PRACTICAL

PLANE GEOMETRY

AND

PROJECTION.

Collins' Advanced Science Series.

PRACTICAL

PLANE GEOMETRY

AND

PROJECTION,

FOR

SCIENCE CLASSES, SCHOOLS, AND COLLEGES,

ADAPTED TO MEET THE REQUIREMENTS OF THE

HIGHER STAGES IN THE SCIENCE AND ART DEPARTMENT SYLLABUS.

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VOL. I. TEXT.

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THIS WORK

IS RESPECTFULLY DEDICATED TO

Rev. J. H. Edgar, M.A.,

OF THE ROYAL SCHOOL OF MINES,

IN TOKEN OF GRATEFUL REMEMBRANCE,

BY HIS FORMER PUPIL,

THE AUTHOR.

PREFACE.

THIS work is intended as a text-book for students wishing to gain such a knowledge of the principles of Practical Geometry as shall be useful to them in the drawing office or in the workshop.

The subject, as a study in Engineering Colleges and elsewhere, has of late years received greatly increased attention, and this is especially to be remarked with regard to that part of it termed "Projection" or "Solid Geometry." The intimate relation existing between a thorough knowledge of its principles and an intelligent appreciation of the methods adopted in preparing working drawings must in the future not only increase its importance as an item in the school curriculum, but must also cause ignorance of it in the workshop to be a fatal impediment to success.

The selection of the Problems is the outcome of the Author's class-teaching for many years. Each solution is intended not only to elaborate a method, but also to teach a principle, and, at the same time, to be progressive, and therefore truly *educational.*

In a certain sense, the work is a sequel to the Elementary one, in the same Science Series, by the same writer, but it has also been made complete in itself, and may be read by a "beginner" who is only slightly acquainted with the earlier books of Euclid's Elements.

The general arrangement has been designed to cover the courses of study pursued at the Royal School of Mines, at the Royal Military Academy, and at the Engineering College at Cooper's Hill, as well as to embrace the major portion of the two higher stages in the syllabus of the Science and Art Department.

It was originally arranged by the Publishers that this work should be prepared by the late F. A. BRADLEY, Esq., in conjunction with the present writer. But the untimely and lamented decease of the eminent professor before even the commencement of the early sheets led to the entire duty devolving upon the Author. Under these circumstances, he has endeavoured to follow the plans and suggestions agreed upon originally, and he trusts that what has been considered a help by his own pupils may now be found of assistance in a larger sphere.

His acknowledgments are due for the aid he has derived from the works of Bradley, Edgar, Heather, Pierce, Winter, and other writers on the subject.

And, in conclusion, he desires also to acknowledge most gratefully the valuable assistance he has received in the preparation of the Plates and Figures from his former pupils, G. Goodwin, R. A. Bond, W. Buck, and T. Treeve. Their able help has much conduced to any merit the work may deserve.

H. A.

Islington Science School,
London, N., *August, 1880.*

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PRACTICAL PLANE GEOMETRY AND PROJECTION.

INTRODUCTION.

PRACTICAL GEOMETRY is an outcome of Theoretical Geometry. The former applies to the solutions of its Problems the Theorems which are proved by the latter.

Certain properties belonging to points, lines, figures, and planes have been enunciated and rigidly demonstrated by geometricians. These are to be found mainly in the several books of Euclid. The practical draughtsman takes advantage of them in making his drawings. Practical Geometry is therefore an *Applied Science.*

Hence, writers upon this subject do not often give rigid proofs of their solutions to problems. It is not considered necessary to do so. In fact, many demonstrations are accepted by them (and correctly so), which would scarcely be deemed sufficient by those who admit only such as are based upon the axioms and postulates of Euclid. But it is of course understood that although a practical geometrician may not desire that his proofs shall be of this exact nature, yet there is no real knowledge which he possesses but what is capable of demonstration. For example, there are some constructions practically applied by the draughtsman whose correctness is indisputable, based as they are upon proofs which may be aptly termed of the *common-sense* kind. The construction of the curves which are required in setting out the teeth of wheels may be instanced, and many others will be noticed by the student in the course of his study of the following pages.

In this book, however, reference will be made at times to the propositions of Euclid upon which the solutions depend, and in some cases the student will find—particularly in those problems included under the head of Practical Plane Geometry—that demonstrations of the methods described are given *in extenso*. The reason for this is that the auhor has found in his teaching that where the methods of solution of some of the given problems are only dogmatically dictated to a student, he very often entirely fails to remember them, and always appears bored in striving to appreciate the meaning of the lines.

Practical Geometry is very well divided into two distinct branches:—1, Plane Geometry; and 2, that part known as Solid Geometry, which is better defined as the Science of Projection. The first of these treats upon the solutions of Problems relating to points, lines, and figures, which are in one plane; whilst the latter enables one to represent upon a certain fixed plane or planes the relation of the points, lines, and surfaces of solid bodies having length, breadth, and thickness.

It will be noticed that the aim of the author has been, in the first branch of his subject, to select a series of characteristic Problems, which shall be of a more advanced kind, rather than to give a detailed set which should carry the student from the elementary knowledge of the bisection of a line completely through the generally recognised course. This has been found necessary, first, because the space at command would be otherwise insufficient; and, secondly, because this volume is intended as a sequel to the elementary text-book on the same science, prepared by the same author, and included in the Elementary Science Series issued by the same publishers. Hence, no description of mathematical instruments and their uses (except scales) is given, and Problems upon the bisection of lines and angles, and others of a similar elementary character are omitted. Again, no attempt is made to give hints upon neatness of execution in the manipulation of the drawings. For all these the student is referred to the elementary volume previously noticed.

HINTS TO THE READER.

1. Divisions of the *inch* are given in all cases as decimals.

2. Feet are indicated by *one* dash following the figure, *inches* by two; thus, 8′ 4″ means eight feet four inches.

3. Reference to the accompanying Book of Plates is made thus:—Plate XII., fig. 2; and where "fig. 5," without the word Plate is used, it is intended to apply to the figures included amidst the text in this volume.

4. In the second part (Projection, or Solid Geometry), the lines in the figures, which are called Projectors, are only partially drawn, to avoid complication, but sufficient of them is always retained so that the student may easily trace them throughout.

5. Some of the figures are drawn to smaller scale than the sizes given in the data of the Problems. Where such is the case, the fact is indicated by the side of the figures; thus, "½ scale," etc.

The student, in making his own drawings, should work to the sizes given, as the attempts of a beginner to execute very small drawings generally result in a failure, both in neatness and exactitude.

PART I.—PRACTICAL PLANE GEOMETRY

CHAPTER I.

ON THE STRAIGHT LINE AND ITS PROPORTIONAL DIVISION.

Problem I.

To divide a given line AB into any number (*n*) of equal parts. (Fig. 1.)

At one extremity of the line as A set out a straight line AE at *any* angle, and starting from A, mark off *n* equal distances (of any length) along it, as at 1′, 2′, etc. Join the last of these to the opposite extremity as *n*′ to B, and through each of the other points draw parallels to *n*′B.*

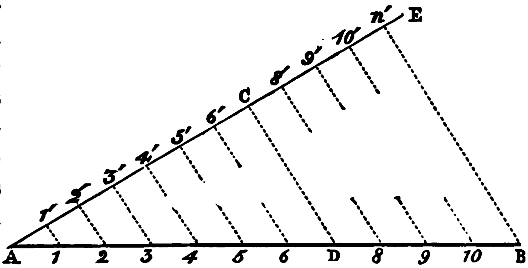

Fig. 1.

Then the line will be divided into *n* equal parts in 1, 2, 3, etc.

This solution depends upon the fact that in similar triangles the corresponding sides are proportional. Hence, the triangle A1′ 1 being similar to A*n*′B, and A1′ being $\frac{1}{n}$ of A*n*′, therefore A1 is also $\frac{1}{n}$ of AB (Euclid vi. 4).

Problem II.

To divide a straight line AB proportionally to a given divided straight line AB′. (Fig. 2.)

* Parallels and perpendiculars are to be set out mechanically by the aid of the Tee and Set Squares.

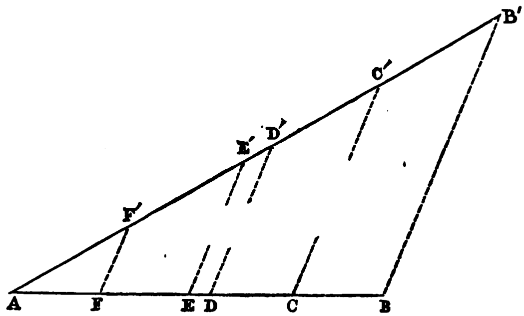

Fig. 2.

At A, in AB, set out the line AB′, with its divisions as given at F′, E′, D′, and C′. Then join B′B. Parallels to this line through the several divisions F′, etc., will divide AB in the required proportion.

The solution of this problem is based upon the same proposition of Euclid as the last.

Problem III.

To find a fourth proportional to three given straight lines A, B, and C. (Fig. 3.)

What is required here is a fourth line *x*, which shall satisfy the proportion, A:B::C:*x* Set out at any angle two indefinite straight lines, as *op*, *oq*. On *oq* measure off the lengths of A and C, as *on* equal to A, and *oz* to C. Then take the length of B, and set it along the other line *op*, as at *om*. Join *mn*, and through *z* draw *zy* parallel to *mn*. Then *oy* is the fourth proportional required. For *omn* and *oyz* are similar triangles; hence, *on* : *oz* : : *om* : *oy*, or A : B : : C : *oy* (Euclid vi. 4).

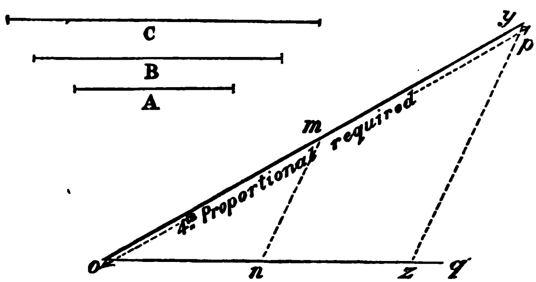

Fig. 3.

If it were desired that the proportion should read C : B : : A to *x*, then *x* would of course be smaller than either given line, and would be determined by joining *z* to *m*, and by drawing a parallel to *mz* through *n*.

Problem IV.

To determine a third proportional to two given straight lines A and B. *(Fig. 4.)*

What is required here is a third line x, such that A : B : : B : x. Putting it in this way, it really resolves itself into another case of the previous problem, and its solution, as shown in fig. 4, is identical with that employed in fig. 3. Thus *op* and *oq* are the indefinite lines along which *on* and *oz* are made equal to A and B, and *oy* also to B. Then the parallel *nm* cuts off *om*, which is the desired third proportional.

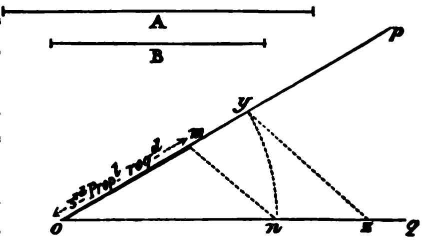

Fig. 4.

The term B is here a *mean* proportional (see Problem V.) between A and the new term found.

Problem V.

Given two straight lines, A and B, to determine (1) their Arithmetic Mean; (2) their Geometric Mean; and (3) their Harmonic Mean. (Figs. 5, 6, and 7.)

(1). The arithmetic mean between two quantities a and b is their average, or their sum divided by 2, and is expressed algebraically as $\frac{a+b}{2}$. Thus, between 6 and 10 it is $\frac{6+10}{2}$, or 8, and the three numbers 6, 8, and 10 form an arithmetical progression, as they proceed by a common difference, 2.

Hence the A. M. between the two lines A and B (fig. 5) is found by placing them in one continuous line and halving their sum. Thus *xy* and *yz* are made equal to A and B respectively, and *ox* is the A. M. required.

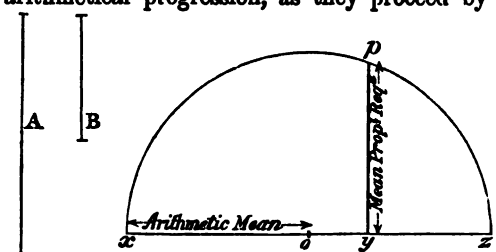

Fig. 5.

(2). The geometrical mean between two quantities a and b is the square root of their product, and is expressed algebraically as $\sqrt{ab}$. Between 4 and 9 it is $\sqrt{4 \times 9}$, or $\sqrt{36}$, or 6, and the three numbers

4, 6, 9 form a geometrical progression, as they proceed by a common ratio, $\frac{3}{2}$; thus, $4 \times \frac{3}{2} = 6$, and $6 \times \frac{3}{2} = 9$. This intermediate term *is generally known in practical geometry as the mean proportional.* It is useful to the draughtsman, as, by finding it, he determines the side of a square equal to a given rectangle. Thus, in the case of the figures quoted, a rectangle 4×9 is equal to a square 6×6.

To determine, geometrically, the mean proportional between A and B (fig. 5), on an indefinite straight line, set off xy and yz equal to A and B respectively. Bisect xz in o, and on it describe a semicircle. At o set out op perpendicular to xy, meeting the semicircle in p. Then py is the desired *mean.*

The proof of the solution depends on the fact that if any two chords of a circle cut one another, the rectangle on the segments of the one is equal to that upon the segments of the other (Euclid iii. 35). For if the whole circle were drawn, and py were produced to meet it in a point q, we should have xy, yz equal to py, qy. But as pq is perpendicular to xy, which passes through the centre of the circle o, therefore $py = qy$ (Euclid iii. 3), and $xy, yz = pq^2$.

(3). Three quantities are said to be in harmonical progression when the *first* is to the *third* as the difference between the *first* and *second* is to the difference between the *second* and *third.* Taking the numbers 10, 12, and 15, for example, they form a H. P.; for $10 : 15 :: 12 - 10 : 15 - 12$, and 12 is termed the harmonic mean between 10 and 15. In algebra this mean between a and b would be written $\frac{2ab}{a + b}$.

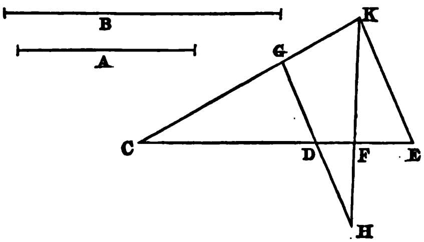

Fig. 6.

To determine this *mean* between two given lines A and B by a *geometrical* construction, proceed as follows (fig. 6):—

Upon any straight line CE mark off CD, CE equal to A and B respectively. Assume any point K, not *in* the line, and join CK, EK. Through D draw GH parallel to EK, and meeting CK in G. Make DH equal to GD, and join KH to intersect CE in F. Then CF is the required harmonic mean.

Proof.—As GDC and KEC are similar triangles:

$$CD : CE :: DG : EK, \text{ or as } DG = DH$$
$$CD : CE :: DH : EK.$$

Again, as KEF and DFH are also similar triangles:

$\therefore$ DH : EK :: DF : FE.

Hence $\frac{CD}{CE}$, $\frac{DH}{EK}$, and $\frac{DF}{FE}$ are equal ratios,

i.e., the first (CD) : the third (CE) :: the difference between the first and second (DF) : the difference between the second and third (FE).

It follows from the proportion CD : CE :: FD : EF (by multiplying extremes and means) that the rectangles CE.DF and CD.EF are equal, and a line is said to be harmonically divided when the rectangle on the *whole* line and the *middle* segment is equal to that upon the two remaining segments.

It is interesting to prove by construction that the G. M. between two lines is also the G. M. between their A. M. and their H. M. If in each of the three cases the two given lines A and B were alike in length, and the A. M. and H. M. being determined, *they* were used as two other given lines whose mean proportional was required, it would by measurement be found that this latter was identical in length with the G. M. between the original pair A and B.

ALGEBRAIC PROOF.—Let $\frac{a+b}{2}$, $\sqrt{ab}$, and $\frac{2ab}{a+b}$, represent the A. M., the G. M., and H. M. respectively.

Then the G. M. between $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$ is the square root of their product, *i.e.*,

$$\sqrt{\frac{a+b}{2} \times \frac{2ab}{a+b}} \text{ is equal to } \sqrt{ab}.$$

There are other methods of construction which may be employed to discover the H. M., one of which is shown in fig. 7, where A and B represent the given terms, which, as before, are arranged in one line along CE (CD = A, and CE = B). On DE a semicircle is described, and CG is drawn to meet this semicircle in *any* point G. Then when G is joined to D, and the angle DGF is made equal to the angle CGD, the point F is discovered, and CF is the required mean.

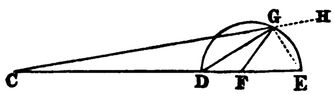

Fig. 7.

PROOF.—This solution depends, first, upon the fact that when the vertical angle of a triangle is bisected by a straight line passing through one of its angular points, that line divides the base, so that the segments are proportional to the other two sides of the triangle; and, secondly, that the line which bisects the exterior angle (formed by producing one of these sides) meets the base produced, so that the whole line and the produced part are also in the same ratio (Euclid vi. 3 and 3*a*).

Now, by our construction, that is exactly what is done, for CGF is a triangle whose vertical angle is bisected by GD. Hence CD : DF :: CG : GF, and by producing CG to H, and joining GE, the exterior angle FGH is bisected. For DGE is a right angle (Euclid iii. 31), and CGD+EGH make one right angle (Euclid i. 13). But CGD=DGF ∴ FGE=EGH. Then CE : EF :: CG : GF. Hence CD : DF :: CE : EF, or the line is harmonically divided.

Problem VI.

To divide a given straight line AB in extreme and mean proportion, or medially. (Fig. 8.)

This problem is identical with the eleventh of the Second Book of Euclid, but the construction usually employed is not the one there adopted, but a modification of it. What is required is to divide the given line in such a manner that a square built upon the greater segment shall be equal to a rectangle made upon the *whole* line and the lesser segment.

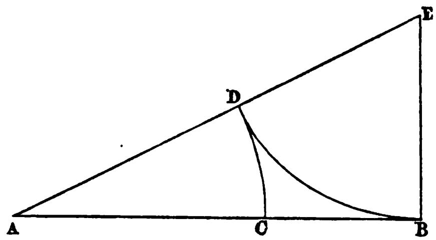

Fig. 8.

At one extremity, as B, of the given line erect a perpendicular BE, equal to *half* the length of AB. Join AE. With E as centre, radius EB, describe the arc BD meeting AE in D. With A as centre, radius AD, describe the arc DC. Then AB will be divided as required, *i.e.*, $AC^2 = AB.BC$.

Proof.—Let the given line be 1 and the greater segment x. Then the lesser segment will equal $1-x$. Then, when divided medially, the following equation will be satisfied:—$x^2 = 1-x$, from which it is found that $x = \frac{\sqrt{5}-1}{2}$ of the whole line (the negative root of the quadratic being inadmissible).

In figure 9, AB=2BE, and $AE^2 = AB^2 + BE^2$ (Euclid i. 47) ∴ AE = $\sqrt{5}$.BE, and AD= $\sqrt{5}$.BE − BE for DE=BE. Hence AD or AC=($\sqrt{5}-1$) BE. But BE = ½AB ∴ AC = $\frac{\sqrt{5}-1}{2}$ AB, which satisfies the equation as shown algebraically.

This problem has one application in the construction of a pentagon, the length of whose diagonal is known, for *the side of the figure is the greater segment of that diagonal medially divided.* Thus, AC (fig. 8) would be the side of a pentagon whose diagonal is AB. (See Problem XXII.)

EXERCISES.

1. A line AB is 4″ long. Divide it into three parts, in the proportion of 2 : 3 : 4.

2. Find by construction the length of $\frac{1}{15}$ of 4·7″

3. Find the fourth term x of the proportion 3″ : 1·7″ :: 4″ : x.

4. Find the term x in the proportion—as 2·5″ : x :: x : 1″. What is the name of this term?

5. The arithmetic mean of two lines is 3″, and the least is 1″ shorter than the greatest. Draw the lines.

6. Divide a line AB, 3·8″ long, in C, and find a point D in it produced so that—

AC : AD :: BC : BD; AC being 2·85″ long.

Hint.—The whole produced line will be harmonically divided, and the problem is really a deduction upon that worked in fig. 6; the given data being sufficient to enable the student, by a line corresponding to KE in that drawing, to find D, which is identical in that case with E.

7. Prove by construction that when a line is divided in extreme and mean proportion the larger segment is a *geometric mean* between the whole line and the lesser segment.

8. The mean proportional between two lines is 3·5″ long, and one of these is ·5″ longer than the other. Determine the two lines.

Hint.—Take any two lines whose difference is ·5″. Find their mean proportional, and, after producing the latter until 3·5″ long, draw a semicircle concentric with the one previously used, and passing through the upper extremity of the produced line.

CHAPTER II.

ON SCALES, PLAIN, DIAGONAL, COMPARATIVE, ETC.

It is very seldom that drawings can be made equal in size to the objects they depict. Sometimes, as in representations of parts of buildings, they are considerably smaller; whilst at other times, as in the drawings of the minute details of the mechanism of a watch, they are larger.

But in all cases it is necessary that every part of the same drawing should bear relatively the same proportion to the size of the corresponding part in the object it represents. To effect this, a scale is used, which consists of a line accurately divided in such a manner as to represent in a smaller space the standards of length used to measure the original object.

As an instance, a line 6″ long, divided into 36 equal parts, may be assumed to represent 36 feet, whilst 3 of the divisions would indicate 1 yard. And a measurement on this scale would be $\frac{1}{72}$ of the equivalent measurement in standard length, for 6″ is $\frac{1}{72}$ of 36 feet. Such a scale would be a *Plain Scale*, and $\frac{1}{72}$ would be its *representative fraction.* It will be seen at once that the expression "6 inches to 36 feet," or "1 inch to 6 feet," is only another way of indicating the same scale as $\frac{1}{72}$.

Scales are of no service unless they be *very* accurate. It is necessary, therefore, that the student should construct them with great care, and also that he should thoroughly test them before using them in making a drawing.

Problem VII.

To construct a plain scale of 7 feet to 1 inch, long enough to measure 40 feet. (Plate I., fig. 1.)

It is not advisable to start with *one* inch and divide it into seven parts, as the work is then so small that inaccuracies are liable to creep in. It is better to determine at once what the total length of the divided line must be. Thus, as our scale is to represent 40 feet, its length must be $\frac{40}{7}$ inches long, which is 5·71″, etc.*

Draw, then, a line of this length, and divide it carefully into four parts. Then each of these will represent 10 feet. Next proceed to divide the extreme left of these into ten equal parts, and *one* of these smaller divisions will indicate 1 foot. By numbering, as shown in the plate, it is rendered unnecessary to minutely divide the other three larger segments. For if 0 be placed at the right hand of the first larger segment, as at A, the tens can be read from left to right, and the units from the same point in the opposite direction. Thus, if it be wished to extract from the scale the length 27 feet, then one leg of the dividers should rest upon 20 and the other upon 7, as from *y* to *x*. It is usual to double the lines, and by the aid of dark and light tints, to give relief to the eye.

PROBLEM VIII.

To show by diagonal division the hundredth part of 1 linear inch. (Fig. 9.)

When very minute division becomes necessary, recourse is had to what is called a Diagonal Scale, the principle of which will be understood by reference to fig. 9, where a line 1 inch long has by its means been divided into 100 parts.

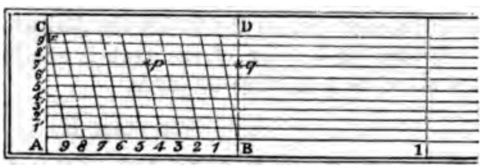

Fig. 9.

To reproduce the drawing proceed as follows:—

First divide the inch accurately into ten parts in the usual manner.

At each extremity of the line erect a perpendicular, and mark off upon one of these ten equal distances (length at pleasure), numbering them as in the figure. Through each of these latter divisions draw horizontal lines, and join C to 9. Then parallels to C9 through each of the divisions of the one-inch line will complete the necessary construction, and $9x$ is $\frac{1}{100}$ of a linear inch. This is evidently true, for CA9 and $C9'x$ are similar triangles, and their homologous sides are therefore proportional. Hence, as $C9'$ is $\frac{1}{10}$ of AC, $9'x$ is $\frac{1}{10}$ of A9, which

* This is quite near enough for general work; and although by taking 1″ at once and dividing by 7 we have not *this* source of inaccuracy, yet the difficulty of execution in that case is likely to lead to an error greater than this one.

is itself $\frac{1}{10}$ of an inch; $9'x$ is then $\frac{1}{10}$ of $\frac{1}{10}$, or $\frac{1}{100}$ of an inch. It must also be noticed that, as the diagonal lines proceed upwards from the base AB, they recede from BD to the amount of one-hundredth of an inch for each horizontal line in succession; hence the length *pq* is 4 tenths added to 7 hundredths, or ·47 of an inch.

NOTE.—Any other minute fraction of a given length can be found by this construction, so long as the denominator of that fraction has two factors. Thus, to obtain $\frac{1}{112}$ it would only be necessary to divide AB into 7, and to use 16 divisions along AC, for $112 = 16 \times 7$.

PROBLEM IX.

To construct a scale of "5 inches to 1 mile" to show (1) furlongs, and (2) lengths of 10 yards, by diagonal division. (Plate I., fig. 2.)

Take a line 5″ long and divide it into eight parts. Then, adding one of these on the left side of the scale, proceed to the second part of the problem.

We have only to show lengths of 10 yards. Now, 10 yards is $\frac{1}{22}$ of a furlong (220 yards). Therefore this added portion must first be divided into two. Then, by the aid of eleven horizontals at equal distances, as shown on the figure, the desired division will be effected. No further explanation of the work is needful. The distance from *x* to *y* would be 4 furlongs 60 yards.

PROBLEM X.

To construct a scale of English yards of $\frac{1}{200}$ and a comparative scale of French metres. (Plate I., fig. 3.)

As different nations adopt different standards of length, and as in the intercourse of trade it is absolutely necessary to use these different standards *in comparison*, the plan is sometimes adopted of making upon a drawing two or more plain scales side by side, upon one of which units of one standard are indicated, and upon the others units of the other standards—each scale, of course, having the same representative fraction.

The problem before us illustrates the principle. In fig. 3, Plate I., an ordinary English scale of yards is shown, the representative fraction being $\frac{1}{200}$. It is long enough to measure 30 yards; hence its length is $\frac{1}{200}$ of 30 yards, or 5·4″.

Beneath (in fig. 4) is a French scale of metres, long enough to *measure* 30 of *them*. To construct this scale it is first necessary to know what length in inches represents $\frac{1}{200}$ of 30 metres. Now a meter is 39·37 inches long, hence 30 metres will measure 39·37 × 30 inches, or 1181·1 inches, and $\frac{1}{200}$ of this will be 5·905 inches, or 5 inches and $\frac{9}{10}$ full.

Take this length from a good scale of inches and tenths, and divide it into three equal parts, each representing 10 metres. Then by dividing the extreme left-hand segment into 10, and numbering as shown, the scale will be complete, and the relation of any length in English yards and French metres can be deduced therefrom.

Problem XI.

To construct a scale showing yards, feet, and inches (by diagonal division), where 7·75 inches represents 31·5 yards; also a comparative scale of paces (1 pace = 32″). (Plate I., figs. 5 and 6.)

Let us assume that the scale is to be long enough to measure 20 yards, the first step is to find what actual length will be required to indicate that amount according to the relation expressed in the problem.

To obtain this, determine the fourth term of the proportion—

As 31·5 yards : 20 yards : : 7·75 inches : x.

This will be 4·92 inches.

Take then this length and divide it into two equal parts, and one of these again into ten, and number as shown in the plate. There it will be noticed that two more yards are added on the left beyond the zero. This makes 22 yards, or 4 poles, or 1 chain. It is not of course necessary, by the wording of the problem, to show this, but it is sometimes advantageous in a scale of the kind to make its length as described. The nearer of these two added yards, being divided into three equal parts, serves to indicate *feet*, and by the aid of twelve equidistant horizontals and diagonals through these three foot-divisions, we are enabled to measure off from our scale, *inches*. Thus, the measurement from x to y represents, according to this scale, 6 yards 2 feet 6 inches. Next, to construct the comparative scale of paces (fig. 6), assume that this scale is to be long enough to measure 25 paces. Then, to discover what length of line is necessary for this purpose, work the proportion—as 31·5 yards : 25 paces : : 7·75 inches. Bringing the first two terms to inches, we have—as 31·5 × 36 : 25 × 32

: : 7·75; or, as 1134 : 800 : : 7·75 : x. Then x will be found to be 5·47 inches. This length, when divided as shown into twenty-five parts, will indicate, as required, *paces* to the same scale as the yards above it.

Problem XII.

How to construct and use the "Scale of Chords." (Fig. 10.)

The "Scale of Chords" is an arrangement by which any required angle can be set out, depending for its principle upon the facts (1) that the chord of an arc which subtends any angle bears a constant ratio to the radius with which that arc is struck; and (2) that the chord of the arc of 60° is equal to the radius.

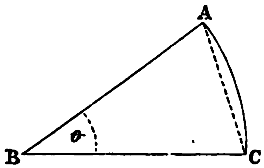

Fig. 10.

Thus, the chord of the angle ABC, as shown in fig. 10, is the straight line AC, and its ratio to BC—the radius—is $\frac{AC}{BC}$; this ratio remaining the same, whatever the length of the radius BC may be.

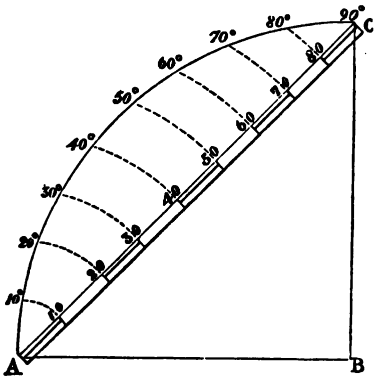

Fig. 11.

1. **To construct the Scale.**—Upon any line AB (fig. 11), describe a quadrant ABC, and divide the curve into nine equal parts, each portion of which subtends therefore 10° at the centre B. Join AC, and with A as centre draw a series of arcs passing through the points of division to intersect the straight line AC, and complete the scale as shown.

The total line AC is clearly the chord of 90°, and the radius with which each arc is struck is the chord of the angle, which is subtended by its relative portion of the quadrant. The student will notice that the divisions on the scale are not equal, but get smaller as the angle approaches 90°. *The chord of* 60° *is equal to the radius,** hence the fact that the radius of a circle steps round the circumference six times exactly.

* For chord $A = 2 \sin \frac{A}{2}$; hence, chord $60° = 2 \sin 30°$, or $2 \times \frac{1}{2}$, or 1.

To use the Scale.—1st. To apply a straight line at any point of a given straight line, so as to include a given angle: for instance, 40°. With the given point as centre, and with a radius equal to the chord of 60°, as shown on the scale, describe an arc, and set off along this arc, measuring, from its intersection with the given straight line, the chord of 40° (from scale). Join the given point and the one determined.

2nd. To discover the number of degrees in a given angle, describe an arc, having its centre at the vertex, and with a radius equal to chord 60°, as taken from the scale. Next, measure the length of the chord intercepted between the legs of the given angle, and determine from the scale the number of degrees required.

THE PROTRACTOR.

This instrument, which in its most common form in the instrument box is a 6″ rectangular piece of boxwood or ivory, is depicted in Plate II., figs. 1 and 2. Around three of its edges on one side (fig. 2) are a series of 180 radiating marks, having their centre at the middle of the fourth unmarked edge. These indicate the degrees of two right angles, and enable one to set out any particular angle desired. They are numbered twice, once from left to right and *vice versa*, for convenience of reference. On the same side the standard inch is divided diagonally, so as to show ·01 of that length (as described in Problem VIII.), and the half-inch is also divided in the same way, so as to show ·005, or $\frac{1}{200}$″. Frequently these two diagonal scales are those of the half-inch and quarter-inch, instead of the two shown in the figure. The reverse side of the protractor (fig. 1) has four plain scales upon it, of 1″, $\frac{3}{4}$″, $\frac{1}{2}$″, and $\frac{1}{4}$″, divided into twelve parts, so that they may be used as representing feet and inches. Thus the uppermost one would avail when a scale of $\frac{1}{4}$″ to 1 foot, or $\frac{1}{48}$, is required, etc.

A scale of chords (see Problem XII.) is also given on the same side. It sometimes occurs that on this same face a series of scales are shown, to which the numbers 20, 25, 30, 35, etc., are attached. To use these, it is necessary to know what the unit in each case is in actual length. This is readily furnished by making a fraction whose numerator is *in every case* 10, and whose denominator is the particular number attached to the particular scale. Thus, for that numbered 35, each principal division is $\frac{10}{35}$, or $\frac{2}{7}$ of a linear inch.

THE SECTOR.

This instrument is depicted in Plate II., fig. 3, and attention is drawn to it here because there are only two scales upon it which are actually of much use in practical geometry, viz., the Line of Lines, marked L in the figure, and the Line of Polygons, marked POL.* The former of these is available when it is required to determine the length of a fourth proportional to three given lines A, B, and C, for which purpose it is used thus :—

The length of one term, as A, is taken by the dividers, and the sector is opened until the distance from L to L on the line of lines is equal to that length.† The next step is to discover, without altering the state of the instrument, where the distance from two *same* numbers upon the scale is equal to the length of B. Suppose this to occur at 8.8. The sector is then reset until the distance from L to L is equal to C. Then the measurement 8.8, under the new condition of opening, gives the fourth term required. The student will see that the principle employed is that of the constant proportion of the sides of similar triangles.

The Line of Polygons enables us to construct upon a given line a regular polygon of any number of sides, or rather, it gives us the radius of the circumscribing circle. Or again, it enables us to determine at once the length of the side of any regular polygon inscribed in a given circle.

For instance, let it be required to construct a regular heptagon of 1″ side. Open the sector until the distance from 7 to 7 is 1″. Then the radius of the circumscribing circle is equal to the distance 6.6 (note that this radius is always taken from 6 to 6), or, in other words, a circle of 6.6 radius being drawn, the chord 1″ will step round it 7 times.

When the circle is given, and the inscribed polygon is to be drawn, the instrument is opened until the distance from 6 to 6 is equal to the given radius. The length of the side is discovered in the distance between those corresponding figures which indicate the number of sides required.

* The instruments usually sold are exceedingly untrustworthy, and the author would consider it better, if possible, for a purchaser to obtain one where *only* these two scales were accurately shown, rather than to have such a confusion of lines upon it as in general is the case.

† This assumes that the term A is greater than B. When this is not the case, *the method* of procedure is reversed, *i.e.*, the term B is taken first.

EXERCISES.

1. Draw a scale for a drawing where 7·5 feet of real magnitude is represented by 1 inch (long enough to measure 50 feet). (May Exam. 1870.)

2. Draw a scale of miles $\frac{1}{633600}$, showing furlongs by the diagonal method. (May Exam. 1870.)

3. A map is constructed so that 1 inch represents 45·5 yards. Draw a scale for it from which paces can be measured (1 pace = 32″). (May Exam. 1873.)

4. On a map a furlong is represented by 1·25″. Draw a scale of *poles* for this map (1 furlong = 40 poles). (May Exam. 1872.)

5. Draw a scale of miles, furlongs, and yards whose representative fraction is $\frac{1}{31680}$, showing 20 yards as the smallest reading. (May Exam. 1872.)

6. Draw a French scale to show metres, $\frac{1}{100}$, and a comparative scale of English yards. (Cooper's Hill Exam. 1876.)

7. Explain the principle of the diagonal division of scales. How should a scale of $\frac{3}{8}$″ = 10 feet be constructed so as to read inches? (May Exam. 1877.)

8. Draw a scale of feet to measure all distances between 70′ and 1′ where $5\frac{1}{2}$ feet is represented by ·52 inches; and, by diagonal division, make this scale available for reading inches. (Cooper's Hill Exam.)

9. Draw a scale of chains $\frac{1}{7920}$ showing poles diagonally. (May Exam. 1874.)

10. The plan of a building is an exact square of 3″ side, the diagonal of which represents 100 feet. Make a scale from which all its other dimensions (in feet only) may be taken. (May Exam. 1875.)

11. Construct a scale of $\frac{1}{72}$ to read decimetres and show 20 metres. A decimetre = 0·328 feet. (May Exam. 1879.)

12. Construct a scale of $\frac{1}{360}$ to show 50 yards. Construct a comparative scale of metres. A decimetre = 0·328 feet. (May Exam. 1878.)

CHAPTER III.*

ON THE CONSTRUCTION OF THE TRIANGLE AND THE REGULAR POLYGONS.

Before commencing the study of this chapter, the reader should recall to his mind the following facts, which will materially assist him in comprehending the solutions of the problems it contains :—

1. A triangle has three sides and three angles, any three of which (except the three angles) being known, the figure is determinable.

2. The three angles together make two right angles (Euclid i. 32).†

3. The altitude of a triangle is the shortest distance between the base and a line parallel to the base drawn through the vertex.

4. The perimeter of any rectilineal figure is the sum of its sides.

5. Similar triangles have equal angles, and their homologous sides are proportional (Euclid vi., Definition 1).

6. A line bisecting any angle of a triangle divides the opposite side into segments, which are in the same ratio as the remaining sides of the figure (Euclid vi. 3).

7. All the internal angles of a regular polygon are equal, and if they be added together, their sum will equal twice as many right angles as the polygon has sides, less four (Euclid i. 32).

Thus, in a pentagon the sum of the degrees of the internal angles is $\left\{(5 \times 2) - 4\right\} 90° = 540°$, and each angle is $\frac{540°}{5}$ or 108°.

8. Lines which bisect and are perpendicular to the sides of any regular polygon meet in *one* point, which is the centre, both of the inscribed and circumscribing circles.

9. One rectilineal figure is said to be inscribed in another when *all* the angular points of the former are contained by the perimeter of the latter.

* The remarks in the latter part of our Introduction apply generally to the problems in this chapter: very elementary cases are omitted.

† From this it follows that the three angles of an equilateral triangle are equal, and each contains $\frac{180°}{3} = 60°$

PROBLEM XIII.

To construct an isosceles triangle, base 1·25″ long, the angles at the base to be double that at the vertex. (Fig. 12.)

Draw the base AB 1·25″ long, and at one extremity A, erect a perpendicular AD, and describe the quadrant BD, radius AB. Divide this quadrant *by trial* into 5 equal parts, and draw the side AC through the fourth division E, counting from the base. The apex C is vertically over the centre of the base, the triangle being isosceles. Join BC. This triangle is a similar one (*i.e.*, it has equal angles) to that formed by joining one corner of a pentagon to the extremities of the side opposite to that corner.

By the solution it is seen that the base angle CAB is $\frac{4}{5}$ of a right angle, and this shows the triangle to be that which is required, for the second base angle and the vertical angle, when added, sum together $\frac{4}{5}+\frac{4}{5}+\frac{2}{5}$, or 2 right angles.

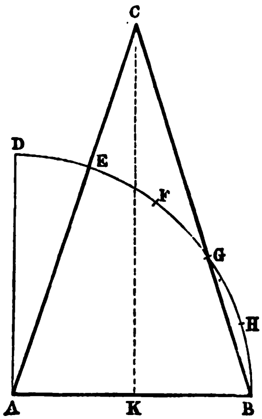

Fig. 12.

PROBLEM XIV.

To construct a triangle whose perimeter (8·5″), altitude (2·5″), and one base angle (50°), are given. (Fig. 13.)

First, draw two parallel straight lines AD and MN at a distance apart equal to the given altitude of the triangle. At A, make an angle of 50° (the given base angle), and produce the line AC until it cuts MN in C. Consider AC as one side of the required triangle, and AD as the indefinite direction of the base. Make AD equal to the remainder of the perimeter after having subtracted

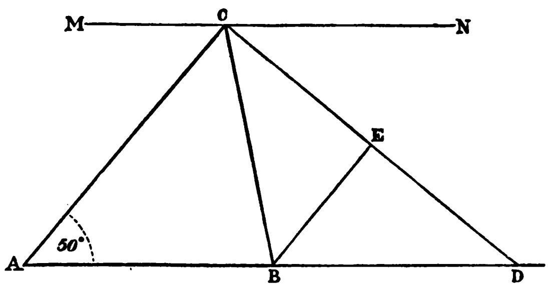

Fig. 13. ($\frac{1}{2}$ scale).

the length of the fixed side AC. Join CD, and at E, its centre, draw BE perpendicular to it, and meeting AD in B. Join BC. Then ABC is the required triangle.

CB is equal to BD (Euclid i. 4), therefore AC+BC+AB=AC+AD, which gives the correct perimeter.

If the base were given instead of the altitude, a construction similar in principle to the above would solve the problem (see Problem xxxiii., Elementary Geometry).

Problem XV.

To construct a triangle whose altitude (2″), vertical angle (42°), and base (2·5″), are given. (Fig. 14.)

All the angles in the same segment of a circle are equal (Euclid iii. 21). That contained by a semicircle is a right angle; whilst a segment less than a semicircle subtends an angle greater than a right angle, and *vice versâ* (Euclid iii. 31).

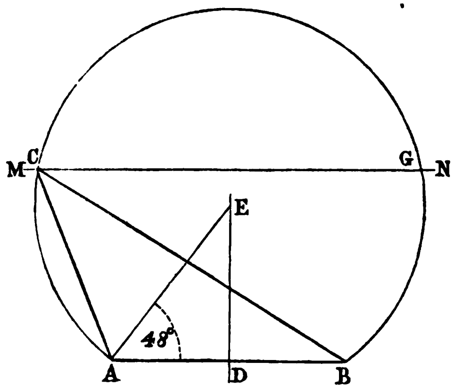

Fig. 14. (½ scale).

Hence, assuming AB 2·5″ long, as the chord of a segment which contains angles of 42°, we shall, when this segment is determined, have a locus which must contain the vertex of the required triangle. Bisect AB perpendicularly by ED, and at H set out the angle BAE of 48° (a right angle, less the given angle);* then E is the centre of the required segment. The arc ACB must therefore contain the apex. Now, MN drawn parallel to the base, at a distance from it equal to the altitude, is also a locus of the apex; and C and G, which are common to both loci, are two points, either of which being joined to A and B, determines the required triangle.

* Note that the angle at E is therefore 42°, or equal to the vertical angle, because AED is a right-angled triangle; and if E were joined to B, AEB would be double the angle at the circumference. Hence, by Euclid iii. 20, E must be the centre of the segment.

PROBLEM XVI.

To construct a triangle whose perimeter (9″), altitude (2·75″) and vertical angle (50°) are given. (Fig. 15.)

Make the indefinite lines AX, AY meeting in A at an angle equal to the given vertical angle. With A as centre, radius equal to the given altitude (2·75″), describe an arc DE. Any straight line tangent to this arc may be the required base, as its perpendicular distance from A measures 2·75″. Next mark off on each of the legs of the angle a length equal to one half the perimeter, as at G and H. From these points set out GK and HK perpendicular to AG and AH respectively. With K as centre, draw the arc GFH. Then the required base is the straight line BC, which is tangential to both the arcs drawn.

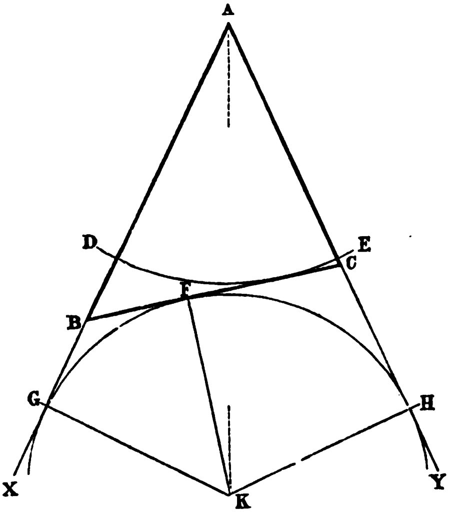

Fig. 15. (½ scale).

To prove that the perimeter is equal to that given in the problem, it should be noticed that AG is a tangent to the arc (centre K) at G, BC being also tangent to the same arc at F, and AH at H. Now, BG is equal to BF, these two lines being the two tangents from a point B outside the circle (Euc. iii. 8.) For the same reason CF is equal to CH. Hence, AB + BC + AC = AG + AH = the given perimeter.

PROBLEM XVII.

To construct a triangle whose vertical angle is 50″, base 2″ long, and such that the line bisecting the vertical angle divides the base into two segments whose ratio is 3 : 2. (Fig. 16.)

From Euclid vi. 3 it is seen that when a line bisects an angle of a triangle it divides the opposite side in segments which are proportional to the remaining sides. Hence the above problem resolves itself into a question where the base, vertical angle, and ratio of the remaining sides are given. At D (Fig. 16) set out an angle of 50°, and from any scale of equal parts mark off, on the legs of the angle, lengths of 3 and 2, as at DB, DE. Join BE. Then DBE is a triangle *similar* to the one required. Next make BC equal to the given base by producing BE, and draw CA parallel to DE to meet BD produced in A. Then ABC is the triangle required. The line AF bisects the vertical angle, and shows the division of the base as required.

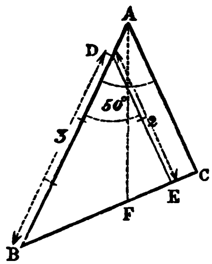

Fig. 16. (½ scale).

Problem XVIII.

To construct a triangle whose base (2″) and altitude (1·8″) are given, the ratio of the remaining sides (3 : 2) also being known. (Fig. 17.)

Draw the base AB 2″ long. and divide it internally in E in the ratio of the sides given (3:2). Produce it to F, making AF: BF : : 3 : 2. On EF construct a semicircle. Draw MN parallel to AB at a distance from it equal to the given altitude. This line

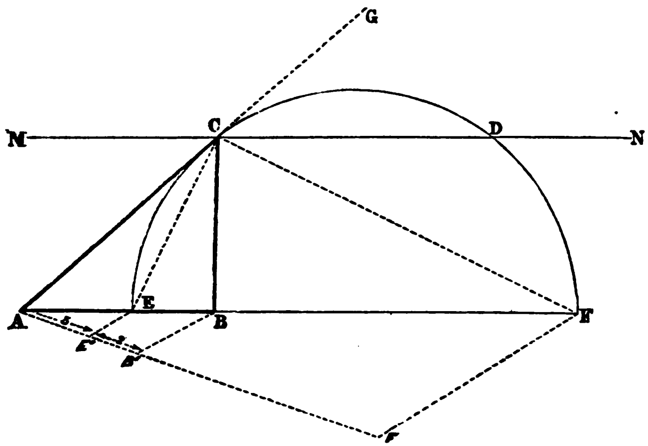

Fig. 17. (½ scale.)

intersects the semicircle in C and D, either of which may be assumed as the vertex of the required triangle.

This solution is based upon the theorems of Euclid vi. 3 and 3A. For because AF : BF : : AE : BE : the exterior angle BCG is bisected by CF, and the interior angle ACB is bisected by CE and AC : BC . : AE : BE, or as 3 : 2.

Problem XIX.

To construct a triangle whose perimeter (7·5″), base (3″), and altitude (1·5″) are given. (Fig. 18.)

Make AB 3″ long, and produce it in either direction. This fixes the base. The sum of the remaining sides must be 4·5″ (7·5″—3″). Make OG equal to one-half of this, or 2·25″. Bisect AB perpendicularly by OF. With A as centre, radius equal to OG, cut OF in F. With centre O draw two semicircles through G and F, the smaller of which will intersect MN (a parallel to AB at a distance of 1·5″ from it) in E. Join OE, and produce to D. From D draw DC perpendicular to MN, and meeting it in C. Join AC, BC, and ABC is the required triangle.

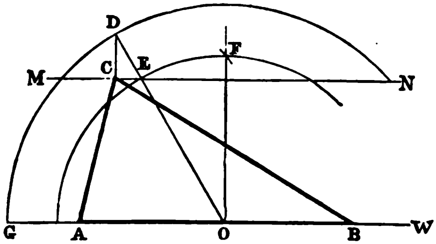

Fig. 18. (½ scale)

The solution here employed depends upon the properties of the ellipse, the application of which is referred to in a foot-note on page 87. The reader will be better able to appreciate its bearing after studying Chapter VI.

Problem XX.

To construct a regular polygon of M sides upon a given base. (Fig. 19.)

In general work, polygons of a greater number of sides than ten are seldom required.

The *equilateral triangle* is constructed by Euclid i., Prop. 1, and the *square* by Euclid i., 46.

The apex of the equilateral triangle is the centre of the circle which circumscribes a *hexagon* built upon the same base.

Further, the properties of the *pentagon* are such as to admit for that figure a particular construction (see fig. 20). Hence, the following method, which is *practically* adopted—although it does not admit of mathematical proof (because of the division by trial)—need apply only to the construction of the heptagon, octagon,* nonagon, decagon, etc.

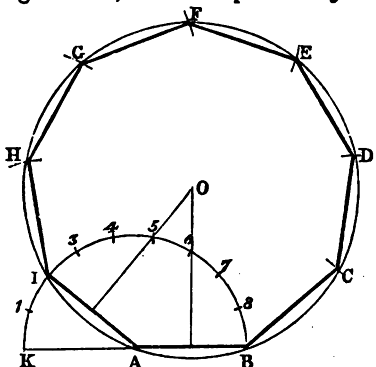

Fig. 19.

Let it be assumed that the polygon is to be a nonagon (9 sides). Draw the base AB, and produce it. With A as centre, describe a semicircle through B, and divide it carefully by trial into 9 (n) equal parts.† Join A to the *second* division mark, counting from K. Then AI, AB are two sides of the polygon. Bisect each of these lines by perpendiculars meeting in O, the centre of the circumscribing circle (see par. 8, page 28). The remainder of the construction is obvious.

Note.—**The accuracy of the solution depends entirely on the correct division of the semicircle.**

Problem XXI.

Upon a given line, AB, to construct a regular pentagon. (Fig. 20.)

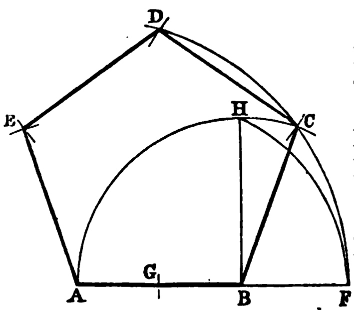

Fig. 20.

The *most* important property of the pentagon is that the diagonal, when divided in extreme and mean proportion, gives in the larger segment the length of the side, as mentioned in Problem VI. Hence, what must be done in this problem is to extend the given side AB, to such a point that the whole produced line AF may be medially divided at the point B. To do this, at B erect BH equal and per-

* The angle at the corner of an octagon is 135°. Hence, by constructing at one extremity of the base a right angle and one-half, a *second* side can be determined and the figure completed. This plan is perhaps best for the octagon.

† The external angles of any straight-lined figure are together equal to 4 right angles (Euclid i. 32, cor.); hence in this case each one is $\frac{1}{9}$ of 4 right angles, or $\frac{2}{9}$ of 180°. This is the basis of the construction.

pendicular to AB. Find G, the middle of AB, and with it, as centre through H, draw an arc HF, to cut AB produced in F. Then AF : AB : : AB : BF, or $AB^2 = AF, BF$. Thus AF is the length of the diagonal. Take this length as radius, and with A and B as centres, describe arcs meeting in D. The determination of the other points E and C needs no further description.

Problem XXII.

Given the diagonal (2″) of a regular pentagon, to construct it. (Fig. 21.)

Draw the diagonal AC, and divide it in extreme and mean proportion in H (Problem VI.). Then AH is the length of the side of the required pentagon. With A and C as centres, radius AH, describe arcs meeting in B. Join AB,BC. Next obtain the circumscribing circle, and complete the figure.

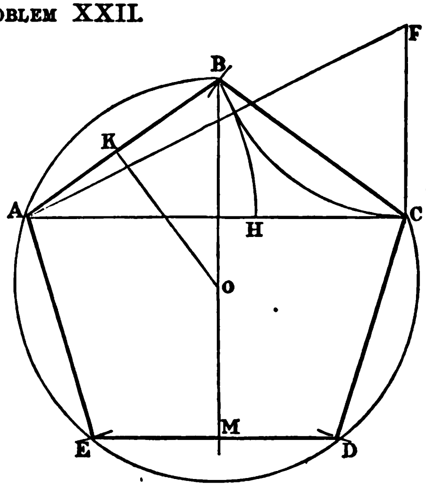

Fig. 21.

EXERCISES.

1. Draw an equilateral triangle having the same perimeter as a square of 1·2″ side.
2. Make an isosceles triangle, altitude 2″, perimeter 6·75″.
3. The base of a triangle is 3″. Its vertical angle is 40°, and it is isosceles. Construct it.
4. A certain circle passes through three points, A, B, and C, which occur upon two parallel lines, 1·5″ apart. A and B are upon one line, and 2″ apart; AC + BC = 5·5″. Determine the radius of the circle.
5. Given the perimeter (7″), base (2″), and one base angle (37°) of a triangle, to construct it.
6. Construct a heptagon on a base 2″ long.
7. To construct a triangle, whose vertical angle is 40°, one base angle 60°, and the altitude 2·5″.
8. To construct a triangle whose perimeter is 8″, its angles being in the proportion of 2 : 3 : 4.
9. To construct a triangle whose perimeter is 8″, its sides in the proportion of 3 : 5 : 6.
10. Construct a pentagon having the same diagonal as a square of 2″ side.
11. Draw a semicircle upon a line AB, 3·5″ long, and find a point C in the curve, such that AC + BC = 4·5 inches.

CHAPTER IV.

ON THE AREAS OF PLANE FIGURES.

Before commencing the study of this chapter, the following fundamental principles, demonstrated in the books of Euclid, are recalled to the student's mind. Upon these a large proportion of the solutions are based:—

1. The area of a figure depends both upon its shape and upon its perimeter.

2. Parallelograms and triangles upon the same or equal bases, and between the same parallels, are equal in area (Euclid i. 35-38).

3. If a parallelogram and a triangle be upon the same or equal bases, and between the same parallels, the area of the former is *twice* that of the latter (Euclid i. 41).

4. The area of a triangle is measured by a rectangle having the same altitude and half the base as sides.

5. Parallelograms or triangles on unequal bases, and between the same parallels, are proportional in area to those bases (Euclid vi. 1.)

6. The squares on the base and perpendicular of a right-angled triangle, are together equal in area to the square on the hypotenuse (Euclid i. 47).

7. *The principle in No. 6 applies equally to other figures made upon the sides of the right-angled triangle, so long as they are similar, i.e.,* have proportional sides (Euclid vi. 31), or equal angles.

8. The areas of *similar* figures are proportional to the squares on their homologous sides (Euclid vi. 19, 20).

9. Circles are to one another in area as the squares upon their respective diameters.

10. Perimeters being equal, the greatest space is enclosed by figures which have equal sides, and the area is increased as the number of sides becomes greater.

11. A circle therefore holds the greatest area, with the shortest boundary line, called in this case *the periphery.*

12. If two triangles have one angle in common, they will be equal in area, if their sides are reciprocally proportional (Euclid vi. 15).

PROBLEM XXIII.

Given a rectangle, to construct a square of equal area. (Fig. 22.)

Attention was drawn to this problem at p. 16, as the mean proportional, or geometric mean between the two sides of the rectangle, is the side of the required square. The figure will sufficiently show the application of the principle.

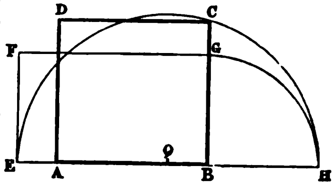

Fig. 22.

PROBLEM XXIV.

Given a square and one side of a rectangle; to complete the latter so that the two figures may be equal in area. (Fig. 23.)

This is a deduction from the previous problem; the mean proportional and one of the terms being given, to find the other one. Let ABCD be the square, and BE the side of the rectangle. Join CE, and bisect it by a perpendicular *p o*, meeting AE in *o*. With *o* as centre, draw a semicircle through C to meet AB in K. Then KB is the second term or length of the unknown side of the rectangle. The figure BEFG can then be completed.

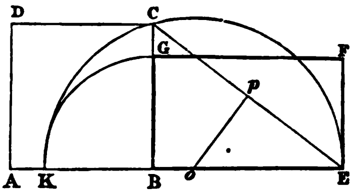

Fig. 23.

PROBLEM XXV.

The perimeter of a rectangle is 5·5″; to construct it, so that its area may be equal to a square of 1·25″ side. (Fig. 24.)

This is also a deduction from Problem XXIII. The length of two contiguous sides is evidently half the total perimeter. Hence we have to divide a line 2·75″ long into two parts, whose geometric mean shall be 1·25″ long. Make AG (fig. 24) 2.75″ long, and describe a semicircle upon it. At G set up GF, 1·25″ long and perpendicular to AG. Imagine this line to move parallel to itself, the point G always remaining in the line AG until the upper extremity meets the semicircle in E. Then B will be the position where BE or GF is the mean proportional between the two segments of the line AG. The rectangle can then be constructed.

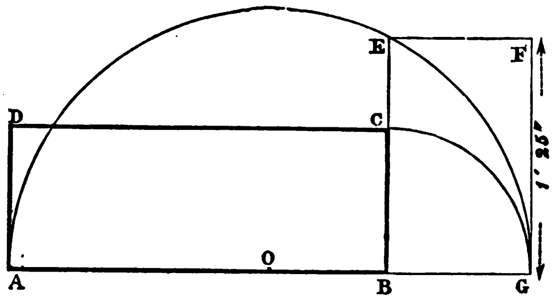

Fig. 24.

Problem XXVI.

To determine (1) a square equal to the sum of two given squares of 1·5″ and 1″ sides respectively; and (2) a square equal to their difference. (Figs. 25, 26.)

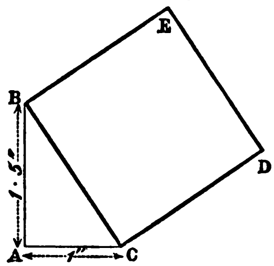

Fig. 25. (½ scale).

This is a direct problem upon the Proposition 47 of the First Book of Euclid. Use the sides of the given squares as the base and perpendicular of a right-angled triangle as AB, AC (fig. 25), and the hypotenuse BC gives the side of a square equal to their sum.

To obtain the square equal to their difference, use the side of the smaller one as the base, as at AE (fig. 26), and draw an indefinite perpendicular AB. With E as centre, radius equal to the side of the larger square, cut AB in B. Then build the required square on AB. For $BE^2 = AE^2 + AB^2 \therefore AB^2 = BE^2 - AE^2$.

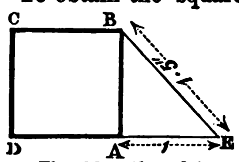

Fig. 26. (½ scale).

Deduction.—A circle equal to the sum of two circles can be found by treating their diameters in a similar manner to the sides of the above squares.

In fig. 27, two pentagons, ABCDE and FGHIK, are supposed to be *given*, and a third, BNOPQ, equal to the two added together, is determined. At A, AM is drawn perpendicular to AB, and equal to FG. Then $AB^2 + AM^2 = BM^2$, and as this principle applies to all *similar* figures, a pentagon having BM for the side will be equal in area to their sum. BN is made equal to BM, and NO is drawn parallel to AE, until it is met by a line through B and E. Notice that the triangles BAE and BNO are similar. This process repeated for the other sides enables one to complete the figure. If the newly determined pentagon were required to be equal to the difference between those given, then AB must be treated as the hypotenuse, and FG as one of the rectangular sides of the summing triangle. This is readily accomplished by constructing a semicircle upon AB, and measuring FG along it from A or B as a chord. The remaining chord gives the length of the corresponding side of the required figure.

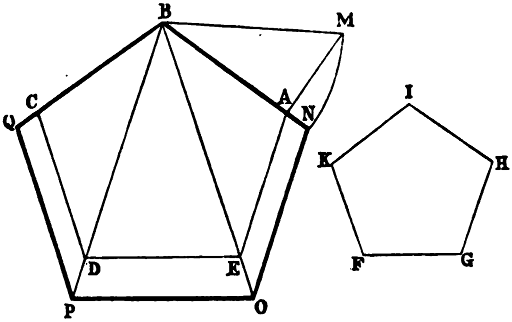

Fig. 27.

This construction could be applied to the finding of any figure similar to a given one, so as to be twice, thrice, or any number of times its area.

Problem XXVII.

To show geometrically $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, and $\sqrt{10}$, to a given unit (1 inch). (Figs. 28 and 29.)

Certain numbers have no square roots, and are called surds. The above are those amongst the first 10 numbers. Hence the side of a square is incommensurable with that of another twice its area, or with its diagonal, *i.e.*, no scale of equal parts could exactly measure both quantities. But, by the aid of the principles shown in the last problem, the actual ratios of such quantities with a standard unit can be readily determined geometrically. Thus, in fig. 28, AB being the standard (1 inch), and AC being perpendicular and equal to AB, BC represents the length of the side of a square of 2 square inches area, for $AB^2 + AC^2 = BC^2$. Its length can only be written as $\sqrt{2}$.

Similarly, BD being perpendicular and equal to AB, CD represents $\sqrt{3}$, and CE $\sqrt{5}$, for DE is $\sqrt{2}$, and $CD^2 + DE^2 = CE^2$. By this means the root of any number may be graphically represented in comparison to unity. In fig. 29, the whole of the roots of the first 10 numbers are shown in a single diagram compared to 1″ as the standard. The construction is as follows: (the compasses only being required). And it is well to notice that more accuracy is ensured by their use alone than by the additional aid of the ruler.

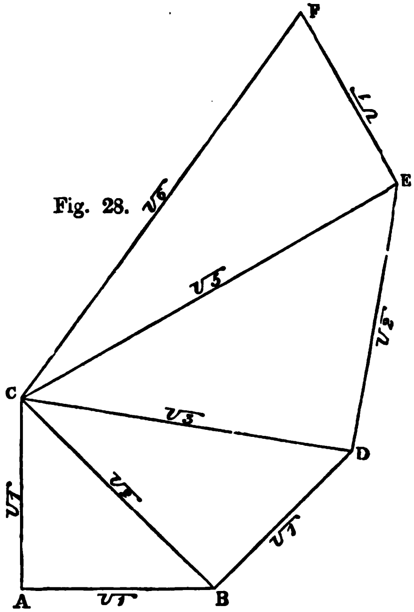

Fig. 28.

With centre O, radius equal to the unit 1″, describe a circle, and divide it in the ordinary way into 6 equal parts in A, B, C, D, E, F.

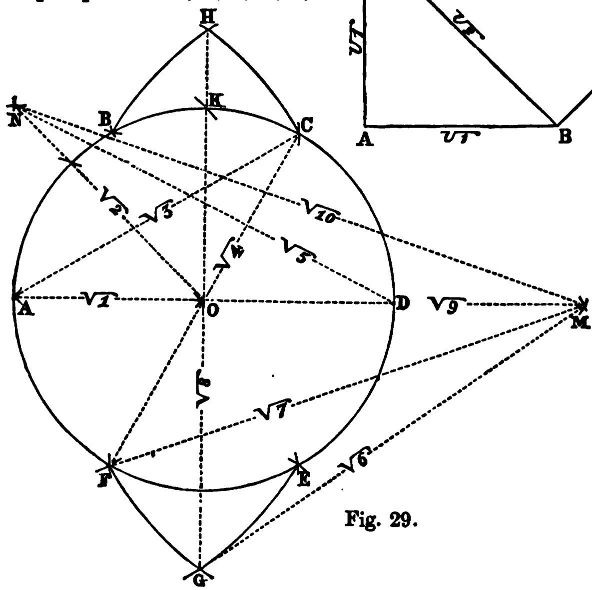

Fig. 29.

With A and D as centres, radius AC, describe two pairs of arcs intersecting in G and H. With A as centre, radius OG, cut the circumference in K. With A and K as centres, radius AO, draw arcs intersecting in N. With E and C as centres, radius EC, draw arcs intersecting in M. The necessary construction will then be complete.

For OA $= 1$ or $\sqrt{1}$.

ON $= \sqrt{2}$ as it is $= \sqrt{AO^2 + OK^2}$ or $\sqrt{1+1}$.

AC $= \sqrt{3}$,, ,, $\sqrt{AD^2 - CD^2}$ or $\sqrt{4-1}$; ACD being a right angle. (Euclid iii. 31.)

CF $= \sqrt{4}$,, ,, twice the radius.

DN $= \sqrt{5}$,, ,, $\sqrt{AD^2 + AN^2} = \sqrt{4+1}$.

GM $= \sqrt{6}$,, ,, $\sqrt{OM^2 + OG^2} = \sqrt{4+2}$ (OM being equal to the radius, and OG to ON).

FM $= \sqrt{7}$,, ,, $\sqrt{FC^2 + CM^2} = \sqrt{4+3}$ (CM being a tangent to the circle).*

GH $= \sqrt{8}$,, ,, 2·OG, or 2 ON, or $2\sqrt{2}$, or $\sqrt{8}$.

AM $= \sqrt{9}$,, ,, 3 times the radius.

and MN $= \sqrt{10}$,, ,, $\sqrt{AM^2 + MN^2} = \sqrt{9+1}$.

To determine the root of a fractional expression, as $\sqrt{2\frac{3}{4}}$, it is most convenient to find the mean proportional between two of its factors. Thus, $\sqrt{2\frac{3}{4}}$ would be found by the construction given in fig. 5, the two terms being 1 and $2\frac{3}{4}$, as they are the most convenient factors.

Problem XXVIII.

To reduce any irregular figure to a triangle of equal area. (Fig. 30.)

It is advisable in working this problem for the first time to assume a figure having only five sides, and, when the principle is well understood, to apply it to more difficult cases. Taking ABCDE (fig. 30) as the given figure, produce the base AB in both directions. Join AD, and thus cut off a triangular portion ADE. Through E draw EF parallel to AD, and join DF. Then we may consider ADF as a compen-

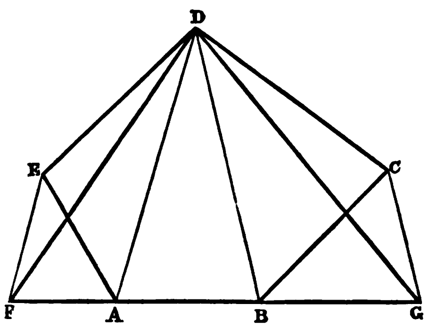

Fig. 30.

* Point C is in a semicircle on OM, whose centre is D. Hence OCM is a right angle (Euclid iii. 31).

sating addition for the loss of the triangle ADE, for they are both on the same base AD, and their apices E and F are in the same parallel EF (Euclid i. 37). After the triangle BDC has been treated in the same manner, DFG will result as the required figure. The student must be careful to notice that this principle applies only to the alteration of *one* triangle at each step, and further, that it can only be used in regular order right and left of the line which is to be assumed as the base of the resulting triangle. In fig. 31 a more complicated case is shown, but the method of procedure is similar to the above, except only, that in starting, the triangle ABC is *added*, and an equivalent triangle ACR is *subtracted.*

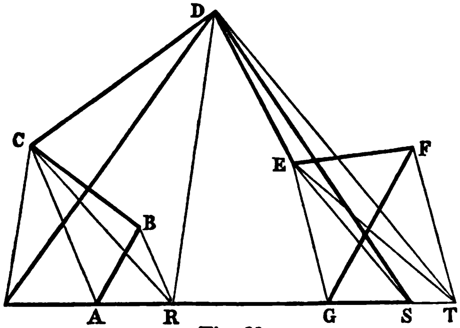

Fig. 31.

PROBLEM XXIX.

To make an isosceles triangle with a vertical angle of 40°, the area of which shall be 3·5 square inches. (Fig. 32.)

Commence by making an isosceles triangle ABC, having a vertical angle of 40°, the lengths of the sides at pleasure, and determine the side of a square BF equal in area to this figure by first constructing a rectangle on the same base AB, and half the height of the triangle, and then finding a mean proportional between the sides of this rectangle.

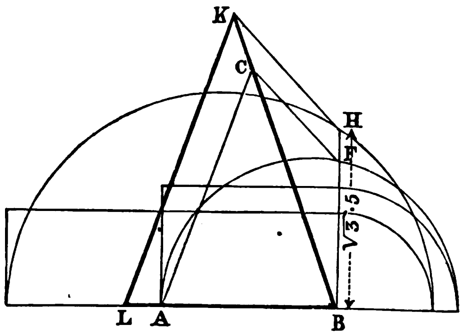

Fig. 32.

This shows the assumed triangle to be of the wrong area. Next produce BF to H, making BH equal to $\sqrt{3\cdot5}$ square inches (Problem XXVII.). Join CF, and draw HK parallel to it to meet BC in K. Complete the similar triangle KLB. It is the one required.

The two similar triangles, ABC and LBK, are in area proportional to the squares on their homologous sides BC, BK, and these two lines are the sides of two other similar triangles BCF, BKH; therefore $BC^2 : BK^2 :: BF^2 : BH^2$; but BF^2 is the actual area of the triangle ABC; hence BH^2, or 3·5 square inches, is the actual area of the triangle LBK.

Problem XXX.

To construct a rhombus (having an angle of 60°) of 5 square inches area. (Fig. 33.)

This problem is solved by first constructing a rhombus having the given included angle, but not necessarily containing the given area. Such is ABCD, fig. 33. The side of the square to which this figure is equal is next deduced by a mean proportional between its base AB and its altitude BE. This is shown at BF. BF is then produced to G, so that BG shall be equal to the side of a square of the required area (5 square inches). Then, by joining CF, and drawing through G a parallel to CF to meet BC produced in H, the side of the required rhombus BH is determined. The point M is most readily obtained by the diagonal through D meeting HM parallel to AB, etc.

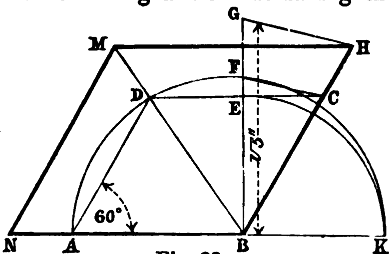

Fig. 33.

Problem XXXI.

To construct an isosceles triangle equal in area to any given triangle, and having one of its angles equal to one of those of the given triangle. (Fig. 34.)

Let it be required to make an isosceles triangle equal to ABC, the angle at A to be common to both figures. Then AF and AG, the equal sides of the required figure, are in length equal to a mean proportional between AB and AC. For as $AF^2 = AB . AC$, therefore $AB : AF :: AG : AC$, or the

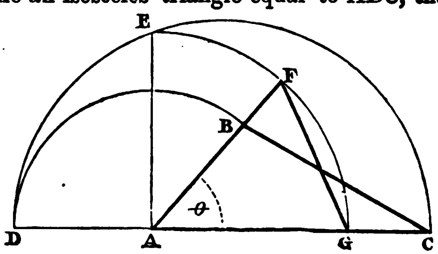

Fig. 34.

sides of the triangles which include the equal angle are reciprocally proportional, hence the areas are equal (Euclid vi. 15).

The construction will be readily understood by reference to the figure; AD is made equal to AB, and AE is the mean proportional, giving the length of AG and AF.

This principle would apply to solve Problem XXX., as a parallelogram with an angle of 60° could be drawn equal to a rectangle of the given area, and then the side of the required rhombus would be a mean proportional between those of the parallelogram.

Problem XXXII.

To construct a regular polygon of any given number of sides, which shall be equal in area to a given triangle. (Plate III., fig. 1).*

A regular polygon can be divided into as many isosceles triangles as the figure has sides by joining its centre to its angular points. Further, a triangle can be divided into any number of equal parts by first dividing one of its sides into that number of parts, and by joining the points of division to the opposite corner of the figure. The problem before us may be solved by taking advantage of the above facts, for the *n*th part of the given triangle having been found, an isosceles triangle having any desired vertical angle (the angle at the centre of the polygon) can be determined equal to it in area by the previous problem, and the polygon of which this isosceles triangle is the *n*th part will be the figure required.

Let ABC (Plate III., fig. 1) be the given triangle which is to be transformed into a regular octagon of equal area.

Divide one of the sides, as BC, into 8 equal parts in 1, 2, 3, etc. Join 1 to A. Then A1B is $\frac{1}{8}$ of the triangle ABC. At B, set out BN, making an angle with AB equal to the angle at the centre of an octagon. This is $\frac{360}{8}°$ or 45°. Through 1 draw 1D parallel to AB to meet BN in D. Then a triangle ADB, obtained by joining D to A, will be equal in area to A1B (Euclid i. 37), and will have one of its angles, ABD, equal to the angle at the centre of the required polygon.† The next step is to convert ABD into an isosceles triangle by finding a mean proportional between its sides AB and BD. The line BH is the

* The drawing is necessarily placed in the book of Plates, as the restricted dimensions of this volume would not allow of its being made sufficiently large to be clear.

† The line joining A to D is omitted in the plate, as it is only required for demonstration, and may be neglected in actual construction.

mean required, and is used as the radius of the circle circumscribing the desired octagon. The rest of the construction will be easily understood from the figure.

A modification of the above solution is employed to obtain an equilateral triangle equal to any irregular triangle.* This is shown in fig. 35, where the irregular triangle ABC is converted into the equilateral one CDE. On one side, AC, of the former, an equilateral triangle AFC is constructed; FC is produced to meet a line through B parallel to AC. Then the mean proportional between FC and CG, *i.e.*, CE, is used as the side of the equilateral triangle required.

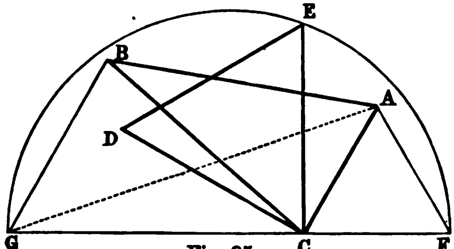

Fig. 35.

The student will notice that by constructing the triangle FAC and producing FC, the angle ACG is 120°, or that at the centre of the polygon required. Further, AGC is equal to ABC (Euclid i. 37). Hence CE, the mean proportional between FC and CG, is the radius of the circumscribing circle, as in the last case. But, not having divided the given triangle into three parts at starting, the inscribed figure (if CE be the radius) will be three times too large. This is remedied by making CE the side instead of the radius, as when an equilateral triangle is circumscribed by a circle the side of the figure is to the radius in the proportion of $\sqrt{3} : 1$.

Problem XXXIII.

Given a triangle ABC, the sides being as follows: AB = 1·5″, BC = 1·2″, and AC = 1·8″. To construct another triangle equal to it in area, two of whose sides shall be 2·3″ and 2·5″ in length respectively. (Fig. 36.)

