frac{54}{3}} \times \underbrace{\frac{1}{34}}_{2} \text{ giving } \frac{1}{3} \times \frac{1}{1} \times \frac{1}{2} = \frac{1}{6}.$$

This result can be shown to be correct if we multiply fractions together without any cancellation and then reduce to lowest terms.

Thus,
$$\frac{2}{3} \times \frac{15}{24} \times \frac{12}{30} = \frac{360}{2160}$$
.

Since numerator and denominator each divide by 360, the fraction in its lowest terms is $\frac{1}{6}$.

When the product of proper fractions and mixed numbers is required, we reduce each mixed number to an improper fraction and then follow the usual method.

Ex. Find product of
$$\frac{7}{8} \times 3\frac{1}{6} \times 2\frac{2}{3} \times \frac{5}{19} \times 4$$
.
Write $3\frac{1}{6} = \frac{19}{6}$; $2\frac{2}{3} = \frac{8}{3}$; $4 = \frac{4}{1}$.

We now have

$$\frac{7}{8} \times \frac{19}{8} \times \frac{1}{8} \times \frac{5}{10} \times \frac{2}{1} = \frac{70}{9}.$$

The result $\frac{70}{9}$ is an improper fraction, which written as a mixed number = 7_6^* .

56. The various rules for multiplying and reducing to their lowest terms arithmetic fractions are equally applicable to algebraic fractions.

Thus.

Ex. 1.
$$\frac{a}{3} \times \frac{2}{x} = \frac{2a}{3x}$$
.
Ex. 2. $\frac{6a}{4a} \times \frac{2b}{3x} \times \frac{6x}{5b} = \frac{72abx}{60abx}$.

This fraction is not in its lowest terms. Divide numerator and denominator by 12abx; the fraction $\frac{72abx}{60abx}$ then reduces to $\frac{6}{5}$ or $1\frac{1}{5}$.

Ex. 3. Find the product of
$$\frac{a^3}{x^2} \times \frac{x^3}{a^2} \times \frac{b^4}{a} \times \frac{a^2}{b^3}$$
.

Here we get $a^3 \times x^3 \times b^4 \times a^2 = a^5b^4x^3$ and $x^2 \times a^2 \times a \times b^3 = a^3b^3x^2$.

The product is thus $\frac{a^5b^4x^3}{a^3b^3x^2}$.

Dividing numerator and denominator by $a^3b^3x^2$, we get $\frac{a^2bx}{1}$ or a^2bx .

This result could have been found by cancellation, as follows:-

$$\frac{a^2}{\frac{\mu^*}{x^2}} \times \frac{\cancel{f}}{\cancel{\mu^*}} \times \frac{\cancel{f}}{\cancel{\mu}} \times \frac{\cancel{h}}{\cancel{\mu}} \times \frac{\cancel{g}}{\cancel{b}^5} = a^2 bx.$$

57. **Multiplication of decimals.** When two decimals or a whole number and a decimal are to be multiplied together, the multiplication is performed exactly as in multiplying numbers, the decimal points in the multiplier and multiplicand being left out of account during the

multiplication.

To ascertain the position of the decimal point in the product, consider first the product of two vulgar fractions, say $\frac{3}{10}$ and $\frac{35}{100}$. The product is $\frac{105}{1000}$. Now $\frac{3}{10} = 0.3$, and $\frac{35}{100} = 0.35$, and $\frac{105}{1000} = 0.105$, hence the product of 0.3 and 0.35 must be 0.105. The figures in the product are the product of 3 and 35, and we have three decimal figures in the product; in the multiplier 0.35 we have two decimal figures, and one in the multiplicand 0.3. Thus, the number of decimal figures in the product is equal to the sum of the numbers of decimal figures in the multiplier and multiplicand.

Ex. Find the product of 2:31 and 5:638.

5·638 2·31 5638 16914 11276 13·02378 There are three decimal figures in the multiplier; hence we have five decimal figures in the product. Count five figures from the right-hand end, and put in the decimal point between the fifth and sixth figures.

58. A convenient method of reducing a quantity represented by a decimal, when expressed in certain units, to other smaller units, is illustrated in the following examples:—

Ex. 1. Find in £. s. d. the value of 0.375 of £5.

0·375
5

1·875 pounds
20

17·5% shillings
12

6·0 pence.

Hence 0.375 of £5 = £1 17s. 6d.

Multiplying by 5 we find 0.375 of £5 = 1.875 pounds.

Multiplying 0.875 by 20 we find that 0.875 pounds = 17.5 shillings; 0.5 shillings = $0.5 \times 12 = 6$ pence.

Ex. 2. Express 35.625 yds, in yards, feet, and inches.

Exercises.

(1) What is the product of \$\frac{3}{8} \times \frac{2}{7} \times \frac{16}{30} \times \frac{12}{25} \times \frac{5}{12} ?
(2) Find the area of a metal plate which shall be three-quarters the weight of a similar plate 5 ft. 6 in. long and 3 ft. 4 in. wide.

(3) What fractional part of 10 in. is one-sixth of a foot?

(4) Find the total value of $\frac{3}{10}$ of £5, $\frac{7}{16}$ of 32 shillings, and $\frac{1}{3}$ of 1 shilling.

(5) 23 pieces of timber each 4 ft. long are required. The timber is obtainable in 14-ft. and 18½-ft. lengths. Which lengths give least waste? Would you order all the same lengths, or would it be preferable to take s me of each length?

(6) Find the total length of picture moulding required to cut 4 lengths of 10% in., 6 lengths of 9% in., 2 lengths of 15% in., and 8 lengths

of 75 in.

(7) 24 pieces of timber each 4 ft. 6 in. long are required. The timber is obtainable in 10-ft, and 15-ft, lengths. Which lengths are preferable as giving the least amount of waste?

(8) What is the product of 17.64 and 3.8, also of 0.00638 and

- 2.4807 ?
 - (9) Convert \(\frac{1}{5}\), \(\frac{1}{8}\), and \(\frac{3}{4}\) into decimals, and find their product.

(10) What is the value of 0.75 of £5?

(11) Find in decimal form the relation of 5 in. to 30 yds.

(12) Fifty railway rails each 30 ft. long are laid end to end and a space of 0.25 in, is left between consecutive rails. Find the actual length covered by the rails.

Answers.

- (7) Both 12 ft. waste. (2) 135 sq. ft. (8) 67·032; 0·015826866. (3) $\frac{1}{5}$. (9) 0.01875. (4) £2 4s. 4d. (10) £3 15×. (5) 18½-ft. lengths; preferable (11) 0.00463. 5 lengths 18½ ft. and 1 of 14 ft. (12) 1501 ft. and 4 in.
 - (6) 16 ft. 13 in.

59. Division of fractions. When we divide one whole number by another, we ascertain how many times the first number contains the second. For example, $60 \div 3 = 20$, *i.e.*, the number 60 contains the number 3 twenty times. Now if we regard the numbers as fractions with 1 as denominator, then $60 = \frac{60}{1}$ and $3 = \frac{3}{1}$ and we see that in order to get the result 20, we invert the fraction $\frac{3}{1}$ which is the divisor, and multiply the first fraction $\frac{60}{1}$ by the inverted fraction, which is now $\frac{1}{3}$.

Thus
$$\frac{60}{1} \times \frac{1}{3} = \frac{20}{1}$$
 or 20.

Now what is true for one fraction is true for all fractions; hence the rule to divide any quantity, whether a whole number or a fraction, by a fraction, invert the fraction which is the divisor, and then

multiply.

As an example, suppose 10 is to be divided by $\frac{1}{2}$. Here the unit is 1, the number 10 represents 10 units, and the fraction $\frac{1}{2}$ represents a half unit, and as 10 units is equal to 20 half-units, the fraction $\frac{1}{2}$ is contained in 10 twenty times.

The same result would be obtained by the above rule, for $10 \div \frac{1}{2} = \frac{10}{10} \times \frac{2}{10} = 20$.

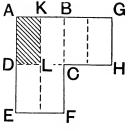


Fig. 24.

Division can be illustrated graphically as follows:—

Take a square ABCD, Fig. 24, as unit, then the figure AGHCFE contains 3 units. Now divide each unit into two parts; each such part AKLD is a half-unit, and we see that the whole figure contains 6 of these parts. Hence $\frac{1}{2}$ is contained in 3 six times. By our rule $\frac{3}{1} \div \frac{1}{2} = \frac{3}{1} \times \frac{2}{1} = 6$.

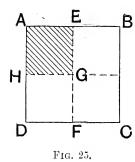
60. The result of dividing one proper fraction by another may be similarly illustrated. For example, $\frac{1}{2} \div \frac{1}{4}$

is by our rule $\frac{1}{2} \times \frac{4}{1} = 2$.

Thus, $\frac{1}{4}$ of a unit should be contained twice in a half-unit. If a square ABCD, Fig. 25, is a unit, then a half-unit is the rectangle AEFD, and a quarter-unit is the square AEGH shown shaded. It will be seen that in the half-unit AEFD

there are two areas each equal to the square AEGH, i.e., two quarter-units.

The figure will also illustrate that a fraction divided by a whole number follows the same rule. If we divide a



half-unit AEFD by 2 we get a quarter-unit AEGH. Thus, $\frac{1}{2} \div 2$ must $= \frac{1}{4}$.

By our rule $\frac{1}{2} \div 2 = \frac{1}{2} \times \frac{1}{2}$ and this $= \frac{1}{4}$.

The process may be extended to any number of fractions, thus, $\frac{1}{2} \times \frac{3}{3} \div \frac{1}{4} \times \frac{2}{3} \div \frac{5}{6}$ would be written out as multiplication with all the dividing fractions inverted thus:—

$$\begin{array}{c} \frac{1}{2} \times \frac{3}{8} \times \frac{4}{1} \times \frac{2}{3} \times \frac{6}{5} \\ \text{and this} = \frac{3}{5}. \end{array}$$

Algebraic fractions are divided in the same manner as arithmetic fractions.

Ex. 1. Divide
$$\frac{a^2}{b^2}$$
 by $\frac{a^3}{b}$.

Invert the divisor and multiply :-

$$\frac{a^2}{b^2} \times \frac{b}{a^3} = \frac{a^2b}{a^3b^2} = \frac{1}{ab}$$

for numerator and denominator will divide by a^2b .

Ex. 2. Simplify
$$\frac{x^2}{y^3} \div \frac{x^9}{y^4} \times \frac{x^2y}{a^3b^2} \div \frac{y}{a^2b}$$
.

To simplify an expression, we must carry out the various operations denoted by the signs and, if a fraction, reduce it to its lowest terms.

Inverting the dividing fractions, we get

$$\frac{x^2}{y^3} \times \frac{y^4}{x^3} \times \frac{x^2y}{a^3b^2} \times \frac{a^2b}{y}.$$

By cancellation these reduce to :-

$$\frac{1}{1} \times \frac{y}{1} \times \frac{x}{ab} \times \frac{1}{1} = \frac{xy}{ab}.$$

61. Division of decimals. When one decimal is to be divided by another, the operation can be expressed in

fractional form; thus, to divide 0.3 by 0.93 may be written $\frac{0.3}{0.93}$. If each decimal is converted into a vulgar fraction, this expression becomes $\frac{3}{10}$ divided by $\frac{93}{100}$ or

$$\frac{\frac{3}{10}}{\frac{93}{100}} \text{ which is } \frac{3}{10} \times \frac{100}{93} = \frac{30}{93} \text{ or } \frac{10}{31}.$$

If we now convert $\frac{10}{31}$ into a decimal fraction, we get 0.32258 correct to five decimal places, and hence 0.3 divided by 0.93 is equal to 0.32258.

We get the same result by making the number of decimal figures in the dividend and divisor equal by adding ciphers, then remove the decimal points in each, and convert the fraction

so obtained into a decimal.

Thus, $\frac{0.3}{0.93}$ becomes $\frac{0.30}{0.93}$, which becomes, after removal of the decimal points, $\frac{30}{93}$ or $\frac{10}{31}$.

Ex. Divide 76.84 by 8.937.

Writing as a fraction $\frac{76.84}{8.937}$ Making decimal figures equal $\frac{76.840}{8.937}$ Removing decimal points $\frac{76.840}{8.937}$ Convert into decimal as shown opposite.

Ex. 1. Divide $\frac{36.947 \text{ by } 0.0004}{10004}$.

This is $\frac{36.9470}{0.0004} = \frac{36.9470}{4}$ by multiplying numerator and denominator by 10,000.

Ex. 2. Divide 0.00064 by 36.23. This becomes $\frac{.00064}{36.23000} = \frac{64}{3623000}$ multiplying numerator and denominator by 100,000. = 0.0000177, etc. Ex. 3. Divide 357.84 by 25.6182.

$$\begin{array}{l} \frac{35784}{25\cdot6182} & \frac{357\cdot8400}{25\cdot6182} \\ & = \frac{357\cdot8400}{25\cdot6182} \\ & = \frac{3578400}{25\cdot6182} & \begin{array}{c} \text{multiplying numerator and} \\ \text{denominator by 10,000.} \\ & = 13\cdot968, \text{ etc.} \end{array}$$

62. When repeating and mixed repeating decimals have to be multiplied together or divided, a more accurate result is obtained if the decimals are first converted into vulgar fractions, then multiply together or divide the fractions, and reconvert the result to decimal form.

Ex. 1. Find the product of 0.142857 and 0.3.

$$0.142857 = \frac{142857}{999999} = \frac{1}{7}.$$

$$0.3 = \frac{3}{9} = \frac{1}{3}.$$

$$\frac{1}{7} \times \frac{1}{3} = \frac{1}{24}.$$

$$\times \frac{1}{3} = \frac{1}{21}$$
.
 $\frac{1}{21} = 0.047619$, etc.

Ex. 2. Find the product of 0.3624 and 0.7.

$$0.36\dot{2}\dot{4} = \frac{(3624 - 36)}{9900} = \frac{3588}{9900},$$

$$0.7 = \frac{7}{9}.$$

$$299$$

$$3388$$

$$3388$$

$$3388$$

$$7 = \frac{2093}{7425} = 0.2818.$$

$$2475 = 3$$

Exercises.

(1) Find the numerical value of the product of 54 and 7 divided by $\frac{5}{8}$ of $\frac{7}{32}$.

(2) The journey from Paddington to Plymouth is 2261 miles, and this is done by the "Cornishman" in 4 hrs. and 7 min. What is the average velocity in miles per hour?

(3) A cycling record made in 1902 is 4 mile in 192 sec.

velocity is this in feet per second?

(4) A skating record made in 1893 is 100 yds. in 9½ sec. velocity is this in feet per second?

(5) A walking record made in 1906 is 1 mile in 6 min. 192 sec. What is the speed in yards per minute?

(6) A running record made in 1904 is 5 miles in 24 min, 33\frac{3}{2} sec. What is the average speed in feet per minute?

(7) Express the sum of money 45 pounds 7 shillings 8 pence in

pounds and decimals of a pound. (B. E., 1907.)
(8) Decimalize £22 13s. 64d. Multiply by 5:273 and divide by 2.156. Give the answer in pounds, shillings and pence; correct to the nearest penny. (B. E., 1909.)

(9) The pitch of a screw thread is the distance, measured parallel to the axis of the screw, from the centre of one thread to the centre of the next. If a screw has 12 threads to the inch, what is the

pitch?

(10) If a screw thread has a pitch of $\frac{3}{32}$ in., how many threads are there per inch?

(11) A screw has a pitch of $\frac{1}{8}$ in. How far does it travel in

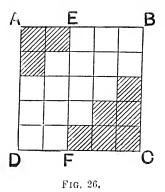
- six revolutions? (12) A square-threaded screw of $\frac{5}{16}$ in, pitch is screwed into a piece of metal in which the tapped (screwed) hole is 3 in. long. How many revolutions does the screw make?
- (13) The screw of a railway carriage coupling has on one end a right-hand thread, and on the other end a left-hand thread. The pitch is 5 in. Find how far the carriages approach or recede from each other for five turns of the screw.

Answers.

- (8) £22.676; £55 9s, 2d, (1) $172\frac{4}{5}$. (2) 55.02. (9) $\frac{1}{12}$ in. (3) 68.04. (10) 10^2_{3} . (4) 30.6. (11) 3 in. (5) 278.3. (12) 93. (6) 1075. (13) 6\(\frac{1}{2}\) in. (7) £45·383.
- 63. Addition and subtraction of fractions. Consider first the case of two fractions which have the same denominator. Suppose a square ABCD, Fig. 26, to be divided into 25 equal squares as shown. The three squares shaded at the corner A represent $\frac{3}{25}$, and the six squares shaded at the corner C represent $\frac{6}{25}$ of the whole square. Now the total number of squares shaded is 9, hence the shaded part represents $\frac{9}{2.5}$ of the whole square, or in other words $\frac{3}{25} + \frac{6}{25} = \frac{9}{25}$. We have simply added together the numerators of the separate fractions to obtain the numerator of the fraction representing the sum.

Similar reasoning can be applied to subtraction. Consider again the square ABCD. There are six small squares shaded at the corner C, and there are 15 small squares in the part EBCF of the square. Now the part EBCF = $\frac{1}{2}\frac{5}{5}$ and the shaded squares represent $\frac{6}{25}$. The nine unshaded squares in the part EBCF represent the difference between $\frac{1}{2}\frac{5}{5}$ and $\frac{6}{25}$, and this is seen to be $\frac{9}{25}$, hence $\frac{15}{25} - \frac{6}{25} = \frac{9}{25}$. Here we have subtracted the numerators to get the numerator of the difference.

64. In practice, we have to deal with fractions in which the denominators are not all alike. By multiplying the numerator and denominator of a fraction by any whole number, we are able to change the denominator without altering the value of the fraction. We can thus make all



the denominators alike, and then add or subtract the numerators.

Ex. 1. $Add \frac{1}{2} + \frac{3}{4} + \frac{7}{8}$. If we multiply the numerator and denominator of $\frac{1}{2}$ by 4 we get $\frac{4}{8}$.

Multiply numerator and denominator of $\frac{3}{4}$ by 2 and we get $\frac{6}{8}$.

The expression $\frac{1}{2} + \frac{3}{4} + \frac{7}{8}$ may now be written $\frac{4}{8} + \frac{6}{9} + \frac{7}{4}$, and as we can now add the numerators we get $\frac{17}{8}$ or $\frac{21}{8}$ as the result.

Ex. 2. Subtract $\frac{3}{4}$ from $\frac{5}{8}$. Convert $\frac{3}{4}$ into $\frac{34}{85}$ by multiplying numerator and denominator by 8. Convert $\frac{5}{8}$ into $\frac{35}{80}$ by multiplying numerator and denominator by 7. We now have $\frac{35}{80} - \frac{25}{80} = \frac{1}{10}$.

65. A convenient method of finding a suitable number for the common denominator of a series of fractions is to cancel any denominator which is seen to be a factor of any of the remaining denominators and then take the product of the remaining denominators as the required number.

Thus, in
$$\frac{1}{3} + \frac{6}{25} + \frac{4}{5} + \frac{7}{9}$$

the denominators are 3, 25, 5, 9. We can cancel 3 and 5, for they are factors of 9 and 25 respectively; we then take 25×9 , or 225, as our common denominator, and we must now write each fraction as an equivalent fraction with 225 as denominator.

This is done by dividing 225 by the denominator of the

fraction we wish to convert, and multiplying the result into the numerator.

Thus
$$\frac{1}{3}$$
 becomes $\frac{75}{225}$ for $\frac{225}{3} = 75$, and $75 \times 1 = 75$.

 $\frac{6}{25}$,, $\frac{54}{225}$,, $\frac{225}{25} = 9$, and $9 \times 6 = 54$.

 $\frac{4}{5}$,, $\frac{180}{225}$,, $\frac{225}{5} = 45$, and $45 \times 4 = 180$.

 $\frac{7}{9}$,, $\frac{175}{225}$,, $\frac{225}{9} = 25$, and $25 \times 7 = 175$.

The expression $\frac{1}{3} + \frac{6}{25} + \frac{4}{5} + \frac{7}{9}$ now becomes

 $\frac{75}{225} + \frac{54}{225} + \frac{180}{225} + \frac{175}{225} = \frac{484}{225}$

66. To find the algebraic sum of a series of fractions, we proceed in a similar manner and finally find the algebraic sum of the numerators of the converted fractions.

$$Ex. \ \ Find \ the \ value \ of \ \frac{1}{2} - \frac{3}{8} + \frac{5}{6} - \frac{7}{12}.$$
 Each denominator is a factor of 24,
$$\text{hence, } \frac{1}{2} - \frac{3}{8} + \frac{5}{6} - \frac{7}{12} \text{ becomes}$$

$$\frac{12}{24} - \frac{9}{24} + \frac{20}{24} - \frac{14}{24}$$
 or
$$\frac{12 - 9 + 20 - 14}{24} = \frac{32 - 23}{24} = \frac{9}{24} = \frac{3}{8}.$$

67. Algebraic fractions are added and subtracted in a similar manner.

Ex. 1. Find the sum of
$$\frac{x}{3} + \frac{2x}{5} + \frac{6x}{9}$$
.

If we write each fraction as an equivalent fraction with 45 as denominator, we get $\frac{15x}{45} + \frac{18x}{45} + \frac{30x}{45} = \frac{63x}{45} = \frac{21x}{15} = \frac{7}{5}x$.

Ex. 2. Find the sum of
$$\frac{3}{x} + \frac{5}{x^2} + \frac{7}{x^3}$$
.

Each denominator is a factor of x^3 , hence $\frac{3}{x}$ is equivalent to $\frac{3x^2}{x^3}$ and $\frac{5}{x^2}$ is equivalent to $\frac{5x}{x^3}$, and thus the expression is equivalent to $\frac{3x^2}{x^3} + \frac{5x}{x^3} + \frac{7}{x^3}$.

$$\frac{1}{x^3} + \frac{1}{x^3} + \frac{1}{x^3}.$$
The sum is
$$\frac{3x^2 + 5x + 7}{x^3}.$$

Ex. 3. Find the sum of
$$\frac{2x}{y^2} + \frac{3}{y^3} + \frac{2x^2}{y} + \frac{2y}{x^3} + \frac{y^2}{x^2}$$
. Make each denominator into x^3y^3 . The fractions then become $\frac{2x^4y}{x^3y^3} + \frac{3x^3}{x^3y^3} + \frac{2x^5y^2}{x^3y^3} + \frac{2y^4}{x^3y^3} + \frac{xy^5}{x^3y^3}$, and the sum is $\frac{2x^4y + 3x^3 + 2x^5y^2 + 2y^4 + xy^5}{x^3y^3}$.

68. Addition and subtraction of decimals. In adding and subtracting decimals, it is only necessary to arrange the decimal points beneath each other and then add and subtract as with ordinary numbers. The decimal point in the result is beneath that in the quantities to be added or subtracted. The student will observe that much less labour is required to add and subtract decimal fractions than vulgar fractions.

Ex. Add together 7:63 + 43:585 + 0:00628 + 417. Arrange as shown.

 $\begin{array}{c} 7.63 \\ 43.585 \\ 0.00628 \\ \underline{417.0} \\ 468.22128 \end{array}$

Subtract 76.842 from 97.31.

97:340 76:842

20.498

Subtract 48:34 from 51:2978.

51.2978

48.3400

2.9578

Make the number of figures in the two decimal parts equal by adding ciphers, thus 97.34 is written as 97.340. These ciphers need not, however, be actually written down. They may be added mentally.

69. When repeating and mixed repeating decimals are to be added, the actual repeating figures should be written down until the number of decimal places is at least equal to those of the longest non-repeating decimal or, better still, exceeds it by one or two places. The answer should be given to the number of figures in the longest non-repeating decimals unless otherwise stated.

Ex. Add 57:82, 0:006394, 326:3, 42:4967.

57.82 0.006394 326.33333333 42:496767.67 426:656494.97

The dotted line shows the number of decimal figures required in the answer.

As the seventh decimal figure is 9, the sixth is increased 1; hence the sum is 426.656495 to six decimal places.

70. If it is required to arrange a series of vulgar fractions in order of magnitude, they must be reduced to equivalent fractions with a common denominator. Their magnitudes may then be compared readily by means of the numerators.

Ex. Arrange in order of magnitude, with the largest fraction first, the following fractions:

Each denominator is a factor of 180.

Hence the equivalent fractions are

 $\frac{135}{180}$, $\frac{150}{180}$, $\frac{144}{180}$, $\frac{120}{180}$, $\frac{160}{180}$, $\frac{165}{180}$, $\frac{156}{180}$

Taking the largest numerators first and arranging in order, we get $\frac{11}{12}$, $\frac{8}{9}$, $\frac{13}{15}$, $\frac{5}{6}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$.

Decimal fractions may be similarly arranged. important figures are those immediately following the decimal point.

The following decimals are in order of magnitude.

0.987, 0.734, 0.429, 0.176, 0.0943, 0.00989.

Exercises.

 Find the total length represented by ½ + 3/8 + 7/16 + 9/2 in.
 A journey of 3,000 miles is performed in 6 days. The part accomplished each day, for five days, is $\frac{1}{7}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{5}{24}$, $\frac{7}{36}$. What part is performed on the sixth day?

(3) A plot of building land is divided into lengths of 60 ft., 80 ft., and 120 ft. A sum of £300 is spent on road making, and the length of the road is 1,040 vds. What part of this cost should be paid by each of three men who own respectively a 60-ft., 80-ft., and 120-ft.

(4) Reduce the following lengths to feet and decimal parts of a foot, and find their sum correct to four decimal figures: 3 ft. 6 in.; 7 ft. 8 in. 2 ft. 4 in.; 11 ft. 7 in.; 4 ft. 2 in.

(5) Find the sum of
$$\frac{a}{x^2} + \frac{3}{x} + \frac{5x}{x^3} + \frac{7}{x^4}$$
.

(6) Simplify
$$\frac{a^2}{x^2} \times \frac{a^3 x}{a^4} \div \frac{1}{ax^2}$$
.

- (7) Find the algebraic sum of $\frac{7}{10} + \frac{5}{8} \frac{3}{4} + \frac{7}{16} \frac{2}{5} + \frac{11}{20}$.
- (8) Simplify $(\frac{9}{3} \text{ of } \frac{9}{7}) + (\frac{3}{3} \times \frac{10}{97} + \frac{15}{33}) (\frac{1}{12} \text{ of } \frac{3}{4}).$ (9) Find the value of 0.00036 + 7.002974 2.36427 + 0.00238.
- (10) Find the sum of $\frac{a^2}{x^2y} + \frac{b}{xy^2} \frac{c}{xy} \frac{a^2}{x^2} \frac{b}{y^2} + \frac{c}{x^2y^2}$

Answers, (1) 1號 in. (6) u^2x , (2) $\frac{11}{2530}$. (3) £5 158,5d,; £7 138,10d,; (7) $1\frac{3}{20}$ (8) $1\frac{65}{1008}$. (9) 4.641444. £11 10s. 9d. (10) $\frac{a^2y + bx - cxy - a^2y^2 - bx^2 + c}{x^2y^2}$ (4) 29·2917 ft,

- $(5) \frac{ax^2 + 3x^3 + 5x^2 + 7}{x^4}.$
- 71. Significant figures. In calculations, we may meet with three kinds of numbers: (a) integral numbers. e.g., 6,970; (b) mixed numbers, e.g., 53:296; (c) fractional numbers, e.g., 0.0027036. In each case, the first figure
- reckoned from the left which exceeds zero is a significant figure, and each succeeding figure, whether a cipher or not, is a significant figure, provided there is to the right of it any figure which exceeds zero, e.g., 6,970 has three significant figures, 53 296 has five, and 0 0027036 has also five.
- 72. Approximate values. The results of calculations are often given approximately, instead of accurately, the degree of accuracy depending upon the nature of the calculation, e.g., if the horse-power of the engines of a ship is computed accurately at 45,324 it would usually be sufficient to give this result as 45,000, or, at any rate, as 45,300. The approximate value 45,300 is correct to three significant figures, for there are three significant figures which are accurate, while the two ciphers remaining are only approximate. A result such as 3,646 could be represented approximately as 3,640 or as 3,650, but, as the latter number is nearer to the correct value than the former (being 4 too large whereas the former is 6 too small), it is the one adopted. Now 3,650 represents the result 3,646 approximately to three significant figures, but it is not correct to three significant figures, since the third figure, 5, is not

accurate.* The number 3,646 written correct to three significant figures would be written 3,640, whereas written approximately to the nearest three significant figures it would be 3,650. If a result is required, say, "to four significant figures," it is understood that the result to the nearest four significant figures is meant.

73. Contracted multiplication. It is not always necessary to multiply out quantities fully. For all practical purposes we only require three or four significant figures and all figures beyond these may be neglected. By contracted multiplication, the labour of multiplying out fully is somewhat reduced.

The process will be illustrated by an example.

Multiply 0.0057869 by 7.384.

By the ordinary method we have

 $\begin{array}{r} 0.0057869 \\ 7.384 \\ \hline 231476 \\ 462952 \\ 173607 \\ 405083 \\ \hline 0.0427304696 \end{array}$

By the contracted method we have

-0057869 (Approximate value is -006 × 7 = -042.)

-040508 × (1736 × 7 = -042.)

-040508 × (1736 × 7 = -042.)

-04273 × (1736 × 7 = -042.)

The ordinary method requires no description; it is set down here for comparison. In the contracted method, the multipliers are taken in the reverse order, and it is advisable first to ascertain the approximate value of the result. Thus, our approximate result is $0.06 \times 7 = 0.042$. Then, multiplying 57,869 by 7, we have 405,083, and, as our approximate result is 0.042, we put a cipher and also a decimal point before the figure 4. As we have reversed the order of the multipliers we step the figures one to the right instead of to the left; thus, when we multiply by 3 we have fully 173,607. Suppose we wish to retain four significant figures in the product, we may in multiplying discard all significant figures beyond the fifth, i.e., all beyond the dotted line; hence, we do not require the figures 0,7 on the right of this second product, nor the figure 3 in the first product.

Now multiply by 8 and we have 462,952. We are to diseard the figures 9,5,2; hence we increase the 2 retained by 1 and write down 463.

Now multiply by 4 and we have 231,476. We discard the figures 1,4,7,6, and, as the first figure discarded is 1, we do not increase the 3 by 1; hence we write down 23.

^{*} Many people maintain that "correct to x significant figures" uneans "corrected to," i.e., "to the nearest x significant figures." This is not the plain meaning of the words, but, if this reading is followed, the xth figure must be corrected when the (x + 1)th figure is 5 or exceeds 5.

Adding, we get '042730, and, as we require four significant figures, we cancel the last figure. If the result is required *correct* to four significant figures, we cancel the fifth significant figure whatever it may be: but if the result is required to the *nearest* four significant figures, or simply "to four significant figures," the fifth significant figure is cancelled and the fourth is increased by 1 if the fifth is 5 or exceeds 5.

After a little practice, the student will not need to write down the cancelled figures: they may be replaced by dots, as shown below, but the discarded figures must be allowed for if the first one (from the left) is equal to, or exceeds, 5, and the number carried over must be the same as if no figures were discarded. Thus,

.0057869	O
7:384	
.040508.3	_
1736	i . •
463	
23.	
·04273\	
04273	

Note.—If x significant figures are required, retain (x + 1) figures in working, and correct the result to x figures.

(Approximate value is $.006 \times 7 = .042$.)

Four significant figures required, hence retain five, viz., 40508.

When multiplying by 8 we have

$$8 \times 9 = 72$$
.
 $8 \times 6 = 48$. Add $7 = 55$

$$8 \times 8 = 64$$
. Add $5 = 69$

$$8 \times 7 = 56$$
. Add $6 = 62$, and allow 1 for the 9 cancelled $= 63$.

$$8 \times 5 = 40$$
. Add $6 = 46$.

74. **Contracted division.** The labour of dividing one number by another may be considerably shortened by rejecting one figure of the divisor at each step in the division. An example will illustrate the method employed.

Divide 7:543 by 2:579.

(Approximate value
$$\frac{7.5}{2.6} = 2.9$$
.)

2579)7543(2.925 Obtain the first figure 2 in the quotient in the ordinary manner, 5158 after making the number of 2385 decimal places equal in dividend and divisor. Then $2579 \times 2 =$ 23215158 and this, when subtracted from 7543, gives 2385 as re-64 mainder. Instead of adding a 51 cipher to 2385, as in the usual method, cancel the right-hand 13 figure 9 in the divisor, or mark 12 it off with a , beneath to indicate 1 its cancellation, and put the decimal point in quotient. Now

we have to divide 2385 by 257 and hence we obtain 9 in quotient

Multiply 257 by 9, but add in the figure 8 which would be carried if the whole of the divisor were left. Thus, $257 \times 9 = 2313$, and adding 8 we have 2321. The remainder is 64. Now cancel the 7 and proceed as before. We have $(25 \times 2) + 1 = 51$. Again subtract and then cancel the figure 5. We now have $(2 \times 5) + 2 = 12$, and the result is 2.925.

If we divide out fully in the ordinary manner the above result would be 2.9247, &c., hence this result is accurate to three significant figures. The fourth figure 5 is approximate, but as the fifth figure is 7, the result 2.925 is to the nearest four significant figures.

75. To find a quotient *correct* to four significant figures by the contracted method, both the dividend and divisor must first be written approximately to five significant figures if they exceed this number, and, generally, if x significant figures are required we first write dividend and divisor with (x + 1) significant figures.

Ex. Find the value of $\frac{.05913 \times 25 \cdot 01}{...}$ to four significant figures $(.03964)^3$ Approximate value $\frac{.06 \times .25}{.00006} = \frac{150000}{6} = 25,000.$.05913 25.01 1.1826(Approximate value 29571 is 1.5.) 00 - 100 - $1.4789! \cdots$ = 1.47903964 +03964.0011892 (Approximate value 3568 = .0016.238i -·0015714 · · = .001571.001571.03964 (Approximate value .00004713 = .000006.1414 . 941. . ·00006227 · · · = .00006227

$$\frac{1\cdot479}{\cdot0006227} = \frac{147900}{6\cdot227} = 20,000 \text{ (approx.)}$$

$$\frac{6227)147900(23751}{23360}$$

$$\frac{23360}{18681}$$

$$\frac{4679}{4359}$$

$$\frac{320}{311}$$

$$\frac{311}{9}$$

$$\frac{6}{3}$$

$$Ans. \text{ required } 23,750.$$

Exercises.

Compute by contracted methods to four significant figures only, and without using logarithms-

(1) 8.102×35.14 and 254.3 + 0.09027 (B. E. 1903).

(2) 3.405×9.123 and $3.405 \div 9.123$ (B. E. 1904).

(3) 34.05×0.009123 and $3.405 \div 0.09123$ (B. E. (2) 1904).

(4) 12.39×5.024 and $5.024 \div 12.39$ (B. E. 1905).

(5) 0.01239×5.024 and $0.5024 \div 0.01239$ (B, E, (2) 1905).

(6) $3.214 \times 0.7423 + 7.912$ (B. E. 1906).

(7) $87.35 \div (0.07568 \times 3.501)$ (B. E. 1907).

(8) 9.325×2.056 and 9.325 - 2.056 (B. E. 1908).

(9) 9.325×0.02056 and $9.325 \div 0.02056$ (B. E. (2) 1908).

(10) 2.573×16.81 and $15.81 \div 2.573$ (B. E. 1909).

(11) 0.02351×63.02 and $63.02 \div 0.02351$ (B. E. (2) 1909).

(12) $5.306 \times 0.07632 \div 73.15$ (B. E. (2) 1910).

Answers.

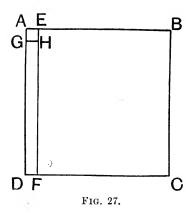
(1) 284.7; 2817. (7) 329·7. (8) 19·17; 4·536. (9) 0·1917; 4·53·6. (2) 31.06; 0.3732. (3) 0.3107; 37.32. (10) 43.25; 6.145. (4) 62·25; 0·4055. (11) 1.482; 2681. (5) 0.06225; 40.55.

(12) 0.005536. (6) 0.3015.

CHAPTER VI.

DUODECIMALS.

76. **Duodecimals.** In Chapter III. it is shown that when we multiply one dimension in feet by a second dimension in feet we get as the product square feet, and, similarly, when we multiply inches by inches we get as the product square inches. In the former case, we get an area expressed in



units, which are square feet, and in the latter case an area

expressed in units which are square inches.

If we multiply a dimension in feet by a second dimension in inches, we get a product or an area expressed in units which are rectangles 1 ft. long and 1 in. wide.

Let ABCD, Fig. 27, be a square of side 1 ft. Let AE be 1 in. and AG be 1 in.

P.M.

Then the square ABCD is a unit of 1 sq. ft, or (ft. \times ft.); the square AEHG is a unit of 1 sq. in. or (in. \times in.); and the rectangle AEFD is a unit 1 ft. long and 1 in. wide or (ft. \times in.).

We know that there are 144 sq. in. in 1 sq. ft.; hence the square

ABCD contains 144 squares each equal to AEHG.

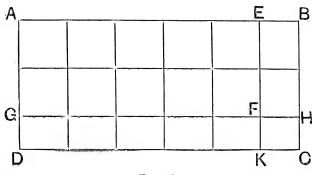


FIG. 28.

Since $AE = \frac{1}{12}AB$, the square ABCD will contain 12 rectangles, each equal to AEFD, and, again, the rectangle AEFD contains 12 squares AEHG or 12 sq. in.

If we now arrange these units in column form, we find that column 1 contains sq. ft., column 2 contains $\frac{1}{12}$ ths sq. ft., and column 3 contains $\frac{1}{14}$ ths sq. ft. or sq. in. Further, each unit in column 2 is $\frac{1}{12}$

1.	2.	3,
ft. × ft. or sq. ft.	$ft. \times in.$ or $\frac{1}{12}ths sq. ft.$	in. × in, or sq. in, or ½ ths sq. ft,

unit of column 1, and each unit of column 3 is 1 unit of column 2.

77. Let us now consider how the area of a rectangle 5 ft. 10 in.

long and 2 ft. 8 in, wide is made up.

If we make AB, Fig. 28, 5 ft. 10. in. long, and AD 2 ft. 8 in. long, and mark off feet from A along AB and AD, and draw lines through each division parallel to AB and AD, we divide the rectangle up into a number of squares and rectangles.

The rectangle AEFG is 5 ft. by 2 ft. and thus contains 10 squares,

each 1 sq. ft. area.

The rectangle GDKF is 5 ft. long and 8 in. wide, and thus contains 5 rectangles, each 1 ft. by 8 in. Each of these rectangles contains 8 units 1 ft. by 1 in., thus rectangle GDKF contains 40 units 1 ft. by 1 in.

Similarly, the rectangle EFHB contains 20 units 1 ft. by 1 in., for

it is 2 ft. by 10 in.

The rectangle FKCH is 10 in. by 8 in., and thus contains 80 sq. in.

The total area thus contains :-

10 sq. ft., (40 + 20) units each 1 ft. by 1 in., and 80 sq. in. Now 12 units 1 ft. by 1 in. are equal to 1 sq. ft., hence 60 such units are equal to 5 sq. ft.

The area of the rectangle is thus (10 + 5) sq. ft. 80 sq. in., or

15 sq. ft. 80 sq. in.

78. In computing the superficial areas of floors, doors, sheets of glass, etc., or the volumes of a number of baulks of timber, etc., a method is used which dispenses with the necessity for reducing all the dimensions to feet, or to inches.

This method is known as the duodecimal system, and by it we can find an area in square feet and square inches when the dimensions are given in feet and inches, and similarly we can find a volume in cubic feet and cubic inches.

Consider the process which we unconsciously apply in ordinary multiplication and addition, e.g., when we multiply, say, 153 by 8, we arrive at the product in the following manner:—8 × 6 = 48, we divide by 10 and get quotient 4 and remainder 8, we put down the remainder 8 in the product and carry 4. We next say $5 \times 8 = 40$, add 4 = 44, and we again divide by 10, getting quotient 4 and remainder 4. We again write in the product the remainder 4 and carry 4. We now say $1 \times 8 = 8$, add 4 = 12, and we write down 12, getting as the complete product, 1248.

In the duodecimal system, we divide by 12 instead of by 10, hence the name "duodecimal."

The method will be illustrated by an example. Let us find the area of the above rectangle, Fig. 28, which is 5 ft. 10 in. by 2 ft. 8 in.

First set down the multiplicand 5 ft. 10 in. and the multiplier 2 ft. 8 in. as in ordinary multiplication.

We set down the products in columns 1, 2, 3, as illustrated in Art. 76, col. 1 at the left being (ft. \times ft.), col. 2 (ft. \times in.) and col. 3 (in. \times in.).

First multiply by 2, and commence with the right-hand end of

5 ft.

2 ft.

Col. 1.

11

3

15

10 ins.

Col. 2.

8

6

10

8 ins.

Col. 3.

8

8

the multiplicand.

Thus 2 ft. \times 10 in. = 20 units (ft. \times in.); \div by 12, and we get 1 sq. ft. and 8 units (ft. \times in.); put down remainder 8 in col. 2 and carry 1.

Now 2 ft. \times 5 ft. = 10 sq. ft. and 1 sq. ft. carried on = 11 sq. ft.

Set this down in col. 1.

Now multiply by 8 in. We have 8 in. \times 10 in. = 80 sq. in. \div 12 = 6 units (ft. \times in.) and

8 sq. in.

Put down 8 sq. in. in col. 3

and carry 6.

8 in. \times 5 ft. = 40 units (ft. \times in.), add 6 = 46 units. \div 12 = 3 sq. ft. and 10 units (ft. \times in.).

Put 3 in col. 1 and 10 in col. 2.

Now add. We get 8 in col. 3, 18 in col. 2, which we + by 12, thus getting 1 sq. ft. to carry and 6 units (ft. x in.) remainder. Put 6 in col. 2 and then add 1 to col. 1 and we get 15 sq. ft. for col. 1.

The area thus = 15 sq. ft., 6 units (ft. \times in.), 8 sq. in. Reducing the 6 units (ft. \times in.) to inches we have 6 \times 12 = 72 sq. in., and add the 8 sq. in. and we get 15 sq. ft. 80 sq. in., which agrees with the area previously found

79. The process can also be extended to fractions of an Each fraction is reduced to an equivalent fraction with 12 as denominator, and the numerator alone is then We shall now get five columns in our result. fourth column is (in. $\times \frac{1}{12}$ ths in.) = $\frac{1}{12}$ ths sq. in., and the fifth column is $(\frac{1}{12}$ ths in. $\times \frac{1}{12}$ ths in.) = $\frac{1}{144}$ ths sq. in.

Ex. Find area of a rectangle 5 ft. $7\frac{1}{2}$ in. by 2 ft. $3\frac{1}{4}$ in. Here $\frac{1}{2}$ in. $=\frac{6}{12}$ in., hence we write 5 ft. 7 in. $6^*(\frac{1}{12}$ ths in.). ,, 2 ft. 3 in. 3 ($\frac{1}{12}$ ths in.). And $\frac{1}{4}$ in. $= \frac{3}{12}$ in. ,, 11 Thus we multiply 5 ft. 7 in. 6 ($\frac{1}{12}$ ths in.). 2 ft. 3 in. 3 ($\frac{1}{12}$ ths in.).

Multiplying by 2 we get 11 sq. ft. 3 (1ths sq. ft.), 0 sq. in. Multiplying by 3 we get 1 sq. ft. 4 ($\frac{1}{12}$ ths sq. ft.), 10 sq. in., 6 ($\frac{1}{12}$ ths sq. in.). Multiplying by second 3 we get 1 ($\frac{1}{12}$ th sq. ft.), 4 sq. in., 10 $(\frac{1}{12}$ ths sq. in.), 6 $(\frac{1}{144}$ ths sq. in.).

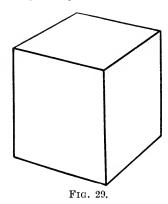
	11	3	0		
	1	4	10	6	
		1	4	10	6
Adding	12	9	3	4	6

This result is 12 sq. ft., 9 ($\frac{1}{12}$ ths sq. ft.), 3 sq. in., 4 ($\frac{1}{12}$ ths sq. in.) 6 ($\frac{1}{14}$ ths sq. in.).

Reducing the 9 units to square inches, we have 108 sq. in., and adding 3 we get 111 sq. in.

We have also
$$\left(\frac{4}{12} + \frac{6}{144}\right)$$
 sq. in
$$= \frac{48 + 6}{144} = \frac{54}{144} = \frac{3}{8} \text{ sq. in.}$$

Hence area is $12 \text{ sq. ft. } 111\frac{3}{8} \text{ sq. in.}$

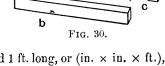


80. Extending this system to volumes, we observe that when we (1) multiply a square foot by a linear foot we get a cubic foot, or (ft. × ft. ×

ft.), Fig. 29.

(2) A square foot multiplied by a linear inch gives a unit volume which is a block 1 ft. square and 1 in. thick, or (ft. \times ft. \times in.). Such a block is $\frac{1}{12}$ cub. ft. and contains 144 cub. in., Fig. 30 (a).

(3) A square inch multiplied by a linear foot is a



a

rod of 1 in. square section, and 1 ft. long, or (in. \times in. \times ft.), and contains 12 sq. in., Fig. 30 (b).

(4) A square inch multiplied by a linear inch is a block of 1 cub. in. or (in. \times in. \times in.), Fig. 30 (c).

Arranging these units in columns as before, we have

Col. 1.	Col. 2.	Col. 3,	Col. 4.
$ft. \times ft.$ $\times ft.$ or cub. $ft.$	ft. × ft. × in.	in. × in. × ft.	in. × in. × in. or cub. in.

and again we see that

12 units of col. 4 are = 1 unit of col. 3.

12 units of col. 3 are = 1 unit of col. 2.

12 units of col. 2 are = 1 unit of col. 1.

The duodecimal system is extended to calculation of volumes as follows. Find the volume of a block 7 ft. 9 in. long, and of rectangular cross section 5 ft. 10 in. by 2 ft. 8 in.

First find the area of the cross section.

This we already know from Art. 78 to be 15 sq. ft., 6 (½ ths sq. ft.), 8 sq. in. or 15 (ft. × ft.), 6 (ft. × in.), 8 (in. × in.).

Now multiply this by the length 7ft. 9 in.

Begin to multiply by 7 and take first the right-hand end of the multiplicand. We have $7 \times 8 = 56$ \div by 12 = 4 quotient and 8 remainder. The 8 units are (in. \times in. \times ft.), and fall in Col. 3.

We nowhave 7×6 = 42, add 4 carried = 46, \div by 12 = 3 quotient and 10 remainder. 5 ft. 10 in. 2 ft. 8 in.

Col. 1.	Col. 2.	Col. 3.	Col. 4.
108 11	10 8	8 0	0
120	6	8	0
ft. × ft. × ft.	ft. × ft. × in.	in. × in. × ft.	in. × in. ×

The 10 units are (ft. \times ft. \times in.), and fall in Col. 2. We now have $7 \times 15 = 105$, add 3 carried = 108. These units are (ft. \times ft. \times ft.) and fall in Col. 1.

Repeat this process by multiplying by 9, and we have—

72 (in. \times in. \times in.), \div by 12 = 6 quotient and 0 remainder.

6 =54, add 6 = 60 (in. \times in. \times ft.), \pm by 12 =5 quotient and 0 remainder.

 $9 \times 15 = 135$, add 5 = 140 (in. \times ft. \times ft.), \div by 12 = 11quotient and 8 remainder.

Adding we get 120 cub. ft., 6 (in. \times ft. \times ft.), 8 (in. \times in. \times ft.), 0 (in. \times in. \times in.).

Reducing the 6 units to cubic inches we have $6 \times 144 = 864$ cub. in. $,, \quad ,, \quad 8 \times 12 = 96 \quad ,,$

81. The volume may be similarly found when each length has a fractional part of an inch. We then have to remember that

$$\begin{array}{c} (\text{in.} \times \text{in.} \times \frac{1}{12} \text{th in.}) = \frac{1}{12} \text{th cub. in.} \\ (\text{in.} \times \frac{1}{12} \text{th in.} \times \frac{1}{12} \text{th in.}) = \frac{1}{14} \text{th cub. in.} \\ (\frac{1}{12} \text{th in.} \times \frac{1}{12} \text{th in.} \times \frac{1}{12} \text{th in.}) = \frac{1}{1723} \text{th cub. in.} \end{array}$$

Ex. Find volume of a block 4 ft. $3\frac{1}{4}$ i.i. by 2 ft. $5\frac{1}{2}$ in. by 6 ft. 73 in.

4 ft.
$$3\frac{1}{4}$$
 in. = 4 ft. 3 in. 3 ($\frac{1}{12}$ ths in.).
Similarly 2 ft. $5\frac{1}{2}$ in. = 2 5 6 (,,).
and 6 ft. $7\frac{3}{4}$ in. = 6 7 9 (,,).

Multiplying by duodecimal system.

62	1 i 1	11	3	0		
6	1	อี	11	1	6	
	7	10	$^{11}_{5}$	10	10	6

In the volume we thus have 69 cub. ft.,

$$(9 \times 144) = 1296 \,\mathrm{cub.\,in.}$$

$$(3 \times 12) = 36 \text{ cub. in.}$$

8 = 8 cub. in.

$$\hat{8} = 8 \text{ cub. in}$$

Total 1340 cub. in., and

$$\frac{0}{12} \text{ cub. in.} + \frac{4}{144} \text{ cub. in.} + \frac{6}{1728} \text{ cub. in.}$$
$$= \frac{48+6}{1728} = \frac{54}{1728} = \frac{3}{96} = \frac{1}{32} \text{ cub. in.}$$

Total volume is thus 69 cub. ft. 1340 cub. in.

In the above example it will be seen that when we have the actual figures of the result, viz.,

69 cub. ft. 9 - 3 - 8 - 0 - 4 - 6,

we reduce to cubic inches and fractions of a cubic inch as follows:— Multiply col. 2 by 144, col. 3 by 12, and add the two products to col. 4 for cubic inches.

Divide col. 5 by 12, col. 6 by 144, col. 7 by 1,728, and add together the three fractions thus obtained, to get the fractional part of a cubic inch in the volume.

Exercises.

- (1) Find the area of a rectangular plate 15 ft. 31 in. long and 10 ft. $7\frac{1}{2}$ in. wide by the duodecimal method.
 - (2) Find the following areas:—
 - (a) 3 ft. 4 in. by 2 ft. 7 in.
 - (b) 2 ft. $6\frac{1}{2}$ in. by 3 ft. $5\frac{3}{4}$ in.
 - (c) 11 ft. $8\frac{3}{4}$ in. by 4 ft. $7\frac{1}{4}$ in.
 - (d) 2 ft. $8\frac{1}{2}$ in. by 1 ft. 9 in.
 - (3) Find the following volumes:— (a) 2 ft. 7 in. by 3 ft. 8 in. by 10 ft. 3 in.
 - (b) 11 ft, 6 in. by 2 ft. 4 in. by 3 ft. 8 in.
 - (c) 4 ft, $8\frac{1}{2}$ in. by 2 ft, $4\frac{1}{4}$ in. by 5 ft. 6 in. (d) 11 ft. $\tilde{7}_{3}^{3}$ in. by 5 ft. $\tilde{6}_{2}^{1}$ in. by 8 ft. 3_{4}^{1} in.
- (4) A window has 20 panes, each 1 ft. 8 in. by 2 ft. 6 in. Find the cost of glazing at 5d. per square foot.
- (5) A tank is 3 ft. 4 in. long, 2 ft. 6 in. wide, and 5 ft. 3 in. deep. What is the cost of lining the tank at 6d. per square foot super.?
- (6) A baulk of timber is 15 ft. long and of section 3 ft. 4 in. by 2 ft. 8 in. Find its volume.
- (7) Calculate to the nearest penny the total cost of glass for the following pictures, at 5d. per square foot:—
 - (1) 3 ft. 3 in. by 1 ft. 8 in. (2) 2 ft. 4 in. by 1 ft. 10 in.
 - (3) 20 in. by 16 in.
- (8) Find the volume of a block of stone whose dimensions are 3 ft. 8½ in., 2 ft. 6¾ in., and 1 ft. 9½ in.

(9) Find the volume of a rectangular pillar of length 10 ft. 6 in., breadth 5 ft. 3 in., and height 35 ft. 8 in. If each cubic foot weighs 50 lb., find its weight.

(10) Find the volume of a rectangular heap of material 15 ft. 63 in.

long, and of sectional area 3 ft. $7\frac{1}{2}$ in. by 2 ft. $10\frac{1}{4}$ in.

(11) A heap of stones is approximately triangular in section having a base 5 ft. 6 in. and height 4 ft. 3 in., and the heap is 12 ft. 6 in. long. What is the volume of the heap?

(12) A railway cutting is of rectangular section 25 ft. 9 in., by 12 ft. 6 in., and is 45½ yds. long. What volume of material was removed to make the cutting?

(13) A reservoir $50\frac{1}{2}$ yds. long, $45\frac{1}{2}$ ft. wide, and 8 ft. 9 in. deep is

full of water. How many cub. ft., &c., of water does it contain?

(14) A hollow column 12 ft. 6 in. high has an outer section 3 ft. 4 in. by 2 ft. 7. in., and an inner section 2 ft. 8 in. by 1 ft. 11 in. What

volume of metal does it contain?

(15) A metal casting measures over all 7 ft. 5 in. by 3 ft. 10 in. by 2 ft. 7 in. In this casting are cut two slots each running the whole length of the casting one being 3 ft. 4 in. by 7 in., the other 3 ft. by 1 ft. 4 in. A slot also runs across the casting being 2 ft. 7 in. long and 5 in. by 3 in. What is the volume and weight of the casting if each cubic foot weighs 460 lbs.

Answers.

(1) $162 \text{ sq. ft. } 36\frac{3}{5} \text{ sq. in.}$

(2) 8 sq. ft. 88 sq. in.; 8 sq. ft. $121\frac{3}{3}$ sq. in.; 54 sq. ft. $0\frac{7}{16}$ sq. in.; 4 sq. ft. $106\frac{1}{2}$ sq. in.

- (3) 97 cub. ft. 156 cub. in.; 98 cub. ft. 672 cub. in.; 60 cub. ft. 1,664\frac{1}{4} cub. in.; 533 cub. ft. 1,343\frac{1}{25} cub. in.
 - (4) £1 14s. 9d.
 - (5) £1 14s. $9\frac{1}{2}d$.
 - (6) 133 cub. ft. 576 cub. in.

- (7) 5s.
- (8) 16 cub. ft. $1429\frac{31}{32}$ cub. in.
- (9) 1,966 cub. ft. 216 cub. in.; 98,3064 lb.
- (10) 161 cub. ft. $26\frac{5}{32}$ cub. in.
- (11) 146 cub. ft. 162 cub. in.
- (12) 43,935 cub. ft. 1,620 cub. in.
- (13) 54,344 cub. ft. 1,548 cub. in. (14) 60,315 cub. ft. 1,548 cub. in.
- (15) 29 cub. ft. 153 cub. in ;
- 13,381 lbs.

CHAPTER VII.

ALGEBRAIC OPERATIONS: INDICES.

82. Use of brackets. When it is required to group together parts of an expression for any purpose, we use brackets. The several forms of brackets in common use are (), { }, []. A line placed over a series of terms has the same effect as enclosing the terms within brackets. Thus, (a + b) and $\overline{a + b}$ mean the same thing.

The whole expression within brackets is treated exactly as if it were a single term, until the brackets are removed. Thus, if it is required to square the expression a + b, we should write this $(a + b)^2$. If we require to multiply (a + b) by 2x we can write it $2x \times (a + b)$, or simply 2x(a+b), in just the same way that we write 2x multiplied

by a as 2xa.

83. We explained, in Chapter I., that an algebraic symbol, say x, really represented $+ 1x^1$, the + sign, the coefficient 1, and the index 1 all being omitted. In the same way, (a + b) really represents $+ 1(a + b)^1$, the corresponding parts being omitted. If we multiply an expression by -1, we change all the signs in such an expression. Thus, a + b multiplied by -1 = -a - b. If then we write a + b in brackets, thus (a + b), and we multiply by -1, this is written -1(a + b) or simply -(a+b). We know the product to be -a-b; hence, the brackets can only be removed from an expression which has a - sign in front of the brackets by changing all the signs within the brackets.

Thus, a - (b + c) is equivalent to a - b - c, the signs of b and c being changed from + to - when the

brackets are removed.

Again, a + b multiplied by + 1 may be written + 1(a + b) or simply (a + b). The product we know to be a + b; hence, we can remove the brackets when they are preceded by a + sign without making any

change.

84. When we wish to multiply (a + b) by some quantity, say 2x, we may write this as 2x (a + b). Now we know that the product of (a + b) and 2x is 2ax + 2bx, by the ordinary rule for multiplication. Thus, 2x(a + b) must = 2ax + 2bx, and hence the brackets may be removed if we multiply every term in the brackets by the quantity or term immediately preceding the brackets. Similar reasoning can be applied to division, for we can

regard $(a + b) \div 3x$ as $(a + b) \times \frac{1}{3x}$, or simply $\frac{1}{3x}(a + b)$; we know by the rules for division that the

result will be $\frac{a}{3x} + \frac{b}{3x}$, which again is equivalent to

multiplying each term in the brackets by $\frac{1}{3x}$, or equiva-

lent to dividing each term by 3x when the brackets are removed.

85. One set of brackets may enclose another set of brackets, for example, $3a\{2x - 5(a+b) - 3(b+c)\}$. In simplifying an expression such as this, first remove the inner brackets (); we then get

$$3a\{2x - 5a - 5b - 3b - 3c\}.$$

Here note that we have multiplied (a + b) by -5 and (b + c) by -3, and hence changed the signs. Now remove the outer brackets $\{\ \}$ by multiplying by 3a and we get

$$6ax - 15a^2 - 15ab - 9ab - 9ac$$

which finally becomes $6ax - 15a^2 - 24ab - 9ac$.

86. Root signs and signs for raising quantities to any required power may be applied to expressions as a whole when the expressions are enclosed within brackets. Thus

the cube root of a + b - c can be written $\sqrt[3]{(a + b - c)}$, and again the fifth power of x - y can be written

 $(x - y)^5$.

87. That two expressions are to be multiplied together, can be expressed by enclosing each expression within brackets and then writing them down together with no sign between them; thus, multiply 5a + 2b by 3a - 5b can be written (5a + 2b)(3a - 5b).

In Chapter I., we have dealt with the multiplication of an expression by a single term. To multiply an expression by any other expression, when either of them contains any number of terms, we proceed as follows, taking as an example the two expressions (5a + 2b) and (3a - 5b):—

Multiply each term in the first expression by each term in the second expression, beginning at the left-hand side. Thus, $3a(5a + 2b) = 15a^2 + 6ab$. Set this down as shown. Again $-5b(5a + 2b) = -25ab - 10b^2$. Set down this expression beneath the first, and place corresponding terms below each other; thus, -25ab is placed below 6ab. The remaining terms are placed to the right. Now add, and the algebraic sum is the required product.

$$\begin{array}{r}
5a + 2b \\
3a - 5b \\
\hline
15a^2 + 6ab \\
- 25ab - 10b^2 \\
\hline
15a^2 - 19ab - 10b^2
\end{array}$$

A further example is shown below for reference; each step is indicated on the left.

$$Ex. \quad Multiply \ 7xy - 10xz + 3 \ by \ xy + 2xz - 2.$$

$$xy(7xy - 10xz + 3) = 7x^2y^2$$

$$-10x^2yz + 3xy$$

$$2xz(7xy - 10xz + 3) = 14x^2yz$$

$$-20x^2z^2 + 6xz$$

$$-2 (7xy - 10xz + 3) = -14xy + 20xz - 6$$

$$14x^2yz - 20x^2z^2 + 6xz$$

$$-14xy + 20xz - 6$$

$$7x^2y^2 + 4x^2yz - 11xy - 20x^2z^2 + 26xz - 6$$

88. There are several results to be obtained by multiplication which it is necessary to remember, as they are often required.

(1)
$$(a + b) (a + b)$$
 or $(a + b)^{2}$.
By multiplication—
$$\begin{array}{c}
a + b \\
a + b \\
a^{2} + ab \\
ab + b^{2}
\end{array}$$
By multiplication—
$$\begin{array}{c}
2x + 3y \\
2x + 3y \\
\hline
4x^{2} + 6xy \\
+ 6xy + 9y^{2} \\
\hline
4x^{2} + 12xy + 9y^{2}
\end{array}$$

Thus, $(a + b)^2 = a^2 + 2ab + b^2$ and, $(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$.

If the form of the product, in each of the above examples, is compared with that of the expression to be squared, we observe that we have the first term, +a, squared (a^2) ; the second term, +b, squared (b^2) ; and twice the product of the two terms (2ab). Similarly, we have $(+2x)^2$ or $4x^2$; $(+3y)^2$ or $9y^2$; and $2 \times (+2x) \times (+3y)$ or 12xy.

Hence, the square of an expression consisting of two terms is = square of first term + twice product of both terms + square of second term. This rule holds good for any expression when the signs of the terms are taken into account, and we can thus write down at once the square of any expression. As an example, we will find the square of (a - b), or $(a - b)^2$.

```
The first term is +a, and the square of first term is +a^2.
```

The second term is -b, and the square of second term is $+b^2$.

Twice the product of both terms is $2 \times (+a) \times (-b) = -2 ab$. Hence $(a-b)^2 = a^2 + b^2 + (-2ab)$

 $=a^2-2ab+b^2$, since this expression is the sum of (a^2+b^2) and -2ab.

This result should be tested by actual multiplication.

Ex. 1. Find $(5m + 6n)^2$.

Square is = 1st term squared + twice product of both terms + 2nd term squared - 25 x 1 60 m + 26 x 2

 $=25m^2+60mn+36n^2.$

Ex. 2. Find $(3x - 5y)^2$.

The first term is (3x) and the second term is (-5y).

Hence the square is $9x^2 - 30xy + 25y^2$.

(2) A further product to be remembered is (a + b) (a - b), or the product of the sum of two terms multiplied

$$\begin{array}{ccc}
a & + & b \\
a & - & b \\
\hline
a^2 & + ab \\
- & ab & - b^2 \\
\hline
a^2 & - b^2
\end{array}$$

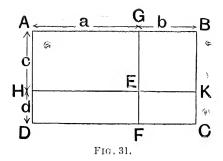
by the difference of the two terms. By multiplication we learn that the product is $(a^2 - b^2)$ or the difference of their squares.

We can thus write down at once a product of two expressions in which the terms are alike but the signs differ.

Thus, (3a - 5b) (3a + 5b)= $9a^3 - 25b^3$.

The converse of each of these results is also useful to remember. They are collected here for reference.

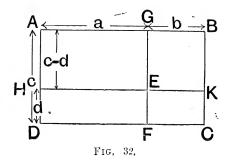
(1) $(a + b)^2 = a^2 + 2ab + b^2$ or $(a^2 + 2ab + b^2)$ may be written $(a + b)^2$. (2) $(a - b)^2 = a^2 - 2ab + b^2$ or $(a^2 - 2ab + b^2)$ may be written $(a - b)^2$. (3) $(a + b)(a - b) = a^2 - b^2$ or $(a^2 - b^2)$ may be written (a + b)(a - b).



89. Algebraic multiplication may be demonstrated graphically. If we have a rectangle of length a and breadth b, the area is ab. Art. 36. If we have a square of length a, the area is a^2 . Now consider a line AB, Fig. 31, of length (a + b) and a second line AD of length (c + d).

The rectangle ABCD has length (a + b) and breadth (c + d); its area is thus = length \times breadth = (a + b) \times (c + d) = ac + bc + ad + bd.

In Fig. 31, the area ABCD = area AGEH + area GBKE + area HEFD + area EKCF = ac + bc + ad + bd, which agrees with the result arrived at by algebraic multiplication.



To show graphically that (a + b)(c - d) = ac - ad + bc - bd.

```
In Fig. 32, let AG = a and GB = b. Then AB = (a + b).

" AD = c and DH = d. Then AH = (c - d).

Area of rectangle AGFD = ac.

" " HEFD = ad.

Hence " " " AGEH = (ac - ad).

Again, the area of rectangle GBCF = bc.

" " EKCF = bd.

Hence " " " GBKE = (bc - bd).

But the rectangle ABKH = (a + b) (c - d).

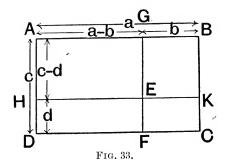
= rectangle AGEH + rectangle GBKE.

= (ac - ad) + (bc - bd).

Hence (a + b) (c - d) = ac - ad + bc - bd.
```

To show graphically that (a - b) (c - d) = ac - ad - bc + bd.

In Fig. 33, let
$$AB = a$$
 and $GB = b$. Then $AG = (a - b)$.
, $AP = c$ and $HD = d$. Then $AH = (c - d)$.



Area of rectangle AGEH =
$$(a - b)(e - d)$$

" ABCD = ae
" HKCD = ad
" GBCF = be
" EKCF = bd
" HEFD = (HKCD - EKCF) = $(ad - bd)$

", ", ", GBKE = (GBCF - EKCF) = (bc - bd).

Now the area of rectangle AGEH = rectangle ABCD - rectangle HEFD - rectangle EKCF - rectangle GBKE,

$$(a-b)(c-d) = ac - (ad-bd) - bd - (bc-bd)$$

$$= ac - ad + bd - bd - bc + bd$$

$$= ac - ad - bc + bd$$
changing signs
when brackets
are removed.

Exercises.

- (1) Find the product of 2a + 3b and 4a 5b.
- (2) Multiply $a^2 + 2ab + b^2 \times 2a 3b$.
- (3) Multiply $x^2 + 6x + 9 \times 2x + 4$.
- (4) Multiply $a^3b^3 + 3a^3b^2 + 3a^2b^3 \times 2a^2b 3ab^2$.
- (5) Multiply $10x + 12xy 15y \times 2x^2 + 3xy 4y$.
- (6) Find the product of $(a^2 + 2ab + b^2)(a + b)(a b)$.
- (7) Write out the squares of (a + 2b), (2a 3b), (-4x + 6y), (-3x 2y), (2a + 3bc), and (2xy + 3ab).
- (8) Represent graphically an area = (2x + 3y) (3x 4y), taking x as 1 in. and y as $\frac{1}{2}$ in.
 - (9) Show graphically that $(a + 2b)(a 2b) = a^2 4b^2$.
 - (10) Find the product of $x^2 + 2xy + y^2$ and x y.
 - (11) Multiply $x^3 3x^2 + 11x 8 \times x^2 + 4x + 5$.
 - (12) Find the value of π (R² r²) when R = 21, r = 20.
- (13) Remove the brackets from $\{3a^2 + 2(a+b) 3(\frac{a}{3} + 2b) (a+3b)\}$.
- (14) Simplify $-[-x^2-2x(y+x)+2xy-3x(x+2y)-3xy\{3-2(x+y)\}]$

Answers.

(1)
$$8a^2 + 2ab - 15b^2$$
, (10) $x^3 - xy^2 + x^2y - y^3$, (11) $x^5 + x^4 + 4x^3 + 21x^2 + (3) 2x^3 + 16x^2 + 42x + 36$, (22) $2a^3b^4 + 6a^5b^3 - 3a^4b^4 - (12) 128 \cdot 8056$, (32) $2a^3b^5 - 9a^3b^5$, (13) $3a^2 - 7b$, (14) $6x^2 + 15xy - 6x^2y - 6xy^2$, (16) $a^4 + 2a^3b - 2ab^3 - b^4$, (17) $a^2 + 4ab + 4b^2$, $4a^2 - 12ab + 9b^2$, $9x^2 + 12xy + 4y^2$, $4a^2 + 12abx + 9b^2c^2$, $4x^2y^2 + 12abxy + 9a^2b^2$,

90. Algebraic subtraction. The difference between two positive terms is found by subtracting one term from the other; thus, if we subtract 15x from 20x the difference is 5x.* Now if we had changed the sign of the term to be subtracted from + to - and then found the algebraic sum of the two terms, we should arrive at the same result, for + 15x becomes - 15x and the algebraic sum of + 20x and - 15x is 5x. This process holds good for an expression

which consists of a number of terms; hence in subtracting 5x + 6y + 11z from 8x + 9y + 10z we write down the first expression with the signs changed and then add as shown.

$$\frac{-\frac{8x+9y+10z}{5x-6y-11z}}{3x+3y-z}$$

Now suppose we have to subtract a negative term from another term. The same rule should suffice; hence to subtract -3x from +5x we change the sign of -3x to +3x and add, and we thus get +8x as the result of sultracting -3x from 5x.† Similarly, to subtract an expression in which the sign of each term is negative from another expression, we change all the signs in the expression to be subtracted and then add the two expressions together.

This rule can be illustrated graphically. Take a sheet of squared paper and draw four lines thereon, a = 30 divisions, b = 20, c = 25, d = 10, as shown in Fig. 34. We regard

^{*} 5x must be added to 15x to make 20x. † + 8x must be added to -3x to make 5x.

lines measured to the right as positive and lines measured to the left as negative. In the upper line AB = a, and BC = b, hence AC = (a + b). The line DE = c, and EF (measured to the left) = -d, hence the line DF = (c - d).

Now suppose we require to find a line to represent $\{(a+b) - (c-d)\}$. We set off AC = (a+b) as shown in the third line from the top in Fig. 34, and we draw to the left from C a line = (c-d), that is, a line = FD, hence the line AD = $\{(a+b) - (c-d)\}$. By our rule given above for subtraction we have, in subtracting c-d from a+b, to change the signs of c and d and add to a+b. We thus get as the difference a+b-c+d, which we can write

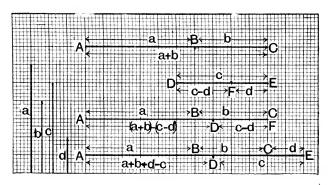


FIG. 34.

a+b+d-c. The line which represents this expression should be equal to AD, since we have found above that AD = $\{(a+b)-(c-d)\}$. To test this, draw AB, BC, CE, = a+b+d, as shown in line four in Fig. 34, and set back from E a line ED = -c, then a+b+d-c is represented by AD, and as the two lengths AD in the third and fourth lines in the figure are equal, we conclude that the rule for changing signs holds good for negative and positive quantities.

By substituting numerical values for a, b, c, d, the result is made clear. Thus the upper line AC = (a + b) = (30 + 20) = 50 divisions on the squared paper. Line DF

= (e - d) = (25 - 10) = 15 divisions. The third line AD = (a + b) - (c - d) = (50 - 15) = 35 divisions,and the fourth line = (a + b + d) - c = (30 + 20 + 10)-25 = 35 divisions also.

Ex. 1. Subtract
$$5x + 6y - 11z$$
 from $2x - 5y - 10z$.

Write down as for addition with the signs of the expression to be subtracted changed,

$$\begin{array}{r}
 2x - 5y - 10z \\
 -5x - 6y + 11z \\
 \hline
 -3x - 11y + z
 \end{array}$$

Ex. 2. Simplify

$$-[3a-2b-(-a+2b+3c)-\{4a-(2a+b)\}].$$

Remove the brackets and change the signs when a - sign precedes a bracket.

Removing () brackets $-\left[3a - 2b + a - 2b - 3c - \left\{4a - 2a - b\right\}\right]$. Removing { } brackets - [3a - 2b + a - 2b - 3c - 4a + 2a + b]. Removing [] brackets - 3a + 2b - a + 2b + 3c + 4a - 2a - b. Collecting like terms -2a + 3b + 3c.

Exercises.

- (1) Subtract -7x + 11y 10z from 3x + 2y + 5z.
- (2) Subtract 3x + 2y from 5x 3y + 10z.
- (3) From 5a + 3b subtract 2a + 4b 2c.
- (4) Show graphically a line equal to 2a + 3b (3c 2d) when $a = 1 \text{ in., } b = \frac{1}{2} \text{ in., } c = \frac{3}{4} \text{ in., } d = \frac{11}{2} \text{ in.}$
- (5) A rectangle has length (x + y) and breadth (2x y). Find its area. A second rectangle has length (x-y) and breadth (x-2y). Find its area, and subtract the second area from the first. Show that your result is correct by putting x = 3 in. and y = 1 in.
 - (6) Simplify 5a + 3b 2(a + b).
 - (7) Simplify $-\left\{5a+4b-2(3a+2b)+2a(b+c)-2b(a-c)\right\}$.
 - (8) Simplify $(2a + 3b)(2a 3b) 4a^2 + 9b^2$.
- (9) Write down algebraically: $x \times$ three times the product of a+b into a-b, and find the value when x=2, a=3, b=4.
- (10) Write down algebraically the difference when three times the sum of x and y is subtracted from the product of a squared and the difference of the squares of x and y.

Answers.

- (1) 10x 9y + 15z.
- (6) 3a + b.
- (2) 2x 5y + 10z. (3) 3a b + 2c.

(7) a - 2ac - 2bc,

- (8) 0.
- $(4) 4\frac{1}{4}$. $(5) x^2 + 4xy - 3y^2$
- (9) 3x(a + b)(a b; -42.(10) $a^2(x^2 y^2) 3(x + y).$
- 91. Division. We have dealt in Chapter I. with the rules for dividing one term by another, and also for dividing

any given expression by a single term. The division of one expression by another whatever number of terms the two expressions may have is performed as shown by the following example:—

Ex. Divide
$$4x^2 + 12xy + 9y^2$$
 by $(2x + 3y)$.

Divide the first term $4x^2$ of the dividend by the first term 2x of the divisor, and the result $\frac{4x^2}{2x} = 2x$ is the first term of the quotient. Now multiply the whole of the divisor by 2x and subtract from the dividend. The remainder is $6xy + 9y^2$.

$$\begin{array}{r}
2x + 3y)4x^{2} + 12xy + 9y^{2}(2x + 3y + 6xy + 6xy + 9y^{2} \\
\underline{6xy + 9y^{2}} \\
\underline{6xy + 9y^{2}}
\end{array}$$

Treat this as the dividend and repeat the operation until we get no remainder; thus, $\frac{6xy}{2x} = 3y$ and $3y(2x + 3y) = 6xy + 9y^2$.

In most exercises usually set in algebra, the dividend divides without remainder by the divisor. If in any case we do get a remainder, it could be written as the numerator of a fraction with the divisor as denominator and added to the quotient, exactly as with ordinary arithmetic.

Cases often occur in which the terms in divisor and dividend are not arranged in any particular order. In such cases, it is first necessary to arrange both expressions in the same order.

Thus, as an example, we may require to find the quotient of $13a^3b^2 + b^5 + 4ab^4 + 13a^4b + 5a^5 + 8a^2b^3 \div 2ab + b^2 + a^2$.

Arrange the two expressions in order with the highest powers of a first, i.e., a^5 , then a^4 , and so on. We thus get

dividend =
$$5a^5 + 13a^4b + 13a^3b^2 + 8a^2b^3 + 4ab^4 + b^5$$
, divisor = $a^2 + 2ab + l^2$.

Now divide, as shown below.

 $a^2 + 2ab + b^2)5a^5 + 13a^4b + 13a^3b^2 + 8a^2b^3 + 4ab^4 + b^5(5a^3 + 3a^2b + 2ab^2 + b^3) \\ 5a^5 + 10a^4b + 5a^3b^2$

$$\begin{array}{r} 3a^4b + 8a^8b^2 + 8a^2b^3 + 4ab^4 + b^5 \\ 3a^4b + 6a^8b^2 + 3a^2b^3 \\ \hline 2a^8b^2 + 5a^2b^3 + 4ab^4 + b^5 \\ 2a^8b^2 + 4a^2b^3 + 2ab^4 \\ \hline a^2b^3 + 2ab^4 + b^5 \\ a^2b^3 + 2ab^4 + b^5 \end{array}$$

ALGEBRAIC OPERATIONS.

The following are two examples in which minus signs occur:— Ex, 1. Divide $(a^2 - b^2)$ by (a - b).

$$\begin{array}{c}
 a - b)a^{2} - b^{2}(a + b) \\
 \underline{a^{2} - ab} \\
 + ab - b^{2} \\
 \underline{ab - b^{2}} \\
 \vdots
 \end{array}$$
i.e

Note that in subtracting $a^2 - ab$ from $a^2 - b^2$ we have $a^2 - b^2$ $a^2 - ab$ $+ ab - b^2$

i.e., subtracting — ab from 0 = +ab and subtracting 0 from — $b^2 = -b^2$.

Ex. 2. Divide
$$(a^3 - 3a^2b + 3ab^2 - b^3)$$
 by $(a - b)$.
$$a - b)a^3 - 3a^2b + 3ab^2 - b^3(a^2 - 2ab + b^2)$$

$$- 2a^2b + 3ab^2 - b^3$$

$$- 2a^2b + 2ab^2$$

$$- ab^2 - b^3$$

$$ab^2 - b^3$$

- 92. The following examples, which illustrate more difficult cases of use of brackets and of finding numerical values of expressions, should be read carefully, particularly those in which a symbol denotes a negative quantity.
- Ex. 1. Simplify the following expression and find its numerical value when x = 2, y = -3, a = 1, b = -2, c = 4.

$$-2[3x + 2y(a + b) - 3x(b + c) - 4\{7x - (x + y)(x - y) + 3(a + c)\}]$$

First remove the brackets in steps, and we get—

$$-2[3x + 2ay + 2by - 3bx - 3cx - 4\{7x - (x^2 - y^2) + 3a + 3c\}\} - 2[3x + 2ay + 2by - 3bx - 3cx - 4\{7x - x^2 + y^2 + 3a + 3c\}\} - 2[3x + 2ay + 2by - 3bx - 3cx - 28x + 4x^2 - 4y^2 - 12a - 12c]$$

 $-6x - 4ay - 4by + 6bx + 6cx + 56x - 8x^2 + 8y^2 + 24a + 24c$. We can now find the numerical value by putting in the values

given for
$$x, y$$
, etc., thus:—
$$-(6 \times 2) - (4 \times 1 \times -3) - (4 \times -2 \times -3) + (6 \times -2 \times 2) + (6 \times 4 \times 2) + (56 \times 2) - (8 \times 4) + \{8 \times (-3)^2\} + (24 \times 1) + (24 \times 4) = -12 + 12 - 24 - 24 + 48 + 112 - 32 + 72 + 24 + 96 = 364 - 92 = 272.$$

In the above example, the following points should be noted:-

- (1) -4ay becomes $-(4 \times 1 \times -3) = -(-12)$, and this becomes +12 when the brackets are removed.
- (2) -4by becomes $-(4 \times -2 \times -3) = -(4 \times 6) = -24$ since $-2 \times -3 = +6$.
 - (3) $8y^2$ becomes $8 \times (-3)^2 = 8 \times 9 = 72$, for $-3 \times -3 = +9$.

Ex. 2. Simplify the following expression:

$$\left\{\frac{a^2 + 2ab + b^2}{a + b}\right\} - \left\{\frac{a^2 - b^2}{a - b}\right\} + \left\{\frac{x^2 - 2xy + y^2}{x - y}\right\} - \left\{\frac{4x^2 - 12xy + 9y^2}{2x - 3y}\right\}$$

Each numerator can be written as the product of two expressions consisting of two terms (Art. 88); thus, $a^2 + 2ab + b^2 = (a + b)$ (a + b); $a^2 - b^2 = (a + b)(a - b)$, and so on.

The whole expression then becomes

$$= \left\{ \frac{(a+b)(a+b)}{(a+b)} \right\} - \left\{ \frac{(a+b)(a-b)}{(a-b)} \right\} + \left\{ \frac{(x-y)(x-y)}{(x-y)} \right\} - \left\{ \frac{(2x-3y)(2x-3y)}{(2x-3y)} \right\}.$$

We can simplify any fraction by dividing numerator and denominator by any expression; thus, $\left\{\frac{(a+b)(a+b)}{(a+b)}\right\}$ becomes =(a+b) if we divide numerator and denominator by (a + b); or simply cancel out as in arithmetic.

The expression then simplifies to

Find the numerical value of $\frac{(2x)^2-(3y)^2}{2x-3y}+\sqrt[3]{\frac{8x}{9x}}$ when

$$x=-1,\,y=3.$$

First simplify to
$$\frac{(2x+3y)(2x-3y)}{(2x-3y)} + \sqrt[3]{\frac{8x}{9y}}$$
$$= 2x + 3y + \sqrt[3]{\frac{8x}{9y}}.$$

Now substitute numerical values and we get $(-2 + 9) + \sqrt[3]{-8}$ $= 7 + \left(-\frac{2}{3}\right)$, for $\left(-\frac{2}{3}\right)^3 = -\frac{8}{27}$ $= 7 - \frac{2}{3}$ = 6..

Exercises.

- (1) Divide $x^2 + 2xy + y^2$ by x + y, and $x^2 2xy + y^2$ by x = y.
- (2) Divide $x^2 y^2$ by x + y, and $x^3 y^3$ by x y. (3) Divide $3x^5 + 18x^4 + 39x^3 + 27x^2 12x 15$ by $x^2 + 4x + 5$. (4) $a^8 + 3a^2b + 3ab^2 + b^3$ is to be divided into (a + b) parts. Find each part,

When a = 4, b = 2, x = 2, y = -2, z = 3, find the numerical value of

(5)
$$x^3 + 3x^2y + 2xy^2 + 4x^3y^3 + 3x^2y^2 + 2xy - 12$$
.

$$\begin{array}{l} (6) \ \ 4x^2 \Big\{ 2y - 3xy + 2x^2y^2 - \left(4xyz - 2x^2y^3z^2\right) + 11x^3y^2z \Big\}. \\ (7) \ \ \frac{(x^2 + 2xy + y^2) \left(x^2 - y^2\right)}{(x + y)^3 \left(x - y\right).} \\ (8) \ \ \frac{(a^2 - 4b^2) \left(a^2 - 2ab + b_2\right)}{(a + 2b) \left(a - b\right)^3.} \end{array}$$

Answers.

$$\begin{array}{lll} (1) \ x + y, x - y. & (5) - 228. \\ (2) \ x - y, x^2 + xy + y^2. & (6) \ 9088. \\ (3) \ 3x^3 + 6x^2 - 3. & (7) \ 1. \\ (4) \ a^2 + 2ab + b^2. & (8) \ 0. \end{array}$$

93. **Square root**. The square roots of certain numbers or of simple algebraic terms can be ascertained by trial, or can be seen at once; thus,

$$\sqrt{36} = 6, \text{ for } 6 \times 6 = 36;$$

$$\sqrt{4a^2b^4} = 2ab^2, \text{ for } 2ab^2 \times 2ab^2 = 4a^2b^4;$$

$$\sqrt{(a^2 + 2ab + b^2)} = a + b, \text{ for } (a + b)(a + b) = a^2 + 2ab + b^2;$$

and so on. In many cases, however, we cannot readily find the square root by inspection or trial.

A method of finding the square root of any number, say, 583,696, is shown below:—

First mark off pairs of figures from the right thus, 96, 36, 58.

Now take the nearest square root of the first pair. This is 7. Put 7 in the result, and the square of 7, i.e., 49, beneath 58, and subtract. Remainder is 9.

Now bring down the second pair of figures 36. Double the number in the result thus, $2 \times 7 = 14$. Place as shown opposite to 936.

$$\begin{array}{c|c}
58'36'96(764)\\
49\\
14 6 & 936\\
876\\
152 4 & 6096\\
6096
\end{array}$$

Now take quotient of $\frac{93}{14} = 6$. Put 6 in the result and 6 after 14 as shown, and multiply 146×6 and subtract from 936. Remainder 60.

Repeat process until we have no remainder.

Thus, bring down 96 making €095.

Double 76, thus, $76 \times 2 = 152$, and write down as shown. Now $\frac{609}{152} = 4$.

Put 4 in result and also after 152, and we have $1524 \times 4 = 6096$. No remainder. Hence $\sqrt{583696}$ is 764.

When we are dealing with a whole number and a decimal fraction, we mark off the pairs of figures to the left and to the right from the decimal point.

Thus,
$$\sqrt{2955 \cdot 0096}$$
.

The pairs to the right are $00, 96$; to the left, $55, 29$.

Put the decimal point in the result when the first pair of figures after the decimal point are brought down.

$$\begin{array}{c|c}
 & 29 \cdot 55 \cdot 0096 \cdot 54 \cdot 36 \\
\hline
 & 25 \\
\hline
 & 104) 455 \\
\hline
 & 416 \\
\hline
 & 1083) 3900 \\
\hline
 & 3249 \\
\hline
 & 10866) 65196 \\
\hline
 & 65196 \\
\hline
\end{array}$$

When the square root does not divide out evenly, we can continue to any required number of decimal places by bringing down two ciphers at a time after the decimal figures are all brought down.

94. Algebraic square roots are found as in the following example:—

Ex. Find the square root of $4x^2 - 12xy + 16x + 9y^2 - 24y + 16$. Take the square root of first term $4x^2$. This is 2x. Put 2x in result and $(2x)^2$ beneath $4x^2$ and subtract. Now bring down the second term -12xy. Double the first term 2x in result, and place opposite -12xy. Now $\frac{-12xy}{4x} = -3y$. Add -3y to result, and also to divisor, and put the product -3y(4x - 3y) beneath -12xy. Bring down the term $9y^2$ to the term -12xy, and then subtract. Now bring down the remaining terms 16x - 24y + 16. Double the result 2x - 3y and put opposite 16x as shown. $\frac{16x}{4x} = 4$. Add 4 to result and to divisor and proceed as before.

$$4x^{2} - 12xy + 16x + 9y^{2} - 24y + 16(2x - 3y + 4)$$

$$4x - 3y) - 12xy + 9y^{2}$$

$$- 12xy + 9y^{2}$$

$$4x - 6y + 4)16x - 24y + 16$$

$$- 16x - 24y + 16$$

Hence the square root is 2x - 3y + 4.

The term $\frac{1}{2}$ is 95. Terms with negative indices. often written as a^{-1} , i.e., a raised to the power minus 1. We can show that $\frac{1}{a}$ is $= a^{-1}$ as follows:—

$$a \times \frac{1}{a} = \frac{a}{a}$$

$$= 1$$

$$a \times a^{-1} = a^{1} \times a^{-1}$$

$$= a^{1-1} \text{ (adding the indices)}$$

$$= a^{0}$$

$$= 1 \text{ (by Art. 13, for } a^{0} = 1 \text{).}$$

In the first case, we multiply a by $\frac{1}{a}$, and get product 1.

In the second case, we multiply a by a^{-1} and get product 1, hence we conclude that $\frac{1}{a}$ and a^{-1} are identical in value.

Similarly, we could show that $\frac{1}{a^2} = a^{-2}, \frac{1}{a^3} = a^{-3}, \frac{1}{a^n} = a^{-n}$. Ex. Find the product of $a^3 \times a^{-4}$. By our rule for indices $a^3 \times a^{-4} = a^{3-4}$ $= a^{-1} = \frac{1}{2}$.

Note also that $\frac{1}{10} = 10^{-1}$; $\frac{1}{100} = \frac{1}{10^2} = 10^{-2}$.

96. Terms with fractional indices. The term \sqrt{a} is often written as $a^{\frac{1}{2}}$ or a raised to the power one-half.

This can be shown as follows:-

$$\sqrt{a} \times \sqrt{a} = a$$
 $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}}$ (by rule for indices)
 $= a^1 = a$.

Hence we conclude that \sqrt{a} must be equal in value to $a^{\frac{1}{2}}$.

Similarly, we can show that $\sqrt[5]{a} = a^{\frac{1}{3}}$, $\sqrt[4]{a} = a^{\frac{1}{4}}$, $\sqrt[n]{a} = a^{\frac{1}{n}}$. Note that \(\square\) 10 may be written 10\frac{1}{2}.

 $\sqrt[3]{10}$... $10^{\frac{1}{3}}$. In accordance with Art. 95, we can also write—

$$\frac{1}{\sqrt{u}} = \frac{1}{u^{\frac{3}{2}}}$$

$$= u^{-\frac{1}{2}}$$

$$\frac{1}{\sqrt{10}} = \frac{1}{10^{\frac{1}{3}}}$$

$$= 10^{-\frac{1}{3}}$$

If we have a term which is raised to some power, and then some root of the term so raised is required, we can write this in two ways, e.g.,—

 $\sqrt[2]{a^3}$ means extract the square root of the cube of a. If a=4, then $a^8=64$, and $\sqrt[2]{a^3}=\sqrt[2]{64}$

Now $\sqrt[2]{x}$ may be written $x^{\frac{1}{2}}$ as shown above.

Let $x = (u^3)$, then $x^{\frac{1}{2}} = (u^3)^{\frac{1}{2}}$ or $u^{\frac{3}{2}}$, since we divide the index to extract a root.

Thus $\sqrt[3]{a^3}$ may be written as a^3 , and similarly 10^3 means $\sqrt[3]{100}$ or $\sqrt[3]{100}$

- 97. The following examples illustrate how a literal statement which expresses a series of operations, or the relationship existing between two or more quantities, can be written down algebraically.
- Ex. 1. To the square of x add the cube of y, and divide the sum by the difference between x and the square of y; extract the square root of the quotient.

The square of x is x^2 , the cube of y is y^3 ; their sum is $(x^2 + y^3)$. The difference between x and the square of y is $(x - y^2)$, and this must divide the sum of x^2 and y^3 . The division is expressed by $(x^2 + y^3)$, $(x - y^2)$.

We must now extract the square root of this quotient, and this is represented by

$$\sqrt{\frac{(x^2+y^8)}{(x-y^2)}}$$
 or $\left\{\frac{x^2+y^3}{x-y^2}\right\}^{\frac{1}{2}}$

Ex. 2. The area of a triangle, when the lengths of the three sides are known, is found as follows:—From half the sum of the three sides subtract each side separately; multiply together the half sum and the three remainders; the square root of the product is the area. Represent this statement algebraically.

If we denote the area by A, the three sides respectively by a, b, c, and the half sum of the sides by s, then

(s-a) = half sum minus the side a. (s-b) = y, y, b. (s-c) = y, z, c. s(s-a)(s-b)(s-c) = product of half sum and three remain lers. $A = \sqrt{s(s-a)(s-b)(s-c)}.$

Ex. 3. The horse power of a steam engine is given by twice the product of the mean pressure of steam on the piston in lbs. per sq. in.,

the length of the stroke in ft., the area of the piston in sq. in., and the number of revolutions per minute, divided by 33,000. Represent this statement algebraically.

Let P denote the mean pressure, L the stroke, A the area, and \mathcal{N} the number of revolutions per minute. Then if H-P denotes the horse-power we have

$$H \cdot P = \frac{2 \times P \times L \times A \times N}{33,000}.$$

Exercises.

Find the square root of—

(1) 582169, 1000000, 36297, 483.

(2) 4637 8246, 369 02174, 0 0023971.

(3) $9x^2 + 12xy + 4y^2$, $4a^2 - 16ab + 16b^2$, $\frac{1}{16}x^2 - \frac{1}{4}xy + \frac{1}{4}y^2$. (4) $a^2 + 4ab - 2av + 4b^2 - 4bc + c^2$.

(5) Write down algebraically: Add twice the square root of the cube of x to the product of y squared and the cube root of z. Divide by the sum of x and the square root of y. Add four and extract the square root of the whole. (B.E. 1903.)

(6) Write down algebraically: Square a, divide by the square of b, add 1, extract the square root, multiply by w, divide by the square

of n. (B.E. 1904.)

- (7) Write down algebraically: The principal P multiplied by 1 + $\frac{r}{100}$ twelve times. (B.E. 1906.)
- (8) Write down algebraically: x is multiplied by the fourth power of y, and this product is subtracted from the square of x multiplied by the cube of y; the cube root of the square of this difference is divided by the square root of the sum of x and y. (B.E. 1907.)
- (9) Express algebraically: Multiply the cube root of the square of a by b, add to this the cube root of a, multiplied by the cube root of the fourth power of b, extract the cube root of the sum and divide by the square root of the sum of the squares of a and b. (B.E. 1908.)

Answers.

(1)
$$763$$
; 1000 ; $190 \cdot 52$; $21 \cdot 93$.
(2) $68 \cdot 1016$; $19 \cdot 209$; $0 \cdot 0489$.
(3) $(3x + 2y)$; $(2a - 4b)$; $(\frac{1}{4}x - \frac{1}{2}y)$.
(4) $a + 2b - c$.
(5) $\left\{\frac{2x^3 + y^2z^3}{x + y^2} + 4\right\}^{\frac{1}{2}}$.
(6) $\left\{\frac{a^3}{b^2} + 1\right\}^{\frac{1}{2}} \cdot \frac{w}{a^2}$.
(7) $P\left(1 + \frac{r}{100}\right)^{12}$.
(8) $\frac{\left\{x^2y^3 - xy^4\right\}^{\frac{2}{3}}}{(z + y)^{\frac{1}{2}}}$.
(9) $\frac{\left\{a^3b + a^3b^4\right\}^{\frac{4}{3}}}{(a^2 + b^2)^{\frac{1}{2}}}$.

CHAPTER VIII.

SIMPLE EQUATIONS AND EVALUATION OF FORMULÆ.

98. An equation is simply a statement that two quantities are equal. It may be a simple arithmetic statement, such as 6+3=9, or it may be an algebraic statement, such as 2x+5=11.

When we have a problem given by data in which there is some unknown quantity, we denote the unknown quantity by a symbol, and then from the given data write down an equation from which the value of the unknown quantity can be determined.

In the second example above, viz., 2x + 5 = 11, the symbol x denotes an unknown quantity, and the solving of an algebraic equation of this type consists in finding a numerical value for the symbol x. In this example, since $(2 \times 3) + 5 = 11$, x must be equal to 3. The number 3, which is the value of x, is called the **root of the equation**.

In the equation x + y = 12, we have two unknown quantities x and y, and we can find many numerical values of x and y which will satisfy the equation.

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Thus, if x is 1 then y is 11, for 1 + 11 = 12; Again, if x is 2 then y is 10, for 2 + 10 = 12; Again, if x is 6 then y is 6, for 6 + 6 = 12.
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From this it will be seen that one equation is not sufficient data for finding two unknown quantities.

If we are given two distinct equations, we can find two unknown quantities. Two such equations are called simultaneous equations.

When the unknown quantity in an equation is only raised to the first power, the equation is called a simple equation,

or an equation of the first degree—for example, x + 3 =

8. Here x is really x^1 .

When the unknown quantity is raised to the second power, the equation is called a **quadratic equation**, or an equation of the **second degree**—for example, $x^2 + 3x = 5$; and when it is raised to the third power it is called a **cubic equation** or an equation of the **third degree**—for example, $x^3 + 5x^2 + 3x = 7$. The degree of an equation is settled by the term containing the highest power of the unknown quantity.

99. All the rules for solving simple equations are covered by one general rule, viz., that whatever we do to one side of the equation, we do not destroy the equality, provided we do the

same to the other side.

Thus we may—

- (1) add the same number to both sides of the equation;
- (2) subtract the same number from both sides of the equation;
 - (3) multiply both sides by the same number or quantity;
 - (4) divide both sides by the same number or quantity;
 - (5) raise both sides to the same power;
 - (6) extract the same root of both sides;
 - (7) change all the signs on both sides; and so on.

Test these statements with an arithmetic equation—for example, 9 + 4 = 13.

- (1) Add 2 to both sides, then 9 + 4 + 2 = 13 + 2, or both sides = 15.
- (2) Subtract 2 from both sides, then 9 + 4 2 = 13 2, or both sides = 11.
- (3) Multiply both sides by 3, then $(9 + 4)3 = 13 \times 3$, or both sides = 39.
 - (4) Divide both sides by 2, then $\frac{(9+4)}{2} = \frac{13}{2}$ or both sides = 6.5.
 - (5) Square both sides, then $(9 + 4)^2 = 13^2$, or both sides = $\frac{169}{100}$.
- (6) Take the square root of both sides, then $\sqrt{(9+4)} = \sqrt{13}$, or both sides = 3.61.
 - (7) Change the signs, then -9-4=-13, or both sides =-13.
- 100. The process of solving a simple equation consists in getting the unknown quantity, say x, by itself on one side, usually the left-hand side, of the equation, and all the numerical quantities on the right-hand side. Provided

that the sign of the unknown quantity is +, then the algebraic sum of the quantities on the right-hand side is the solution or root of the equation. If the sign of the unknown quantity is -, we make it + by multiplying both sides of the equation by - 1.

Thus, to solve x + 6 = -3x + 7.

We require to bring -3x from the right to the left side. If we add +3x to both sides we get

x+6+3x=-3x+7+3x and this simplifies to x+6+3x=74x+6=7.

We now require to bring the 6 from the left to the right side.

Subtracting 6 from both sides we get 4x + 6 - 6 = 7 - 6,

and this simplifies to 4x = 7 - 64x = 7 - 64x = 1

We now require to transpose the 4 from the left to the right side. Dividing both sides by 4 we get

$$\frac{4x}{4} = \frac{1}{4}$$

which simplifies to $x = \frac{1}{4}$, hence $\frac{1}{4}$ is the root, or solution, of the equation, for we now have + x by itself on the left-hand side of the equation.

We can test the result by putting $x = \frac{1}{4}$ in the equation. The two

sides should then be equal; thus, x + 6 = -

$$\begin{array}{rcl}
 x^{2} + 6 & = & -3x + 7 \\
 \frac{1}{4} + 6 & = & -(3 \times \frac{1}{4}) + 7 = (7 - \frac{3}{4}) \\
 & 6\frac{1}{4} = 6\frac{1}{4}.
 \end{array}$$

A similar test should be applied to all equations.

From the above example, the following facts are observed:—

(1) A negative term on either side of the equation may be transferred or transposed to the other side if we change its sign from - to +.

Thus x+6=-3x+7 becomes x+6+3x=7, the -3x on the right becoming +3x when transposed to the left.

(2) A positive term on either side of an equation may be transposed to the opposite side if we change its sign from + to -.

Thus 4x + 6 = 7 becomes 4x = 7 - 6, the + 6 on the left becoming - 6 when transposed to the right.

(3) Λ quantity which multiplies all the terms on one side of an equation may be transposed to the other side if it is made to divide all the terms on that side.

Thus 4x = 1 becomes $x = \frac{1}{4}$, for the multiplier 4 on the left

becomes a divisor 4 on the right.

Similarly, we can show that a quantity which divides all the terms on one side of an equation may be transposed to the other side if it is made to multiply all the terms on that side,

Take as an example
$$\frac{(x+5)}{2} = 3$$
.

If we multiply both sides by 2 we get

$$2\frac{(x+5)}{2} = 3 \times 2$$

or $(x+5) = 3 \times 2$.

Here the divisor 2 on the left becomes a multiplier 2 on the right.

The following examples illustrate the methods of solving equations:—

Ex. 1. Solve
$$3x + 4 - 2x = 5x + 12$$
.

Transposing $5x \cdot 3x + 4 - 2x - 5x = 12$

Transposing $4 \quad 3x - 2x - 5x = 12 - 4$

Simplifying $-4x = 8$

Transposing $4 \quad -x = \frac{8}{4}$

Multiply both sides by -1 , and hence $x = -2$

In an equation involving fractions, first eliminate the denominators by multiplying both sides of the equation by a multiple of all the denominators.

Ex. 2. Solve
$$\frac{(3x+2)}{4} = (5x+1)$$
Transposing 4 we get $(3x+2) = 4(5x+1)$

$$= 20x+4$$
Transposing 20x and 2. $3x-20x = 4-2$
Simplifying $-17x = 2$
Transposing 17 and changing signs $x = -\frac{2}{17}$

Ex. 4. Solve
$$\frac{2}{3}(x+6) - \frac{3}{4}(4-x) = \frac{(2x+3)}{12} + 2.$$
 his may be written
$$\frac{2(x+6)}{3} - \frac{3(4-x)}{4} = \frac{(2x+3)}{12} + \frac{2}{1}$$

This may be written

Multiplying both sides by 12 . 8(x+6) - 9(4-x) = (2x+3) + 24. 8x + 48 - 36 + 9x = 2x + 3 + 24.Simplifying

Transposing all terms containing x to the left and all others to the right-

> 8x + 9x - 2x = 3 + 24 + 36 - 48. Simplifying 15x = 15.

Ex. 5. Solve
$$\frac{(x+2)}{(x+3)} + \frac{4(x-5)}{(x-6)} = 5$$
.

Multiplying both sides by
$$(x+3)(x-6)$$
 we get $(x+3)(x-6)\frac{(x+2)}{(x+3)} + (x+3)(x-6)\frac{4(x-5)}{x-6} = 5(x+3)(x-6)$

This simplifies to (x-6)(x+2) + 4(x+3)(x-5) = 5(x+3)(x-6)Multiplying out the terms in brackets-

$$x^2 - 4x - 12 + 4(x^2 - 2x - 15) = 5(x^2 - 3x - 18)$$

 $x^2 - 4x - 12 + 4x^2 - 8x - 60 = 5x^2 - 15x - 90$

Simplifying and transposing terms we get

$$x^{2} - 4x + 4x^{2} - 8x - 5x^{2} + 15x = -90 + 60 + 12$$

$$- 4x - 8x + 15x = -18$$
Again simplifying . $3x = -18$
Transposing 3 $x = -\frac{18}{3}$
Hence $x = -6$.

101. In an equation of the type $\frac{x}{u} = \frac{a}{b}$, we can multiply successively by b and y, thus transposing b to the left side and y to the right side of the equation.

Multiplying by b we get $\frac{bx}{u} = a$.

Multiplying by y we get bx = ay.

Thus we can write $\frac{x}{y} = \frac{a}{b}$ as bx = ay, or generally, when each side of an equation consists of a single fraction, we can multiply the numerator of each fraction by the denominator of the other and write the products as equal;

thus
$$\frac{x}{y} \times \frac{a}{b}$$
 becomes $xb = ay$.

An example of such an equation is

$$\frac{(x-6)}{(x+4)} = \frac{(x+10)}{(x-3)}$$

By analogy to $\frac{x}{u} = \frac{a}{b}$ written as xb = ay, we at once put

$$(x-6)(x-3) = (x+10)(x+4).$$

Multiplying out $x^2 - 9x + 18 = x^2 + 14x + 40$

 $x^2 - x^2 - 9x - 14x = 40 - 18$ Transposing

 $\begin{array}{rcl}
-23x &=& 22\\
x &=& -\frac{22}{23}
\end{array}$ Simplifying

In some equations, it is advisable to simplify both sides of the equation before commencing to solve the equation. The following is an example:-

Ex. 5. Solve
$$\frac{(x-1)}{(x-2)} - \frac{(x-2)}{(x-3)} = \frac{(x-4)}{(x-5)} - \frac{(x-5)}{(x-6)}$$

Add together the two fractions on left of equation.
$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)} = \frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)} \begin{cases} \text{Add together the two fractions on the right.} \end{cases}$$

This simplifies to $\begin{cases} \frac{x^2 - 4x + 3 - (x^2 - 4x + 4)}{(x^2 - 5x + 6)} = \frac{x^2 - 10x + 24 - (x^2 - 10x + 25)}{(x^2 - 11x + 30)} \\ \frac{x^2 - 4x + 3 - x^2 + 4x - 4}{(x^2 - 5x + 6)} = \frac{x^2 - 10x + 24 - x^2 + 10x - 25}{(x^2 - 11x + 30)} \\ \frac{-1}{(x^2 - 5x + 6)} = \frac{-1}{(x^2 - 11x + 30)} \end{cases}$ This simplifies to

We now have an equation of the type $\frac{x}{y} = \frac{a}{b}$ in which x and a both

equal -1, $y = (x^2 - 5x + 6)$ and $b = (x^2 - 11x + 30)$.

Hence by analogy to xb = ay, we can write—

 $-1(x^2 - 11x + 30) = -1(x^2 - 5x + 6)$ Multiplying out. $-x^2 + 11x - 30 = -x^2 + 5x - 6$

 $-x^2 + x^2 + 11x - 5x = -6 + 30$ Transposing

Simplifying 6x = 24

Ex. 6. Solve $x^2 + 3x = 5x^2 + 4x$.

Simplifying

In this equation, every term contains x or x^2 ; hence we divide both sides by x and thus obtain an equation in which x^2 does not occur. The equation then becomes—

x+3=5x+4Dividing by xTransposing terms x - 5x = 4 - 3-4x = 1

Transpose 4 and change signs $x = -\frac{1}{4}$

P.M.

Exercises.

Solve the following equations: -

(1)
$$x + 4 = 2x - 2$$
. (2) $x + 3 - 5x + 7 = 3x - 4$.

(4)
$$x^2 + 3x + 7x = -5x - 2x^2 + 13x$$
.

(1)
$$x + 4 = 2x - 2$$
. (2) $x + 3 - 5x + 7 = 3x - 4$.
(3) $2x - 7 = x - 1 + 3x - 3$.
(4) $x^2 + 3x + 7x = -5x - 2x^2 + 13x$.
(5) $\frac{x}{3} - \frac{x}{4} = \frac{x}{6} + \frac{x}{8} + \frac{1}{2}$. (6) $\frac{x}{2} - \frac{x}{5} - \frac{3}{4} = \frac{x}{6} - \frac{2}{15}$.

$$(7) \frac{x+1}{3} + \frac{x-2}{4} = \frac{2}{4} + \frac{x}{8} - \frac{2x+3}{9}.$$

(8)
$$\frac{a-3}{5} + \frac{4+2a}{10} = \frac{3a+7}{15} - \frac{2a+4}{30}$$
.

$$(9) \ \frac{3\frac{1}{4}}{5+2\frac{2}{5}} + \frac{3x+4}{37} = \frac{2x-11}{74} + 2.$$

$$(10) \ \frac{3}{2+x} - \frac{4}{2-x} = \frac{6}{4-x^2}.$$

(11)
$$\frac{2+x}{3-x} + \frac{3+x}{x+4} = \frac{7}{12-x-x^2}$$

$$(9) \frac{3\frac{1}{4}}{5 + 2\frac{2}{5}} + \frac{3x + 4}{37} = \frac{2x - 11}{74} + 2.$$

$$(10) \frac{3}{2 + x} - \frac{4}{2 - x} = \frac{6}{4 - x^2}.$$

$$(11) \frac{2 + x}{3 - x} + \frac{3 + x}{x + 4} = \frac{7}{12 - x - x^2}.$$

$$(12) \frac{2x - 3}{2x - 4} - \frac{2x - 4}{2x - 5} = \frac{2x - 6}{2x - 7} - \frac{2x - 7}{2x - 8}.$$

Answers.

- $\begin{array}{cccc} (4) & -\frac{2}{3}. & (7) & \frac{24}{49}. \\ (5) & -2\cdot 4. & (8) & 2. \\ (6) & 4\frac{5}{8}. & (9) & 24\frac{1}{8}. \end{array}$ $(10) - 1\frac{1}{7}$ $(11) - 1\frac{3}{4}$
- 102. In expressing data or given conditions of a problem in the form of an equation, the symbol x is usually adopted for the unknown quantity. Suppose we take x as a certain number, then a number exceeding the given number by 3 would be represented by (x + 3); a number which is 3 less than the given number would be represented by (x-3); a number equal to 5 times the given number is 5x; if we divide the given number into 5 parts each part is $\frac{x}{z}$ and so on.

If we know that the distance between two railway stations is 50 miles, and that a train travels at a rate of x miles per hour from one station to the other, then the time occupied by the train is $\frac{50}{x}$ hours.

Again, if the train travels 20 miles per hour and the distance is x miles, then the time taken is $\frac{x}{20}$ hours.

If a rectangular plate is x ft. long and y ft. wide, its area will be xy sq. ft., and its perimeter is 2(x + y) ft.

The electric current C in ampères passing through a circuit of resistance R ohms, with an electromotive force of E volts, is known to be equal to the electromotive force divided by the resistance. Hence we can represent this

statement by
$$C = \frac{E}{R}$$
.

The kinetic energy of a weight W lb. moving with a velocity of V ft. per second, is known from mechanics to be equal to half the product of the weight multiplied by the square of the velocity and divided by the acceleration due to gravity.

If g = acceleration in feet per second per second, then the kinetic energy is represented by $\frac{1}{2} \frac{WV^2}{y}$.

Ex. 1. The space described or passed through by a falling body is known from mechanics to be given by one-half the product of the acceleration due to gravity and the square of the time in seconds during which the body is falling. Express this algebraically, and find the depth of a well if a stone dropped down it reaches the bottom in 9 secs.

Let s = space in feet, or depth of well in feet.

g = acceleration due to gravity in feet per second per second.

t = time in seconds.

Then from the given relation we write $s = \frac{1}{2}gt^2$. The value of g is usually taken as 32.2 ft. per second per second,

Hence, when t = 9 $s = \frac{1}{2} \times (32 \cdot 2) \times (9)^2$ $= \frac{32 \cdot 2 \times 81}{2}$ $= 130 \cdot 1 \text{ ft.}$

Ex. 2. If a train starts from a point distant a feet from a terminus, and moves with an initial velocity V ft. per second, and an acceleration of f ft. per second per second, and is moving for a time t sees, then at the end of the time, its distance from the terminus is equal to the sum of (1) its distance at the time of starting; (2) the product of the initial velocity and the time; and (3) the product of half the acceleration and the square of the time. Express this algebraically

and find the distance when a=12 ft., V=20 ft. per second, f=30 ft. per second per second, and t=25 secs.

Let s = distance at end of time t.

Then product of initial velocity and time = 17.

And the product of half the acceleration and the square of the time $=\frac{1}{2}fl^{2}$.

Hence from given data $s = a + Vt + \frac{1}{2} ft^2$

Putting in the numerical values for a, V, t and f $s = 12 + (20 \times 25) + (\frac{1}{2} \times 30 \times 25 \times 25)$. = 12 + 500 + 9375= 9.887 ft.

102A. The following examples illustrate methods of solving problems which involve simple equations:—

Er. 1. If 15 be added to the product of 5, and a certain number, the sum is equal to 8 times the number. Find the number.

Let x = the number, then 5x = the product of 5 and the number, and (5x + 15) = the product with 15 added.

Also 8x = 8 times the number.

Hence 5x + 15 = 8x from which x = 5.

Ex. 2. The length of a rectangular plate is 12 ft., and the area is 60 sq. ft. Find the width.

Let x feet be the width. Then $12 \times x = \text{area}$. Thus 12x = 60 $x = \frac{60}{12} = 5$ ft. width.

Ex. 3. A column 10 ft. square is to be built as high as possible, but the total weight must not exceed 1,050 tons. Each cubic foot of the material weighs 145 lb. What will be the safe height to build?

The area of base of column $= 10 \times 10 = 100$ sq. ft.

Let h= height of column in feet. Then $100\times h$ will be the number of cubic feet in the column. $100\times h\times 145=$ weight of column in pounds.

Hence $100 \times h \times 145 = 1050 \times 2240$ $h = \frac{1050 \times 2240}{100 \times 145}$ = 162.2 ft.

Exercises.

(1) One foot-pound of work is done when a force of 1 lb. is exerted through a distance of one foot. If a force of 50 lb. is exerted through a distance of half a mile, how much work is done, and what would be the horse power of a motor which did this amount of work every half minute? One horse power = 33,000 ft.-lb. of work per minute.

(2) If $t = 2\pi \sqrt{\frac{l}{g}}$, find t when l = 2.5 ft., $\pi = 3.1416$, and g = 32.19. Also find l when t = 2.

- (3) $F = \mu N$. Calculate F when $\mu = 0.25$ and N = 250. (4) $H = \frac{(N M)r}{33000}$. Find H when N = 2M and N = 200 lb., $r \approx 2 \times \pi \times 100$.
 - (5) $K = \frac{W}{2g}(r^2 V^2)$. Find K when W = 50, V = 25, and r = 30.
- (6) $T = \frac{\pi}{16} f d^3$. Calculate d when $T = 10 \times 2240 \times 24$, and f = 9000.
- (7) If $F = EI\pi^2 + 4l^2$; if $I = bl^3 \div 12$; if $E = 3 \times 10^7$, $\pi = 3.142$, l = 62, b = 2, t = 0.5, find F. (B.E. 1904.)
 - (8) From the formula, Young's modulus $=\frac{\text{load} \times \text{original length}}{\text{constant}}$ $area \times \left\{\begin{array}{l} extension \ or \\ compression \end{array}\right\}$

find how much a column 15 ft. long will shorten under a load of 125 tons if Young's modulus = 12,500,000 lb. per sq. in. Area = $\pi \left\{ 12^2 - (9.5)^2 \right\}.$

- (9) $W = \frac{1}{6} \times c \times h \times \frac{hd^2}{l}$. Find W when l = 25 ft., h = 10 in. d = 14 in., c = 1, k = 6680. (Note b, d, and l must be in inches.)
 - (10) $d = 3.3 \frac{\sqrt{H}}{N}$. Find d when H = 5 and N = 100.
- (11) $I = \frac{BH^3}{3}$ and $k = \frac{H}{\sqrt{3}}$. Find I and k when B = 3 and H = 10.
- (12) $I_1 = \frac{BH^3}{12}$ and $I_2 = BH(\frac{H^2}{12} + R^2)$. Calculate I_1 and I_2 when B = 3, H = 10, and R = 7.
 - (13) If $I = \frac{\pi D^4}{32}$, find its value when $D \approx 12$ ft.
- (14) $Q = 0.62 A \sqrt{2gh}$. Find Q when $A = \pi \times \frac{1}{4} \times \frac{1}{4}$, g = 32.2,
- (15) $Q = 0.61 \times \frac{2}{3} BH \sqrt{2gH}$. Find Q when B = 6, $H \approx 3$, and q = 32.2.
 - (16) Simpson's rule for the area A of an irregular figure is

$$A = \frac{x}{2} \left\{ (h_1 + h_1) + 4(h_2 + h_4 + h_6) + 2(h_3 + h_4) \right\}$$

where h_1 , h_2 , etc., are the lengths of ordinates drawn across the figure at equal distances x. Calculate the area when x = 0.75 in., and

height, and hence find the area. Compare the two results.

(17) Simpson's rule for the volume V of a prismoid is

$$V = \frac{x}{3} \left\{ A_1 + 4 \left(A_2 + A_4 + A_6 \right) + 2(A_3 + A_5) + A_7 \right\}$$

where A_1 , A_2 , etc., are sectional areas at equal distances x.

Find V when x = 2 ft.

Plot A and x on squared paper, find the average value of A, and hence calculate the volume. Compare the two results obtained.

(18)
$$F = \frac{W\pi^2 N^2 R}{900 \times g}$$
. If $W = 50$, $R = 3$, $N = 150$, $\pi = 3.1416$, and $q = 32.2$, find F .

(19) Divide 12:56 into two parts, one of which is $3\frac{1}{2}$ times the

other. (B.E. 1909.)

(20) À rectangular garden has one side 28 yds. longer than the other; if the smaller side were increased by 40 yds, and the greater diminished by 34 yds., the area would remain unaltered. What are the lengths of the sides? (B.E. 1907.)

(21) Some men agree to pay equally for the use of a boat, and each pays 15 pence. If there had been two more men in the party, each would have paid 10 pence. How many men were there, and how

much was the hire of the boat? (B.E. 1905.)

(22) If a, b, and c represent the lengths of the three sides of a triangle, and if $s = \frac{1}{2}(a + b + c)$, the area of the triangle is given by the formula:—Area = $\sqrt{s(s-a)(s-b)(s-c)}$. Calculate the area of a triangle which has sides 12, 10, and 8 ft, long.

(23) Write down algebraically:—Square a, divide by the square of b, add 1, extract the square root, multiply by w, divide by the square

of n. (B.E. 1904.)

(13) 2034.72.

(1) 139 000 ft -1b + 8 HP

(24) The ages of a man and his wife added together amount to 72:36 years; 15 years ago the man's age was 2:3 times that of his wife; what are their ages now? (B.E. 1904.)

Answers.

(1.1) 3.78

(1) 102,000 16,-10., 0 111.	(14) 5 10.
(2) 1.75 sec.; 3.26 ft.	(15) 101.73.
(3) 62·5 lb.	(16) 30·1.
(4) 1·904,	(17) 359·1.
(5) 213·5.	(18) 1,149.
(6) 6毫.	$(19) \ 2.79 \ ; \ 9.77.$
(7) 401.2 .	(20) 40; 68,
(8) 0.024 in.	(21) 4 men; 5 shillings.
(9) 7273·6.	(22) 39.7.
(10) 1·2.	• /
(11) 1,000 : 5.77.	(23) $\frac{w}{bu^2} \sqrt{a^2 + b}$.
(12) 250; 1,720.	(24) 44:52; 27:84.
(10) 2004 50	(21) 11 02 , 21 01.

CHAPTER IX.

GEOMETRICAL DEFINITIONS AND FUNDA-MENTAL CONSTRUCTIONS AND PROBLEMS.

103. A **geometrical point** has position but not magnitude. In making drawings, it is best represented by two fine lines drawn across each other, thus X.*

A geometrical line has length but not breadth. All lines should, therefore, be as thin as is consistent with distinctness.

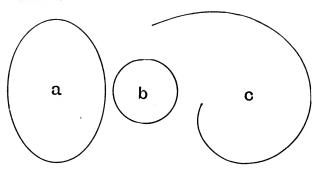


Fig. 35.

Lines may be either straight or curved.

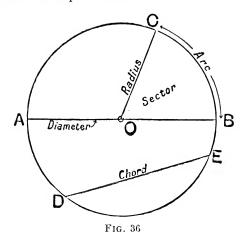
Intersecting lines are lines which cross each other.

The **plane** is a perfectly straight or flat surface and one on which a straight-edge will lie evenly, no matter how placed.

^{*} In the diagrams illustrating this book, many particular points are represented by small circles or round dots. This device is employed to make such points conspicuous and more easily located by the reader.

Plane curves are curves which lie wholly in one plane, such, for example, as any curve traced on a perfectly true drawing board. See a, b, c, Fig. 35.

A circle is a plane figure bounded by a curved line called the circumference, every point of which is equally distant from a point within called the centre. The circumference is sometimes referred to as the circle when no reference is being made to the space enclosed.



A radius is a straight line drawn from the centre to the circumference.

A diameter is a straight line drawn through the centre,

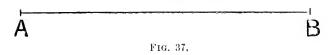
and terminated both ways by the circumference.

Thus, in Fig. 36, O is the centre of the circle; OA, OB, OC, are radii; AOB is a diameter. From the definition of a circle, it is evident that all the radii of the same circle are equal; and all diameters are equal, and each doubte the radius.

An arc of a circle is any part of the circumference, as BC. A chord is the straight line which joins any two points in the circumference, as DE. A chord divides a circle into two parts, each of which is called a segment.

A sector of a circle is the space enclosed by an arc and two radii drawn from the centre of the circle to the extremities of the arc.

104. Locus. Every line may be considered as the path



traced by a moving point. Thus the line AB, Fig. 37, traced by a pencil point moving from A to B, marks every

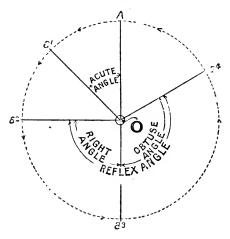


FIG. 38.

position occupied by the point, and is called a locus; the moving point is said to **generate** its locus.

Constant Direction. If a point situated at A, Fig. 37, move in a straight line to a new position B, it moves all the time in the same direction, viz., along the straight line AB towards B: and thus the direction of motion of the point is constant. Hence, a straight line is the locus of a point the direction of which is constant,

105. **Angles.** Suppose a straight wire OA, Fig. 38, secured to a drawing-board at O by a pin joint so that it can rotate about the point O. Let OA represent the initial position and direction of the wire. When the wire has turned

into the position Oa^1 , its direction is Oa^1 .

The line Oa^1 is said to be inclined at an angle to OA, and the inclination of one line (Oa^1) to the other (OA) is called, in general terms, the angle a^1OA . The two lines which form an angle are called the arms of the angle, and the point in which the two arms meet is called the angular point, or vertex.

It must be clearly understood that an angle denotes the inclination, or slope, of one line to another line, but has nothing to do with the length of any line. The size of an angle depends solely on the difference in the directions of

its arms.

If the rotating wire makes a complete turn, the point A will describe a circle and return to the starting point. When it has moved through half a complete turn into the position Oa^3 , then a^3O , OA are in the same straight line, and divide the circle traced by A into two equal parts. When the wire has made one-quarter of a turn, taking up the position Oa^2 , it makes equal angles with OA and Oa^3 , in which case the lines are said to be **square** or **perpendicular** to each other, and each of the equal angles is called a **right-angle**.

If the rotating wire OA be supposed to describe a complete turn in 360 equal stages, the circle will be divided into 360 equal parts, and if lines were drawn marking these different positions the angle between adjacent lines would be constant and equal to one-three hundred and sixtieth part of a complete turn. One-three hundred and sixtieth part of a complete turn is called a **degree**. The degree is symbolized by a small

circle placed as an index, thus °.

Since there are 360° in a complete turn, it follows that a right-angle contains 90°; one-half of a right-angle, 45°; one-third of a right-angle, 30°; and two-thirds of a right-angle, 60°.

Mathematicians divide the degree into sixty equal parts called **minutes**, symbolized by one dash, thus '; and the minute is again subdivided into sixty equal parts called

seconds, symbolized by two dashes, thus ". In Practical Geometry and Practical Mathematics, it is better to divide the degree into tenths and hundredths. Angles of 90°, 60°, 45°, and 30°, are extensively used, and these are the angles to which ordinary set-squares are made.

An Acute Angle is an angle less than a right-angle.

An **Obtuse Angle** is an angle greater than a right-angle. A **Reflex Angle** is an angle greater than two right-angles; it is sometimes called a **re-entrant angle**.

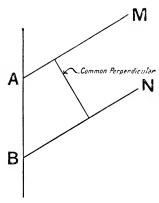


FIG. 39.

The **complement** of an angle is the difference between the angle and a right-angle.

The **supplement** of an angle is the difference between the angle and two right-angles.

A straight line which intersects two or more lines and makes a right-angle with each is called a common perpendicular. (See Fig. 39.)

106. Parallel lines are lines which have the same direction.

Parallel straight lines. If from two points A, B, Fig. 39, in a straight line there are drawn straight lines AM, BN, each making the same angle with AB, these two lines

have the same direction and are, therefore, parallel. Any two or more parallel lines, whether straight or curved, possess two characteristic properties, viz.:—

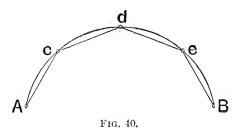
(a) A straight line perpendicular to a given line is perpen-

dicular to all lines which are parallel to the given line.

(b) The distance apart of two parallel lines is constant throughout their entire length, such distance being always

measured along common perpendiculars,

107. **Measuring straight lines**. In measuring straight lines, always apply the scale (*i.e.*, measure) direct to the line, taking care that one end of the line coincides with one of the unit division marks on the scale. If the line is not exactly measured by inches and tenths of an inch (which

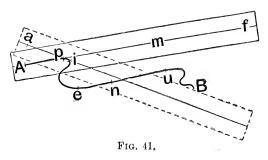


may be the smallest graduation on the scale), fractions of the smallest graduation can be measured visually. After a little practice, such fractions as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, etc., of a $\frac{1}{10}$ -in. division are easily estimated at sight.

Measuring curved lines. The lengths of circular arcs and many other curves may be determined arithmetically by formulae, and graphically by constructions deduced from certain properties which the curves are known to possess. The draughtsman, however, has frequently to measure curves about which he knows nothing beyond what the curves reveal, and often a tedious analysis leads no nearer to accuracy than can be attained much more quickly by approximations which are easily understood without a knowledge of pure mathematics.

108. Practical method of measuring any plane curve. Consider the circular arc AB, Fig. 40. Mark

a number of points e, d, e, on the arc and join them up by the chords Ac, cd, de, eB. The length of the curve is greater than the combined lengths of the chords. If more points in the curve are taken, the chords will be more numerous and will approach more nearly to the enrve, and if it were possible to take points in the curve indefinitely near to each other, the chain of chords would become indefinitely near to the curve: in fact, the two would coincide. Now similar reasoning may be applied to the irregular curve AB, Fig. 41, or to any curve. It is possible, therefore, to obtain a near approximation to the exact length of any curve by finding the length of a chain of chords taken of such lengths



as the curvature of the line and the importance of exactitude render necessarv.

Ex. Measure the length of the irregular curve AB,

Fig. 41.

Draw a straight line on a strip of stout tracing-paper. Mark a point a on the line and place the tracing-paper over the curve AB with the point a exactly over the end A. Insert the pricker at A, and rotate the tracing-paper so as to bring a portion of the straight line from a into the closest possible agreement with the curve at that end. Now hold the tracing-paper quite still and transfer the pricker to a point p—which marks the end of the segment from A in agreement with the straight line—and repeat the operation of turning the tracing-paper so as to bring a further portion of the straight line into agreement with the curve at pi. Then transfer the pricker to i, and continue until the whole line has been measured.

The length of the curve is then shown at Af. This

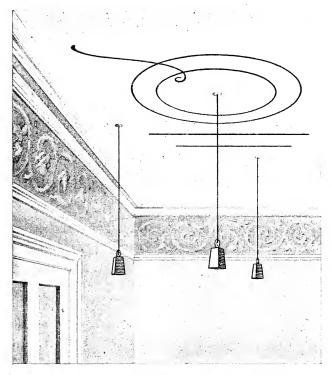


Fig. 42.

method, if applied with care and judgment, will give very good results. In selecting each step along the curve, attention must be paid to the curvature of the line at the particular part being measured. Where the curvature is sharp it is necessary to take short steps; but where the

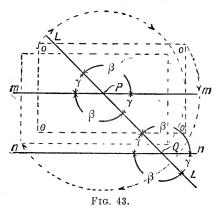
curve is flat, as between Ap, eu, much longer steps can be taken without increasing the error to an appreciable extent.

Exercise.—Describe a circle, radius 1 in., and measure its circumference by the method just explained.

The length of this circumference is known to be $6\frac{\circ}{7}$ in.

How near to this is the result you obtain?

109. **Vertical lines and horizontal lines.** Fig. 42 is a sketch showing three strings, with weights attached, suspended from points in a *flat level* ceiling. We know from experience that these strings hang in vertical lines,



and all have the same direction. Hence, all vertical lines

are parallel straight lines.

A perfectly level line is called a horizontal line. The sketch, Fig. 42, shows two parallel straight lines, two concentric circles, and a sinuous curve, all of which lie on the surface of the flat level ceiling. We gather from this diagram that horizontal lines may be either straight or curved, and, further, may be parallel or non-parallel.

110. **Experiment.** Fig. 43. Draw two parallel lines mm, nn, and draw any straight line LL across the parallels. Make a tracing of the lines mm, and LL, and by rotating this tracing, first about the point P, and secondly about the point Q, show that the angles which are marked by the

same Greek letters are equal. In this way we obtain, by the method of superposition, a proof of the following:—

Theorem. If a straight line crosses two parallel straight lines, it makes (1) the alternate angles equal; (2) the exterior angle equal to the interior opposite angle on the same side of the line.

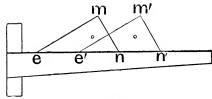


FIG. 44.

111. **Experiment.** Fig. 44 Place a 60° set-square with its long side against the T-square, as at *enm*, and draw lines along the edges *em* and *mn*. Move the set-square along the T-square to a new position e'm'n'. Then since *en* and e'n' have the same direction, and the set-square remains unaltered in form, therefore e'm' is parallel to *em*, and m'n' is parallel to *mn*. Hence the following:—

Theorem. If any figure move in a plane so that one line in it has always the same direction, then all lines of the figure

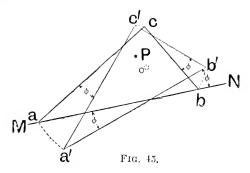
remain parallel to their original directions.

112. **Experiment.** Fig. 45. Draw a straight line MN, and place a set-square on the line, as at abc. Make a tracing of the set-square and the line MN in that position. Rotate the tracing through any angle about any point P, into a new position as a'b'c'. Prick through the corners a',b',c', and join them up, making a drawing of the set-square in the new position. Make a tracing of the angle between MN and a'b' and apply this tracing over all the angles marked ϕ . Observe that all these angles are equal to one another.

From this experiment we conclude that the theorem in Art. 111 may be extended as follows:—

Theorem. If any figure be moved in its plane so that a line in it makes an angle ϕ with its original direction, then

all lines in the figure also make the angle ϕ with their original directions.



113. Experiment. No Figure. Describe a circle, and draw a diameter AB.

On a piece of tracing-paper draw two straight lines Oa, Ob, perpendicular to each other. Place this tracing over the circle with the meeting point (O) of the two lines on the circumference of the circle. Insert a pricker at the point O, and rotate the tracing until the line Oa passes through Λ . Then observe that Ob passes through the opposite end B of the diameter AB. Repeat the experiment, placing O in different positions on the circumference, and observe that when Oa passes through A, then Obpasses through B.

In this way we verify the following:---

Theorem. All angles in a semicircle are equal to one another, and are right-angles.

114. Experiment. Fig. 46. Draw a straight line ER (use the T-square in drawing ER). Place the 45° setsquare against the T-square and draw from points E, R lines which intersect in i. Next, use the 60° set-square, and draw from the same points (E, R) lines which intersect in pairs at o and v. Observe that the points o, v lie on the line through i perpendicular to ER. This perpendicular intersects ER in n. Measure En, nR and verify that $\mathbf{E}n = n\mathbf{R}$.

Take the compasses, and with any point p in on as centre, describe a circle of radius pE. This circle passes through R. Repeat the test by describing a circle having q for centre, and radius qE. Hence,

The perpendicular bisector of a straight line is the locus of points equidistant from the ends of the line, and conversely.

115. Experiment. No Figure. Describe any circular arc AB; draw the chord AB and find the perpendicular bisector

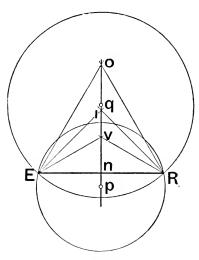


FIG. 46.

of this chord as explained in the last experiment. This bisecting line intersects the arc in a point M. Verify by measurements that the chords MA, MB are equal, and also that the arcs MA, MB are equal.

116. Two or more parallel straight lines are sometimes called a system of parallels. Any straight line which intersects a system of parallels is called a transversal.

Experiment. Fig. 47. On a straight line OH mark off a number of equal distances 1, 2, 3, 4, 5. Draw through

H and the divisions 1, 2, 3, 4, 5 parallel straight lines, making any convenient angle with OH. Through any

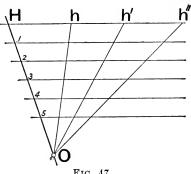


Fig. 47.

point O in OH draw transversals, as Oh, Oh', Oh". Verify by measurements the following:—

Theorem. Any straight line drawn across a system of equidistant parallels is divided by the parallels into a number of equal parts.

This principle is extensively used in the solution of problems. See Art. 117.

Exercises.

1. Make a drawing to a reduced scale of a 60° set-square, and letter the corners a, b, c, as in Fig. 45. Rotate the figure about the corner c until ca is perpendicular to its original direction. By measurement, verify that cb and ab are each perpendicular to their original directions.

2. Draw a system of seven parallel lines 1 arart, and draw a transversal ab across and perpendicular to them. Find the position of two lines ac, ad, on opposite sides of ab, such that equal distances of $\frac{3}{4}$ " are cut from them by the parallels, and show that the two lines ac, ad are equally inclined to ab.

3. Mark a point a on your paper, and on a line bc, 5'' long, as diameter, construct a semicircle to pass through a. Measure ab, ac, and show that, in one case, ab and ac may be, respectively, 4" and 3" long.

117. Problem. Fig. 48. Divide the given straight line

CD into a number—say five—of equal parts.

Through the ends C, D of the line draw parallels Co, Dr (making use of the most convenient angle on your setsquares). Place the zero (0) point of a scale on one of the parallels (Co in the diagrams) and turn the scale until the unit division which corresponds to the given number of parts required—5 in this case—falls on the other parallel. Mark the positions of the division marks 1, 2, 3, 4 direct from the scale either with a pencil or pricker. Then draw through these points parallels to Co, Dr, which determine by their intersection with CD the required points of division.

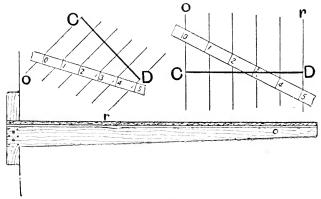


Fig. 48.

Exercises.

(1). Make a tracing of the curved part of Fig. 133 (page 251), and employ the method of Art. 117 to divide the figure into ten equal

(2). Draw any curved figure you please, making its greatest length 7" and its greatest width 2½". Divide it crosswise into nine equal

parts, and lengthwise into seven equal parts.

CHAPTER X.

ANGLES, MEASUREMENT AND CONSTRUCTION OF.

118. In measuring angles, any magnitude which has always the same value may be employed as the unit. The unit employed in constructive geometry is the degree, which is the angle subtended at the centre of the circle by one-three-hundred-and-sixtieth part of the circumference, and is therefore always the same (see Art. 105). The magnitude of an angle is therefore given by the number of degrees contained in the angle.

119. Another unit of angular measure is the *right-angle*. This has always a fixed value, viz., 90°, and the magnitude of any angle may be expressed in terms of the *unit right-angle*.

If the number of degrees in an angle be given, or determinable, it is only necessary to divide that number by 90 in order to obtain a measure of the angle in terms of the unit right-angle. The quotient may, or may not, be a whole number, but it can be obtained correct to any desired number of decimal places.

Ex. Express in terms of the unit right-angle (a) an angle of 76.5° , (b) an angle of 113° .

(a)
$$\frac{76.5^{\circ}}{90}$$
 = 0.85 right-angle.

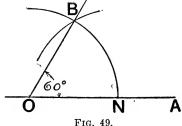
(b)
$$\frac{113^{\circ}}{50}$$
 = 1.25 right-angles.

120. Certain angles can be readily constructed by means of intersecting arcs. Thus, an angle of 45° is obtained by first setting out an angle of 90° and then bisecting this angle. Again, if the angle of 45° is bisected, we obtain angles of $22\frac{1}{2}^{\circ}$.

ANGLES, MEASUREMENT AND CONSTRUCTION OF.

Ex. To construct an angle of 60°. Fig. 49.

With the point O as centre, and any radius, describe an arc intersecting OA in N. With N as centre and the same radius describe an



arc intersecting the previously drawn arc in B. Join OB, then AOB is the required angle of 60°. (This construction is due to an important fact, viz.: The chord of 60° is equal to the radius of circle. See Art. 142.)

If the angle AOB be bisected we obtain angles of 30°.

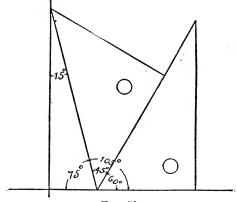


Fig. 50.

121. In practical drawing, we never use a geometrical construction to obtain a result which can be obtained more easily by mechanical means. Thus, angles of 90°, 60°, 45°, 30° are drawn direct by using the usual set-squares; and by arranging these same tools as shown in Fig. 50, angles of 75°, 105°, and 15° may be drawn.

122. In setting out and measuring angles generally, we use a protractor.

A protractor is a scale of degrees; it is usually either semicircular or oblong in form. See Figs. 51 and 52. In all protractors, lines representing the degrees radiate from a point on, or near to, the base, which is distinguished by a * or some other conspicuous mark.

On good quality protractors, the degrees are numbered in

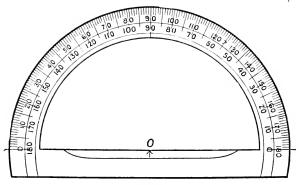


Fig. 51.

both directions, giving to each division two numbers; this enables the user to read off degrees either from left to right or vice versâ.

123. How to use a protractor:

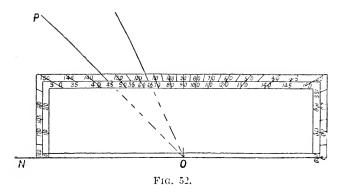
- (a) to measure a given angle.
- (b) to set out a given angle at a given point.
- (a) To measure the given angle NOP, Fig. 52.

Place the base of the protractor along the arm NO with the centre of the protractor at the angular point O. The number of degrees in the angle NOP is then indicated by the figures on the protractor at the point where OP crosses the outer edge. In the diagram, OP crosses at the double reading 45, 135. As the angle NOP is less than a right-angle, the first of these numbers—viz., 45—indicates the number of degrees between ON and OP, and therefore the angle NOP is 45°.

(b) Fig. 52. To draw from a point O in a given straight

line ON, a line inclined at 65° to ON.

Place the base of the protractor on the line NO with the centre of the protractor at the point O. Mark a point on the drawing corresponding to the 65th division on the protractor reading from N in the direction of the arrow,



and draw a line from O through this point. This line will be inclined to NO at 65°.

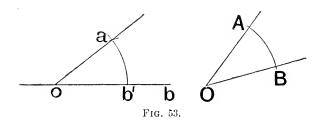
124. In the two examples above, we have measured to a number of whole degrees and on most protractors the degree is the smallest measure shown; but we have frequently to treat of angles containing some fraction of the degree, such as 0.5° or 0.2°, and a very close approximation of these parts of the degree can be guessed, or determined visually, after a little practice.

Some protractors are wholly unreliable, and many others, even among the higher-priced instruments, have the first

degree division inaccurately marked.

125. A method of testing the protractor is as follows: Draw two lines at right-angles to each other and

meeting in a point O. Insert the pricker at O and place the centre of the protractor against the needle, and observe whether there is always a difference of 90° between the readings of the protractor when it is rotated about the point O into various positions. Any variation is evidence of error in the scale. When the first degree division only is inaccurate, this division may be dispensed with by placing the line $O-10^\circ$ over an arm of the angle. If this expedient be employed, the magnitude of an angle will be 10° less than the number shown on the scale.



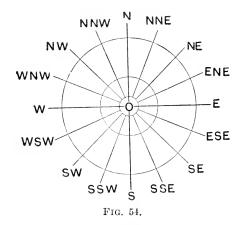
126. **Problem.** Fig. 53. At a point o in a line ob set out an angle equal to a given angle AOB.

By far the best way to copy an angle is to place a piece of tracing-paper over the figure and mark the position of the angular point and a point on each arm of the angle. Then transfer the tracing to the desired position, prick through the points, and join up.

The problem may also be done as follows: Make point O the centre of an arc of convenient radius cutting OB, OA in B, A. With centre o, and radius OB or OA, describe an arc cutting ob in b'. With centre b', and radius BA, describe an arc cutting the arc previously drawn in a. Join ao.

127. **Points of the compass.** On the compass there are four cardinal points, viz., North, East, South, and West, and a number of intermediate points at regular intervals. The directions North, East, South, and West are indicated in Fig. 54 by ON, OE, OS, and OW

respectively, which form 4 rt Z^s at the point O. The direction of any intermediate point we may denote by specifying the angle which that direction makes with the direction of either cardinal point. Thus, the direction E.N.E. is completely given as $22\frac{1}{2}^{\circ}$ N.E. (read "North of East"); or as $67\frac{1}{2}^{\circ}$ E.N. (read "East of North").



This notation is convenient where we wish to indicate several directions.

Ex. A road runs due North from a place A to a place B; from B to a place C its direction is 32° East of North. At C its direction is again changed by an anticlockwise turn of 48°. What angle does the direction of the road from C make with the direction AB; and what with the direction due East?

Draw a line AN, Fig. 55, representing the direction due North.

Draw AE I to AN representing the direction due East.

From any point B in AN draw BC making 32° with AN and to the right. From any point C in this line draw CO making 48° with BC and to the left, and let this line extend in both directions so as to intersect AN in O and AE in O. Then α and ϕ are the angles which CO makes with the directions AN and AE respectively.

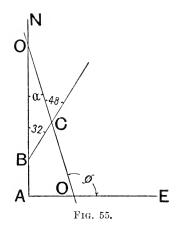
For angle α we have $48^{\circ} - 32^{\circ} = 16^{\circ}$. For angle ϕ we have $90^{\circ} - 32^{\circ} + 48^{\circ} = 106^{\circ}$.

Ex. A surreyor is making a map on which he wishes to locate two inaccessible objects H and K situated towards the North, He lans

off a base line DE, 20 chains long, going due East. When stationed at D, he measures the angles EDK, KDH, by means of a sextant, and finds them to be 51° and 55°. When stationed at E the angles DEH and HEK measure 48° and 62°.

Plot the points D, E, H, K to a scale of 8 in. to the mile. Measure the distance and direction of HK. (B, E, 1904.)

Fig. 56. Draw the base line DE 2 in, long. Draw Dk making



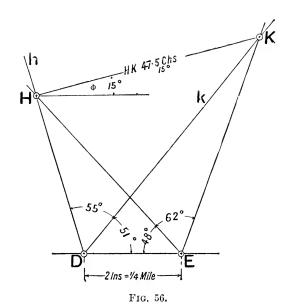
the given angle 51° with DE. Draw Dh making the given angle 55° with Dh. Draw EH making the given angle 48° with DE and produce to meet Dh in H. Draw EK making the given angle 62° with HE and produce to meet Dh in K. Then H and K mark the position of the two objects. Draw through H a parallel to DE and measure HK and the angle ϕ . HK measures 4.75 in. = 47.5 chains; ϕ measures 15°. Hence K is 47.5 chains from H in direction E, 15° N.

128. The positions of places or fixed objects relatively to one another are conveniently given by specifying the distance of each place or object from some fixed point, and the magnitude of the angle which the line drawn from the fixed point to the place or object makes with a fixed direction.*

^{*} If the fixed direction be regarded as an axis of reference, and the fixed point as the origin, and if the distance be denoted by r and the angle by θ , the measurements r, θ locate the point, and are called the **polar** or r, θ co-ordinates of the point. See also Art.

Thus, for example, in Fig. 56, the position of the place K relatively to H is completely defined as 47.5 chains in direction E. 15° N. Similarly, the position of K relatively to D may be given as DK distance in direction E. 51° N.

129. In practice, it is usual to take the fixed direction or direction of reference due East, and instead of writing



HK = 47.5 chains in direction E. 15° N.

we write $HK = 47.5_{15}$, chains,

the direction of reference being understood to be due East, the first numbers indicating the distance, and the smaller numbers, written as a subscript, indicating the angle which the direction of the object makes with the direction of reference.

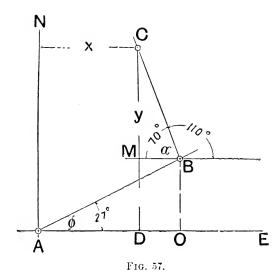
130. Angles when measured in an anticlockwise direction are considered **positive**; when measured in the opposite,

or clockwise direction, they are considered **negative**. Thus the position of K is given as:—

$$HK = 47.5_{150}$$
 chains, or $HK = 47.5_{-3450}$ chains.

Note carefully that the minus sign is given to the angle measure only; the distance is positive in both cases.

From the above illustration, we see that when a direction is given by a positive angle the same direction may be expressed as a minus angle by subtracting the number of



degrees in the positive angle from 360° ; the remainder is the number of degrees in the minus angle, e.g., 360 - 15 = 345; and since 15 is + we have - 345.

Ex. In a traverse survey, lengths and directions are measured by using a chain and a magnetic compass. A, B, C being three trees, the following measurements are made:—

AB = 8.35 chains in direction E. 27° N. = 8.35₂₇₀ chains. BC = 6.24 chains in direction E. 110° N. = 6.24₁₁₀₀ chains. Plet AB and BC to a scale of 1 cm, to 1 chain.

Measure and calculate how much C is to the East of A, and how

much to the North of A. (B. E. 1905.)

Draw lines of reference AE, AN, Fig. 57, at right-angles. AB 8:35 cm. long, and making the given angle 27° with AE. Draw BC 6.24 cm, long and making 110° with the direction East, draw x and y perpendicular to AN and AE.

By measurement x = 5.3 cm. . . . C is 5.3 chains East of A.

By measurement y = 9.7 cm. nearly ... C is nearly 9.7 chains North of A.

*By calculation :-

$$x = AD = AO - DO = AO - MB$$

= $(AB \cos 27^\circ) - (BC \cos , 70^\circ)$
= $(8.35 \times 0.891) - (6.24 \times 0.342)$
= 5.306 chains
 $y = CD = CM + MD = CM + BO$
= $(BC \sin , 70^\circ) + (AB \sin , 27^\circ)$
= $(6.24 \times 0.9397) + (8.35 \times 0.454)$
= 9.65 chains,

Exercises.

(1) A ship S is observed from a station O to be at a distance of 3.2 miles in the direction 16° northwards of East; after 1 hour it has moved to the position S' distant 5.6 miles from O, in the direction 23° N. of E. Plot the points O, S, S' to the scale of I in. to 1 mile. Measure the magnitude and direction of SS', the change of position of the ship. What is the average velocity of the ship during this time? (B.E. 1904.) Ans. 2.45 miles E. 22° N.; 9.8 miles per hour.

(2) The following are the field notes of the survey of a triangular plot of ground ABC. Bearing means the direction from one station to the next succeeding station, and N. 31\frac{1}{2}\circ W. indicates the direction

31% West of North:

Stations.	Bearings.	Distances.		
A	N. 31½° W.	10 chains.		
В	N. 623° E.	8.25 chains.		
C	· .			

Plot the survey to a scale of 1 cm. to 1 chain. Find and measure the bearing and distance from C to A. What is the enclosed area in square chains? (B.E. 1908.) Ans. 12:65 chains W. 79° S.; 82:25 square chains.

(3) Two steamers leave a pierhead together, one going in a direction 22° N. of E. at the rate of 8 miles an hour, the other going in a direction 10° South of East at the rate of 11 miles an hour. How far

are the ships apart $1\frac{1}{2}$ hours after the start? Ans. 9 miles.

(4) At a point A in a straight road which runs due North, a road branches off in the direction N.W. Find the distance apart of two places, one being 7 miles on the main road North of A; the other being on the branch road 31 miles from A. Ans. 5.3 miles.

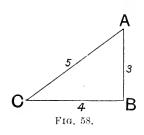
^{*} This calculation may be deferred until Chap. XI. has been read.

CHAPTER XI.

RATIO AND PROPORTION.

131. In order to specify the magnitudes of quantities, we choose some quantity as the unit, and then specify the magnitude of any particular quantity by saying that it contains a certain number of units.

The magnitude may be represented by a whole number, or by a fractional number. For example, if we take as unit a length of 1 in., then, as 1 yd. contains 36 in., the magnitude of 1 yd., when expressed in inches, is 36; or again, if



1 lb. is the unit, then the magnitude of 1 cwt. expressed in lbs. is 112. If 1 ft. is the unit, then 1 in. expressed in feet is $\frac{1}{12}$, or 0.083 of a unit.

To compare magnitudes, it is necessary that (a) the magnitudes must be of the same kind; (b) they must be expressed in the same units. We cannot compare lbs. and inches, for they are not

quantities of the same kind, and again we cannot compare a length in inches with a length in yards without first making the units alike, *i.e.*, we must either reduce the yards to inches or the inches to yards.

If we have two like magnitudes reduced to the same units, we can then say how many times one magnitude is contained in the other; the number expressing this, whether it is a whole number or a fraction, is called a ratio.

(1) A ratio may be expressed as a single whole number, e.q., the ratio of 36 in. to 6 in. is 6.

(2) A ratio may be expressed as a fractional number, e.g., the ratio of 1 cwt. to 1 ton is $\frac{1}{20}$.

(3) A ratio may be expressed as a decimal fraction, e.g., the ratio of one side of a triangle 3 in. long to a second side 5 in. long is $\frac{3}{5}$ or 0.6.

(4) A ratio may be written in fractional form, thus ³/₅ may be read as the ratio of 3 to 5.

(5) A ratio may be written with two dots separating the terms, thus, 3:5 means the ratio of 3 to 5.

It will assist the student very considerably if he remembers that when a ratio is expressed by a single integral, fractional, or decimal number, the number 1 is the second term of the ratio, but is not written down. Thus, the ratio of 36 to 6 is the ratio 6 to 1 or simply 6.

Algebraic ratios are written in either of the forms indicated at (4) and (5) above; thus, the ratio of x to y may be written

$$\frac{x}{y}$$
 or $x:y$.

Ex. 1. Find the ratio of 1 yd. to 50 cm., being given that 1 in. = 2.54 cm.

Here we have 1 yd. = 36 in.

$$50 \text{ cm.} = \frac{50}{2.54} = 19.69 \text{ in.}$$

Hence ratio is $\frac{36}{19.69}$ or 36: 19.69 or 1.83.

Ex. 2. The sides of a right-angled triangle ABC, Fig. 58, are respectively AB = 3 in., BC = 4 in., AC = 5 in. Find the ratio of AB to AC, BC to AC, and AB to BC.

Ratio
$$\frac{AB}{AC} = \frac{3}{5} = \frac{0.6}{1} = 0.6$$

Ratio $\frac{BC}{AC} = \frac{4}{5} = \frac{0.8}{1} = 0.8$
Ratio $\frac{AB}{BC} = \frac{3}{4} = \frac{0.75}{1} = 0.75$.

132. The ratios found in the above example will be the same whatever the lengths of the sides of the triangles, provided the angles remain the same. This statement can be tested by drawing any triangle having a right-angle B, Fig. 59, and angles C and A respectively, equal to those

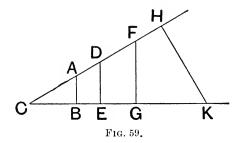
in the above triangle ABC. Then if we measure the sides of the triangles ABC, DEC, FGC we find:—

Ratio
$$\frac{AB}{AC}$$
 = ratio $\frac{DE}{DC}$ = ratio $\frac{FG}{FC} = \frac{3}{5}$.

Ratio $\frac{BC}{AC}$ = ratio $\frac{EC}{DC}$ = ratio $\frac{GC}{FC} = \frac{4}{5}$.

Ratio $\frac{AB}{BC} = \frac{DE}{FC} = \frac{FG}{GC} = \frac{3}{4}$.

We can carry this process further. Draw HK per-



pendicular to CH. We now have the right-angle at H, and by measurement we can show that

$$\frac{AB}{AC} = \frac{KH}{KC}; \frac{BC}{AC} = \frac{HC}{KC}; \frac{AB}{BC} = \frac{KH}{HC}.$$

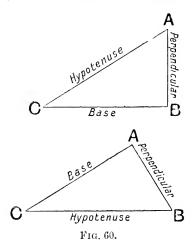
If we take any right-angled triangle ABC, Fig. 59 or 60, having a right-angle at B, and call the two sides forming the right-angle respectively the perpendicular and the base of the triangle, and the side opposite to the right-angle the hypotenuse, then whatever the lengths of the three sides are, provided the angles C and A remain unaltered, we have:—

 $\begin{aligned} & \text{Ratio } \frac{AB}{AC} \text{ or } \frac{\text{perpendicular}}{\text{hypotenuse}} \text{ is always the same number.} \\ & \text{Ratio } \frac{BC}{AC} \text{ or } \frac{\text{base}}{\text{hypotenuse}} \text{ is always the same number.} \\ & \text{Ratio } \frac{AB}{BC} \text{ or } \frac{\text{perpendicular}}{\text{base}} \text{ is always the same number.} \end{aligned}$

P.M.

If we change the right-angle from B to A, but keep the angle C the same, we do not alter any of the ratios.

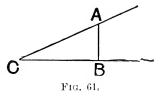
When the ratio of two quantities is always equal to the same number, the ratio is said to be *constant*.



183. The ratios of the sides of a right-angled triangle are of the greatest importance in mathematics, and are called

trigonometrical ratios. Every angle has six definite trigonometrical ratios.

Take any angle C, Fig. 61, bounded by two lines, and consider the six trigonometrical ratios for that angle. From any point A in either of the lines bounding the



angle, draw ΛB perpendicular to the other line. We now have a triangle ΛBC with a right-angle at B. (Suppose your 60° set-square to be this triangle, and that the angle Λ

is 60%) The side AC opposite to the right-angle is the hypotenuse (the longest side of the set-square), the side AB opposite to the given angle C is the perpendicular (the shortest side of the set-square), and the remaining side BC is the base.

The ratio	perpendicular hypotenuse	(AB) (AU)	is called the ${\bf sine}$ of the angle ${\bf C}$.
The ratio	base hypotenuse	(BC) (AU)	is called the cosine of the angle

e **cosinë** of the angle C:

(BC) The ratio perpendicular (AB)

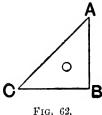
The ratio perpendicular (AB) is called the tangent of the angle G

The ratio hypotenuse

is called the cotangent of the angle C.

is called the secant of the angle C.

The ratio $\frac{\text{hypotenuse}}{\text{perpendicular (AB)}}$ is called the **cosecant** of the angle C_i



These six ratios are abbreviated to sin. C, cos. C, tan. C, cot. C, sec. C, cosec. C; and their numerical value depends only upon the angle C. Every angle has a definite value for its sin., cos., etc., and tables are prepared from which these values can be found.

The trigonometrical ratios for certain angles can be readily ascertained. Consider your 45° set-square, shown in Fig. 62. The

angle C = 45°, the angle A = 45°, and hence AB = BC. If we call AB and AC each 1, then

since
$$AC^2 = AB^2 + BC^2$$

 $AC = \sqrt{AB^2 + BC^2}$
 $AC = \sqrt{1 + 1} = \sqrt{2}$

Hence perpendicular = 1, base = 1, hypotenuse = $\sqrt{2}$.

and
$$\sin . 45 = \frac{1}{\sqrt{2}}$$
.
 $\cos . 45 = \frac{1}{\sqrt{2}}$.
 $\tan . 45 = 1$.
 $\cot . 45 = 1$.
 $\sec . 45 = \sqrt{2}$.
 $\csc . 45 = \sqrt{2}$.

Similarly for the trigonometrical ratios of angles of 60° and 30°, we first draw an equilateral triangle ACD, Fig. 63. Each angle is 60°: if we bisect the angle C by CB drawn perpendicular to AD we get a triangle ACB in which the angle BCA = 30° , angle A = 60° ; and angle $B = 90^{\circ}$. This triangle is exactly like your 60° set-square.

If each side of the equilateral triangle = 2, then, since B bisects AD, AB = 1; and since AC^2 = CB2 + AB2, hence

$$CB^2 = AC^2 - AB^2$$

$$CB = \sqrt{AC^2 - AB^2}$$
$$= \sqrt{4 - 1} = \sqrt{3}.$$

For an angle of 60°, we have

For an angle of
$$60^{\circ}$$
, we have $\sin 60^{\circ} = \frac{\sqrt{3}}{2}$, $\cos 60^{\circ} = \frac{1}{2}$, $\cot 60^{\circ} = \sqrt{3}$, $\sec 60^{\circ} = 2$, $\csc 60^{\circ} = \frac{2}{\sqrt{3}}$.

For an angle of 30°, we have :-

$$\sin 30^{\circ} = \frac{1}{2}$$
, $\cos 30^{\circ} = \frac{\sqrt{3}}{2}$, $\tan 30^{\circ} = \frac{1}{\sqrt{3}}$, $\cot 30^{\circ} = \sqrt{3}$, $\sec 30^{\circ} = \frac{2}{\sqrt{3}}$, $\csc 30^{\circ} = 2$.

All the above ratios can be worked out to decimal quantities. As an example :—

$$\cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866.$$

134. In any right-angled triangle, as ABC, Fig. 62, right-angled at B, the angles C and A are complementary, *i.e.*, $C + A = 90^{\circ}$.

And since sin.
$$C = \frac{AB}{AC}$$

,, ,, $\cos A = \frac{AB}{AC}$

Therefore the sine of an angle is equal to the cosine of the complementary angle.

Again, tan.
$$C = \frac{AB}{BC}$$

cot. $A = \frac{AB}{BC}$

Therefore the tangent of an angle is equal to the cotangent

of the complementary angle.

Consequently, if we set out, as in Table 3, Appendix, the values of the sines and tangents of angles from 0° to 45°, these values will also be the cosines and cotangents of angles from 90° to 45°.

In Table 3, Appendix, we have in col. 1 degrees from 9° to 45° , and in col. 10 degrees from 90° to 45° , as shown in the extract below.

Ang	gle.	Chord.	Sin.	Tan.	Cot.	Cos.			
Degrees.	Radians								
0° 1° 43° 45°	•0175 •7505 •7854	·017 ·733 ·765	·0175 ·6820 ·7071	*0175 *9325 1:0000	57·2900 1·0724 1·0000	-9998 -7314 -7071	1·402 ·797 ·765	1:5533 :8203 :7854	90° 89° 47° 45°
	Con	Cos.	os, Cot.	Tan,	Sin.	Chord.	Radians.	Degrees.	
			cos.	330.	1411,	om.	Chora.	Angle.	

If any value of an angle be read off in col. 1, the value $(90^{\circ} - \text{angle})$ is found in col. 10. Thus, 43° in col. 1 and $(90^{\circ} - 43^{\circ}) = 47^{\circ}$ in col. 10 are in the same line.

At the top of col. 4 we find sin. and at the bottom cos.

Suppose we require to know the value of sin. 43°. Look for 43° in col. 1 and in col. 4 with sin. at top, read "sin. 43° = 0.6820." If we require the value of cos. 47°, look in col. 10 for 47°, and read in col. 4 which has cos. at the bottom, the value "cos. $47^{\circ} = 0.6820$."

Thus we see from the Table, \sin 43 = \cos 47 = 0.6820. Above we found that \sin 43 = \cos (90 - 43) = \cos 47.

In reading any ratio from the Table, if the angle is in col. 1 we take the column with sin., cos., etc., at the top agreeing with the particular ratio we require; whereas, if the angle is found in col. 10, take the column with sin., cos., etc., at the bottom.

Ex. 1. Find tan. 15°.

15° is in col. 1, hence read in col. 5 "tan. 15° = 0.2679."

Ex. 2. Find tan. 78°.

78° is in col. 10, hence read in col. 6 " tan. $78^{\circ} = 4.7046$."

135. The trigonometrical table is very useful in calculation, particularly in calculating distances or heights which are found inconvenient to measure directly.

Ex. We wish to know the height of a church tower, or a cliff, or Measure a distance AB, Fig. 64, say 300 ft. from the foot of the tower. At the point B we take a box with a hinged lid and holding the box horizontally we open the lid until looking along

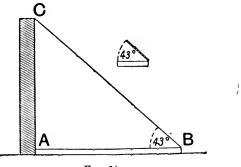


FIG. 64.

it we just see the top C of the tower. Now measure the angle the lid makes with the box. Suppose this to be 43°.

We now have $\frac{AC}{AB} = \tan 43^{\circ}$.

We know AB = 300 ft., and from the table we find tan, 43° = 0.9325, hence $\frac{AC}{300} = \frac{0.9325}{1}$

or AC =
$$300 \times 0.9325$$

= 279.75 ft.

The point B is at the level of the observer's eye. Suppose this is 5 ft. 3 in. above the ground.

Hence height of tower = (AC + 5.25) ft. = 279.75 + 5.25 ft.

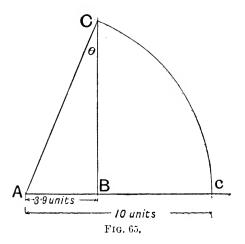
= 285 ft. high.

136. Each trigonometrical ratio is a measure of one side of a right-angled triangle in terms of another side, and all these ratios are interdependent, their values depending on the magnitude of the angle; thus, when the magnitude of the angle is fixed, the values of all the ratios are

fixed. Conversely, if the value of either ratio be given, then each of the other ratios has, at the same time, been definitely fixed; and the magnitude of the angle also fixed. It is therefore a determinate problem to construct an angle when given the value of one of the above ratios. The solution of this problem involves the drawing of a right-angled triangle.

Ex. The sine of an angle (θ°) is 0.39; construct the angle.

Sin. $\theta = \frac{\text{perpendicular}}{\text{hypotenuse}}$... perpendicular = $\begin{cases} \sin. \theta \times \text{hypotenuse} \\ 0.39 \times \text{hypotenuse}. \end{cases}$ Draw a straight line Ac, Fig. 65, 10 units—say centimetres—in



length. Take a point B on Ac, such that $AB = 3\frac{a}{10}$ cm. (the unit being 1 cm.).

At B draw a perpendicular to Ac. With centre Λ and radius Ac describe an arc meeting the perpendicular from B in C. Join CA.

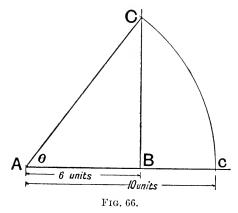
Then $\frac{AB}{AC} = \sin \theta = \frac{3.9}{10} = 0.39$. \therefore ACB is the required angle.

Ex. 2. The cosine of an angle (θ°) is 0.6; construct the angle. Cos. $\theta = \frac{\text{base}}{\text{hypotenuse}}$ \therefore base = $\begin{cases} \cos. \theta \times \text{hypotenuse}, \\ 0.6 \times \text{hypotenuse}. \end{cases}$

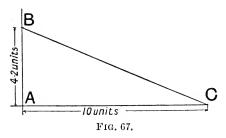
Draw a straight line Ac, Fig. 66, 10 (units) em. in length. Take a point B on Ac such that AB = 6 cm. At B draw a perpendicular to Ac. With centre A and radius Ac describe an arc meeting the

perpendicular from B in C. Join CA. Then $\frac{AB}{AC} = \cos \theta = \frac{6}{10} = 0.6$. \therefore BAC is the required angle.

Ex. 3. The tangent of an angle (θ°) is 0.42; construct the angle. Tan. $\theta = \frac{\text{perpendicular}}{\text{base}}$: perpendicular = $\begin{cases} \tan x \times \text{base} \\ 0.42 \times \text{base} \end{cases}$



Draw AC, Fig. 67, 10 (units) cm. in length. At A draw AB perpendicular to AC; and make $AB=4\frac{1}{5}$ cm. Draw BC. Then ACB is the required angle.



137. In problems similar to the three preceding, angles may be plotted direct from a protractor, if we have a table of values similar to Table 3, Appendix. Thus, being given sin. $\theta^{\circ} = 0.39$, we search the trigonometrical table

for the measure in degrees of the angle whose sine is 0.39. This we find is 23, hence the required angle is 23°, and can be drawn as explained on p. 136.

Similarly, being given cos. $\theta = 0.6$, reference to the

table shows $\theta = 37^{\circ}$.

Again, given tan. $\theta = 0.42$, the table shows $\theta = 23^{\circ}$. 138. Graphical representation of the change in magnitude of the sin., cos., and tan. of an angle from

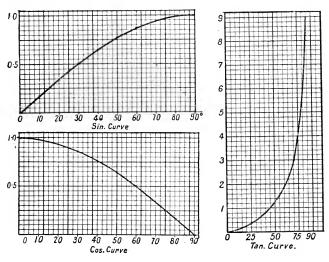


Fig. 68.

0° to 90°. If we plot on squared paper corresponding values of the angle in degrees, and the sin., cos., or tan. of the angle, the graph so obtained is called a sine, cosine, or a tangent curve. First make out a table of corresponding values of angles increasing by 10°, and values of the sin., cos., and tan., using the trigonometrical table.

Plotting angles horizontally and values of the sin., cos., or tan. vertically, we get the graphs shown in Fig. 68. Note, in the tangent graph since tan. 90° is infinite in value,

and as we cannot represent infinity by any finite ordinate, we adopt the convention of making the curve parallel to the vertical line through the point at which $x = 90^{\circ}$ to represent that the value of the tangent becomes infinite for an angle of 90° .

From these curves we can read off values of the sin., cos., and tan. of angles. Thus tan. $45^{\circ} = 1$, and so on. Similar curves can be constructed for the cot., see, and cosec. of an angle.

Exercises.

(1) What are the ratios of 5 to 6, 4 to 25, 3 to 17, x^6 to x^5 , 5 lb. to $\frac{1}{2}$ ton, $1\frac{1}{2}$ ft. to $\frac{3}{4}$ mile, 25 cm. to $3\frac{1}{2}$ yds., 25 sq. ft. to an acre, 800 cub. in. to 1 cub. yd., and 11 gallons to 100 litres. (See p. 100.)

(2) State exactly what we mean by the sin., cos., and tan. of an angle. Write down the value of sin. 60, cos. 30, and tan. 45. (B.E. 1907.)

(3) In a triangle ABC, C being a right-angle AB is 14.85 in. and AC is 8.32 in. Compute the angle A in degrees, using your tables. (B.E. 1903.)

(4) ABC is a triangle, C being a right-angle. The side BC is 12.4 ft, and the angle A is 65°; find the other sides and angle, using

the tables. (B.E. (2) 1905.)

(5) Write down the values of the sine, cosine, and tangent of 37° . Explain, from the definitions, why sin $37^{\circ} + \cos 37^{\circ} = \tan 37^{\circ}$. Try by division if this is so. (B.E. (2) 1905.)

(6) The altitude of a tower observed from a point distant 150 ft.

horizontally from its foot is 26°; find its height. (B.E. 1905.)

(7) In a right-angled triangle ABC, C being the right-angle, the side AC is 20 in.. the angle BAC is 55°; find the side BC and the area of the triangle. (B.E. 1909.)

(8) A rectangular plate is 10 ft. long and the diagonal makes an angle of 35° with this side; find the length of the short side, and the

area.

(9) A force F of 100 lb. acts at an angle $\theta = 75^{\circ}$ to the horizontal. Its vertical component is F sin. θ and its horizontal component is

F cos. θ. Calculate the values of the components.

(10) A person walking along a straight road sees a house in a direction at right-angles to the road at the point A, and a mile further on at B he sees it at an angle of 60° with the road; find the distance of the house from the points A and B.

(11) If $\sin (A - B) = \sin A \cos B - \cos A \sin B$. Calculate this

when $A = 75^{\circ}$ and $B = 12^{\circ}$. (B.E. 1909.)

(12) Plot on squared paper a curve from which the value of the cot. of any angle from 0° to 90° may be obtained, using your tables. Repeat this for a sec. curve and a cosec curve given that sec. A =

 $\frac{1}{\cos A}$ and cosec. $A = \frac{1}{\sin A}$. Read from these curves the values of cot, 55°, sec. 70°, and cosec, 35°,

(13) The area of cross-section of a prism is 92.30 sq. in.; what is the area of a section making an angle of 25° with the cross-section?

(The cross-section is the smallest section.) (B.E. (2) 1904.)

(14) There is a district in which the surface of the ground may be regarded as a sloping plane; its actual area is 3.246 sq. miles; it is shown on the map as an area of 2.875 sq. miles; at what angle is it inclined to the horizontal?

There is a straight line 20.17 ft. long which makes an angle of 52° with the horizontal plane; what is the length of its projection on

the horizontal plane? (B.E. 1904.)

(15) ABC is a triangle, C being a right-angle. The side AB is 15 34 in., the side BC is 10 15 in. What is the length of AC? Express the angles A and B in degrees. What is the area of the triangle in square inches? (B.E. 1904.)

Answers.

- (1) 0.83; 0.16; 0.176; x; (8) 7.002; 70.02. 0.00116; 0.00038; 0.08; 0.00057; (9) 96·59; 25·88. 0.46; 0.5. (10) 1.732; 2, (2) 0·866; 1. (11) 0.891. (3) 56°. (13) 101.9.(4) 25°; 5.78; 13.68. (14) 28°; 12·418. (6) 73.15. (15) 41°: 49°: 11 5: 58.4. (7) 28.562; 285.62.
- 139. Important ratios connected with the circle and angles. The length of the circumference of the circle. Draw two straight lines. Take a centimetre as a unit, and make AB on one line equal 7 units, i.e., 7 cm. Take 1 in. as a unit, and make CD on the other line equal 7 units, i.e., $7 \times \frac{1}{4}$ in. $= 1\frac{3}{4}$ in. Describe circles on diameters AB, CD, and by the method of Art. 108 carefully measure the circumference of each. The length of circumference of the circle on AB will be found approximately 22 cm., i.e., $\frac{22}{7}$ × AB.

The length of circumference of circle CD will be found approximately $5\frac{1}{2}$ ins., i.e., $\frac{22}{7}$ × CD. Thus, in each case the circumference is approximately 3½ times the diameter.

The student may draw other circles, and by careful measurements satisfy himself that the circumference of a circle is always approximately 31 times its diameter.

Hence, the ratio length of circumference is the same for all circles, and it is universally denoted by the letter π (pi). The value of this ratio we have given above as $\frac{22}{7}$. is the simplest approximate value of π ; the exact value cannot be stated by any number. More nearly the ratio is 3 141592, etc., so that $\frac{22}{7}$ is too large. The error, how-

ever, is only about $\frac{1}{2500}$ of the value.

For graphical work, the value $\frac{22}{7}$ may be used. arithmetical work, the number 3.1416 is usually taken.

In this book π is taken as 3.1416. Whence we have:—

- (1) length of circumference of circle = $\pi \times \text{diameter}$ $= 2\pi \times \text{radius} = 2 \times 3.1416 \times r \text{ (where } r =$ radius of circle):
- (2) semicircumference = $\pi \times r$;
- (3) $\frac{1}{6}$ of circumference $=\frac{1}{6}\times 2\pi r$; and generally where n equals any number,
- (4) $\frac{1}{n}$ of circumference $=\frac{1}{n}\times 2\pi r$.

Ex. 1. In order to ascertain the diameter at the base of a circular chimney shaft, a steel tape measure is folded around the chimney at. the base level. The tape shows the circumference of the shaft to be 39 ft, 7 in. What is its diameter?

Here we have $\pi \times \text{diameter} = \text{circumference} = 39.58 \text{ ft.}$

∴ diameter =
$$\frac{39.58}{3.1416}$$
 ft.
= 12.5 ft. = 12 ft. 6 in.

Ex. 2. The diameter of a bicycle wheel is $22\frac{1}{2}$ in. How far does the bicycle travel in one revolution of the wheel?

(Note.—In one revolution the wheel measures its circumference along the track.)

Circumference =
$$\frac{\pi}{3.1416 \times 22.5}$$
 in.
= 3.1416 × 22.5 in.
= 70.686 in.

- ... Bicycle travels 5 ft. 10.7 in. during one revolution of the wheel,
- 140. The length of the circumference of a circle may be determined graphically by the following construction:

Fig. 69. Draw a diameter BCA; and a tangent to the circle at A. Draw CO making 30° with BA meeting the tangent in O. Make OD = 3r (r being the radius of circle). Join BD. Then we have:—

$$BD^{2} = BA^{2} + AD^{2}$$

$$= BA^{2} + (OD - OA)^{2}$$

$$= (2r)^{2} + (3r - r \tan, 30)^{2}, \text{ for } OA = AC \tan, 30^{\circ}.$$

$$= 4r^{2} + \left(3r - r \frac{1}{\sqrt{3}}\right)^{2}$$

$$= 4r^{2} + r^{2}\left(3 - \frac{1}{\sqrt{3}}\right)^{2}$$

$$= r^{2}(4 + 5.869)$$

$$= 9.869 r^{2}.$$

Hence BD = $3.1415 \times r$; it is therefore an exceedingly

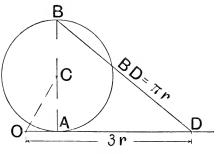


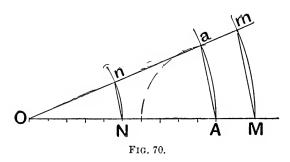
FIG. 69.

near approximation to the correct value πr . Hence, $2BD = 2\pi r = \text{length of circumference of circle.}$ And $\frac{1}{2}BD = \text{length of a quadrant.}$

141. Chords of angles. Fig. 70. On a straight line OM mark a point A such that OA = 10 units (any unit). With centre O and radius OA describe an arc. With centre A and radius equal 4 units, say, describe an arc intersecting the previously drawn arc in a. Draw Oa and the straight line aA. Then the chord Aa is called the chord of the angle AOa; it is said to subtend that angle.

For the ratio
$$\frac{\text{length of chord}}{\text{radius}}$$
 we have $\frac{Aa}{OA} = \frac{4}{10}$ or 0.4.

Now take any points NM on OA and describe arcs from centre O, to pass through these points intersecting the radial Oa in n and m. Draw the chords Mm, Nn. Let θ°



denote the angle between the radials OM, Om, then, the ratio $\frac{\theta^{\circ}}{360}$ is the same for the three arcs centred in O.

- : (i.) Ratio length of arc is the same in all the circles.
- (ii.) Ratio $\frac{\text{length of chord of angle}}{\text{radius of are}}$ is the same in all the circles.

Whence,

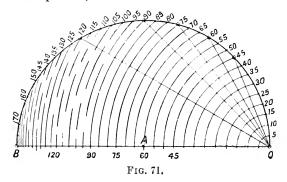
from (i.)
$$\frac{\text{Arc N}n}{\text{ON}} = \frac{\text{Arc M}m}{\text{OM}} = \frac{\text{Arc A}a}{\text{OA}} = \text{constant};$$
from (ii.) $\frac{\text{Chord N}n}{\text{ON}} = \frac{\text{Chord M}m}{\text{OM}} = \frac{\text{Chord A}a}{\text{OA}}$
 $= 0.4 \text{ constant}.$

142. The value of the ratio chord radius depends solely on the magnitude of the angle at O. For, in an angle less than MOm, the chord in the arc through M is less than Mm whilst the radius OM remains the same; consequently, the

ratio $\frac{\text{chord}}{\text{radius}}$ is less than $\frac{Mm}{OM}$. Again, in an angle greater than MOm, the chord is greater than Mm, and therefore the ratio is greater than $\frac{Mm}{OM}$.

If a semicircle, diameter OB, were divided into 180 equal parts, and lines drawn from O an extremity of the diameter, to each division on the are, such lines would be *chords* of angles advancing by single degrees from 1° to 180°.

If all these *chords* were set off along a straight line from the same fixed point O, such line would be a scale of chords.



In Fig. 71, we show how to construct a scale of chords. Draw a straight line BO, and on it describe a semicircle, centre A. Mark off degrees on the circular arc, making use of a protractor. Intervals of 5° are shown on the diagram. With centre O describe arcs bringing the degree points on the semicircle down on to diameter BO, and number the divisions from O towards B. Then OB is a scale of chords a lyancing in steps of 5° from 0° to 180°.

From the figure, it is evident that-

- (i.) In an angle of 60°, $\frac{\text{chord}}{\text{radius}} = 1$ (unity).
- (ii.) In an angle greater than 60°, $\frac{\text{chord}}{\text{radius}}$ is greater than unity.
- (iii.) In an angle less than 60° chord radius is less than unity.

The student, when using a scale of chords, must remember that the chords are chords of a circle having radius equal to 0—60 on the scale.

A scale of chords is engraved on many of the boxwood and other scale rules sold by mathematical instrument makers. It is distinguished by the mark "Cho."

A scale of chords may be used—

- (a) To measure a given angle.
- (b) To set out a given angle.

(a) Fig. 72. Let MON be the angle we are required to measure. Take the distance 0—60 from your scale of chords in a pair of compasses, and with centre O describe an arc intersecting OM, ON

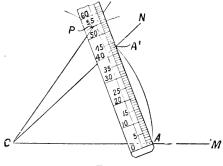


FIG. 72.

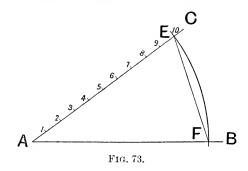
(produced if necessary) in A, A' respectively. Apply your scale to the chord AA', placing the zero point (0) exactly to the point A. Then the number of degrees in the angle MON is indicated by the reading (43) on the scale at the point A'. Therefore the angle MON is 43°

(b) From a point O, Fig. 72, in a line OM, set out an angle of 55°. Make OA on OM = 0-60 from your scale of chords. With centre O and radius OA describe an arc; with centre A and radius 0-55 (taken from the scale) describe an arc intersecting the arc previously drawn in P. Draw OP. Then MOP is the required angle of 55°.

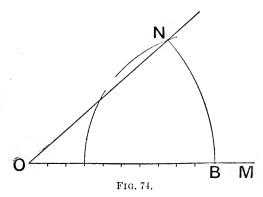
143. In Table 3, Appendix, is given the numerical value of the ratio $\frac{\text{chord}}{\text{radius}}$ for angles advancing by single degrees from

1° to 90°. This Table may be used in measuring angles.

E.e. 1. Let it be required to measure the given angle BAC, Fig. 73. Select some convenient unit, and with the centre A and radius = 10 units describe an arc intersecting AC, AB (produced if necessary) in E and F respectively. Measure the chord FE, employing the



same unit as used for the radius AF. Suppose this done, and we fin l ehord EF = 6.2 units. Then ratio $\frac{FE}{AE} = \frac{6.2}{10} = 0.62$. Hence the numerical value of the ratio is 0.62. Consulting the table, we find the



number nearest to 0·62 is 0·618, which is opposite the degree number 36. Therefore the nearest measure in whole degrees of the angle BAC is 36° .

We now give an example showing how an angle may be constructed

without the aid of a properly constructed scale of chords, or reference to a table of chords.

Ex. 2. Fig. 74. The chord of angle is 0.7; construct the angle.

Draw a straight line OM, and take a point B on OM such that OB = 10 cm. (any unit). With centre O and radius OB describe an are; with centre B and radius equal 7 units in OB (0.7 being the given ratio) describe an arc intersecting the arc from B in N. Draw ON. Then MON is the required angle.

Exercises.

(1) The driving wheel of a locomotive is 7.5 ft. in diameter, and it makes 250 revolutions per minute. Find the speed of the locomotive in miles per hour.

(2) A piece of metal 3 in. diameter is being turned in a lathe, the spindle of which makes 25 revolutions per minute. Find the speed of cutting the metal. If it is desired to turn a 2-in, piece of metal at the same cutting speed, how many revolutions must the

spindle make per minute?

(3) A shunt-coil in an electric arc lamp is 8 in, long, and the diameter of the tube upon which the wire is wound is $\frac{3}{4}$ in. The wire with its insulating-covering is $\frac{1}{20}$ in. in diameter, and there are 20 layers to be wound upon the tube. What length of wire is wound

upon the tube, and what is the diameter of the outer layer?

(4) In a bicycle the pedal makes 1 revolution while the wheel makes a greater number, say, x. The gear of the bicycle is the product $x \times \text{diameter of driving wheel.}$ If a bievele has a 28-in. wheel, and its gear is 84, what is the value of x? In a 3-speed gear for this bicycle, the gears provided are 70, 98, and 126. What is the value of x in each case?

(5) The pitch circle of a spur wheel having 80 teeth is 12:73 in.

diameter. Find the pitch of the teeth.

(6) A pinion having 12 teeth of 3 in, pitch is to gear with a wheel having 80 teeth. Find the diameter of the pitch circle of the wheel.

(7) A wheel is 3.45 ft. in diameter; it makes 1,020 revolutions rolling along a road; what is the distance passed over? (B. E. 1906.)

(8) In working this question employ a decimal scale of 1 in. to 1 unit.

Draw a circular arc, radius 10 units, centre O. Mark a chord AB of this arc, 3.47 units long, and draw the radii OA, OB. Measure the angle AOB in degrees.

From B draw a perpendicular BM on OA, and at A draw a tangent to meet OB produced in N. Measure carefully BM and AN (on the above unit scale), and calculate the sine and the tangent of the angle AOB.

Give the correct values of the angle, the sine and the tangent, taken directly from the examination tables supplied. (B. E. 1906.)

Answers.

(1) 66:94. (4) 3; 2:5; 3:5; 4:5. (2) 19:635 ft, per min.; 37:5 revs. (5) 0:5.

(2) 19·635 ft. per min.; 37·5 revs. (5) 0·5. (6) 19·09. dia. (7) 2·09 mls,

144. **Proportion.** When four quantities are so related that the ratio of the first two quantities is equal to the ratio of the second two quantities, the four quantities are said to form a proportion.

As an example, consider two rods, of equal sectional area but of different lengths, made of similar material. If the first rod is 1 yd. long, and the second rod 5 yds. long, then the ratio of the lengths of the rods is $\frac{1}{5}$. If each rod weighs 1 lb. per foot of length, the first rod weighs 3 lb., and the second rod weighs 15 lb., and the ratio of their weights is $\frac{3}{15}$ or $\frac{1}{5}$.

Hence, the ratio of the lengths of the rods is equal to the ratio of the weights of the rods, and so the four quantities, viz., the two lengths and the two weights, form a proportion. This is expressed mathematically as follows:—

$$({\rm Ratio}) \frac{{\rm Length~of~first~rod}}{{\rm Length~of~second~rod}} \ = \ ({\rm Ratio}) \frac{{\rm Weight~of~first~rod}}{{\rm Weight~of~second~rod}}$$
 or $\frac{1}{5} = \frac{3}{15}$.

It may also be expressed as follows:-

$$\begin{array}{cccc} I. & & II. \\ (Length of 1st rod) & : & (Length of 2nd rod) :: \\ III. & & IV. \\ (Weight of 1st rod) & : & (Weight of 2nd rod). \\ \end{array}$$

The sign :: means the same as = (equals).

The first and fourth terms of the proportion are called the *extremes*, and the second and third terms are called the *means*.

When four magnitudes form a proportion, the product of the extremes = the product of the means. This may be tested as follows:

The ratio
$$\frac{3}{5}$$
 is equal to the ratio $\frac{9}{15}$,
Hence, $3:5::9:15$ or $\frac{3}{5}=\frac{9}{15}$,

and 3 and 15 are the first and fourth terms, or extremes; 5 and 9 are the second and third terms or means.

Product of extremes $3 \times 15 = 45$ Product of means $5 \times 9 = 45$.

Expressed algebraically—

Suppose the ratio
$$\frac{a}{b}$$
 is given = ratio $\frac{x}{y}$.

Then a, b, x, y form a proportion, and ay = bx.

From this equation, we can find any one term of a proportion if we are given the other three terms; thus, given

$$b$$
, x and y , then, since $ay = bx$, $a = \frac{bx}{y}$.

Ex.1. In mechanics it is shown that the strength of a beam is proportional to the width of the beam. A beam 5 in, wide carries a maximum load of 3 tons; what load will a similar beam as regards length and depth, but $7\frac{1}{2}$ in, wide, earry?

We have $\frac{\text{Width of 1st beam}}{\text{Width of 2nd beam}} = \frac{\text{Maximum load of 1st beam}}{\text{Maximum load of 2nd beam}}$ If y tons is the maximum load for the second beam, then

$$\frac{5 \text{ in.}}{7 \cdot 5 \text{ in.}} = \frac{3 \text{ tons}}{y \text{ tons}}$$

$$5 \times y = \frac{3 \times 7 \cdot 5}{3 \times 7 \cdot 5} = \frac{22 \cdot 5}{5} = 4 \cdot 5 \text{ tons.}$$

145. We have seen in Art. 139 that an arc of a circle whose length is $\frac{1}{n}$ th part of the whole circumference = $\frac{1}{n} \times 2\pi r$ (where r = radius).

Now, such an are will subtend at the centre of the eircle an angle $=\frac{1}{a} \times 360^{\circ}$.

Hence, the length of a circular are is proportional to the angle subtended by that are at the centre of the circle.

Ex. 2. In a circle 4 ft, diameter, fed the length of the are which subtends an angle of 135° .

The circumference of the circle = 4π ft.

The angle subtended by the whole circumference = 360°.

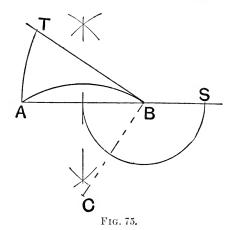
Hence
$$\frac{\text{Length of arc}}{\text{Length of circumference}} = \frac{135^{\circ}}{360^{\circ}}$$
.

Let the length of the are be denoted by a ft.

Then
$$\frac{a}{4\pi} = \frac{135}{360}$$
.
 $a = \frac{4 \times \pi \times 135}{360} = 4.7124 \text{ ft.}$

146. The length of an arc AB may be found graphically.

At an extremity B, Fig. 75, draw BT tangent to the are; draw the chord AB and produce to S such that $BS = \frac{1}{2}AB$. With S as centre, and radius = SA, describe an arc meeting the tangent



from B in T. Then BT is very nearly equal to the length of the arc AB.

When the given are is greater than a quadrant, first find πr , i.e., the length of the semicircumference; then find the length of a segment, which together with the given are equals a semicircle, and subtract this length from πr .

147. We already know that the magnitudes of angles may be measured in units called degrees. If we choose as a unit the angle subtended at the centre of a circle by an arc equal in length to the radius, such an angle or unit is called a radian, and the magnitude of an angle expressed in radians is called the circular measure of the angle.

The comparison between the magnitudes of angles, when

expressed in degrees, and when expressed in radians, is an example in proportion.

Ex. 1. Find the number of degrees in one radian.

Consider a complete circle. The angle subtended by the circumference is 360° . Now, since the circumference is equal to 2π times the radius, there will be 2π arcs in the circumference, each equal to the radius, and hence if the extremities of each arc equal in length to the radius be joined to the centre of the circle we should have 2π angles, each of which would be 1 radian. It follows that 360 degrees is equal 360

to 2π radians, hence the number of degrees in one radian is $\frac{360}{2\pi} = 57.3$ degrees nearly, or $\frac{360}{2\pi}$ radians = $\frac{\text{number of degrees in a radian}}{\text{one radian}}$.

148. The area of a sector of a circle is proportional to the angle of the sector, whether expressed in degrees or radians. See Art. 222.

Ex. A circle is 3-ft. radius. Find (1) the area of a sector having an angle of 140 degrees, and (2) the area of a second sector having an angle of 2.5 radians.

(1)
$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Angle of sector}}{\text{Angle of circle}}.$$

$$\frac{\text{Area of sector}}{\pi \times 3 \times 3} = \frac{140^{\circ}}{360^{\circ}}.$$

$$\text{Area of sector} = \frac{\pi \times 9 \times 140}{360}.$$

$$= 10.9956 \text{ sq. ft.}$$
(2)
$$\frac{\text{Area of sector}}{\pi \times 3 \times 3} = \frac{2.5 \text{ radians}}{2\pi \text{ radians}}.$$

$$\text{Area of sector} = \frac{2.5 \times 3 \times 3 \times 3}{23}$$

$$= \frac{1}{2} \times (3)^{2} \times 2.5$$

$$= 11.25 \text{ sq. ft.}$$

A general expression may be obtained for the area of a sector of a circle when the radius is known, and also the angle of the sector in *radians*.

Thus, let $\phi =$ angle in radians, and r = radius of circle

Then
$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{\text{angle of circle}}.$$

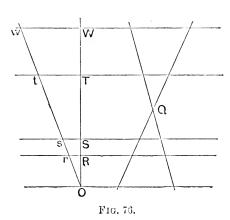
$$\frac{\text{Area of sector}}{\pi r^2} = \frac{\phi}{2\pi}.$$
Area of sector $=\frac{\phi \cdot \lambda_{1} \cdot r^2}{2 \cdot \lambda_{2}}.$

$$= \frac{1}{2} \cdot r^2 \phi.$$

$$= \frac{1}{2} \cdot (\text{radius})^2 \times (\text{angle in radians}).$$

Exercises.

- (1) If $\frac{x}{y} = \frac{7}{16}$, find the value of y when x = 21.
- (2) Find a fourth proportional to 7:9::12: .
- (3) If the ratio $\frac{x}{y} = 0.6$, find the value of x when y is 30.
- (4) The extension of a steel bar is proportional to the load carried by the bar. A load of 2 tons causes the bar to stretch 1 in.; what extension will be produced by a load of 4 tons 5 cwt.? If the original length of the bar is 10 ft., and the ratio extension is called the strain, find the strain in the bar for each of the above loads.



(5) A circular arc is 4.817 in. long, and the radius is 12 in. What is the angle subtended by the arc at the centre in radians and in degrees? (B.E. 1907.)

(6) A beam 5 in, wide carries a load of 3 tons; what load is carried by a similar beam 8 in, wide if the load is proportional to the width?

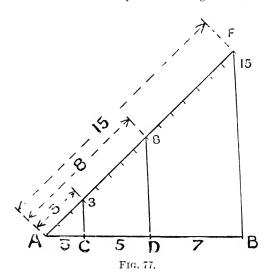
(7) Find the area of a sector of a circle 6 in. diameter, if the angle of the sector is 200 degrees. What is this angle in radians?

Answers.

- (1) 48. (4) $2\frac{1}{8}$ in.; $\frac{1}{120}$; $\frac{17}{960}$. (7) 15.7; 3.49.
- (2) 15³/₂. (5) 0 401; 23. (6) 4.8 tons,

149. Problems of proportion can be solved graphically by constructions based on an important property of parallel lines.

Experiment. Fig. 76. Draw any straight line through a point O, and mark off along same, OR = 0.5 in.; RS = 0.25 in.; RS = 0.25 in.; RS = 0.25 in. RS = 0.25 in. Draw through these points a series of lines parallel to each other and inclined at any convenient angle to OW.



From O draw any straight line as Ow across the parallels, intersecting same in r, s, t, w.

Then the triangles ORr, OSs, OTt, OWw are equiangular. They are also similar.

i.e., $Or = 2rs = \frac{1}{2}st = \frac{2}{3}tw$, or, Or : rs : st : tw = OR : RS : ST : TW, also, Rr : Ss : Tt : Ww = OR : OS : OT : OW.

Verify the above statements by measurements.

Mark any other point as Q, from which draw two or more straight lines across the system of parallels. And verify that the parallels divide all transversals in the ratio, OR: RS: ST: TW.

Ex. To divide a given straight line AB into three parts at C and D such that

AC : CD : DB = 3 : 5 : 7.

From A. Fig. 77, draw any convenient line AF, and mark off along AF 15 equal parts (always take the sum of the given numbers in this case 3+5+7=15). Join B—15, and draw \parallel s through the 3rd and 8th divisions (counting from A) intersecting AB in C and D.

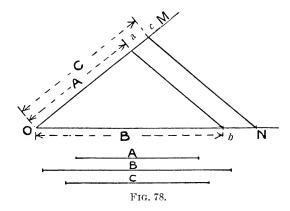
Then the given line is divided as required at C and D.

For, evidently AC : CD : DB = 3 : 5 : 7.

Exercise. Determine a given fractional length, say $\frac{5}{8}$, of a given straight line AB. (Divide AB at D such that AD : AB = 5 : 8.)

Ex. The three lines (A. B, C) represent the three terms A, B, C, of the proportion A: B:: C: X. Determine X.

Draw any two intersecting lines as OM, ON, Fig. 78. Set off



along OM from O the given lengths of A and C. Set off along ON from O the given length of B. Join ab, and draw $cN \parallel ab$ meeting ON in N.

Then, since the △s are similar A:B::C:ON
∴ ON is the required term X.

150. A particular case of proportion occurs when the eccond and third terms (or the means), are equal in magnitude. For example, when the ratio

$$\frac{A}{B}$$
 is equal to the ratio $\frac{B}{C}$
Then $A:B::B:C$

and $A \times C = B^2$ or $B = \sqrt{A \times C}$.

The magnitude B is called a mean proportional to the magnitudes A and C.

The magnitude C is called a third proportional to A and B.

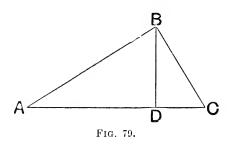
If we are given two terms of the three, we can determine the third from the equation $A \times C = B^2$, e.g., given A and

B, then since
$$A \times C = B^2$$
 we have $C = \frac{B^2}{A}$.

An example occurs in finding the length x of the side of a square which shall be equal in area to that of a given rectangle, say, one having sides 16 in, and 4 in, long respectively.

Then since
$$16 \times 4 = \text{area of rectangle, and}$$
 $x \times x = x^2$ is area of square, we have
$$64 = x^2$$
or $x = 8$.

151. In a right-angled triangle, if a perpendicular be drawn from the right-angle on to the hypotenuse, it can be



shown that, the perpendicular divides the figure into two triangles similar to the original and to each other.

From this proposition some useful facts can be deduced.

Let ABC, Fig. 79, be a triangle having a right-angle at B. Let BD be the [f on AC.

Then (i.)
$$\frac{AD}{DB} = \frac{DB}{DC}$$
 or $DB^2 = AD$. DC

i.e., DB is a mean proportional between AD and DC.

(ii.)
$$\frac{AD}{AB} = \frac{AB}{AU}$$
 or $AB^2 = AD$. AC

i.e., AB is a mean proportional between AD and AC.

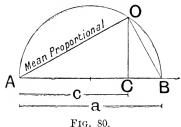
(iii.)
$$\frac{DC}{BC} = \frac{BC}{AC}$$
 or $BC^2 = DC$. AC

i.c., BC is a mean proportional between DC and AC.
 (iv.) Since, AB² = AD. AC, and BC² = DC. AC

$$AD^2 + BC^2 = AC(AD + DC) = AC^3$$

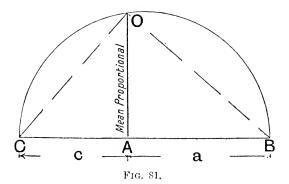
i.e., if squares be constructed on the hypotenuse, and the two sides of a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the two sides.

We have thus available some useful constructions.



Ex. To find a mean proportional (b) to two given straight lines a and c, i.e., find a line b, such that $\frac{a}{b} = \frac{b}{a}$.

First method. Make AB, AC, Fig. 80, on a straight line equal to a and c respectively. Draw a semicircle on diameter AB. Draw



CO perpendicular to AB to meet the semicircle in O. Draw AO, then triangles AOB, ACO are right-angled at O and C respectively, and have a common angle at A. The triangles, therefore, are similar and $\frac{AB}{AO} = \frac{AO}{AC}$.. AO is the required mean proportional.

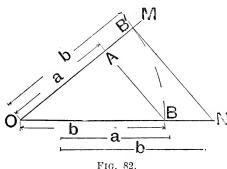
Second method. Set off a, c, Fig. 81, on opposite sides of A on a

straight line. On BC describe a semicircle and draw AO perpendicular to BC, meeting the semicircle in O. Then AO is the required mean proportional; for in the right-angled triangles BAO, OAC,

angle AOB =
$$90^{\circ}$$
 - angle AOC = angle ACO;
 \therefore \triangle s BAO, OAC are similar,
and $\frac{AB}{AO} = \frac{AO}{AC}$.

To find (graphically) a third proportional to two given lines a and b, i.e., to find a line c such that a : b : : b : c.

Make OA, OB', Fig. 82, on a straight line OM, equal to the given



lines a and b. Draw any line ON, and make OB on this line equal Join AB, and draw B'N | AB. Then ON is the line required, OB'.

For, OB = OB', and
$$\frac{AO}{OB} = \frac{OB'}{ON}$$
.

153. If in the above figure we take OA = 1, we have $1 \times ON = OB$. OB', and since OB = OB'... ON = OB2.

Hence the construction may be employed to raise a line of given length, or a given number, to any power.

Ex. To find (graphically) the values of (i.) $(1.7)^2$; (ii.) $(1.7)^3$. Draw two intersecting lines AB, AC, Fig. 83. Select some convenient unit—say $\frac{1}{2}$ in.—and make AE on AC = 1·7 units. Make AD on AB = 1 unit. Join DE; make Ae on AB = AE; draw eF | DE. Then

Since $\frac{AF}{Ae} = \frac{AE}{AD}$ or $AF \times AD = AE \times Ae$; and AD = 1 we have

$$AF \times 1 = AE \times Ae$$

$$AF = AE^{2}$$

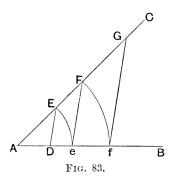
$$= (1.7)^{2}.$$

By repeating the construction, any power of the given length AE may be determined.

Thus,
$$AG = AE^3$$
.

Measure and check by calculation.

154. **Percentages.** Another particular case of proportion occurs when the fourth term in the proportion is 100, and we require to find the third term, the first and second terms being given. The third term, when found, is called a percentage. We can express any ratio as a percentage if we write the ratio as a fraction and then convert it into an equivalent fraction with 100 as the



denominator. The numerator of the second fraction is the number which denotes the percentage.

Ex. 1. In a steam engine, the indicated horse power is 540, but it is known that 60 horse power is lost in overcoming friction, etc., in the engine. What percentage of the horse power is used for useful work?

Total horse power
$$= 540$$

Lost horse power $= 60$

Hence useful horse power 480

The ratio $\frac{480}{540}$ thus represents the ratio $\frac{\text{useful horse power}}{\text{total horse power}}$. We require to know what number (say x) has the same ratio to 100 that 480 has to 540.

Hence
$$\frac{480}{540} = \frac{x}{100}$$

or $x = \frac{480 \times 100}{540} = 88.89$.

The number 88.89 represents the percentage horse power that is used for useful work.

Ex. 2. A man in purchasing stone slabs, knows that 5 per cent. are unfit for use. If he requires to use 950, how many must be purchase? Out of 100 slabs he can only use 95. Hence, for each 95 wanted he purchases 100.

Then, if x = number to be purchased,

$$\frac{95}{100} = \frac{950}{x} \text{ or } x = \frac{100 \times 950}{95}$$
= 1,000 slabs.

The correct value of a certain quantity is 56.5, but by measurement it is found to be 56.48. What is the percentage error in the measured value?

We now require to find the quantity, say x, which has the same ratio to 100, that 0.02 (the error) bears to 56.5 (the correct value).

Hence
$$\frac{0.02}{56.5} = \frac{x}{100}$$
$$x = \frac{100 \times 0.02}{56.5}$$

= 0.035 per cent. too low.

Ex. 4. A mixture consists of 25 parts sand, 35 parts cement, an 120 parts water. What is the percentage composition of the mixture? Total number of parts is 25 + 35 + 120 = 180.

Hence,
$$\frac{25}{180} = \frac{x}{100}$$
; $\frac{35}{180} = \frac{x'}{100}$; $\frac{120}{180} = \frac{x''}{100}$

where x, x', and x'' represent respectively the percentage of sand, cement, and water.

x = 13.9, x' = 66.7, x'' = 19.4,

Exercises.

(1) A steam engine has an efficiency of 75 per cent. If the engine develops an indicated horse power of 3,000, what horse power is available for external work?

(2) A pulley-block lifting-tackle has an efficiency of 45 per cent. Neglecting friction, etc., it is calculated that a load of \(\frac{1}{2}\) ton should be lifted by a certain pull on the lifting-chain. What load will actually be lifted?

(3) In a piece of coal there was found to be 11.30 lb. of carbon, 0.92 lb, of hydrogen, 0.84 lb, of oxygen, 0.56 lb, of nitrogen, 0.71 lb. of ash. There being nothing else, state the percentage composition of the coal. (B.E. 1905.)

(4) On board a ship there were 1,312 men, 514 women, and 132 children. State these as percentages of the total number of persons. (B.E. 1906.)

- (5) Two men measure a rectangular box; one finds its length, breadth, and depth in inches to be 8·54, 5·17, and 3·19. The other finds them to be 8·50, 5·12, and 3·16. Calculate the volume in each case. What is the mean of the two? What is the percentage difference of either from the mean? (B.E. (2) 1906.)
- (6) When x and y are small, we may take $\frac{1+x}{1+y}$ as being very nearly equal to 1+x-y. What is the error in this when x=0.02 and y=0.03? (B.E. 1906.)

(7) When x is small, we may take $(1 + x)^n$ as being nearly equal to 1 + nx. What is the percentage error in this when x = 0.01 and

n = 2? (B.E. 1907.)

- (8) If p is the pressure in a thin pipe of outside diameter d and thickness t, the greatest tensile stress being f, then t = pd + (p + 2f). A formula not quite so correct is t' = pd + 2f. Calculate from both formula the thickness of a pipe such that f = 4,000, d = 8, p = 500. What is the percentage error? (B.E. 1909.)
- (9) A lump of alloy contains 3.41 lb. of copper, 0.97 lb. of zinc, 0.31 lb. of lead, and 0.26 lb. of other material. What are the percentages of copper, zinc, and lead in the alloy? (B.E. 1909.)

(10) Find the chord of 45°, when the radius is 2 in.

(11) Take from the tables the chord and tangent of 22°. Construct an angle of 22° by using the value of the chord, and a second angle of 22° by using the tangent. Verify the results by measuring the two angles with your protractor, writing down to the first decimal, what each angle measures. (B.E. 1907.)

Answers.

- (1) 2.250. (5) 139·184; 1·2 per cent. (2) 504. (6) 0·0003 too small.
- (3) 78.85; 6.42; 5.86; 3.907; (7) 0.0098.
- 4.95. (8) $\frac{8}{17}$; $\frac{1}{2}$; 6.25 too great. (9) 68.88; 19.59; 6.26.

CHAPTER XII.

LOGARITHMS.

155. **Logarithms** are used to simplify calculation. By their use, the process of multiplication is reduced to addition; division is reduced to subtraction; involution, or the raising of numbers to given powers, becomes multiplication; and evolution, or the extraction of roots, becomes division. In ordinary arithmetic processes for involution and evolution, we are limited to powers or roots represented by integral numbers; but, by using logarithms, we can raise a number to any power, or extract any root, whether integral or fractional. Thus, by using logarithms we can find the value of 7 raised to the power 1.743, or we can find the value of 7^{2} or $\sqrt[2]{86.49}$, just as readily as we can find the value of 7^{2} or $\sqrt[2]{86.49}$.