Having drawn the first figure ABC through B, set out BD parallel to AC, and with A as centre, radius 2·3″, describe an arc to meet this

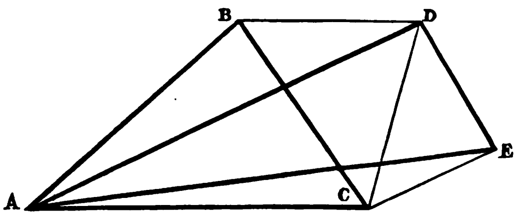

Fig. 36.

* The construction given would of course apply directly to this case, as the equilateral triangle is a regular polygon.

parallel in D. Join AD. Next through C draw CE parallel to AD, and with A as centre, radius 2·5″, describe an arc cutting this parallel in E. Join AE and DE. AED will be the required triangle.

The proof of the construction is based on Euclid i. 37, as ABC=ADC, the common base being AC, and AC, BD the parallels. Similarly, AED=ADC, the common base being AD, and AD and CE the parallels.

Problem XXXIV.

To divide a given triangle into two equal areas by lines drawn parallel to one of its sides. (Fig. 37.)

Let it be required to bisect the triangle ABC by a line parallel to the base BC. On one of the other sides, as AC, describe a semicircle ADC, and bisect it in D. Join D to A, and with A as centre, radius AD, draw an arc DE, meeting AC in E. Then EF drawn through E parallel to BC bisects the triangle.

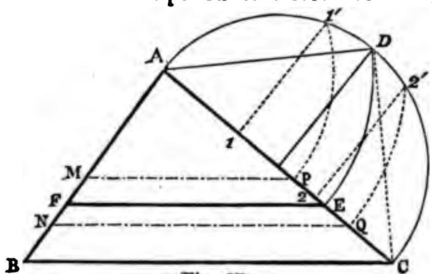

Fig. 37.

Proof.—Join AD, DC. Then ADC is a right angle (Euclid iii. 31), and $AD^2+DC^2=AC^2$ (Euclid i. 47), but AD=DC (Euclid i. 4), therefore $AD^2=\frac{1}{2}AC^2$. Again, AFE and ABC are similar triangles. Hence their areas are as AE^2 or $AD^2 : AC^2$, or 1 : 2.

On the same figure, in dotted lines, the construction is shown for dividing the triangle into three parts by parallels to BC. The principle is the same as before, AC being divided into three parts instead of two. The construction is applicable therefore for any number of divisions.

Problem XXXV.

To bisect a given triangle ABC, by a line perpendicular to one of the sides, as AB. (Fig. 38.)

Bisect the base AB in T, and from C drop a perpendicular CE to meet the base in E. Then find a mean proportional between the half of the base and the portion intercepted between one extremity and the foot of the perpendicular. This is shown at AF, which is the geometrical mean between AD and AT (AD being equal to AE.) Then AG is made equal to AF, and GH is drawn perpendicular to AB.

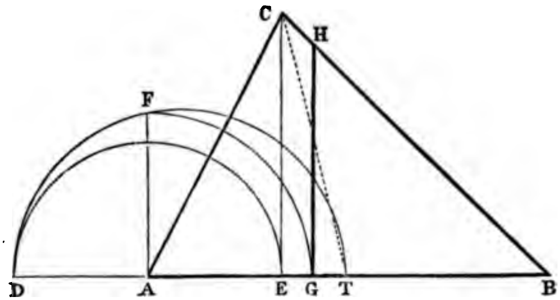

Fig. 38.

If it be required to divide the triangle into three or any other number (n) of equal parts, the base must be divided first into n equal parts, and the mean proportional used must be taken between $\frac{1}{n}$ of the base and the intercepted portion as before.

PROOF.—The triangles EBC and GBH are similar. Hence BE : BG : : BC : BH, but by the construction adopted $BE.BT = BG^2$, or BE : BG : : BG : BT. Hence BC : BH : : BG : BT, *i.e.*, the two triangles BTC and BGH have one angle common to both, and the sides about that angle reciprocally proportional. They are therefore equal in area (Euclid vi. 15). But the triangle BTC is one half of BAC (Euclid i. 37). Hence BGH is also half of the same triangle.

PROBLEM XXXVI.

To divide a given triangle ABC (fig. 39), into three equal parts, by lines drawn through a given point P in one of the sides.

Divide the base AB into three equal parts, in points G and F. Join PG and PF; and through C draw CD parallel to PG and CE to PF. Join PD and PE.

If C be joined to G, it is clear that CAG is one-third of the whole triangle, and CDG is eqnal to CDP, for they are on the same base CD, and between the same parallels CD and PG.

This solution is of course applicable for any number of parts.

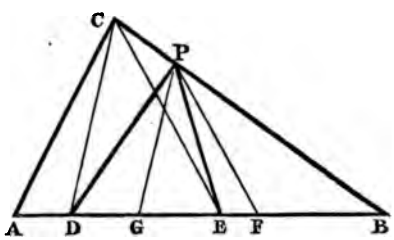

Fig. 39.

Problem XXXVII. (*a*).

Given any triangle; to bisect its area by a straight line passing through *any* given point within or without it. (Fig. 40*a*).

Let ABC be the triangle, and D the given point. Through D draw a line DE parallel to the side CB (the furthest from D), and bisect AC and BC. Join F to H, and through G draw GK parallel to FH, and KE parallel to AC, to meet DE in E. On ED describe a semicircle, and with D as centre, radius DF, draw the arc FT. Measure the distance ET, and from K mark off KL, *equal to it, along the side* BC. Join DL. Then the portion SL will divide the figure into two equal areas. *

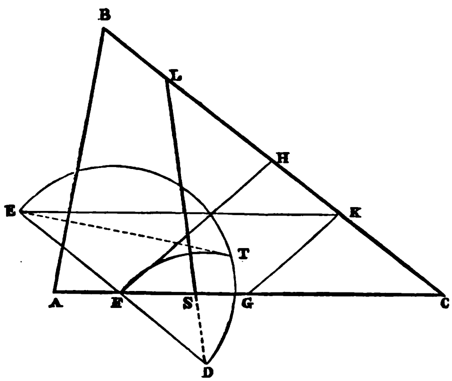

Fig. 40*a*.

When the given point is inside the triangle, the same solution can be adopted.

Problem XXXVII. (*b*).

To divide the rectangle ABCD into three equal areas by straight lines drawn through the point P. (Fig. 40*b*.)

Divide the side containing P into three equal parts, and draw perpendiculars from each of the points of division. Bisect these perpendiculars, and draw lines from the givèn point through each of their centres.

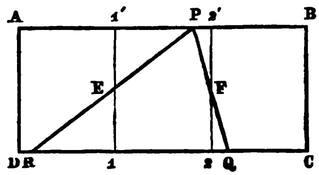

Fig. 40*b*.

When the given point D falls so that either of the dividing lines does not meet the base until it is produced, it will not be correct for that one. In such a case the irregular quadrilateral, forming the space, containing two divisions, must be halved by the construction described in the next problem.

* A proof of this construction may be found in Bradley's *Elements of Geometrical Drawing*, Part I.

PROBLEM XXXVIII.

To bisect any irregular quadrilateral figure by a line drawn through one of its angular points. (Fig. 41.)

Let ABCD be the given figure, and A the given point. Join BD, and bisect it in E. Then a figure AECB would be half the quadrilateral, for AEB = $\frac{1}{2}$ ADB, and BEC = $\frac{1}{2}$ BDC. Join AC, and through E draw EF parallel to it to meet DC in F. Join AF, and the figure will be bisected for AFC = AEC (Euclid i. 37).

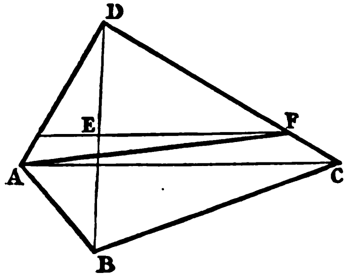

Fig. 41.

PROBLEM XXXIX.

To divide a given irregular figure ABCDEFG (Plate III., fig. 2) **into any number** (11) **of equal parts, by lines drawn through one of its angular points, as** A.

The irregular figure must first be reduced to an equivalent triangle, having its apex in the point A, from which the dividing lines are to radiate by the construction described at Problem XXVIII. In the plate, AMN is the triangle equivalent to the figure A......H. Next, the base of this triangle must be divided into the number of parts required; and it is evident that, if two consecutive points be conceived to be joined to A, the triangle thus formed will be $\frac{1}{11}$ of AMN, and therefore $\frac{1}{11}$ of the given figure. Hence all those points which occur within EF may at once be joined to A. For those which fall outside EF, proceed as follows:—Taking the cases of f and g, join EA, and through these two points draw f, f^1 and g, g^1 parallel to EA, until they meet ED in f^1 and g^1. Join these to A. It will be seen that the parallels to the same line through h and i do not meet ED until it is produced; hence it is necessary to join AD, and at h^1 and i^1 to draw another pair of parallels to AD meeting CD in h^2 and i^2. The point K, under the conditions given, requires yet another system of parallels, and in some cases it would be necessary to repeat the construction many times to obtain the division points on the contour of the figure.

The principle of the construction depends on Euclid i. 37; for, taking the case of the portion $AeEf^1$, it is evident that f^1f being parallel to AE, the triangle AEf^1 is equal to *a* triangle AEf completed by joining f to A. Hence, adding AeE to both, the whole figure $AeEf^1$ is equal to Aef; but ef is $\frac{1}{11}$ of the base MN, therefore $AeEf$ is $\frac{1}{11}$ of the triangle AMN, or of the given irregular figure.

Problem XL.

To divide a circle into any number of equal parts by concentric circles. (Fig. 42.)

Let it be required to divide the circle, centre C (fig. 42), into five equal parts, by other circles having the same centre. Draw a radius CD, and divide it into five equal parts. On CD construct a semicircle, and at each of the points of division in the radius raise lines perpendicular to CD to meet the semicircle in 1′, 2′, 3′, 4′. Then, taking C as centre, draw a series of circles through these points, and the figure will be divided as required.

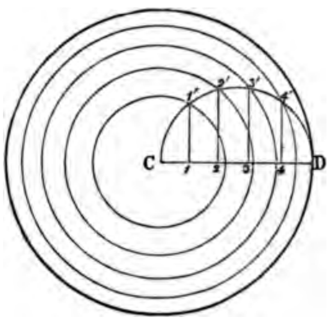

Fig. 42.

The proof is analogous with that explained in the case of the triangle in Problem XXXIV.

Problem XLI.

To construct a parallelogram equal in area and perimeter to a given triangle. (Fig. 43.)

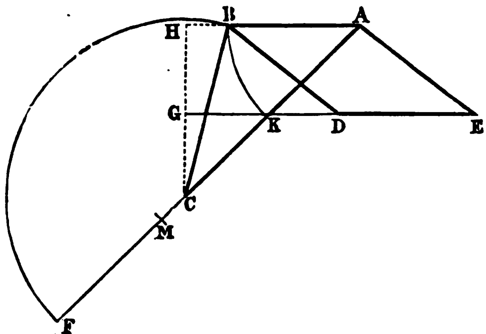

Fig. 43.

Let ABC (fig. 43) be the given triangle. Draw an indefinite parallel to AB, at one-half the height, CH. Then any parallelogram on the base AB, and between the two parallels, will be equal to the triangle. Next, with regard to the perimeter, it is evident that the side opposite

AB being equal to it, the other sides must each be in length equal to one-half (AC + BC − AB). Produce AC to F, making CF = CB. Mark off upon AF, AK equal to AB, and bisect the remainder KF in M. Then KM is the length of the remaining sides of the parallelogram, and BD and EA can be set off from B and A to intersect the parallel EG.

Problem XLII.

To construct a triangle equal in area and perimeter to a given parallelogram, ABCD (fig. 44), whose opposite sides are 3″ and 2″ long respectively, and which has one of its angles 50°.*

In fig. 44, the parallelogram ABCD is that supposed to be given, and ABE is the triangle which is equal to it in area and perimeter, being determined as follows:—The line ME is first drawn parallel to AB, and at a distance equal to twice the vertical height of the given figure. Then the apex of the required triangle must fall on this line (AB being assumed as its base); and the problem resolves itself into finding a point on ME, such that the sum of its distances from A and B shall equal the sum of the remaining three sides of the parallelogram. The method of finding this point was described in Problem XIX.

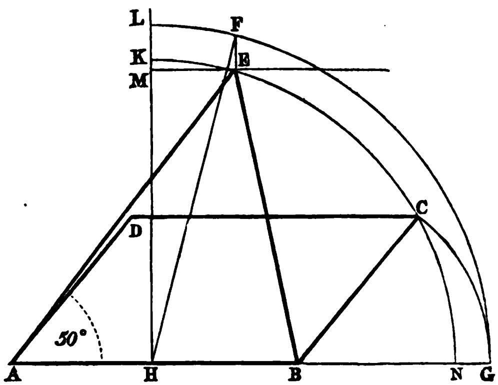

Fig. 44. (½ scale).

* It must be noticed here, that this problem cannot be considered general in its application, as it would be impossible in certain cases to find the triangle which should be equal both in area and perimeter to an assumed parallelogram. For instance, a square cannot be so transformed; for being a *regular* figure, it contains a larger area than any other figure of equal perimeter, and having a less number of sides. The possibility of the problem depends therefore on the irregularity of the assumed parallelogram; that is why the data are inserted of a proposed possible case.

Problem XLIII.

Given an obtuse-angled triangle, ABC, to draw a line BP through the obtuse angle B, so that it may divide the opposite side, in such a manner, that the rectangle on the two segments may equal the square on the dividing line; or so that AP.CP may equal BP². (Fig. 45.)

First circumscribe the triangle by a circle (centre D). Join BD, and on it describe a circle cutting AC in P and P′. Either of these points being joined to B, the triangle will be divided as required.

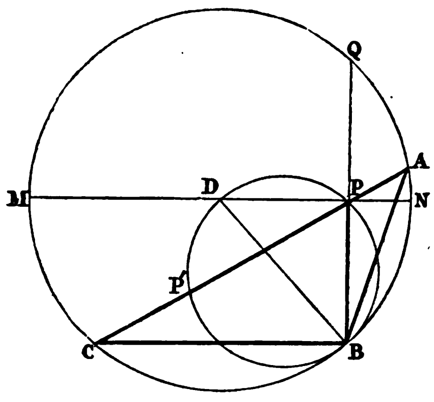

Fig. 45.

Proof.—If P be joined to D, and produced to meet the circle in M and N, and BP be also produced to meet the circle in Q, it will be seen that the rectangles AP.PC, MP.PN, and BP.PQ are all equal, for they are made upon the segments of intersecting chords of the same circle (Euclid iii. 35). But BP is equal to PQ (Euclid iii. 3), for MN passing through the centre meets BQ at a right angle (BPD being the angle in a semicircle, Euclid iii. 31). Hence BP.PQ = BP^2 = AP.PC.

Problem XLIV.

To divide a triangle into any number of proportional parts by lines drawn through a point P within it. (Plate III., fig. 3.)

Let the triangle ABC (Plate III., fig. 3) be the one given, and P the point within it, through which the dividing lines are to be drawn. Also let the number of parts be four, and their ratios 3 : 5 : 4 : 2. Divide the base AB proportionally to the given numbers, as at 3, 5, 4, etc. Join 3 to P, and make CD parallel to 3P. Join PD. Repeat this construction to obtain EP. Join P4, and draw CG parallel to P4 to meet the base AB produced in G. Join PB, and draw GF parallel to it. Join PF. Then the triangle is divided as required by the lines radiating from P.

The proof of this construction does not need detailing, as it is similar to others already demonstrated, and depends upon Euclid i. 37.

Problem XLV.

To divide an irregular polygon into any number of proportional parts (as, for instance, 5 : 4 : 7 : 6 : 9), by lines drawn through a given point P within it. (Plate III., fig. 4.)

Take one side, as AB, and produce it, using this line for the base of a triangle GEH, equal to the given polygon in area (Problem XXVIII.). Next, obtain a second triangle IPK equal to the first GEH, but having its apex in P. This is done by joining P to G, and drawing a parallel to PG through E, meeting GH produced in I; K is determined similarly.*

Divide IK in the given proportion, as at *a*, *b*, *c*, *d*, *e*. Join P to B, and draw parallels to it through *c*, *d*, *e* to meet BC produced in *f*, *g*, *h*. As *f* is in BC itself, join it to P. Next join P to C, and through *g* and *h* draw parallels to PC to meet CD, or CD produced in *i* and *k*. Join *i* to P. From *k* draw a line parallel to PD, to meet DE in *l*. Join P*l*. The remaining points are found in the same way upon the other side of the figure.

This problem depends also for its solution upon the theorem of Euclid i. 37, and therefore needs no further detailed explanation.

It will be convenient, at this point, to draw the student's attention to the application of practical geometry in the solution of certain algebraical equations. It is frequently possible, by a simple geometrical construction, to obtain the value of the unknown quantities from such equations, supposing always that the known terms are understood to represent, to a given unit, quantities measured in length and area, etc.

Thus, in the equation $x = \frac{a+3b}{7}$, assuming that a represents a length of 3″, and b of 2″, then x can be discovered by drawing a line whose total length is 3″ (a) + 3 × 2″ ($3b$) or 9″, and by taking one-seventh of the

* These lines are not shown in the Plate; only the results are given.

whole. Or, assuming a to represent 2 square inches, and b 3 square inches, then the value of x would be expressed geometrically by a figure (most easily a rectangle) equal in area to the seventh part of 11 square inches (2 square inches + 3 × 3 square inches).

These two very apparent cases will give an idea as to the nature of the few problems upon this subject which follow.

Problem XLVI.

To determine, by a geometrical construction, the value of x in the following equations, the unit being *one* linear inch. (Figs. 46, 47.)

$$(1)\ x=\sqrt{8};\quad (2)\ x=\frac{\sqrt{3}}{2};\quad (3)\ x=\frac{\sqrt{5}+1}{2};$$

$$(4)\ x=\frac{1}{\sqrt{3}};\quad (5)\ x=\sqrt{\frac{5}{6}};\quad (6)\ x=\sqrt{\frac{8}{9}\times 1\tfrac{1}{3}}.$$

1. In the first equation, the quantity expressed by $x=\sqrt{8}$ can be determined by first drawing two lines each 2″ long, and perpendicular to one another, as in fig. 46. Then, by completing the right-angled triangle, the hypotenuse AC will represent the unknown quantity required, for $\sqrt{2^2+2^2}=\sqrt{8}$ (Euclid i. 47.)

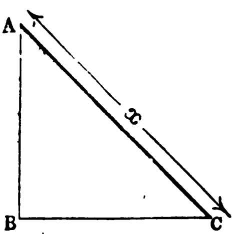

Fig. 46. (½ scale).

2. To solve the second equation, find the value of $\sqrt{3}$ by Problem XXVI., and one-half of this will be the quantity required.

3. First find $\sqrt{5}$ by drawing a right-angled triangle, base 2″, perpendicular 1″. The hypotenuse will be $\sqrt{5}$. To this add 1″, and take one-half the result. This is shown in fig. 47, where CE represents the value of x.

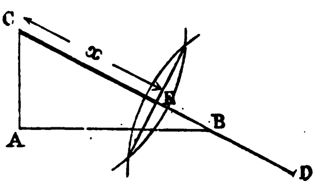

Fig. 47. (½ scale).

4. In this equation, it is first necessary to get rid of the surd denominator. To do so, multiply both terms of the fraction by that denominator; thus—$\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$. Then find $\sqrt{3}$ by Problem XXVI., and take one-third of its length for the value of x required.

5. In this case, as $\sqrt{\frac{5}{6}}=\sqrt{\frac{5}{6}\times 1}$, find a mean proportional between

two lines, one of which is 1″ long, and the other $\frac{5}{6}$″ long. This will be the required value of x.

6. This case is similar to the last, the result being the side of a square equal in area to a rectangle $1\frac{8}{9}$″ by $1\frac{1}{3}$″.

PROBLEM XLVII.

To determine, by a geometrical construction, the value of x in the following equations—the value of a being 2″, of b 1·5″, and of c 1″. (Figs. 48 and 49.)

$$(1)\ x = \frac{2a}{\sqrt{3}};\quad (2)\ x = \sqrt{a^2+b^2};\quad (3)\ x = \sqrt{a^2-b^2};\quad (4)\ x=\sqrt{ab};$$

$$(5)\ x = \sqrt{\frac{ab}{3}};\quad (6)\ x = \frac{a^2+c^2}{b};\quad (7)\ x=a^2+2ab+b^2.$$

1. Multiplying both numerator and denominator by $\sqrt{3}$ to avoid the surd divisor, we get $\frac{2a\sqrt{3}}{3}$ as the algebraic value of x, which, by substituting the known value of a, reads as $\frac{4\sqrt{3}}{3}$. Find first, then, the length of $\sqrt{3}$ by Problem XXVI., and take $\frac{4}{3}$ of it to discover the value of x, required.

2. In this case x is the side of a square, whose area is equal to the sum of the areas of squares made upon a and b, and this will be readily found by Euclid i. 47.

3. Here the value of x is the side of a square, equal to the difference of the areas of a square on a and another on b. But as $a^2 - b^2$ is equal to $(a+b)$ $(a-b)$, the length representing x can be found by determining a mean proportional between 3·5″ $(a+b)$ and ·5″ $(a-b)$.

4. In this case the value of x is the mean proportional between a and b, or their geometric mean, as described in Problem V.

5. To solve the equation $x=\sqrt{\frac{ab}{3}}$, we must first get rid of the surd divisor, which we can do by squaring both sides, so that $x^2 = \frac{ab}{3}$. Then, as ab means a rectangle (sides a and b respectively), that figure must be drawn first and divided into three equal parts. Then, as one of these by the equation

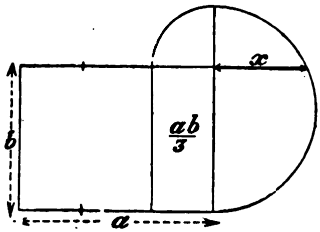

Fig. 48.

will be equal to x^2, the mean proportional between its sides will be the value of x. This is shown in fig. 48.

6. The sum of the squares upon a and c can be determined by Problem XXVI. Then the meaning of the equation is, that x is the unknown side of a rectangle whose other side is equal to b, and whose area is equal to the square (a^2+c^2). This will be apparent if both sides of the equation be multiplied by b. It will then read $bx=a^2+c^2$. The construction is shown in fig. 49, where AB and AD, equal to a and c respectively, are drawn at right angles. Then BD is the side of a square equal to a^2+c^2. Next, BE is drawn perpendicular to BD, and equal to it, BD being produced to C, so that BC shall be equal to b. Then a semicircle through C and E intersects BD, produced in F. BF is the value of x required.

For $BC,BF=BE^2$ or $bx=a^2+c^2$.

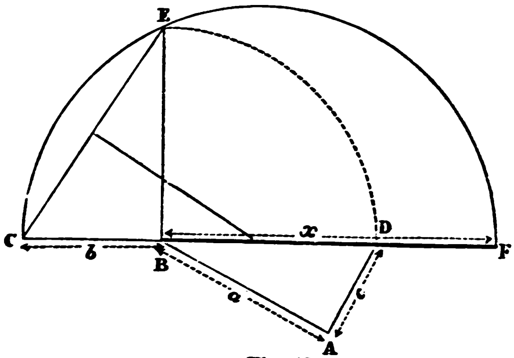

Fig. 49.

7. In this case, as $a^2+2ab+b^2=(a+b)\ (a+b)$. The value of x is a square drawn upon a line whose length is equal to that of $a+b$.

Problem XLVIII.

To construct a triangle ABC, base AB = 3″, and the sines of two of its angles to be $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$ respectively. (Fig. 50.)

First determine the value of $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$,* as shown in fig. 50 (a). Then these lines will represent to radius 1″ the relative values of the sines of the two base angles. Next draw the base AB 3 inches long, and at distances equal to the lines just determined draw MN and

* Note that $\frac{1}{\sqrt{2}}$ is equal to $\frac{\sqrt{2}}{2}$.

PQ parallel to AB. With A and B as centres, radius 1″, draw arcs to intersect these parallels in D and E. Then the triangle is completed by joining AD and DE, and producing these lines until they meet in C. For DG = AD sin DAG $= 1'' \times \frac{1}{\sqrt{2}}$. It will be noticed that the angles having these sines are 60° and 45° respectively.

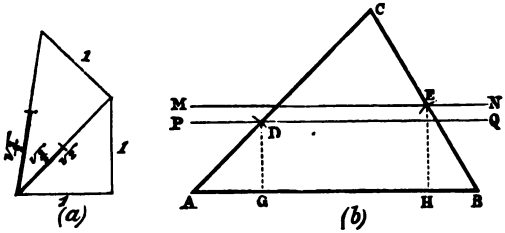

Fig. 50. (½ scale.)

EXERCISES.

1. Determine a square equal in area to a rectangle 3″ by 2″, and a triangle 3″ by 2″ by 2½″ added together.

2. Given any parallelogram with sides 3 : 2, to construct an equal parallelogram having the same angles, but whose sides shall be as 2 : 1. (May Exam. 1871.)

3. Divide a hexagon of 1·5″ side into eight equal parts by lines drawn through one of its angular points. (May Exam. 1877. Honours.)

4. Draw a semicircle of 4″ diameter, and another one-third of its area.

5. Given a triangle ABC; AB=2″; BC=3″; AC=2·5″. Required, a right-angled triangle equal in area, and having one of its angles equal to ABC. (Honours.)

6. Make an equilateral triangle of 6 square inches area.

7. Draw a square of 2·5″ side and a heptagon equal to it in area.

HINT.—Convert the square first into a triangle, and proceed by Problem XXXII.

8. Make an equilateral triangle equal in area to the difference between a square of 4″ side and a pentagon of 2″ side.

9. Find the value of x in the following expressions, the unit being 1 inch:—

$$x=\sqrt{5}+1; \quad x=\frac{2\sqrt{3}}{6}; \quad x=\sqrt{5\tfrac{5}{8}}; \quad x=\frac{7}{\sqrt{3}}; \quad x=\sqrt{2\tfrac{1}{2}\times 1\tfrac{1}{3}}.$$

10. Construct a square equal in area to the sum of the areas of three squares of 1″, 1·5″, and 2·25″ side. (May Exam. 1877.)

11. In a square of 3″ side, inscribe another three-fourths of its area. The corners of the required square must be in the sides of the given one. (May Exam. 1870.)

Hint.—Having found the *side* of the required square, determine the radius of its circumscribing circle, then describe this circle so as to cut the sides of the given square.

12. Produce a line 3·5″ long, so that the rectangle contained by the whole produced line and the part produced may equal the square of the given line. (May Exam. 1870.) (Honours.)

Hint.—The produced line will be medially divided at one extremity of the given line.

13. In a circle of 3″ diameter inscribe a rectangle of 4 square inches area.

Hint.—Using the diameter as a base, inscribe a triangle in each semicircle equal to half the area of the rectangle, the height of this triangle being 4 square inches divided by 3″, or $1\frac{1}{3}$″.

14. Divide a line 3·5″ long into two segments, so that the area of a rectangle contained by these segments may be $1\frac{1}{2}$ square inches.

Hint.—The $\sqrt{1\frac{1}{2}}$ is evidently the mean proportional between the two segments. Hence a semicircle must be made upon the given line, and a point in it found which is $\sqrt{1\frac{1}{2}}$ distant from the given line perpendicularly.

15. Construct a square of 1·25″ side and a hexagon of 1″ side. Construct a triangle having an angle of 60° and an area equal to the sum of the areas of the two figures. (Cooper's Hill Engineering College Exam. 1876.)

16. A parallelogram, whose sides are 4″ and 1·2″ and included angle 50°, is given. Determine a rhombus of equal area and having the same included angle. (May Exam. 1871.)

Hint.—The required side is a mean proportional between the two different sides of the parallelogram.

17. Draw a triangle whose sides are AB=2·5″; BC=3″; AC=3·5″. Take a point P in AC, ·75″ from A, and divide the triangle into 7 equal parts by lines drawn through P. (May Exam. 1871.)

18. Draw any irregular pentagon, and bisect it by a line drawn through one of its corners.

19. Divide an equilateral triangle 3″ side into 4 equal parts by perpendiculars to one of the sides.

20. Make a nonagon equal in area to a pentagon of 2·5″ side.

21. Draw an equilateral triangle on a base 2″ long, and divide it into 3 parts, in the proportion of the numbers 2, 3, and 4, by lines passing through P—a point 1″ from one corner and 1·5″ from another. (Honours.)

22. Construct a square equal in area to the difference between an equilateral triangle of 2″ side and a scalene triangle 2″ by 1″ by 1·5″.

CHAPTER V.

ON CIRCLES IN CONTACT AND THEIR TANGENTS.

THE following theorems demonstrated by Euclid are recalled to the reader's recollection before commencing the problems of this chapter, as many of the methods of solution employed are based upon those theorems :—

1. When a straight line touches a circle, it does so in *one* point only, and the radius drawn through that point is perpendicular to the touching line or tangent (Euclid iii. 16 and 18).

2. When two circles touch one another, they do so in *one* point only, and the straight line which joins their centres passes through the point of contact (Euclid iii. 11, 12, and 13).

3. Only two tangents can be drawn from one point outside a circle, and the angle included between these two tangents is the supplement of the angle made at the centre by the two radii, drawn through the points of contact.

4. If from a point outside a circle two straight lines be drawn, one to cut the circle and the other to touch it, then the rectangle on the whole secant, and the part of it outside the circle, is equal to the square on the tangent line (Euclid iii, 36).

5. From the last theorem it follows that the rectangles on all secants and their relative outside segments, if they be drawn from *one* point, are equal, as each is equivalent to the square on the tangent line through the same point.

6. If two chords of a circle intersect, the rectangles upon the segments of each chord are equal (Euclid iii. 35).

SECTION I.—CIRCLES AND STRAIGHT LINES IN CONTACT.

PROBLEM XLIX.

To draw a straight line tangent to a given circle, and passing through a given point; (1) upon its circumference, and (2) without the circle. (Fig. 51.)

1. As the tangent is perpendicular to the radius at the point of

contact, the line BD drawn at right angles to AC (the radius through the given point A) will be the line required.

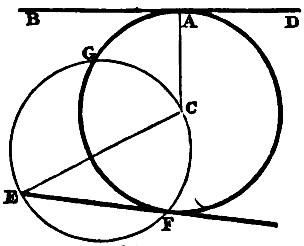

Fig. 51.

2. Assuming E (fig. 51) as the given point, it will be at once seen that there can be two solutions. Join E to the centre C, and describe a circle having EC for its diameter. The points F and G, where this circle intersects the given one, are the points of contact of the two tangents referred to. Join EF or EG.

NOTE.—If C were joined to F, then CFE would be a right angle, as it would be contained by a semicircle (Euclid iii. 31.)

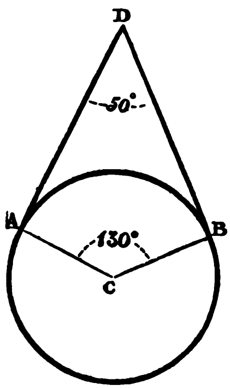

Fig. 52.

PROBLEM L.

To draw two tangents to a given circle, to meet at a given angle. (Fig. 52.)

Assuming the meeting angle as 50°, the contact points can be at once determined by drawing two radii whose included angle is 180°, less the given angle. In this case, therefore, it will be 180° − 50°, or 130°. The required tangents are perpendicular to these radii.

PROBLEM LI.

Given two unequal circles, to determine the four straight lines which shall be tangent to both. (Fig. 53.)

Join the two centres A and B, and describe a semicircle on AB. On each side of K make K*a* and K*b* equal to the radius of the smaller circle. Then A*a* will be the *difference* and A*b* the *sum* of the radii of the two given circles. Through *a* and *b*, with centre A, describe arcs to meet the semicircle in *d* and *e*. Join A*d* and produce it to D. Join A*e*. Draw BC parallel to AD, and BH parallel to A*e*. Then C and D are the two points of contact of the tangent which touches both circles on the same side, whilst G and H are the points of contact of a tangent which passes between them. The two others, EF and MN, are shown in the drawing, and, being symmetrical about AB with those already determined, they can be found without further description.

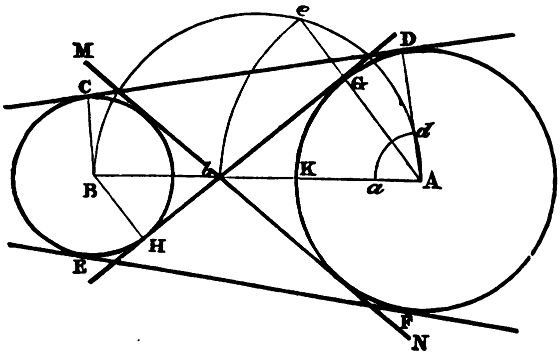

Fig. 53.

The principle of the construction is that the smaller circle is supposed to be reduced to a point B, and the larger one is diminished in radius as much as the smaller. If B were joined to *d*, it would be seen at once that B*d* would be a tangent to this diminished circle. Then, by the construction, this line is, as it were, removed through a distance equal to that by which the given circles were reduced, *i.e.*, CD is parallel to B*d*, at a distance from it equal to the radius of the smaller circle. The same principle applies to GH, which is parallel to a line B*e*, not shown in the figure.

Problem LII.

Given a circle of 1″ radius to find a point in the diameter produced, such that the tangent from that point may be 1·5″ long. (Fig. 54.)

At one extremity, as A, of the given diameter, set out AB a tangent to the circle, and in length equal to the one required. With centre C and radius BC, describe an arc to meet the diameter produced in F. This is the required point, for the arc BF is a locus of all points, from which tangents to the given circle are equal in length to AB.

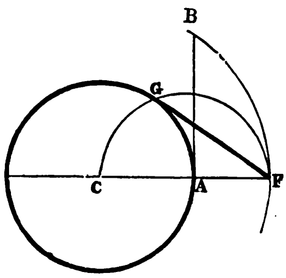

Fig. 54.

Problem LIII.

To describe a circle through three given points which are not in one straight line. (Fig. 55.)

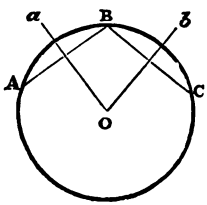

Fig. 55.

Let ABC be the given points. Join them, and draw two lines perpendicular to and bisecting AB and BC respectively. The intersection of these two lines is the centre of the circle required.

Problem LIV.

To describe a circle which shall pass through two given points, and which shall touch a given straight line. (Fig. 56.)

This problem is possible with any given relative positions of the points and line, so long as the former are on *one* side of the latter.

If the two points are equidistant from the straight line, then a third point in the circumference (the point of contact of circle and given line), is found by a perpendicular bisecting the line joining the given points. But if, as in fig. 56, they are not equidistant, then proceed as follows : — Join the given points A and B, and produce the line AB in one direction till it meets the given line in C. This line AC is then a secant of the required circle, meeting a tangent to it in C, and by the Principle 4, cited at the commencement of this chapter, AC.BC must equal CE^2. Make CE, therefore, a mean proportional between AC and BC, as shown. Then, having A, B, and E as 3 points, proceed to describe the required circle through them.

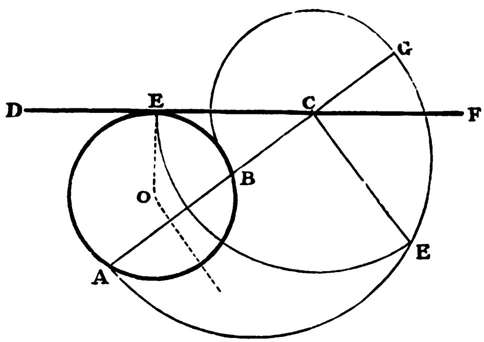

Fig. 56.

Problem LV.

To describe a circle which shall pass through two given points and touch a given circle. (Fig. 57.)

In fig. 57 A and B are the given points, and the circle (centre C) the given one. Join AB, and bisect by an indefinite perpendicular DE. This line is at all points equidistant from A and B. Take then any point, as F, upon it, and describe a circle to pass through A and B, so as to intersect the given circle in two points, H and K. Then HK produced will meet AB produced in T. From this point draw the tangent TM to touch the given circle in M.* From C draw CM perpendicular to TM, and produce it to meet DE in Q. Then Q is the centre of a circle ABM, which satisfies the conditions of the problem. There is, of course, a second solution, in which the required circle passes through A and B and includes the given one. Its centre is at P, which is found by drawing the second tangent TN, and producing the radius NC to meet DE in P.

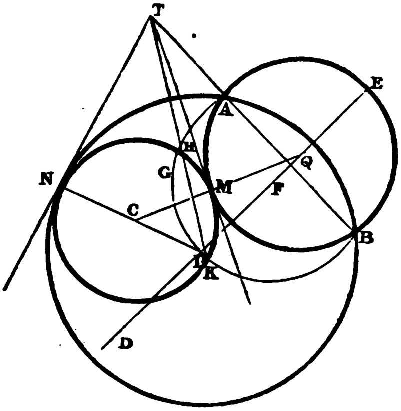

Fig. 57.

The property of the chord and tangent (Euclid iii. 36), described in the previous problem, is taken advantage of in the above solution. For it will be readily seen that, as BT.AT is equal to TK.TH (both being secants of the same circle), and as also TK.TH is equal to TM^2, therefore TM must be a tangent to the required circle.

Problem LVI.

In a given angle to inscribe a circle (1) of given radius; (2) to pass through a given point P within the angle. (Fig. 58.)

1. Let AED (fig. 58) be the angle in which a circle of ·3″ radius is to

* The student should notice that if A and B had been equidistant from the given circle, the chords HK and AB would be parallel. The tangent in that case would also be parallel to these two lines.

be inscribed. Bisect the angle as at EF, and draw a line HK parallel to one of the legs of the angle, and at a distance from it equal to the required radius. As each of the lines EF and HK is a locus of points, satisfying one of the two conditions of being equidistant from AE and DE, and of being ·3″ from one of them, it is clear that the point O satisfies both; it is therefore the centre required.

2. On the same figure, the solution of the second portion of the problem is shown. P is the given point. Any circle, as SMT, is first drawn, touching both legs of the angle. Then P being joined to E, PE meets the circle SMT in M. Then M on this circle corresponds to P on the required one. Join M to N (the centre), and draw PQ parallel to MN. Then Q is the centre required. This needs no further proof.

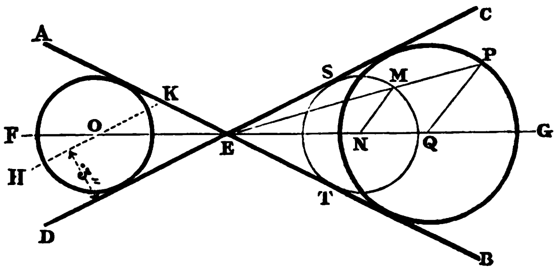

Fig. 58.

Problem LVII.

To inscribe a succession of circles in a given angle, each one touching two others and the lines of the angle. (Fig. 59.)

In fig. 59, the given angle is ABC, and one circle (centre D) is supposed to be a fixed or given one, the problem being to find the centres of two others (one on each side) which shall touch the fixed one, and be inscribed in the angle. From D set out a radius DE, perpendicular to AB. At F draw FG, perpendicular to BQ, meeting AB in G. Then GE and GF are two tangents drawn to one circle (centre D) from one point G; hence (Principle 3, page 59) they are equal. Take then G for centre, and with GF as radius describe an arc EFH, to meet AB in H. Then, as GH and GF must both be tangential to the next smaller circle, and as they must also be equal, H must be the point of contact, and the radius through H, perpendicular to AB, therefore discovers the required centre K. The larger circle is determined in the same manner.

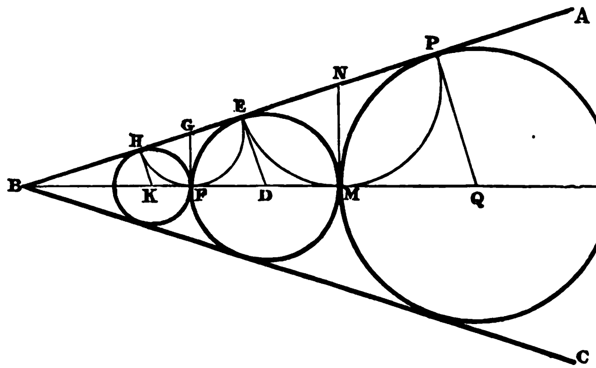

Fig. 59.

Problem LVIII.

Given a circle and a straight line, to describe another circle of given radius to touch both. (Fig. 60.)

Generally two different solutions of this problem are possible—one, where the required circle touches the given one exteriorly; and the other, where it includes it. Both cases are shown in the figure. Further, the given line may cut the given circle, and with such a condition one of those required must necessarily *be included by the given one.*

Case 1. Let AB be the given straight line, and the circle (centre C) the given circle; and let another circle be required, of radius ·5″, to touch the line and circle *exteriorly*. At a distance of ·5″ (the given radius) from AB, draw a line *ab* parallel to it. This is a *locus* of all points ·5″ from AB; hence the centre of the required circle is in this line. Next draw the arc of a circle, *gh*, concentric with the given circle, but having a radius ·5″ larger, and intersecting the line *ab* in F. Then F will be a point ·5″ away from both given circle and line. It is therefore the centre of the required circle. The points of contact are R and S, R being contained by a radius FR, perpendicular to AB, and S being in the line joining C to F.

CASE 2. In this case the required circle (1·75″ radius) is to include the given circle, *i.e.*, it must touch it *interiorly*. As before, draw *a′b′* parallel to AB, and 1·75″ from it. Next draw a line *yz* through the centre of the circle, and make it 1·75″ long, measured from *y*. With C as centre, radius C*z*, describe an arc *st* to meet *a′b′* in G. Note that *st* is an arc of the circle, which is the *locus* of all points that are 1·75″ from the *opposite* side of the given circle. The required circle is LK, centre G.

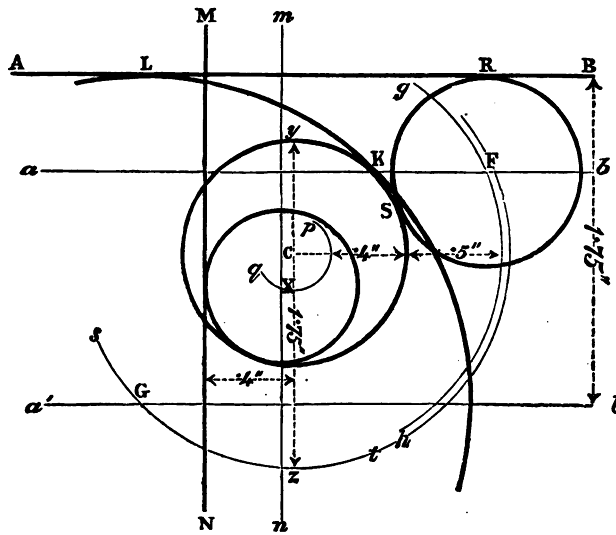

Fig. 60.

A *third case* is shown on the figure, in which MN is the given line cutting the given circle. In such a case it is evident that the required circle cannot *include* the given one, but may be included by it. The radius is taken as ·4″ in this instance, *mn* being parallel to MN, and the arc *pq* being ·4″ nearer to the centre C than to the given circle. The intersection of *mn* and *pq* gives X, the centre of the required circle.

PROBLEM LIX.

To draw a circle of given radius, to touch two given circles, centres A and B (1) exteriorly; (2) interiorly. (Fig. 61.)

CASE 1. This is a similar problem to the previous one. The first case is solved by the circle (centre C), as *mn* and CD are arcs concentric with the two given circles A and B respectively, and at a distance from them equal to the given radius, which is taken as ·5″.

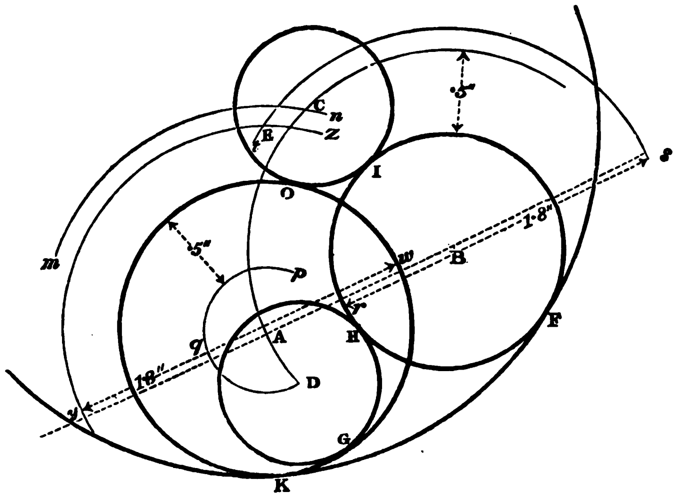

Fig. 61.

CASE 2. The second case is satisfied by the large circle, centre E. This point is found as follows :—A being joined to B, and produced in both directions, *wy* and *rs* are each made equal to the given radius (1·8″.) Then taking A and B as centres in turn, two arcs, *yz* and *st*, are drawn through *y* and *s*. The intersection of these in E determines a point which is 1·8″ from the opposite sides of each of the given circles. It is therefore the centre required. The circle (centre D) is drawn to illustrate the case in which the required circle is to touch one circle exteriorly, and the other interiorly. The diagram explains itself sufficiently as to how the centre D is obtained.

PROBLEM LX.

To describe a circle which shall be tangential to a given straight line at a given point, and shall also touch a given circle. (Fig. 62.)

As in the two previous problems, so in this one, there may be several cases for solution resulting from the relative positions of the given data.

Case 1. Here the required circle is to touch the given one (centre C) exteriorly, AB being the given straight line, and P the given point. Draw QH through P, perpendicular to AB, and take PQ *above* the line equal to the radius of the given circle. Join QC, and bisect it by a perpendicular *pq*. This line will intersect QH in F, the centre of the required circle.

Proof.—If F be joined to C, then Q*f*F and C*f*F will be equal and similar triangles (Euclid i. 4); hence QF=CF; but PQ, a part of QF, is equal to CG, a part of CF; therefore the remainder PF=FG.

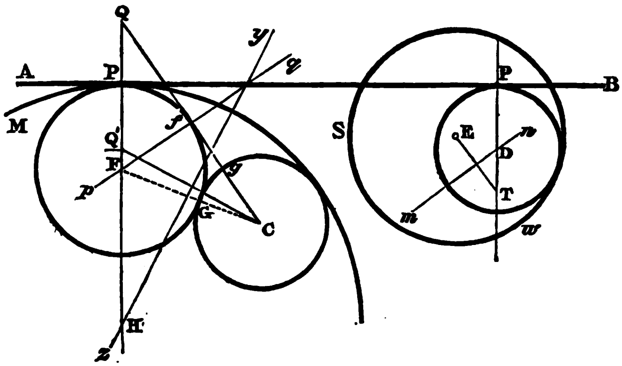

Fig. 62.

Case 2. Here the required circle is to touch the given one interiorly. In this case take PQ′ *below* AB equal to the radius of the given circle, and proceed as before.

Case 3. A third case is shown in the figure, where the given line cuts the given circle. In this instance, the circle to be determined *must* be included by the given one. The construction is similar to the second case.

Problem LXI.

To describe a circle which shall be tangential to a given circle at a given point, and also to a given straight line. (Fig. 63.)

In the figure, AB is the given straight line, and C and M are two given circles, P and P′ being given points upon their circumferences. Three solutions are shown, in the first of which the circle determined (*centre* O) touches the given one *exteriorly;* in the second, it (centre E) touches it *interiorly;* and lastly, the given line is shown cutting the given circle, and, in this instance, the one determined (centre N) is necessarily included in the given one.

Case 1. As the fixed point P is the point of contact of the given circle and the required one, the centre of the latter must be in a line joining E to P produced (Euclid iii. 12). Hence a line PQ, perpendicular to CP, must be a common tangent to both circles. Then as PQ and AB are both to touch the required circle, the line OQ, bisecting the angle between them, must contain the centre. Hence O is the centre.

Case 2. If the angle AQP be bisected by QE, the intersection of this line with PC produced will give the centre E of a circle fulfilling the conditions of the problem, and touching the given circle *interiorly.*

Case 3. As before, MP′ produced must contain the centre, and P′T is a tangent to the required circle. Hence TN, bisecting the angle ATP′, gives N, the required centre, upon MP′.

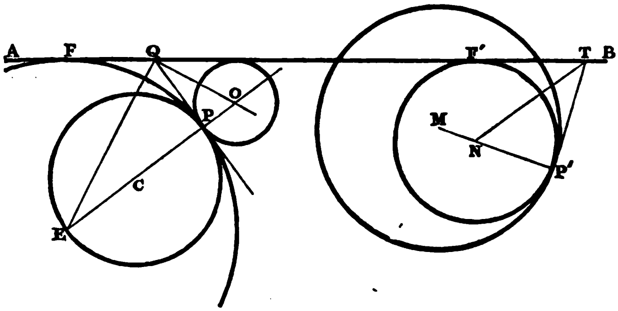

Fig. 63.

Problem LXII.

To describe a circle to touch two given circles, one of them in a given fixed point. (Fig. 64.)

Let A and B (fig. 64) be the centres of the two given circles, and P the fixed point upon one of them.

Case 1. *Where the required circle touches both and includes them.*—Join A to B and P to A. Through B draw BC parallel to AP. Join PC and produce, to meet the circle B in D. Join BD and produce, to meet AP, also produced in E. Then describe the required circle with E as centre.

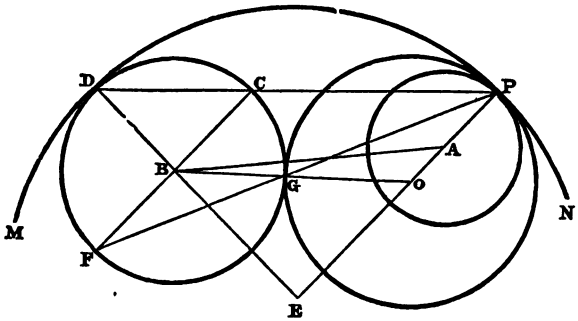

Fig. 64.

Case 2. *Where the required circle touches both and includes one of them.*—Proceed as before to obtain BC. Produce it to F, and join FP. Then G, the intersection of FP with the given circle, is the point of contact of the required circle with it. Join BG, and produce to meet PE in O. Then O is the centre of the required circle.

Proof.—The angle DCB is equal to DPE (Euclid i. 29), but DCB is equal to CDB, for BD=BC. Hence EDP is equal to DPE, the triangle is therefore isosceles, and ED is equal to EP. Further, D must be the point of contact, as it is in the line ED passing through the centres of both circles. A similar proof applies to the second case.

Problem LXIII.

Given a circle, a straight line, and a point between them, to describe a second circle which shall pass through the given point, and be tangential to both line and circle. (Fig. 65.)

Through C, the centre of the given circle, draw EF perpendicular to AB, the given line. Next describe a circle to pass through P, G, and F. Join E to P. Then the point Q, where EP meets the circle just described, is a second point in the circumference of the one required. Proceed then by Problem LIV. to find T, the point of contact of the line and circle, and hence obtain the centre D.*

Note.—If HT were measured along AB on the other side of H, we should obtain the tangent point of another and larger circle, which would pass through P and Q, and touch the given circle in a point further from AB.

* The whole of the construction lines are left in the figure.

PROOF.—Join ET, and draw GM perpendicular to it. Then ETF and EGM are similar triangles, both being right-angled, one at F and the other at M (Euclid iii. 31), and having one angle TEF in common. Hence ET : EF :: EG : EM, or ET . EM = EF . EG. But EP . EQ = EF . EG. Therefore EP . EQ = ET . EM, and hence a circle can be drawn through P, T, M, and Q (Euclid iii. 36).

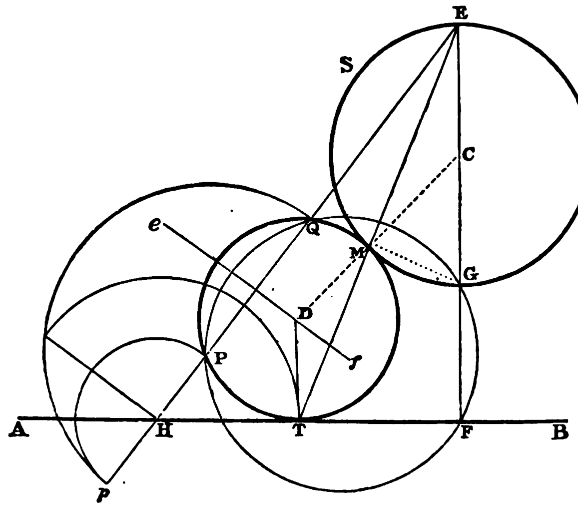

Fig. 65.

Next, to prove that M is the point of contact of the given and required circles, join DM, MC. Then in the two isosceles triangles, ECM, MDT, the angle MEC = DTM (Euclid i. 29). But EMC = MEC, and DMT = DTM; therefore DMT is equal to EMC, and DC is a straight line (Euclid i. 15), and M is the point of contact, as it is in the straight line joining the centres of the circles (Euclid iii. 12).

Several problems have been omitted from the chapter which would seem to most suitably find their place at this stage. Such questions, for instance, as the following :—

a. "To describe a circle which shall touch three given circles."

b. "To describe a circle to be tangential to two given circles and a given straight line."

These problems have been omitted because they are generally solved by methods which do not admit of mathematical proof. For example, the former of the two quoted above would be determined as follows:— A curve would first be obtained which at all its points is equidistant from two of the given circles (see Problem LXXXVII., Chapter VI.). Then a second curve would be found which is equidistant from one of these and the third circle. The intersection of these two curves would give the centre of the circle required, as it would be a point equidistant from each of the three given ones.

There are solutions, however, to some of these problems which are based on the properties of the straight line and circle only, and which can be proved from Euclid's theorems, but they are intricate, and, although ingenious, are not so suitable to the practical draughtsman as the methods previously referred to. In the following chapter the determination of these and other curves is demonstrated, and the student will, after its perusal, be enabled to more satisfactorily appreciate the gist of the preceding remarks.

SECTION II.—SOLUTIONS OF PROBLEMS UNDER PECULIARLY RESTRICTED CONDITIONS.

It frequently happens that the draughtsman requires solutions to simple geometrical problems which he cannot obtain by the ordinary methods. For lines may meet at points so remote as to be practically "out of bounds," or parts of circles may have to be struck whose centres are inaccessible. It is proposed therefore to insert here a few solutions of characteristic questions of this class.

PROBLEM LXIV.

Given two straight lines which are not parallel, to bisect the angle between them without using the angular point. (Fig. 66.)

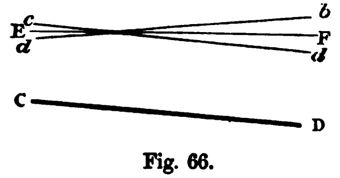

Fig. 66.

Let AB and CD (fig. 66) be the given lines. Draw two parallels, *ab* and *cd*, to the given lines, and equidistant from them, choosing such a distance that the parallels may meet. Then bisect the angle between them by the line EF, which will be the one required.

PROBLEM LXV.

Given two straight lines which approach each other, given also a fixed point; through the point to draw a third straight line which would meet the two given ones in their intersection, that intersection being inaccessible. (Fig. 67.)

Let AB and CD (fig. 67) be the given lines, and P a given point. Draw any line *ab* to meet the given lines, and join *a*P, *b*P. Next draw any parellel *cd* to *ab*, and through *c* and *d* draw parallels to *a*P and *b*P respectively, meeting in T. Then the straight line passing through P and T is the one required.

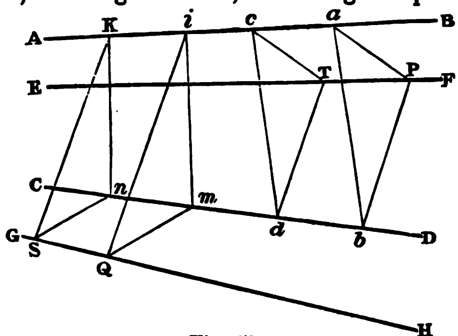

Fig. 67.

When the given point is outside the given lines, as at Q, the solution depends on the same principle, *im*Q and K*n*S being similar triangles.

PROBLEM LXVI.

Given two straight lines which approach, and a point between them; through the given point to draw a straight line which shall meet the two given lines at equal distances from the given point. (Fig. 68.)

Let AB and CD be the given lines, and P the given point. Through P draw any line GH, meeting CD in H. Then make PG equal to PH, and through G, draw GF parallel to CD, to meet AB in F. Join FP, and produce it to meet CD in E. Then EF is the line required. Note that the triangles GPF and EPH are equal and similar. Hence PF = PE.

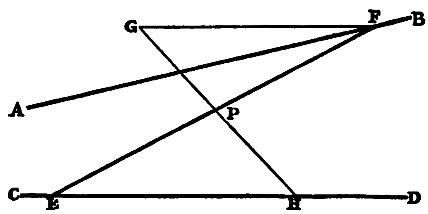

Fig. 68.

Problem LXVII.

To describe the arc of a circle which shall pass through three given points not in the same straight line, the centre being inaccessible. (Fig. 69.)

Assuming it to be desired to draw through A, B, and C (fig. 69), the arc of a circle, the centre being too remote to be available, points in it can be found as follows, and the curve can then be traced through them. Join AB and BC, producing the lines as shown. With A and C as centres, radius equal to AC, describe the arcs AE and CD. From G and H set off along the arcs, on each

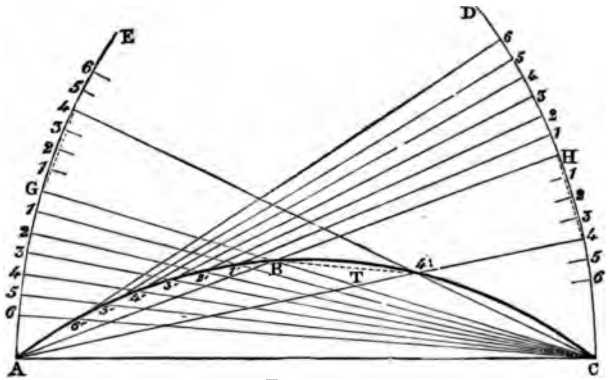

Fig. 69.

side of the two points, equal distances, as 1, 2, 3, 4, 5, etc. Join the points on each arc to the centre with which that arc was struck. Then the intersections of the corresponding lines, taken on alternate sides of G and H, will give points in the arc required. Thus 1C *below* G meets 1A *above* H in 1′, and so on. The points are marked for obtaining the arc from B to C, but only one pair of intersecting lines is shown, to avoid confusion.

This construction depends upon the building of a series of triangles (upon the chord AC), all having the same vertical angle. Thus the angle ABC is equal to the angle $A4_1C$. They can be proved to be equal thus:—Join G4 and H4, as shown by dotted lines. Then angle GC4 = angle HA4 (Euclid iii. 27). But the angles of a triangle together make two right angles. Hence the angles ABT + BAT + BTA = $C4_1T + 4_1TC + TC4'$. But 4′TC = BTA (Euclid i. 15), and therefore the angle ABT = $A4_1C$, and as this can be proved for each triangle, *the curve* passing through their apices is a circle (Euclid iii. 21).

Problem LXVIII.

Given the arc of a circle to draw a tangent to the arc (1) to pass through a point in it; (2) through a point outside it; the centre of the circle not being available. (Figs. 70 and 71.)

1. Let AB be the arc, and C the given point. Mark off equal distances on each side of C, as at E and F. Join EF, and draw CG perpendicular to EF. Then ST, the required tangent, will pass through C perpendicular to CG.

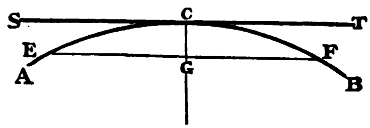

Fig. 70.

2. In this case, let F (fig. 71) be the given point. Draw through F a secant FGH. Then a mean proportional between FG and FH will give the length of the tangent to the arc from F. This is shown at FC′, FG′ being equal to FG. With F as centre, FC′ as radius, describe an arc meeting the given one in C, and draw the required tangent through F and C.

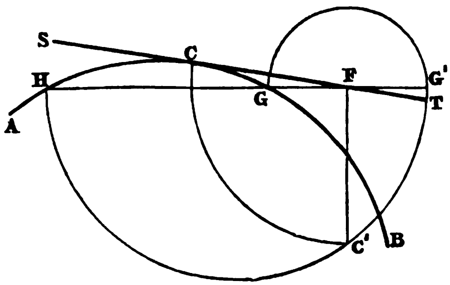

Fig. 71.

Problem LXIX.

Given two straight lines which approach, and a point between them; to describe a circle which shall pass through the given point and touch the given lines; the meeting point of the given lines being unavailable. (Fig. 72.)

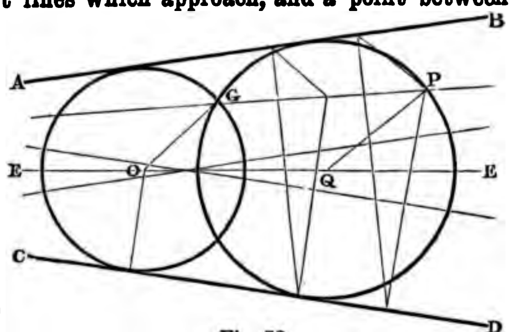

Fig. 72.

To solve this problem, first find the line bisecting the angle between those given (Problem LXIV.). Next proceed, as in Problem LVI., to determine the required circle, the direction of the line PG being found by the method shown in Problem LXV.

Problem LXX.

Given three parallel lines and a fixed point upon one of them, to draw an equilateral triangle having an angular point on each given line (one of them being the given point). (Fig. 73.)

Let AB, CD, and EF (fig. 73) be the given parallels, 0 being the fixed point on one of them. Draw any line *ab* perpendicular to those given, and mark off on it *ad* equal to *bc*. With *a* and *b* as centres, radius equal to *ab*, describe arcs meeting in *e*. Join *ed*. Then draw through 0 a line PO perpendicular to the direction of *ed*. This is one side of the required triangle.

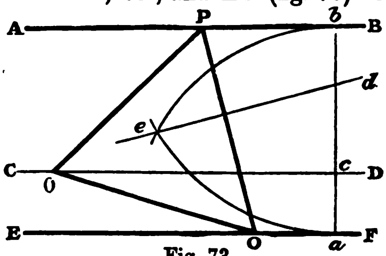

Fig. 73.

Note.—This problem is of special value when the projectors of a vertical equilateral triangle are given, and an elevation of the figure is desired (see Solid Geometry).

EXERCISES.

1. Given a point, a straight line, and a point in the line, to describe a circle which shall pass through the first point and touch the straight line in the other point.

2. Three circles have radii ·5″, ·75″, and ·9″ respectively. Each is in contact with the other two. Show them.

3. Two circles of 1·75″ and 1″ diameter have their centres 1·5″ apart. Draw a circle of 3·5″ diameter to touch both the former and include them.

4. Draw any three unequal circles not in contact, and determine all the straight lines which can be tangential to two of them.

5. Draw an isosceles triangle, base 1·5″, sides 3″ long, and inscribe three circles in it (not equal), each touching the other two.

6. Take any three lines which make angles with each other, and describe a circle tangential to the three without using the angular points.

7. Draw a semicircle on a line 2.5″ long. In it inscribe a circle.

8. Two straight lines meet at 60°. Draw a touch to touch one of them, and to pass through the other in two points at distances of 1·4″ and 2·75″ from the angle.

9. Draw a circle of 2·5″ diameter. Through the centre draw a line, and mark two points upon it at distances of 1·5″ and 2·25″ respectively from the centre. Describe a circle to pass through the two points and touch the given circle. (May Exam. 1869.)

10. The centre A of a circle of 1·5″ radius is 1·75″ from a line; a point P in the line is 2″ from A. Draw a circle to touch the line in P and to touch the circle, but to contain it within it. (May Exam. 1875.)

11. Draw a small portion of a large circle, and find points in the continuation of the curve, not having recourse to the use of the centre.

12. Draw a circle of 3″ diameter, and two others to be included by the larger one, and to touch each other. The radii of the smaller circles to be ·5″ and ·75″ respectively.

CHAPTER VI.

ON PLANE CURVES, THEIR TANGENTS AND NORMALS.

THERE are certain curves, the determination of which is frequently necessary in preparing geometrical and mechanical drawings. Amongst these are the ellipse, parabola, and hyperbola, the cycloid and its allied curves, the spiral, the involute, etc. These cannot be drawn by the ordinary compass, and attempts to substitute small arcs of circles with ingeniously contrived systems of centreing, should be avoided. The draughtsman should therefore adopt the device of finding accurately a series of points in the line required, and should then trace it through these points, either by hand, or with the aid of flat moulds of wood or ebonite, termed French curves.

The known properties of the lines should form the bases of his constructions. These properties are demonstrated by mathematicians. The accuracy of the result would of course depend partly upon the number of points found, but greatly also upon his own judgment and the manual skill shown in drawing the final line.

It should be mentioned here, that practice will after a time enable one to instantly detect any imperfection in the shape or continuity of the curve, and it will then be found that a few points, truly determined, will be considered quite sufficient to obtain the desired line. It is of special importance, where possible, to draw some of the tangents to these curves,* as they are of the greatest utility in acting as guide-lines. A few of these can be rendered of more service than even many points, as, by their aid, the eye is led to picture to the mind the direction and sweep of the whole curve.

The student, before commencing to solve any of the problems of this chapter, should make himself well acquainted with the properties of the lines under consideration. He will find that all of these are given which are necessary for him to know. If he desires the proofs of the statements made, he is referred to mathematical works upon the subject.

* The circle is the only curve whose tangent can be drawn at once from a point upon it. In all other instances, a second point has to be determined by constructions based upon the known properties of the curves.

DIVISION I.—THE CONIC SECTIONS.

There are three curved lines which are deducible from plane sections of the right conical surface. These are termed the Conic Sections. They are the Parabola, the Ellipse, and the Hyperbola.

Generally speaking, the common notion of a right cone is that of a pyramid with circular base. Euclid defines it as being generated by the revolution of a right-angled triangle about one of the sides containing the right angle. It is best for our purpose, however, to conceive of the surface being developed by the rotation of a straight line which always meets a fixed one at a constant angle, as in fig. 74, where the line CD is supposed to revolve about the fixed line AB, the two meeting in E at the same angle, at all periods of the rotation. By this conception, the conical surface becomes doubled and bisymmetrical on either side of E. The three curves are then deducible by three plane sections of the surface; thus:—

1. *The Parabola.*—The cutting plane is taken parallel to one of the positions of the moving line CD, and perpendicular to the plane containing this line and the fixed axis, as at GFH. The curve thus formed, it will be readily seen, is unlimited, and can have only one branch, for the conical surface is supposed to be indefinite, and the section plane only intersects one portion of it.

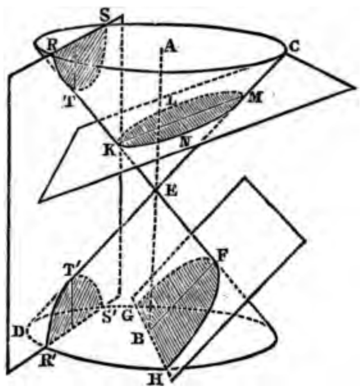

Fig. 74.

2. *The Ellipse.*—The cutting plane is taken in such a manner as to intersect the rotating line CD in all its positions. This gives the ellipse (KLMN.) The curve is necessarily a closed one, and limited.

3. *The Hyperbola.*—The cutting plane is taken parallel to the fixed line AB, as at RST. The section in this case gives the hyperbola; but, as the plane cuts both portions of the conical surface, the curve will be double, *i.e.*, it will have two branches, and be unlimited.*

* The student will notice that two sections of the cone are here omitted, one where the cutting plane contains the axis, and the other where it is perpendicular to it. In the former case, two lines meeting in E, and in the latter a circle, result as the sections. It is usual to consider the first as a particular case of the hyperbola, and the second of the ellipse.

The student will find that in Chapter XI., Solid Geometry, each of these curves is deduced from the right cone as above described, and the necessary constructions are there given. But it is proposed at this stage to treat these lines in quite a different manner. It is more convenient to discuss the plane geometry of them, by considering them as being generated by the movement of a single point under restricted conditions over a plane surface. For instance, the parabola is generated by a point moving in a plane so that in all its positions it is equidistant from a fixed straight line and a fixed point. This will be therefore the definition of the curve adopted in the constructions described, and the other two will be treated in similar ways.*

THE PARABOLA. (Plate IV., fig. 1.)

The following definitions and properties should be understood before commencing the problems:—

1. The parabola (YAZ, Plate IV., fig. 1) is a curve which, at all its points, *a*, *b*, *c*, etc., is equally distant from a fixed point S, called the *focus*, and from a fixed straight line XX, called the directrix. Thus S*c* = *ct*.

2. DE is the axis; A is the vertex; *e*L and *f*K are ordinates of the axis; LA is an abscissa of the axis relative to the ordinate *e*L; GH is a tangent at M; MN, perpendicular to GH, is a normal; any straight line parallel to the axis, and intersecting the curve, as RT, is a diameter.

3. A tangent meets the curve in a single point as at M, and bisects the angle between the two lines which measure the distances of the contact point from the directrix and the focus respectively. Thus (see fig.) GH bisects the angle SMT.

NOTE.—The other two conic sections have double foci (see pp. 85 and 91); and if from a point in either of these curves, two straight lines be drawn to the foci, these lines are known as the "focal distances." Practically the parabola has but *one* focus; but it is convenient to consider the curve as possessed of a second focus infinitely distant. For then the general rule can be applied, that the tangent to either of these three curves bisects the angle between the focal distances at the point of contact (MT being drawn parallel to the axis.)

4. If the intersection of the tangent with the directrix be joined to

* The circle is a curve traced by a point moving in a plane at a constant *distance from* a fixed point.

the focus, and the focus to the point of contact, the angle at the focus is a right angle. Thus SM is perpendicular to SH.

5. The tangent meets the axis at a point such that its distance from the vertex is equal to the distance between the vertex and the intersection of the axis with an ordinate through the point of contact. Thus WA (see fig.) is equal to AP.

6. The normal is perpendicular to the tangent at the point of contact, and bisects the exterior angle formed by the focal distances. Thus NM (see fig.) is perpendicular to GH, and bisects the angle OMT.

7. The axis divides the curve into two symmetrical portions. Hence ordinates to the axis are bisected by it.

8. If a tangent line to a given parabola be given, and a perpendicular to it be drawn through the focus, these two lines will meet in a point which is contained by the tangent through the vertex. Thus (see fig.) ST′, perpendicular to JV, meets it in T′, which is contained by AT′.

9. Any diameter bisects all lines drawn parallel to the tangent, passing through the intersection of that diameter with the curve. Thus R′R (Plate IV., fig. 2) bisects *ab* and *cd*. These lines are called ordinates to the diameter R′R.

Problem LXXI.

Given the directrix and focus, to describe the curve of a parabola. (Plate IV., fig. 1.)

Let S be the focus, and XX the directrix. Through S draw DE perpendicular to XX. This is the axis. Bisect SQ in A. Then A is evidently a point in the curve, being equidistant from directrix and focus. It is the vertex. Next draw a series of indefinite lines perpendicular to the axis, and find points upon them as *a*, *b*, *c*, etc., in the following manner:—Taking the line through L, for example, measure the distance LQ, and with S as centre, radius equal to LQ, cut the ordinate in *e*. Then the distance of *e* from S is equal to its distance from XX. It is therefore a point in the required curve. Proceed in a similar manner to obtain a series of points, and draw the line through them. Note that the ordinate through the focus would meet the curve at a distance from it equal to SQ.

Problem LXXII.

To determine tangents and normals to a given parabola, its focus and directrix being known—1. Through a point in the curve; 2. Through a point outside it. (Plate IV., fig. 1.)

1. From the properties of the parabola and its tangents, previously described, it will be seen that there are several ways of solving the first part of the problem.

Let M (see fig. 1) be the given point. Join M to the focus, and draw MT parallel to the axis. Then the required tangent can be found by bisecting the angle TMS (par. 3).*

Or, from M draw the ordinate MP, and make AW equal to AP, and join WM (par. 5).

Or, join MS, and draw SH perpendicular to it, to meet the directrix in H. Then HM is the required tangent (par. 4).

The normal is in each case determined by drawing a perpendicular to the tangent.

It should be noticed that the directions of the normals give the lines of the side joints of the voussoirs or stones forming a parabolic arch.

2. When the given point is outside the curve, as at J (see fig.), the following construction may be used:—

Join J to the focus, as JS. On this line describe a semicircle, and through A, the vertex, set out AT′ perpendicular to the axis, and meeting the semicircle in T′. Then JT′ is the tangent required, which, being produced, meets the curve in V (par. 7).

The point V can be found exactly by producing JT′ to meet the directrix, and proceeding by the aid of the principle described in par. 4.

Problem LXXIII.

Given the curve of a parabola, to discover its axis, focus, and directrix. (Plate IV., fig. 2.)

Let YAZ be the given curve. Draw any two parallel lines each to meet the curve in two points, as *ab*, *cd*. These lines may be considered as ordinates to a diameter which passes through their centres (par. 8). Bisect them in M and N. Join and produce in both directions.

* *These* references relate to the principles tabulated at pages 11 and 12.

Then RR′ is a diameter. Draw *ef* perpendicular to RR′. It will be an ordinate to the axis which therefore bisects it. Hence the point P (centre of *ef*) is in the axis. A second point I is found by using another ordinate, or the axis can be drawn at once parallel to R′R.

To determine the focus, draw a tangent (parallel to *ab*) through T to meet the axis in W, and a normal TK perpendicular to it. Next make the angle KTS equal to the angle KTR′. Then TS intersects the axis in S, which is the focus.

To determine the directrix, mark off AQ along the axis, and outside the curve equal to AS. Then XX drawn through Q, perpendicular to the axis, is the directrix required.

Problem LXXIV.

Given an ordinate of the axis and the tangent at one extremity of the ordinate, to determine the curve of a parabola. (Plate IV., fig. 3.)

Let A′T be the given tangent, making an angle θ with the ordinate A′B′. Bisect A′B′ by a perpendicular PT. This line must be the axis of the curve. Bisect PT in A. This point is the vertex (see par. 5). Through A draw AT′ perpendicular to PT to meet A′T in T′. From T′ set out T′S perpendicular to A′T′ to meet PT in S. Then S is the focus of the parabola required (see par. 7), and the curve can be completed as described in Problem LXXI.

The parabola is the curve described by a projectile moving in a vacuum when fired at an angle less than 90° with the horizon. Hence this problem would be useful in determining the path of a ball fired from a mortar, etc., the inclination of the piece being given, and either the range (the length of the ordinate) or the vertical rise (the abscissa of the axis).

Problem LXXV.

To inscribe a parabolic curve in a given rectangle. (Plate IV., fig. 4.)

Let ABCD (fig. 4) be the given rectangle. Bisect AB in F and CD in E. Join EF. Divide AD and AF into the same number (n) of equal parts, as at 1 1′, 2 2′, 3 3′, etc., and number as shown, starting from A. Through 1, 2, 3, 4, draw parallels to AD, and through 1′, 2′, 3′, 4′, draw lines converging to E. The intersections of 11′, 22′, 33′, etc., give a series of points in the required curve.

The parabola is the locus of centres of circles passing through a fixed point, and touching a fixed straight line. The curve equidistant from a circle and a straight line is also a parabola. Such a curve is shown at HK (Plate IV., fig. 5), for at all its points it is equidistant from the line FG, and from the circumference of the circle, centre C. The focus of this curve is C, and its directrix would be a line parallel to FG, at a distance from it equal to the radius of the generating circle (CD).

Problem LXXVI.

To determine the locus of centres of circles which shall be tangential to a given straight line and to a given circle. (Plate IV., fig. 5.)

Through C draw CF perpendicular to FG. Bisect DF in H. This is the vertex of the required parabola. To discover points in the curve, proceed as follows, *a* being taken as an example. On either side of H set off H1′ and H1 at equal distances from H. Through 1 (the more remote from FG) draw a parallel to FG. Through 1′, with centre C, describe an arc to meet the parallel in *a*. Note that F1 must therefore equal D1′. Proceed in a similar manner to obtain other points, and trace the curve through them. Two circles having their centres on this line are shown. They are tangential to GF and the fixed circle, and touch one another. The centre of the second circle (*a*) can be found by trial, or by tracing a second parabola equidistant from the circle, centre H and the line FG. This is shown in the figure. The intersection of the two parabolas gives the centre.

The student will now be able to solve some problems specially mentioned as being omitted at page 71. A few of these are inserted here as suggestions for his practice.

1. Given a circle of 2″ radius, and a chord of the circle 2″ long.

Required, a second circle to pass through a point within the first one (not the centre), and to touch it as well as the given straight line.

Hint.—Find first the parabolic curve equidistant from the given point and the straight line. Secondly, determine a similar curve equidistant from the straight line and the circumference of the circle. Their intersection gives the centre.

2. Draw any two converging lines meeting at an inaccessible point, and also a circle between them, but not touching the lines. Determine a circle to touch the given one and the two straight lines.

3. Given a circle and any two chords which would meet if produced outside the circle. To describe another circle to touch both lines and the given circle internally.

4. Draw a quadrant of a circle 4·5″ radius, and the two tangents at its extremities. In the figure formed inscribe three circles—one large and two small, each of the latter to touch the former.

THE ELLIPSE.

It has been already shown how the ellipse can be traced by a plane section of a right cone. Our next step is to demonstrate how it may be generated by the motion of a point on a plane surface.

Definition.—If a point moves in such a manner that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line (being nearer to the point than to the line), the curve traced by the moving point is an ellipse. Thus, in Plate V*a.*, fig. 1), if XX represent the fixed line, S the fixed point, and A a second point nearer to S than to XX, if A be transferred to positions I, II, III, etc., still maintaining the same *relative* distances from S and XX, these positions will be in the curve of an ellipse. The line XX is a *directrix*, and the point S a focus.

Properties, etc.—The ellipse has two foci and two directrices; for another point S′ can be found on *aa′*, whose distances from XX and S are in the same ratio as $\frac{a\text{A}}{\text{AS}}$. Hence the curve is symmetrical about two lines, one of which contains the two foci, and is called the conjugate or major axis (AA′), and another, which is parallel to and equidistant from the directrices, and called the transverse or minor axis (BB′). These are mutually perpendicular. The extremities of the major axis (A and A′) are called the vertices.

1. Any straight line perpendicular to the major axis is called an ordinate, as GH.

2. The sum of the focal distances of any point on the curve is always equal to the conjugate or major axis. Thus $TS + TS^1 = AA^1$. Hence, (Plate V*a*, fig. 3*a*) $PF^1 + PF^2 = QF^1 + QF^2$.

This property of the ellipse is invaluable to the practical draughtsman. Many of the solutions of problems on the ellipse are based upon this fact.

3. The tangent MN (Plate V*a*, fig. 1) and the normal RT bisect the angles between the focal distances at the point of contact. Thus MN bisects KTS, and RT bisects STS^1.

4. If two lines meeting in the centre of an ellipse be so placed that each is parallel to the tangent at the extremity of the other, these two lines are said to be conjugate to one another. Thus in the diagram PP^1 and TT^1 are conjugate diameters, for PP^1 is parallel to MN, etc.

Note.—It is highly important that this definition be understood, for in certain problems in Part II. (Solid Geometry) such diameters can be determined readily, and from them the entire ellipse. But the student is cautioned that although *any* two lines mutually bisecting may be assumed as conjugate diameters of an ellipse to be determined, yet if the figure be already known, each line passing through the centre has its *corresponding* conjugate diameter, which *cannot* be assumed at random.

5. If the intersection of the tangent with the directrix be joined to the focus, and the focus to the point of contact, the angle at the focus is a right angle; thus TS^1 is perpendicular to SN.

6. If two tangents to an ellipse meet, and lines be drawn from the two contact points to a focus, the angles subtended by the tangents at the focus are equal. Thus in Plate V*b*., fig. 1, the angle QF^2T is equal to the angle QF^2S.

Problem LXXVII.

Given the major and minor axes of an ellipse, to determine points in the curve. (Plate V*a*.)

There are several methods of solving this problem, based upon the properties of the ellipse just described.

1st Method. (Plate V*a*, fig. 3*a*.)

Let CD, EF (fig. 3*a*) be the given axes. Determine the foci in the following manner. With E as centre, radius equal to CO, describe an arc, intersecting CD in F^1 and F^2. Next obtain a flexible thread equal in length to CD, and fix its extremities to pins inserted at F^1 *and* F^2, *allowing* the whole to hang loosely. Then adjusting the pencil so as to slightly stretch the thread, trace the curve, being careful not to allow the pencil to *pass over* the thread, and not to pull it. This is a mechanical method frequently adopted in the workshop, etc. It is based upon the property explained in par. 4.

2nd Method.

Let AA′ and BB′ (Plate V*a*, fig. 1) be the given major and minor axes. Describe a circle on each of these, and assume any points as 1, 2, 3, etc., on one of them, and find by means of the radii produced the corresponding points upon the other, as at 1′, 2′, 3′, etc. From the points upon the smaller circumference, draw parallels to AA′ to meet parallels to BB′ through the corresponding points upon the greater circumference. The intersection of each pair of parallels is a point in the curve required.*

3rd Method. (Plate V*b*, fig. 1.)

As the sum of the focal distances of all points on the ellipse is equal to the major axis, if the foci be first determined the following device will give four points in the curve. Divide the major axis into two unequal portions, and using the lengths of each of these parts in turn as radius, draw intersecting arcs having the foci as centres. This is shown in Plate V*b*, fig. 1, where the major axis is divided in *g*; C*g* and D*g* being used as radii to describe arcs having F^1 and F^2 as centres, and meeting in *a, b, c, d*, four points in the curve required. Mere multiplication of this process is all that is necessary to obtain sufficient guides to draw the ellipse.

4th Method. (Plate V*a*, figs. 2 and 3.)

Again assuming CD and EF as the major and minor axes given, construct a rectangle by drawing WZ and XY parallel to EF through C and D respectively, and WX and YZ parallel to CD through E and F respectively. Divide CW and CO into the same number (*n*) of equal parts, and index them as shown at 1·1′, 2·2′, etc. Join the points of division on CW to E, and through F draw a series of straight lines passing through the points of division on CO to meet them. The intersections of corresponding pairs of lines give *a, b, c, d*, etc., points in the required ellipse.

* This method of describing an ellipse is the basis of the construction for Problem XIX., Chapter III.

This construction may be applied where two conjugate diameters other than the major and minor axes are given. For instance, in Plate V*a*, fig. 3, MN and PQ are to be assumed as conjugate to each other (see par. 6). The points *a*, *b*, *c*, etc., are determined by a mere modification of the construction just described. The drawing will sufficiently explain itself. In Part II. frequent use is made of this latter problem.

NOTE.—The student must be very careful here not to confound the problem where the major axis is given and *one* point in the curve. Those are not the data given in fig. 3.

5th Method. (Plate V*b*, fig. 4.)

A very simple plan frequently adopted by the draughtsman to discover points in the ellipse is as follows:—A strip of paper or card with a straight edge has marked upon it the lengths of the semi-major and semi-minor axes, so arranged as to show the difference of their lengths between them. Thus, in the figure, these markings are shown at *c'*, *d'*, and *e'*, *e'd'* being equal to OE and *c'd'* to OD; *c'e'*, therefore, representing their difference. Now, keeping *e'* and *c'* upon the two given axes, the point *d'* must always be in the ellipse. Hence, by moving the paper into different positions (always under the condition of *e'* and *c'* being upon the axes), a series of points can be marked, and the curve drawn through them. This principle is the basis of the action of the carpenter's trammel.

6th Method. (Plate V*a*, fig. 1.)

If the directrices and the foci be known, the following method may be adopted, based upon the definition of the ellipse given at the commencement. The distance of A from S is less than its distance from XX in a certain ratio. At *n*, any point in the axis, draw an indefinite ordinate *n*1'. Then find a fourth proportional to A*a*, AS, and *an*. This will give the focal distance from F1 of a point on *n*1'. In fig. 1*a* the necessary lines for finding these fourth proportionals are shown. There *cf* is made equal to AS, and an indefinite perpendicular is drawn through *f*. Then *cg* is made equal to A*a*, and the two lines are produced. If, then, the distance of any assumed ordinate from AB be measured along *co*, the corresponding focal distance for the points upon it is determinable by parallels to *fg* upon *cp*.

Problem LXXVIII.

Having given the curve of an ellipse, to determine its major and minor axes, and its foci. (Plate V*b*, fig. 2.)

First draw two parallel lines to cut the ellipse as *ab*, *cd*, and bisect each of these in *e* and *f* respectively. Join *ef*, and produce it in both directions to meet the ellipse in *g* and *h*. Then *gh* is the diameter to which *ab* and *cd* are ordinates. If it be bisected in 0, that point will be the centre of the curve. Then, taking 0 as centre, describe a circle of such a radius as to intersect the ellipse in four points, P, Q, R, and S. Join PS. It is an ordinate of the axis, and CD (perpendicular to it) and EF (parallel to it) are the major and minor diameters required. The foci can then be determined as previously described.

Problem LXXIX.

Given any two conjugate diameters of an ellipse, to determine its major and minor axes. (Plate V*b*, fig. 3.)

This problem could be solved by completing a rhomboid about the diameters given, and (after inscribing an ellipse in the figure) by using the construction of the preceding problem. But a direct method may be employed as follows:—Assuming AB and CD (see figure) as the given conjugate diameters, produce OD to E so that OD : OA :: OA : DE. Through D draw GT, indefinite in length, and parallel to AB. Bisect OE in *p*, and set out *p*M perpendicular to OE, to meet GT in M. With M as centre, describe a circle passing through E, and intersecting GT in G and T. Each of these points is in *one* of the required axes. Lines through TO and GO give, therefore, their directions. From D set out two perpendiculars (D*a* and D*b*) to OT and OG respectively. Then the lengths of the semi-axes (OY and OX) are mean proportionals between OT . O*a* and OG . O*b*. The ellipse is not shown in the figure, but would of course pass through AZCW, etc.

Problem LXXX.

To determine the tangent to an ellipse passing (1) through a point on the curve, and (2) through a point outside it. (Plate V*b*, fig. 1.)

1. When the given point is on the ellipse, the tangent and normal can be found most directly by bisecting the angles between the focal distances of the points. This is shown in Plate V*a*, fig. 1, and has been already described.

2. When the given point is without the curve, as at Q (Plate V*b*, fig. 1), first determine the major and minor axes and foci of the ellipse (Problem LXXVIII.). Then, with Q as centre, describe a circle BA to pass through F1 (either focus may be used). Then, with the major axis CD as radius, and the other focus F1 as centre, draw a second arc to intersect the former in A and B. Join A and B to F2. Then S and T, the intersections of these lines with the ellipse, are the contact points of the two tangents from Q.

The construction adopted here depends upon the property mentioned in par. 8, proved in works on conic sections, viz., that the angles QF_2T and QF_2S are equal. By the construction we make them so, for in the two triangles, QBF_2, QAF_2, the three sides are correspondingly equal, and hence the contact points must be in BF_2 and AF_2.

Problem LXXXI.

Given the major axis of an ellipse, and a point in the curve, to determine the minor axis. (Plate V*b*, fig. 2.)

Let us assume that CD and S in the figure are the data given. On CD describe a semicircle, and from S set out SN parallel to EF, and meeting this semicircle in N. Join ON, and from S draw SN′ parallel to CD, and intersecting ON in N′. With O as centre, and ON′ as radius, describe an arc to meet EF drawn through O at right angles to CD in F. This is the converse of the construction shown in Plate V*a*, fig. 1.

In Plate V*b*, fig. 5, it is shown that if one straight line is caused *to move in* such a manner that its extremities are always in two fixed straight lines at a right angle with each other, the centre of the moving line describes the arc of a circle, and any other point the quadrant of an ellipse, the semi-major and semi-minor axes being equal respectively to the two portions of the moving line. Detailed description is barely necessary. The movable line is shown in several positions, and the points in the lines are measured separately along each, and the curves traced through them.

A curve is said to be parallel to an ellipse when all the normals to the latter are intercepted by that curve at equal distances. This figure is sometimes used in arch construction instead of the true ellipse, as it gives greater headway at the two vertices. It can only be drawn by first determining the true ellipse, and then the normals.

THE HYPERBOLA.

It has been already shown how the hyperbola is generated by a plane section of a cone, and that it is a double curve having two branches which are unlimited.

Definition.—If a point moves in such a manner that its distance from a fixed point is always greater than its perpendicular distance from a fixed straight line, in a constant ratio, the curve traced by the moving point is an hyperbola. The fixed point is the focus, and the fixed straight line is the directrix.

Referring to Plate VI., fig. 1, let us assume that point A is situated nearer to XX than to S, in the ratio of 2 : 3. Next suppose A to move on the surface of the paper, in such a way that at positions *a*, *b*, *c*, *d*, etc., the ratio of the distances from XX and S is constantly 2 : 3. Then UAW is one branch of an hyperbola, S being its focus, and XX its directrix.

Properties and Functions.—The properties and functions of an hyperbola are similar to those of an ellipse.

1. It has two foci, S_1S_2, and two directrices, XX and X′X′.

2. The line joining the foci divides the curve symmetrically, and the portion of this line between the two branches is the major axis (AA′), the extremities of which (A and A′) are the vertices; MN is a tangent, TT′ a normal; C is the centre, and PQ is the direction of the minor axis; *aa′*, *bb′*, etc., are ordinates of the axis, and AL is an abscissa of the axis relative to the ordinate *f*L.

3. The tangent bisects the interior angle between the focal distances at the point of contact; thus STM is equal to MTS′.

4. The normal bisects the exterior angle between the focal distances.

5. The tangent meets the directrix in such a manner, that if their intersection and the point of contact be joined to the focus, the angle at the focus is a right angle; thus MS′ is perpendicular to S′T.

6. The curve equidistant from the circumferences of two unequal circles is an hyperbola.

7. If a straight line continually approaches an hyperbola, but never meets it, it is termed an asymptote to the curve.

Problem LXXXII.

Given a focus, directrix, and vertex of an hyperbola, to determine points in the curve. (Plate VI., fig. 1.)

Let XX be the directrix, S the focus, and A the vertex. Draw ZZ′ perpendicular to AB indefinitely. At A draw A*p* perpendicular to ZZ′, and equal to AS. Join O*p*, and produce it. Next set out a series of indefinite lines perpendicular to ZZ′, as *a.a′*, *b.b′*, etc., meeting O*p* in 1′, 2′, 3′, etc., and ZZ′ in 1, 2, 3, etc. Measure the length of 11′, and with S as centre, using 11′ as radius, cut the ordinate through 1 in *a* and *a′*. This gives two points in the curve, for by the principle of similar triangles, OA : A*p* :: O1 : 11′, or S*a*.

The radius for cutting the ordinate passing through 2 is 22′, and so on. The other branch of the curve is the reverse of this one, and a method of finding S′ will be described after the problem upon the tangent is given.

Problem LXXXIII.

Given the curve of an hyperbola, with its directrix and focus, to determine a tangent and normal at a given point upon it. (Plate VI., fig. 1.)

The construction for this problem is shown on the left branch of the curve, U′A′W′; S′ and X′X′ being the focus and directrix, and T the given point on the curve. Join S′T, and draw S′M perpendicular to S′T, to meet X′X′ in M. Then a line through M and T is the tangent required, and TT′ perpendicular to it is the *normal.*

This construction is based upon the property described in par. 5, and by the aid of that mentioned in par. 3 the second focus can be found. Thus, by making an angle MTS at T, equal to MTS′, the point S on the axis produced is determined as the focus required, and the second directrix XX is as far from S as X′X′ is from S′.

Problem LXXXIV.

Having given the double curve of an hyperbola, with its directrices and foci, to determine the asymptotes. (Plate VI., fig. 1.)

Let UAW and U′A′W′ be the given curve, and XX, X′X′ its directrices, and S, S′ the foci. Join the foci, and bisect by a perpendicular PQ, meeting the axis in C the centre. On CS describe a semicircle, and mark off SG as a chord of the semicircle equal to CA. With C as centre, CG as radius, describe an arc to meet PQ in B. Next draw AK perpendicular to the axis, and BK parallel to it, to intersect each other in K. Join CK, and produce it. It is an asymptote to the curve, *i.e.*, it would never meet it although continually approaching it. The second asymptote can then be determined very readily.

It has been already stated that AA′ is called the major axis. The line CB is the semi-minor axis, the whole minor axis being BB′. These two points (B and B′) may be assumed as the vertices of a second hyperbola, the lips of which would fill the remaining angles between the asymptotes. Such a curve is said to be conjugate to the first one, and the construction used for obtaining the asymptotes depends upon the fact that these lines are the diagonals of a rectangle made up of the four tangents through the vertices of the conjugate hyperbolæ.

Problem LXXXV.

Given the double curve of an hyperbola, with its foci and directrices, to determine a tangent to it from a given point outside. (Plate VI., fig. 1.)

Assuming the curves, directrices, and foci, in fig. 1, to be those given, and M as the given point outside, proceed as follows:—With M as centre, draw a circle passing through S. With S′ as centre, radius equal to the major axis AA′, describe an arc intersecting the circle in *t* and K. Join TS′. Then T, the intersection of this line with the curve, is one point of contact, and MTN is the tangent required. This construction is analogous with that employed in Problem LXXX., the ellipse and hyperbola having similar properties.

Problem LXXXVI.

To construct an hyperbola, its semi-axis major AB, an abscissa of the axis BC, and the corresponding ordinate CD being given. (Plate VI., fig. 2.)*

Through B draw BE′ parallel and equal to CE, and join DE′. Divide CD and DE′ each into *n* equal parts, numbering as in the figure from D. Join the division points on DC to A, and those on DE to B; the intersections of the corresponding lines give points in the required curve. The other half can be found similarly.

Note.—A is the centre, corresponding to C in fig. 1.

Problem LXXXVII.

To determine a line which, at all its points, shall be equidistant from the circumferences of two unequal circles. (Plate VI., fig. 3.)

Let the circles (centres A and B) be those given. Join the centres, and bisect the distance intercepted between them in V. On each side of V set off equal short distances, as 11′, 22′, 33′, etc. With A as centre, draw arcs passing through 1, 2, 3, etc., to meet arcs having B as centre, passing through 1′, 2′, 3′, etc. The intersections of these arcs, *aa*′, *bb*′, etc., are points in the required curve. Very little consideration is necessary to see that the points *a*, *b*, etc., are equidistant from both circles.

The line determined is the locus of centres of circles touching both of those given. Constructions similar to this one are required to solve certain problems on circles in contact, which were mentioned but omitted in Chapter V. The student is advised to apply the principle of the above construction to the following:—

1. Three unequal circles, radii ·7″, 2″, and 1″, have their centres A, B, and C at the angular points of a triangle, whose sides are 3·2″, 3·5″, and 4″ long. Describe a circle to touch each of those given (1) exteriorly, (2) interiorly, or including them.

* It should be recalled to the student's mind, that it is a property of the *hyperbola, proved* in works on Conic Sections, that $CS^2 = AC^2 + BC^2$.

2. Draw a triangle made up of three curved lines, and inscribe a circle in it.

3. Draw a semicircle on a line 4″ long, and in it place a circle of ·5″ radius, not touching either of the lines. Then determine a circle to be tangential to the arc of the semicircle, to the given small circle, and to the diameter.

DIVISION II.—THE CYCLOID AND ITS ALLIED CURVES; THE SPIRAL, ETC., ETC.

When a circular disc is caused to revolve, the centre remaining stationary, any point in the circumference, or on the disc, or outside of it, and rigidly connected to it, traces the curve of a circle.

Again, if the disc be moved forward without rotation, so that its centre is constantly in any given line, then either of these points traces a line similar and corresponding to the motion of the centre, *i.e.*, if the latter moves in a straight line, any point in the disc, or connected with it, will do the same.

But if these two motions be combined, then the curve traced by the point is compounded of the two directions, and a line is generated, depending for its form upon the nature of the movement of the centre of the disc.

Several curves, much used in mechanical construction, are produced in this way. It is to their consideration we now proceed.

THE CYCLOID.

Definition.—The cycloid is a curve traced by a point in the circumference of a circle, which rolls along a fixed straight line, the motion being in one plane.

The moving circle is the *generator*, and the fixed line the *director*. The point tracing the curve is the *generating point.*

PROBLEM LXXXVIII.

Given the generating circle and the director, to determine points in the cycloid curve. (Fig. 75.)

Let the circle centre C be the generator, the line AB the director, and P the generating point. Through C draw CC_6 parallel to AB. This line contains the centre in all positions of the generator. Next draw the diameter PQ, and from Q measure along AB a distance, QP_6, equal to πr (r being the radius of the generating circle). This distance being equal to the semi-circumference, it is evident that the point P will reach the line AB at P_6, after half a revolution.* At P6 raise a

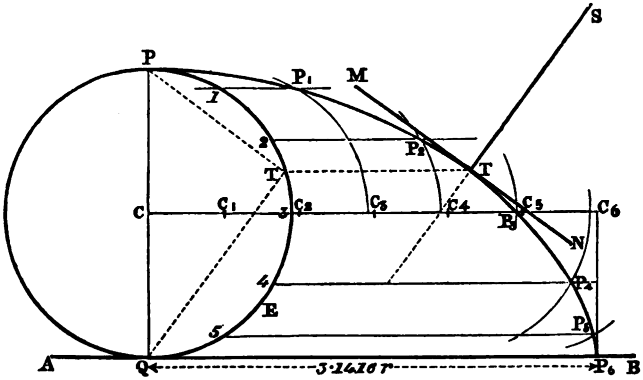

Fig. 75.

perpendicular to AB to meet the locus of centres in C_6. Then C_6 is the position of the centre of the generating circle, when P is at P6, or after half a revolution.

Divide $C.C_6$ into any number (6) of equal parts as at C_1, C_2, etc. Also divide the semicircle PEQ into the same number of equal parts, in points 1, 2, 3, 4, 5, etc.

Through these points draw parallels to the director AB, and with C_1, C_2, etc., in turn as centres (radius equal to that of the generating circle), describe arcs to cut the parallels in P_1, P_2, etc., thus determining points in the required cycloid.

The principle of this construction is not difficult to understand, for it is self-evident that when the generating circle has performed one-sixth of a revolution, its centre will have advanced one-sixth part of CC_6. Hence the point P will have lowered toward AB so much as to be on the horizontal through 1, and the arc (centre C_1) represents part of the circle in its new position after having completed one-sixth of its journey. The intersection of the two lines must therefore be a point in the curve.

The semi-cycloid must then be traced through P_1, P_2, etc. The other half of the curve is similar, and therefore can easily be completed.

* Practically, it is found sufficient to take $3\frac{1}{7}r$ for the distance QP_6. This is *better than* measuring short chords of the circle along the straight line.

To determine the tangent and normal to the curve at any point T, proceed as follows:—Draw TT′ parallel to the director AB, *until it meets the generating circle* in T′. Join T′P and T′Q. These chords of the generator are parallel to the tangent and normal at T. Make, therefore, MN parallel to T′P, and S′T parallel to T′Q.

NOTE.—In all curves of the cycloidal class, the tangent and normal are referable to the chords of the semicircle in the generator, which meet in the point corresponding to the point of contact. This will be noticed in future problems.

PROBLEM LXXXIX.

Given two points, A and B, to determine the semi-cycloid joining them. (Fig. 76.)

Join AB, and on it describe a semicircle. From a scale of equal parts mark off on AB, or AB produced, twenty-two parts, as at AC. At C erect a perpendicular, making it in length equal to fourteen parts on the same scale.* Join A to D. Then AD, or AD produced, meets the semicircle in E. Join E to B, and on it describe a circle. Then AE is the director, and EB the diameter of the generator, and the semi-cycloid required can be drawn by the construction described in the previous problem.†

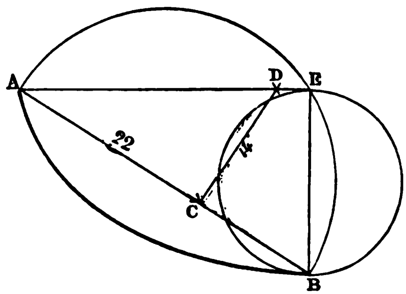

Fig. 76.

THE TROCHOID.

Definition.—The trochoid curve is traced by a point rigidly connected to the circumference of a circle, which rolls along a straight line. (Plate VII., fig. 1.)

* These lengths should be as $\frac{\pi}{2} : 1$; but 22 : 14 is practically near enough.

† This curve (the semi-cycloid) is the line of quickest descent from A to B (see works on Theoretical Mechanics).

The cycloid is also the curve described by the centre of oscillation of an isochronous simple pendulum.

Problem XC.

Given the generator, generating point, and director, to determine points in the trochoid curve. (Plate VII., fig. 1.)

Let AB be the director, PQ the generating circle, X the point connected with it, but not in the circumference. Draw CC_6 parallel to AB. Join C to X, and produce to Y. With C as centre, draw a circle through X, and divide one-half of it into equal parts in points 1, 2, etc.

Along AB from Q, set off a distance, QP_6, equal to half the circumference of the generator $\left(\frac{\pi r}{2}\right)$, as described in the case of the cycloid, and make CC_6 equal to it. Divide CC_6 into the same number of equal parts as the semi-generator. With C_1, C_2, etc., as centres, describe arcs to meet parallels to AB through 1, 2, 3, 4, etc. The intersections of these straight lines and arcs give X_1, X_2, etc., points in the curve. When the generating point is outside the circle, as at X, the curve is called the superior trochoid. If the second half be drawn as in the figure, it forms a loop dipping below the director. But if the generating point, as X′, be within the circumference, the curve traced is called the inferior trochoid. In this case no such loop is formed.

The tangent and normal at any point T are determined as follows :—The point T is referred (by a parallel to AB) to the circle through X at T′. Then T′ being joined to the contact point of the generator and director at Q, T′Q gives the direction of the normal at T. Then ST drawn parallel to T′Q, and MN perpendicular to ST, are the two lines required. A second tangent is shown in the figure, the construction for which is analogous to that already described.

THE EPICYCLOID.

Definition.—The epicycloid curve is the line traced by a point in the circumference of a circle, which rolls upon another circle, and outside it, the two circles being constantly in the same plane.

The rolling circle is the *generator*, and the fixed one the *director*.

PROBLEM XCI.

Given the generator, director, and generating point, to determine the epicycloid curve. (Plate VII., fig. 2.)

Let the circle (centre C) be the generator, and the larger circle (centre O) the director, and P the generating point.

It is first necessary to cut off from the circumference of the director an arc equal to the semi-circumference of the generator, *i.e.*, 6Q must be made equal to P, 3, 6. The angle which this arc subtends at the centre O, bears the same ratio to 180° that the radius of the generator does to that of the director. Calling the former radius r, and the latter R, the proportion would read thus: As $R : r :: 180^\circ : x^\circ$, the angle to be set out at O. In the drawing shown in the plate, the radius of the generator is 1″, and that of the director 3″. Hence, to find the angle 6OQ, the proportion reads: As 3″ : 1″ :: 180° : 60°. The arc 6Q is therefore equal to half the circumference of the generator, and, after a semi-revolution to the left, the point P reaches Q.

A circle drawn through C, having O for its centre, is the path of the centre of the generator as it rolls round the director, and C_6 (the intersection of this line with OQ produced) is necessarily the position of that centre when P has reached Q. Divide $C.C_6$ into any number (6) of equal parts, and consider the points obtained as the positions of the centre of the generator at successive stages of its revolution. Divide the semicircle PQ into the same number of equal parts, and, with centre O, describe indefinite arcs through the points of division. Next take points C_1, C_2, etc., in turn as centres, and, with a radius equal to that of the generating circle, describe small arcs to meet the larger ones just drawn through 1, 2, 3, etc. Thus, the arc through 1 meets that having its centre at C_1 in P_1, a point of the curve. The principle of the construction is therefore similar to that described in the case of the cycloid.

The tangent and normal at any point (T) can be determined as follows:—Refer the point of contact T on to the original position of the generator by describing an arc through it, having its centre at O. Then T′P is the proportionate amount of the revolution which P has made when it has reached the given position T. Join T′6. With T as centre, radius equal in length to T′6, cut the circumference of the director in R. Then RT is the direction of the normal, and the tangent MN is perpendicular to it.

THE HYPOCYCLOID. (Plate VII., fig. 2.)

Definition.—The hypocycloid is the line traced by a point in the circumference of a circle which rolls upon another circle and inside it, the two circles being constantly in the same plane.

The hypocycloid differs from the epicycloid in its generation in one condition only. *The generator rolls on the concavity of the director.* In form it is a flatter curve than the epicycloid, and when the diameter of the generator is equal to the radius of the director, it becomes a straight line. The method of obtaining points in the curve is analogous with that used in the case of the epicycloid and in the figure, the same letters are used in both cases, those appertaining to the hypocycloid being accompanied by a dash. It is, therefore, unnecessary to describe the construction in detail.

It should be noticed that the same curve, QP′, shown in the figure, would be traced if the generating circle were in diameter equal to the difference between the diameters of the two circles employed. In other words, if the director were completed, and a generator were used which touched the director at its lowest point (in 6O produced), and passed through P′, the rolling of this circle would cause P′ to trace exactly the same curve.

THE EPITROCHOID AND HYPOTROCHOID. (Plate VIII., figs. 1 and 2.)

These two curves bear the same relation to the epicycloid and hypocycloid that the trochoid does to the cycloid, *i.e.*, the point which traces either of these trochoid curves is not on the circumference of the generator, but is within it or without it, the terms inferior and superior being used to distinguish between the two cases. Thus, the superior epitrochoid is traced by a point outside, but connected with the circumference of a circle rolling upon the convexity of a second circle, like the extremity of a pointer attached to a pinion which gears with a large fixed wheel. In Plate VIII., the superior epitrochoid, with its tangent and normal, is shown in fig. 1, and the superior hypotrochoid in fig. 2.

The construction in each case is only a modification of that adopted to obtain the trochoid. The director has its centre at O, the generator at C, and the generating point is P.

As the hypotrochoid is perhaps the more difficult of the two, it is advisable to describe the construction in detail, it being understood that *the method* adapted to determine the one is applicable to solve the other.

Referring then to fig. 2, it is first necessary to determine the arc QQ′, equal to half the circumference of the director. The proportion $R : r = 180° : x°$ gives in the fourth term, as described in a former problem, the angle to be set out at 0, so as to determine Q′. Then CC_6 is the path of centres, the equal divisions along it marking the positions of C at successive stages of the revolution.

With centre C, a circle drawn through P gives the path of the generating point when C is stationary. Half of it should be divided into the same number of equal parts as CC_6 in points 1, 2, 3, 4, etc., and with 0 as centre arcs should be drawn through the division points. Next, taking C_1, C_2, etc., in turn as centres, and with radius equal to CP, a series of other arcs should be described, giving, by their intersections with those previously drawn, points in the required curve. Thus the arc through 4 meets the arc, centre C_4 (radius equal to CP) in P_4.

The tangent and normal are obtained thus:—With centre O, the arc TT′ refers the given contact point to the circle through P. Then T′Q being measured, the director is cut by an arc (centre T and radius equal to T′Q) in R. This enables one to draw RTS as a normal, and MN perpendicular to it as the tangent.

A remarkable property should be noticed here, viz., that one half the hypotrochoid is a quadrant of an ellipse when the diameter of the generator is one-half that of the director.

THE INVOLUTE OF A CURVE.

Definition.—When a perfectly flexible thread is gradually unwound from a curve, the unwound part being kept constantly stretched, the extremity of the thread describes a line termed the involute of the curve from which it is unwound, the latter being known as its evolute.

The involute of the circle is much employed in shaping the teeth of wheels, as are also the cycloid, epicycloid, and hypocycloid. The adaptation of these to the purpose mentioned is admirably discussed in a companion volume of this series upon Machine Construction, to which the student is recommended to refer.

PROBLEM XCII.

To determine the involute of a given circle. (Fig. 77.)

Let the circle O (fig. 77) be the generator, and A the generating point. Draw a diameter through A, and a tangent G_6 at its other extremity. Next mark off upon G6 a portion equal to the semicircle A36. This can be readily determined from a scale of equal parts, as πr, or about $3\frac{1}{7}$ of the radius. Divide G6 and the semicircle into the same number of equal parts as at 1, 2, etc., 1′, 2′, 3′, etc. At each of the points of division on the semicircle set out tangents to it, as 1B, 2C, etc. Then, if the point A were the extremity of a thread wound round the semicircle, 1B would represent a portion of it, unwound as far as 1. Hence 1B must equal in length the arc A1, or $\frac{1}{6}$ of G6. Take then the distance 61′, and cut the tangent at 1 in B.

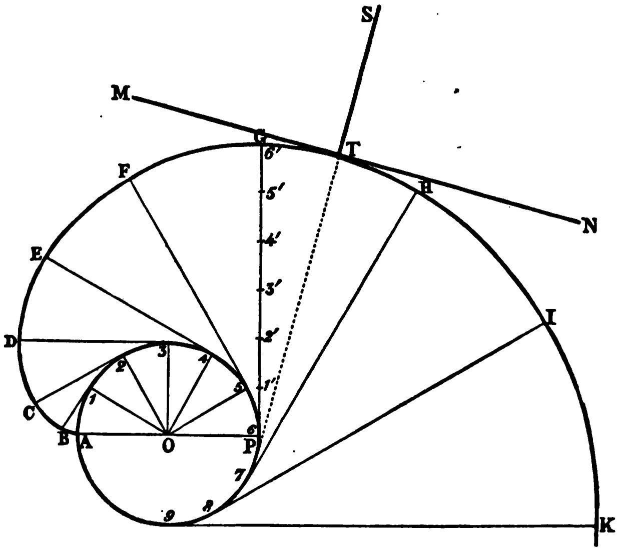

Fig. 77.

Take in length two divisions, and set it out along the tangent through 2, and so on. The curve traced through the points B, C, D, etc., is the involute of the circle. To determine a tangent to the curve at a given point as T, draw TP through the given point tangential to the original circle. The direction of this line is a normal to the involute, and the tangent is determined by being perpendicular to it. The curve has been traced further than G by the same construction, the lengths of the tangents to the circle increasing in length in order one division of G6. Thus H7 is in length equal to 7 divisions, I8 to 8 divisions, etc.

THE SPIRAL.

Definition.—The spiral is a curved line, whose consecutive points continually and uniformly approach or recede from a certain fixed point, called *the pole*. If any point in the curve be joined to the pole by a straight line, this line is called a radius. Hence the radii constantly differ, their respective lengths being related by an equation expressed in terms of the angle between them. The curve may make any number of complete revolutions before reaching the pole, or, as in the case of the logarithmic spiral, may never do so. Hence a spiral may be described as of one, two, or any number of revolutions.

The *Archimedean* spiral approaches the pole by equal quantities measured along consecutive radii, as in fig. 78. The *Logarithmic* spiral is such that two consecutive radii, enclosing a given angle at the pole, differ in length, so that the shorter one is equal to the longer multiplied by a given proper fraction. Thus, if this fraction and angle were ·9 and 10° respectively, and one radius were 10″ long, the next shorter would be 10″ × ·9, or 9″, the next 9″ × ·9, or 8·1″, and so on, the radii all meeting at the pole at equal angles of 10°. The student will see that, as these lengths form a constantly decreasing geometrical series which never reaches zero, the logarithmic spiral is one of an indefinite number of convolutions, and that it never reaches its pole.

There are several other forms of spiral, but all of them are deducible from the relations of their respective radii.

In mechanical contrivances, the spiral is frequently employed for the production of peculiar motions from continual circular motion. Detailed descriptions of them would be out of place in a work on Geometry, but in the problems which follow notes will be found hinting at some of these adaptations.

Problem XCIII*a*. (Plate IX., fig. 1.)

To determine points in an Archimedean spiral of two convolutions, its extreme radius being 1·2″ long. (Fig. 78.)

Let O be the pole, and OQ the given greatest radius. On OQ describe a complete circle, and determine equidistant radii (the number of these being a matter of choice), as OA, OB, OC, etc. Divide OQ into as many equal parts as the given number of convolutions (in this case two). Then it is evident that, after the first turn, the curve will have arrived at *q*. Next, divide Q*q* into as many equal parts as the number of radii drawn, as at 1, 2, 3, etc. With O as centre, radius equal to O1, describe an arc to meet OA in *a*. Then, as the curve approaches the pole by equal quantities measured along consecutive radii, O*a* must be the length of the radius after $\frac{1}{8}$ of a turn; *a* is therefore a point in the required spiral. The remainder of the construction is obvious. The second convolution being parallel to the first, its points can be determined most readily by measuring the distance Q*q* along the radii from *a*, *b*, etc., towards the pole; thus—*as*, *bt*, are equal to Q*q*.

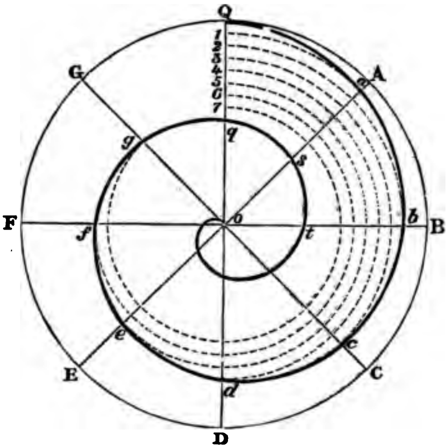

Fig. 78.

The Archimedean spiral is a curve frequently employed in shaping the peripheries of cams used in mechanical contrivances to produce a *uniform* reciprocating motion from a constantly uniform circular one. In Plate IX., fig. 1, the form (PCBDP) is shown, consisting of two half turns of the Archimedean spiral reversed to each other, and known in the workshop as the heart-plate (the construction of the rim being similar to that just described).

In the ordinary crank and connecting-rod arrangement for obtaining reciprocating straight line motion from circular, the result is not a uniform rate of movement. But, if the heart cam be used, it is easy to see that there can be no variation in the speed. For, if P (see figure) be supposed to represent a *point* set in motion by the revolution of the rim PCBDP from right to left, about the centre O, when $\frac{1}{8}$ of a turn has been completed, point *a* will have arrived at *a*′, which, according to the construction employed, is $\frac{1}{8}$ of PQ, and P will have been moved upwards to that distance. Hence $\frac{1}{8}$ of the one motion is performed in the same time as $\frac{1}{8}$ of the other. The speed is therefore uniform. The distance

PQ gives the entire rise and fall of the moving point, and is termed "*the travel.*" Hence, if the greatest radius of the plate and the travel be given, the curve can always be determined.

The student must notice here that the conditions just described only obtain so long as P is a geometrical point; for, if it be a pin with an appreciable radius (which it must always be in practice), the motion would then be that of the centre, and not of a point on the rim. If PE (see figure) be assumed as the radius of such a pin, then the necessary curve ceases to be a true spiral, but a parallel to it, sometimes called its envelope.

This new curve is best deduced by a series of small tangential arcs equal in radius to that of the pin, and having their centres in the true spiral. One-half of such a cam is shown in the plate, on the right side of fig. 1, by a dotted line.

The Archimedean spiral is also employed in constructing what is known as the snail cam. In Plate IX., fig. 1, one half of the curve is shown as continued (BCPGA) until one revolution is completed. Then the shape BCPGAB is that of the snail cam, which in practice gives a uniform lifting motion with a sudden fall. Such a motion is required in punching and stamping machines.

Again, if a half of a heart cam be used as CBD, and the remainder be replaced by a circle, as shown by the fine line DKC, the movement obtained is one of uniform rise and fall so long as the spiral is the actuating rim, but one of rest whilst the circle is passing. This form of cam is employed in some punching machines, to give a period of rest while the feed is being adjusted.

PROBLEM XCIII*b*.

To construct a logarithmic spiral, the following data being given: the greatest radius, the angle between assumed consecutive radii, and the proportion of length of one radius to that which follows it. (Plate IX., fig. 2*a*.)

This curve has already been defined (page 103). In the drawing shown in Plate IX., fig. 2*a*, the radii are taken as enclosing angles of 30°, the greatest radius is 1·5″, and the ratio of two succeeding radii is $\frac{9}{10}$. Having drawn the twelve radii, as OQ, OA, etc., make a supplementary figure, as in fig. 2*b*, by which $\frac{9}{10}$ of each succeeding radius may be deduced. To do this, set out O′Q′ and O′1′ at any angle, and mark off on one of these lines a length equal to OQ (fig. 2*a*). Next, make O′1′ equal to $\frac{9}{10}$ of OQ, and join 1′Q′. With O′ as centre, describe an arc through 1′ to meet O′Q′ in A. Afterwards draw A2′ parallel to Q′1′, then O2′ is $\frac{9}{10}$ of O′1′. By proceeding in this way, the lengths of successive radii are shown at O′1′, O′2′, O′3′, etc. Transfer these in order to the lines OQ, OA, OB, etc., and points 1, 2, 3, etc., in the desired curve will result. Only one turn is shown in the diagram, but (as was previously noted) the number of these is infinite, as this spiral never reaches its pole.

It should be understood here, that the logarithmic spiral meets successive radii at the same angle. Thus, the angle Q1O is the same as 12O, and this property would enable us to determine its points. On the left of the figure, a second spiral of the same kind is shown in a dotted line QST, etc., the points S, T, etc., being obtained by setting out lines from Q and S to make a right angle with the next radius. Of course, any angle may be assumed. According to the principle of similar triangles (QSO and STO being equiangular) TO : TS : : OS : OQ, which is the similar property to O1 being $\frac{9}{10}$ of OQ and O2, $\frac{9}{10}$ of O1, etc.

Irregular variations of quantity may frequently with great advantage be depicted geometrically by two lines, one a fixed straight line, the other a variable crooked or curved one, the relation of these two, as measured by a system of ordinates (or perpendiculars to the fixed line) being arranged to correspond with the variable quantities represented. For instance, barometric pressure is often depicted in the newspapers by an irregular curved line, the points of which approach or recede from a fixed horizontal line in correspondence with the fall and rise of the mercurial column. This method of representation is valuable as assisting the mind by the aid of the eye, and is very commonly adopted.

One very good instance only of the kind can be inserted here to serve as an example.

Problem XCIV*a*.

In a steam cylinder, diameter 12″, length 24″, the steam is admitted for one quarter of the stroke only, the remainder of the movement of the piston being due to expansion, to depict geometrically the variation of pressure of the steam during the entire stroke, scale $\frac{1}{12}$. (Fig. 79.)

Let ABGF (fig. 79) represent an outline diagram of the cylinder to the scale required. Make AC $\frac{1}{4}$ of AB. Then C represents the point of "cut off," and a perpendicular CD may be supposed to indicate the full pressure of steam at that point. Next, divide CB into any number (12) of equal parts, and at each of the points of division erect perpendiculars to AB. Then, when the piston has moved to a_1, the volume of the steam will be $\frac{5}{4}$ of what it was when at C, for AC is equal to four times Ca_1, hence the pressure (according to Boyle's law) will be $\frac{4}{5}$, or ·8. Take off, therefore, along *aa* a length equal to ·8 of CD. When the piston reaches *b*, the volume is $\frac{6}{4}$ of what it

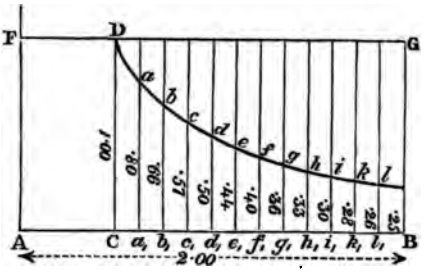

Fig. 79.

was at C, hence the pressure is $\frac{4}{6}$, or ·66, which gives the length of *b'b*. The series of ordinates having been determined in this way, as 1·00, ·80, ·66, ·57, ·50, etc., as shown, the curve traced through them illustrates pictorially the variation of the pressure throughout the stroke. In this case the curve is hyperbolic.

Another common adaptation of plane geometry may with advantage be introduced to the notice of the student. When combinations of moving parts are shown in a drawing, it is often desirable to trace the motion of a particular point in the arrangement; and, to do this, all that is necessary is to draw the whole combination in several of its positions, and hence deduce the movement of the required point. For instance, it can be practically shown by the geometrician that the points of attachment of the pump-rods and piston-rod in Watt's parallel motion, do not appreciably vary from a straight line whilst the beam and the bridle-rod are swinging on their respective centres of motion through a small angle.

Space will not allow of our enlarging upon this subject, or of taking a number of instances; but one very instructive example is shown in Plate IX., fig. 3, which must serve as a type of the class of problem referred to.

PROBLEM XCIV*b*.

Three links, AB, CD, and BD, are connected, as shown in Plate IX., fig. 3; AB and CD being free to turn about points A and C, whilst at B and D they are attached by loose pins, allowing free movement of the connecting link BD over the two swinging bars AB and CD. Required to determine the path of a point P in BD throughout the entire possible movement of the combination. (Plate IX., fig. 3.)

In a problem of this kind, the first duty is to settle the limits of the possible motion of the arrangement; and secondly, by drawing the whole combination in several of its principal intermediate positions to deduce points in the required line.

1. *As to the limits of movement in the case given.*—With A and C as centres, describe complete circles through B and D respectively. These circles are locii of the free extremities of the two swinging bars, AB and CD, in all positions. The distance of the fixed centres A and C apart being less than the length of the connecting link BD, there is a portion of each circle never traversed by the points B and D.* In the figure, these portions are 6, T, 18 on the smaller circle, and 21, S, 10 on the larger. When the arrangement reaches the limit of its movements, the long connecting link coincides in direction with one or other of the swinging bars. Two of these positions are shown by dotted lines in the plate. Thus, when D in its right-hand travel reaches 6 on the small circle, the position of the combination is such that 66 represents the link BD covering the bar A6 representing AB. Such positions are recognised by the term "*dead points*," and the motion can only be continued by the connecting link sliding over to the other side of the bar it covers. To determine these dead points, of which there are four, proceed as follows:—With C as centre, radius equal to the *difference* between CD and BD, describe an arc intersecting the circle, centre A in 21 and 10. It is evident that when BD covers CD, as it does at M, then M21 must be equal to BD, hence C21 is equal to BD, less the radius CM; M and N are therefore the two dead points on the smaller circle. With A as centre, radius equal to the difference between BD and AB, describe arcs to cut the circle (centre C) in 6 and 18. Then R and W are the dead points on the larger circle.

2. *To obtain points in the desired line.*—Starting from D, conceive

* *This* would also be the case if AC were *greater* than BD.

the bar CD to have moved forward (to the right) until D is at 1. Then 11 will be the position of the link BD, and 1′ that of the point P. For 1′1 is equal to BD, and 11′ to BP. Assuming a series of positions as at 1, 2, 3, 4 on the larger circle, draw the straight lines representing the connecting link for each case, and assign the relative positions of P in them, as at 1′, 2′, 3′, 4′, etc. The dead point 6′ having been reached, it will be found that, if AB is to continue its swing, the bar CD, being at C6 on the small circle, must reverse, and D gradually return, until point 10 on the larger circle is gained. Here the smaller link CD continues its left-hand motion, and the longer link AB reverses and travels upwards towards the right to points 11, 12, etc. By reasoning the matter carefully in this way, the peculiar curve shown can be determined as the required path of the point P.

The crossing point of the cusps at 5′ is in a straight line, joining the two centres A and C.

EXERCISES.

1. Define a cycloidal curve. What would the curve traced by a point in the spoke of a carriage wheel be called when the latter is *rolling along a level road?*

2. Describe a half ellipse with axes 4″ and 2·5″. Draw a sufficient number of normals to the curve, and produce them externally. Through points on these ·5″ from the curve, draw a second curve "parallel" to the first. (Cooper's Hill Engineering College Exam.)

3. Construct one convolution of a logarithmic spiral from the following conditions:—The longest radius is 4″, and another radius which meets this one, at an angle of $22\frac{1}{2}°$, is $\frac{7}{8}$ of its length.

4. A parabola has its focus ·75″ from its vertex; determine the curve and a tangent to it from a point outside, which is 2″ from the vertex and 2″ from the focus.

5. The distance between the foci of a certain ellipse is 3·5″, the minor axis is 2·25″. Determine the curve.

6. Two diameters of an ellipse are 2″ and 3″ in length, they contain an angle of 60°; draw the curve. (May Exam. 1875.)

7. Determine a normal at any point of an ellipse (not at the extremity of an axis). Of what practical use is this normal? (May Exam. 1875. Honours.)

8. A circle of 2″ diameter rolls on a straight line. Draw the curve traced in one revolution by a point on the circumference of the circle. (May Exam. 1877.)

9. A curve line passes through the extremities of all tangents drawn to a circle of 1·5″ radius, the length of each tangent from its point of contact being equal to the *arc* of the circle included between that point and another fixed point A on the circumference; the curve to be continued until it meets the diameter through A produced. (May Exam. 1869.)

10. A circle 1·5″ diameter rolls along a straight line. Draw the curve traced in one revolution by a point in the plane of the circle, and rigidly connected to it, and distant 1·25″ from its centre. (Cooper's Hill College Exam. 1875.)

11. A and C are two points 3·25″ apart. B is a third point, 1·25″ from A, 2·75″ from B. Given that A and C are the foci of an ellipse, and that B is a point in the curve. Draw that curve. (May Exam. 1870.)

12. AB, 3″ long, is the *major axis* of an ellipse; C is a point in the curve, 1·25″ from A, and 2·25″ from B. Draw the curve and determine the tangent at C. (May Exam. 1870. Honours.)

13. The paddle-wheel of a steamer is 15 feet in diameter. The speed of the vessel is such that it progresses 30 feet for each revolution. Required the path of a point on the rim of the wheel. Scale, $\frac{1}{10}$ inch = 1 foot.

14. Describe the curved line whose ordinates, at intervals of ·25″ apart, represent the decreasing series $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc.

15. The centre pin of a wheel 2″ diameter is constrained to move in a slot, which is in form 60° of the arc of a circle of 3·5″ radius. The wheel turns twice round in the same time as its centre completely traverses the slot. Determine the entire curve traced by a point in the rim of the wheel, the motion being uniform throughout.

16. Show by a geometrical diagram that the motion of a reciprocating piece, actuated by a crank and connecting-rod arrangement, is not uniform if that of the crank be so. Use the following lengths:—Crank 1″, connecting link 4″.

17. It is desired to lift vertically a moving piece at a uniform speed through 3 inches; further, it is desired that the return motion should be at one-half this speed, but still uniform. Construct the shape of a cam which will enable one to effect the above purpose by the means of continuous equable circular motion. The moving piece, or friction roller, is to be ·25″ radius, its centre never approaching nearer to that of the cam than 2·5″.

HINT.—The upward motion must be effected in $\frac{1}{3}$ of a revolution, the downward motion taking $\frac{2}{3}$.

PART II.—ORTHOGRAPHICAL PROJECTION,

OR

SOLID GEOMETRY.

CHAPTER VII.

INTRODUCTION, LINES, POINTS, PLANES, ETC.

THE dimensions of a solid body are generally estimated in three directions, mutually perpendicular, and are recognised as *length*, *breadth*, and *thickness*.

The principles and rules of Orthographical Projection enable us to depict these on one flat surface, such as a sheet of drawing paper.

Two planes* are assumed, as in fig. 80, mutually perpendicular, and therefore forming four corners or dihedral † angles. Upon these co-ordinate planes two projections or drawings of the solid are made, one called an *elevation*, and the other a *plan*.

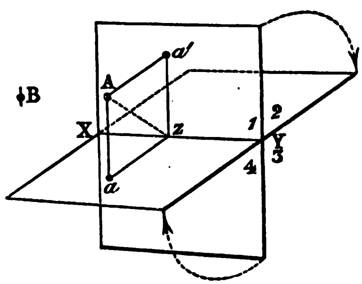

Fig. 80.

As the boundaries of solids are surfaces, and of surfaces, lines; and as, further, lines may be considered to consist of aggregations of points, it is best to illustrate the first principles of the subject by reference to the projections of a point. Let point A, fig. 80, exist as shown in perspective. Conceive an imaginary line Aa', perpen-

* The terms "plane," "line," "point," etc., are used here without definition, for which see Euclid.

† The term "dihedral" is used to distinguish angles formed by two surfaces meeting, from rectilineal angles formed by the crossing of two straight lines.

dicular to the plane represented as vertical, and passing through A. Such a line would intersect that plane in a', and similarly another line Aa, perpendicular to the other plane, would give by its penetration therewith the point a. The lines Aa' and Aa are called *projectors*, the assumed planes are the *planes of projection*, and $a'a$ are the *projections* of the point A. In orthographic projection the two planes are always perpendicular, and the projectors are assumed as at right angles to the planes. From these first principles, it follows that every point has two projections, one upon each plane. It is usual to distinguish one plane as the *vertical* and the other as the *horizontal*, and the projection upon the former is called the *elevation*, whilst that on the latter is the *plan*. The student will see that it is necessary to use two planes and two projections, as the position of the point is *not* fixed by one only.

Now, as four angles are formed by these assumed planes, a point in space may be in position in either of these, and its projections will be sufficient to determine which. Let us suppose an observer to be placed in a position, such as point B, fig. 80, so that A is in the same dihedral angle; then the four angles would be numbered as shown. A point like A would therefore be in angle 1, or in front of the vertical plane, and above the horizontal plane.* Further, as drawings are usually made upon a single flat sheet of paper, it is necessary to imagine one of the planes rotating upon the intersecting line (which is sometimes called the ground line, and is generally indicated by the letters XY), until it coincides with the other. It does not matter which, but it is better to conceive the v. p. as rotating until its upper part coincides with the back part of the h. p. This movement is indicated by the arrows in fig. 80.

When this rotation takes place, the projections will be found to take positions *in the same straight line perpendicular to* XY.

To demonstrate this, let two lines a'Z and aZ (fig. 80) be drawn perpendicular to XY, then the whole figure Aa'Za will be a rectangle whose plane is perpendicular to the ground line. As the rotation takes place, this rectangle will gradually alter in shape, becoming a narrower and narrower rhomboid, until, when the planes coincide, it is a straight line perpendicular to XY.

In fig. 81 the points A, B, C, and D are shown as occurring in different dihedral angles. Point A is similar to the one previously discussed. Point B is behind the v. p. and above h. p., and the student

* In all future references to these co-ordinate planes, h. p. will indicate the *horizontal* plane, and v. p. the vertical plane.

will see that its projections, after the usual rotation of the v. p., occur both above XY. By following out the details of the other diagrams carefully, the student will be able to understand the relative positions of the projections of points occurring in the other two angles.

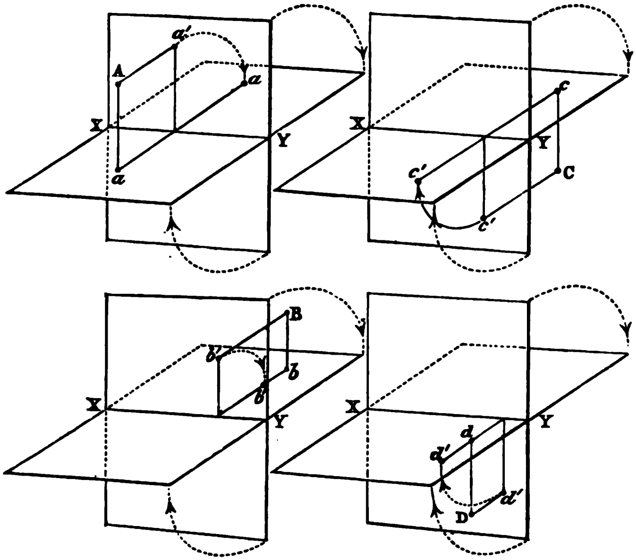

Fig. 81.

If a point be *in one* of the co-ordinate planes, one projection will be on XY, and *vice versa,* if the plan or elevation is on XY, the point itself is *in* the vertical or the horizontal plane.

The real distance of a point from XY is deducible from the lengths of its projectors, as it is equal to the hypotenuse of a right-angled triangle, having the projectors for the other two sides; thus, in fig. 1, $AZ = \sqrt{a'Z^2 + aZ^2}$ (Euclid, Book i., Prop. 47).

A line may be assumed as a sequence of points; hence the projections of a line would consist of the projections of its points. If the line be straight, the plan and elevation must be straight also. Therefore, having determined the projections of the *ends* of a finite line, those of the line itself can be found. For instance, in fig. 82, the projections of the extremities of the line AB are *a'* and *b'* on the v. p., and *a* and *b* on the h. p. By joining these, as *a'b'* and *ab*, the elevation and plan of the whole line are determined.

The plane which contains *all* the projectors of a straight line is called its projecting plane, or, if the line be curved, its *projecting surface.*

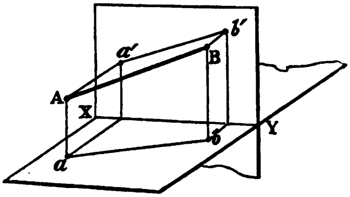

Fig. 82.

A line may be parallel to either or both of the co-ordinate planes; if so, its projection on such plane or planes will be equal in length to the line itself (see p. 92, Elementary Series).*

The line, when not parallel to the projecting plane, must either meet it or would do so if prolonged, and the projection must be shorter in length than the line itself. It is then said to be *inclined* to that plane. If to the h. p., the angle of inclination (generally marked with the Greek letter θ) is measured *between the line itself and the plan;* and if to the v. p., by the angle (generally indicated by ϕ) *between the line itself and the elevation.* Thus, in fig. 83, the line AB is inclined to both the v. p. and the h. p. These inclinations are measured by the angles a'BA and BAb.

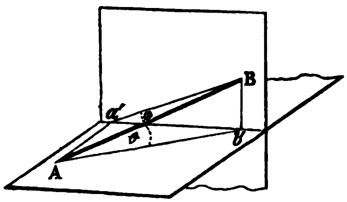

Fig. 83.

The student should be careful to thoroughly understand this last point before proceeding, as he will perceive that after the v. p. is rotated as usual, these angles will not exist on the drawing in either case, and require a special method (to be explained further on) for their determination.

Again, the length of the projection of an oblique line varies as the position alters, being less as the inclination increases, until the line is perpendicular to the plane, when its projection is a point. But the length of a line, the length ot its projection, and the angle of inclination to the plane of projection, are three mutually dependent quantities, any two of which being given, the other is determinable. Referring to fig. 83, the length of the plan *ab* is equal to AB, cos. θ, and the elevation to AB, cos. ϕ. In fact, a right-angled triangle, made up with its base equal to the projection of the line, and its hypotenuse equal to the line itself, determines at once (by the angle between them) the in-

* Where reference is made to the volume on Geometry, Elementary Series, the author has not thought it necessary to elucidate further the very easy principles there taught.

clination. This principle will be frequently referred to in future problems.

In the position of $a'b'$, ab, shown in fig. 84, the sum of the inclinations to the co-ordinate planes equals 90°, and *this sum cannot be exceeded.* Further, in that case, if the projections were not lettered,* the line would not be determined, because the question as to which end was uppermost would be ambiguous. Again, the line may be in such a position that its projections coincide, as is illustrated by the line CD, fig. 84. In all other cases, the plan and elevation of a line are sufficient to determine its position and length.

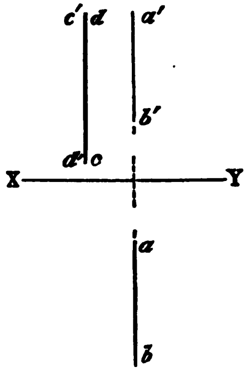

Fig. 84.

When a line is inclined to a plane, it either passes through it, or would do so if produced. These points of penetration are called traces, and are recognised as the horizontal trace (h. t.), or the vertical trace (v. t.), according as the line intersects the h. p. or the v. p.

It will be readily seen that a line parallel to both co-ordinate planes has *no* trace upon them, and, therefore, that the trace or traces of a line depend for existence upon that line being inclined to one or both of the co-ordinate planes.

Problem XCV.

Given a line by its plan and elevation; required its traces, real length, and inclination to both co-ordinate planes. (Fig. 85.)

Case 1. Let $a'b'$ and ab (fig. 85) be the projections of a line AB, then if $a'b'$ be produced until it intersects XY in T, this will be the elevation of the line produced until it meets the h. p. The plan of T will be on ab produced, and will be found by a projector from T. The point marked HT on the diagram indicates the position where the given line penetrates the h. p., and is called its horizontal trace.

Conversely, if ab be produced until it meets XY in the point S, then a line from S, perpendicular to XY, and meeting $a'b'$ produced,

* Our system of lettering is that usually adopted. Capital letters to indicate points in space; corresponding italics with dashes for elevations, and without dashes for plans.

will give VT, the vertical trace, or the point where the line AB enters the v. p.

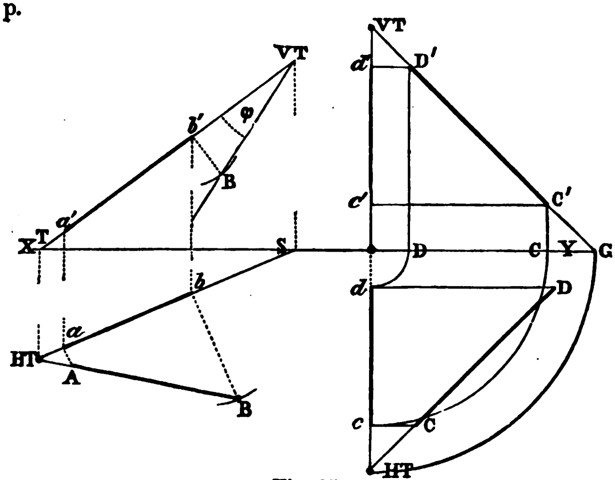

Fig. 85.

To obtain the real length, conceive the projectors of the plan as supporting the line AB; then rotating the whole arrangement on *ab*, the lines *a*A and *b*B, at right angles to *ab*, and each equal in length to the heights of the points *a'* and *b'* above XY, will represent the projectors after the rotation referred to. Join A and B, and AB will be the true length of the line. Further, if AB and *ab* be produced until they meet, they will do so in the h. t., and the angle between them will be the inclination to the h. p. The same principle carried out upon the v. p. gives the real length again, and also the v. t., and inclination to the v. p. (see p. 96, Elementary Series). The student will notice that in fig. 85 only point *b'* has been treated in the elevation, as the v. t. was known previously.

CASE 2. The line CD (fig. 85) has both its projections perpendicular to XY, *i.e.*, its projecting plane is perpendicular to both the co-ordinate planes. The principle for obtaining the true length and traces of the line is the same as in the previous example, but in finding the v. t. the diagram shows a method of rotating* this projecting plane upon its intersection with the v. p. The arcs *d*D and *c*C are loci of the points *c* and *d* during this rotation.

* This principle of rotating a plane upon a line belonging to it is generally *known as "constructing"* it about that line.

CASE 3. The line AB in Plate X., fig. 1, has its extremities in different dihedral angles. Point A is below the h. p. and behind the v. p., whilst point B is above the h. p. and in front of the v. p. Hence the line *itself* passes through both planes. The elevation crosses XY at the point *k*, and this point is the elevation of the h. t. of the line. From *k*, therefore, set out a projector to intersect the plan in the h. t. The v. t. is found in a similar manner. To determine the real length of the line, draw *a*A and *b*B perpendicular to *ab*, and equal to the respective distances of A and B from the h. p., as shown by *ca'* and *tb'*. (Notice that *a*A and *b*B are taken on opposite sides of *ab*.) Join A to B, and the line will have been rotated or "*constructed*" into the h. p., the only point which is stationary being the h. t. Further, the angle of inclination (θ) is shown between the plan and the "constructed line." By a similar proceeding upon the elevation, we obtain the inclination to the v. p. (ϕ).

NOTE.—Only point B is constructed there, as the v. t. is known.

CASE 4. The line AB (fig. 86) is entirely behind the v. p., but is parallel to it. It has, therefore, no v. t. Its h. t. is shown, and, as it is parallel to the v. p., the elevation gives us the real length. The inclination to the h. p. (θ) is also shown at *m*.

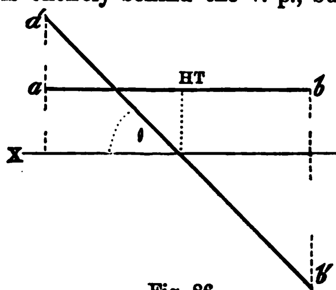

Fig. 86.

The student should at this stage suggest to himself a series of lines in different positions, and work the above problem in each case. Further, he is advised to solve the series of problems on lines in pp. 92 to 101 in the Elementary Book, if he has not already done so.

PROBLEM XCVI.

A line AB is inclined 40°, *a'b'* is its elevation, and *a* is one extremity of the plan. Complete the plan. (Fig. 87.)

Conceive the extremity A of the line as the vertex of an inverted cone, of which $a'c'$ is the axis, having an angle at the base of 40°. Set out $a'd'$ to meet a horizontal through c'. Then $a'c'd'$ is the elevation of half the cone, and the arc bdb_1 is part of the plan. Now, all straight lines lying on the surface of this cone will be inclined 40°, and the elevation given tells us that the point B is either on the front or the back surface. Then a projector from b', intersecting the plan in b or b_1, gives the two solutions to the problem of finding the other extremity.

NOTE.—This solution involves a knowledge of the cone, etc., which it is presumed the student has gained from elementary sources.

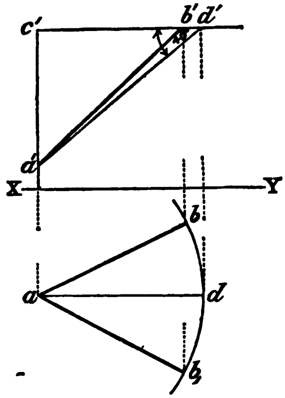

Fig. 87.

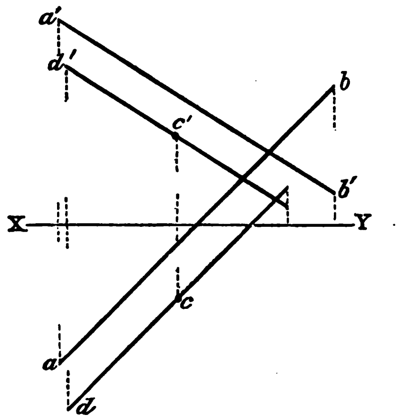

Fig. 88.

PROBLEM XCVII.

Through a given point $c'c$ to draw a line parallel to a given line $a'b'$, ab. (Fig. 88.)

The projections of parallel lines upon any plane are themselves parallel (Euclid xi. 16). Therefore the elevation of the required line will be $c'd'$ (fig. 88) parallel to $a'b'$, and the plan will be cd parallel to ab.

PROBLEM XCVIII.

Given the inclinations of a line to each co-ordinate plane, to determine its projections. (Fig. 89.)

We have already noticed that the inclination of a line to a projecting plane settles the ratio of its real length to its projection on that plane. Further, that if a line has one extremity fixed whilst the other moves in such a way as to preserve this ratio constant, the conical surface generated will be a *locus* of all lines having the same inclination to the projecting plane, and passing through the fixed point. Let us suppose that in the problem before us the inclination to the h. p. is to be 55°, and to the vertical 22°. Further, that the line is 3″ long. Draw B*a*′ (fig. 89) 3″ long, and making an angle of 55° with XY. Find the plan *a* of *a*′. Then B*a*′ and *ba* are the plan and elevation of a line inclined 55° to h. p., but *in* the v. p. Now, what we have to do is to move the extremity B round the circle B*b* (*a*′ being stationary) until it reaches such a position that the line will make 22° with the v. p. If, then, the line BA be set out, making 22° with XY and 3″ long, then BA′ must be the length of the elevation. Take this distance from *a*′ to cut XY in *b*′, then *b*′ will be the elevation of the extremity B, and *b* on the circle below will be the plan. The student will notice that there are more solutions than *one*, according as point B is in front or behind the v. p. —to the right or to the left, etc. —but the relative *lengths*, etc., of the projections must be as shown. The sum of the inclinations θ and ϕ cannot exceed 90°. When that sum is 90°, the projecting plane of the line is perpendicular to XY.

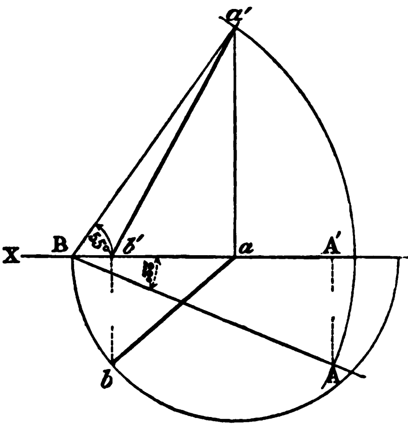

Fig. 89.

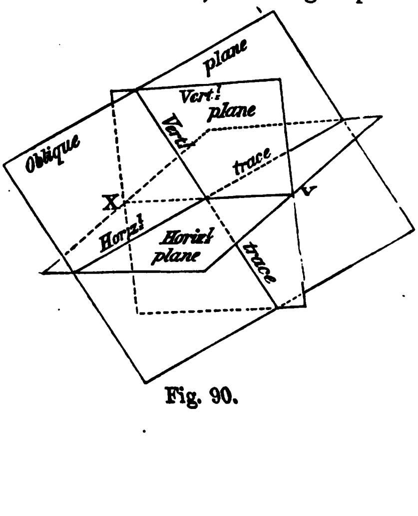

Fig. 90.

OBLIQUE PLANES.

Planes, other than the vertical and horizontal planes of projection, are represented by their *traces, i.e.*, by their intersections with th planes. Thus, in fig. 90, an oblique plane is represented as meeti the co-ordinate planes in two lines crossing upon XY. These two li are the traces of the plane, and their relative positions—*i.e.*, the ang they make with XY—are dependent upon the inclinations of the pla to the co-ordinate planes.

Any two lines meeting upon or parallel to XY, *can be assumed as traces of a third plane.* (Fig. 91.)

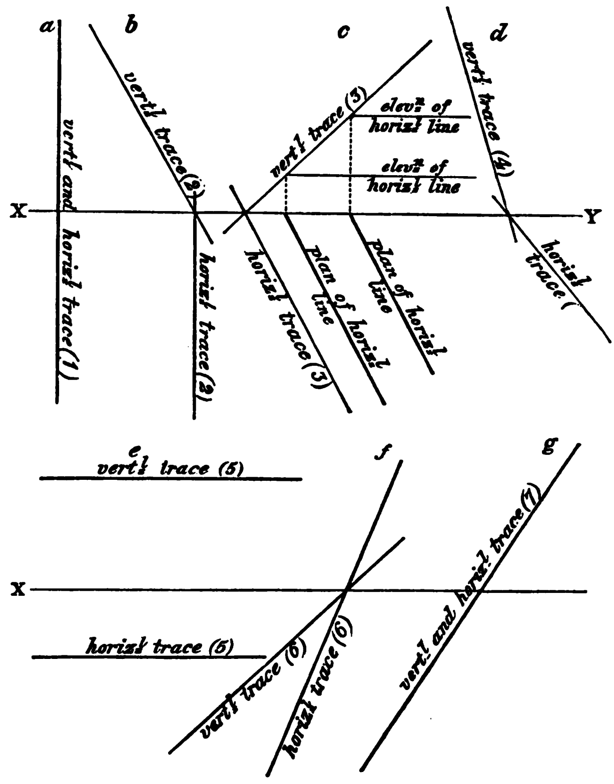

Fig. 91.

-t principles should be understoo

2. The traces are always either parallel to XY or intersect upon XY.

3. The angle which the vertical trace makes with XY is a measure of the inclination of the plane only when the horizontal trace is at right angles to XY, as in fig. 91(*b*).

4. In other cases, *i.e.*, when both h. t. and v. t. make other angles with XY, these angles are not measures of the inclinations of the plane.

5. In such a case as the above, a line upon the plane perpendicular to the h. t. will give, with its plan, the inclination to the h. p.; and similarly, a line on the plane perpendicular to the v. t. will make, with its elevation, the angle of inclination to the v. p.

6. All horizontal lines on such planes are *parallel* to their horizontal traces. See fig. 91(*c*).

7. A line cannot be contained by a plane of less inclination than the line itself.

8. From 7, it follows that the inclination (θ) of an oblique plane being given, the inclinations of all lines upon it must be between θ and zero.

9. After the conventional rotation of the v. p., the angle shown between the traces is not that which exists upon the oblique plane. A special solution is required for its determination.