The contraction "log." is usually written for the word "logarithm."

156. **Definition.** The logarithm of a number to any given base is the index of the power to which the base must be raised to give the number.

Suppose we take the number 10 as a base; we have $1{,}000 = 10^3$. The index 3 gives the power to which we must raise the base 10 to make the number $1{,}000$, and, by the above definition of a log,, we can also say that 3 is the log, of $1{,}000$ to the base 10.

As a general statement, if a denote the base, and N a number such that $a^x = N$ (i.e., a raised to the power x = N), then, by the above definition, log. N to the base a = x. This is written $\log_a N = x$, the base being written as a suffix to the word log. Any number could be used as the base of a system of logarithms, and further, the

same number may be the logarithm of two different numbers if the bases are different, thus

```
3^{3} = 27; 5^{3} = 125;
hence log_3 27 = 3 or log_3 27 to base 3 = 3
and \log_5 125 = 3 or \log_5 125 to base 5 = 3
```

Common logarithms are calculated to the base 10, and, when common logarithms are being spoken of, the bise is omitted, thus $\log_{10} 1,000 = 3$ may be written simply $\log 1.000 = 3$

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Ex. 1. Find the logs, of 64 to bases 8, 4, and 2 respectively.
                   8^2 = 64 hence \log_8 64 = 2.
                   4^3 = 64 , \log_4 64 = 3.
                   2^6 = 64
                             \log_2 61 = 6.
```

Ex. 2. Find the logs, of 100, 1,000, and 1,000,000 to the base 10. $10^2 = 100$ hence log. 100 $10^3 = 1,000$, log. 1,00

 $\log \log 100 = 2.$ $10^6 = 1,000,000$, $\log 1,000,000 = 6.$

(Note. In this example, the base is omitted after the word log., as the base is that of common logarithms.)

157. We shall now confine our attention to common logarithms; hence when a log. is indicated it must be understood that the base is 10, unless otherwise stated.

From Art. 13, we know that $a^0 = 1$. If a = 10, then $10^{\circ} = 1.$

Since
$$10^0 = 1$$
, therefore $\log 1 = 0$, and $10^1 = 10$, $\log 10 = 1$, $\log 10^2 = 100$, $\log 100 = 2$.

Suppose we require the logarithm of a number such as 55, which lies between, 10 and 100. As log. 10 = 1, and $\log 100 = 2$, it follows that the \log of a number such as 55 will be greater than 1 and less than 2; hence, the log. must consist of the integer or integral part 1, and a fractional or decimal part, which may be extended to any required number of decimal figures. In fact, the log. of 55 is actually 1.7404 correct to four places of decimals, and this means that 10 raised to the power 1.7404 is equal to 55, written $10^{1.7494} = 55$.

P·M·

A logarithm thus consists of two parts, viz. :-

(a) An integral part, which is called the characteristic.

(b) A decimal part, which is called the mantissa.

In the logarithm of 55 given above, the characteristic is 1 and the mantissa 0.7404.

158. The Table 1, in Appendix, is the log. Table commonly used by students, the logs being given to four decimal figures. It must be understood, however, that the decimal figures beyond the fourth have been omitted, the fourth figure being raised by 1 if the fifth figure is equal to, or greater than, 5. Thus, if the log. of a number to five decimal places is 2.76379, then in a four-figure table the log. is given as 2.7638, whereas, if the five-figure log. is 2.76372, the four-figure log. is 2.7637.

If we write each log. as consisting of an integral part and

a decimal part, then since

 $10^{0} = 1$, \log . 1 = 0.0000 $10^{1} = 10$, \log . 10 = 1.0000 $10^{2} = 100$, \log . 100 = 2.0000 $10^{3} = 1,000$, \log . 1,000 = 3.0000

 $10^4 = 10,000$, $\log 10,000 = 4.0000$, and so on.

From this table, we can see that the characteristic of the log. of any numbers between 10 and 100 is 1; that for any number between 100 and 1,000 is 2; and that for any number between 1,000 and 10,000 is 3, and so on: or we can say that the characteristic is one less than the number of figures in the integral part of the number, and is positive. The integral part of a number is the part on the left of the decimal point; hence we get the following rule:—

The characteristic of the log. of a number greater than 1 is positive and less by one than the number of figures to the left of the decimal point.

Ex. The characteristic of 100 is 2; of 106.78 is 2; of 763.9 is 2; of 74.28 is 1; and of 7.634 is 0.

159. The mantissa is found from the log. tables as set out in Art. 160. It is independent of the position of the decimal point in the number. The decimal point only affects the

characteristic, and after the characteristic has been ascertained, the decimal point may be left out of account, thus,

The log. of 8,638 is 3.9364, for the characteristic is 3 (according to rule). The mantissa is .9364. Log. 863.8 = 2.9364; log. 86.38 = 1.9364; log. 8.638 = 0.9364, the mantissa being the same for each, but the characteristic different.

160. To find the mantissa from the Table, consider the extract from the Table shown below.

												Difference columns.							
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	s	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11 12 13																			
14 15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6 ->	8	11	14	17	20	22	23
16																			

1 1 1

In col. 1 of the Table will be found numbers from 10 to 99. Along the top line are numbers from 0 to 9, representing ten columns of figures, and a second set of numbers from 1 to 9, representing nine additional columns of figures. The second set are called the difference columns.

Suppose we require to know the mantissa of the log. of 153.2. Ignore the decimal point and consider only the figures 153.2. Look in col. 1 for the number corresponding to the first two figures (15) in the number whose log. is required. Now read along the line against 15 until you come to the column corresponding to the third figure in the number whose log. is required. In this case, it is the column with 3 at the top. The number found in this column, and against number 15 in the first column, is 1847. Now read the number in the difference column corresponding to the fourth figure in the number whose log. is required. This is the difference column 2, and the number in line

with 15 of the first column is 6. Adding the 6 to 1847 we get 1847 + 6 = 1853, and this is the required mantissa, which, being a decimal, is written 1853. The characteristic of the log. of 153.2 is 2, according to our rule, hence log. 153.2 = 2.1853. In the above extract from the Table, the arrows indicate the process to be followed.

Ex. 1. Find the logs. of 7,896, 7896, 7896, and 7.896.

Log. 7,896 has characteristic 3. For mantissa, find 78 in col. 1, read along line 78 and under col. 9, and we find 8971; along same line and in difference col. 6 we find 3, and since 8971 + 3 = 8974, hence, log. 7,896 = 3.8974.

By similar reasoning, log.
$$789.6 = 2.8974$$

, , , log. $78.96 = 1.8974$
, , , log. $7.896 = 0.8974$.

Ex. 2. Find the log. of 2,001.

Characteristic = 3.

Mantissa =
$$\cdot 3012$$
 (Read line 20, col. 0) = 3.010 (difference col. 1) = $\frac{2}{3.012}$

hence, $\log_{10} 2.001 = 3.3012$.

When the log, of a number containing less than four figures is required, we can make it up to four by adding a decimal point and ciphers. Thus 70 may be written as 70 00; 3 may be written 3 000; and 2 3 as 2 300.

Ex. 3. Find logs, of 7 and 9.

Writing 7 and 9 as 7.000 and 9.000 respectively, we have, for log. 7, characteristic = 0, mantissa for 7,000 = .8451,

hence, $\log 7 = 0.8451$, similarly, $\log 9 = 0.9542$.

When we have five figures in the number whose log, we require, we take the nearest four figures. Hence, if the fifth figure is equal to or greater than 5, increase the fourth by 1; if not equal to 5, the fifth figure may be cancelled.

Thus for log. 86738, read log. 86740, which is 4.9332. For log. 76.293, read log. 76.29, which is 1.8825.

161. When we know the log of a certain number, we require some ready means of ascertaining the number corresponding to it, and for this purpose we make use of the table of **antilogarithms**, Table 2, Appendix, which is drawn up on a similar plan to that of the table of logarithms. An example will make the method clear. Suppose we are given 2.7694 as the log of a number and we require to know the number. Take the mantissa part only .7694,

and in col. 1 of the table of antilogs, find '76, read along this line to col. 9 (the third figure in mantissa), and here we find 5,875; read on along the same line to difference col. 4 (the fourth figure in mantissa), and here we find figure 5. Add 5875 + 5 and we get 5,880 as the figures of the number having '7694 for log. mantissa. We now require to fix the position of the decimal point. Since the characteristic is 2, the number must have three figures in the integral part, hence the required number is 588.0 or 588.

To check this result, find the log. of 588 from the table of

logs. This is found to be 2.7694.

It is seen from the above that the mantissa only is used to ascertain the actual figures in the number, while the characteristic is used to fix the position of the decimal point.

Conversely, in finding the log of a number, the mantissa depends only upon the actual figures, and the characteristic only upon the position of the decimal point.

It often happens that when we read the number corresponding to a given log, from the antilog, table, and then check this result by reading the log, of the number so found from the log, table, that the fourth figure in the mantissa of the second log, does not agree with that in the given log. For example, find the number whose log, is 0.7853. Reading from the antilog, table, we find the number to be 6.099. Now read the log, of 6.099 from the log, table and we find that it is 0.7852. The reason for this apparent discrepancy will be clear if we remember that the logarithms of any numbers as found from the tables are corrected to four figures of decimals, and that the decimal figures beyond the fourth are omitted, the fourth figure being increased by 1 if the fifth be equal to or greater than 5.

162. By using logs., the process of multiplication is reduced to addition.

If a and b represent two quantities, then their product is ab, and by above statement

log. $ab = \log$, $a + \log$, b. In order to test this statement, let a = 95, b = 5, then ab = 475 and hence log. 475 should equal log. $95 + \log$, 5.

From log. table $\begin{cases} \log_{5} 95 = 1.97777 \\ \log_{5} 5 = 0.6990 \end{cases}$

the sum = 2.6767 and from log, table log. 475 is seen to be 2.6767.

If we repeat this test by taking a series of values of a and b, we shall find that $\log_a ab$ always = $\log_a a + \log_a b$; hence the statement is assumed to be correct.

Hence if we have to multiply together a series of numbers, we take the logs, of each of the numbers, add them together, and the sum gives the log, of the product. The product can then be ascertained from the table of antilogs.

Ex. 1. Find the product of $47.96 \times 3.43 \times 2.764$. Log. of product = log. 47.96 + log. 3.43 + log. 2.764 = 1.6808 + 0.5353 + 0.4415

= 2.6576.

From table of antilogs., we find that this is the log. of 454.5; hence product is 454.5.

163. By using logs, the process of division is reduced to subtraction.

Let a and b represent two quantities, then $\frac{a}{b}$ is their quotient, and

by above statement $\log \frac{a}{b} = \log a - \log b$.

In order to test this statement, let a=5,275 and b=25, then $\frac{a}{b}=211$, and hence log. 211 should equal log. 5,275 $-\log$. 25.

From log. table
$$\begin{cases} \log. 5,275 = 3.7222 \\ \log. 25 = 1.3979 \end{cases}$$

the difference = 2.3243

and from log, table log. 211 is seen to be 2.3243.

By repeating the test for various values of a and b, we can show that this statement is correct.

Hence to divide one number by another, subtract the log. of the divisor from the log. of the dividend, the difference is the log. of the quotient. The quotient is then found from the table of antilogs.

Ex. Divide 78:96 by 7:589.

Log. of quotient =
$$log. 78.96 - log. 7.589$$

= $1.8974 - 0.8832$
= 1.0172 .

From table of antilogs, this is seen to be the log, of $10^{\circ}4$; hence the quotient is $10^{\circ}4$.

Ex. Find the value of $\frac{43.69 \times 52.77 \times 3.849}{5.763 \times 49.38 \times 2.767}$

In examples such as these, find the sum of the logs, of all numbers in the numerator, and the sum of the logs, of all numbers in the denominator; subtract the latter sum from the former for the log, of the result, since product of numerators must be divided by product of denominators.

and this is log. of 11.27.

Hence result is 11.27.

Ex. The horse power of a steam engine is given by the formula

$$\text{H-P} = \frac{2 \times P \times L \times A \times N}{33,000}$$
 where $P = \text{mean pressure in}$

pounds per sq. inch, L = length of stroke in feet, A = area of piston in sq. inches, and N = number of revolutions per minute. If P = 116.5, L = 1.5, A = 95.8, N = 150, find the H-P, using logs, for the calculation.

$$HP = \frac{2 \times 116.5 \times 1.5 \times 95.8 \times 150}{33,000}$$

log. H-P = log.
$$2 + \log$$
. $116.5 + \log$. $1.5 + \log$. $95.8 + \log$. $150 - \log$. $33,000$
= $0.3010 + 2.0664 + 0.1761 + 1.9814 + 2.1761 - 4.5185$
= 2.1825 .

From antilog, table, the corresponding number is 152.3; hence, the horse-power is 152·3.

164. The two rules given above for multiplication and division respectively by using logs follow at once from a consideration of the rules for dealing with indices in algebra.

In algebra, $x^2 \times x^3 = x^5$, since we add the indices 2+3 for the index 5 of the product.

Similarly,
$$10^2 \times 10^3 = 10^5$$
, or $100 \times 1,000 = 100,000$.

But by definition, the log, of 100 = 2 and the log, of 1,000 = 3. Further, the log, of 100,000 = 5, and this is seen to be the sum of the logs, of 100 and 1,000.

As a further illustration,

$$25 \times 50 = 1,250$$

 $\log. 25 = 1.3979, \log. 50 = 1.6990, \text{ and}$
 $\log. 1,250 = 3.0969,$
hence $25 = 10^{1.999}, 50 = 10^{1.699}, \text{ and } 1,250 = 10^{8.096}.$

Thus
$$25 \times 50 \approx 1,250$$
 may be written $10^{1.6259} \times 10^{1.6990} = 10^{6.6990}$.

and this is seen to agree with the rules for indices.

Similarly for division.

In algebra, $\frac{x^3}{x^2} = x$, since we subtract the indices 3 - 2 for the index 1 of the quotient.

$$\frac{10^3}{10^2} = 10$$
, or $\frac{1000}{100} = 10$.

By definition, log. 1,000 = 3, log. 100 = 9, and log. 10 = 1, and the log. of 10 is seen to be log. $1,000 = \log 100$.

Is a further illustration,

$$\frac{50}{25} = 2,$$

$$50 = 10^{1.0590}, 25 = 10^{0.059}, 2 = 10^{0.0010},$$
hence $\frac{f_0}{25} = 2$ may be written
$$\frac{10^{1.0990}}{100^{1.0999}} = 16^{0.0010},$$

and this is seen to agree with the rules for indices, subject, of course, to the fourth decimal figure differing by 1 from the value of log. 2 as given in the tables.

165. So far, we have only dealt with the logs of numbers greater than 1, and we have found that the characteristic of the log. of such numbers is positive. Considering numbers less than 1; for example, such numbers as 0.006798,

we know from Art. 95 that
$$\frac{1}{a} = a^{-1}, \frac{1}{a^2} = a^{-2}$$
, etc.

Similarly,
$$\frac{1}{10} = 10^{-1}$$
, $\frac{1}{100} = \frac{1}{10^2} = 10^{-2}$, etc.
As $10^0 = 1$, hence, $\log 1 = 0.000$
also $10^{-1} = \frac{1}{10} = 0.1$, hence, $\log 0.1 = -1.000$
 $10^{-2} = \frac{1}{10^2} = 0.01$, hence, $\log 0.01 = -2.000$
 $10^{-3} = \frac{1}{10^3} = 0.001$, hence, $\log 0.001 = -3.0000$

$$10^{-4} = \frac{1}{10^4} = 0.0001$$
, hence, $\log 0.0001 = -4.0000$

The characteristic of all numbers less than 1 is thus negative, and is one greater than the number of ciphers between the decimal point and the first significant figure; thus, when the decimal point immediately precedes the first significant figure, as in 0.1, the log. is -1; when there is one cipher immediately after the decimal point, as in 0.01, the log. is -2, and so on.

166. The mantissa of a log. is always positive, but **the characteristic may be positive or negative.** If, then, as in the number '006798 we have a negative characteristic — 3, and a positive mantissa '8324, we cannot write the log. of '006798 as — 3.8324, for that would imply that the whole quantity 3.8324 was negative, whereas, only the 3 is negative, the quantity '8324 being positive. If written fully, the log. of 0.006798 is (— 3.0000 + 0.8324). In order to indicate that only the characteristic is negative, the minus sign is always written above and not in front of, the characteristic. Thus, log. '006798 is written 3.8324.

$$\begin{array}{rcl}
\log & 0.1 & = & \widehat{1}.0000 \\
\log & 0.001 & = & \overline{3}.0000.
\end{array}$$

In reading the mantissa for a number less than 1, we proceed as before, omitting the decimal point and also the ciphers immediately following it. Thus, the mantissa for '006798 corresponds to that for 6,798. The student should check this from the tables.

- 167. Summarizing, we now see that:
- (1) The characteristic of 1 is 0.
- (2) The characteristic of all numbers greater than 1 is positive, and one less than the number of figures to the left of the decimal point.
- (3) The characteristic of all numbers less than 1 is negative and one greater than the number of ciphers between the decimal point and the first significant figure.
- (4) The mantissa is always positive and is quite independent of the decimal point and also independent of the ciphers immediately following the decimal point.

Examples to be checked by student. First find the logs, of the given numbers, then take the logs, and find the corresponding numbers from antilog, table.

 $= \overline{2}.8280.$ log. :06729 $\log 67,290 = 4.8280.$ log. '006729 = 3.8280, $\log.6729 = 3.8280.$ $\log \cdot 0006729 = \overline{4.8280}$ $\log. 672.9 = 2.8280.$ log. 5760 = 3.7604. $\log. 67.29 = 1.8280.$ log. 576 = 2.7604. $\log.6.729 = 0.8280.$ $\log \cdot 0.6729 = \overline{1.8280}$. log, 0.00385 $= \bar{3}.5855$.

 $\log_{10} 0.0007 = 4.8451 \text{ (read as .00070)}.$

Ex. 1. Find the product of 0.00649 \times 73.28 \times 0.00237.

Log. of product = log. 0·00649 + log. 73·28 + log. 0·00237 , , , = $\overline{3}\cdot8122 + 1\cdot8650 + \overline{3}\cdot3747$, , , , = $\overline{3}\cdot0519$.

Note. Sum of mantissæ 2.0519Algebraic sum of characteristics = $\frac{5.0000}{3.0519}$ Hence nett sum $\frac{3.0519}{3.0519}$

From antilog, table, the corresponding number is found to be 1,127, and as the characteristic is 3, we must have two ciphers immediately after the decimal point.

Hence, product is 0.001127.

Ex. 2. Find the value of $\frac{0.006395}{0.07538}$.

 $\log 0.006395 = \overline{3}.8058$ $\log 0.07538 = \overline{2}.8773$.

To obtain the difference of these logs,, we must treat the characteristics and mantissæ quite separately, thus:—

 Note -3 - (-2)= -3 + 2= -1;

Hence, \log , of quotient = 2.9285.

also the 1 to be carried when subtracting '8773 from '8058 is put down as I.

From the table of antilogs, 2.9285 is seen to be the log. of 0.08482.

Hence, $\frac{0.006395}{0.07538} = 0.08482$.

Exercises.

Answers.

(1) 8·281. (2) 54·76. (3) 6·613. (4) 5694.

168. By using logs, the process of involution is reduced to multiplication.

Let a represent a given quantity.

Then a^n represents the quantity a raised to the power n.

And $\log_{\bullet} a^n = n \log_{\bullet} a$.

In order to test this statement,

Let
$$a = 15$$
 and $n = 2$.
Then $a^n = 15^2 = 225$.

hence log. 225 should equal $2 \times \log_2 15$.

From log. table, log. 15 = 1.1761; hence $2 \times \log_{\bullet} 15 = 2.3522$.

And also from table $\log_{10} 225 = 2.3522$.

By repeating the test with various values of a and n, we can show that the statement is correct.

Ex. 1, Find the value of (26.39)1.5.

Log.
$$(26\cdot39)^{1\cdot5} = 1\cdot5 \times \log. 26\cdot39$$

= $1\cdot5 \times 1\cdot4215$
= $2\cdot1322$,

hence, from antilog. table $(26.39)^{1.5} = 135.6$.

Ex. 2. Find the value of (0.00374)3

Log.
$$(0.00374)^3 = 3 \times \log_0 0.00374$$

= 3×3.5729
= $3 (-3.0000 + 0.5729)$
= $(-9.0000 + 1.7187)$

= 8.7187 (treating characteristics separately) hence, from antilog. table $(0.00374)^3 = 0.00000005232$.

Note.—A fraction raised to a power is less in value than the original fraction.

169. By using logs, the process of evolution is reduced to division.

Let a represent a given quantity.

Then $\sqrt[n]{a}$ represents the *n*th root of *a*.

and log.
$$\sqrt[n]{a} = \frac{1}{n} \log_a a$$
.

In order to test this statement

let
$$a = 225$$
 and $n = 2$
then $\sqrt[n]{a} = \sqrt[2]{225} = 15$.

hence log. 15 should equal \(\frac{1}{2}\) log. 225.

From table log. 15 = 1.1761

log. 225 = 2.3522, and hence
$$\frac{1}{2}$$
 log. 225 = 1.1761.

By repeating this test with various values of a and n, we can show that the statement is correct.

Ex. 1. Find the vilue of $\sqrt[1.5]{10.89}$.

Log.
$$\sqrt[15]{76.89} = \frac{1}{1.5} \times \log. 76.89$$

= $\frac{1.8859}{1.5} = 1.2573$.

hence, from antilog. table $\sqrt[15]{76.89} = 18.08$

Ex. 2. Find the value of $\sqrt[3]{0.007985}$.

Log.
$$\sqrt[3]{0.007985} = \frac{1}{3} \times \log_{10} 0.007985$$

= $\frac{3.9023}{3}$
= 1.3008

hence, from antilog. table $\sqrt{0.007985} = 0.1999$

Note.—Any root of a fraction is greater than the original fraction.

170. The rules given above for involution and evolution by using logs, also follow from the algebraic rules for dealing with indices.

By algebra $(a^2)^3 = a^6$, since for index of powers we multiply index 2 of quantity by index 3, representing the required power.

Also $(100)^3 = 1,000,000$;

Or
$$(10^2)^3 = 10^6$$
, which = 1,000,000.

But log. 100 = 2, and log. 1,000,000 = 6, and since $2 \times 3 = 6$, it is seen that log. of a quantity raised to any power = log. of quantity tself \times index representing the required power.

Similarly, $\sqrt[3]{a^6} = a^{\frac{6}{3}} = a^2$ by rules of indices.

$$\sqrt[3]{1,000,000} = 100$$

 $\sqrt[3]{10^6} = 100 = 10^2$.

But $\log 1,000,000 = 6$, and $\log 100 = 2$, and $\frac{1}{3} \log 1,000,000 = \frac{6}{3} = 2$, hence, the \log of the root of any quantity is equal to the \log of the quantity divided by the index representing the root.

Some miscellaneous examples will now be given, and difficulties which arise in working with logs, will be explained as they occur.

5.8373

Ex. 1. Evaluate
$$\frac{0.494 \times 0.00286 \times 0.579}{0.00197 \times 3.49}$$
Sum of logs. of numerator = $\frac{4.9128}{3.8373}$
Difference = $\frac{1.0755}{1.0755}$
log. 0.494 = $\frac{1.6937}{1.6937}$
log. 0.494 = $\frac{1.6937}{1.6937}$
log. 0.69286 = $\frac{3.4564}{3.4564}$
log. 0.579 = $\frac{1.7627}{4.9128}$

hence, from antilogs, value required is 0:119.

Note.—In subtracting characteristics, treat them algebraically; thus -4 - (-3) = -4 + 3 = -1.

Ex. 2. Evaluate
$$\frac{5.31 \times 0.364 \times 7.62}{0.00831 \times 4.61 \times 0.0478}$$
.

Sum of logs, of numerator . 1:1682 Sum of logs, of denominator . 3:2627

Difference . 3.9055

Hence, from antilogs. value = 8044.

Note,—Again treat characteristics algebraically, thus +1 - (-3) = +1 + 3 = 4.

We thus get 4.1682 - 0.2627 = 3.9055.

Ex. 3. Evaluate
$$\begin{array}{c|c} 63.2 \times \sqrt[5]{3642} \\ \log 63.2 = 1.8007 \\ \frac{1}{5} \log 3642 = 0.7123 \\ \operatorname{Sum} = 2.5130 \end{array} \qquad \begin{array}{c|c} \log 2 \times \sqrt[5]{3642} \\ \log 3642 = 0.6405 \\ \operatorname{Sum} = 7.9809 \end{array}$$

Sum of logs, of numerators 2:5130 Sum of logs, of denominators 7:9809

Difference 6:5321

From antilogs, value is 0.000003405.

E.e. 4. Find the value of $\sqrt{0.972}$

Log.
$$\sqrt{0.9} = \frac{\frac{1}{2} \log_{10} 0.072}{\frac{1}{2} \frac{1}{2}}$$

Remembering that the mantissa is positive and the characteristic is negative, we are now in a difficulty as to dividing by 2. What we really require is $\frac{-1\cdot0000+0\cdot9877}{2}$. This would give a number less than 1 for characteristic if divided out. To overcome the difficulty add -1 to characteristic and +1 to mantissa. The whole value is thereby unaltered, and we now get $\frac{-2\cdot0000+1\cdot9877}{2}$, and this is equal to 1·9938, which is the log. we require.

From antilogs, the value is 0.9858.

Rule. In every case of extracting a root of a quantity less than 1 by using logs., add to the negative characteristic a negative quantity sufficient to make the index of the root, divide the characteristic without remainder, and add a corresponding positive quantity to the mantissa to equalize the value.

Thus,
$$\sqrt[3]{0.763} = \frac{1}{3}\log.0.763$$

$$= \frac{1.8825}{3}$$
By adding $\frac{7}{2}$ and $\frac{7}{2$

Ex. 5. Find the value of $\sqrt{0.00867}$.

$$\left(\frac{\bar{4} + 1.9380}{2}\right) = \frac{1}{3} \cdot \frac{1}{9380} = \frac{1}{3} \cdot \frac{1}{9380} = \frac{1}{2} = \frac$$

From antilogs, value = 0.09311.

Ex. 6. Evaluate
$$(0.5)^{0.25}$$
.
Log. $(0.5)^{0.25} = 0.25 \times \log .0.5$.
 $= 0.25 \times 1.6990$.
 $= 0.25 (-1.0000 + 0.6990)$.
 $= (-0.25 + 0.1747)$.
 $= -0.0753$.

We now have a negative mantissa. The mantissa, however, must always be positive. In order to make this negative mantissa into a positive one, add +1 to the mantissa and -1 to the characteristic. We thus get $\{(\bar{1} + (1.000 - 0.0753)\} = 1.9247$ as the required log. Hence, value is 0.8409.

Ex. 6 may be checked as follows:-

$$(0.5)^{0.25} = (0.5)^{\frac{1}{4}}$$

Taking 1 as a power, we have-

Log.
$$(0.5)^{\frac{3}{4}} = \frac{1}{4} \log_{5} \cdot 0.5$$

$$= \frac{\overline{1} \cdot 6990}{4} = \left\{ \frac{\overline{4} + 3 \cdot 6990}{4} \right\}$$

$$= \overline{1} \cdot 9247$$

Hence, as overleaf, value = 0.8409.

- **Rule.** In every case of a negative mantissa, make it positive by adding +1 to the mantissa, and by taking $\overline{1}$ as a characteristic to equalize the value.
- 171. A system of logs. called Napierian or hyperbolic logs., calculated to a base denoted by the letter e, of the value 2.7183 to four decimal places, is in use. A log. in this system is written $\log_{e} N = x$ where $N = e^{x}$, according to the definition of a logarithm.

Common logs. (calculated to the base 10) may be converted into Napierian logs. by multiplying by 2.3026.

Ex. Find log.e 76:37.

$$\begin{array}{l} \log_{10} 76 \cdot 37 = 1 \cdot 8829 \\ \log_{\cdot 6} 76 \cdot 37 = 2 \cdot 3026 \times \log_{\cdot 10} 76 \cdot 37 \\ = 2 \cdot 3026 \times 1 \cdot 8829 \\ = 4 \cdot 3356. \end{array}$$

172. By taking numbers from 1 to 100 as abscissæ, and the logs, of the numbers as ordinates, we can draw a graph from which the log, of a number may be found, and also which shows how the log, varies with the number. In the table below, x is a number and y the log, of the number.

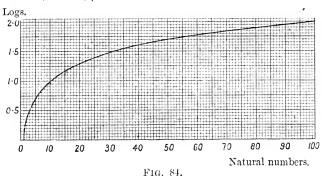
_		-							-										
x	1	2	3	4	6	8	10	15	20	25	30	35	40	50	60	70	80	90	166
_	_																		
y		0.3	0.48	0.6	0.78	6.0	1.0	1.17	1.3	1:397	1.48	1.54	1.6	1.698	1.778	1.845	1.9	1.62	2.0

In plotting the graph, we plot x horizontally to a scale of 1 in. = 10, and y vertically to a scale of 1 in. = 0.5, Fig. 84. The graph is called a log. curve, or the graph $y = \log_x x$,

Exercises (a).

Note.—Exercises 2 to 15, 18, and 22, pages 116, 117, and 118, will be found suitable for practice in logarithms. The answers obtained by logs, may differ slightly from those given on p. 118.

- (1) Find the product $3.862 \times 0.00296 \times 2.784$. Ans. 0.03182.
- (2) Evaluate 7.384 $\times \sqrt{4.782} \div 36.97$. Ans. 0.4368.
- (3) Evaluate $11.76 \times \pi \times 3.761 \times 2.87 \times e$. Ans. 1084.
- (4) Find the ratue of $(a-2b) 3c \div 2 (a-x)^2$ when a=51.6, b=7.8, c=11.29, x=0.0637. Ans. 0.2295.
 - (5) Evaluate $\frac{7\cdot29\times(3\cdot621)^2\times\sqrt{4\cdot36}}{2\cdot824\times0\cdot0297}$. Ans. 2380.
- (6) Evaluate $\frac{5 w x^2 \cos \theta}{3 a b^3 \tan \phi}$ when w = 56, x = 7.94, a = 12.71 b = 4.29, $\theta = 55^{\circ}$, $\phi = 0.4887$ radians. Ans. 6.324.



Exercises (b).

- (1) Given the values of logs. 2, 3, 5, and 7, make out a log. table for the logs. of numbers from 1 to 10.
 - (2) Calculate $\sqrt{0.9736}$; $\sqrt{0.01758}$; $\sqrt[3]{0.01758}$; $\sqrt[3]{27.83}$; $\sqrt{0.00867}$.
 - (3) Calculate $\frac{53.89 \times \sqrt[3]{0.07629} \times (3.761)^8}{2.597 \times \sqrt{0.00638} \times (0.7689)^2}$.
 - (4) Evaluate (0.56)0.87.
 - (5) Calculate $\frac{0.36 \times (4.23)^{2.5} \times \sqrt[3]{76.41}}{(5.27)^{1}}$
 - $0.00364 \times (5.27)^{\frac{1}{2}}$
 - (6) Find the value of $\frac{7.3 \times 0.00641 \times (0.58)^2 \times \sqrt[3]{0.366}}{8.21 \times 0.0003625 \times (2.38)^3}$

(7) The ratio of the tension N in the tight side of a driving belt to the tension M in the slack side is given by $\frac{N}{M}=e^{\mu\theta}$

where $\mu=$ coefficient of friction between belt and pulley and θ is the angle of lap in radians. Take value of e as 2.72. Calculate the value of $\frac{N}{M}$ when $\mu=0.25$ and $\theta=2.5$. If N=250 lb., find M.

(8) At speeds greater than the velocity of sound, the air resistance to the motion of a projectile of the usual shape of weight w lb., diameter d inches, is such that when the speed–diminishes from r_1 ft. per second to r, if t is the time in seconds and s is the space–passed over in feet,

$$t = 7000 \frac{w}{d^2} \left(\frac{1}{r} - \frac{1}{r_1} \right).$$

$$s = 7000 \frac{w}{d^2} \log_e \frac{r_1}{r}$$

If r_1 is 2000, find s and t when v = 1500 for a projectile of 12 lb. whose diameter is 3 in. (B.E. 1903.)

(9) Find the value of $t_1 - t_3 - t_3 \log_e \frac{t_1}{t_3} + l_1 \left(1 - \frac{t_3}{t_1}\right)$ if $t_1 = 458$, $t_3 = 373$, and $l_1 = 796 - 0.695$ t_1 .

[Convert common logs. into Napierian logs. by multiplying by 2·3026.] (B.E. 1903.)

(10) It has been found that if P is the horse power wasted in air friction when a disc d feet diameter is revolving at n revolutions per minute $P = cd^{5.5}$ $a^{3.5}$. If P is 0.1 when d = 4 and n = 500, find the constant c. Now find P when d is 9 and n is 400. (B.E. 1904.)

constant c. Now find P when d is 9 and n is 400. (B.E. 1904.)
(11) Write down the values of cos. 35°, tan. 52°, sin. $^{-1}$ 0.4226, \log_{10}

14·36, log_e 14·36.

[Note. Sin. ^{-1}n means the angle whose sin. is n.] (B.E. 1905.)

(12) If $y = \frac{2}{x} + 5 \log_{10} x - 2.70$, find the values of y when x has

the values 2, 2.5, 3. Plot the values of y and x on squared paper, and draw the probable curve in which these points lie. State approximately what value of x would cause y to be 0. (B.E. 1905.)

(13) Using the tables, find the number of which $0^{\cdot 2}$ is the Napierian logarithm.

If
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + &c.$$

ealculate e^x when x = 0.2, to three decimal places.

After how many terms are more of them useless in this case where we only need three decimal places?

(Note that
$$\lfloor 5 \rfloor$$
 means $1 \times 2 \times 3 \times 4 \times 5$.) (B.E. (2) 1905.)

(14) If
$$p_1 r_1^{1\cdot 13} = p_2 r_2^{1\cdot 13}$$
, and if $\frac{r_2}{r_1}$ be called r . If $p_2 = 6$, find r if $p_1 = 150$. (B.E. 1905.)

(15) Given
$$A = P\left(1 + \frac{r}{10}\right)^n$$
 find A when $P = 200$, $r = 4$, and $n = 12$. (B.E. 1906.)

(16) If $pu^{1.0646} = 479$, find p when u is 3.25. (B.E. 1906.)

(17) If $pu^{1.0646} = 479$, find u when p is 120. (B.E. (2) 1906.)

(18) A disc varies in thickness so that when running at a certain speed the radial and hoop stresses may be the same and constant everywhere. The thickness x at the radius r is such that

$$ar^2 = \log \frac{x_1}{x}$$

If a is 0.04 and x_1 is 0.3 find r for the following values of x, 0.2, 0.1, 0.05, and draw a section of the disc. The logarithms are common logarithms. (B.E. 1908.)

(19) If $xy^{2\cdot b} = 350$, find x if y is 12. (B.E. 1909.)

(20) If
$$y = 2.5 \log_{10} x + \frac{x^2}{100} - 6.35$$
.

Find y for a number of values of x between 15 and 20, and represent these on squared paper. For what value of x is y = 0? Use common logarithms. (B.E. 1909.)

(21) Plot the following values of p and u, and then plot $\log p$ against $\log u$. Note the second graph is a straight line.

p	6.86	14.7	28.83	60.4	101.9	163:3	250.3
\overline{u}	53.92	26:36	14.00	6.992	4.28	2.748	1.853

Answers (b).

(2) 0·9868; 0·1326; 0·26; 3·031; 297·0; 0·09311. (3) 35·75. (4) 0·8069. (5) 6725. (6) 1·963. (7) 1·869; 133·8. (8) t = 1·5; s = 2679·6. (9) 97·04.	(12) 2·38. (13) 1·221; 4 terms. (14) 17·26. (15) £320·8. (16) 136·6. (17) 3·669. (19) 0·7017. (20) 18.
(10) $\left(1746 \times \frac{1}{10^{19}}\right)$; 0.3958.	

CHAPTER XIII.

SQUARED PAPER.

173. We are already familiar, from Chapter IV. and from subsequent examples, with the use of squared paper for plotting points and curves, and for representing graphically the relationship between quantities. We have also seen from Chapter VIII. that the relationship between quantities may be represented by an algebraic equation. In the present chapter, we shall show that the graphic and algebraic representations agree one with the other, and we shall also show, in many cases, how the one representation may be derived from the other.

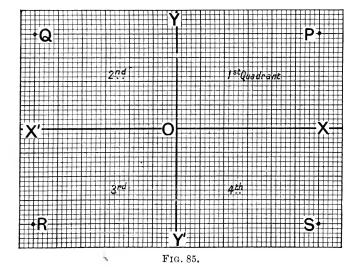
Reverting to the plotting of a point we found that we required to know its distance from each of two fixed axes of reference OX, OY. So long as these fixed axes are at, or near, the left-hand and bottom edges of the paper these two distances would definitely fix the point. Suppose, however, that we take the two axes as shown in Fig. 85 along the two centre lines of the paper, and that we require a point 20 divisions from the horizontal axis and 30 divisions from the vertical axis. Each of the four points P, Q, R, S will agree with these two dimensions. There is thus an ambiguity as to which of the four points is the one required.

In order to remove this ambiguity, we call those distances measured upwards from the horizontal axis, and those measured to the right of the vertical axis, positive; while those measured downwards from the horizontal axis and those measured to the left of the vertical axis are negative. All distances are measured from one or other of the two

axes, which are now lettered XOX' and YOY', thus dividing the paper into four quadrants, called the 1st, 2nd, 3rd, and 4th quadrants respectively as shown in Fig. 85.

The distances or co-ordinates of P are thus + 30, + 20.

,,	,,	Q	,,	_	30,	+	20.
,,	,,	${ m R}$,,	-	30,	_	20.
••	••	S	••	+	30,	_	20.



In each case, the horizontal distance or x co-ordinate is given first.

A point is often defined by its co-ordinates as follows, $P_{(30,20)}$ meaning the point P in quadrant 1 in Fig. 85.

In general language, $P_{(x, y)}$ is the point whose coordinates are x and y respectively, Fig. 86.

Ex. Plot the points whose co-ordinates are respectively (13, 14) (-15, 13), (-13, -16), (14, -12) on squared paper.

Taking each $\frac{1}{10}$ in. division as unity, then the points are respectively P, Q, R, S, Fig. 87.

The following examples should be carefully plotted by the student

before proceeding further with this chapter, and the result of each example should be particularly noted:—

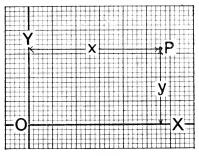


FIG. 86.

Fx. 1. Plot the points (5, 7), (-5, 7), (-5, -7), (5, -7), and find the area enclosed by the four lines joining the points

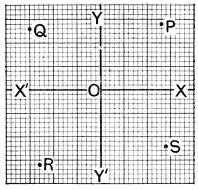


FIG. 87.

Ex. 2. Plot the points (5, 2), (5, 4), (5, 6), (5, -2), (5, -4), (5, -6), and note that they all lie on a line parallel to, and distant 5 units from, YOY'.

Ex. 3. Plot the points (-6, 3), (-4, 3), (-2, 3), (0, 3), (2, 3), (6, 3), and note that they all lie on a line parallel to, and distant 3 units from, XOX'.

Ex. 4. Plot the points (0, 2), (0, 4), (0, -2), (0, -4), also the points (3, 0), (1, 0), (-2, 0), (-4, 0), and note that all these points lie on one or other of the two axes of reference.

Ex. 5. Plot the points (4,4),(2,2),(0,0),(-2,-2),(-4,-4), and note that all these points lie on a line passing through the origin

and inclined at 45° to the axis XOX'.

Ex. 6. Plot the points (4, 0), (2, 3.5), (0, 4), (-2, 3.5), (-4, 0), (-2, -3.5), (0, -4), (2, -3.5), and note that a circle with centre at the origin and radius equal to 4 units practically passes through all these points.

Ex. 7. Plot the points (4, 0), $(2, 2^{\circ}6)$, (0, 3), $(-2, 2^{\circ}6)$, (-4, 0), $(-2, -2^{\circ}6)$, (0, -3), $(2, -2^{\circ}6)$. Observe that all the plotted

points lie on a closed curve, called an ellipse.

Ex. 8. Plot the points (1, 16), (2, 8), (4, 4), (8, 2), (16, 1), and draw a curve through the points. Now take any points on the curve and draw from each point lines parallel to the axes of reference to meet the axes, thus forming a series of rectangles. Show that the areas of all these rectangles are equal.

It will be observed in working the above examples that it is possible to select a series of points so that, when plotted, they lie along a straight line which may, or may not, pass through the origin, or may be parallel to either axis, and further that such points may lie along a circle, or an ellipse, or along a curve (such as that given by Ex. 8) called a rectangular hyperbola.

174. Graphical method of solving a simple equation. If we take any simple equation whatever and place all the terms (whether they contain the unknown quantity or not) on one side of the equation, then the other side of the equation is = 0.

Thus,
$$x - 5 = 3x + 2$$
.
Transposing x we get $-5 = 3x - x + 2$.
Transposing -5 (i.e., add 5 to each side) $-5 + 5 = 3x - x + 2 + 5$ we get $-5 + 5 = 3x - x + 2 + 5$

Now put y in place of 0, and we get y = 2x + 7. If in this equation we put x = 0, 1, 2, 3, 4, etc., we can calculate a series of corresponding values of y.

Thus, when
$$x = 0$$
, $y = 2x + 7$
 $= (2 \times 0) + 7$
 $= 0 + 7$
 $= 7$.
Again, when $x = 1$, $y = (2 \times 1) + 7$
 $= 2 + 7$
 $= 9$.

Repeating	this	process	we	get	the	Table	shown	below	:

\overline{x}	0	1	2	3	4	5
y	7	9	11	13	15	17

If we now plot these values of y and x on squared paper, putting values of y vertically and values of x horizontally, as shown in Fig. 88, we get a **graph which is a straight line.** By producing this graph until it crosses the horizontal axis of reference XOX' on either side of the origin, we

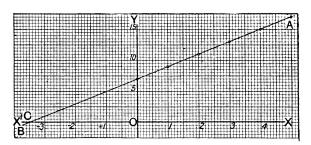


Fig. 88.

can find the value of x at the point at which the graph intersects the axis, and this value is the solution of the equation. It must be remembered that if the graph cuts the horizontal axis on the right of the origin, the value of x is positive, and if on the left it is negative. See Art. 173.

Thus, in the above example, the graph AB cuts the axis at a point C, $3\frac{1}{2}$ divisions to the left of the origin, and hence $x = -3\frac{1}{2}$ is the solution of the equation. By solving the equation algebraically, we also get $x = -3\frac{1}{2}$.

In solving the equation y = 2x + 7 graphically, we have found that when the graph crosses the axis XOX', i.e., when y = o, the corresponding value of x gives the required solution of the equation. Now, our original equation reduced to 0 = 2x + 7, and in solving this by the algebraic method, we transpose the terms thus:—

$$2x = -7$$
hence $x = -3\frac{1}{2}$.

If in the equation y = 2x + 7, we put $x = 3\frac{1}{2}$, then $y = (-3\frac{1}{2} \times 2) + 7$ = -7 + 7= 0.

Thus, the value of x which makes the value of y = 0 corresponds to the value of x which is the solution of the equation, and hence the above rule for solving simple equations graphically.

Ex. Solve
$$5x + 10 = 3x + 22$$
 graphically.
We have $5x - 3x + 10 - 22 = 0$,
or $2x - 12 = 0$.

Put y = 2x - 12. Our Table is now:—

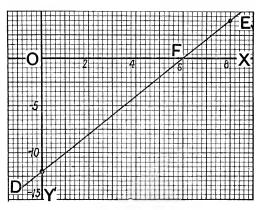
\boldsymbol{x}	0	8
y	- 12	4

and the graph is DE, Fig. 89.

This cuts the axis at F at which x = 6.

Hence x = 6 is the solution required.

Note. We only require two points to fix a straight line; hence



F1G. 89.

when we know that the equation gives a straight-line graph we only find the co-ordinates of two points as in the above Table.

175. Simple equations are often called linear equations, by reason of the fact stated above—that if we take ann

simple equation whatever and plot it on squared paper, as set out in the above Arts., we get a graph which is a straight line. This graph is called the **graph of the equation**. As the value of y depends upon that of x, y is said to be a

function of x, and the graph is often called the graph of the function.

As we give to x any value we choose in calculating the table of corresponding values of x and y from which we finally plot the graph, the quantity x is called the **independent variable**, whereas, since the value of y depends upon that chosen for x, the quantity y is called the **dependent variable**.

176. General equation of a straight line. Consider the equation x - 5 = 3x+ 2, used in Art. 174. We transformed this equation to 0 = 2x + 7, and then put y = 2x + 7. Now if we do this with a number of simple equations, we shall find that the coefficient of x (in this example 2), may be any number whatever, integral or fractional, and further it may be + or -. Similarly, the number added (in this example 7) may have any magnitude and may be + or -. These two numbers are called

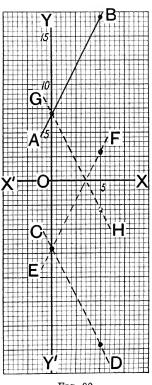


FIG. 90.

constants, and are denoted respectively by m and c. Hence, the equation in its most general form may be written $y = \pm mx \pm c$, which reads "y is equal to plus or minus mx plus or minus c," and means that y may equal (1) + mx + c, or (2) - mx + c, or (3) + mx - c, or

(4) -mx - c. This is the **general equation** of a straight line, and will represent *all* straight lines. The equation of any particular straight line is found when we know the numerical value and sign of both m and c.

Thus, when m and c are both positive we have y = 2x + 7.

", " is positive and c negative we have y = 2x - 7.

", ", "m is negative and e positive ", y = -2x + 7. If we calculate a table for each line, as in Art. 174, and plot on squared paper, we get the four lines AB, CD, EF, GH, Fig. 90, and we observe the following facts:—

(1) When m is + (lines AB and EF) the line is inclined to OX at

an angle less than 90°.

(2) When m is - (lines CD and GH) the inclination exceeds 90°.
(3) When c is + (lines AB and GH) the line cuts the axis YOY'

above O, and when negative (lines CD and EF) it cuts below O.

By giving to m and c in the general equation $y = \pm mx \pm c$ various values, we can write down a series of equations, and we will now do so to illustrate further the various forms the lines or graphs take with certain particular values of m and c.

Ex. 1. Plot the graphs of the equations (1) y = x; (2) y = x + 3;

(3) y = x - 3.

The Tables are as follows:—

	(1)											
\boldsymbol{x}	0	1	2	-2	-1							
y	0	1	2	-2	-1							

	(2)												
x	0	1	2	-2	-1								
y	3	4	5	+1	+2								
	,												

$$\begin{array}{c|c}
x & 0 & 1 & 2 & -1 \\
\hline
y & -3 & -2 & -1 & -5 & -4
\end{array}$$

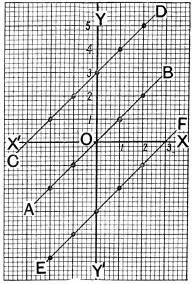


FIG. 91.

In Fig. 91, AB is graph (1), CD graph (2), and EF graph (3).

The student will note from this example that when m = +1 and c = 0, then y = x, and the graph passes through the origin and is inclined at 45° .

When m = +1 and c = +3, then y = x + 3, and the graph is still inclined at 45°, but it cuts the axis of y at the point +3.

When m = +1 and c = -3, then y = x - 3, and the graph is still inclined at 45°, but it cuts the axis of y at the point y = -3.

The distance from the origin to the point at which a line cuts either axis is called the **intercept** of the line on that axis.

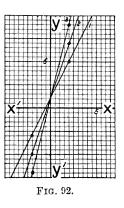
Thus, the intercept of line AB on each axis is 0.

Also note, the intercept on the axis of y is the value of y when x = 0, and the intercept on the axis of x is the value of x when y = 0.

Ex. 2. Plot the graphs of the equations: (1) y = 2x + 1; (2) y = 3x + 1; (3) y = 4x + 1.

The Tables are :-

(1)	<i>x y</i>	0	5	- 2 - 3
(2)	<i>x y</i>	0	7	- 2 - 5
(3)	<i>x y</i>	0	9	- 2 - 7



The lines are plotted in Fig. 92, and it will be observed that they all have the same intercept on the axis of y, viz., 1, while as the value m increases, the inclination of the line to the axis of x increases. The line (3) when m=4 has a greater inclination than the line (2) when m=3, and so on.

Ex. 3. Plot the graphs of (1) y = 1.5x + 4; (2) y = -2x + 4.

Tables are :-

(1)	<i>x y</i>	0 4	4
(2)	$\frac{x}{y}$	0 4	$\frac{2}{0}$

The graphs are (1) AP, (2) PQ, Fig. 93.

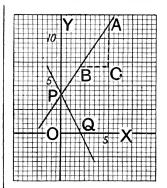


Fig. 93.

Note.—Since m is negative in the line PQ (m=-2), the line is inclined to the axis OX at an angle greater than 90°, and as m is positive in the line AP (m=1.5), the line is inclined at an angle less than 90°. When m is positive, y increases as x increases, and when m is negative, y decreases as x increases.

Ex. 4. Plot the graphs (1) y = 3, (2) x = 3.

Every point on the line (1) has an ordinate = 3, hence the graph AB, Fig. 94, is parallel to OX.

Every point on the line (2) has an abscissa = 3, hence the graph CD is parallel to OY.

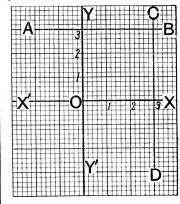


Fig. 94.

177. We have now demonstrated that every simple equation may be written in the form $\mathbf{y} = \pm mx \pm c$, and also that every such equation corresponds to a graph which is a straight line. The converse of this is true, viz., every graph which is a straight line corresponds to some simple or linear equation of the form $\mathbf{y} = \pm m\mathbf{x} \pm \mathbf{c}$, and we shall now show how the equation is to be found when the graph is given.

Consider first a particular case. In Ex. 3 of Art. 176 above we plotted two graphs from their given equations. See Fig. 93. Graph (1) corresponds to the equation y = 1.5x + 4. The general equation for any straight-line graph is $y = \pm mx \pm c$. In this particular graph, m is +1.5, and c is +4. If we read off the intercept, on the axis of y, we find that this is +4, and apparently corresponds to c in the general equation. Again, if we select any two points, as A and B, on the graph, and take the length AB as the hypotenuse of a right-angled triangle ABC, we find that BC = 3 divisions, and AC = 4.5 divisions, and

hence that $\frac{AC}{BC} = \frac{4.5}{3} = 1.5$. This value apparently corre-

sponds to m. If we repeat this investigation for graph (2)

we again find the intercept on the axis of y = 4, and the value of m, taking the right-angled triangle OPQ formed by the axis OX and

OY, is
$$\frac{\overrightarrow{OP}}{\overrightarrow{OQ}} = \frac{4}{2} = 2$$
, but as

this graph is inclined at a greater angle than 90° to the axis XOX', we must take the negative value for m, hence m = -2.

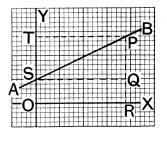


FIG. 95.

Now take a general case.

We know that the co-ordinates of every point on the given line will satisfy the equation $y = \pm mx \pm c$. Suppose we take the co-ordinates of the point P, Fig. 95, as x and y; the ordinate y is the length PR, and the abscissa x is the length PT.

The intercept on the axis of Y is OS. We require to find the values of c and m in the general equation $y = \pm mx \pm c$, which represents any line, to make it into a particular equation for the particular line AB. As the line is inclined at less than 90° to XOX', we take m as +. If we put x = 0 in the equation,

$$y = mx \pm c$$
, we get
 $y = 0 \pm c$
hence $y = \pm c$.

The value of y when x = 0 is the intercept OS; hence the value of c is the intercept OS of the given line on the axis of Y, and for the line AB this is +c.

Again, to find the value of m, we now have the equation

$$y = mx + c$$
 from which $m = \frac{y - c}{x}$.

In Fig. 95,
$$y = PR$$

 $c = OS = QR$
hence $(y - c) = (PR - QR) = PQ$.
Also $x = TP = SQ$

hence $\frac{y-c}{x} = \frac{PQ}{SQ}$, and this value is the required value of m.

As the ratios of the lengths of corresponding sides of similar triangles are equal, we can find the value of m by drawing any right-angled triangle having the given line as hypotenuse, and the two sides containing the right-angle parallel respectively to the axes of reference, and by calculating the value of the ratio $\frac{\text{length of side parallel to axis of Y}}{\text{length of side parallel to axis of X}}$ This value or ratio is sometimes called the "slope of the line."

As an example, find the equation of the given line AB in Fig. 95. The equation is of the form $y = \pm mx \pm c$.

The value of c is OS = +5.

The value of m is the ratio $\frac{PQ}{QS} = \frac{9}{19} = 0.47$, and it is positive, hence the equation of the line AB is y = 0.47x + 5.

This can be verified by plotting the graph of the equation $y = 0.47 \ x + 5$; it will be found to correspond to the given line AB. The student should also note that in all linear equations in which he has hitherto found the graphs that

the coefficient of x corresponds to the value m, while the constant c corresponds to the intercept on the axis of y.

Ex. Find the equation of each of the given lines AL, CP, EU, Fig. 96.

 For line AL. Form of equation is $y = \pm mx \pm c$. c = intercept on axis OY = -3.

$$m = \text{ratio} \frac{\text{LM}}{\text{MN}} = \frac{7}{3.5} =$$

2, and is +.

Hence equation of AL is y = 2x - 3.

(2) For line CP. Form of equation is y $= \pm mx \pm c$.

c = intercept on axis $O \cdot Y = 0$.

$$m = \text{ratio} \frac{PQ}{RQ} = \frac{3.5}{5} =$$

Hence equation of CP is y = 0.7x.

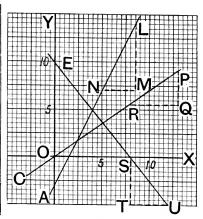


Fig. 96.

(3) For line EU.

Form of equation is $y = \pm mx \pm c$. c = intercept on axis OY = +10.

 $=\frac{5}{4}=1.25$, and is negative (since angle ESX $m = \text{ratio } \frac{1}{\text{TU}}$ between OX and the line EU exceeds 90°).

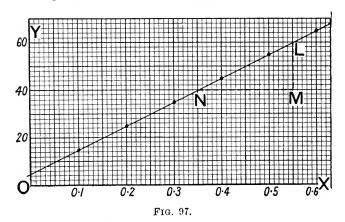
Hence equation of EU is y = -1.25 x + 10.

178. In the preceding examples in this chapter, we have plotted graphs in which the same scale is used for the ordinates and abscissæ. Now, in practice, as we already know from Chapter IV., the co-ordinates of points to be plotted may differ very considerably in magnitude, and hence we often adopt a different scale for ordinates from that used for abscissæ. As an example, consider a series of values of x and y as shown in the following Table:—

x	0.1	0.5	0.3	0.4	0.2	0.6
y	15	25	35	45	55	65

In this case, our greatest abscissa or x value is only 0.6, whereas the corresponding ordinate or y value is 65. In order to get all the points within an ordinary sheet of squared paper, we adopt the scales shown in Fig. 97, in which for values of x each inch division is taken as 0.1, and for values of y each inch division is 20.

The equation to a graph in which different scales are used is,



as before, of the form $y = \pm mx \pm c$, and also the value of c is the intercept on the axis OY, and the value of m is the ratio \overline{LM} but we must observe that the value of c is the magnitude represented by the intercept on the scale adopted along OY, i.e., 5 (not 2.5, since each division = 2 units along OY). Similarly, in finding the value of m, the value of LM must be the magnitude represented by a length LM on the scale marked along OY, and the value of LM must be the magnitude represented by a length LM on the scale marked along OY.

Thus, LM = 10 divisions = 20 units on scale along OY. MN = 20 , = 0.2 , , , OX.

Hence $\frac{LM}{MN} = \frac{20}{0.2} = 100$, and this value is the value of m.

The equation is thus y = 100x + 5.

179. When corresponding values of two variable quantities are obtained experimentally, the data obtained will not be free from error. When these corresponding values are plotted on squared paper, the graph is obtained by drawing a line or curve to lie as evenly as possible among the points, passing through some, and leaving the others some on one side and some on the other side of the graph. With a little practice the student will readily determine which line or curve lies best within the plotted points. These graphs are extensively used in laboratory and workshop experiments.

By plotting on squared paper data obtained experimentally, we not only obtain a graphical illustration of the way in which one quantity varies relatively to another, but we are also enabled (1) to correct errors in our experimental observations, (2) to ascertain approximately values within the range of our experiment which we have not actually observed, and (3) to predict approximately values which lie outside the range of our experiment. As an example, the force P in pounds necessary to lift a weight W in pounds with an experimental screw-jack is tabulated below, and the corresponding graph is shown in Fig. 98.

\overline{P} lb.	0.5	1	1.5	2	2.5	3	3.5	4
W lb.	4	11.2	25	34	43	56	66	77

The values of P and W differ considerably in magnitude, so we adopt different scales. The vertical scale for P is 1 in. = 1 lb., hence each $\frac{1}{10}$ in. represents 0·1 lb.; whereas for W each inch = 20 lb., hence each $\frac{1}{10}$ in. represents 2 lb.

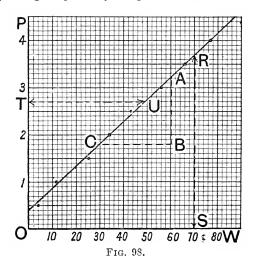
The values of P and W having been obtained experimentally, it is to be expected that the points will not all lie accurately upon a straight line; and it will be observed that where W=4, 11.5, 25, and 43 the values of P are not on the graph, but the other values of P lie accurately along the graph.

By analogy to the equation $y = \pm mx \pm c$, the equation of this graph is $P = \pm mW \pm c$, and we must find numerical release for m and a form the graph.

values for m and c from the graph.

The intercept c on the vertical axis is positive, and corresponds to 0.4 lb., and $m = \frac{AB}{BC} =$ is positive. Hence the equation of the graph, or the Law of the Machine, is P = 0.047 W + 0.4.

From this equation we can calculate values of P and W corresponding respectively to given values of W and P



which lie outside the range of the graph—for example, we can find the force which will lift half a ton, or the load lifted by a force of 15 lb.

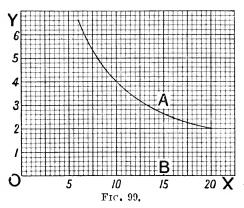
Where $W = \frac{1}{2}$ ton or 1,120 lb., substitute this value for W and we = 0.047 W + 0.4get $= (0.047 \times 1120) + 0.4$ = 52.64 + 0.4

This force lifts 1 ton.

= 53.04 lb.When P = 15 lb., substitute this value for P and we get 15 = 0.047 W + 0.4

$$\frac{15 - 0.4}{0.047} = W.$$
310.6 = W, hence a force of 15 lb, lifts 310.6 lb.

Values of P and W which do lie within the range of the graph may be obtained directly from the graph, or they may be calculated from the equation; thus, the force required to lift 70 lb. is shown graphically by RS which = 3.69 lb., and the load lifted by a force of 2.7 lb. is shown graphically by TU which = 49 lb. By calculation, if we put W = 90 lb. in the equation we get $P = (0.047 \times 70) + 0.4 = 3.69$ lb., and also by putting P = 2.7 we get $2.7 = (0.047 \times W) + 0.4$ from which W = 49 lb.



180. Two variable quantities may be related in such a manner that when corresponding values are plotted on squared paper, the graph obtained by joining the points is not a straight line. The methods adopted for finding the equations of such graphs are discussed in Chap. XXIII., but a value of one variable corresponding to a given value of the other variable can always be obtained (within the limits of the curve) by direct measurement from the graph.

As an example, plot the following values of x and y and find the value of y when x = 15:—

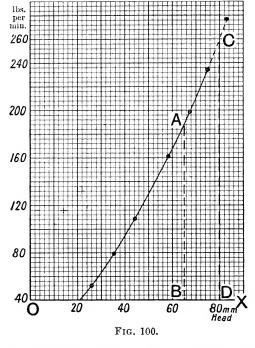
<i>x</i>	6	8	10	12	14	16	18	20
y	6.6	5	4	3.3	2.8	2.5	2.2	2

From curve, Fig. 99, it is seen that when x = 15, y = AB = 2.66.

As an exercise, the student should plot a curve from which the cube roots of numbers can be read, taking the numbers as abscissæ and the cube roots as ordinates, as shown in the Table below.

x	0	1	5	8	10	20	27	40	50	64	70	80	90	100
$y = \sqrt[3]{x}$	0	1	1:71	2	2.12	2.71	3	3.42	3.68	4	4.12	4:31	4.48	4.64

From the graph, read off the cube root of 2.7, 12.69, and 77.4. Note that the values of x are not equidistant.



As a practical example, the following corresponding values of the number of pounds of water passing over a weir, 2 in.

wide, per minute, and the height of the surface of the water (head of water) above the weir were found experimentally:—

Head of water in mm	26.1	35.44	44.22	52.43	58.1	67.2	75.1
Lb. of water discharged per min	52.9	79:33	109.4	140.5	161.25	199.2	235

In plotting these values on squared paper, we plot the head horizontally and the discharge vertically, as in Fig. 100, and we commence the vertical scale at 40, as our smallest ordinate is 52.9. Let each inch or 10 divisions horizontally represent a head of 20 mm., and each inch or 10 divisions vertically represent a discharge of 40 lb. per minute. Each division horizontally then represents a head of 2 mm., and each division vertically represents a discharge of 4 lb., per minute. Now draw an even curve through as many as possible of the points. Any points lying off the curve are those in which errors have been made in observation or measurement during the experiment. From this curve we can obtain by measurement the discharge for a head of, say, 65 lb., lying within the range of the experiment, by reading off the value AB, which is seen to be 188 lb. per minute. can also estimate the probable discharge for a head of, say, 80 mm., lying outside the range of the experiment, by producing the curve, as shown dotted, and reading off the value of CD, which is 260 lb. per minute.

Exercise. The discharge in lb. per minute through an orifice in the bottom of a tank and the corresponding head of water in the tank are as follows (obtained experimentally):—

Head in feet.		0.5	1.0	1.2	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Discharge per in lb	minute	12.1	18.4	22.5	25.5	28.0	30.5	33.0	35.0	36.2	39.0

Plot on squared paper, find from the curve the discharge for a head of 4.25 ft., and the probable discharge for a head of 6 ft.

Ans. 36; 42.2 lb. per min.

Exercises (a).

Each of the following exercises is to be solved by plotting on squared paper.

(1) Plot points whose co-ordinates are respectively (+2, +3), (+2, -3), (-2, -3), (-2, +3), and find the area of the figure obtained by joining the four points. Give the area in number of squares enclosed.

(2) Find the equation of the line joining the two points whose co-

ordinates are -2, 1 and 6, 7.

(3) By experiment and subsequent plotting upon squared paper, the law of a screw-jack is found to be $P = \frac{1}{7} W + 3.5$; P being the lifting force and W the load lifted in lbs. Plot the probable graph represented by the above law. Find also the load lifted when P = 20 lb.

(4) Plot the following values of P and W. They relate to a lifting

appliance.

P (force in lbs.) .	4	4.5	5	5.6	6.1	6.4	7.0	8.2	9.0
W (load lifted in lbs.)	20	30	40	50	60	70	80	100	120

Find the value of P when W is 1 cwt.

(5) The following values of F and R were obtained experimentally. Plot a graph connecting F and R, and find its equation.

${m F}$	3.1	4.2	5	6.1	7.3	9	11.2
\overline{R}	12	21	36	48	60	81	108

(6) The circumference c of a circle is proportional to the radius r. When r=3, c=9.42, and when r=10, c=31.42. Plot these two points. Join them by a straight line, and from this graph find the value of r when c=78.54.

(7) A spiral spring carries a marker moving over a graduated scale. When a load of 5 lb, is on the spring the marker reads 5.3 in., and when the load is increased to 12 lb, the marker reads 5.72 in. From this data, draw a straight-line graph which will represent the relation between the load and the scale reading. What does the marker read when there is no load on the spring, and, when there is 25 lb.

(8) The percentage efficiency of a Worthington steam pump at various speeds was determined experimentally as follows:—

Doublestrokes per minute .	10	20	30	40	50	60	70	80.	90	100
Percentage efficiency .	25	40	48.5	55	59	63.5	66	68:5	71	72.5

Plot a graph, and find the probable efficiency when the number of double strokes per minute is 55.

(9) In a centrifugal pump experiment, the relation between the motor borse-power and the number of lbs. of water delivered per minute was found to be as follows:—

м.н.р	1.15	1.21	1:32	1.5	1.55	1:58
lbs. delivery	245	255	296	360	400	420

Find the probable delivery when the M.H.P. is 1.28,

(10) The relation between pressure P and temperature T of steam shown in the Table below was found experimentally. Find the value of T when P is 105, and the value of P when T is 300.

P lbs. per sq. inch .		10	15	20.5	27	31	36	44	50	60	70	80	90	100	110	120
T Fah	235	243	251	260	270	276	282	290	296	306	314	322	3.9	336	342	348

(11) The deflection D in inches at the centre of a beam, and the span L in inches are given below, the load being constant. Find the value of D for a span of 32 in.

L	15	20	25	30	35	40	44
D	0.025	0.075	0.165	0.275	0.43	0.65	0.84

(12) With an experimental beam, the deflection at the centre, and the load carried at the centre, are observed. Find the probable deflection for a load of 45 lb.

Load in lbs.	0	4	8	12	15	20	24	28	32	36	40
										_	
Deflection in inches.	0	0.05	0.08	0.12	0.16	0.19	0.23	0.27	0.31	0.35	0.33

(13) A copper wire was loaded gradually until it broke, and the extension for each load was measured, with the following results. Find the probable extension for a load of 40.5 lb.

Load in lbs.	0	5	10	15	20	25	30	32	35
Extension in inches.	0	.02	.03	.01	.06	•08	•29	.38	.82

Readings continued on next page.

Load in lbs.	37	40	41	42	43	44	45	46	47
Extension in inches.	1.32	2.43	2.97	3.64	4.24	4.95	6.05	7.7	wire broke

(14) With an experimental apparatus to illustrate the turning-effort on a crank-pin through 180° from the inner dead centre, the following readings were taken. Plot a curve to show this variation.

Degrees . Turning-effort	!-	$\frac{10}{2.5} \frac{20}{4.2}$!_	30 ·75	40 8·5		10.5	70	80	90
Degrees . Turning-effort	•	100	9.0	-	75	130	140 5·5	150	160	170

(15) The number N of coils in a spiral spring, and the number P of lbs. required to produce an extension of 1 in. are obtained experimentally. Find the probable value of P when N is 10 and also 13.

N	7	8	9	11	12
\overline{P}	14.6	13.0	11.3	10.0	8.75

(16) The load and extension for a loaded wire was found experimentally. Find the probable extension for a load of 29 lb.

Load in lbs.	0	3	6	8	10	12	14	16	18
Extension in inches.	0	0.02	0.01	0.02	0.055	0.06	0.07	0.09	0.1
Load in lbs.	20	22	24	26	28	30	32	33	34
Extension in inches.	0.11	0.12	0.13	0.15	0.17	0.19	0.26	1.61	wire broke

(17) Plot the following values, which were obtained experimentally, and find the twisting-moment required for a rod 0.275 in. in diameter:

Diameter of rod i inches	n 0.185	0.25	0.312	0 344	0 375
Twisting-moment to produce a twist of		12:1	28:6	42:0	56:0
l radian .	. 3.19	12.1	28.6	42.0	96.0

(18) The number N of double strokes per minute of a pump, and the number W of lbs. of water delivered by the pump are found experimentally.

N	26	34	58	72	83	92	100	104	154
11.	6.2	10	20.25	25.25	29	33	35.5	37	43.25

Find the probable delivery when N = 50.

(19) Plot the graphs of 3y = 4.8x + 0.9 and y = 2.24 - 0.7x. What are the co-ordinates of the point where they cross? Measure the angle each line makes with the axis of x and the angle between the graphs. (B.E. 1903.)

(20) The following values represent the areas of cross section of a body perpendicular to its axis:—

A sq. in	250	292	310	273	215	180	135	120
x in, from end	0	22	41	70	84	102	130	145

Plot A and x on squared paper. What is the probable cross section 50 in. from the end? What is the average cross section and the whole volume? (B.E. 1903.)

(21) A British man or woman of age x years may, on the average, expect to live for an additional y years.

Age x		70	60	50	40	30
Expected further	Man .	8.27	13.14	18.93	25.30	32.10
	Woman .	8.95	14.24	20.68	27:46	34.41

Plot a curve for men and one for women, and find the expectations of life for a man and for a woman aged 54 years. (B.E. 1904.)

(22) A series of soundings taken across a river channel is given by the following table, x feet being distance from one shore, and y feet the corresponding depth. Draw the section. Find its area. (B.E. 1904.)

		1	1	1			43						1
\overline{y}	5	10	13	14	15	16	14	12	8	6	4	3	0

(23) A feed pump of variable stroke driven by an electromotor at constant speed; the following experimental results were obtained:—

Electrical horse-power .	3.12	4.5	7.5	10.74
Power given to water .	1.19	2.21	4.26	6.44

Plot on squared paper, and state the probable electrical power when the power given to the water was 5. (B.E. 1905.)

(24) Mr. Scott Russell found that at the following speeds of a canal boat the tow-rope pull was as follows:—

Speed in miles per hour	6.19	7.57	8.52	9.04
Tow-rope pull in pounds	250	500	400	280

What was the probable pull when the speed was 8 miles per hour? There was reason to believe that the pull was at its maximum at 8 miles per hour, because this was the natural speed of a long wave in that canal. (B.E. 1905.)

(25) The angular position D of a rocking shaft at any time t is measured from a fixed position. Successive positions at intervals of $\frac{1}{10}$ sec. have been determined as follows:—

Time t sec	0.0	0.05	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18
•										
Position D radians .	0.106	0.208	0.337	0.487	0.651	0.819	0.978	1.111	1.201	1.222

Find the change of angular position during the first interval from t=0.0 to t=0.2; calculate the mean angular velocity during this interval in radians per second, and set this up on a time base as an ordinate at the middle of the interval. Repeat this for other intervals, tabulate the results, and draw the curve connecting angular velocity and time. Read from the curve the angular velocity when t=0.075 sec. (B.E. 1905.)

(26) In the following table, A is the area in square feet of the horizontal section of a ship at the level of the surface of the water when

the vertical draught of the ship is h ft. When the draught changes from 17.5 to 18.5 ft., what is the increased displacement of the vessel in cubic feet? (B.E. 1906.)

h	15	18	21
A	6020	6660	8250

(27) x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. Describe why it is that the slope of the curve shows the speed; where, approximately, is the speed greatest and where is it least? (B.E. 1906.)

\overline{x}	0	1.5	6.0	14.0	19:0	21.0	21.5	21.8	23.0	24.7	26.8
t	0	•1	•2	.3	.1	.5	.6	.7	.8	.9	1.0

(28) The speed of a ship in knots (nautical miles per hour) has been noted at the following times:—

Speed	11.23	12.56	13.20	14.11	14:53
O'clock	4	5	6	7	8

Plot on squared paper. What is the distance passed through during the hour after 6 o'clock? (B.E. 1906.)

(29) The following corresponding values of x and y are given in a Table:—

x	1.22	1:37	1.20
\overline{y}	5.88	8.32	9.71

What is the probable value of x when y is 8? (B.E. 1906.)

(30) The following results were obtained by experiment of water discharged through an orifice:—

Duration of experiment.	15 min.	15	15	15	15	15	10	10
Actual discharge	576 lb.	660	733	827	915	1011	737	738
Head of water	1.5 in.	2.0	2.5	3.27	4.01	5.0	6.0	7.0

Plot a curve showing the relation between discharge in lbs. per

minute and head in inches. Determine also the discharge in gallons per hour when the head is $5\frac{1}{2}$ inches. (B.E. 1907.)

(31) The following values of x and y are thought probably to fulfil the law y = a + bx. Try if this is so. Find the most probable values of a and b. What is the probable error of each value of y? (B.E. 1907.)

\boldsymbol{x}	10	20	30	40	50	60	70	so
\overline{y}	180	303	492	603	807	919	1131	1200

(32) The area of the horizontal section of a reservoir is A sq. ft. at the height h ft. from its lowest point. When h is 30 what volume of water is in the reservoir? (B.E. 1907.)

h	0	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25	27.5	30
A	0	2510	3860	4670	5160	5490	5810	6210	6890	7680	8270	8620	8780

(33) Plot the following values of x and y on squared paper. What is the average value of y between x = 0 and x = 50? (B.E. 1907.)

x	5	16	26	35	50
y	13	22	24	27	31

(34) At the following draughts in sea water a particular vessel has the following displacements:—

Draught in feet	. 15	12	9	6.3
Displacement in tons	2098	1512	1018	586

What are the probable displacements when the draughts are 11 ft. and 13 ft. respectively?

(35) Plot the following values of x and y on squared paper. They are thought to follow the law y = a + bx. If so, find a and b. What is the value of x when y is 50 and the value of y when x is 17.5?

\boldsymbol{x}	3	6	9	12	15	18	21
y	10	16	22.5	28.5	34	40.2	46.2

⁽³⁶⁾ A dynamo is in two parts whose weights are x and y. The cost of the machine is z = y + 4x. The usefulness is $v = x^2 + 3xy$.

If z is 10, express v in terms of x alone. Now take various values of x (say from 0.5 to 2) and calculate v. Plot v and x on squared paper. For what value of x is v a maximum? (B.E. 1907.)

(37) The population of a country (in millions) in 1880 was 29.8; in 1890 it was 33·1; in 1900 it was 37·2. What was the probable population in 1894? What was the average rate of increase in the population per year from 1880 to 1890? What was this average rate from 1890 to 1900? (B.E. 1908.)

(38) If d is the number of days after the birth of a baby, and w is the observed weight of the baby in pounds; show the relation of w to d on squared paper. What was the average rate of increase in weight (pounds per day) during the first 50 days? (B.E. 1909.)

d	0	21	42	56
w	6.1	6.75	8.00	9.1

(39) A straight line passes through points plotted on squared

paper; it is y = a + bx.

If one of the points is y = 2, x = 1, and if another of the points is y = 6, x = 2.5, find a and b. What is the value of y when x = 1.753? What is the value of x when y is 5.3? (B.E. 1909.)

(40) A sliding piece is at the distance s ft. from a point in its path at the time t sec. Do not plot s and t. What is its average speed in each interval of time? Assume that this is really the speed in the middle of the interval, and plot time and speed on squared paper. (B.E. 1909.)

8	0	1.2	2.8	4.8	7.2	10.0
\overline{t}	0	0.1	0.5	0.3	0.4	0.2

(41) In a table of values of x and y I find the following entries:-

\boldsymbol{x}	10	11	12	13
\overline{y}	·1115	1128	·1150	1202

What is the probable value of y when x is 11.5? What is the probable value of x when y is 1165? (B.E. 1909.)

(42) The cross section of a tree (A sq. in.) at distance x in, from one end is as follows:—

\boldsymbol{x}	10	30	50	70	90	110	130	150
A	120	123	129	129	131	135	142	156

What is the volume of the tree in cubic inches and in cubic feet, its total length being 160 in.? (B.E. 1909.)

(43) A man finds that if he uses x horses, his daily expenditure in pounds is

$$0.3x + \frac{7.3}{1+x}$$

Calculate this for various values of x and find what value of x makes his daily expenditure a minimum. (B.E. 1909.)

(44) The following numbers give v the speed of a train in miles per hour at the time t hours since leaving a railway station. In each interval of time, what is the distance passed over by the train? At each of the times tabulated what is x, the distance from the station? Tabulate your answers.

v	0	2.4	4.7	7.2	9.6	12.0	14.3	16.9	18.9	20.7	22.2	23.4	24.3	24.9
$\frac{-}{t}$.00	.04	.08	.12	.16	.20	•24	·2S	•32	.36	•40	•44	•48	*52

The average speed in the interval from t = 0.04 to 0.08, may be taken as the mean value of the speeds 2.4 and 4.7, that is, $\frac{1}{2}(2.4 + 4.7)$. (B.E. 1908.)

(45) \hat{x} is distance in chains measured along a straight line AB from the point A, the values of y are offsets or distances in chains measured at right-angles to AB to the border of a field. Draw the shape of this border. Find the area in square chains between the first and last offset and the straight line and border. Notice that the intervals in x are not equal. (B.E. 1908.)

x	0	1.50	3.00	5.00	7:50	9.00
y	0.53	0.27	0.46	0.42	0.35	0.52

(46) The following numbers give x, the distance of a place in miles from a railway station, and t the time in hours, taken by a train to reach that place. Find r, the average speed (in miles per hour) of the train in each interval.

x	0	0.16	0.48	0.94	1.58	2.40	3.34	4.46	5.74	7.14	8.64	10.20	11.78	13.38
t	.00	.04	•98	•12	•16	.20	•24	-28	.32	.36	·40	•44	·48	.52

Show by a curve how the speed changes as t increases. The average speed in an interval may be taken to be the speed in the middle of that interval. (B.E. 1908.)

Answers (a).

- (1) 24. (2) y = 0.75x + 2.5. (3) 1237.5 lb. (4) 8.6 lb. (5) $F = \frac{1}{12}R + 2$. (6) 25. (7) 5; 6.5 in. (8) 61.5. (9) 274 lb.
- (8) 61·5. (9) 274 lb. (10) 339; 55. (11) 0·34 in. (12) 0·43 in. (13) 2·7 in. (15) 10·5; 8·4. (16) 0·18.
- (17) 16. (18) 17. (19) (0.8, 1.65); 58°, 145°; 87°. (20) 304 sq. in.; 226.6 sq. in.;
- 32,857 cub. in. (21) 16·53; 17·98. (22) 756 sq. ft.

- (23) 8·5. (24) 510.
- (25) 8·3. (26) 6,660 cub. ft. (27) 0·25; 0·65.
- (28) 15.91 mls. (29) 1.345, (30) 423.
- (31) 20; 15.4. (32) 174,498 cub. ft.
- (33) 22. (34) 1,350; 1,700. (35) 2; 4; 23; 39
- (36) $v = 30x 11x^2$; 1·36. (37) 34·7; 0·33; 0·41.
- (38) 0.172 lb. (39) $\frac{8}{3}$; -0.6; 4.075; 2.27.
- (41) 0·1138; 12·45. (42) 20,448 cub. in. (43) 4.
- (45) 3.66 sq. chains.

Exercises (b).

The following exercises may be deferred until Chap. XXII. has been read.

(1) A straight line passes through a point whose co-ordinates are (4,3), and makes an angle of 35° with the axis of x. Calculate the value of c and find also the value of m in the equation $y=\pm mx\pm c$ for this line. Verify your result by plotting. Ans. c=0.2; m=0.7.

(2) A straight line passes through the points whose co-ordinates are, respectively, (2,7) and (6,10). Show that the value of m in the equation for this line is 0.75, and find the value of c. Ans. 5.5.

(3) The bending moment at any point distant x feet from the free end of a cantilerer is $\frac{w}{2}x^2$, when w is the load in lbs. per foot run of the cantilerer. Plot a curve from which the bending moment M at any point of a cantilerer 12 ft. long can be determined, if w = 20 lbs.

(4) An arch across a roadway is shaped to the form of a semi-ellipse whose major axis is 100 ft. (the span of the arch) and minor axis is 60 ft. (the rise being thus 30 ft.) The equation for this ellipse is $\frac{x^2}{2.500} + \frac{y^2}{900} = 1$, when the origin is at the centre of the ellipse. Find by calculation, and by plotting, the height of the arch above a point 20 ft.

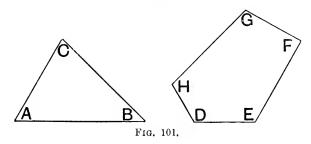
from the centre of the road. Ans. 27.48 ft.

CHAPTER XIV.

PLANE RECTILINEAR FIGURES.

181. **Definitions and properties.** Plane figures bounded by straight lines are called **rectilinear figures**. The lines which bound the figure are called **sides**. If the figure has 3 sides, it is called a **triangle**; if 4, a **quadrilateral**; if more than 4, it is called a **polygon**.

Particular names are given to polygons according to the number of their sides: a pentagon has 5 sides, a hexagon 6,



a heptagon 7, an octagon 8, a nonagon 9, a decagon 10, etc.

The sum of the sides or length of outline of a figure is called the perimeter.

A line joining any two non-consecutive corners of a

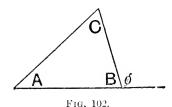
rectilinear figure is called a diagonal.

In order to refer definitely to a particular figure or part of a figure, we denote each angular point or vertex by a distinct letter, and call the figure by these letters; thus, the triangle in Fig. 101, we call the triangle ABC; and the polygon we refer to as the figure DEFGII.

Greek letters are sometimes used to represent angles; thus, the angle EDH may be denoted by β (beta). Sometimes the angle is represented by the letter attached to the angular point; thus, the angle Λ would mean the angle BAC.

If any side, say AB, in the triangle, Fig. 102, be produced, this line and the adjacent side BC form two angles at B; one (B) an *interior* angle, and one (ϕ) an *exterior* angle. When the term *angle* is applied to a figure, an internal angle is always understood, in the absence of any statement to the contrary.

If every angle in a figure is less than 180°, it is called a convex figure. If all the angles are equal to one another, it is said to be equiangular. If all the sides are



equal in length, it is called **equilateral**. A figure having all its sides equal and all its angles equal is called a **regular** figure.

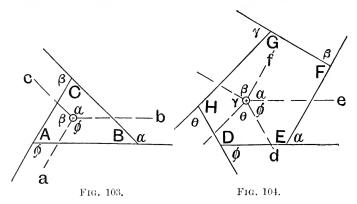
Regular figures have a definite centre O, equidistant from the corners and sides, so that (a) all the angular points of a regular figure lie on the circumference of a circle centred in O; (b) a circle centred in O and touching one side touches every side of the figure.

182. Experiment. Draw any triangle ABC, Fig. 103, and extend each side, as shown in the diagram. Denote the exterior angles by a, β , ϕ . From any point O, draw Ob parallel to the side AB; draw Oc parallel to the side BC; draw Oa parallel to the side CA. Then

angle bOc = a; angle $rOa = \beta$; angle $aOb = \phi$. And since the sum of the angles bOc, rOa, aOb is 369° , $\therefore a + \beta + \phi = 360^{\circ}$. Now draw any convex polygon, as DEFGH, Fig. 104. Extend each side as above described and denote the exterior angles by α , β , γ , θ , ϕ . From any point O, draw Oe parallel to the side DE; draw Of parallel to the side EF; . . and Od parallel to the side HD. Then, the angles which are marked by the same letters in Fig. 141 are equal to one another, and the sum of the exterior angles is equal to the sum of the angles about the point O, viz., 360°, and the same is true of Fig. 104.

Hence the **Theorem**. The sum of the exterior angles of

any convex rectilinear figure is 360°, or 4 right-angles.



If we denote the interior angles of the triangle, Fig. 103, by Λ , B, C, we have—

 $A + \phi = 2$ right-angles; B + a = 2 right-angles; $C + \beta = 2$ right-angles;

and, by addition,

 $\Lambda + B + C + \phi + \alpha + \beta = 6$ right-angles. Again, if the interior angles of the polygon in Fig. 104 be denoted by D, E, F, G, H, we have

D + ϕ = 2 right-angles; E + α = 2 right-angles; F + β = 2 right-angles, etc.;

and, by addition,

D + E + F + G + H + ϕ + α + β + γ + θ = 10 right-angles.

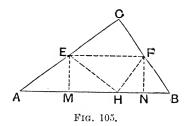
Comparing these results we find: The sum of the interior and exterior angles of any rectilinear figure is = twice as many right-angles as the figure has sides.

And, since the exterior angles always equal 4 right-angles, we have: (a) The interior angles of a convex rectilinear figure of n sides = (2n - 4) right-angles. (b) The sum of the interior angles of a triangle is 2 right-angles.

In Fig. 105, we show how this last important theorem

may be verified by folding paper.

Draw any triangle ABC and bisect AC, CB in E and F Join EF and draw perpendiculars EM, FN on to AB. Cut out the triangle



ABC, and fold the corner ECF along EF, bringing C on to AB in H. Fold the corners EAM, FBN along EM, FN; then A and B should coincide with H, and AE and BF should coincide with HE and HF.

When we have two figures, as, for example, the triangles ABC, abc, Fig. 106, or the polygons DEFGH defgh, which we know are alike in some respects, and we wish to compare them, we call the points which are denoted by the same letters corresponding points. Lines which join two points in one figure and the corresponding points in the other figure we call corresponding lines. Thus, for example in the triangles; AB, ab; BC, bc; and in the polygons; DE, de; DF, df are corresponding times. The angles between corresponding lines in the two figures are corresponding angles.

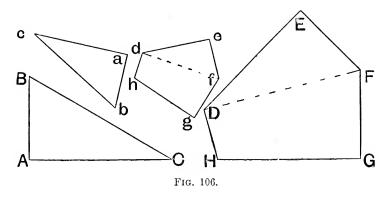
If any two figures, as, for example, the triangles or polygons in Fig. 106, have their corresponding angles equal, i.e., A = a, B = b, C = c, and D = d, E = e, . .

H = h, the figures are equiangular.

(It must be noticed that in this case we are not comparing one angle with another angle in the same figure, but with an angle in another figure. The angles in the same

figure may, or may not, be equal.)

If two or more equiangular figures are such that every line in one figure is equal in length to the corresponding line in the other figure, the figures are alike in all respects, and are said to be **congruent**. Congruent figures can always be superposed one on the other so as to make the two wholly coincide; and the practical test of **congruency** in plane figures is to make a tracing of one figure and



place this tracing over the other figure and see whether all lines on the tracing coincide with corresponding lines on the figure beneath.

183. Similar figures. When two or more figures have the angles in one figure equal to the corresponding angles in the other, and corresponding lines in the figures are proportionals, the figures are called similar figures.

The triangles ABC, abc, Fig. 106, and the polygons DEFGH, defyh, are examples of *similar figures*. In each pair of figures, corresponding angles are equal, and, in the triangles, linear dimensions are in the ratio $\frac{3}{2}$. In the polygons, the ratio is $\frac{3}{4}$. Test this statement by

measuring the angles and sides of each pair of figures. You will find—

$$\begin{split} \frac{\text{AB}}{ab} &= \frac{\text{BC}}{bc} = \frac{\text{CA}}{ac} = \frac{3}{2} \\ \text{and} &\frac{\text{DE}}{de} = \frac{\text{EF}}{cf} = \frac{\text{FG}}{fg} = \frac{\text{GH}}{gu} = \frac{\text{HD}}{hd} = \frac{\text{FD}}{fa} = \frac{2}{1} \,. \end{split}$$

This constant ratio is called the ratio of similitude of the two triangles (or polygons).



Circles, squares, and equiangular triangles are always similar figures, because in these figures corresponding lines are in a constant ratio. But rectilinear figures of more than three sides may be equiangular without having sides in a constant ratio, e.g., the square and the rectangle, Fig. 107.

Also, two figures may have sides in a constant ratio without being equiangular, e.g., the square and the rhombus, Fig. 107.

Exercises.

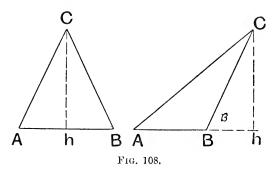
- (1) In a triangle ABC, the exterior angle at A is double that at B while the exterior angle at C is 150°. What are the exterior angles at A and B, and the three interior angles? Ans. 140°; 70°; 40°; 110°; 30°.
- (2) Six lines radiate from a point O; the angles between them are, respectively, 60, 40, 20, 70, 50, and \(\theta\) degrees. What is the value of \(\theta\)? A six-sided polygon has sides parallel to the six radial lines. What are the values of the six interior angles of the polygon? Ans. \(\theta\) = 120.