10. If a plane be equally inclined to both planes, its traces make equal angles with XY.

11. A plane, oblique to both the co-ordinate planes, but parallel to XY, has its traces parallel to XY, as in fig. 91(*e*), and the sum of its inclinations θ and $\phi = 90°$.

12. When a plane is perpendicular to both co-ordinate planes, the sum of its inclinations, θ and ϕ, is, of course, equal to 180°. Fig. 91(*a*).

13. In all other cases, except those in 11 and 12, the sum of the inclinations θ and ϕ varies between 90° and 180°.

14. If two planes intersect, they do so in a straight line common to both. Horizontal lines of the same height, lying upon the two planes, meet upon their intersection.

15. The angle between two planes (the dihedral angle) is measured by the angle between two lines upon them made by a third plane cutting them perpendicularly to their intersection.

16. If a *line* be perpendicular to a plane, its projections will be perpendicular to the traces of that plane—the plan to the h. t. and the elevation to the v. t.

17. Parallel planes have parallel traces. *The distance between the*

***traces does not determine the* actual *distance between the two parallel planes.* (*See Problem CI.*).**

In fig. 91 a series of planes are represented by their traces, and the student would do well to illustrate for himself, by the aid of pieces of paper, the positions indicated and the practical proofs of the propositions enunciated above.

NOTE.—If the student is just commencing the subject, he is advised to read pp. 127-140 in the Geometry, Elementary Series, as the description of first principles is necessarily not so detailed here.

PROBLEM XCIX.

To determine the inclinations of a given plane to both planes of projection, also the real angle between its traces. (Fig. 92.)

CASE 1. In fig. 92 a plane $v'fh$ is given, inclined to both the v. p. and the h. p. If a semi-cone be conceived, having its axis $a'a$ in the v. p., its apex a' in the v. t. of the plane, and its base Bb upon the h. p. tangential to the h. t., then the given plane would touch a line AB upon this cone. This line would be perpendicular to the h. t., and therefore equal in inclination to that of the plane. Now, all straight

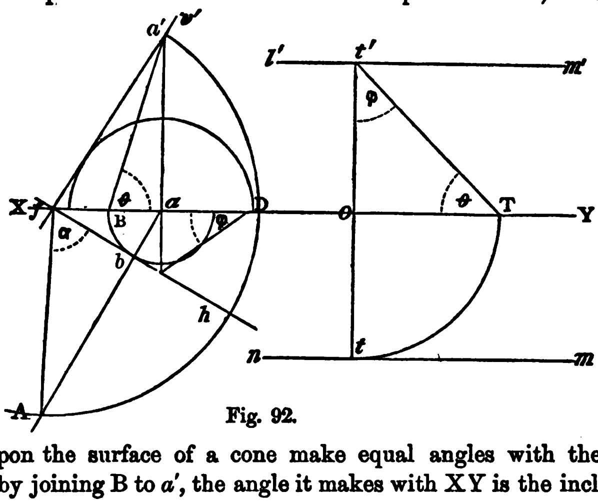

Fig. 92.

lines upon the surface of a cone make equal angles with the base; hence, by joining B to a', the angle it makes with XY is the inclination θ required. To obtain the inclination to v. p. the same principle is *adopted, except that* the semi-cone has its axis in the h. p., and the elevation of its base is tangential to the v. t. The result is shown at D. The angle between the traces is obtained by conceiving the oblique plane to revolve upon its h. t. (this is called constructing a plane) until it coincides with the h. p.

When a plane is constructed upon one of its traces, then all points in that plane move in planes perpendicular to the trace. Hence the point A will, after construction, be in A*a*, which is perpendicular to *fh*. In fact, A*a* is the h. t. of the vertical plane in which A moves. The length A*b* is equal to the length B*a*′, or the point A may be found by centreing at *f* and describing an arc through *a*′ to meet *b*A.

CASE 2. The line *l*′*m*′, *mn* (fig. 92), has its traces parallel to XY. A right-angled triangle, its plane being perpendicular to XY, would measure the inclinations θ and ϕ by its acute angles. In *t*′O we have the perpendicular; *ot* is its base; and, by taking *o* as centre and *ot* as radius, the arc *t*T will find a point T on XY, such that *t*′T is the hypotenuse constructed into the v. p. The angles θ and ϕ are then known, and are together equal to 90°. Of course the angle between the traces has not to be determined.

CASE 3. In Plate X., fig. 2, the plane given is *v*′*f*, *fh*, and here the semicones required to determine the inclination would be in different dihedral angles to those in Case 1. The semicone *a*B*a*′ is *below* the h. p., and the apex *a*′ is in *v*′*f* produced. Similarly, the semi-cone *c*D*c*′ is behind the v. p., and the plan *c* of its apex is on *fh* produced. In all other respects the principle is the same.

If a piece of paper be cut to fit the traces, it will be found to be greater than a right angle; and the student will be able to reason that the angles between the traces, in two dihedral angles next to each other, together make 180°. To find, then, the angle between the traces, proceed as before to produce *ab*, and with *f* as centre, radius *fa*′, draw the arc *a*′A. Then join *f*A, and the angle *af*A will be that between the traces in the fourth dihedral angle.

The supplement *cfa*, therefore, is the angle between the traces in the first dihedral angle, *i.e.*, above the h. p. and in front of the vertical plane.

The student will notice a line, v^2f, on the right-hand side of the perpendicular *ef*, making the same angle with *ef* as *v*′*f*. This may be assumed as the v. t. of a plane having the same inclinations as the given one *v*′*fh*, and the problem is then similar to Case 1, and is more easily solved.

PROBLEM C.

Given either trace of an oblique plane and one of its inclinations; to find the other trace.

The student will see that this is only the converse of the preceding problem, and will find no difficulty in its solution.

PROBLEM CI.

Given the angle between the traces of an oblique plane, and also the angle which the h. t. makes with XY; required the v. t. (Fig. 93.)

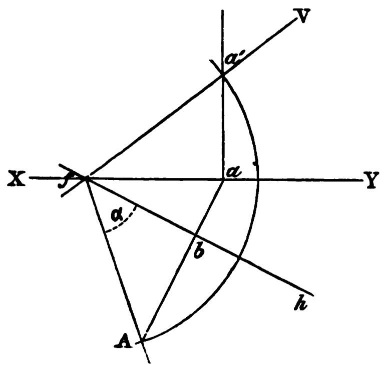

Fig. 93.

Let *fh* (fig. 93) be given with regard to XY. Further, let the angle *α* be that between the traces of a plane having *fh* for its h. t. Take any point A, and draw A*a* perpendicular to *fh*. With *f* as centre, radius *f*A, draw the arc A*a'*, intersecting a line through *a* perpendicular to XY. Join *fa'* for the v. t. required.

PROBLEM CII.

The angle between the traces of an oblique plane is 50°. The plane is equally inclined to both co-ordinate planes. Determine its traces. (Fig. 94.)

As the plane is to be equally inclined to both v. p. and h. p., its traces will make equal angles with XY. Conceive it in position, and then imagine it intersected at any part by a plane perpendicular to XY. The line of intersection on the oblique plane, together with the two traces of the assumed vertical plane, will form an isosceles triangle with a right angle at its vertex, whilst the two traces of the oblique *plane with the* intersection will form an isosceles triangle having the given angle of 50° at the vertex. Take *gfh* (fig. 94) at 50°, and assume two points, *a* and *b*, equidistant from *f*. Join *ab*. On *ab* construct a semicircle (which contains a right angle), and find the middle point C. Then *b*C is the length of the base of the right-angled triangle, made by the cutting vertical plane with the co-ordinate planes and the given one. Or, in other words, the triangle *a*C*b* would fit, under the oblique plane, with the point C on XY. With *b* as centre draw an arc through C, and make *fc* tangent to this arc. This fixes XY. Then *fv'* makes the same angle with XY as *fh*.

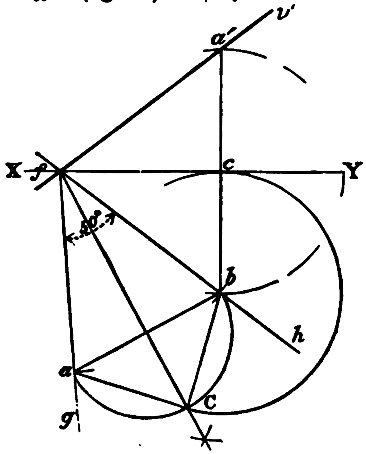

Fig. 94.

Problem CIII.

Given the inclinations of an oblique plane to both the co-ordinate planes; required its traces. (Fig. 95.)

Let the given angles, for instance, be 60° to the h. p. and 42° to the v. p.

To solve the above, it is necessary to determine two cones whose generatrices shall make the given angles of inclination with the co-ordinate planes. These two cones must have their axes *in* these planes, meeting in one point upon XY. They must also envelope a common sphere, having its centre in XY, at the point where the axes of the cones meet. Then the plane which touches both these cones is that required.

Draw a line *a'b* perpendicular to XY, and at any point *c* on XY make *ca'* at 60°. Conceive *a'ca* to be the elevation of half a vertical cone. Describe the arc *cd* to represent part of the plan of this cone. Then, with *a* as centre, describe a circle *egn*, which shall be tangent to the line *a'c*. This is the elevation of the sphere enveloped by the cone.

Make the line *kb* tangent to the arc *ge*, and at an angle of 42° with

XY. Then the triangle *kab* will be the plan of the horizontal cone, and *kst* will be its elevation (one-half).

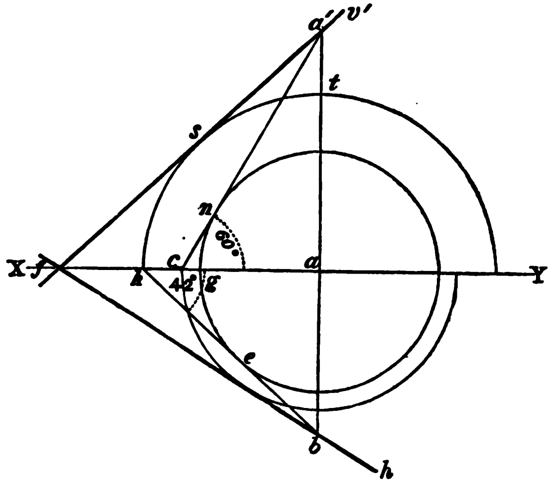

Fig. 95.

Then *v'f*, drawn through the point *a'* and touching the arc *kst*, will be the required v. t., and *fh*, passing through *f* and *b*, will be the required h. t.

Note.—If the sum of the two inclinations were 90°, the traces would be parallel to XY; and if it were 180°, the traces would be perpendicular to XY.

Problem CIV.

Given any three points (not in the same straight line);* required the traces of the plane containing them. (Figs. 96, 97.)

Case 1. The straight lines joining the points must lie wholly in the plane, hence the traces of such lines will be in the traces of the plane. In fig. 96 the given points are *a'a*, *b'b*, and *c'c*. The vertical traces of the lines AB and BC are shown at *v'* and *t'*; and *v'f*, drawn through these points, gives the v. t. of the required plane. As the point *f* is known, the h. t. of one only of the lines need be determined: *k* is the h. t. of the line BC. Then *fh*, drawn through *f* and *k*, is the h. t. of the required plane.

* The restriction given in the brackets is necessary, as an infinite number of planes may contain any given straight line.

CASE 2. When two of the given points are in a straight line, parallel to either of the co-ordinate planes, the student will find that, by joining a third pair of those given, the conditions will be similar to those in Case 1, *i.e.*, he will have two lines, each of which has a h. t.

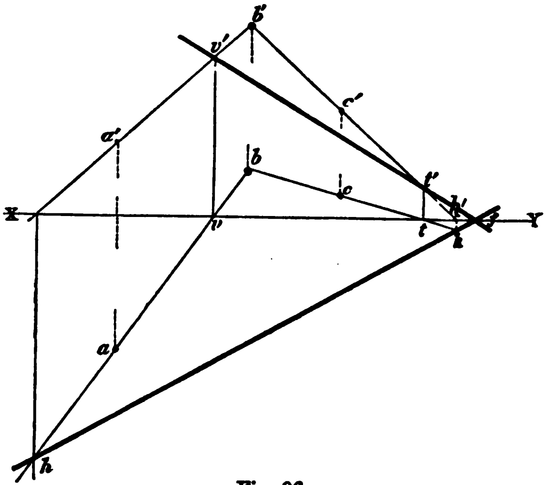

Fig. 96.

But this need not be so solved, as the trace of the required plane, upon that co-ordinate plane to which the line is parallel, will be parallel to the projection of the line, and the trace of *one* other line only need be found. For instance, if *a* and *b*, when joined, are in a line parallel to the h. p., it is clear that AB must be a horizontal of the plane, and the h. t. must therefore be parallel to its plan. If, then, the h. t. of *bc* or of *ac* be found, the h. t. of the plane can be drawn parallel to *ab*.

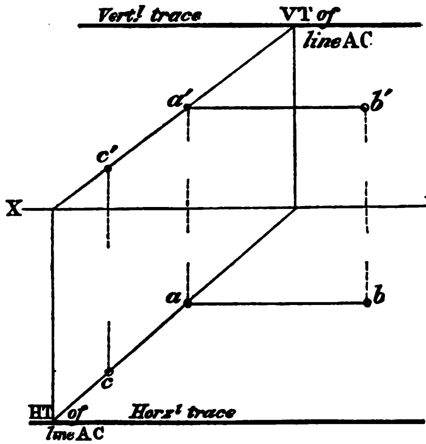

Fig. 97.

CASE 3. When two of the given points are in a straight line, parallel to *both* the co-ordinate planes, the plane required will be parallel to XY, and it will be only necessary to find the h. t. and v. t. of the line joining two others of the points, and to draw lines parallel to XY through these, for the traces of the required plane. This is shown in fig. 97.

CASE 4. When *two* of the lines are parallel to either co-ordinate plane, then the plane containing them is also parallel to that plane, and there is only *one* trace—on the other one.

NOTE.—It will be seen that if any two straight lines meet each other, a plane can contain them, and the conditions would be similar to the above problem.

PROBLEM CV.

Given the projections of two lines which are parallel; required the traces of the plane containing them. (Plate X., fig. 3.)

The lines $a'b'$, ab, and $c'd'$, cd, are parallel, and a plane therefore can be found to contain them. Find the horizontal traces of the lines for points in the horizontal trace of the required plane. The elevations cross XY at t and t', and the traces are found by perpendiculars to XY, from these points intersecting the plans. The line fh passing through these h. t.s is the h. t. of the plane required. Only one v. t., that of CD is necessary, and $v'f$ is the vertical trace of the required plane.

PROBLEM CVI.

To determine the intersection of two given planes. (Figs. 98-101.)

The intersection of two planes is the only line contained by both. Therefore *all* the points common to both planes lie in the intersection. All horizontal lines in one plane meet the horizontals in the other plane which have the same height, upon the intersection. Hence the point where the v. t. of one plane meets that of the other is in the intersection, and the point where the horizontal trace of the one meets that of the other is also in the intersection. Further, these points are the relative traces of the line of intersection, so that the problem resolves itself into that of having the traces of a line given, to show its plan and elevation.

CASE 1. In fig. 98 the intersections of planes are shown in three cases. *In each* of them the vertical traces intersect at the point a'

and the horizontals at *b*. Hence *a* joined to *b* gives the plan of the intersection, whilst *b'* joined to *a'* gives the elevation. In the first instance no determination of the elevation is required, as all points and lines which occur upon the plane *vfh* (its h. t. being perpendicular to XY) have their elevations upon the v. t. of the plane.

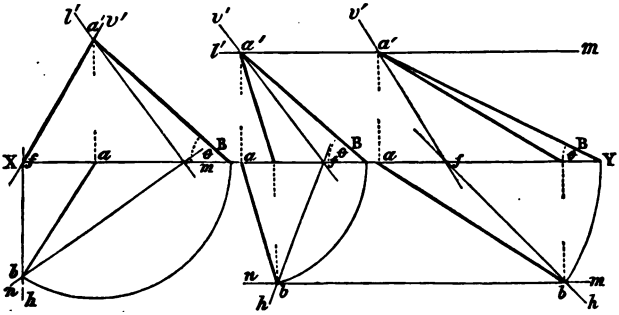

Fig. 98.

Keeping the point A in position, and constructing the line AB into the vertical plane, the inclination of the intersection to the h. p. is shown. This is done by taking *a* as centre and *ab* as radius, and describing the arc *b*B. Then *a'*B is the elevation of the intersection constructed into the v. p., and *θ* is the angle of inclination, which, of course, is not greater than that of either of the intersecting planes.

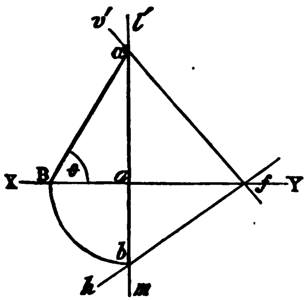

Fig. 99.

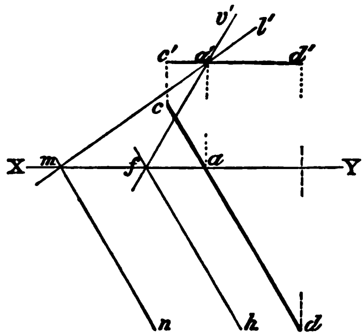

Fig. 100.

Case 2. In this case (fig. 99), one of the planes being perpendicular to XY, the projections of the intersection coincide with the traces of that plane. The angle which *a'*B makes with XY gives the inclination of the intersection.

CASE 3. In fig. 100 the intersecting planes have their horizontals parallel, although they are of different inclinations. Then the intersection is also a horizontal line, and consequently parallel to the horizontal traces. From a' drop $a'a$ perpendicular to XY, and draw cd through a parallel to fh or mn. The elevation of the intersection is $c'd'$ drawn through a' parallel to XY.

CASE 4. In fig. 101 both the intersecting planes are parallel to XY. In such a case it is necessary to obtain a profile view of them. This is effected by cutting them perpendicular to XY, as shown by the line $t's$. With o as centre, and os and ot as radii, draw the arcs sS and tT. Join Tt' and Ss'. Then the point a' is a profile view of the intersection, or its projection on a plane perpendicular to both coordinate planes, and its height $a'a$ is exposed, also its distance aA from the v. p. Through a' draw the elevation $c'd'$ parallel to XY, and with o as centre, oa as radius, draw the arc ab. Then cd, passing through b, and parallel to XY, will be the plan of the intersection.

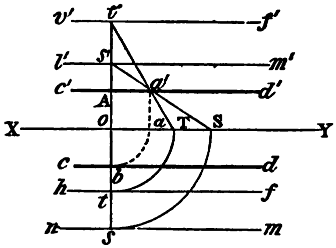

Fig. 101.

CASE 5. In Plate XI., fig. 1, the intersecting planes meet in a point in XY, and the traces do not cross in any other place. Now, as this point must be in both the plan and elevation of the intersection, what we require to find is the *direction* of each of these lines. It will be sufficient, therefore, to determine *one* other point in the intersection.

To do this, assume a plane perpendicular to XY (represented in the figure by its traces as $v'm$). Then the intersections of this vertical plane with the two oblique planes will meet in a point which is in *their* intersection. "Construct" the points a and m into the v. p. by the arcs aA and mM (centre o). Join Av' and bM. Then C'C is the height, and C'c' is the distance in front of the v. p. of the point C, common to both intersections. Draw the plan and elevation of the intersection required through fc and fc'.

Another solution of a similar problem is shown in Plate XI., fig. 2. A horizontal line is assumed in each plane of the same height. These lines meet in a point common to both planes, and therefore in their intersection. The elevations of the two lines *coincide*. *CD* is the horizontal line in the first plane $v'fh$, and AB is the one in the second plane *l′fm*. (The projections of these lines should give no difficulty, *ab* and *cd* being parallel to the h. t.s of the planes they are upon, and their v. t.s being on the v. t.s of these planes.) The plans of these lines intersect in *c*, and their elevations in *c′*. Hence the common line to both planes passes through C and F, *cf* being the plan, and *c′f* the elevation.

Problem CVII.

To find the point* which is the intersection of three given planes. (Fig. 102.)

Two planes intersect in a line. If the three given therefore be taken two and two together, their relative intersections will meet in the point required.

ab (fig. 102) is the plan of the intersection of the plane *v′fh* with the plane *k′st*, and *cd* is the plan of the intersection of the plane *l′mn* with the plane *k′st*. These lines meet at *i*, which is the plan of the point required. The elevation *i′* is of course upon *c′d′*, and a projector from *i* determines its position.

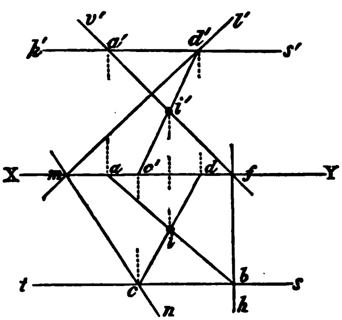

Fig. 102.

Problem CVIII.

Given one projection of a point contained by a given plane, required to find the other projection. (Fig. 103.)

Case 1. If a point be contained by a plane, it will be upon some horizontal line in the plane. Let *v′fh* (fig. 103) be the given plane, and *a* the plan of a point in it. Then a line *ad*, parallel to the h. t. of the line, and passing through *a*, will be the plan of the horizontal of the plane containing the point. To

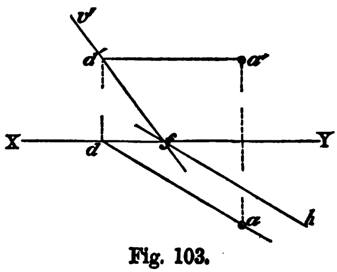

Fig. 103.

* Of course three planes may meet in a line, but this problem assumes that the given planes do not do so.

determine the elevation of this line, its height above the h. p. is required. As the point *d* is upon the v. p., and it is also on the oblique plane, it must be upon the vertical trace *v'f.* A projector from *d* gives *d'*, and *d'a'*, drawn parallel to XY, is the elevation of the horizontal line. The elevation of the point *a'* is determined from *a*.

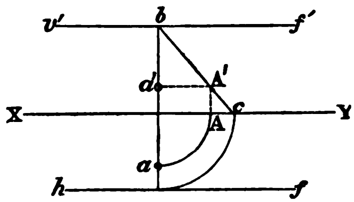

Fig. 104.

Case 2. In fig. 104 the same problem is worked when the plane given is parallel to XY. In this case a profile view of the plane is taken at *bc*, and the height of the point is determined upon it at AA'. The diagram will explain itself sufficiently.

Problem CIX.

To find the intersection of a given line and plane. (Fig. 105.)

Case 1. The line *a'b'*, *ab* (fig. 105) is parallel to the h. p. If, therefore, a horizontal line be found *in* the plane, and at the same height as the given line, the point where these lines meet will be the intersection required. The point *c'* on the v. t. is in the plane, and at the same height as AB. Through its plan *c* draw *cd* parallel to the h. t. Then *cd* is the plan of a line of the same height as AB. The point *d*, where the two plans meet, is therefore in the given plane, because it is on CD, and in the given line; *d'd* is therefore the intersection required.

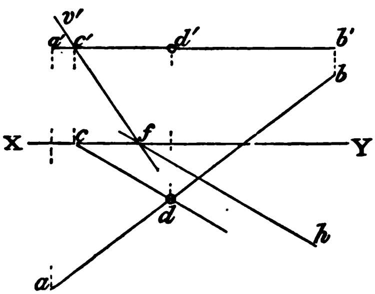

Fig. 105.

Case 2. When the given line is inclined to both co-ordinate planes, the above solution is not applicable. In fig. 106, the line *a'b'*, *ab*, is assumed as inclined to both v. p. and h. p.

If a vertical plane be taken containing the line AB, the inter*section* of the given plane and the assumed one will give all the *points common* to both. Then, as the line AB is in one of these planes, viz., the vertical one, if it intersects the other plane, it must do so in the line common to both planes, *i.e.*, in *their* intersection.

The traces *c'cb* represent the assumed vertical plane. *c'g'* is the elevation of the intersection of this plane with the given one *v'fh*. The elevation of the given line (*a'b'*) meets *c'g'* in *k'*. This point, therefore, is in both planes, and is also on the line AB. It is, therefore, the required intersection of the line AB and the plane *v'fh*. The part KB of the line is *above* the plane, the other part being *below*.

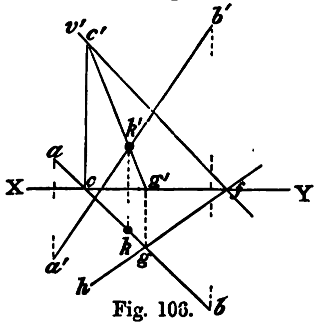

Fig. 106.

PROBLEM CX.

From a given point, to draw a line perpendicular to a given plane, to show its intersection with that plane, and to determine the length of the portion intercepted between the given point and plane.

CASE 1. When the given plane is inclined only to one plane, as in fig. 107, the solution is very simple. On page 121 it is stated, that when a line is perpendicular to a plane, its projections are perpendicular to the traces of the plane, the plan to the h. t., and the elevation to the v. t. If, then, a line *a'b'* be drawn perpendicular to *v'f* from the elevation of the given point A, and another line *ab* be made perpendicular to *fh* through *a*, then the projections required will be determined. As all points and lines on such a plane have their elevations on the vertical trace, *b'* must be the elevation of the intersection of AB with the given plane, and *b* is its plan. Again, the line AB being perpendicular to a plane which is at right angles to the v. p., this line must be parallel to the v. p.; hence the elevation *a'b'* gives the true length of the intercepted portion between point and plane.

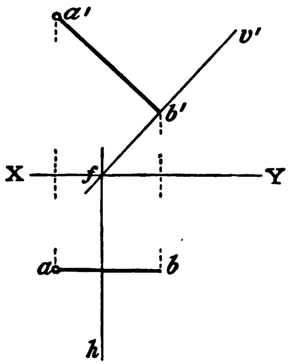

Fig. 107.

CASE 2. In fig. 108 the given point is $a'a$, and the plane $v'fh$. Draw $a'c'$ and ac perpendicular to the traces. These lines are the projections of an indefinite perpendicular to the plane passing through A. The intersection of this line with the plane is worked as in Case 2 of the last problem, ckk' being the assumed vertical plane containing the line, and $k'm'$ the elevation of its intersection with the given oblique plane. The point b' is the elevation of the intersection of the line AB with the plane, and b is its plan. The true length of this line can be determined as follows:—With b as centre, radius ba, draw the arc aA (Ab being parallel to XY). Through a' draw a line parallel to XY, and determine A′ by a projector through A. Then A′b' is the elevation of the line when constructed into a plane parallel to the v. p., and gives, therefore, the true length required.

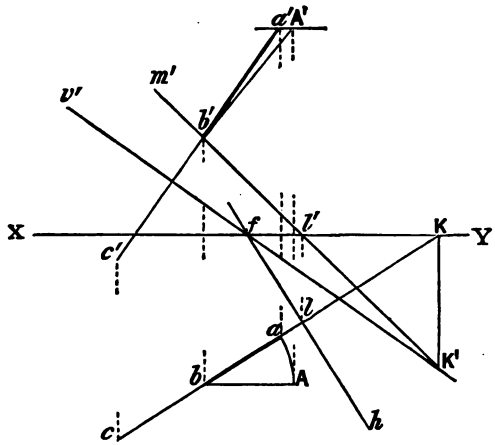

Fig. 108.

The solution given above is involved in the case of a plane and point being given (the latter not being in the former), and the projections of a sphere being required, which, having the point for its centre, shall be tangential to the plane. For the radius passing through the point of contact of sphere and plane is a perpendicular or normal to the latter, and its real length gives the radius of the circles, which are the required projections.

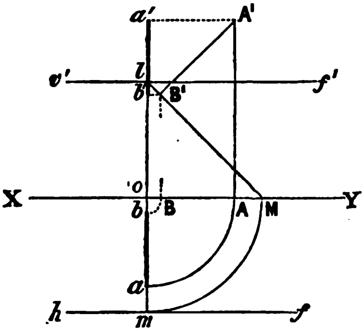

Fig. 109.

CASE 3. In fig. 109 the given plane is parallel to XY, and a' a *is the given point.* Then $a'b'$ and ab are the projections of the line required. To obtain the point of intersection, take a profile view of the line and plane. With o as centre, draw arcs mM and aA. Join Ml, and at A raise a perpendicular to meet (at A′) a parallel to XY through a'. Then Ml is a profile view of the plane, and A′ gives the relative position of point A. Draw A′B′ perpendicular to lM, intersecting it in B′. Project B′ on to $a'm$ and XY. Then b' is the elevation of the intersection of the line and plane, and b (determined on ao by the arc bB) is the plan. The real length of the perpendicular is given at A′B′.

Problem CXI.

To determine the real distance between two parallel planes. (Fig. 110.)

Let $v'fh$ and $l'mn$ be the traces of the given planes. Proceed as if to find the inclination of the plane $l'mn$. Produce aa^1, beyond a^1 to a^2, and with a as centre, describe an arc tangential to fh, meeting XY in b. Join ba^2. Then the plans and elevations of two semicones, fitting under the given planes, and having a *common axis*, will have been determined. The perpendicular distance op between these cones will be the required distance between the parallel planes.

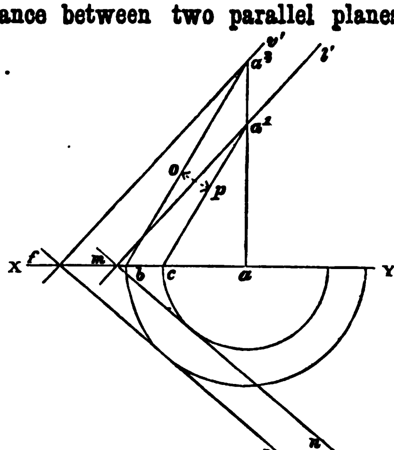

Fig. 110.

Problem CXII.

To determine the traces of a plane parallel to the given plane at a given distance (·375″) from it. (Fig. 111.)

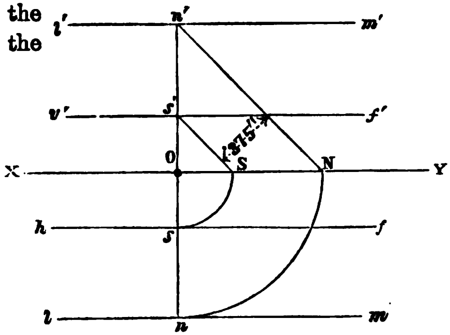

Fig. 111.

If the given plane were

inclined as in fig. 110, this problem would be *exactly* the converse of the previous one, but in fig. 111 the given plane has its traces parallel to XY. The solution of the problem, nevertheless, is identical; *n'*N and *s'*S may be considered as elevations of semicones, or profile views of the planes. Of course, as parallel planes have parallel traces, the lines *l'm'* and *lm* are drawn parallel to *v'f'* and *fh* through the points *n'* and *n*.

Problem CXIII.

Through a given point, *a' a*, to draw a plane parallel to a given plane, *v'fh*. (Fig. 112.)

When two planes are parallel, their horizontals are also parallel. Then *ab* drawn through *a*, parallel to the h. t. of the plane, and *a'b'* drawn through *a'* parallel to XY, will be the projections of a horizontal line upon the required plane. This line meets the vertical plane in *b'*. That point is, therefore, in the vertical trace required. Draw then *l'm* parallel to *v'f* through *b'*, and *mn* parallel to XY through *m*.

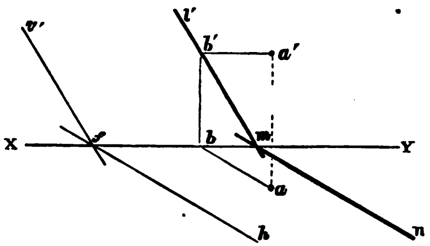

Fig. 112.

Note.—If the given plane be parallel to XY, a profile view of it will be required, and the traces can then be readily determined.

Problem CXIV.

To find the angle between two given planes. (Fig. 113.)

Case 1. By the angle between two planes, is meant the angle measured by two lines (one on each plane) perpendicular to, and meeting upon their intersection. Let *v'fh* and *l'mn* be the given planes. The plan and elevation *ab*, *a'b'*, of their intersection must first be determined. Then a plane perpendicular to AB must be assumed, and the intersections of this plane with the given ones will contain the required angle. The two lines of intersection, and the h. t. of the assumed plane, will form a triangle, and this figure must be constructed into the h. p. to give the real angle between the two given planes.

Draw *pq* perpendicular to *ab*, and meeting the h. t's of the given planes in *p* and *q*, and intersecting *ab* in *o*. This line, *pq*, is the base of the triangle mentioned above. With *a* as centre, radii *ao* and *ab*, describe the arcs *b*B and *oo'*, and join *a'*B.

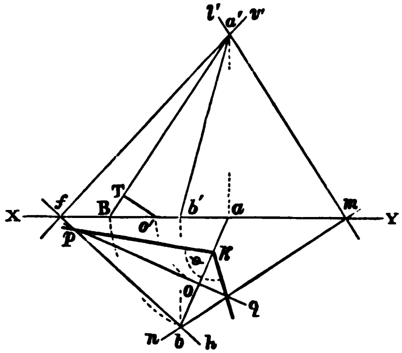

Fig. 113.

By this means we construct the intersection of the two given planes into the v. p. The altitude of the triangle is measured by a line O'T perpendicular to *a'*B. From *o* in *ab*, mark off *o*K equal to OT, and join K*p*, K*q*. Then the angle *p*K*q* will be that between the given planes.

Case 2. In Plate X., fig. 4, another case of the above problem is worked. Here the plane perpendicular to the intersection of the given planes, cuts one of these in a horizontal line. The steps of the solution are exactly similar to those in the last case, except that the line *po*, perpendicular to the intersection, is parallel to the h. t. *fn*. The student will see that when he has obtained point *k* as before, he must join *kp*, and draw *kq* parallel to *fn*. The angle *gkp* is the dihedral angle between the planes.

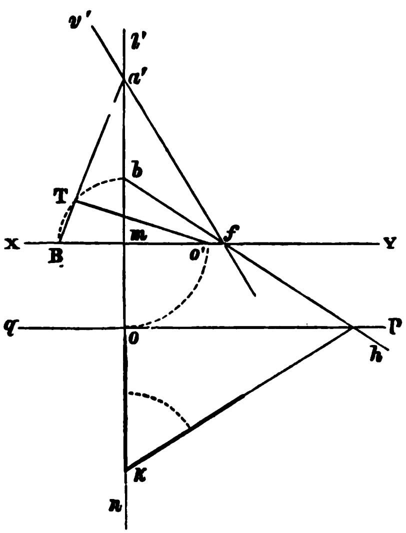

Fig. 114.

Note.—This angle, together with the inclinations of the two planes, must be equal to or exceed 180°. When the given planes are both parallel to XY, a profile view of the planes will disclose all three angles.

Case 3. A more difficult case of the same problem is shown in fig.

114, which the student would do well to investigate. One of the planes is there taken perpendicular to XY. The intersection of the two planes therefore coincides with the traces of one of them. The line *qp* is taken perpendicular to *mn*, and the point *b* in the intersection is constructed as before into the v. p. Then *a*′B is the intersection. The line O′T, as before, gives the length of OK, and *p*K*o* is the angle between the two planes.

Problem CXV.

To find the angle between two given lines which meet. (Fig. 115.)

Case 1. In fig. 115, the lines *a′b′*, *ab*, and *a′c′*, *ac*, meeting in the point A, are *both* inclined to *each* of the co-ordinate planes. Take two points, *p′p* and *q′q* (one on each line), having the same vertical height. Join *pq*, and conceive the lines PA, AQ, of the triangle PAQ to be rotated upon the horizontal line PQ until A attains the same level as the other two points. Then it will be seen that the true shape of the triangle will be obtained, and thence the angle between the two given lines.

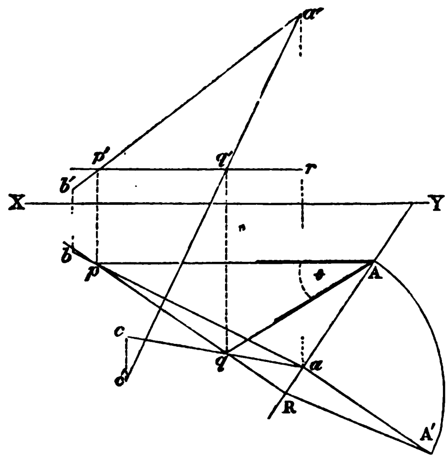

Fig. 115.

The point A will move in a vertical plane perpendicular to *pq*. Draw R*a*, and produce it indefinitely to represent the path of the plan *a*, as the triangle is rotated. The point itself will describe the arc of a circle having R for its centre, the radius of the arc being equal to the hypotenuse of a right-angled triangle, having R*a* for base, and *ra′* for perpendicular. Make *a*A′ perpendicular to R*a* equal to *ra′*, and join RA′. With centre R, radius RA′, describe the arc *A′A*. *This* obtains the point A, the position of the vertex of the triangle, when all its sides are horizontal. Join Ap and Aq, and the angle required (θ) will be determined.

We could, of course, attain the same object by finding the horizontal traces of each of the lines, and using the line joining these points as hinge, *instead* of assuming two points of equal height on the lines; but it is perhaps better for the student to adopt the method shown, as the horizontal traces of the lines in some cases are very remote, and are not practically of use. That principle is shown on p. 161 of the Elementary Book in this series.

Another solution would be, to find the real lengths of the lines AP, AQ, and to form a triangle with PQ as base, and the other sides equal to AP, AQ.

Case 2. A second case is shown in fig. 116, in which each line is parallel to *one* co-ordinate plane. In this case the real lengths of the lines are known; and we have only to produce one of them (ca) and rotate the other line, AB, until B is on the same level as A and C.

The line Gb, perpendicular to Gc, and passing through b, is the path in which the plan of B must move; and as the line AB will be horizontal when its plan is equal to its real length, take a as centre, radius equal to $a'b'$, and cut the line GB in B. Then caB is the angle required.

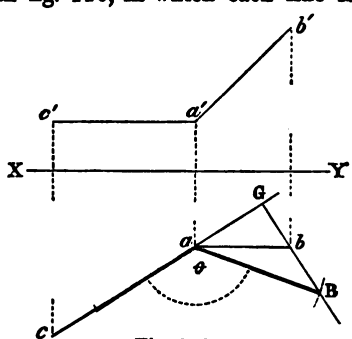

Fig. 116.

Problem CXVI.

To determine the angle between a line and a plane. (1) When the line is horizontal; (2) when inclined to both co-ordinate planes.

The angle between a line and a plane is the angle between the line and its orthographic projection on that plane, and as such projection is obtained by projectors perpendicular to the plane through the points of the line, the angle between the line and one of these projectors is the complement of that between the line and the plane. In other words, the line, its projection, and a perpendicular to the plane through any point in the line, form a right-angled triangle, the right angle being at the point where the perpendicular meets the plane; the other two angles together making 90°. Hence the angle between the given line and the perpendicular is the complement of that between the line and plane. By the aid of this principle the above problem is solved, and the modifications which the solution may require, depend only upon the relative positions of line and plane.

CASE 1. In fig. 117, the line AB is horizontal, and as the angle between the line, and a perpendicular to the plane meeting the line, is the complement of that required, set out at any point *b′b*, the lines *b′c′* and *bc* perpendicular to the traces of the plane. Then obtain the angle between the lines AB and AC, knowing that AB is horizontal. As this presents a slight difference in the position of the two lines to that shown in Case 2 of the previous problem, a description of the working in fig. 117 is appended. AB is taken as the axis of a conical surface, having its apex at B, and containing the line BC.

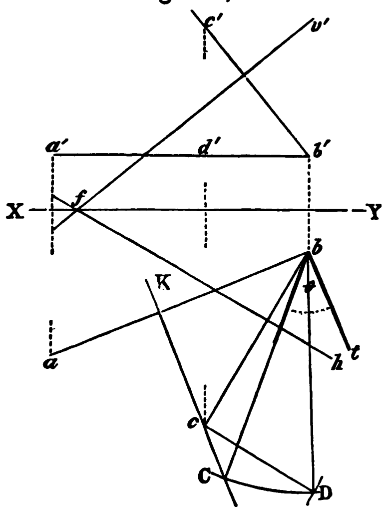

Fig. 117.

Therefore, if BC were to move round this surface, it would, in all positions, make the same angle with AB.

Then *c*K drawn perpendicular to *ab* is the path of the plan of point C, as this rotation proceeds. When BC is in plan equal to its real length, the line will be horizontal, and the angle between AB and BC (both then being horizontal) will be discovered.

The true length of BC is shown at *b*D (*c*D is perpendicular to *bc*, and equal to *c′d′*). The arc CD, struck with centre *b*, gives, by its intersection with KC, the point C. Then the angle K*b*C is that between the given line and the perpendicular to the plane; and C*bt*, which completes the right angle, is the angle required between the line AB and the plane *v′fh*.

Another Method.

In Plate XI., fig. 3, another solution of this problem is shown. The line *a′b′*, *ab*, is horizontal. Find its intersection *e′*, *e*, with the given plane *v′fh*. From any point *b′b*, in the line, set out a perpendicular to the plane. To do this, draw *b′c′* and *bc* perpendicular to *v′f* and *fh* respectively. Find the intersection of this line with the plane. Its projections are *d′d*. Join *ed*. Then *edb* is the *plan* of a right-angled triangle made up of part of the given line (*eb*), the perpendicular to the plane (*bd*), and the projection of the line on the plane (*ed*). Of these three lines *eb* is in plan, full length, because it is horizontal. Opposite to it, at the point *d*, the angle is a right angle. Construct therefore on *eb* a semicircle, and through *d* draw *d*D perpendicular to *eb*, intersecting the semicircle in D. Join *e*D, and the triangle *eb*D will have been rotated upon the line *eb* until the point D is on the same level as *e* and *b*. The angle *be*D is therefore that between the *line* and its projection on the plane, or, as we have shown above, the required angle between the line and the given plane.

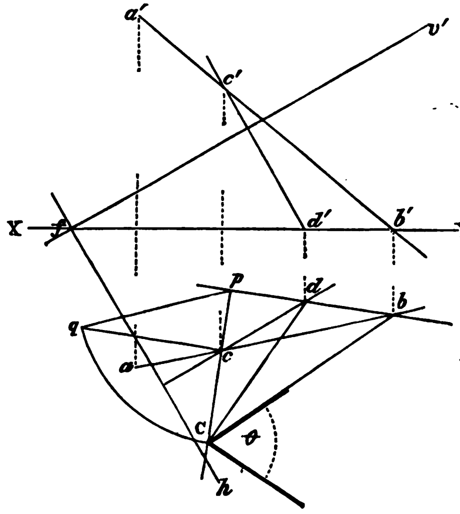

Fig. 118.

Case 2. When the given line is inclined to both co-ordinate planes, as in fig. 118, the angle between the line and a perpendicular to the plane must be determined as in Problem CXV. BC and CD are the two lines, CD being perpendicular to the plane. The points *c* and *b* are the horizontal traces of the lines, and *d*C, *b*C, are the same lines "constructed" into the h. p. The solution of this part of the problem is clearly shown in the diagram. The angle *b*C*d* is that between the lines; its complement, as shown, is that between the line AB and the plane *v′fh*.

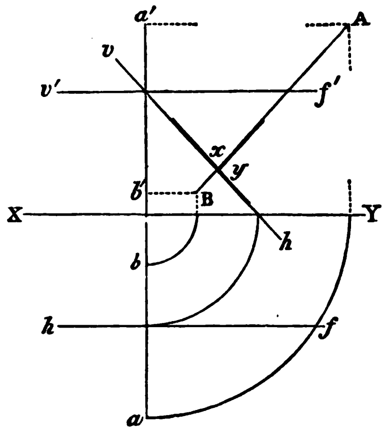

Fig. 119.

Case 3. When the line and plane are situated as shown in fig. 119, a profile view of them exposes the angle between them, because a vertical plane through the line contains its projection on the given plane. The solution of the problem is shown in the figure, and the line *vh* represents the profile view of the plane, and AB that of the line. The angles *x* and *y* are those required.

Problem CXVII.

Through a given point in a straight line, to draw a plane perpendicular to the given line. (Fig. 120.)

When a line is perpendicular to a plane, its projections are perpendicular to the traces of that plane, so that, as the projections of the line are given, the traces of the plane will be at right angles to them. At the given point *c′*, *c*, set out *c′d′* and *cd*—the former parallel to XY, and the latter perpendicular to *ab*. These lines will be the projections of a horizontal on the plane. The point *d′* is the v. t. of this horizontal, and must be in the vertical trace of the plane. Draw *v′f* through *d′* perpendicular to *a′b′*, and *fh* perpendicular to *ab*.

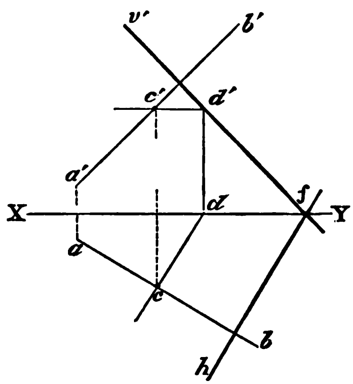

Fig. 120.

Note.—The given point need not necessarily be in the given line. The same *construction is* used for any point.

Problem CXVIII.

Through a given point, to draw a line to meet a given line at a given angle.

Let *a'b'*, *ab*, be the projections of a line AB, and *c'c* those of a point; it is required to draw the projections of a line CD passing through C, and meeting the line AB at an angle of 120°.

It is clear that both lines (the given one and the required one) must lie in one plane. Find then the plane containing the points A, B, and C. In fig. 121, EF is the h. t. of this plane. It is found as in Problem CIV. It is not necessary to find the v. t. Construct then

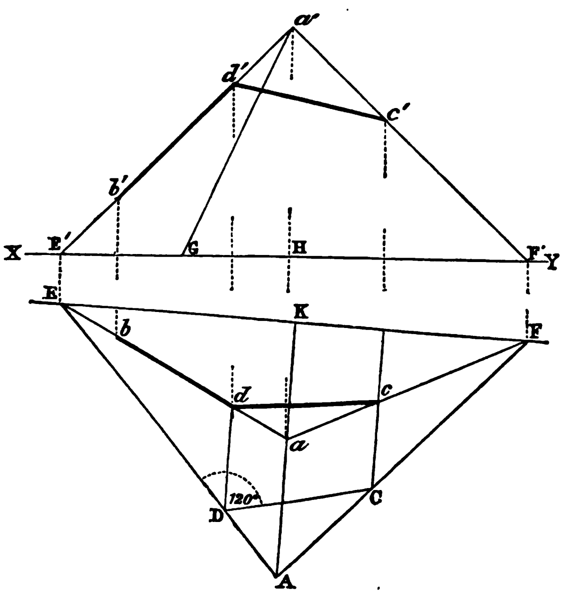

Fig. 121.

this plane about its h. t. into the h. p., and show the line AB and the point C so "constructed." To do this, through point *a* set out K*a*A perpendicular to EF. The point A will move in a vertical plane, having KA for its h. t. The line KA is therefore a locus of the plan of the point A at all stages of this rotation. The point A will reach the h. p. when KA is equal to the real distance of K from the point A on the plane. To obtain this distance, set out GH on XY equal to Ka. Join GA′. Then GHa' is a right-angled triangle, which would fit under the oblique plane, having its base coincident with Ka, and the apex a' with the point A. Then if KA be made equal to GA′, and E and F be joined to A, the line AB will have been constructed into the h. p. As the point C is on AF, a perpendicular to EF through C will give C its "constructed" plan. Set out CD at 120°, with AB intersecting that line in D, and refer the point D back to the original plan by drawing Dd perpendicular to the h. t. Join cd, and after finding d', join $c'd'$, and the projections of the required line will have been determined.

When the angle which the required line is to make with the given one is a right angle, it is best to find the plane through the given point perpendicular to the given line, and their intersection will be the second point required.

Problem CXIX.

To draw the projections of a straight line of given inclination contained by a given plane. (Fig. 122.)

When a line is contained by a plane, all its points are in that plane. Hence, if two points of a straight line, lying in a plane, be determined, the whole line can be drawn. It has already been shown that the inclination of any line contained by a plane can be equal to, but cannot exceed, that of the plane itself.

Again, there can be two lines, and two only, passing through the same point, and contained by the same plane, and having a given inclination.

Having assumed any point in the plane, conceive that point to be the apex of a vertical cone, having an angle at the base equal to the inclination of the required line. This cone will either be tangential to the plane (in which case the line of contact is the solution required), or it will be cut by the plane. In the latter case, the traces of the conical surface upon the plane will be the two lines which satisfy the conditions of the problem. Further, the h. t. of the plane will, in such case, meet the circle, which is the plan of the cone in two points.

The application of the above principles is shown in three cases in *fig. 122.*

The plane $v'fh$ is perpendicular to the v. p., and a point A is assumed by its projections as being upon the v. t. of the plane. The line a'B is set out, making the given angle of inclination with XY, and $a'a$ is assumed as the axis of the vertical cone, and $B'b_1b_2$ as its plan. This circle crosses the h. t. in two points, b_1 and b_2. Then these points being joined to a, ab_1 and ab_2 are the plans of two lines, which are inclined θ in virtue of being contained by the surface of the cone, and are in the given plane, because the point A is on the v. t., and B is on the h. t.

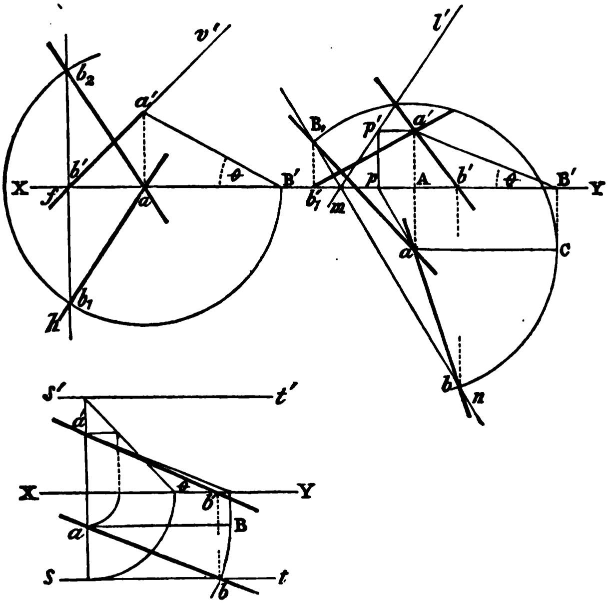

Fig. 122.

The student should be aware that all lines of equal inclination on the given plane must have their projections parallel to those found.

NOTE.—This construction is necessary in obtaining the projections of a figure, the inclination of one of its surfaces, and that of a line in that surface, being given. (See Problem CLVIII.)

In the second case, the traces of the plane make acute angles with XY, and the starting point A is *not* in the vertical plane. The student will see that the point A is *in* the plane, as it is on AP, one of its horizontals.

The solution is similar: $a'AB'$ is the elevation of half the cone; bCB_1 is its plan, and the two lines are in plan $B_1\,a$ and ab, and in elevation b'_1a' and $a'b'$. The student must be careful that in striking the circle, he takes a as centre, and that the radius is equal to $b'B'$, or to aC.

In the third case, the plane given is parallel to XY, and the point A is shown by construction (see Problem CVIII., Case 2), to be in the plane. The solution is exactly similar to the preceding, but only *one* line is shown, $a'b$, ab.

Problem CXX.

Given a line, contained by a plane, required a second line also contained by the plane, but making an angle of 35° with the given one. (Exam. Paper, 1875.)

Let $v'fh$ and $a'b'$, ab (fig. 123), represent the conditions given. Then if the plane be constructed into the h. p. about its h. t., the point B will move in a vertical plane through b, having bB for its h. t. Draw bB perpendicular to fh. Make DG equal to the height of point B above XY, *i.e.*, equal to bb'. Make DB equal to bG, and the point B will be the "constructed" plan of b. Set out the angle aBc of 35°, and the line Bc will represent the new line required (but in the h. p.). When c is joined to b, the line bc will be the plan of this line when lying on the plane, and as c is in the h. p., its elevation c' is on XY. Join $c'b'$, and the required projections will have been determined.

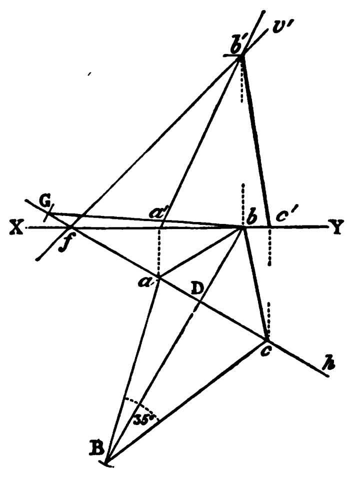

Fig. 123.

Problem CXXI.

Through a given straight line, to draw a plane of a given inclination.

Of course, the inclination of the plane must equal or exceed that of *the line.* Let $a'b'$, ab, fig. 124, be the given line, and the angle of the plane 65°. As the plane is to pass through the line, the h. t. of the line must be in that of the plane. Find *c*, the h. t. of the line AB. Then at any point *a'*, set out the elevation of a vertical cone, having a base angle of 65°. In the figure, *a'e'd'* represents half the cone. Its plan is the circle *de*, having *a* for its centre. All planes which are tangential to this cone will be inclined 65°. Hence, if the h. t. of the plane be drawn through C, and tangential to the circle *de*, there will only remain the condition of that plane passing through the apex to fulfil entirely those of the problem. Assuming *cf*, therefore, to be the h t. of the plane, and *a'a* a point in it, the line *a'g'*, *ag*, is a horizontal of the plane, and its v. t. (*g'*) is in the v. t. of the plane. Draw then *g'f*, and thus complete the required traces.

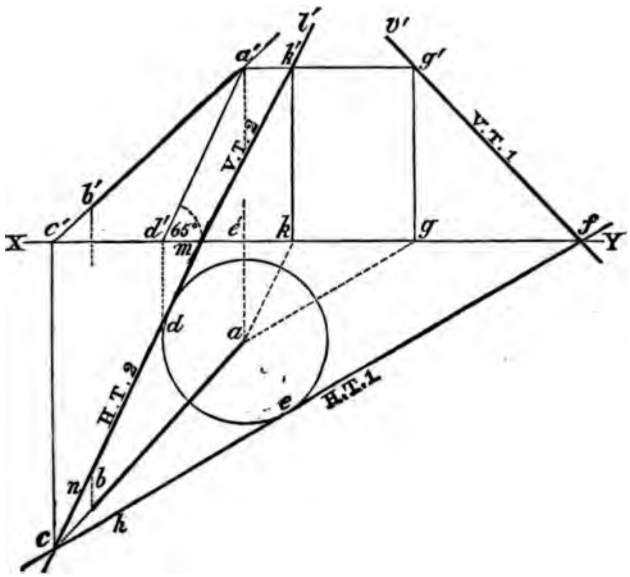

Fig. 124.

There are two solutions to this problem, as shown in the figure. The plane *l'mn* also contains the given line, and fulfils the condition of being inclined 65°, as its h. t. *mn* touches the base of the cone, and the v. t. of the horizontal through A is in the v. t. *l'm* of the plane.

Problem CXXII.

Given the inclinations of two straight lines which meet, and the angle between them; required the plane containing them; also their projections.

The sum of the inclinations of the lines, together with the angle included between them, must not exceed 180°. When it equals that angle, the plane containing them is perpendicular to the h. p. This is obvious, as the three angles of a triangle are together equal to two right angles, and the lines have their greatest inclinations when they are contained by a vertical plane. The angle between them is in that case the vertical angle of the triangle.

Let us suppose that the two lines are to be inclined at 34° and 40° respectively, and that they are to contain an angle of 60°.

If a piece of paper be cut exactly like that in fig. 125, and it be folded upon the lines AB and BC until the points D and D_1 coincide, the student will have before him a model of the conditions suggested in the problem. The model should rest on the lines AB and BC, and the inclined surface of the triangle ABC will be the plane required. It will be noticed that the longer edge will be the less inclined, and the h. t. of the two lines will give the h. t. of the plane. Further, if a v. p. be assumed perpendicular to this trace, the elevation on that plane of the triangle ABC will be a straight line, coinciding with the v. t. of the plane of the triangle. The method of procedure now to be explained is based upon this principle, and by the aid of such a model the student will see a reason for each step and line in the solution.

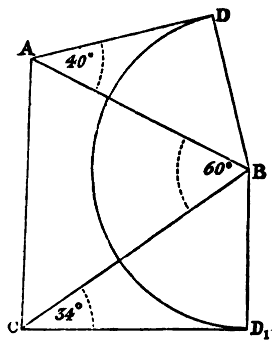

Fig. 125.

Draw two lines AB, BC (fig. 126), at an angle of 60°. Assume a point A in AB, and draw AD′ at an angle of 40° with AB. At the point B make BD′ perpendicular to AD′. Then the distance BD′ is the height to which the point B must be raised in order that AB may meet the h. p. in A at the inclination of 40°. Now, as the second line *BC meets the* first in B, it is clear that BD must be equal to AD′.

Describe, then, an arc having B for centre, and radius equal to BD′. It follows, then, that wherever BD is to be drawn, it must be a radius of this arc. At any point E, set out an angle of 34° (the inclination of the second line BC), and draw a line parallel to EF and tangential to the circle DD′. If BD be drawn perpendicular to this tangent, the lines of the model will be clearly perceived. Join AC, which may be assumed as the h. t. of the plane required. In fact, all lines parallel to AC are horizontals of the plane, and any one may be taken for the h. t. Draw XY perpendicular to AC, and, knowing that BD′ represents the height of the point B above the h. p., draw *b*B′ perpendicular to XY, and with *a*′ as centre, describe the arc B′*b*′, marking the point *b*′ at a height above XY equal to BD′. (The line K*b*′ is drawn parallel to XY at a height equal to BD′.) Then *a*′*b*′ is the elevation of the triangle ABC, and *a*′*v*′ is the v. t. of the plane containing it. The plan of point B is found by a projector through *b*′ meeting a line through B parallel to XY, because, when the triangle rotates on AC, the point B moves in a vertical plane perpendicular to that line.

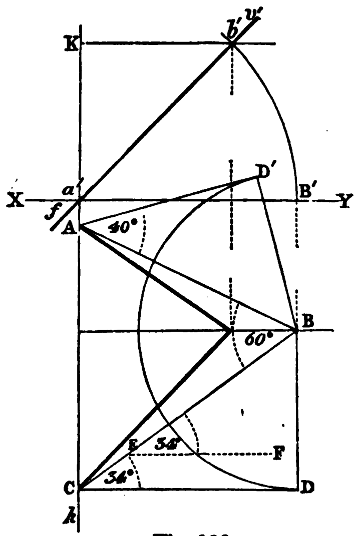

Fig. 126.

NOTE.—This problem is required in obtaining the projections of a figure, the inclinations of two of its lines and the angle between them being known. (Problem CLXII.)

PROBLEM CXXIII.

To draw a plane through a given straight line parallel to another given straight line.

Let *a*′*b*′, *ab*, and *c*′*d*′, *cd*, represent the lines, through the former of which a plane is to be drawn parallel to the latter. This problem is always determinate, except in the case of the lines being themselves parallel, when *any* plane containing the one will be parallel to the other.

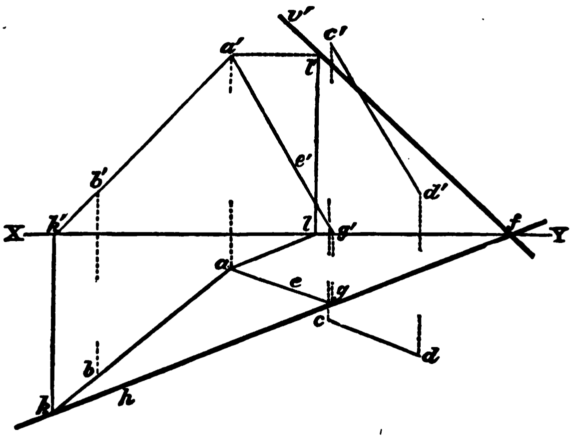

Fig. 127.

At any point *a'a*, in the line AB set out the projections *a'e'*, *ae* of a line parallel to CD. Then find the plane containing the two lines AB and CE, as in Problem CIV., and that plane will satisfy the condition of being parallel to CD.

Problem CXXIV.

To determine the traces of two planes, mutually perpendicular, when their inclinations to the h. p. are given.

The sum of the inclinations of the two planes, together with the angle between them, cannot be less than 180°. If the sum be 180°, the horizontals of the planes are parallel, and their intersection is a horizontal line.

If a plane contains any line which is perpendicular to another plane, the two planes will be perpendicular to each other. This is readily seen by placing a pencil perpendicular to a sheet of paper, and then placing another sheet in such a position that the pencil may lie upon it. But there are an infinity of planes which would satisfy this condition, as *will be* seen by rotating the second sheet round the pencil. Hence, *for this problem* to be determinate, the inclination of the second plane is given. If, then, a right *cone*, having for its base angle that of the given inclination of this plane, be conceived to have its apex in the point where the pencil meets the first sheet and its base upon the h. p., it will be seen that the second sheet of paper can be placed in two positions where it contains the perpendicular to the first plane, and also touches the surface of the cone. In virtue of the former fact

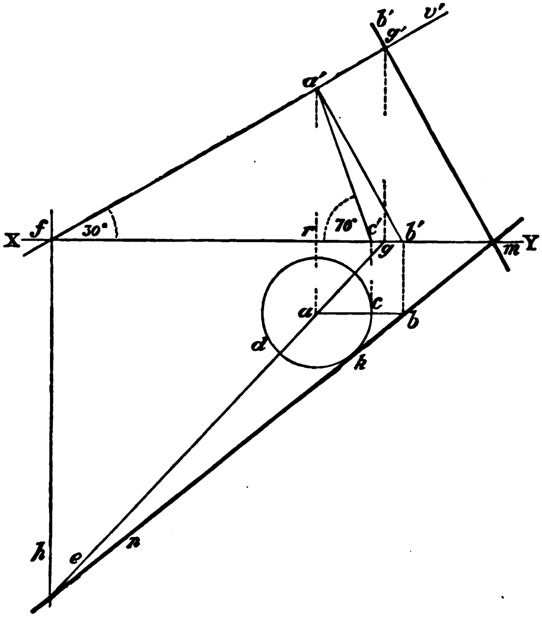

Fig. 128.

(containing the perpendicular), this sheet will be perpendicular to the first sheet, and in virtue of the latter (being tangential to the cone), it will have the required inclination. The horizontal trace of this plane will pass through the h. t. of the perpendicular, and will touch the circle, which is the trace of the cone.

In fig. 128 *v'fh* represents the traces of a plane inclined 30°, and it is required to draw the traces of a plane inclined 70°, which shall be perpendicular to *v'fh*. Assume any point, *a'a*, *in the plane*, and set out *a'b'*, *ab*, the projections of a perpendicular to this plane. The triangle *a'c'r* is the elevation of a semicone having a base angle of 70° (the required inclination), and its apex in the point A, whilst the circle *ckd* represents the whole plan. The point *b* is the h. t. of the line AB, and *bke*, drawn through *b*, and tangential to *ckd*, is the h. t. of the required plane.

As the apex of the cone is in both planes, it must be a point in their intersection; hence, by drawing *ea* and producing it to *g*, we shall have the plan of the common line to both planes. The point *g'*, determined from *g*, is in both vertical traces, as it is the v. t. of the intersection. Then *l'm*, passing through *g* and *m*, is the v. t. of the plane required.

NOTE.—This problem is required in obtaining the projections of solids, having faces which are perpendicular, when the inclinations of two of those faces are given (see Problem CLXVIII.)

PROBLEM CXXV.

To determine by its traces a plane, making a given angle with a given plane, and having a given inclination. (Plate XII., fig. 1.)

This problem differs from the preceding in one condition, *i.e.*, the angle between the given and required planes is not a right angle.

Any point in space can be assumed as the common apex of two cones, one with axis vertical, and its generatrices making an angle with the h. p. equal to the inclination of the required plane; and the other with its axis perpendicular to the given plane, and its generatrices making an angle with it equal to the given dihedral angle. The horizontal traces of these cones, when determined, will be in the former case a circle, and in the latter an ellipse. Then a plane which is tangential to each of these cones (*i.e.*, whose h. t. touches the circle and the ellipse, and which passes through the common apex), will be inclined to the h. p. as desired, in virtue of being tangential to the vertical cone, and will make the given angle with the given plane, because it is also tangential to the oblique cone. In Plate XII., fig. 1, *v'fh* is the given plane, inclined 42°, and the planes determined fulfil the two conditions of being inclined 78°, and of making an angle of 73° with *v'fh*.

The point *a'a*, is taken as the common apex of the two cones; the vertical one *a'b'c'* making 78° with the h. p., and the oblique one, *a'e'd'*, with axis perpendicular to *v'f*, making 73° with the plane. The circle *bc* is the trace of the former, and the semi-ellipse *rxp* is that of the oblique cone. The description of the construction employed to find this semi-ellipse belongs to Chapter XI., and will be found in Problem CLXXX., to which the student is referred.

The h. t. of any plane which touches both these cones will be tangential to the circle, and also to the ellipse. Four such lines can be drawn, and the student will see that there are four planes which would satisfy the conditions of the problem. Two planes would touch the cones—one in front and one behind, and two others would touch them and pass between them. The traces of all four are shown in the figure.

The vertical traces are determined by finding the v. t.'s of a horizontal line in each plane, passing through the point A, as it is known that that point is in the intersection of the given plane with those required.

Problem CXXVI.

Two planes are mutually perpendicular. Their intersection is inclined 40°; one of them is inclined 50°; show them by their traces. (May Exam., 1871.)

It is better in working this problem to assume the given plane with its traces, each making an acute angle with XY. In fig. 129, *v′fh* is the given plane (the proper inclination of 50° is ensured by the plane being tangential to a cone of 50°). On this plane a line *p′k′*, *pk*, is shown inclined 40°. Draw then SQ perpendicular to the intersection, and, knowing the true length and inclination of the intersection, as shown at *k′*P, find the length of a line TR perpendicular to it. This is shown at TR′, the construction being the converse to that of finding the angle between two given planes (Problem CXIV.). Make TR equal to T′R′, and join RS. Then RQ, at an angle of 90° with RS, gives a point Q upon the h. t. of the second plane; and as P is another point in that trace,

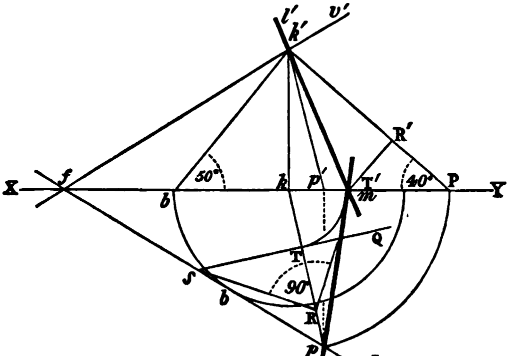

Fig. 129.

and *k′* is in both vertical traces, the plane is determined by its traces *l′m*, *mn*.

NOTE.—This problem is really an exercise upon Problems CVI. and CXIV., and the student will see that other dihedral angles, instead of a right angle, could have been given.

PROBLEM CXXVII.

Two planes are inclined at angles of 50° and 57° respectively. Their intersection is inclined 40°; show them by their traces. (Fig. 130.) (Sc. Exam., May.)

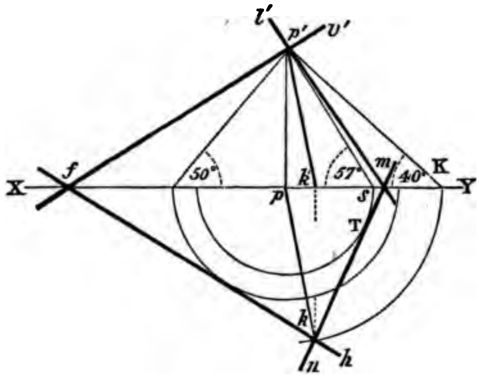

Fig. 130.

The plane *v′fh* is inclined 50°, and the line *p′k′*, *pk*, contained by it is inclined 40°. Assume a cone having vertical axis and base on h. p., the base angle to be 57°, and the apex in *p′p*. The semicircle ST is plan of one-half of this cone, and *p′ps* is the elevation. Then *mn* drawn through *k*, tangential to ST, and meeting XY in *m*, is the h. t. required, and *lm* drawn through *p′* and *m* is the v. t.

PROBLEM CXXVIII.

Through a given point to draw a straight line parallel to two given planes. (Fig. 131.)

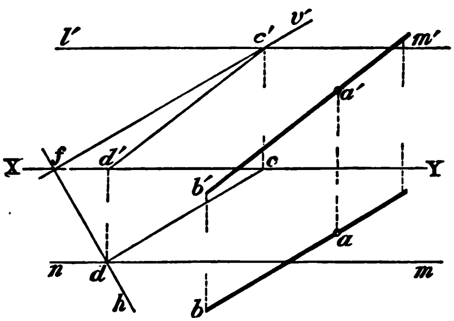

Fig. 131.

When a line is parallel to two planes, it is parallel to their intersection. In fig. 131, two planes, *v′fh* and *l′mn* are assumed, and through the point *a′a*, it *is required to* draw a line parallel to both. Their intersection is *c'd'*, *cd* (Problem CVI.). Then *a'b'*, *ab*, drawn parallel to *c'd'*, *cd*, respectively, are the projections of the line required.

Problem CXXIX.

Through a given point to draw a plane perpendicular to two given planes.

When a plane is perpendicular to two other planes, it is perpendicular to their intersection, and has its traces perpendicular to the projections of that intersection. Find then the intersection of the two planes (Problem CVI.), and through the given point draw the plane as required (*i.e.*, perpendicular to the line found, Problem CXVII.).

Problem CXXX.

From two given points, not contained by a given oblique plane, to draw two lines meeting in that plane, and making equal angles with it.

This is a similar problem, in Solid Geometry, to that of finding, by Plane Geometry, two lines PR and QR to pass through two given points, P and Q, and to meet a given line AB, and make equal angles with it. The construction of this latter question is shown here, at fig. 133, as it is so intimately connected with the question before us.

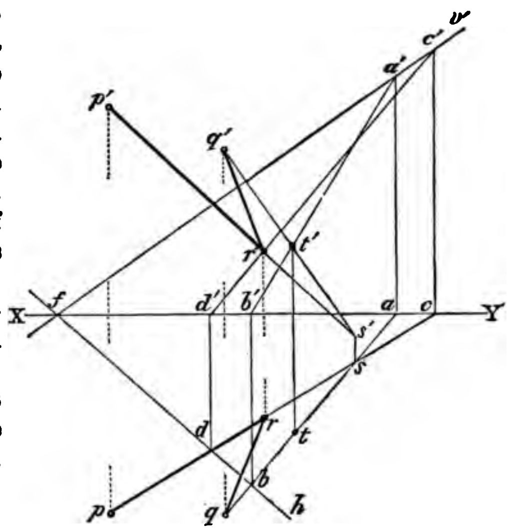

Fig. 132.

Draw QS at right angles to AB. Make ST = QT. Join PS, meeting AB in R. Join RQ.

The proof is very simple. QTR and STR are equal and similar triangles; therefore the angle QRT = TRS, and PRB is the opposite angle to TRS, and hence is equal to it (Euclid i. 14), and therefore to QRT.

In fig. 132, the given plane is *v′fh*, and from the two points, *p p*, *q′q*, it is desired to draw two lines, *p′r′*, *pr*, and *q′r′*, *qr*, to meet in a point *r′r* upon the planes, and to make equal angles with that plane, *i.e.*, with their projections on that plane.

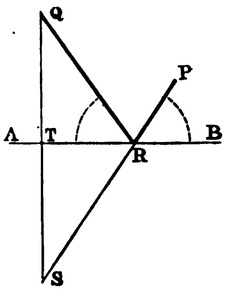

Fig. 133.

Through one of the points, as *q′q*, draw *q′s′*, *qs*, perpendicular to the plane (Problem CX.), and find their intersection *t′t*. Make *t′s′*, *ts*, equal in length to *q′t′*, *qt*. This line plays the same part here as the line QS in the *plane solution* referred to above. Join *s′* to *p′*, and *s* to *p*. Then find the intersection of the line PS with the given plane. This gives *r′r*, which is the point on the plane where the lines from P and Q will make equal angles with it, and *p′r′*, *q′r′*, and *pr*, *qr*, are the projections required.

There can be an infinite number of pairs of lines meeting on the plane as required, but the two determined are together less in length than any other pair. The locus of the meeting points of all the pairs is a circle, the point R being at one extremity of a diameter, whose other extremity is the trace on the plane of a line joining P and Q. This is seen by the following proof:—

If a line AB (fig. 134) be divided in the point C in any ratio, and produced to D, so that AC, AB, AD are in harmonical progression, *i.e.*, if AC : BC :: AD : BD, and if a circle CPD be described on CD, then any pair of lines drawn from A and B, and meeting in this circle, will be in the constant ratio of $\frac{AC}{BC}$ (Euclid vi. 3, 4, and *b*).

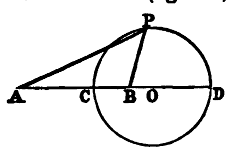

Fig. 134.

In the perspective view (fig. 135), let MN represent the plane in the problem, and A and B the two given points. From A and B draw two perpendiculars to the plane (AE and BF), and find point D, as shown in the problem. Join AB, and produce it to meet the plane in C. This line will also meet EF produced, as EF is the *orthographic* projection of AB on the plane.

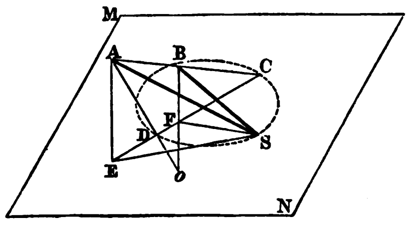

Fig. 135.

Then AE, BF, AC, and EC are in one plane, and from the properties of similar triangles,

EC : FC : : AE : BF; but (Euclid vi. 2)
ED : DF : : AE : BF; hence ED : DF : : EC : CF,

or EC is harmonically divided. Therefore, as shown above, if a circle DSC be described on CD, the ratio of any two lines, ES, FS, drawn from E and F, to meet the circle, is constant and equal to ED : DF. Join AS and BS. Then in each of the two triangles, AES, BFS, there is a right angle, and the sides containing them are proportional. Hence the triangles are similar, and the angle ASE = BSF.

When the two given points are equidistant from the given plane, the locus is a straight line, which is the intersection of a plane perpendicular to and bisecting PQ, with the given plane.

Problem CXXXI.

Through a given point, to draw a straight line to meet two given straight lines.

This problem can always be solved, except in the case where the two given lines are in one plane and the given point is without it.

1st Solution.

Let the given lines be *a′b′*, *ab*, and *c′d′*, *cd* (Plate XIV., fig. 1), and the given point *i′i*. Find, first, the plane containing the given point and *one* of the given lines; secondly, the plane containing the given point and the other line. Then the line of intersection of these two planes will be that required in the problem.

Referring to the figure, the plane *v′fh* is that which contains the point I and the line AB (Problem CIV.); and the student will notice, that as the horizontal traces of lines joining I to A and B would be at so great a distance, two points, *e′e* and *g′g*, which are *in* the line *a′b′*, *ab*, have been used instead. Of course, the plane which contains these three points will also contain the points A and B. Similarly, the plane *l′mn* is that containing the point *i′i*, and the given line *c′d*, *cd.** The intersection of these two planes, $\alpha\beta''$, $\alpha\beta$ (Problem CVI.), is the line required.

Note.—The lines do not meet upon the drawing, but would do so if produced. It is possible that the line found may be parallel to *one* of the lines, and then it is recognised as meeting that line at a point infinitely distant.

* The points *r′r* and *s′s* are taken instead of *c′c* and *d′d*, because the vertical traces, which determine the v. t. of the plane, fall within the sheet.

2nd Solution.

An ingenious method of solving this problem is shown in the same plate, fig. 2, where *a'b'*, *ab*, and *c'd'*, *cd*, are the projections of the given lines, and *p'p* those of the given point. Any two points, as E and G, are taken in the line AB, and two lines from P are drawn through them to intersect a vertical plane, containing the other line CD, in the points S and T. The line ST is therefore the intersection of a plane containing P and the line AB, with the vertical plane containing CD. Hence all lines passing through P, and *a point in* AB, will intersect this plane in ST, or ST produced. But ST crosses CD in Q, therefore a line joining P to Q will be in both these planes, and will pass through AB and meet CD.

Take *e'e* and *g'g* on *a'b'*, *ab*, and through *p'* and *p*, and draw *p'e'*, *p'g'*, *pe*, *pg*, to meet *c'd'* and *cd* in *s't'* and *st*. Join *s't'*, intersecting *c'd'* in *q'*. Then *p'q'* is the elevation of the line required; and, by determining *q*, the other projection of point Q, the line *pq* can be drawn, which is its plan.

Problem CXXXII.

Given the projections of two parallel straight lines, required those of a third line which shall meet them, making alternate angles of 40°. (Plate XII., fig. 2.)

First, find the plane containing the two lines, and construct it about its horizontal trace into the h. p. Next, construct also the parallel lines which lie on the plane. Then join them by a secant, meeting them alternately at 40°. Transfer the points of intersection of the three lines to the plan and elevation, and complete the projections of the line required.

In Plate XII., fig. 2, the given parallel lines are *a'b'*, *ab*, and *c'd'*, *cd*. The plane *v'fh* is the one containing these lines (Problem CIV.). To construct this plane, with the lines on it, into the h. p., a point *a* in the plan of *ab* is taken, and *a*A is drawn perpendicular to the h. t. Then *a*A is the h. t. of the vertical plane in which this assumed point A moves during the rotation of the plane *v'fh*.* The length OA is determined by constructing a right-angled triangle *a*A'O, such that *a*A' is equal to the height of point A above the h. p., as shown in the eleva-

* The plane revolves *towards* XY.

tion. Then the hypotenuse A′O determines the real distance of A from point *o*.

The point *b* in AB does not move during the rotation, so that *b*A is the representation of the "constructed" line, and *tr*, drawn parallel to *a*B, is that of the "constructed" line CD. At any point draw the secant PQ, meeting these lines alternately at 40° in P and Q. Transfer these intersecting points by perpendiculars to the h. t. on to the plans at *p* and *q*, and by projectors obtain *p′* and *q′*. Then *pq* and *p′q′* will be the projections of the required line.

NOTE.—A construction similar to the above will solve the problem "to show the centre of a circle to pass through three points whose projections are known," as the plane containing them having been found, they can be constructed into the h. p., and the centre determined. Then that point can be transferred, like P and Q in the above problem, into its position on the plane of the three given points.

PROBLEM CXXXIII.

To determine the projections of a common perpendicular to two given lines.

This problem is required when the positions of the axes of two right cylinders touching each other are known, and it is desired to find the point of contact between them.

If the given lines meet, its solution is impossible, and if they be parallel, the common perpendicular required will be in the plane containing them, and the solution would be similar to that in the last problem. But in the case where the given lines are neither parallel nor meet, a common perpendicular can always be determined.

In Plate XIII., fig. 1, the given lines are *a′b′*, *ab*, and *c′d′*, *cd*. The plane must first be found which contains one of the lines (AB), and is parallel to the other (CD). The solution of this part of the problem is shown, and its explanation was given at Problem CXIII. The traces are *v′f′* and *fh*. From any point, as *r′r*, in the line CD a perpendicular to the plane *v′fh* is first set out, and then its intersection *t′t* with that plane is found. The projections of the line are *r′s′*, *rs*, and those of the intersection are *t′t*. Now, as the line CD is parallel to the plane *v′fh*, all its points are equidistant from that plane, and the line RT measures that distance. If, then, the line RT be conceived to move along the line CD, so that its extremity R is always in that line, then the point T would trace upon the plane a line *t′p′*, *tp*, parallel to CD. Now, this line must meet AB, because they are both in one plane, and by the conditions of the problem they cannot be parallel; *p′p* is therefore the point of meeting, and *p′q′*, *pq*, drawn parallel to *r′t′*, *rt*, is the common perpendicular required. The line PQ is perpendicular to AB, because it is perpendicular to a plane containing AB. This line is the shortest distance between the two given ones.

Problem CXXXIV.

Two lines, inclined 40° and 50° respectively, do not meet. They are 2″ apart where they are nearest to each other, and the common perpendicular to both is inclined 28°. To show their projections. (May Exam.)

The common perpendicular to both given lines measures the shortest distance between them. Hence the line of 28° inclination is 2″ long. Draw the projections of this line, as *a′b′*, *ab* (fig. 136), assuming it to be parallel to the v. p., and therefore of full inclination and length in the elevation. Take *v′fh* as the traces of a plane perpendicular to the line AB. Then every line in this plane which has one extremity in B will necessarily be perpendicular to the line AB. In this plane, therefore, determine a line inclined 50° (Problem CXIX.). The projections of this line are *c′d′*, *cd*. A similar proceeding at the extremity A would give the other required line, but in the figure a line inclined 40° is shown in the plane *v′fh*, its plan is *bk′*, and the projections of the line ST through A are parallel to *lk* and *b′f*. Of course, as two lines can always be found of given inclina-

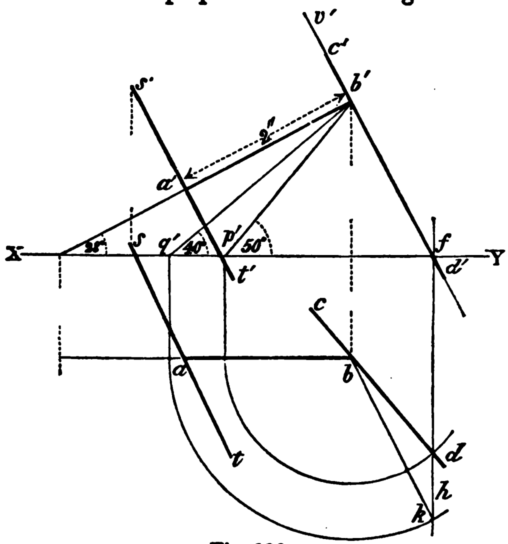

Fig. 136.

tion to pass through a point in a given plane, there are four solutions to this problem (according to the inclined *lines* used.)

PROBLEM CXXXV.

A straight line inclined 33° in a plane inclined 50° is the orthographic projection upon that plane of another straight line making an angle of 40° with it. Determine the plan of this latter line and its inclination. (May Exam., 1871.)

In fig. 137 the line whose plan is *ab* is inclined 33°, and is contained by the plane *v'fh* (50°). This being the orthographic projection of another line upon this plane, its points must be in a projecting plane passing through the line itself, and perpendicular to the plane *v'fh*; and as the angle which a line makes with a plane is measured by the angle between the line itself and its projection on the plane, we have to conceive a right-angled triangle resting with its base on the line AB, and its plane perpendicular to *v'fh*, with a base angle of 40°. This triangle is shown in the figure, constructed into the h. p., *i.e.*, the line AB is so constructed, and the triangle A*b*D built upon it. This discovers at once the height of point D perpendicularly above the plane. Set out the projections of a perpendicular to *v'fh* passing through A. These are *a'd'*, *ad*, and make *a'd'* equal to AD, and join *b'd'*. This is the elevation of the line required, and *bd* is its plan. The inclination (θ) is obtained by constructing the line into the h. p. as shown.

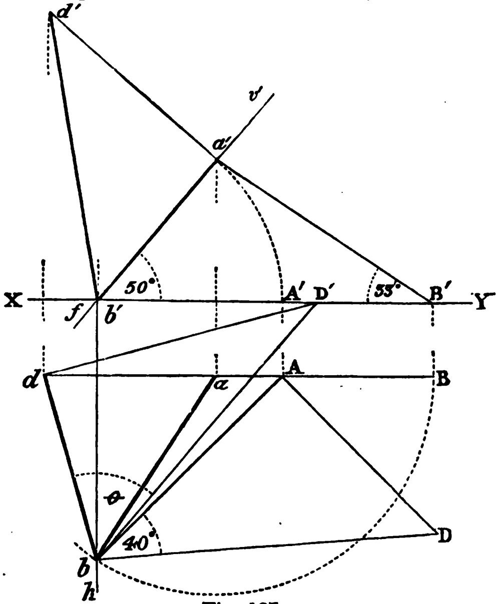

Fig. 137.

PROBLEM CXXXVI.

To draw a line to meet two given lines, to be parallel to, and ·75″ from a given plane. (May Exam., 1876.)

This problem can be solved only when neither given line is parallel to the given plane, or one or both being parallel to it are at the required distance from the plane.

Find the traces of a plane parallel to the given one ·75″ away from it. Then find the intersections of the given lines with this plane. Now, as all lines in the newly found plane will fulfil the condition of being parallel to the plane, and ·75″ from it, that found by joining the intersections mentioned above will satisfy the conditions of the problem. In Plate XIII., fig. 2, *v′fh* is the given plane, and *a′b′*, *ab*, *c′d′*, *cd*, are the projections of two given lines. The plane *l′mn* is found by the solution in Problem CXII., and the intersections of the lines AB and CD with that plane are found by Problem CIX. to be *p′p*, *q′q*. Then *p′q′*, *pq*, is the line required, as it is in the plane *l′mn*, and therefore parallel to *v′fh*. It is ·75″ from that plane, and meets both the given lines.

PROBLEM CXXXVII.

Given the projections of four points not in the same plane, to determine the radius of the sphere having these points upon its surface. (Plate XV.)

This problem in Solid Geometry corresponds to that in Plane Geometry of finding a circle to pass through three given points, and, as in that case the three points cannot be in the same straight line, so, in this one, the four points on the sphere must not be contained by the same plane, except a circle can be passed through them, and then the problem is indeterminate.

In the case of a circle, if any two points be joined, and a line be found bisecting the chord at right angles, it contains the centre. Similarly, if two points be joined on the surface of a sphere, the *plane* which bisects this line perpendicularly *also* contains the centre, as it is a locus of all points equidistant from those so joined. Now, as this principle must apply to *any* two points, the centre of the sphere *must be in* all planes which bisect a chord perpendicularly. The common point to all these planes is the centre. ***The intersection of three such planes*** is required to fix their common point, and the radius is determined by finding the real length of the line joining the centre found to either of the given points.

In Problem CXXVII. the construction was shown which is needed here.

In Plate XV., fig. 1, the projections of four points, A, B, C, and D, are assumed, and *v'fh* is the plane which bisects the chord BD; but in fig. 2 this construction is repeated, isolated so that the student may not be confused whilst having to pick out the lines from the more complicated drawing.

A case similar in position to that of the line BD is taken, and in the supplementary figure *a'b'*, *ab*, are the projections. Join them in plan and elevation, and determine the point *i'i* in the centre of the line.

Draw *i'w'* and *iw* as the projections of a horizontal in the plane. Its v. t. *w'* will be in the v. t. of the plane.

Then *v'f* drawn through *w'* perpendicular to *a'b'*, and *fh*, perpendicular to *ab*, will be the traces of the plane bisecting AB at right angles.

The other planes shown are *l'mn*, bisecting AD, and *r'st*, bisecting BC.

The projections of the intersections of these planes meet in O'*o*. That, therefore, is the centre of the required sphere.

Next, the real length of either of the lines OA, OB, OC, or OD will give the radius of the circles, which are the projections required.

But as the point C is on a horizontal great circle of the sphere (being at the same height as O), the plan OC gives the radius.

As this is not always available, the true length of OA is shown constructed into a horizontal plane on the same level as the centre, *i.e.*, *a*A is equal to the difference in height of O' and A'.

EXERCISES.

1. Show the projections of any two points A and B, their projectors being 2″ apart, measured along XY; A is 1·2″ above the h. p., and *in* the v. p., whilst B is ·5″ below the h. p. and ·75″ behind the v. p. Then find the real length and inclinations of the line joining them.

2. The plan of a line is 2″ long, and its elevation is 3″ long. The projectors of its extremities are 1″ apart, measured along XY. What is its true length and inclination?

3. A line AB is inclined 45″ to h. p. and 30° to v. p. Show its projections.

4. Two lines which meet are at right angles to one another, and are equally inclined to the h. p. Their plans contain an angle of 110°. Draw an elevation of them on a ground line parallel to the plan of one of them.

5. The horizontal and vertical traces of a plane make angles of 30° and 55° respectively with the ground line. A line, parallel to both planes of projection, is 3″ above the horizontal, and 2″ in front of the vertical. Find its intersection with the given plane.

6. Draw the plan of an isosceles triangle, whose base is 2·5″ and sides 3″, when that base is inclined at 40°, and one of the sides at 48°.

7. Given *fh*, a line making 30° with XY as the h. t. of two planes inclined 45°. Show their vertical traces. Show also the real angle between two lines which meet and lie one on each plane, and which are inclined 30°.

8. An indefinite line is inclined at 60°. At any point in this line draw two others which shall be at right angles to it, and to one another.

HINT.—These two latter lines will be in a plane perpendicular to the first line.

9. Two horizontal lines, AB, AC, contain an angle of 56°, a plane inclined at 30° contains AB, another inclined at 65° contains AC. Draw two lines passing through A, inclined at 20°, and lying one in each plane. Determine the angle between these two lines.

HINT.—It is convenient to assume the two horizontal lines above the h. p.

10. Draw a plane passing through a given line at right angles to a given plane.

HINT.—Draw from any two points in the line two perpendiculars to the plane. Then the plane required is that containing these two perpendiculars. Use only one perpendicular and the given line.

11. Determine the real angle between the traces of a plane when the horizontals make angles of 30° with XY, the vertical trace 40°.

12. Represent a plane inclined at 55°, parallel to XY, and 1″ distant from it.

13. Through a given (horizontal) line AB draw a plane parallel to a line CD.

HINT.—The plane required contains AB, and any line BE parallel to CD.

14. Represent an indefinite line inclined at 60°, and three other lines meeting in one point of it, each making an angle of 25° with the given line, and equal angles with the other two.

HINT.—These lines lie on the surface of a cone having a vertical angle of 50°, and the first line (inclined 60°) as its axis.

15. Two planes contain a right angle; one of them is inclined to the horizontal plane at 60°, and their intersection is inclined at 50°. Represent these planes.

16. AB is a line parallel to the h. p., AC is a line parallel to the vertical plane. The angles *bac*, between the plans, and *b′a′c′*, between the elevations, *are each 120°.* What is the real angle between the lines?

17. Determine the traces of three planes which are mutually at right angles, two of them being inclined at 35° and 70° respectively.

HINT.—The traces of the third plane are perpendicular to the intersection of the other two.

18. Represent by its traces a plane inclined at 40° to the h. p., 65° to the vertical, and another at right angles to this, and inclined at 60°.

19. An indefinite line inclined at 40° is the intersection of two planes, one inclined at 60°, and the other perpendicular to the first. Show them by their traces.

20. Each of three lines meeting in a point O is perpendicular to the plane containing the other two; two of them are inclined at 30° and 45°. Show them by a plan and an elevation when the point O is 3″ above the paper, and in the plane of the elevation.

HINT.—This exercise is based upon the principles in Problem CXXII.

21. Through a given point to draw a line inclined 35°, and parallel to a given plane.

22. Show the projections of two lines, each inclined 60°, to pass through a point P, and to make an angle of 70° with a given plane inclined 30°.

HINT.—The lines required are the intersections of two cones, each having its apex in P, one making an angle of 70° with the plane at its base, and the other an angle of 60°, with h. p. at its base.

23. Given the projections of any three points, A, B, and C, not in a straight line; required those of the circle passing through those points.

24. Find the projections of a line which lies in the same plane as two given lines, and which bisects the angle between them.

HINT.—Find the plane containing the given lines, and construct these lines about the h. t. into the h. p., then bisect the angle between them, and transfer the required line to the plane.

25. On the plane *v′fh* (fig. 132) show the projections of a straight line, such that if any two straight lines be drawn through A and B to meet in this line, these two lines shall be of equal length; or, in other words, required the locus of all points on the plane equidistant from A and B.

CHAPTER VIII.

ON THE PROJECTION OF THE FIVE REGULAR SOLIDS.

A REGULAR solid has all its edges of equal length, all its faces similar, and, necessarily from the first condition, regular polygons, and all its dihedral angles, or those between the contiguous faces, equal. All its angular points are equidistant from a fixed point within it; hence it can be inscribed in a sphere, of which the fixed point mentioned is the centre.

There are five such regular solids:—

1. The tetrahedron, whose four faces are each equilateral triangles. 2. The cube, whose six faces are squares. 3. The octahedron, with eight faces, all equilateral triangles. 4. The dodecahedron, with twelve faces, all regular pentagons. 5. The icosahedron, with twenty faces, all equilateral triangles.

For the determination of the elementary projections of these, only one condition of dimensions need be known, that is, the length of the edge.*

PROBLEM CXXXVIII.

To determine the projections of a tetrahedron when resting with one face upon the horizontal plane (edge 1·75″). (Plate XVI., fig. 1.)

Draw first the equilateral triangle *abc*, which is the plan of the face upon the h. p. The elevation of this face is $a'b'c'$ upon XY. Then as the three edges, which have their extremities terminated in *a*, *b*, and *c* respectively, meet in a fourth point, and are each equal in length, its plan *d* is determined by bisecting the angles of the equilateral triangle. The lines *ad*, *bd*, *cd*, are the plans of the three edges. For the elevation, it is necessary to know the height above the h. p. of the point D. A vertical plane passing through either of the sloping edges will contain the axis. The line *cf* is the h. t. of such a plane contain-

* The figures of this chapter will all be found on plates.

ing the edge CD. Construct this plane, about the h. t., into the h. p., by setting out *d*D perpendicular to *cf*. Then with *c* as centre, and with a radius equal to the length of an edge, cut *d*D in D. Join *c*D and D*f*, and *c*D*f* is the shape of a section of the solid made by the vertical plane assumed. Now this section exposes not only the height of the apex, but also the dihedral angle (θ)* between the faces *abc*, *adb*, as the cutting plane is perpendicular to the common edge *ab*. The elevation of the solid is completed by the projection d' being joined to a', b', and c'.

Problem CXXXIX.

To determine the projections of a cube with one face on the h. p. (edge 1·4″). (Plate XVI., fig. 2.)

The cube has six faces, all squares. Commence by drawing the square *abcd*. This is the complete plan. The elevation of this square is $a'b'c'd'$. As the vertical edges are all équal to the side of the square drawn, the elevation can give no difficulty. The line *ac* on the plan is *the diagonal of a face;* and from Euclid i. 47, the length of this line, compared to that of the edge of the solid, is $\sqrt{2}:1$; $a'g'$ is the elevation of a line joining opposite corners, and which is called *the diagonal of the solid*. A vertical plane containing this line would also contain the edge CG, and the diagonal of a face *ac*. The three lines form a right-angled triangle; and by Euclid i. 47, we see that the lines $a'g'$, $g'c'$, are in the ratio of $\sqrt{3}:1$. Hence the edge of a cube: a diagonal of a face: a diagonal of the solid as $1:\sqrt{2}:\sqrt{3}$. As the XY in the figure is taken parallel to the line *ag*, the diagonal of the solid is shown full length in elevation, and the angle $a'g'c'$ is the real angle between that diagonal and a contiguous edge.

Problem CXL.

To determine the projections of an octahedron when one of its axes is vertical (edge 1·5″). (Plate XVI., fig. 3.)

The octahedron has eight faces, all equilateral triangles, and may best be understood by conceiving it as made up of two square pyramids placed base to base. It has *three axes* or diagonals, mutually perpendicular, and equal in length; and hence the plane which contains any

* This angle is between 69° and 70°.

two is perpendicular to the third. When one axis is vertical, the plane of the others is horizontal, and the plan of the solid is a square with its diagonals. Draw this first, as *abcd*, in the figure; then *ef* is the plan of the vertical axis. The elevation of the axis is *e′f′*, made equal to *bd* or *ac* in plan; and the horizontal plane of the square *abcd* bisects the axis, as shown in the elevation.

PROBLEM CXLI.

To construct the projections of an octahedron (edge 1·5″) when resting with one face on the h. p. (Plate XVI., fig. 4).

The opposite faces of this solid, as well as those of the dodecahedron and icosahedron, are parallel, and the polygons have their angular points alternating. Hence the plan will consist of two equilateral triangles, *abc*, *def*, as arranged in the figure. The other edges in the plan will be found by joining the corners of the triangles as shown, and the contour of the whole will be a hexagon. In the elevation the two horizontal faces will be represented as straight lines, the distance between them being determined thus: Project point *d* to *d′* in XY, and through *b* draw the indefinite projector *b*, *b′*. Then as the plane of the square, *abde*, is perpendicular to the v. p., and one edge is on the h. p., the line *d′b′* must be made equal to the edge of the solid. The remainder of the construction will not be found difficult. The angle *f′d′c′* measures the dihedral angle between two contiguous faces.

PROBLEM CXLII.

To determine the projections of a dodecahdron (edge 1″) when resting with one face on the h. p. (Plate XVII., fig. 1).

All the faces of a dodecahedron are pentagons, and the opposite ones are parallel, the angles of the two polygons alternating. Draw first, then, a pentagon *abcde*, and a second pentagon *fghik*, as shown in the figure. On the lines *ae* and *ed*, construct two other pentagons, *ae*RQP and *ed*TSR$_1$. These represent two contiguous faces folded into the h. p., rotating on their edges *ae* and *ed*. Now, it is clear that the angular points R and R$_1$ must coincide in the solid. Draw RJ and R$_1$U perpendicular respectively to *ae* and *ed*. Consider these lines as the horizontal traces of the vertical planes in which R and R$_1$ must move when the two pentagons are folded upon their horizontal edges. The intersection of these lines gives *r*, the plan of the angular point R. With O as centre, radius O*r*, describe a circle, and project points P and Q by lines perpendicular to *ae*, until they meet that circle in *p* and *q*. Then, as the solid is symmetrical about the centre, the remaining points in plan will be found equidistant around the circle. Join the angular points as shown.

By assuming RJ as a ground line (perpendicular to *ae*), the inclination of the plane of the pentagon *ae*RQP can be shown. With J as centre, radius RJ, describe an arc to meet a projector through *r* perpendicular to RJ. Then R′J*a* is the plane of the pentagon. Further, this gives the height of point R above the h. p., and as Q is in the plane, its projection Q′ gives, by its distance from RJ, the height of point Q above the h. p. With regard to the elevation, all the points of the solids are in four distinct levels. The horizontal pentagon ABCDE is on the h. p., and the heights of points R and Q are shown upon the plane Q′J*a*, RJ representing the ground line. Now these heights give those of the two levels 3·4 and 1·2 in the elevation. The height of the horizontal pentagon FGHIK, above the level 1·2, is equal to that of the level 3·4 above the h. p. Having drawn, then, these four lines, project the points in elevation upon these lines by projectors from the plan, remembering that the highest points of the pentagonal faces having their lowest edges around the pentagon *abcde* will be on the same level as Q, whilst the other two intermediate points will be on the same level as R and P.

Problem CXLIII.

To determine the projections of a dodecahedron (edge 1″) when one of its axes is vertical. (Plate XVII., fig. 2.)

When an axis is vertical, three pentagonal faces meet in the lowest point of that axis, and three also meet in the highest point. These pentagons are equally inclined to the h. p., and the points of the upper three alternate in plan with those of the lower three. Now it is clear their common edges must meet in plan in three lines, making 120°, with each other. Further, if these pentagons be folded into the h. p., keeping their highest edges horizontal, the lines bisecting these pentagons will also, when thus constructed, meet at 120° with each other. Draw then the pentagon ABC*d*E, having its edge 1″ long. Bisect it by the line J*d*, and set out *d*V at 120°. Bisect the angle J*d*V by the line *d*U. Then when the pentagon ABC*d*E is folded upwards, keeping the edge AB horizontal, its point C will move in a vertical plane, whose h. t. is parallel to J*d*, and the plan of C will be on *d*U at *c*.

The next proceeding is to find the plane of the pentagon; thus:—Take X′Y′, parallel to J*d*, as the intersecting line of the h. p. with an assumed v. p. Then as point *d* is stationary during the rotation, set out *dz* perpendicular to X′Y′ as the h. t. of the required plane. With *z* as centre, radius *z*C, draw an arc to meet a projector through *c* in *c″*. Then *dzc″* is the plane of the pentagon ABC*d*E. By means of this plane, the plan of the remaining points of the face can be determined. Then the plans of the other two pentagons which meet in *d* can be obtained readily, as they are exactly similar to the first. The three pentagons meeting in the upper extremity of the axis, have their angular points alternating in plan with those just determined, and the remainder of the plan is completed by joining the points, as shown in the figure.

As all the points of the solid fall upon four levels, of which *d* and *d′* are upon the lowest and highest respectively, the others must occur upon the two intermediate, and it is necessary to obtain the vertical distances of these planes or levels from each other. The height of point *b* above the h. p. is shown at b_{11}B′, and this gives the level 3·4 in the elevation. Then as (for example) Q and R are upon the levels 3·4 and 1·2 respectively, set out *q*Q perpendicular to *qr*, and make *r*Q equal to the length of the edge *rs*, which, being horizontal, is projected full length in the plan. Then *q*Q gives the difference in height of the levels of the planes upon which points *r* and *q* are situate. This fixes the line 1·2. The rest of the construction is not difficult.

Problem CXLIV.

To determine the projections of an icosahedron (edge 1″) when one face is horizontal. (Plate XVIII., fig. 1.)

This solid has twenty faces, all equilateral triangles, and the upper and lower faces are parallel, their angular points being alternate. Commence the plan by drawing the two equilateral triangles *abk*, *dmn*, inscribed in a circle of 1″ radius, as shown in the figure.

Now, these two faces in the solid are connected by three pentagonal pyramids. The student will understand this by looking at the finished *plan and mentally* isolating the pyramid, having *abcde* for base and *f* for its vertex. This pyramid connects the base AB of the lower triangle with point D of the upper one. There are two others which complete the solid.

Conceive the pentagon *abcde* constructed into the h. p. about its edge *ab*. We should then have the figure *ab*CDE, and we know that it is to be rotated back until the point D is in plan at *d*. Assume a v. p. (X′Y′) perpendicular to *ab*, and after projecting D to meet X′Y′, describe an arc with *o* as centre, radius *oz*. Through *d* take a projector to meet this arc in D′. Then D′*ob* are the traces of the plane of the pentagon upon the assumed v. p. The plans of the remaining points of the figure are easily determined, as shown.

To obtain point *f*, which represents the vertex of the pyramid, find the centre of the horizontal pentagon, and construct it into the oblique plane D′*ob* at F′. There set out $F'F'_1$ perpendicular to the plane. This represents the axis of the pyramid. Its length is determined at $F.F_1$ by drawing a perpendicular to FC, and making CF_1 equal to the edge of the solid.* Then point *f* in plan is vertically under F′, and in a line through F parallel to X′Y′. The remaining points in the plan are situate upon a circle through *efc*, and the contour lines form a hexagon.

For the elevation, as in the dodecahedron, the points fall upon four levels, which are determined by the heights of points D′, E′, and F′, above the assumed X′Y′, one level being the original XY. The levels being drawn, the points can easily be projected upon them.

Problem CXLV.

To construct the projections of the icosahedron (edge 1″) when one of its axes is vertical. (Plate XVIII., fig. 2.)

In this case two of the pentagonal pyramids mentioned in the last problem have their axes coinciding with that of the solid, and their pentagonal bases horizontal, with the angular points of those bases alternating. Commence the plan, then, by drawing two pentagons, *abcde*, *fghik* (edge = 1″), with alternate corners, as shown in the figure. One of these (the lower) is hidden, and therefore dotted. The plan is completed by drawing lines from all the points to the centre, and by joining the outside points of the pentagons. In the elevation the bases of the pyramids will be represented by horizontal lines, and the height

* This is the same method as that used to obtain the height of the tetrahedron.

of these lines above XY is determined thus:—At *v* in plan make *v*V perpendicular to *vd*. Make *d*V equal to the edge of the solid. Then *d*V gives the height of the vertex of the pentagonal pyramid above the plane of its base, *i.e.*, it gives the level of the points of the pentagon *fghik* above the h. p., and also the height of the uppermost point of the solid above the plane of the pentagon *abcde*. To obtain the vertical distance between the two pentagons, take the plan of a line having its extremities, one in each figure, such as KA. Set out *a*A perpendicular to *ak*, and make *k*A equal to the edge of the solid. Then *a*A gives the height of the point A above the h. p. passing through *k*. The further construction of the elevation can give no difficulty.

CHAPTER IX.

PROBLEMS INVOLVING THE USE OF A VARIABLE PLANE OF PROJECTION.

Many problems, having for their object the determination of the projections of solids under given conditions of position, are solved very readily in the following manner :—

Simple projections are first obtained upon co-ordinate planes, so taken as to be parallel or perpendicular (as may be most convenient) to the lines or surfaces involved in the conditions.

Next, a new plane of projection is assumed, which, by its relative position with regard to these lines or surfaces, satisfies the data of the problem.

The required projection is then made upon this plane.

The assumed plane being always taken perpendicular to *one* of the original co-ordinate planes, the change is really effected merely by the alteration of the intersecting line or XY.

The elucidation of the principles which guide the solutions of this class of geometrical problems will be best understood by reference to the characteristic questions solved in the ensuing pages.*

Problem CXLVI.

To determine the plan of a heptagonal pyramid when the plane of one of its triangular faces is horizontal. (Fig. 138.)

Obtain first the plane of the solid when resting with its base upon the h. p. This is the heptagon *a......g* in fig. 138. Assume a v. p. (upon which to draw an elevation) such that the plane of one of the triangular faces may be perpendicular to it, or, in other words, take XY at right angles to one edge of the base, as *ed*. When the elevation is finished, assume a plane of projection containing, or parallel to, the face VED. This is effected by drawing X′Y′ coinciding with *v′e′*.

* For a series of very elementary problems on "alteration of the ground line," the reader is referred to page 110 of the elementary treatise in this series.

Then the projection of the pyramid upon that plane which intersects the original v. p. along the line X′Y′ will be the one required. This is easily understood by folding the paper upon X′Y′ until the enclosed angle is a right angle. The elevation will then be seen to be that of the pyramid lying on the triangular face VED.

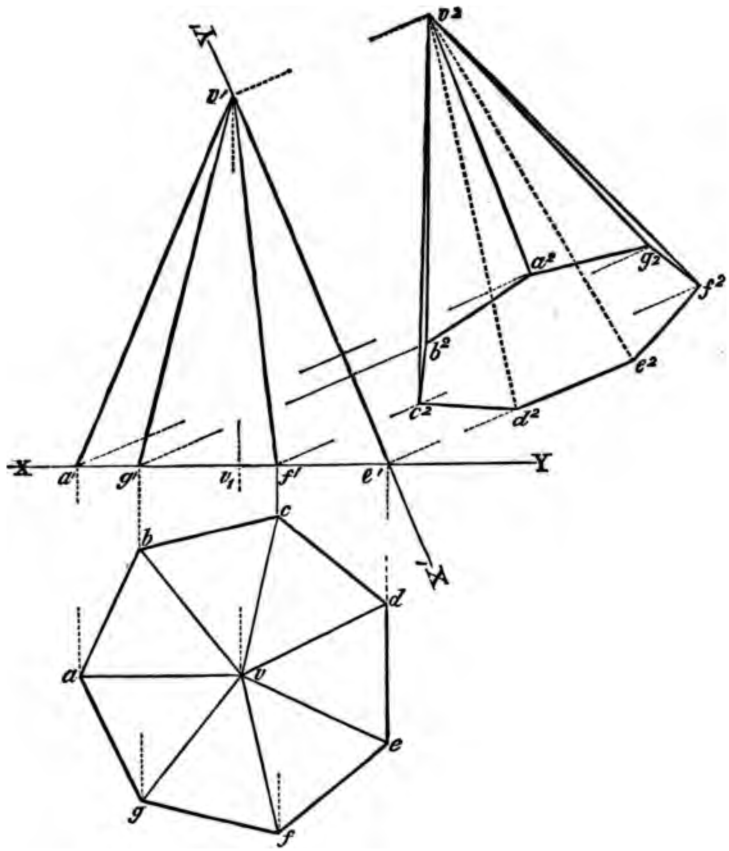

Fig. 138.

To obtain the plan, set out projectors through all the points in the elevation perpendicular to the new ground line. Then, as the assumed plane is still perpendicular to the original v. p., the projections of all *the points of* the solid upon it will be as far from X′Y′ as they are in the original plan from XY. Thus the vertex V is shown in the first plan at a distance v, v_1 from XY, and its second plan v^2 is the same distance from X′Y′ along the new projector. The other points are obtained in the same way, $e'e^2$ is equal to $e'e$; $e'd^2$ is equal to $e'd$, and so on. As it is the face VED which is supposed to be upon the new h. p., the edges v^2d^2 and v^2e^2 are dotted, as they would be hidden by the other parts of the solid.

The principle which the student will learn by this problem is that if the conditions involve the position of a face or surface, that face or surface must *be so arranged at starting as to be represented in elevation by a single line, or the plane of that surface must be perpendicular to one of the co-ordinate planes.*

Problem CXLVII.

To determine the projections of a hexagonal pyramid when it is freely suspended from one corner of the base. (Fig. 139.)

Under such conditions the solid will rest in stable equilibrium when the centre of gravity (C. of G.) is vertically *under* the point of suspension A, or, in other words, when the line passing through A in the solid and its C. of G. is vertical. Now, the C. of G. of a right pyramid is on its axis at a distance $\frac{1}{4}$ of the length from the base. Here, then, the line from a' to C. of G. ($\frac{1}{4}$ of $v'v$) is to be a vertical one.

Draw the plan as shown in fig. 139, and take XY for the elevation parallel to av. By this means the elevation of the line involved in the condition shows its full length, as the v. p. is parallel to it. *And this is always necessary when the condition refers to the relative position of a given line in the solid to the plane of projection.*

Determine the simple elevation and join a' to C. of G. producing the line. Then assume X^2Y^2 perpendicular to this line as the intersecting line of a new plane of projection, which, being perpendicular to the original v. p., and hence co-ordinate to it, may be considered as a new horizontal plane.

Obtain, then, the new plan by projectors through the points of the elevation, taking the distances from X^2Y^2 along these projectors equal to the corresponding lengths in the first plan, in exactly the same way as described in the previous problem.

The edge v^2d^2, being in the same plane as a^2 and the C. of G., and, being *below* them, will be hidden, and must therefore be dotted. This determines practically the other dotted edges in the new projection.

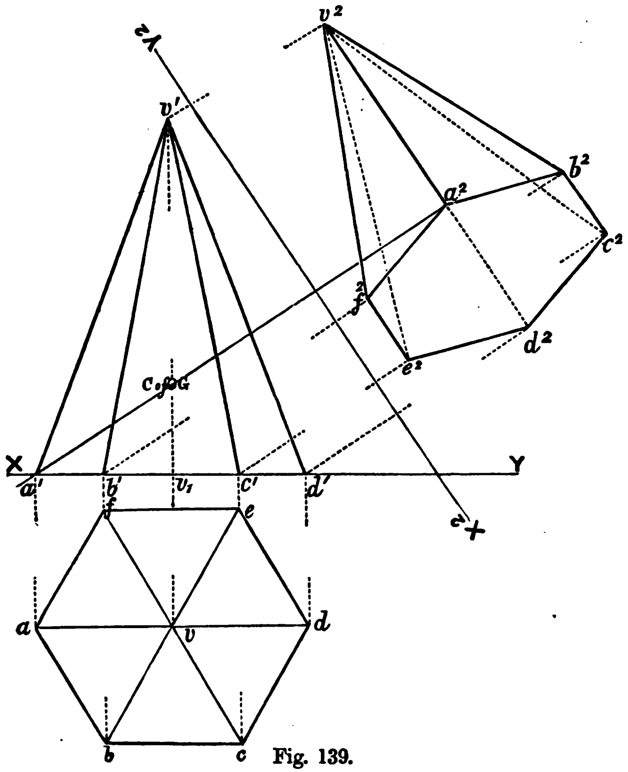

Fig. 139.

PROBLEM CXLVIII.

A pentagonal prism pierces a circular slab, and protrudes equally upon either side of it. The axes of the two solids are in one straight line perpendicular to their parallel faces. To determine a plan of the whole when one corner of the prism and an edge of the slab rest upon the h. p. (Plate XIX.)

The solution of this problem is shown in Plate XIX., and is given *here, not so* much as involving any new principle, but as an excellent instance of obtaining from original projections others under given conditions of position. The simple plan is shown when the common axis of the two solids is vertical, and an elevation also—the XY being taken parallel to *h.b* in the plan. Then, as it is a condition of the problem that an angular point of the prism and an edge of the slab should be on the h. p., the new X^2Y^2 is drawn, joining h'_3 and b'. Regarding this line, then, as the intersecting line of two co-ordinate planes, the elevation becomes that of the group under the required conditions. Then projectors are taken through every point of the solids perpendicular to X^2Y^2, and the points of the new projection upon them are obtained by their distances from X^2Y^2 being equal to the distances of the plans of the same points from XY. As there are several points having the same letter in the first plan, they are distinguished in the other two drawings by figures below them. To obtain the projections of the two circles, those of the extremities of the major and minor axes of the ellipses only are necessary, as, when these are known, the curves may be drawn by any of the methods described in the Plane Geometry (Chapter VI.).

Problem CXLIX.

To determine the true shape of a section of a solid made by a cutting plane inclined to ONE only of the co-ordinate planes of projection.

In the case shown (Plate XX., fig. 1) an octahedron is taken as an instance, and its projections being known, a cutting plane STJ, perpendicular to the v. p., is supposed to make a section of the solid. The problem is, to determine the true shape of the cut.

As the whole figure of the section will be contained by the cutting plane, the line 1′4′, which is part of its v. t., must be the elevation of it. Assume, then, a plane of projection parallel to the plane STJ, and the projection upon that plane will give the true shape. The line X^2Y^2 is taken parallel to ST. The elevation shows which edges of the solid are cut, and as the plane of the points *abcd* is horizontal, the student will notice that 3′ and 6′ represent in elevation the points of intersection of the edges AB and CD with the cutting plane.

Proceed to determine the plan of the section upon that of the octahedron. This is done most easily by reasoning thus, taking point 2′ as an instance:—The elevation tells us that it occurs upon the edge

VD. Then a projector through 2′ gives by its intersection with *vd* the plan point 2.

Having finished the plan, proceed to obtain the new projection upon the assumed plane X^2Y^2 by drawing projectors (perpendicular to X^2Y^2) through 1′, 2′, 3′, 4′, 5′, and 6′, and by measuring along these lines distances from X^2Y^2 equal to the relative distances of 1, 2, 3, 4, 5, and 6 from XY.

Another plane of projection has here been assumed as parallel to the cutting plane; this is not absolutely necessary, for if the plane of section be constructed on its v. t. into the v. p. of projection, taking with it the figure required, the true shape would be obtained. The only difference this would make in the drawing would be that the distances, instead of being measured from X^2Y^2, would be taken from ST.

Problem CL.

To obtain a sectional plan of a solid given by its projections, and cut by a given plane of section. (Plate XX., fig. 2.)

It is first necessary to understand that what is meant by a sectional plan or elevation of a solid is a projection of the section on the cutting plane, or upon one parallel to it with the addition of the projections of the other parts of the solid attached to it. This is advantageous as not only giving the true shape of the section, but also showing its relation to the other parts of the object.

The case taken in Plate XX., fig. 2, is a hollow square prism, and the plane of section is shown by its traces STJ. The true shape of the section is found in a manner similar to that described in the last problem. As the plane passes through the h. p. it cuts four edges of the solid in points 3′4′9′8′, all of which have their elevations in one point. The distances to be measured along the projector in that case are shown on the plan from XY to the points 3, 4, 9, 8 respectively. To complete the solution, project upon the same plane the remainder of the back portion of the solid. This is done in the figure, and, taking point C as an example, its new projection is found by a projector through *c*′ (its elevation), the distance R*c*″ being made equal to *c*′*c*. The drawing is so lettered that, by following line for line and point for point through the three projections, the student will find little or no further difficulty in reproducing it.

PROBLEM CLI.

A pentagonal pyramid has one of its triangular faces vertical, one long edge of that face being inclined 40°. To determine its plan. (Plate XXI., fig. 1.)

This problem can be solved indirectly by alteration of the planes of projection in the following way :—

First, make a plan and elevation of the solid when its axis is horizontal and perpendicular to the v. p., one triangular face being vertical.

Next, make a projection (an elevation) on a new v. p. either containing the face mentioned above or parallel to it.

Lastly, from this projection obtain another (the required plan) on a plane whose intersecting line makes 40°, with one of the edges of the vertical face as directed.

The pentagon a'......e' is the elevation under the first conditions. The axis is horizontal, as shown at v'. The triangular face ABV is vertical, and therefore is represented by a straight line in the plan.

Then, assuming X^2Y^2 as the intersecting line of a new plane of projection containing the face ABV, the drawing on that plane (an elevation) can be determined by projectors through the plan distances being measured along them, obtained from the first elevation, as described in previous problems.

Having thus deduced an elevation of the solid with one of its triangular faces contained by the v. p., another plane of projection can then be taken, whose intersecting line makes the required angle of 40° with one of the edges of that face.

The line X^3Y^3 in the figure satisfies the above conditions as a ground line for obtaining the required plan, the new projectors being taken through the points of the last elevation, and the distances measured along them being equal to the distances of these points in the last plan from X^2Y^2.

PROBLEM CLII.

To determine the plan of a right hexagonal pyramid which has one of the edges meeting in the apex horizontal, and a triangular face containing that edge inclined 30°. (Plate XXI., fig. 2.)

This problem can also be solved, as in the last case, by deducing a sequence of projections from those of the solid in a simple position in the following manner :—

First.—Obtain the plan and elevation when the hexagonal base is on the h. p., and one triangular face is perpendicular to the v. p.

Secondly.—Assume a plane of projection containing the triangular face mentioned above, and deduce the projection of the solid on this plane. By this means the plan of the pyramid, when one face is horizontal, is obtained.

Thirdly.—Thence obtain a projection of the solid upon a plane perpendicular to one of the long edges of the horizontal face. This will be an elevation of the pyramid with one face horizontal, and one long edge of that face perpendicular to the v. p.

Lastly.—Deduce from this elevation a plan of the solid upon a plane of projection, making the required angle of 30° with the face VDE, which was before horizontal.

The hexagon *abcdef...v* is the first plan, and XY is taken at a right angle with *ed*, hence the elevation of the face VED is a single line $v'e'd'$.

The new X^2Y^2 is drawn passing through this line $e'v'$, and the plan $a_1......e_1v_1$ is obtained from the first elevation. Next, X^3Y^3 is taken perpendicular to the plan of the edge VD, and the elevation $a''......e''v''$ is obtained from the plan. The student will readily see, then, that as the face VED is represented by a single line, if he assumes X^4Y^4 making the angle of 30° with $e''d''$, the elevation will then be that of the pyramid, with its face VED inclined to the new h. p. at 30°.

Projectors through all the points $a''......e''v''$, perpendicular to X^4Y^4, and distances along them, measured equal to the distances of the plans $a_1......e_1v_1$ from X^3Y^3, will give the required projection of the solid under the conditions given.

Problem CLIII.*

An irregular pyramid has for its base a regular hexagon of 1″ side, and three of its consecutive faces, which meet in the apex, make angles of 60°, 62°, and 70° with the base. To determine the projections of the solid when resting with the hexagonal base on the h. p. (Plate XXII., fig. 1.)

* This and the following problems of this chapter are given to illustrate principles of construction connected with the manipulation of the surfaces and edges of solids, which may perhaps be wrongly classed as variation of the planes of projection, but it is more advisable to study them here than further on in the subject.

Commence by drawing a hexagon *abcdef* as the plan of the base. Then produce *af*, and take XY perpendicular to it. Assume *af* as the h. t. of a plane inclined 60°. The v. t. of this plane will make the full angle of 60° with XY, as it is perpendicular to the v. p. Then *r'a't* is the plane of one face.

Next, produce the edge *ab* to meet XY, and consider it as the h. t. of the plane of the next face, and determine the v. t., so that the plane may be inclined 62°. Then *l'p*S is the plane of a second face. By a similar proceeding the plane *l'mt* is determined as that of a third face inclined 70°.

The intersections of these planes will give the edges of the solid, and as the apex of the pyramid is a point common to all the edges and faces, the meeting-point of two intersections will be sufficient to determine it. On the figure the intersection of the planes *l'p*S and *l'mt* is *l'b'*, *lb*. The direction of the edge through *b* thus becomes known. The intersection of the planes *l'p*S and *r'a't* would be inconvenient to determine, as their v. t.'s do not meet within the limits of the sheet, but as the intersections of all the planes of the faces must pass through the apex, that of the planes *r'a't* and *l'mt* is shown. The projections of this intersection are *k'a'* and *kt*. Then, as the plans *kt* and *lb* meet in *v*, that must be the plan of the apex, and its elevation can be obtained by a projector through *v*, meeting *r'a'* in *v'*. The remainder of the drawing requires no description.

Problem CLIV.

An irregular pyramid (Plate XXII., fig. 2) with triangular base, ABC, has the following dimensions:—AB = 2·5″, BC = 3·25″, AC = 1·75″, BV = CV = 3·25″, and AV = 2″. To determine a plan of the solid with the face ABC horizontal, and an elevation on a v. p. not parallel to either of the slant edges. (May Exam., 1874.)

As the base ABC is to be horizontal, commence by drawing a triangle ABC, the sides being equal to the lengths given. On AC construct a triangle A*v'*C, the sides A*v'* and *v*C being according to the conditions given. This triangle will represent that face of the pyramid which meets the base in AC, folded about its edge AC into the h. p. Similarly, the triangle BCV represents another face meeting the base in BC. Now, if these two triangles be conceived to rotate about their base edges until they meet the plans of the two points, V′ and V will travel in vertical planes perpendicular to the edges AC and BC respectively. The lines V′*v* and V*v* are the h. t.'s of these vertical planes, and their intersection *v* gives the plan of the apex, and *v*A, *v*B, *v*C, those of the edges. To obtain the elevation upon the XY chosen, the height of the apex must be known.

To determine this, set out vV^2 perpendicular to C*v*, and with C as centre, CV as radius, describe an arc meeting this perpendicular in V^2. Then vV^2 is the height of the pyramid. For CvV^2 represents a right-angled triangle, which, in the solid, would be made up of the axis as a perpendicular, the edge CV as the hypotenuse, and the projection of that edge on the plane of ABC as a base; this triangle being folded about the line C*v*.

To find the angle between two contiguous faces, proceed as follows:—Taking the faces ACV and BCV, for example, draw *rt* at a right angle with C*v*, meeting that line in S. Set out *s*S perpendicular to CV^2; and mark off the length of *s*S from *s* to S′. Join S′ to *t* and *r*.

The principle involved is, that *r*S′*t* is a triangle made up of the intersections of a cutting plane, perpendicular to the edge C*v*, with the two faces and the h. p. This is really the same proceeding as shown in Problem CXIV. The length *s*S gives the altitude of the triangle, so that it may be folded into the h. p.

Problem CLV.

Given the indefinite plans of three edges of a cube, meeting in a point; to determine the entire plan of the solid (edge 2″). (Plate XXIII., fig. 1.)

The three given lines in the figure are supposed to be *ab*, *bc*, and *bd*, meeting in *b* at angles of 110°, 120°, and 130°.

As the three lines, of which these are the projections, form a solid right angle, each of them is perpendicular to the plane containing the other two. Again, the traces of a plane are always at right angles to the projections of lines perpendicular to it. Draw then, at pleasure, the line h. t. perpendicular to *bd*, and assume it as the h. t. of the plane containing AB and BC. Construct this plane (with the lines mentioned) about its h. t. into the h. p. This is effected readily by drawing a semicircle upon h. t. (ABC being a right angle, and therefore contained by a semicircle), thus obtaining B, and by joining B to *t* and *h*. Mark off upon these lines lengths from B equal to the edge of the cube, and obtain the projections of points *a* and *c* from A and C, *aA* and *cC* being, of course, parallel to *bd*.

Next, taking *d*F as a ground line, make an elevation of the plane of AB and BC. To do this, set out at *b* a perpendicular *b.b′*, and with *f* as centre, radius *fb*, describe the arc B*b′*. Then as point *b′* is in both planes, it must be in the v. t. of the one required. Draw *v′f′* through *b′*, and *v′f′h* will be the plane of one face of the cube.

The edge BD is perpendicular to this face, and will therefore have its elevation perpendicular to *v′f′*, and being in the plane of projection, will be shown full length. Draw, then, *b′d′* at a right angle with *v′f′*, and 2″ long, and project point *d* from *d′*, *d′d* being perpendicular to *bd*. Thus far, then, the plans of AB, BC, and BD, are determined. Now, in the case of the cube, each of the remaining edges is parallel to *one* of these; and as the projections of equal and parallel lines upon any plane are themselves equal and parallel, the plan is finished by drawing *df* and *ck* parallel to *ab*, meeting *af*, and *ak* is made parallel to *bd* and *bc* respectively, etc.

Assuming that B is the uppermost corner of the solid, the edges meeting in G would he hidden.

The student will notice that the three lines given meet at obtuse angles, and the construction described would be applicable in all problems where such was the case. But if the angles given had been two right angles, the construction would not apply, as a right angle can only be projected as such, when one of the legs of it, at least, is horizontal. Then, because one edge of a cube being horizontal, the face meeting it is vertical, the plan could not be determined as shown.

Further, the construction employed is applicable where *one* of the given angles is *acute*,* except when either of the lines forming this angle is a horizontal of the plane containing two of them, in which case the same remark applies as in the last instance.

Problem CLVI.

The plan of a building is an irregular four-sided figure, ABCD; its slopes are inclined to the horizon at 55°, 40°, 45°, and 45° respectively; to determine the plan of the hips and ridge. (Plate XXIII., fig. 2.)

This problem is inserted to illustrate the principle of obtaining

* The three lines forming a solid right angle cannot be projected orthographically into three acute angles, hence the converse, that three acute angles cannot be given as the projection of the corner of a cube.

the projections of the intersecting lines of the faces of solids, by contouring them with a level on each plane of the same height. These levels will, of course, meet in points which are in the intersections desired. Commence by drawing the plan of the outline of the building. In fig. 2, Plate XXIII., ABCD is the plan.

Draw *pq* perpendicular to AB, and assume it as the intersecting line of a vertical plane of projection. At *f* draw *v'f*, making an angle of 55° with *pq* (the inclination of the slope containing AB). Then *v'f* is the plane of that slope shown by its trace on the v. p. assumed. Take point *k'* (at some known height at pleasure) as the elevation of a horizontal of the plane, and project its plan *kt* (parallel to AB). Next proceed to show similarly the plane of the slope containing BC, and determine the plan of a horizontal of this plane of the same height as *kt*. The traces of the plane are shown at *l'm*B, and *k''* is the elevation, and *ck* is the plan of the line. The two horizontals meet in *k*, and thus give a second point in the plan of the hip. The hip CE, which meets this one, is found in an exactly similar manner. Then, as the slopes from CD and AD are each 45°, the intersecting line DF bisects the angle CDA. To obtain a second point in AF, the horizontal *kt* is produced to meet a horizontal on the plane FAD at *t* (*st* being parallel to AD). Then EF is the plan of the ridge. The true length and inclination of the hip EB is shown on the drawing at E'B, the height of point E being determined from its projection on the plane *v'f'*A.

Problem CLVII.

An inclined embankment, gradient 1 in 10, the road on the top having a uniform width of 18 feet, has one of its sides sloping at an angle of 45°, and the other at 60°, to the horizon. It is intersected by a cutting, having a width of 22 feet at the bottom, the sides sloping 1 in 6 from the vertical, and the centre line making an angle of 60° with that of the embankment. The lowest point of the upper edges of the cutting is 12 feet above the base. To draw the plan and an elevation on a plane parallel to the centre line of the embankment. Scale $\frac{1}{10}$ in. to 1 foot. (Plate XXIV.).

Commence by drawing two parallel lines, 18 feet apart, to represent the plan of the top of the embankment, and mark the centre line

3·4. Assume XY parallel to 3·4, and draw a line above it, having a gradient of 1 in 10, as *m'a'*, in the plate. Determine a point in the elevation, 12 feet high, as *a'*, and project its plan *a*. Then, assuming that the slope from *an* is 60°, and from a_1m_1 is 45°, draw *a'k'* and *a'q'*, making those angles with XY. Then *a'k'a"*, and *a's"a"*, represent two semi-cones, having the elevations of their apices in *a'*, and with base angles of 60° and 45° respectively. The circles, which are the plans of these, are shown partly at *ke* and *qi*. Another point, *m'*, being taken as the elevation of the apices of two other cones of the same description, the lines *ve* and *h.i.*, drawn tangential to their plans, are the traces of the slopes upon the h. p.

We have next to show the plan of the cutting. As its direction is to be at an angle of 60° with that of the embankment, assume an auxiliary *v.p.* at right angles to this direction, *i.e.*, at 30° to *ma*, and project point A upon it. In the drawing, X^2Y^2 is the ground line, and A is the projection. Then draw A*t'*, and A*s'*, at such an angle on each side as to slope with the vertical at 1 in 6. From *t'* draw *ge* perpendicular to X^2Y^2, and thus obtain the trace of one side of the cutting. Join *ae*; and to get the point *c*, find the elevation of a line from *p* parallel to *ta*, thus:—Obtain the elevation of *t* at *t"*, and join *a't"*. This is the projection on the vertical plane of a line joining *t* and *a*. Now, a line joining *p* and the unknown point C is parallel to this line, and therefore would have its projection *p'c'* parallel to *a't"*. Thus we get *c'*, and *c* is projected from it.* Next draw the centre line of the cutting, 1, 2, and the other trace, their width apart being 11'0", or 22'0 in all. Then points *b* and *d* are found by a similar construction to that used in obtaining *c*; for *rb* and *od* are two lines parallel to *as*, and would have their elevations parallel to *a's"*. Then from *r* and *o* obtain *r'* and *o"*, and draw *r'b'* and *o'd'* parallel to *a's"*. Project the plans *b* and *d*, and join to *f* and *h* respectively. The points *b*, *c*, and *d*, could also have been determined by finding the intersection of the lines *an* and *qm*, with the planes of the slopes of the two sides of the cutting.

* The line *ac* being parallel to *eg*, it could in this case be drawn at once; but if the upper surface were sloped from *a* towards *c*, the student would require the construction described before he could determine *c*. Hence the insertion of its description.

EXERCISES.

1. Draw the projections of an octahedron, when resting upon one of its triangular faces, and obtain an elevation upon any v. p. not parallel to either edge.

2. A pyramid has a square base of 2 inches side, and its vertex is perpendicularly over one corner (axis 3″); show its plan, when the shortest of its edges, meeting in the apex, is inclined 45°.

3. Draw the plan of a cube, when hung up by one of its corners (edge 3″).

4. A cube has a square pyramid upon each of its faces (edge of the cube 1½″, base of pyramid 1½″ square, and height 2″.) Draw a plan of the group, when resting on any two points of the pyramids.

5. A solid has an equilateral triangle of 3″ edge for its base, and the three other sides slope at angles of 40°, 50°, and 45° with the base; determine the height of the apex.

6. Draw the plan and elevation of a triangular base pyramid, $ab=2''$, $bc=2{\cdot}5''$, $ac=2{\cdot}75''$, $ad=3''$, $bd=3{\cdot}5''$, $cd=4''$—the face, *abc*, of the solid to stand on the h. p., and the edge *ab* to make 45° with the v. p.

7. A pentagonal pyramid, with the side of the base 1·75″, and axis 4″ long, rests on one edge, and has two adjacent faces equally inclined. Draw its plan, and add an elevation on a ground line perpendicular to the horizontal edge.

8. The base of an embankment is 46′ 0″ wide, its sides slope at angles of 40° and 56°, and its height is 16′ 0″. Through it a cutting is made 20′ 0″ wide at bottom, and having sides which slope at 1 in 5, the centre line of the cutting making 48° with that of the bank. Draw plan, scale ⅛″ equals 1 foot.

9. A cone (base 1″ radius, axis 3″ long) is to be shown in plan and elevation, when resting with its side on the horizontal plane, the plan of the axis to make 30° with XY.

10. A tetrahedron (3″ edge) has one face vertical, and the line bisecting that face inclined 50°. Draw its plan.

11. Draw the plan of the pyramid in Question 7, when one triangular face is vertical, and one edge of that face horizontal, and perpendicular to the v. p.

12. A solid is formed by removing the upper part of a right pyramid, the axis being halved by a plane parallel to the base. The base is a hexagon of 1·5″ side, axis 4″ long. Draw its plan, when one of the trapezoid faces is vertical, the edge of the base in that face being inclined 40°.

CHAPTER X.

ON THE DETERMINATION OF THE PROJECTIONS OF SIMPLE SOLIDS, UNDER GIVEN CONDITIONS OF POSITION.

"WHEN a solid is bounded by planes, and therefore has edges which are straight lines, its position with respect to a plane of projection can only be determined from two data or assumptions. These must be either the inclinations of two of its planes, or of two of its lines, or the inclinations of a plane and a line in that plane; but it is not necessary that either the plane or line should be that of a face or edge of the solid." *

As such is the case, it will be best to study the principles elucidated in this chapter under three divisions.

1. Given the inclinations of a plane of the solid, and of a line contained by that plane.

2. Given the inclinations of two lines connected with the solid, the angle between them being known; and

3. Given the inclinations of two planes of the solid, the dihedral angle between these planes being known. †

DIVISION I.

PROBLEM CLVIII.

An isosceles triangle (base ·75″, sides 1″) has the plane of its surface inclined 50°, whilst one of its equal sides is inclined 25°; to determine its projections.

It was shown in Problem CXIX. how the projection of a line lying in a given plane could be obtained. By that method, the line *ab* (fig. 140) is determined as inclined 25°, and contained by the

* Bradley's *Elements of Geometrical Drawing*. Part I. Plate XVI.

† The last two chapters in the Elementary Geometry of this series discuss simple instances belonging to the first two cases, which the student is recommended to peruse.

plane *v′fh*, having an inclination of 50°. The plane is folded about its h. t., until it is brought to coincide with the h. p. Then point *a′* having moved in the v. p. in an arc with *f* for its centre, and point *b* having remained stationary, A*b* will represent the line (which was before inclined 25°), after its "construction" into the h. p. Upon this line, a portion must be marked off equal to the desired length of the side of the triangle, and the figure must be completed as at RST. Afterwards, the plane is brought back into its inclined position by rotating upon *fh*. It is clear, then, that point T moves in a vertical plane perpendicular to the h. t., or, what is the same thing, parallel to the original v. p. Draw, then, TT′ perpendicular to XY, and with *f* as centre, describe the arc T′*t′*. From *t′*, draw *t′t* to meet T*t*. This gives the projection of point T of the triangle. Then as points R and S are on the line inclined 25°, their plans are easily found by referring by parallels to XY from R and S. The elevation of the triangle is *r′s′t′*.

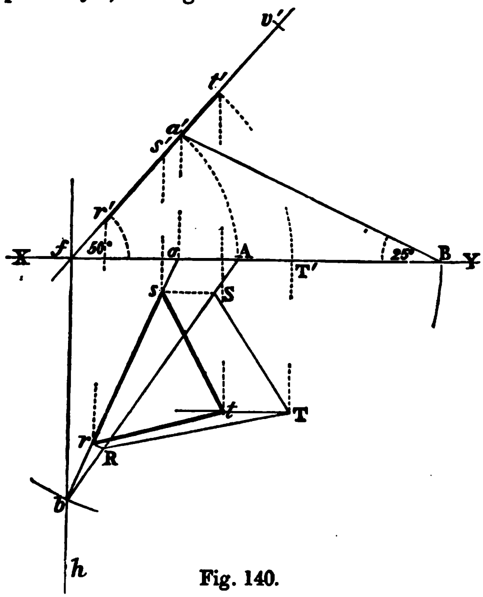

Fig. 140.

Problem CLIX.

A cube of 1·5″ edge has the plane of one face inclined at 50°, and one edge of that face at 30°; to determine its plan and elevation. (Plate XXV., fig. 1.)

Commence by determining the plan of the square, the position of which is given. As this is really only a similar question to that in the preceding problem, no special explanation is necessary. The plans and elevations of the four edges, which are perpendicular to the plane *of this square*, will be at right angles to the traces of that plane, and the elevations of these edges, $a'e'$, $b'f'$, $c'g'$, and $d'h'$, are made equal to their real lengths, as they are parallel to the v. p., and therefore have their projections upon that plane fully equal to the originals themselves. Then the remaining points, *efgh*, can be obtained from the elevations, e', f', g', and h', and the plan can be completed. That point which is nearest to the plane of projection in each case, is hidden by the other surfaces. Thus a' in the elevation shows that the three lines proceeding from *a* are hidden in plan, and *b* in the plan settles that $b'f'$ is to be dotted in elevation. The student will notice that the plan of the face, EFGH, is a parallelogram, with sides parallel to ABCD; and this should be so, as in the solid these faces are equal and parallel squares. Hence, if the length of one edge only in plan be deduced as *bf*, the others can be set off as equal, and the whole projection thus finished.

Problem CLX.

A cube of 1′ 5″ edge has the plane containing two of its diagonals inclined 60°, while one of those diagonals is inclined 40°. To determine its projections. (Plate XXV., fig. 3.)

This and the next problem are given as instances where the conditions of inclination affect not the faces or edges of the solid themselves, but other planes or lines connected with them.

The plane which contains *two diagonals of the solid* bisects it, and the method of determining this section is shown in fig. 2. ABCD represents a face of the cube, and AC its diagonal. Now, this diagonal of a face is the longer side of the rectangle which is the section of the solid containing two of its diagonals, the contiguous edge being its other side. Draw then CG equal to the edge of the cube, and complete the rectangle ACGE. Then, assuming that the plane of the section ACGE is to be inclined 60°, while the line EC is to be inclined 40°, draw the traces of the plane *l′mn*, and determine in it a line inclined 40°, as described in Problem CXIX. In the figure *st* is the plan of the line. Construct this line into the plane of projection, as at T*s*, and on it build a rectangle similar to ACGE, and having its diagonal coincident with it. Next, obtain the plan of this figure, *acge*, in the manner described in the first problem of this chapter. The elevations of the four points are shown at a', c', g', and e'. Now, as the points, F, B, D, and H, are not in the plane of the diagonals of the solid assumed, some special method is required to obtain their projections. It will be seen that B and D are two extremes of one diagonal of the face ABCD; and further, that BD is perpendicular to the plane containing the two given diagonals of the solid. Then, as the two diagonals, AC and BD, bisect each other, find the centre 1 of AC, and through it draw a line perpendicular to the h. t. to represent the indefinite plan of the line BD. Determine from 1 the elevation 1′, and draw another line through that point perpendicular to *l′m*. This will give the indefinite elevation of BD.

Again, as BD is parallel to the plane of elevation (the h. t. of the assumed plane being taken perpendicular to XY), its projection *b′d′* must be made equal to its original length, which can be obtained from fig. 2. We are now able to project in plan the points *b* and *d* upon the indefinite line through 1, and, by joining *a* and *c* to *d* and *b* respectively, to complete the plan of the face ABCD; the elevation of that face being *a′b′c′d′*.

By a similar construction, as regards the diagonal FH of the opposite face EFGH, the cube can be completed.

Problem CLXI.

A tetrahedron of 2″ edge has the plane, containing one edge and bisecting the opposite face, inclined 50°, whilst the inclination of that edge is 30°. To determine its plan and elevation. (Plate XXVI., fig. 1.)

This is another problem of a similar kind to the preceding, and, at starting, it is necessary, as in that case, to determine the true shape of the section of the solid made by the plane whose inclination is given. This is shown in fig. 3. A′B′C′ is an equilateral triangle representing one face of the solid, and E′C′ is the trace of the cutting plane upon that face. The point D′ is obtained by arcs, B′D′ and C′D′, having their centres at C′ and E′, the radii being equal to B′C′ and E′C′ respectively. Then, joining D′E′ and D′C′, the triangle E′D′C′ is the true shape of a section bisecting the face A′B′C′ and containing the edge C′D′, and, therefore, also bisecting the edge A′B′. The student will notice in this triangle that E′D′ represents a line bisecting the face A′B′D′.

This preliminary being settled, the problem is solved as follows:—

The plane *l′mn*, inclined 50°, is shown by its traces, and a line *kt*, *inclined* 30°, is first determined on that plane, and then the whole is as before, constructed about the h. t. into the h. p. On this line the figure EDC is built equal and similar to E'D'C', the section of the solid (the line CD representing an edge of the solid, being coincident with *kt*, as required by the conditions of the problem), and its plan *ecd* is obtained as in previous cases.

Then, referring to fig. 3, it is readily seen that the edge A'B' of the tetrahedron is perpendicular to the plane of section, and intersects that plane at point E'. At *e'* and *e*, therefore, lines perpendicular to the traces of the plane must be drawn to represent the indefinite projections of that edge. Further, as this line is parallel to the original v. p., the lengths *e'b'* and *e'a'* (each equal to one-half the edge of the solid), must be cut off, and the projection *a'b'* upon the v. p. thus determined. From these the plans *a* and *b* can be obtained. The student should be very careful in joining the points whose projections have been found, and in settling the edges which are dotted. The line *ec* on the plan is only shown to enable one to readily appreciate the section *ecd*, which is inclined 50°.

DIVISION II.

Problem CLXII.

A regular pentagon of ·75″ base has one side inclined 37°, whilst one of its diagonals which meets that side is inclined 27°. To determine the projections of the figure.

In Problem CXXII. a method is shown for obtaining the plane of two lines which meet when the angle between them and also their respective inclinations to the planes of projection are known. This method must be adopted in the case before us. ABCDE (fig. 141) is the pentagon, and the edge AB and the diagonal are required to be in such a position as to have inclinations of 37° and 27°. The plane *v'fh* is that of the figure, and as a detailed description of the construction has been given, it is assumed that the student is able to follow it thus far. The remainder of the solution consists in obtaining the projections of the five points of the pentagon when they have been constructed into the plane *v'fh*. Taking E as example, E' is its elevation upon XY before the rotation, and E*e* is the trace of the v. p. in which E moves during the rotation. The arc E'*e'* is struck with *f* as centre, and *e'* is the elevation of point E when in the plane $v'fh$, and e is determined from e'. The rest of the points are obtained similarly.

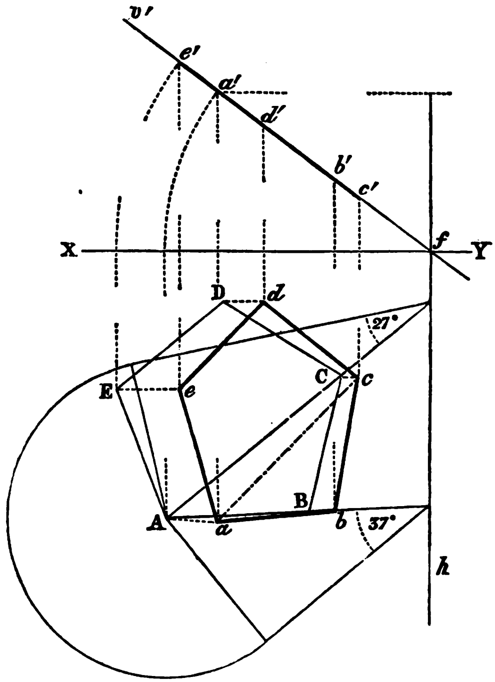

Fig. 141.

The student should notice that the plan of the diagonal *ac* would, if produced, meet AC in the h. t. of the plane.

Problem CLXIII.

To draw the projections of a cube when two of its edges which meet are inclined at angles of 27° and 40°. (Plate XXVI., fig. 2.)

If the student quite understands the principles involved in finding the plane of two meeting lines of given inclination, as described in Problem CXXII., and the determination of the plan of a figure when in this plane, as shown in the previous problem; and further, how to build the cube when the projections of one of its faces is obtained, as *given* in Problem CLX., he will be able to follow the lines of the drawing in the figure without further description, as these principles, and these only, are involved in the construction.

Problem CLXIV.

A hexagonal prism (base ·75″ side, 2″ long) is so placed that one of its long edges is inclined 40°, and the edge of the base meeting it at 30°. To determine the plan and elevation of the solid. (Plate XXVII., fig. 1.)

1st Solution.

Commence by drawing the rectangle BCHG, which is the true shape of the face, the inclinations of two of whose sides are given. Proceed, then, as in the last problem, to find the plane of the figure, and then its projection when the edges indicated are in the required position. In the figure, *l′mn* is the plane, and *bchg* is the projection; *bc* being inclined 30°, and *bg* 40°. On this rectangle the solid is to be built, and it is this part of the problem which admits of two distinct methods of solution. The plane of the hexagonal base, ABCDEF, makes a right angle with that of the face already determined, and two of the points, E and F, are in perpendiculars to the latter plane passing through C and B respectively. If, then, a hexagon be constructed upon BC, and the figure be considered as the base folded into a horizontal plane about that line, the heights of the points, E and F, above the plane of the face will be shown at CE and BF. Through A and D, the other two points of the figure, two perpendiculars to BC produced, will discover not only the heights of *those* points above the same plane, but will indicate at A_1 and D_1 the relative positions (on BC produced) of the feet of these perpendiculars, *i.e.*, their intersections with the plane of the rectangle. Construct the points D_1 and A_1 into the plane *l′mn*, as shown at D″ and A″, and at each of the points, D″, *c′*, *b′*, and A″, draw perpendiculars to the v. t.—*l′m*. These lines will be the projections on the v. p. of the perpendiculars previously spoken of. Then D_1d, C*c*, B*b*, and A_1a, drawn perpendicular to the h. t., will be the indefinite projections of the same lines upon the h. p. Now, as these perpendiculars to the plane *l′mn* are necessarily parallel to the v. p., the points, *d′*, *e′*, *f′*, *a′*, are readily obtained on the elevations, for their respective heights above the plane of the rectangle are shown on the auxiliary hexagon previously described, *i.e.*, D″*d′* is equal to D_1D, etc.

The plans of these points can then be determined from their elevations.

The projections of the hexagon at the other end of the solid could be determined in the same way; but as the long edges are equal and parallel, their projections also must be so; and further, as two of these are already determined (*ch*, *c'h'*, and *bg*, *b'g'*), lines parallel and equal to these, through the other points of the hexagon, *abcdef*, will enable the student readily to finish the required plan and elevation. The principles regulating the dotting of the hidden edges, are applied as explained in a previous problem.

2nd Solution.

In Plate XXVII., fig. 2, the above problem is solved by a different method to that just described. The construction, in so far as finding the traces of the plane of the face, whose sides are given, and the projections of the figure, is the same. But in the building of the base and other edges upon this face it differs. The plane of the face is *l'mn*, and the plan of the rectangle is *bchg* as before. Through *b'b*, a line perpendicular to the plane *l'mn* is first determined, and its trace upon the h. p. found. In the figure, *b't'*, *bt*, are the projections of this line. Then any plane containing this line will be perpendicular to the plane *l'mn*. Next, the trace of *bc* (*h'*) is joined to *t*, and assumed as the horizontal trace of a new plane containing BC, and, (as shown above), perpendicular to *l'mn*. This is the plane of the hexagonal base, and an elevation of it is made on any assumed v. p. (X'Y') by showing the elevation on that plane of either point B or C, the height of their projections above the new ground line being equal to that of their first elevations above the original XY. Then, having arranged X'Y' perpendicular to the horizontal trace *bt*, the vertical trace must pass through the elevation of the point just determined. Hence *v'f'h* is the plane of the base. Having found, then, the plane *v'f'h'* and the projection *b''c''* of BC, the whole is constructed into the h. p. about *f'h'*, and on the line BC the hexagon is built. This represents the base folded about the h. t. of its plane into the h. p. The plan is then obtained in the usual way, and the remaining long edges are projected in exactly the same manner as in the first solution, by making them parallel and equal to the two already known. Of course, an elevation upon the original v. p. could be easily *obtained* by transferring the heights of the points from the elevation already determined to the projectors of the required one. This is not shown in the drawing.*

PROBLEM CLXV.

An octahedron of 2″ edge has two of its diagonals inclined at 25° and 50°; to determine its projections. (Plate XXVIII., fig. 1).

The plane containing two diagonals of an octahedron divides the solid into two equal square pyramids, their altitude being in length, one-half the other diagonal. Find, then, as before, the plane *l′mn* of the square ABCD, the two diagonals AC and BD meeting in K, being the lines which are to have the given inclinations. Determine also the projection of this square upon the h. p. when "constructed" into *l′mn*. Proceed, then, to build upon this square the two pyramids, of which the solid is made up, by drawing through *k′* and *k* lines at right angles to the traces to represent the indefinite projections of the other diagonal of the solid. Make *k′v′* and *k′w′* equal in length to AK, and obtain the plans of V and W from the elevations. The projections can then be completed by joining these points to the corners of the square.