- (3) What is the magnitude of each of the interior angles of a regular pentagon, heptagon, nonagon, and decagon? Ans. 108°; 128.6°; 140°; 144°.
- (4) You are required to set out two consecutive sides only of a regular octagon of 25 ft. side. How would you proceed to do this?

CHAPTER XV.

RIGHT-ANGLED TRIANGLE.

184. A triangle has 3 sides and 3 angles; these are called elements of the figure. Thus a triangle has 6 elements.

When two sides of a triangle are of equal length, the triangle is called **isosceles**. In an *isosceles* triangle, the unequal side is called the *base*. In other forms, any side may be called the base. If we call one side the base, the



angles adjacent to that side are called *base angles*, and only the angle opposite to the base is called the vertex.

The perpendicular distance of the vertex of a triangle from the base line is called the *height or altitude*. Thus, in Fig. 108, Ch is the height of each of the triangles on the base AB.

If A denotes the angle between a side AC and the base line AB, we have—

Height $Ch = AC \times \sin A$.

185. A triangle having one side perpendicular to another

side is called a **right-angled triangle**. The side opposite the right-angle is called the **hypotenuse**. The sides forming the right-angle are sometimes called legs; either of these sides may be called the base.

Experiment.

Draw any triangle, and by carefully measuring the 3 sides and angles verify the following—



Fig. 109.

Theorem. The shortest side of a triangle is opposite the smallest angle.

Hence, we see—(a) in an equilateral triangle, the three angles are equal; (b) in an isosceles triangle, the angles opposite to the equal sides are equal.

If a side AB, Fig. 109, of a triangle

ABC be produced, the adjacent interior and exterior angles at B, when added together = 2 right-angles. And, since the 3 interior angles together = 2 right angles (Art. 182) we have angle B + angle ϕ = angle B + angle C + angle A. Subtracting common angle B, angle ϕ = angle C + angle A, i.e., An exterior angle of a triangle is

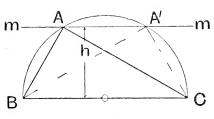


Fig. 110.

equal to the sum of the two interior and opposite angles. In a right-angled triangle, the angles adjacent to the hypotenuse are complementary.

186. We have learned from experiment, Art. 113, that any right-angled triangle has its right-angle on the circumference of the circle drawn on the hypotenuse as diameter. It follows, therefore, that when we know the hypotenuse of a triangle we also know the *locus* of the right-angle.

Construct right-angled triangles having given the following duta:—

(i.) Fig. 110. Given the hypotenuse a and its distance h

from the opposite angle A, say 3.5 in. and 1.5 in.

Draw side BC of given length a=3.5 in. Describe a circle on diameter BC. Draw mm parallel to, and at the given distance h above, BC, intersecting the semicircle in two points A and A'. Join AB, AC, or A'B, A'C. There are thus two solutions in this case. For limitations, see note to problem, Art. 199.

(ii.) No Figure. Given the two sides including the right-

angle, say b and c, 3 in. and 7 in.

Draw AC, AB at right-angles and equal to the given lengths, viz., 3 in.; 7 in. Join BC.

В

FIG. 111.

(iii.) Given one side b and the hypotenuse a, say 3 in. and 43 in.

In this case, we may draw the hypotenuse of given length a, describe a semicircle on same and ent off a chord of given length, b. The drawing of the semicircle is, however, unnecessary if we use the following method:—

Fig. 111. Draw AC of given length b=3 in. Erect a perpendicular at A. With C as centre

and radius of length a=4.5 in, describe an arc intersecting the perpendicular from A in B. Join BC.

(iv.) No Figure. Given the hypotenuse a and one adjacent

angle, say 5 in. and 57°.

Draw side BC of given length a=5 in. At C set out the given angle 57°. From B draw BA perpendicular on to the line from C.

187. All right-angled triangles possess the following

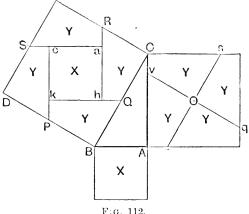
property:-

If squares be constructed on each of the three sides, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This is a very useful theorem. In Fig. 112, we show how

the square on the hypotenuse may be divided and rearranged to form the squares on the other two sides.

Draw any triangle ABC with a right-angle at Λ . Draw squares on each of the 3 sides of the triangle. Through the mid-points of the sides of the square on the hypotenuse draw parallels to the sides of the triangle, i.e., Sa, Qk parallel to BA; Pr, Rh parallel to AC. Through the centre O of the square on long side AC draw qv parallel to BD, rs parallel to BC. Make a tracing of DScP, and by superposing this



F.G. 112.

tracing show that each of the figures lettered Y are equal, and that the figures lettered X are equal.

If we denote the two legs of the right-angled triangle by b and c, and the hypotenuse by a, the theorem may be expressed thus:-

$$a^2 = b^2 + \epsilon^2$$
; or $b^2 = a^2 - c^2$; or $\epsilon^2 = a^2 - b^2$.

These equations enable us to determine the length of cither side when the other two sides are given.

Ex. 1. Given
$$b = 3$$
 in., $c = 4$ in. Then
$$a^{2} = 3^{2} + 4^{2}$$

$$a = \sqrt{9 + 16}$$

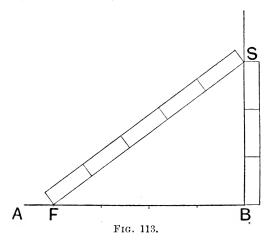
$$= \sqrt{25} = 5$$
 in.

Ex. 2. Given
$$a = 9$$
 in., $c = 7$ in. Then
$$b = \sqrt{9^2 - 7^2}$$

$$= 5 \cdot 657.$$
Ex. 3. Given $a = 5$ in., $b = 3$ in.
$$c = \sqrt{5^2 - 3^2}$$

$$= 4$$
 in.

188. In the practical work of building and surveying, right-angles have frequently to be set out, and it is an advantage to have all three sides of the triangle whole numbers, so that the foot, or the *link*, will serve to measure



the figure. In example 1 above, we have a useful series, viz., 3, 4, 5, which can be used to set out a right-angle.

Mark off from B, Fig. 113, along a straight line AB the point F 4 ft.; place at B one end of a rod 3 ft. long, and at F one end of a rod 5 ft. long; the other ends of the rods will meet in S, a point in the perpendicular from B.

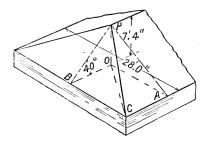
There are other sets of three integers which form the sides of right-angled triangles, but in many of these two of the dimensions are almost equal, whilst the third is very much smaller, and consequently are not so convenient in practice as the series 3, 4, 5. If larger numbers are

desired, it is only necessary to multiply each number by some other number; we then obtain a new series as convenient and well-proportioned as the first set.

Thus, multiply by 2 and we have sides 6.8.10

and so on.

189. Practical applications. Fig. 114 is a dimensioned sketch of a portion of a hipped roof; the span of the roof, and the height of the ridge above the level of supporting walls are given by figures; the slope of the end portion is given in degrees. Determine (a) the length of the common rafters for the main roof and the herels to which they must be cut to make them fit against the ridge at top, and



Frg. 114.

the top of supporting wall at the bottom (i.e., the length of the hypotenuse of the triangle having OA, OP for legs, and the angles at P and A); (b) the length of the rafter BP; (c) the length of the hip rafter CP.

(1) Graphical method. Select a convenient scale and construct the right-angled triangle POA, making PO = the given length 7 ft. 4 in., and OA = $\frac{1}{2}$ of 28 ft. = 14 ft. by problem 186 (ii.).

Then measure the hypotenuse and the angles it makes with the sides OP, OA.

To find the length of BP, construct the right-angled triangle having one side = PO and the angle OPB = $(90 - 40) = 50^{\circ}$.

To find the length of the hip CP, first find the length of OC. which can be determined as the hypotenuse of a right-angled triangle having legs respectively equal to AO, BO. With OC and OP known, the length of CP is found in the manner explained in Art. 183 (ii.).

(2) By calculation.

$$AP = \sqrt{OP^2 + OA^2}$$

$$= \sqrt{88^2 + 168^2 \text{ in.}}$$

$$= 15 \text{ ft. } 10 \text{ in.}$$

$$DP = \frac{OP}{\sin \cdot 40^9}$$

$$= \frac{88}{0.643}$$

$$= 11 \text{ ft. } 5 \text{ in.}$$

$$CP = \sqrt{OP^2 + OC^2}$$

$$= \sqrt{OP^2 + (OA^2 + AC^2)}$$

$$= \sqrt{OP^2 + (OA^2 + OB^2)}$$

$$= \sqrt{OP^2 + (OA^2 + BP^2)}$$

$$= \sqrt{OA^2 + BP^2}$$

$$= \sqrt{168^2 + 137^2 \text{ in.}}$$

$$= 18 \text{ ft. } 03 \text{ in.}$$

Exercises.

(1) The height of a wall is 20 ft.: required the length of a ladder which will reach to the top, when the foot of the ladder is 5 ft. from the wall. Draw a figure and measure the ladder and its inclination to the ground.

20.6 ft.; 80.5°. Answer.

(2) A flagstaff on a tower is kept in a vertical position by three wire cable stays fastened to the staff at a height of 18 ft, above the

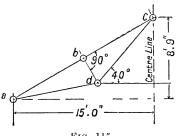


Fig. 115.

level of paraget walls to which the cables are secured. The horizontal distances of the staff from points convenient for securing the cables are respectively 12 ft., 17 ft., and 9 ft. What is the length of each stay?

Answer. 21 ft. 7½ in.; 24 ft. 9 in.; 20 ft. 3 in.

(3) Fig. 115 is a line diagram of one-half of a roof truss which is

symmetrical about the centre line (shown dotted). The small circles indicate the positions of pin joints connecting different bars in the framework. The bars ab, bc are equal in length. Set out this truss to scale and measure the length of each bar from centre to centre of

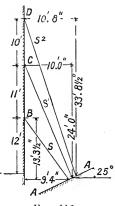


Fig. 116.

the pins. Also calculate the length of each bar, making use of the figured data given on the diagram.

Answer. ab = bc = 8.65ft.; da = dc

= 8.8 ft.; bd = 1.65 ft.

(4) Fig. 116 shows the centre lines of a system of raking shores supporting a wall BCD. The heights of the needles BCD against which the heads of the shores abut are shown by figures, and the position of the foot of each shore on the sole-piece AA is also given by figures.

What length timbers are required for

the three shores S, S1, S2?

Answer. 16 ft. 3 in.; 26 ft. 0 in.; 35 ft. 4 in.

Set out the shores above to scale and measure the bevels to which the head and foot of each shore must be cut to fit the wall and sole-piece AA.

(5) A tower stands on a rock; a man on the sea at 100 yds, from the foot of the rock finds the angle of elevation

of the foot of the tower 15°; he then rows 100 yds. further off and finds the elevation of the top of the tower 15°. Find the heights of the rock and of the tower.

The term **angle of elevation** is applied to the angle through which a horizontal line level with the spectator's eye must be rotated upwards to come into line with the elevated object.

Answer. 26 yds. 2 ft. 3 in.; 53 yds. 1 ft. 6 in.

(6) At the top of a mountain, the angle of depression of the peak of a neighbouring mountain is 5° ; if the difference of the heights of the two peaks be known to be 500 ft., find the distance from the centre of the base of one mountain to that of the other. Scale $\frac{1}{2}$ in. = 500 ft.

The term angle of depression is applied to the angle through which a horizontal line level with the spectator's eye must be rotated downwards to come into line with the object below.

Answer. 5,715 ft.

CHAPTER XVI.

CONSTRUCTION OF TRIANGLES FROM GIVEN DATA.

190. A triangle can in general be drawn to satisfy three geometrical conditions. The given conditions, or data, must comprise three *independent elements*.

Only two angles of a triangle can be chosen arbitrarily, since the three angles added together $= 180^{\circ}$. If we are

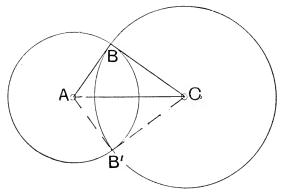


FIG. 117.

asked to draw a triangle ABC having given angle $A=46^\circ$; angle $B=64^\circ$; angle $C=70^\circ$, this problem is possible, because $46^\circ+64^\circ+70^\circ=180^\circ$. But we find we can draw innumerable triangles having angles of the given magnitude, and some other condition is necessary to make the problem definite.

Sometimes the given data is sufficient to fix the size of the figure and the position of one side, and yet not wholly

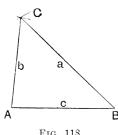


Fig. 118.

fix its position, consequently more than one figure can be found. Fig. 117 is an example of this type.

Again, three independent magnitudes chosen arbitrarily may, and sometimes do, constitute a problem to which two or more solutions can be given. Such problems are called ambiguous. Examples of this type will be found in problems 195 and 199.

We will now consider some

of the conditions which suffice to determine a triangle. We denote the sides of the triangle by a small italic letter corresponding to the capital letter denoting the angle opposite. Thus, a denotes the side BC which is opposite the angle A. And the three sides of a triangle ABC are denoted by the letters a, b, c.

Experiment. Fig. 117. Draw any triangle ABC. With centres A and C describe circles through the corner B. These circles intersect in a second point B'; and evidently AB' = AB, CB' = CB, i.e., the triangles AB'C, ABC are congruent. We learn from this diagram two things, viz., (i.) a triangle is given in magnitude when its three sides are given; (ii.) how to construct a triangle when its three sides are given.

191. Problem. Fig. 118. Construct a △ when given a, b, c, say 4 in., 2.8 in., 3.2 in. Draw one side, say, AB, of given length c = 3.2 in.With centre A, and radius b = 2.8 in., describe an arc. With centre B and radius a = 4 in, describe an arc cutting the arc previously drawn in C. Join CA, CB.

The above problem is always possible if the sum of any two sides is

greater than the third side.

192. Problem. To copy a given triangle in a new position. This may be done as explained in the last problem, but a much better way is to lay a piece of tracing-paper over the triangle, and mark on this tracing-paper the positions of the corners of the triangle; then lay the tracing in the desired position and prick through the corners, and join up the points thus found. Adopt this method for copying any rectilinear figure.

193. If we mar's the positions of the three corners of a triangle ABC on a piece of tracing-paper, as explained in the last problem, and join AB, AC, we see that these two sides completely fix the size of the triangle. From this we conclude that a triangle is given in magnitude if two sides and the included angle are given. It is also evident that one side and two angles suffice to fix the magnitude of a triangle. When two sides and an angle opposite one side are given the triangle is determined, except when the given angle is opposite the smaller of the given sides, in which case two triangles can be drawn.

To illustrate these cases, we show in the following problems

now to construct triangles from given data:-

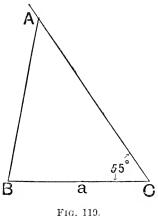


FIG. 113

194. **Problem.** Fig. 119. Given two sides a, b and the included angle C. Let a=3 in., b=4.2 in., $C=55^{\circ}$.

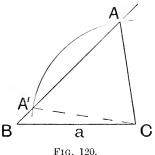
Draw side BC of given length a=3 in. From C draw side CA of given length b=4 2 in., and making C° = 55° with BC. Join BA.

195. **Problem.** Fig. 120. Given two sides a, b and an angle B opposite one side. Let a, b, b = 2·5 in, 2·2 in, 45° respectively.

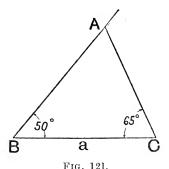
Draw side BC of given length a=2.5 in. From B draw a straight

CONSTRUCTION OF TRIANGLES FROM GIVEN DATA, 241

line making $B^{\circ} = 45^{\circ}$ with CB. With C as centre and radius = b (2.2 in.) describe a circle. In the diagram the circle intersects the line from B in two points; either point may be taken as vertex A; and there are thus two solutions.



If side b be reduced so that the circle centre C only touches the line from B, there will be but one solution. And if b be further reduced, the line from B will have no point in common with the circle, and in this event the problem is impossible.



Observe, also, that when a is less than b the circle of radius bintersects the line from B in one point only on the same side of LC. and there is but one solution.

196. Problem. Fig. 121. Given one side a and the two adjacent angles B, C. Let a, B, C = 3 in., $50^{\circ}, 65^{\circ}$ respectively.

 \mathbf{R}

Draw side BC of given length a = 3 in. From B, C draw lines making the given angles B°, C° with BC intersecting in A.

197. **Problem.** Fig. 122. Given one side a, the opposite angle A and one adjacent angle C. Let a, A, C = 2.5 in., 50° , 70° respectively.

In this case, since the three angles $B + C + A = 180^{\circ}$,

.: angle B =
$$180 - (A + C)$$

= $180 - (50^{\circ} + 70^{\circ})$
= 60° .

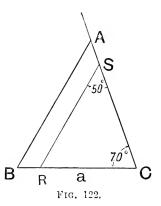
With the angle A known, the problem is reduced to that immediately preceding.

Or, proceed as follows:—

Draw side BC of given length a = 2.5 in. From C draw CS making $C^{\circ} = 70^{\circ}$ with BC. From any point S in this line draw SR making $A^{\circ} = 50^{\circ}$ with CS. Draw BA | RS meeting CS in the point A.

198. Problem. Fig. 123. Given the base a, one base angle C and the height h, Let a, C, h =2.5 in., 42°, 1.7 in. respectively.

Draw base BC of given length a = 2.5 in. From C draw CA making $C = 42^{\circ}$ with BC. Draw a parallel to BC at given distance h = 1.7 in, above same, meeting CA in A. Join AB.



a Fig. 123.

199. Problem. Fig. 124. Given the base a, the angle opposite A, and the height h. Let a, A, and h = 2.5 in., 70° , 1.7 in. respectively.

It is shown in Art. 215 that the locus of the vertex A is a segment of the circle which stands on the given base a and contains the given angle A. It has also a second locus parallel to the base at the given distance h above the base.

These two loci may intersect in two points, in one point, or not at all.

Draw base BC of given length a=2.5 in. Describe a segment of a circle on BC to contain the given angle $A = 70^{\circ}$ by problem 220.

Draw a line parallel to, and at the given distance h = 1.7 in., above, BC.

CONSTRUCTION OF TRIANGLES FROM GIVEN DATA. 243

In the diagram, this line cuts the circle in two points A, A', either of which may be taken as the vertex of the required triangle.

Had the line been tangent to the circle there would have been but

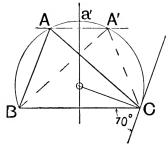


Fig. 124.

one solution, i.e., the triangle a'BC, which is isosceles. Again, if the two loci have no point in common, there is evidently no solution.

200. Experiment. Fig. 125. Draw any straight line and on it take four points A¹, B, C, A². With centre B, and radius BA¹ describe

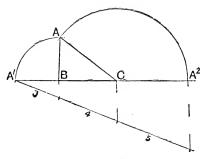


Fig. 125.

a circle. With centre C, and radius CA² describe a circle meeting the circle centre B in A. Join AB, AC. Then the perimeter of the triangle ABC is equal to A¹A². We have thus available a construction for drawing a triangle when given its perimeter and the ratio of its sides.

Problem. Fig. 125. The perimeter of a triangle is 5.5 in.; the sides e, a, b are in the ratio of 3:4:5. Lay off on a straight line the

perimeter A^1A^2 of given length = 5.5 in. Divide A^1A^2 at B and C in the given ratio (see Art. 149), and complete as above.

Exercises.

(1) A triangle has sides respectively 3, 5, and 6 in. long. Construct the triangle, measure each angle in degrees, find the length of the perpendicular falling on the longest side, and calculate the area.

(Ans. Angles 29°, 56°, 95°; perp., 2.5 in.; area, 7.5 sq. in.)

(2) Construct a triangle having base 2 in. long, height $1\frac{3}{4}$ in., and one base angle 75°. Measure and write down the lengths of the other two sides, and calculate the area. Ans. 2.32 in.; 1.82 in.; 1.75 sq. in.

(3) Draw a triangle ABC, base AB = 3 in., AC = 2.5 in., BC = 2.75 in. Measure the height of the triangle and calculate its area; also measure the angle ABC and take out its sine from the tables. Calculate the area of the triangle making use of the formula, area = $\frac{1}{2}$ AB. AC sin. BAC.

 $area = \frac{1}{2} AB \cdot AU \sin \cdot BA$ Find the mean of the two answers.

(4) In any triangle, if A, B, C, be the angles, and a, b, c the opposite sides, then,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Calculate cos. C when a = 6 cm., b = 5 cm., c = 4 cm.

Now verify the formula (and your answer) by construction, measurement and if you like the use of the tables. (B.E. 1910)

measurement, and, if you like, the use of the tables. (B.E. 1910.)

(5) A person, 6 ft. high, walking along a road with the sun directly behind him, observes that at a place where the road is level his shadow measures $7\frac{1}{2}$ ft., and at a place where the road has an upward gradient, his shadow measures $5\frac{1}{4}$ ft. What is the angle which the sun's rays make with the horizontal? What is the inclination of the road? (B.E. 1910.)

(Ans. 38° , 40'; 24° , 32'.)

Exercises to Chap. XVIII.

(1) A steam engine piston is 20 in diameter, and the rod is 2 in diameter. The steam pressure on this side of the piston is 200 lb. per square inch. Find the total pressure on the piston. Ans. 62,203 lb.

(2) The area of a circle is 100 sq. in. What is its diameter? (B.E.

1909.) Ans. 11.3 in.

(3) A circular path has an outer diameter of 50 ft, and an inner diameter of 40 ft. Find the cost of laying it at one shilling per square foot. Find also the cost of fencing it on both sides at five shillings per yard. Ans. Cost of laying the path, £35 6s. 10d. Cost of fencing, £23 11s. 3d.

CHAPTER XVII.

RECTILINEAR FIGURES OF MORE THAN THREE SIDES.

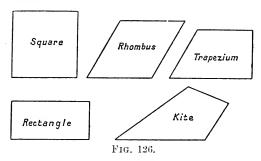
201. Quadrilaterals. Any figure enclosed by four straight lines is called a quadrilateral.

A quadrilateral in which two sides only are parallel is called a **trapezium**.

A quadrilateral having pairs of adjacent sides equal is called a kite.

A quadrilateral having its opposite sides parallel is a parallelogram.

It is evident from Expt. Art. 110, that the opposite angles



in a parallelogram are equal, and any two consecutive angles are supplementary. Hence, if one angle in a parallelogram is a right-angle, then every angle is a right-angle.

If the angles in a quadrilateral are right-angles, the figure

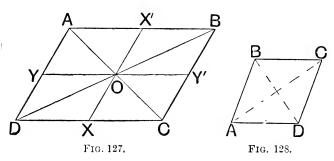
is a rectangle.

If all sides of a rectangle are equal, the figure is a **square**. A parallelogram having all sides equal, but not rectangular, is called a **rhombus**.

Some Properties of Parallelograms.

Experiment. Fig. 127. Draw any parallelogram ABCD. Draw a diagonal DB. Then, because AB = CD and AD = CB, and the angles at A, C are equal, the triangles DAB, BCD are congruent.

This may be verified with tracing-paper. Draw the diagonal AC, and satisfy yourself that this line also divides the parallelogram into two equal parts. The two diagonals meet in O. Verify also that the two diagonals are bisected in O, and that the triangles AOB COD are congruent; also AOD, COB are congruent. Next, through O, draw XX', YY', respectively, parallel to AD, AB. Show that XX', YY', divide ABCD into four parallelograms exactly alike. Also verify the following statements: The triangles YAO, Y'CO, X'OA, XOC are congruent, also YDO, Y'BO, XOD, X'OB are congruent.



Draw any *rhombus*, a rectangle, a square, and apply the tests described above. Also verify—

- (1) In a *rhombus*, and a square, the diagonals are at right-angles and each diagonal is an axis of symmetry.
 - (2) A circle can be inscribed in a rhombus.
- (3) The four corners of a rectangle are on the circumference of a circle which has its centre in the meeting point of diagonals, *i.e.*, a rectangle can be inscribed in a circle.
- (4) A square can be inscribed and circumscribed by circles centered in the meeting point of diagonals.
- 202. The area of a rhombus may be found, if the lengths of the diagonals are known.

In a rhombus, the diagonals bisect each other at right-angles. See Fig. 128.

Hence the area of rhombus ABCD = sum of areas of triangles ABD and CBD.

Area of triangle ABD = $\frac{1}{2}$ BD × AE (since AE is $\underline{\Gamma}$ to BD). Similarly, area of triangle BCD = $\frac{1}{2}$ BD × CE.

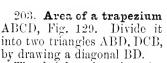
Hence area of rhombus =
$$(\frac{1}{2}BD \times AE) + (\frac{1}{4}BD \times CE)$$

= $\frac{1}{2}BD (AE + CE)$
= $\frac{1}{2}BD \times AC$

or, area of rhombus = $\frac{1}{2}$ product of diagonals.

Exercise. The diagonals of a parallelogram are respectively 4 and 3 in. long, and one side is $1\frac{3}{4}$ in. long. Construct the figure and find its area,

Ans. 5 sq. in.



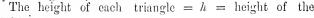


Fig. 129.

trapezium.

Area of triangle DCB =
$$\frac{1}{2}$$
 DC × h .
Area of triangle ABD = $\frac{1}{2}$ AB × h .
Then total area of trapezium = $(\frac{1}{2}$ DC × h) + $(\frac{1}{2}$ AB × h).
= $\frac{1}{2}$ (DC + AB) h .
= $\frac{1}{2}$ sum of parallel sides × perpendicular distance between them.

Ex. In a trapezium, the parallel sides are 10 and 8 ft. long respectively, and they are 5 ft. apart. Find its area,

Area = $\frac{1}{2}$ sum of parallel sides × perpendicular distance between

them.
=
$$\frac{1}{2} (10 + 8) \times 5$$
.
= 9×5 .
= 45 sq. ft.

204. Consider the regular polygon ABCDEFG, Fig. 130. If we join the angular points to the centre of the figure, these lines divide the angle at the centre O into a number of equal parts. Consequently, the angle between the lines radiating to consecutive corners of the polygon is equal

to $\frac{360}{n}$, where n = the number of sides. In our example,

the angle AOB = $\frac{360}{7}$ = 51.428°.

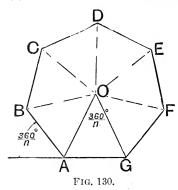
Now we know from Art. 182 that the exterior angle of any regular polygon of n sides is equal to $\frac{360}{n}$, and therefore the angle subtended at the centre by the side of 3

regular polygon is equal to each of the exterior angles

of the polygon.

205. The area of any polygon is found by dividing the figure into a number of triangles, finding the area of each triangle separately, and adding all the areas together.

In the case of a regular polygon of n sides, we can find the centre of the figure and join to each corner. We have then a number (n) of triangles all equal in area, and the area of the polygon = the area of one triangle $\times n$.



Ex. To find the area of the regular polygon ABCDEFG, Fig. 139. Determine (O) the centre of the polygon. Join OB, OA, etc.

Then the area of polygon = $7 \times \text{area of } \triangle \text{ OAB}$.

The area of a regular hexagon can be shown to be equal to $2.598 \times (\text{length of side})^2$.

Examples.

The area to be distempered is the area of the four walls less the

rea of door and window.

⁽¹⁾ A room is 15 ft. long, 12 ft. wide, and 9 ft. high. It has a door 6 ft. high and 3 ft. 6 in. wide, and a window 8 ft. long and 4 ft. 6 in. high. Find the cost of distempering the walls at 1d. per square foot; also cost of papering the walls with paper 27 in. wide at 1s. 6d. per dozen yards. Find also the length of earpet 2ft. wide, which would - be necessary to cover the floor.

Area of four walls =
$$2\{(15 \times 9) + (12 \times 9)\}$$
 = 486 sq. ft.
Area of door = $(6 \times 3.5) = 21 \text{ rq. ft.}$
Area of window = $(8 \times 4.5) = 36 \text{ sq. ft.}$
Area to be distempered = $\{486 - (21 + 36)\} = 429 \text{ sq. ft.}$
Cost at 1*d.* per sq. = 429 pence
= 429 pence
= 429 pence
= 429 pence
= 429 pence

The area to be papered is, as the area to be distempered, 429 sq. ft. To find the number of yards of paper 27 in, wide required, we note that 1 yd. of such paper covers an area 3 ft. by $2\frac{1}{4}$ ft. or $6\frac{3}{4}$ sq. ft.

Hence number of yards required is
$$\frac{429}{6\frac{3}{4}} = \frac{429 \times 4}{27}$$

The number of "dozens" of yards is $\frac{429 \times 4}{27 \times 12}$

The cost at 1s. 6d. per dozen =
$$\frac{429 \times 4}{27 \times 12} \times \frac{3}{2}$$
 shillings $\frac{143}{18} = 8$ shillings (approximately).

Area of floor to be covered by earpet is $(15 \times 12) = 180$ sq. ft. Carpet is 2 ft. wide.

One yard length of carpet thus covers $2 \times 3 = 6$ sq. ft. of floor.

Hence number of yards required is $\frac{180}{6} = 30$ yds.

(2) A eistern 10 ft. long, 6 ft. wide, and 3 ft. deep, is to be lined internally with lead which weighs 5 lb. per square foot. Find the weight of lead necessary.

Area of two long sides of eistern $(2 \times 10 \times 3) = 60$ sq. ft. ", ", short ", ", $(2 \times 6 \times 3) = 36$ sq. ft. Area of bottom of eistern $(10 \times 6) = 60$ sq. ft. Total area = (60 + 36 + 60) = 156 sq. ft. Weight $= 156 \times 5$ lb. = 780 lb.

(3) A hexagonal plate of 5-ft. side has a rectangular hole 2 ft. by 3 ft. punched from it. Find the area of the remainder of the plate.

Area of plate = area of hexagon - area of rectangle,
=
$$(2.598 \times 5 \times 5) - (2 \times 3)$$

= $64.95 - 6$
= 58.95 sq. ft.

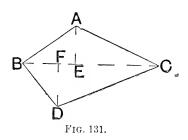
(4) A parement 1 mile long and 8 ft, wide is to be pared with tiles 9 in, by 4 in. How many are required?

Total area of pavement =
$$(5280 \times 8)$$
 sq. ft.
= $(5280 \times 8 \times 114)$ sq. in.
Area of each tile = $(5280 \times 8 \times 114)$ sq. in.
= $(9 \times 4) = 36$ sq. in.
Sumber required = $(5280 \times 8) \times 114$ sq. in.

206. Area of any quadrilateral ABCD, Fig. 131. Divide it into two triangles by drawing the diagonal BC, and draw from the corners A, D the perpendiculars AE, DF. Then

Area of triangle ABC = $\frac{1}{2}$ BC × AE ,, ,, BCD = $\frac{1}{2}$ BC × FD Total area of quadrilateral = $(\frac{1}{2}$ BC × AE) + $\frac{1}{2}$ (BC × FD) = $\frac{1}{2}$ BC (AE + FD)

 $=\frac{1}{2}$ diagonal \times sum of perpendiculars from opposite corners upon the diagonal.



Ex. In a quadrilateral, the diagonal is 12 ft. long, and the lengths of perpendiculars from the opposite corners of the parallelogram upon this diagonal are 5 ft. and 8 ft. respectively. Find the area,

Area = $\frac{1}{2}$ diagonal × sum of perpendiculars = $\frac{1}{2} \times 12 \times (5 + 8)$ = 6×13 = 78 sq. ft.

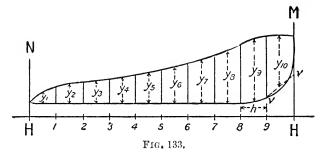
207. To find approximately the area of a plane figure bounded by any curve.*

As an example, take Fig. 133, which is a reproduction of an actual indicator diagram for a steam engine.

Draw two parallel lines HN, HM tonching the curve, and draw the line IIH perpendicular to HN and HM. Divide IIII into a convenient number of equal parts (in the diagram 10 divisions are shown). Through these points of division

draw lines parallel to IIN, thus dividing the area into 10 strips each of width $h = \frac{1}{10}$ IIII. (The intercepts of these parallels are conveniently referred to as *ordinates* of the figure.)

Now the area of a strip of the figure between two consecutive ordinates is approximately equal to the distance between the ordinates multiplied by the length of an ordinate placed midway between them; thus, the area of the strip between the ordinates 8 and 9 is approximately $h \times y_9$. Hence, if y_1, y_2, y_3, \ldots are the mid-ordinates of the various strips, the total area is equal to $\{(y_1 \times h) + (y_2 \times h) + (y_3 \times h) + \text{etc.}\}$. And since, in this example, $h = \frac{1}{10}$ IIII,



we have, total area = $\frac{1}{10}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10})$ IIII. The area is thus equal to a rectangle on base IIII and having height equal to the average height of the mid-ordinates of all the strips.

Where the curvature is sharp, as in the strips at the ends of Fig. 133 we can replace the curves by a straight line which will, to the best of our judgment, cut from, and add to, the strip portions equal in area. Such a line is shown in vv; it is called an **equalising line**. The area of the trapezoid is then taken as equivalent to the actual strip, and the mid-ordinate terminates on the line vv as shown.

208. The weights of two or more sheets of cardboard, zinc, or other material of uniform thickness, are, provided the composition of the material is also uniform throughout, proportional to the size of the sheets. This fact may be

utilized to determine approximately the area of any plane figure. Cut the figure out of cardboard, sheet-zinc or other available material, and compare its weight with the weight of a known unit of area of the same material.

Exercises.

NOTE. The diagrams used in the following exercises should be re-drawn to twice their lineal dimensions.

(1) What measurements would you make in order to find the area of a quadrilateral and a trapezium? The side of a hexagon is 5 in.

Find its area. Ans. 65 sq. in.
(2) ABCDE are the corners of an irregular pentagonal piece of ground. The side AB is 200 chains long, BC 300, CD 400, DE 450, and EA 200, whilst the angles ABC, BCD, CDE DEA, and EAB are 90°, 100°, 120°, 110° and 120° respectively. Make a drawing of the ground, using a scale of 1 in. to 100 chains.

(3) The plan of a hall is given (Q. 3), scale \frac{1}{2} in. to 10 ft.; L is the platform and K the body of the hall. Find the area of K in square feet. Calculate the seating accommodation of K, 30 per cent. of its area being occupied by passages, and allowing one person to

every 4 sq. ft. of the remainder. (B.E. 1904.)

(4) A trench is dug of the shape and size shown in Q. 4. The scale of the figure being 1 cm. to 1 ft. Find the cross-sectional area

of the trench in square fect.

The material expands 10 per cent. in bulk, and is piled alongside the trench in an embankment of triangular section with its sides sloping at 45°. Draw this embankment to scale. State its height. (B.E. 1905.)

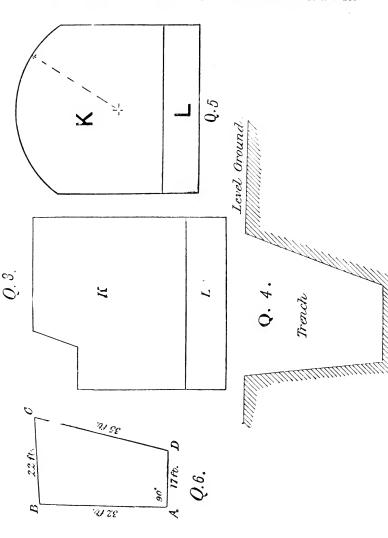
(5) Substitute Q. 5 for Q. 3 in Exercise 3, then solve the

problem.

- (6) The diagram Q. 6 gives the dimensions of a plot of level land. Draw this figure to a scale of $\frac{1}{8}$ in. to 1 ft. Measure the lengths of the two diagonals in feet. Determine the area of the plot in square yards. (B.E. 1910.)
- (7) The sides of a pentagon ABCDE have the following lengths: $AB = 1\frac{1}{2}$ in., $BC = 1\frac{1}{4}$ in., $CD = 1\frac{5}{8}$ in., $DE = 1\frac{3}{4}$ in., EA = 2 in., and the diagonals AU and CE are 21 in. and 21 in respectively. Construct the figure, and write down the length of the diagonal BD.
- (8) Make a tracing of Fig. 133, and find the area of the diagram in square inches. What is the average height? If the indicator spring used for this diagram requires 80 lb. to compress it so that the marking pencil moves one inch, what is the mean pressure throughout the stroke? If the stroke HH of the engine from which this diagram is taken is 18 in., and the cylinder is 10 in. diameter, what is the horse-power when the engine makes 150 revolutions per minute? (Refer to the example on p. 183.)

Ans. Area, 1.212 sq. in.; height, 0.373 in.; pressure, 29.84 lb.;

h.-p., 31.94.



CHAPTER XVIII.

THE CIRCLE.

CHORD AND TANGENT PROPERTIES AND CONSTRUCTIONS.

209. In this chapter, our attention will be confined to circles, and problems directly connected with circles.

The perpendicular bisector vv of any chord AB of the circle, Fig. 134, is the locus of all points equidistant from Λ

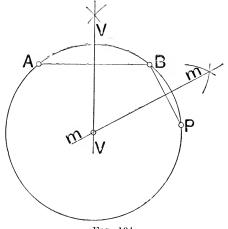


Fig. 131.

and B (Art. 114); and consequently passes through the centre of every circle which contains AB. This property provides a solution of the following:—

210. **Problem**. Determine the centre of a circle to pass through any three given non-colinear points A, B, P, Fig. 134.

Join AB, BP and draw the perpendicular bisectors vv, mm of the two lines. Then the line vv is the locus of all points equidistant from A and B; and the line mm is the locus of points equidistant from B and P. Therefore the point in which the lines vv and mm intersect, is a point equidistant from the three points Λ , B, P, and is the centre of the required circle.

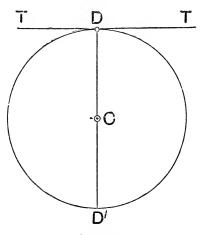


Fig. 135.

211. Fig. 135 shows a circle centre C, a diameter DD', and a line TT which passes through D and is perpendicular to CD. The line TT is said to be **tangent** to the circle at D; D is called the **point of contact**. DC, the perpendicular from the point of contact D, is called a **normal**.

All normals of a circle pass through the centre of the circle. This property is characteristic of the circle, and thus, the tangent at any point D on the circumference of a circle is perpendicula, to the radius DC.

- 212. **Experiment.** Fig. 136. Describe any circle and draw any chord as xx. Draw cv a perpendicular on to the chord xx from c. With c as centre and radius cv describe a circle. Make a tracing of the chord xx and the line cv. Insert a pricker at c and rotate the tracing about this point. Observe—
- (a) Both ends of the line xx fall always on the circumference of the larger circle, or, in other words, xx is always a chord of the large circle.

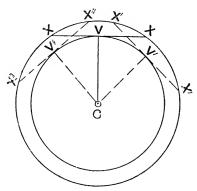


Fig. 136.

(b) xx is always tangent to the smaller circle at the point v. Hence—

Theorem. Equal chords in a circle are equidistant from

the centre of the circle.

213. On a straight line, Fig. 137, mark points A and B 4 in. apart. With A as centre, and radius R equal to $2\frac{1}{2}$ in., describe the circle S. With B as centre, and radius r equal to 2 in., describe a circle Z. The circles S, Z intersect in two points L, M, and two only. Join LM; then LM is a chord of both circles and perpendicular to AB, Art. 210. Hence, the common chord of two circles is perpendicular to the line joining their centres, and is bisected in that line: also, if more than two circles have a common chord, all the centres lie

on the same straight line. If the radius r of the circle, centre B, be increased repeatedly, the common chord will get longer and longer until it coincides with the diameter DD of the circle S; but after that position is reached, the chord gets shorter and shorter until it coincides with the point Q. Again, if the original value of r be reduced, the common chord gets shorter and shorter until it coincides with the point o. There is thus a maximum and a minimum value

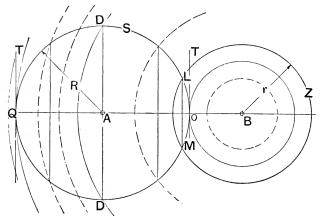


Fig. 137.

of r between which limits the two circles always intersect in two finite points, and two only.

Maximum r = AB + RMinimum r = AB - R

In each limiting case, the two circles have a common point represented in the former case by Q; and in the latter by o. These common points lie in the respective tangents QT, oT. Hence, where r is a maximum, QT is tangent to both circles at the point Q. Similarly, where r is a minimum, oT is tangent to both circles at the point o.

P.M.

214. Parallel circles. Parallel lines have been defined (Art. 106) as lines which have the same direction.

A circle, or indeed any curve, may be defined as the locus

of a point which continually changes its direction.

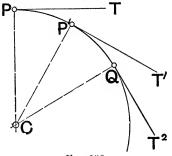
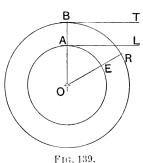


Fig. 138.

Let a point P, Fig. 138, turn about a fixed point C as centre into a new position Q, generating the circular arc PQ. The moving point P is constrained to remain at a fixed



distance from the fixed point C, and therefore at any instant it can move in one direction only, viz., at right-angles to the Thus, when the generant is at P its direction is along the tangent PT; when at P^1 , its direction is along P^1T^1 ; and when at Q, along QT².

(It will be understood that the moving point P occupies any particular position in its path, as P1, for a period of time infinitely small, and therefore

the tangent P1T1 indicates the direction of the moving point only for an infinitely small period at the particular instant when the point occupies the position P1.)

Consider two concentric circles AE, BR, centre O, Draw OB which is normal to both curves. Fig. 139.

Imagine the circles as generated by points A and B rotating about the fixed point O; then the corresponding points A and B have the same direction, viz., along the parallel tangents AL, BT. Now, let OR be drawn from the common centre O, terminating at R on the larger circle, and crossing the smaller circle in E. Then OR is normal to the two circles at the respective points R and E, and consequently the two circles have the same direction at those points, from which we conclude that circles which have a common centre, have (a) always the same direction, and are parallel curves; (b) either circle is the locus of points in the plane of the circles at a fixed distance from the other; inside or outside as the case may be.

Angle Properties of Circles.

215. Experiment. Fig. 140. Draw any circle centre C and any chord PQ. Take any point B on the smaller are and join to the extremities of the chord PQ. Let PB, QB respectively extend to

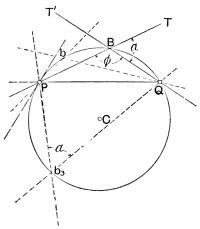


FIG. 140.

T, T'. Denote the angle PBQ by ϕ , and the adjacent angle by α . Now make a tracing of the chord PQ and the lines PBT, QBT'; move the tracing and place B in a new position b on the arc PQ; insert a

pricker at the point b to secure the tracing at that point, and rotate the tracing so as to make the line T'BQ pass through the fixed point Q on the drawing; then observe, that TBP on the tracing passes through the fixed point P. Repeat this operation, taking different positions on the arc PQ, and observe that no matter what point on the arc be taken, the angle formed by lines drawn from the point to the extremities of the fixed chord PQ is always the same and equal to ϕ .

Suppose the tracing to move with the line T'BQ always passing through Q, and TBP always passing through P. As B moves from its initial position towards the fixed point P, it traces out the circular arc BP. When the moving point arrives at P, the line BQ on the

tracing falls along the fixed chord PQ.

Let B continue moving beyond P; then the segment BT of the moving line PBT will be passing always through the fixed point P, and at the instant when the moving point is at b_3 , the angle Pb_3Q is equal to the angle PPQ, i.e., $\alpha = 180 - \phi$. Apply the tracing in several positions, placing the point B on the tracing always on the larger circular are PQ, the line T'BQ always passing through Q: and observe that BT passes always through the fixed point P. You have thus verified the following important theorems.

Theorem 1. The angles in the same segment of a circle are constant.

Theorem 2. The opposite angles of any quadrilateral

inscribed in a circle are supplementary.

It will be observed that the smaller segment contains the larger angle, and therefore if the two segments are equal the included angles are equal, each being a right-angle. Now, when the chord divides a circle into two equal segments, that chord is a diameter of the circle. Hence,

Theorem 3. The angle in a semicircle is a right-angle.

Theorem 4. The angle in a segment greater than a semicircle is less than a right-angle.

Theorem 5. The angle in a segment less than a semi-

circle is greater than a right-angle.

When the tracing in Fig. 140 occupies its initial position with the point B over the point B on the drawing, PB, BQ are chords of the circle, PB being the longer of the two. As the point B moves towards P, the chord BQ increases, and the chord BP decreases continually, until when B arrives at P, BQ lies along PQ; and the line PBT on the tracing is tangent at P. This experiment verifies an

important **Theorem.** If at an extremity of a chord of a circle a tangent of the circle be drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments. Thus, the small angle a is the angle in the larger segment Pb_sQ , and the larger angle ϕ is the angle in the smaller segment PBQ.

216. **Theorem.** The angle which an arc of a circle subtends at the centre is double the angle it subtends at any point on the circumference. For, let β and ϕ , Fig. 141, be

the angles subtended by the arc PBQ at the centre and circumference respectively. Then the triangle CPQ is isosceles and the angles marked $\underline{\alpha}$ are equal.

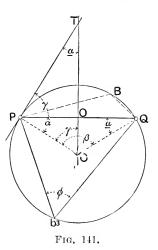
Let the perpendicular from C on the chord PQ meet PQ in O, and the tangent from P in T. Then the right-angled triangles CPO, CTP, PTO are similar, and the angles which are similarly marked in the diagram are equal.

The angle $TPQ = \phi$ Art. 215

 \therefore The angle $PCO = \phi$

but ", ", PCO =
$$\frac{1}{2}\beta$$

 \therefore ", $\beta = 2\phi$.



217. Fig. 142 shows a mechanical device used to draw a circular arc by continuous motion. PQ is the clear width between the supports of a segmental arch; B is the highest point in the curve measured above the chord line PQ; and P, Q are points in the curve. The point B is equidistant from P and Q. We desire to draw the circular arc through the three points P, B, Q.

Let a light wooden triangle be made having two straight edges, and shaped to the angle PBQ. Round nails, or other suitable guiding-pins, are driven into the drawing-board at points P, Q. The triangle is placed in position

against these pins and given a sliding motion, the sides of the triangle being kept constantly against the pins; a pencil is held close to the apex of the moving tool, and thus traces out the required circular arc through P B Q.

An important line in arch problems is the *normal* at the extremity of the arc; this is obtained by placing a square on the triangle when it occupies the limiting position shown in the diagram.

The curve we have just dealt with forms the soffit line

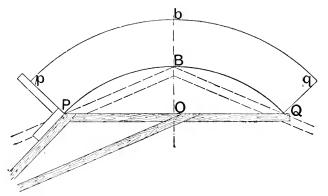


Fig. 142.

of the arch, and in general another curve parallel to this one has to be drawn to form the outer curve or extrados of the arch.

The triangle already used will serve to draw the outer curve, or, indeed, any number of arcs parallel to the first. For, consider the parallel arc pbq; then the sectors CPQ, Cpq are similar, and the angles PBQ, pbq are equal.*

The distance between the supports indicated in Fig. 142 by PQ is called the **span of the arch**; the height of the mid-point B above the chord line PQ is called the **rise**.

218. In actual work, the span and rise of the arch are nearly always given by the designer. The draughts-

^{*} The centre C is not shown in the diagram.

man, when preparing a full-size drawing, prefers to calculate the length of the radius from these known values. A formula for this purpose may be deduced as follows:—

Fig. 143. Let
$$S = PO = \frac{1}{2}$$
 the span = 1 ft. 9 in.
, $V = OB =$ the rise = 9 in.
and $R = BC =$ the radius.

Let the perpendicular from C on to PB meet same in E.

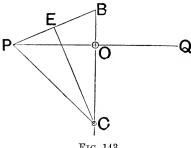


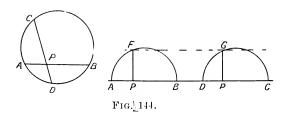
Fig. 143.

Then the triangles POB, CEB are similar and right-angled at O and E respectively.

$$\begin{array}{cccc} \text{Whence} & \text{OB}: \text{BP}:: \text{EB}: \text{BC} \\ & \text{BC} &= \frac{\text{BP} \cdot \text{EB}}{\text{OB}} \\ \text{But} & \text{EB} &= \frac{1}{2} \text{PB} : \\ & \text{hence,} & \text{BC} &= \frac{\text{PB}^2}{2\text{OB}} = \frac{\text{OP}^2 + \text{OB}^2}{2\text{OB}} \\ & \therefore & R &= \frac{S^2 + V^2}{2 \cdot V}. \\ \text{By substitution,} & R &= \frac{21^2 + 9^2}{2 \cdot \sqrt{9}} = 2 \text{ ft. 5 in.} \end{array}$$

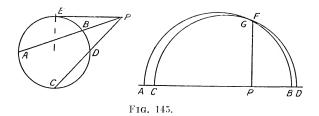
219. **Theorem.** If from any fixed point two straight lines be drawn intersecting the circle in two points, the product of the segments of one line shall equal the product of the segments of the other, and equal the square on the tangent from the point when the point is without the circle.

(1) Let the fixed point P, Fig. 144, be within the circle, and let the two lines be the chords AB, CD.



By Problem 151, find PF a mean proportional to AP and PB, and similarly find PG a mean proportional to CP and PD. Then $AP.PB = PF^2$ and $CP.PD = PG^2$, and by measurement it can be seen that PF = PG.

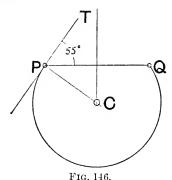
(2) Let the fixed point P, Fig. 145, be outside the circle, and let the two lines be PBA and PDC so that the segments are now AP, PB and CP, PD.



By Problem 151, find PF a mean proportional to AP and PB, and also PG a mean proportional to CP and PD. Then by measurement show that PF = PG = PE, when PE is a tangent to the circle from the point P.

220. Problem. Fig. 146. On a given line PQ as chord, construct a segment of a circle which shall contain a given angle, say 55°.
At one end of the chord as P, draw PT making the given angle 55°

with QP. Then PT touches the required circle at P, and its centre lies on the perpendicular from P. The centre also lies on the perpendicular bisector of PQ (Art. 210). It is therefore at C, where the two loci intersect.



221. Area of circle expressed in terms of its radius. In Fig. 147 (a), the circle centre O is divided into 12 equal sectors.

In Fig. 147 (b), these sectors are arranged in a form somewhat resembling a parallelogram. Now, suppose the circle

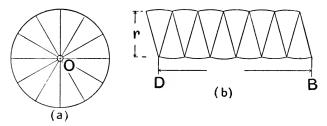


FIG. 147.

divided into 24 equal parts, separated as before described and arranged as in Fig. 147(b). The various circular arcs, being now more numerous, consequently approach more nearly to a straight line. If this process of division is continued

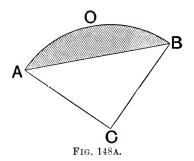
until the number of sectors becomes greater than any assignable number, the arc of each sector will not differ from a straight line by an assignable magnitude, and the sectors, if arranged as in Fig. 147 (b), would form a rectangle having for length, DB = semi-circumference of circle; and for breadth, r = radius of circle. Hence

Area of circle =
$$r \times \pi r$$
.
= πr^2

Ex. What would be the cost of paving a circular courtyard the diameter of which is 33 yds., at 5s. per square yard?

Area of yard =
$$\pi r^2$$
 sq. yds.
= $3 \cdot 1416 \times (16 \cdot 5)^2$
= $3 \cdot 1416 \times 272 \cdot 25 = 855 \cdot 3$ sq. yds.
... Cost = $\pounds \frac{855 \cdot 3}{4} = \pounds 213 \cdot 148 \cdot 6d$.

222. Area of a sector of a circle. If a circle be divided into two or more sectors, the areas of the sectors will be proportional to the lengths of their arcs; and, since the



angles at the centre of a circle are proportional to the arcs on which they stand, it follows that the area of a sector of a circle is proportional to the angle of the sector, whether expressed in degrees or radians. (An example is worked out on p. 166.)

The area of a segment AOB, Fig. 148A, of a circle is equal to the difference between the area of the sector AOBC and the area of triangle ABC.

223. To find the area of an annulus, or the space included between two concentric circles. (Fig. 148B.)

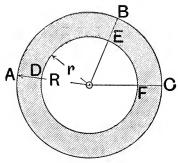


FIG. 148B

Let R = radius of the outer circle ABC.

r = ,, ,,inner , DEF.

A = the area of the annulus.

Then, $A = \pi R^2 - \pi r^2$ (or area of larger circle – area of smaller circle.

 $= \pi (R^2 - r^2)$

 $=\pi(R+r)(R-r)$

 $= \pi \times \text{(sum of radii)} \times \text{(difference of radii)}.$

Ex. Surrounding the circular base of a monument 24 ft. in diameter is a path 12 ft. wide. What would be the cost of constructing this path if each yard super, cost 12s, 6d,?

Making use of above formula

A =
$$\pi$$
(R + r)(R - r)
= 3·1416 × (24 + 12) × (24 - 12) sq. ft.
= $\frac{3·1416 \times 36 \times 12}{9}$ sq. yds. = 150·8 sq. yds.

As each yard super costs 12s. 6d., or £5, the cost of path $= £(150.8 \times 5) = £94.58.0d.$

224. The sectorial area of the annulus lying between the parallel arcs BC, EF may be shown to be equal to

$$\frac{(C+c)h}{2}$$
, where C = length of outer arc BC, $c = \text{length}$

of inner arc EF, and h = distance DA between them.

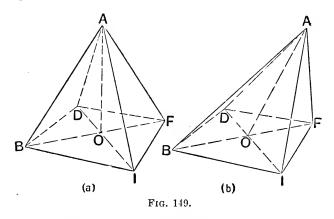
Note. - Exercises for this Chapter are on p. 244.

CHAPTER XIX.

SURFACES AND VOLUMES OF SOLIDS.

225. Any geometrical figure that has length, breadth, and thickness is called a solid.

In Fig. 149, we have a sketch of a figure bounded entirely by planes, viz., the square DBFI, and the triangles ABD, ADF, AFI, AIB. This figure is called a *pyramid*. The square is called the *base*. The triangles which meet in A



are called *sides* or *faces*. The boundaries of the faces are *edges*. The point (A) in which all the sloping edges intersect is called the *apex*. The line AO joining the apex to the centre of the base is called the *axis*.

A pyramid may have any polygon for base. If the base is a triangle, it is called a *triangular pyramid*; if a square, it is called a *square* pyramid; if a pentagon, it is called a *pentagonal* pyramid, and so on.

A pyramid is said to be **right**, or **oblique**, according as its axis is, or is not, perpendicular to the base. See Fig. 149, (a) and (b).

When the base of a right pyramid is a regular polygon the figure is called a regular pyramid; and all the sloping

faces are congruent triangles.

226. If we regard a circle as a regular polygon of innumerable sides which have no finite length and imagine lines drawn from the apex A, in Fig. 150, to points on the circumference of the circle centre O, and close together, we

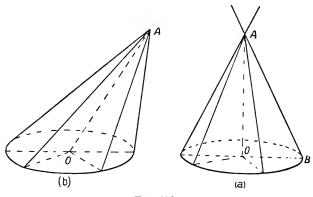


FIG. 150.

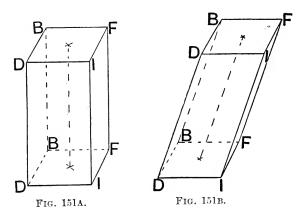
obtain a figure having a sloping surface which is not made up of a number of triangles, but which is one regular curved surface upon which no straight lines can be drawn except from the apex to points on the circular base. This figure is called a *cone*.

A cone is said to be *right* or *oblique* according as its axis is, or is not, perpendicular to the base. See Fig. 150, (a) and (b).

Another conception of the right cone. Fig. 150 (a). Let one side OA of a set-square occupy a fixed position, and let the set-square revolve on this line as an axis through a complete turn. Then the side OB describes a circle in a

plane perpendicular to the line OA. The locus of B is the circumference of the circle, and the hypotenuse BA generates the surface of the cone, which has OA for axis and the circle centre O, radius OB, for base.

227. If parallel lines of equal length be drawn from each corner of the square DBFI, Fig. 151, and the extremities of these lines be joined in proper order, we obtain a figure bounded entirely by planes, viz., the two parallel squares DBFI, DBFI, and four parallelograms DBBD; BFFB; FIIF; IDDI. This figure is called a prism. The squares



are called the *ends*, and either end may be called the *base* of the prism. The parallelograms are called *sides* or *faces*. The line joining the centres of the two ends is called the *axis*.

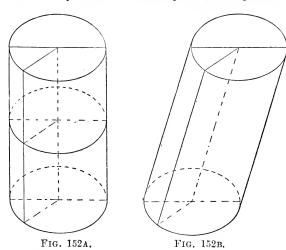
A prism is said to be *right* or *oblique* according as its axis is, or is not, perpendicular to the base. Fig. 151A shows a *right* prism; Fig. 151B shows an *oblique prism*.

In a right prism, all the faces or sides are rectangles.

Prisms are named according to the shape of their base or end, *i.e.*, triangular, square, pentagonal, or hexagonal, according as their base is a triangle, square, pentagon, or hexagon, etc.

A cube is a solid bounded by 6 square faces.

228. If the sides of the polygonal base of a prism be continually increased in number, and reduced in length, then finally the plane faces disappear and are replaced by a regular curved surface on which there are no straight lines except those joining corresponding points on the periphery of the two ends of the solid, *i.e.*, parallel to the axis of the prism. See Fig. 152. This figure is called a *cylinder*. When the base is a circle and the axis is perpendicular to the base, the cylinder is called a *right circular cylinder*.



229. Superficial areas of solid figures.

This part of our subject is illustrated by showing how the various solids may be built up of stout paper. The student should copy the diagrams given in the book to a larger scale; cut the whole diagram from the paper with a sharp knife and cut half through along the dotted lines about which the paper is folded to form the model. The narrow strips attached are intended to wrap around the edges and to be gummed or pasted to secure the correct form of the model.

Apart from the immediate purpose the making of these

models will serve, they will be of great assistance to the student and save him much mental labour when engaged on any problem connected with the particular solids.

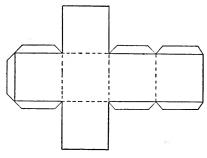
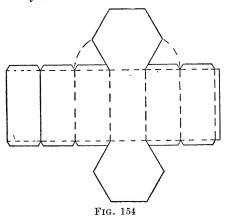


Fig. 153.

The superficial area of a solid is equal to the total area of the separate faces.



230. The cube. The surface of a cube consists of 6 squares. The diagram, Fig. 153, if cut half through along the dotted lines and folded, will make a cube.

If a in, is the length of the edge of the cube, then the area of each face = a^2 in. And hence

the superficial area of cube = $6a^2$ sq. in.

231. Right prism. The superficial area of any right prism is equal to the area of the two ends added to the rectangles which form the faces or sides of the prism. There are as many rectangular faces as there are sides to the polygonal base.

The diagram, Fig. 154, if cut half through along the

dotted line, and folded, will make a hexagonal prism.

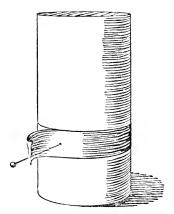


Fig. 155.

232. Right cylinder. The superficial area is equal to the area of the two circles forming the ends added to that

of the curved surface.

Suppose we have a short length of a round iron bar or other cylindrical object, placed on end, as in Fig. 155. Wrap a strip of paper tightly round the object and insert a pricker where the paper overlaps. When this paper is unfolded, the distance between the two pin-holes is the circumference of the circle which forms the base of the cylinder. If we take a strip of paper of width equal to the height of the cylinder and length equal to the distance

between the two pin-holes in the diagram, we get a rectangle having sides respectively equal to the height of the cylinder, and the circumference of the circular base. This rectangle, if applied to the curved surface of the cylinder, will be found to cover it exactly. Hence for a cylinder of diameter d in., and height h in.,

Area of each end
$$= \frac{\pi}{4}d^2 \text{ sq. in.}$$
Area of curved surface = circumference of cylinder × height = $\pi d \cdot h \cdot \text{sq. in.}$

$$\therefore \text{ Superficial area} = \frac{\pi d^2}{4} + \frac{\pi d^2}{4} + \pi dh$$

$$= \pi d \left(\frac{d}{2} + h\right) \text{ sq. in.}$$

Ex. Find the cost of painting the exterior of a closed cylindrical tank which is 10 ft, in diameter, and 15 ft high at 13d, per sy ft.

Area of each end
$$= \left(\frac{\pi}{4} \times 10^2\right) \text{ sq. ft.}$$
Area of curved surface
$$= (\pi \times 10 \times 15) \text{ sq. ft.}$$
Superficial area
$$= 2\left(\frac{\pi}{4} \times 100\right) + (\pi + 150)$$

$$= 628 \text{ sq. ft.}$$

Or, using above formula:-

Superficial area =
$$\pi d \left(\frac{d}{2} + h \right) = \pi \times 10 (5 + 15) = 628 \text{ sq. ft.}$$

$$\text{Cost} = \frac{1.5 \times 628}{12 \times 20} = £3 18s. 6d.$$

233. Micrometer screw caliper. Fig. 156 shows a common form of micrometer gauge used in making exact measurements of the thickness of metal plates, the diameter of wire, and other objects. The spindle *e* is attached to the thimble E, and is threaded to fit a screw on the inside of the hollow cylindrical sleeve D, which forms part of the frame A. As the thimble is turned between the thumb and finger, the spindle revolves with it, moving through the threaded sleeve approaching or receding from the anvil B, according to the direction in which the thimble is being revolved.

In order to allow of the distance between the anvil and the end of the spindle being read off with facility, scales are engraved on the sleeve D of the frame and on the bevelled edge of the thimble. These scales are not the same on all micrometers. Some tools are graduated to measure millimetres and fractions of the millimetre. The tool shown in Fig. 156 is made to measure thousandths of an inch. The pitch of the screw-threads on the spindle is $\frac{1}{40}$ in. One complete

revolution of the spindle therefore moves it along the sleeve $\frac{1}{40}$ (or 0.025) of an inch. The sleeve is marked with lines $\frac{1}{40}$ inch apart so that each division on the sleeve corresponds to one complete turn of the spindle and represents a movement of the spindle equal to 0.025 in. Each fourth line, representing $\frac{4}{40}$ (or 0.1) of an inch, is longer than the others. The zero (0) mark on the sleeve coincides with the position of the edge of the bevel on the thimble when the micrometer is closed. Hence, when the anvil and spindle are in contact no part of the scale on the sleeve D is visible. Backward rotation of the spindle through one complete revolution brings the edge of the thimble to the first line on the sleeve reading from Further rotation through one more complete revolution brings the edge of the thimble to the second line from o, thus exposing to view two of the scale divisions on the sleeve; and so on. since every part of the spindle and thimble recede equally from the anvil B, the distance between the anvil and spindle is always equal to the visible portion of the scale on the sleeve.

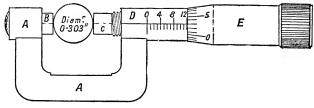


FIG. 156.

The bevelled edge of the thimble is marked with a line 0-, which coincides with the scalar line on the sleeve when the micrometer is closed, and is divided into twenty-five equal parts; every fifth division line being numbered 5, 10, etc. Rotating the thimble from one of these marks to the next moves the spindle along the sleeve of 0.025 in., or one-thousandth of an inch. A rotation of two divisions moves the spindle two-thousandths of an inch and so on. Rotation through a complete revolution corresponds to a lateral movement of 0.025 (or $\frac{1}{40}$) of an inch. The fraction of a complete revolution through which the spindle has been rotated, is indicated by the particular mark on the thimble scale coinciding with, or nearest to, the long scale line on the sleeve. Hence, to read the micrometer we have the rule: Multiply the number of divisions visible on the sleeve scale by 25, and add the number of divisions on the thimble scale, counting from 0 to the line which coincides with the scale line on the sleeve.

Ex. The object being measured in Fig. 156 is seen to be twelve divisions of the sleeve scale and a fraction of the next division,

i.e., $(12 \times 0.025 \text{ in.}) + \text{a fraction of } 0.025 \text{ in.}$

The scale line on the sleeve coincides with the *third mark* from 0 on the thimble scale, hence the measure of the fraction above is $\frac{3}{25}$, or three-thousandths of an inch.

The object measures (12 \times 0.025 in.) + 0.003 = 0.303 in.

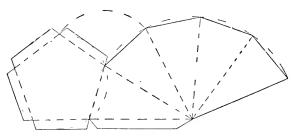


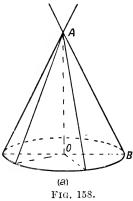
Fig. 157.

234. **Right pyramid.** The superficial area of any pyramid is obtained by adding the area of the base to that of the triangular faces. There are as many of these triangles as there are sides to the polygonal base.

The diagram, Fig. 157, if cut and folded along the dotted lines will form a right penta-

gonal pyramid.

235. Circular cone. If a piece of paper be wrapped, without crumpling or tearing, round the curved surface of a cone (Fig. 158), and cut with a knife along the edge of the base and the line AB, and then folded out, the paper will be a sector of a circle (Fig. 159) of radius AB equal to the slant height of the cone, and the length of the arc BD will be equal to the circumference of the base of the cone.



If we know the perpendicular height h, and radius of base r; then since angle AOB is a right-angle, the slant height $AB = \sqrt{r^2 + h^2}$.

The arc BD = $2\pi r$.

The angle DAB in radians =
$$\frac{\text{length of are}}{\text{radius}} = \frac{2\pi r}{\text{AB}}$$

If l = AB, we have—

Area of sector = $\frac{1}{2}$ (radius)² × (angle in radians). See p. 166.

$$= \frac{1}{2}l^2 \times \frac{2\pi r}{l} = \pi r l.$$

Ex. Find superficial area if a cone of radius of base r=3 in., and slant height l=5 in.

(Superficial area = area of base + area of curved surface,)
=
$$\pi r^2 + \pi r^l$$
.
= $\pi r(r + l)$ sq. in.
= $\pi 3(3 + 5)$.

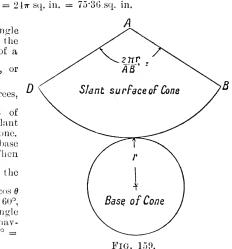
Note. — The angle DAB, Fig. 159, of the developed surface of a cone is $\frac{2\pi r}{l}$ radians, or

$$\left(360^{\circ} \times \frac{r}{l}\right)$$
 degrees,

when r = radius of base and l = slant height AB of the cone.

Let θ denote the base angle of the cone. Then $\frac{r}{l} = \cos \theta$, hence the angle DAB = $360^{\circ} \cos \theta$

angle DAB = $360^{\circ} \cos \theta$ degrees. Thus, if $\theta = 60^{\circ}$, $\cos \theta = \frac{1}{2}$, and the angle DAB for a cone having a base angle $60^{\circ} = 360^{\circ} \times \frac{1}{2} = 180^{\circ}$.



236. Volumes of solids. Expressions for the volumes of some geometrical solids are given below:—

A cube, length of edge a ft. Volume = area of one face \times height hence $V = a^2 \times a = a^3$ cub. ft. A square prism, height h ft., edge of base a ft.

 $Volume = area of base \times height$

hence $V = a^2h$ cub. ft.

A triangular prism. height h ft., sides of base a, b, and c ft.

 $Volume = area of base \times h$

hence
$$V = \left\{ \sqrt{s(s-a)(s-b)(s-c)} \times h \right\}$$
 cub. ft.

where $s = \frac{a+b+c}{2}$.

A hexagonal prism, height h ft., sides of base a ft.

 $Volume = area of base \times h$

hence $V = 2.598a^2h$ cub. ft.

A circular cylinder, radius of circle r ft., height h ft.

 $Volume = area of circular section \times h$

hence $V = \pi \ell^2 h$ cub. ft.

237. To find the volume of a hollow body we may find the volume as if the body were a solid one, then find the volume of a solid which would just fill the void in the hollow body, and subtract the latter volume from the former. The result is the required volume.

Ex. A cylindrical hollow column has an outer diameter D in., and an inner diameter d in., and is h in. high. Find an expression for its

volume.

Sectional area
$$= \frac{\pi}{4} D^2 - \frac{\pi}{4} d^2$$
.
 $= \frac{\pi}{4} (D^2 - d^2) = \frac{\pi}{4} (D - d)(D + d)$.
Volume $= \frac{\pi}{4} (D - d)(D + d)h$ cub. in.

If, in the above example, D = 12 in., d = 10 in., and h = 10 ft., find the weight of the column if each cubic inch weighs 0·3 lb.

Volume
$$= \frac{\pi}{4} (D - d)(D + d)12 h \text{ cub. in. (note, } h \text{ is given in feet)}$$
substituting
$$= \frac{\pi \times 2 \times 22 \times 12 \times 10}{4} \text{ cub. in.}$$
weight
$$= \frac{\pi \times 2 \times 22 \times 12 \times 10 \times 0.3}{4} \text{ lb.}$$

$$= 1243.44 \text{ lb.}$$

238. To find a rule for the volume of a pyramid, construct a cube ABCFGHK on the base ABCD of the pyramid in Fig. 160, and suppose the point E to be at the exact centre of the cube. The cube would contain six right square pyramids similar to ABCDE. The volume of the

cube is a^3 cub. in. if AB = a in. long; hence the volume of each pyramid is $ABCE = \frac{a^3}{6}$ cub. in. The area of the base of each pyramid is a^2 sq. in., and since E is in the centre of the cube, the length of the axis or the height of each pyramid is $\frac{a}{2}$ in. If we multiply the area of the

base (a^2 sq. in.) by one-third of the height $(\frac{1}{3} \times \frac{a}{2})$, we get

 $a^2 \times \frac{a}{6}$ or $\frac{a^3}{6}$, which agrees with the volume of the pyramid as found above. The rule for finding the volume of a pyramid may thus be stated: multiply the area of the base of the pyramid by one-third of the height and the product is the volume.

239. It must be noted that the height in every case must be measured perpendicular to the base of the pyramid. In a right pyramid, the height

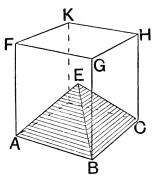


Fig. 160.

is equal to the length of the axis, but in an oblique pyramid this is not so.

Ex. Let
$$r = \text{radius of base of a cone in inches.}$$

 $h = \text{perpendicular height AO in inches.}$

Then volume = area of base
$$\times \frac{1}{3}$$
 height
= $\pi r^2 \times \frac{1}{3} h$
= $\frac{\pi r^2 h}{2}$ cub, in.

If the cone is 4 ft. high and has a base of radius 3 ft.

Volume =
$$\frac{\pi \times 9 \times 4}{3}$$
 = 37.68 cub. ft.

240. The formulæ for volumes of prisms given in Art. 236 will become the volumes of corresponding pyramids if each

is divided by 3. For example, volume of a square pyramid side of base a and height h is $\frac{1}{3}a^2h$.

241. The volume of a sphere of radius $r = \frac{4}{8}\pi r^3$ cub. in.

The superficial area of a sphere of radius $r = 4\pi r^2$ sq. in. These results should be remembered. See also Art. 249 and Chap. XXVIII.

Ex. 1. Show that the rolume of a sphere is equal to two-thirds of the volume of the circumscribing cylinder,

r = radius of sphere

Then r = radius of base of circumscribing cylin ler

and 2r = height of cylinder, since each end just touches the sphere at opposite ends of a diameter.

Volume of sphere
$$= \frac{4}{3}\pi r^3$$
Volume of cylinder = area of base × height = $\pi r^2 \times 2r$ = $2\pi r^3$

Ratio Volume of sphere $\frac{4}{3}\pi r^3 = \frac{4}{6} = \frac{2}{3}$.

Ex. 2. Show that the superficial area of a sphere is equal to that of the curved surface of the circumscribing cylinder,

Let r = radius of sphere

Then r = radius of cylinder, and 2r = height. Superficial area of sphere = $4\pi r^2$.

Area of curved surface of cylinder = length of circumference of base \times height, $2\pi r \times 2r = 4\pi r^2$.

Ex. 3. A cylinder, a sphere, and a cone, are of equal diameter, and the height of cone and cylinder = diameter of sphere. Show that the volumes of cylinder, sphere, and cone are in the ratio of 3, 2, 1.

Let r = radius of sphere, cone, and cylinder.

Then height of cone and cylinder = 2r.

Volume of cylinder =
$$\pi r^2 \times 2r = 2\pi r^3$$

Volume of sphere = $\frac{4}{3}\pi r^3$
Volume of cone = $\frac{1}{3}\pi r^2 \times 2r = \frac{2}{3}\pi r^3$
But $2\pi r^3 = 3 \times \frac{2}{3}\pi r^3$
And $\frac{4}{2}\pi r^3 = 2 \times \frac{2}{3}\pi r^3$

Hence the volumes are in ratio of 3, 2, 1.

242. If a ring be cut across radially and opened out into a straight piece, it becomes a prism of length equal to the length of the mean circumference of the ring.

The volume of a ring of any section is thus equal to the sectional area multiplied by the circumference of a circle of

radius equal to the mean radius of the ring.

Fig. 161. Let r = inner radius of a ring of rectangular section.

Let R =outer radius of the ring and t =thickness.

Then mean radius
$$=\frac{R+r}{2}$$

Area of section =
$$(R - r)t$$

Hence volume =
$$(R - r)t \times 2\pi \frac{R + r}{2}$$

= $(R - r)t \times \pi(R + r)$.

The superficial area of a ring is equal to the perimeter of any radial section multiplied by the length of the mean circumference. The perimeter of a section of the ring in Fig. 161 = 2 (R - r) + 2t.

Length of mean circumference = $\pi(R+r)$.

Hence superficial area
$$= \pi(R + r) \times 2 \{(R - r) + t\}$$

Logarithms should be used wherever possible for solving examples in mensuration. Two such examples are shown below:—

suration. Two such examples are shown below:—

Ex. 1. Find the volume and superficial area of the rim of a fly-wheel of

point area of the rim of a fly-wheel of circular section, the outer diameter being 12 ft. and the radius of the section of the rim 6 in.

Mean radius of rim
$$= \frac{6+5}{2} = 5.5 \text{ ft.}$$

Mean circumference of rim =
$$2\pi$$
 radius = $2 \times \pi \times 5^{\circ}5 = 11\pi$
Area of section of rim = $\pi \times (0^{\circ}5)^2$ sq. ft.
Volume = $\pi \times (0^{\circ}5)^2 \times 11\pi$ cub. ft. = $\pi^2 \times (0^{\circ}5)^2 \times 11$

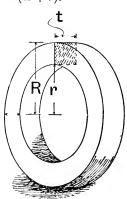


FIG. 161.

Log. volume = $2 \log. \pi + 2 \log. 0.5 + \log. 11$ = $(2 \times 0.4972) + (2 \times \overline{1}.6990) + 1.0414 = 1.4338$ Hence volume = 27.15 cub. ft. Perimeter of section of rim = $(\pi \times 1)$ ft. Superficial area of rim = $(\pi \times 1) \times 11\pi$ = $11\pi^2$ Log. superficial area = $\log. 11 + 2 \log. \pi$ = $1.0414 + (2 \times 0.4972)$ = 2.0358

Hence superficial area = 108.6 sq. ft.

Ex. 2 Find the weight of a hollow east-iron ball of outer diameter 15 ft. and inner diameter 12 ft., if each cubic inch weighs 0.26 lb. Weight of 1 cub. ft. of east iron = (0.26 × 1728) lb.

Volume of sphere
$$= \frac{4}{3}\pi (15)^3 - \frac{4}{3}\pi (12)^3$$

$$= \frac{4}{3}\pi (15^3 - 12^3)$$

$$= \frac{4}{3}\pi \times 1647 \text{ cub. ft.}$$
Weight of sphere
$$= \frac{4}{3}\pi \times 1647 \times 0.26 \times 1728$$
Log. of weight
$$= \log. 4 + \log. \pi + \log. 1647 + \log. 0.26 + \log. 1728$$

$$- \log. 3$$

$$= 0.6021 + 0.4972 + 3.2166 + \overline{1}.4150 + 3.2375$$

$$- 0.4771$$

$$= 6.4913$$
Weight
$$= 3,099,000 \text{ lb.}$$

Exercises.

(1) A reservoir is 50 ft. long, 30 ft. wide, and 8 ft. deep, and it is filled with water. What weight of water does it contain? How many gallons does it contain? 1 c. ft. = 62.3 lbs.

(2) A swimming bath is 30 yds. long and 12 yds. wide, and its depth varies gradually from 7 ft. at one end to 3 ft. at the other. How many gallons and how many tons of water does it contain?

(3) A column is 5 ft. diameter and 12 ft. high. Find its superficial area (both ends included) and its weight if each cubic foot

weighs 100 lb.

(4) Find the superficial area, volume, and weight of a hexagonal pyramid of height 10 ft, the side of hexagon being 3 ft. long. Each cubic inch of the material weighs 0.26 lb. The length of the line joining the centre of the base line of each triangular face to the apex must be found graphically.

(5) Find the volume of a cone 15 ft. high, the base having a

diameter 3 ft. 6 in. Use logs.

(6) The base of a cone is a circle of 3.25 in. diameter. The vertical height is 5.24 in. It is made of cast iron, which weighs 0.26 lb. per cubic inch. What is the weight of the cone? (B.E. 1909.)

(7) A vessel is shaped like the frustum of a cone, the circular base is 10 in. diameter, the top is 5 in. diameter, the vertical axial height is 8 in. By drawing, find the axial height to the imaginary vertex of the cone. If x is the height of the surface of a liquid from the bottom, plot a curve, to any scales you please, showing for any value of x the area of the horizontal section there. Three points of the curve will be enough to find. (B.E. 1905.)

(8) A hexagonal bar of 3 in. side and 1 ft. long has a circular hole 11 in. diameter bored through it longitudinally. Find its weight if

each cubic inch weighs 0.26 lb.

(9) A hollow cylindrical column is 50 ft. high, outer diameter is 5 ft. and inner diameter 3 ft. Find its weight if each cubic inch weighs 0.26 lb. Find the cost of painting it externally and internally at $\frac{1}{2}d$, per square foot.

(10) A hollow iron column is 12 in. external diameter, and the metal is 1\frac{1}{4} in. thick. It carries a load of 125 tons. Find the compressive stress, i.e., the load upon each square inch of sectional area

of the column.

- (11) A brass tube, 8 ft. long, has an outside diameter 3 in., inside 2.8 in. What is the volume of the brass in cubic inches? If a cubic inch of brass weighs 0.3 lb., what is the weight of the tube? (B.E. 1905.)
- (12) A sphere of stone is 3 ft. in diameter, the weight of the stone per cubic foot is 120 lb.; what is the weight of the sphere? (B.E. 1907.)
- (13) A circle is 3 in. diameter, its centre is 4 in. from a line in its plane. The circle revolves about the line as an axis and so generates a ring. Find the volume of the ring, also its surface area. (B.E. 1905.)

Answers.

- (1) 747,600 lb.; 74,760 galls.
- (2) 450.5 tons; 100,926 galls.
- (3) 227.6 sq. ft.; 23,562 lb.
- (5) 48.11 cub. ft.
- (6) 3.767 lb.
- (8) 67·44 lb.

- (9) 282,400 lb.; £2 12s. 4d.
- (10) 2.96 tons per square inch.
- (11) 87.48 cub. in.; 26.24 lb.
- (12) 1,696 lb.
- (13) 177.7 eub. in.; 236.9 sq.

CHAPTER XX.

MENSURATION OF IRREGULAR FIGURES, ETC. CENTRE OF GRAVITY: CURVATURE.

243. In the present chapter will be given some further useful rules in mensuration; the mensuration of irregular figures and solids, and of solids of revolution; practical methods of estimating volumes; methods of finding centres of gravity; some useful problems

dealing with loci; involutes evolutes; and curvature.

Area of a Triangle. Fig. 162. When the base and height are given, the area, as explained in Art. 38, is $= \frac{1}{2} \times \text{base} \times \text{height, or } \frac{1}{2} bh$.

As the height $h = c \times \sin A$, the area = $\frac{1}{2}$ bc sin. A, or generally, area

 $=\frac{1}{2}$ product of two sides and the sine of included angle (as b, c, A).

Fig. 162.

When the three sides are given, the area is found from the formula.

Area =
$$\sqrt{s(s-a)(s-b)(s-c)}$$
, in which $s = \frac{a+b+c}{2}$, or half the sum of the three sides.

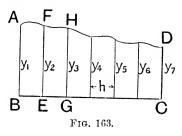
Ex. Find the area of a triangle having sides 12, 10, and 8ft. fl. $s = \frac{12 + 10 + 8}{2} = 15.$ $(s - a) = (15 - 12)^{e} = 3$ (s - b) = (15 - 10) = 5(s - c) = (15 - 8) = 7.

244. Area of an Irregular Figure. One method of

finding the area of an irregular figure is to divide the figure into a number of strips of equal width, and regard each strip as a rectangle whose width is equal to the width of the strip, and whose length is equal to the central ordinate of the strip. This method has already been dealt with in Art. 207, when finding the area of an indicator diagram.

Another method is to draw the irregular figure upon squared paper to scale, and count the number of squares enclosed. The area of each square depends upon the scales adopted. Part of a square may be neglected, if less than half the area of a square, and counted as a whole square if it exceeds the half.

A further method involves for its proof the area of a trapezoid, or trapezium, and is known in its application



as the **trapezoidal rule**. The area of a trapezium is shown in Art. 262 to be equal to half the sum of the parallel sides × perpendicular distance between the sides.

Let the figure ABCD (Fig. 163) represent a plot of land bounded by straight roads at right-angles on the

sides AB, BC, and let CD be parallel to AB, and AD an irregular boundary.

To find the area, divide it into any number n of strips of equal width h, and measure the lengths of the ordinates y_1 , y_2 , etc. In Fig. 163, n=6, and we have (n+1) or 7 ordinates. Each strip can be regarded as a trapezoid.

The area of the strip ABEF is $\frac{1}{2} (y_1 + y_2)h$.

The area of the strip FEGH is $\frac{1}{2} (y_2 + y_3)h$.

The whole area is

$$\frac{1}{2} h \left\{ (y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + \dots + (y_6 + y_7) \right\}$$

$$= \frac{1}{2} h \left\{ y_1 + 2 y_2 + 2 y_3 + 2 y_4 + 2 y_5 + 2 y_6 + y_7 \right\}$$

$$= h \left\{ \frac{y_1 + y_7}{2} y_2 + y_3 + y_4 + y_5 + y_6 \right\}$$

$$= \begin{cases} 1 \text{ sum of first and last exclinates the sum of first and last exclinates.}$$

= $\left\{\frac{1}{2} \text{ sum of first and last ordinates} + \text{ sum of intermediate}\right\}$

286 MENSURATION OF IRREGULAR FIGURES, ETC.

ordinates $\}$ multiplied by the distance between consecutive ordinates. The mean height or ordinate $=\frac{\text{area}}{nh}$.

245. Simpson's Rule for finding Area of an Irregular Figure. Divide the area into an even number of strips; then, referring to Fig. 163,

Area =
$$\frac{h}{3}$$
 { $y_1 + y_7 + 4 (y_2 + y_4 + y_6) + 2 (y_3 + y_5)$ }

or in words: Add together the first and last ordinates, four times the sum of the even ordinates, and twice the sum of the odd ordinates (omitting the first and last). Multiply the sum by one-third of the

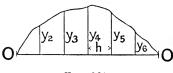


Fig. 164.

sum by one-third of the distance between consecutive ordinates, and the product gives the area.

When the first and last ordinates are zero, as in Fig. 164, the area is $\frac{h}{3}$ { 4 $(y_2 + y_4 + y_6) + 2 (y_3 + y_5)$ }.

246. Volume of an Irregular Solid. Any of the above rules for irregular areas can be applied to volumes, if we measure the sectional areas at equidistant points, say h apart, and substitute "area" for "ordinate" in each rule.

Denoting the sectional areas by A_1 , A_2 , A_3 , etc., then Simpson's Rule gives

Volume $=\frac{h}{3}\left\{A_1+A_7+4\left(A_2+A_4+A_6\right)+2\left(A_3+A_5\right)\right\}$ when the volume is divided into six parts. As an example, the volume of a tree trunk could be estimated by measuring the girth at, say, seven equidistant points, treating each section as a circle, and thus estimating the area of each section.

When ordinates of areas, or sectional areas of volumes, are given at points which are not equidistant, the ordinates, or sectional areas, should be plotted on squared paper, and a graph drawn through the plotted points. The ordinates or sectional areas at equidistant points can then be measured from the graph.

Similarly, we can estimate the volume of an irregular heap of material, or a girder of variable section, by making measurements from which the sectional area at various

points can be determined.

The capacity of a trench, cutting, reservoir, river bed, etc., can also be found if we ascertain the sectional areas at various points, and apply the given rules. When the method of counting squares, or the mid-ordinate rule is to be used, the ordinates of the figure represent to scale the areas at the points chosen.

If three sections only are taken, Simpson's Rule becomes

Volume
$$= \frac{h}{3} \left\{ A_1 + A_3 + 4 A_2 \right\} \cdot \frac{\text{(This is often called the Prismoidal Formula.)}}{\text{MOIDAL FORMULA.)}}$$

The average section is $\frac{\text{volume}}{\text{length}} = \frac{\text{volume}}{2 h}$,

$$= \frac{\hbar}{3 \times 2 h} \Big(A_1 + A_3 + 4 A_2 \Big),$$

$$=\frac{1}{6}$$
 { first section + last section + 4 central section }.

247. As an example of each of the above methods, the following exercise will be worked:—

x is the distance in chains measured along a straight line O9 from the point O, the values of y are offsets or distances in chains measured at right-angles to O9 to the border of a field. Draw the shape of this border. Find the area in square chains between the first and last offset, and the straight line and border. Notice that the intervals in x are not equal.

(B. E., 1908.)

x	0	1.50	3.00	5.00	7:50	9.00
<i>y</i>	0.53	0.27	0.46	0.42	0.35	0.52

First plot values of x and y on squared paper, as in Fig. 165. From the graph AB we can measure equidistant ordinates.

^(1.) By counting the squares, the area is estimated to be (908 × 0.004) = 3.63 sq. chains. There are 908 squares; the area of each square is 0.2 × 0.02 = 0.004 sq. chain.
(2.) By the Mid-Ordinate rule (Art. 208), the average ordinate is

found to be 0.4 chain when the area is divided into nine equal strips

Hence area = width × average ordinate.

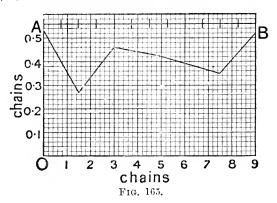
$$= 9 \times 0.4$$
 sq. chains.
= 3.6 sq. chains.