PROBLEM CLXVI.

A hexagon, ABCDEF, of 2″ side, is so placed that the points A, C, and E, are at heights of 1·7″, ·9″, and ·5″, above the paper respectively; to determine the plane of the figure, and afterwards its plan. (Plate XXVIII., fig. 2).

A plane is defined when the heights of three of its points are known, as the real distance between any two points being given, and also their heights, the inclinations of the lines joining them are determinable. Thus, then, this problem is really a modification of those of the Second Division of this chapter; and one could, by setting out on auxiliary drawings the heights of the points and the elevations of the lines, obtain directly these inclinations, and proceed as before. But a better method is to obtain a horizontal of the plane required by deduction from the data given. The way this is done will be understood by reference to the figure. ABCDEF is the hexagon. Join A to E, *i.e.*, the lowest point given to the highest. Now, it is clear that if A

* Through a slight inaccuracy in setting out the angles, the plans are not exactly alike, which of course, they should be.

be 1·7″ high, and E ·5″, there must be some point upon AE that will be ·9″ high. The difference of level of the two extremes is 1·7″ – ·5″ = 1·2″, and the difference between the height of points E and C = ·9″ – ·5″ = ·4″. Hence the point upon AE, which will be at the same level as C, must be at a distance from E equal to $\frac{4}{12}$, or $\frac{1}{3}$ of the line itself. Divide AE therefore into three equal parts, and the point *p* will be discovered, such that *p*C will be a horizontal of the plane of the figure. The same point could be obtained by the principal of the proportions of sides of similar triangles. At A, set up AA′ perpendicular to AE, and equal to the difference in the heights of A and E. Join A′E, and mark off upon AA′ the difference of the heights of E and C. In the case before us, AA′ is equal to ·8″, and AP to ·4″. Draw PP′ parallel to AE, to meet A′E, and from P′ make P′*p* perpendicular to AE. Then A′P : P′*p* :: A*p* : *p*E, or the line is divided in the ratio of the differences of the heights of the given points above E.

Assume, then, that the line joining *p* and C is the h. t. of the plane of the figure, and take any plane of elevation perpendicular to it, as XY. Draw AA′, perpendicular to XY, and with *c*′ as centre, radius *c*A′, describe an indefinite arc, A′, *a*′, to represent upon this v. p. the path of the point A, as the figure revolves upon *pc*. It is necessary here to find a point upon this arc, equal in height above XY, to the difference of level of A and C (·8″). Make, therefore, A″*c*′ equal to ·8″, and draw A″*a*′, parallel to XY, to meet the arc in *a*′. Then *q*′*c*′*k* will be the plane of the figure.

The remainder of the construction consists in finding the plans of the several points when the figure is brought into the plane just determined. It will be noticed that, because the plane of projection has been assumed as containing C, the points E and D fall below this plane. Of course, this is quite allowable in construction, as the projection would be the same whether the h. t. were at level ·9″, or at any other height. The construction for determining the plan of D is shown in the figure, and needs no special description.

Problem CLXVII.

A pentagonal pyramid, 1·25″ side, 2·5″ high, has three of the corners (A, B, and C), of the base, at heights 1″, 1·75″ and ·5″ respectively; to determine its plan and elevation. (Plate XXIX., fig. 1.)

The method of determination of the plane of the base is similar to that adopted in the preceding problem, except that the h. t. of the plane is taken as at a level zero, instead of at level 1″. The student will see that, by producing A′C′ and AC until they meet, he obtains a point *q*, which may be assumed as upon the h. p. Then *mn* drawn through *q*, parallel to the horizontal 1″, is the h. t. of the plane of the figure. The position of the axis is obtained by finding the centre of the pentagon, and constructing this point into the plane *l′mn*, as at *s″*. Then *s″s′* is the elevation of the axis, and as that axis is parallel to the v. p., *s″s′* is made equal to its real length. The remainder of the construction is obvious.

DIVISION III.

Problem CLXVIII.

To determine the plan and elevation of a cube (side 1·5″), when the planes of two of its faces which meet are inclined 28° and 76° respectively. (Plate XXIX., fig. 2.)

In Problem CXXIV., Chapter VII., the construction was shown to obtain a plane perpendicular to a given plane, and inclined at a given angle. Now, the two contiguous faces of a cube are perpendicular to each other, and hence the first step in the solution of our problem will be to determine the planes of the faces given, then to find the intersection of these planes, and to use that intersection as the edge of the cube. The plane of one face inclined 28° * being assumed, any point R in that plane is made the apex of a cone having its base on the h. p., and also the extremity of a perpendicular to the given plane. The traces of the cone and line are then found, the circle having *r* for centre being that of the former, and the point *s* that of the latter. Then *ks*, passing through and touching the circle, is the h. t. of a plane tangential to the cone, and therefore inclined 76°, and perpendicular to the first plane, because it contains RS. This plane is therefore the plane of a second face of the cube. The v. t. need not be determined, as it is the intersection of the planes which is required to become the edge of the solid, and that can be obtained in another way. As the two h. t.'s cross in *k*, that must be *one* point in the intersection, and as both planes pass through the apex of the

* It is more convenient to start with that plane, which has the lesser inclination.

cone, that must be another point. If, therefore, *r* be joined to *k*, then *rk* will be the indefinite plan of the edge of the cube. This line, with the plane *l'mn*, must be constructed about the h. t. into the h. p. With *m* as centre, *mr'* as radius, the arc *r'*R' is struck, and R is found, from R', upon *r*R, parallel to XY. Then upon R*k*, a portion AB is taken, equal to the edge of the cube, and the square ABCD is built upon it. The projection of this square is obtained as usual, and the cube is constructed upon it as shown in previous problems.

This solution for determining the projection of a solid, having given the inclinations of two of its planes, is applicable only when the dihedral angle between the two planes is a right angle. The next problem will show the student how to proceed in other cases.

Problem CLXIX.*

To determine the projections of a square pyramid (side of base 1·5″, height of vertex 2·25″), when the plane of the base is inclined 30°, and that of one of the faces at 73°. (Plate XXX., fig. 1.)

In this case, the first proceeding is to find the real angle between the planes whose inclinations are given. Secondly, a plane must be obtained fulfilling the two conditions of (1) being inclined at a given angle; and (2) making with a given plane (that of the base, for instance) the angle determined.

The angle between a face and the base of the solid is obtained thus, as shown in fig. 2:—A right-angled triangle, GVV', is drawn, having its base GV equal to half the length of a base edge, and GV' equal to the length of the axis. This is, therefore, half the true shape of a section of the pyramid containing the axis, and bisecting one edge of the base. The angle *a*, therefore, is the dihedral angle between the planes of the face and base. Next, to obtain the two planes by their traces, proceed as follows:—Assume *l'mn* as the traces of a plane inclined 30° (the plane of the base), and take any point, *v'v*, as the common apex of two cones, one of which is to be inclined 73° to the h. p., as *v'r's'*, and the other to make an angle *a* with the plane *l'mn*, as *v'f'g'*. Find the h. t.'s of these cones; that of the first is the circle *ro*, and of the latter the ellipse *qpy*.† Then *yt* drawn tangential

* This problem should not be attempted until Problem CLXXX. has been studied.

† The method of obtaining this ellipse is shown in Problem CLXXX., Chapter XI.

to both these curves, may be considered as the h. t. of the plane of the second given face. The intersection of the two planes must necessarily pass through *t* and the vertex V of the cones; hence *tv* is the plan of that intersection, and, therefore, of the edge of the base. The remainder of the solution consists in "constructing" *t*W into the h. p., building the square upon it, and thence obtaining the projection of the base, and afterwards of the entire pyramid in the usual manner.

It is most advisable that the student should thoroughly understand the limits as to the inclinations of planes and edges of solids in the three cases just studied. In the first divison, **the inclination of the line cannot exceed that of the plane.** *In the second division,* **the sum of the two inclinations, together with the angle between the lines, cannot exceed 180°;** *and in the third division,* **the sum of the inclinations, together with the dihedral angle between the planes, cannot be less than 180°, nor more than 360°.**

EXERCISES.

1. Draw the plan of a hexagon of 1·5″ side, when the plane of the figure is inclined 70°, one side being inclined 40°.

2. Determine the plan of a tetrahedron of 3″ side, when two of its edges are inclined at angles of 30° and 57°.

3. A cube of 3″ edge has the plane of one face inclined 60°, whilst two opposite corners of that face are ·8″ and 2·3″ high. Draw plan and elevation.

HINT.—Obtain the inclination of the diagonal of the face by an auxiliary drawing.

4. A tetrahedron of 3″ edge has two of its faces inclined 45° and 75°. Draw plan and elevation.

HINT.—This exercise is solved similarly to the construction in Problem CLXIX.

5. An octahedron has one edge inclined 40°, whilst the plane of a face containing that edge is inclined 65°. Determine its projections.

HINT.—Having obtained the projection of the face given, draw the plan and elevation of the solid upon the "constructed" triangle when the face is horizontal, and hence obtain the positions of perpendiculars to the plane of that face, and containing the other points of the solid.

6. Two diagonals of a cube are inclined 27° and 50°. Draw the plan of the solid (edge 2·5″).

7. A rectangular box, 10 feet × 6 feet × 4 feet, is to be drawn in plan, when one surface (10′ × 6′) is inclined 25°, and another (6′ × 4′) is inclined 80°. Scale, $\frac{1}{48}$.

8. A hexagonal prism (base 1·5″, edge 3″ long), has its base inclined 24°, and one of its rectangular faces 76°. Draw plan and elevation.

9. An isosceles triangle, 3″ altitude, 1·8″ base, is so placed that its apex is 2″, and the other corners are 1·2″ and 1·5″ above the paper. Find the plane containing it, and its plan.

10. A square pyramid (base 1″, edge 4″ long), has its axis inclined 60°, and one edge of the base 47°. Draw its projections.

11. An irregular triangular pyramid, ABCD, has the following dimensions:—AB = BC = 3″; AC = 2·5″; AD = CD = 3″; BD = 2·5″. Draw its plan when AB is inclined 45° and the plane of ABC is inclined 60°.

Hint.—The only difficulty in this problem is to find the position of the vertex. This is done by first determining the plan of ABC when horizontal, and the position upon this plan (and also the length) of a perpendicular to the plane of the figure through D. See Problem CLIV.

12. A cube of 2″ edge is so placed that the centres of three contiguous faces are 1·8″, 2·7″, and 2·2″ above the h. p. Draw plan and elevation.

13. A square pyramid, base 2″ edge, axis 3″ long, has its apex 1″ above the h. p., one corner of its base *upon* the h. p., and an edge of the base containing this corner inclined 20°. Show plan and elevation.

14. An octagonal pyramid (base 1·5″ edge, axis 4″), has the plane of the base inclined 25°, whilst that containing the axis, and two of the long edges, is inclined 78°. Show its projections.

Hint.—The two planes given are perpendicular, and their intersection coincides with a diagonal of the base.

15. A pentagonal pyramid, size at pleasure, is so placed that the axis is inclined 27°, whilst one edge of the base is inclined 35°. Show its plan.

Hint.—The inclination of the axis is the complement of that of the base.

16. An octahedron of 3″ edge, has the plane containing two of its diagonals inclined 27°, whilst that containing one of these, and the other diagonal, is inclined 80°. Show its projections.

Hint.—The planes given are perpendicular, and the problem is solved by the construction described in Problem CLXVIII.

CHAPTER XI.

ON THE PROJECTION AND DEVELOPMENT OF FLAT AND CURVED SURFACES, INTERSECTED BY CUTTING PLANES.

As in this chapter, and others to follow, continual reference will be made to surfaces of revolution, such as the cone, the sphere, the cylinder, and the ellipsoid, it will be convenient at this point to draw the attention of the student to certain functions and properties of these surfaces, sufficient for him to investigate intelligently the solutions of the problems connected with them.

1. The surface of a sphere is generated by the revolution of a semicircle about its diameter.

2. All plane sections of a sphere are circles; if they contain the centre, they are *great circles*.

3. A right cylindrical surface is generated by a straight line, moving around, and continually at the same distance from, and parallel to a fixed straight line, which is called its axis; or it may be developed by the revolution of a rectangle about one of its sides. The moving line is called the *generatrix*, and the fixed one the *director*.

4. The section of a right cylinder, made by any plane, inclined to the axis, is in true shape an ellipse, the minor axis of which is equal to twice the distance of the generatrix from the director.

5. The surface of a right cone is generated by the revolution of a movable line which meets a fixed one, in such a way that at all points of the rotation, the angle between them is constant; the meeting-point of the two lines is the vertex of the conical surface, and the fixed line is the axis.

NOTE.—Generally speaking, whenever a cone is indicated in these problems, only one-half of the above surface is meant, such as would be generated by the revolution of a right-angled triangle, about one of the sides containing the right angle; but as will be seen in future problems, it is better to consider the surface as generated by the two lines.

6. The sections of the right cone are as follows :—

   a. When the cutting plane contains the axis, the true shape is a double isosceles triangle with a common apex.

b. When the cutting plane is perpendicular to the axis, the true shape is a circle, the radius depending upon the distance of the plane from the vertex.

c. When the cutting plane is neither parallel to a generatrix nor to the axis, and is wholly on one side of the vertex, the true shape is an ellipse.*

d. When the cutting plane is parallel to a generatrix, and perpendicular to the plane containing that line and the axis, the true shape is a parabola.

e. When the cutting plane is parallel to the axis, it will intersect the surface on each side of the vertex, the true shape of the section being two conjugate hyperbolæ.

7. The ellipsoid is a surface of revolution, generated by the rotation of an ellipse about either its major or its minor diameter, which then becomes the axis. Its sections are of two kinds—1. When the cutting plane is perpendicular to the axis, they are circles; and 2, When parallel or inclined to it, they are ellipses.

The student is referred to works upon Conic Sections for proofs of the above statements, as they are merely collected here for convenient reference whilst studying the constructions used for the determination of the projections of sections in the problems of this chapter.

Problem CLXX.

To determine the projections and true forms of the three principal conic sections.

Case I. (Plate XXX., fig. 2).—In this Plate the cone AVC, shown by its projections, is cut by a plane represented in elevation by the line *s't'*. According to the principles described at page 79, the true shape of this section is an ellipse, and it is for us to determine its major and minor axes. The plans of points S and T will be found very readily upon those of the generatrices *av*, *cv*, by projectors through *s'* and *t'*. Bisect *s't'* in *o'*, and as *s't'* discovers the real length of the major axis (the points *s'* and *t'* being the highest and lowest of the section, and upon generatrices parallel to the v. p.), it is clear that the minor axis must be a horizontal line passing through *o*, the plan

* The properties of these curves, as plane figures, are given in Chapter VI., which the student is recommended to thoroughly investigate before proceeding *to the solution* of the problems which follow.

of which can be drawn at once, indefinite in length, through *o*. To discover the real length of this axis, imagine a second section of the cone passing through *o*, perpendicular to the axis. The elevation of this section would be the line *k'l'* and its shape a circle. Conceive one-half of this circle to revolve upon a diameter until it was brought into a v. p. This would be shown in the drawing by *k'p'l'* having 1' for its centre. Then *o'p'*, taken perpendicular to *k'l'*, would represent the half of a horizontal line belonging to both sections (the circle and the ellipse). Hence point P must be on the surface of the cone, and *o'p'* must be the required length of the semi-minor axis of the ellipse. To complete the plan, make *op* and *oq* each equal to *o'p'*, and to draw the true shape of the section, as shown at PSQT, make the major axis equal to *s't'*, and the minor to *pq*.

The properties of the ellipse, as a plane figure, were discussed at page 85; and it will be interesting to the student to learn how he may discover, from the section of the cone, the foci and directrices of the figure. This is shown in the Plate. The circle *m'n'*F" is the elevation of a sphere, which touches the cutting plane, and is inscribed in the upper frustum of the cone. The point of contact of the sphere and plane is the focus of the elliptical section, and *f"* is obtained by a parallel to PQ through F". Further, the plane of the circle of contact of cone and globe intersects the plane of section in a straight line, *which is the directrix of the ellipse* belonging to the focus just found. The line *m'n'* (the elevation of the circle of contact) produced, meets *s't'* also produced in g_1, and the directrix *g'h'* can be obtained as shown in the drawing. Of course, the same principle holds good for determining the other focus and directrix. The sphere *i'*F'*k'*, in the other frustum, is only partly shown in the figure.

Case II. (Plate XXXI., fig. 1).—In this case the cutting plane is taken parallel to the generatrix VA, and perpendicular to the plane containing that line and the axis. The curve traced by this plane upon the conical surface is therefore a parabola, the elevation of which may be very conveniently assumed as a straight line *s't'*. The points of the plan, and afterwards those of the true shape, are found by taking a series of horizontal sections of the cone, the plans and true shapes of which are circles. The cutting planes intersect the given one in lines which are chords of the circles and ordinates of the parabola. Taking, for instance, the horizontal section of which *p'q'* is the elevation, its plan is a circle, with *v* as centre and *vp* as radius.

The elevation of the intersection of the two planes, *i.e.*, the given one $s't'$ and the assumed one $p'q'$, is g', and the indefinite plan of this line is *wz*, projected from g', and meeting the circle in *g* and *h*. Then *gh* is a chord of the circle, and also a double ordinate of the parabola. Proceeding in a similar manner to find others of the points in plan, and noting that the point *s* and its companion *r* fall upon the base of the circle, whilst the point *t* projected from t' occurs upon the plan of the generatrix *vc*, the curve can be traced, and the entire horizontal projection of the section determined. With regard to the true form, the lengths of the ordinates shown in the plan can be set off on either side of an assumed axis taken parallel to $s't'$, the proper relative positions of these lines being obtained by perpendiculars to the section through their elevations. Thus, MN is an axis taken parallel to $s't'$; the point T, the vertex of the parabola, is found by drawing t'T perpendicular to $s't'$; and the ordinate GH is fixed in position by another perpendicular to $s't'$ through g', and in length it is equal to *gh*, one-half being taken each side of MN. It is assumed that the student will be able to set out the entire drawing without further explanation.

The directrix and the focus of the curve can be found in an exactly similar manner to that adopted in the elliptical section. The circle $m'n'$F$'$ being the elevation of a sphere inscribed in the frustum of the cone, and touching the plane of section, the point of contact F$'$ enables us to obtain F, the focus of the curve, and o', the elevation of the intersection of the plane containing the circle of contact, and the cutting one settles the position of the directrix KO.

Case III. (Plate XXXI., fig. 2).—In this case the cutting plane is parallel to the axis; the curve traced upon the surface is therefore an hyperbola. The elevation of the cut in the drawing is $s't'$, and, as the plane of section is a vertical one, its plan is a straight line *rs*. The method adopted to obtain other points in the projection, and afterwards the true form, is similar to that used in the preceding case. Horizontal sections are taken, and the ordinates of the curve are determined in position and length as before. Taking the section $p'q'$ for an instance, the circle in plan through *p* cuts *rs* in two points, *g* and *h*. Then *gh* is the length of the double ordinate at that height. The true form is shown at RST, and the construction used for obtaining GH shows the principle involved throughout. The focus is F, and the directrix is *OK, obtained* as described in the other two cases.

PROBLEM CLXXI.

To determine the projections of the section of a sphere made by a given cutting plane—(*a*) When the latter is inclined only to one plane of projection; (*b*) When it is inclined to both. (Plate XXXII., figs. 1 and 2.)

CASE 1. All plane sections of the sphere are circles, and hence their orthographic projections are either straight lines, ellipses, or circles, and, as the first and last would present little difficulty, two cases only are taken, where one or both the projections become ellipses. In the first case, the two circles, having *c'* and *c* for centres, are the projections of a sphere resting upon the h. p., and *v'fh* are the traces of a cutting plane inclined only to the h. p. The elevation of the section is therefore the line *p'q'*, and, as that section is a circle contained by the oblique plane, its plan must be an ellipse. The minor axis *pq* of this ellipse is determined from *p'q'*. Then the major axis will be in a horizontal of the plane represented in plan indefinitely by *tg* bisecting *pq* in *o*. This axis will therefore be projected equal in length to the diameter of the circular section, and that length can be taken from the elevation *p'q'*. Make then *or* and *os* equal to *o'p'*, and construct the ellipse. The points of contact of this curve with the contour circle in plan may be found by drawing the elevation of the horizontal great circle of the sphere. The elevation *m'* of the intersection of this circle with the cutting plane is that of the required points of contact, and the plans *m* and *n* are projected from it. More than half the circle of section can be seen in the horizontal projection, the dotted or hidden portion being the shorter arc between these two points of contact.

CASE 2. In Plate XXXII., fig. 2, the same problem is solved, except that, in that case the cutting plane is inclined to both the planes of projection. The circle of section is therefore projected as an ellipse in both plan and elevation, and the construction required is that for obtaining points in the conjugate diameters of these ellipses.

The circles whose centres are *c'* and *c* are the projections of the given sphere, and *l'mn* is the cutting plane. A vertical plane *s'st* is first assumed, containing the centre of the sphere, and having its h. t. perpendicular to *mn*. This plane intersects the solid in a great circle, and the given plane in a straight line *s't'*, *st*. The new plane is then constructed about its h. t. into the h. p., with the circle and straight line just mentioned. To do this, *st* is assumed as a ground line, c'' is projected from *c* at a distance from *st* equal to that of c' above XY, and the great circle is drawn. Next, *s*S is made perpendicular to *st* and equal to $s's$. Then S*t* represents the constructed intersection of the cutting plane and the assumed vertical one. This line is seen to meet the circle in p'' and q'', and the plan *pq* of the minor axis of the ellipse is obtained (considering the plane to be folded back into its vertical position) by drawing projectors from p'' and q'' to meet *st*. The elevation $p'q'$ is then directly projected upon $s't'$ from *pq*. The length of $p''q''$ is that of the diameter of the circle of section. If *pq* be bisected in *o*, and *wz* be drawn parallel to the h. t. of the cutting plane, that line will represent the indefinite plan of a second diameter of the circle, which, being horizontal, must be equal in length to $p''q''$; *ow* and *oz* are therefore made equal to $o''p''$, and $o'w'$ and $o'z'$ are projected upon the elevation. The ellipse in plan can be readily drawn, as the major and minor axes are known, but the four points, p', w', q', z', in the elevation do not give the extremes of the principal axes of the curve. The lines $p'q'$ and $w'z'$ are conjugate diameters, for each bisects all chords of the figure parallel to the other. Hence, as shown in Problem LXXVII., Plane Geometry, the figure can be accurately obtained by the aid of the circumscribing parallelogram 1, 2, 3, 4. The ellipse touches the contour circle in elevation in points a' and b', the positions of which may be readily discovered by drawing EF, the plan of the vertical great circle of the sphere. For $f'b'a'$, parallel to $l'm$, is the elevation of the intersection of the plane of this great circle with the cutting plane. A converse proceeding on the plan would discover the points of contact of the ellipse and circle.

Problem CLXXII.

To determine the projections and true form of the section of a vertical prism made by a cutting plane inclined to both planes of projection, and to show the development of the cut surfaces of the solid. (Plate XXXIII., fig. 1.)

In the figure the given solid is the pentagonal prism ABCDE, and the cutting plane is $v''f''h''$. As the faces of the solid are all vertical planes, their projections or traces in plan will also be the projections of any lines upon them. Hence, if every face of the solid were cut, the entire plan of the section would be the pentagon *abcde*. In the case before us, however, one edge of the prism is uncut, hence the plan of the section is the figure *aedgfb*. It remains, therefore, to find the elevations of the points where the cutting plane meets the other vertical edges. The problem is thus resolved into finding the intersection of a vertical line and an oblique plane (Problem CIX., Chapter VII.). Taking the edge through A as an example, draw a line aa_1 parallel to $f''h''$, to represent in plan a horizontal line passing through the point of section on the vertical edge $a''a'''$. The v. t. of this line (as it is contained by the plane of section) will be in the v. t. of the plane. $A''a'$ is the elevation of the horizontal, and its intersection a' with $a''a'''$ gives the vertical projection of the point where the cutting plane meets that edge. The other points, $b'e'd'$, are found in a similar manner, and the whole elevation is completed by joining these points together with the elevations of F and G, which are of course upon XY.

There is another construction for obtaining the vertical projection of the section, which depends upon the assumption of v. p.'s containing the edges of the solid, and using the intersections of these planes with the cutting one to obtain the required points. But this is not so convenient a method as that shown in the figure.

To determine the true shape of the section, the cutting plane is constructed about either its h. t. or v. t. into one of the planes of projection. It is generally solved by adopting the former plan, as being perhaps more readily understood.

Taking the case of the point E, the student will see that when the plane of section is "constructed" as described, that point will move in a v. p. perpendicular to $f''h''$—the hinge of rotation—eE being the trace of this plane upon the h. p. Further, it will move in a circle in this plane, having t for its centre, the radius of the curve being equal to the actual distance of the point from the h. t. of the cutting plane. To obtain this distance, a right-angled triangle must be conceived as fitting beneath the plane of section, so that its base coincides with et, and its perpendicular with the edge containing E. Then the hypotenuse will give the desired distance. This triangle is shown folded about its base into the h. p. thus :—eE′ is drawn perpendicular to et, and equal to the height of point e' above XY, E′ is joined to t; E′t is the radius of the circle of rotation. Then, taking t as centre, and tE′ as radius, the arc E′E gives, by its intersection with eE, the point E. All the other points having been constructed in the same way, and noting that f and g, being on the hinge-line, do not move, the true shape of the section is found by joining them as shown in the figure.

By the development of the faces of the solid is meant the bringing them all into one plane by, as it were, unfolding them about their edges. This is shown in the case before us, in fig. 2, where E_1 E_1 is first drawn equal in length to the sum of the five edges of the base, and, at the equidistant points, E_1 D_1 C_1 B_1 A_1 and E_1 lines perpendicular to E_1E_1 are set out equal in length to the long edges of the solid. The complete rectangle $E_1k_1k_1E_1$ then represents the surface, which, if refolded on the four lines, could be made up into the original pentagonal prism. The points of section upon the edges are then set along the perpendiculars at distances from E_1E_1 equal to their heights above the h. p., as shown in the elevation. Thus, E_1E_{11} is equal to the height of e' above XY, etc. If all the edges were cut, the development of the section would be completed by joining these points together, noting that E_1E_{11} has to be taken twice, once at each extreme of the rectangle, so that they may coincide when refolded. But two of the section points occur upon base edges, and, to show these points on the development, G_1D_1 must be made equal to *gd* in the plan, and F_1B_1 to *fb*. The points must then be joined as shown.

Problem CLXXIII.

To determine the projections and true form of a section of a prism (axis horizontal) made by a given cutting plane. (Plate XXXIV., fig. 1.)

This is really a very elementary problem, the construction required being only that for finding the points of intersection of the edges of the prism with the given plane. These edges being horizontal lines, the method adopted is that shown in Problem CIX., Case 1. The edge EL intersects the plane in *p'p*, for it there meets a horizontal 3·2 in the plane of the same height above the h. p. as itself. The edges AH and BI being on the h. p., the h. t. of the cutting planes gives the line of section on the face ABIH. The true shape of the section is shown at OPQ....V, where the plane has been "constructed" about its v. t. into the v. p. in the same way as described in the previous problem.

Problem CLXXIV.

To determine the projections and true form of a section of a cylinder (axis horizontal) made by a given cutting plane. (Plate XXXIV., fig. 2.)

The construction employed in the case of the prism is applicable to the cylinder, as the generatrices of the latter can be treated like edges of the former. But, as the section is known to be an ellipse, all that is necessary is to determine two conjugate diameters in each projection, and to complete the curves by inscribing them in two rhomboids whose sides are parallel to those diameters. Thus, in fig. 2 the traces on the plane of the highest and lowest and two intermediate generatrices only are determined. This gives four points, *g′g*, *h′h*, *p′p*, and *q′q*, which are at the extremities of conjugate diameters * of the projected ellipses, and the curves are inscribed in the parallelograms 1, 2, 3, 4, and 5, 6, 7, 8, whose sides are parallel to these diameters. The true form can be deduced, as at GPHQ, by constructing the plane of section with the two conjugate diameters, GH and PQ, about its v. t. into the v. p. For the ellipse inscribed in the parallelogram 9, 10, 11, 12, (having its sides parallel to GH and PQ), is the true form of the section required.

Problem CLXXV.

To determine the projections and true shape of a section of a vertical right pyramid, by a cutting plane inclined to both planes of projection, and to show the development of the cut surface of the solid. (Plates XXXV. and XXXVI.)

The student will see that in this problem he has to deal with edges which are inclined to both planes of projection; hence the construction he was able to adopt in the previous problems, where the edges or generatrices were parallel to one plane, cannot be used here. There

* That these are conjugate diameters can be readily proved by assuming that the four generatrices used are the lines of contact of a circumscribed square prism. The cutting plane would then give the parallelograms as projections of the section of this prism, and the ellipses would have the four lines in each case for tangents. But if a diameter of an ellipse be parallel to the tangents at the extremities of another diameter, these two diameters are conjugate (see Plane Geometry).

are several methods of solution applicable, and the student will see in the cases set before him that the selection of either of these depends upon the conditions of relative position of the pyramid, the co-ordinate planes, and the plane of section.

Case I. (Plate XXXV., fig. 1).—The pyramid given is hexagonal, and the plane of section is *v′fh*. The principle upon which the construction used in this case is based is the following:—Vertical planes are assumed containing the edges of the solid. The intersections of these planes with the given cutting plane pass through those edges, giving the required points of section. Thus the assumed plane *w′wx* is a vertical one, and it contains the edges IE, IB. The elevation of the intersection of this plane with the given one *v′fh* is *w′x′*, and the points *n′*, *q′*, where this line meets the elevations *i′e′* and *i′b′*, are the vertical projections of the points of intersection of the cutting plane with those edges whence the plans *n* and *q* can be obtained. In the same way all the edges of the pyramid could be treated, and the entire projections of the section determined. But it would often be extremely inconvenient to use *both* traces of these vertical planes, as their intersections with those of the given one would be so remote. Then the following expedient may be adopted, which at once gets rid of the difficulty:—

It will be seen that, in using the intersection of the vertical plane *w′wx* with *v′fh*, we have not only discovered the points of intersection of the edges IB and IE with the cutting plane, but also that of the axis. *Hence i″ is the elevation of a point in the intersections of all the vertical planes with the given one.* Therefore only one other point in each of those intersections need be determined. Taking the case of the edges AI, DI, the h. t. of a v. p. containing them is *ay*, and *yy′* is the v. t. Hence, as *yy′* meets *v′f* in *y′*, that point is in the vertical projection of the intersection required. If, then, from *y′* a line be drawn through *i″*, it will be the elevation of the intersection of the plane containing AI and DI with the plane *v′fh*, and will determine the vertical projections of two more points, *o′* and *r′*. Again, in the case of the edges CI and GI, it is most convenient to use the h. t.'s of the two planes which meet in *z*; for as that is the plan of one point in their intersection, the elevation *z′* enables us to complete the vertical projection of the entire intersection by drawing *z′i″* to meet *c′i′* and *g′i′* in *p′* and *m′*.

The true form of the section is obtained in exactly the same manner as in the previous problem, *i.e.*, the section is brought into the h. p. by revolving the cutting plane about its h. t.

The development of the pyramid is shown in fig. 3, where all the triangular faces are brought into one plane, by unfolding them, as it were, about their edges. To make this drawing, the real length of one of the slant edges must be used as the radius of a curve along which the series of base edges must be stepped as chords, and afterwards joined as shown. The right-angled triangle I'IB (fig. 2) is the true form of a semi-section of the pyramid made by a v. p. containing the edge IB, I'B being equal to *ib* and I'I to the axis. Hence I'B gives the real length of the edge. Then, with I (fig. 3) as centre, and a radius equal to I'B, the arc G_1G_1 is drawn, and G_1A_1, A_1B_1, ... and E_1G_1 are made equal in length to the base edges, and joined. Then from I_1 lines are drawn to these points, and the whole figure $I_1G_1G_1I_1$ is the development of the surface of the pyramid. To mark upon this figure the line of section of the cutting plane, it is necessary to find the real distance of the points of section along the edges from the base. This is readily done by drawing from the vertical projections of those points parallels to XY, meeting the true edge I'B. Thus the parallel from *m'* to M' discovers upon I'B the distance of point M from the vertex or from the base. Then, as that point occurs upon GI, G_1M_1 in fig. 3 is made equal to BM^1, and so on with regard to the others. Then, by joining these points, when found, the crooked line obtained gives the development of the cut required.

CASE 2. In Plate XXXVI., fig. 1, the projections of a square pyramid and the traces of a cutting plane are so arranged that, in adopting the same construction as was used in the last case for determining the projections of the section, the assumed vertical planes containing the slant edges would in one instance be parallel, and in the other perpendicular to the original vertical plane of projection. In such an example a slight modification of the construction is necessary. The vertical plane containing the edges VA, VC cuts the solid in a figure which is an isosceles triangle, and *a*V*c* is a drawing of this figure brought into the h. p. by rotating it about its base *ac* (*v*V being made equal to *vv'*). The intersection of this plane with the cutting plane is a line parallel to the v. t. *l'm*, and *l''m''* represents this line brought into the h. p. By this means E and F, two of the required points of section, are obtained, and also their heights above the h. p. The plane *e* and *f* are projected from E and F, and the elevations *e'* and *f'* are obtained as usual.* Again, the vertical plane containing the edges BV, DV is perpendicular to the original v. p., and the traces coincide indefinitely, therefore, with the projections of the intersection of the planes. But if the section of the solid and the intersection of the two planes be brought into the h. p. by folding them about *bd*, the other points of section G and H can be determined at once. The triangle *b*V'*d* is the section of the solid, and S*t* is the intersection of the planes "constructed" as described, the arc S'*s* being struck with centre *t'*. The plans *g* and *h* are projected from G and H, and the elevations *g'* and *h'* from *g* and *h*.

Case 3. Another solution of the same problem, which is ingenious and sometimes very convenient (especially when the pyramid is an irregular one), is shown in Plate XXXVI., fig. 2, where an hexagonal pyramid is supposed to be cut by a plane of section.

A horizontal section of the solid is first taken at any convenient height. Thus, in the figure *s't'* is the trace of a horizontal cutting plane, and the hexagon 1, 2, 3, 4, 5, 6 is the plan of the section. Next a horizontal line in the given cutting plane is determined, *of the same height* as the assumed h. p., or, in other words, the intersection of the two planes is obtained. In the drawing *ft* is the horizontal projection of this line. Now, it will be seen that 2, 3 meets this line in *z*; and as that point is in the plane of the face VCD, and also in the plane *lmn*, it must be one point in their intersection. Then, by producing *cd* to meet *mn*, another point *y* is obtained, and *yz* is the indefinite projection of the intersection of the two planes. The portion *ik* is therefore the plan of one line in the required section of the solid. Again, *i* must be a point in the section upon the next face VBC; and by producing *bc* (the h. t. of the plane of that face), to meet *mn* in *w*, a second point is obtained, and *wih* can be drawn to determine *ih*. In this way the whole plan can be completed, and the elevation projected from it.

A very interesting problem can be suggested at this stage, viz., "to draw a sectional elevation of a frustum of the pyramid projected upon the plane of section." This is shown at v_1, a_1, b_1, c_1, etc., where the

* The points *e'* and *f'* could be also obtained by finding the elevation of point *m''* on XY, and drawing a line through it parallel to *l'm'*. This is really the *same construction* as the above, but saves drawing the triangle *a*V*c*.

lower frustum of the solid is supposed to rotate upon the v. t. of the cutting plane until the section coincides with the v. p. The true form of that section is first found by "constructing" the plane with the figure upon it into the v. p.

To obtain the projections of the points of the base, the following construction is employed:—A new plane, $r'x'x$, is assumed, containing the axis and perpendicular to the cutting plane, and its intersection, $r'x'$, rx, with the latter is determined. This intersection passes through the axis at the point o', where it is cut by the given plane of section. Next, these two planes are conceived to maintain their relative positions during the "rotation" of the given one upon its v. t.; and, of course, when that rotation is completed, the assumed plane becomes perpendicular to the original vertical plane, and can be represented by its trace thereupon. If, therefore (knowing that point r' remains stationary), the position of point o_1 be obtained by the same construction as that adopted for the other points of the section, and $r'o_1$ be joined, that line will represent on the v. p. the trace of the assumed plane containing the vertex. The vertex is then projected at v_1 upon this plane from v'.

Having thus obtained the projection of point V, it is known that the projections of the edges of the solid must all pass through that point, and also through the points of the section. Draw, then, through v_1 a series of lines passing through g_1, h_1, i_1, k_1, p_1, and q_1, to represent the indefinite projections of these edges. Again, the corners of the base of the solid move in planes perpendicular to the hinge of rotation $l'm$. If, then, through the elevations a', b', c', d', e', and f' perpendiculars to $l'm$ be drawn, these lines will represent the traces of the planes in which these points move, and the intersection of either of them with the edge upon which the corresponding base corner occurs will give the desired projection of that point; thus, $a'a_1$ is made perpendicular to $l'm$, and meets the edge v_1a_1 in a_1, which is the projection of A upon the plane of the section.

Problem CLXXVI.

To obtain the projections and true form of the section made by a given plane cutting a given vertical right cylinder; and to develop the cut surface. (Plate XXXVII., figs. 1 and 2.)

If the cylinder were treated as a prism with an infinite number of edges (the generatrices of the surface being considered as representing those edges), the construction adopted in Problem CLXXII. would apply in this case. The curves, which would be the required projections, could then be traced through the several points determined, their accuracy depending on the number used.

But as it is known that the section of the cylinder must be an ellipse (the cutting plane being inclined to the axis), it is really only necessary to determine conjugate diameters of the ellipses which are the projections of the curve, and the figures can then be completed by constructions explained in the Plane Geometry.

In the plate the projections of a cylinder are given, and the plane of section is *v'fh*.

Commence by drawing *bd* and *ac* upon the plan, respectively perpendicular and parallel to the h. t. of the given cutting plane. These two lines will be the indefinite horizontal projections of the major and minor axes of the elliptical section, for the circle *abcd* is the plan of the cut, and its diameter *bd* contains its highest and lowest points.

The elevations of these two lines can then be determined as contained by the plane *v'fh*, the four points *a'*, *b'*, *c'*, and *d'* occurring upon generatrices of the cylinder, as shown. Or a vertical plane K'K*t* may be assumed, perpendicular to *v'fh* and containing *bd*. The intersection of this plane with the cutting one can then be shown in elevation, as K'*t'* and *b'd'* can be projected from *b* and *d* upon this line. AC is a horizontal of the plane; its elevation is therefore a horizontal line, *a'c'*. As the vertical projections *b'd'* and *a'c'* are not the major and minor axes of the ellipse in elevation, the curve can best be constructed by using the circumscribing rhomboid, as *lmnp*, having its sides parallel to *a'c'* and *b'd'*, and by proceeding as shown in the Plane Geometry.

The two principal axes of this figure are given in the drawing, and the method of obtaining both their direction and length is explained in Problem LXXIX., Plane Geometry.

The student will notice that the ellipse in elevation *b'a'd'c'* touches the extreme generatrices in points *r'* and *s'*, and he can find the exact points of contact by drawing *rs* in plan parallel to XY, to meet the h. t. in *q*; the elevation *q'* gives a point from which a parallel to *v'f'* can be drawn, meeting the generatrices mentioned in *r'* and *s'*, the points of contact required. The explanation of this will suggest itself to the reader by reference to Case 2 in Problem CLXX., as the principle involved is the same.

The true form of the section is given at ABCD, where the major axis BD is obtained by constructing the plane of section into the h. p., in exactly the same way as shown in previous problems, and AC is made equal to the diameter of the cylinder. In fig. 2 the development of the surface is shown as a rectangle, one side, D_1D_1, being made equal to the circumference of the circle *abcd*,* and D_1D_{111} to the length of the cylinder. Points in the development of the curve of section are obtained by assuming a series of generatrices at equal intervals around the surface, and showing these upon the rectangle by dividing D_1D_1 into a corresponding number of parts, and drawing parallel to D_1D_{111} through the points of division. These generatrices are also shown in elevation, and also the heights above XY that the curve meets, these several lines are then taken upon their corresponding lines in the development, and the desired curve traced upon it. Thus, upon the generatrix through *b*, the point of section is seen in elevation to be at a distance ($b'b_1$) from XY. That distance is marked from B_1 to B_{11} in the development. Only four of these are given in the drawing, so as not to confuse; but the greater the number the more accurate the curve.

Problem CLXXVII.

To determine the projections and the true form of the section of a given vertical right cone, made by a given cutting plane; and to develope the conical surface with the curve of section. (Plate XXXVIII.)

This problem could be solved by adopting the construction used in the case of the pyramid, if generatrices of the cone were assumed as the edges of such a solid, and the curves required could be traced through the points of section so determined; but as the cutting plane is parallel to no generatrix of the surface, and as the section is entirely on one side of the vertex, we know that the form of the cut is an ellipse, and that its projections are also ellipses. Hence the better method is that which enables us to find conjugate diameters of the curves directly. This can be done by the following construction:—

Assume a vertical plane containing the axis, and perpendicular to

* Generally speaking, it is sufficient to divide the circle *abcd* into a large number of parts, and to step the chords of the arcs along a line to obtain this side of the rectangle; but it is rather better to take the length, $2\pi r$, from a good scale.

the cutting plane, and find their intersection: $s'st$ is the assumed plane, and $s't'$, st the projections of the intersection. Part of this line is the major axis of the ellipse, and the extremities of it are in the two generatrices contained by the assumed vertical plane. Hence their elevations $i'a'$ and $i'b'$, meeting $s't'$ in the two points r' and p', determine the vertical projection of the major axis, and rp is its plan. Bisect rp in o, and draw wq through o parallel to the h. t., as the indefinite plan of the minor axis, and determine its elevation as a horizontal of the cutting plane. To discover the length of this minor axis, conceive a horizontal section of the cone at the same height as the line just found. The plan of this section would be a circle of radius (vV); and as the line WM meets the conical surface in two points, and further, as this circle contains all the points on the surface of the same height as the line, the points w and q must be the horizontal projections of the extremities of the minor axis required, and w' and q' are the elevations.

It so happens in the drawing given that the two axes in the plan are equal, and hence the projection is a circle. The ellipse in elevation is inscribed in the rhomboid 5, 6, 7, 8 as in previous problems, and the points of contact x and z with the extreme generatrices $i'g'$ and $i'k'$ are discovered as in the last problem.

The true form of the section is shown at PQRW, where the plane is folded as usual about its h. t. into the h. p.; but it will be noticed that, as PQ is in a horizontal of the plane, W and Q can be found on a line bisecting PR at right angles, its length being equal to wq.

To develop the surface, take i' as centre and $i'k'$ as radius, and draw an indefinite arc ($k'k''$). Cut off from this arc a portion ($k'k''$) equal in length to the circle which is the base of the cone.* Then the sector $i'k'k''$ is the development of the entire conical surface. To show upon it the curve of section, a series of generatrices must be assumed at equal intervals. Six of these are shown in the drawing in elevation, obtained from six points in the plan, stepped at equal distances,

* To do this the general method is to divide the circle in plan into a convenient number of parts, and to step the chords of one of the small arcs along the curve; but this must necessarily be inaccurate. The following is a better plan:—Remembering that arcs of circles struck with the same radius are proportional to the angles subtended at the centre, and also that arcs subtending the same angle at the centre are proportional to the radii, a proportion as $i'k' : ik :: 360° : \phi$ discovers in its fourth term ϕ, the angle which, if made at i', cuts off the arc required.

commencing from *a*.* These are *a'i'*, 1'*i'*, 2'*i'*, *b'i'*, 3'*i'*, and 4'*i'*. The arc *k'k''* is then divided into six equal parts and lines, joined from the points of division to *i'*, to represent these generatrices on the developed cone. The distance from the apex to the point of section upon any generatrix is found by drawing a line through the elevation of that point parallel to XY, until it meets *i'k'* or *i'g'*. As these two generatrices are projected full length in elevation, the distance of the point of section from *i'* or *k'* gives the length which must be cut off from the corresponding developed generatrix. Thus 3'*i'* is cut at 3. Then a line 32, parallel to XY, gives at 3'' a point which tells us that *i'*3'' is the distance from the vertex that that generatrix meets the section. Then upon *i'*, III (the development of that line), a portion ($i'3_{11}$) is cut off, and hence a point is found in the developed curve of section. The student will, it is hoped, be able by the drawing to appreciate the remainder of the construction.

PROBLEM CLXXVIII.

To determine the projections of the section of an ellipsoidal surface, made by a given cutting plane. (Plate XXXIX.)

In the case shown the ellipsoid has one of its principal axes vertical. The section is in true form an ellipse, and so are its projections; hence we proceed to determine the plan and elevation of the major and minor axes of the curve. The major axis is in a vertical plane containing the long axis of the solid, and perpendicular to the cutting plane, and is part of the intersection of those planes. Proceed then to find the projections of the intersections of the planes *v'fh* and *s'st*. Then *s't'*, *st* contains the major axis of the section, and shows that the axis is cut at point *c''c*. Next proceed to find where this intersection meets the surface of the solid. To do this it is necessary to bring the line *s't'*, *st* into a plane parallel to the v. p., as the generatrices cut by this line are in a plane inclined to it. With *c* as centre, radius *ct*, draw the arc *t*T until it meets a parallel to XY through *c*. Project T' from T, and join *c''*T'. From the elevation it will then be seen, that the line of intersection found passes through the generatrices of the ellipsoid at certain heights shown by P' and Q'. Transfer these heights by horizontal lines to *p'* and *q'*, and project the plans *p* and *q*. Then *p'q'*, *pq* is the major axis of the section; the minor axis being

* In the plate, the distances 1_1 and 2_1 have been inadvertently measured slightly unequal; they should be alike.

a horizontal line bisecting PQ. Draw, therefore, lines through a', a to represent the indefinite projections of this horizontal. To find where it pierces the ellipsoid surface, conceive a horizontal section of the solid at the same height as this line. Its plan is a circle, radius equal to $a'6$, and having its centre at c. This circle discovers points m and n, the horizontal projections of the extremities of the minor axis, from which m' and n' can be projected. The elevation of the curve is obtained by the aid of the circumscribing rhomboid, or the major and minor axes of the ellipse could be obtained from the two conjugates. The points x and y, where the ellipse is in contact with the elevation contour of the solid, is found in a similar way to that adopted in previous problems. The intersection of a vertical plane containing the generatrices forming this contour, with the given cutting plane, passes through these points of contact. Thus cz is the h. t. of such a plane, and $z'y$ is the elevation of its intersection with $v'fh$. Then x and y are the vertical projections of the points of circle, and i and r in the plan where the ellipse meets the plan is found by the converse construction. A horizontal plane containing the great circle of the solid intersects the cutting plane in a line whose plan passes through these points.

PROBLEM CLXXIX.

To determine upon a plane of projection the trace of any given right cylindrical surface whose axis is inclined to that plane. (Plate XL., fig. 1.)

This problem really amounts to finding the true shape of a section of the cylinder made by the plane of projection, or by any plane parallel to it. In Plate XL., fig. 1, the axis of a right cylinder is inclined to the h. p.; hence its surface must be intersected by the h. p. in an ellipse, the points of which would be the horizontal traces of all the generatrices of that surface. The h. t. of the axis AB is at b, giving the centre of the ellipse.

The two extremes of the major axis, f and e, are the horizontal traces of GI and AK, the uppermost and lowest generatrices. The minor axis must be cd passing through b at right angles to ef, and as its length is equal to the diameter of the cylinder, mn and st being produced determine points c and d. This problem will be found to be of great service in questions upon shadows and tangential planes *to curved* surfaces, and frequently the student may have the cylindrical surface inclined to both planes of projection, instead of, as in the plate, only to one. In such a case he must make an auxiliary elevation parallel to the axis, and proceed as just directed, as the problem would then be reduced to the instance here given, or he must find the traces of any four generatrices, which divide the surface into four equal parts, and of the axis, and get the ellipse by using the circumscribing rhomboid, whose sides would then be parallel to conjugate diameters passing through these traces.

Problem CLXXX.

To determine the horizontal trace of a given right conical surface, whose axis is inclined to the h. p. (Plate XL., fig. 2.)

By the definitions on the first page of this chapter, the student will see that this trace must be an ellipse. It is therefore the aim of the construction to discover the two axes of the figure. The h. t. of the generatrices AC and AD are at C and D, and these two points are the extremes of the major axis required. Bisect CD in E, and draw E_1E_{11} through that point perpendicular to CD. This will be the indefinite plan of the minor axis; and the student must here carefully note that the h. t. of the axis AB is *not* the centre of the elliptical trace, as it was in the case of the cylinder; neither do the traces of those generatrices which are intermediate between AD and AC occur upon the minor axis. Produce E_1E_{11} to meet XY in E^1; then $a'E^1$ is the vertical projection of two generatrices AE^1 and AE^2 (the former in front, and the latter at the back of the conical surface), whose horizontal traces occur upon the minor axis. Then the question resolves itself into finding the distance to which these two lines spread, on either side of a vertical plane containing the axis. To determine this, conceive a section of the cone perpendicular to the axis passing through E^1 : $k'm'$ is the elevation of such a section. If, then, one half of this section, which is in true form a circle, be folded about its diameter until it is vertical, as partly shown in the arc $k'E''$, having its centre at b'', a perpendicular to $m'k'$ through E^1, and meeting this arc in E'', will give $E'E''$, the distance that the generatrix AE^1 spreads in front of the bisecting vertical plane, and therefore the length of the semi-minor axis required. This length is then transferred to EE_1 and EE_{11}, and the elliptical trace completed.

This problem has been referred to, and is required in the solutions of Problem CXXV., Chapter VII., and Problem CLXIX., Chapter

X. It will be found of great service (like that upon the cylinder) in many problems on shadows and tangent planes; and the same remark applies here, as was used about that question in regard to the case when the axis is oblique to both planes of projection. An auxiliary elevation must then be made on a plane parallel to the axis, and the trace can be obtained by the above construction. And the student must be careful to note that, as the axis of the cone does not give by its trace the centre of the ellipse, he could not here adopt the method of the circumscribing rhomboid, because conjugate diameters must pass through that centre.

Problem CLXXXI.

On a given vertical right cylinder, to determine, by its projections, a helix of a given pitch. (Plate XLI., fig. 1.)

If a line upon a right cylindrical surface cuts all the generatrices at the same angle θ, that line is called a *helix*. In the mechanical arts, it is recognised as the "screw thread," and the term "pitch" is used to indicate the distance between two consecutive winds of the same thread measured along a generatrix. When the surface and the helix are developed on a single plane, the latter becomes a straight line, as will be seen in the plate. In fact, if the right-angled triangle ARi′ were made to envelope the cylinder in such a manner that the base Ai′ should coincide with the circular base of the solid, the hypotenuse AR would be identical with the helix, and the perpendicular of the triangle would measure the "pitch."

The projection upon a plane, of the curve of a helix, is known as the "sinusoid," and can only be determined from a series of points in it. Hence, the greater the number of these obtained, the more accurate the entire performance. To project the helix, then, proceed as follows:—

Divide the circle, which is the plan of the cylinder, into any convenient number (16 in the figure) of parts, and thence obtain the vertical projections of a series of equidistant generatrices. At *a′r′* mark off the length of the desired "pitch" or "rise," and divide it also into the same number of equal parts, as at 1, 2, 3, etc. Through these points of division, take horizontal sections of the cylinder, and these sections will meet the generatrices in a certain order, thus:—*The section* through 1 meets the generatrix through *b′* in 1′; that through 2 meets the next one through c' in 2′, and so on; one half of the curve is on the front of the cylinder, and the other half on the back, these two halves being symmetrical and reversed. The development can give little trouble, after the explanation in Problem CLXXVI. The helix becomes a right line meeting all the generatrices at the angle θ.

It is very interesting, by the aid of a paper model of the triangle wound round a wooden cylinder, to illustrate to one's self the following fact:—

If the triangle be *partly* unwound, by commencing with the point A, the free portion being always flat or plane, that part of the hypotenuse not in contact with the cylindrical surface would be tangential to the curve at its point of departure. The angle A would describe, on the horizontal plane, a path which would be the involute to the plan circle (the method of construction of this curve is given in the Plane Geometry), and the plan of the free hypotenuse at any one of its sequence of positions, would be a normal of the involute.

Further, the elevation of the free part of the hypotenuse would give the tangent to the sinusoid, which is the projection of the helix. This is not shown in the drawing, as it would complicate the figure too much.

Problem CLXXXII.

To determine the projections of a helix traced upon the surface of a vertical right cone. (Plate XLI., fig. 2.)

This is a companion problem to that just described, and before proceeding to demonstrate its construction, it would be advisable to note in what respects the properties of the one curve differ from those of the other. In the first case, all its points were equidistant from the axis of the surface of revolution; but in the other, the points of the curve gradually approach the axis. The helix upon the cylinder cuts all the generatrices at the same angle; that upon the cone does not do so. When it is developed together with the conical surface, it becomes an equable spiral, gradually approaching the vertex of the sector by equal increments as measured along equidistant generatrices. Thus, if there be m of these lines conceived at equal intervals around the surface, the amount of approach between any following pair would be $\frac{1}{m}$ of the pitch.

This last property enables us to construct very readily the projections of the curve as follows:—Develope the surface of the cone as $v'Ra'$ (Plate XLI., fig. 2), and divide it into any number of equal parts by lines through v' (16 in the figure). Divide the circle, which is the plan of the cone, into the same number of equal parts, and show the plans, and thence the elevations of a series of generatrices of the solid. At $a'r'$ set off the desired pitch,* and divide it into the same number of equal parts, and take horizontal sections of the cone through each of the points of division. Then proceed as in the last problem to mark points in the elevation of the curve, by noting where the first section meets the generatrix through b', and the second through c', and so on. This gives points 1′, 2′, 3′, 4′, etc. The plans of these being projected on the plans of the lines on the cone, the equable spiral 16_1, 12_1, 8_1, can be drawn, which is the horizontal projection of the helix. To develope the curve, take v' as centre and through each of the points of division between a' and r', draw arcs to meet the lines which represent the developed generatrices. Thus, as point 8′ occurs on the generatrix through i', the arc through 8″ must be arrested at its intersection with v'I. The drawing, it is hoped, will give any further explanation required.

EXERCISES.

1. A vessel is in the form of a hollow cone, 10 feet long, and having a base 4 feet in diameter, material 2″ thick. It is tilted, so that its axis is inclined 60°, and is partly filled with liquid, which reaches to a point one-third of its length from the base, measured along the axis. Draw its plan, and that of the watermark upon the inside of the vessel. Scale $\frac{1}{24}$.

2. Draw the projections of a tetrahedron of 3″ edge, with one face in the h. p., one edge of that face being perpendicular to the vertical plane of projection. Determine the true form of a section, made by any plane, inclined to both planes of projection.

3. A cone 4″ long, base 3″ radius, lies with its side upon the horizontal plane, so that its axis is parallel to the v. p., and 3″ from it. The h. t. of a plane of section bisects the line of contact with the h. p., and meets the v. p. at an angle of 65°; the v. t. makes 30° with XY, both traces tending towards the base. Give the projections and true form of the section.

* Frequently this problem is set in such a way that the helix may make a certain number of turns in reaching the apex; if so, the whole line $v'a'$ must be divided into that number of equal parts to obtain the pitch.

HINT.—Take a series of vertical planes, each passing through the apex of the cone. These planes will contain generatrices of the surface, and their intersections with the given plane will pass through these generatrices, giving points in the desired section.

4. An ellipsoidal surface is generated by the revolution of an ellipse, 3″ by 2·5″ about its shorter diameter, which is vertical. A cutting plane has its h. t. tangential to the plan, and meeting XY at 45°, whilst the v. t. meets the same line at 30°. Draw plan and elevation, and true form of the section. The centre of the plan is 2″ from XY.

5. Arrange a cone, standing on a vertical cylinder, with their axis in the same straight line and vertical. Dimensions of base and height in both cases to be 2″ diameter and 2·5″ respectively. Make the h. t. of a plane of section to meet XY at 67°, and to be ·5″ from the plan, and arrange the v. t. to pass through the elevation of the apex of the cone. Find the true shape of the section of the compound solid. (The centre of the plan should be 1·5″ from the v. p.)

6. A cone has its axis horizontal, but inclined to the v. p. at 70°. (Dimensions: base, 3″ diameter; height, 3″.) Required its trace upon the v. p. of projection, when the apex is 3·5″ from the v. p.

7. A spiral spring, axis vertical, is of the form of a square screw thread; size of square, ½″; external diameter on the plan, 3″; pitch, 2·25″. Draw the elevation of *one* complete turn of the spring.

HINT.—This is really a projection of four helices; two on the external cylinder of 3″ diameter, and two on a second cylinder, having the same axis, but of 2″ diameter. The pair on each surface are parallel curves, half an inch apart, measured along the generatrices.

CHAPTER XII.

THE PROJECTION OF SOLIDS, INSCRIBED IN, OR CIRCUMSCRIBING, THE SURFACES OF OTHER SOLIDS.

PHYSICALLY, it would of course be impossible to place one solid within another; but, geometrically, a solid is said to be inscribed within another one, when all the points of the former are contained by the surface of the latter. In this sense a tetrahedron, or a cube, can be inscribed in a sphere, an octahedron in a cube, etc., etc. A surface of revolution, such as a sphere, would be said to be inscribed in a cube when all the planes of the faces of the latter were tangential to the surface of the former.

The five regular solids, viz., the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, can all be inscribed in the sphere, as the angular points in each are equidistant from a fixed point within, which, therefore, coincides with the centre of the spherical surface. A certain relation exists between the diameter of the sphere and the length of the edge of any inscribed regular polyhedron. By a few simple exercises on their trigonometrical properties, the following ratios and relations can be demonstrated:—

1. The diameter of the sphere : the edge of the inscribed tetrahedron $:: \sqrt{3} : \sqrt{2}$.

2. The diameter of the sphere : the edge of the inscribed cube $:: \sqrt{3} : 1$.

3. The diameter of the sphere : the edge of the inscribed octahedron $:: \sqrt{2} : 1$.

4. It follows from 1 and 2, that, in the same sphere, the edge of the inscribed tetrahedron is equal to the diagonal of the face of the inscribed cube.

5. The vertical distances between two parallel and opposite faces of a dodecahedron, and of an icosahedron, inscribed in equal spheres, are the same.

6. The edge of the inscribed dodecahedron is equal in length to the major segment of the edge of the inscribed cube, when medially divided, or the two lengths are as $1 : \frac{1}{2}(\sqrt{5}-1)$.

7. The edge of the inscribed cube is equal to the diagonal of a face of the inscribed dodecahedron.

8. The octahedron has three principal plane sections passing through the centre (each containing two axes), which are squares. When the solid is inscribed in a sphere, these squares are inscribed in great circles of its surface.

But the comparative lengths mentioned above can be arrived at geometrically in a very simple manner, by constructing diagrams similar to those shown in figs. 2, 3, and 4, Plate XLII., thus :—

Take any line, AB (fig. 2), equal in length to the diameter of the circumscribing sphere, and upon it construct a semicircle. Divide AB into three equal parts in C and D, and into two equal parts at E, and raise perpendiculars to it through D and E to meet the semicircle in F and G. Join AF, BF, and AG.

Then if AB be the diameter of a sphere, AF gives the length of the edge of the inscribed tetrahedron, BF of the inscribed cube, and AG of the inscribed octahedron; and it can be proved that these lengths are numerically in the ratios shown in the paragraphs numbered 1, 2, 3, of this chapter.

Again, turning to figure 4, AB representing as before the diameter of the sphere, at A raise AC, a perpendicular to AB, and equal to it. Join C to E (the centre of the semicircle), to meet the arc in D. Join AD. Then AD is the length of the edge of the inscribed icosahedron.

To obtain the edge of the inscribed dodecahedron (fig. 3), divide B^1F^1, the length of the edge of the inscribed cube, medially (Problem VI., Plane Geometry) in H. Then the greater segment, B'H, is what is desired.

Bearing these facts in mind, we proceed to investigate the constructions of a few problems with which they are intimately connected.

Problem CLXXXIII.

Given the projections of a tetrahedron, to determine those of the inscribed and circumscribing spheres. (Plate XLII., fig. 1.)

The centre of the *inscribed* sphere is a point equidistant from each of the faces of the tetrahedron, whilst that of the circumscribing one is equidistant from all its angular points. By arranging the tetrahedron, so that its base is horizontal, and one of its other faces is per-

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pendicular to the v. p., as illustrated in the plate, the construction is much facilitated.

Taking the case of the inscribed sphere first, it is clear the centre must be on the vertical axis of the solid $d''d'$, as that line is a locus of all points equidistant from the three faces meeting in d''. Then it is also evident that the centre must fall upon any plane which bisects the dihedral angle between two faces. Such a plane is shown by its traces, $v'b'h$ ($v'b'$ bisecting the angle $d''b'c'$). The intersection o' of this line with the axis $d''d'$, is the elevation of the centre of the required sphere.* The radius is determined by a perpendicular through o' to $d''b'$. The plan has its centre at d, and is of course of the same radius as the elevation. The points of contact of the sphere with the four faces of the tetrahedron are shown in the plan at d, e, f, and g; d is that which occurs upon the base ABC, whilst e is projected from e' upon the line bisecting the angle *adb*, and f and g, which are upon the same horizontal circle of the sphere represented in elevation by $e't'$, are determined by a circle through e, having d for its centre, their positions upon this circle being obtained by lines which bisect the angles *adc* and *cdb* respectively. The elevation of G coincides with that of F, being immediately in front of it.

The circumscribing sphere, *i.e.*, that which passes through the four angular points of the tetrahedron, has the same centre as the inscribed one, and as the edge CD is parallel to the v. p., the vertical great circle of the sphere which determines the elevation passes through d'' and c'; but the circle, which is the plan, does not contain a, b, and c, for those points are not upon the horizontal great circle, but have positions below it upon a smaller but parallel circle of the sphere.

Problem CLXXXIV.

Given the projections of a sphere; to determine (1) those of the inscribed tetrahedron; and (2) of the circumscribing tetrahedron. (Plate XLII., fig. 5.)

As in the given problem there is no restriction regarding the position of the required solids relative to the planes of projection, it will be most convenient to assume their axes as vertical, because, by

* The centre of the inscribed and circumscribing spheres coincides with the centre of gravity of the tetrahedron, and hence is on the axis, at a point ¼ of its *length measured* from the base. (See Treatises on Mechanics.)

doing so, it is assured that at least *one* face of each will be parallel to the h. p.

Proceeding, then, to determine the *inscribed* solid first, draw $d'd$ through o' to represent the indefinite elevation of the axis. Divide this line, $d'd$, into three equal parts, and at point 3 set out $b'c'$ perpendicular to it, to meet the circle in c'. Join $c'd'$, and, according to the construction explained at the commencement of this chapter, this line determines the length of the edge of the inscribed tetrahedron. Assume it as the elevation of one edge required, and obtain its plan *oc* through *o*. Set out *oa* and *ob* at angles of 120° with *oc*, and equal to it in length. Join *ab*, *bc*, *ac*, to complete the entire plan. The elevation of the horizontal face ABC is $b'c'$, the point b' being projected from *b*, and $b'd'$ completes the entire vertical projections of the solid. The faces BAD, ABC, being perpendicular to the v. p., are projected thereupon simply as straight lines.

The circumscribing tetrahedron can be assumed very conveniently to have its edges and faces parallel to those of the inscribed one. Hence, $f'k'$ is drawn tangential to the elevation circle, and parallel to $b'd'$, meeting the axis produced in k', and then $f'k'$ and $f'g'$ become the elevations of two faces, and the point g' is fixed by $k'g'$ being made parallel to $c'd'$, as that line is the vertical projection of an edge of the circumscribing solid parallel to CD. The plan of KG being obtained, those of the edges KF and KE can be deduced by the same construction as that adopted in the case of the inscribed solid.

The points of contact of the sphere with the four faces of this tetrahedron could be obtained in the same way as shown in fig. 1.

Problem CLXXXV.*

To complete the plan of a cube inscribed in a given sphere, when that of ONE corner, and of an edge containing that point are known. (Plate XLIII., fig. 1.)

In the figure shown, the circle EFZ, having *o* for its centre, is the plan of the given sphere, *b*, of one point on its surface, and *bt* of the indefinite edge of the required inscribed cube.

1. A vertical section of the sphere is taken through *bt*. In true

* It is advisable to defer the study of this problem and the next until the student has read the chapter upon Horizontal Projections, but they are inserted here to complete a sequence.

shape this section would be a small circle of that sphere, whilst the edge of the cube, which is given in direction, would be a chord of this circle. On PQ a semicircle is described, which represents one half of the above small circle, constructed into a h. p., and B′ (obtained by a perpendicular to PQ through *b*) gives the position upon this circle of the known corner B. In fig. 3, the length of the edge of the cube is deduced at BC, as explained at the commencement of the chapter. This length, when transferred from B′ to C′ (fig. 1), decides the position upon the small circle of the second point C of the given edge BC; and *c*, its plan, completely determines *one* line of the required projection of the cube.

2. The plane of the face contiguous to BC, and containing B, is perpendicular to that line. Hence, *v′f* drawn through B′, perpendicular to B′C′, is the v. t. of that plane upon the plane of section, and *fh* is the h. t. This plane also cuts the sphere in a small circle passing through three known points, B, E, and F. If, then, the point B be brought into the h. p., by rotating the plane *v′f′h* about its h. t., and a circle be determined passing through B, E, and F, that circle (having *w* for its centre) will be the one which circumscribes the square face perpendicular to the edge BC.

3. The square BAGM is inscribed in this circle, and then its plan is determined when brought into its correct position upon the plane *v′fh*. This gives *bagm;* and each of the other edges of the cube being parallel to one of the three proceeding from *b*, its projection can be finished without further determination of the planes of the faces.

The student will notice that the corners of the cube in plan are indexed;* and further, that no separate elevation is given. If he will defer the consideration of these points until he has read the chapter upon Horizontal Projection, he will then understand the object of the figures, and why an elevation is not required.

Problem CLXXXVI.

To complete the plan of a tetrahedron inscribed in a given sphere, when that of one angular point, and of an edge containing that point, are known. (Plate XLIII., fig. 2).

In the plate, the circle EFT is the plan of the given sphere; *a*, of the given angular point, and *as* of one of its edges, indefinitely.

* The unit assumed is ·1″, and the centre of the sphere is supposed to be contained by the horizontal plane, to which the indices refer.

The principle upon which the construction is based is as follows :—

1. The edge AC, whose indefinite plan is known, is made finite, by obtaining the real length of that edge, and projecting the unknown point *c* from C′ in exactly the same manner as that adopted in the case of the cube.

2. The axis being known to pass through the centre of the sphere, a plane is determined through C perpendicular to that axis. This must be the plane of the face opposite to the given angular point A, *i.e.*, regarding A as the vertex of the tetrahedron, this plane would contain the base BCD.

3. The small circle in which this plane intersects the circumscribing sphere is constructed into the h. p., and the equilateral triangle representing the base BCD is inscribed within it.

4. The plan of this face is obtained when brought into the plane determined, and the whole projection is completed by joining its several points to *a*. The following explains the detailed steps of the solution :—

The length of the edge is determined at AC, in fig. 3. The indefinite plan, *a.s.*, cuts the contour circle in E and F. On EF a semicircle is described, and the position of point A′ is obtained by a perpendicular to EF through *a*. Then the chord A′C′ is made equal to AC, and *c* is projected on *ac* from C′. Through *o* and *a*, the line *ao* is drawn as the plan of the axis, and this line is taken as the intersecting line of a plane of elevation, so that the original contour circle in plan becomes also the contour circle in elevation. Then as the point A is on this circle, and its height above the plane of the centre is shown at *a*A′, that height is transferred to *a*′, and the line through *a*′ and *o*, becomes the indefinite elevation of the axis AO.

The elevation of point C, upon the assumed v. p., is at *c*′, determined thus :—Through *c*, a projector *c*, *c*″, is drawn perpendicular to *xy*, and *c*″*c*′ is made equal to *c*C′, because that distance represents the height of point C above the h. p. through the centre. Then, if *v*′*f* be drawn through *i*, perpendicular to *a*′*o*, and *fh* be made perpendicular to *xy*, the plane represented by its traces, *v*′*fh*, will be that of the base BCD, because it passes through one of the angles of that base, C, and is perpendicular to the axis. This plane cuts the sphere in a circle, of which PQ is a chord. If, then, point C be constructed into the h. p. by rotating the plane *v*′*fh* upon its h. t., a third point in the circle will be obtained, and the whole can be described passing through PQ and C. In this circle, the equilateral triangle CBD is inscribed, and its plan obtained when folded back into the plane *v'fh*, giving the horizontal projections of points B, C, and D, which are then joined severally to *a* to complete the required plan.

Problem CLXXXVII.

Given the projections of an octahedron, to determine those of the inscribed and circumscribing spheres. (Plate XLIV., fig. 1.)

In the figure, the given octahedron has its axis vertical, and it follows from this that the square ABCD, which contains the other two axes, is horizontal. Again, this square is inscribed in a great circle of the circumscribing sphere; hence, *e* being taken as centre, and *ea* as radius, the circle GH becomes the horizontal projection of that sphere. The elevation G'H' has its centre at *o*, and passes through the two extremities of the vertical projection of the axis *e'f'*.

To obtain the inscribed sphere, draw *e*M perpendicular to *ab*, to represent the trace of a vertical plane bisecting the face AEB. At *e* draw *e*N perpendicular to *e*M, and join MN. Then MN will be an elevation of the face AEB on a v. p. perpendicular to it. At *e* set out *e*G″ perpendicular to MN, and describe a circle, with *e* as centre, and passing through G″. This circle will be the plan of the inscribed sphere, for *e*G″ gives the length of a line perpendicular to the face AEB, through the centre of the octahedron. There are eight points of contact, the projections of which are shown in the figure. The axis being vertical, the upper four of these are immediately over the lower four. The plan *g* is projected from G″ upon *e*N, and *h*, *i*, and *k* occur upon a circle through *g*, and also upon lines bisecting the angles formed at *e*. The elevations are obtained by drawing the two horizontal lines, which represent the vertical projections of the small circles of the sphere, of which *g*, *h*, *i*, *k* is the horizontal projection, and determining upon them the points *g'*, *g″*, *h'*, *h″*, etc., from the corresponding plans.

Problem CLXXXVIII. (Honours Exam. Paper, 1870.)

The centre *o* of a sphere of 1·5″ radius is 1·75″ above the paper. A point *a* on the surface, and 2·7″ high, is the plan of one corner A of an octahedron inscribed in the sphere. The edge AB is inclined at 45°, and its plan *ab* passes through *a* and is 1·1″ from O. To complete the projections of the inscribed solid. (Plate XLIV., fig. 2.)

First draw the circles to represent the projections of the sphere.

Next, find a point a' on the elevation 2·7″ above XY, and assume it as that of the given corner of the octahedron, and determine its plan a on a line through O, parallel to XY. Next, draw a line through a in plan, tangential to a circle, centre O and radius 1·1″, to represent the indefinite plan of the given edge AB. To fix point b, consider the line just obtained as the intersecting line of a new v. p., and make an elevation on it (A′B′) of the edge whose inclination (45°) is given, its length being that of the edge of an octahedron inscribed in the given sphere (see fig. 1, Plate XLII.), and project b from B′. Through a draw ao, and produce it, making $of = oa$, and consider af as the plan of an axis of the octahedron. Its elevation is $a'f'$. The plane containing its other four angular points is perpendicular to this axis, and is represented by its traces $r'st$, $r's$ being drawn through o', perpendicular to $a'f'$. The point whose plan is b is contained by this plane; and as by the arrangement the plane is perpendicular to the v. p., the elevation b' is upon its v. t. This plane should then be folded about its h. t. into the horizontal plane, by constructing two of its points, o' and b'. The circle which contains the four points of the square, of which B is one, should then be drawn, and the square inscribed in it. When folded back into its place on the plane $v'fh$, the projections can be completed.

Problem CLXXXIX.

Given the projections of an irregular four-faced pyramid, to determine those of the circumscribing sphere. (Plate XLV., fig. 1.)

This problem is really identical with that solved in Chapter VII., p. 162, where the construction was described to determine a sphere passing through four points given by their projections, and that method of solution would be preferable in most cases to any other; but the pyramid shown in Plate XLV. has one of its faces, ABC, upon the h. p.; and that being so, another and shorter solution can be adopted.

First, find the circle upon the plan which passes through a, b, and c. Its centre is o. This circle is necessarily upon the surface of the required sphere, and its plane is perpendicular to the line joining o to the centre of that sphere. Hence o is the plan of the centre, and its elevation must be contained by a perpendicular ($s't'$) to XY through o. Again, conceive a v. p. parallel to the original v. p. passing through the vertex of the pyramid. Such a plane would cut the sphere when determined in a vertical circle. The horizontal trace of a plane answering this description is shown at EF; and because this plane meets the horizontal circle through *a*, *b*, and *c* in E and F, those two points must be contained by the vertical circular section spoken of, and their elevations E′ and F′, when joined to *d*′, give E′*d*′ and F′*d*′, two of its chords. By bisecting either of these at right angles as at HO′, the elevation O′ of the centre is discovered, for that point is the vertical projection of a line perpendicular to the plane of one circle of the sphere and passing through its centre. The real distance of the point O from either A, B, C, or D has next to be determined. This is shown at OA, where *o* is joined to *a*, and *a*A perpendicular to *oa* is made equal to the difference in heights of O and A, and *o*A is used as the radius of the required projections. The chord F′*d*′ is also bisected and GO′ drawn, to show the student that either of these would serve the same purpose.

Problem CXC.

Given the projections of an irregular four-faced pyramid, to determine those of the inscribed sphere. (Plate XLV., fig. 2.)

The plane which bisects the dihedral angle between two of the faces must contain the centre of the required sphere, as such a plane is a locus of all points equidistant from the planes forming the dihedral angle. If, then, three planes be found, each of which bisects the angle between two contiguous faces, the common intersection of these three planes must be the centre of the required sphere. By arranging the pyramid so that one of its faces is on the h. p. and another is perpendicular to the v. p., the solution is facilitated.