(3.) By the **Trapezoidal rule**, dividing the area into nine equal strips we have 10 ordinates, and h = 1 chain.

Area =
$$1\left\{\frac{0.53 + 0.52}{2} + 0.35 + 0.34 + 0.46 + 0.44 + 0.42 + 0.39 + 0.36 + 0.41\right\}$$

= 3.695 sq. chains.

(4.) By Simpson's Rule, dividing the area into 12 strips each of



0.75 chain wide, we have 13 ordinates. (The twelve divisions are shown by a line at the top of Fig. 325.)

Area =
$$\frac{0.75}{3}$$
 ($0.53 + 0.52 + 4$ ($0.41 + 0.37 + 0.45 + 0.41 + 0.37$)
= 0.25×14.65 = 0.66 sq. chains.

248. Centre of Gravity.* If a solid body, say a thin plate of irregular shape as shown in Fig. 166, be suspended from any point A, it will come to rest in some definite position. Let AC be a vertical line drawn on the plate through A in this position. Suspend the plate from any other point B, and let BD be the vertical line then drawn through B. These lines AC, BD intersect in a

^{*} See also Chapter XXVIII.

point O. If we suspend the plate from any other point, and draw a vertical line through the point of suspen-



FIG. 166,

sion, this line will also pass through O. The point O is called the centre of gravity of the plate. If we take an irregular solid body, other than a thin plate, e.g., a piece of stone, we cannot find the position of the centre of gravity by the experimental method set above, for the vertical lines pass through the material and hence cannot be seen;

nevertheless, they intersect in the point which is the centre of gravity.

It is shown, in works on Mechanics, that the following method locates the position of the centre of gravity. The method is here described as applied to an irregular area, e.g., a thin plate A. Fig. 167. Draw two lines OX, OY, at right-angles, and each touching

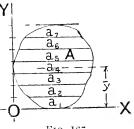


Fig. 167.

the boundary line of the plate. Divide the plate into a number of strips of small width y, parallel to one line, say OX.

Find the area of each strip, say, a_1 , a_2 a_3 , etc.

The centre line of area a_1 is distant $\frac{y}{2}$ from OX.

,,
$$a_2$$
 ,, $\frac{3 y}{2}$,, OX, etc.

Then if y be the distance of the centre of gravity from OX, and the total area be A,

$$\mathbf{A} \times \tilde{y} = \left(a_1 \times \frac{y}{2}\right) + \left(a_2 \times \frac{3}{2} \frac{y}{2}\right) + \left(a_3 \times \frac{5}{2} \frac{y}{2}\right) + \dots \cdot \left(a_7 \times \frac{13}{2} \frac{y}{2}\right)$$

or $\overline{g} = \begin{cases} \text{Sum of products of each area into the distance of its } \\ \text{centre from OX} \end{cases}$

By dividing the plate into a number of strips of width x parallel to OY, we can find in a similar manner the distance \overline{x} of the centre of gravity from OY. The intersection of the two lines gives the position of the centre of gravity, and hence \overline{x} , \overline{y} are the co-ordinates of the centre of gravity, with OX, OY as axes of reference.

For a solid body other than a thin plate, we take three planes of reference which intersect in lines OX, OY, OZ, mutually perpendicular, as in Art. 41; divide the solid into small slices parallel to each plane in turn, and calculate, as

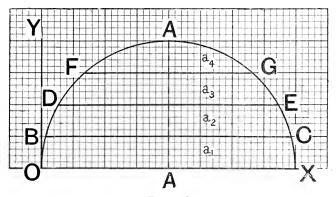


FIG. 168.

above, the distances \bar{x} , \bar{y} , \bar{z} , of the centre of gravity from each plane. These distances are the co-ordinates of the centre of gravity with OX, OY, and OZ as the three axes of reference.

If an area is symmetrical about any axis, the centre of gravity lies in that axis, hence, only \bar{x} or \bar{y} need be determined, for we can take OX or OY parallel to the axis of symmetry.

Ex. Find the centre of gravity of a semicircular area of radius r. Fig. 168. The semicircle is symmetrical about AA; hence the centre of gravity lies in AA, and we take OY parallel to AA. We require \overline{y} , or the distance of the centre of gravity from OX.

Divide into four strips, by lines BC, DE, FG.

Then by counting squares, area $a_1 = 198$ squares.

,,
$$a_2 = 184$$
 squares.
,, $a_3 = 154$ squares.
,, $a_4 = 90$ squares.

The centres of the areas $a_1,a_2,$ etc., are 2.5, 7.5, 12.5, and 17.5 squares respectively from OX; the whole area = 628 squares. Hence

ence
$$\overline{y} = \frac{(198 \times 2.5) + (181 \times 7.5) + (154 \times 12.5) + (90 \times 17.5)}{626}$$

= 8.59 squares from OX.

We have taken r = 20 squares, hence

$$\bar{y} = \frac{8.59}{20}r = 0.429 r$$
 from the diameter OX.

If we take more strips than four, we obtain a more accurate result. The correct value of \overline{y} , and a useful form in which to remember it, is $\frac{4 r}{3 \pi}$, which = 0.4244 r.

Table of Centres of Gravity.

If an area is symmetrical about two axes, the centre of gravity is at the intersection of the axes, e.g.—

In a circle it is at the centre. .

Ellipse, at the intersection of the major and minor axes.

Parallelogram, at the intersection of the axes.

Any triangle . . . at intersection of medians.

Semicircular area . . . $\frac{4 \text{ radius}}{3 \pi}$ or 0.4244 radius

from diameter.

Segment of parabola $y^2 = 4 ax$. $\frac{2}{3}$ height from base.

Semicircular wire . . . $\frac{\text{diameter}}{\pi}$ from centre.

249. A solid of revolution is the solid traced out by any curve revolving about an axis in the same plane as the curve—e.g., if we rotate a semicircle about the diameter we trace out a sphere.

Guldin's Theorems. (1) The area of a surface traced out by the revolution of a curve (including a straight line or lines) about an axis in its own plane is equal to the product of the perimeter of the curve, and the length of the path of its centre of gravity during the revolution.

(2) The volume generated by the revolution of a curve about an axis in its own plane is equal to the product of the area enclosed by the curve and the length of the path of its centre of gravity during

the revolution.

The results of the following examples should be compared with the results obtained in Chap. XIX.

(1) Circle of radius r. A circle is generated by a line rotating about an axis at one end.

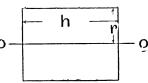


Fig. 169.

The path of C, of G, is $2 \pi \frac{r}{2} = \pi r$.

 $\therefore \text{ Area } = r \times \pi r = \pi r^2.$

(2) Cylinder of radius r, height h. Fig. 169.

Volume. The rectangle $(h \times r)$ is the area which, rotating about 00, generates a cylinder of radius r and height h.

Centre of gravity is $\frac{r}{2}$ from OQ.

In one revolution, C. of G. moves a distance of

$$\left(2 \pi \times \frac{r}{2}\right)$$
, or πr .

Volume of cylinder = Area × Path of C.G. = $hr \times \pi r$. = $\pi^{-2}h$.

Superficial Area. A line h distant r from OO generates the curved surface,

 \therefore Area of curved surface = $h \times 2 \pi r = 2 \pi r h$.

The circular ends are each generated by line r.

 $\therefore \text{ Each end has area } r \times \frac{2 \pi r}{2} = \pi r^2.$

Total surface = $2 \pi r^2$ (two ends) + $2 \pi rh$ (curved surface).

(3) Ring of circular section. Fig. 170.

Outer diameter of ring D.

Sectional area of diameter d.

Volume. Generated by rotation of circular area, shown shaded, about OO.

Area of section
$$=\frac{\pi}{4} d^2$$
.

C. G. =
$$\left(\frac{D}{2} - \frac{d}{2}\right)$$
 from OO.

... Path of C. G. =
$$2 \pi \frac{(D-d)}{2} = \pi (D-d)$$
.

Hence volume =
$$\frac{\pi}{4} d^2 \times \pi (D - d)$$

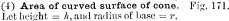
= $\frac{\pi^2 d^2}{4} (D - d)$.

Superficial Area. Generated by revolution of circumference of the circular section about OO.

Length of circumference = πd .

Path of C.G. =
$$\pi$$
 (D - d).

Hence superficial area =
$$\pi d \times \pi$$
 (D - d).
= $\pi^2 d$ (D - d).



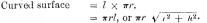
The curved surface is generated by line AO rotating about OO.

The centre of gravity C is $\frac{r}{2}$ from OO.

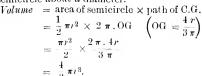
Length of path of C.G. =
$$2 \pi \frac{r}{2} = \pi r$$
.

Length of AO =
$$l = \sqrt{r^2 + h^2}$$
.

Curved surface
$$= \iota \times \pi r$$
.
 $= \pi r l$ or $\pi r \sqrt{\iota^2 + \iota l^2}$



(5) Sphere. Fig. 172. A sphere is generated by the revolution of a semicircle about a diameter.



Superficial area, generated by the semicircular are rotating about diameter.

Length of arc = πr .

Position of C.G. = $\frac{2r}{\pi}$ from diameter.

Path of C.G.
$$= 2 \pi \frac{2 r}{\pi} = 4 r$$
.

Superficial area = $\pi r \times 4 r = 4 \pi r^2$.

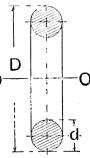


FIG. 170.

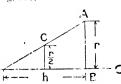
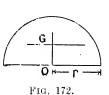


FIG. 171.



or

250. The Theorems of Guldin are useful for finding the position of the centre of gravity of curves and areas; thus, for a semicircular wire we have length of wire $= \pi r$ if r is the radius. The surface traced out by the wire rotating about the diameter we know to be a spherical surface whose area $= 4 \pi r^2$. Let the distance OG (Fig. 170) of the Centre of Gravity from the diameter be \bar{x} . Then length of path of C.G. $= 2 \pi \bar{x}$.

By Guldin's Theorem—
(Area) $4 \pi r^2 = \pi r$ (length of wire) $\times 2 \pi \bar{x}$, $\bar{x} = \frac{4 \pi r^2}{2 \pi^2 r} = \frac{2 r}{\pi}$ from diameter.

A semicircular area generates a sphere of volume $\frac{4}{3}$ πr^3 , when rotated about its diameter.

The area of semicircle $=\frac{1}{2} \pi r^2$.

Let $\overline{x} = \text{distance}$ of C.G. from diameter. Then length of path of C.G. = $2 \pi \overline{x}$ and

$$\frac{4}{3} \pi r^3 = 2 \pi \overline{x} \times \frac{1}{2} \pi r^2,$$

$$\overline{x} = \frac{4 \pi r^3}{3 \pi^2 r^2} = \frac{4 r}{3 \pi} \text{ from diameter.}$$

251. Estimating Volume by Weighing in Water. If the material of a body is heavier, bulk for bulk, than water, we may ascertain the volume by weighing, for a body loses weight upon immersion in water by an amount equal to the weight of the water displaced. Hence if the body weighs

x lbs. in air, and x_1 lbs. in water, its volume is $\frac{x-x_1}{62\cdot 3}$ cu. ft. for 1 cu. ft. of water weighs 62·3 lbs.

The density of a body or substance is the mass of unit volume. The unit employed is always specified, e.g., castiron weighs 0.26 lbs. per cubic inch.

We can thus find the weight of any body if we know its

volume and density for

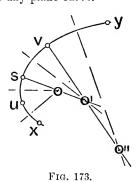
Weight = volume \times density.

The specific gravity of a body (Art. 268) is often called the relative density. It states the ratio

weight of given volume of body weight of equal volume of water; the specific gravity of water is thus taken as unity. Hence, the weight of a body may be determined when we know the volume and the specific gravity or relative density for

Weight = $\begin{cases} \text{Volume in} \\ \text{cu. ft.} \end{cases} \times \begin{cases} \text{Specific gravity or} \\ \text{Relative density} \end{cases} \times 62.3 \, \text{lbs.}$

(The factor 62.3 is the weight of a cu. ft. of water.) 252. A curve can be built up of circular arcs closely approximating to any plane curve.



Ex. Fig. 173. Given a curve xvy; determine three centres from which a curve closely resembling the given curve can be described.

Select any three points x, u, s in the curve, and not too far apart. Find the centre o of the circle that can be drawn through the points x, u, s. Select some point v in the curve and draw the \mathbf{r} bisector of sv meeting so prolonged in o'. Draw the \mathbf{r} bisector of vy meeting vo' prolonged in o''. A line formed of arc sv described from o, arc sv described from o', and arc vy described from o'', will lie close to the given curve xvy.

The greater the number of centres employed, the nearer will the new line lie to the original.

Loci.

253. In modern geometry, a curve is considered as made up of an infinite number of points placed in order along the eurve.

Any one of these elemental points is indefinitely near to the next consecutive point.

A right line drawn through two consecutive points in a curve is called a tangent.

An illustration may help to make this definition clear. Let a line Pt (Fig. 174) intersect the curve SS in points A

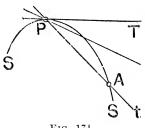


Fig. 174.

and P. Imagine the point A to move along the curve toward the fixed point P, and the line to pass always through P. The moving line changes position the point A continually as approaches the fixed point P, and ultimately, when A coincides with P, the line Pt lies in t the tangent PT.

254. Lines tangent, or normal, to most of the mathematical

curves met with in the constructive arts can be drawn from known properties of the curve. If the known data is insufficient to provide a direct solution of the problem, a locus may be used.

Through a fixed point P draw a tangent Problem.

to a given curve mn.

First Case. Fig. 175 (a). The point P is on the curve.

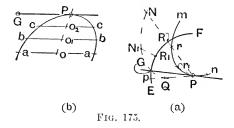
With centre P and any radius describe a circular are EF. Draw PR intersecting the curve in r and the are EF in R. Make RN on PR produced = Pr. Through a second point r, on the curve draw Pr, meeting the arc EF in R1. Make $R_1N_1 = Pr_1$. Draw lines from P through other points on the curve, one of which, n, is situated on the right of P. Set off along these lines, measured from the arc EF, lengths equal to the respective chords of the curve mn, and to the right or left of the arc EF according as the line intersects mn on the right or left of P. Thus in the Fig. pQ = Pn.

Draw a fair line through the points $N, N_1 \dots Q$, which intersects the arc EF in G. The point G is in the tangent of P.

For, imagine NP to rotate about the fixed point P in the curve, and let PN, PN₁. P_p represent different positions of the moving line. Then the curve NN₁GQ is the locus of points which are in the moving line, and at a distance from the circle EF equal to that of P from the second and variable point r in which the line intersects the curve, and on the left or right of the are, according as NP cuts the curve on the left or right of P. Consequently the point G where the locus NN₁GQ crosses EF corresponds to that position of the moving line where it does not cut the curve on either side of P. Hence, GP is tangent at P.

Second Case. Fig. 175 (b). Let G be the given point not on the curve.

Place a pencil point on G, slide a straight edge against the pencil, and rotate so as to bring the edge as near as



the eye can judge in contact with the curve. Draw a line along the edge; this is the required tangent. The point of contact P may be found thus: Draw parallels to the tangent meeting the curve in aa, bb, cc; bisect aa in o, bb in o_1 , cc in o_2 . Draw a fair line through o, o_1 , o_2 , and produce it to meet the curve in P, which is the point of contact of the tangent from G.

For, the curve oo_1o_2 is the locus of the mid-points of chords parallel to the tangent, and P marks the position where the gradually shortening ehord is reduced to two consecutive points in the curve.

255. **Problem**. Fig. 176. To draw a normal of a curve SS to pass through a fixed point,

First Case. Let Q be the point, and not on the curve. With Q as centre describe circular arcs intersecting the

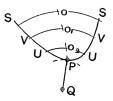


Fig. 176.

curve in SS, VV, UU. Bisect these arcs (or their chords) in o, o_1 , o_2 . Draw a fair line through these points and produce to meet the curve SS in P, which is a point in the normal from Q; for, the circle centre Q and radius QP passes through P and not through any other point in the curve SS.

Second Case. If it is required to draw a normal from a point situate on

the curve, determine first the tangent of the point (Art. 254),

and draw a perpendicular thereto.

256. Involute and Evolute. PE roll, without sliding, along the fixed curve OO_2 , any fixed point P in the tangent has a definite locus $P, P_1 \dots P_3$, which is called an involute of the fixed curve. The fixed curve is called the evolute of $P, P_1 \dots P_3$.

Since each point in the rolling tangent has a definite locus, there may be any number of involutes of a given curve.

257. **Problem**. Given a fixed curve $0, 0_1 \dots 0_2$ and a fixed point P, draw an involute of the curve to pass through P.

First Method. Draw PE

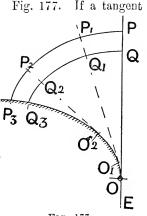


Fig. 177.

tangent to the curve and determine the point of contact $O.^*$ Draw OP on a piece of tracing-paper. Insert a pricker at O and rotate the tracing through a *small* angle, then prick through the new position of P. Transfer the pricker to the new point of contact O_1 , and further rotate the tracing and prick through P. Continue in this way until the line on the

^{*} Employ the construction of Art. 254 if knowledge of the evolute provides no easier method.

tracing measures itself along the curve O... O_2 . A fair line drawn freehand (or aided by French curves) through the different positions occupied by P is the required involute.

Second Method. To draw the involute by continuous motion. Determine O the point of contact of the tangent from P as before. Insert pins close together along the curve O. O₂, or cut a templet to fit the curve. Fasten a flexible and inextensible thread to the drawing-board at O. Make this thread taut in the direction OP and attach a pencil to the thread at P. If this pencil be moved in an anti-clockwise direction, the thread, if kept taut, is wrapped against the curve O. O₂, and the pencil traces the involute P, P₁. P₃.

258. If we consider the curves $P cdots P_3$, $Q cdots Q_3$, Fig. 177, as the paths of definite points P, Q in the tangent PE as it rolls along the fixed curve $Q cdots Q_2$, we see that the direction of the curves at corresponding points— P_2 , Q_2 —must be the same as that of the generating points P, Q when they occupy

those particular positions.

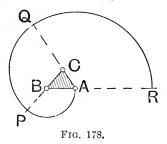
Now clearly at any particular instant the generating points P and Q can move in one direction only, viz., at right-angles to the direction of EP at the particular instant. Thus the curves PP_3 , QQ_3 have the same direction and are consequently parallel. From this we conclude that all involutes derived from a fixed evolute are parallel lines.

259. Since the rolling tangent of the *evolute* is always normal to the *involute*, the evolute may be defined as the locus of the intersection of normals at consecutive points in a curve. As there can be but one such line for any particular

curve, it follows that any plane curve has a definite

evolute, and one only.

260. In Fig. 178, APQR is the involute of the triangle ABC. This curve consists of three circular arcs: AP, centre B, radius BA; PQ, centre C, radius CP = CB + BA, and QR, centre A, radius AQ = AC + CB + BA, or



the perimemeter of the triangle. The *involute* of *any* rectilinear figure is made up of tangential arcs, and conversely, every curve comprised of arcs of circles has a rectilinear evolute.

261. Curvature.* The curvature of a line is its deviation from a straight line. The curvature of a circle is the same at every point, and in order that the curvatures of different circles may be compared, curvature is measured by the reciprocal of the radius of the circle, *i.e.*, by $\frac{1}{r}$ where r is the radius. It follows that, of two circles that which has the greater radius has the smaller curvature.

The curvature of curves other than circles is not constant,

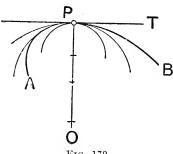


Fig. 179.

than circles is not constant, but varies continually. The amount of curvature at any point is equal to that of the circle which passes through the point and has the most intimate contact with the curve at that point. In Art. 253, we have seen that any plane curve and its tangent have two consecutive points in common. Consider now the curve APB, Fig. 179, and its tan-

gent PT. Clearly any circle having centre on the normal PO and passing through P is tangent to the curve AB, and has two consecutive points in common with the curve AB. An infinite number of such circles can exist; there must, however, exist one circle which has closer agreement with the curve AB at P than any other circle can possibly have. This particular circle is considered as passing through the point P and two other points one on either side of, and infinitely close to, P. This circle is said to osculate the curve at P, and is called the osculatory circle,† and its

^{*} See also Art. 375.

than a curve of varying curvature, the curvature at a point on one side of a given point P is, in general, greater than at P, whilst at a point on the opposite side of P the curvature is less than at P. Hence the esculatory circle of a point P always crosses the curve at P unless the curvature is equal on both sides of the point, in which case there is a cusp in the evolute at the centre of curvature of the point. Examples of this type occur at the ends of the axis of a conic.

curvature is equal to that of the curve at the point P. The centre and radius of the osculatory circle are respectively the centre of curvature, and radius of curvature of the curve at the point P.

Since the osculatory circle of a point P passes through three consecutive points in the curve, it follows that its centre must be at the meeting point of normals drawn from points consecutive at P; but, the locus of all such points is the evolute of the curve, hence the centre of curvature of any point in a plane curve must lie on the evolute of the curve.

To find the locus of centres of curvature of any plane curve. In the case of a known mathematical curve, an algebraic expression for the radius of curvature at definite points can, in general, be found by analysis. These expressions may often be interpreted graphically. When a curve is given and nothing is known of its properties, its evolute may be found by drawing normals at different points, and then drawing the envelope of these normals. The evolute is the locus required.

To determine the centre of curvature at a given point on a curve. Draw from the point a tangent to the evolute and find its point of contact which is the required centre.

It is stated above that the curvature of a curve at any point is equal to that of the circle which osculates the

curve at that point, and that the osculatory circle is the circle passing through the point, and two other points, one on either side of, and infinitely close to, the point. Let P, Fig. 180, be a point on a

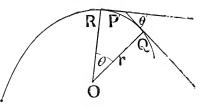


Fig. 180.

curve, and let R and Q be two points close to P, and on either side of P. Let O be the centre of a circle which passes through R, P, and Q, and lies practically along the curve from R to Q. Then if r is the rad us of the circle, $\frac{1}{n}$ is the

curvative of the circle. Now draw tangents to the circle at R and Q, and let θ be the angle between these tangents. The angle QOR will also $= \theta$. The tangent at R gives the direction of the circle at R, and that at Q gives the direction at Q (Art. 214); hence the angle θ gives the change in direction of the circle from R to Q. The angle θ \div length of arc RQ gives the change in direction per unit length of the arc, or the average change in direction from R to Q. But the length of the arc RQ is $r\theta$, as the angle θ is usually given in radians. Hence

average change in direction average change in direction $=\frac{\theta}{1}=\frac{\theta}{r\theta}=\frac{1}{r}$ which is the curvature of the circle. The curvature of a circle being constant, the change in direction per unit length is also constant, but for any other curve the curvature is not constant, and hence the change in direction per unit length is not constant. The average change in direction of a change in direction, and this curve is, however, taken as length of curve quantity is also called the average or mean curvature of the curve, for it gives the reciprocal of the radius, or the curvature, of a circle which would be tangential to the curve at each end of the length of curve under consideration, and hence would have the same direction as the curve at its end points, and a constant change of direction between the two points.

If a railway runs due north to a point A, and then curves from A to another point B, and then runs east, and if A and B measured along the curve are 2 miles apart, the direction has changed 90° or $\frac{\pi}{2}$ radians in 2 miles. The average change in direction per mile between A and B is $\frac{\pi}{2} \div 2$ or $\frac{\pi}{4}$ radians per mile. Hence $\frac{\pi}{4}$ or 0.7854 is the reciprocal of the radius of a circular are joining A and B, or if r miles is the radius of the curve, $\frac{1}{r} = 0.7854$ and $r = \frac{1}{0.7854} = 1.273$ miles. The average curvature is $\frac{1}{r}$ or 0.7854. The units for the angle and the distance should always be stated.

Exercise. A railway train changes in direction 15° in a distance of 300 ft.; what is the average curvature of the line? The curvature of a circular arc means the reciprocal of the radius.

(B. E. 1911.)

Exercises.

(1) State Simpson's rule. An area is divided into ten equal parts by eleven equidistant parallel lines 0.2 in, apart, the first and last touching the boundary curve; the lengths of these lines or ordinates or breadths are, in inches:—

 $0,\, 1\cdot 24,\, 2\cdot 37,\, 4\cdot 10,\, 5\cdot 28,\, 4\cdot 76,\, 4\cdot 60,\, 4\cdot 36,\, 2\cdot 45,\, 1\cdot 62,\, 0.\quad \text{Find the area}$

in square inches. (B.E. (2) 1905.) Ans. 6.25 sq. ins.

(2) In the following Table, A is the area in square feet of the horizontal section of a ship at the level of the surface of the water when the vertical draught of the ship is h ft. When the draught changes from 17.5 to 18.5 ft., what is the increased displacement of the vessel in cubic feet? (B.E. (2) 1906.)

h, A; 15, 6020; 18, 6660; 21, 8250. Ans. 6660 cu. ft.

(3) The sections of the two ends of a barrel are each 12·35 square feet; the middle section is 14·6 square feet; the axial length of the barrel is 5 ft., what is its volume? (B.E. (2) 1910.) Ans. 69·25 cm. ft.

(4) x being distance in feet across a river measuring from one side, and y the depth of water in feet, the following measurements were made:—

x	. 0	10	25	33	40		co	70
y	0	4	7	8	10	9	6	4

Find the area of the cross-section. If the average speed of the water normal to the section is 3.2 ft. per second, what is the quantity flowing in cubic feet per second? (B.E. (2) 1906.) Ans. 447.3 sq. ft; 1,431 cu. ft. per sec.

(5) If A, B and C are the angles of a spherical triangle, $A + B + C - 180^\circ$, divided by 330°, is equal to the area of the triangle divided by the area of the hemisphere. The sphere is 8,000 miles in diameter. What is the area of the hemisphere? If $A = 55^\circ$, $B = 62^\circ$, $C = 64^\circ$. What is the triangle in square miles? (B.E. 1998.) Ans. 100-53 \times 106; 33-51 \times 106.

See also Exercises 16, 17, pp. 117-118,

CHAPTER XXI.

VARIATION.

262. Many quantities are related to each other in such a manner that any change in the magnitude of one quantity produces a change in the magnitude of the other. For example, consider a pavement of given width. The area of the pavement will depend upon its length. If the width is 6 ft., then the area of a length of 300 ft. of such a pavement would be 1800 sq. ft., whereas if we take a length of 600 ft. of the pavement we get an area of 3600 sq. ft., and so on. The area is said to vary with the length.

Now take the case of a square of given length of side, say 6 ft.; its area is 36 sq. ft. If we make the side 12 ft. long, we make the area 144 sq. ft., and again we say the area varies with the length of the side.

Again, take a cubical block of stone, say of length of side 2 ft. Its volume is 8 cub. ft. If we now make the length of each side of the block 4 ft., we make the volume 64 cub. ft. Here also the volume varies with the length of the side.

From the three examples set out above, it will be noted that the change in the magnitude of the second quantity is not always proportional to the change in that of the first quantity.

In the first example, when we double the length of the pavement, we double the area. In this kind of variation, the change in the magnitude of the second quantity (area) is proportional to the change in that of the first (length), and the second quantity is said to vary directly as the first.

Put into general terms, we may say that one quantity y

varies **directly** as a second quantity x, when any change in the magnitude of y produces a proportional change in the magnitude of x. (The word change includes an increase or a decrease.)

In the second of the above examples, when we double the length of the side of the square, we make the area four times as large—i.e., increasing the side from 6 to 12 ft. increases the area from 36 to 144 sq. ft. The ratio of the areas $\frac{144}{36}$ or 4 is equal to the square of the ratio of the

lengths $\frac{12}{6}$ or 2. When this is the case, the quantity y is said to vary directly as the square of the quantity x.

In the third example, when we double the length of the side of the cube, we make the volume eight times as great. The ratio of the volumes $\frac{64}{8}$ or 8 is the cube of the ratio of

the lengths $\frac{12}{6}$ or 2. In this case, the quantity y is said to vary directly as the cube of the quantity x.

The same kind of reasoning may be extended to the 4th, 5th, or any power of x.

In all cases of *direct variation*, it will be noted that increasing one quantity increases the other, and decreasing one quantity decreases the other. If we take y and x as the two quantities before change, and y^1 and x^1 as the quantities after the change, then

(1) If y varies directly as x, we have
$$\frac{y^1}{y} = \frac{x^1}{x}.$$

(2) ,, ,, as the square of
$$x$$
, we have $\frac{y^1}{y} = \left(\frac{x^1}{x}\right)^2$.

(3) , , , as the cube of
$$x$$
, we have $\frac{y^1}{y} = \left(\frac{x^1}{x}\right)^3$.

From these equations, we can determine any one of the four quantities, provided the other three are given.

Ex. It is known that the strength of a beam varies directly as the

square of the depth.

A beam of given length and width and 6 in. deep carries a maximum load of 5 tons. What load will a beam of the same width and length, but 12 in. deep, carry?

Let $x = \max \max$ load of second beam, then

$$\frac{x}{5} = \left(\frac{12}{6}\right)^2$$

$$x = 5 \times \left(\frac{12}{6}\right)^2$$

$$= 5 \times 4$$

$$= 20 \text{ tons.}$$

The statement y varies as x is written $y \propto x$, the symbol \propto being used to denote "varies as."

263. Certain quantities are so related that an increase in one produces a decrease in the other, and *vice versā*. For example, if a beam 10 ft. long carries a maximum load of 10 tons, it is known that a beam 20 ft. long but otherwise similar to the first beam will only carry 5 tons. Here the two related quantities concerned are the length of, and the maximum load carried by, the beam, and we see that if we double one we halve the other. This kind of variation is called inverse variation, and the maximum load is said to vary inversely as the length of the beam.

Putting this statement into general terms, one quantity y is said to vary **inversely** as a second quantity x when any increase in the magnitude of y produces a corresponding decrease in the magnitude of x, and $vice\ versa$.

As a further example, we may take the case of a series of rectangles all of the same area, say 36 sq. ft., but of different lengths and breadths. If we choose any magnitude less than 36 for the length, then we can ascertain at once the breadth by dividing the area by the length.

Thus, we can take

When the breadth is 3, the length is 12; if we double the breadth, *i.e.*, make it 6, we halve the length, *i.e.*, make it 6; hence the breadth is inversely proportional to the length.

From the above figures it will be observed that the length is always equal to $\frac{1}{\text{breadth}}$ × area, thus

$$12 = \frac{1}{3} \times 36$$

$$18 = \frac{1}{2} \times 36 \text{ and generally}$$

$$y = \frac{1}{x} \times 36$$

and again, the ratio of the lengths

$$\frac{12}{18} \text{ is equal to } \underbrace{\left(\frac{1}{3} \times 36\right)}_{\left(\frac{1}{2} \times 36\right)} \text{ which simplifies}$$

$$\text{to } \frac{2 \times 36}{3 \times 36} \text{ or } \frac{2}{3} \text{ an 1 is seen to be}$$

the inverse ratio of the breadths.

Now if we put y = 12, $y^1 = 18$, x = 3, $x^1 = 2$, then we get as a general statement $\frac{y}{y^1} = \frac{x^1}{x}$, and this proportion expresses generally the condition when one quantity y varies inversely as another quantity x.

264. The statement that y varies inversely as x is written

$$y \propto \frac{1}{x}$$
, hence $\frac{y}{y^1} = \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^1}\right)} = \frac{x^1}{x}$.

As in direct variation, we can determine any one of the four magnitudes y, y^1 , x, x^1 , provided the other three are given.

We may note here in passing that when one quantity y varies inversely as another quantity x, the product xy is always a constant quantity. This can be tested by referring to the example above, where the length \times breadth = area of rectangle, the latter quantity always being equal to 36, which is a constant quantity.

A further example will make this inverse variation clear. The pressure of steam in an engine cylinder varies inversely as the volume. If a volume of 50 cub. in. is at a pressure of 100 lb.

per square inch, what will be the pressure per square inch when the volume increases to 75 cub. in. ?

Let p = original pressure, and $p^1 = \text{new pressure}$. , r = original volume, and $r^1 = \text{new volume}$.

Then
$$p \propto \frac{1}{r}$$

$$\frac{p}{r^{1}} = \frac{\frac{1}{v}}{\frac{1}{v^{1}}} = \frac{v^{1}}{v}$$

$$\frac{100}{p^{1}} = \frac{75}{50}$$

$$t^{1} = \frac{50 \times 100}{75} = 66.6 \text{ lb. per sq. in.}$$

265. When we were discussing direct variation, we dealt with cases in which y varied directly as x, or as x^2 , x^3 , and so on. In the above articles, we have dealt with the case in which y varies inversely as x, and examples are numerous in which y varies inversely as the square of x, or as the cube of x, and so on.

Thus,
$$y$$
 may $\propto \frac{1}{x^2}$

$$y \text{ may } \propto \frac{1}{x^2}$$

$$y \text{ may } \propto \frac{1}{x^{2^3}} \text{ etc.}$$

When $y \propto \frac{1}{x^2}$, it will be seen that, if we double x we reduce y to one-quarter of its original value or $\left\{\frac{1}{2^2} \times \text{ original value}\right\}$; if we make x three times its

value, we reduce y to one-ninth or $\left\{\frac{1}{3^2} \times \text{ original value}\right\}$.

As an example, the deflection of a beam under a given load is inversely proportional to the square of the depth. If a given beam carrying a certain load is 5 in deep and has a deflection at the centre (due to the load) of 2 in., what deflection will be produced by the same load if the depth be 8 in.?

Let x and x^1 = depths of beams. ., y and y^1 = deflections of beams.

Then since
$$y \propto \frac{1}{x^2}$$

 $\frac{y}{y^1} = \frac{(x^2)^2}{(x^2)^2}$
 $\frac{2}{y^1} = \frac{(8)^2}{(5)^2}$
 $y^1 = \frac{2 \times 25}{64} = 0.78 \text{ in.}$

266. When y varies inversely as the cube of x, or $y \propto \frac{1}{x^3}$, it will be seen that, if we double x we make $y = \frac{1}{x^3}$

one-eighth of its original value, or $\left(\frac{1}{2^3} \times \text{original value}\right)$.

As an example, suppose we have a certain compressible material in such a state of compression that the weight of a cube having each side 2 in, long is 560 lb. The weight of each cubic inch is then $\frac{560}{8} = 70$ lb., since we have 8 cub. in, of the material. If we now increase each side of the cube to 3 in., the weight per cubic inch is $\frac{560}{27} = 20\frac{29}{4}$ lb., and we see that the weight per cubic inch of the material is inversely proportional to the cube of the length of the side (the total weight remaining constant), for if so, the ratios:—

Weight per cubic inch for 2-in, side, viz., 70 lb.

Weight per cubic inch for 3-in, side, viz.,
$$20\frac{20}{10}$$
 lb.

or simplifying
$$\frac{70}{\frac{560}{27}} = \frac{1}{\frac{27}{8}}$$

$$\frac{27 \times 70}{\frac{560}{8}} = \frac{27}{8}$$

267. In all the preceding examples on variation, we have been concerned only with comparing magnitudes; for example, in the case of the maximum loads carried by beams of given depths we compared the loads and compared the depths.

Now it is often desirable to know the exact relation between the magnitudes of two quantities, in addition to the way in which one depends upon, or varies with, the other.

In an example, say y varies as x^2 ; if we make x=2 it does not follow that y will be =4. y may have any magnitude, say 20, when x has the value 2.

Then we can say
$$y \propto x^2$$
 and $y = 20$ when $x = 2$.

If now we make x = 4, then y must equal

80, for
$$\frac{y^1}{y} = \frac{(x^1)^2}{(x)^2}$$

 $\frac{y^1}{20} = \frac{(4)^2}{(2)^2}$
and $y^1 = \frac{16 \times 20}{4} = 80$.

We now know that

$$y \propto x^2$$

 $y = 20$ when $x = 2$ and $y = 80$ when $x = 4$
 $y = 20$ when $x^2 = 4$ and $y = 80$ when $x^2 = 16$.

The ratio $\frac{y}{x^2} = \frac{20}{4} = \frac{80}{16} = 5$, for all corresponding values of y and x^2 .

$$\therefore \frac{y}{x^2} = 5 \text{ for all magnitudes of } y \text{ and } x,$$

and hence $y = 5x^2$.

This equation expresses the exact relation between y and x, *i.e.*, it states that whatever value we give to x, then, if we square that value and multiply by 5, the product is y. The figure 5 is called **a constant**.

To obtain an exact relation between x and y, we must always determine the value of the constant. If we know any two corresponding values of x and y, and if we know how x and y are related, we can find this constant.

In the above example, let K = the constant.

Then
$$y \propto x^2$$

 $y = Kx^2$
 $20 = K(2)^2$ and $K = \frac{20}{4} = 5$.

In a similar manner, we can always replace the statement

$$y \propto x^{2}; \frac{1}{x^{2}}; x^{8}, \text{ etc., by}$$

 $y = Kx^{2}; K. \frac{1}{x^{2}}; K. x^{8}, \text{ etc.,}$

and from the two given values of y and x determine the value of Λ .

Ex. The pressure of a gas varies inversely as the volume. When the volume is 20 cmb, in, the pressure is 50 lb, per square inch. Find the exact relation between pressure and volume.

Let p = pressure and r = volume.

Then
$$p \propto \frac{1}{e}$$

 $p = K \times \frac{1}{e}$
 $50 = K \times \frac{1}{20}$
 $K = 50 \times 20 \approx 1,000$

and hence $p = 1,000 \times \frac{1}{v}$ is the exact relation

required. If, now, we are given any value of p or r, we can find the corresponding value of r or p.

268. The following miscellaneous examples will illustrate the various principles of ratio, proportion, and variation.

Ex. 1. The specific gravity of a substance is the ratio between the weight of a given volume of the substance and that of an equal volume of water.

If the specific gravity of oak is 0.93, find the weight of an oak beam 20 ft. long 1 ft. deep and 8 in. wide, given that a cubic foot of water weighs 62.3 lb.

Volume of beam =
$$\left(20 \times 1 \times \frac{8}{12}\right) = \frac{40}{3}$$
 cub. ft.
Weight of $\frac{40}{3}$ cub. ft. of water = $\left(\frac{40}{3} \times 62 \cdot 3\right)$ lb.

$$\frac{\text{Weight of } \frac{40}{3} \text{ cub. ft. of oak}}{\text{Weight of } \frac{40}{3} \text{ cub. ft. of water}} = \frac{0.93}{1}$$

Weight of oak = $\left\{\frac{40}{3} \times 62.3 \times 0.93\right\}$ lb. = 772.52 lb.

Ex. 2. In a map drawn to a scale of 2 in, to a mile, an island covers an area of 7 s₄, in. Find the area covered by the same islan I when the scale is 5 in, to a mile.

Area are proportional to the squares of corresponding lengths.

Area of island on large scale map

Area of island on small scale map $\therefore \frac{\text{Area of island on small scale map}}{7} = \left(\frac{\text{Scale of large map}}{\text{Scale of small map}}\right)^2$ $\therefore \frac{\text{Area of island on large scale map}}{7} = \left(\frac{5}{2}\right)^2$ $\therefore \text{Area of island on large scale map} = \frac{7 \times 5 \times 5}{2 \times 2} = \frac{175}{4}$ = 43.75 sq. in.

Ex. 3. Two spheres have radii of 8 and 2 in, respectively. the ratio of their volumes.

Volumes are proportional to cubes of corresponding lengths.

reportional to cubes of corresponding len
$$\frac{\text{Volume of large sphere}}{\text{Volume of small sphere}} = \left(\frac{8}{2}\right)^3$$

$$= \frac{8 \times 8 \times 8}{2 \times 2}$$
Retio of volumes = $\frac{6 \times 2}{2}$

Ratio of volumes = 64

Ex. 4. The velocity of a moving body is the space described in unit time. The time taken to describe a given space is inversely proportional to the velocity. Find the ratio between the velocities of two moving bodies A and B which take respectively 5 and 15 sec. to travel a given distance.

Let v_A and v_B be velocities respectively of A and B, and t_A and t_B

be times.

$$\frac{v_A}{r_B} = \frac{\frac{1}{t_A}}{\frac{1}{t_B}}; \ \frac{v_A}{v_B} = \frac{t_B}{t_A} = \frac{15}{5} \text{ or } v_A = 3v_B.$$

The pressure per square inch on a hydraulic ram is inversely proportional to the square of the diameter, if the total load on the ram is constant.

If a load supported by a hydraulic ram of 10 in, diameter gives a pressure per square inch of 50 lb., find the pressure per square inch on a 3-in, ram which supports an equal load.

Pressure per square inch on 10-in. ram Pressure per square inch on 3-in. ram
$$= \frac{\left(\frac{1}{10}\right)^2}{\left(\frac{1}{3}\right)^2} = \left(\frac{3}{10}\right)^2$$

 $= \frac{50 \times 10^2}{3^2}$ $= \frac{5000}{9}$ Pressure per square inch on 3-in, ram

 $=455\frac{5}{9}$ lb. per sq. inch.

- A standard beam 20 ft, long, 6 in, wide, and 10 in, deep, carries a safe load of 8 tons. Find the safe load carried by the following beams of the same shape and material:—
 - (1) Beam 20 ft, long, 12 in. wide, 10 in. deep,
 - (2) Beam 20 ft. long, 6 in. wide, 20 in. deep, (3) Beam 40 ft. long, 6 in. wide, 10 in. deep,

given that the safe load varies directly as the width, directly as the square of the depth, and inversely as the length of the beam.

Beam 1. Width of standard beam (6 in.)
$$= \frac{8 \text{ tons}}{x \text{ tons}}$$
. $\therefore x = \frac{12 \times 8}{6} = 16 \text{ tons}$.

Ex, 7. A quantity y varies inversely as the cube of a quantity x and when y is 56, x is 2. Find the exact relation between y and x.

$$y \propto \frac{1}{x^3}$$

 $y = K \frac{1}{x^3}$ where K is a constant
 $56 = K \frac{1}{(2)^3} = \frac{K}{8}$
 $K = 56 \times 8 = 448$.

Thus, $y = 448 \cdot \frac{1}{x^3}$ is the exact relation required.

Ex. 8. $y \propto x^n$ and y = 40 when x = 2, and $y = 5{,}000$ when x = 10. Find the exact relation between y and x.

$$y = Kx^n$$
 where K is a constant
 $40 = K2^n$ (Equation 1)
 $5000 = K10^n$ (Equation 2)
 $\therefore \frac{5000}{40} = \frac{10^n}{2^n} = 5^n$
 $125 = 5^n$,
But $5^3 = 125$

From Equation 1— n = 3.

$$\begin{array}{l}
40 = K2^{n} \\
40 = K2^{3} \text{ since } n = 3 \\
\frac{40}{8} = K
\end{array}$$

Exact relation is $y = 5x^3$.

To find n generally, in cases where its value cannot be ascertained by inspection, e.g., in the above case in which $125 = 5^n$.

Take logs. of both sides of equation, hence

$$\log 125 = n \log 5.$$

$$\therefore n = \frac{\log 125}{\log 5} = \frac{2.0969}{0.0990} = 3.$$

Ex. 9. The absolute pressure p lb. per sq. in., and the volume n enh. ft. of saturated steam are such that $p \propto u^n$. When p = 10, u = 37.87, and when p = 50, u = 8.352. Find the exact relation between p and u.

$$p = Ku^n$$
 where K is a constant $10 = K 37.87^n$ (Equation 1) $50 = K 8.352^n$ (Equation 2) $\therefore \frac{10}{50} = \left(\frac{37.87}{8.352}\right)^n$.

Taking legs.
$$\log 10 - \log 50 = n (\log 37.87 - \log 8.352)$$
.

$$n = \frac{\log 10 - \log 50}{\log 37.87 - \log 8.352} = \frac{1 - 1.6990}{1.5783 - 0.9218}$$

$$= \frac{0.6990}{0.6565} = -1.0647.$$

From Equation 1.
$$10 = K$$
. $37.87^{-1.0647}$
 $\therefore K = 10 \div 37.87^{-1.0647} = 10 \times 37.87^{1.0517}$
 $= 479$.

Hence $p u^{1.0647} = 479$ is the relation required.

Exercises.

(1) The deflection of a beam is directly proportional to the load and to the cube of the span, inversely proportional to the width and to the cube of the depth. Express this algebraically.

If a beam of 15 ft. span, 3 in. wide, and 4 in. deep has a deflection of $\frac{1}{10}$ in. under a certain load, what deflection will the same load produce in a beam of 20 ft. span, 4 in. wide, and 6 in. deep?

(2) The strength of a beam varies directly as the breadth, directly

as the square of the depth, and inversely as the length.

A beam is 16 ft. long, 6 in. broad, and 8 in. deep. A second beam is 7 in. broad and 9 in. deep. Find the length of the second beam if the two are to be of equal strength.

(3) Compare the strengths of two beams each loaded at the centre and each of rectangular section 6 in. by 3 in., the first beam having the longer side vertical and of 20 ft. span, the second having the shorter side vertical and of 10 ft. span.

(4) A beam 20 ft. long, 5 in. wide, and 10 in. deep carries a load of 2 tons at the centre. What load will be carried if the width and

depth are doubled?

(5) A test bar 1 in, \times 1 in, \times 20 in, breaks under a load of 350 lb. Under what load will a beam 6 in, wide \times 10 in, deep \times 20 ft, long break?

(6) Compare the strengths and the weights of two shafts, one 12 in diameter and solid, and the other 12 in diameter with a 6 in hole through it, given that the strengths are as $D^3: \frac{D^4 - d^4}{D}$, D

being external diameter and d internal diameter.

(7) The volumes of spheres are proportional to the cubes of the

diameters. Compare the volumes of three spheres whose respective diameters are 3.76, 4.89, and 6.71 ft.

(8) The electrical resistance of copper wire is proportional to its length divided by its cross section. Show that the resistance of a pound of wire of circular section all in one length is inversely proportional to the fourth power of the diameter of the wire. (B.E. 1903.)

(9) The value of a ruby is said to be proportional to the $1\frac{1}{2}$ power of its weight. If one ruby is exactly of the same shape as another, but of 2.20 times its linear dimensions, of how many times the value

is it? (B.E. 1904.)

(10) If u is usefulness of flywheels, $u \propto d^5 n^2$, if d is the linear size (say diameter) and n the speed. We assume all flywheels to be similar in shape. I wish to have the usefulness 100 times as great, the speed being trebled; what is the ratio of the new diameter to the old one? (B.E. 1905.)

(11) There are two maps, one to the scale of 2 in. to the mile, the other to the scale of $\frac{1}{2}$ in. to the mile. The area of an estate on the first map is 1.46 sq. in.; what is the area of this estate on the second

map? (B.E. 1906.)

(12) The horse power of the engines of a ship being proportional to the cube of the speed; if the horse power is 2,000 at a speed of 10 knots, what is the power when the speed is 15 knots? (B.E. 1906.)

(13) If y varies as the square root of x, and if y is 5 when x is 3, state the true formula connecting x and y. What is y if x is 9? (B.E. 1907.)

(14) If y is proportional to the square root of x, and if y is 3.5when x is 4, express y in terms of x. What is y when x is 25?

(B.E. 1908.)

(15) The electrical resistance of a wire is $R \propto l/a^2$ where l is length and d is diameter; its weight W $\propto ld^2$. Show that the resistance of a wire $Ik \propto W/d^4$. If a pound of wire of diameter d = 0.06 in. has a resistance of 0.25 ohms, what is the resistance of a pound of wire of the same material, the diameter being 0.01 in.? (B.E. 1907.)

Answers.

(1) 0.053 in.	(9) 34.74.
(2) 23 ft, $7\frac{1}{2}$ in.	(Ì0) 1·619.
(3) Equal strength.	(11) 0.091 sq. in.
(4) 16 tons.	(12) 6750.
(5) 17,500 lb,	(13) 9.88 1/4. 8

(6) 100: 93.75 strengths; 100:

75 weights.

(7) 53·16; 116·7; 302·1.

(13) $2.88 \sqrt{x}$; 8.64. (14) $y = 1.75 \sqrt{x}$; 8.75.

(15) 324 ohms.

CHAPTER XXII.

SIMULTANEOUS SIMPLE EQUATIONS. QUADRATIC AND OTHER EQUATIONS.

269. We can find two unknown quantities if we are given two distinct and separate equations in which the unknown quantities occur. Thus, if x and y are unknown quantities, and if we know that

$$3x + 4y = 18$$
; and also that $5x + 6y = 28$;

then we have two equations, and the required values of x and y must satisfy each equation simultaneously.

There are two algebraic methods of solving such equations:—

(1) By elimination.

(2) By substitution.

270. The **method of elimination** consists in multiplying each equation by a number which will make either the x terms or the y terms equal. We can then subtract or add the two equations, and the equal terms will cancel out or become eliminated. We thus derive one simple equation from which the value of the other unknown quantity can be found. Thus, in the above example—

$$3x + 4y = 18$$
 (Equation 1)
 $5x + 6y = 28$ (, 2).

In order to make the x terms equal,

Multiply Eq. 1 by 5. 15x + 20y = 90 (Equation 3) , Eq. 2 by 3. 15x + 18y = 84 (, 4) Now subtract

Eq. 4 from Eq. 3

y = 3.

To find x, substitute this value for y in either of the equations 1 or 2.

Taking Eq. 1, we have
$$3x + 4y = 18$$
.
Putting $y = 3$, we have $3x + (4 \times 3) = 18$
 $3x = 18 - 12$
 $= 6$
 $x = 2$.

Thus, x = 2 and y = 3 are the values which satisfy both equations.

This result can be tested by putting x = 2 and y = 3 in both equations.

271. By the **method of substitution** we find from either equation the value of one unknown quantity in terms of the other, and then substitute this value in the other equation. We thus get a simple equation from which we can determine one unknown quantity.

Thus,
$$3x + 4y = 18$$
 (Equation 1)
 $5x + 6y = 28$ (, , 2).
From Equation 1, $3x = 18 - 4y$
hence, $x = \frac{(18 - 4y)}{3}$

Substitute this value for x in Equation 2, and we get

$$5x + 6y = 28$$
$$5\left(\frac{18 - 4y}{3}\right) + 6y = 28$$

Multiply both sides by 3.
$$5(18 - 4y) + 18y = 84$$

Simplifying. $90 - 20y + 18y = 84$
 $- 20y + 18y = 84 - 90$
 $- 2y = -6$

$$2y = -6$$

 $y = \frac{-6}{-2} = 3.$

As before, put this value for y in Equation 1 or 2 and we get x = 2,

272. When the x and y terms are not on the left-hand side of the equation, they must be transposed before we begin to solve the equations.

Thus, the equations 5x = 45 - 6y4y = -7x + 415x + 6y = 45 (Equation 1) are first rearranged to 7x + 4y = 4135x + 42y = 315 (Multiply Eq. 1 by 7. 35x + 20y = 205 (,, 2 by 5. 4) Subtract Eq. 4 from Eq. 3 22y = 110

y = 5.hence By substituting this value for y in Equation 1, we find

5x + 6y = 455x + 30 = 455x = 15hence x = 3.

273. If the two equations have fractional x and y terms

it is advisable first to multiply each by a number which will climinate the denominators of the fractions.

Thus,
$$\frac{x}{5} + \frac{y}{4} = 5$$
 (Equation 1)
$$\frac{x}{2} + \frac{y}{3} = 9 \quad (, , 2)$$
Multiply Eq. 1 by 20.
$$y, y, 2$$
 by 6.
$$x^2 + \frac{y}{3} = 9 \quad (, , 2)$$
Multiplying Eq. 3 by 3.
$$x^2 + 2y = 54 \quad (, , 4)$$
Now eliminate x by—
$$x^2 + \frac{y}{3} = 9 \quad (, , 3)$$
Multiplying Eq. 3 by 3.
$$x^2 + 2y = 54 \quad (, , 4)$$
Now eliminate x by—
$$x^2 + \frac{y}{3} = 9 \quad (, , 3)$$
Multiplying Eq. 3 by 3.
$$x^2 + 2y = 54 \quad (, , 5)$$

$$x^2 + 3y = 216 \quad (, , 6)$$
Subtract Eq. 6 from Eq. 5.
$$x^2 + 3y = 216 \quad (, , 6)$$
Substitute this value for y
in Eq. 3.
$$x^2 + \frac{y}{3} = 9 \quad (, 2)$$

$$x^2 + 3y = 100$$

$$4x + 5y = 100$$

$$4x + 60 = 100$$

$$4x = 40$$
hence
$$x = 10$$

274. There are many problems which require the use of two simultaneous equations for their solution, and some examples follow which illustrate methods of writing down algebraically the data of the problem.

If we have two numbers, one of which exceeds the other by 4, and if x is the greater and y the smaller number, we can represent this statement algebraically by x - y = 4.

If one number is one-sixth of the other, then $x = \frac{y}{6}$, or again, if one number is less by 5 than 7 times the other, we have x = 7y - 5 or 7y - x = 5.

If x pounds are to be divided amongst y people, then each will receive $\frac{x}{y}$ pounds.

If two trains are travelling at x and y miles per hour respectively, then their times in travelling 100 miles will be $\frac{100}{x}$ and $\frac{100}{y}$ hours respectively.

If P lb. is the force applied to a lifting-appliance to lift a weight W lb., and if P is always greater by 5 lb. than one-twentieth of W, this is represented algebraically by $P = \frac{1}{2\alpha} W + 5$.

The following examples illustrate problems involving simultaneous equations:—

Ex. 1. The greater of two numbers exceeds the smaller by 30, and the greater number is equal to two and a half times the less.—Find the numbers.

Let x =the larger number, and y =the smaller number

Then w - y = 30, since the greater exceeds the lesser by 30

also $x = \frac{5}{2}y$ or 2x - 5y = 0, since the greater is $2\frac{1}{2}$ times the lesser;

hence x - y = 30 (is Equation 1)

2x - 5y = 0 (is , , 2)Multiply Eq. 1 by 2, 2x - 2y = 60 (is , 3)

Subtract Eq. 2 from Eq. 3, 3y = 60

hence y = 20.

Also from Eq. 1 x - y = 30

Substitute 20 for y, then x - 20 = 30hence x = 50.

The two numbers are thus 50 and 20.

Ex. 2. A bar 100 ft, long is to be divided into two parts such that one part is half as long again as the other part. Find the length of each part.

Let x = length of larger part, and y = length of smaller part.

Then x + y = 100, since the whole bar is 100 ft. long.

and $x = \frac{3}{2}y$, since one part is half as long again as the other part.

Thus, x + y = 100 (Equation 1)

2x - 3y = 0 (, 2)

Multiply Eq. 1 by 2. 2x + 2y = 200 (3) Subtract Eq. 2 from Eq. 3. 5y = 200

hence y = 40 ft.From Eq. 1. x + 40 = 100

hence x = 60 ft.

The two parts of the bar are thus 60 and 40 ft. long.

Ex. 3. Two posts together weigh 5 cwt., and one is three times as heavy as the other. Find their weights.

Let \vec{x} lb. be the weight of the heavier post, and y lb. be the weight of the lighter post.

Then $x + y = (5 \times 112)$ lb. since their total weight is 5 cwt.

also x = 3y, for one post is three times as heavy as the other post.

hence x + y = 560 (is Equation 1) x - 3y = 0 (is ... 2)

Subtracting Eq. 2 from Eq. 1, 4y = 560 lb.

hence y = 140 lb.

Put this value for y in Eq. 1 and we get

x + y = 560x + 110 = 560

x + 110 = 560x = 420 lb.

The heavier post thus weighs 420 lb, and the lighter post weighs 140 lb.

Ex. 4. In a pulley-block lifting-tackle a force of 15 lb, will lift a load of 100 lb., and a force of 35 lb. will lift a load of 300 lb. If the force (P lb.) and the load (W lb.) are related by an equation of the form P = mW + K, find the values of m and K.

Here we can write down from the given data two simultaneous

equations from which m and K can be found.

We have
$$P = mW + K$$

 $15 = m \ 100 + K \text{ (Equation 1)}$
 $35 = m \ 300 + K \text{ (} \text{ ,} \text{ } \text{ 2)}$
Subtract Eq. 1 from Eq. 2. $20 = m \ 200$
hence $\frac{20}{200} = m$
 $\frac{1}{10} = m$.

Put this value for m in Eq. 1, and we have

$$15 = m100 + K$$

$$15 = \left(\frac{1}{10} \times 100\right) + K$$

$$15 = 10 + K$$

$$15 - 10 = K$$

$$5 = K$$
thus, $m = \frac{1}{10}$ and $K = 5$.

The equation P = mW + K becomes

 $P = \frac{1}{10}W + 5$, and this is called the "Law of the

Machine " (see p. 210).

Ex. 5. One-third of a certain number exceeds 5 times a second number by 12, and the second number is less by 21 than one-sixth of the first number. Find the numbers.

Let x = large number, and y = smaller number.

Then $\frac{x}{3}$ = one-third of larger number.

And since one-third the larger number exceeds 5 times the smaller number by 12,

$$\frac{x}{3} = 5y + 12$$
 (Equation 1).

Also, since the smaller number is less by 21 than one-sixth of the larger number,

$$y = \frac{x}{6} - 21 \quad \text{(Equation 2)}$$
Hence from Eq. 1 $-\frac{x}{3} - 5y = -12 \quad (-, -3)$
and from Eq. 2, $-\frac{x}{6} + y = -21 \quad (-, -4)$
Multiply Eq. 3 by 3. $x - 15y = -36 \quad (-, -5)$

Multiply Eq. 2 by 6.
$$-x + 6y = -126$$
 (Equation C)
Add Eqs. 5 and 6. $-9y = -90$ hence $y = \frac{-90}{-9} = 10$.
Put this value for y in Eq. 5, and we get

x - 15y = 36

x - 150 = 36x = 186.

Hence the two numbers are 186 and 10.

275. Simultaneous equations may be solved graphically by plotting on squared paper. If we first arrange each equation with the η term by itself on the left-band side, and the remaining terms on the right, we can then make out a table of corresponding values of x and y for each equation. as explained in Art. 174.

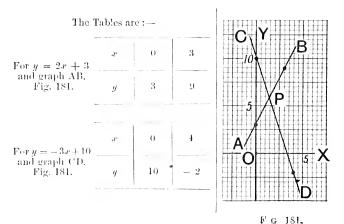
As an example, to solve graphically
$$6x + 9 = 3y$$

$$4y = -12x + 40 \text{ (Eq. 2)}$$

we first arrange each equation to have y by itself on the left-hand side. Thus, 6x + 9 = 3y becomes 3y = 6x + 9, and dividing by 3 becomes y = 2x + 3 (Eq. 3).

Also by dividing by 4, the equation 4y = -12x + 40

becomes y = -3x + 10 (Eq. 4). We now plot y = 2x + 3, and y = -3x + 10 on squared paper.



Y

The two graphs intersect at P, and the co-ordinates of P are x = 1.4, y = 5.8.

Solving the two equations algebraically, we also find that x = 1.4 and y = 5.8; hence two simultaneous equations are solved graphically by plotting their graphs on squared paper, and reading from the graphs the co-ordinates of the point of intersection. The x co-ordinate is the value of x, and the y co-ordinate is the value of y, which satisfies the two equations.

This result is to be expected, for, the co-ordinates of every point on the graph AB must fit the equation y = 2x + 3, and also the co-ordinates of every point on the graph CD must fit the equation y = -3x + 10. Now the co-ordinates of the point of intersection must fit both equations, for this point is on both graphs. We know that the values of x and y which give the solution of the two equations must also satisfy both equations, hence the co-ordinates of the point of intersection are the values of x and y which satisfy both equations.

276. If two quantities are connected by a **linear law**, the law may be found by solving two simultaneous equations, if two sets of corresponding values of the quantities are known.

Thus, as an example we may take temperature on a Centigrade and on a Fahrenheit seale.

If we put F for degrees Fahrenheit,

C for degrees Centigrade,

and let m and K be constants to be found from solving two simultaneous equations, we know that

Now when
$$F = 212^{\circ}$$
 $F = mC + K$.

Also when $F = 32^{\circ}$ $C = 100^{\circ}$ (Boiling point)

We thus have $F = mC + K$ (Freezing point).

We thus have $F = mC + K$ (Freezing point).

 $212 = m100 + K$ (Equation 1)

 $32 = mO + K$ (, 2).

Subtract Eq. 1 from Eq. 2. 180 = m 100,

hence $m = \frac{180}{100}$

$$= \frac{9}{5}$$
The results of the second state o

From Equation 1 we have, substituting this value for m, $\begin{array}{c} 212 = (\frac{a}{5} \times \tilde{1}00) + K \\ 212 = 180 + K \end{array}$

$$L = 160 + R$$

= $212 - 180$
= 32 .

Hence $F = \frac{9}{5}C + 32$, or

degrees Fahrenheit = $\binom{9}{5} \times \text{degrees Centigrade} + 32$.

Exercises.

(1) Solve
$$3x + 4y = 18$$
 (2) Solve $15x + 10y = 35$ $6x - 11y = -16$. (3) Solve $\frac{1}{2}x + \frac{1}{2}y = 5$ (4) Solve $2x + 10y = 3.74$ $5x - y = 0.25$.

(5) Two quantities are connected by a law y = mx + c. When x = 1, y = 11, and when x = -1, y = 1. Find the values of m

and c, and the value of y when x is 30.

(5) m = 5, c = 6; 156.

(6) If apples cost sevenpence per dozen and pears ninepence, a man would pay 142 pence for the fruit he intends to buy; whereas, if apples cost eightpence per dozen and pears tenpence, he would pay 160 pence; how many apples and pears does he intend to buy? (B.E. 1908.)

(7) The total cost C of a ship per hour (including interest and depreciation on capital, wages, coal, etc.) is $C = a + bs^3$, where s is

the speed in knots (or nautical miles per hour).

When s is 10, C is found to be 5.20 pounds. When s is 15, C is found to be 7.375 pounds.

Calculate a and b. What is C when s is 12? (B.E. 1905.) (8) $y = ax^2 + bx^3$. When x = 1, y is 4.3, and when x = 2, y is 30; find a and b. What is y when x is 1.5? (B.E. 1906.)

(9) If $V = 2\pi^2 x y^2$ and $A = 4\pi^2 x y$, and if V = 240 and A = 170,

find x and y. (B.E. 1907.) (10) Find two numbers such that if four times the first be added to two and a half times the second the sum is 17.3, and if three times the second be subtracted from twice the first the difference is 1.2. (B.E. 1903.)

Answers.

(10) 3.229; 1.753.

- (6) 120 apples, 96 pears. (1) x = 2, y = 3.(2) x = 1, y = 2.(7) 4.3; 0.0009; 5.8552. (3) x = 4, y = 8. (8) 1.1; 3.2. (4) x = 0.12, y = 0.35. (9) 1.5 ; 2.81.
- 277. Quadratic and cubic equations. An equation in which there is only one unknown quantity, and in which the highest power of the unknown quantity is the second power, is called a quadratic equation, e.g., $x^2 + 2x - 3 = 0$. Here the unknown quantity is x, and the highest power of the unknown quantity is x^2 . If in this equation we put x = 1, we get 1 + 2 - 3 = 0, and hence the value x = 1is a solution of the equation. Again, if we put x = -3, we get 9 - 6 - 3 = 0, and hence -3 is also a solution. There are thus two values of x which satisfy the equation, viz., x = 1 and x = -3, and each of these values is called

a root of the quadratic. The process of solving a quadratic equation consists in finding these two roots, and we may state here that every quadratic equation has two roots; the roots may be unequal, or equal, in value, or they may be imaginary quantities, such as $\sqrt{-1}$.

We shall limit ourselves in this chapter to a simple algebraic method, and to a graphical method, of solving a quadratic e pation; the latter method will also be found applicable to **cubic equations** (i.e., equations in which the unknown quantity is raised to the third power, e.g., $x^3 + 3x^2 + 2x + 1 = 0$), and also to equations containing the fourth power, or any higher power of the unknown quantity. Other algebraic and graphical methods of solution will be discussed in Chapter XXIII.

278. In Art. 88, we gave a rule to obtain the square of an expression consisting of two terms, thus:—

$$(x + 3)^2 = x^2 + 6x + 9$$

 $(x + 4)^2 = x^2 + 8x + 16$, and so on.

If we inspect the form of the square, we see that the term which does not contain either x or x^2 is the square of half the coefficient of the term containing x, provided that the coefficient of the term containing x^2 is 1 and is +. Thus, in $x^2 + 6x + 9$ the coefficient of x is 6. One-half of this coefficient is 3, and 3^2 or 9 is the term not containing x or x^2 . Similarly, for $x^2 + 8x + 16$. One-half of 8 = 4 and $4^2 = 16$.

An expression containing an x^2 term with a coefficient +1, and an x term with any coefficient + or - (such, for example, as $x^2 + 8x$), may be made into the square of an expression containing x and a numerical term equal to half the coefficient of x in the original expression (such, for example, as x + 4), by adding to it a term equal to the square of half the coefficient of the original x term. Thus, $x^2 + 8x$ can be made into the square of (x + 4) if we add to it 16, for $x^2 + 8x + 16 = (x + 4)^2$. Similarly, $x^2 + 7x$ can be made into the square of $(x + \frac{7}{2})$ by adding $\frac{4}{9}$ for $x^2 + 7x + \frac{4}{9} = (x + \frac{7}{2})^2$.

279. In one method of solving a quadratic equation, we first arrange the x^2 and x terms on the left-hand side of the equation, and the remaining term, not containing x^2 or x, on the right-hand side. Now if the coefficient of x^2 is — and is greater than 1, we can make it into + 1 by dividing

both sides of the equation by this coefficient and changing the signs. Thus,

 $-3x^2 - 15x = 18$ would become $x^2 + 5x = -6$, if we divide by 3 and change the

signs.

We now have on the left an expression which can be made into the square of $\left(x+\frac{5}{2}\right)$ by adding to it $\left(\frac{5}{2}\right)^3$ or $\frac{25}{4}$, and to preserve the equality of the two sides of the equation we must also add $\frac{25}{1}$ to the right-hand side.

We now get
$$x^2 + 5x + \frac{25}{4} = -6 + \frac{25}{4}$$
$$= -\frac{24}{4} + \frac{25}{4}$$
$$= \frac{1}{4}$$

(These + and $x + \frac{5}{2} = \pm \frac{1}{2} \begin{cases} \frac{1}{\sin x} & \text{eigns must} \\ \frac{1}{\sin x} & \text{other neg} \end{cases}$ If we extract the square root of each side of the equation, we get from which $x = \frac{1}{2} - \frac{5}{2} = -\frac{4}{2} = -2$ or $x = -\frac{1}{2} - \frac{5}{2} = -\frac{6}{5} = -3$.

Thus, x = -2 and x = -3 are the two roots of the equation.

280. The process set out in detail above is available for the solution of any quadratic equation, and may be summarized as follows:-

(1) Arrange the x^2 and x terms on the left, and the remaining term on the right of the equation.

(2) Make the coefficient of the x^2 term = + 1.

(3) Make the left-hand side into a perfect square, i.e., into the square of $x \pm (\text{half the coefficient of the } x \text{ term})^2$, and add a similar quantity to the right-hand side.

(4) Extract the square root of each side.

(5) Transpose terms to find the two values of x.

Ex. 1. Solve
$$8x^2 - 2x - 15 = 0$$
.

The steps are numbered to be read with the above summary.

Step 1.
$$8x^{2} - 2x = 15$$

$$x^{2} - \frac{1}{4}x = \frac{15}{8}$$

$$x^{2} - \frac{1}{4}x + \left(\frac{1}{8}\right)^{2} = \frac{15}{8} + \frac{1}{64}$$

$$= \frac{120 + 1}{64}$$

$$= \frac{121}{64}$$

$$x - \frac{1}{8} = \pm \frac{11}{8}$$

$$x - \frac{1}{8} = \pm \frac{11}{8}$$

$$x = +\frac{11}{8} + \frac{1}{8} = -\frac{12}{8} = -1\frac{1}{2}$$
or $x = -\frac{11}{8} + \frac{1}{8} = -\frac{10}{8} = -1\frac{1}{4}$.

In this example, both roots are real quantities, and they are unequal in magnitude.

Ex. 2. Solve
$$x^2 - 6x + 9 = 0$$
.
Step 1. $x^2 - 6x = -9$
 $x^2 - 6x + 9 = -9 + 9$
 $x^2 - 6x + 9 = 0$
 $x - 3 = \pm 0$
 $x - 3 = \pm 0$
 $x - 3 = 4$
 $x - 3 = 3 = 0$ both of which = 3.

In this example both roots are equal.

Ex. 3. Solve
$$x^2 - 6x + 11 = 0$$
.
Step 1. $x^2 - 6x = -11$
 $x^2 - 6x + 9 = -11 + 9$
 $x^2 - 6x + 9 = -11 + 9$
 $x^2 - 6x + 9 = -11 + 9$
 $x^2 - 6x + 9 = -11 + 9$
 $x^2 - 6x + 9 = -11$
 $x^2 - 3x + \sqrt{-2}$
 $x^2 - 3x + \sqrt{-2}$
 $x^2 - 3x + \sqrt{-2}$

In this example, both roots are imaginary, for we cannot find a quantity to represent the square root of -2.

281. Graphical solution of a quadratic equation. In solving such equations graphically, we write all the

terms on one side of the equation and put y on the other side. e.g., $8x^2 - 2x = 15$ would be written $y = 8x^2 - 2x - 15$.

Now give to x a series of values both + and -, and calculate corresponding values of y, and then plot the x and y values on squared paper. The graph may-

- (1) Cut the axis of x twice:
- (2) Touch the axis of x;
- (3) Lie entirely above the axis.

The first case corresponds to two real roots, as in Ex. 1 above, and the x co-ordinates of the two points in which the curve crosses the axis give the two roots, just as in the case of the graphical solution of a simple equation, Art. 174.

The second case corresponds to two equal roots, as in

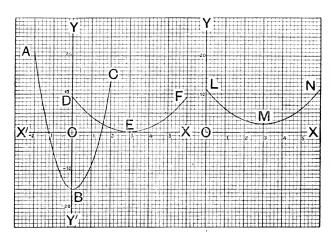


Fig. 182.

Ex. 2 above, the equal roots being given by the x coordinate of the point at which the curve touches the axis of x.

The third case corresponds to two imaginary roots, which cannot be found graphically.

Examples 1 and 2 above are solved graphically in Fig. 182, and the graph for Ex. 3 is also shown. The co-ordinates of the plotted points are shown in the Tables below.

Graph ABC.	x	2	1 5	1	0.2	0	-0.5	-1	-1.5	-2
	у	13	0	-9	-13.5	-15	-12	-5	6	21

Graph DEF.	- a		5		· ·	2	1 -)
Graph LMN.	<i>x</i>	0 11	$\frac{1}{6}$	$\frac{2}{3}$	3	4	5	6

282. Graphical solution of cubic, etc., equations. A cubic equation has three roots, and if all the roots are real the corresponding graph will cut the axis of x three times. The values of the x co-ordinates at these points are the required roots. Imaginary roots cannot be found graphically.

Similarly, an equation containing x^4 has four roots, one containing x^5 has five roots, and so on. The graphs cut the axis of x once for each real root.

Ex. Solve $x^3 - 7x^2 + 4x + 12 = 0$ graphically. Preparing our Table we get—

				٠	· · · · · · · · · · · · · · · · · · ·						
æ	_ 2	— 1	0	1	2	3	4	5	6	7	
y	— 22	0	12	10	0	- 12	— 20	- 18	0	40	

The graph is plotted in Fig. 183.

The roots are x = -1, x = 2, x = 6, corresponding to the x coordinates of the points L, M, N.

283. Maximum and minimum values.* When a continuous graph is of such a shape that one value is greater than any neighbouring values on either side of it, that value is said to be a maximum value, and a value which is less than any neighbouring value on either side of it is called a minimum value. In the graph, Fig. 183, the ordinate PQ is a maximum, and the ordinate RS is a minimum, value.

Exercises.

Solve the following equations graphically:-

(1)
$$x^2 + x - 12 = 0$$
.
(2) $x^2 - 6x + 9 = 0$.
(3) $x^2 + 0.5x - 3 = 0$.
(4) $x^2 - 4 = 0$.

^{*} See also Arts. 331 and 374.

- $(5) x^3 5x^2 2x + 24 = 0.$
- (6) Plot the graph of $y = x^2 3.4x + 2.73$. Take x = 1, 1.2, 1.3, etc., to 2.2. Find the values of x which make y = 0. (B.E. 1906.)
- (7) If $y = x^3 10x^2 + 40x 35$, find y for a number of values of x between 1 and 2, and represent these on squared paper to as large a scale as your paper will allow. You need not show the whole of x, as you need no values less than 1. For what value of x is y = 0? (B.E. 1908.)
- (8) A cast-iron flywheel rim (weighing 0.26 lb, per cub. in.) weighs 13709 lb. The rim is rectangular in section, the thickness radially

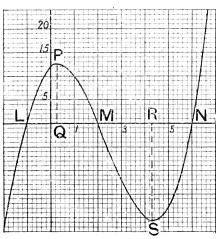


Fig. 183.

is x, the size the other way is 1.6x. The inside radius of the rim is 14x. Find the actual sizes. (B.E. 1903.)

- (9) The sum of the areas of two squares is 92:14 sq. in., and the sum of their sides is 13 in. Find these sides. (B.E. (2) 1904.)
- (10) If $y = 2x + \frac{1.5}{x}$; for various values of x calculate y, plot on squared paper; state approximately the value of x which causes y to be of its smallest value. (B.E. 1904.)

Answers.

- (6) 1.3; 2.1,
 - (7) 1.2.
- (8) 7·124; 11·393; 99·7 in.
 - (9) 8.45, 4.55,
 - (10) 0.87.

(1) -4 or +3.

(2) 3.

(3) 1.5 or - 2.

(4) 2, -2.

(5) 3, 4, and -2.

CHAPTER XXIII.

FACTORS; BINOMIAL THEOREM;
MISCELLANEOUS EQUATIONS AND GRAPHS;
DETERMINATION OF LAWS; POLAR COORDINATES: APPROXIMATIONS.

284. **Factors.** Many algebraic expressions are the result of multiplying together two or more simple expressions, e.g.,

 $(x + 4) \times (x - 3) = x^2 + x - 12.$

The two expressions (x + 4) and (x - 3) are called the factors of the expression $x^2 + x - 12$, and the process of factorizing any given expression consists in finding the two or more simple expressions of which the given expression is the product.

285. If every term in an expression is divisible by any quantity, then that quantity is a factor of the expression, thus $5 x^3 + 10 x^2 - 15 x = 5 x (x^2 + 2 x - 3)$, and the first step in factorizing must be the separation of any such common quantity from every term in the expression.

286. Algebraic expressions are frequently of some standard form or type which is readily recognised as the product of certain factors.

Ex. $(a + b)(a + b) = a^2 + 2ab + b^2$.

 $(a-b)(a-b) = a^2 - 2ab + b^2.$

In each of the above products, the first term is the square of a, the third term is the square of b, and the second term is twice the product of a and b. The square of any expression consisting of two terms is of this form (Art. 88). The + sign before the third term b^2 indicates that the signs in the factors are alike, and the sign before the second term $2\ ab$ indicates whether the signs are + or -. Hence, any expression consisting of three terms, comprising two symbols squared (as a^2 and b^2) and a term which is twice the product of the two symbols (as $2\ ab$), can be factorized, the factors being the square root of the first term plus or minus the

square root of the third term, according as the sign before the second term is + or -.

Ex. The factors of
$$x^2 + 6x + 9$$
 are $(x + 3)(x + 3)$, and of $4a^2 - 12ab + 9b^2$ are $(2a - 3b)(2a - 3b)$.

Again (a + b) $(a - b) = a^2 - b^2$. Here the product is the difference of two squares, viz., a^2 and b^2 , and the factors are respectively the sum of the two terms (a + b), and the difference of the two terms (a - b). Hence, any expression comprising the difference of two squares can be factorized.

Ex. The factors of
$$x^2 - y^2$$
 are $(x + y)(x - y)$, and of $16 \ a^2 - 9 \ b^2$ are $(4 \ a + 3 \ b)(4 \ a - 3 \ b)$.

287. If we multiply together any two expressions consisting of two terms in which the same symbols are used, and inspect the product, we shall find rules which are available for factorizing the products.

Ex. (1)
$$(a + 5 b) (a + 3 b) = a^2 + 8 ab + 15 b^2$$
.
(2) $(a - 5 b) (a - 3 b) = a^2 - 8 ab + 15 b^2$.
(3) $(a - 5 b) (a + 3 b) = a^2 - 2 ab - 15 b^2$.
(4) $(a + 5 b) (a - 3 b) = a^2 + 2 ab - 15 b^2$.

Calling + a the first term in each factor, and + 5 b or - 3 b the second term, we note that in the product

- (1) The first term is the product of the first terms of the factors.
- (2) The third term is the product of the second terms of the factors.
- (3) The symbol of the second term is the product of the symbols of the two terms in either factor, while the coefficient of the second term is the algebraic sum of the coefficients of the second terms of the factors.

Thus, in (3) above, the product of first terms is $a \times a = a^2$.

The product of second terms is $-5 b \times 3 b = -15 b^2$.

The product of second term symbols is ab, and the co-efficient is 3-5=-2.

(4) Where a + sign precedes the third term in an expression, the sign between the terms in each of the factors is the same as the sign preceding the second term of the expression. See Examples (1) and (2) above.

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(5) When a minus sign precedes the third term, the signs between the terms in the factors are unlike, and the sign preceding the second term in the product gives the sign of the factor having the larger co-efficient for the second term.

Thus in (3) above, the sign of the term -2ab denotes that 5 b is and 3 h is + in the factors.

Similarly in (4), the sign of the term +2 ab denotes that 5 b is +and that 3 b is – in the factors.

To factorize any expression of the type (1), (2), (3), or (4) above, we proceed by trial and error to discover the factors.

As examples, factorize

(1) $x^2 + xy - 42y^2$.

We require two co-efficients for y such that their product is -42and their sum +1. As we have a - sign before the third term, and a + sign before the second, the co-efficients are unlike in sign and the greater is +. Try -3 and +14. Their product is -42, but their sum is +11. They are **not** the required co-efficients. Again try -6and +7. Their product is -42, and their sum is +1, hence they are the required co-efficients. The required factors are thus (x-6y)(x+7y).

- (2) Factors of $x^2 ax 6a^2$ are (x 3a)(x + 2a).
- ", " $m^2 8 mn + 15 n^2$ are (m 5 n) (m 3 n), ", ", $a^2 + 10 x + 21$ are (x + 7) (x + 3)."
- 288. The above rules apply for the case in which the coefficient of the first term in each factor is not unity if we substitute therein "cross-products" for "co-efficients of the second term."

Thus
$$(7 \ x + 2 \ y) (2 \ x - y) = 11 \ x^2 - 3 \ xy - 2 \ y^2$$
.

The "cross products" are -7 and $+14$.

Their algebraic sum is -3 .

Thus, in factorizing $14 x^2 - 3 xy - 2 y^2$, the - sign in the term -2 y^2 indicates that the signs of the factors are unlike; the sign of the term -3xy indicates that the greater "cross-product" is negative. The co-efficients of xare two factors of 14 both +; the co-efficients of y are two factors of 2, one + and the other -; the algebraic sum of the cross-products must be -3. From these conditions we must find by trial and error the factors. They are (7 x + 2 y) (2 x - y).

Ex. Factorize $56 a^2 - 58 ab - 18 b^2$.*

The signs of the factors are unlike, for we have $a = \text{sign before } 18 \ l^2$. The greater cross-product is -, for we have $a = \text{sign before } 58 \ ab$. Their sum is -58.

Try factors 14a and 4a for $56a^2$, and -6b and 3b for $-18b^2$.

Cross-products are
$$\frac{11}{4} \times \frac{7-6}{3} = 42 - 21 = 18.$$

We require, however, -58 for the sum; hence these factors are unsuitable.

Repeat the process with 14a and 4a, and 6b and -3b.

This gives -42 and 24, with sum =-18.

Try again with 8a and 7a, and 2b and -9b. The cross-products are -72 and +14, and the sum is -58. This is the required sum, hence the factors are (8a + 2b)(7a - 9b), or 2(4a + b)(7a - 9b).

289. A general method of factorizing any expression of the form $ax^2 + lx + c$, for example $8x^2 + 16x + 6$,* consists in finding an equivalent expression in the form of the difference of two squares. This can be done as follows:—

- Make the co-efficient of x² = 1 by placing the given co-efficient (8) outside a bracket.
- (2) Add and subtract the square of half the co-efficient of x (i.e., square of 1).
- (3) Write as the difference of two squares.
- (4) Factorize as the sum and difference of two terms and simplify.

$$8\left(x^2 + 2x + \frac{3}{4}\right)$$

$$8\left(\left(x^2+2\;x+1\right)\;-\;\left(1\;-\;\frac{3}{4}\right)\right)$$

$$8\left(\left(x+1\right)^{2}-\left(\frac{1}{2}\right)^{2}\right)$$

$$8\left(\left(x+1+\frac{1}{2}\right)\left(x+1-\frac{1}{2}\right)\right)$$

$$= 8 \left(x + \frac{3}{2} \right) \left(x + \frac{1}{2} \right)$$

$$= 2 \left(x + \frac{3}{2} \right) \times 4 \left(x + \frac{1}{2} \right)$$

$$= (2 x + 3) (4 x + 2)$$

$$= 2 (2 x + 3) (2 x + 1),$$

or again, S
$$\left\{ \left(x + \frac{3}{2} \right) \left(x + \frac{1}{2} \right) \right\} = 4 \left(x + \frac{3}{2} \right) \times 2 \left(x + \frac{1}{2} \right)$$

= $(4x + 6)(2x + 1)$
= $2(2x + 3)(2x + 1)$.

The common factor 2 may be removed first.

Ex. Factorize $x^2 - 3.25 x + 1.56$. (B. E., 1911.)

Following the above rules, we have (1) $x^2 - 3.25 x + 1.56$.

- (2) Add and subtract square of half co-efficient of x, i.e., $(1.625)^2$ = $\{(1.625)^2 1.56\}$ square, i.e., $(1.625)^2$.
- (3) Write as differ- $(x 1.625)^2 (1.04)^2$. ence of two squares.
- (4) Factorize as the sum and difference of two terms, $\{(x-1.625)+1.04\}$ $\{(x-1.625)-1.04\}$.

=(x-0.585)(x-2.665).

This result should be checked by multiplying together the two factors obtained.

290. A convenient method of finding the factors of an expression such as that in the above example is to plot on squared paper the graph $y = x^2 - 3.25 x + 1.56$. (Art. 281.) This graph cuts the axis of x in two points, whose abscisse, with the signs changed, correspond to the second term in each factor. Thus, the graph $y = x^2 - 3.25 x + 1.56$ cuts the axis of x at points +0.585 and +2.665. The factors are thus (x - 0.585) (x - 2.665).

290A. In addition to the results given in the preceding Arts., the following should be remembered in connexion with factors.

$$(a - b) (a^2 + ab + b^2) = (a^3 - b^3),
(a + b) (a^2 - ab + b^2) = (a^3 + b^3),
(a + b)^3 = a^3 + 3 a^2b + 3 ab^2 + b^3,
(a - b)^3 = a^3 - 3 a^2b + 3 ab^2 - b^3.$$

Any expression of the form of one of the expressions on the right above can be factorized into factors corresponding to those on the left.

291. **Binomial Theorem.** A binomial is an algebraic expression consisting of two terms, e.g., (x + a) or (x - a).

$$(x+a)^4 = x^4 + 4 x^3 a + 6 x^2 a^2 + 4 x a^3 + a^4,$$

 $(x+a)^3 = x^3 + 3 x^2 a + 3 a x^2 + a^3,$
 $(x+a)^2 = x^2 + 2 x a + a^2,$

and, similarly, for other powers.

From these results we note that

(1) When the power is 2, we have 3 terms, ,, ,, ,, 3, ,, 4 terms, ,, ,, ,, 4, ,, 5 terms, and if the power is *n*, ,, ,, (*n* + 1) terms. (2) The sum of the indices in each term is constant and

- equal to the index of the power to which the binomial is to be raised, provided the indices of x and a are each unity.
- (3) The powers of x are in descending order, e.g., x^4 , x^3 , x^2 , x.

(4) The powers of a are in ascending order, e.g., $a, a^2, a^3, a^4.$

(5) The co-efficients of terms at equal distances from the ends of the expression are alike, provided the co-efficients of x and a are unity, e.q., the second term from each end of the first result above has co-efficient 4. The co-efficients follow a law which is indicated in the general expression below.

It can be shown that

$$(x+a)^{n} = x^{n} + nx^{n-1} \cdot a + \frac{n(n-1)}{2} \cdot x^{n-2} \cdot a^{2} + \frac{n(n-1)(n-2)}{3} \cdot x^{n-3} \cdot a^{3} + \dots nxa^{n-1} + a^{n},$$
where "factorial 3" or $\frac{1}{3} = 3 \times 2 \times 1$, and $\frac{1}{2} = 2 \times 1$, etc.,

provided that n is a positive integral or fractional number, x and a being either positive or negative, and either integral or fractional.* From this general expression we can write down at once the result of raising any binomial to a given power n. This process is called "expanding."

Ex. Expand
$$(x + a)^5$$
. Here $n = 5$.

$$(x + a)^5 = x^5 + 5x^4 \cdot a + \frac{5 \times 4}{2 \times 1}x^3 \cdot a^2 + \frac{5 \times 4 \times 3}{3 \times 2 \times 1}x^2 \cdot a^3 + \frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2 \times 1}x \cdot a^4 + a^5 = x^5 + 5x^4 \cdot a + 10x^3 \cdot a^2 + 10x^2 \cdot a^3 + 5x \cdot a^4 + a^5.$$

^{*} For a proof of the binomial theorem, and for cases in which n is negative, the student should consult one of the many excellent treatises on algebra. The theorem is true for n negative, provided the series on the right is convergent. A convergent series is one that tends to a finite sum, e.g., $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\cdots$...

In the case in which the sign is negative, e.g., $(x-a)^n$, regard (-a) as the second term. Similarly, in a case such as $(3 \ x - 4 \ a)^n$, regard $(+3 \ x)$ as the equivalent of x, and $(-4 \ a)$ as the equivalent of a in the general expression.

Ex. Expand
$$(2x-3a)^3$$

 $(2x-3a)^3 = (2x)^3 + 3(2x)^2(-3a) + \frac{3\times 2}{2\times 1}(2x)(-3a)^3$
 $+ (-3a)^8$
 $= 8x^3 - 36x^2, a + 51x, a^2 - 27a^3.$

Any expression of the form given above can be factorized as $(x + a)^n$, or the product of n factors each = (x + a).

292. Quadratic equations involving only one unknown quantity may (in addition to the general method of solution given in Arts. 277—280) be solved by factorizing.