In Plate XLV., fig. 2, an irregular pyramid ABCD is given by its projections, and is so placed that the face ABC is upon the h. p., and ADB is perpendicular to the v. p. Hence the two lines *d*′*a*′ and *a*′*c*′ measure the dihedral angle between those two faces, and *v*′*a*′ is the v. t. of a plane bisecting that angle. The centre of the required sphere must therefore be contained by this plane. At *x*′*y*′ a new plane of elevation is assumed perpendicular to the face BDC, and *s*′*c* is the vertical trace of the plane of that face, and *t*′*c* of the plane bisecting the angle between that face and the base. By a similar proceeding at *x*″*y*″, the traces of the planes of the face ADC and the corresponding bisecting plane are determined. The intersection of these three bisecting planes gives the centre of the inscribed sphere required. This intersection is found by determining the plans of level lines at the same height, one upon each plane, in the same way as was described at Chapter IX., Problem CLVI., *kl* being the horizontal projection of one of these upon plane *v'a'*, while *mn* and *gh* are the plans of those upon the two other planes *t'c* and *f'a*. Then *n* being a point in the intersection of *v'a* with *t'c*, the line *bo* drawn through *b* and *n* determines in plan the complete intersection of those planes. Similarly *ao* is the plan of the intersection of the planes bisecting two other faces, and *o* is therefore the horizontal projection of the centre of the sphere required. The elevation *o'* upon *v'a'* and *o'p'*, perpendicular to *a'd'*, discovers the length of the radius, enabling us to complete the plan and elevation. The points of contact of the three sloping faces of the pyramid with the sphere are shown at *p'.p*, *r'.r*, and *q'.q*. Taking one, for example, from the centre of the sphere in plan, *ow* is drawn perpendicular to the edge *bc*, which is the h. t. of the face BCD. This is the plan of a line perpendicular to that face through *o*, and its elevation *o''*Q, upon the auxiliary v. p. *x'y'*, gives Q, the intersection of that line with the face. Its plan (*q*) is projected from Q, and the elevation *q'* is obtained from this plan.

Problem CXCI.

Given the projections of a sphere, also those of the axis of a circumscribing cylinder; to complete the projections of the latter, and to determine the circle of contact of the two surfaces. (Plate XLVI., fig. 1.)

As the generatrices of the cylinder are at all points equidistant from the axis, the line of contact of the two surfaces is a great circle of the sphere whose plane is perpendicular to the axis; hence the centre of the circle coincides with that of the sphere.

Under the conditions arranged in Plate XLVI., fig. 1, the axis of the required cylinder is oblique to both planes of projection, so, to simplify the construction, a new elevation is made upon a v. p. containing the axis. The circle having C' for its centre is the new projection of the sphere, and C'*t* that of the axis, *t* being its h. t. Under this aspect, therefore, the axis and the generatrices of the cylinder being parallel to the plane of projection, and the plane of contact being perpendicular to these, its v. t. will be at right angles to C'*t*. Hence through C' a line (*r'z*) is drawn to represent this v. t., and *r'p'*, part of this line, is the elevation of the circle of contact, whilst *r'b* and *p'a*, taken parallel to C'*t*, are the projections of the uppermost and lowermost generatrices of the cylinder; *b* and *a* being the extremities of the major axis of the ellipse, which is its horizontal trace. The plan of the circle of contact is the ellipse *opqr*. The minor axis *pr* is projected from *p'* and *r'*, and *oq*, bisecting *pr*, is equal to the diameter of the sphere, being a horizontal line. The plan of the circumscribing cylinder is completed by parallels to *ct* through *o* and *q*, and the elliptical horizontal trace has its minor axis equal to the diameter of the cylinder, as described in a previous chapter. The elevation can be obtained by a reverse construction to that just described,* *i.e.*, the elevation may be treated as a plan. This needs no further description, the construction lines being shown on the plate.

The traces of the plane of contact are obtained thus:—Through *z* (the h. t. of *p'r'*) the h. t. of the plane is drawn, for PR is a line in the plane, and the v. t. passes through *w*, both lines being of course perpendicular to the projections of the axis.

This construction is required when the shadow of a sphere is to be projected, the rays of light being parallel. That is why certain parts are shaded, the explanation of which will be found in a coming chapter.

Problem CXCII.

Given the projections of a sphere, also those of the vertex and axis of a cone enveloping the sphere; to determine the projections of the circle of contact. (Plate XLVI., fig. 2.)

This is a companion problem to the preceding, and its solution is based upon principles similar to those described in that case, except in one point. The necessary modification arises from the fact that the generatrices of a cone are not at all points equidistant from the axis; hence the circle of contact is not a *great* circle of the sphere, and its plane does not therefore contain the centre of the solid.

In the auxiliary elevation R'P' represents the line of contact, and *r* and *p*, the extremities of the minor axis of the ellipse in plan, are projected from it as before. The major axis *oq* is then drawn indefinitely at right angles to and bisecting *pr*, and is made equal to *p'r'* (which gives the length of the diameter of the circle), for it is a horizontal

* The elevation of the circle of contact could be easily obtained by determining that of the four points OPQR (their height, above the h. p. being known from the auxiliary elevation), and passing an ellipse through them, as they are at extremities of conjugate diameters.

line, and is therefore projected full length on the h. p. The student will see that the two points *o* and *q* do not fall on the contour circle which is the plan of the sphere, but that the ellipse is in contact with that circle at *s* and s_1. These two points may be determined by drawing upon the auxiliary elevation a line through C′ parallel to the plan of the axis of the cone. This line is the elevation of the contour circle in plan, and *s′*, its intersection with *p′r′*, is the vertical projection of the two points where that circle meets the circle of contact; then *s* and s_1 are projected from it. It should be noticed that the plans of the extreme generatrices of the cone meet the circle of contact in the same points *s* and s_1.

The ellipse, which is the h. t. of the cone, is shown in the plate; but as the construction necessary for its determination was given in Problem CLXXX., Chapter XI., it is not repeated here. The elevation of the circle of contact upon the original v. p. could, of course, be obtained in the same manner as the plan; but in the drawing four of its points have been determined at the extremities of conjugate diameters, viz., *o′*, *p′*, *q′*, and *r′*, and the curve has been traced as inscribed within the parallelogram 1, 2, 3, 4.

EXERCISES.

1. Draw the projections of a cube of 3″ edge in any position at pleasure, so that neither of its edges is horizontal, and determine the inscribed and circumscribing spheres.

2. If the edge of a tetrahedron inscribed in a certain sphere be 3″ long, determine that of the cube inscribed in the same sphere.

3. Show the octahedron inscribed in a cube of 3″ edge, whose angular points are in the centres of the faces of the latter, and determine by a geometrical diagram the comparative lengths of the edges of the two solids.

4. Draw the projections of an octahedron, inscribed in a sphere of 2″ radius, when one of its axes is inclined 30° to the h. p. of projection.

5. Show the projections of an octahedron circumscribing a sphere of 2″ radius, in such a manner that one face of the circumscribing solid is vertical and inclined 30° to the v. p. of projection.

6. Show how to obtain the projections of a tetrahedron circumscribing a given sphere, so that one of its edges is parallel to a given line.

HINT.—Take a section of the sphere by a plane containing the given line, and proceed, as in Problem CLXXXIV. of this chapter, to circumscribe the required solid about the given one.

CHAPTER XIII.

ON THE INTERPENETRATION OF GIVEN SOLIDS.

WHEN one solid pierces another, the surfaces give rise, by their intersection, to lines of interpenetration, which are crooked or curved, according as the surfaces forming them are plane or rounded.

Should a portion of one solid be entirely hidden within the other, *i.e.*, should the exposed part of the one consist of two completely detached pieces, then two lines of interpenetration result, as in Plate XLVII., fig. 1, where the horizontal cylinder, by passing through the vertical one, traces on the surface of the latter two closed curves, one of entrance, and a second of exit. But if the solids be so placed that one is only partially buried within the other, like the pair of cylinders shown in fig. 2 of the same plate, then only one continuous line of interpenetration is generated, which reflects back upon itself as shown.

The problems of this chapter are intended to illustrate to the student some of the most approved methods adopted for determining the projections of these lines which are more or less complicated, according to the conditions of position and form of the penetrating solids.

Generally speaking, a series of sections is taken, the characters of which are such that they are projected either as circles or straight lines; and the student will see, after perusing the coming cases, that this is settled entirely as a matter of convenience. These sections give traces upon the surfaces of the solids which meet in the required line or lines of interpenetration.

Frequently, too, advantage is taken of auxiliary projections, so arranged that they may assist in discovering points in the line required. But to give a general rule for working these problems would be impossible, as the nature of the surfaces, and their relative positions to one another, as well as to the planes of projections, can be so varied that the construction to be adopted must be that which, by its convenience of draughtsmanship, renders itself most suitable to the case under *consideration*.

Problem CXCIII.

To determine the projections of the line of interpenetration of two given right cylinders, the axis of one of them to be perpendicular to a plane of projection. (Plate XLVII., figs. 1 and 2.)

Two cases of this problem are illustrated, in the first of which the one cylinder completely pierces the other, giving a line of entrance and another of exit; whilst in the second, it is only partially embedded, and a single line of interpenetration therefore results. Further, in the first case, the axes of the two solids meet, and the consequence of this is, that the curves of penetration are bisymmetrical on either side of a vertical plane containing those axes; and if the elevation plane were taken parallel to this one, the portions of the curves which occur upon the front of the vertical cylinder would cover those on the rear, and the elevation of the two halves would coincide.

The construction adopted for obtaining points in the interpenetration is the same in both instances. In fig. 1, vertical sections of the horizontal cylinder are taken parallel to its axis. The section planes are so arranged that they trace upon the surface a succession of lines or stripes at equal distances around its circumference*, and the positions of these in the plan and elevation are determined by folding one half of the circle of the base into a plane parallel to the respective plane of projection. Thus, a semicircle on 5′.13′, represents half the base folded about its diameter into a vertical plane. This semicircle being divided into any number of equal parts (eight in the figures), the points 1′, 2′, 3′, ... 16′ on the ellipse, are projected from the divisions on the semicircle. A similar proceeding in plan obtains the corresponding projections, 1, 2, 3, . . . 16. Lines then drawn through each of these points in plan and elevation, parallel to the projections of the axes, determine the series of stripes which are really the traces on the surface of the cylinder of the assumed planes of section previously mentioned.

In numbering these, it is most important to notice that the uppermost and lowermost traces in elevation are projected as the centre one in plan, and *vice versa.*

* Of course, the foremost and rearmost lines would be traced on the cylinder, not by *cutting* planes, but by vertical tangent planes, of which these are the lines of contact.

As the other cylinder is vertical, all points on its surface are projected horizontally upon the circle, which is its entire plan; hence the arcs *eo* and *sw* (fig. 1) are the projections of the two curves of interpenetration, whilst *d*, *c*, *b*, *a*, etc., are those of an intermediate series of points in them.

The elevations *a′*, *b′*, *c′*, *d′*, etc., are obtained from the plans upon the vertical projections of the lines upon which they occur.

From each point in the plan, except *e* and *o*, *two* points in the elevation are deduced, because the stripe which passes through a point on the upper surface is, by the arrangement of the divisions, vertically over one in the lower surface. Thus, from *c* we get *c′* and *g′*, the former on the stripe through 7′, and the latter on that through 11′. The two kidney-shaped curves are then traced through the points found. To determine which part of these lines should be dotted, consider that all that occurs upon the vertical cylinder behind T_1 and T_2, and also all that occurs upon the horizontal one behind its uppermost and lowermost stripes, represented in plan by the line *ua*, would be hidden. Hence, on the left-hand curve, all is seen in elevation that is represented in plan between *u* and *w*; and on the right-hand one between T_2 and *o*.

The points of contact of the curves with the vertical contour lines in elevation are obtained thus:—A section, parallel to the original v. p., through *v* is taken, and through the points T_1 and T_2, where its trace meets the circle, two extra stripes, T_1t_1 and T_2t, are drawn. The elevations of these pass through K′ and K″ (M′K′ and M′K″ being made equal to K*t*), and intersect the vertical contour lines of the cylinder in *t′* and *t″*.

The same construction is adopted in the case shown in fig. 2, where the curve is single, and the only point requiring further explanation is that regarding the determination of A′ and A″, where two loops of the curve reach their highest and lowest points. A plane, tangential to the vertical cylinder, and parallel to the axis of the oblique one, is shown by its trace AB on the h. p. This plane cuts the latter cylinder in two lines or stripes, whose elevations pass through B′ and B″ (C′B′ and C″B″ being equal to CB). Then through the point of contact A, in plan, a projector obtains A′ and A″ upon these elevations. From considerations, described in the previous case, it is to be inferred that in elevation the portions of the curve between *k′* and *e′* and between *k″* and *e″* are seen, and the remainder hidden, and therefore dotted.

Problem CXCIV.

To determine the projections of the curve of interpenetration, when a cone pierces a vertical right cylinder. (Plate XLVIII.)

The construction used to solve this problem is analogous to that described in the preceding one, *i.e.*, the surface of one solid (the cone) is divided into an equal number of parts by stripes or lines, which are the traces upon it of a series of cutting planes. These assumed planes are vertical, and all pass through the apex. The position of the points 1′ 1, 2′ 2, 3′ 3, etc., on the base being obtained by the same construction as described in the case of the cylinder, the projections of the lines or stripes are completed by joining to those of the apex; or if, as in the plate, only a frustum of the cone be used, the other base is divided in the same way as the first, and the lines are drawn through the corresponding points of division. The remainder of the construction for obtaining the curve needs no further explanation. The points *t*′ and *t*″, where the vertical contour lines of the cylinder are tangential to the line of interpenetration, are determined in the same way as in the case of the cylinder.

If the cone had only partially embedded itself in the cylinder, the curve would have been single, and the points corresponding to A′ and A″, in Plate XLVII., fig. 2, would be found in the same way as shown in that drawing, except that the vertical tangent plane to the cylinder would have to pass through the apex of the cone. The student should set himself cases of this kind, varying the positions of the two solids.

Problem CXCV.

To determine the projections of the line of interpenetration of a cone with axis vertical, and a cylinder with axis horizontal. (Plate XLIX.)

The principle upon which the construction used for this case of penetration depends, may be described as follows:—The solids are each cut by a series of horizontal planes, the cone in a number of circles, decreasing in diameter towards the apex, and the cylinder in lines parallel to its axis. Taking a single section plane, for example, the plan of its traces on the surfaces of the two solids would be a circle and a rectangle; and if that section plane passed through the line of interpenetration, the points where the sides of the rectangle met the circle would evidently be in that line.

In the figure, A is the plan of the axis of the cone, which is therefore the centre of all the circles which are the horizontal projections of the assumed sections of it.

First, the surface of the cylinder is divided, as in the previous cases, into equal parts by horizontal lines. Thus, the line passing through 15′, 11′ in the elevation, is the vertical projection of the rectangle traced upon the surface by a horizontal cutting plane at that height. This same plane cuts the cone in a circle, having PQ for its radius, and this circle is shown on the plan in full. Hence, it is readily seen that four points in the plan of the line of penetration are discovered in *c*, *g*, *i*, and *n*, where the sides of the rectangle meet the circle. The elevations *c*′, *g*′, *i*′, and *n*′, are projected from these upon the lines through 15, 11.

All the desired points could be found in this way except *v*′*v*, and *h*′*h*; and the student will see that it is highly important to determine these exactly, as they are the extremes in the loops of the curve. To obtain them, a section of the solids is made by a vertical cutting plane passing through the axis of the cone at right angles to that of the cylinder. This section is then folded about its h. t. into the h. p. In the figure, AA″B is half the section of the cone, and the circle having T′ for centre is that of the cylinder. Then V and H are the points on the section whose projections on the original drawing are required; *v* and *h* are determined on AB, and *v*′ and *h*′ from these, their heights above the h. p. being obtained from the constructed section. The dotted lines passing through the projections *v*′ and *h*′, in the figure, are the elevations of the tangent lines on the cylinder to the curve of penetration at those points.

The points of contact of the contour lines are not determined in the drawing for fear of complication; but the principle for obtaining them has been described in previous cases.

Problem CXCVI.

To determine the projections of the line of interpenetration of a given sphere with a cone whose axis is vertical. (Plate L., fig. 1.)

Horizontal sections of this group would trace circles on the surfaces *of both* solids, the meeting points of which would be in the interpenetration line. As the plans of these would also be circles, that kind of section is best adapted for the solution required.

To discover the two levels which embrace the portions of the solids engaged in the interpenetration, an auxiliary elevation is taken on an assumed v. p., containing the centre of the sphere and also the axis of the cone. This is shown in Plate L., fig. 1, below the plan, where ST, which passes through *v* and *w*, is taken as the intersection line of the new plane of projection, and *v*TV represents half the elevation of the cone on that plane, whilst the circle having w'' for its centre represents the sphere. Then it is readily seen that the portion of the two solids concerned in the penetration lies between points K′ and A′, and from the heights of these two above ST, the levels between which the horizontal sections must be taken can be inferred. The plans *a* and *k* of the highest and lowest points in the curve are projected at once from A′ and K′.

Upon the original elevation a series of horizontal section lines, numbered 1, 2, 3, etc., represent cutting planes which have been assumed between the two respective heights just found, and the plans of these sections are circles in pairs, having *w* and *v* for their centres. Thus, taking that which is numbered 2 for an example, the circle having *w* for its centre, and a radius equal to 2,*x*, is the plan of the section of the sphere, and the circle having *v* for centre, and radius equal to $y'z'$, is the plan of that of the cone. These two circles meet in *b* and *s*, thus giving two points in the horizontal projection of the curve of interpenetration. The other points are found similarly, and the elevations are projected from the plans upon the corresponding horizontal lines representing the traces of the different section planes. The points of contact of the projections of the curve with the contour lines in plan and elevation can be found as before, by sections parallel to the planes of projections taken through the centres of the solids. Again, this is not shown in the plate, to prevent crowding of lines.

Problem CXCVII.

To determine the projections of the curve of interpenetration of a given right cone, having its axis vertical with a given oblique cone, whose horizontal trace is a circle. (Plate L., fig. 2.)

This is given as another good instance of the principle involved in using horizontal sections of two solids to obtain points in their line of interpenetration. In Plate L., fig. 2, the projections of two cones are shown which mutually penetrate. That having the vertical axis is a right cone, and the circle AB, centre *w*, is its plan. The other, having V.12. for its axis, is supposed to be generated by a line revolving around *v*.12 in such a way that one point in it is fixed at *v*, whilst another point is continually contained, by the circle ST (centre 12), on the h. p. All sections of the latter solid parallel to the h. p. will be in true form, therefore circles, as will the plans also; but the centres will not coincide, but be found at intermediate points along the plan of the axis. Its elevation *v*′12′ is intersected by a series of horizontal lines numbered from 1′ to 12′. These are the traces of the assumed cutting planes. The centres of the several circles which are the horizontal projections of these sections of the oblique cone are at 1, 2, 3, etc.......12 on *v*.12 the plan of the axis. The remainder of the construction is similar to that adopted in the last problem.

The student will find, in drawing the figure, that the intersecting pairs of curves, by which the points in the plan of the interpenetration line are determined, make angles which gradually become more acute, until in two places they are absolutely tangential. This occurs at *a* and *l*, which are the horizontal projections of the highest and lowest points in the interpenetration. To ensure obtaining these accurately, a line should in each case be drawn through the centres from which the pairs of circles are struck. Thus a line through 11 and *w* gives *l*, whilst another through 1 and *w* gives *a*.

Problem CXCVIII.

To determine the interpenetration of a given vertical prism with a given sphere. (Plate LI., fig. 1.)

This problem is intended to illustrate the interpenetration of plane with curved surfaces. The principles adopted are the same as described in the two preceding problems. Horizontal sections of the group are taken, which are circles on the sphere. The horizontal projections of these are seen in the plan to penetrate the faces of the prism at certain points, from which the elevations can be projected. It is best to draw the circles *on the plan* at starting, to ensure that certain sections are assumed passing through the main points of each curve. Thus each angle of the polygon in plan should be contained by one circle, so that the elevation of the points, where the edges of the prism meet the sphere, may be determined. In the *figure*, the circle passing through *a* (partly shown) is the plan of two horizontal sections, one above and one below the horizontal great circle of the sphere, and its radius is *o*.1. From 1—1′ and 1″ are projected on the contour circle in elevation, and horizontal lines through these two points determine the vertical projections of the two sections. Then *f*′ and *f*″ are the elevations of the two points where the vertical edge *a*′*a*″ meets the sphere.

Through *g*, *h*, and *k*, three points on the plan of the face AB of the prism, other circles are drawn, and their elevations obtained in the same way, thus determining three more points in each curve of interpenetration on that face. Further, as it is most desirable to know the uppermost and lowermost point of each loop before drawing it, the following construction may be adopted for that purpose:—Draw from the centre of the circle in plan a line, as *oi*, perpendicular to the plan of the face of the prism. This must be assumed as the h. t. of a vertical plane, which therefore intersects the plane of the face in a vertical line, passing through the two points whose positions are desired. With *o* as centre, *oi* as radius, draw a circle in plan* as the projection of two horizontal sections of the sphere, and determine their elevations. The vertical line previously mentioned passes through these circles in the desired points, *i.e.*, *i*′ and *i*″ are projected on them from *i*.

NOTE.—This problem may be solved by another method. As all plane sections of a sphere are circles, each face of the prism may be considered as a portion of a cutting plane, thus giving five small verticle circles on the sphere, whose projections on the v. p. would be ellipses, the entire interpenetration lines on the elevation being made up of loops, which are portions of these ellipses. For instance, if *a.b* in the plan be produced to meet the circle in *s* and *t*, then *st* becomes the horizontal projection of one of these sections, and *s*′*t*′ is the elevation of its horizontal diameter, whilst *i*′*i*″, the vertical diameter, is equal in length to *s.t*. In this way, the five ellipses having been completed, the portions of them required can be settled by inspection.

PROBLEM CXCIX.

To determine the line of interpenetration of two prisms, one of which has its axis vertical, the other horizontal. (Plate LI., fig. 2.)

When the surfaces of two penetrating solids are all plane, the line or lines in which they meet are crooked, and the angular points are generated, by the entrances and exits of the edges of each solid into and from the faces of the other.

* This circle is omitted in the figure.

In the case drawn in the figure, as one prism is assumed to be vertical, the square contains the plan of the line of interpenetration; and it is evident that, as the edges *d,d'* in the one solid, and *lm,l'm'*, in the other, are not concerned in the problem, this line is single, *i.e.*, there is not a complete hole on each side. The points 1, 3, 11, 9, 5, 7 are the plans of the entrances and exits of the edges of the horizontal prism into and from the faces of the vertical one, and their elevations can be projected at once, as at 1', 3', 11', 9', 5', and 7'.

The three vertical edges, whose plans are *a*, *b*, and *c*, penetrate certain faces of the horizontal prism; and "which" and "where" can be readily seen when a new auxiliary elevation of that solid, as E', I', L', G', is made (perpendicular to its axis).

Through *a*, *b*, and *c* draw lines parallel to the axis of the horizontal prism. These will be the plans of lines on the surfaces of this prism, passing through the vertical edges A, B, and C. Their elevations on the auxiliary drawing are at VI, VIII, II, XII, IV, and X. From this it is clear that the edge A penetrates the face EFHG at VI, and the face EFKI at VIII; and thus the heights of these points are known by their distances from *x'y'*. These heights should be transferred to the original elevation of the edge *a'a"*. The points of entrance and exit of the other two edges are determined in the same way.

There is generally some difficulty experienced by students in discovering how these several points are to be rightly joined; and sometimes, where the line of penetration is complicated, this gives much trouble. For the sake of making one case clear, and thereby helping in many other instances, the following reasoning is given, as perhaps the best method of solution:—Regard the plan, and trace the line of interpenetration around the horizontal prism, thus: Starting at 1, it runs over the face EGHF, until it meets the vertical edge through *c* at 2; it then continues to rise over the same face to 3, and afterwards to descend over the next face, GHML, till it meets the vertical edge through *b* in 4. (Notice that the point where *ki* meets *bc* is not to be used at present, but is required when tracing the curve on the *under* surfaces.) The line then returns from 4—still on the face GHML—to 5, and thence, on the face EFHG, to *a* or 6, which is the trace of the other vertical edge A'A" on that surface. It then passes on to 7, on the edge EF; whence it proceeds, upon the under face EIKF, first to meet the vertical edge A again in 8, and then to 9, upon the horizontal edge IK. From 9 it rises, on the face IKML, to 10, which is on the vertical edge B, and to 11, on the horizontal edge IK. Lastly, it proceeds, upon the lower face EFKI, first to 12, where C emerges from that face, and then to 1, the starting point.

PROBLEM CC.

To determine the projections of the line of interpenetration of a vertical pyramid with a horizontal prism. (Plate LII.)

In this case, as in the last, the line required is a crooked one, whose angular points are generated by the entrances and exits of the edges of one solid into and from the faces of the other. Having drawn the entire projections of the two solids, proceed to discover where the sloping edges of the pyramid enter and emerge from the faces of the prism, thus :—Make an auxiliary elevation of the whole on a v. p. perpendicular to the edges of the prism; then that solid will be projected as a triangle (three lines representing its faces). Hence the points where the auxiliary elevations of the edges of the pyramid meet these three lines, will be the projections of their intersections with the prism. Thus V′C′ is seen to enter the face, whose elevation is F′G′ at VI, and to emerge at I, and the plans of these points can be obtained at once at 6 and 1; their projections on the original elevation can be determined at 6′ and 1′.

Having completed this half of the work, proceed next to discover the points where the horizontal edges of the prism intersect the faces of the pyramid. This is best solved as follows :—On the auxiliary elevation draw lines from the apex to pass through the corners of the prism, which are seen to be concerned in the penetration. Thus, in the figure, these lines are V′G′ and V′E′, none being required through F′. Consider these as the projections of lines on the faces of the pyramid, and determine their plans, thus :— Taking V′S′, for example, it represents one line on the face V′D′A′, and another on the face V′B′A′; find, therefore, the plans s and s_2 from S′, on the base edges *da* and *ab* respectively, and join to *v*. Then vs and vs_2 are the horizontal projections of these two lines, and their intersections (4 and 8) with the edge *e.h.* are the projections on the h. p. of the points where that edge enters and leaves the pyramid; and thence their elevations 4′ and 8′ are obtainable. Points 2 and 10 are found in a similar manner. The joining of these angular points of the line of interpenetration can be settled by the same process of reasoning described in the last case, *i.e.*, by mentally tracing the line around the horizontal solid in plan.

In fig. 2, the development of the pyramid, and that of the interpenetration line is given, the only explanation required being the construction used to find upon that development the positions of points 4_1, 2_1, 10_1, and 8_1. Taking 4_1 and 8_1 first, they are known to be generated by the penetration of the edge EH with the faces VDA and VAB. Produce, therefore, F′E′ and G′E′ in the auxiliary view, until they meet the elevation of the edge VA, which is common to both these faces of the pyramid, in t' and r', and, by horizontals through these points, meeting the line V′A, which is the true length of that edge, in R and T, obtain their distances from the apex. If, then, these lengths be transferred to V_1A_1 in the development on each side at t and r, by joining them to 5_1 and 3_1 in the one case, and to 7_1 and 9_1 in the other, the crossing of these pairs of lines will determine the correct positions of 4_1 and 8_1. Another method is shown for obtaining 2_1 and 10_1. It is known that these points are generated by the edge GK intersecting the faces VCD and VCB; and further, that they are contained by the two lines vw_1 and vw_2, which were used to obtain their plans. Show these lines upon the development by making C_1W_1 and C_1W_2 equal to cw_1 and cw_2, and joining to V_1. Conceive, then, a horizontal line on the two faces of the pyramid passing through the two points of intersection, and determine on the true-length line the distance of this horizontal from the base edges. This is shown by a line parallel to $x'y'$ through G′, meeting V′A in W. On the development draw lines parallel to C_1B_1 and C_1D_1, meeting at a distance along V_1C_1 equal to AW′, and their intersections with V_1W_1 and V_1W_2 necessarily give the desired points 10_1 and 2_1.

Problem CCI.

To determine the lines of interpenetration of a prism, axis vertical, and a pyramid whose axis is oblique to both planes of projection. (Plate LIII.)

The construction for obtaining the projections of the pyramid is not shown in the plate, it being assumed that the student is familiar with it from previous study. The only new difficulty in the determination of the line of interpenetration is that of finding the points in which *the vertical* edge of the prism, of which a is the plan, enters the faces of the pyramid. It is clear that this edge enters VEH above and emerges from VGF below. On the plan draw the projection of a line on each of these faces parallel to its corresponding base edge, and passing through *a*; *mn*, *pq* represent these lines on the figure. Determine the elevations *m'n'* and *p'q'* by projecting from *m* and *n* upon *e'v'* and *h'v'*, and from *p* and *q* upon *v'f'* and *v'g'*, respectively. Then, because these lines MN and PQ lie upon the two faces of the pyramid, and meet the vertical edge A'A" in 1' and 4', those two points must be the intersections required. The development of the prism, and the interpenetration line thereupon, is shown in fig. 2. The rectangle, which is the development of the prism, needs no description. The angular points of the penetration line are obtained by setting out parallels to the edges, at distances from them, taken from the plan. Thus, points 3_1 and 2_1 are discovered by making $A'2_1$ and $A'3_1$ equal to *a*.2 and *a*.3 respectively. Then 3_1III and 2_1II are made equal to the heights of the points 3' and 2' above XY in the original elevation. The student will find no difficulty in this which needs any further description.

Problem CCII.

Given the plans of two or more irregular pyramids; to discover upon them the line or lines of interpenetration, by contouring them at different levels.* (Plate LIV., figs. 1 and 2.)

In Plate LIV., fig. 1, the plans of two triangular pyramids, *vabc* and *odef*, are given, which penetrate. The bases of both are supposed to be upon the horizontal plane, the heights of the apices being *v*—3" and *o*—4". If a horizontal section were made of both solids, it would trace upon their surfaces lines parallel to the edges of the bases. Such a section is shown upon the drawing, taken at a height of 2" above the h. p., and marked level 2". The position of this level is found by halving any sloping edge of the 4" pyramid, and taking two-thirds of that of the 3" one. Now, it is evident that the meeting point of any two base edges is in the line of intersection of the two sloping surfaces containing those edges; for instance, *m* is a point in the intersection of the two faces *eof* and *vac*. Further, the levels drawn upon these faces meet in a second point of the same line of intersection. Hence, as the two levels shown on the drawing, belong-

* It should be noticed that no vertical projection is used in these problems.

ing to the faces just mentioned, meet in 6, the line *mq* drawn through *m* and 6, and produced *until arrested by an edge* of one of the solids, is the plan of the intersection of those two surfaces. Similarly, *gr* is the plan of the intersection of the faces *eof* and *avb*; and *rq*, joining the two points where these lines are stopped by the edges, represents the meeting line of the face *vcb* with *eof*. The other intersection lines are found in the same way. Sometimes these levels have to be produced to intersect exteriorly to the surfaces of the solids, and then the meeting point is used to obtain the *direction* of the line, which is drawn only so far as these two limited faces of the solids are in intersection.

A more complicated case is given in fig. 2, where an irregular triangular pyramid is penetrated by two others, one regular and hexagonal, and the other regular and pentagonal. The principle adopted to obtain the intersections is the same as just described; and, as every necessary working line is shown, the student will find it a very interesting and not particularly difficult exercise to trace the solution without further help.

Problem CCIII.

To determine the interpenetration of two irregular prisms, whose edges are inclined to both planes of projection. (Plate LV.)

Previous cases have shown the student methods for obtaining the interpenetration of prisms and pyramids, where their principal lines have been parallel to one or both planes of projection. This problem is intended to illustrate the construction necessary, under the more difficult conditions of the edges being *inclined* to both these planes. The example selected is that of a four-sided and a triangular prism. The first step in the solution is to find their traces *abcd* and *efg* upon one of the co-ordinate planes. A horizontal section of both solids is then made by an assumed cutting plane at any height, and the projections are shown upon the plans. Thus the plane whose v. t. is ST cuts the four-sided prism in a figure whose plan is ABCD, the sides of this figure being, of course, parallel to the lines of the h. t. *abcd*, and the triangle EFG is the plan of the section of the other solid.

Each of the lines in which this assumed plane cuts a face is contained by that face, and the intersection of any two of these (one upon each prism) gives one point in their common intersection; *the meeting* point of the corresponding horizontal traces giving the necessary second point, by which the entire intersection can be determined.

For instance, the assumed cutting plane meets the face AIKB in AB, and the face EOPG in EG, and these two lines intersect when EG is produced in XI, which, being therefore in the planes of both faces, is a point in their intersection. Again, the two horizontal traces *eg* and *ab* meet in I, hence the line I, XI, is the indefinite plan of the intersection of the planes of those two faces, and that portion of it 6.7, intercepted between the edges which bound these faces, is one line in the plan of the interpenetration required.

By finding systematically, in this way, the line of intersection of each face of the triangular prism with each face of the other solid, the whole plan can be obtained, and from that the elevation can be deduced. When the number of faces is large, the problem is very complicated, and only by an orderly and well-arranged notation can the solution be satisfactorily accomplished. Further, it is highly advisable in such a case to tabulate the results as they are found, in some such way as follows (for the case just described, and shown in the plate), the face having *eg* for its h. t. being selected for example:—

The face EGPO meets the face ABIK in a line through XI and I, giving 6.7.

The face EGPO meets the face BCLK in a line through VII and XIII, giving 1.2.

This face EGPO is not further concerned in the interpenetration, as can be discovered from the fact that both 6.7 and 1.2 lie between the edges EO and PG.

The face FGPN meets the face BCLK in a line through IV and XIV, giving 2.3.

The face FGPN meets the face ABKI in a line through VI, giving 5.6, etc., etc.*

It frequently happens that the traces of the faces of the solids meet at remote distances; sometimes they are nearly parallel. It is in such cases advisable either to use the *vertical* traces of these faces for obtaining second points in the intersections, or, if that fail, to make two sections of the group, one parallel to each plane of projection; and having determined the projection of the unknown point on one co-ordinate plane, to use its other projection on XY in the second drawing.

* Notice that in this case advantage is taken of the fact that point 6 is already known, for the meeting point of the two horizontal traces *fg* and *ab* does not fall within the limits of the plate.

As one edge BK is free of the interpenetration, the line reflects back upon itself, and there are not two closed figures of entrance and exit. For more difficult examples of this kind of construction, the student is referred to Plate XXII. of Bradley's *Elements of Geometrical Drawing*; but the writer strongly recommends every student to attempt such a simple case as has been described, before investigating the more difficult examples there shown.

Problem CCIV.

To determine the interpenetration line of two given surfaces of revolution whose axes intersect. (Plate LVI.)

A very simple device is illustrated in Plate LVI., which can always be adopted in cases where the interpenetration of two elementary curved surfaces is required, whose axes are in one plane. The example selected includes an ellipsoid generated by the revolution of an ellipse about its minor diameter, which is assumed to be vertical, and a cone whose axis meets that diameter in V. In the first place, the elevation should be made upon a plane parallel to, or containing the two axes. It is so arranged in the plate.

A series of concentric spheres, having *v'v* for their common centre, would intersect the two given surfaces in circles perpendicular to their respective axes, the elevations of which would hence be straight lines.

Taking, for example, the arc 5.*z'*, let it be conceived as part of the vertical projection of a sphere whose centre is at V. Then 5.5′ would be the elevation of the intersection of this sphere with the cone (the points 5.5′ being determined by the arc cutting the elevation contour of the cone in those two points), and *y'z'* that of its intersection with the ellipsoid. These sections would be circles, traced upon each of the curved surfaces. They would meet in two points, one in front and one in rear, that must necessarily belong to the interpenetration line, as they are common to both surfaces. The elevation of the two points generated by the intersection of the circles selected is at *h'*. As the axis of the ellipsoid is vertical, the plans of its section-circles are also circles having their centres at *v*, their respective diameters being equal to the lines which are their projections upon the elevation. This being the case, the plans of the several points can be determined directly on those of the circles upon which they occur. Thus *h* and *i* are obtained from

h' upon the circle passing through y, whose centre is v, and whose radius is equal to $p'y'$. The elevation $s'h't'$ of the curve of interpenetration is a single line, for it is bisymmetrical on either side of the plane containing the two axes, and that part which is in front covers that which is at the rear. The points s and t are projected from s' and t' upon the line joining the two centres of the plans. To discover the points of contact g, g_1 of the plan of the curve with the contour circle of that of the ellipsoid, assume one of the sections, as $a'b'$, to pass through the centre of its axis. This section will be the greatest circle of the ellipsoid, of which the contour circle ab is the plan. Through a' describe the arc $a'4$, centre v, and draw 4,4 perpendicular to the axis of the cone, to meet $a'b'$ in g'. From this point g and g_1 can be projected on the circle ab. The student will see that the entire arcs and sections are not shown upon the figure, and that only those portions are given which are sufficient for the determination of the points of the desired curve.

Problem CCV.

To determine the interpenetration of two right cylinders, whose axes are inclined to each co-ordinate plane, and do not meet. (Plate LVII.)

The method generally adopted for solving this problem, is to assume a series of cutting planes which shall be parallel to both axes. Such planes intersect each surface in straight lines corresponding to different positions of the generatrices, and these straight lines by their intersections give points in the curve. This is certainly more convenient than taking horizontal sections of them, which would necessitate the drawing of a large number of ellipses. It is necessary, therefore, first to proceed to find the h. t. of a plane parallel to both the axes, $p'q'$, pq, and $s't'$, st. This is done by taking any point $w'w$ in one of them, and through it drawing a line $w'z'$, wz, parallel to the other. The h. t. required passes through the horizontal traces t and z of these two lines. It is not necessary to show the vertical traces of these planes, as all those which will be used to cut the two cylinders will do so in lines parallel to the respective axes, and their projections will hence be parallel to $p'q'$, pq, $s't'$, and st, so that, if one point only in each is determined, the positions of these lines can be assumed. This one point for each line is that in which the h. t. of any assumed cutting plane meets the ellipses, which are the traces on the h. p. of the two cylindrical surfaces. These ellipses must therefore be determined by the method shown in Problem CLXXIX.

It will be seen that the h. t. of the plane which we have just found, *e.g.*, that marked E, passes through these ellipses in E1, E2, E3, E4, and through these points the lines marked E1.E1, E2.E2, on one cylinder, and E3.E3, E4.E4, on the other, give the horizontal projections of the traces of this plane on the two curved surfaces. Their intersections at *e*1, *e*2, *e*3, and *e*4, are the plans of four points in the curve of interpenetration. Notice that E2E2 is on the under side of its cylinder, and so is E3.E3. This helps to settle what part of the curve, when formed, should be dotted as a hidden line. A series of h. t's, ABCDEFGH and I, are taken, which are parallel to E, the one determined, and from each of their intersections with the ellipses, other lines upon the cylinders are obtained, and thence other points on the plan of the curve. It is very important to consider the cases of those two planes, which, cutting one cylinder, are tangential to the other. These are lettered I and A in the plate. The line A_1A_2 cuts the ellipse (centre *q*) in A_1 and A_2, and touches the other ellipse (centre *t*) in A_3. The line upon the one surface, passing through A_3, gives rise, by its intersections with the two corresponding lines through A_1 and A_2, to two points on the other cylinder, and these latter lines are tangential to the curve of interpenetration at those two points. Similarly, the plane marked I gives corresponding points i_2 and i_1 on the other side. It is highly desirable to determine these exactly, as the curve is more easily and correctly drawn when the tangents at its extremes in any direction are known.

The cutting planes lettered B and F contain the lines which give the contour of one cylinder in elevation, and those lettered D and F contain the lines which give the contour of the other. It is advisable always to use these to obtain the points of the interpenetration on those lines. A similar remark applies as regards the plan.

The readiest way of obtaining the projection of the curve on the elevation is to show those of the lines traced by the cutting planes on one curved surface, and to project upon them from the plan for the corresponding points. Thus from A_3 the stripe A_3', A_3' in elevation is deduced, and a_1' and a_2' are projected upon it from a_1 and a_2 in plan.

PROBLEM CCVI.

To determine the curve of interpenetration of a cylinder and cone, the axes of both surfaces being inclined to each co-ordinate plane. (Plate LVIII.)

The constructions employed in this and the next problem are analogous to that described in the last, *i.e.*, secant planes are assumed, which cut both surfaces in straight lines, the intersections of which are points in the desired curve. The only modification is, that to ensure the sections being straight lines, the cutting planes must each pass through the apex of the cone, and be parallel to the axis of the cylinder. This will be effected if all the h. t.'s are arranged to pass through the h. t. of a line $c'e'$, ce containing the apex c'_1c of the cone and parallel to the axis $a'b'$, ab of the cylinder. The point mentioned is beyond the limits of the plate, but part of the line is shown. The remainder of the solution will be understood by reference to the drawing, which has not been lettered, to avoid want of clearness. The same remarks apply in this case as in the last about the desirability of using the two secant planes, which give the tangents to the curves, as well as those containing the contour lines. Notice that the cone completely pierces the cylinder, giving a curve of entrance and another of exit.

PROBLEM CCVII.

To determine the curve of interpenetration of two cones whose axes are each inclined to both co-ordinate planes. (Plate LIX.)

The only instruction the student will require is that the secant planes in this case must necessarily contain both apices. Hence their h. t.'s will in every instance pass through that of the line joining those apices. Again, this point is outside the limits of the sheet, but it is the line $a'e'$, ae which gives the desired position. In this case the curve is single, and reflects back upon itself. The method of obtaining the two ellipses, which are the h. t.'s of the two surfaces, is omitted in the drawing, but is described in Problem CLXXX.

EXERCISES.

NOTE.—*Those exercises marked thus * are considered most advanced.*

1. A hollow vertical pipe, external diameter 3″, thickness ½″, is pierced by a cylindrical hole 2″ diameter; the axes of the hole and the pipe intersect, and the former is inclined 40° to the v. p. Show the entire elevation. (Exam. Paper, Cooper's Hill Engineering College, 1874.)

2. Arrange an example of a tetrahedron with its base on the horizontal plane, edge 4″, and show upon its projections those of a horizontal square hole of 2″ side, whose lowest edge is ½″ above the base.

3. Show the plan of the line of interpenetration of a cylinder, whose axis is horizontal, diameter 2″, and resting upon the h. p., with a sphere of 3″ diameter, also touching the h. p., the plan of whose centre is 1·75″ from that of the axis of the cylinder. (May Exam. 1875.)

*4. Take three straight lines in space, no two being parallel, and no two intersecting, and determine a point which shall be ·75″ distant from each line. (Honours, May Exam. 1875.)

HINT.—This point is the common intersection of three cylinders, having the three given lines for their axes, their radius being ·75″. There are, of course, two solutions.

5. Draw the plans of two intersecting half cylinders of unequal diameters; the half cylinders to rest on the horizontal plane, their axes being in that plane. Draw the plans of the curves of intersection, and make a sectional elevation on a ground line, making 30°, with the plan of one axis. (Honours, May Exam. 1877.)

6. A cylinder of 2″ diameter and 4″ axis has the middle point of the latter coinciding with the centre of a sphere of 3″ diameter. The axis of the cylinder is inclined at 35°. Draw the plan of the intersection of the surfaces. (May Exam., Adv. Stage, 1871.)

*7. Arrange an example of a square pyramid, base on h. p., and a cone also on h. p., axis vertical, the plans of the apices to be ½″ apart. Dimensions of pyramid, 3″ square by 4″ high; of cone, base 2·5″ diameter, height 3″. Determine the plan of their intersection by contour lines, or "lines of level."

HINT.—More than one level will be required here besides those which are the edges on the h. p., as the intersection is made up of curved lines, the accuracy of their determination depending on the number of points found.

8. A right prism, base an equilateral triangle of 2″ sides, edges 4″ long, is in the following position:—Its long edges are horizontal, and inclined 30° to the v. p.; one face is inclined 20° to h. p. It is penetrated by a cone, axis vertical, diameter of the base 2·5″; one edge at least of the prism being free from the conical surface, and one edge at least penetrating it. Determine plan and elevation. (Royal School of Mines Exam. 1873).

9. Give the projections of a vertical right cone intersected by a cylinder, the axis of which cuts the axis of the cone 2″ from its base, and is parallel to both the h. p. and v. p. of projection. The cone to be 3″ diameter at base, the cylinder 2″ diameter. (Cooper's Hill Engineering College Exam. 1875.)

*10. Arrange two oblique four-faced pyramids, whose axes do not meet, and determine their interpenetration.

11. Determine the intersection of two right prisms, with square bases of 2·5″ side, under the following conditions:—One is placed with its lowest face inclined 28°, its long edges horizontal, and making 40° with the v. p.; the other stands on its base, with one side at 20° with *xy*, and its axis passing through the highest horizontal edge of the other. Determine the elevation of the line of interpenetration. (Royal School of Mines Exam. 1876.)

12. Arrange an example of sphere and right pyramid, intersecting under the following conditions:—

Pyramid—Axis 3″ and vertical; base an equilateral triangle of 2″ side.

Sphere—Diameter 2″; horizontal line through centre of sphere and axis of the pyramid inclined 40° to the v. p.; centre of sphere vertically over one horizontal edge of the pyramid, and 1·5″ above its base.

Show the projections of the line of interpenetration.

13. A hollow sphere 3½″ exterior diameter, ⅜″ thick, is pierced by a horizontal cylindrical hole 1¼″ in diameter. The axis of the boring cylinder is ½″ vertically above the centre of the sphere, and makes 40° with the vertical plane. Draw the plan and elevation of the sphere. (Cooper's Hill Engineering College Exam. 1876.)

14. An ellipsoidal surface is generated by the revolution of an ellipse, 4″ by 2·5″ about its longer axis, which is vertical. A right cylindrical surface, whose axis is inclined 70°, meets that of the ellipsoid at a point 3″ from the centre. The diameter of the cylinder is 3″. Draw the projections of the curve of interpenetration.

CHAPTER XIV.

ON THE THREE ELEMENTARY CURVED SURFACES IN CONTACT AND THE DETERMINATION OF PLANES TANGENTIAL TO THEM.

BEFORE proceeding to the consideration of the problems of this chapter, it is well to invite attention to the following facts, proofs of which are given in works on Theoretical Geometry:—

1. If two spheres be in contact, they touch in one point, which is in the straight line joining their centres.

2. If a sphere touches a conical surface exteriorly, it does so in a point which is contained by the plane passing through the axis of the cone and the centre of the sphere.

This is also true in the case of the sphere and cylinder.

3. If two cones touch one another, their vertices coinciding, they do so in a straight line, which is in the plane containing the two axes.

Similarly, if two cylinders touch, giving a *line* of contact, that line is in the plane of the two axes.

4. A plane is said to be tangential to a curved surface, when it contains the tangents to all the curves which can be drawn upon that surface through the point of contact.

5. When two of the elementary curved surfaces meet, they have a common tangent plane which passes through the point or line of contact.

6. The tangent plane to a sphere is perpendicular to the radius passing through the point of contact; or, in other words, the radius is a normal to the plane at that point.

7. The tangent plane to a cylindrical surface touches it in a straight line which is in the plane perpendicular to the tangent plane and the axis.

This is also true of the cone; further, all tangent planes to a conical surface must contain the vertex.

8. If a cylinder and a cone, or two cylinders, or two cones, be in contact at a *point* only, the common tangent plane at that point is perpendicular to two other planes, one containing one axis and the

point of contact, and the other containing the other axis and the point of contact.

9. If a plane be tangential to a given cylindrical or conical surface, the traces of the surface and of the plane upon *any* plane of projection are also tangential, *i.e.*, the trace of the tangent plane is a tangent to the figure which is the trace of the curved surface.

10. If a cone or a cylinder envelope a sphere, the tangent planes to the former are also tangent planes to the latter, the points of contact always being contained by the circle of contact of the two curved surfaces.

Problem CCVIII.

To determine the traces of a plane tangential to a given vertical right cone, to pass through a given point on its surface. (Fig. 142.)

In fig. 142 the projections of a cone are given with axis vertical, and through $p'p$, a point on its surface, it is required to determine a tangent plane. It would be well here to pause and consider how the one projection of a point on the surface of a cone can be determined *if the other projection only* be known. Let us suppose that in the case before us p' were given and p were to be obtained. Through the elevation of the vertex draw $b'p'$, the vertical projection of two lines upon the surface, one on the rear and the other on the front.* The front line has q for its h. t., which is determined from q'. Then bq must contain the point p, and the plan p is found upon bq by a projector through p'.

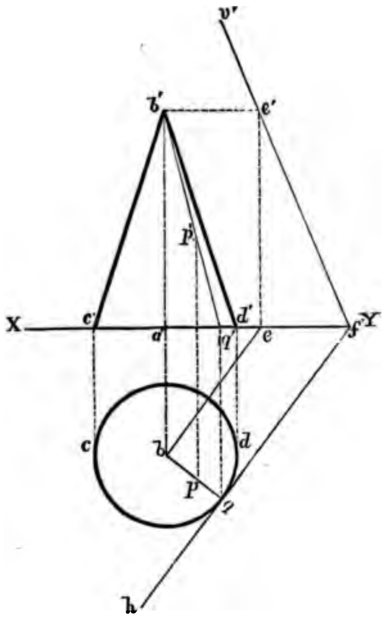

Fig. 142.

* The plan of the line on the rear part of the cone, whose elevation is also $b'q'$, is not shown, as the point P is assumed to be on the front surface.

The tangent plane required must necessarily contain the apex : $b'q'$, bq must therefore be the line of contact, and its h. t. must give the point where the h. t. of the plane is tangent to that of the cone. At q, therefore, set out fh perpendicular to the radius bq to give the h. t. of the required plane. To obtain the v. t. let it be remembered that the tangent plane necessarily contains the apex $b'b$, hence be drawn parallel to fh is the plan of a horizontal of the plane, and $b'e'$ is its elevation. The vertical trace of this line, viz. e', is in the v. t. of the plane, therefore $v'f$ completes the determination of the tangent plane required.

Problem CCIX.

To determine the traces of a tangent plane to a given vertical right cone, to pass through a given point without it. (Fig. 143.)

There are two solutions to this problem, for two planes can be found to contain the given point and touch the cone, one in front and one in rear.* The principle involved in the construction may be described thus:—As the plane must necessarily contain the vertex, its h. t. must pass through that of the line joining the vertex to the given point. Hence t (fig. 143) is one point in the h. t. Again, the h. t. of the plane is a tangent to the trace of the cone; therefore, through t draw hf to touch the circle in plan. This is the h. t. required, and the v. t. can be found, knowing that the plane passes through W and P by a horizontal line,

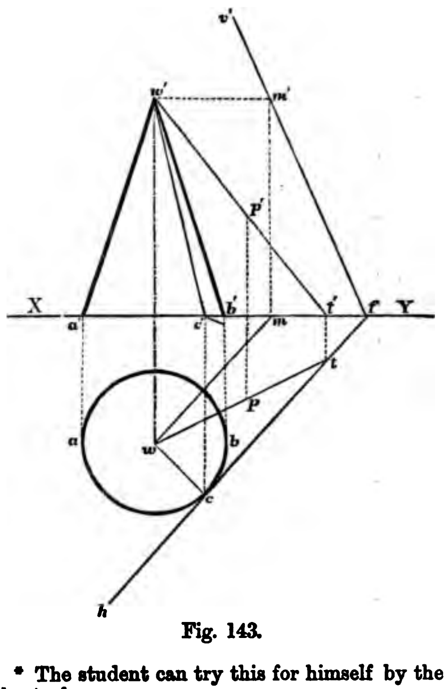

Fig. 143.

* The student can try this for himself by the aid of a model of a cone and a *sheet of paper.*

as *w'm'*, *wm*, through either of them. The line of contact is in plan *wc* drawn through the centre *w*, and perpendicular to the h. t. Its elevation is *w'c'*. The second solution is not shown, but the h. t. would pass through *t*, and touch the plan circle in the rear.

Problem CCX.

To determine the traces of a plane tangential to a given sphere, to pass through a given point on its surface. (Figs. 144, 145.)

1st Solution.

Let the circles *c.'c* be the projections of the sphere, and *p'* the elevation of a point on its surface (in front). To obtain the plan of this point, conceive a horizontal section *a'b'* of the sphere to pass through *p.'p*. The plan of this section will be a circle *ab*, and as P is contained by this circle, project *p* from *p'*. Through the point *p.'p* it is desired to determine a plane tangential to the given sphere. This plane must be perpendicular to a line passing through the centre and the given point *p.'p*. Join, therefore, *p'c'* and *p.c.*, and through *p* draw *p.q* perpendicular to *pc*, and through *p* draw *p'q'* parallel to *xy*. This determines a horizontal of the desired plane, and the v. t. *q'* of this horizontal is in the v. t. of that plane. Then *v'f*, drawn through *q'* perpendicular to *p'c'*, and *fh*, through *f*, perpendicular to *pc*, are the traces of the tangent plane, touching the given sphere at *p'.p.**

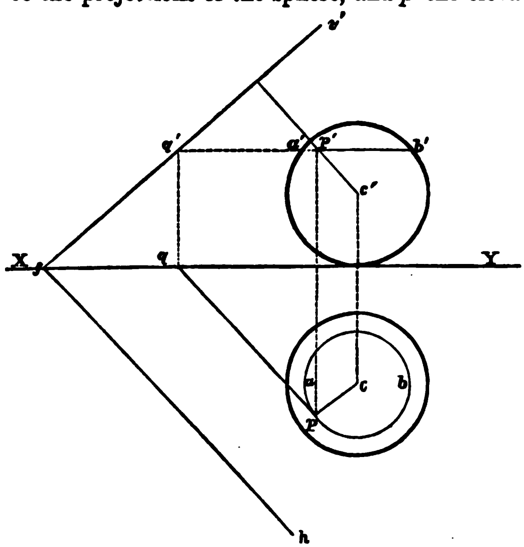

Fig. 144.

* If the given point were upon the horizontal great circle, the tangent plane would be vertical, and its h. t. would touch the plan circle.

2nd Solution.

In fig. 145 another solution of this problem is given. A conical surface having a vertical axis is arranged to envelope the given sphere, the circle of contact passing through the given point *p'.p*. To obtain the elevation of the cone, take a horizontal section of the sphere through the given point. Its elevation is *a'b'*; its plan is the circle *ab*. Join *c'* to *b'* and draw *n'w'* through *b'* to meet the elevation of the axis *w'w''* in *w'*. Complete the elevation, and obtain the plan of the cone. Then through *c* and *p* draw *cp*, the plan of a line upon the conical surface passing through the given point. The trace of this line will be one point in the h. t. of the plane, which will, of course, be tangential to the circle *m.n*. The v. t. is found as in previous cases.

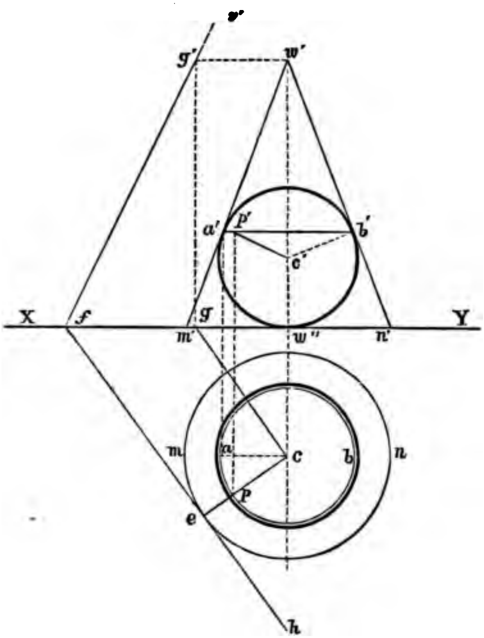

Fig. 145.

PROBLEM CCXI.

To determine a plane tangential to a given vertical right cone and a sphere. (Fig. 146.)

Over the given sphere centre *a',a* place a cone *x'p'q'*, having a vertical angle equal to that of the given one. Then the tangent plane to the two cones will be the one required. The circle *pq* is the trace of such a cone upon the given sphere *a',a*, and the line *fh* touching both circles is the h. t. of *one* tangent plane, which touches the given cone in the line *w'k'*, *wk* and the sphere *a',a* in *g',g*. The method of obtaining *these points* and the v. t. needs no further description.

There are *four* planes which would satisfy the conditions given; only one of these is shown. A second would have its h. t. touching the same

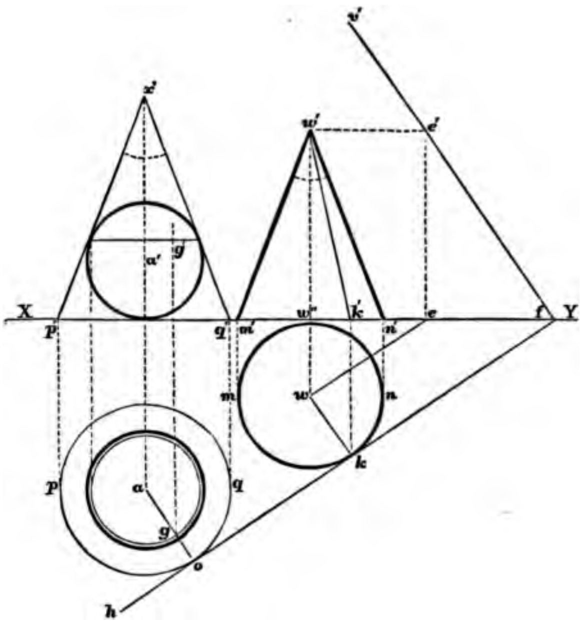

Fig. 146.

two circles on the rear of each, and the v. t. would be found as before. The third and fourth would require for their determination that the enveloping cone just described should be drawn vertex downwards, as they touch the sphere in points below its horizontal great circle. Their horizontal traces would be tangent to the h. t. of this cone and the given one, and pass between them, touching one on the rear and the other in the front, and *vice versâ.* The v. t.'s would be found as usual by a horizontal through one of the vertices.

Another problem is very frequently quoted whose construction depends on the principles just described—viz.: To determine a tangent plane to two given spheres and having a given inclination. Enveloping vertical cones having their base angles equal to the inclination given would trace circles on the h. p. to which the h. traces of the planes required would be tangential.

It should be noticed also in such a case that there are four planes satisfying the conditions, and that to determine two of them the enveloping cone over one of the spheres must be taken vertex downwards.

Problem CCXII.

To determine a plane tangential to a given cylinder (whose axis, *a′b′*, *ab*, is horizontal), and to pass through a given point *p′*,*p* on its surface. (Fig. 147.)

If a plane be tangential to a cylindrical surface, it touches that surface in a line parallel to the axis; hence *c′d′*, *cd* drawn through *p′p* is the line of contact of the plane and cylinder. Take an elevation on a vertical plane perpendicular to the axis. In the figure the circle centre *a″* is that of the cylinder, and *c″* that of the line. Then E′*g*, drawn perpendicular to *a′c″* and through *c″*, will be the trace upon the assumed v. p. of a tangent plane to the cylinder touching it in the line just determined, and the h. t. of this plane will be *gh* taken perpendicular to *x′y′*, the new ground line. If, then, *gh* be produced until it meets the original ground-line XY, one point in the v. t. of the tangent plane upon the original v. p. will be discovered. This point does not fall within the limits of the drawing.

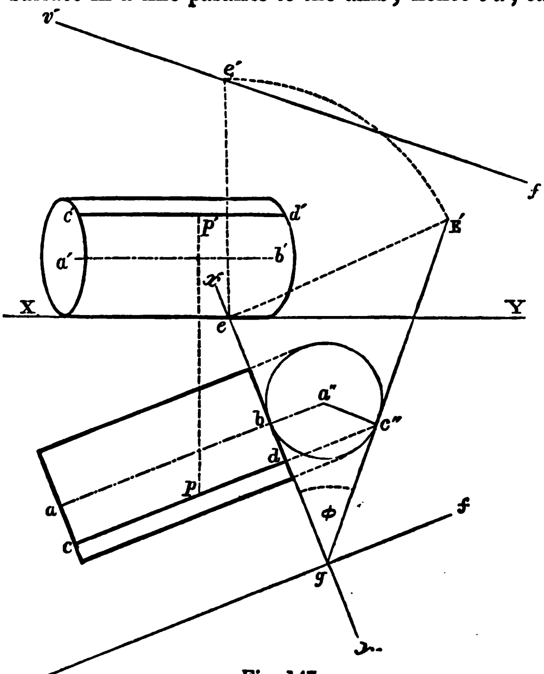

Fig. 147.

A second point in this trace is found thus:—At the intersection *e* of the two ground lines perpendiculars are set out, and *e*,*e′* is made equal in length to *e*E′. Then *e′* is a point in the desired v. t. For if the triangle *e*E′*g* were revolved about its horizontal line *eg* until it became vertical, the two lines *ee′* and *e*E′ would coincide; and as E′ is a point in the tangent plane, *e′* when coinciding with E′ would be one point in its v. t. on the original plane of elevation.

This will no doubt suggest to the student a means of obtaining the traces of a plane where the meeting-point upon XY is so remote as to be inaccessible; for if *two* vertical sections (like that at *x′y′*) be made, two points in the v. t. can be found by the same construction as that used to obtain *e′*.

Had the inclination of the plane been included in the given data instead of a point of contact, the vertical trace on the auxiliary plane would have had to be arranged to touch the circle and to meet *x′y′* at the given angle, the remainder of the solution being like the above.

Problem CCXIII.

Three spheres of different dimensions rest upon the h. p., and each is in contact with the other two. To determine—1st, their projections; and, 2ndly, the traces of a tangent plane which shall touch the three upon their upper surfaces. (Plate LX.)

Draw first the two circles, centres *a′* and *a*, which are the projections of one of the spheres, preferably the largest. If then a second sphere be placed, its centre *b′b* being equidistant with *a′a* from the v.p., its elevation will be a circle touching that of the first one, because the v. p. containing the two centres cuts the spheres in the great circles which give the vertical projections, and which contain the point of contact. To draw this circle accurately, take a line parallel to and above XY at a distance from it equal to the radius of the sphere *b′b*, and intersect it by an arc with *a′* as centre, radius equal to that of the circle *a′*, increased by that of the second sphere. This gives *b′* the centre required, and from it the plan *b* is obtained by a projector meeting a line through *a′* parallel to XY. When the plan of the second sphere is drawn it will be found to be overlapped by that of the first. This is as it should be, for the diameter being smaller, the point of contact is below the horizontal great circle of the larger solid. The point of contact is lettered *n′n* in the drawing, *n′* being determined at once by a line joining *a′* to *b′*, whilst its plan *n* is projected from the elevation upon the line joining *a* and *b*. (See paragraph 1, page 256.)

To obtain the projections of the third and smallest sphere, first make two auxiliary elevations of it as at C′ and C″, assuming that it takes up, for the time being, positions such that it touches *first* the largest sphere A and *then* the medium sphere B, the centres of all three in each case being equidistant from the v. p. The points C′ and C″ are determined in the same way as *b*′. The plans of these are C_1 and C_{11}. Next, conceive the small sphere when in one of these positions, as C″, to commence revolving around that which it touches. It is evident then that an arc having *a* for centre, and passing through C_{11}, will be the locus of the plan of its centre C at all parts of its journey.

Similarly an arc centre *b*, and passing through C_1, is the locus of the plan of the centre, if it revolved in the same way around the sphere B. Hence the intersection of the two arcs at C_1 gives the plan of the centre of the rolling sphere when it touches *both* the others, and its elevation C′ can be projected from it. Of course the same two arcs would intersect at the rear if continued, and give the plan of the centre of the smallest sphere when behind the other two.

The point of contact *m′m* of the sphere C with A can be found thus:—Join *ac* and *a*′C″. Project M_1 from M′. With *a* as centre describe an arc through M_1 to meet *ac* in *m*. The reason of this construction will be apparent. The elevation *m*′ is projected from *m* at a height above XY equal to M′. The point of contact *p′p* of the sphere C with B is found similarly.

There is another method of solving this problem, the principle of which is as follows:—The three lines joining the centres of the spheres when in contact form an irregular triangle, the sides being equal to the sums of their respective radii taken two and two together. The heights of the angular points—*i.e.*, of the centres above the h. p.—being known, the horizontal projection of this triangle is found (see Problem CLXVI., Chapter X.), and the circles which are the required plans are drawn, having their centres in the projections of the three corners.

To determine the traces of the tangent plane, consider two of the spheres as enveloped by a conical surface. This surface will have its vertex upon the h. p., because the spheres rest upon that plane. Further, any plane tangential to this cone will touch both spheres, and its h. t. will necessarily pass through the plan of the vertex. Hence the point *d*, which is the plan of the vertex of a cone enveloping the spheres A and C, is one point in the h. t. of the tangent plane required. Similarly, *e* is another point, as it is the plan of the vertex of a second cone which envelopes spheres A and B. Then *fh*, drawn through *e* and *d*, is the h. t. required.

The vertical trace is found thus:—An auxiliary elevation of one of the spheres is made upon a v. p. perpendicular to the trace just determined. The vertical trace of the tangent plane upon this new vertical plane is a line touching the circle, which is the new elevation of the sphere. Thus *t″g* is the v. t. of the tangent plane required upon the v. p., having X′Y′ for its intersection line, and the drawing shows that the elevation of either sphere would be sufficient, as this line touches all three. At *i*, the two perpendiculars *i,i′* and *i,i″* are set up, and *i″* is found as a point in the v. t. upon the original v. p.

The points of contact of the plane with the spheres are shown at *t′,t*, *r′,r*, and *s′.s*, the plans being determined from *t″*, *r″*, and *s″* upon lines drawn through the centres of the plans of their respective spheres perpendicular to the h. t., and the elevations being found in projectors through the plans, the heights above XY being taken from the auxiliary projections.

Problem CCXIV.

Given a cylinder (axis inclined to h. p.) and a point upon the surface, required the traces of a plane tangential to the given cylinder, and passing through the given point. (Plate LXI., figs. 1 and 2.)

1st Solution (Fig. 1).

As a plane in contact with a cylindrical surface must necessarily touch it in a line parallel to the axis, the line in this case can be determined at once by drawing *s′t′*, *s.t* through the given point *p′.p*. Again, when a plane and a cylindrical surface in contact with it are intersected by another plane, the straight line and the curve which are their traces upon the cutting plane are tangential at a point which is the trace upon that plane of the line of contact. Conceive, then, the given cylinder to be produced until it meets the h. p. in the ellipse *gikl*. Then the h. t. is a tangent to this ellipse at a point *q*, which is the h. t. of the line of contact *s′t′*, *st*. Then *fh* is the h. t. required.* The vertical trace will in this case be parallel to the axis, because the axis itself is parallel to the v. p. and to the tangent plane. Had this

* The method of drawing a tangent to an ellipse at a point on its surface, being described in the plane geometry construction, is not shown in the plate.

not been so, a point in the v. t. could have been obtained by the aid of a horizontal line through the known point *p'p* parallel to the h. t. just found, as described in previous problems.

2nd Solution.

The method employed in the first solution given necessitates the drawing of an ellipse, and the correctness of the traces found depends upon the accuracy of that curve. Hence it is, if possible, desirable to adopt a construction which avoids the use of any curves but circles. The following method solves this problem without the ellipse, and is shown in fig. 2 of the same plate, the conditions being exactly as before:—Assume a new ground line *x'y'* perpendicular to the elevation of the axis, and make an auxiliary plan on the new h. p. This plan is a circle, having B for centre (the distance of B from *x'y'* being equal to that of *b* from XY), and R is the trace of the line of contact—SR being drawn through the given point P. The required tangent plane is, as regards the new horizontal plane, a vertical one, its h. t. being a line which touches the circle at R, and its v. t. *gv'* being parallel to the elevation of the axis; *v'gt* is therefore the plane required, shown by its traces upon the original v. p. and the assumed h. p. Produce the v. t. until it meets XY in *f;* and as it is only the h. p. which has been changed, this v. t. will still be the one required, and only the new h. t. (upon the original h. p.) needs to be determined. Find then *q*, the trace of the line of contact *s'r'*, *sr* through *p'.p*. This point must be in the h. t. of the plane, and as *f* is known, join *ft*. Then *v'fh* is the tangent plane required.

Problem CCXV.

To determine the traces of a plane tangential to a given cone (axis inclined to the h. p.), passing through a given point upon the surface. (Plate LXII., figs. 1 and 2.)

This problem is a companion one to the last, and the same kind of constructions are used. In the first case (fig. 1) the ellipse is determined which is the h. t. of the given cone, and at *e* (the h. t. of the contact line *w'e'*, *we*, passing through the given point *p'.p*), a tangent is drawn, which is the h. t. of the required plane, the v. t. being obtained by discovering one of its points *i'*, where a horizontal of the tangent plane *w'i'*, *wi*, containing the vertex of the cone, meets the v. p.

In the second figure the same problem is solved without the use of the ellipse, the construction being exactly like that in the second solution of the preceding problem. It is not considered necessary to give any detailed description.

PROBLEM CCXVI.

Given a plane by its traces and a point by its projections. Assuming the given point as the centre of a sphere which touches the given plane, required to determine its radius and projections. (Plate LXIII., fig. 1.)

Let *v′fh* be the given plane, and *p′p* the given point. As the plane is to be in contact with the sphere, the line joining the point of contact to the centre will be perpendicular to it. Determine, then, *p′q′* and *pq*, the indefinite projections of a line through *p′p* at right angles to *v′fh*. Find its intersection *t′,t* with that plane (Problem CX., Chapter VII.), and determine the true length *p*T of the line joining *p′p* and *t′t*. This gives the radius of the sphere, and the projections can be described at once. The point of contact of sphere and plane is *t′t*.

PROBLEM CCXVII.

The projections of a cone being given, lying upon the h. p., its axis being parallel to the v. p., required the projections of a sphere of given radius which shall rest upon the h. p. and be in contact with the cone at a point which is at a given distance from the vertex. (Plate LXIII., fig. 2.)

The conditions involved in this problem are satisfied, as regards the cone by its projections, upon Plate LXIII., and it is there assumed that the sphere to be determined shall be of ·75″ radius, and shall touch the cone at a point 1″ from the vertex *v′v′*. . Having completed the projections of the cone, take a point *p′*,1″ from *v′*, and through it draw S*p′* perpendicular to the axis, and consider it as the v. t. of a plane perpendicular to the original v. p. This plane cuts the cone in a circle, which is a locus of all points 1″ from the vertex. The point of contact, therefore, lies in this plane and in the circle. Next describe a circle of ·75″ radius which shall touch the line *v′a′* in *q′*, and consider it as the elevation of a sphere, fulfilling the conditions of the problem in all respects *except* that it does not rest upon the h. p. Let the sphere then be imagined to revolve around the surface of the cone until it reaches the h. p. either in front or behind, the point of contact always remaining in the plane whose v. t. is Sp'. The centre of the sphere will then move in another plane parallel to this one, whose vertical trace is *mn* taken parallel to Sp' and passing through F′, and as, when it reaches the h. p., the centre must be at a height above that plane equal to the radius, draw a line parallel to XY and ·75″ from it to meet *mn* in f'. This point is the vertical projection of the centre of the sphere.

Next, to obtain the plan. Make an auxiliary plan of the circle of contact on a plane perpendicular to the axis of the cone; $x'y'$ is the intersecting line of the new plane of projection,* and the circle (centre *o*) is the plan mentioned. Next consider that the plane of this circle cuts the sphere in one of its smaller circles. This will be readily seen in the elevation of the first sphere, centre F′, where the line Sp' cuts it in Sq'. Now, as the sphere rolls around the cone the auxiliary projection of the point of contact will move around the circle (centre *o*), and that of the centre of the section of the sphere (which is at G in its first position) will move in another circle passing through G. Further, the auxiliary plan of the centre of the sphere in all positions of the revolution coincides with that of the centre of the section. Hence a projector through f' perpendicular to $x'y'$ meets the arc through G just described in two points, F_1 and F_2, which are the auxiliary plans of the centre of the sphere under the given conditions (first, when in front of the cone, and, secondly, when behind it). From these two points the distances f_1 and f_2 are obtained, along the projector through f' perpendicular to XY.

Again, from the auxiliary plan the positions of the projections of the points of contact can be determined: t' from T and T_1, and t and t_1 from t' at distances from XY equal to those of T and T_1 from $x'y'$.

Problem CCXVIII.

To determine the traces of the two planes which, being parallel to a given line, shall be tangential to a given cylindrical surface, the axis of which is inclined to the horizontal plane of projection. (Plate LXIV., fig. 1.) †

In the plate the cylindrical surface has $c'd'$, cd for its axis, and the given line is $a'b'$, ab.

* The distance of $x'y'$ from c' is equal to the distance of vb from XY.

† *The inclination of the line must not be less than that of the axis.*

There are always two planes which will satisfy the conditions of being tangential to a cylinder, and parallel to a given line. These planes when determined are parallel to two lines, viz., the given one and the axis, for the line of contact of either plane must necessarily be parallel to the axis. Hence any plane which is parallel to both *a'b'*, *ab* and *c'd'*, *cd* will either be one of the two required or parallel to them. The required traces, therefore, will be parallel to those of any such plane. Proceed, then, first to determine the h. t. of a plane parallel to both *c'd'*, *cd* and *a'b'*, *ab*. To do this at any point *p'.p* set out *p'q'*, *pq* parallel to *c'd'*, *cd*, and find the horizontal traces of *a'b'*, *ab* and *p'q'*, *pq*. Then *bq* passing through these traces is the h. t. of a plane, satisfying the condition of being parallel to both lines. Next obtain the ellipse which is the h. t. of the cylindrical surface, and draw the two tangents to it parallel to *bq*, producing them till they meet *xy* in *t* and *n*. These two lines are the horizontal traces of the tangent planes required, as they touch the trace of the curved surface, and are parallel to that of the plane just obtained.

As the axis of the cylinder is parallel to the v. p. (and this condition was purposely arranged so that the elliptic trace could be determined directly*) the vertical traces are parallel to the elevation of the axis. Hence *v'tr* and *k'nl* are the planes required.

The lines of contact can be determined at once by drawing through the projections of the points of contact of the horizontal traces and the ellipse parallels to the projections of the axis. Thus *ss* is drawn through *s* where *tr* meets the ellipse parallel to *cd*, and $s's'_1$ is the elevation parallel to *c'd'*.

These are the projections of the line in which the plane *v'tr* touches the cylinder; $m'm'_1$ and mm_1 are obtained in a similar manner.

This problem and the next are concerned when it is required to show the separation line of light and shade upon the respective curved surfaces, the direction of the light being known and the rays being parallel. Of this further mention will be made in the chapter upon Shadows.