```
Ex. 1. Solve x^2 - x = 6.

Write this as x^2 - x - 6 = 0.

Eactorizing (x - 3) (x + 2) = 0.

The product (x - 3) (x + 2) can only = 0 if (x - 3) = 0, or x = 3; or if (x + 2) = 0, or x = -2.

Hence, the roots of the equation are x = 3, x = -2.

Ex. 2. Solve 6x^2 + 17x = -12.

Write as 6x^2 + 17x + 12 = 0.

Factorizing (3x + 4)(2x + 3) = 0.

Hence (3x + 4) = 0, or 3x = -4, and x = -133, or (2x + 3) = 0, or 2x = -3, and x = -15.

The roots are x = -133 and x = -15.
```

293. Graphical Solution of a Quadratic Equation. In Art. 281, one graphical method of solving the quadratic equation $8x^2 - 2x = 15$ is given; the graph $y = 8x^2 - 2x = 15$ is plotted, and the roots of the equation are shown to be the values of x for the points in which the graph cuts the axis of x. A further, and in some cases a more convenient, graphical solution is as follows:—

The equation $8 x^2 - 2 x = 15$ may be written $8 x^2 = 2 x + 15$. If we plot a graph $y = 8 x^2$ we have the parabola AOB (Fig. 184). If we plot the graph $y_1 = 2 x + 15$, we have a straight line CD. At the points of intersection of the straight line and parabola, we have $y = y_1$, i.e., the ordinate CE or DF for the line = the ordinate CE or DF

for the parabola. Hence, $8 x^2 = 2 x + 15$ for the values of x corresponding to the points of intersection, and as the roots of the equation are the values of x which make $8 x^2 = 2 x + 15$, the abscissa, or x values, for the points C and D give the required roots of the equation. From Fig. 184, these values are seen to be x = 1.5 or -1.25, which values agree with those found in Art. 281 and shown in Fig. 182.

If the roots are equal, as in the equation $x^2 - 6x + 9 = 0$,

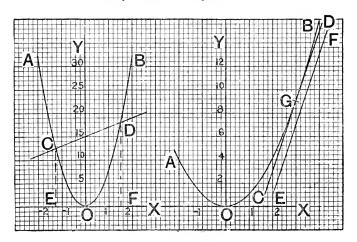


Fig. 184. Fig. 185.

solved graphically in Art. 281, the line CD (Fig. 185), or y = 6x - 9, will be tangential to the parabola AOB, or $y = x^2$, at the point G at which x = 3, which value gives the root of the equation. Again, if the roots are imaginary, as in the equation $x^2 - 6x + 11 = 0$ of Art. 281, the line and parabola will not intersect. The line EF, or y = 6x + 11, is shown in Fig. 185, and it is seen that it does not touch the parabola AOB.

294. Cubic equations, and equations of higher degree, may be similarly solved. As an example, the cubic equation

 $x^3 - 7x^2 + 4x + 12 = 0$, solved graphically in Art. 282, may also be solved by plotting the graphs (1) $y = x^3$, (2) $y = 7x^2 - 4x - 12$. These graphs will intersect in three points corresponding to the three real roots of the equation. The student should plot these graphs as an exercise, and note that the points of intersection are x = -1, x = 2, x = 6, which are the three roots of the equation.

295. In eases in which two unknown quantities are to be found, we require two equations, called simultaneous

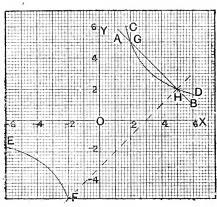


Fig. 186.

equations (Arts. 98, 269). The method of solving such equations when the unknown quantities are of higher power than the first consists in finding, from the given data, the sum and difference of the two unknown quantities, e.g., (x+y) and (x-y). We can then find x and y respectively, by adding and subtracting these two equations. Such equations can also be **solved graphically** by plotting the corresponding graphs, the points of intersection giving, by their co-ordinates, the values of x and y, which are the roots of the equations (Art. 275). In each of the following examples, the graphical solutions are given with the algebraic solution.

$$\begin{array}{c} x+y=7,\\ xy=10. \end{array}$$

Algebraic Solution.

$$x + y = 7 (1)$$

 $xy = 10 (2)$

Square (1)

$$a^{2} + 2 xy + y^{2} = 49$$

Multiply (2) $4 xy = 40$
by 4.

Subtract.
$$x^2 - 2xy + y^2 = 9$$

Extract sq. root. $x - y = \pm 3$
We now have $x + y = 7$
 $x - y = \pm 3$

$$\therefore \text{ adding} \qquad 2 \stackrel{\mathcal{I}}{x} = \stackrel{1}{\cancel{1}} \text{ or } 4$$

$$x = 5 \text{ or } 2$$

From (1)
$$y = 2 \text{ or } 5$$
.
Exercise. Solve $x - y = 3$
 $xy = 10$.

Note.—In the graphical solution, both parts CD and EF of the graph xy = 10 will be required. The dotted line FH (Fig. 186) is the graph x - y = 3. The roots of the equations are x = 5 or -2, y = 2 or -5. The points of intersection are F, H.

Ex. 2. Salve
$$\begin{array}{ccc} x - y &= 1, \\ x^2 + y^2 &= 25. \end{array}$$

A'gebraic Solution.

$$\begin{aligned}
 x - y &= 1 (1) \\
 x^2 + y^2 &= 25 (2)
 \end{aligned}$$

S juare (1)

$$a^2 - 2xy + y^2 = 1$$
 (3)
Subtract (3) from (2)

$$2xy = 24 (4)$$
Add (4) to (2)

$$x^2 + 2xy + y^2 = 49$$

Extract sq. root. $x + y = \pm 7$

We now have
$$x - y = 1$$

 $x + y = \pm 7$

Adding
$$2x = 8 \text{ or } -6$$
$$x = 4 \text{ or } -3$$

From (1)
$$y = 3 \text{ or } -4$$
.
Exercise. Solve $x + y = 7$

$$x^2 + y^2 = 25$$
.
Note.—The line $x + y = 7$
intersects the circle $x^2 + y^2 = 25$
in points $x = 4$, $y = 3$ an $1 = 3$, $y = 4$. It is not shown in Fig. 187.

Graphical Solution.

The graph x + y = 7 is a straight line. Written in the form $y = \pm mx \pm c$, it is y = -x + 7. In Fig. 186, the graph is AB.

The graph xy = 10 is the rectangular hyperbola CD. For negative values of x, we have a graph EF in the third quadrant, as corresponding values of y are negative. As the line AB does not enter the third quadrant, this part of the graph xy = 10 is not required. The co-ordinates of G and H are the roots of the equations.

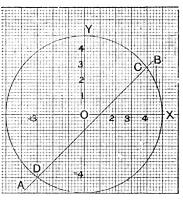


Fig. 187.

Graphical Solution.

The graph x - y = 1 is the straight line AB (Fig. 187).

The graph $x^2 + y^2 = 25$ is a circle, centre at O, and radius = 5. The co-ordinates of the points of intersection C, D of the line and circle give the roots of the equation. At C, x = 4, y = 3, and at D, x = -3, y = -4.

Ex. 3. Solve
$$x^3 + y^3 = 28$$
.
 $x + y = 4$.

Algebraic Solution.

$$x^{3} + y^{3} = 28 (1)$$

$$x + y = 4 (2)$$
Square (2)
$$x^{2} + 2 xy + y^{2} = 16 (3)$$

$$\div (1) \text{ by (2)}$$

$$x^2 - xy + y^2 = 7$$
Subtracting
$$3 xy = 9$$

$$\therefore 4 xy = 12$$

$$x^{2} - 2 xy + y^{2} = 4$$
Extract sq. root $x - y = \pm 2$
We now have $x + y = 4$

$$x - y = \pm 2$$

Adding
$$2x = 6 \text{ or } 2$$

 $x = 3 \text{ or } 1$
From (2) $y = 1 \text{ or } 3$.
Exercise. Solve $x^3 - y^3 = 26$

x-y=2Note. The graph $x^3-y^3=26$ is similar to the graph AB, but inverted relatively to OX.

Ans.
$$x = 3 \text{ or } -1$$

 $y = 1 \text{ or } -3$.

Graphical Solution.

To plot the graph $x^3 + y^3 = 28$, we have $y = \sqrt[3]{28 - x^3}$. From this we deduce the following facts, which the student should verify.

When
$$x = 0$$
, $y = \sqrt[3]{28} = 3.036$
,, $x = 1$, $y = 3$
,, $x = 2$, $y = 2.71$
,, $x = 3$, $y = 1$
,, $x = 3.036$, $y = 0$.

For negative values of x, y is + and is greater than 3.036, and increases as x increases in magnitude.

For positive values of x greater than 3.036, y is negative and increases as x increases. The graph is shown at A, B (Fig. 188). The graph x + y = 4 is the line CD. The co-ordinates of the points of intersection E, F are the rests of the equations.

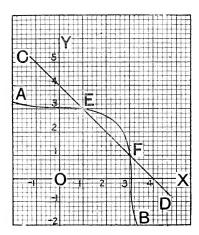


Fig. 188

$$a^2 + y^2 = 25.$$

 $xy = 12.$

Algebraic Solution.

$$x^{2} + y^{2} = 25 (1)$$

 $xy = 12 (2)$

Multiply (2) by 2. 2
$$xy = 24$$
 (3) Add (1) and (3)

$$x^2 + 2 xy + y^2 = 49$$

Subtract (3) from (1)

$$x^2 - 2 xy + y^2 = 1$$

Extract sq. root of each
$$x +$$

of each
$$x + y = \pm 7$$

 $x - y = \pm 1$
Adding. $2x = +7 + 1 = 8$

or
$$-7 - 1 = -8$$

or $+7 - 1 = 6$
or $-7 + 1 = -6$

$$\therefore x = 4, -4, 3, \text{ or } -3$$

From (2),
$$y = 3, -3, 4, \text{ or } -4$$
.
Exercise. Solve $x^2 - y^2 = 16$

xy = 15. Note. The graph $x^2 - y^2 = 16$ comprises four branches, each similar in form to CD and EF and lying one in each quadrant. Graphical Solution.

The graph $x^2 + y^2 = 25$ is a circle AB (Fig. 189), centre at origin O, and radius $= \sqrt{25} = 5$.

The graph xy = 12 is the rectangular hyperbola CD, or EF, for negative values of x give negative values of y.

The points of intersection are marked 1, 2, 3, 4.

The co-ordinates of (1) are
$$x = 4, y = 3.$$

$$x = -4, y = -3.$$

The eo-ordinates of (3) are $x = 3, y = 4.$

The co-ordinates of (4) are
$$x = -3, y = -4$$
.

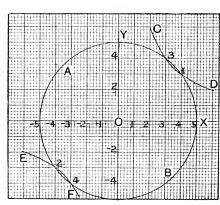


Fig. 189.

- 296. Problems which involve quadratic equations are frequently met with. The following examples illustrate methods of solving such problems.
- Ex. 1. A rectangular plot of land has an area of 12,000 sq. ft, and the length of the diagonal is 208.8 ft. Find the length and breadth of the plot.

If x = length and y = breadth of plot in ft.

Then xy = area in sq. ft.

and
$$\sqrt{x^2 + y^2} = \text{length of diagonal.}$$

We have

$$\sqrt{x^2 + y^2} = 208.8$$
 it.

 $\sqrt{x^2 + y^2} = 208.8$

(1) $x^2 + y^2 = 43600$ (correct to 3 significant figures)

 $\begin{array}{rcl} (2) & xy & = 12000 \\ (3) & 2 xy & = 24000 \end{array}$

Add (1) and (3) $x^2 + 2xy + y^2 = 67600$. Subtract (3) from (1) $x^2 - 2xy + y^2 = 19600$.

 $x + y = \pm 260$ (Negative signs may be) $x - y = \pm 140$ (neglected in this problem.)

Adding

$$2 x = 400$$

$$x = 200 \text{ ft. length.}$$

$$y = 60 \text{ ft. breadth.}$$

Ex. 2. Two cubical blocks of stone together contain 370 c. ft, and their combined height is 10 ft. Find the length of the side of each cube.

Let x ft. be side of one cube, and y ft. that of the other

Then $x^3 + y^3 = 370 \text{ c. ft.}$ x + y = 10 ft.

Square (2) x + y = 10 ft. (2) $x^2 + 2 xy + y^2 = 100$ (3)

 \div (1) by (2) $x^2 - xy + y^2 = 37$ Subtracting 3xy = 63

3 xy = 63 $\therefore 4 xy = 84$

Subtract from (3). $x^2 - 2 xy + \tilde{y^2} = 16$ $x - y = \pm 4$.

We now have x + y = 10

Adding $x - y = \pm 4.$ 2 x = 14 or 6

 $\begin{array}{rcl} x = & 7 \text{ or } 3 \\ y = & 3 \text{ or } 7. \end{array}$

Hence, the cubes are 7 ft. and 3 ft. side respectively.

Er. 3. The perimeter of a right-angled triangle is 18 ft., and the length of the hypotenuse is 7.5 ft. Find the remaining sides.

Let x and y be lengths of sides required.

Then x + y = 18 - 7.5

= 10.5 ft. (1) (Hypotenuse)² = $a^2 + y^2 = 56.25$ (2)

Square (1) $x^2 + 2xy + y^2 = 110^{\circ}25$ (2) Subtract (2) from (3) 2xy = 54.

Subtract (2) from (3) 2 xy = 54. Subtract from (2) $x^2 - 2 xy + x^2 = 2.25$

 $\therefore x - y = \pm 1.5.$

We now have
$$x + y = 10^{\circ}5$$

 $x - y = \pm 1^{\circ}5$.
Adding $\therefore 2x = 12 \text{ or } 9$.
 $x = 6 \text{ or } 4^{\circ}5$.
From (1) $y = 4^{\circ}5 \text{ or } 6$.

Hence, the sides required are 6 ft. and 4 ft. 6 ins.

Ex. 4. The base of a triangle exceeds half the height by 3 ft. The area is 88 sq. ft. Find the base and height.

Let
$$x = \text{base and } y = \frac{1}{2} \text{ height.}$$

Then $x - y = 3$ (1)

 $xy = 88$ (2) (for area = $\frac{1}{2} \text{ base } \times \text{ height)}$,

From (1)

 $x^2 - 2 xy + y^2 = 9$.

From (2)

 $4 xy = 352$.

Adding

 $x^2 + 2 xy + y^2 = 361$.

 $\therefore x + y = \pm 19$,

We now have

 $x + y = 19$
 $x - y = 3$
 $\therefore 2 x = 22$
 $x = 11$.

and from (1) y = 8. Hence base=11 and height=8 ft, The negative value of x + y is neglected, for the length of base + half the height cannot be negative.

- 297. Equations of the form kx = a, where values of k, n, and n are given, are solved by taking logarithms of each side of the equation.
- Ex. The volume of water passing over a triangular or V-shaped notch in a weir is given by the formula $Q=2.34\ x^{2.5}$, where x is the height (or head) in ft. of the water above the bottom of the notch, and Q is the discharge in c, ft. per second. Find the head which will give a discharge of 5,800 c, ft. per minute.

$$Q = 2.34 \ u^{2.5}$$
Required discharge per sec. = $\frac{5800}{60}$ e. ft., hence $\frac{5800}{60} = 2.31 \ u^{2.5}$.

Taking logs.
$$\begin{array}{ll} 60 \\ \log .5800 - \log .60 = \log .2^{\circ}34 + 2^{\circ}5 \log .x \\ 3.7634 - 1.7782 = 0.3692 + 2^{\circ}5 \log .x, \\ 1.6160 = 2^{\circ}5 \log .x \\ -\frac{1.616}{2^{\circ}5} = \log .x \\ 0.6164 = \log .x. \\ \text{But 0.6164} = \log .4^{\circ}43, \text{ hence} \\ x = 4.43 \text{ ft. head.} \end{array}$$

298. Equations of the form $kn^x = a$, where values of k, n, and a are given, are similarly solved. (Art. 268, Ex. 8.)

Ex. The formula $\frac{N}{M} = e^{\mu\theta}$ relates to driving-belts. N is the tension in the tight side, and M that in the slack side. θ is the angle of lap of the belt on the pulley in radians, μ the co-efficient of friction between the belt and pulley, and e=2.72. If N=250 lb., M=133.8 lb., $\mu=0.25$, find θ .

we find
$$\frac{N}{M} = a$$
, $k = 1$, $\mu \theta = x$, and $n = c$.)

$$\frac{N}{M} = e^{\mu\theta}.$$
Taking logs. log. N - log. M = $\mu\theta$ log. e.
log. 250 - log. 133.8 = 0.25 × θ × log. 2.72
$$2.3979 - 2.1265 = 0.25 × θ × 0.4346.
$$\therefore \theta = \frac{0.2714}{0.25 \times 0.4346}$$
= 2.5 radians.$$

299. In an equation of the form $y = ax^n$, we can find the values of a and n if we are given two corresponding sets of values of x and y.

Thus, let y = 6.75 when x = 3, and

also y = 2 when x = 2.

To find a and n.

We have $y = ax^n$.

Substitute given values
$$6.75 = u \times 3^n$$
 (1)

$$2 = a \times 2^n \tag{2}$$

Take logs, of each equation (1) and (2),

$$\log_{10} 6.75 = \log_{10} a + n \cdot \log_{10} 3$$
 (3)
 $\log_{10} 2 = \log_{10} a + n \cdot \log_{10} 2$ (4)

 $\log_{10} 2 = 10$ Subtract (4) from (3),

$$\log_{10}(6.75) = \log_{10}(2) = n \text{ (log. 3 - log. 2)}$$

$$\therefore n = \frac{\log_{10}(6.75 - \log_{10}(2))}{\log_{10}(3 - \log_{10}(2))}$$

$$= \frac{0.8293 - 0.3010}{0.4771 - 0.3010}$$

$$= \frac{0.5283}{0.7761}$$

$$= 3. \qquad \therefore n = 3.$$

$$2 = a \times 2^{3}$$

From (2)

$$a = 0.25$$
, $a = 0.25$

The equation is thus $y = 0.25 x^3$.

Graph of the equation $y = ax^n$, *n* being greater or less than 1.

 $= a \times 8$

The form of this graph will depend upon whether n is integral or fractional, or positive or negative, and also upon

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whether a is positive or negative. If n=1, we have the straight line y=ax, and if n=-1, we have $y=ax^{-1}$, or $y=\frac{a}{x}$, or xy=a, which is a rectangular hyperbola. (Art. 173.)

If n is integral and even, the graph is symmetrical about

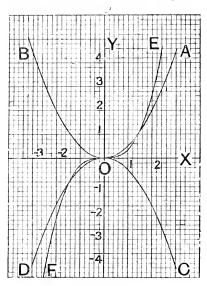


Fig. 190.

the axis of y, as AB (Fig. 190), which is the graph $y = 0.5 x^2$, or CD, which is the graph $y = -0.5 x^2$.

If n is integral and odd, e.g., $y = \frac{1}{3} x^3$, the graph is of the form EF, for y is negative when x is negative. The graph $y = -\frac{1}{3} x^3$ is the graph EF inverted with respect to the axis of x.

When n is fractional (and either greater or less than 1), it is not usual to deal with values of x which are negative. In all practical applications of this kind, x would be positive.

When $n = \frac{1}{2}$, e.g., $y = \pm 2$ $x^{\frac{1}{2}}$, we have $y^2 = 4$ x, or $x = \frac{1}{4}y^2$. The value of x will be + whether y is + or -, hence this graph, shown at AB (Fig. 191), is symmetrical about the axis of x.

When $n = \frac{1}{3}$, e.g., $y = 2x^{\frac{1}{3}}$, we have $y^3 = 8x$, and y will be + or - according as x is + or -. This graph is shown at EOF (Fig. 191); it touches the axis of y at the origin.

If, in the preceding case, the value of a were negative,

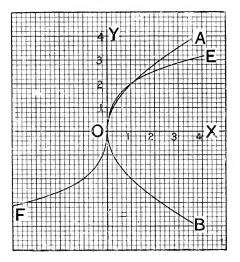


Fig. 191.

e.y., $y = -2 x^{\frac{1}{3}}$, the graph EOF (Fig. 191), would be inverted with respect to the axis of y.

When n is negative, and either integral or fractional, we shall deal only with positive values of x.

(1) Let
$$n = -2$$
, e.g., $y = 0.5 \ x^{-2}$, or $y = \frac{1}{2 \ x^2}$.

(2) Let
$$n = -\frac{1}{2}$$
, e.g., $y = 0.5 \ x^{-\frac{1}{2}}$, or $y = \frac{1}{2 \ \sqrt{x}}$. In each case, when $x = 0$, y is infinite,

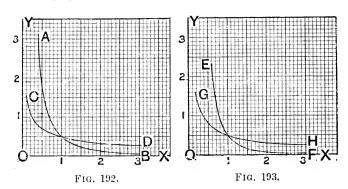
and when x is infinite, y = 0. For all + values of x, y is +, and as x increases y decreases. The graph (1) is shown at AB (Fig. 192), and (2) at CD.

(3) Let
$$n = -3$$
, e.g., $y = 0.5 x^{-3}$

$$= \frac{1}{2 x^3}.$$
(4) Let $n = -\frac{1}{3}$, e.g., $y = 0.5 x^{-\frac{1}{3}}$

$$= \frac{1}{2 \sqrt[3]{x}}.$$

The graphs for (3) and (4) are EF and GH (Fig. 193)



respectively, and the same remarks apply as to the graphs for cases (1) and (2).

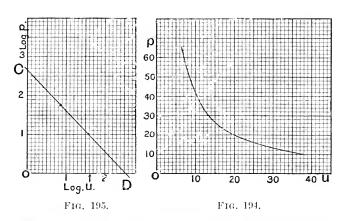
300. Practical Example. When saturated steam expands in an engine cylinder, in such a way that heat neither enters nor leaves the cylinder during the expansion, the steam is said to expand adiabatically. If the absolute pressure p in lbs. per sq. in. and volume u in c. ft. of 1 lb. of steam at each point of the expansion be measured, it is found that p and u are connected by the law $pu^{1.0646} = 479$. This may be written

$$p = \frac{479}{u^{1.0646}}$$
 or $p = 479 \ u^{-1.0646}$.

Comparing with $y = ax^n$, we have y = p, x = u, a = 479, and n = -1.0646. By giving to p any series of values, we can calculate

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corresponding values of u and plot on squared paper. The graph is shown in Fig. 194.



7:034	8:352	10:30	13:50	19:75	37.87	•		и
60	50	40	30	20	10			ľ
0.8472	0.9218	1:0128	1:1303	1:2956	1.5783		u ,	log.
1.7782	1.6990	1.6021	1.4771	1:301	1		p.	log.

If we plot corresponding values of $\log p$ and $\log u$, we get a straight line, as shown in Fig. 195, for

$$p = 479 \ u^{-1.0646} \ . \ . \ . \ (1)$$

 $\therefore \log_{*} p = \log_{*} 479 - 1.0646 \log_{*} u$. (2) Equation (2) is of the form y = -mx + c, where $y = \log_{*} p$,

-m = -1.0646, $x = \log n$, and $c = \log 479$. In every equation of the form $y = ax^n$, we get a straight line if we not corresponding values of $\log x$ and $\log x$. We can thus ascertain

In every equation of the form $y = ax^n$, we get a straight line if we plot corresponding values of $\log y$ and $\log x$. We can thus ascertain whether a given set of corresponding ralues of y and x follow a law of the kind $y = ax^n$ by plotting the $\log x$, of corresponding ralues of y and x. If the $\log x$, when plotted, lie on a straight line, the ralues do follow this law.

To ascertain the values of a and n from the graph of a given set of readings, we will refer to Fig. 195, and assume that the ordinates are values of $\log_{x} y$ and abscissa values of $\log_{x} x$. The equation of the line CD is of the form y = -mx + c, or, as we plot logs, of y and x, $\log_{x} y = -m \cdot \log_{x} x + c \quad .$

The law is $y = ax^n$, and by taking logs, we get

 $\log y = n \log x + \log a$.

Comparing (1) and (2) above, we see that n corresponds to (-m), and c to \log . a.

But in the graph, (-m) is the slope of the line, and c is the intercept on the axis of y. The slope thus gives the value of n in the law, and the intercept gives the log. of the constant u, from which the actual value of a can be found.

In Fig. 195, the intercept OC = 2.6803. This is log. a, hence a =

Again, the slope n is $\frac{OC}{OD}$ and is negative,

 $\frac{OC}{OD} = -\frac{2.6803}{2.5177} = -1.0646.$ and

Hence, n = -1.0616, and the law is $y = 479 x^{-1.0646}$. $yx^{1.0646} = 479.$

The values of u and u can also be found by taking two corresponding values of \log , y and \log , x from the graph CD (Fig. 195), as follows:— When $\log_{10} y = 1$, $\log_{10} x = 1.5783$.

When $\log_{x} y = 1.7782, \log_{x} x = 0.8472$.

From log. y = u, log. $x + \log a$, we have by substitution $1 = (n \times 1.5783) + \log_{10} a$ $1.7782 = (n \times 0.8472) + \log_{10} a$ $-0.7782 = n \times 0.7311$ $\therefore n = \frac{-0.7782}{0.7311} = -1.0614.$

Subtracting

To find $\log a$, substitute the calculated value for n in

1.7782 =
$$(n \times 0.8472) + \log_{\bullet} a$$
.
Then 1.7782 = $(-1.0644 \times 0.8472) + \log_{\bullet} a$.
= $-0.9018 + \log_{\bullet} a$.
 $\therefore 1.7782 + 0.9018 = \log_{\bullet} a$
 $2.6800 = \log_{\bullet} a$
 $a = 479$.

from which

Equation of a Straight Line in Various Forms.

The equation of a straight line in the form y = + mr + r(Art. 177) is called the tangent form of the equation, for the constant m gives the slope of the line, and is equal to the tan, of the angle of inclination of the line to the axis of x, measured anti-clockwise, when the scales for y and xare equal.

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The equation of a line may be obtained in other forms.

(1) Equation of a line passing through two points whose co-ordinates are respectively $x_1, y_1; x_2, y_2$. Fig. 196.

Let A be the point $x_1, y_1,$ B be the point $x_2, y_2,$

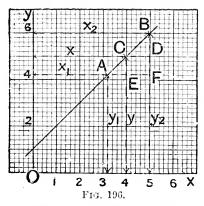
and C be any point having co-ordinates x, y.

The triangles ABF, ACE are similar.

$$\frac{\text{CE}}{\text{BF}} = \frac{\text{AE}}{\text{AF}}.$$
But CE = $y - y_1$, AE = $x - x_1$
BF = $y_2 - y_1$, AF = $x_2 - x_1$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$
(1)

is the equation of the line passing through the given points.



Ex. Let the co-ordinates of the points be A = 3, 4; B = 5, 7. Then $y=4,\,y_2=7,\,x_1=3,\,x_2=5,$ and we have

$$\frac{y-4}{7-4} = \frac{x-3}{5-3},
\frac{y-4}{3} = \frac{x-3}{2}$$

This can be transformed to the tangent form,

for
$$2(y-4) = 3(x-3)$$

 $2y-8 = 3x-9$
 $2y = 3x-1$
 $y = 1.5x-0.5$.

(2) Equation of a line which cuts off intercepts c, b respectively from the axes of y and x. Fig. 197.

Let the co-ordinates of any point D on the line be x, y.

Then
$$\frac{DE}{AO} = \frac{EB}{BO}$$
.

But DE =
$$y$$
, AO = c , EB = $(b - x)$, BO = b .

$$\therefore \quad \frac{y}{c} = \frac{(b-x)}{b} \text{ is the required equation.}$$

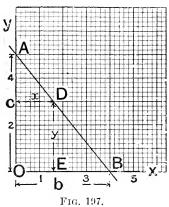
Simplifying,
$$yb = bc - xc$$

÷ both sides by
$$bc$$
, and re-arrange $\frac{x}{b} + \frac{y}{c} = 1$. (2)

(This equation can also be deduced from Equation (1) by substituting c, o for y_1 , x_1 and o, b for y_2 , x_2 .)

The line AB (Fig. 197) is the line
$$\frac{x}{4} + \frac{y}{5} = 1$$
.

302. We often obtain experimentally a series of corresponding values of two related quantities, say, x and y, and



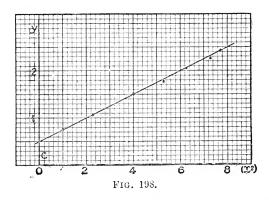
by plotting upon squared paper we can obtain a corresponding graph. this graph is a straight line, the determination of its equation presents no difficulty. If the graph is not a straight line, the determination of the equation is not usually an easy matter.

It may happen that if we plot values other than corresponding values of y and x we can obtain a straight line, e.g., in Art, 300 we have seen that when the quantities

are connected by an equation of the form $y = ax^n$ we get a straight line by plotting values of log. y and log. x, and from this line we can determine the values of a and n. The selection of values which will give a straight line requires a certain amount of practice. If y and x are connected by an equation such as $y = a + bx^2$, we get a straight line by plotting values of y and values of x^2 .

Ex.1. The following corresponding values of x and y were measured. There may be errors of observation. Test if there is a probable law $y = a + bx^2$, and, if this is the case, what are the probable values of a and b.

<i>x</i>	1.00	1:50	2.00	2:30	2.50	2.70	2.80
$\frac{y}{\text{Va}}$	0.77	1.05 2 are as fo	2.03	2·25 2·12 (B. E., 1911.)			
x2	1:00	2.25	4.00	5.29	6.25	7:29	7.84



If we plot y vertically and values of x^2 horizontally, we get practically a straight line. Fig. 198. The equation is $y = m(x^2) + c$, (x^2) corresponds to x in the usual linear equation, $m = \frac{2}{8} = 0.25$; c = 0.5, hence the law is y = 0.25 $x^2 + 0.5$.

Ex. 2. It is thought that the following observed quantities, in which there are probably errors of observation, follow a law like $y = a_0 b x$. Test if this is so, and find the most probable values of a and b.

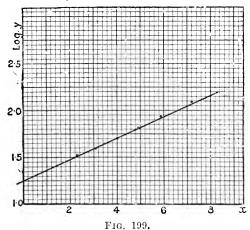
x	2.30	3.10	4.00	4.92	5.91	7.20
y	33.0	39·1	50.3	67.2	85.6	125.0

(B. E., 1903.)

From $y = ae^{bx}$, we obtain $\log y = bx \log e + \log a$. Now $\log e = \log 2.7183 = 0.4343$

 $\log y = (0.4343 \ b) \ x + \log a \ . \tag{1}$

If we plot corresponding values of $\log y$ and x, we obtain a straight line graph (Fig. 199). The slope m is = 0.4343 b, from which b can be determined. When x = 0, $\log y = \log a$, hence the value of $\log a$ is the intercept on the axis of y, and, from this, a can be determined. a could also be found from $y = ac^{bx}$, which gives y = a when x = 0 (for e^{bx} then $= e^0 = 1$).



The values of log. y are 1.5185, 1.5922, 1.7016, 1.5274, 1.9325, 2.0969. The intercept log. a=1.25, hence a=17.78.

The slope is $\frac{13.5 \times 0.05}{6}$ (Each square vertically = 0.05.)

hence, 0.4343
$$b = \frac{13.5 \times 0.05}{6}$$

 $b = \frac{13.5 \times 0.05}{6 \times 0.4343} = 0.26$,

The law is thus

 $\Lambda \Lambda$

If tested by putting x = 4, we calculate y = 50.31, which agrees with the observed value of y.

The following suggestions, in addition to the examples already given, may prove of use in selecting suitable quantities to give a straight line when plotted.

(1) If
$$y = axy + b$$
, plot values of y and of (xy) .

(2) If
$$y = \frac{a}{x} + b$$
, , , y and of $\frac{1}{x}$.

(3) If
$$y^2 = 4 \, ax$$
, ,, $y^2 \text{ and } x$.

(4) If
$$y^2 = nx^2 + c$$
, , , y^2 and x^2 .

(5) If
$$x = ay^{\theta}$$
, , , log. x and log. y .

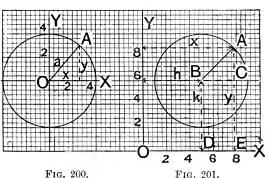
x and

(6) If
$$x = av$$
, , , log. x and of y .

(7) If
$$x = \frac{a}{y} + b$$
, , , , (8) If $y = an^x$, , , ,

(8) If
$$y = an^x$$
, , , $\log y$ and x .

(9) If
$$y^2 = b^2 \left(1 \pm \frac{x^2}{a^2} \right)$$
,, y^2 and x^2 .



304. The Circle. Fig. 200. If we take the centre of a circle as the origin, and two diameters mutually perpendicular as axes of x and y, then if the co-ordinates of any point A on the circle are x and y, we have $OA^2 = x^2 + y^2$. But OA is constant; call it a.

The equation of a circle is thus $x^2 + y^2 = a^2$.

Ex. $x^2 + y^2 = 16$ is the circle, Fig. 200, of radius $\sqrt{16}$ or 4.

If the centre of the circle is at a point whose co-ordinates are h, k (Fig. 201), and if the co-ordinates of any point A on the circle are x, y, then

$$AC = AE - CE$$

$$= AE - BD$$

$$= y - k.$$

Similarly BC = (x - h), and $(x - h)^2 + (y - k)^2 = a^2$, where a is the radius AB of the circle. (AB² = BC² + CA².)

This simplifies to

$$x^{2} + y^{2} - 2 hx - 2 ky + h^{2} + k^{2} = a^{2}$$
 (1).

The general form of the equation to a circle is usually given as $x^2 + y^2 + 2 yx + 2 fy + c = 0$ (2).

By comparing (1) with (2), it will be seen that g = -h, or h = -g. Similarly k = -f, and $c = h^2 + k^2 - a^2$.

The co-ordinates of the centre in the general equation are thus -g, -f, and the radius is $a = \sqrt{g^2 + f^2 - c}$.

Ev. The equation of a circle is $x^2 + y^2 - 6 x - 4 y - 3 = 0$. Find its radius, and the co-ordinates of the centre.

The general equation is $x^2 + y^2 + 2 gx + 2 fy + c = 0$. Our equation is $x^2 + y^2 - 6 x - 4 y - 3 = 0$.

Our equation is Hence 2

$$\begin{array}{lll}
2 & g & = & -6, g & = & -3 \\
2 & f & = & -4, f & = & -2, \text{ and } c & = & -3. \\
a & = & \sqrt{g^2 + f^2 - c} & = & \sqrt{9 + 4 + 3} \\
& = & \sqrt{16} \\
& = & 16
\end{array}$$

 \therefore . The co-ordinates of the centre are -g, -f, or 3, 2, and the radius is 4.

Examples of this kind may also be solved by making the left hand side of the equation into the sum of two squares of the form $(x - h)^2$ and $(y - k)^2$, thus

$$x^2 + y^2 - 6x - 4y - 3 = 0$$
 may be written $(x^2 - 6x) + (y^2 - 4y) = 3$.

Make the expressions in brackets into complete squares.

$$(x^2 - 6x + 9) + (y^2 - 4y + 4) = 3 + 9 + 4$$

 $(x - 3)^2 + (y - 2)^2 = 16$

(9 + 4 is added to the right side to balance that added to left.)

By comparison with the equation $(x-h)^2 + (y-k)^2 = a^2$, it is readily seen that the co-ordinates of the centre are 3, 2, and the radius 4. This method is equivalent to shifting the axes of reference so that the origin is at the centre of the circle. All ordinates then become (y-k), and abscissæ

(x - h), for the axis OY moves a distance k upwards, and

OX moves a distance h to the right.

305. **Polar Co-ordinates**. Refer to Fig. 240. The polar co-ordinates of any point P are (1) its distance OP from a fixed point O, called the pole or origin; (2) the angle θ (XOP in Fig. 240) which OP makes with a fixed line OX (called the initial line) passing through the origin. The distance OP is called the **radius vector**, and is usually denoted by r. The angle θ is always measured counterclockwise. Hence the polar co-ordinates of a point are often called the " $r\theta$ co-ordinates." (See Art. 128.)

Any equation of a curve in rectangular co-ordinates may be readily transformed into one in polar co-ordinates, if the initial line and axis of x coincide, by substituting $r \cdot \cos \theta$ for x and $r \cdot \sin \theta$ for y.

Thus, the equation of a circle in rectangular co-ordinates is $(x-h)^2 + (y-k)^2 = a^2$ when the radius is a, and the centre is at a point h, k. For a circle of radius a, with centre on the axis of y, we have h=o, and k=a. Its equation is thus $(x-o)^2 + (y-a)^2 = a^2$, in rectangular co-ordinates. By writing x=r. $\cos\theta$ and y=r. $\sin\theta$ we have the corresponding equation in polar co-ordinates, $(r \cdot \cos\theta)^2 + (r \cdot \sin\theta - a)^2 = a^2$. This simplifies to $r^2 \cdot (\cos^2\theta + \sin^2\theta) = 2r \cdot a \cdot \sin\theta$.

 $r^2 = 2 r. a. \sin \theta$ $r = 2 a \sin \theta$

which is the polar equation of a circle of radius a.

Select any value for a, and calculate r for values of θ from 0° to 180°, and plot the resulting curve, thus satisfying yourself that it is a circle of radius a.

306. **Approximations.** When x and y are small, the product of (1 + x) (1 + y) may be taken as 1 + x + y instead of 1 + x + y + xy, which is obtained by multiplying out fully. The term xy which is neglected is called a small quantity of the second order, since it is the product of two small quantities. As an example, if x = 0.02 and y = 0.003, then xy is 0.00006 and is negligible.

This approximation may be extended to the product of three or more factors having small second terms. Thus if x, y, and z are small, (1 + x)(1 + y)(1 + z) is taken as 1 + x + y + z, all the terms neglected being small.

Ex. Find the product of $1.006 \times 1.024 \times 1.037$.

 \sim Let x = 0.006, y = 0.024, z = 0.037.

Then the terms are (1 + x), (1 + y), and (1 + z) and the product is 1 + x + y + z = 1 + 0.006 + 0.024 + 0.037 = 1.067.

If x, y, and z are equal, we get (1 + x) (1 + y) (1 + z) =(1+x)(1+x)(1+x) = 1+x+x+x=1+3xor $(1 + x)^3 = 1 + 3x$.

Similarly $(1+x)^4 = 1+4x$, and generally $(1+x)^n =$ 1 + nx, whether x be + or - and integral or fractional.

Ex. (1) Find an approximate value for

$$\frac{(1+x)}{(1+y)^3(1+z)}.$$

This can be written $(1 + x)(1 + y)^{-3}(1 + z)^{-1}$ (Art. 95). The value of $(1 + y)^{-3}$ is 1 + (-3y) = 1 - 3y, $(1+z)^{-1}$ is 1+(-z)=1-z,

hence, approximate value is 1 + x - 3y - z.

Let x = 0.02, y = 0.03, z = 0.04.

Then
$$\frac{1.02}{(1.03)^3 (1.04)} = 1 + 0.02 - 3 (0.03) - 0.04$$

= $1 + 0.02 - 0.09 - 0.04$
= 0.89 .

(2) Find an approximate value of $\frac{1}{\sqrt{1.003}}$.

This may be written $1 \times (1.003)^{-\frac{1}{2}}$ (Art. 96), $(1.003)^{-\frac{1}{2}} = (1 + 0.003)^{-\frac{1}{2}}$ $=1-\frac{0.003}{2}$ = 0.9985.

When θ is small we can take sin. θ , tan. θ , and the angle θ in radians as equal.

From the Trig. Table it will be seen that for an angle of 2°,

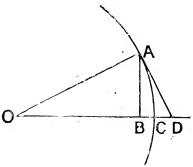


FIG. 202.

 $\sin 2^{\circ} = 0.0349$, $\tan 2^{\circ} = 0.0349$, and $2^{\circ} = 0.0349$ radians.

From Fig. 202, $\sin \theta = \frac{AB}{OA},$

tan. $\theta = \frac{AD}{OA}$

if AD is pr to OA.

 θ (radians) = $\frac{1}{OA}$

When θ is small, AB, AC, and AD are approximately equal in length,

307. **Evaluation of Formulæ**. A complex-looking formula is easy to evaluate if numerical values of the quantities are given. Such symbols as sin. a, cos. a, tan. a, log. a, etc., merely direct the student to refer to certain tables.

Certain conventions are used in formulæ with which the student should be familiar. Some have already been

explained. The following are collected for reference.

(1) 520,440 may be written as $5 \cdot 204 \times 10^5$, for $5 \cdot 204$ when multiplied by 100,000 is 520,400. Similarly, any numerical quantity may be written as $10^n \times$ a quantity

which is $=\frac{\text{given quantity}}{10^n}$.

The index n may be + or -.

Thus
$$0.005204 = \frac{5.204}{1000} = \frac{5.204}{10^3} = 5.204 \times 10^{-3}$$
.

(2) Logs. calculated to the base e are written as $\log_{e} x$ where x is the number whose $\log_{e} x$ is required.

 $\log_{e} 500 = (\log_{10} 500) \times 2.3026$.

(3) Sin. ^{-1}n means the angle whose sine is n. Similarly for $\cos ^{-1}n$, $\tan ^{-1}n$, etc.

(4) [4 or 4! means "factorial four," and is

$$= 4 \times 3 \times 2 \times 1.$$

(5) In terms such as sin. (nt + g), the value of (nt + g) when calculated usually gives the angle in radians, and not in degrees, unless otherwise stated.

(6) When an angle is given, say as ϕ , and in the same formula either sin. ϕ , cos. ϕ , etc., is used, ϕ must be taken to be in radians.

The following examples illustrate methods of calculation.

Ex. 1. If x = a (ϕ – sin. ϕ) and y = a (1 – cos. ϕ), find x and y when a is 10 and ϕ = 0.5061 radians. (B. E., 1905.)

From the Trig. Table it will be seen that 0.5061 radians = 29°.

Hence
$$x = a (\phi - \sin \phi)$$

$$x = a (\phi - \sin \phi)$$

$$= 10 (0.5061 - \sin 2.29^{\circ})$$

$$= 10 (0.5061 - 0.4848)$$

$$= 10 \times 0.0213$$

$$= 0.213.$$

$$y = a (1 - \cos \phi)$$

$$= 10 (1 - \cos 2.29^{\circ})$$

$$= 10 (1 - 0.8746)$$

$$= 10 \times 0.1254$$

$$= 1.254.$$

Ex. 2. If x is in radia

$$\sin x = x - \frac{x^3}{13} + \frac{x^5}{15} - \frac{x^7}{7}$$
, etc.

is the angle in degrees?

Find sin. x correct to four significant figures* when
$$x = 0.3$$
. What the angle in degrees? (B.E., 1904. sin. $x = 0.3 - \frac{(0.3)^5}{3 \times 2} + \frac{(0.3)^5}{5 \times 4 \times 3 \times 2} - \frac{(0.3)^7}{7 \times 6 \times 5 \times 4 \times 3 \times 2}$, etc.
$$\frac{\frac{0.3 \times 0.3 \times 0.3 \times 0.3}{3 \times 2} = 0.0045}{\frac{0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3}{5 \times 4 \times 3 \times 2}} = 0.00002.$$

We need not calculate any further, as we only require four significant figures, hence we can neglect 0.00002.

$$\therefore \sin x = 0.3 - 0.0045,$$

= 0.2955.

The angle in degrees and min, is found by interpolation (Art. 343) from the Trig. Table to be 17° 12'.

Ex. 3. There are two formulæ for calculating φ;

(1)
$$\phi = log.e \frac{t}{273}$$
, which is approximate, and

(2)
$$\phi = 1.0565 \log_{10} \frac{t}{273} + 9 \times 10^{-7} \left(\frac{t^2}{2} - 503 t\right) + 0.0902$$

If $t = \theta + 273$ and $\theta = 57$, find the two answer which is correct. and the percentage error in using the approximate formula. (B, E., 1904.)

(1)
$$\phi = \log_e \frac{57 + 273}{273} = \log_e \frac{330}{273}$$
$$= 2.3026 \log_{10} \frac{330}{273}$$
$$= 2.3026 (\log_{10} 330 - \log_{10} 273)$$
$$= 2.3026 (2.5185 - 2.4362)$$

$$= \frac{6.1895.}{0.1895.}$$
(2) $\phi = 1.0565 \log_{10} \frac{57 + 273}{273} + 9 \times 10^{-7} \left(\frac{(57 + 273)^2}{2} - 503 (57 + 273) \right) + 0.0902.$

We have $\log_{\cdot}e^{\frac{57+273}{273}} = 0.1895$ from (1) above.

$$\frac{1.0565 \times 0.1895 = 0.200207 \text{ correct to six places of decimals.}}{9 \times 10^{-7} \left(\frac{330 \times 330}{2} - 503 \times 330\right)} = \frac{9}{10^{7}} \left(\frac{54450 - 165990}{54450 - 165990}\right) \\
= \frac{9 \times (-111540)}{10^{7}} \\
= -\frac{1003860}{107} \\
= -\frac{0.100386}{1007}$$

^{*} To ensure a correct answer to the required number of significant figures, all terms which have less than the same number of zeros after the decimal points must be included.

Hence
$$\phi = 0.200207 - 0.100386 + 0.0902$$

= 0.19.

The correct value is 0.19.

The approx. value is 0.1895.

The difference is 0.0005 too small.

Hence percentage error is $\frac{0.0005 \times 100}{0.19} = 0.26$ ° too small

Extreises.

- (1) The difference of x and y is 3.14; the sum of x^2 and y^2 is 140; find x and y. (B.E. (2) 1908.)
 - (2) What are the factors of $x^2 8.92x + 18.37$? (B.E. (2) 1908.)
- (3) If $y = ax^{1.46} + bx^{2.5}$; if y = 6.3 when x = 1, and if y = 1.33 when x = 2, find a and b. (B.E. (2) 1908.)
- (4) If $\frac{x}{y} = e^{\mu\theta}$ and if $\mu = 0.25$, $\theta = 3$, find $\frac{x}{y}$. It is known that x y = 1000, find x and y. (B.E. (2) 1908.)
- (5) The following quantities measured in a laboratory are thought to follow the law $y = ab^{-x}$. Try if this is so, and, if so, find the most probable values of a and b. There are errors of observation.

x	0.1	0.2	0.4	0.6	1.0	1.5	2.0
y	350	316	120	63	12.86	2.57	0.425

(B.E. (2) 1908.)

(6) A steam engine when its load is P (horse-power) uses W lb. of steam per hour. The following measurements were made:—

P	124.6	62	19.8
W	2750	1490	640

Is there any apparent law connecting W and P? If so, what is it (B.E. (2) 1910.)

(7) The following figures show the indicated horse-power P on a vessel at the speed of v knots. At these speeds the wave-resistance is considerable.

r.	20.1	24.9	30.2
P	1054	2135	3850

It is thought there is a law like P = arn. Try if this is so, and state the most probable value of n. (B.E. (2) 1910.)

(8) If $xy^{1.87} = 25$. If x = 4, find y. (B.E. (2) 1907.) (9) L being length in feet and H the height in feet of still water level above the sill of a thin-edged rectangular notch for measuring water, Q being cubic feet per second flowing; it is known that Q= $a \text{ LH}^{\frac{3}{2}} - b \text{ H}^{\frac{5}{2}}$.

A notch of length 10 ft. was experimented with. When H was 0.51, Q was found to be 5.82, and when H was 0.98, Q was found to be 32.10. What are the values of a and b? What is Q when H is 1.21? (B.E. (2) 1907.)

(10) The following numbers are authentic; t seconds is the record

time of a trotting (in harness) race of m miles:

-	,						,	,		
m	1	2	3	4	5	10	20	30	50	100
t	119	257	416	598	751	1575	3505	6479	14141	32153

It is found that there is approximately a law $t = am^b$, where a Test if this is so, and find the most probable and b are constants. values of a and b. The average speed in a race is $s = \frac{m}{t}$; express s in terms of m. (B.E. (2) 1907.)

(11) The model of a ship, when being drawn at the following speeds r (in feet per minute), offered the following resistances R (in pounds) to motion :-

V	233	287	347	406	466	525	588	646
\overline{R}	1.08	1:76	2.93	4.26	6.33	9.52	12.74	15.16

It is to be remembered that there are small errors in such measurements.

If we assume a law like $R = ar^n$, find n for the smallest and highest speeds; for what value of r does n seem to be greatest?

(Suggestion, plot log. R and log. r on squared paper.) (B.E. (2) 1905.)

(12) In the equation $y = 4.5 x^3$ put x = 1, 2, 3, 4, 5, etc., and calculate corresponding values of y. Now plot x and y on squared paper, and plot also $\log x$ and $\log y$. Find the law connecting $\log x$. x and log. y, and from that deduce the original equation.

Answers.

- (1) 9.63; 6.49 or -6.49; -9.63. (2) (x-5.69)(x-3.23).
- (3) 11.81; 5.51. (4) 2.116; 1,806.5; 806.5.
- (5) a = 485.5; b = 31.62. (6) W = 20 P + 250.
- (7) a = 0.1214; n = 3.1. (8) 3.81. (9) a = 0.26; b = -26.2.
- (10) a = 119; b = 1.22. (11) a = 0.4259; n = 2.75.
- (12) Log. $y = 3 \log_{10} x + 0.6532$,

CHAPTER XXIV.

VECTORS.

308. In Mathematics, Mechanics, and Physics we have to deal with two distinct types of quantities. When we speak of 50 shillings, meaning a sum of money, we have fully specified the quantity we are speaking of, for we have specified the kind of quantity, viz., money, and we have specified the amount or magnitude of the quantity, viz., 50 shillings. If we move a body from any point to another point, say 50 ft. away, we say that the body has had a displacement of 50 ft.; here we have stated the kind of quantity, viz., displacement, and also the amount, viz., 50 ft.; but, before we can specify the point to which the body has been moved, we still require to state a further characteristic, viz., in what direction the displacement has been made. A quantity such as a displacement, then, is not fully specified until we know quantity, kind, and direction.

Quantities which are fully specified by kind and amount only and are independent of direction are called **Scalar Quantities**; examples are numbers, length, mass, time,

temperature, etc.

Quantities which have a definite direction, and hence are fully specified only by kind, amount, and direction, are called **Vector Quantities***; examples are forces, displace-

ments, velocities, accelerations, momentum, etc.

309. Both scalar and vector quantities can be represented geometrically by a line drawn to scale. The length of the line, in terms of the unit employed for the scale, represents in each case the amount of the quantity. If we are dealing with a scalar quantity, we may draw the line in any direction on the paper. If we are dealing with a vector quantity, we cannot draw the line in any direction we please, because the vector quantity has direction as well as magnitude, and

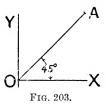
^{*} Vector quantities are often referred to simply as vectors.

hence we must draw the line on the paper parallel to the given direction of the vector quantity.

Thus, a line 1 in. long drawn anywhere, e.g., OA (Fig. 203), will represent 50 shillings money; if our scale is 1 in. = 50 shillings,

but it will only represent, to the same scale a displacement of 50 ft., provided the displacement is along the direction OA, i.e., 45° to the horizontal reference line OX.

When a vector, e.g., a displacement is represented by a line, either drawn on paper, or specified in space, we are still in doubt as to whether displacement is from O to A or from A to O,



i.e., we require to know which way along the given direction line the displacement takes place. This is usually called the **sense** of the vector quantity, and it is denoted by an arrow-head placed anywhere on the direction line. Other terms used for sense are **ort** and **clinure**.

310. Summarizing, we can represent fully a vector quantity by a line if

(1) The length represents to scale the amount of the

quantity.

(2) The direction of the line corresponds to that of the vector quantity.

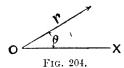
(3) The sense, ort, or clinure is shown on the line.

311. Some vector quantities require not only direction to be specified, but also their actual position on the paper, or in space, e.g., a force is not completely specified until we know its magnitude, direction, sense, and a point through which its direction passes. A force is thus a particular kind of vector quantity, for it has a definite position as well as direction, and hence it is called a localized vector or a rotor.

312. A vector quantity may be specified numerically if we adopt a fixed line of reference and specify the angle, measured anti-clockwise, between the reference line and the direction of the vector quantity. The line of reference is usually taken as horizontal.

Let r units be the magnitude of the vector, and let θ be the angle its direction makes with the horizontal reference line OX (Fig. 204). Then the vector is specified numerically

as the vector r_{θ} , the symbol r denoting the magnitude, and

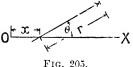


the suffix θ the direction with the fixed reference line. The sense is understood as being away from O.

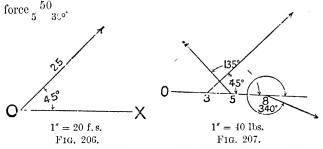
The displacement OA (Fig. 203), would be written as a displacement 50_{450} .

Similarly, to specify numerically a rotor or localized vector, we adopt a fixed line of reference, usually horizontal, and a fixed point O in that line (Fig. 205). Then, a rotor of magnitude τ and inclination θ and passing through a point distant τ from O

through a point distant x from O measured along OX is written as $x^{r_{\theta}}$. The distance x is called an intercept, and is + or - according as it is to be measured to the



right or left of O. As an example, a force of 50 lbs. whose direction is inclined at 30° to the horizontal, and which passes through a point 5 ft. from O, would be written as the



Ex. Represent graphically and numerically the following vectors—
(1) A velocity of 25 ft. per second in a direction North-east.

(2) Three forces (a) 50 lbs. acting North-east, (b) 30 lbs. acting North-west, and (c) 25 lbs. acting 20° South of East and passing through points 3, 5, and 8 ft. respectively from O.

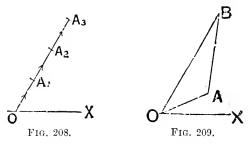
Mumerically.

(1)
$$\frac{25}{45^{\circ}}$$
 (2) $\frac{50}{3}$ $\frac{50}{45^{\circ}} + \frac{30}{5}$ $\frac{20}{340^{\circ}}$.

Graphically, in Figs. 206 and 207.

313. Consider a vector, say a displacement OA_3 (Fig. 208). If we divide it into three equal parts, OA_1 , A_1A_2 , A_2A_3 , and consider each of these parts separately as a displacement, then the displacement OA_3 may be regarded as the sum of three separate displacements, $OA_1 + A_1A_2 + A_2A_3$.

Similarly, the sum of any number of parallel vectors of the same sense is a single vector, parallel to each of, and having the same sense as, the separate vectors, but having a magnitude equal to the sum of the magnitudes of the separate vectors. The vector sum is itself a vector joining the beginning of the first vector to the end of the last vector when all the vectors are placed end to end. If we regard



the separate vectors as a series of steps to be taken in order, then the vector sum is a single step equal in effect to the sum of the separate steps.

314. Now consider vectors which are not parallel, and consider each vector as a step. If we have two vectors, say displacements OA and AB (Fig. 209), then by taking the step OA we arrive at A, and by a further step AB we arrive at B; a single step OB, equivalent to the two separate steps, would be the step OB.

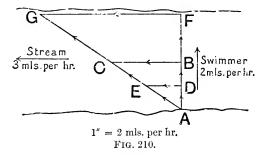
The vector OB is the **vector sum** of OA and AB, and may be defined as that vector which is obtained by placing the separate vectors end to end, with the beginning of the second in contact with the end of the first, and then joining the beginning of the first to the end of the second.

The vector OB = vector OA + vector AB.

^{*} Due to O. Henrici.

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This rule for the sum of two non-parallel vectors will be clear if we consider a swimmer crossing a wide river in which the stream is flowing at a certain velocity, say, 3 miles per hour. Let us assume that the swimmer can swim 2 miles per hour in still water, and that he heads straight across the stream. The swimmer's direction is AB (Fig. 210), and that of the stream is BC. Let AD be ½ mile. Then in ¼ hour the swimmer would be at D, if swimming in still water. But, in the same time, the stream has carried him ¾ mile or a distance DE down the stream; hence, at the end of ¼ hour the swimmer is actually at E. If AB is 1 mile, the swimmer would arrive at B, in still water, in ½ hour, but the stream carries him ½ miles, or a distance BC, hence his actual position is C. If we plot his position in this way for different intervals of time, and join up all



points as E, C, etc., we find the actual path of the swimmer to be AG. Now find the path by the rule above for the sum of two vectors, one a displacement of 2 miles, and the second a displacement of 3 miles—perpendicular to the first. The vector sum has the direction AC.

Again, as the vector AF is the swimmer's velocity in still water in miles per hour, and as the vector FG is the velocity of the stream in miles per hour, the resultant velocity (Art. 316) is given in magnitude and direction by the vector AG.

Since AF = 2 and FG = 3,
$$\therefore$$
 AG = $\sqrt{\frac{1+9}{13}}$ = $\sqrt{13}$ miles per hour.

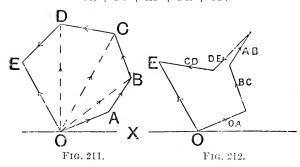
This result means that the actual velocity of the swimmer, relatively to the bed of the stream, or as seen by a person standing on the bank, is $\sqrt{13}$, or 3.6, miles per hour in the direction AG.

Similarly, the sum of any number of vectors is the vector obtained by placing the end of the first vector in contact with the beginning of the second, the end of the second in contact with the beginning of the third, and so on, and then joining the beginning of the first to the end of the last.

$$\begin{array}{l} \text{Vector OE} = \text{OA} + \text{AB} + \text{BC} + \text{CD} + \text{DE}, \\ \text{for OB} = \text{OA} + \text{AB}, \\ \text{OC} = \text{OB} + \text{BC} = \text{OA} + \text{AB} + \text{BC}, \\ \text{OD} = \text{OC} + \text{CD} = \text{OA} + \text{AB} + \text{BC} + \text{CD}, \\ \text{OE} = \text{OD} + \text{DE} = \text{OA} + \text{AB} + \text{BC} + \text{CD} + \text{DE}, \\ \end{array}$$

· 315. It does not matter in what order we take the vectors, we obtain the same vector for the sum.

Thus, in the above example, Fig. 211, shows that OE = OA + AB + BC + CD + DE, and Fig. 212 shows that OE is also the sum, say, of OA + BC + AB + DE + CD.



The above statement agrees with the corresponding algebraic law that

 $a+b+c+d+e= \operatorname{say}(a+c+b+e+d),$ or, the order is immaterial in addition.

316. The vector sum of a series of vectors is called the resultant, and the two or more vectors which collectively give the vector sum or resultant are called components.

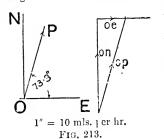
We may require to find the resultant of a given series of vectors, or we may be given a single vector and be required to find its components in any given directions.

The latter operation is called resolving the given vector in the given directions.

Ex. I. A ship steers due North at 10 miles per hour, and is moving in a current which travels East at 3 miles per hour. What is the actual velocity and direction of the ship?

ON, Fig. 213, represents the direction in which the ship steers; OE the direction of the current. Then OP is the actual direction of

the ship. The direction OP is determined by drawing the vector diagram. The vector on is 10 units long and parallel to ON, the



vector θe is placed at the end of the vector θn , and is 3 units long and parallel to OE. The vector sum is then θp , and by measurement this is 10.4 units long. The actual direction of the ship is OP parallel to θp , and the actual velocity is 10.4 miles per hour.

This result can be found by calculation, for on is $\underline{\mathbf{I}}^{\mathbf{r}}$ to oc, hence

$$op = \sqrt{10^2 + 3^2} = \sqrt{109}$$

= 10.44 miles per hour.

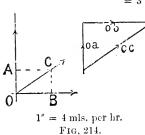
The angle EOP is an angle whose tangent is $\frac{10}{3} = 3.3333$. This angle is found from the Trig. Table to be 73.3 degrees.

Ex. II. A swimmer wishes to cross a stream 880 yards wide, which is flowing at the rate of 3 miles per hour. He heads straight across the stream, and can swim 2 miles in 1 hour in still water. At what point will he land; what will be his actual velocity and path, and how long will he take to cross?

Let OA, Fig. 214, be the direction across the stream, and OB the direction in which the stream flows. In the vector diagram, oa = 2 and is parallel to OA, ob = 3 and is parallel to OB; then oe gives the velocity and OC the direction of the swimmer.

$$oc = \sqrt{4 + 9} = \sqrt{13}$$

= 3.6 miles per hour.



The angle BOC = angle whose tangent is $\frac{2}{3}$ or 0.6666, i.e., 33.6° . If OA = $\frac{1}{2}$ mile (880 yards), then OB = $\frac{3}{4}$ mile, and OC = $\frac{\sqrt{13}}{4}$ miles = 0.9 miles. The swimmer thus lands $\frac{3}{4}$ mile down stream and relatively to the bed of the stream he swims 0.9 miles. He has to swim the equivalent of $\frac{1}{2}$ mile in still water, and he can swim 2 miles per hour. Hence

the time taken is $\frac{1}{4}$ hour. We get the same result if we find the time taken to swim the path OC at the velocity oc, for

OC =
$$\frac{\sqrt{13}}{4}$$
 miles. Velocity $oc = \sqrt{13}$ miles per hour.

Hence time to swim OC =
$$\frac{\sqrt{13}}{4}$$
 ÷ $\sqrt{13}$ = $\frac{1}{4}$ hour.

Ex. III. A cyclist travels due South at 12 miles per hour, and the wind actually blows from the South-west at 5 miles per hour. What is the apparent velocity and direction of the wind to the cyclist?

If a cyclist travels 12 miles per hour due South in a calm atmosphere, he will experience a wind resistance due to his own motion equivalent to that of a wind having a velocity of 12 miles per hour due North. Hence the cyclist in this example really experiences the resultant of two winds, one blowing due North at 12 miles per hour, and represented by ON, Fig. 215, and the other blowing from South-west to Northeast at 3 miles per hour and represented by OP.

The vector diagram is thus
$$on = 12$$
 parallel to ON $op = 5$, OP.

The vector sum is oq = 15.93 mls. per hr. The direction OQ is 13° East of North.

*To calculate the magnitude and direction of OQ, we have

Angle between on and op = 135°.
oq² =
$$(on)^2 + (op)^2 - 2$$
 (on) (op)
cos. 135°.
= $12^2 + 5^2 - (2 \times 12 \times 5 \times - \cos.45^\circ)$.
= $144 + 25 + (120 \times 0.7071)$.
= $144 + 25 + 84.9$.
= 253.9 .

oq = 15.93. The angle NOQ is given by

n by $\frac{\sin \text{ NOQ}}{\sin 135} = \frac{5}{15\cdot 93}$. 1'' = 16 mls. per hr. Fig. 215.

 $\sin 135$... Sin. NOQ = 0.2219,

and \therefore angle NOQ = 13° (correct to the nearest degree).

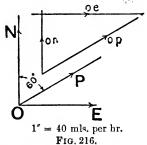
Ex. IV. An aeroplane is travelling 30° North of East at a velocity of 50 miles per hour. What are the components of this velocity in the directions North and East?

Let on, oe, op, Fig. 216, be the vector diagram. We know that op, drawn parallel to OP must = 50. We require to find two vectors

on and oe parallel respectively to ON and OE, which will give op as their vector sum. Hence from the ends of op draw on, oe to intersect as shown. Then on = 25 is the northerly component and oe = 43.3 is the easterly component.

To calculate on and oe we have Angle NOP = 60° .

Hence
$$oe = op \times \sin 60^\circ$$
,
 $= 50 \times 0.866$,
 $= 43.3 \text{ mls. per hr.}$
 $on = op \cos 60$,
 $= 50 \times 0.5$,
 $= 25 \text{ mls. per hr.}$

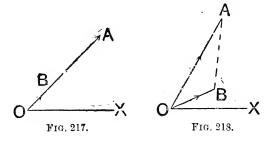


• This calculation may be deferred until Chap. XXXV. has been read.

317. To determine the difference of two vector

quantities.

Let us consider, first, two parallel vectors, say, displacements. One body is moved 50 ft. from a given point O in the direction OA to the point A, Fig. 217. A second body is moved 30 ft. in the direction OA to B. The difference between the two displacements, or vectors, OA and OB is a displacement or vector BA, or that vector (BA) which must be added to the vector (OB) to be subtracted, in order to



make the vector sum (OB + BA) equal to the given vector

(OA).

Similarly, when OA and OB, Fig. 218, are not parallel. The difference between the vectors OA and OB is the vector BA, which is the vector to be added to the vector OB to make the sum = OA. Hence, to find the difference between any two vector quantities OA, OB, set out the two vectors from a common point O and join the ends B and A of the two vectors. This gives the vector BA, which must be added to the vector OB to give OA as the vector sum.

[The student should note that the vector BA = OA - OB, whereas the vector AB would = OB - OA for OB + BA = OA and OA + AB = OB.]

If we again consider the illustration of the swimmer in a stream (Art. 314), the difference between the vector AG (resultant velocity) and the vector AF (velocity of the swimmer) is the vector FG (or velocity of stream) necessary to make the path AG.

Ex. I. Two ships at sea are travelling respectively 25 knots per hour North-cast and 20 knots East, and their positions are such that they are approaching each other. What is their velocity of approach?

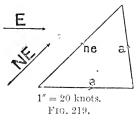
The directions of the ships are shown at E. and N. E., Fig. 219.

The vector diagram is

nc = 25 and parallel to N.E. c = 20 and parallel to E.

The vector a = 17.8 is the vector difference of the two given velocities and hence gives the velocity with which the ships approach each other.

Note.—The student will get a better idea of this kind of problem by assuming that the ships are travelling in the



same direction at different speeds, say 25 and 20 knots. Their **velocity of approach** is 5 knots, or the vector which added to the velocity 20 gives the vector 25. The two steps 20 and 5 make up the single step 25. Similarly, in Fig. 219, the two steps e = 20 and a = 17.8 make up the single step ne = 25. Hence a is the velocity and direction of approach. If the ships do not ultimately collide, there is a certain time at which they are at a minimum distance apart; after that they separate with a velocity equal to their velocity of approach.

Ex. II. A jet of water travels at a velocity of 50 ft. per second in a direction AB, Fig. 220, and impinges upon an inclined plate B which has a relocity of 20 ft. per second in the direction BC. Find the relocity

A B C bc

1" = 50 f.s.

Fig. 220.

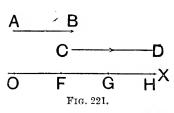
of the water relatively to the plate, assuming that the jet impinges upon the plate without shock.

The water is travelling faster than the plate and hence is approaching the plate and will ultimately impinge upon and run along it. The velocity of the water relatively to the plate is the velocity at which the water approaches the plate. The vector sum of the velocity of the water relative to the plate and the actual velocity of the plate must = the velocity || AB.

Hence, find the vector difference of velocities \parallel AB and \parallel BC. This is r in vector diagram and = 33.5 ft. per second.

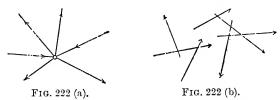
318. **Rotors.** A vector quantity which must occupy a definite position has already been referred to (Art. 311) as a localized vector or a rotor, and a force has been given as an example of a rotor quantity.

The line representing a vector quantity other than a rotor may be moved anywhere parallel to itself. The line repre-



senting a rotor quantity must not be moved except along its own direction, e.g., a displacement of 50 ft. in a direction West to East may be represented by the line AB (Fig. 221), or by CD, or by any line parallel to AB or CD, whereas a

force of 50 lbs. acting from West to East and passing through O may only be represented by OF or FG or GH, or some segment along the line OX. We shall now confine our attention to those rotors which are forces and we shall use the term "force" instead of the more general term "rotor quantity." It must be remembered, however, that the various rules for finding the vector sum and vector difference of a series of forces apply equally to other rotor quantities.



319. A series of forces which act in the same plane are called **co-planar forces**.

A series of forces which act at the same point are called **concurrent forces**, whether they are co-planar or not.

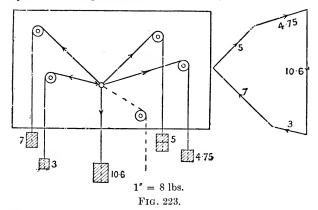
We shall deal only with co-planar forces, and consider

(1) Concurrent co-planar forces, Fig. 222 (a).
(2) Non-concurrent co-planar forces, Fig. 222 (b).

320. Concurrent Forces. If a number of forces act at a point, it can be shown experimentally that a single force,

acting at the same point, will produce equilibrium, i.e., will prevent motion of the point due to the acting forces.

Take a drawing-board and four small pulleys, as shown in Fig. 223. To a small ring attach five strings, and pass four of the strings over the four pulleys, allowing the fifth string to hang freely. Now place weights, say of 3, 7, 5, and 4.75 lbs., on the strings, and by pulling on the fifth string in various directions, note that the small ring may be made to assume different positions. The magnitude of the force or pull in each string is equal to the weight attached to the end. The directions of the various forces exerted on the ring are indicated by the part of the string between the ring and the pulley over which the



string passes, and each force or pull passes through the centre of the

ring. The sense of each force or pull is away from the ring.

By pulling the fifth string vertically, move the ring to some definite position on the board, and secure it there by an ordinary pin. (If the pin is removed, the position of the ring will alter.) Now add to the fifth string a weight just sufficient to retain the ring in its position when the pin is removed. This weight or pull in the fifth string is a single force which balances the four forces 3, 4.75, 5, and 7 lbs. and it is called the "equilibrant" of the four forces. It also acts away from th ring.

We c n determine the magnitude, direction, and sense of this equilibrant by means of a vector diagram, or as it is called in the case of

forces, a force polygon.

From any point draw a vector equal and parallel to the force 3, from the end of the vector 3 draw a second vector equal and parallel to the force 7, and in turn vectors equal and parallel to the forces 5, 4·75. Now join the end of the vector 4·75 to beginning of vector 3.

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This vector gives the magnitude and direction of the equilibrant, and in the present example it is vertical and equal to 10.6 lbs.

If the fifth string be now passed over a pulley, as shown in dotted lines, and a weight attached to its end, the position of the ring, and hence the directions of all pulls, will alter. Now draw a second force polygon as before, and note that the resultant vector is again equal to, and parallel to, the pull in the fifth string.

We conclude from the above experiment that, when a series of forces act at a point and are in equilibrium (i.e., the point does not move under the action of the forces) they can be represented in magnitude and direction by the sides of a closed polygon called the polygon of forces.

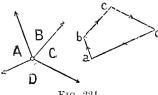


Fig. 224.

We can regard any one of the series of forces as the equilibrant of the remainder of the series, since each force may be regarded as balancing the remainder. Again (as in a "tug of war," a pull along a rope in one direction will balance an exactly equal pull in the opposite direction), it

will be seen that the four forces or pulls 3, 7, 5, 4.75 lbs. (Fig. 223) are equal in effect to a single vertical pull upwards of 10.6 lbs., for this pull would just balance the equilibrant pull of 10.6 lbs. downwards. The single force which is equal in effect to a series of forces is called the resultant force, and the separate forces of the series are called component forces. It will be also seen that the closing line of the force polygon gives the magnitude and direction of the resultant, while the sense of the resultant is opposite to that of the equilibrant.

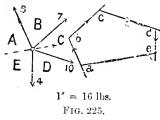
321. Notation.—It is usual in drawing force diagrams to denote each force by two letters placed on opposite sides of the force; thus the four forces shown in Fig. 224 would be called the forces AB, BC, CD, DA.

When we draw the force diagram, we letter the ends of the vector which is drawn parallel to the force AB as ab, that parallel to BC as bc, and so on, the capital letters denoting the forces in the one diagram, often called the position diagram, and the corresponding small letters denoting the vectors in the vector diagram.

Ex. I. Forces of 5, 7, 10, and 4 lbs. act at a point O. Find a single force (resultant) which has the same effect at the point O, and a force (equilibrant) to balance the four given forces.

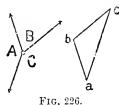
Let the forces 5, 7, 10, and 4 be denoted, as shown in the position diagram, Fig. 225, by letters AB, BC, CD, DE.

Draw the vector diagram abcde. Then the vector ae gives the vector sum of the given forces, and hence is the resultant force. This resultant force is shown in dotted lines in the position diagram and it acts at O. If we reverse the sense, we have the vector ea, which



is the equilibrant, and which would also act at O.

322. Fig. 226. If we have only three forces acting at a given point and they are in equilibrium, the vector diagram

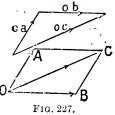


becomes a closed triangle, hence the following theorem, known as the triangle of forces.

If three concurrent co - planar forces can be represented in magnitude and direction by the sides of a triangle, the forces are in equilibrium.

323. Parallelogram of

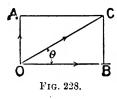
Forces.—The resultant or equilibrant of two concurrent forces is given in magnitude and direction by the diagonal



of a parallelogram, the sides of which are equal and parallel respectively to the given forces.

Thus if OA, OB, Fig. 227, represent to scale two forces acting at O, their resultant is given in magnitude and direction by the diagonal OC, and the equilibrant would be CO; for if we draw the usual vector diagram to the same scale we have oa = OA, ob = OB = AC, and hence oc =OC.

This construction is convenient, as it may be readily reversed to find the components of a given force. Thus, let OC, Fig. 228, be a given force. To find components acting horizontally and vertically.



Set out OC to scale and draw OB and OA respectively horizontal and vertical. Draw CA parallel to OB and CB parallel

to OA.

Then OA and OB represent to scale two forces which have OC for their resultant and hence are the horizontal and vertical components of the force OC.

By calculation OB = OC cos. θ : OA =

 $BC = OC \sin \theta$.

324. The resultant or equilibrant of any number of forces acting at a point can be found by repeated applications of

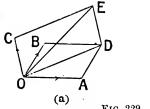
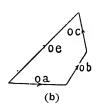


FIG. 229.



the parallelogram of forces. Fig. 229 (a) shows three forces, OA, OB, and OC.

$$OD = OA + OB$$

 $OE = OD + OC$
 $= OA + OB + OC$.

Fig. 229 (b) shows the vector diagram drawn to the same scale, and it will be seen that it gives the same result (oe = OE).

The resultant of two concurrent forces OA, OB, Fig. 230, can be calculated if the angle θ between the forces is known.

$$OC^{2} = OD^{2} + DC^{2}$$

$$OB = AC \text{ and angle DAC} = AOB = \theta$$

$$OD = OA + AD$$

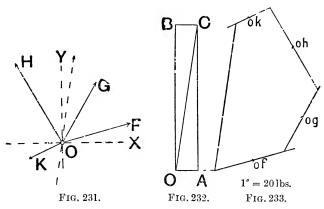
$$= OA + AC \cdot \cos \theta$$

$$= OA + OB \cdot \cos \theta$$
also, $DC = AC \sin \theta = OB \sin \theta$

$$\therefore OC^{2} = (OA + OB \cdot \cos \theta)^{2} + (OB \cdot \sin \theta)^{2}$$

$$OC = \sqrt{(OA + OB \cdot \cos \theta)^{2} + (OB \cdot \sin \theta)^{2}}$$

325. The resultant or equilibrant of a number of forces acting at a point can be calculated by resolving all the forces



horizontally and vertically, finding a single horizontal force equal to all the horizontal components, a single vertical force equal to all the vertical components, and finally the resultant of the two forces thus found.

Ex. Let the given forces be OF, OG, OH, OK, = respectively to 15, 15, 20, and 8 lbs., as shown in the diagram, Fig. 231.

The horizontal components are

The normal components are
$$15 \cdot \cos . 15 + 15 \cos . 60 + 20 \cos . 120 + 8 \cos . 205.$$

$$= (15 \times 0.966) + (15 \times 0.5) + (20 \times -0.5) + (8 \times -0.9063).$$
The single horizontal force is thus
$$= 14.49 + 7.5 - 10 - 7.25$$

$$= 4.74 \text{ lbs. Call this OA.}$$
The vertical components are
$$15 \sin . 15 + 15 \sin . 60 + 20 \sin . 120 + 8 \sin . 205$$

 $15 \sin. 15 + 15 \sin. 60 + 20 \sin. 120 + 8 \sin. 205$ = $(15 \times 0.2588) + (15 \times 0.866) + (20 \times 0.866) + (8 \times -0.4226)$, The single vertical force is thus

$$3.882 + 12.99 + 17.32 - 3.38$$

= 30.812 lbs. Call this OB.

The resultant of the two single forces is (Fig. 232)

 $OC = \sqrt{(OA)^2 + (OB)^2} = \sqrt{(4.74)^2 + (30.812)^2} = 31.34 \text{ lbs.}$

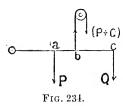
The direction θ which OC makes with OX is such that

$$\sin \theta = \frac{AC}{OC} = \frac{OB}{OC} = \frac{30.812}{31.34} = 0.9832,$$

hence $\theta = 79.5$ degrees.

The result agrees with that found by the vector diagram shown in Fig. 233.

326. **Non-concurrent Forces.**—If a bar oc, Fig. 234, be pivoted to a pin at O, and weights P and Q be applied at the points a and c respectively, the bar would tend to rotate



about O in a clockwise direction.

This tendency to rotate can be exactly counteracted by an upward pull at any point of the bar to the right of O, the amount of the upward pull depending upon the distance from O at which it is applied.

At some point b between a and e the upward pull would be found

to be equal to P + Q, and experimentally it can be shown that the distance Ob is such that $(P + Q) \times ob = (P \times oa) + (Q \times oc)$.

The product of any force and its perpendicular distance from any point is called its moment about the point; it gives a measure of the tendency of the force to produce rotation at the point. The above statement thus shows that the position of the equilibrant of a series of parallel forces is such that the moment of the equilibrant is equal to the sum of the moments of the separate forces about any point in the same plane as the forces.

It must be noted that the equilibrant is equal in magnitude to the vector sum of the separate forces, in this case (P+Q), but is opposite in sense. Although we could prevent rotation at O by a less or greater force than (P+Q) if applied respectively to the right or left of b, we could not prevent motion of the bar oc parallel to itself, if the pivot

at O were free to move, unless the upward pull (P + Q)were just equal to the sum of the downward pulls P and Q.

A series of parallel forces such as P, Q, and (P + Q) will

be in equilibrium if

(a) There is no tendency of the bar oc to move parallel to itself either upwards or downwards.

(b) There is no tendency of the bar oc to rotate about O,

or any other point in the same plane as the forces.

327. The equilibrant or resultant of any series of parallel forces can be found graphically, and also the point at which it must act, as in the following example.

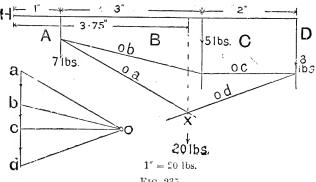


Fig. 235.

Take three forces, AB, BC, CD, Fig. 235, equal respectively to 7, 5, and 8 lbs. The force polygon is abcda. A single force da = 20 lbs. is the equilibrant.

We now require to find a point through which the

equilibrant passes.

Choose any point O, called the pole of the vector diagram, and join oa, ob, oc, od. Between the forces AB, BC in the position diagram draw a line parallel to ob, which joins the pole to the point between the lines ab, bc in the vector diagram. Similarly, draw between forces BC and CD a line parallel to oc. From the end of oc draw od parallel to od in the vector diagram, and from the beginning of ob draw oa parallel to oa. The point X in which these lines intersect is a point through which the resultant or the equilibrant passes.

We can satisfy ourselves that this construction does give a point through which the resultant or the equilibrant passes, by choosing any

point, say H, and drawing through H a line perpendicular to the forces. Measure the distance from H to each force and ascertain whether the moment of the resultant is equal to the sum of the moments of the separate forces, thus—

Moment of AB =
$$(7 \times 1)$$

BC = (5×4)
CD = (8×6)
Sum = $7 + 20 + 48$
= 75

Moment of resultant = $(20 \times 3.75) = 75$.

The diagram *abcda* is the force or vector polygon, and the diagram drawn between the forces AB, BC, CD, and the equilibrant in the position diagram is called the **link or funicular polygon**.

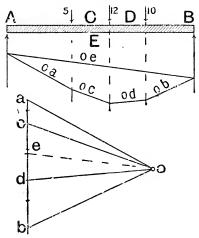


Fig. 236.

It will be noticed that the forces being in equilibrium, both these diagrams are closed. These are the necessary conditions for the equilibrium of a system of co-planar non-concurrent forces. They are utilized in solving various problems.

Ex. A beam 20 ft. long carries loads of 5, 12, and 10 cwts, at points 7, 11, and 15 ft. from the left hand end. Find the magnitude of each

supporting force, assuming that they act vertically upwards.

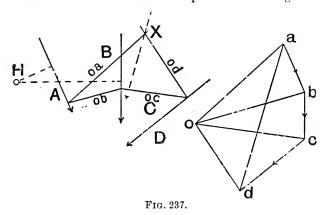
Let AC, CD, DB, Fig. 236, be the three loads, and BE and EA the supporting forces in the position diagram. The force polygon is acdba; we know that the two supporting forces together must be equal to (5+12+10) cwts.; and we have to find the magnitude of

each. Take any pole O, and draw oa, oc, od, ob. Commence from any point on the supporting force EA and draw between the forces in the position diagram lines parallel to oa, oc, od, ob of the vector diagram. We thus obtain four sides of a link polygon. As the whole system of forces must be in equilibrium (for the beam does not move) the link polygon must close. Hence draw oe, the closing line from the force EA to the force BE. From O, the pole, draw a line (shown dotted) parallel to this closing line oe. This line intersects the line acdb in e, and determines the magnitude of each of the supporting forces, for the force BE is given by be, and the force EA by ea. Thus BE = 15·85 cwts., EA = 11·15 cwts.

This result can be checked by calculation. The moment of the force EB about the end A must be equal to the sum of the moments of forces AC, CD, and DB.

Thus
$$EB \times 20 = (5 \times 7) + (12 \times 11) + (10 \times 15)$$
.
 $EB = \frac{35 + 132 + 150}{20}$.
 $= \frac{317}{20}$.
 $= 15.85 \text{ cwts.}$
Total load $= 27 \text{ cwts.}$, hence $EA = (27 - 15.85)$
 $= 11.15 \text{ cwts.}$

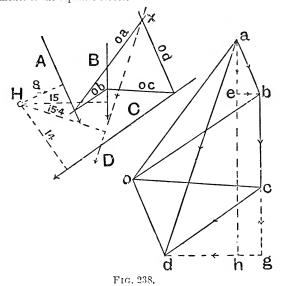
328. If a series of non-concurrent forces are not parallel, we can find their resultant or equilibrant in magnitude



and direction by means of a force polygon; and a point through which it passes by means of a link polygon, exactly as for parallel forces. In Fig. 237, the forces are AB, BC, CD. The force polygon is abcda. The resultant is ad, the equilibrant is da.

The link polygon is begun on the force AB. The two lines ob, oc are first drawn, then oa, od intersecting at X, which gives a point through which the resultant or equilibrant passes.

This construction can be shown to be correct by resolving each of the three forces AB, BC, CD, and the resultant, in any two directions perpendicular to each other, say horizontally and vertically; the component of the resultant in each direction would be found equal to the sum of the three components of the separate forces in that direction. Also, by taking moments about any point such as H, the moment of the resultant is found to be equal to the sum of the moments of the separate forces.



Ex. Forces of 5, 8, and 10 lbs. act as shown at AB, BC, CD, Fig. 238.

The resultant is 19.5 lbs. and acts through X, found by a force polygon and a link polygon, as shown.

The vertical component of AB = ae,

and of resultant and
$$ah = ac + cb + cq$$
.

Similarly, the horizontal component of the resultant is -h d = horizontal components of AB, BC, and CD.

Note horizontal component of ab is + eb.

,, of
$$bc$$
 is a as force BC is vertical.
,, of cd is $-gd$, as it acts to the left.

The algebraic sum is -hd.

Taking moments about II we note that each force tends to produce clockwise rotation about II, hence we can regard all moments as having the same sign. In measuring distances of each force from II, $\frac{1}{16}$ has been taken as unit.

The moment of force AB about
$$H = (5 \times 8) = 40$$
.
BC $_{n} = (8 \times 15) = 120$.

" "
$$(10 \times 14) = 140$$
.

The sum of these is 300 units.

The moment of the resultant about $H = (19.5 \times 15.4)$ = 300

If we reverse the sense of the resultant, we have the equilibrant. The sum of the moments of the three forces and of the equilibrant about H is then zero, and the algebraic sum of the horizontal and vertical components of the forces and equilibrant is zero.

Exercises.

(1) An aeroplane steers due North with a velocity of 50 miles per hour, and is blown by the wind in a direction 25° North of East at 10 miles per hour. In what direction and at what actual speed does the aeroplane travel?

(2) A balloon rises vertically with a velocity of 40 miles per hour, and is blown due South at 15 miles per hour. What is the inclination

to the horizontal of the path taken by the balloon?

(3) The wind is blowing due North with a velocity of 10 miles per hour, and an aviator, who can travel 50 miles per hour in a calm atmosphere, wishes to reach a town situated 20° North of East. In what direction should he steer?

(4) A man wishes to swim across a river 300 yards wide, in which the stream is running at 15 yards per minute. He can swim 45 yards in a minute. At what point will he land if he swims in a direction at right angles to the flow of the stream? If he wishes to land at a point directly opposite to his starting-point, in what direction must he swim and how long will he be in crossing?

(5) A ship steams North at a speed of 20 knots per hour, but the current is taking the ship East at the speed of 4 knots per hour. How far is the ship from its starting-point in $2\frac{1}{2}$ hours, and what is the

direction?

- (6) A smooth board is inclined at 30° to the horizontal, and a metal roller rests on the board, being supported in position by a spring balance which exerts a pull in a direction inclined at 50° to the horizontal. If the balance indicates a pull of 20 lbs., what is the weight of the roller?
- (7) A picture weighing 50 lbs, is supported by a wire passing over a peg which is 5 ft, above the line of attachment of the wire to the picture. The points of attachment are 3 ft, apart. What is the tension in the picture wire?

(8) In a steam engine, the piston is 20 in. diameter, and the mean steam pressure is 50 lbs. per square inch. The crosshead is guided between two parallel guide bars. The connecting-rod is 6 ft. long, and the crank is 1 ft. long. What is the stress in the connecting-rod, the pressure on the lower guide-bar, and the force on the crank-pin tangential to the crank-pin circle when the crank has rotated 35° clockwise from the inner dead centre?

(9) Taking the data in the previous example, plot a curve showing how the turning-force on the crank-pin varies as the crank makes a complete revolution. What is the maximum turning-force? Show that the upper guide-bar for the crosshead has no pressure upon it during one half revolution of the crank from the inner dead centre, and that the lower guide-rod has no pressure upon it for the other half

revolution.

(10) A ship is steering North-east with a speed of 15 knots, and a torpedo, travelling 30 knots, in a direction 10° East of North, is fired from a second ship. What is the velocity, in magnitude and direction, at which the torpedo approaches the first ship. If the torpedo hits the ship 3 minutes after it is fired, how far apart were the ships at the time of firing, and what was the bearing of the first ship relatively to the second?

(11) An aeroplane is in flight during a steady wind which blows at 20 miles per hour from the South. It is propelled relatively to the wind at a speed of 40 miles per hour. In what direction must the pilot apparently steer in order that his actual course shall be due

West? At what speed will he travel West? (B.E. 1910.)

(12) A steamer is moving at 20 ft. per second towards the East; the passengers notice that the smoke from the funnel streams off apparently towards the South with a speed of 10 ft. per second. What is the real speed of the wind, and what is its direction? (B.E. 1910.) (13) There is a triangular roof truss ABC; AC is horizontal and 10 ft. long. The angle BCA is 25°, and BAC is 55°; there is a vertical load of 5 tons at B. What are the compressive forces in BA and BC? What are the vertical supporting forces at A and C? (B.E. 1910.)

(14) The radial speed of the water in the wheel of a centrifugal pump is 6 ft. per second. The vanes are directed backwards at an angle of 35° to the rim. What is the real velocity of the water relatively to the vanes? What is the component of this which is tangential to

the rim? (B.E. 1910.)

(15) The weight of a span of telegraph wire is 12.7 lbs. At one end the wire makes an angle of 5° and at the other an angle of 7° with the horizontal. What are the pulling forces at these ends?

(B.E. 1909.)

(16) The positions of two points A and B in a horizontal plane, referred to an origin O, are defined by the vectors $OA = 3''_{27^{\circ}}, OB = 2''_{68^{\circ}}$ angles being measured anti-clockwise from the East. Choose an origin O, mark the Eastward direction, and plot the points A and B. If C is the middle point of AB, verify that $\frac{1}{2} (3''_{27^{\circ}} + 2''_{68^{\circ}})$, the vector mean of OA and OB, is equal to the vector OC. Measure the length and direction of OC. (B.E. 1909.)

CHAPTER XXV.

TRIGONOMETRY AND APPLICATIONS.

Trigonometry.

329. Plane Trigonometry in its widest sense comprises all algebraical investigations relating to plane angles, whether they form elements of a triangle or not. In its restricted sense, it comprises investigations of the relations existing between the elements (sides and angles) of triangles. By the use of trigonometry, we can calculate three of the six elements of a triangle if we are given the remaining three, one at least of the given elements being a side. This is called **solving a triangle**. In practice, e.g., surveying, civil and mechanical engineering, and building, trigonometry is especially useful. The present chapter is devoted to a consideration of the principles of trigonometry as applied to the solution of plane triangles of any shape, and to the application of trigonometry to many practical problems.

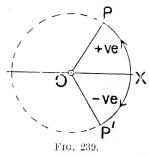
330. We have already discussed, in Chap. XI., the six trigonometrical ratios of an angle less than 90°, and have utilized them in connexion with certain problems involving right-angled triangles. We frequently meet with angles which exceed 90°. Triangles can be constructed with one angle greater than 90° and less than 180°. Again, if we are considering the angle turned through by the crank of an engine, say from the inner dead centre, we can have any angle between 0° and 360°, the latter comprising a

complete revolution.

331. Any angle can be considered as generated by a straight wire OP (Fig. 239), secured by a pin at O so that it can rotate about that point. We require some convention to specify the direction in which the wire is to rotate, *i.e.*, whether clockwise or counter-clockwise. The convention

usually adopted is to consider counter-clockwise rotation as +, and clockwise rotation as -.

Thus, if the initial position of the wire is OX (Fig. 239), and the



new position OP (the wire having rotated counter-clockwise) the angle XOP is +, whereas if the new position is OP' and the wire has rotated clockwise, the angle XOP' would be -. The reflex angle XOP' would, however, be +. In the following pages, all angles are to be regarded as + unless otherwise stated.

332. Now let the wire rotate to a new position OP (Fig. 240), so that the angle XOP is less than 90°. Then, by drawing PM perpendicular to OX, we have a

right-angled triangle POM from which we can determine the trigonometrical ratios for the angle XOP, exactly as in Chap. XI. We have

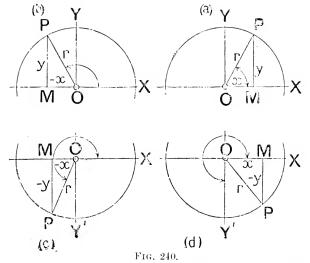
$$\begin{array}{ll} \text{Sin. XOP} = \frac{\text{PM}}{\text{OP}} & \text{Cos. XOP} = \frac{\text{OM}}{\text{OP}} & \text{Tan. XOP} = \frac{\text{PM}}{\text{OM}} \\ \text{Cot. XOP} = \frac{\text{OM}}{\text{PM}} & \text{Sec. XOP} = \frac{\text{OP}}{\text{OM}} & \text{Cosec. XOP} = \frac{\text{OP}}{\text{PM}}. \end{array}$$

If we consider the initial position OX of the wire as the axis of x, and the point O as the origin, and draw YOY_r to XOX' to represent the axis of y, the co-ordinates of the point P for any position in which the angle XOP is less than 90° are x, y; for OM = x, PM = y. If we denote OP by r, we see that sin. XOP = $\frac{y}{r}$, cos. XOP = $\frac{x}{r}$, tan. XOP = $\frac{y}{r}$.

r, we see that sin.
$$XOP = \frac{y}{r}$$
, cos. $XOP = \frac{x}{r}$, tan. $XOP = \frac{y}{x}$ cot. $XOP = \frac{x}{y}$, sec. $XOP = \frac{r}{x}$, cosec. $XOP = \frac{r}{y}$.

The rectangular co-ordinates of the point P, and the distance r of P from the origin, thus furnish a ready means of defining the six trigonometrical ratios of an angle less than 90°. They also furnish a means of defining the six ratios for an angle of any magnitude whatever, the usual convention being followed as to the signs of the co-ordinates, i.e., x is + when measured to the right, and — when measured to the left of YOY'; similarly y is + when measured above, and — when measured below, XOX' (Art. 173).

The following table shows the six trigonometrical ratios for angles from 6° to 360° , with the signs for y and x. The length OP, or r, is always regarded as +, and the student is again reminded that the angle XOP, in each case, is measured counter-clockwise.



Trigonometrical Ratio.		Angle between								
		Fig. 240 (a), 0° & 90°.	Fig. 240 (b), 90° & 180°.	Fig. 240 (c), 180° & 270°.	Fig. 240 (d), 270° & 360°.					
Sin.	ZOP	$\frac{y}{r}$	$\frac{y}{r}$	$\frac{-y}{r}$	$\frac{-y}{r}$					
Cos.	ZOP	$\frac{x}{r}$	$\frac{-x}{r}$	<u>-3'</u>	$\frac{x}{r}$					
Tan.	ZOP	$\frac{y}{x}$	$\frac{y}{-x}$	$\frac{-y}{-x}$	$\frac{-y}{4}$					
Cotan	.XOP	$\frac{x}{y}$	-x y	$\frac{-x}{-y}$	$\frac{x}{-y}$					
Sec.	XOP	$\frac{\ddot{r}}{w}$	$\frac{\ddot{r}}{-x}$	$\frac{r'}{-x}$	$\frac{r}{x}$.					
Cosee	.XOP	$\frac{r}{y}$	$\frac{r}{y}$	$\frac{r}{-y}$	$\frac{y}{-y}$					

From the above Table, or from Fig. 241, we can compile a table which gives the sign of any trigonometrical ratio for any angle, e.g.,

For angles less than 90°, tan. XOP = $\frac{y}{x}$, which is +. (Art. 12)

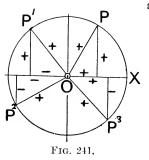
For angles between 90° & 180°, tan. XOP = $\frac{y}{-x}$, which is -. ,,

For angles between 180° & 270°, tan. XOP = $\frac{-y}{-x}$, which is +. ,

For angles between 270° & 360°, tan. XOP = $\frac{-y}{x}$, which is -. ,,

Ratio.	0° to 90°.	90° to 180°.	180° to 270°.	270° to 360°.
Sin. or cosec.	+	+	-	_
Cos. or sec.	+	_	_	+
Tan. or cot.	+	_	+	_

The trigonometrical ratios for angles of 0° , 90° , 180° , 270° , and 360° , are determined by considering the *limiting values* of any of the ratios set out in the Table on page 387 when y or x is very small, or when either y or x is equal to x. As an example, take an angle of 90° . For this angle, y = r, and $x = \theta$.



$$\frac{y}{r} \text{ or sin. } 90^{\circ} = \frac{r}{r} = 1.$$

$$\frac{r}{y} \text{ or cose. } 90^{\circ} = \frac{r}{r} = 1,$$

$$\frac{x}{r} \text{ or cos. } 90^{\circ} = \frac{0}{r} = 0.$$

$$\frac{r}{x} \text{ or see. } 90^{\circ} = \frac{r}{0} = \infty.$$

$$\frac{y}{x} \text{ or tan. } 90^{\circ} = \frac{y}{0} = \infty.$$

$$\frac{x}{y} \text{ or cot. } 90^{\circ} = \frac{0}{y} = 0.$$

The following Table gives the values of the ratios for angles 0° , 90° , 180° , 270° , 360° found as in the above example.

Angle.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec,
x = r, y = 0	0	1	0	x	1	æ
x = 0, y = r	1	0	x	0	œ	1
$x = -\frac{180^{\circ}}{r, y} = 0$	0	-1	0	- x	– I	œ
x = 0, y = -r	-1	0	- ∞	0	%	-1

333. For each of the four angles XOP in Fig. 240, we have a triangle OPM from which the values of the six

trigonometrical ratios are determined. These triangles lie, respectively, one in each of the four quadrants formed by the two axes of reference. If we take any position OP in the first quadrant, we can find a corresponding position for OP in each of the other quadrants so that the four triangles POM shall be congruent.

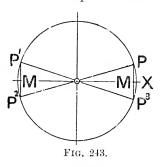
In Fig. 242, OP, OP', OP², and OP³ are four such positions. The

P' (1) P X (3) (4) M X

triangles OPM, OP'M, OP'M, and OP^3M are congruent; hence, the magnitudes of the x and y co-ordinates for the four points P, P', P^2 , P^3 are alike, although the signs of these co-ordinates will differ. The numerical value (apart from the sign) of any particular ratio for the angle XOP will be

equal to the numerical value for the same ratio for each of the three angles XOP', XOP², XOP³. Hence, when we are given the numerical value of a trigonometrical ratio, e.g., the cosine, of any angle, we can find four angles having that numerical value for cosine. One of the four angles is less than 90°, one between 90° and 180°, one between 180° and 270°, and one between 270° and 360°. Two are + and two are - for any ratio; hence, if we are given also the sign of the ratio (i.e., + or -), we can find two angles less than 360° to satisfy this. We can only decide which of the two angles is the one required from other conditions of the problem, e.g., if we are solving a triangle, an angle greater than 180° would not be admissible.

334. In the Trigonometrical Tables, we are only given the values of the ratios for angles from 0° to 90°, and they are all +; hence, when we are given any particular ratio, and we are required to find the corresponding angle, we



first find the angle less than 90° having the given numerical value for its ratio; let this angle be θ . Now refer to Fig. 242, and let the angle XOP be θ . The angles XOP', XOP², XOP³ have the same numerical value for any ratio as the angle XOP. But the angles XOP, P'OM, MOP² and P³OM are all equal, and have the value θ . Hence angles XOP', XOP², XOP³ are respectively $(180^{\circ} - \theta)$.

 $(180^{\circ} + \theta)$, $(360^{\circ} - \theta)$. We must now decide which two of these angles have the required sign. This may be done from Table 2 of Art. 332, and Fig. 241. Finally, we decide which of the two angles having the required sign and numerical value is the one for the problem we are solving.

Ex. 1. Find two angles less than 360° having 0.2588 for their sine. From the Table of sines, we find sin. $15^{\circ} = 0.2588$. In Fig. 243, angle XOP=15°.

The angles XOP, XOP', XOP2, XOP3 all have the same numerical value for sine, as the triangles POM, P'OM, P2OM, P3OM are congruent.

Angle XOP' = $(180^{\circ} - 15) = 165^{\circ}$ and sign is +.

Angle XOP² = $(186^{\circ} + 15)$ = 195° and sign is -. Angle XOP³ = $(360^{\circ} - 15)$ = 345° and sign is -.

The two angles required are thus 15° and 165°.

Ex. 2. Find the angle θ in a triangle for which $\cos \theta = -0.9063$. From the Table of cosines, we find $\cos .25^{\circ} = 0.9063$. Let XOP (Fig. 243) be 25°. Then angles XOP', XOP², XOP³ are respectively 155°, 205°, 335°. The cosines of angles XOP' and XOP² are negative.

Hence $\cos .155^{\circ} = -0.9063$, also $\cos .205^{\circ} = -0.9063$. But an angle of 205° cannot be found in a triangle.

... the required angle is 155°.

335. The variation in numerical value and sign of the sin., cos., or tan. of an angle from 0° to 360° is shown in the following graphs. The numerical values for angles up to 90° are read from the Trig. Tables, and the signs and numerical values for other angles are found according to the preceding rules. In Fig. 244, OAB is a sin. curve, CDE is a cos. curve; Fig. 245 shows a tan. curve. At 90° and 270° , the tan. is ∞ , hence, the curve becomes parallel to the ordinate at those points.

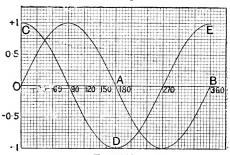


FIG. 244.

Ang'e.	0	300	60:	90°	120°	150°	180°	210°	240°	2700	300°	330°	300
Sin.	0	0.5	0.866	1	0.863	0.5	6	-0.5	-0.866	-1	-0·866	+0.2	0
Cos.	1	0.866	0.5	0	-0.2	-0.866	+1	+0.866	+0.2	0	0.5	0.866	1
Tan.	0	0.5774	1.7321	- o	-1.7321	+0.5774	0	0.5774	1:7321	· ·	+1:7321	-0.5774	0

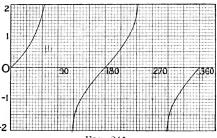


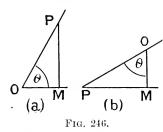
FIG. 245.

Relations between the Trigonometrical Ratios.

336. The difference between an angle and 90° is called the *complement of the angle*.

The difference between an angle and 180° is called the supplement of the angle.

The sin., tan., and sec. of any angle θ are equal



respectively to the cos., cot., and cosec. of the complement **OPM**. (θ + **OPM** = **90**°.)

$$\begin{aligned} & \text{Fig. 246 (a).} & \text{Fig. 246 (b).} \\ & \sin \theta = \frac{\text{PM}}{\text{OP}} & \cos \theta \text{ OPM} = \frac{\text{PM}}{\text{OP}} \\ & \tan \theta = \frac{\text{PM}}{\text{OM}} & \cot \theta \text{ OPM} = \frac{\text{PM}}{\text{OM}} \\ & \sec \theta = \frac{\text{OP}}{\text{OM}} \csc \theta \text{ OPM} = \frac{\text{OP}}{\text{OM}} \end{aligned}$$

(Note.—For cos., cot., and cosec. of angle OPM, the origin is at P, and PM corresponds to the axis of x.)

537. **Tan.**
$$\theta = \frac{\sin \theta}{\cos \theta}$$
 for any angle θ . Refer to Fig. 246 (a).
$$\sin \theta = \frac{PM}{OP} \text{ and } \cos \theta = \frac{OM}{OP}.$$

$$\frac{\sin \theta}{\cos \theta} = \frac{PM}{OP} \div \frac{OM}{OP} = \frac{PM}{OM}, \text{ which is tan. } \theta.$$

Similarly cot. $\theta = \cos \theta \div \sin \theta$.

This result may be tested by using the Trigonometrical Tables. Let $\theta = 35^{\circ}$.

$$\sin \theta = 0.5736$$
 $\cos \theta = 0.8192$
 $\frac{\sin 35^{\circ}}{\cos 35^{\circ}} = \frac{0.5736}{0.8192} = 0.7002$
 $\tan \theta = 0.7002$

338. Sin.² θ + cos.² θ = 1, for any angle θ .

The index 2 in $\sin^2 \theta$ or $\cos^2 \theta$ indicates that the ratio $\sin \theta$ or $\cos \theta$ is raised to the second power. read as "sine squared θ ." Refer to Fig. 247.

read as "sine squared
$$\theta$$
." Refer to Fig. 247.
$$\sin \theta = \frac{PM}{OP} \qquad \cos \theta = \frac{OM}{OP}.$$

$$\therefore \sin^2 \theta = \frac{(PM)^2}{(OP)^2} \qquad \cos^2 \theta = \frac{(OM)^2}{(OP)^2}.$$

$$\sin^2 \theta + \cos^2 \theta = \frac{(PM)^2}{(OP)^2} + \frac{(OM)^2}{(OP)^2}.$$
Since M is a right-angle,
$$\frac{(PM)^2 + (OM)^2}{(OP)^2}.$$

Since M is a right-angle,

$$(OM)^2 + (PM)^2 = (OP)^2$$
 = $\frac{(OP)^2}{(OP)^2}$.

This result may be tested by considering the actual values of the sin, and cos, of a given angle, thus, let $\theta = 60^{\circ}$.

Then
$$\sin. 60 = \frac{\sqrt{3}}{2}, \cos. 60 = \frac{1}{2} \text{ (Art. 133)}$$

 $\sin.^2 60 = \frac{3}{4}; \cos.^2 60 = \frac{1}{4}$
 $\sin.^2 60 + \cos.^2 60 = \frac{3}{4} + \frac{1}{4}$
 $= 1.$

(1)
$$\sin \theta = \frac{1}{\csc \theta}$$
. (2) $\cos \theta = \frac{1}{\sec \theta}$. (3) $\tan \theta = \frac{1}{\cot \theta}$.

(4) cosec.
$$\theta = \frac{1}{\sin \theta}$$
. (5) sec. $\theta = \frac{1}{\cos \theta}$. (6) cot. $\theta = \frac{1}{\tan \theta}$.

Refer to Fig. 240 and Table 1 of Art. 332.

Let angle $XOP = \theta$. Then in any column of the Table we see

$$\sin \theta = \frac{y}{r}; \text{ cosec. } \theta = \frac{r}{y}, \text{ which is } 1 \div \frac{y}{r}.$$
 Similarly,
$$\cot \theta = \frac{x}{y}; \tan \theta = \frac{y}{x}, \text{ which is } 1 \div \frac{x}{y}.$$

The whole of the relations (1) to (6) above can be similarly shown. The relations (5) and (6) above are useful when the values of sec. θ or cosec. θ are required, for, in the Trigonometrical Tables, values of sec. and cosec. are omitted; hence, when we require, say, sec. 35°, we read cos. 35° from

the Tables and take sec.
$$35^{\circ} = \frac{1}{\cos .35^{\circ}} = \frac{1}{0.8192} = 1.2208$$
.

339. From the preceding relations we can readily deduce other relations, thus

(1)
$$\sin^2 \theta + \cos^2 \theta = 1$$

 $\sin^2 \theta = 1 - \cos^2 \theta$ (transposing $\cos^2 \theta$)
(2) $\therefore \sin \theta = + \sqrt{1 - \cos^2 \theta}$.

(3)
$$\cos \theta = + \sqrt{1 - \sin^2 \theta}$$
 (transposing $\sin \theta^2$).

If we divide across by $\sin^2 \theta$, we get

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

(4) \therefore **1** + cot.² θ = cosec.² θ , and by dividing by cos.² θ we get

(5)
$$\tan^2 \theta + 1 = \sec^2 \theta$$
.

Ex. Given the value of the sin. of any angle less than 90° deduce the values of the other trigonometrical ratios.

Let sin. A = 0.5
(By 3 above) cos. A =
$$\sqrt{1 - \sin^2 A}$$

= $\sqrt{1 - \frac{1}{4}}$
= $\frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866$
(By Art. 337) tan. A = $\frac{\sin A}{\cos A} = \frac{0.5}{0.866} = 0.5774$
(By Art. 338) cot. A = $\frac{1}{\tan A} = \frac{1}{0.5774} = 1.7321$
(By Art. 338) sec. A = $\frac{1}{\cos A} = \frac{1}{0.866} = 1.1547$
(By Art. 338) cosec. A = $\frac{1}{\sin A} = \frac{1}{0.5} = 2$.

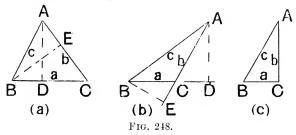
Sec. A and cosec. A could also be found from (4) and (5) above. These results may be checked from the Trigonometrical Table; as the angle baying 0.5 for sin. is 30°.

Solving Triangles.

340. A triangle is said to be completely solved when we know the magnitudes of its six elements, viz., three angles and three sides. In Chap. XVI., it is shown that we can construct, and hence solve, a triangle when we are given three elements, one at least being a side, and geometrical constructions are given for the various cases that may arise. The student should read again the chapter referred to before proceeding, as it will assist him considerably in understanding the following trigonometrical solutions.

The four cases that may arise are :-

- (1) Given two angles and any side (see Art. 196).
- (2) Given two sides and the angle between them (see Art. 194).
 - (3) Given the three sides (see Art. 191).
- (4) Given two sides and an angle opposite to one of them (see Art. 195).
- 341. Certain formulæ, or relations between the sides and the trigonometrical ratios of the angles of a triangle are necessary for each of the above cases, and these formulæ will be deduced as required.



In any triangle, the sides are proportional to the sines of the opposite angles.

Adopting the usual notation (Art. 190) of denoting the length of any side of a triangle by a small letter, say b, corresponding to the capital letter, say B, used for the opposite angle we have—

In Fig. 248 (a),

AD =
$$c$$
 sin. B, also AD = b sin. C.

 c c sin. B = b sin. C,

or

$$\frac{c}{\sin C} = \frac{b}{\sin B}.$$
Again

BE = c sin. A, also BE = a sin. C.

 c sin. A = a sin. C.

or

$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$
But

$$\frac{c}{\sin C} = \frac{b}{\sin B} \text{ (shown above)},$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

or the sides a, b, c, are proportional to the sines of the opposite angles A, B, C.

In Fig. 248 (b),

Sin. $ACD = \sin BCA$, which is angle C.

Hence

$$AD = c \sin B,$$

 $AD = b \sin ACD = b \sin C,$

and as before

$$\frac{c}{\sin.\,\mathrm{C}} = \frac{b}{\sin.\,\mathrm{B}}.$$

Similarly, by drawing the perpendicular BE we have

$$\frac{\frac{c}{\sin \cdot C} = \frac{a}{\sin \cdot A}}{\frac{a}{\sin \cdot A}} = \frac{a}{\sin \cdot B} = \frac{c}{\sin \cdot C}$$

and

In Fig. 248 (c), the angle C is a right-angle. \therefore sin. C = 1.

By definition, sin. $A = \frac{a}{c}$; sin. $B = \frac{b}{c}$.

By the above rule $\frac{\sin A}{\sin C} = \frac{a}{c}$; $\frac{\sin B}{\sin C} = \frac{b}{c}$, and as $\sin C = 1$, the rule also applies to right-angled triangles.

This relation between the sides and sines of angles of a triangle is very important and should be remembered. It is utilized for solving any triangle when the given data agree with Case I. above.

Case I. Ex. Two observers at points B and C 300 ft. apart observe an aeroplane A (in the same vertical plane as B and C) at an angle of 55° from B and 60° from C. Find the actual distance of the aeroplane from B and C and also the angle BAC.

In Fig. 249, angle $A = 180^{\circ} - (B + C)$, for the three angles of a triangle together $= 180^{\circ}$, $\therefore A = 65^{\circ}$.

We now know angles A, B, C and side a.

Fig. 249.

To find b, we have

$$\frac{b}{\sin . B} = \frac{a}{\sin . A}$$

$$\therefore b = \frac{a \sin . B}{\sin . A} = \frac{300 \times \sin . 55^{\circ}}{\frac{300 \times 0.8192}{0.9063}} \text{ ft.}$$

$$= \frac{300 \times 0.8192}{0.9063} \text{ ft.}$$

$$= 271.2 \text{ ft.}$$
To find c we have
$$\frac{c}{\sin . C} = \frac{a}{\sin . A}$$

$$c = \frac{a \cdot \sin . C}{\sin . A} = \frac{300 \times \sin . 60^{\circ}}{\sin . 65^{\circ}},$$

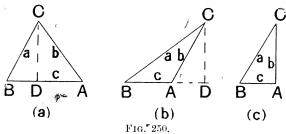
$$\frac{300 \times 0.866}{300 \times 0.866}$$

342. In any triangle ABC, $a^2 = b^2 + c^2 - 2$ bc cos. A.*

ft.

0.9063

= 286.7 ft.



In Fig. 250 (a),

$$BC^{2} = CA^{2} + AB^{2} - 2 BA \cdot AD.$$
 (Euclid, II. 13.)
 $a^{2} = b^{2} + \epsilon^{2} - 2 cb \cos A$ (for $AD = b \cos A$).
 $a^{2} = b^{2} + \epsilon^{2} - 2 bc \cos A$.

or $a^2 = b^2 +$ In Fig. 250 (b).

$$BC^2 = CA^2 + AB^2 + 2BA \cdot AD.$$
 (Euclid, II. 12.)

 $a^2 = b^2 + c^2 + 2 c$ (b cos. DAC).

$$a^2 = b^2 + c^2 + 2 bc \cos . DAC$$
 Note. Cos. DAC $a^2 = b^2 + c^2 - 2 bc \cos . A.$ = -cos. A.

In Fig. 250 (c),

$$a^2 = b^2 + c^2$$
, hence 2 $bc \cos A$ should = 0.
As $\cos A = \cos 90^\circ$ and $\cos 90^\circ = 0$,
 $\therefore 2 bc \cos A = 0$.

^{*} Similarly $b^2 = a^2 + c^2 - 2 ac \cos B$. $c^2 = a^2 + b^2 - 2 ab \cos C$.

This formula is used for solving all triangles in which the given data agree with cases (2) and (3).

Case II. Ex. A ship leaves a pier-head and steams due North for three miles. It then steams North-east for five miles. What is then its distance and its direction from the pier-head?

In all examples agreeing with Case II., the given angle must be called angle A.

In Fig. 251, c = BA drawn due North = 3. b = AC drawn North-east = 5.

Angle $A = 135^{\circ}$ (as North-east is 45° East of North).

To find BC or a, we have

$$u^{2} = b^{2} + c^{2} - 2 bc \cos. A$$

$$= 25 + 9 - (2 \times 5 \times 3 \times \cos. 135^{\circ})$$

$$= 25 + 9 - (2 \times 5 \times 3 \times - \cos. 45^{\circ})$$

$$= 25 + 9 + (30 \times 0.7071) \text{ (Note change of sign.)}$$

$$= 34 + 21.213$$

$$= 55.213.$$

 $a = \sqrt{55 \cdot 213}$ = 7.43 miles.

Either of the angles B and C can be found from the formulæ

(1) sin. B =
$$\frac{b}{a}$$
 sin. A.
(2) sin. C = $\frac{c}{a}$ sin. A.

(Note, b, c, a and angle A are known.) As we require the angle B, we have

sin. B =
$$\frac{b}{a}$$
 sin. A
= $\frac{5 \times \sin. 135^{\circ}}{7.43}$
= $\frac{5 \times 0.7071}{7.43}$
= 0.4758

Hence, the angle B is that angle which has 04758 for sine. From the Tables, we see that

$$\sin 28^{\circ} = 0.4695$$

 $\sin 29^{\circ} = 0.4848$.

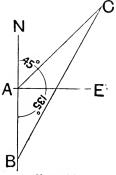


Fig. 251.

The angle B must therefore lie between these two values. If we require a result to the nearest degree, we must now decide whether our angle B is nearer in value to 28° than it is to 29°. This can be done as follows.*

^{*} The method employed here and in Art. 343 assumes that for differences less than I in angles the difference in the sin., cos., etc., is proportional to the difference in the angle, although, strictly, this is not so.

The difference in value between sin. 29° and sin. 28° is (0.4848 - 0.4695) = 0.0153. The difference between sin. B and sin. 28° is (0.4758 - 0.4695) = 0.0063. Now 0.0063 is less than $\frac{0.0153}{2}$ or .0076, hence, we conclude that angle B is less than 28.5°, and in giving a result to the nearest

less than 28.5°, and in giving a result to the nearest degree we should give angle B as 28°.

The ship is thus 7.43 miles from the pier-head in a direction 28° East of North.

The angle C, if required =
$$180^{\circ} - (A + B)$$

= $180^{\circ} - (135 + 28)$
= 17° .

343. **To find an angle** corresponding to a given trigonometrical ratio **to the nearest minute** (*i.e.*, one-sixtieth part of a degree), we require more elaborate Tables than those given in Appendix I. We can, however, find such angles to a fair degree of accuracy from these Tables by interpolation, as in the following example.

Find the angle B if
$$\sin$$
, B = 0·4746.
 \sin , 29° = 0·4848 \sin , B = 0·4758,
 \sin , 28° = 0·4695 \sin , 28° = 0·4695 0 00063.

For 1°, or 60 min., above 28° we have a difference in the value of the sine of 0.0153, and we require to know how many minutes will give a difference of 0.0063.

Let x = required number of minutes, then

$$\begin{aligned} \frac{x}{60} &= \frac{0.0063}{0.0153}, \\ x &= \frac{60 \times 0.0063}{0.0153} = 25 \text{ min,} \end{aligned}$$

The angle is thus 28° 25'.

It must be remembered that the sine and tangent of an angle increase in value with the angle from 6° to 90° , hence an angle corresponding to a given tangent is also found as above; whereas a cosine decreases in value as the angle increases from 0° to 90° . Hence, to find an angle corresponding to a given cosine, we must subtract the proportional difference in minutes from the larger of the

two angles in degrees between which the required angle is seen to lie. An illustration occurs in the following example.

344. Case III. Ex. An axiator wishes to make a triangular circuit of three towns A, B, and C. The towns C and B are shown on a map to be 15 miles apart; from B to A is 11.5, and from A to C is 8.5 miles. The direction CB is 45° East of North, and A is partly North and partly West of C. Find the directions to be taken in flying round the triangle CAB in the order named.

We must first find the three angles of the triangle CAB, Fig. 252.

We have, from Art. 342, $a^2 = b^2 + c^2 - 2 bc \cos A$, which may be written

$$\cos. \Lambda = \frac{b^2 + c^2 - a^2}{2 \, bc}.$$

In our example, $b = 8^{\circ}5$, $c = 11^{\circ}5$, a = 15, ∴ $\cos A = \frac{(8^{\circ}5)^2 + (11^{\circ}5)^2 - (15)^2}{2 \times 8^{\circ}5 \times 11^{\circ}5}$ $= \frac{72^{\circ}25 + 132^{\circ}25 - 225}{195^{\circ}5}$ $= -\frac{20^{\circ}5}{195^{\circ}5} = -0.1049.$

The minus sign indicates that the angle A is greater than 90°, and as it is an angle of a triangle it must also be less than 180°.

A 11°
45°
C E
FIG. 252.

From the Tables we have

$$\begin{array}{cccc} \cos & 33^{\circ} &= 0.1219 & \cos & \Lambda^{\circ} &= 0.1049 \\ \cos & 84^{\circ} &= 0.1045 & \cos & 84^{\circ} &= 0.1045 \\ \hline 0.0174 & 0.004 & 0.0004 \end{array}$$

For 60 min. less than 84° we have a difference in cosine of 0.0074; how many minutes less will correspond to a difference of 0.0004?

Let x = the number of min.

Then

$$\frac{x}{60} = \frac{0.0004}{0.0174}$$
$$x = \frac{60 \times 0.0004}{0.0174}$$
$$= 1 \text{ min.}$$

The angle having for cosine 0·1049 is thus 1 min. less than 84° , or 83° 59'.

The angle having for cos. -0.1049 is thus $(180^{\circ} - 83^{\circ} 59') = 96^{\circ} 1'$. Hence angle A is $96^{\circ} 1'$.

To find angle B, we have
$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
,
or $\sin B = \frac{b}{a} \sin A = \frac{8 \cdot 5 \times \sin .96^{\circ} 1'}{15}$

$$= \frac{8 \cdot 5 \times \sin .83^{\circ} 59'}{15} \left\{ \text{ for } \sin A = \sin . (180^{\circ} - A) \right\}$$

$$= \frac{8 \cdot 5 \times 0.9944}{15}$$

$$= \frac{8 \cdot 5 \times 0.9944}{15}$$

$$= 0.9925 + \left(\frac{59}{60} \times .0020\right)$$

$$= 0.9925 + \left(0.9020\right)$$

$$= 0.9925 + \left(0.9020\right)$$

To find angle C, we have
$$C = 180^{\circ} - (A + B)$$
.
= $180^{\circ} - (96^{\circ}1' + 34^{\circ}18')$.
= $49^{\circ}41'$.

The direction CA thus makes $(45^{\circ} + 49^{\circ} 41')$ or $94^{\circ} 41'$ with CE, or it is $4^{\circ} 41'$ West of North.

The angle CAE = $(90^{\circ} - 4^{\circ}41') = 85^{\circ}19'$.

∴ B = 34° 18′

.. Direction AB is (96°1′ - 85°21′) = 10°40′ North of East.

The direction BC is $(49^{\circ} 41' - 4^{\circ} 41') = 45^{\circ}$ West of South.

This example has been worked out fully to the nearest minute as an exercise in this method. Generally, it will be sufficient to take the angles C, A, and B, respectively, as 50, 96, and 34 degrees. The directions then become $CA = 5^{\circ}$ W. of N., $AB = 11^{\circ}$ N. of E., $BC = 45^{\circ}$ W. of S.

Note.—In order to avoid mistakes in solving a triangle when the three sides are given and when using the formula cos. $\Lambda = \frac{b^2 + c^2 - a^2}{2 \ bc}$, the side a should be taken as the longest side of the triangle. This ensures the discovery of an angle greater than 90° if there is such an angle in the triangle.

345. Case IV. Given two sides and an angle opposite to one of them.

Two types may occur, for the side opposite to the given angle may be (1) greater than, (2) less than, the other given side.

Given the angle B and sides b, c (Fig. 253).

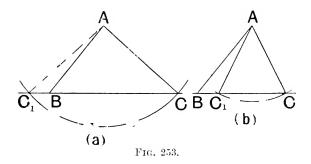
In drawing this triangle geometrically, set out AB = c. Draw BC inclined at the given angle B to AB. With centre A and radius = given side c, draw an arc cutting BC in points C and C₁.

If C and C_1 lie on opposite sides of B, as in Fig. 253 (a), only one triangle ABC satisfies the given conditions, for the triangle ABC₁ would not have the required angle

at B.

If C and C₁ lie on the same side of B, as in Fig. 253 (b), then both triangles ABC and ABC₁ satisfy the given conditions.

In order that it may be possible to construct a triangle at



all, AC or b must be equal to, or greater than, the perpendicular from A to BC, which is = AB sin. B or c sin. B. If it = c sin. B, we have a right-angled triangle with the right-angle at C. If AC = AB, we have an isosceles triangle, and C_1 coincides with B.

• Ex. In a wall crane, the jib AB is 12 ft. long and is inclined at 40° to the wall. A tie rod 8:5 ft. long is to be used and this may be attached to the wall in either of two points C, C₁, vertically above B. Find the distance of each point above B and the inclination of the tie to the jib and to the wall in each case, the inclinations to be to the nearest degree.

The diagram is shown in Fig. 254 (a). The given data is AB, AC or AC₁, and angle B. We require to know the lengths BC₁, BC, the angles BC₁A, BAC₁, and the angles BCA, BAC. This is equivalent to

solving the two triangles BC₁A, BCA, and Fig. 254 (a) is similar to Fig. 253 (b).

By geometry, we have angle $BCA = CC_1A$ (for $AC = AC_1$).

Hence angles BCA and BC1A are supplementary angles.

To find angle BCA, or angle C, we have

$$\frac{c}{\sin C} = \frac{b}{\sin B}, \quad \therefore \quad \sin C = \frac{c \sin B}{b},$$

$$= \frac{12 \times \sin A0}{85},$$

$$= \frac{12 \times 0.6128}{85}$$

Angle BC₁A or
$$C_1 = 180^{\circ} - C$$
.

$$= 180^{\circ} - 65^{\circ}.$$

= 115°.

In triangle BCA, angle BAC = $180^{\circ} - (40^{\circ} + 65^{\circ})$. = 75° .

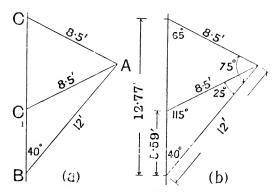


FIG. 254.

Also in triangle BC₁A, angle BAC₁ = $180^{\circ} - (40^{\circ} + 115^{\circ})$ = 25° .

To find lengths BC and BC1, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
 or $a = \frac{b \sin A}{\sin B}$.

For length BC, $A = 75^{\circ}$.

For length BC₁, $A = 25^{\circ}$.

$$\therefore \text{ For BC we have } a = \frac{8.5 \times \sin.75}{\sin.40}$$

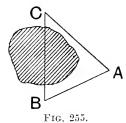
$$= \frac{8.5 \times 0.9659}{0.6428}$$

$$= 12.77 \text{ ft.}$$
For BC, we have $a = \frac{8.5 \times \sin.25}{\sin.40}$

$$= \frac{8.5 \times 0.4226}{0.6428}$$

$$= 5.59 \text{ ft.}$$

These results are best exhibited in the diagram, Fig. 254 (b),



which is a dimensioned sketch of the crane. The same formulæ are used when there is only one triangle. An example of this kind, to be worked by the student, is as follows:—

Two points B and C (Fig. 255) on a proposed railway on opposite sides of a piece of wet marsh land are to be connected by a bridge BC. A base line BA is measured = 300 ft., the angle B is 65°, and the length AC is found to be 400 ft. Calculate the length of the bridge.

Ans. 420 ft.

Applications of Trigonometry.

246. To find the length of each member in a roof truss. Fig. 256.

Given span l, height h, pitch of member AF, *i.e.*, angle CAF, height m, and also that AD = DE = EB.

By symmetry, angle BAC = angle BCA, angle ABF = angle CBK.

The angle BAC can be calculated, for tan. BAC = $\frac{h}{\frac{1}{2}l} = \frac{2h}{t}$

The length AB is given by $\frac{h}{AB} = \sin BAC$.

thus,
$$AB = \frac{h}{\sin BAC} = h \text{ cosec. PAC.}$$

We now find AD \rightleftharpoons DE = EB = $\frac{AB}{3}$.

The length AF is given by $\frac{m}{AF} = \sin$. CAF,

or AF =
$$\frac{m}{\sin \text{, CAF}} = m \text{ cosec. CAF.}$$

The angle DAF = (angle BAC - angle CAF).

Hence, in the triangle DAF, we now know the sides AD, AF and the included angle DAF.

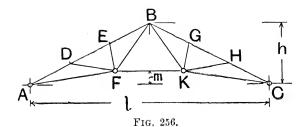
By Art. 395 (Case II.) we have

 $DF^2 = DA^2 + AF^2 - 2 DA \cdot AF \cos DAF,$

from which we find the length DF.

The angle ADF is next found from

$$\frac{\mathrm{DF}}{\sin.\,\mathrm{DAF}} = \frac{\mathrm{AF}}{\sin.\,\mathrm{ADF}}$$
 or $\sin.\,\mathrm{ADF} = \frac{\mathrm{AF}\sin.\,\mathrm{DAF}}{\mathrm{DF}}$.



[Note.—This angle will probably exceed 90°, hence select angle from second quadrant having the given value.]

The angle $EDF = (180^{\circ} - \text{angle ADF}).$

In the triangle DEF, we now know DE, DF, and the included angle, hence

 $EF^2 = ED^2 + DF^2 - 2 ED . DF cos. EDF,$

from which we find EF.

The length $FK = l - AF \cos CAF - KC \cos KCA$.

The height of triangle BFK = (h - m),

hence, tan, BFK =
$$\frac{(h-m)}{\frac{1}{2} \text{ FK}} = \frac{2 (h-m)}{\text{FK}}$$
,

from which we find the angle BFK.

Then
$$\frac{h-m}{BF} = \sin BFK$$
,
or $BF = \frac{(h-m)}{\sin BFK} = (h-m)$ cosec. BFK.

The lengths of BG, GH, HC, CK, HK, GK, BK are found by symmetry.

In this example, the student will do well to take an actual case, and follow the given steps. Thus let l = 60 ft., h = 14 ft.,

angle CAF =
$$9^{\circ}$$
, and $m = 3$ ft.

347. **Steam Engine Mechanism**. To find the piston or crosshead displacement corresponding to any position of the evank. Fig. 257.

Let r = length of crank.

5 r = length of connecting-rod.

 θ = angle turned through by erank from the inner dead centre.

P denote the piston. A the crosshead.

D the inner dead centre.

When the crank-pin B is at D, the crosshead A is at its greatest distance from O, and this = 6 r, for OA then = OB + BA.

In any other position OB of the crank, the crosshead is distant OA = OC + CA from O,

But $OC = r \cos \theta$, which is — when θ is between 90° and 270°. $CA = BA \cos BAC$ or 5 $r \cos BAC$.

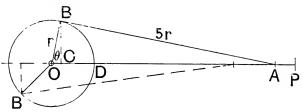


Fig. 257.

Hence, as the crosshead has travelled a distance = (5 r – OA) towards O, the crosshead displacement is = 6 r – OA

which is $= 6 r - (r \cos \theta + 5 r \cos BAC)$.

The piston displacement is given by the same formula.

The angle BAC may be found from Sin. BAC = $\frac{\text{BC}}{\text{BA}} = \frac{r \sin{\theta}}{5 r}$.

Note, that when θ is between 90° and 270°, e.g., when the crankpin is below AO, the value cos. θ is negative, hence r cos. θ is negative, and $(r \cos \theta + 5 r \cos BAC)$ is less than BA cos. BAC or $5 r \cos BAC$.

To test this formula. Let $\theta = 180^{\circ}$. We see from the figure that the piston has then moved 2 r from its initial position.

The formula gives $6 r - (r \cos \theta + 5 r \cos BAC)$

$$\theta = 180^{\circ}; \text{ BAC} = 0^{\circ}.$$
Hence, $r \cos \theta = r \cos 180^{\circ} = r \times (-1) = -r$.
$$5 r \cos \text{ BAC} = 5 r \cos \theta = 5 r \times 1 = 5 r$$
,
and displacement = $6 r - (-r + 5 r)$

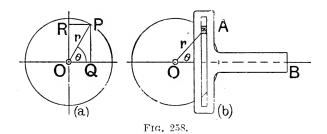
$$= 2 r$$
.

which agrees with the above result.

348. Simple Harmonic Motion. Fig. 258 (a). If a point P moves uniformly round a circle, the projection Q, or R, on a diameter of the circle is said to move with a simple harmonic motion. It will be seen that Q, or R, moves backwards and forwards along the diameter. The centre O of the circle is the mean position of Q or R. The distance OQ, or OR, gives the displacement from the mean position O. Let $\theta =$ angle turned through by OP.

Then $OQ = OP \cos \theta$. $OR = PQ = OP \sin \theta$.

The angle turned through by OP in one second measured in radians is called the **angular velocity** of OP and is usually denoted by ω . The angle turned through in t seconds will then be ωt radians, and the displacement of Q after t seconds is OP cos. ωt , while the displacement of R is OP sin. ωt .



To calculate these displacements, the value ωt in radiars must be converted into degrees.

A **practical embodiment** occurs in the mechanism of a donkey pump, or of a small slotting or shaping machine, shown diagrammatically in Fig. 258 (b). The crank r carries a block which works in a slotted link A guided so as to reciprocate parallel to OB as the crank rotates. The cutting-tool, or the pump piston, is carried by, or connected to, the slotted link.

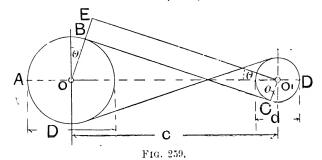
If θ is the angle between OA and OB, then $r \cos \theta$ gives the displacement of the tool, etc., from the mean position, and $r (1 - \cos \theta)$ gives the displacement from either extreme position.

349. Length of Belts.

(1.) Crossed Belt. Fig. 259. Let the diameters of the pulleys be respectively D, d, and the distance between the centres OO_1 of the pulleys be c. As the part BC is tangential to both pulleys, the radii BO, CO_1 are parallel, and the angles BOA, DO_1C are equal in magnitude. Let each angle $= \left(\frac{\pi}{2} + \theta\right)$ radians. EO₁ is parallel to BC, and the angle EO₁ O is also $= \theta$.

The length ABCD is equal to half the length of the belt,

AB =
$$\frac{D}{2} \begin{pmatrix} \pi \\ \frac{\pi}{2} + \theta \end{pmatrix}$$
,
BC = EO₁ = $c \cos \theta$,
CD = $\frac{d}{2} \begin{pmatrix} \frac{\pi}{2} + \theta \end{pmatrix}$.



The angle θ in degrees is such that $\sin \theta = \frac{EO}{OO_1} = \frac{OB + BE}{OO_1}$ $= \frac{D}{2} + \frac{d}{2}$ $= \frac{D + d}{2a}$

... Whole belt = 2 (AB + BC + CD)
$$= 2 \left\{ \frac{D}{2} \left(\frac{\pi}{2} + \theta \right) + c \cos \theta^{\circ} + \frac{d}{2} \left(\frac{\pi}{2} + \theta \right) \right\}$$

$$= (D + d) \left(\frac{\pi}{2} + \theta \right) + 2 c \cos \theta^{\circ}.$$

(2.) Open Belt. Fig. 260.

For an open belt, angle AEB = $\left(\frac{\pi}{2} + \theta\right)$ radians, angle CO₁D = $\left(\frac{\pi}{2} - \theta\right)$ radians,

an l as CO_1 is perpendicular to EO_1 , the angle EO_1O is $= \theta$.

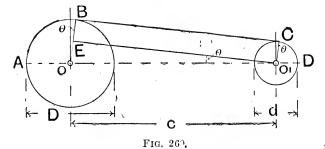
$$AB = \frac{D}{2} \left(\frac{\pi}{2} + \theta \right).$$

$$CD = \frac{d}{2} \left(\frac{\pi}{2} - \theta \right).$$

$$BC = EO_1 = c \cos \theta^2.$$

$$\theta \text{ in degrees is given by } \sin \theta^2 = \frac{OE}{OO_1} = \frac{OB - BE}{OO_1}.$$

$$\frac{D}{2} - \frac{d}{2} = \frac{D - d}{2 e^{-d}}.$$



$$\frac{1}{2} \text{ length of belt} = AB + BC + CD,$$

$$= 2 (AB + BC + CD)$$

$$= 2 \left\{ \frac{D}{2} \left(\frac{\pi}{2} + \theta \right) + c \cos \theta + \frac{d}{2} \left(\frac{\pi}{2} - \theta \right) \right\}$$

$$= D \left(\frac{\pi}{2} + \theta \right) + 2 c \cos \theta + d \left(\frac{\pi}{2} - \theta \right)$$

$$= (D + d) \frac{\pi}{2} + 2 c \cos \theta + (D - d) \theta.$$

Note.—In these two examples, the angle θ should be taken in radians wherever it occurs, except where it is written as θ° , e.g., cos. θ° .

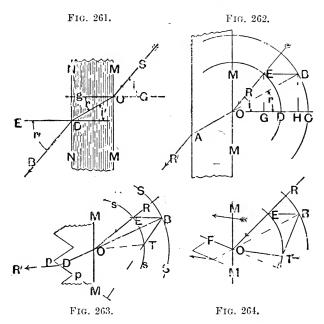
350. **Refraction of Light**. It is shown in works on Light that a luminous ray is propagated in a straight line through the ether, or through any homogeneous medium, such as glass.

The velocity of light depends on the medium through which it is passing; its velocity through good quality glass is about two-thirds of its velocity through the air.

Light changes its direction when it passes obliquely from

one medium to another; this change in direction is called refraction.

If SO, Fig. 261, indicates a ray of light falling on a plane sheet of glass at O, and OD is the path of the ray through the glass; DB the direction of the ray after it emerges from the glass; and OG, DE respectively are normal to the



surfaces MM and NN; then SO is called the *incident* ray; OD is called the *refracted* ray; DB is called the *emergent* ray; the angle SOG (i) is called the angle of *incidence*; the angle DOg (r) is called the angle of *refraction*; the angle BDE (r') is called the angle of *emergence*. It can be shown that

$$\frac{\sin . \, i}{\sin . \, r} = \frac{\sin . \, r'}{\sin . \, i'} = \frac{\text{velocity of light in air}}{\text{velocity of light in glass}} = \lambda$$

The above ratio λ is called the index of refraction, or refractive index, and sin. $i = \lambda \sin r$.

Problem. Fig. 262. A luminous ray R falls at O on the surface MM of a plate of glass, with parallel faces. Trace the path of the ray through the glass and show its direction after it emerges from the glass, having given the angle of incidence $(i) = 50^{\circ}$, refractive index = 1.55.

With centre O and any radius OD describe a circle: with centre O and radius OC = 1.55 OD describe a second circle. From E, the point in which the incident ray intersects the smaller circle, draw EB r to MM. Draw BOA, which is the path of the ray through the glass.

For $\sin i = \frac{EG}{EO}$ $\sin r = \frac{BH}{BO}$ This may be calculated also, for $\frac{EG}{EO} = BH$. $\therefore \frac{EO}{EO} = \frac{\sin i}{\sin r}$. This may be calculated also, for $\sin i = \frac{EG}{\sin i} =$

To find the direction which the ray takes on leaving the glass,

draw AR' || RO.

Problem. Fig. 263 is a section through a piece of glass, the outside face of which is plane, whilst prisms are cut on the inside. RO indicates a ray of light falling on the glass at O. Determine the path of the ray after it leaves the glass; having given the refractive index of the glass = 1.53.

With O as centre describe two circles ss, SS the ratio of their radii being equal to the refractive index. Draw EB $\underline{\mathbf{r}}$ MM meeting the arc SS in B. Draw BO and produce to meet pp in D. This is the path of the ray in the glass. Draw BT perpendicular to the face pp of the prism, meeting the smaller circle in T. Draw DR' \parallel TO. Then DR' is the path of the emergent ray.

Problem. Fig. 264. Having given a surface MM upon which an incident ray RO falls, determine the angle between this surface and a surface which will refract the ray in a direction parallel to that of

the given arrow, having given the refractive index = 1.53.

From centre O describe two circles the ratio of whose radii are equal to the given in lex 1.53. Draw EB \(\mu^r\) MM. Draw OT \(\psi\) to the direction indicated by the arrow, meeting the smaller circle in T. Join BT and draw OF \(\mu^r\) BT. OF is parallel to the refracting surface required.

- 351. **Reflection of Light**. When a ray of light meets a highly polished surface, part of the light is *reflected* according to the following laws:—
 - 1. The angle of reflection is equal to the angle of incidence.
- 2. The incident and the reflected rays are in one plane, which is perpendicular to the reflecting surface.

If an object is held in front of a plane mirror, it appears

behind the mirror at a distance equal to the distance of the object in front of the reflecting surface; this reflection of the object is called an **image**.

When an object is placed between two plane mirrors, it is reflected in both mirrors. The image of an object in a plane mirror is determined geometrically by aid of the fore-

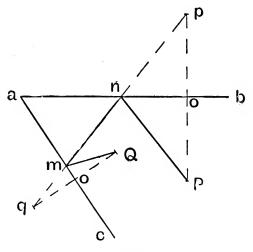


Fig. 265.

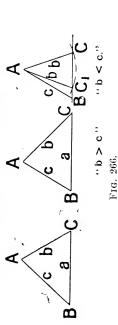
going principle which applied to a point becomes —. The image of a point is on the perpendicular from the point on to the mirror and at a distance behind the mirror equal to that of the point in front.

Problem. Fig. 265. A ray of light from P is reflected to Q after impinging successively on ab and ac. Show the path of the ray.

(B. E. 1892.)

Draw $Pp \ \underline{\mathbf{r}} \ ab$ and make op = oP. Draw $Qq \ \underline{\mathbf{r}} \ ac$ and make oq = oQ. Join p, q meeting ab in n and ac in m. Join Pn, nm, mQ, which is the path of the ray.





Case III.

For angle A, the included angle, a, b, C. Given two sides and Case II.

Given two angles and

any side, B, C, a. For angle A,

Case I.

For side
$$c$$
,
 $c^3 = a^2 + b^2 - 2 ab \cos C$.

 $A + B + C = 180^{\circ}$.

For angle A,
$$\frac{\sin A}{\sin C} = \frac{a}{c}.$$
 For angle B,
$$A + B + C = 180^{\circ}.$$

sin. A.

gin. B

For side b,

sın. C.

sin. B

For side c,

For angle C, $A + B + C = 180^{\circ}$.

Gase III.

Given three sides,
$$a, b, c$$
.
For angle A ,

cos. $A = \frac{h^2 + c^2 - a^2}{2bc}$.

Given tree sides and an angle a .

The b is $a = c$ one solution.

If b is $a = c$ or one solution.

For angle $a = c$ in $a = c$

 $b^2 + c^2 - a^2$

 $\cos A = 7$

of those given.)

For angle B. sin. B

sin. A

Exercises.

- (1) Solve the following triangles, using logs for the calculations:—
 - (1) Side a = 10 ft., angles B, $C = 75^{\circ}$, 60° .
 - (2) Sides a, b, c = 6, 10, 7 ft. respectively.
 - (3) Sides a, b = 45, 37 ft., angle $C = 50^{\circ}$.
 - (4) Sides a, b = 10, 6 ft., angle $A = 45^{\circ}$.
 - (5) Sides a, b = 8, 10 ft., angle $A = 45^{\circ}$. (Ambiguous case. Two solutions.)
- (2) Calculate the length of each member in the truss shown in Fig. 267.
 - (3) Write down the values of sin. 207°, cos. 123°, tan. 325°. (B.E., 1908.)
- (4) Write down the values of sin. 52°, cos. 140°, tan. 220°, cos. 340°, sin. 340°. (B. E., 1909.)
- (5) The lengths of a degree of latitude and longitude, in centimetres, in latitude l are—
 - $(1111.317 5.688 \cos l) 10^4$ and
 - (1111.164 cos. l = .950 cos. 3 l) 10^4 .

The length of a sea mile (or 6082 ft.) is 185:380 cm.

What are the lengths of a minute of latitude and of a minute of longitude in sea miles in the latitude 52°? (B. E., 1910.)

(6) In a derrick crane (Fig. 268), the post AB is 30 ft. high, and two stays EB, DB are employed. The points

the angle EAD is 115°. Find the lengths of each stay, the distance between E and D, and the length and inclination to the horizontal of a stay joining the point B to the centre of the line ED.

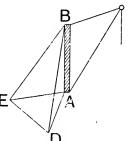


Fig. 267.

---iŏ" 30′

Fig. 268.

(7) If y=2 sin. A + 3 cos. A - 3·55, take A as 20°, 23°, 26°, etc., finding in each case the value of y. Plot y and A on squared paper. For what value of A is y just 0? (B. E., 1907.)

(8) If \sin (A - B) = \sin $A \cos$ $B - \cos$ $A \sin$ B. Calculate this when $A = 75^{\circ}$ and $B = 12^{\circ}$. (B. E., 1909.)

(9) In the equation $\frac{a}{\sin A} = \frac{b}{\sin B}$, you are given $A = 41^{\circ}$, $B = 72^{\circ}$, a = 7.6 cm. Determine b by graphical construction and verify your answer by numerical calculation, using the Tables. (B. E., 1909.)

(10) If $x = \tan \theta \div \tan \theta$, $(\theta + \varphi)$ where φ is always 10°, find x when θ has the values 30°, 40°, 50°, 60°, and plot the values of x and of θ on squared paper. About what value of θ seems to give the largest value of x? (B. E., 1903.)

(11) From the opposite ends of a ship 500 ft. long, a distant ship is sighted at angles 88° and 89° respectively with the centre line of the

ship. Find the distance apart of the two ships.

(12) A gun situated at a point A is to be directed upon an invisible target C on the opposite side of a hill to that of the gun. It is desired to know the horizontal distance between gun and target, An observer at a point B can see both the gun and the target. The distance AB is 500 ft., the angle ABC is 55°, and by means of a range-finder the distance BC is found to be 1,500 feet. Find the range AC and the angle which the centre line of the gun must make with AB.

(13) A crane ABC is mounted on a platform 50 ft, high. The height AB of the crane post is 20 ft., the length of the jib is 30 ft., and the tie BC is made up of steel ropes passing round two pulleys so arranged that the tie may be shortened, thus varying the inclination of the jib AC to the post AB. If this inclination must not fall below 30°, what is the greatest height to which the crane can lift an object, and what is the length and inclination of the tie to the horizontal in this position?

(14) The span of a roof is 30 ft., the lengths of the rafters on one side are 20 ft., and on the other side 15 ft. Find the pitch of each side of the roof, and the height of the ridge above the supporting walls.

(15) A road AC runs East and West. A point B is distant ½ mile North-east of A. What is the shortest distance from B to the road? How many paths each 750 yards long can be made from B to end on the road, and how many paths 1,000 yards long? What is the angle between AB and each path? Make a scale drawing of the roads and the point B.

(16) A searchlight A illuminates an area included between two lines AB, AC inclined at 55°. Two ships B, C are 1 mile apart and are equidistant from A. What is the smallest distance of each ship

from A if both ships are to be visible at the same time?

(17) Two pulleys are respectively 3 ft. and 1 ft. diameter, and their centres are 10 ft. apart. Find the length of a crossed belt and the length of an open belt to connect them, and the angle of lap on each pulley in radians.

(18) In a steam engine mechanism, the crank is 18 ins. long and the connecting-rod is 8 ft. long. Find the distance travelled by the piston when the crank has turned through 55° from the inner dead

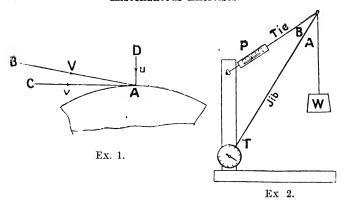
centre, and also the obliquity of the connecting-rod.

(19) A point P moves round a vertical circle centre O, at a uniform angular velocity, and Q is the projection of the point on a horizontal straight line at any instant. OP is 3 ft. long. Assuming that P and Q commence from the same point, what angle has OP turned through when Q has travelled 5 ft.? If the line OP has an angular velocity of 5 radians per second, how far is Q from the centre after one minute?

(20) A mountain railway has a gradient of 1 in 5 for a 4 mile, a gradient of 1 in 4 for the next ½ mile, a gradient of 1 in 8 for the next ½ mile,

a gradient of 1 in 3 for the next 300 yards, and a final gradient of 1 in $2\frac{1}{2}$. If the vertical height of the terminus above the starting point is 638 yards, find length of last section, and inclination of line joining terminus to starting point.

Miscellaneous Exercises.



(1) Water enters a turbine wheel with a velocity V of 60 ft. per second along the given line BA; what are the components u and v of this velocity, radially and tangentially, that is, in the directions CA and DA? Adopt a scale of $\frac{1}{2}$ in. to 10 ft. per second.

Measure the angle BAC with your protractor, and by using the tables, verify your answers by numerical calculation. (B.E. 1909.)

- (2) In a laboratory experiment on a jib crane, the angle B between tie and jib, and the angle A between jib and cord are measured, and the pull P and thrust T in the tie and jib are registered by spring balances. If W were 24 lbs., and if the angles B and A measured 25° and 32°, what would you expect P and T to be? Adopt a force scale of 1 in. to 20 lbs. (B.E. 1905.)
- (3) Draw to scale a line AE 20 ft. long horizontal, and on this construct a polygon ABCDE, such that AB=5, BE=18, BC=0, CE=13, CD=8, DE=7 ft. At the point B a force of 50 lbs. acts inclined at 30° to the left of the vertical through B; at Ca vertical force of 100 lbs. acts, and at D a force of 80 lbs. inclined at 20° to the right of the vertical through D. A supporting force acts vertically at E, and a second supporting force acts at the point A. Find the magnitude of the force at E and the magnitude and direction of the force at A.

Hint: Commence link polygon at A.

CHAPTER XXVI.

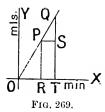
RATES OF INCREASE.

352. If two quantities, say y and x, are so related that the value of y depends upon that of x, then y is said to be a function of x; thus, if $y = ax^2 + bx + c$, y is said to be a function of x. The general method of indicating that yis a function of x is y = f(x), which is read as "y is a function of x." (Art. 175.) If we plot corresponding values of y and x on squared paper, a graph is obtained from which corresponding values of y and x, not already calculated, can be ascertained. From this graph we can also ascertain the increase (or decrease) in value of y corresponding to any increase in the value of x. In the present chapter, we discuss the ratio of the increase (or decrease) in the value of y to the corresponding increase in value of x, and particularly the value which this ratio assumes when the amount of increase in the value of x is made indefinitely small.

As an actual example, consider a railway train. If we time the train over a measured distance, say 5 miles, and find that it travels this distance in 5 minutes, we say that "the train travels at the rate of 1 mile per minute." If we time the train over shorter distances, say of 1 mile each, we may find that the train travels exactly 1 mile in each minute of the five, or we may probably find that it travels in the following manner—it may travel 0.75 mile in the first minute, 0.85 mile in the second, 1.05 miles in the third, 1.25 miles in the fourth, and 1.1 miles in the fifth, thus travelling 5 miles in 5 minutes. In the former case, we say that the train travels at a uniform rate of 1 mile per minute, while in the latter case we are only entitled to say that its average rate for 5 minutes is 1 mile per minute. Generally, when a rate is being

spoken of in everyday language, it is an average rate that is meant, but the student must now begin to think of the strict meaning of the term "rate"; he must also think of it as applied to quantities other than distance travelled and time taken, and he must also remember that rates may be rates of increase or rates of decrease.

353. Let us now consider more carefully the term "uniform rate." If we time the train over each $\frac{1}{2}$ mile, and find that it travels each one in $\frac{1}{2}$ minute, and then time it over each $\frac{1}{4}$ mile and find that it travels each one in $\frac{1}{4}$



minute, and so on, gradually making the timed distances smaller and smaller, and if, finally, we take any distance, long or short, and find that the distance in miles divided by the time taken in minutes gives as the result a constant quantity, then we have a strictly correct case of a uniform rate of increase. In our example, taking the distances as 5, 0.5, 0.25 miles,

and the corresponding times as 5, 0.5, 0.25 minutes, we have

or,
$$\frac{5 \text{ miles}}{5 \text{ minutes}} = \frac{0.5 \text{ mile}}{0.5 \text{ minute}} = \frac{.25 \text{ mile}}{0.25 \text{ minute}} = 1,$$

$$\frac{\text{increase in distance}}{\text{increase in time}} = 1.$$

or, rate of increase is uniform and = 1.

In the above example, let us denote distance by y and time by x and plot distances and time. We obtain a *straight line* graph, as shown in Fig. 269.

x miles	0	0.25	0.5	1	õ
y minutes	0	0.25	0.2	1	5

Let the abscissa OR represent 3 minutes and the corresponding ordinate PR represent 3 miles. Increase the time by a small amount RT, say 0.25 minute. (Fig. 269)

is not drawn to scale.) The distance travelled in time OT, or 3.25 minutes, is represented by QT = 3.25 miles. This exceeds the distance PR (travelled in time OR or 3 minutes) by an amount QS, or 0.25 mile, hence

 $\frac{\text{QS}}{\text{RT}} = \frac{\text{QS}}{\text{PS}} = \frac{\text{increase in distance}}{\text{increase in time}} = \frac{0.25 \text{ mile}}{0.25 \text{ minute}}$

= rate of increase. It is usual to denote an abscissa, such as OR, by x, and a small increase in this value as δx (read "delta x"). The ordinate PR corresponding to x is denoted by y, and the increase QS in y corresponding to the increase δx in x is denoted by δy . Then $\frac{\text{increase in }y}{\text{increase in }x} = \frac{\delta y}{\delta x}$ is the

rate of increase of y with respect to x. The student must note that δy or δx is not to be regarded as $(\delta \times y)$ or $(\delta \times x)$, but as one quantity, meaning a very small increase in the value of y, or a very small increase in the value of x; hence, we cannot cancel out the symbols δ in the ratio $\frac{\delta y}{\delta x}$ as would at first sight appear possible.

We can thus see that if the value of $\frac{\delta y}{\delta x}$ reckened for any value of δx is a constant ratio, we have a uniform rate of increase of y with respect to x. This is true when y and x denote, respectively, any two related quantities. If they denote, respectively, distance and time, then $\frac{\delta y}{\delta x}$ is velocity; if y denotes velocity and x time, then $\frac{\delta y}{\delta x}$ would denote rate of increase of velocity with respect to time, which is called acceleration.

354. In the preceding example, it will also be seen that for any value of x the value of $\frac{\delta y}{\delta x}$ for the small interval of time immediately following x is constant, and it will be found that, in all cases in which the graph obtained by plotting corresponding values of y and x is a straight line, the ratio $\frac{\delta y}{\delta x}$ is constant.

Ex. Let the graph AB, Fig. 270, be y=3 x+4. It is not drawn to scale in the Fig., as δy and δx are made sufficiently large to be visible.

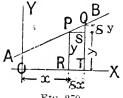


Fig. 270.

Take
$$x = 2$$
, then $y = (3 \times 2) + 4 = 10$
or $OR = 2$, $PR = 10$.

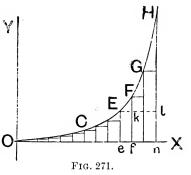
Increase
$$x$$
 by 0.1. Then $\delta x = 0.1$;
OT = $x + \delta x$; QT = $y + \delta y$; and
QT = $(3 \times \text{OT}) + 4$
 $\therefore y + \delta y = 3(x + \delta x) + 4$
= $(3 \times 2.1) + 4$
= 10.3 .

The increase δy thus = 0.3, and $\frac{\delta y}{\delta x} = \frac{0.3}{0.1} = 3.$ δx

Repeat this process, taking x = 1, 3, 5, etc., and take $\delta x = 0.1$, or 0.2, etc. In each case, it will be found that $\frac{\delta y}{\delta x} = 3$, or the rate of increase of y relative to x is constant.

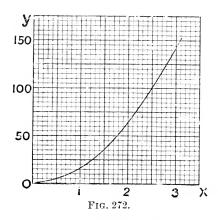
355. When the graph obtained by plotting corresponding values of x and y is a curve, it will be found that, in general, a different value of $\frac{\delta y}{\delta x}$ is obtained for each value of x, and, further, that for any particular value of x

the value of $\frac{\delta y}{\delta x}$ varies according to the magnitude of the increase δx in the value of x. Consider a graph such as that shown in Fig. 271. From the point O to the point C, the ordinates increase slowly comparison with ofthe increase scissæ; from C to E the increase is more



rapid, and from E to II still more rapid. Again, if we take Oe as x, and ef as δx , then $\frac{\delta y}{\delta x} = \frac{Fk}{Ek}$, and if we take Oe as x, and eh as δx , then $\frac{\delta y}{\delta x} = \frac{\Pi t}{E t}$, which ratio has clearly a much greater value than $\frac{Fk}{E t}$. It will thus be evident that, when y and x are so related that the graph is a curve, we do not get a constant rate of increase.

Consider a concrete example, viz., that of a stone falling from a height under the action of gravity. It is shown, in works on



Mechanics, that the relation between the distance y in feet through which the stone has fallen, and the time x in seconds during which it has been falling, is $y = 16 x^2$. Plotting this graph (Fig. 272) we obtain a parabola.

y ft.	0	1	4	16	64	144
r sec.	0	0.25	0.2	1	2	3

Take x=1 second, then y=16 ft. Increase x by 0.01 second, and find the corresponding increase in y. δx is thus 0.01 second and $(x+\delta x)$ becomes 1.01, and $y+\delta y=16$ $(x+\delta x)^2=16$ $(1.01)^2=16.3216$.

The increase δy corresponding to the increase δx of 0.01 is thus (16.3216-16) or 0.3216.

The value
$$\frac{\delta y}{\delta x} = \frac{0.3216}{0.01} = 32.16.$$
 Now let $x = 2$; $y = 16 \times 2^2 = 64$. Increase x by 0.01 so that $\delta x = 0.01$. Then
$$y + \delta y = 16 (x + \delta x)^2 = 16 (2.01)^2 = 64.6416.$$
 The value
$$\frac{\delta y}{\delta x} = \frac{0.6416}{0.01} = 61.16.$$
 If we repeat the calculation for $x = 3$ we find that
$$\frac{\delta y}{\delta x} = \frac{0.9616}{0.01} = 96.16.$$

It is evident that we obtain a different value of $\frac{\delta y}{\delta x}$ for each value of x taken; hence, the rate of increase of y relative to x (or the velocity of the falling stone) is not constant. It appears to be gradually increasing in value as x, or the distance fallen, increases.

Next, consider how the value of $\frac{\delta y}{\delta x}$ varies when the increase δx is varied, the value of x remaining constant. Let x = 2, then y = 16 $x^2 = 61$.

Let
$$\delta x = 0.01$$
, then $x + \delta x = 2.01$.
 $y + \delta y = 16 (x + \delta x)^2$
 $= 16 (2.01)^2$
 $= 64.6416$,
hence,
 $\delta y = (64.6416 - 64) = 0.6416$,
and
$$\frac{\delta y}{\delta x} = \frac{0.6416}{0.01} = 64.16.$$
When
$$\delta x = 0.001, \text{ we have}$$

$$y + \delta y = 16 (x + \delta x)^2$$

$$= 16 (2.001)^2$$

$$= 16.064016$$
and
$$\delta y = 0.064016$$
hence,
$$\frac{\delta y}{\delta x} = \frac{0.064016}{0.01} = 64.016.$$

Repeating the calculation, we find that

when
$$\delta x = 0.0001, \frac{\delta y}{\delta x} = 64.0016.$$

$$\delta x = 0.00001, \frac{\delta y}{\delta x} = 64.00016.$$

$$\delta x = 0.000001, \frac{\delta y}{\delta x} = 64.00016.$$

From the above results, it will be seen that the value $\frac{\delta y}{\delta x}$ has different values according to the magnitude of the increase δx in the value of x. The student will thus

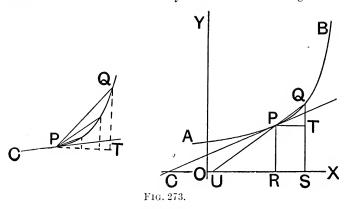
wonder what particular value for δx must be taken in calculating the rate of increase for any particular value of x. Now, if the increase δx is a finite amount, the corresponding increase in y divided by the increase δx gives the average rate of increase during the interval δx , and not the actual rate of increase for any particular value of x. We require, however, the actual rate of increase for a particular value of x, and not the average rate for an interval δx ; hence, we must reduce the interval δx indefinitely, i.e., we must make δx so small as to be practically negligible. results show that, as δx is made smaller and smaller, the value $\frac{y}{8x}$ approaches nearer and nearer to the value 64, and, by making the value of δx sufficiently small, we can bring the value $\frac{\delta y}{8x}$ as near to 64 as we please; for we can, in this way, make the decimal fraction added to 64 as small as we please. This value (64, in this example), to which approaches as δx is made smaller and smaller, is called the limiting value of $\frac{\delta y}{\delta x}$, and this limiting value is also the rate of increase of y with respect to x when the interval δx is so small as to be negligible, and hence gives the actual rate of increase for the particular value of x. The student will readily see that a value of $\frac{\delta y}{\delta x}$, such as 61.000000016, obtained when δx is 0.000000001, is not really different from the value 64. 356. When δx is made so small that it is negligible

as 64 000000016, obtained when δx is 0 000000001, is not really different from the value 64.

356. When δx is made so small that it is negligible it is usual to write \mathbf{dx} , instead of δx , for this very small quantity. The corresponding increase in y is written as \mathbf{dy} , and the ratio $\frac{\delta y}{\delta x}$ is now written as $\frac{\mathbf{dy}}{\delta x}$, which represents the limiting value of $\frac{\delta y}{\delta x}$ when δx is made indefinitely small. The symbols δx and δx must each be regarded as one symbol, not δx or δx .

hence the symbols d must not be eancelled in the ratio $\frac{d\eta}{dx}$. In the ease of the falling stone, when $y=16~x^2$, we have seen that when x=2, the value $\frac{\delta \eta}{\delta x}$ approaches the value 64 as δx decreases; hence the value of $\frac{d\eta}{dx}$ when x=2 is 64, or, at the end of 2 seconds the falling stone has a velocity of 64 ft. per second.

357. The **geometrical meaning** of the value which $\frac{\delta y}{\delta x}$ assumes when δx is made very small is seen from Fig. 273.



Let the co-ordinates of any point P on the graph AB be x and y; then OR = x; PR = y. Let $RS = \delta x$; $QT = \delta y$. Then $\frac{\delta y}{\delta x} = \frac{QT}{PT}$. Produce the chord QP to cut the axis OX in U. Then $\frac{QT}{PT} = \frac{QS}{SU}$ (similar triangles) and $\therefore \frac{\delta y}{\delta x} = \frac{QS}{SU}$, which is $\tan \theta$, where θ is the inclination of the chord PQ to the axis OX.

If we make RS, or δx , smaller and smaller, as shown in the enlarged Fig., the point Q approaches P and ultimately is

so near to P as to be practically coincident with it. The chord PQ then becomes shorter and shorter, and becomes more and more nearly parallel to the tangent to the curve at P. The sides QT, PT of the triangle also become shorter and shorter. When the points P and Q are so close together as to be practically coincident, the chord PQ has the same direction as the tangent PC to the curve at P. The ratio $\frac{QT}{PT}$, or $\frac{\delta y}{\delta x}$, must now be written as $\frac{dy}{dx}$ (for δx is indefinitely small), and the ratio $\frac{QT}{PT}$ for the very small triangle QPT which we obtain when P and Q are very close together does not differ appreciably from the slope of the tangent PC. Hence, the ratio $\frac{dy}{dx}$ for any point P on a curve is the slope of the tangent to the curve at that point. The slope of the tangent at that point.

358. A tangent may be drawn at any point on a curve whose equation is known by calculating the value of $\frac{d\eta}{dx}$ for the point, as this value gives the slope of the tangent. An example will be found in Art. 363.

359. So far we have only considered cases in which an increase in x produces an increase in y. There are, however, many cases in which an increase in x produces a decrease in y, as in the graph shown in Fig. 274. In this case, when x increases from OT to OS (i.e. by δx), y decreases by an amount PR or δy , hence, $\frac{\delta y}{\delta x} = \frac{\text{PR}}{\text{RQ}}$, and when δx is made

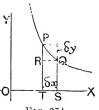


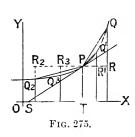
FIG. 274.

very small so that P and Q practically coincide, we get $\frac{dy}{dx}$ = slope of tangent to graph at P. It is usual to denote a small increase in x or y by a positive (+) sign, and

a small decrease by a negative (-) sign. Thus in Fig. 274, PR or δy is negative, RQ or δx is positive, hence

$$\frac{\delta y}{\delta x} = \frac{-PR}{+RQ}$$
 and is negative, and $\therefore \frac{dy}{dx}$ is negative.

360. Fig. 275. It does not matter whether we ascertain



the value of $\frac{dy}{dx}$ for any point P on a curve by taking the small interval δx as an increase or as a decrease in the value of x at the point P; as we get the same value either way, thus

if $PR = \delta x$, $\frac{\delta y}{\delta x} = \frac{+QR}{+PR}$, if $PR' = \delta x$, $\frac{\delta y}{\delta x} = \frac{+Q'R'}{+PR'}$,

and if δx is indefinitely small, $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$ and is the slope of the tangent PS, *i.e.*, $\frac{\text{PT}}{\text{PS}}$.

Again, if
$$-PR_2 = \delta x$$
, $\delta y = -Q_2R_2$
and $\frac{\delta y}{\delta x} = \frac{-Q_2R_2}{-PR_2} = \frac{Q_2R_2}{PR_2}$.
If $-PR_3 = \delta x$, $\frac{\delta y}{\delta x} = \frac{-Q_3R_3}{-PR_3} = \frac{Q_3R_3}{PR_3}$, and finally,

when δx is indefinitely small, $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$ and is the slope

of the tangent PS, *i.e.*, $\frac{PT}{TS}$.

As a numerical example, calculate the value of $\frac{dy}{dx}$ for the curve $y = 3x^2$ when x = 3, (1) by taking + values of δx , (2) by taking - values of δx .

$$y=3x^2$$
, hence, when $x=3$, $y=27$.
(1) Let $\delta x=0.2$; then $y+\delta y=30.72$ and $\frac{\delta y}{\delta x}=\frac{3.72}{0.2}=18.6$
,, $\delta x=0.1$; then $y+\delta y=28.83$ and $\frac{\delta y}{\delta x}=\frac{1.83}{0.1}=18.3$
,, $\delta x=0.05$; then $y+\delta y=27.91$ and $\frac{\delta y}{\delta x}=\frac{0.91}{0.05}=18.2$

(2) Let
$$\delta x = -0.2$$
; then $y - \delta y = 23.52$ and $\frac{\delta y}{\delta x} = \frac{-3.48}{-0.2} = 17.4$
,, $\delta x = -0.1$; then $y - \delta y = 25.23$ and $\frac{\delta y}{\delta x} = \frac{-1.77}{-0.1} = 17.7$
,, $\delta x = -0.05$; then $y - \delta y = 26.11$ and $\frac{\delta y}{\delta x} = \frac{-0.89}{-0.05} = 17.8$

From the above results, it will be seen that we can make $\frac{\delta y}{2\pi}$ approach as nearly as we please to the value 18, whether we make $\delta x + \text{or } -$; hence, we conclude that when x = 3, the value of $\frac{dy}{dx}$ is 18.

361. The slope of a curve at any point has already been

defined as the slope of the tangent to the curve at that point. As the tangent, or slope, represents also the rate of increase of the ordinate (y) with respect to the abscissa (x), the variation in the rate of increase is indicated by the variation in slope of the tangent. In Fig. 276, the graph FG has

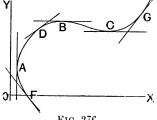


Fig. 276.

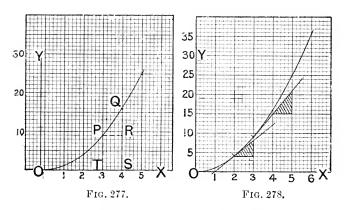
various slopes at different points, e.g., at A the tangent is vertical and $\frac{dy}{dx}$ is infinite. At B and C the tangent is

horizontal and $\frac{dy}{dx} = 0$, hence at a maximum or minimum value (Art. 283) the rate of increase of y with respect to x is zero. At F the tangent is such that $\frac{dy}{dx}$ is negative. At

D and G the tangent is such that $\frac{dy}{dx}$ is positive, and as the slope is steeper at G than at D, the rate of increase is greater at G than at D.

362. The ratio $\frac{dy}{dx}$ is called the differential co-efficient of y with respect to x. The process of finding this value is called differentiating; and the Differential Calculus is concerned chiefly with finding this value when y is any

given function of x. A complete discussion of the Differential Calculus is beyond the scope of the present book; the following method of differentiation from first principles will, however, be readily understood.



Let $y = x^2$ be the graph shown in Fig. 277.

Take OT = x; PT = y; $TS = \delta x$; $QR = \delta y$.

The co-ordinates of P are thus x, y.

The co-ordinates of Q are $(x + \delta x)$, $(y + \delta y)$, and, as both points P and Q are on the graph,

$$y = x^{2}. Eq. 1.$$

$$(y + \delta y) = (x + \delta x)^{2}$$

$$= x^{2} + 2x \cdot \delta x + (\delta x)^{2}. Eq. 2.$$
Subtract Eq. 1 from Eq. 2. $\delta y = 2x \cdot \delta x + (\delta x)^{2}.$

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta x} \left\{ \begin{array}{l} \delta y \\ \delta x \end{array} = \frac{2 \cdot x \cdot \delta x}{\delta x} + \frac{(\delta x)^2}{\delta x} \\ = 2 \cdot x + \delta x. \end{array} \right.$$

Now make
$$\delta x$$
 so small as to be negligible $\frac{\delta y}{\delta x}$ then becomes $\frac{dy}{dx}$ $\frac{dy}{dx} = 2 x$.

This result shows that when $y = x^2$, the value of $\frac{d\eta}{dx} = 2x$ at any point on the graph, x being the abscissa of the

point. As the value $\frac{dy}{dx}$ for any point is the slope of the tangent to the graph at that point, it follows that at a point at which x=2 the tangent has a slope 2 x=4. At the point x=3, the slope is 2 x=6, and so on. These statements can be tested geometrically, as shown in Fig. 278, which represents the graph $y=x^2$ and the tangents at the points x=2 and x=3.

It can be shown that **if** $\mathbf{y} = \mathbf{x}^n$, the value of $\frac{d\mathbf{y}}{d\mathbf{x}} = \mathbf{n}\mathbf{x}^{n-1}$, whether n is + or -, or integral or fractional.

Ex. When
$$y = x^{12}$$
, $\frac{dy}{dx} = 12 x^{11}$ for $n = 12$.

363. If we are dealing with related quantities which are denoted by symbols other than y and x, e.g., if we denote space by S and time by T, we write $\frac{\delta S}{\delta \Gamma}$ for the ratio of a small increase in space to the corresponding increase in time, and $\frac{dS}{d\Gamma}$ for the value of this ratio when the interval δT is so small as to be negligible.

Ex. The space S in feet passed through by a body in T seconds is $S=10\ T+8\ T^2$. If T=2, find S. Now if T=2+t, find the new S. The extra space (new $S-old\ S$) divided by t is evidently the average velocity in the short interval of time t; write out what is its value. Now imagine t to be smaller and smaller without limit, what is the velocity? (Board of Education, 1907.)

Velocity is rate of increase of space with respect to time, or the

value $\frac{dS}{dY}$.

$$S = 10 \text{ T} + 8 \text{ T}^{2}.$$
When
$$T = 2, S = (10 \times 2) + (8 \times 4)$$

$$= 52 \text{ ft., or "old S."}$$
When
$$T = 2 + t, S = 10 (2 + t) + 8 (2 + t)^{2}$$

$$= 20 + 10 t + 32 + 32 t + 8 t^{2}$$

$$= (52 + 42 t + 8 t^{2}) \text{ ft., or "new S."}$$

$$= (60 \times 8 - \text{old S})$$

$$= (52 + 42 t + 8 t^{2}) - 52$$

$$= 42 t + 8 t^{2}.$$

Average velocity in interval $t = \frac{42 t + 8 t^2}{t} = (42 + 8 t)$ ft. per second.

When t becomes smaller and smaller without limit, the velocity is $42 + (8 \times 0) = 42$ ft. per second.

Plot the graph $S = 10 \text{ T} + 8 \text{ T}^2$, as shown in Fig. 279. At the point P, T = 2. Imagine the blank triangle to be

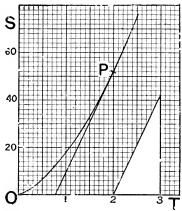


Fig. 279.

a set-square, the inclined side having a slope 42. This side is parallel to the tangent at P, for $\frac{dS}{dT} = 42$, when T = 2, and the slope of the curve (or tangent to the curve) at the point at which T = 2 is 42.

Exercises.

- (1) The space S in feet passed through by a body in T seconds is $S = 10 \text{ T}^2$. If T = 3, find S. Now take T = 3 + t, find the new S. The extra space (new S old S) divided by t, is evidently the average t to become smaller and smaller without limit; what is the velocity?

 (B.E., 1908.)
- (2) The space S ft. passed through by a body in the time t seconds is S = 12 t^2 . When t = 2, find s. When $t = 2 \cdot 01$ find s. When $t = 2 \cdot 01$ find s. When $t = 2 \cdot 01$ find s. In each case find the speed in the interval after t = 2. What is the speed at the time $t = 2 \cdot ?$ (B.E., 1909.)
- (3) A body has moved through the distance s ft. in the time t seconds, and it is known that $s = bt^2$ when b is a constant. Find the

distance when t is 4. Find the distance when the time is $4 + \delta t$. What is the average speed during the interval δt . As δt is imagined to be smaller and smaller, what does the average speed become?

(B.E. (2), 1906.)

(4) The following Table records the heights in inches of a girl A (born January, 1890), and a boy B (born May, 1894). Plot these records. The intervals of time may be taken as exactly four months.

Year.	1900.	1901.				1903.		
Month.	Sept.	Jan.	May.	Sept.	Jan.	May.		Jan.
A	54.8	55.6	56:6	58.0	59.2	6 2	60.9	61.3
В	48.3	49.0	49.8	50.6	51.5	52.3	53.1	53 9

Find in inches per year the average rates of growth of A and B during the given period. At about what age was the growth of A most rapid? State this rate; divide it by her average rate.

(B.E., 1903.)

(5) In any such question as the above (No. 4), where points on a curve have co-ordinates, like h (height) and t (time), show exactly how it is that the slope of the curve at a point represents there the rate of growth of h as t increases.

(B.E., 1903.)

- (6) $y=a+bx^n$ is the equation to a curve which passes through these three points, x=0, y=1.24; x=2.2, y=5.07; x=3.5, y=12.64; find a, b, and n. When we say that $\frac{dy}{dx}$ is shown by the slope of the curve, what exactly do we mean? Find $\frac{dy}{dx}$ when x=2.

 (B.E. (2), 1903.)
- (7) The following are the areas of cross-section of a body at right angles to its straight axis.

	1				l		l	
A. sq. ins.	250	292	310	$2^{7}3$	215	180	135	120
v ins. from one end.	0	22	41	70	84	102	130	145

What is the whole volume from x=0 to x=145? At x=50, if a cross-section slice of small thickness δx has the volume δr , find $\frac{\delta r}{\delta x}$ (B.E. (2), 1903.)

(8) The following numbers give x ft. the distance of a sliding piece measured along its path from a certain point to the place where it is at the time t seconds; what (approximately) is its acceleration at all the tabulated times except the first and last? Show in a curve how the acceleration depends upon t. (B.E. (2), 1908.)

x	1.000	2:736	4.420	6.090	7.428	8.660	9.660	10:397	10.848	11.000
\overline{t}	0	0.1	0.5	0.3	0.4	0.5	0.6	0.7	0.8	0.9

(9) If $s = 12 t^2$ where s is the space in feet which has been passed through by a body in t seconds, find s when t = 10. Find the space when t = 10 + m. What is the distance passed through in the interval of m seconds after t = 10? What is the average speed during this interval? What is this as m gets to be very small t (B.E. (2), 1909.)

(10) At a certain place where all the months of the year are assumed to be of the same length (30.44 days each), at the same time in each month the length of the day (interval from sunrise to sunset in hours) was measured, as in this Table.

Nov.	Dec.	Jan.	Feb.	Mar.	April.	May.	June.	July.
8.35	7.78	8.35	9.87	12	14.11	15.65	16.22	15.65

What is the average increase of the length of the day (state in decimals of an hour per day) from the shortest day, which is 7.78 hours, to the longest, which is 16.22 hours? When is the increase of the day most rapid, and what is it? (B.E., 1903.)

(11) A sliding piece is at the distance s ft. from a point in its path at the time t seconds. Do not plot s and t. What is the average speed in each interval of time? Assume that this is really the speed in the middle of the interval, and now plot time and speed on squared paper.

8	1.0000	1.1054	1.2146	1:3268	1.4432	1.5624	1.6857	1.8118
t	0	0.1	0.2	0.3	0.4	0.2	0.6	0.7

What is the approximate increase in speed between t = 0.25 and t = 0.35? What is approximately the acceleration when t = 0.3? (B.E. (2), 1910.)

Answers are on p. 493.

CHAPTER XXVII.

DIFFERENTIATION.

364. The method of differentiation from first principles described in the preceding chapter will now be applied to several examples which occur frequently in integration.

Ex. 1. $y = x^n$, when n is negative, or when n is fractional.

(1)
$$y = x^{-1} \text{ or } \frac{1}{x}$$
(2)
$$y + \delta y = \frac{1}{(x+\delta x)}$$
Subtracting (1) from (2).
$$\delta y = \frac{1}{(x+\delta x)} - \frac{1}{x} = \frac{-\delta x}{x(x+\delta x)}$$

$$\div \text{ by } \delta x. \qquad \frac{\delta y}{\delta x} = \frac{-\delta x}{x(x+\delta x)\delta x} = \frac{-1}{x(x+\delta x)} = \frac{-1}{x^2 + x^2 \delta x}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x^2}.$$

$$y = \sqrt{x} \text{ or } x^{\frac{1}{2}}$$

$$(2) \qquad y + \delta y = (x + \delta x)^{\frac{1}{2}}$$
Subtracting (1) from (2).
$$\delta y = (x + \delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

$$\delta y = (x + \delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

$$= \frac{x + \delta x - x}{(x + \delta x)^{\frac{1}{2}} + \delta x^{\frac{1}{2}}}$$

$$= \frac{\delta x}{(x + \delta x)^{\frac{1}{2}} + \delta x^{\frac{1}{2}}}$$

$$\vdots \text{ by } \delta x.$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x + \delta x)^{\frac{1}{2}} + x^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Both these examples follow the rule $y = x^n$; $\frac{dy}{dx} = nx^{n-1}$.

P.M.

$$Er. 2. (1) y = \sin x
(2) y + \delta y = \sin (x + \delta x)
\text{Subtracting. } \delta y = \sin (x + \delta x) - \sin x
\Rightarrow \text{ by } \delta x. \quad \frac{\delta y}{\delta x} = \frac{\sin (x + \delta x) - \sin x}{\delta x}
= \frac{2 \sin \frac{\delta x}{2} \cos (x + \frac{\delta x}{2})}{\delta x} = \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \cos (x + \frac{\delta x}{2})
= \cos (x + \frac{\delta x}{2}) \qquad [\text{for } \frac{\sin x}{x} = 1 \text{ when } x \text{ is small. } \text{See Art. } 306]$$

$$\therefore \frac{dy}{dx} = \cos x
Ex. 3. (1) y = \cos x
(2) y + \delta y = \cos (x + \delta x)
\text{Subtracting. } \delta y = \cos (x + \delta x) - \cos x
\Rightarrow \text{ by } \delta x. \quad \frac{\delta y}{\delta x} = \frac{\cos (x + \delta x) - \cos x}{\delta x} = -\frac{2 \sin (x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}}{\delta x}
* = -\sin (x + \frac{\delta x}{2}) \cdot \frac{\sin \frac{\delta x}{2}}{\delta x} = -\sin (\delta x + \frac{\delta x}{2})$$

365. To differentiate sums and differences.

 $\therefore \frac{dy}{dx} = -\sin x.$

Subtracting.
$$\delta y = \delta u + \delta v$$
 $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u + \delta v$ $\Rightarrow \delta y = \delta u - \delta v$ $\Rightarrow \delta u = \delta u - \delta v$ $\Rightarrow \delta u = \delta u - \delta v$ $\Rightarrow \delta u = \delta u - \delta v$ $\Rightarrow \delta u = \delta u -$

Hence, differentiate each term in an expression separately, and find the algebraic sum of the results.

Ex. 1.
$$y = x^3 + \sin x$$
 Ex. 2. $y = \sqrt{x} - \cos x$
$$\frac{dy}{dx} = 3x^2 + \cos x$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - (-\sin x)$$

$$= \frac{1}{2\sqrt{x}} + \sin x$$

366. Constants.

(1) If y = constant, say 3, it is clear that $\frac{dy}{dx} = 0$, for ybeing constant, $\frac{dy}{dx}$, which is the rate of increase of y with respect to x, must be zero.

(2) If $y = mx^2$, where m is a constant, then $y + \delta y = m (x + \delta r)^2$. (Subtracting and \div by δx) $\frac{\delta y}{\delta x} = \frac{m(x + \delta x)^2 - mx^2}{\delta x}$ $= \frac{2 mx \cdot \delta x + m (\delta x)^2}{\delta x}$ $= \frac{2 mx + m \cdot (\delta x)}{\delta x}$ and $\therefore \frac{dy}{dx} = 2 mx$.

When $y = x^2$, we have $\frac{dy}{dx} = 2x$, hence the constant m multiplies the value of $\frac{dy}{dx}$ for the variable x^2 . It can be shown that this rule holds good generally, hence $\frac{d}{dx}$ (mx^p) $= m \cdot \frac{d}{dx} x^n$. We thus differentiate as if the constant were not present, and multiply the result by the constant.

(3) If $y = x^n + \epsilon$, where ϵ is a constant, we have $\frac{d\tilde{y}}{dx} = \frac{d}{dx}(x^n) + \frac{d}{dx}(r) \qquad \text{(differentiating a sum)}$ $=\frac{d}{d}(x^n)+0$ (by (1) above).

Geometrically this is equivalent to saving that the addition of a constant moves the graph upwards or downwards relatively to the axis of x, but does not affect the slope at any particular point on the graph. This should be tested by plotting graphs, e.g., $y = x^2$; $y = x^2 + 3$; $y = x^2 - 3$, &c.

Exercises.

Differentiate (1) $x^2 + x + 3$. (2) $\sin x - e^x - 4$. (3) $3x^2 + 2x - \sin x$. (4) $ax^3 + bx^2 + cx - d$. (5) $2ax + 3\sin x + 2e^x$.

(1) $2 \ x + 1$. (2) $\cos x - e^x$. (3) $6 \ x + 2 - \cos x$. (4) $3 \ ax^2 + 2 \ bx + c$. (5) $2 \ a + 3 \cos x + 2 \ e^x$.

367. e^z. **Definition and differentiation.** If we find the sum, to any number of terms, of a series of terms following the law

$$1 + \frac{1}{\lfloor \frac{1}{4}} + \frac{1}{\lfloor \frac{2}{4}} + \frac{1}{\lfloor \frac{3}{4}} + \frac{1}{\lfloor \frac{4}{4}} + \dots$$
 &c., we obtain a

number which is usually denoted by the letter e. When 9 terms are taken the value is 2.7183, and this is the value of e adopted in practical work.

Text-books on algebra prove that, e being defined as

$$1 + \frac{1}{\lfloor \frac{1}{2} + \frac{1}{\lfloor \frac{2}{2} + \frac{1}{\lfloor \frac{3}{2} + \frac{1}{\lfloor \frac{4}{2} + \frac{x^4}{\lfloor \frac{x}{2} + \frac{x^3}{2} + \frac{x^4}{\lfloor \frac{x}{2} + \frac{x^3}{2} + \frac{x^4}{\lfloor \frac{x}{2} + \frac{x^4}{2} + \dots}}$$
, then $e^x = 1 + x + \frac{x^2}{\lfloor \frac{x}{2} + \frac{x^3}{2} + \frac{x^4}{\lfloor \frac{x}{2} + \frac{x^4}{2} + \dots}}$. &c. This is called the **exponential series.**

If $y = e^x$, then to find $\frac{dy}{dx}$ we have

$$\frac{d}{dx}(e^{x}) = \frac{d}{dx}\left(1 + x + \frac{x^{2}}{2} + \frac{x^{3}}{3} + \frac{x^{4}}{4} + \dots & \&c.\right)$$

$$= 0 + 1 + x + \frac{x^{2}}{2} + \frac{x^{3}}{3} + \dots & \&c.$$

$$= e^{x} \text{ (by definition of } e^{x}\text{)}.$$

Thus, when $y = e^x$; $\frac{dy}{dx} = e^x$. This result is very important.

368. **Log.** e **x**. This term is frequently used in works on the calculus without the suffix e. It is advisable to add the suffix in all cases, thus denoting that it is the log. of x in the Naperian system of logs. (see Art. 171).

If
$$y = \log_{e} x$$
, then to find $\frac{dy}{dx}$ we have $e = x$ (by definition of a log.) $\frac{dx}{dy} = \frac{d}{dy} e^{y} = e^{y} = x$.

Hence $\frac{dy}{dx} = \frac{1}{x} \left(\text{inverting } \frac{dx}{dy} = x \right)$ or $\frac{d}{dx} \log_{e} x = \frac{1}{x}$.

Exercises. Differentiate (1) $e^x + \log_{e} x$. (2) $\sin_e x + e^x - \log_{e} x$.

Answers. (1)
$$e^x + 1/x$$
. (2) cos. $x + e^x - 1/x$.

369. Differentiation of a Product.

(1) Let y = uv.

(2) Then
$$y + \delta y = (u + \delta u)(r + \delta r)$$

 $= ur + u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
Subtracting. $\delta y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.
 $\dot{\delta} y = u \cdot \delta r + r \cdot \delta u + \delta r \cdot \delta u$.

Hence the rule: -The differential coefficient of a product of two factors is equal to the product of the first factor and the differential coefficient of the second plus the product of the second factor and the differential coefficient of the first.

Ex. 1.
$$y = x^2 \cdot \sin x$$

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2)$$

$$= x^2 \cdot \cos x + 2x \cdot \sin x$$

$$= x (x \cdot \cos x + 2\sin x).$$
Ex. 2. $y = \epsilon^x \cdot \log_{\epsilon} x$

$$\frac{dy}{dx} = \epsilon^x \cdot \frac{1}{x} + \log_{\epsilon} x \cdot \epsilon^x$$

$$= \epsilon^x \left(\frac{1}{x} + \log_{\epsilon} x\right).$$

Exercises on Art. 369.

Differentiate (1)
$$e^{x}$$
. sin. x . (2) x^{2} . $\log_{e} x$. (3) $ax(x^{3} - b)$. (4) \sqrt{x} . sin. x . (5) $\frac{1}{x}$. e^{x} . (6) $(e^{x} - 3)$ sin. x . Ans. (1) e^{x} (cos. $x + \sin x$). (2) $x(1 + \log_{e} 2x)$.

(3)
$$4 ax^3 - ab$$
. (4) $\sqrt{x} \cdot \cos x + \frac{1}{2}x - \frac{1}{2} \cdot \sin x$.

(5)
$$e^x \left(\frac{1}{x} - x^{-2}\right)$$
. (6) $(e^x - 3)\cos x + e^x$, $\sin x$.

Exercises on Art. 370.

Differentiate (1)
$$\frac{x^2}{\cos x}$$
. (2) $\frac{(u-x)}{(2x+3)}$. (3) $\frac{\cos x}{(x^2+a^2)}$.

(4) $\frac{\sqrt{x}}{(x+a)}$. (5) sec. x . (6) cosec. x .

Ans. (1) $\frac{2x\cos x + x^2\sin x}{\cos^2 x}$. (2) $\frac{-(2a+6)}{4x^2+12x+9}$.

Ans. (1)
$$\frac{2 x \cos x + x^2 \sin x}{\cos^2 x}$$
. (2) $\frac{-(2 a + 6)}{4 x^2 + 12 x + 9}$. (3) $\frac{-\sin x (c^x + x^2) + \cos x \cdot (c^x + 2x)}{(e^x + x^2)^2}$. (4) $\frac{a - x}{2 \sqrt{x} (x + a)^2}$

(5) sec.
$$x$$
 . tan. x . (6) $-\csc x$. cot. x .

370. Differentiation of a Quotient.

(1) Let
$$y = \frac{u}{v}$$
.

(2) Then $y + \delta y = \frac{u + \delta u}{v + \delta v}$.

Subtracting. $\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$

$$= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$$

$$= \frac{vu + v \cdot \delta u - uv - u \cdot \delta v}{v(v + \delta v)}$$

$$= \frac{v \cdot \delta u - u \cdot \delta v}{v(v + \delta v)}$$

$$= \frac{v \cdot \delta u - u \cdot \delta v}{v(v + \delta v)}$$

$$\div \text{by } \delta x. \qquad \frac{\delta y}{\delta x} = \frac{v \cdot \frac{\delta u}{\delta x} - u \cdot \frac{\delta v}{\delta x}}{v(v + \delta v)}$$

$$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} \qquad (v \delta v \cdot \text{ultimately vanishes.})$$

Hence the rule:—From the product of the denominator and **D.C.** of numerator subtract the product of the numerator and **D.C.** of denominator, and divide by the denominator squared.

$$Ex. 1. \quad y = \frac{x^3}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x}{\sin^2 x} \cdot \frac{\frac{d}{dx}(x^3) - x^3}{\sin^2 x} \cdot \frac{\frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{3x^2 \cdot \sin x - x^3 \cos x}{\sin^2 x}.$$

$$Ex. 2. \quad To differentiate tan. x which = \frac{\sin x}{\cos x}. \qquad (Art. 337)$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx} \left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x}{\frac{d}{dx}(\sin x) - \sin x} \cdot \frac{\frac{d}{dx}(\cos x)}{\frac{d}{dx}(\cos x)} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x. \qquad (See Art. 339)$$

$$Similarly \frac{d}{dx}(\cot x) = \frac{d}{dx} \left(\frac{\cos x}{\sin x}\right) = -\csc^2 x.$$

371. Differentiation of a function of a function.

Suppose y is a function of u, (say $y = u^{\frac{5}{3}}$), and that u is a function of x, (say $u = x^3 + 2$), and that we require the value $\frac{dy}{dx}$. If we increase y by a small increment δy , we

have a corresponding increase δu in the value of u, and an increase δu in the value of u is accompanied by a corresponding increase δv in the value of x.

 $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$, and this identity holds good however small δy , δu , and δx may be. When the increments are made indefinitely small, the identity becomes

 $\frac{dy}{dx} = \frac{d}{d}\frac{y}{u} \cdot \frac{d}{d}\frac{u}{x}$. It must be observed that, in the final form, this is not a mere algebraic expression, but a statement connecting three operations, viz.: that when y = f(u) and u = F(x), then the rate of increase of y with respect to x is equal to the product of the rate of increase of y with respect to u and the rate of increase of u with respect to u.

In the above example, $u = x^3 + 2$, $\frac{du}{dx} = 3x^2$, $y = u^{\frac{5}{3}}$, $\frac{dy}{du} = \frac{5}{3}u^{\frac{3}{3}}$. hence $\frac{d}{dx}(x^3 + 2)^{\frac{5}{3}} = \frac{5}{3}u^{\frac{2}{3}} \cdot 3x^2$ (i.e., $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$) $= \frac{5}{3}(x^3 + 2)^{\frac{3}{2}} \cdot 3x^2$. (putting $u = x^3 + 2$) $= 5 \cdot x^2 \cdot (x^3 + 2)^{\frac{3}{2}}$.

Ex. 1. If
$$y = e^{kx}$$
, find $\frac{dy}{dx}$.
Let $u = kx$, then $y = e^{u}$.

$$\frac{du}{dx} = k, \frac{dy}{du} = e^{u}$$
hence $\frac{dy}{dx} = e^{u} \cdot k$ or ke^{kx} .
Ex. 2. $y = \sin^5 x$, find $\frac{dy}{dx}$

Let
$$u = \sin x$$
, then $y = u^5$

$$\frac{dy}{du} = 5 u^4 \text{ and } \frac{du}{dx} = \frac{d}{dx} \sin x = \cos x.$$

Hence, $\frac{d}{dx}(\sin^5 x) = 5 u^4$, cos. $x = 5 \sin^4 x \cos x$.

Ex. 3.
$$y = \sqrt{a^2 - x^2}$$
, find $\frac{dy}{dx}$.
Let $u = u^2 - x^2$, then $y = u^{\frac{1}{2}}$:
$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}} \text{ and } \frac{du}{dx} = \frac{d}{dx} (u^2 - x^2) = -2 x.$$
Hence, $\frac{d}{dx} (a^2 - x^2)^{\frac{1}{2}} = \frac{1}{2} u^{-\frac{1}{2}} \cdot (-2x)$

$$= -\frac{x}{\sqrt{u}} = -\frac{x}{\sqrt{a^2 - x^2}}.$$

[Note, $-y = \sqrt{\frac{-x}{a^2 - x^2}}$ is the equation of circle, hence $\frac{-x}{\sqrt{a^2 - x^2}}$ gives the slope of the tangent at any point x. When x=a , $\cos 45^\circ = \frac{a}{\sqrt{2}}$, the tangent is inclined at 135°, and $\frac{dy}{dx} = -\frac{a}{\sqrt{2}} \div \frac{a}{\sqrt{2}} = -1$, which agrees with tan. 135°.]

Similarly (by calling the expression in brackets u)

$$y = (ax + b)^{n}; \frac{dy}{dx} = an (ax + b)^{n-1}$$

$$y = \sin. (ax); \frac{dy}{dx} = a \cdot \cos. ax$$

$$y = \sin. (ax + b); \frac{dy}{dx} = a \cdot \cos. (ax + b)$$

$$y = \log_{e} (ax + b); \frac{dy}{dx} = \frac{a}{(ax + b)}.$$

Exercises.

Differentiate (1) $(2 \ x + 3)^3$; $\sin . 5 \ x$; $\sin . (2 \ x + 4)$. (2) $\log_{e}(2 \ x + 4)$; $\sqrt[4]{4 - x^3}$; $\sqrt[3]{a^2 - x^2}$.

- (3) $(x^2+2)^{\frac{3}{2}}$; e^{-2x} ; $(e^x+a)^3$. (4) Cos. (3x-6); tan. (2x-3); cot. (5x+4).
- (5) Tan. 2 x; $\log_{e} ax$; $(a^2 + 2x) \cdot \sin^2 x$.
- (6) Differentiate with respect to t, a, $\sin (wt + a)$; b, $\cos (wt + a)$.
- (7) If $y = 3 \sin 2 u$, show that $\frac{dy}{du} = 6 \cos 2 u$.

(1)
$$6(2x+3)^2$$
; $5\cos 5x$; $2\cos (2x+4)$

(1)
$$6(2x+3)^2$$
; $5\cos 5x$; $2\cos (2x+4)$.
(2) $\frac{1}{x+2}$; $\frac{-3x^2}{2\sqrt{4-x^3}}$; $\frac{-2x}{3(a^2-x^2)^3}$.

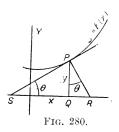
(3)
$$3 x (x^2 + 2)^{\frac{1}{2}}$$
; $-2 e^{-2x}$; $3 e^x (e^x + a)^2$.

(3)
$$3 x (x^2 + 2)^{\frac{1}{2}}$$
; $-2 e^{-2x}$; $3 e^x (e^x + a)^2$.
(4) $-3 \sin. (3 x - 6)$; $2 \sec.^2 (2 x - 3)$; $-5 \csc.^2 (5 x + 4)$.

(5)
$$2 \sec^2 2x$$
; $\frac{1}{x}$; $\left\{ (a^2 + 2x)\cos x + \sin x \right\} 2 \sin x$.

(6)
$$aw \cos (wt + a)$$
; $-b \cdot w \cdot \sin (wt + a)$

372. Geometrical application of the differential coefficient. If y = f(x), we have already shown (Chap. XXVI.) that $\frac{dy}{dx}$ gives the slope of the curve at any point, i.e., gives the value $\tan \theta$, where θ is the inclination of the tangent to the axis OX.



 $\cdot \cdot \cdot \frac{dy}{dx} = \tan \theta.$

In Fig. 280, PS is the tangent to the graph y = (fx) at the point P. PR is the normal. The length SQ is called the sub-tangent; and QR the sub-normal. The angle QPR also $= \theta$. Let QQ = x, PQ = y.

The lengths of PS, PR, QS, QR can be expressed in terms

of x, y, and $\frac{dy}{dx}$ (for the point P).

(1) Tangent PS = PQ . cosec.
$$\theta$$
 = PQ . $\sqrt{1 + \cot^2 \theta}$ (Art. 339)
= PQ $\sqrt{1 + \frac{1}{\tan^2 \theta}}$.
= PQ $\frac{\sqrt{\tan^2 \theta + 1}}{\tan \theta}$.
= $y = \left\{ \left(\frac{dy}{dx} \right)^2 + 1 \right\}^{\frac{1}{2}} \div \frac{dy}{dx}$
(2) Normal PR = PQ . sec. θ = PQ $\sqrt{\tan^2 \theta + 1}$

 $= y \left\{ \left(\frac{dy}{dx} \right)^2 + 1 \right\}^{\frac{1}{2}}$

(3) Sub-tangent QS = PQ · cot ·
$$\theta = \frac{\dot{PQ}}{\tan{\cdot}\theta} = y \div \frac{dy}{dx}$$

(4) Sub-normal QR = PQ. tan. $\theta = y \cdot \frac{dy}{dx}$.

Ex. 1. In the parabola $y^2 = 4ax$, show that the sub-normal is constant.

Sub-normal =
$$y \cdot \frac{dy}{dx} = 2a^{\frac{1}{2}}x^{\frac{1}{2}} \times 2a^{\frac{1}{2}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

= $2a$, and is thus constant.

Ex. 2. A semi-elliptic arch has a span of 40 ft, and a rise of 10 ft. Find the distance of the foot of the normal at the point x = 10 from the centre of the ellipse.

Equation of ellipse is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
 $\therefore y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$.
Sub-normal $= y \cdot \frac{dy}{dx} = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} \cdot \frac{b}{a}\left(\frac{-x}{(a^2 - x^2)^{\frac{1}{2}}}\right)$

$$= \frac{-b^2}{a^2}x.$$

when x = 10; sub-normal $= -\frac{100 \times 10}{20 \times 20} = -2\frac{1}{2}$ ft., hence distance required $= (10 - 2\frac{1}{2}) = 7\frac{1}{2}$ ft.

373. Successive differentiation: notation.

If
$$y = f(x)$$
, say $y = 3x^3$, then $\frac{dy}{dx} = 9x^2$.

If we differentiate this result, we have $\frac{d}{dx}(9x^2) = 18x$.

If we again differentiate, we have $\frac{d}{dx}(18x) = 18$.

 $\frac{dy}{dx}$ is the differential coefficient of y, or the 1st derivative. $\frac{d}{dx}(9x^2)$ is called the 2nd derivative, and is written $\frac{d^2y}{dx^2}$.

 $\frac{dx}{dx}$ (18x) is called the 3rd derivative, and is written $\frac{d^3y}{dx^3}$, and, generally, if y = f'(x) the first operation of differentiating

y is written $\frac{dy}{dx}$, differentiating this result, or the second

differentiation is written $\frac{d^2y}{dx^2}$, and so on, the *n*th differentiation.

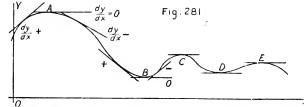
tion being written $\frac{d^n y}{dx}$.

Ex. A moving body starts from a point distant k ft. from a datum point o along a line ox, and moves along the line in such a way that its distance s from o in any time t. seconds is given by $s = ut + \frac{1}{2}gt^2 + k$, u and g being constants. Find the velocity and acceleration.

Velocity = Rate of increase of space = $\frac{ds}{dt}$ Acceleration = Rate of increase of velocity = $\frac{dv}{dt} = \frac{d}{dt} \cdot \frac{ds}{dt} = \frac{d^2s}{dt^2}$

Hence
$$\begin{aligned} \frac{ds}{dt} &= \frac{d}{dt} \left(ut + \frac{1}{2} g t^2 + k \right) = u + gt \text{ (velocity)} \\ \frac{d^2s}{dt^2} &= \frac{d}{dt} \left(u + gt \right) = g \text{ (acceleration)}. \end{aligned}$$

It is thus seen that g in the given formula is acceleration, and u is initial velocity of body. v=u+gt is velocity after t seconds.



374. **Maxima and minima.** In Art. 283, a maximum value of y in a continuous graph y = f(x), is defined as a value greater than any neighbouring values on either side of it, and a minimum value as one less than neighbouring values on either side. Applying this definition to the graph shown in Fig. 281, it will be seen that there are maxima at A, C, and E, and minima at B and D, and it must be observed that there may be more than one maximum or minimum value. The value of $\frac{dy}{dx}$ at any point on a graph gives the slope of the tangent at that point (Art. 361); at a maximum or minimum value the tangent is parallel to OX, hence $\frac{dy}{dx} = 0$.

As x increases, $\frac{dy}{dx}$ changes sign from + to - at a maximum value (as at A), and it changes from - to + at a minimum value (as at B), or it may be said that at a maximum value the curve lies below the tangent, and at a minimum value it lies above. We can thus ascertain values of x for which y is a maximum or minimum by finding $\frac{dy}{dx}$ for y = f(x), equating $\frac{dy}{dx}$

 $\frac{dy}{dx} = 0$, and finding the values of x which satisfy the resulting equation. By taking values of x a little smaller and a little larger than the determined values, we can find whether

changes from + to - or from - to +, and hence whether the value is a maximum or a minimum. If the angle of inclination of the tangent at any point on the graph to the axis OX be denoted by ϕ , then $\frac{dy}{dx} = \tan \theta$ and $\frac{d}{dx}(\tan \theta)$, or $\frac{d^2y}{dx^2}$, gives the rate of increase of $\frac{dy}{dx}$ or tan. ϕ with respect to x. Now, if the inclination is changing from + through zero value to - as x increases as at A, the rate of increase of tan. ϕ or $\frac{dy}{dx}$ must be negative, hence $\frac{d^2y}{dx^2}$ will, in general, be negative at a maximum value, and conversely, if, as at B, the inclination is changing from — through zero value to + as xincreases, the rate of increase of $\frac{dy}{dx}$ or tan. ϕ must be positive, hence $\frac{d^2y}{dx^2}$ will, in general, be positive for a minimum value. We can thus decide whether the value of x, which makes $\frac{dy}{dx} = 0$ is a maximum or a minimum by substituting the value of x thus found in the expression for $\frac{d^2y}{dx^2}$, and noting whether the result is - or +. This rule fails when $\frac{d^2y}{dx^2} = 0$ for the same value of x, for which $\frac{dy}{dx} = 0$. In such cases, the former must be adopted.

Ex. $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x + 1$. Find values of x for which y is a maximum or a minimum.

$$\frac{dy}{dx} = x^2 - 5x + 6. \quad \text{For } \frac{dy}{dx} = 0, \text{ we must have } x = 3 \text{ or } 2.$$

$$\frac{d^2y}{dx^2} = 2x - 5. \quad \text{Put } x = 3 \text{ and } \frac{d^2y}{dx^2} = +1.$$

$$\text{Put } x = 2 \text{ and } \frac{d^2y}{dx^2} = -1.$$

Hence x = 3 is a minimum, and x = 2 a maximum value for y. Check by plotting a graph for y.

375. Curvature; radius of curvature. It has been explained in Art. 261 that the average curvature for a small length PQ of an arc is the change in direction of the tangent from P to Q ÷ length of arc PQ. To find the curvature at any point P, we require the limiting value of change in direction of tangent or, if PQ is \delta s and the change

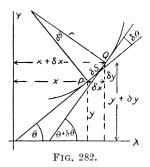
in direction of the tangent is $\delta\theta$, we require the value of $\frac{d\theta}{ds}$

In Fig. 282, consider a very small are PQ. Let the co-ordinates of P and Q be x, y, and $(x + \delta x)$, $(y + \delta y)$ respectively. Let arc $P Q = \delta s$.

Then
$$(\delta s)^2 = (\delta y^2) + (\delta x)^2$$
 and $\frac{ds}{dx} = \left(\left(\frac{dy}{dx}\right)^2 + 1\right)^{\frac{1}{2}} \left\{\begin{array}{l} \text{PQ is small and does not} \\ \text{differ appreciably from a} \\ \text{straight line.} \end{array}\right\}$

Let θ be the inclination of the tangent at P, and $(\theta + \delta\theta)$ the inclination of the tangent at Q to OX. The tangent changes in direction $\delta\theta$ for an arc δs , and $r = \delta s \div \delta\theta$, where r = radius of curvature for PQ.

In the limit
$$r = \frac{ds}{d\theta}$$
 :. $\frac{1}{r} = \frac{d\theta}{ds}$.



A general expression for $\frac{1}{r}$ in terms of x and y will involve $\frac{dy}{dx}$ and As θ is a function of s, tan. θ , which $=\frac{du}{dx}$ is also a function of s.

$$\tan \theta = \frac{dy}{dx}.$$
Differentiate each side $\frac{1}{ds}$ (tan. θ) = $\frac{d}{ds}$ ($\frac{dy}{dx}$)
$$\frac{d}{d\theta}$$
 (tan. θ) : $\frac{d\theta}{ds} = \frac{d}{dx} \left(\frac{dy}{dx}\right) \cdot \frac{dx}{ds}$ (Function of $\frac{d\theta}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds}$.

$$\sqrt{1 + \tan^2 \theta} \cdot \frac{d\theta}{ds} = \frac{d^2y}{dx^2} \cdot \frac{1}{\frac{ds}{ds}}.$$

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{d\theta}{ds} = \frac{d^2y}{dx^2} \cdot \frac{1}{\left(\frac{dy}{dx}\right)^2 + 1} \cdot \frac{1}{\frac{1}{2}}$$

$$\therefore \frac{d\theta}{ds} \text{ or } \frac{1}{r} = \frac{d^2y}{dx^2} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{3}{2}$$

Exercises.

- Art. 372. (1) In a parabola $y^2 = 4ax$, show that the subtangent for any point P (x, y) is = 2x, and hence that OQ = OS in Fig. 280.
 - (2) Show that the sub-normal to a velocity-space curve represents acceleration, and that, if acceleration is constant, the velocity-space curve is a parabola.
- Art. 373. (3) A point moves according to the law x=a, $\sin (wt+a)$, where x is the displacement at time t, and w and a are constants. Show that the velocity is v=aw cos. (wt+a) and that the acceleration is $-w^2x$. (Such motion is simple harmonic motion.)
- Art. 374. (4) Find the maximum and minimum values for (1) $y = \sin \theta$ between 0 and 2π radians, also for (2) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 6x + 4$.

Ans. (1)
$$\frac{\pi}{2}$$
; $\frac{3\pi}{2}$. (2) -3; 2.

- (5) Show that a square is the rectangular figure of greatest area for a given perimeter.
- (6) Show that the rectangular figure of greatest perimeter that can be inscribed in a circle is a square.
- (7) Show that the rectangle of greatest area that can be inscribed in a parabola $y^2 = 4ax$, bounded by the ordinate x = c, is of width $\frac{2}{3}$.

- (8) Show that the greatest cylinder that can be inscribed in a sphere has a radius $\sqrt{\frac{2}{3}}$. r, where r is radius of sphere, and that the ratio of volumes of cylinder and sphere is 0.5773.
- Art. 375. (9) Show that the curvature of a circle is constant.
 - (10) Find an expression for the radius of curvature of the curve $y = 3x^2 + 2x$, and the numerical value of this at the origin.

Ans.
$$\left\{1+(6x+2)^2\right\}^{\frac{3}{2}}\div 6$$
; 1.87.

Miscellaneous Exercises.

- (1) Differentiate $x^3 + \sin x \sqrt{x}$, also $1/x x^3 + \cos x$. Ans. $3x^2 + \cos x - \frac{1}{2}x^{-\frac{1}{2}}$; $-x^{-2} - 3x^2 - \sin x$.
- (2) If $s = \left(ut + \frac{1}{2}ft^2\right)^{\frac{1}{2}}$, show that $\frac{ds}{dt} = (u + ft) \div t \ (2 \ u + ft)$.
- (3) If x = ...a, sin. (wt + a) find $\frac{dx}{dt}$, and differentiate your result again with respect to t. Express the second result in terms of w and x.

 Ans. $-u^2x$.
- (4) If $y = 2 x + \epsilon x + \sin x$, find $\frac{d^2y}{dx^2}$. Ans. $\epsilon^x \sin x$.
- (5) If $y = \cos x$, find $\frac{d^3y}{dx^3}$. Ans. $\sin x$.
- (6) If $y = u^{3} + 3bt^{2} + 2at + 2c$, show that $\frac{d^{4}y}{dt^{4}} = 0$.

CHAPTER XXVIII.

INTEGRATION.

376. An idea of what is meant by integration will be obtained by considering the method of finding an area bounded on one side by a curve, the equation of which is known, and on the other sides by the axis OX and two ordinates respectively. Let OA (Fig. 283) be the graph $y=3x^2$, and suppose we require the area ABCD. Consider a small strip EFGH of the area. Let OE be denoted by x and EH by y, then, since $y=3x^2$, EF = 3 (OE)². Increase OE by a small element EH, and denote EH by δx . Then HG is $(y+\delta y)$, and again $(y+\delta y)=3(x+\delta x)^2$ or HG = $3\cdot (\mathrm{OH})^2$. If we regard the strip EFGH as a trapezium, the area will be $\frac{1}{2}\left\{y+(y+\delta y)\right\}\cdot\delta x$ or $\left(y+\frac{1}{2}\cdot\delta y\right)\cdot\delta x$.

Y G A B B B B

FIG. 283.

This value is in excess of, or is less than, the true area EFGH according as the curve FG is concave upwards (as shown), or concave downwards, and it will represent the true area when FG is a straight line, and consequently the mid-ordinate is $\left(y + \frac{1}{2} \delta y\right)$ The smaller we make the distance EH or δx , the nearer will the arc FG approach a straight line, and con-

sequently the more accurately will

 $\left(y + \frac{1}{2} \delta y\right)$ · δx represent the true area of the strip.

By making δc and consequently δy indefinitely small, and writing δy and δc as dy and dx respectively, the area of the resulting narrow strip becomes accurately

$$\left(y + \frac{1}{2} \cdot dy\right) \cdot dx \cdot r \left(y \cdot dx + \frac{1}{2} \cdot dy \cdot dx\right).$$

The second term $\frac{1}{2} \cdot dy \cdot dx$ involves the product of two indefinitely small quantities, and hence is negligible in comparison with $y \cdot dx$, so that we may now regard the area of each small strip as $y \cdot dx$. The whole area bounded by the curve will be the sum of the areas of an infinite number of these strips, *i.e.*, the sum of an infinite number of products $y \cdot dx$. The sign used to denote the sum of an infinite number of such

small quantities is an elongated S, thus \int , hence the area

bounded by the curve is written as $\int y \cdot dx$, and is read as "the integral $y \cdot dx$." But $y = 3x^2$, and therefore the area $= \int 3x^2 \cdot dx$. To be of any practical use, this expression must be evaluated.

Let us now consider the steps necessary for the evaluation of $\int 3x^2 \cdot dx$. Obviously we cannot divide the area mechanically into an infinite number of strips, find the area of each, and add together the results, but we may be able to discover some known value which is also $=\int 3x^2 \cdot dx$, and if such a

value can be found, we conclude that it is the area bounded by the curve $y = 3x^2$.

Refer to the rule for differentiating x^n . When $y = x^n$, we have $\frac{dy}{dx} = nx^{n-1}$, hence, when $y = x^3$, we have $\frac{dy}{dx} = 3x^2$. This equation may be written $dy = 3x^2 \cdot dx$, meaning P.M.

that a small element of y, viz., dy is = the product of $3x^2$, and the corresponding small element of x, viz., dx. The sum of an infinite number of small elements dy, is written as $\int \cdot dy$, and similarly the sum of an infinite number of small products $3x^2 \cdot dx$ is written as $\int 2x^2 \cdot dx$. If a quantity y be divided into an infinite number of small elements each = dy, then $\int \cdot dy$, or the sum of an infinite number of these small elements, must be y.

Hence
$$\int \cdot dy = y$$
, and since $\int \cdot dy = \int 3x^2 \cdot dx$, it follows that $y = \int 3 \cdot x^2 \cdot dx$.

But $y = x^3$, hence $x^3 = \int 3x^2 \cdot dx$, that is, the quantity (x^3) which, when differentiated with respect to x, gives the quantity appearing between the sign \int and dx (in this case $3x^2$) is the required integral. The area bounded by the curve $y = 3x^2$ is thus x^3 , and by an inspection of Fig. 283, it will be seen that the numerical value of such area must depend upon the positions of the last and first ordinates BA, DC, and consequently upon the final and initial values OB, OC, of x. The value x^3 represents the area between the curve, the axis OX, and an ordinate corresponding to any value of x. If OC = a, and OB = b, then the area ODC = a^3 and the area OAB = b^3 , and hence the area ABCD, which = (area OAB — area OCD), must = $b^3 - a^3$. Again, if OB = 20 and CC = 10, the area ABCD = $(20^3 - 10^3) = 7,000$ units.

377. The general value of an integral, such as x^3 above, is called an **indefinite integral**. The two limits, as OC and OB above, between which the area lies, and hence between which integration is required; are indicated by writing their values at the lower and upper ends of the integral sign

thus, $\int_a^b 3x^2 \cdot dx$ means that the integration is to be per-

formed from x = a to x = b, and as we have already seen, this result, called the **definite integral**, is obtained by first finding the indefinite integral, then substituting the two values of x, between which integration is required, and finally subtracting from the expression containing the upper value that containing the lower value.

Repeating the above process when y is any continuous function of x, say y = f(x), we can show that the area of any strip bounded by the graph y = f(x) is $f(x) \cdot dx$, and that

the whole area is $\int f(x) dx$. We next find a value, say z,

which, when differentiated with respect to x, gives f(x), and z is the indefinite integral for the area. The actual area is found by putting in the two limits between which integration is required, and subtracting.

378. It will thus be seen that the process of integrat on is one of summation of a series of small elements which follow some definite law, and further that integration is the inverse operation to differentiation, and consists in solving the problem:—Given the value of $\frac{dy}{dx}$, find y. It will also be observed that an integral can only be found for expressions which result from differentiating some other expression; hence, many of the rules for integrating are derived from the corresponding rules for differentiating, and one of the first things to be done is to make a standard list of differentiations to be used as a guide in the inverse operation of integration.

379. Additive constant of integration. If we differentiate $y = x^3 + c$, where c is any constant, we have $\frac{dy}{dx} = 3x^2$, for the differential coefficient of an additive constant is zero (Art. 366). In the process of integrating $3x^2$, in which we have to find what value of y will, when differentiated, give $3x^2$, we may have an innumerable number of expressions for the indefinite integral, i.e., $y = x^3 + c$ is the indefinite integral, and c may have any value. The constant c is called "the **constant of integration**," and should, strictly, always be added when writing down an indefinite integral. The value of c will, in any particular problem, be determined from the conditions given, as in the following example.

Ex. A body falling freely due to gravity has an acceleration g=32 (ft. per sec.) per sec. Find its velocity v at the end of t seconds, if its initial celecity is 20 ft. per sec.

Acceleration $g = \text{rate of increase of velocity} = \frac{dv}{dt}$

Hence
$$\frac{dv}{dt} = g = 32$$

$$dv = 32 \cdot dt$$

$$\int dv = \int 32 \, dt$$

$$\int dv = \int 32 \, dt$$

$$\int dv = \int 32 \, dt$$

$$\int 32 \cdot dt = 32 \, t.$$

 $\therefore v = 32 \cdot t + \text{constant}.$

To find the value of the constant, we have v=20 when t=0. Hence, $20=(32\times 0)+$ constant, thus the value of the constant is 20. Hence, $v=32\cdot t+20$ is the velocity required.

In the case of a definite integral, the constant of integration will vanish when the integration is performed between limits, thus the integral of x^3 . dx between the limits a and b is

$$\int_{a}^{b} x^{3} \cdot dx = \frac{1}{4}x^{4} + c \text{ (indefinite integral)}$$

$$= \left(\frac{1}{4}b^{4} + c\right) - \left(\frac{1}{4}a^{4} + c\right)$$

$$= \frac{1}{4}\left(b^{4} - a^{4}\right)$$

STANDARD TABLE.

У	$\frac{dy}{dx}$	2	\int_{z} . dx .
d B	n _d n - 1	nx^{n-1} x^{n} (except when $n = -1$)	$a^n + c$ (c is constant of integration) $\frac{1}{n+1} x_{n+1} + c.$
$(ax+b)^n$	$an(ax+b)^{n-1}$	$(ax+b)^n$	$\left \frac{1}{a(n+1)} \left(ax + b \right)^{n+1} + c \right $
\boldsymbol{x}	1	1	x+c.
$\log_e x$	$\frac{1}{4}$	$\frac{1}{x} (\text{or } x^{-1})$	$\log_e x + c.$
$\log_{\theta}(ax+b)$	$\frac{a}{ax+b}$	$\frac{a}{x+a}$	$a \log_e (x+a) + c.$
$\sin x$.	cos. a	cos. a	$\sin x + c$.
$\sin (ax+b)$	$a \cdot \cos(ax+b)$	$\cos(ax+b)$	$\frac{1}{a}\sin.(ax+b)+c.$
$\cos x$	$-\sin x$	sin. x	$-\cos x + c$.
$\cos(ax+b)$	$-\frac{1}{a}\sin(ax+b)$	$\sin(ax+b)$	$-\frac{1}{a}\cos((ax+b)+c,$
tan. x	$\sec^2 x$	$\sec^2 x$	$\tan x + c$.
cotan. x	$-\csc^2 x$	cosec.2 d	$-\cot x + c$
e^{kx}	ke^{kx}	ι ^{kx}	$\frac{1}{h}e^{hx}+c.$
$(k=1)$ e^x	ϵ^x	ϵ^x	$\iota^x + c$.
$(k=-1) e^{-x}$	- c + x	e-x	$=e^{-x}+c.$

380. Simple problems in integration should be tested by differentiating, thus

$$\int x^4 \cdot dx = \frac{1}{5} x^5 + c \text{ (by rule for } x^n)$$
Differentiating $\left(\frac{1}{5}x^5 + c\right)$ we have $\frac{d}{dx}\left(\frac{1}{5}x^5\right) + \frac{d}{dx}(c) = \frac{1}{5} \cdot 5x^4 + 0$

$$= x^4.$$
Exercise.

Exercise. Show that
$$\int x^3 \cdot dx = \frac{1}{3}e^{-3x} \cdot dx = \frac{1}{3}e^{-3x}$$
.

But $\int x^{n-1} dx = \frac{1}{n-1+1} e^{n-1} dx$.

But $\int x^{n-1} dx = \int x^{n-1} dx = \int x^{n-1} dx$.

But $\int x^{n-1} dx = \int x^{n-1} dx = \int x^{n-1} dx$.

A constant as a multiplier may thus be placed outside the integral sign, and will multiply the integral of the variable, thus $\int k \cdot x^n \cdot dx = k \int x^n \cdot dx = \frac{k}{n+1} x^{n+1} + \text{constant}.$

382. Rule for a sum or difference of terms. The differential coefficient of a sum or difference of terms is equal to the sum or difference of the differential coefficients of the separate terms. By inversion, the integral of a sum or difference of terms will be the sum or difference of the integrals of the separate terms.

Thus
$$\frac{d}{dx}\left(x^4 + x^3 + x^2 + x\right) = \frac{d}{dx}x^4 + \frac{d}{dx}x^3 + \frac{d}{dx}x^2 + \frac{d}{dx}x$$

$$= 4x^3 + 3x^2 + 2x + 1$$
hence $\int (4x^3 + 3x^2 + 2x + 1) dx = \int 4x^3 dx + \int 3x^2 dx + \int 2x dx + \int 1 dx$

$$= 4 \cdot \frac{1}{4}x^4 + 3 \cdot \frac{1}{3}x^3 + 2 \cdot \frac{1}{2}x^2 + x + \text{const.}$$

$$= x^4 + x^3 + x^2 + x + \text{const.}$$

Exercises.

(1)
$$Fixd \int 3x^3 \cdot dx$$
; $\int 2\sin x \cdot dx$; $\int 4e^x \cdot dx$.
 $Ans. \frac{3}{4}x^4$; $-2\cos x$; $4e^x$.
(2) $Find \int (x^3 + 2x^2 + x) \cdot dx$; $\int (2x^2 + \sin x) dx$.

Ans
$$\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{3}x^2$$
; $\frac{2}{3}x^3 - \cos x$.

383. Many expressions can be simplified, or written in an equivalent form, before integration, thus simplifying the work.

Ex. 1.
$$\int_{-x}^{x^2 - a^2} dx dx$$

$$\frac{x^2 - a^2}{x + a} = \frac{(x - a)(x + a)}{(x + a)} = (x - a). \text{ (Simplifying.)}$$
Hence
$$\int_{-x}^{x^2 - a^2} dx dx = \int_{-x}^{x^2 - a^2} dx = \int_{-x}^{x^2 - a^2} dx - a \int_{-x}^{x^2 - a^2} dx dx = \int_{-x}^{x^2 - a^2} dx - a \int_{-x}^{x^2 - a^2} dx dx = \int_{-x}^{x^2 - a^2} dx + \text{const.}$$

$$Ex. \ 2. \int \frac{(x+2)^2}{2x} \cdot dx$$

$$\frac{(x+2)^2}{2x} = \frac{x^2 + 4x + 4}{2x} = \frac{x}{2} + 2 + \frac{2}{x} \cdot \text{ (Simplifying.)}$$
Hence
$$\int \frac{(x+2)^2}{2x} \cdot dx = \frac{1}{2} \int x \cdot dx + 2 \int dx + 2 \int \frac{dx}{x}$$

$$= \frac{1}{4} x^2 + 2 x + 2 \log x + \text{const.}$$

Ex. 3.
$$\int (x+2) (x+3) dx$$
. Multiply out, then the problem becomes $\int (x^2 + 5x + 6) dx = \frac{1}{3} x^3 + \frac{5}{2} x^2 + 6x + \text{const.}$

384. A notation commonly used when a definite integral is to be found is to enclose the indefinite integral in square brackets with the limits placed after the right-hand bracket prior to performing the subtraction, as follows:—

$$\int_{a}^{b} (x^{3} + 3x + 2) dx = \left[\frac{1}{4} x^{4} + \frac{3}{2} x^{2} + 2x \right]_{a}^{b}$$
$$= \frac{1}{4} (t^{4} - a^{4}) + \frac{3}{2} (t^{2} - a^{2}) + 2 (b - a).$$

Exercises.

(1) Find
$$\int \frac{t^2 - a^2}{t - u} \cdot dt$$
; $\int (x - u)(x + u) dx$.
Ans. $\frac{1}{2} \cdot 2 + at + \text{const.}$; $\frac{1}{3} \cdot x^3 - a^2 x + \text{const.}$

(2) Find
$$\int_{2}^{4} (x^{2} + ax + b) dx$$
; $\int_{0}^{90^{\circ}} \cos \theta \cdot \delta \theta$.

Ans. $18\frac{2}{3} + 6a + 2b$; 1.

385. Integration by substitution. Many integrals are found by a method of substitution, illustrated by the following examples.

Ex. 1. Find
$$\int \sin ax \cdot dx$$

Let $u = ax$, then $\frac{du}{dx} = a$ and $\frac{du}{a} = dx$.
Substituting, $\int \sin ax \cdot dx = \frac{1}{a} \int \sin u \cdot du$
 $= -\frac{1}{a} \cos u \cdot = -\frac{1}{a} \cos ax$.
Thus, when $a = 2$ we have $\int \sin 2x \cdot dx = -\frac{1}{2} \cos 2x$. Similarly, $\int \cos ax \cdot dx = \frac{1}{a} \sin ax$; $\int \sin (ax + b) dx = -\frac{1}{a} \cos (ax + b)$, and $\int \cos (ax + b) = \frac{1}{a} \sin (ax + b)$.
Ex. 2. Find $\int (2x + 3)\frac{1}{3} \cdot dx$.
Let $u = (2x + 3)$, then $\frac{du}{dx} = 2$ and $\frac{du}{2} = dx$.
Substituting $\int (2x + 3)\frac{1}{3} \cdot dx = \frac{1}{2} \int u\frac{1}{3} \cdot du$
 $= \frac{1}{2} \cdot \frac{3}{4} u\frac{4}{3} = \frac{3}{8} u\frac{4}{3}$
 $= \frac{3}{8} (2x + 3)\frac{4}{3}$.
Ex. 3. $\int e^{kx} \cdot dx$
Let $u = kx$, then $\frac{du}{dx} = k$ and $\frac{du}{k} = ax$.
Substituting, $\int e^{kx} \cdot dx = \frac{1}{k} \int e^{u} du = \frac{1}{k} \cdot e^{u}$
 $= \frac{1}{k} e^{-x}$
Ex. 4. $\int e^{u} \sqrt{a^{2} - x^{2}} \cdot dx$

Let
$$x = a$$
. sin. θ , then $\frac{dx}{d\theta} = a \cdot \cos \theta$, and $dx = a \cdot \cos \theta$. $d\theta$

$$\int \sqrt{a^2 - a^2} \cdot dx = \int \sqrt{a^2 (1 - \sin^2 \theta)} \cdot a \cdot \cos \theta \cdot d\theta.$$
 (Substituting.)

$$= \int a \cdot \cos \theta \cdot a \cdot \cos \theta \cdot d\theta. \quad (\text{See Art. 339.})$$

$$= a^2 \int \cos^2 \theta \cdot d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta) \cdot d\theta. \quad (\text{Appendix L})$$

$$= \frac{a^2}{2} \int d\theta + \frac{a^2}{2} \int \cos 2\theta \, d\theta = \frac{a^2}{2} \left\{ \theta + \frac{1}{2} \sin 2\theta \right\}.$$

If the limits of x are θ and a, the corresponding limits for θ are θ and 90° or $\frac{\pi}{2}$ radians (for sin. 0 = 0 and sin. $90^{\circ} = 1$), and then we

have
$$\int_0^a \sqrt{u^2-x^2}$$
 , $dx=\frac{a^2}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\frac{\pi a^2}{4}$.

This integral is required when finding the area of a circle.

386. Integration of rational fractions. In this method, the fraction to be integrated is written as the algebraic sum of two or more partial fractions, * i.e., a number of fractions, called partial fractions, are first found having denominators of lower dimensions than the given fraction and whose algebraic sum is = the given fraction.

Ex. Find
$$\int \frac{1}{x^2 - 9}, dx$$
,

To resolve $\frac{1}{x^2 - 9}$ into partial fractions, proceed as follows:

 $\frac{1}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}$ (factorizing denominator)

 $\therefore 1 = A(x + 3) + B(x - 3)$ (equating numerators).

Put $x = 3$, then $A = \frac{1}{6}$; also put $x = -3$, then $B = -\frac{1}{6}$,
and $\therefore \frac{1}{x^2 - 9} = \frac{1}{6(x - 3)} - \frac{1}{6(x + 3)}$.

Thus $\int \frac{1}{x^2 - 9} dx = \frac{1}{6} \int \frac{dx}{x - 3} - \frac{1}{6} \int \frac{dx}{x + 3}$.

 $= \frac{1}{6} \log_{e}(x - 3) - \frac{1}{6} \log_{e}(x + 3)$
 $= \frac{1}{6} \log_{e}(x - 3) - \frac{1}{6} \log_{e}(x + 3)$

^{*} For a discussion of partial fractions, text-books on algebra should be consulted. The example here given is typical of those met with in integration,

387. Integration by parts. This method consists in inverting the formula—

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \text{ (Art. 369)}.$$
Integrating of the sides
$$uv = \int u \cdot \frac{dv}{dx} \cdot dx + \int v \cdot \frac{du}{dx} \cdot dx$$

$$\therefore \int u \cdot \frac{dv}{dx} \cdot dx = uv - \int v \cdot \frac{du}{dx} \cdot dx.$$

$$Ex. 1. \int_{x \cdot e^{x}} dx.$$

Put x = u and $\frac{dv}{dx} = e^x$, then $v = e^x$ and $\frac{du}{dx} = 1$

hence, substituting in above formula,

$$\int x \cdot e^x \cdot dx = x \cdot e^x - \int e^x \cdot 1 \cdot dx$$

$$= xe^x - e^x.$$

$$Ex. \ 2. \int \cdot \log_e x \cdot dx \cdot \text{Regard this as } \int \log_e x \cdot 1 \cdot dx.$$
Put $\log x = u$ and $\frac{dv}{dx} = 1$, then $v = x$ and $\frac{du}{dx} = \frac{d}{dx} \log_e x = \frac{1}{x}$
hence, substituting, $\int \log_e x \cdot dx = (\log_e x) \cdot x - \int x \cdot \frac{1}{x} \cdot dx$

$$= x \cdot \log_e x - x.$$

It will be observed that this method depends upon recognizing one of the two factors of the product to be integrated as the differential coefficient of some known quantity, and further that its utility depends upon the integral in the resulting expression being easier to find than the original integral.

388. Change of variable. If y = f(u) and (u) = f(x), we have $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (Art. 371).

The corresponding formula for integration is

$$\int f(u)\frac{du}{dx} \cdot dx = \int f(u) \cdot du,$$

hence, if the expression to be integrated is a product of a function (say of u), and $\frac{du}{dx}$, we may write it as $\int f(u) \ du$, which may be much easier to find. The following examples illustrate the process.

Ex. 1.
$$\int \sin^2 x \cdot \cos x \cdot dx. \text{ Let } u = \sin x, \text{ then } \frac{du}{dx} = \cos x \text{ and}$$

$$\int \sin^2 x \cdot \cos x \cdot dx \text{ becomes } \int u^2 \cdot \frac{du}{dx} \cdot dx = \int u^2 \cdot du$$

$$= \frac{1}{3} u^3 = \frac{1}{3} \sin^3 x$$

or, since $\frac{d}{dx}$. sin. $x = \cos x$, we may write

$$\int \sin^2 x \cdot \cos x \cdot dx \text{ as } \int \sin^2 x \cdot \frac{d}{dx} (\sin x) \cdot dx = \int \sin^2 x \, d \text{ (sin. } x)$$

$$= \frac{1}{3} \sin^2 x \cdot (\text{compare with } \int x^2 \cdot dx = \frac{1}{3} x^3).$$

Ex. 2.
$$\int \frac{dx}{\sqrt{(2-x^2)}}$$
.

This is equivalent to $\int \frac{x \cdot dx}{x^2 (2-x^2)}$, and if $u = x^2$, $\frac{du}{dx} = 2x$ and $x \cdot dx = \frac{du}{2}$

hence, substituting,
$$\int \frac{x \cdot dx}{u^2 (2 - u^2)} \text{ becomes } \frac{1}{2} \int \frac{du}{u (2 - u)}$$

$$= \frac{1}{2} \int \left(\frac{1}{2u} + \frac{1}{2 (2 - u)}\right) du$$

$$= \frac{1}{4} \int \frac{du}{u} + \frac{1}{4} \int \frac{du}{2 - u}$$

$$= \frac{1}{4} \log_e u - \frac{1}{4} \log_e (2 - u)$$

$$= \frac{1}{4} \log_e \frac{u^2}{(2 - u^2)}$$

$$= \frac{1}{4} \log_e \frac{x^2}{(2 - u^2)}$$

CHAPTER XXIX.*

SOME APPLICATIONS OF INTEGRATION.

389. To find an area bounded by a curve the equation of which is known. This was explained fully in Art. 376. If y = f(x) is the equation of the curve, then the indefinite integral for the area is $\int f(x) \, dx$, and the actual

area, if the limits or extreme ordinates are x = a and x = b

is
$$\int_a^b f(x) \cdot dx$$
.

Ex. 1. To find the area of a parabola $y^2=4\,ax$, and its relation to the area of the circumscribing rectangle (Fig. 284)

Area AOB =
$$\int y \cdot dx$$

= $\int 2 \sqrt{ax} \cdot dx$
= $2 \sqrt{a} \int x^{\frac{1}{2}} \cdot dx$
= $2 \sqrt{a} \cdot \frac{2}{3} x^{\frac{3}{2}} + \text{const.}$
Fig. 284.

When x = v, area = v, hence const. = v. Area = $\frac{2}{3} \times \sqrt{4ax}$ = $\frac{2}{3} xy$. (substituting y for $\sqrt{4ax}$).

Let OA = x, then AB = y, and $Area AOB = \frac{2}{3} OA$. $AB = \frac{2}{3} OABE$.

Total area of parabola OBC = 2 area $AOB = \frac{2}{3}$ rectangle EBCD.

Area of a parabola $= \frac{2}{3}$ area of circumscribing rectangle.

^{*}Note.- Exercises for this chapter are on pp. 490, 491 and 492,

Er. 2. Find the area of a parabola $y^2 = x$ when x = 16.

Fig. 284. Let
$$OA = 16$$
, then $AB = \sqrt{16} = 4$

Area =
$$\frac{2}{3}$$
 rectangle EBCD
= $\frac{2}{3} \times 16 \times 8 = \frac{256}{3} = \frac{853}{3}$

Ex. 3. Area of a circle of radius r. Fig. 285.

Consider any strip of radius x and width δx . The length of its mean circumference is $2\pi \left(x + \frac{\delta x}{2}\right)$, and its area is $2\pi \left(x + \frac{\delta x}{2}\right)\delta x$.

If we make each strip indefinitely narrow, the area is $2\pi\left(x+\frac{1}{2}dx\right)dx$

or $2\pi x$. dx, neglecting the term 2π . $\frac{1}{2}dx$. dx.

The sum of all strips from $= \int_{\sigma}^{r} 2\pi x \cdot dx,$ $= 2\pi \int x \cdot dx.$

 $= 2\pi \int_{0}^{\infty} x \cdot dx$ $= \left[2\pi \cdot \frac{1}{2} a^{2}\right]_{0}^{r}$

F1G. 285.

... Area = πr^2 . Fig. 285.

Another method is to take the equation of the circle as $x^2 + y^2 = r^2$ or $y = \sqrt{r^2 - x^2}$.

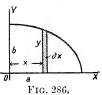
or
$$y = \sqrt{r^2 - x^2}$$
.
Area of a quadrant of circle
$$\begin{cases} -r & \text{of } x = \frac{1}{4}\pi r^2 \text{ (see Art. 385).} \end{cases}$$

 \therefore Area of whole circle = $4 \times \frac{1}{4} \pi r^2 = \pi r^2$.

Ex. 4. Area of an ellipse having axes 2a, 2b, Fig. 286.

The equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{t^2} = 1$... $y = \frac{b}{a} (a^2 - x^2)^{\frac{1}{2}}$

Area of a quadrant
$$\left. \begin{cases} \begin{cases} a \\ a \end{cases} \right\} = \int_{a}^{a} y \cdot dx = \int_{a}^{b} \int_{a}^{a} (a^{2} - x^{2})^{\frac{1}{2}} \cdot dx \end{cases}$$



$$= \frac{b}{a} \cdot \frac{1}{4} \pi . \ell^2 \text{ (see Art. 385)}.$$

$$= \frac{1}{4} \pi a b.$$

,, whole area = $4 \cdot \frac{1}{4} \pi ab = \pi a b$.

Ex. 5. Area bounded by the curve p | r = constant, Fig. 287.

This curve is a rectangular hyperbola and is that for a gas expanding according to the law pressure \times volume = constant. The product p. δr gives the work done by a gas at pressure p when volume increases by δr .

$$pr = k$$

$$p = kr^{-1}$$

$$Area = \int p \cdot dr \text{ (indefinite integral)}$$

$$Area = \int p \cdot dr \text{ (indefinite integral)}$$

$$Area = \int p \cdot dr \text{ (indefinite integral)}$$

$$= k \int_{a}^{b} p \cdot dr$$

$$= k \int_{a}^{b} e^{-1} \cdot dr$$

$$= \left[k \log_{a} r + c \right]_{a}^{b}$$

$$= k (\log_{a} b - \log_{a} a)$$

$$= k \log_{a} e^{-1}$$

Note that k = product of p and r for any point on the graph if the graph is given, and that $\int_a^b p \cdot dr$ gives the work done in expanding from volume a to volume b.

Ex. 6. Area bounded by the curve $px^n = k$ or p = k, x^{-n}

Area =
$$\int p \cdot dr$$
 (indefinite integral)
= $k \int r^{-n} \cdot dr$
= $k \cdot -\frac{1}{n+1} r^{-n+1}$

If the limits are r = a and r = b,

area =
$$\frac{k}{1-n} \left(b^{1-n} - a^{1-n} \right)$$
.

This example is of importance for finding the work done per lb. of steam, when used saturated, during adiabatic expansion. For adiabatic expansion $p \ r^{1.0646} = 479$ (see p. 347), where p = pressure in lbs. per sq. in., r = volume in r ft. of 1 lb. of steam. Hence, if 1 lb. of steam occupying a volume a (r ft.) expands to r (r ft.), the work done is

$$479 \int_{a}^{b} r^{-1.0646} \cdot dr = \frac{479}{-0.0646} \left(h^{-0.0646} - a^{-0.0646} \right) \text{ ft. lbs.}$$

 E_{ℓ} , 7. Simpson's Rule for determining the area of an irregular figure (or volume of an irregular solid) has already been given on p. 286. It is derived as follows :-

Fig. 288. Divide the area into an even number of strips of width h, and bounded by ordinates y_1, y_2 , etc., and consider the area ABCD of two strips. It is assumed that the curve DC is part of a parabola having as its equation $y=a+bx+cx^2$. Taking the axis OY coincident with y_2 , for the ordinate y_2 , we have x = o, and hence $y_2 = a$. For y_3 , we have x = h and $y_3 = a + bh + ch^2$, and for y_1 we have x = -h and $y_1 = a - bh + ch^2$. From these three equations, we derive

$$y_{3} = y_{2} + bh + ch^{2} \\ y_{1} = y_{2} - bh + ch^{2}$$
 Adding and transposing we get
$$y_{1} + y_{3} - 2y_{2} = 2ch^{2}.$$
Area ABCD =
$$\int_{-h}^{+} (a + bx + cx^{2}) dx$$

$$= \left[ax + \frac{1}{2} bx^{2} + \frac{1}{3} cx^{3} \right]^{h}$$

$$= ah + ah + \frac{1}{3} ch^{3} + \frac{1}{3} ch^{3}$$

$$= 2ah + \frac{2}{3} ch^{3}$$

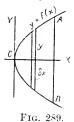
$$= \frac{h}{3} (6a + 2ch^{2}).$$
Substituting the above value of
$$\begin{cases} \frac{h}{3} (6y_{2} + y_{1} + y_{3} - 2y_{2}) \\ \frac{h}{3} (y_{1} + 4y_{2} + y_{3}) \end{cases}$$

Treating areas BGHC and GEFH in the same manner, we have

If the ordinates y_1 , y_2 , etc., represent sectional areas of a solid, then area ADFE represents graphically the volume of the solid.

390. Volumes of solids of revolution (Art. 249) may be found by integration, if the equation of the generating curve be known.

Let y = f(x) be the equation of the curve AOB, Fig. 289, which rotates about its axis OX to generate a solid. Consider the solid as made up of a number of circular disks of radius y and thickness δx . Then,



area of each disk = πy^2 volume of each disk = πy^2 . δr

volume of solid $= \int \pi y^2 \cdot dx$ $= \pi \int \left(f(x) \right)^2 \cdot dx$

If the limits are x = a and x = b, $= \pi \int_{a}^{b} (f(x))^{2} dx$.

Ex. 1. Volume of a paraboloid of revolution, generated by a parabola $y^2 = 4ax$, when a = 1 and limits are x = a and x = 20.

Volume =
$$\pi \int_{0}^{20} y^{2} dx = 4 \pi a \int_{0}^{20} x \cdot dx$$

= $4 \pi a \left[\frac{1}{2} x^{2} \right]_{0}^{20}$
= $2 \pi a (20)^{2}$
= $2 \pi \times 1 \times 20 \times 20$ (when $a = 1$)
= 800π .

Ex. 2. Volume of a cone of height h and radius of base r. A cone is generated by a triangle AOB rotating about OB. Fig. 290

Volume $=\int_{0}^{h} \pi y^{2} dx$. For any disk distant x from u, we have, by similar triangles, $\frac{y}{x} = \frac{r}{h} \cdot y = \frac{rx}{h}$ $\frac{r}{h} \cdot y = \frac{rx}{h}$ $\frac{r}{h} \cdot y = \frac{rx}{h}$ $\frac{r}{h} \cdot y = \frac{rx}{h}$ Fig. 290.

Ex. 3. Volume of a sphere of radius r, Fig. 291.

A sphere is generated by a semicircle rotating about a diameter.

The volume of any disk = $\pi y^2 \cdot \delta x$.

But $y^2 = r^2 - x^2$ for a circle (see Art. 304).

... Volume
$$\pi \int y^2 \, dx = \pi \int (r^2 - x^2) \, dx$$
, and

since the limits are from A to B, i.e., from -r to + r,

Volume of sphere
$$= \pi \int_{-r}^{+r} (r^{2} - x^{2}) dx$$

$$= \pi r^{2} \int_{-r}^{+r} dx - \pi \int_{-r}^{+r} dx$$

$$= \left[\pi r^{2} x \right]_{-r}^{+r} - \frac{1}{3} \pi \left[x^{3} \right]_{-r}^{+r}$$

$$= \left\{ \pi r^{3} - (-\pi r^{3}) \right\} - \frac{1}{3} \pi \left[r^{3} - (-r^{3}) \right]$$

$$= 2 \pi r^{3} - \frac{2}{3} \pi r^{3}$$

$$= \frac{4}{3} \pi r^{3}.$$

391. Centre of gravity. The centre of gravity of an area, and of a mass, has already been explained in Art. 248, and also an approximate method of finding its position. regard a mass as made up of a large number of elements m_1 , m_2 , etc., and consider the pull of gravity on each of the various elements as a force proportional to the magnitude of each element, we have a series of parallel forces, the resultant of which is equal to Σ (m) (read sigma m, meaning the sum of m_1 , m_2 , etc.). By referring to Art. 327, it will be seen that the moment of the resultant of a series of non-concurrent forces about any point is equal to the sum of the moments of the separate forces. The point at which the resultant pull of gravity acts is the position of the centre of gravity. To find this point, consider an area AOB, Fig. 292, symmetrical about the axis OX and bounded by a graph Divide it into a series of strips of width δx , and consider one of these strips.

Area of strip = $2y \cdot \delta x$

Moment of force on this strip about $OY = 2 y \cdot x \cdot \delta x$

Total moment =
$$\int 2y \cdot x \cdot dx = 2 \int f(x) \cdot x \cdot dx$$
.
The resultant force on the whole area = $2 \int f(x) \cdot dx$, and

if x be the distance of the centre of gravity along OX from O, the moment of the resultant force about OY =

$$\vec{x} \times 2 \int f(x) dx.$$

$$\therefore \vec{x} = \frac{\text{total moment}}{\text{resultant force}} = \frac{2 \int f(x) x dx}{2 \int f(x) dx}$$
Fig. 292.

When the area is not symmetrical about an axis OX, as in Fig. 293, it is necessary to find separately the distance of the centre of gravity from OX and from OY, thus locating the position.

If the curve has equation y = f(x), the distance of the centre of gravity from OY is given by the above formula.

To find the distance from OX, consider strips parallel to OX.

Area of each strip =
$$x \cdot \delta y$$

Moment about $OX = x \cdot y \cdot \delta y$

Total moment =
$$\int x \cdot y \cdot dy.$$

Let \overline{y} be distance of C.G from OX.

Then $y = \frac{\int xy \, dy}{\cot a \cdot \cos a}$. In evaluating $\int x \cdot y \cdot dy$, we

must write x as a function of y, e.g., if the curve is

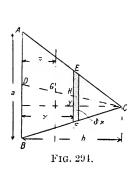
$$y = \frac{k}{x^n}$$
, then $x^n = \frac{k}{y}$ and $x = \left(\frac{k}{y}\right)^{\frac{1}{n}}$.

Ex. 1. To find the centre of gravity of any triangle, Fig. 294. Divide the triangle ABC into strips EF of width δx and length y parallel to the base AB. Let axis OY coincide with the base.

$$\frac{y}{a} = \frac{h - x}{h}$$
 ... $y = \frac{(h - x)}{h} a$ (By similar triangles.)

The area of each strip EF thus $=\frac{(h-x)}{h}a$. δx .

The moment of each strip about the base $=\frac{(h-x)}{h}a \cdot x \cdot \delta x$.



The total moment
$$\begin{cases} = \frac{a}{h} \int_{0}^{h} (h - x) x \cdot dx, \\ = \frac{a}{h} \left[\frac{hx^{2}}{2} - \frac{x^{3}}{3} \right]_{0}^{h} \\ = \frac{a}{h} \left[\frac{h^{3}}{2} - \frac{h^{3}}{3} \right] \\ = \frac{ah^{2}}{6}. \end{cases}$$

Let *e* be distance of C.G. from base.

Moment of triangle = area $\times \overline{x} = \frac{ah\overline{x}}{2}$

$$\therefore \frac{ah\overline{x}}{2} = \frac{ah^2}{6}.$$

$$\overline{x} = \frac{1}{3}h \text{ from base.}$$

The centre of gravity of each strip EF is at the centre H of the strip. Hence the centre of gravity of the whole triangle must lie along the median CD at a perpendicular distance $\frac{h}{3}$ from the base,

which distance, from similar triangles, is seen to be also $\frac{1}{3}$ CD from the base, measured along CD.

Exp. 2. A narabila $v^2 = 4ax$ Fig. 295. Find distance of CC.

Ex. 2. A parabola $y^2 = 4ax$, Fig. 295. Find distance of C.C. from apex, if last ordinate is distant h from apex.

н н 2

Ex. 3. Find the distance of C.G. of a semicircle from the diameter Fig. 296.

$$y = f(x) = \sqrt{r^2 - x^2}$$

$$\therefore \overline{x} = \frac{2\int_{0}^{r} (r^2 - x^2)^{\frac{1}{2}} \cdot x \cdot dx}{2\int_{0}^{r} (r^2 - x^2)^{\frac{1}{2}} \cdot dx} = \frac{\frac{1}{3}r^3}{\frac{\pi r^2}{4}} = \frac{4r}{3\pi}$$

Note.—The value of $\int_0^r (r^2 - x^2)^{\frac{1}{2}} x \cdot dx$ may be found by evaluating the moment for area OAB about OA, thus, each strip $= y \cdot \delta x$ in area, and its centre of gravity from OA $= \frac{y}{2}$; hence, moment of each strip is $\frac{1}{2} y^2 \cdot \delta x$.

Whole moment =
$$\int_{0}^{r} \frac{1}{2} y^{2} dx = \frac{1}{2} \int_{0}^{r} (r^{2} - x^{2}) dx$$

= $\frac{1}{2} \left[r^{2}x - \frac{1}{3} r^{3} \right]_{0}^{r}$
= $\frac{1}{2} r^{3} - \frac{1}{6} r^{3}$
= $\frac{1}{2} r^{3}$.

Fig. 296.

The moment about OB must have the same value.

392. The position of the centre of gravity of a solid of revolution is found by regarding the solid as composed of small circular disks. Regard the mass of each disk as equal to its volume, *i.e.*, assume the density is 1.

Fig. 297. Let y = f(x) be the equation of the generating curve. A disk distant x from o has a radius y = f(x), and hence volume $= \pi \left\{ f(x) \right\}^2 \delta x$. The moment of this disk about OY is $\pi \left\{ f(x) \right\}^2 x . \delta x$, and the sum of such moments from o to h is

$$\int_0^h \pi \left\{ f(x) \right\}^2, x, dx$$

Let \overline{x} be distance of C.G. from O.

The whole volume is $\int_{a}^{b} \pi \{f(x)\}^{2} dx$

Hence
$$\overline{x} = \frac{\int_{0}^{h} \pi \left\{ f(x) \right\}^{2} \cdot x \cdot dx}{\int_{0}^{h} \pi \left\{ f(x) \right\}^{2} dx}$$



FIG. 297.

Ex. 1. A cone, Fig. 293.

Equation of generator is y = mx.

Volume of each disk = $\pi y^2 \delta x = \pi (mx)^2$. δx .

Moment about OY = $\pi (mx)^2 \cdot x \cdot \delta x$.

$$\vec{x} = \int_{0}^{h} \frac{\pi \cdot m^{2} \cdot x^{3} \cdot dx}{\int_{0}^{h} \pi \cdot m^{2} \cdot x^{2} \cdot dx} = \frac{\int_{0}^{h} \pi \cdot m^{2} \cdot x^{2} \cdot dx}{\int_{0}^{h} \pi \cdot \frac{1}{4} h^{4}} = \frac{3}{4} h \text{ from apex of cone.} \quad \text{Fig. 298.}$$

Ex. 2. A paraboloid of revolution generated by parabola $y^2 = 4ax$, from x = 0 to x = h, Fig. 299.

Volume of each disk πy^2 . $\delta x = \pi \cdot 4 ax \delta x$. Moment of each disk about $OY = 4 \pi a x^2 \cdot \delta x$.

$$\therefore \vec{x} = \frac{\int_{0}^{h} 4\pi x \, x^{2} \cdot dx}{\int_{0}^{h} 4\pi x \, x \cdot dx} = \frac{4\pi x \cdot \frac{1}{3} \, h^{3}}{4\pi a \cdot \frac{1}{2} \, h^{2}}$$
$$= \frac{2}{3} h \text{ from apex.}$$

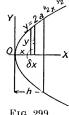


Fig. 299.

Ex. 3. A solid hemisphere, To find distance of C.G. from the base, Fig. 300.

A hemisphere is generated by quadrant AOB of a circle rotating about OB. Equation of curve AB is $x^2 + y^2 = r^2$.

Volume of each disk πy^2 . $\delta x = \pi (r^2 - x^2) \cdot \delta x$.

Moment of each disk about OY = $\pi (r^2 - x^2) x$. δx .

Moment of each disk about OY =
$$\pi(r^2 - x^2) x \cdot \delta x$$
.

$$\frac{1}{x} = \frac{\int_0^r \pi(r^2 - x^2) x \cdot dx}{\int_0^r \pi(r^2 - x^2) dx}$$

$$= \frac{\pi \int_0^r r^2 \cdot x \cdot dx - \pi \int_0^r x^3 \cdot dx}{\frac{2}{3} \pi r^3}$$
[Volume of sphere = $\frac{4}{3} \pi r^3$]

$$=\frac{\pi r^2 \cdot \frac{1}{2} \, r^2 - \pi \cdot \frac{1}{4} \, r^4}{\frac{2}{3} \, \pi r^3} = \frac{\pi r^4 \left(\frac{1}{2} - \frac{1}{4}\right)}{\pi r^3 \cdot \frac{2}{3}} = r \cdot \frac{\frac{1}{4}}{\frac{2}{3}} = \frac{3}{8} \, r \text{ from base.}$$

The centre of gravity of a circular arc AB, Fig. 301, is found as follows.

Take as axis OX the middle radius of the arc, and the centre of the circle as the origin.

Let angle XOA be a. Consider any small element of the arc, such as CD, and let angle XOC be θ radians, and angle DOC be $\delta\theta$.

The length CD = $r \cdot \delta\theta$

The moment of CD about an axis OY perpendicular to OX and in the same plane as the arc is

$$r \cdot \delta\theta \times OE = r \delta\theta \times r \cdot \cos \theta$$

= $r^2 \cos \theta \cdot \delta\theta$.

Sum of moments for whole arc AB is

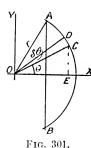
$$\int_{-\alpha}^{+\alpha} r^{2} \cdot \cos \theta \cdot d\theta = \left[r^{2} \cdot \sin \theta \right]_{-\alpha}^{+\alpha}$$

$$= r^{2} \left\{ \sin \alpha - \sin (-\alpha) \right\}_{\beta}^{+\alpha}$$

$$= r^{2} \left\{ \sin \alpha - \sin \alpha - \sin \alpha \right\}_{\beta}^{+\alpha}$$

Let \overline{x} be distance of C.G. from O.

Then
$$\tilde{x} = \frac{\text{sum of moments}}{\text{length of are}} = \frac{2 r^2 \cdot \sin \alpha}{2r\alpha}$$
$$= \frac{r \sin \alpha}{2r\alpha}$$



Note,—For a semicircular arc,
$$\alpha = \frac{\pi}{2}$$
, $\sin \frac{\pi}{2} = 1$, hence $\tilde{x} = \frac{2r}{\pi}$.

The above result may be utilised in finding the C.G. of a circular sector, AOB, Fig. 302.

Let the radius of sector be r and angle 2 α radians. Consider the sector as made up of small sectors OCD, each of which can be regarded as a triangle. The centre of gravity of each is then $\frac{2}{3}r$ from θ , and if we regard each $\underline{\mathcal{Q}}$ sector as a particle placed at its centre of gravity, we get an arc of radius $\frac{2}{r}r$. centre of gravity of sector will coincide with that of the arc, which we know from the above

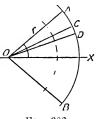
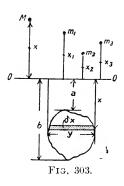


Fig. 302.

$$= \frac{\text{radius of are } \times \sin \alpha}{\alpha}$$
Hence C.G. of sector
$$= \frac{\binom{2}{3} r}{\alpha} \frac{\sin \alpha}{\alpha} = \frac{2}{5} \cdot \frac{r \sin \alpha}{\alpha}$$

393. Moment of inertia. If a particle of mass M (Fig. 303), is at a distance x from a line OO, then the product of M and x^2 is called the "moment of inertia" (usually



the "moment of mertia" (usually denoted by I) of M about the axis OO. If a number of separate particles or elements m_1 , m_2 , m_3 , etc., are at distances x_1 , x_2 , x_3 , etc., respectively from OO, then

$$I = m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^2 + \text{etc.}$$

= $\sum mx^2$.

If the elements form part of a continuous surface or body, e.g., an area or a solid body, then if each element of area is δA , and of mass δM , and its distance from OO = x,

$$I = \Sigma \cdot \delta A \cdot x^2$$
 for an area $I = \Sigma \cdot \delta M \cdot x^2$ for a mass.

In Fig. 303 let y = the length of a strip or element of width δx of the area A, and x the distance from OO of the centre line of the strip. Then area of strip $= y \cdot \delta x$ and

I =
$$\int_{a}^{y} x^{2} dx$$
. If the area is such that $y = f(x)$ is

the graph of the boundary, then $I = \int_{a}^{b} f(x) x^{2} dx$.

Similarly for a solid of revolution, if y = f'(x) when plotted gives the generating curve, then assuming the solid divided into disks, each is a circle of radius f(x) and area $\pi(f'(x))^2$ and mass $\pi m(f'(x))^2 \cdot dx$, and

$$I = \int \pi m (f(x))^2 \cdot x^2 \cdot dx$$
 taken between the proper limits

to include the whole volume, m being the density of the material.

Ex. 1. Find 1 for a rectangle of breadth b and depth h when axis passes (1) through the centre of gravity (2) coincides with the base, (3) is at a distance k from the centre of gravity and parallel to the base, Fig. 304.

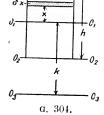
(1) About axis O₁O₁ passing through centre of gravity.

$$I = \int_{-\frac{h}{2}}^{+\frac{h}{2}} b \cdot x^{2} \cdot dx, \text{ taking } O_{1}O_{1} \text{ as axis of } y.$$

$$= \frac{1}{3} b \left[\left(\frac{h}{2} \right)^{3} - \left(-\frac{h}{2} \right)^{3} \right]$$

$$= \frac{1}{3} b \left[\frac{h^{3}}{8} + \frac{h^{3}}{8} \right]$$

$$= \frac{1}{12} bh^{3}.$$



(2) About the base O_2O_2 which is taken as axis of ψ .

$$\mathbf{I} = \int_{a}^{b} b \cdot x^{2} \cdot dx = \frac{1}{3} bh^{3}.$$

(3) About the axis O_8O_8 distant k from O_1O_1 . Take axis O_8O_3 as axis of y. Then limits are

$$k \pm \frac{h}{2} \text{ and } I = \int_{k - \frac{h}{2}}^{k + \frac{h}{2}} x^2 \cdot dx = \frac{1}{3} b \left[x^3 \right]_{k - \frac{h}{2}}^{k + \frac{h}{2}}$$

$$= \frac{1}{3} b \left\{ k^3 + 3k^2 \cdot \frac{h}{2} + 3k \cdot \frac{h^2}{4} + \frac{h^3}{8} - \left(k^3 - 3k^2 \cdot \frac{h}{2} + 3k \cdot \frac{h^2}{4} - \frac{h^3}{8} \right) \right\}$$

$$= \frac{1}{3} b \left\{ \frac{6}{2} k^2 h + \frac{2}{8} h^3 \right\}$$

$$= bk^2 h + \frac{1}{12} bh^8.$$

394. The above result (3) illustrates a very important theorem for the value of I about an axis parallel to an axis through the centre of gravity for any area or volume.

$$bk^2h = \text{area } (bh) \times k^2.$$

 $\frac{1}{12}bh^3 = \text{I about } O_1O_1.$

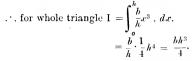
Hence, to find the value of I for an area or mass about any axis O_8O_8 parallel to that O_1O_1 through the centre of gravity—Add to the value of I about O_1O_1 the product of the area (bh), or mass, and the square (k^2) of the distance between the axes

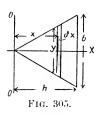
Ex. 2. Find I for a triangle about (1) an axis OO through the apex and parallel to the base: (2) about the base; (3) about an axis through the centre of gravity, Fig. 305.

We have
$$\frac{y}{x} = \frac{b}{h} \cdot \cdot \cdot y = \frac{b}{h} x$$
.

Each element of area = $y \cdot \delta x = \frac{b}{b} x \cdot \delta x$.

and I for each element of area $= \frac{b}{b} x \cdot x^2 \cdot \delta x$.





(2) If OO coincides with the base, then,

taking base as axis of reference we have $\left(\frac{y}{h-x} = \frac{b}{h}\right)$ and

...
$$I = \int_{\bar{h}}^{h} (h-x) x^{2} . dx = \frac{1}{12} bh^{3}.$$

(3) To find I for a triangle when axis passes through the centre of gravity, apply the rule of Art. 394 and show that $I=rac{bh^3}{2a}$.

Ex. 3. To find I for a square about a diagonal, treat as the sum of I for two triangles about their bases.

Let a = side of square; then diagonal = $a\sqrt{2}$, and each triangle has height $\frac{1}{2} a \sqrt{2} = \frac{a}{\sqrt{2}}$, then $I = 2 \left\{ \frac{1}{12} \cdot \sqrt{2} \cdot a \cdot \left(\frac{a}{\sqrt{2}} \right)^3 \right\} = \frac{a^4}{12}$.

Ex. 4. To find I for a circular area about an axis passing through the centre perpendicular to the plane of the area.

Consider the circle divided into concentric rings of width δr and radius r, Fig. 306.

Area of each ring
$$= 2\pi r \cdot \delta r.$$
 I for each ring
$$= 2\pi r^3 \cdot \delta r.$$
 and . · . I for whole area
$$= \int_{\sigma}^{R} 2\pi r^3 \cdot dr$$

$$= 2\pi \cdot \frac{1}{4} R^4$$

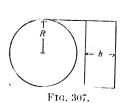
$$= \frac{\pi R^4}{2}.$$



If D = diameter of circle, R = $\frac{D}{2}$ and I = $\frac{\pi D^4}{82}$.

Ex. 5. To find I for a cylinder of mass M lbs. per cubic inchabout the longitudinal axis when cylinder has radius r" and height h", Fig. 307.

Divide into concentric hollow cylinders or rings.



Volume of each ring =
$$2\pi r \cdot h \cdot \delta r$$
.

Mass of each ring = $2\pi r h \cdot M \cdot \delta r$.

= $2\pi r h \cdot M \cdot \delta r$.

= $2\pi h M r^3 \cdot \delta r$.

$$\therefore I = \frac{\pi R^4 h M}{2} \text{ (by analogy to } Ex. 4)$$

$$= \frac{\pi R^2 h M}{2} \cdot R^2$$
= Mass of cylinder $\times \frac{1}{2} \cdot L^2$.

Ex. 6. Find I for a cone about its axis, Fig. 308.

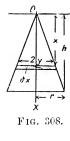
Regard the cone as made up of circular disks, each of radius y, and such that $y = f(x) = \frac{r}{h}x$.

Volume of each disk
$$= \pi y^2 dx = \pi \frac{r^2 x^2}{h^2} \delta r.$$
Regarding each disk as a cylinder,
$$= \frac{\pi y^4}{2} \cdot \delta x$$

$$= \frac{\pi r^4 x^4 \delta x}{2h^4}$$

$$\therefore \text{ I for whole cone } = \int_0^h \frac{\pi r^4 x^4 dx}{2h^4}$$

$$= \frac{\pi r^4}{2n^4} \frac{1}{5} h^5 = \frac{\pi r^4 h}{10}.$$

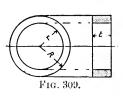


Ex. 7. Find 1 for a hollow cylinder about its axis, e.g., a fly-wheel rim of outer radius R, inner radius r, and thickness t, and of rectangular section, Fig. 309.

Treat as the difference of two cylinders.

For outer cylinder I
$$=\frac{\pi R^4 t}{2}$$

For inner cylinder I $=\frac{\pi r^4 t}{2}$
 \therefore For fly-wheel rim I $=\frac{\pi^4}{2}(R^4 - r^4)$.



395. **Theorem**. The moment of inertia of an area about an axis perpendicular to its plane is equal to the sum of the moments of inertia about two rectangular axes in the plane which axes intersect in the perpendicular axis; thus

Consider any element of area &A, Fig. 310, distant x from OX, y from OY, and r from the origin.

Then I about OY =
$$\delta A \cdot a^2$$

I about OX = $\delta A \cdot y^2$
and I about an axis through O = $\delta A \cdot r^2$.
But $r^2 = (x^2 + y^2)$
 $\cdot \cdot \cdot \delta A \cdot r^2 = \delta A \cdot (x^2 + y^2)$
= $\delta A \cdot x^2 + \delta A \cdot y^2$.
i.e., I about \mathbf{L}^r axis through O = I about

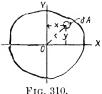
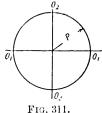


Fig. 310.

This rule is useful for finding I for a circle about its diameter. From p. 473, Ex. 4, we have I for circle about 1 axis through its centre = $\frac{\pi R^4}{2}$. Let I be moment of inertia about diameter O_1O_1 and I_2 about O_2O_2 , Fig. 311.

Then
$$I_1 = I_2$$
 and $I_1 + I_2 = \frac{\pi R^4}{2}$
Hence $I_1 = I_2 = \frac{\pi R^4}{4}$
Or since $R = \frac{D}{2}$, if $D = \text{diameter}$,
$$I_1 = I_2 = \frac{\pi}{4} \left(\frac{D}{2}\right)^4 = \frac{\pi D^4}{64}.$$

OY + I about OX.



If I is the moment of inertia for an area A about any axis, and $I = Ak^2$, the quantity k is colled the radius of gyration for the area about that axis. Similarly for a mass M, if $I = Mk^2$, the quantity k is the radius of gyration of the mass.

396. Pressure on an area immersed in a liquid. The intensity of pressure at any point in a liquid is equal to the weight of unit volume of the liquid × depth, e.g., a cubic foot of water weighs 62.3 lbs., hence the pressure per square foot at a depth of 100 ft. is the weight of 100 cub. ft. of water = 6,230 lbs. If a surface acted upon by the liquid is horizontal, the pressure at every point is the same, and hence the total pressure = intensity of pressure × area of surface. As liquid pressure is normal to the surface immersed, and is transmitted equally in all directions, it follows that the pressure intensity on any surface not horizontal will not be uniform, but will vary with the depth of each part below the free surface of the liquid.

The total pressure on any surface is found as follows:—

Consider a surface ABCD, Fig. 312, immersed with its upper edge at a depth h feet below the free surface, and its lower edge at a depth (h+a) feet. Divide the surface into strips of width δx with edges parallel to the free surface. A strip at depth x has an area $y \cdot \delta x$ (if y represents an ordinate of the area at depth x), and if w is the weight per unit volume of the liquid, the intensity of pressure on this strip is $w \cdot x$. The whole pressure on strip is intensity \times area, and thus $w \cdot x \cdot y \cdot \delta x$. Summing up all such pressures for the whole area we

have $\int_{h}^{h+a} wxy \cdot dx$, as the total pressure. If the immersed surface is

bounded by a curve y = f(x), then total pressure is $\int_{h}^{h+a} w \cdot f(x) \cdot dx$.

Ex. 1. A rectangle of breadth b and length a with upper surface at depth b.

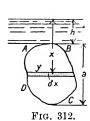
Total pressure =
$$\int_{h}^{h+a} w \cdot x \cdot b \cdot dx = wb \left[\frac{1}{2} x^2 \right]_{h}^{h+a}$$

= $\frac{1}{2} wb \left[h^2 + 2ah + a^2 - h^2 \right]$
= $\frac{1}{2} w \cdot ab \cdot (2h + a)$
= $w \cdot ab \cdot \left(h + \frac{a}{2} \right)$.

The above result shows, for the case of a rectangle, that the area (ab) × pressure at depth of centre of gravity of area $\left\{w\left(h+\frac{a}{2}\right)\right\}$ gives the total pressure on the surface, or that the average pressure is that at the centre of gravity.

This can be shown to be true generally, for, referring to Fig. 312, the total pressure $=\int w \cdot x \cdot y \cdot dx$; the whole area is $=\int y \cdot dx$, and the depth \bar{x} of the centre of gravity of the area below the free surface

is given by $\bar{x} = \frac{\int y \cdot x \cdot dx}{\int y \cdot dx}$. Intensity of pressure at C.G. gravity is $w\bar{x}$,



Hence, whole area
$$f(y) = \int y \cdot dx \times \frac{\int y \cdot x \cdot dx}{\int y \cdot dx} \times w$$

$$= w \int y \cdot x \cdot dx, \text{ which}$$

agrees with the above formula for the total pressure. (The limits will be the same for area, total pressure, and centre of gravity, and hence are omitted.)

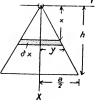
Ex. 2. Find the total pressure on an isosceles triangle having base a, height h, immersed in water with its plane vertical and its apex in the surface of the liquid.

Take the origin at the apex, and axis OX, Fig. 313, along the median of the triangle. Then $y = \frac{a}{2h}x$. (Similar triangles.)

Area of each strip
$$=2y$$
, $\delta x=\frac{2a}{2h}x$, $\delta x=\frac{ax\cdot\delta x}{h}$.
Pressure on each strip $=\frac{wxax\cdot\delta x}{h}$.

Total pressure =
$$\frac{wa}{h} \int_{0}^{h} x^{2} \cdot dx$$

= $\frac{wa}{h} \frac{1}{3} h^{3} = \frac{wah^{2}}{3}$.



[Test.

Depth of C.G. of $\Delta = \frac{2}{3}h$. Area of $\Delta = \frac{1}{2}ah$. Fig. 313.

... total pressure =
$$w \cdot \frac{2}{3} h \cdot \frac{1}{2} \cdot ah = \frac{wah^2}{3}$$
.]

Ex. 3. A reservoir has sides inclined at 30° to the vertical, and has rertical ends, and is 20 ft. deep and 100 ft. wide at the free surface of the liquid. Find the pressure on each end wall.

Fig. 314. Taking OA as axis of y and OX as axis of x, then

e quation of side AB is
$$y = mx + r$$
,
 $r = 50$ and $m = -\tan 30^{\circ}$

$$\therefore y = 50 - \frac{1}{\sqrt{3}} x.$$

Consider a strip at depth x.

Area is
$$y \cdot \delta x = \left(50 - \frac{x}{\sqrt{5}}\right) \delta v$$
.

Pressure is

$$62.3 \times x \times \left(50 - \frac{x}{\sqrt{\delta}}\right) \delta x.$$

Total pressure

$$= 62.3 \int_{0}^{20} \left(50 - \frac{x}{\sqrt{3}} \right) x \cdot dx$$

$$= 62.3 \left\{ \int_{0}^{20} 50x \cdot dx - \frac{1}{\sqrt{3}} \int_{0}^{20} x^{2} \cdot dx \right\}$$

$$= 62.3 \left\{ \left(50 \times \frac{1}{2} \times 20 \times 20 \right) - \left(\frac{1}{\sqrt{3}} \times \frac{1}{3} \times 20^{3} \right) \right\}$$

= 623 (10.000 - 1.540)

= 527.058 lbs. = 235.3 tons.

397. Centre of pressure. If the pressure on each element of an immersed surface be regarded as a force perpendicular to the surface, the forces will be parallel and will vary in magnitude according to the depth of the element. The point at which the resultant of these parallel forces acts is called the centre of pressure. To find its depth below the free surface, we take moments about the line where the plane of the immersed surface intersects the free surface of the liquid.

Consider an immersed surface A, Fig. 315, perpendicular to the free

surface, and let OO be the intersection.

Divide the area A into strips δx in width. Let y be the ordinate at depth x.

Then area of each strip is $y \cdot \delta x$.

Pressure on each strip is $w \cdot x \cdot y \cdot \delta x$.

Moment of this force about OO = $w \cdot x^2 \cdot y \cdot \delta x$.

The sum of these moments = $\int_{-\infty}^{\infty} w \ y \ x^2 dx$.

Let \overline{x} be depth of centre of pressure below OO. Then resultant of forces, if placed at \bar{x} from OO has a moment = sum of moments of separate forces.

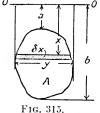
Re ultant of forces = whole pressure on area $\Lambda = \int_0^a w \cdot y \cdot x \cdot dx$

Hence
$$\overline{x} \times \int_a^b w \cdot y \cdot x \cdot dx = \text{moment of re ultant,}$$
and $... \overline{x} = \frac{\int_a^b w \cdot y \cdot x^2 \cdot dx}{\int_a^b w \cdot y \cdot x \cdot dx}$

If the area immersed is bounded by a curve y = f(x) then

$$\frac{1}{a} = \frac{w \int_{a}^{b} f(x) \cdot x^{2} \cdot dx}{w \int_{a}^{b} f(x) \cdot x \cdot dx} = \frac{\text{I for area about intersection of its plane with free surface.}}{\text{Area} \times \text{depth of C.G.}}$$

and if the area is symmetrical about a line perpendicular to the intersection, the centre of pressure lies in this line at a depth x.





Ex. 1. Find the position of the centre of pressure on a rectangular plate of breadth a and length b if the upper edge is at a depth b below the surface, the plane of the rectangle being vertical. Fig. 316. Area of each strip = $u \cdot \delta x$; pressure = $ax \cdot \delta x$; moment about

surface = $ax^2 \cdot \delta x$. Put (b+h) = k, then limits are h and k.

and
$$\bar{x} = \frac{\int_{h}^{k} ax^{2} dx}{\int_{h}^{k} ax dx} = \frac{\left[\frac{1}{3}x^{3}\right]_{h}^{k}}{\left[\frac{1}{2}x^{2}\right]_{h}^{k}} = \frac{\frac{1}{3}(k^{3} - h^{3})}{\frac{1}{2}(k^{2} - h^{2})}$$

If the upper edge is in the free surface, then h = a and k = b and

$$\bar{x} = \frac{\frac{1}{3} b^3}{\frac{1}{2} b^2} = \frac{2}{3} b.$$

Ex. 2. A triangle with apex in surjace of liquid. Fig. 317.

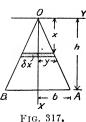
Let side OA have equation y = mx. If base

of triangle is 2b, then $\frac{y}{x} = \frac{b}{b}$ and $y = \frac{b}{b} x$, or

$$m=\frac{b}{\bar{h}}$$

Each strip has area $2y \cdot \delta x = 2mx \cdot \delta x$. Pressure on strip = $2mx^2$. δx , and moment about OY = $2mx^3$. δx

$$\bar{x} = \frac{\int_{0}^{h} 2mx^{3} \cdot dx}{\int_{0}^{h} 2mx^{2} \cdot dx} = \frac{\frac{1}{4}h^{4}}{\frac{1}{3}h^{3}} = \frac{3}{4}h \text{ from O.}$$



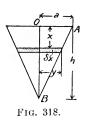
Ex. 3. A triangle with base 2a in surface; height h. Fig. 318.

The equation of side AB of the triangle is $y = a - \frac{a}{L}r$. Each strip

thus has area $2\left(a-\frac{a}{h}x\right) \delta x$; presture $=2\left(a-\frac{a}{h}x\right)x \cdot \delta x$; moment

$$= 2\left(a - \frac{a}{h}x\right)x^2 \cdot \delta x$$

$$\therefore \bar{x} = \frac{2\int_{0}^{h} \left(ax^{2} - \frac{ax^{3}}{h}\right) \cdot dx}{2\int_{0}^{h} \left(ax - \frac{a}{h}x^{2}\right) dx} = \frac{\frac{1}{3}ah^{3} - \frac{1}{4}ah^{3}}{\frac{1}{2}ah^{2} - \frac{1}{3}ah^{2}}$$
$$= \frac{\frac{1}{12}ah^{3}}{\frac{1}{6}ah^{2}} = \frac{1}{2}h.$$



Ex. 4. Find the position of the centre of pressure in the problem Ex. 3, p. 478. Fig. 319.

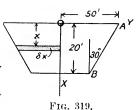
Equation of AB is $y = (50 - \frac{x^2}{\sqrt{2}})$.

Moment of each strip about OA is

$$2 \times 62.3 \left(50 - \frac{x}{\sqrt{3}}\right) \iota^2.\delta x.$$

Total moment $2 \times 62 \cdot 3 \left(50 - \frac{x}{\sqrt{\pi}} \right) x^2 \cdot dx$.

Total pressure =
$$2 \times 62.3 \int_0^{20} \left(50 - \frac{x}{\sqrt{3}}\right) x.dx$$



$$\therefore \quad \bar{x} = \frac{2 \times 62 \cdot 3 \int_{0}^{20} \left(50 \ x^{2} - \frac{x^{3}}{\sqrt{3}}\right) dx}{2 \times 62 \cdot 3 \int_{0}^{20} \left(50 \ x - \frac{x^{2}}{\sqrt{3}}\right) dx} = \frac{\left(\frac{50}{3} \cdot 20^{8} - \frac{20^{4}}{4 \cdot \sqrt{3}}\right)}{\left(\frac{50}{2} \cdot 20^{2} - \frac{20^{8}}{3 \cdot \sqrt{3}}\right)} \\
= \frac{(133,333 - 23,010)}{(10,000 - 1,540)} = \frac{110,323}{8,460} \\
= 13 \text{ ft. from O along OX.}$$

398. A useful theorem for cases in which the upper edge of the area is below the surface of the liquid is:—the depth of the centre of gravity multiplied by the difference between the depths of the centre of gravity and centre of pressure is a constant.

Proof. Let A, Fig. 320, be the area, m the depth of centre of gravity, and n the depth of centre of pressure. Take moments about a line parallel to the surface of liquid, and through the centre of gravity.

The whole pressure on area = A m w (Art. 396). Total moment = A m w (n-m).

Consider a strip of length k, width δx , and distant x from axis through C.G.

Pressure on strip =
$$k \cdot \delta x \cdot w \cdot (m + x)$$

Moment about C.G. = $k \cdot \delta x \cdot w \cdot (m + x) \cdot x$

Total moment = $kw \int (m + x) \cdot x \cdot dx$

= $kwm \int x \cdot dx + kw \cdot \int x^2 \cdot dx$
 $\Rightarrow kwm \int k \cdot x \cdot dx + w \int kx^2 \cdot dx$
 $\therefore A mw \cdot (n - m) = mw \int kxdx + w \int kx^2 \cdot dx$

or $m \cdot (n - m) = m \int kr \cdot dx + \int kx^2 \cdot dx$

P.M.

But $\int kx \cdot dx = \text{moment}$ of whole area about C.G. and this = o, also $\int kx^2 \cdot dx = I$ for area about an axis through C.G. Hence, the whole expression on the right-hand side is constant and $\therefore m (n-m)$ is constant, and $= \frac{I}{A}$, = moment of inertia of area about C.G. \div area, or = the square of the radius of gyration for the area about an axis through the C.G.

In any example, the value of the constant is determined by finding $\frac{I}{A}$, or by finding m (n-m) when the upper edge of the area is in the surface of the liquid, as follows:—

For a triangle having base b and height h, the value $1 \div A$ is $\frac{bh^3}{36} \div \frac{bh}{2} = \frac{1}{18}h^2$, or taking the apex in surface of liquid, the depth m of C.G. is $\frac{2}{3}h$, and the depth n of

C.P. is
$$\frac{3}{4}h$$
. Hence $m(n-m) = \frac{2}{3}h\left(\frac{3}{4}h - \frac{2}{3}h\right) = \frac{1}{18}h^2$.

To find C.P. if the apex is d ft. below the surface, we have $m = \left(d + \frac{2}{3}h\right)$, and if we call (n - m) = x, then

$$m \cdot x = \frac{1}{18}h^2$$

$$x = \frac{1}{18}\frac{h^2}{m} = \frac{1}{18}\frac{h^2}{\left(d + \frac{2}{3}h\right)}.$$

Thus C.G. is below C.P. by the distance x, and knowing x the distance of C.P. below surface is obtained.

A few examples will now be given to illustrate the use of integration in solving problems generally.

399. If the pull in the tight side of a driving-belt is T_1 lbs, and that in the slack side T_2 lbs, the angle of lap of the belt on the pulley θ radians, and μ the exciticent of friction between the belt and vulley, show that $\frac{T_1}{T_2} = e^{\mu\theta}$. Fig. 321.

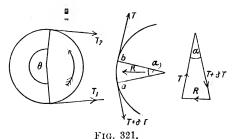
Consider a small length ab of the pulley. The belt on this element of the pulley is kept in equilibrium by forces T, $(T + \delta T)$, and R, where R lbs. = reaction of pulley on belt. A triangle of forces for T, $(T + \delta T)$, and R is shown in the Fig., the angle between T and $(T + \delta T)$ being = that between normals to the pulley at a and b, say a radians.

Since α is small, we may write $R = T\alpha$.

The friction between belt and pulley = $R\mu = T\alpha\mu$ or, when the belt is about to slip on the pulley,

$$\left\{ (T + \delta T) - T) \right\} = \delta T = T \alpha \mu$$

$$\therefore \frac{\delta T}{T} = \alpha \mu.$$



If we now make α indefinitely small, and write it $=d\theta$, δT becomes dT and we have $\frac{dT}{dt}=\mu . d\theta$.

As the angle of lap is θ , we must sum up all the products $\mu d\theta$ between limits θ and θ radians, and also the quantities $\frac{dT}{T}$ from

$$\begin{aligned} & \mathbf{T_2} \text{ to } \mathbf{T_1}, \\ & \text{hence} \int_{\mathbf{T_2}}^{\mathbf{T_1}} \frac{d \, \mathbf{I'}}{\mathbf{T}} = \int_{o}^{\theta} \boldsymbol{\mu} \; . \; d\theta \\ & \left[\log_{\cdot \mathbf{e}} \mathbf{T} \right]_{\mathbf{T_2}}^{\mathbf{T_1}} = \left[\boldsymbol{\mu} \boldsymbol{\theta} \right]_{o}^{\theta} \\ & \text{or } \log_{\cdot \mathbf{e}} \mathbf{T_1} - \log_{\cdot \mathbf{e}} \mathbf{T_2} = \boldsymbol{\mu} \boldsymbol{\theta} \\ & \log_{\cdot} \; \frac{\mathbf{T_1}}{\mathbf{T_2}} = \boldsymbol{\theta} \boldsymbol{\mu} \quad \therefore \frac{\mathbf{T_1}}{\mathbf{T_2}} = \epsilon^{\boldsymbol{\mu} \boldsymbol{\theta}} \end{aligned}$$

400. Torsion of a shaft. To find an expression for the twisting

moment on a shaft. Fig. 322.

Consider first a tube of length l, fixed at one end and acted upon by a turning couple at the other end. Let a line AB turn to AB¹ through an angle ϕ radians. The shearing stress q (i.e., the force shearing the tube per unit area) produced in the tube is proportional to the strain ϕ , and $q = C\phi$, where C is a constant called the Modulus of Rigidity for the material of the tube. If t is the thickness and r the mean radius of the tube, the total twisting moment $T = q \times 2\pi rt$ or (force \times area) = T. At a distance t from the fixed end, a point t moves to t and if t is the centre of the section of the tube a radial line originally at t t moves to t0 moves to t0 moves to t0 moves to t1 through an angle t1 radians.

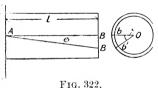
Then $bl^1=ri$, and $bl^1=\mathrm{BB}^1=l\phi$. Thus $l\phi=ri$, and as $\phi=\frac{q}{\mathrm{C}}$

we have
$$i = \frac{l\phi}{r} = \frac{lq}{rC}$$
.

A solid shaft may be considered as made up of a number of tubes, each of radius r and thickness δr . If q is the stress on each tube, the torsional force $=q \cdot 2\pi r \cdot \delta r$, and the twisting moment about the centre is $q \cdot 2\pi r^2 \cdot \delta r$ or (force $\times r$). The total twisting moment is

$$T = \int_0^R q \cdot 2 \pi r^2 \cdot dr \text{ where } R = \text{outer radius of shaft. Now } q \text{ is a}$$

variable depending upon r, but as $i = \frac{lq}{Cr}$ and as i, l, and C are constant



for any section of the shaft, $\frac{q}{r}$ must be a constant, say k, then q = kr. Substituting for q we have

 $T = 2 \pi^{\frac{1}{k}} \int_{0}^{R} r^{3} \cdot dr = 2 \pi k \left[\frac{1}{4} r^{4} \right]_{0}^{R}$ $= \frac{\pi k R^{4}}{2}$

Let f = the shear stress at the surface of the shaft, i.e., at a radius R, then $\frac{f}{R} = k$, and substituting for k we have $T = \frac{\pi f R^3}{2} = \frac{\pi}{16} f D^3$

where D = diameter of shaft. For a hollow shaft, having an inner radius R_1 and outer radius R_2 , the integration is performed from

$$\frac{f}{R_2} \text{ is substituted for } k, T = \frac{\pi f}{2 \, R_2} \left(\, R_2^4 - \, \overline{R}_1^4 \right) = \frac{\pi f}{32 \, D_2} \left(\, D_2^4 - \, \overline{D}_1^4 \right)$$

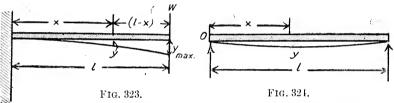
where D_1 and D_2 are respectively the inner and outer diameters of the shaft.

401. To find the deflection at any point of a cantilever loaded at the end.

The curvature of any arc is given by $\frac{1}{R} = \frac{d^2y}{dx^2} \div \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$ (Art. 375). In the case of loaded beams and cantilevers, the deflection is small and hence the curvature is small, so that if y is the deflection at a distance x from one end, $\frac{dy}{dx}$ will be small, and $\left(\frac{dy}{dx}\right)^2$ will

be negligible. We may thus take $\frac{1}{R} = \frac{d^2y}{dx^2}$ for beams and cantilevers. It can also be shown that $\frac{1}{R} = \frac{M}{RR}$, where R is the radius of curvature

of the bent beam, M the bending moment, I the moment of inertia of the sectional area about an axis through the centre of gravity of the section, and E the value of Young's Modulus of Elasticity for the material of the beam.



In the case of a cantilever of length l, Fig. 323, carrying a load W at the end, the value of M* at any point distant x from the fixed end is -W(l-x). Combining the two formulæ for $\frac{1}{R}$ we get

$$\frac{d^2y}{dx^2} = \frac{M}{E1} = -\frac{W}{E1}(l-x).$$

To obtain the deflection y, we must integrate twice, thus obtaining (1) $\frac{dy}{dx}$ and (2) y.

(1)
$$\frac{dy}{dx} = -\frac{W}{EI} \int (l-x) dx$$

= $-\frac{W}{EI} (lx - \frac{1}{2}x^2 + \text{constant}).$

To find the value of the constant, we have $\frac{dy}{dx} = o$ at the fixed end of the cantilever (for the beam is horizontal there), i.e., $\frac{dy}{dx} = o$ when x = o, hence, substituting in equation (1), $o = -\frac{W}{EI}(o - o + \text{const.})$, and $\cdot\cdot\cdot$ constant = o,

^{*} M is + when the forces tend to bend the beam concave upwards, and - when convex upwards.

(2)
$$y = -\frac{W}{EI} \int \left(lx - \frac{1}{2}x^2 \right) dx = -\frac{W}{EI} \left\{ \frac{1}{2}lx^2 - \frac{1}{6}x^3 + \text{const.} \right\}$$

When x = o, y = o (i.e., no deflection at fixed end of beam) \therefore constant = o.

$$\therefore \text{ deflection } y = -\frac{W}{EI} \left\{ \frac{1}{2} lx^2 - \frac{1}{6} x^3 \right\}$$

At the free end of the beam, the deflection is a maximum.

Here
$$x = l$$
 and $\therefore y = -\frac{W}{EI} \left(\frac{1}{2} l^3 - \frac{1}{6} l^3 \right)$
$$= -\frac{W}{3} \frac{l^3}{EI}.$$

The - sign indicates that y is a distance below the axis OX, *i.e.*, the free end of the beam is bent downwards.

402. A beam supported at each end and loaded with w lbs. per foot run. Fig. 324.

In this case, M at any point distant x from o is $\frac{wlx - wx^2}{2}$.

Hence
$$\frac{d^2y}{dx^2} = \frac{1}{\text{EI}} \left\{ \frac{wlx - wx^2}{2} \right\}$$

Integrating (1) $\frac{dy}{dx} = \frac{1}{\text{EI}} \left(\frac{wlx^2}{4} - \frac{wx^3}{6} + \text{const.} \right)$

To find constant, we have $\frac{dy}{dx} = v$ when $x = \frac{l}{2}$ (i.e., the tangent to the bent beam is horizontal at the centre).

Hence
$$o = \frac{1}{\text{EI}} \left\{ \frac{wl^3}{16} - \frac{wl^3}{48} + \text{const.} \right\}$$
 (substituting in Eq. 1)
 $= \frac{1}{\text{EI}} \left\{ \frac{wl^3}{24} + \text{const.} \right\}$... constant $= -\frac{wl^3}{24}$
Thus $\frac{dy}{dx} = \frac{1}{\text{EI}} \left\{ \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24} \right\}$
Integrating again (2) $y = \frac{1}{\text{EI}} \left\{ \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + \text{const.} \right\}$

To find the constant, we have y = v when x = v, i.e., deflection = v at the end of the beam, hence, substituting in Eq. 2,

$$o = \frac{1}{\text{EI}} \left\{ o - o - o + \text{const.} \right\} \quad \therefore \text{ const.} = o$$
and thus $y = \frac{1}{\text{EI}} \left\{ \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} \right\}$

The maximum deflection occurs when $x=\frac{l}{2}$ (at centre of beam), and then $y=\frac{1}{\mathrm{EI}}\left\{\frac{wl^4}{96}-\frac{wl^4}{384}-\frac{wl^4}{48}\right\}=-\frac{wl^4}{\mathrm{EI}}\cdot\frac{5}{384}$, or, if $wl=\mathrm{W}$ (whole load on beam), then $y=-\frac{\mathrm{W}l^3}{\mathrm{EI}}\cdot\frac{5}{384}$.

403. Similarly for a cantilever carrying w lbs. per foot run and of length l, we have $M = -\frac{w (l-x)^2}{2}$ at a point distant x from the fixed end, and $\therefore y = -\frac{1}{8} \frac{wl^4}{\text{EI}}$, or $-\frac{Wl^3}{8 \text{ EI}}$, at the free end; for a beam of length l, having a load W at the centre, we have $M = \frac{Wx}{2}$ at a point distant x from one end, and $\therefore y = -\frac{Wl^3}{48 \text{ EI}}$.

404. To find the time taken to empty a tank or reservoir, or to lower the level by a given amount.

Let the original level be h feet above the discharge orifice; sectional area of reservoir be A sq. ft.; and area of outlet a sq. ft. The coefficient of discharge may be written as m.

Then when head is h feet, velocity of discharge is $\sqrt{2gh}$ per sechence quantity Q discharged in time δt seconds is $(m,a,\sqrt{2gh})$ t) c. ft. The fall in head δh in time δt is—

$$\delta h = \frac{m \ a \sqrt{2 \ gh} \cdot \delta t}{A} \qquad \text{(for A . δh = quantity discharged.)}$$

$$\therefore \delta t = \frac{A \cdot \delta h}{m \ a \sqrt{2 \ g} \cdot \sqrt{h}}.$$

For the head to fall to o, we must integrate from o to h, then time $\int \cdot \, \delta t \, = \, t$

and
$$t = \int_{0}^{h} \frac{A}{m \cdot a \cdot \sqrt{2g}} \cdot \frac{1}{h^{\frac{1}{2}}} \cdot dh$$

$$= \left[\frac{A}{m \cdot a \cdot \sqrt{2g}} \cdot \frac{1}{-\frac{1}{2} + 1} h^{-\frac{1}{2} + 1} \right]_{0}^{h}$$

$$= \frac{2A\sqrt{h}}{m \cdot a \cdot \sqrt{2g}}.$$

If the head is to fall by $(h - h_1)$ ft. then $t = \int_{h_1}^{h} \frac{A}{m \cdot a \cdot \sqrt{2g}} \cdot \frac{1}{h^{\frac{1}{2}}} \cdot dh$

$$= \frac{2 \text{ A}}{m \cdot a \cdot \sqrt{2g}} \left[\sqrt{h} - \sqrt{h_1} \right] \text{ seconds.}$$

From the above result, it is readily seen that the time to empty the tank is double that required to run off the same volume if the head is kept constant, since the latter time is $=\frac{Ah}{m~a~\sqrt{2~gh}}=\frac{A~\sqrt{h}}{m~a~\sqrt{2~g}}$ secs.

Exercises for Chaps. 28 and 29.

Arts. 376 (1) Plot carefully the graph $y^2 = 9x$ from x = 0 to x = 25. Divide the enclosed area into 25 equal strips; call the width of each strip δx . Find the area of each strip, (1) treating it as a rectangle of length = the left-hand ordinate, (2) equal to the right-hand ordinate. Adding the areas of the strips we have two values for whole area: (1) obviously too large, (2) too small.

Again, find the area (3) taking each rectangle of length = its mid-ordinate (4) by Simpson's rule. Compare your results.

Now find the area by integrating $2\int_{0}^{30} 3\sqrt{|x|} dx$.

Ans. Correct value, 500 units.

(2) Initial velocity of a body is 12 ft. per sec., and acceleration is 15 ft. (per sec.) per sec. Find velocity after t secs. Ans. 15t + 12.

velocity after
$$t$$
 sees.

Ans. $15t + 12$.

(3) Evaluate $\int x^{10} \cdot dx$; $\int \sqrt{x} \cdot dx$; $\int \frac{dx}{x^2}$; $\int 3u^2 \cdot du$; $\int \frac{1}{3}z^{\frac{3}{2}} \cdot dz$.

Ans. $\frac{1}{11}x^{11}; \frac{2}{3}x^{\frac{3}{2}}; -\frac{1}{x}; -\frac{1}{2x^2}; \frac{2}{1\cdot55}x^{1\cdot56}; \frac{5}{4}x^4; \frac{at^3}{3};$
 $u^3; \frac{2}{15}z^{\frac{3}{2}}$.

(4) $\int (x+4)^2 \cdot dx$; $\int (3x^2+2x-1) \, dx$; $\int (2t+3t^2+4t^8) \cdot dt$.

Ans. $\left(\frac{1}{3}x^3+4x^2+16x\right)$; (x^3+x^2-x) ; $(t^2+t^3+t^4)$.

(5) $\int \frac{(x^2-a^2)}{x-a} \cdot dx$; $\int \frac{(x+3)^5}{3x} \cdot dx$; $\int \frac{4a(x+2x^3)}{x^2} \cdot dx$; $\int (t+3)(2-t^2) \cdot dt$.

Ans. $\frac{1}{2}x^2+ax$; $\frac{x^3}{9}+\frac{3x}{2}+9x+9\log x$;

4a. $\log_{10} x + 8ax$; $-\frac{1}{4}t^4 - t^3 + t^2 + 6t$.

Art. 385. (6)
$$\int \sin_{1} 3x \cdot dx ; \int \cos_{1} 3x \cdot dx : \int \sin_{1} ax \cdot dx ;$$

$$\int \cos_{1} \theta \cdot d\theta .$$

$$Ans. = -\frac{1}{3} \cos_{1} 3x ; \frac{1}{3} \sin_{1} 3x ; -\frac{1}{a} \cos_{1} ax ;$$

$$\frac{1}{b} \cdot \sin_{1} l \theta .$$
(7)
$$\int \sin_{1} (2x+1) dx : \int \cos_{1} (3x+2) dx ; \int \sin_{1} (2\theta+\epsilon) \cdot d\theta ;$$

$$\int \cos_{1} (nx+t) dx .$$

$$Ans. = -\frac{1}{2} \cos_{1} (2x+4) ; \frac{1}{3} \sin_{1} (3x+2) ; -\frac{1}{2} \cos_{1} (2\theta+\epsilon) .$$
(8)
$$\int (2x+4)^{\frac{1}{2}} dx ; \int (2t+3) dt ; \int e^{\mu\theta} \cdot d\theta .$$

$$Ans. = \frac{1}{3} (2x+4)^{\frac{3}{2}} ; \frac{1}{4} (2t+3)^{\frac{3}{2}} ; \frac{1}{\mu} e^{\mu\theta} .$$
Art. 386. (9)
$$\int \frac{dx}{x^{2}-4} ; \int \frac{dx}{x^{2}+5x+6} ; \int \frac{dx}{x^{2}-x-6} .$$

$$Ans. = \frac{1}{4} \log_{10} \frac{x-2}{x+2} ; \log_{10} \frac{x+2}{x+3} ; \frac{1}{5} \log_{10} \frac{x-2}{x+3} .$$
(10) Show that
$$\int \frac{dx}{x^{2}-3x+2} = \log_{10} \frac{x-2}{x-1} ;$$
and that
$$\int \frac{3}{3+x-2x^{2}} = \frac{3}{5} \log_{10} \frac{x+1}{3-2x} .$$
Art. 387. (11)
$$\int x^{3} \cdot e^{x} \cdot dx ; \int x \cdot e^{2x} \cdot dx ; 2 \int x \cdot \log_{10} x \cdot dx .$$

$$Ans. \quad e^{x} (x^{2}-3x^{2}+6x-6) ; \frac{e^{3x}}{3} (x-\frac{1}{3}) ; \quad x^{2} \left(-\frac{1}{2}+\log_{10} x\right),$$
(12)
$$\int x \cdot \sin_{1} x \cdot dx ; \int x \cdot \cos_{1} x \cdot dx .$$

Ans. $\sin x = x \cdot \cos x$; $\cos x + x \cdot \sin x$

Art. 388. (13)
$$\int \sin^3 x \cdot \cos x \cdot dx$$
; $\int \sin x \cdot \cos x \cdot dx$; $\int \log_{e} x \cdot dx$.
Ans. $\frac{1}{4} \sin^4 x$; $\frac{1}{2} \sin^2 x$; $\frac{1}{2} \left(\log_{e} x \right)^2$.
(14) $\int \frac{\sin x}{1 + \cos x} \cdot dx$; $\int \frac{\sin x}{\cos^4 x} dx$; $\int \frac{\tan x}{\cos^4 x} \cdot dx$.
Ans. $-\log_{e} (1 + \cos x)$; $-\frac{1}{3 \cos^3 x}$; $-\frac{1}{3} \sec^3 x$.

- Art. 389. (15) Find the area bounded by the curve pr = 26,220, when the limits are r = 3.4 and 10.2 c. ft.; p is then lbs. per sq. ft.

 Ans. 28,816.
 - (16) If a gas at pressure p lbs. per sq. ft. and volume r.c. ft. expands according to some given law, then $\int_{r_1}^{r_2} dv$ gives the work done in ft.-lbs., if the gas expands from a volume r_1 to a volume r_2 . For isothermal expansion, the law is $pv = k_1$, and for adiabatic expansion, the law is $pv^2 = k_2$, where γ is the ratio of the two specific heats for the gas, and k_1 and k_2 are different constants. A well-known law in Physics is $\frac{pr}{T} = k$ where T is absolute temperature of the gas and k a constant. Show that for a pound of air at volume r_1 and temperature T_1 expanding adiabatically to volume r_2 and temperature T_2 the

work done is $=\frac{k(\Gamma_1-\Gamma_2)}{(\gamma-1)}$, and that for isothermal expansion the work done is $=p_1r_1\log_{10}R_1$ where $R=r_2|r_1$.

- Art. 390. (17) A parabola $y^2 = 9x$ rotates about its axis. Find the volume swept out if the last ordinate is x = 20.

 Ans. 5.654.9.
 - (18) An ellipse having major and minor axes = 2a, 2b respectively rotates about the major axis. Show that the volume (called a prolate spheroid) generated is $\frac{4}{3}\pi b^2 a$, and that, if it rotates about the minor axis, the volume (called an oblate spheroid) is $\frac{4}{3}\pi a^2 b$.
- Art. 391. (19) Find the C.G. of a parabola $y^2 = 9x$, if the apex is 8 ft. from the base, and also of a triangle having sides 8, 10, and 12 ft., regarding the longest side as the base.

 Ans. (1) 4.8 from 0; 2.2 ft. \mathbf{r} from base.
- Art. 392. (20) Find C.G. of curve $y = \sin x$ between $\liminf_{x \to 1} 0$ and π radians.

 Ans. $\pi/8 = y$.

- A curve xy = 4 rotates about the axis of x. The limiting values of x are 1 and 12. Find the C.G. of the solid generated.
 - Ans. $\bar{x} = 1.62$... C.G. is 0.62 from base of solid.
- (21) Find C.G. for the circular arc and sector which subtends an angle of 50° and has radius 12 ft.

 Ans. 10.5 and 7 ft. from 0.
- Art. 393. (22) Find I for a rod of circular section of radius r.

 length I and density m. about an axis coincident.
 - length l, and density m, about an axis coincident with one end of the rod, and \underline{r} to the longitudinal axis.

 Ans. $\frac{\pi m r^2 l^3}{3}$.
 - (23) In a beam of I section, the flanges are 8" × 3" and the web is 9" × 2". Find the value of I about the neutral axis, and also about the base of the section. Ans. 1,885:5; 2,238.
 - (24) A hollow shaft has an inner diameter of 10" and an outer diameter of 14". Find I for the section of the shaft about the axis of rotation. Ans. 2,789.7.
- Arts. 396 (25) Find the whole pressure and the centre of pressure on to 398.

 a rectangular sluice gate of depth 6 ft. and breadth 4 ft. if the upper edge is 12 ft. below the free surface of water.

 Ans. 22,428 lbs.; 2·4 ins. below centre.
 - (26) A tank has a semicircular cross section and vertical end faces. The diameter of the semicircle is 12 ft. If it is full of water, what is the pressure on each end face? Ans. 8,979 lbs.
 - (27) An elliptical disk is immersed in water with its major axis horizontal and at a depth of 20 ft. below the surface. The semi-axes are 4 and 3 ft. long respectively. Find the total pressure on the disk and the depth of the centre of pressure.
 - Ans. 20.5 tons; 1.35" below centre of ellipse.
 - (28) A submerged pier has a rectangular base 20 ft. by 12 ft., and at the surface of the water has a rectangular section 20 ft. by 8 ft., the pier being 12 ft. high. Find the total pressure on each face of the pier, and the depth of the centre of pressure on each end face.

 Ans. 40.6 tons; 20.03 tons; 8.4 ft.
- Arts. 399 (29) A driving belt passes round $\frac{2}{3}$ of a pulley and the ratio to 403. $T_1/T_2 = 2$. What is the coefficient of friction μ ?
 - (30) If f = 9,000 lbs. per sq. ins., what diameter d of solid shaft will transmit a torque of 5 ft.-tons, and what
 - is the equivalent hollow shaft if $D_1 = \frac{1}{2}D_2$?

 Ans. d = 4.236''; $D_2 = 5.45''$; $D_1 = 2.73''$.

- Arts. 401 (31) Calculate the maximum deflection in a beam 20 ft. to 403. long and $10'' \times 4''$ section supported at each end, and carrying a load of 50 lbs. per foot. Take $E=29\times 10^6$ lbs. per sq. in.

 Ans. 0·112 in.
 - (32) Find the deflection at the free end of a cantilever 12 ft. long carrying 40 lbs. per foot, and having a section $6" \times 3"$. E as in question above.

Ans. 1.37 ins.

- Art. 404. (33) Find the time taken to lower the level by 9 ft. in a reservoir having an area of 3,000 sq. ft., the original depth being 25 ft., and the discharge orifice being 6 sq. ft. in area. Take m as 0.62. Ans. 201-6 secs.
- Miscellaneous. (34) A cable of length l ft. and weighing w lbs. per ft. hangs vertically, and is wound on a drum. Show that the work done in winding the cable completely on the drum is $=\frac{1}{2} w l^2$ ft.-lbs.
 - (35) A rod of length l has a uniform section a but its density varies as the distance from one end. Find an expression for the weight of the rod, the position of the C.G., and also its moment of inertia about the lighter end. (Take the density at any point as mx, where x = distance from lighter end.)

Ans. $\frac{am}{2} l^2$; $\frac{2}{3} l$; $\frac{am}{4} l^4$.

Answers.

Pages 383 and 384.

- (1) 55 mls. per hr.; 81° N. of E. (2) 70°. (3) 5° N. of E.
- (4) 300 yds. down stream; 19.5°; 7.1 mins.
- (5) 51 sea miles; 78° 41′ N. of E. (6) 38 lbs. (7) 26 lbs.
- (8) 15,900 lbs.; 1,500 lbs.; 10,450 lbs.
- (10) 19.5 knots; 17° W. of N.; 0.975 knots; 17° W. of N.
- (11) 36° S. of W.; 34.5 mls. per hr. (12) 22.5 f.s.; 27° W. of S.
- (13) 4·75; 3·7; 3·375; 1·625 tons. (14) 10·46; 8·6 f.s.; (15) €0 6; 60·9 lbs,

Pages 414 and 415.

- (1) (1) 45°; 13.66 ft.; 12.25 ft.
 - (2) 100° 17′; 36° 11′; 43° 22′.
 - (3) 35·4 ft.; 76° 54'; 53° 6'.
 - (4) 13·3 ft.; 25° 6'; 109° 54'.
 - (5) 62° 7′; 72° 53′; 10·8 ft. or 117° 53′; 17° 7′; 3·3 ft.
- (2) Lower triangles, 8.5 ft.; 10 ft.; 4 ft. Upper triangles, 4 ft.; 8.5 ft.; 8.6 ft. Horizontal member, 9.6 ft.
 - (3) = 0.4540; = 0.5446; = 0.7002.
 - (4) 0.788; -0.766; 0.8391; 0.9397; -0.342.
 - (5) 995.6; 615.8 sea miles.
 - (6) EB = BD = 33 ft.; ED = 23.6 ft.; central stay, 30.9 ft.; 76°.
 - (7) 23.6°. (8) 0.8909. (9) 11.02. (10) 40°. (11) 9,912 ft.
 - (12) AC = 1,281 ft.; 73° 36′. (13) 77 ft.; 16·1 ft.; 16°.
- (14) 26°; 36°; 8.77 ft. (15) 622 yds.; two; one; 96.5°; 79°; 11°.
- (16) 1,996 yds. (17) 26.7 ft.; 26.38 ft.; 3.56 radians; for open belt, 3.35 radians and 2.93 radians.
 - (18) 0.74 ft.; 9°. (19) 132°; 0.93 ft. (20) 300 yds.; 13° 14'.

Miscellaneous, p. 416.

- (1) 10¹4; 59¹1 f.s. (2) 29 lbs.; 46 lbs.
- (3) 122 lbs. at E; 98 lbs. at A inclined 88° to horizontal.

Pages 430, 431, and 432.

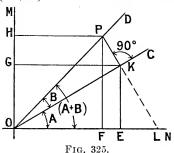
- (1) 60 + 10t; 60 f.s. (2) 48 f.s. (3) 8b.
- (6) a = 1.24; b = 1.123; n = 1.85; $\frac{dy}{dx} = 3.7$.
- (7) 34,000 e.ins.; $\frac{\delta v}{\delta \cdot v} = 300$. (9) 240 + 12 m; 240 f.s.
- (10) 0.046 hr. per day; Feb.—Mar.; 0.069 hrs. per day.
- (11) 0.042 f.s.; 0.42 f.s.s.

APPENDIX I.

THE following trigonometrical formulæ will be found useful:-

(1) Addition Formulæ, giving the trigonometrical ratios for the sum, and difference, of two angles, in terms of the ratios for the separate angles.

I. \sin . $(A + B) = \sin$. A. \cos . B + \cos . A. \sin . B. II. \cos . $(A + B) = \cos$. A. \cos . B - \sin . A. \sin B.



Proof.

Let angle NOC = A and angle DOC = B, then angle NOD = (A + B). Take any point P in OD, and draw PK $\underline{\Gamma}$ OC; PH and KG are $\underline{\Gamma}$ OM, and PF and KE $\underline{\Gamma}$ ON. Angle PKG = PLO = $(90^{\circ} - A)$.

$$OH = OG + GH$$
, also $OH = PF$ and $OG = KE$.
 $OP.\sin.(A + B) = OK \sin. A + PK. \sin. (90 - A)$.
 $= (OP.\cos. B) \sin. A + (OP.\sin. B)$. cos. A.
(Art. 134.)

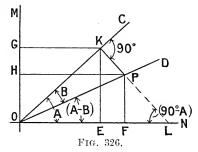
 \div by OP. \sin (A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B. Again OF = OE - EF.

$$OP \cdot cos. (A + B) = OK \cdot cos. A - PK \cdot cos. (90 - A).$$

= $(OP \cdot cos. B) \cdot cos. A - OP \cdot sin. B \cdot sin. A.$

sin. 30.

÷ by OP. cos. (A + B) = cos. A cos. B - sin. A. sin. B. For example, sin. 75° = sin. $(45^{\circ} + 36^{\circ})$ = sin. 45 cos. $30 + \cos$. 45 sin. 30. cos. $75 = \cos$. $(45^{\circ} + 30^{\circ}) = \cos$. 45. cos. $30 - \sin$. 45. III. $\sin \cdot (A - B) = \sin \cdot A \cdot \cos \cdot B - \cos \cdot A \cdot \sin \cdot B$. IV. $\cos \cdot (A - B) = \cos \cdot A \cdot \cos \cdot B + \sin \cdot A \cdot \sin \cdot B$.



Angle NOD now = (A - B). PK is drawn from any point in OD $\underline{\mathbf{r}}$ to OC. Angle HPK = OLK = $(90^{\circ} - A)$. OH = OG - GH.

OP .
$$\sin \cdot (A - B) = OK \sin \cdot A - PK \sin \cdot (90 - A)$$
.
= $(OP \cdot \cos \cdot B) \sin \cdot A - PK \sin \cdot A - B$

 $\begin{array}{c} \div \text{ OP sin. B) sin. (90 - A).} \\ \sin. (90 - A) \\ = \cos. A. \end{array}$ sin. $(A - B) = \sin. A \cos. B - \cos. A \sin. B.$

Again
$$OF = OE + EF$$
.

OP.
$$\cos (A - B) = OK \cdot \cos A + PK \cdot \cos (90 - A)$$
.

 $= (OP \cdot \cos. B) \cos. A + OP \cdot \sin. B \cdot \sin A.$ $\div \text{ by } OP \cdot \cos. (A - B) = \cos. A \cos. B + \sin. A \cdot \sin. B.$ For example,

 $\sin 15^{\circ} = \sin (45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30.$ $\cos 15 = \cos (45 - 30) = \cos 45 \cos 30 + \sin 45 \sin 30.$

Corresponding formulæ for tan. (A+B) and tan. (A-B) are derived from the above formulæ as follows:—

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cdot \cos B + \cos A \cdot \sin B}{\cos A \cdot \cos B}$$

$$\frac{\sin A \cdot \cos B + \cos A \cdot \sin B}{\cos A \cdot \cos B}$$

$$\frac{\sin A \cdot \cos B}{\cos A \cdot \cos B} + \frac{\cos A \cdot \sin B}{\cos A \cdot \cos B}$$

$$\frac{\cos A \cdot \cos B}{\cos A \cdot \cos B} = \frac{\sin A \cdot \cos B}{\cos A \cdot \cos B}$$

$$\frac{\cos A \cdot \cos B}{\cos A \cdot \cos B} = \frac{\sin A \cdot \cos B}{\cos A \cdot \cos B}$$

$$V. : \tan. (A + B) = \frac{\tan. A + \tan. B}{1 - \tan. A \cdot \tan. B}.$$

Similarly, VI.
$$tan. (A - B) = \frac{tan. A - tan. B}{1 + tan. A \cdot tan. B}$$

From formulæ I., II., and V. we obtain, by making B = A, the following formulæ:—

$$\sin 2 A = \sin (A + A) = \sin A \cdot \cos A + \cos A \cdot \sin A$$

VII. \therefore sin. 2 A = 2 sin. A. cos. A. also, cos. 2 A = cos. (A + A) = cos. A. cos. A - sin. A. sin. A

VIII.
$$\therefore \cos 2 \mathbf{A} = \cos^2 \mathbf{A} - \sin^2 \mathbf{A}$$
.
and, $\tan 2 \mathbf{A} = \tan (\mathbf{A} + \mathbf{A}) = \frac{\tan \mathbf{A} + \tan \mathbf{A}}{1 - \tan \mathbf{A} \cdot \tan \mathbf{A}}$.

$$IX. \ \ \therefore \ \tan \cdot 2A \equiv \frac{2 \ \tan \cdot A}{1 \ - \ \tan^{\cdot 2} A.}$$

By writing an angle A as $\left(\frac{A}{2} + \frac{A}{2}\right)$, we derive

$$X. \sin A = \sin \left(2 \cdot \frac{A}{2}\right) = 2 \sin \frac{A}{2} \cdot \cos \frac{A}{2}$$

XI.
$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

 $2 \tan \frac{A}{2}$

XII.
$$\tan A = \frac{2A}{1 - \tan^2 A}$$

From XI., we derive
$$\cos A = 1 - 2\sin^2\frac{A}{2}$$
, $\left(\text{for } \cos^2\frac{A}{2} = 1 - \sin^2\frac{A}{2} \right)$
or $\cos A = 2\cos^2\frac{A}{2} - 1$, $\left(\text{for } \sin^2\frac{A}{2} = 1 - \cos^2\frac{A}{2} \right)$

From formulæ I. to IV., we obtain useful formulæ, as follows: --

(1)
$$\sin (A + B) = \sin A \cos B + \cos A \sin i$$
.

(2)
$$\sin \cdot (A - B) = \sin \cdot A \cos \cdot B - \cos \cdot A \cdot \sin \cdot B$$
.

Adding (1) and (2).

(3)
$$\sin \cdot (A + B) + \sin \cdot (A - B) = 2 \sin \cdot A \cdot \cos \cdot B$$
.

Subtracting (2) from (1).

(4)
$$\sin (A + B) - \sin (A - B) = 2 \cos A \cdot \sin B$$
.

Now put
$$(A + B) = C$$
 and $(A - B) = D$, then $A = \frac{C + D}{2}$ and $B = \frac{C - D}{2}$

Substitute these values in (3) and (4), and we obtain

XIII.
$$\sin C + \sin D = 2 \sin C + \frac{D}{2} \cos C - \frac{D}{2}$$
 (from 3)).

XIV.
$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2} (from (4)).$$

Again, (5) eos.
$$(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$
.
(6) $\cos (A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$.

Adding (5) and (6).
(7)
$$\cos. (A + B) + \cos. (A - B) = 2 \cos. A$$
. eos. B.

(1) $\cos(A + B) + \cos(A - B) = 2 \cos A \cdot \cos B$. Subtracting (6) from (5). (8) $\cos(A + B) - \cos(A - B) = -2 \sin A \cdot \sin B$

Substituting

XV.
$$\cos C + \cos D = 2 \cos C \cdot \frac{C+D}{2} \cos C \cdot \frac{C-D}{2} (from (7)).$$

XVI. cos.
$$C - \cos D = -2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}$$
 (from (8)).

APPENDIX II.

MISCELLANEOUS EXERCISES.

CHAPTER XX.

(1) A piece of land has one straight edge 12 chains long, and a curved edge given by end ordinates of 3 and 2 chains respectively and equidistant intermediate ordinates of 5, 7, 9, 6, and 4 chains. Find the area.

Answer. 68.6 sq. chains.

(2) A rectangular plate 3 ft. by 2 ft. has a hole punched out of elliptical shape with major and minor axes respectively 6 and 4 ins. The major axis is parallel to and midway between the long sides of the plate. The centre of the ellipse is 9 ins. from the centre of the plate. Find the position of the centre of gravity of the remainder.

Answer. 0.2 in, from centre.

CHAPTER XXIII.

(3) The area of the square on the diagonal of a rectangle is 25 sq. ft. and the area of the rectangle is 12 sq. ft. What are the lengths of the sides?

Answer. 4; 3 ft.

(4) The sides of a right-angled triangle are in the ratio of 5 to 2, and the area of the square on the hypotenuse is 145 sq. ft. Find the length of each side of the triangle.

Answer. 4.47; 11.17 ft.

(5) If $y = a + bx^{3/2}$, and if y = 1.62 when x = 1, and y = 5.32 when x = 4, find a and b.

Answer, 1:0915; 0:5205.

(6) If $y = ax^{3/2} - bx^{5/2}$, and if y = 5.82 when x = 0.51, and if y = 32.10 when x = 0.98, find a and b. (B.E., 1907.)

Answer, a = -0.26; b = -34.04.

(7) Because of centrifugal force due to want of balance of a wheel, the stress y in a certain shaft when rotating n times per second is $y = \frac{0.026 n^2}{1 - 0.012 h^2}$. What is the critical speed of the shaft (i.e., the speed for which y becomes indefinitely great)? (B.E., 1907.) Answer. 9.1 revs. per sec.

CHAPTER XXIV.

(8) A steel cable 460 ft. long connects two points which are 450 ft. apart and the same height above sea level. A trolley loaded to 1 ton passes along the cable. Find the stress in the cable when the trolley is at the centre.

Answer. 2.75 tons.

(9) Five forces of 10, 15, 20, 18, and 14 lbs. respectively act at a point, and the angles between the forces are 30°, 40°, 80°, and 120°. Find the magnitude and direction of the resultant.

Answer. 25.75 lbs.; 56° with force 10.

(10) In a small crane the jib is 20 ft. long, the tie-rod 12 ft., and the vertical post 14 ft. A weight of 2 tons is supported by the crane, the rope passing down the jib. Find the stress in the jib and tie.

Answer. 4.9 tons; 1.9 tons.

(11) In a direct-acting steam engine, the connecting-rod is equal in length to four times the crank, and has turned through 45° clockwise from its inner dead centre. A force of 12,200 lbs. acts on the piston. Find graphically the thrust in the connecting-rod, and also the force acting upon the crosshead slide bar. (B.E., 1905.)

Answer. 12,400 lbs.; 1,062 lbs.

(12) In the previous example, determine the force tangential to the crank-pin circle centre O, for angles of 0° , 30° , 60° , . . . 180° from the inner dead centre. Mark off to scale, along the crank, distances measured from O proportional to the tangential force for the position of the crank. Join the points thus obtained by a fair curve. Verify that the radius vector of this curve, in any position, gives the tangential force for the same position of the crank.

(13) Draw any co-planar system of five non-concurrent forces; state what conditions are necessary for this system to be in equilibrium. Illustrate your answer by actually drawing the necessary diagrams for a beam supported at each end and carrying three

loads at intermediate points.

(14) A cyclist travels due south at 12 miles per hour and the wind actually blows from the south-east at 5 miles per hour. What is the apparent velocity and direction of the wind to the cyclist?

Answer. 16 m.p.h. 13° N. of E.

(15) A cyclist travels due south at a certain speed. The wind blows from the N.W. at 5 miles per hour. At what speed must the cyclist travel (a) if the apparent direction of the wind is to be from the W.; (b) if from the S.W.? What is the apparent velocity of the wind in each case?

Answer. (a) 3.5; (b) 7 m.p.h. Wind, 3.5; 5 m.p.h.

(16) In a hailstorm, the particles of hail travel vertically at a velocity of 50 miles per hour, and a railway train passes through the storm at 60 miles per hour due West. What is the apparent velocity and direction of the particles of hail to the passengers in the train?

Answer. 77 m.p.h. 41° to horizontal.

CHAPTER XXV.

(17) A hill summit subtends an angle of 60° at a point A and an angle of 35° at B, A and B being in the same horizontal plane and 700 ft. apart. What is the height of the hill?

Answer. 823 ft.

(18) A triangle ABC has sides c = 5 ins. and angles A and C respectively 35° and 40°. Find the sides a, b, and the area.

Answer. 4:46 ins.; 7:51 ins.; 10:74 sq. ins.

(19) Is it possible to construct a triangle having $\Lambda = 50^{\circ}$, a = 15,

b = 20. Give reasons for your answer.

(20) Three towns ABC are so situated that the distance from A to B is 12 miles, B to C 18 miles, and C to A 14 miles. If BC is due east, find the bearing of A relatively to B and C, and the area enclosed by the triangle ABC.

Answer. A is 51° N. of E. from B and 138 N. of E. from C. Area

84 sq. miles.

(21) If a, b and c are the three sides of a triangle, and $s = \frac{1}{2}(a+b+c)$ it can be shown that $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ and

that cos. $\frac{A}{2} = \sqrt{\frac{s(s-a)}{hc}}$. Find angle A when a = 6, b = 5, c = 3. Answer. $A = 96^{\circ}$ 6'.

(22) At two points A and B on a river bank ½ ml. apart, a point C on the opposite bank is seen at angles of 50° and 75° respectively. Find the width of the river.

Answer. 795 yds.

(23) An observer at a point A sees a balloon at an elevation of 66°, and a second observer at B sees it at an elevation of 75°. A, B, and balloon are in the same vertical plane. Make a diagram showing two possible positions for B, and find the distance of B from A in each position.

Answer. 161 ft., 429 ft.

(24) Two houses A and B are situated on a straight road running E. and W. From a point C on a parallel road, one mile from first road, the angles are 75° to A and 65° to B. Find the distances AC, BC, and AB.

Answer. 1823, 1942, 1293 yds.

(25) A rectangular reservoir ABCD has one short side AB 60 ft. long and accessible, and the remaining sides are inaccessible. A point E at right-angles to the centre of the long side AD is seen from A at an angle of 60°, and from B at an angle of 30°. Find the length of the reservoir, and the perpendicular distance of E from the side AD.

Answer. 103.9 ft.; 30 ft.

EXAMINATION PAPER.

BOARD OF EDUCATION.

PRACTICAL MATHEMATICS.

LOWER EXAMINATION (1912).

- 1. The four parts (a), (b), (c), and (d) must all be answered to get full marks :—
 - (a) Without using logarithms compute by contracted methods

$$3.207 \times 0.01342 \div 9.415$$
.

- (b) Using logarithms compute the square root of $62.41 \times 0.1352 \div 2.416$.
- (c) State the values of the sine, cosine, and tangent of 230°.
- (d) State the value of the Napierian logarithm of 13520.
- 2. The three parts (a), (b), and (r) must all be answered to get full marks:—
 - (a) A hollow circular cylinder of cast iron is 10 inches long and 3 inches inside diameter; what is the outside d'ameter if the cylinder weighs 30 lb.? [One cubic inch of cast iron weighs 0.26 lb.]
 - (b) ABC is a right-angled triangle, C being the right angle. If AC is 4 inches and the angle A is 40°, find BC and the area of the triangle.
 - (e) If x is 1.201, find $\frac{1}{2}(e^x + e^{-x})$.
- 3. The three parts (a), (b) and (r) must all be answered to get full marks:—
 - (a) The sum of x and y is 5:17 and the sum of their squares is 14:25, find x and y.
 - (b) What is the area of the curved surface of a right cone if its base is 3 inches in diameter and vertical height 5 inches?

- (c) There are two perfectly similar statues of marble; the height of one is 2.13 times the height of the other; the smaller weighs 20 lbs. What is the weight of the other?
- 4. If $y = x^2 3.39x + 1.95$, for values of x from 0 to 3, plot sufficient points of the curve on squared paper to show for what values of x, y is 0. What are these values of x?
- 5. When the pointer of a planimeter is guided once round the boundary line of a plane figure, the reading of the instrument R is such that the area A is CR, where C is some constant.

If R is 22.48 for a circle of 3 inch radius, what is C, the area being required in square inches? On applying the instrument now to an indicator diagram, R is found to be 3.77, what is the area? The length of the diagram being 4.11 inches, what is its average breadth?

6. A disc whose outside radius is r_0 and inside radius r_1 is rotating; the radial stress P and the hoop stress Q, at any radius r_0 are

$$\begin{split} \mathbf{P} &= r_0{}^2 + r_1{}^2 - r^2 - \frac{r_0{}^2 r_1{}^2}{r^2} \\ \mathbf{Q} &= r_0{}^2 + r_1{}^2 + \frac{r_0{}^2 r_1{}^2}{r^2} - \cdot 538 \ r^2. \end{split}$$

If $r_0 = 10$, $r_1 = 4$, write out the expressions for P and Q. Now calculate the values of P and Q for the following values of r := 4, 6, 8, 10, and show them in two curves.

7. The insulation resistance R of a piece of submarine cable is being measured; it has been charged and the voltage v is diminishing according to the law

$$v = be^{-t/KR}$$
,

where b is some constant, t is the time in seconds; K is known to be 0.8×10^{-6} .

If v is noted to be 30 and in 15 seconds afterwards it is noted to be 26.43, find R.

8. x and y are as tabulated. It is known that

$$u = 5y + 10\frac{dy}{ax}:$$

find u approximately in the middle of each interval. Show y and u as two curves, x being abscissa:—

x	0	-1	•2	-3	•4	*5	*6	•7	·s	•9	1
y	5.000	5.736	6.428	7.071	7:660	8:192	8.660	9.063	9:397	9:659	9.848

u and y need not be plotted to the same scale.

9. A pin at one end of a horizontal lever is 42 inches from the fulcrum. If the lever turns upwards 70°, find

(i.) the length of the path traversed by the pin,

(ii.) its distance from its original position,

(iii.) its height above its original level.

What would these answers have been had the angle turned through been only 7°?

10. There is a probability that if a man stands at so short a distance as d from the muzzle of a gun which discharges a projectile of weight w, his sense of hearing will be hurt. If d is proportional to the sixth root of w and if

d is 10 feet for the discharge of a 64 lb. shot, what is d for the discharge of a 9 lb. shot?

11. A square whose side is 4 inches has one diagonal parallel to an axis which lies in the plane of the square at the distance of 3 inches from the diagonal. The square revolves about the axis and generates a ring. What are the

volume and area of the ring?

12. The ends of a round barrel are 40 inches in diameter and the mid section is 48 inches in diameter: the barrel is 60 inches long. What is its volume? For what shape of barrel is your approximate rule quite exact?

13. A ship going at 21 knots changes its direction steadily from due north to north-west in two minutes: what is the

radius of its path?

14. The lengths of the intercepts of a plane on the three mutually perpendicular axes are OA = 3, OB = 4, OC = 5. Find the length OP of the perpendicular from the origin on the plane.

APPENDIX III.

The following Tables are those supplied to candidates at examinations conducted by the Board of Education.

EXAMINATION TABLES.

USEFUL CONSTANTS.

1 inch = 25.4 millimetres.

1 gallon = '1604 cubic foot = 10 lb, of water at 62° F.

1 knot = 6080 feet per hour = 1 Nautical mile per hour.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453.6 grammes.

1 cubic foot of water weighs 62.3 lb.

1 cubic foot of air at 0° C. and 1 atmosphere, weighs .0807 lb.

1 cubic foot of hydrogen at 0° C. and 1 atmosphere, weighs .00559 lb.

1 foot-pound = 1.3562×10^7 ergs.

1 horse-power-hour = $33,000 \times 60$ foot-pounds.

1 electrical unit = 1,000 watt-hours.

Joule's equivalent to suit Regnault's II, is $\begin{cases} 774 & \text{ft.-lb.} = 1 \text{ Fah. unit.} \\ 1,393 & \text{ft.-lb.} = 1 \text{ Cent. unit.} \end{cases}$ 1 horse-power = 33,000 foot-pounds per minute = 746 watts.

Volts \times ampères = watts.

1 atmosphere = 14.7 lb. per square inch = 2.116 lb. per square foot = 760 mm. of mercury = 106 dynes per square centimetre nearly.

A column of water 2.3 ft. high corresponds to a pressure of 1 lb. per square inch.

Absolute temp., $t = \theta^{\circ} C. + 273^{\circ} \text{ or } \theta^{\circ} F. + 460^{\circ}$.

Regnault's II = $606.5 + .305 \, \theta^{\circ} \, \text{C.} = 1,082 + .305 \, \theta^{\circ} \, \text{F.}$ $p u^{1.0646} = 479.$

 $\log_{10} p = 6.1007 - \frac{B}{I} - \frac{C}{I^2}$

where $\log_{10}B = 3.1812$, $\log_{10}C_{10} = 5.0882$.

p is in pounds per square inch, t is absolute temperature Centigrade, u is the volume in cubic feet per pound of steam.

 $\pi = 3.1416$.

One radian = 57.3 degrees.

To convert common into Napierian logarithms, multiply by 2:3026,

The base of the Napierian logarithms is e = 2.7183.

The value of g at London = 32 182 ft. per second per second.

LOGARITHMS.

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10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	056 9	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2310	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
72	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3781	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	392 7	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
81	4914	4928	4912	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
83	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551		5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670		5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	P 9 10
41 42 43	6128 6232 6335	6138 6243 6345	6149 62 5 3 6355	6160 6263 6365	6170 6274 6375	6180 6284 6385	6191 6294 6395	6201 6304 6405	$6212 \\ 6314 \\ 6415$	6222 6325 6125	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	4 5 6 4 5 6 4 5 6	7 8 9 7 8 9 7 8 9
44	6435	6414	6454	6464	6474	6484	6493	6503	6513	6522	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	4 5 6	7 8 9
45	6532	6512	6551	6561	6571	6580	6590	6599	6609	6618		4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712		4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8
60	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 E
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
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LOGARITHMS.

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65	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56 57 88	7482 7559 7634	7490 7566 7642	7497 7574 7649	7505 7582 7657	7513 7589 7664	7520 7597 7672	7528 7504 7679	7536 7612 7686	7513 7619 7694	7551 7627 7701	1 1 1	2	2 2 2	3 3 3	4 4	5 5 1	5 5 5	6 6 6	7 7 7
69 60 61	7709 7782 7853	7716 7789 7860	7723 7796 7868	7731 7803 7875	7738 7810 7882	7745 7818 7889	7752 7825 7896	7760 7832 7903	7767 7839 7910	7774 7846 7917	1 1 1	1	2 2 2	3 3 3	4 4	4 4	5 5 5	6 6 6	7 6
63 64	7924 7993 8062	7931 8000 8069	7938 8007 8075	7945 8014 8082	7952 8021 8089	7959 8028 8096	7966 8035 8102	7973 8011 8109	7980 8048 8116	7987 8055 8122	1	1 1 1	2 2 2	3 3 3	3 3	4 4	5 5 5	6 5 5	6 6 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66 67 68	8195 8261 8325	8202 8267 8331	8209 8274 8338	8215 8280 8344	8222 8287 8351	8228 8293 8357	8235 8299 8363	8241 8306 8370	8248 8312 8376	8254 8319 8382	1	1 1 1	2 2 2	3 3 3	3 3	4 4 4	5 5 4	5 5 5	6 6
69 70 71	8388 8451 8513	8395 8457 8519	8401 8463 8525	8407 8470 8531	8114 8476 8537	8120 8482 8543	8426 8488 8549	8432 8194 8555	8439 8500 8561	8445 8506 8567	1	1 1 1	2 2 2	2 2 2	3 3 3	4 4	4 4 4	5 5 5	6 6 5
72 73 74	8573 8633 8692	8579 8639 8698	8585 8645 8704	8591 8651 8710	8597 8657 8716	8603 8663 8722	8609 8669 8727	861 <i>5</i> 8675 8733	8621 8681 8739	8627 8686 8745	1	1 1 1	2 2 2	2 2 2	3 3	4 4 4	4 4	5 5 5	5 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
78 77 78	8808 8865 8921	8814 8871 8927	8820 8876 8932	8825 8882 8938	8831 8887 8943	8837 8893 8949	8842 8899 8954	8848 8904 8960	8854 8910 8965	8859 8915 8971	ī	1 1 1	$\frac{2}{2}$	2 2 2	3 3	3 3 3	4 4 4	5 4 4	5 5
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82 83 84	9138 9191 9243	9143 9196 9248	9149 9201 9253	9154 9206 9258	9159 9212 9263	9165 9217 9269	9170 9222 9274	9175 9227 9279	9180 9232 9284	9186 9238 9289	1	1 1 1	2 2 2	2 2 2	3 3 3	3 3	4 4 4	4 4 4	5 5 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86 87 88	9345 9395 9445	9350 9400 9450	9355 9105 9455	9360 9410 9460	9365 9415 9465	9370 9420 9469	9375 9125 9474	9380 9430 9479	9385 9435 9484	9390 9440 9489	0	1 1 1	2 1 1	2 2 2	3 2 2	3	4 3 3	4 4	5 4 4
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93 93	9638 9685 9731	9643 9689 9736	9647 9694 9741	9652 9699 9745	9657 9703 9750	9661 9708 9754	9666 9713 9759	9671 971 7 9763	9675 9722 9768	9690 9727 9773	0	1 1 1	1 1 1	2 2 2	2 2 2	3 3 3	3 3 3	4 4	1 1
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96 97 98	9823 9868 9912	9827 9872 9917	9832 9877 9921	9836 9881 9926	9841 9886 9930	9845 9890 9934	9850 9894 9939	9854 9899 9943	9859 9903 9948	9863 9908 9952	0	1 1	1 1 1	2 2 2	2 2 2	3 3 3	3 3 3	4 4	4 4
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·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	011	1	1	2	2	2	3
·11 ·12 ·13	1288 1318 1319	1291 1321 1352	1294 1324 1355	1297 1327 1358	1300 1330 1361	1303 1334 1365	1306 1337 1368	1309 1340 1371	1312 1343 1374	1315 1346 1377	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	1 1 1	2 2 2	2 2 2	2 2 2	2 2 3	333
·14 ·15 ·16	1380 1413 1445	1384 1416 1449	1387 1419 1452	1390 1422 1455	1393 1426 1459	1396 1429 1462	1400 1432 1466	1403 1435 1469	1406 1439 1472	1409 1442 1476	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	1 1 1	2 2 2	2 2 2	2 2 2	3 3 3	333
·17 ·18 ·19	1479 1514 1549	1483 1517 1552	1486 1521 1556	1489 1524 1560	1493 1528 1563	1496 1531 1567	1500 1535 1570	150 3 1528 1574	1507 1542 1578	1510 1545 1581	$ \begin{array}{ccccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} $	1 1 1	2 2 2	2 2 2	2 2 3	3 3 3	333
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1	2	2	3	3	3
·21 ·22 ·23	1622 1660 1698	1626 1663 1702	1629 1667 1706	1633 1671 1710	1637 1675 1714	1641 1679 1718	$^{1644}_{1683}_{1722}$	1648 1687 1726	1652 1690 1730	1656 1694 1734	$ \begin{array}{ccccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} $	2 2 2	2 2 2	2 2 2	3 3	3	3 4
·24 ·25 ·26	1738 1778 1820	1742 1782 1824	1746 1786 1828	1750 1791 1832	1754 1795 1837	1758 1799 1841	1762 1803 1845	1766 1807 1849	1770 1811 1854	1774 1816 1858		2 2 2	2 2 2	2 2 3	3 3 3	3 3 3	444
·27 ·28 ·29	1862 1905 1950	1866 1910 1954	1871 1914 1959	1875 1919 1963	1879 1923 1968	1884 1928 1972	1888 1932 1977	1892 1936 1982	1897 1941 1986	1901 1945 1991	$ \begin{array}{ccccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} $	2 2 2	2 2 2	3 3 3	3 3	3 4 4	4 4 4
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·31 ·32 ·33	2042 2089 2138	2046 2094 2143	2051 2099 2148	2056 2104 2153	2061 2109 2158	2065 2113 2163	2070 2118 2168	2075 2123 2173	2080 2128 2178	2084 2133 2183	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	2 2 2	2 2 2	3 3 3	3 3	4 4	4 4
·34 ·35 ·36	2188 2239 2291	2193 2244 2296	2198 2249 2301	2203 2254 2307	2208 2259 2312	2213 2265 2317	2218 2270 2323	2223 2275 2328	2228 2280 2333	2234 2286 2339	1 1 2 1 1 2 1 1 2	2 2 2	3 3 3	3	4 4	4 4	5 5 5
·37 ·38 ·39	2344 2399 2455	2350 2404 2460	2355 2410 24 66	2360 2415 2472	2366 2421 2477	2371 2427 2483	2377 2432 2489	2382 2438 2495	2388 2443 2500	2393 2449 2506	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	2 2 2	3 3	3 3	4 4	4 5	5 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2	3	4	4	5	5
·41 ·42 ·43	2570 2630 2692	2576 2636 2598	2582 2642 2704	2588 2649 2710	2594 2655 2716	2600 2661 2723	2606 2667 2729	2612 2673 2735	2618 2679 2742	2624 2685 2748	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	2 2 3	3 3 8	4 4	4 4	5 5 5	566
·44 ·45 ·46	2754 2818 2884	2761 2825 2891	2767 2831 2897	2773 2838 2904	2780 2844 2911	2786 2851 2917	2793 2858 2924	2799 2864 2931	2805 2871 2938	2812 2877 2944	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	3 3 3	3 3 3	4 4	4 5 5	5 5 5	666

ANTILOGARITHMS.

3

9 123 4 5 6 7 8 9

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·60	3162	3170	3177	3184	3192	3199	3206	3214	3221	3 228	112	3	4	4	5	6	7
•51 •52 •53	3236 3311 3388	3243 3319 2396	3251 3327 3404	3258 3334 3412	3266 3342 3420	3273 3350 3428	3281 3357 3436	3289 3365 3443	3296 3373 3451	3304 3381 3459	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	3 3 3	4 4	5 5 5	5 5 6	6 6	7 7 7
·54 ·53 ·56	3467 3548 3631	3475 3556 3639	3483 3565 3648	3191 3573 3656	3499 3581 3664	3508 3589 3673	3516 3597 3681	3524 3606 3690	3532 3614 3698	3540 3622 3707	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	3 3 3	4 4	5 5 5	6 6	6 7 7	7 7 8
•57 •58 •59	3715 3802 3890	3724 3811 3899	3733 3819 3908	3741 3828 3917	3750 3837 3 926	3758 3846 39 3 6	3767 3855 3945	3776 3864 3954	3784 3873 3963	3793 3882 3972	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	3 4 4	4 5	5 5 5	6 6 6	7 7 7	8 8 8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	123	4	5	6	6	7	8
·61 ·62 ·63	4074 4169 4266	4083 4178 4276	4093 4188 4285	4102 4198 4295	4111 4207 4305	4121 4217 4315	4130 4227 4325	4140 4236 4335	4150 4246 4345	4159 4256 4355	1 2 3 1 2 3 1 2 3	4 4	5 5 5	6 6 6	7 7 7	8 8 8	9 9
·64 ·65 ·66	4365 4467 4571	4375 4477 4581	4385 4487 4592	4395 4498 4603	4406 4508 4613	4416 4519 4624	4426 4529 4634	4436 4539 4645	4446 4550 4656	4457 4560 4667	1 2 3 1 2 3 1 2 3	4 4	5 5 5	6 6	7 7 7		9 10
·67 ·68 ·69	4677 4786 4898	4688 4797 4909	4699 4808 4920	4710 4819 4932	4721 4831 4943	4732 4842 4955	4742 4853 4966	4753 4864 4977	4764 4875 4989	4775 4887 5000	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	4 5	5 6 6	7 7 7	8 8 8	9 1 9 1 9 1	O.
•70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	124	5	6	7	8	9 1	11
•71 •72 •73	5129 5248 5370	5140 5260 5383	5152 5272 5395	5164 5284 5408	5176 5297 5420	5188 5309 5433	5200 5321 5445	5212 5333 5458	5224 5346 5470	5236 5358 5483	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	5 5 5	6 6 6	7 7 8	9	10 1 10 1	11
·74 ·75 ·76	5495 5623 5754	5508 5636 5768	5521 5649 5781	5534 5662 5 7 94	55 46 56 7 5 5808	5559 5689 5821	5572 5702 5831	5585 5715 5848	5598 5728 5861	5610 5741 5875	1 3 4 1 3 4 1 3 4	5 5 5	6 7 7	8 8 8	9	10 1 10 1	12
·77 ·78 ·79	5888 6026 6166	5902 6039 6180	5916 6053 6194	5929 6067 6209	5943 6081 622 3	595 7 6095 623 7	5970 6109 6252	5984 6124 6266	5998 6138 6281	6012 6152 6295	1 3 4 1 3 4 1 3 4	5 6 6	7 7 7	8 8 9	10 10 10	11 1	13
-80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	134	6	7	9	10	12 1	13
·81 ·32 ·83	6457 6607 6761	6471- 6622 6776	6486 6637 6792	6501 6653 6808	6516 6668 682 3	6531 6683 6839	6546 6699 6855	6561 6714 6871	6577 6730 688 7	6592 6745 6902	2 3 5 2 3 5 2 3 5	6 6 6	8 8 8	9 9	11 11 11	12 1	14
·81 ·85 ·86	6918 7079 7244	6934 7096 7261	6950 7112 7278	6966 7129 7295	6982 7145 7311	6998 7161 7328	7015 7178 7345	7031 7194 7362	7047 7211 7379	7063 7228 7396	2 3 5 2 3 5 2 3 5	6 7 7		10 10 10	11 12 12	13 1	15
·87 ·88 ·89	7413 7586 7762	7430 7603 7780	7447 7621 7798	7464 7638 7816	7482 7656 7834	7499 7674 7852	7516 7691 7870	7534 7709 7889	7551 7727 7907	7568 7745 7925	2 3 5 2 4 5 2 4 5	7 7 7	9	10 11 11	12 12 13	14	16
.80	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2 4 6	7	9	11	13	15	17
·91 ·92 •93	8128 8318 8511	8147 8337 8531	8166 8356 8551	8185 8375 8570	8204 8395 8590	8222 8414 8610	8241 8433 8630	8260 8453 8650	8279 8472 8670	8299 8492 8690	2 4 6 2 4 6 2 4 6		9 10 10		13 14 14	15]	17
•94 •95 •96	8710 8913 9120	8730 8933 9141	8750 8954 9162	8770 8974 9183	8790 8995 9204	8810 9016 9226	8831 9036 9247	8851 9057 9268	8872 9078 9290	8892 9099 9311	2 4 6 2 4 6 2 4 6	8	10 10 11	12	14 15 15	16 I 17 I	19
•97 •98 •99	9333 9550 9772	9354 9572 9795	9376 9594 9817	9397 9615 9840	9419 9638 9863	9441 9661 9886	9462 9683 9908	9484 9705 9931	9506 9727 9954	9528 9750 9977	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	9	11 11 11	13	15 16 16	18 9	20

A	Angle.								
De- grees.	Radians.	Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
0	0	0	0	0	x	1	1.414	1.5708	90°
1	.0175	.017	.0175	.0175	57:2900	:9998	1:402	1.5533	89
2	.0349	035	.0349	0349	28.6363	9994	1.389	1.5359	88
3	0524	.052	0523	0524	19 0811	9986	1:377	1.5184	87
4	.0698	.070	0525	0699	14:3007	19976	1.364	1.5010	86
	0020	1770	0000	.0050	14.9001	2010	1 504	1.9010	80
5	.0873	.087	.0872	.0875	11:4301	-9962	1:351	1.4835	85
6	.1047	.105	1045	.1051	9.5144	.9945	1.338	1.4661	84
7	.1222	.122	1219	1228	8.1443	9925	1.325	1:4486	83
8	.1396	.140	1392	.1405	7:1154	.0903	1.312	1:4312	82
9	1571	157	1564	1584	6.3138	9877	1.209	1.4137	81
10	1745	174	1736	1763	5:6713	9848	1.286	1:3963	80
								ļ	
11	1920	192	1908	1944	5.1446	.9816	1.272	1.3788	79
12	.2094	209	2079	2126	4.7046	.9781	1.259	1:3614	78
13	*2269	*226	2250	.5309	4.3312	'9744	1.532	1:3439	77
14	.2443	.244	*2419	.2493	4.0108	.9703	1.531	1.3265	76
15	2618	261	*2588	2679	3:7321	9659	1:218	1.3090	75
16	2793	*278	2756	2867	3.4874	9613	1.204	1 2915	74
17	·2967	296	2924		3.2709		1.190	1.2741	
18	3142	313	3090	3057	3.0777	19563	1.170		73
19	3142	330	3090	*3249 *3443	2.9042	9511	1·176 1·161	1.2566	72
	- 2910		3236	3443	219042	9455	1.101	1.5395	71
20	*3491	*347	*3420	3640	2:7475	•9397	1.147	1.2217	70
21	*3665	.364	3584	.3839	2.6051	.9336	1.133	1.2043	69
22	.3840	*382	'3746	.4040	2.4751	9272	1.118	1.1868	68
23	.4014	.399	3907	*4245	2.3559	9205	1.104	1.1694	67
24	4189	416	4067	.4452	2.2460	.9135	1.089	1.1519	66
25	.4363	•433	*4226	4663	2:1445	.0063	1.075	1.1345	65
26	*4538	*450	*4384	.1077	2:0503	.0000	1.060	1:1170	64
27		467		·4877		*8988			
28	·4712 ·4887	484	4540	.5095	1:9626	*8910	1.045	1:0096	63
			*4695	5317	1.8807	8829	1.030	1.0821	62
29	*5061	.501	*4848	.5543	1.8040	*8746	1.012	1.0647	61
30	*5236	•518	•5000	.5774	1:7321	.8660	1.000	1:0472	60
31	.5411	.534	.5150	.6009	1.6643	.8572	.985	1:0297	59
32	*5585	551	•5299	.6249	1.6003	*8480	970	1.0153	58
33	.5760	.568	5446	.6494	1.5399	8387	954	9948	57
34	5934	.585	5592	6745	1.4826	8290	939	9774	56
35	*6109	•601	*5736	7002	1:4281	*8192	-923	'9599	55
36	-0300	.010		:7305				10.105	
	16283	.618	.5878	7265	1:3764	8090	.908	19425	54
37	*6458	*d25	6018	7536	1.3270	7986	.892	*9250	53
38	6632	:651	6157	.7813	1.2799	.7880	.877	9076	52
39	*6807	*668	.6293	*8098	1.2349	.7771	·861	.8901	51
40	6981	.684	6428	.8391	1.1918	·7660	.845	·8727	50
41	.7156	.700	.6561	.8693	1.1504	.7547	.829	.8552	49
42	.7330	717	.6691	.9004	1.1106	.7431	.813	.8378	48
43	*7505	.733	6820	9325	1.0724	.7314	.797	8203	47
44	7679	.749	*6947	•9657	1.0355	7193	781	*8029	46
45°	·7854	.765	.7071	1.0000	1 0000	7071	765	•7854	45°
			Cosine.	Co-tangent.	Tangent,	Sine.	Chord.	Radians.	De-
			Coame	Co-tangeno.	Tangent,	cune.	Chora.	Itaans.	grees.
			1		l j	1 1			

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