Another construction is sometimes employed to solve this problem, based upon the following principles :—

As the two tangent planes required must necessarily be parallel to

* If the axis had been inclined to *both* planes of projection, the construction would be the same as the above, except that the elliptical trace would require an auxiliary elevation on a v. p. parallel to the axis for its determination.

the axis as well as to the given line, the plane of the base, *i.e.*, any plane perpendicular to the axis, must cut the cylinder in a circle and the two planes in two lines touching this circle, each of which must be parallel to the intersection of the base plane with that of *a′b′*, *ab* and *p′q′*, *pq*, or any other plane parallel to it. Hence, a point being assumed in the axis, and a line being drawn through this point parallel to the given one, the plan of the intersection of the plane containing the axis and this line, with the plane of the base, will give the direction in which two tangents to the ellipse (the horizontal projection of the base) can be drawn, thus enabling one to determine the entire lines in which the required planes meet the cylinder, for these lines must pass through the points of contact and be parallel to the axis.

This method saves the drawing of the elliptical trace of the solid. Its lines are not shown in the figure, but the student will find the same construction adopted in Problem CCXXXII., Chapter XV., to obtain the shadow of a cylinder; and he is referred to the accompanying Plate LXX., fig. 1, for its illustration.

Problem CCXIX.

To determine the traces of a plane parallel to a given line, and tangential to a given cone (axis inclined). (Plate LXIV., fig. 2.)

In this problem the conditions are similar to those of the preceding, except that the curved surface is a cone instead of a cylinder, and the constructions employed in the two instances are somewhat analogous.

In the figure *a′b′*, *ab* is the given line. Through the vertex of the cone *v′c′*, *vc* are drawn as the projections of a line parallel to the given one. Then through this line two planes can be passed which shall be tangential to the cone, and they will necessarily satisfy the conditions of the problem, as they will contain a parallel to *a′b′*, *ab*. Further, their traces on the h. p. will be tangential to the elliptical trace of the cone. Find this trace *mynz*, and through *c*, which is the h. t. of *c′v′*, *cv*, draw the horizontal traces *cp* and *cs* of the two required tangent planes. The vertical traces and the lines of contact are found by methods adopted in previous problems, and therefore requiring no further special explanation.

A second construction analagous to that explained in the last problem may be adopted to solve this one. It has been illustrated in the plate.

Point *k′k* is the intersection of the line through the vertex (parallel to the given one *a'b'*, *ab*) with the plane of the base. From k_1 two tangents to the ellipse in plan give, by their points of contact, the extremities of the lines in which the two planes required touch the given cone. Only one is shown, and *l'l* is joined to *v'v*, the horizontal trace of the plane being determined by finding the h. t.'s of *v'k'*, *vk* and *v'l'*, *vl*.

PROBLEM CCXX.

Through a given straight line to draw a plane tangential to a given sphere. (Plate LXV., figs. 1 and 2.)

This problem can always be solved when the given line does not pass through the sphere. When it is quite clear of it, there are two planes which satisfy the required conditions. If it *touches* the sphere, there is only *one* solution, and the tangent plane is then perpendicular to the radius which passes through the point of contact.

The conditions given in both cases upon the plate admit of two planes, and there are two distinct constructions for finding the traces, which may be described as follows :—

1st Solution (fig. 1).

First. Conceive one of the required tangent planes to be in its correct position. *Secondly.* Conceive a plane to pass through the centre of the sphere perpendicular to the given line. This plane will cut the line in a point, and the sphere in one of its great circles. The intersection of the two planes passes through this point on the line, and touches the great circle of section on the sphere. If, then, the point of contact be known, the tangent plane can be determined, for the extremities of the given line give two other points in it.

Turning to the plate (fig. 1), *a'b'*, *ab* is the given line, and the two circles, centres *c'* and *c*, are the projections of the sphere, and *r'st* is the plane which passes through *c'c* and is perpendicular to *a'b'*, *ab*.* The point where this plane intersects the given line is *g'.g*. Now, as it is known that the plane *r'st* cuts the sphere in a circle, and that the intersection lines of this plane and the required tangent planes are tangents to that circle, and in the same plane, revolve *r'st* about its h. t. into the h. p., and show thereupon the circle and tangent lines through

* All the lines requisite for determining the constructions are left in the plate, and the student is reminded that the explanations of methods of obtaining elementary conclusions like these should be familiar at this stage.

g'g which have been described. Thus G. is the constructed point *g'g.*, and C is the "constructed" centre *c'c* of the sphere. Then GF and GE, drawn as tangents to the circle, give E and F, which are two "constructed" points, one in each required tangent plane. It remains, therefore, to find the projections of these two points when in their correct positions. Taking F, for example, produce GF to meet the h. t. of the plane in *t*. Then, if the proper position of the plan of GF is known when the plane is rotated back into its place, it is evident that the plan of F must fall upon that line at the point where a perpendicular to the h. t. through *g.* meets it. Hence *f* is its plan, and its elevation *f'* is upon *g'f'* the elevation of *gf.* Then the plane containing A, B and F is one of those required. The other contains A, B and E, the projections of the last point being found by a similar construction.

The student will notice that both h. t.'s must pass through *b* (that of the given line), and that one contains *t*, the h. t. of *g'f'*, *gf*, and the other contains K, the h. t. of *g'e'*, *ge*. This solves the problem without drawing an ellipse, which is a great advantage. Usually the great circle of section is projected in two ellipses.

2nd Solution.

If any two points in the given line be conceived to be the vertices of cones, each enveloping the given sphere, it is evident that any plane which touches one of the cones will necessarily touch the sphere, and hence the two planes which touch both cones will be those required, as they must contain the line in virtue of containing the two assumed vertices. The circles of contact of the two enveloping cones with the sphere intersect in two points, one of which lies in the first tangent plane and the other in the second. If the two vertices be assumed at random, the projections of these two circles of contact may all be ellipses, and if the drawing of these can by any means be partly dispensed with, the correct solution will be facilitated. This can be done by assuming the vertex in each instance at a point on the given line equidistant with the centre of the sphere from one plane of projection. Thus, in the drawing *d'd* is the same distance from the v. p. as the centre *c'c*. Then the elevation of the circle of contact of that cone having *d'd* for vertex, is a straight line *e'f'*, of which the ellipse *egfh* is the plan. Again, the cone having *i'i* for vertex, that point being equidistant with *c'c* from the h. p., has the straight line *kl* for its plan, and *k'ml'n'* is its elevation. By thus arranging the vertices, two ellipses are dispensed with. Now, it is evident that these circles of contact intersect in two points $p'p$ and $o'o$, the former upon the under side and the latter upon the upper side of the sphere. One of these two points is contained by each required tangent plane, and the respective traces can be determined at once, remembering that one plane contains AB and P, whilst the other contains AB and O. The traces of the latter meet the ground line in a point too remote to be shown upon the drawing.*

The student should note that a series of cones having their vertices in the given line $a'b', ab$ could be assumed, or rather a conical surface could be imagined to move with its vertex always in the given line, and always enveloping the sphere. The circles of contact would all pass through the points $p'p$ and $o'o$; hence the chord $p'o', po$ is in all their planes, and is their common intersection. This chord of contact is perpendicular to the plane containing the given line and the centre of the sphere. From this fact the principle of the first solution given is derived, as the extremities of the chord of contact are found by drawing tangents to the great circle of section through the point of intersection on the given line.

Problem CCXXI.

Given three spheres of unequal dimensions and height, to determine the traces of planes tangential to them. (Plate LXVI., fig. 1.)

Imagine a flat piece of paper resting upon the three spheres to represent a tangent plane touching them upon their upper surfaces. Then conceive two of the spheres to be enveloped by a cone whose vertex is on the line of their centres produced. It is evident that the tangent plane touches the conical surface in a line corresponding to one of its generatrices, and, therefore, containing the vertex. This would be equally true with regard to a second cone enveloping one of these two spheres and the other one. Hence it is seen that the tangent plane, because it passes through both vertices, contains the line joining them, and the problem is reduced to the preceding one, *i.e.*, through a certain line a plane is to be determined which shall touch that sphere which is enveloped by both cones. At this point it might be suggested that if a third cone be used enveloping a third pair of spheres, its vertex

* This problem is frequently so arranged that both planes of projection are made to contain the centre of the sphere. By this means one circle is made to serve for both plan and elevation of the sphere, and the drawing is more concise.

would give a third point in the plane by which the traces could be discovered directly. But this would not help us, for all the three vertices would be in one straight line, and an infinity of planes would contain it. This is evident from the fact that the plane containing the three centres also contains the three vertices, as they are in the lines joining these centres. Consequently they (the vertices) are in the intersection of the two planes (the tangent plane and the plane of the centres), and, therefore, in one straight line. Again, the student will see that, as the circles of contact of the enveloping cones upon the sphere covered by both meet in two points, there are, as in the last problem, two planes containing both vertices and touching this sphere. But this is not all. One can always imagine two different cones to envelope two given spheres, for one may have its vertex on the line joining the centres produced, and the other on the same line and between the centres. (See fig. 2.) Hence there are six conical surfaces which can envelope the three spheres, three of them covering them exteriorly, taken two and two together, and the other three with their vertices between them. Now, any three of the vertices will be in one line, and two planes can be determined containing this line and touching the sphere over which their respective cones pass. Then, as the changes of three can be rung four times, it is clear that eight tangent planes may exist in contact with the three spheres. Two of these touch the three spheres on the same side, and the other six have one sphere on one side and two upon the other side.

In the figure the three given spheres have *a′a*, *b′b* and *c′c* for centres, and only two of the planes out of the possible eight are shown, as the drawing would cease to be clear if all were determined. The cone having *p′p* for vertex is first drawn in plan and elevation. This cone envelopes spheres A and B. A new elevation on a v. p. having *ap* for its ground line, is then made at AP, and the line IK, joining the points of contact of circle and tangents, is the elevation of the circle of contact from which the plan *ilkz* is deduced. Next, the cone having *q′q* for vertex is drawn, enveloping the spheres B and C, and the plan of its circle of contact is determined from an auxiliary elevation, as in the previous case. This is not shown in the drawing. These two circles of contact upon the sphere A intersect in *d* and *e*, and the elevations of these points are at *d′* and *e′*, the heights being obtained from D and E on the auxiliary drawing. Then the plane whose *traces* are *v′f* and *fh* is deduced from the projections of the points P.Q and D, and the other one, whose traces are *l'm* and *mn*, from those of P,Q and E.

Of course *d'd* and *e'e* are the projections of the points of contact of their respective tangent planes with sphere A. Those upon the other spheres are found thus:—Taking the case of B, for example, it is known that the points of contact must lie upon the generatrices of the enveloping cone which pass through *e'e* and *d'd*. Join, then, *e'p'*, *ep* and *d'p'*, *dp*. Further, they are in the radii of the sphere which are perpendicular to the tangent planes. Hence, *bg* drawn perpendicular to *fh* gives *t* by its intersection with *dp*, and *br* drawn perpendicular to *mn* gives *s* by its intersection with *ep*, the former being the contact point of *v'fh*, and the latter of *l'mn* with the sphere *b'b*. The contact points on sphere C could be determined in the same way.

On the small drawing (fig. 2) the projections of a cone enveloping two spheres are shown, the vertex being between their centres; and this principle, applied to A, B and C, as previously described, would enable us to find the other six planes mentioned.

Problem CCXXII.

To determine the traces of a plane tangential to a given right cone (axis oblique), and which shall have a given inclination. (Plate LXVII., fig. 2.)

Having obtained the projections and the ellipse which is the horizontal trace of the given conical surface, assume a second cone, its vertex coinciding with that of the given one, its axis being vertical and the inclination of its generatrices being equal to that of the required tangent plane. Then any plane which touches both cones will satisfy the conditions of the problem. There is a limit, however, to the given inclination, for it cannot be less than that of the generatrix of the given cone which is uppermost upon its surface, and is, therefore, least inclined. With this inclination there is but one tangent plane, of which the generatrix mentioned is the line of contact. If the inclination exceeds this angle, but lies between it and the inclination of the lowermost generatrix, then there are two such tangent planes, one in front and the other in the rear. Further, when the given inclination is greater still but yet less than 90°, there may be four planes, two of which touch the cone in lines upon its upper surface, one in front and one in rear, and two which touch it in corresponding lines upon its lower surface. If the tangent planes were desired to be vertical, there would be two solutions, each of which would contain a vertical line through the vertex. But these latter conditions of inclination could only exist when all the generatrices of the given cone are on one side of this vertical line. This will be evident after the construction employed is described in detail.

In Plate LXVI., fig. 1, the given cone is represented as having *v'v* for its vertex and *v'm'*, *v.m* for its axis, and the assumed inclination of the required tangent planes is θ. Through *v'* draw *v'd* perpendicular to XY, and consider it as the axis of a vertical cone having $\theta°$ for its base angle, *v'c'd'* is its half elevation, and the circle (radius *cv*) is its entire plan. Then any line which touches the ellipse *agb* and this circle may be considered as the h. t. of a plane tangent to both cones, and it will be at once seen that there are four of them. Hence, under the given conditions, there are four solutions. The vertical trace of each is determined in the usual manner by a horizontal of the plane passing through *v'v*, as that point is, of course, in all of them. Those numbered I. and II. pass between the given cone and the assumed one, whilst III. and IV. touch them in front and rear respectively.

It will be evident from this construction that the radius of the circle *cv*, as it depends upon the given inclination, could be so extended that the ellipse would be entirely surrounded by the circle. The problem ceases to be possible when that occurs, as the two curves would then have no common tangent. Hence the inferences drawn and previously described.

The line of contact of plane No. I. with the cone is shown at *v'f'*, *vf*, the point *f* being where the h. t. of the plane meets the ellipse. The other contact lines are not given.

Problem CCXXIII.

To determine the traces of planes of a given inclination which shall be tangential to a given cylinder (axis inclined). (Plate LXVII., fig. 2.)

In this case, as in the last, there is a limit to the possible inclination of the planes. *It cannot be less than that of the axis.*

Assume any point in the axis as the vertex of a vertical right cone, the generatrices of its surface being inclined at the given angle. Then lines drawn through the h. t. of the axis, and touching the circle, which is the plan of the assumed cone, may be considered as the h. t.'s of planes containing that axis and inclined at the given angle. The tangent planes required will have their traces parallel to these lines and touching the ellipse, which is the trace of the cylinder.

In the figure the ellipse *cd* is the h. t. of the given cylinder, and *b* is that of its axis. At *a'a*, a point in the axis, a vertical right cone is set out; its base angle being θ°. The circle *fg* is its plan, the radius being equal to *e'f'*. Then *bi* and *bk*, two tangents to this circle through *b*, are the traces on the h. p. of two planes containing the axis and inclined θ. Then the h. t. I, which is tangent to the ellipse and parallel to *bk*, is that of one of the required planes, for it touches the cylinder and is parallel to one of the planes just determined. The v. t. would be parallel to the elevation of the axis, but is not shown in the plate. The student will see that another line parallel to *bk*, and touching the ellipse on the other side, would be the trace of a second tangent plane fulfilling the conditions, and that there are two other traces which are parallel to *bi* and in contact with the ellipse. These four planes all satisfy the data of the problem. The contact line of one only is shown, viz., where plane I touches the cylinder. The h. t. meets the ellipse in *z*, and *z'* is its elevation. Through these points the projections of the contact line are drawn parallel to those of the axis.

Problem CCXXIV.

Given the traces of a plane and the indefinite projections of a line contained by it. To determine the projections of a cone having a given vertical angle which shall touch the given plane in the given line. (Plate LXVIII.)

Let *l'mn* be the given plane and *a'v'*, *av* the given line contained by it, which is to be the line of contact of the required conical surface with this plane. Further, let the vertical angle of the cone be θ° and the length of its axis 3·25″.

The axis and the line of contact are contained by a plane perpendicular to the given tangent plane. Proceed first, then, to discover the h. t. of this plane by assuming any point *f'f* in the given line *a'v'*, *av*, and by setting out through it a line *f'g'*, *fg* perpendicular to *l'mn*. Then *g*, the h. t. of this line, is in the trace of the plane, and *v*, the h.t. of the given line, is a second point in it. It is not necessary to determine the v. t.

Next, conceive this plane to be "constructed" into the h. p. about its

h. t. Point *v*. remains stationary, whilst $f'f$, removes to F. Then F*v* is the "constructed" line of contact. At *v* set out the semi-vertical angle $\frac{\theta}{2}$, and *v*R will represent indefinitely the "constructed" axis, and FR (a perpendicular to F*v*), will be a portion of the line $f'g', fg$ also "constructed" into the h. p. Hence, conceiving the plane to be revolved again into its correct position, it is evident that from R point *r* can be obtained upon *fg*, and this is the plan, therefore, of a point in the axis of the cone, its elevation r' being projected from *r* upon $f'g'$. If the vertex were taken at any other point on $a'v'$, *av*, instead of at its h. t., as shown in the figure, it would be necessary to "construct" that point into the h. p., and then set the semi-vertical angle as before.

Having obtained the *indefinite projections* of the axis, make an elevation of it on a vertical plane parallel to it. In the figure this is shown at vR$'$ (rR$'$ being perpendicular to *vr* and equal to the height of r' above XY). About this axis set out the elevation *v*PQ of the entire cone, making the vertical angle equal to θ° and the axis vO$'$ 3·25$''$ long. Thence obtain the plan of the base, and from that the elevation, the ellipse in the latter instance being inscribed in a parallelogram, as explained in previous problems.

Problem CCXXV.

Given the traces of a plane and the projections of a line, not contained by the plane nor parallel to it, to determine the conical surface, which, having the given line for its axis (given length), touches the given plane. (Plate LXIX.)

Let $l'mn$ be the traces of the given plane, and $a'b'$, *ab* the projections of the given line. Further, let the axis of the cone be 3·25$''$ in length.

First determine the intersection $b'b$ of the given line and plane, for it is evident that that point must be the vertex of the required cone. Next cut off from the indefinite axis a part $a'b'$, *ab* of the required length, 3·25$''$ measured from the vertex. This completes the axis, and the problem resolves itself into finding the semi-vertical angle of the cone. This would be measured by two lines which meet in $b'b$, viz., the axis and the contact line of the cone and plane. It is therefore necessary to determine this line of contact. Any line which passes through a point on the axis, and is perpendicular to the tangent plane, *will intersect* that plane in a point which is in the desired line of contact.

At $a'a$, therefore, set out $a'k'$, ak perpendicular to $l'mn$, and determine their intersection $k'k$. Then $b'k'$ and bk are the projections of the contact line.

The plane of the base is perpendicular to the axis, and passes through the known point $a'a$. Further, the intersection of this plane and the given tangent plane is a line which touches the base circle in a point at one extremity of the contact line. Determine, therefore, the plane of the base, which is shown and described in the figure, and its intersection $s't'$, st with $l'mn$. Then produce $b'k'$, bk until it meets $s't'$, st, in i',i. This is the extremity of the line of contact, and hence is a point in the circumference of the base, and from this it is inferred that $a'i'$, ai is a radius of the base, the true length of which can be determined, and which is shown at aI. Lastly, draw an elevation of the entire cone, using ab as the ground line, and making AP and AQ equal to the radius of the base just discovered. From this elevation complete the plan, and thence obtain the elevation as before.

There are several other problems upon the subject of curved and plane surfaces in contact, which are reserved for a future chapter upon Horizontal Projection, as the constructions necessary are very much simplified when worked upon the principles there described.

EXERCISES.

N.B.—Those marked * are the more difficult, and should only be attempted after the others have been worked.

1. Two spheres, radii 1·5″ and 1″, rest in contact on the h. p. Determine a plane tangential to both, and having an inclination of 50°. (Honours, May Exam. 1870.)

2. A right cone whose vertical angle is 40° and axis 3·5″ has the latter inclined at 60°. Show a plane tangential to the surface and inclined 70°. (Advanced May Exam. 1870.)

3. A cone, axis vertical; a cylinder, axis horizontal, and a sphere, all rest upon the h. p., mutually touching each other, dimensions as follows:—Cone, base 2·5″ diameter, axis 3″ long; cylinder 2″ diameter; length at pleasure. Sphere 1·25″ diameter. Show the projections of the whole.

4. A sphere of 2″ diameter rests upon the h. p. A cylinder 3″ diameter, axis horizontal, also rests upon the h. p. The plan of the axis is 2″ from that of the centre of the sphere. Show traces of a tangent plane touching the upper surfaces of both.

5. Three spheres of radii ·4″, ·7″, and 1·5″ respectively, stand upon the h. p., each one touching the other two. Draw the plan of them. (Advanced May Exam. 1873.)

6. A circle of 2·5″ diameter is the plan of a sphere, centre 2″ high. A line 3″ from the centre is the h. t. of a tangent plane to sphere. Determine the inclination of the plane and the plan of the point of contact.

7. Draw two lines meeting in XY at 35° with it. Consider these as the traces of an oblique plane. Show projections of sphere of 1·5″ radius which touches this plane in a point 2″ from each plane of projection.

HINT.—When the given point has been found, set out a perpendicular to the plane through it. Cut off the length of the given radius.

8. A sphere radius ·5″ stands on the h. p. A vertical cone of 1·2″ radius has the plan of its axis 3″ from that of the centre of the sphere. Show the traces of all the planes which could be tangential to both curved surfaces. The axis of the cone to be 3·5″ long.

*9. Show how to obtain the tangent plane to a given sphere which shall pass through two given points of unequal heights, outside the spherical surface. What are the limits to their positions? (Royal School of Mines Exam. 1873.)

*10. Draw a circle of 1·5″ radius, and take within it a point 1″ distant from the centre. The circle is the plan of a sphere whose centre is 2·5″ above the h. p., and the point is the plan of one situated on the upper surface of the sphere. Through this point draw a straight line which shall touch the sphere and be inclined at 30°. (Honours, May Exam. 1871.)

HINT.—First obtain the tangent plane at the given point, and then through the point draw a line inclined 30°, contained by the tangent plane.

*11. The plan of the axis of a cone is 3″ long, the vertex is ·7″ high, and the centre of the base 3·2″ above the h. p. Draw the traces of a plane tangential to the surface, and inclined at 75°. The radius of the base is 1·5″. (Honours, May Exam. 1874.)

*12. A cylinder 2″ diameter has its axis inclined 40°. Show traces of a plane touching its surface, inclined 65°. Also, line of contact.

*13. Given two spheres, centres 3·6″ and 2″ above the h. p., and a point 2″ high. The plans of the two centres a and b and that of the point c form a triangle $ab = 5''$, $bc = 2{\cdot}25''$, and $ac = 3{\cdot}5''$. The diameter of sphere A is 2·75″ and B 1.5″. Draw the traces of a plane passing through the given point, and touching both the spheres.

HINT.—Draw a cone enveloping both spheres. Its apex is one point in the plane. Draw another cone whose apex is in the given point and touching the sphere whose centre is at the same height as that point. The intersection of the lines of contact (one of them being a straight line in plan) of the two cones upon the small sphere gives two other points, and either of these can be taken as the third point in the desired tangent plane, as there are two solutions satisfying the condition.

CHAPTER XV.

ON THE PROJECTION OF SHADOWS.

1. THE term "*Shadow*" may be used to denote either of two perfectly distinct conceptions. It may in one sense mean the *space deprived of light* by placing an opaque body before a luminous source. For instance, if light be proceeding from a point, as in the case of the carbon of an electric lamp, its rays—diverging in all directions—fill the entire space around. But if a stone marble be placed near the lamp, a certain space beyond the object—taking the form, in this instance, of an enveloping cone—is deprived of light, and this space may be termed the shadow of the marble. In this sense a pea could be placed so as to be "*in the Shadow*" of the object.

But the same word may also mean the space upon a certain surface deprived of light by interposing an opaque body before the source which would otherwise illuminate it. With this conception, one speaks of the shadow of a stick upon the ground. Such a shadow is therefore the *outline of the projection* of the object upon the receiving surface made by a system of imaginary projectors coinciding in direction with the rays of light.

Again, shadows are not of equal intensity; the umbra is of different tint to the penumbra, etc. It is well, therefore, to understand that in discussing the application of the principles of Descriptive Geometry to the determination of shadows, we use the term in the latter sense as described above, and that difference of tint is not regarded.

It is, of course, further assumed that the medium through which the rays of light are passing is uniform, and hence, that they are straight lines.

2. The principles of Orthographic Projection necessitate the use of two drawings of any solid object (one upon each of two assumed planes not parallel to one another) before the three principal dimensions—length, breadth, and thickness—can be demonstrated.

In two of the chapters which follow this one, devices for dispensing with one of these drawings will be explained and discussed; but it is most interesting to note at this point, that if only *one* projection of an object be given, together with the outline of its shadow cast upon the plane of projection, the direction of the rays of light being known, that is sufficient to determine all the data generally obtained from a plan and elevation. For instance, let *ab*, fig. 148, be the plan of a line AB and *bc* its shadow; also, let the arrow X denote the direction of the rays of light which are parallel to each other and inclined 45° to the plane of projection. It is clear that the plan *ab* gives no indication of the heights of A and B above the paper. But having the shadow, we are able at once to see that as it meets the plan in *b*, that end of the line must be upon the plane of projection. Then, again, *ac* drawn parallel to the arrow X, through *a*, must be the plan of the ray which gives the extremity *c*, and if *c*A′ be drawn, making 45° with *ac* to meet *a*A′ perpendicular to it, that line represents the ray constructed into the plane of projection, and A′*a* indicates the height of A above the paper. From this the true length (A*b*) of the line can be determined.

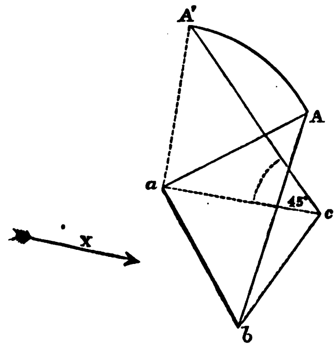

Fig. 148.

Similarly, it could be shown that the whole circumstances of position and dimensions of any solid may be obtained from one projection and the shadow. This really amounts, though, to two projections on one plane—one by projectors perpendicular to that plane, and the other (the shadow) by projectors making some other angle with it.

The latter, if made by parallel projectors, has been termed an "orthogonal projection." Hence such shadows may be defined as orthogonal projections of the objects themselves.

3. The rays of light which, by their interruption, give rise to shadows, may proceed from a luminous point near to the object, or from one at an infinite distance. In the former case the rays would diverge, but in the latter they would be practically parallel. Thus, the rays of the sun's light are to all intents and purposes parallel when they reach us. In mechanical drawings it is generally understood, that if shadows are shown they are produced by the interruption of "parallel rays;" and indication is given of the direction of these by the plan and elevation of one of them (most usually represented by two

arrows).* A few instances of shadows where the light is divergent will be discussed in this chapter; but, without it be otherwise stated, "parallelism" is to be inferred.

4. When a solid throws a shadow, one portion of itself must be in darkness, and the line which marks the boundary of the illuminated portion is called the "*line of separation.*" The student will find that this line plays a most important part in helping him to solve problems upon shadows, as from the fact that the light beyond this line always escapes interruption, it is the *projection of the line of separation which gives, the contour of the cast shadow.*

Simple methods for at once assigning the position of this line will be discussed in the problems; but it cannot be too firmly impressed upon the beginner's mind, that he has constantly to look for this line in a drawing before he can intelligently proceed with the determination of the cast shadows.

5. Sometimes the shadow of a solid falls wholly upon one of the co-ordinate planes, and at other times it occurs partly on each. This, of course, depends entirely upon the size and position of the object, and the direction of the rays of light. But it should be understood that when the latter occurs, only the portions of shade which are within the first dihedral angle are drawn, and the two parts, therefore, meet along XY. Instances of this are given in some of the problems about to be discussed.

6. The part of a solid not illuminated is sometimes called its Shadow Proper, to distinguish from the Cast Shadows to which it may give rise.

DIVISION I.

SHADOWS OF FIGURES AND SOLIDS BOUNDED BY STRAIGHT LINES AND FALLING UPON ONE PLANE.

It is our intention, first to call the student's notice to shadows cast by solids bounded by plane surfaces. Now, as the edges of such solids must be straight lines, and as straight lines may be considered as an assemblage of points, it is best—although, of course, involving the

* These directions are nearly always 45° with XY, both in plan and elevation, the former proceeding from the bottom and the latter from the top left-hand corners of the drawing.

idea of a physical impossibility—to begin by determining the shadow of a point and of a line, and then to apply the principles learned to the cases of solid bodies.

Problem CCXXVI.

Given the projections of a line, to determine its shadow upon the horizontal plane—1. When the rays of light are parallel; 2. When they diverge from a fixed point. (Figs. 149-151.)

Before proceeding to the solution of the present problem, it is well to call attention to the elementary case of a single point illustrated in fig. 149. If $a'a$ be taken as the projections of a fixed point A, and R′R as the direction of parallel rays, all that is required to find the shadow of A is to pass a ray through it in the given direction, and then to find the horizontal trace of that ray. This is shown at A_1. Of course, if the light proceeded divergently from a fixed point S′S, the ray would then be taken through A and S, and the horizontal trace would give the shadow as before. This exemplary case illustrates the method used in all instances.

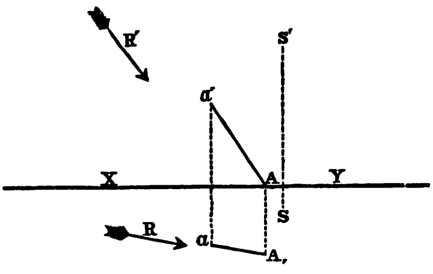

Fig. 149.

Let $a'b'$, ab (fig. 150) be the given line, and R′R the direction of the light. Through each extremity pass a ray parallel to R′R, and find their horizontal traces, as just described in the case of the single point. The line A_1B_1 joining these traces is the shadow required.

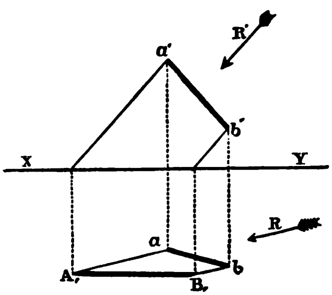

Fig. 150.

The shadow of any straight line may also be considered as the intersection of the plane upon which it is cast, with a second plane containing the line and parallel to the given ray.

In fig. 151 the same problem is solved, the light being supposed to emanate from point S′,S, and to diverge. The only modification required in the solution is that the rays passing through the extremities of the line both go through S′S. In this case the plane, which by its intersection with the h. p. gives the shadow, contains the line AB and the point S′S.

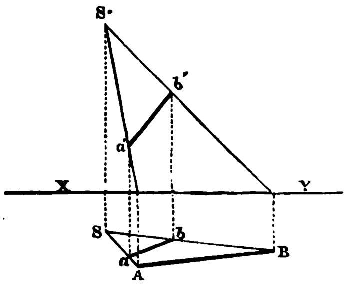

Fig. 151.

PROBLEM CCXXVII.

To determine the shadow cast by a plane figure, the rays of light being divergent, and proceeding from a given point. (Fig. 152.)

Let $a'b'c'$, abc, fig. 152, be the given figure, and S′S the source of the divergent rays. Proceed, as in the previous cases, to pass a ray from S′S through each of the points $a'a$, $b'b$, and $c'c$, and find the traces of these rays, which in the case chosen fall upon the vertical plane. The shadow $A_1B_1C_1$ is therefore on that plane instead of on the horizontal one.

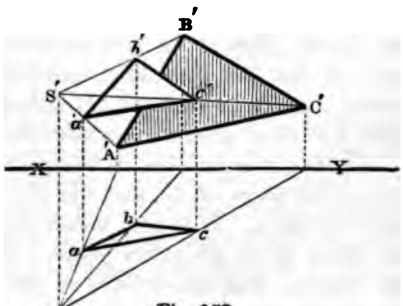

Fig. 152.

PROBLEM CCXXVIII.

To determine the parts in shade and the shadow cast by a given cube, the direction of the rays being given. (Fig. 153.)

To simplify the case, the cube is shown in one of its most elementary positions. It will be at once seen that certain of the vertical faces of the solid are illuminated. These are the two which contain *ad* and *cd*. The plan settles this, for if two parallels to R be drawn passing through the extreme corners, *a* and *c* will represent the horizontal traces of two vertical planes parallel to the given ray, and embracing the solid between them. The two faces, therefore, on the same side as the light is proceeding from will be illuminated, and the other two will be in shade. And in all cases of *vertical* surfaces which meet, the following rule will serve to tell which is in shade and which is illuminated:—Through the point which is the plan of the intersection of the two surfaces draw a line parallel to the plan of the ray, or through the plan of the luminous point if the light be divergent. Then, if this line has the plans of both surfaces on the same side of it (like pA_1 in the figure, which has *ad* and *ab* both below it), one face is in the light and the other in shade. But if the two surfaces have their plans one on each side of the line (like qB_1, which has *ab* on one side and *bc* on the other), then both surfaces are either in light or both in shade. This rule will be of great service in the case of prisms, etc.

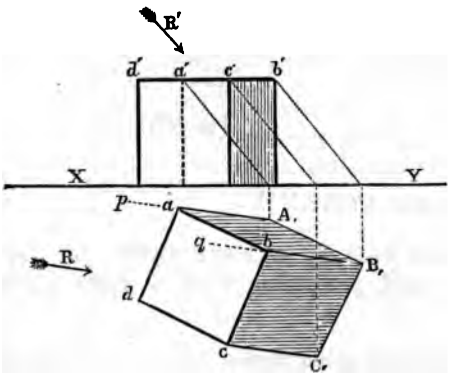

Fig. 153.

The line of separation in the problem before us is made up of the two upper edges *ab*, *bc*, and the two vertical edges, of which *a* and *c* are the plans. It is only necessary, therefore, to project *a*,*b*, etc., on to the h. p. by projectors parallel to R′R, and to join, as shown in the figure. The face *bc* being in shade, it is necessary to indicate this in the elevation by shade lines.

Problem CCXXIX.

To determine the line of separation and the shadow cast by a solid of the pyramidal form. (Fig. 154.)

Let the hexagonal pyramid (fig. 154) be the given solid. Through the vertex pass a ray parallel to R′R, and find its h. t., as at V_1. *This, then,* is the projection of the vertex in shadow.

From V_1 draw two straight lines which shall embrace between them the extreme angular points of the plan, as V_1c, V_1e. These two lines may be considered as the horizontal traces of planes containing the vertex, parallel to the given ray, and touching the solid along its two sloping edges V_1c and V_1e. All light included between these two planes is reflected from the surface of the pyramid; hence the faces V_1cd, V_1ed must be in shade, and V_1cde must be the cast shadow.

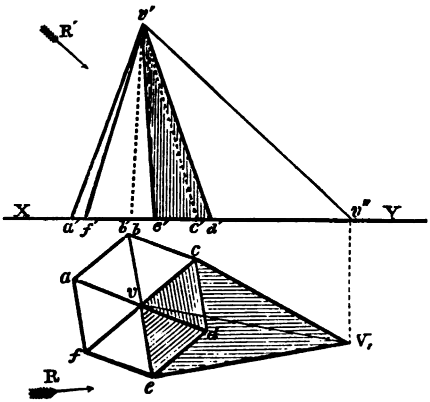

Fig. 151.

It is highly important to notice here, that in solids of the pyramidal form it is best in all cases to first determine the shadow cast by the vertex, as that must be one point in every trace of the planes containing those rays which pass through the line of separation on the solid.

For want of space, no example is given of simple solids lifted above the h. p. In such cases a set of lines, including some of the edges of the base and corresponding to those in the upper surface, forms part of the line of separation, and these lines have to be orthogonally projected before the entire contour of the cast shadow can be completed. Such cases, though, present no great difficulty.

DIVISION II.

ON THE SHADOWS OF CURVED SOLIDS CAST UPON ONE PLANE.

In the case of curved surfaces the line of separation is sometimes a curved line. Hence, a totally different treatment to that used for plane surfaced solids is rendered necessary. The rays of light, when shining around and upon a curved surface such as a sphere, are either reflected from the illuminated portion or pass on uninterrupted. But a certain set of rays merely touch the curved surface in single points, and each of these rays is a line contained by a plane which is tangential to that curved surface at the contact. Hence, in the determination both of the line of separation and the cast shadows of solids having curved surfaces, recourse is had to many of the principles discussed in our chapter on tangent planes. The problems which follow will serve to illustrate some of these applications.

Problem CCXXX.

To determine the line of separation and the shadow cast upon the horizontal plane by a vertical cylinder and cone; the direction of the rays of light being given. (Fig. 155.)

Taking the cylinder first (fig. 155), it is evident that the upper base is in the light, and that part of the circle forming it must be in the line of separation. Proceed first to project this circle on the h. p. Being in a plane parallel to the h. p., this circle will, under any conditions of parallel projection, be projected as a circle. It is necessary, therefore, to pass a ray through $a'a$ and to find its h. t. A_1. This will give the centre of a circle p_1q_1, the shadow which would be cast by the upper base if the remainder of the solid were removed.

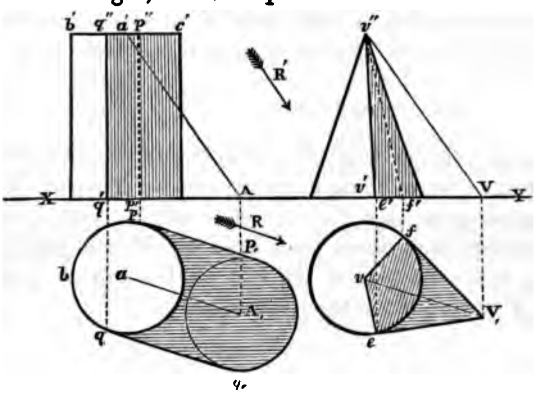

Fig. 155.

Next draw pp', qq', tangents to this circle and the plan. These are the horizontal traces of vertical planes tangential to the cylinder and parallel to the given ray.

Outside of these planes the light is uninterrupted, hence the two vertical lines of contact, whose plans are *p* and *q*, and whose elevations are shown at *p'p''* and *q'q''*, are in the line of separation of light and shade. No further description is considered necessary.

In the case of the cone it is best to begin by finding the trace of the ray through *v'v*, the vertex. This is shown at V_1. Then V_1e and V_1f, drawn through this point as tangents to the plan-circle at *e* and *f*, may be considered as the horizontal traces of the two tangent planes to the cone, which are parallel to the given ray (because they contain a line parallel to it), and between which the light illuminates one side of the solid, leaving the opposite side in shade. The lines of contact of these planes are *ve,vf*, from which *v'e'* and *v'f'* can be deduced. These two lines give the separation contour on the cone.

NOTE.—*If the cone were projected (point downwards) a larger portion would be in darkness than would be illuminated.*

PROBLEM CCXXXI.

To determine the line of separation and the shadow cast by a given sphere, the direction of the ray of light being known. (Plate XLVI., figs. 1 and 2.)

This is a practical application of the principles described in Problem CXCI., p. 233, and illustrated in Plate XLVI. The method was there shown which enables one to determine the cylinder which, having its axis in a given direction, envelopes a given sphere.

The rays of light (when parallel) which just pass over a sphere, each touching it on one point only, form an enveloping cylindrical surface, and the circle of contact of this cylinder is the line of separation of light and shade. Reverting to the figure, let the circles having centres *c'* and *c* be the projections of a given sphere, and *c't',ct* those of a given ray. Then the circle of contact of the cylinder and sphere, determined in plan and elevation, will separate the light from the shade on the solid, and the ellipse *afbc*, which is the horizontal trace of the cylinder, will give the shadow cast by the sphere on the h. p., for it is the projection of the circle of contact. The details of the solution, having been previously described, are not here repeated.

In the same plate (fig. 2) a cone is shown enveloping a sphere, and having its axis parallel to a given direction. Now, if the light proceeded from a point, as from the vertex *v'v*, this cone, by its circle of contact with the sphere, would give the line of separation upon it.

and, as in the previous instance, the cast shadow would be the large ellipse which is shown as its trace.

PROBLEM CCXXXII.

To determine the line of separation and the shadow cast by a cylinder, its axis being inclined to one co-ordinate plane but parallel to the other, the direction of the rays of light being given. (Plate LXX., fig. 1.)

Let the cylinder (axis *a'b'*, *ab*) be that which is given, and R'R the direction of the rays.

The circular end having *a'a* as centre is at once seen to be in shade, and the opposite end illuminated.

The lines EF and GH parallel to the axis, which separate upon the curved surface the light from the shade, are the lines of contact of two tangent planes, which are parallel to the given ray. To determine these, proceed as follows:—Assume any point *c'*,*c* upon the axis, and through it draw the projections of a line *c'd'*,*cd* parallel to R'R. Next draw the traces of the vertical plane *l'mn*, which contains the circular base in shade. Determine *d'd*, the intersection of the line *c'd'*,*cd*, with this plane, and join *d'a'*. Then this line will be the vertical projection of the intersection of the plane of the base with another plane which contains the axis and the line *c'd'*,*cd*. Hence, as this second plane (because of its containing *c'd'*, *cd*) is parallel to the rays of light, it must also be parallel to the two tangent planes, which, by their lines of contact, give the required separation of light and shade. Therefore, the intersection of these two tangent planes with the plane of the base must be parallel to *a'd'*,*ad*, and they must also touch the circle of the base in the two extremities of the lines of contact. This principle was described in Problem CCXVIII., Chapter XIV. Hence if two lines *i'n'* and *r'q'* be drawn parallel to *a'd'*, and tangential to the ellipse which is the elevation of the circle in shade, they will be the indefinite vertical projections of the intersection of the two tangent planes referred to with the plane of the base, and the points *e'* and *g'*, where they meet the ellipse, must necessarily be in the two parallel lines of separation, and these can therefore be drawn at once, the plans being deduced from the elevations.

As this method depends upon the use of a true ellipse, inaccuracy may be involved by bad drawing. In this instance the difficulty may be avoided thus:—Construct the circle of the base about the h. t. of its plane into the h. p. (see circle, centre A). Determine D, the elevation of *d′,d,* using *mn* as ground line (*d′* being below XY, D is equally distant on the *opposite* side of *mn*), and join AD. This line represents *a′d′*, *ad* also constructed into the h. p. about *mn*. Draw IN and RQ parallel to AD and tangential to the circle at E and G respectively. Project *e* from E and *g* from G, and thence the elevations *e′* and *g′*.

The entire line of separation of light and shade includes (referring to elevation) *e′f′*, the curve *f′p′h′*, the straight line *h′g′*, and the opposite curve *g′s′e′*. This entire line is illustrated in the supplementary figure 3 separated from the elevation. The cast shadow is its orthogonal projection on the h. p. The straight lines should be projected first, and then the two curves, points of which may be taken at random and treated as in the first problem of this chapter.

Perhaps a simpler solution of the above problem could be obtained as follows:—

First project the cylinder on a plane parallel to the bases, and determine the correct elevation of the given ray upon the same plane. Next, draw two tangents to the *circle* parallel to the new elevation of the ray. The contact points will give at once the vertical projections on the assumed plane of the two parallel lines of separation on the cylinder. The remainder of the solution would be the same as before. But this method, besides being indirect, would not apply if the axis of the given solid were inclined to both planes of projection, whereas the method previously described would be applicable in all cases.

Problem CCXXXIII.

To determine the shadows, proper and cast, of a cylinder, whose axis is inclined to both planes of projection, the direction of the rays of light being given.

The solution adopted for the last problem is applicable in this case, the only modification being that the plane of the base will have traces, neither of which will be perpendicular to XY. It therefore has been deemed unnecessary to explain the solution of this problem further, or to illustrate it.

PROBLEM CCXXXIV.

To determine the line of separation and the shadow cast by a cone whose axis is inclined to both planes of projection, the direction of the rays of light being given. (Plate LXX., fig. 2.)

The line of separation must contain two generatrices of the cone, and one of the intercepted portions of the circular contour of the base.

To determine these generatrices, pass a ray *v't'*, *vt* through the vertex parallel to the one given, and find its intersection *t',t* with the plane of the base. From this point draw two tangents to the base circle. These will meet that circle in the extremities of the required generatrices.

In the figure *l'mn* is the plane of the base (determined by Problem CXVII., p. 142), and *v't',vt* are the projections of the ray through the vertex, and parallel to R'R, *t',t* being the intersection of this line with the plane of the base. Then the two tangents to the ellipses, which are the projections of the base, viz., *t'a'*, *ta* and *t'b',tb*, give *a'a* and *b'b* (the points of contact), which are the extremities of the two straight lines meeting in the vertex, to form part of the line of separation, the remainder of that line being the uppermost curve included between these two points.

The rationale of this construction may be explained as follows:—The ray passing through the vertex is the intersection of the two tangent planes to the cone, which, being parallel to the given ray, touch the solid along the lines dividing light from shade. The traces of these two tangent planes upon the plane of the base are, of course, the straight lines touching the base circle and meeting in the point common to the three planes, *i.e.*, in the intersection of the ray through the vertex with the plane of the base.

DIVISION III.

ON SHADOWS, CAST PARTIALLY ON EACH CO-ORDINATE PLANE.

In each of the cases previously described, the entire shadow has fallen upon one or other of the co-ordinate planes. It is now proposed to consider a few selected instances where the shadow is cast partly upon one and partly upon the other. When this is the case, *the two* portions of the co-ordinate planes, which include the first dihedral angle, are treated as if they were opaque, and were capable of receiving a shadow, the two parts of which, therefore, necessarily meet along XY.

Problem CCXXXV.

To determine the shadow cast by a given straight line, the direction of the rays of light being known. (Fig. 156.)

Let *a′b′*, *ab* be the projections of the line, and R′R those of one ray. Pass a ray through *b′b*, and find its h. t. at *d*. This must be one extremity of the shadow on the h. p. But if a ray be also passed through *a′a*, it will be found that it pierces the vertical plane before reaching the horizontal. Thus the vertical trace of this ray gives at *e′* one extremity of the shadow on the v. p. But as the two extremities must not be joined to give the complete shadow, it is necessary to determine its direction on one plane, as if the other were absent. Thus *c* is the h. t. of the ray *a′c′*, *ac*, and *cd* would be the entire shadow on the h. p. if no v. p. were used. The portion *df*, therefore, is the part really cast upon the h. p., and by joining *f* to *e′* the whole is completed, for that part is cast upon the v. p.

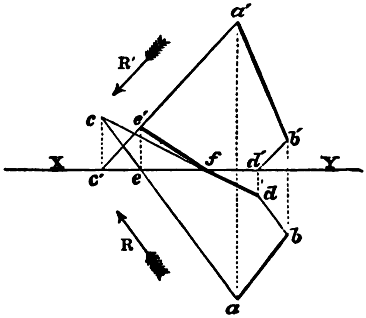

Fig. 156.

In fact, the entire shadow coincides with the traces of an oblique plane which contains the given line, and is parallel to the given ray.

Problem CCXXXVI.

To determine the shadow cast by a vertical prism, the direction of the rays of light being such as to throw part of that shadow upon each co-ordinate plane. (Fig. 157.)

Let the pentagonal prism, whose projections are shown in fig. 157, be the subject of the problem, and R′R the direction of the light. In the first place, it is evident that the separation line on the solid includes the two vertical edges, of which *b* and *e* are the plans, and the whole line is completed by the three edges of the upper base, BC, CD, DE. If these edges be projected in the usual way, it will be found that B, C, and D, will cast their shadows on the v. p.; but that E′, the v. t. of the ray through *e′e*, occurs upon the lower portion of the v. p. The shadow, therefore, of the edge DE is a broken line, one portion on the v. p., and the other on the h. p. This line must be treated in the manner described in the previous problem, *i.e.*, its vertical trace E′ must be found and joined to D_1. Then D_1F is the portion of the projection on the v. p., and FE_{11} that upon the h. p.

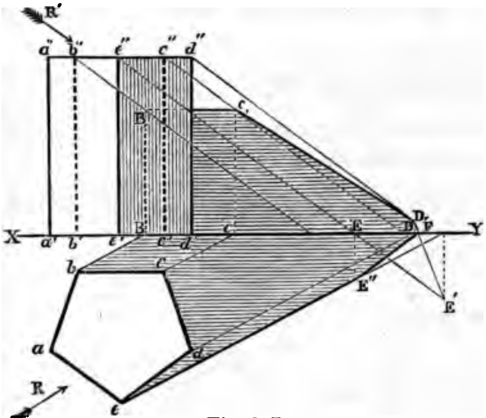
Fig. 157.

PROBLEM CCXXXVII.

To determine the line of separation and the shadow cast by the vertical cone (fig. 158), the direction of the light being so arranged as to throw part of the shadow on the v. p.

First, determine the h. t. of the ray through the vertex. This will fall upon the rear portion of the h. p. From this point draw two tangents to the base circle, meeting it in *c* and *d*, and the ground line in *m* and *n*. Then *cm* and *dn* are the shadows cast by the lines of separation as far as they fall on the h. p. Next, determine the v. t. of the ray through the vertex at V″. Join this point to *m* and *n* to complete the entire shadow. Then *cm*V″, and *dn*V″, are the traces of the two tangent planes to the cone, which are parallel to the given ray.

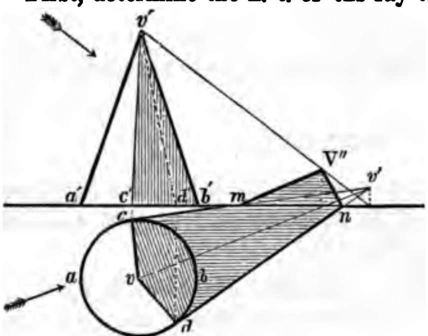
Fig. 158.

Problem CCXXXVIII.

A circular disc being so situate with regard to the co-ordinate planes, that with a given direction of parallel rays of light, the shadow cast is partly on each plane, required to determine that shadow. (Fig. 159.)

Every parallel projection of a circle is either a straight line, a circle, or an ellipse. If, therefore, the rays are inclined to the plane of the circle, as in the figure, the shadow cast on either co-ordinate plane must be elliptical in contour. Hence, should both planes be concerned in receiving this shadow, it evidently must be made up of portions of two distinct ellipses. In the figure, $D_1A_1B_1C_1$ is that which occurs upon the h. p., whilst $D_{11}A_{11}B_{11}C_{11}$ is that upon the v. p. These two curves meet in XY at p and q. In drawing this figure, it is well to remember that the major and minor axis of any ellipse, when projected orthographically, always become conjugate diameters of the elliptical projection.

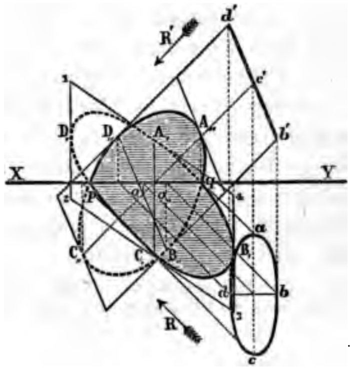

Fig. 159.

Hence, the horizontal traces of the rays through $a'a$, $b'b$, $c'c$, and $d'd$, are the extremities of two conjugate diameters of the ellipse $A_1B_1C_1D_1$, which would be the shadow of the circle if no v. p. were to intervene. Similarly, the vertical traces $A_{11}B_{11}C_{11}D_{11}$ of the same rays determine like diameters of the elliptical shadow on the v. p. These two ellipses are best completed by the construction explained in Problem LXXIX. of the Plane Geometry. If accurately worked, p and q, the meeting-points of these, will be found to fall on XY. The student will notice that the centre O is shown in both the projections of the circle. This is not absolutely necessary, but it promotes the accuracy of the work, as the conjugate diameters of the ellipses must pass through these two projections.

Problem CCXXXIX.

To determine the line of separation and the shadow cast by the cone whose projections are given in Plate LXXI., fig. 1, R'R being the direction of the parallel rays of light.

The cone is so arranged that one of its generating lines is on the h. p. The lines of separation on the curved surface are determined as in Problem CCXXXIV., but the student must notice that the point where the ray through the vertex enters the plane of the base is below the h. p., as shown at *t't*. Hence the two tangents to the ellipse in plan must be drawn from *t* (above XY). Having obtained *vm* and *vn*, and the corresponding elevations *v'm'* and *v'n'*, it will be clear that the entire separation line consists of these two and the curve *mdbn* (see plan) which connects them. And it would be enough to obtain the projections of these three lines to complete the contour of the shadow, but in the plate the entire ellipses on both planes have been left, so that the student may see the connection of this problem and the one which precedes it.

Note.—The scoring of the shadow is purposely omitted, except in the case of the base, so as to keep the construction lines clear.

DIVISION IV.

SHADOWS CAST BY SOLIDS ON OTHER SOLIDS.

In groups of solids certain of them may cast their shadows upon the surfaces of the others. In all cases it is the line of separation on the one throwing the shadow, which, being projected on the plane or curved surfaces of the others, gives the necessary contour lines. Hence, most of these problems are solved by first finding the lines of separation concerned, and then passing a series of rays parallel to the given one through points in this line, and then finding their intersections with the receiving surfaces. These intersections are points in the outline of the cast shadows required.

Problem CCXL.

To determine the entire shadows (proper and cast) of the group whose projections are given in Plate LXXII., fig. 1, the direction of the parallel rays of light being indicated by R'R.

The shadow of the lower plinth on the h. p. being first determined, proceed to discover that cast by the upper one upon the lower; thus :— Assume $i'.l'$ as the vertical trace of a new h. p. containing the uppermost surface of the lower plinth, and through the three corners $f'.f$, $g'g$ and $h'h$ pass rays parallel to R'R, and find their horizontal traces on the newly assumed plane. Join as shown in the plate.

The shadow cast by the pyramid requires special care. For the extreme planes of light which contain the two sloping edges $b'v'$, bv and $d'v'$, dv, give distinct traces on each of the three horizontal planes, viz., those of the upper surfaces of the blocks and that of the paper.

To discover these traces, determine the intersections of a ray through the vertex $v'v$, with each of these planes in turn. These are shown at v_4, v_3, v_2. Next produce the edges $b'v'$, $d'v'$ in the elevation until they meet the traces of the successive horizontal planes in p',s' and q',t', and obtain the plans of the intersections, as at $p.s.$ and $q.t$ on bv and dv produced. Join v_4 to s and t, v_3 to p and q, and v_2 to b and d. The portions of these lines which fall within the boundaries of the upper surfaces of the two plinths within the plan give the contour of the shadow cast by the pyramid upon them, for they are the horizontal traces of the two planes containing the separation lines on the pyramid upon the successive horizontal planes of the lower solids.

Problem CCXLI.

To determine the entire shadows (proper and cast) of the group (Plate LXXII., fig. 2), R'R being the direction of given parallel rays.

The only part of this problem which requires explanation here is the determination of the shadow cast by the square capital upon the vertical cylinder. Its contour line on the elevation of the curve surface is the vertical projection of the intersection of planes parallel to the given ray, and containing the edges of the capital which are concerned in casting the shadow. But as these intersections are curved lines, it is necessary to assume a series of rays passing through the edges mentioned, and to find where these rays intersect the cylinder, thus obtaining points in the curved shadow-line. Taking one case, for example, the ray through $p'p$ grazes the edges $e'f'$, ef of the square capital (this edge being included in the separation line), and, according to the plan, it is seen to intersect the cylinder at q. Hence, q' projected from q upon $p'q'$ must be one point in the elevation of the shadow-contour. The others are found in a similar manner.

The student will see that he need only assume points between *g* and *h* to solve the problem, as, beyond those positions, the cast shadow is either on the rear portion of the cylinder or is immersed in its shadow proper. It is best to ascertain these points *g* and *h* at starting, by a tangent line *h.d* to the plan-circle and a second straight line *bg* meeting the same circle in *b*, the extremity of a diameter parallel to XY, both these lines being parallel to R, the plan of the given ray. It should also be noted that the ray through the corner *e'e.* gives the meeting point on the cylinder of the two curves.

The remainder of the solution depends upon principles described in previous problems.

PROBLEM CCXLII.

To determine the shadow cast by a sphere upon an oblique plane given by its traces, the direction of the parallel rays being known. (Plate LXXI., fig. 2.)

This problem requires no new principle for its solution, as the shadows which it casts are merely the projections of the separation line or circle of contact on the h. p. and on the given oblique plane *v'fh.*

To determine the portion of the shadow which falls upon the h. p., proceed as in Problem CCXXXI. to discover the elliptical trace of the cylinder which envelopes the given sphere, and whose axis is parallel to the given ray. Only a portion of this ellipse is required, viz., from *g* to C_1 and *p*, where it meets the h. t. of the given plane.

The shadow upon the plane *v'fh*, as it is generated by the intersection of that plane with the enveloping cylinder previously mentioned, can best be found by determining the intersection of straight lines on the cylindrical surface with the given plane.

But as all orthographic projections of a circle must necessarily be ellipses (if not straight lines or circles), it is most convenient to select four points at extremities of two diameters of the contact-circle which are at right angles, and through these points to pass rays parallel to the one given, and then to determine the four intersections of these rays with the given plane, for these points must be at the extremities of conjugate diameters of the elliptical projections. The complete curves can then be inscribed in rhomboid figures, having their sides parallel to these conjugate axes, as has been described in previous problems.

Thus, in the figure *c'd'*, *cd* and *e'f*, *ef* are the projections of two diameters of the circle of contact. The rays passing through *c'c*, *d'd*, *e'e* and *f'f* meet the plane (Problem CIX., Chapter VII.) in *k'k*, *m'm*, *r'*,*r* and *s's*; and the rhomboids 1, 2, 3, 4, and 5, 6, 7, 8 are used to obtain the two ellipses which are the plan and elevation of the shadow on the given plane. Only those portions of the ellipses are required which occur above the h. p. The two portions of the shadow should, of course, meet on the h. t. *fh*.

Problem CCXLIII.

To determine the shadow cast by a vertical cone on a horizontal cylinder, R'R being the direction of the parallel rays. (Plate LXXIII., fig. 1.)

First obtain the projections *v'd'*, *vd* and *v'c'*, *vc* of the lines of separation on the cone. The tangent planes to the cone, of which these are the lines of contact, will give, by their intersection with the cylindrical surface, the outline of the required shadow.

Draw two tangents to the circle which is the elevation of the cylinder, parallel to R'. These meet the circle in *a'* and *b'*, the elevations of the separation lines on the cylinder. From these points obtain *aa* and *bb* in plan. Further, these tangent lines cross the elevations *v'd'* and *v'c'*, giving two points upon each line, and only between these points are the separation lines on the cone concerned in casting a shadow upon the cylinder. Thus, in the case of *v'c'* this part lies between 1′ and 6′ in elevation, and between 1 and 6 in plan.

Assume, therefore, a series of points in the projections of this portion as 2′2, 3′,3, 4′,4, etc., and through each of them draw rays parallel to R'R. These rays will be seen to intersect the cylinder at certain points, as shown by their elevations, and from these elevations the plans 1_1, 2_1, 3_1, etc., can be deduced. The rays through 1 and 6 are tangential to the plan curve on the cylinder, and it is better to make a horizontal section of the cone at 1′, and thence deduce the point 1 by the circle passing through *e*, rather than to obtain 1 from 1′ directly, as any inaccuracy in finding point 1 destroys the use of the tangent through it in filling in the curve. The same remark applies to the point 6.

To discover the exact position of the point of contact of the shadow-curve with the straight contour line in plan, proceed as follows:—

Through *p'* in the elevation draw a special ray to meet *v'c'* and *v'd'* in *q'* and *r'* respectively. Thence obtain *q* and *r* in plan (by circular sections of cone). Through these points draw the plans of rays parallel to R to meet the cylinder in *p* and *p*.*

Problem CCLXIV.

To determine the shadow cast by a vertical cone on a vertical cylinder, R′R being the direction of the parallel rays of light. (Plate LXXIII., fig. 2.)

The principles of construction employed in this problem are similar to those described in the previous one, and do not require special explanation. All the necessary lines are left in the plate, so that the student may be able to follow the working throughout.

Note that a small portion of shadow is cast upon the upper surface of the cylinder.

The limits of the plate prevent the whole of the shadow upon the h. p. being shown.

EXERCISES.

In these exercises the direction of the rays of light is to be assumed as making 45° with XY, both in plan and elevation, without it be otherwise stated; the elevation proceeding from the top left-hand corner, and the plan from the bottom left-hand corner of the paper. This will save cumbersome explanation.

1. Determine the shadow of a square pyramid, axis vertical and apex on the h. p. Other conditions at pleasure.

2. After obtaining the projections of a cube, in a position where none of its edges are horizontal, determine the shadow it casts upon the plane or planes of projection.

3. Draw the projections of a horizontal cylinder and those of a straight line, inclined to both planes of projection; the line to be above the cylinder. Determine the shadow of the line upon the cylinder.

Hint.—By rays through two points in the given line discover the h. t. of a plane parallel to the given ray and containing the given line. Next determine the projections of the section of the cylinder made by this plane.

4. Arrange a straight line and vertical cone as in last question, and determine the shadow of the latter on the former.

5. An octahedron of 2″ edge rests upon a square slab 4″ × 4″ × 1″, with one face on the upper surface. Obtain the shadows of the group.

* The shadow of the cylinder on the h. p. is purposely omitted.

6. A circular slab is so arranged that its faces are parallel to the h. p., but above it. A cylinder lying upon the h. p. has its axis perpendicular to the v. p., and is so placed as to receive part of the shadow of the slab. Complete the projections of the shadows proper and cast.

7. Two adjacent edges of a tetrahedron, edge 2·5", are inclined at 25° and 33° respectively. Determine the plan and the shadow cast upon the h. p., the rays of light being parallel and inclined 35°.—(Cooper's Hill Engineering College Exam. 1874.)

8. Draw a circle of 1" radius and take a point 2·5" from its centre. The circle is the plan of a sphere resting on the h. p.; the point is the plan of a luminous point 4" above that plane. Show the shadow of the sphere cast on the h. p. by rays of light diverging from the point.—(May Exam. 1870.)

9. A solid is formed by the junction of a right cone and a right cylinder. Diameter of the base of each, 2·5"; axis of cylinder, 3"; of cone, 1·75". Draw plan of the solid when the base of the cylinder touches the h. p., and the common axis is inclined at 59°. Add the shadow thrown by rays of light, making in direction an angle of 40°, with the plan of the axis, and having an inclination of 55°.

Note.—The separation lines on the cone do *not* meet the separation lines on the cylinder. Treat each solid as a distinct case.

10. A sphere of 2·5" diameter rests on the h. p. Its shadow, produced by rays of light diverging from a luminous point, is a parabola, the distance between whose focus and vertex is ·75". Determine the position of the luminous point and draw a part of the curve.—(May Exam. 1875. Honours.)

Hint.—The point must be at the same height as the highest point of the sphere, as the h. p. then cuts a cone of light parallel to one of its generatrices (see chapter on Parabola). The position of the luminous source may be deduced by the principles explained in the problems on the sections of the cone.

CHAPTER XVI.

ON ISOMETRIC PROJECTION.

If a cube be so placed that its diagonal is perpendicular to a plane of projection, the three faces meeting in the lowest point will evidently be equally inclined to that plane, and as each of the other three faces is parallel to one of these, the whole of them must necessarily have the same inclination.

Similarly, the three edges meeting in the same point, and forming the solid right angle, will also be equally inclined,* and as each of the others is parallel to one of these three, the inclination of any one is that of them all.

But equally inclined lines are equally foreshortened in their projections upon the same planes; hence, if 1 inch be measured along each of the edges of the cube, the projections of these measured lengths must also be equal.

Professor Farish of Cambridge suggested that advantage might be taken of these facts, to establish a system of projection where, from one drawing alone, the three rectangular measurements—length, breadth, and thickness—might be directly deducible.

This system of projection, which is called isometrical, has one disadvantage. It will only apply to those figures and solids which are rectangular in form. But its utility is not greatly compromised by this fact, for nearly all mechanical structures are arranged about three principal axes mutually perpendicular, and their main lines are generally parallel to these, so that those measurements which are of the most importance can readily be taken from their isometric projections.

The foundation of the system is the ordinary orthographic projection of the edges of a solid right angle upon a plane equally inclined to the three intersection lines forming it. Their isometric representation consists of three lines meeting in one point at equal angles of $\frac{360^\circ}{3}$, or 120°. Hence, in starting to make an isometrical representation of any suitable object, the setting out of these three lines must necessarily be

* The inclination of the edges is not that of the faces.

the first proceeding. They are called the *Isometric Axes*, and their intersection is the "regulating point."

On all lines parallel to these three, equal measurements correspond to the same length on the lines in the object they represent, but this is not true of any others, as will be noticed in the case of the cube. Hence, the want of applicability of this kind of projection to forms other than rectangular.

Any plane parallel to two of the isometric axes is called an isometric plane.

PROBLEM CCXLV.

To project a cube (1″ side) isometrically. (Fig. 160.)

This problem is, as has been previously explained, identical with the determination of the plan of the solid when its diagonal is vertical. Let A represent the projection of the vertical diagonal. Draw AC, AB, and AE, each making angles of 120° with the other two. From the principles already explained, it is clear that these three lines must be the indefinite projections of the three edges meeting in A, the upper extremity of the diagonal. At its lower extremity A_1, three other edges meet, and each of these is in the same vertical plane as one of the three already drawn. Hence AB, AC, and AD, being produced, the directions of the indefinite projections of the three lower edges are known. The student should be careful to note, that when the diagonal of a cube is vertical, there are three planes which bisect the cube, and contain two of its edges. Thus, AH and AB are contained by one such plane.

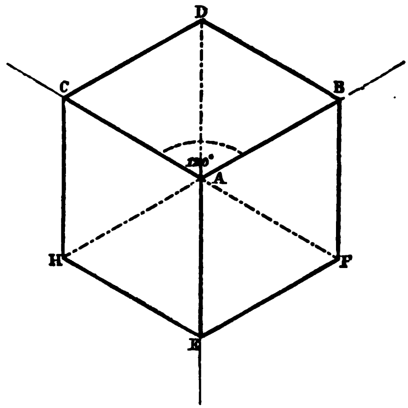

Fig. 160.

Next, to determine the length upon these projections which corresponds to 1″ on the edges of the cube.

Conceive of a vertical section of the solid containing AB and AH. Such a section would, in true shape, be a rectangle, two of whose opposite sides would be AB and A_1H, whilst the other two would be the diagonals of the faces A_1DBF and ACEH, and the diagonal of this rectangle would be AA_1, that of the entire solid.

The section is shown in fig. 161, constructed as follows:—AB and BF, each 1″ long, are drawn perpendicular to each other. Then AF gives the length of the diagonal of a face. The rectangle ABA_1H is then made upon AB, AH being equal to AF. This completes the section.

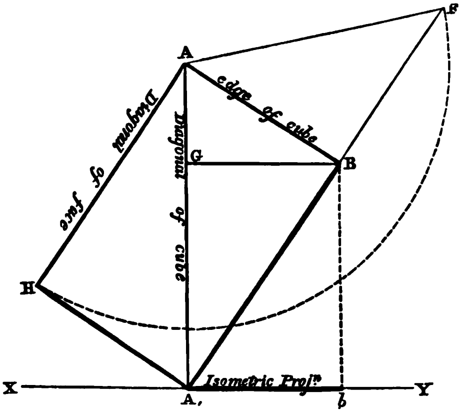

Fig. 161.

Join AA_1, and draw XY perpendicular to it; and from B let fall B*b*, also perpendicular to XY. Now, consider the rectangular section with regard to XY, as indicating the position of that section in the solid relative to the h. p. Then A_1b must be the length of the projection of AB; and the ratio of these two must be that which is constant in all cases of isometrical projection. Draw BG perpendicular to AA_1. Then BG is equal to A_1b.

But ABA_1 and ABG are similar right-angled triangles, for the angles in each are the same. Hence, $AA_1 : A_1B :: AB : BG$, but $AA_1 : A_1B :: \sqrt{3} : \sqrt{2}$ (Euclid i. 47) $\therefore$ AB : BG, or to A_1b, as $\sqrt{3} : \sqrt{2}$. The inclination of the edges is therefore given by the equation, cos. $\theta = \frac{\sqrt{3}}{\sqrt{2}}$, or about 35° 16′, and that of the face is $90° - \theta°$, or 54° 44′.

It is highly important that this numerical relation should be thoroughly understood, as it is the basis upon which the isometric scale is constructed. To complete the projection of the cube (fig. 160), measure off upon the six radiating lines a distance equal to A_1b, and join as shown. The diagonals of the faces are not projected of equal length. Thus BC is greater than AD. This is because BC is a horizontal of the isometric plane containing it, whilst AD is a line having the full inclination of that plane. Hence, measurements cannot be taken from such lines by a single scale. This proves the remark previously made, that this kind of projection is only serviceable in the delineation of rectangular forms.

The right angles which meet at A are all projected as 120°, and the other right angles become either 120° or 60°.*

Problem CCXLVI.

To construct an isometric scale showing inches divided into eighths and tenths. (Fig. 162.)

As any real length is to its isometric projection :: $\sqrt{2}$: $\sqrt{3}$, an ordinary scale is to its corresponding isometric scale in the same ratio. Hence all that is necessary to solve this problem is to compare two assumed lengths in the proper proportion to each other, and then by the principle of similar triangles, or by actual division, to obtain the smaller or larger units of the scale required.

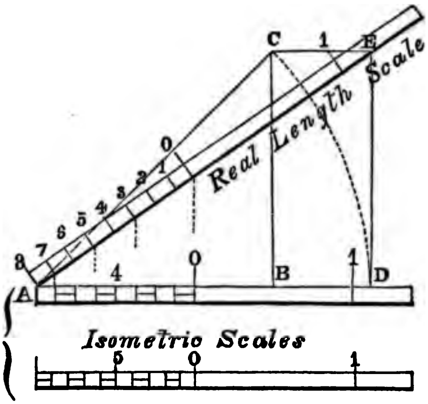

Fig. 162.

Referring to fig. 162, draw any two lines, AB and BC, perpendicular to each other and equal in length. Join AC. Then AB : AC = 1 : $\sqrt{2}$ (Euclid i. 47). Make AD equal to AC, and at D set out DE perpendicular to AD and equal to AB. Join AE. Then AE : AD as $\sqrt{3}$: $\sqrt{2}$, and any length measured along AE being taken to represent a real dimension, its isometric equivalent can at once be found on AD by parallels to DE. Hence any ordinary scale, like the one shown, where standard inches are divided into eighths, can be transferred to the isometric line directly, or one single unit may be so treated, and the remainder of the scale completed by actual division. The isometric tenths are given comparatively in the lowest scale.

At times it is necessary to show isometric scales having a representative fraction, such as $\frac{1}{48}$ or $\frac{1}{4}''$ to 1 foot, etc. In such cases the

* It is usual in practice to take advantage of the 60° set square, and the T square for setting out the directions of the isometric axes.

ordinary scale must first be drawn on AE, and the isometric one corresponding to it can be deduced at once on AD.

For instance, if the small divisions on the lowermost scale were read as feet, the representative fraction would be $\frac{1}{120}$ or 10 feet to 1 inch.

As from the principles of Isometric Projection the same length measured upon either axis corresponds to one length on the object, it is clear *that a reducing scale may be totally dispensed with*, and then the measurements taken along these axes would be the actual dimensions on the solid represented. This does not alter *the form* of the drawing. But it must be remembered that, if this plan be adopted, the projection is that of an object larger than the one it represents. But the great advantage of being able to apply an ordinary scale to the drawing, without reference to a second one, more than compensates for the evil mentioned.

The drawings illustrating this kind of projection in the Book of Plates are in all instances but one, which will be noted, worked without the isometric scale.

In Plate LXXIV. several examples are given illustrating the usual class of objects to which this kind of projection is most frequently applied.

In fig. 1 two rectangular timbers are shown meeting at a right angle, and jointed by "halving," like wall plates, to receive the ends of joists supporting a floor. The drawing speaks for itself as to the method of obtaining the lines.

In fig. 2 a mortise and tenon joint is shown, the tenon being free from the mortise. To the representation of this kind of structure isometric projection is particularly applicable.

In fig. 3 a group of two square plinths and a superposed pyramid is given. The dimensions are placed upon the drawing, and the scale adopted is $\frac{1}{120}$ *after being reduced to the proper isometric ratio of* $\frac{\sqrt{3}}{\sqrt{2}}$ —*i.e.*, the scale of tenths in fig. 162 has been used for this drawing, the divisions being assumed to represent feet. The student will see that to obtain the small square, which is the base of the upper plinth, the measurement of its sides is taken along the edges of the upper surface of the lower block, as at *ab*, leaving equal distances at each end, and that the whole is completed by drawing parallels to these edges. The sloping lines of the pyramid not being parallel to the three isometric axes, their measurements cannot be taken along their projections. Hence the apex is determined by projecting the axis of the solid, and by cutting off the length of it from that projection.

Fig. 4 represents a box with lid half open, and furnished with a separating vertical partition. The small squares *abcd* and *efgh* are used to obtain the lines representing the inside edges, so that the lid and the bottom of the box may each have a thickness. Students sometimes draw this object incorrectly, leaving no thickness at all at these places.

In fig. 5 a regular octagonal prism is shown by its isometrical projection, and although this kind of drawing is not applicable to any but rectangular forms, yet it is frequently necessary to project such solids in the midst of a complicated arrangement. Thus hexagonal nuts must be shown in the isometrical projection of a plummer-block.

To draw the subject, commence by constructing an octagon, and circumscribe it by a square, as in fig. 5*a*. Determine the isometrical projection of this square, and on its edges mark off the corner of the octagon. The long edges of the prism, being perpendicular to the plane of the octagon, are projected as parallel to that isometric axis which does not enter into the construction for the circumscribing square.

In all cases where it is necessary to discover the isometric projection of an object not rectangular in form, it must first be surrounded and intersected by lines at right angles to each other, so arranged as to catch each of its corners. Then these lines being projected as usual, they, by their intersection with each other, give the projections of the angular points in the irregular figure required.

Problem CCXLVII.

To determine the isometrical projection of the circle. (Fig. 163.)

When a circle is inclined to a plane, its projection upon that plane is an ellipse, the major axis of which is equal to the diameter, and the minor axis varies with the angle of inclination. Hence the isometric projection of the circle is an ellipse, and as the plane of projection makes a fixed angle with the plane of the circle, the minor axis must bear a fixed relation with regard to the major. For the latter is the projection of a horizontal diameter, and the former is that of a second diameter perpendicular to the first, and which, being inclined fully as much as the plane, becomes the most foreshortened.

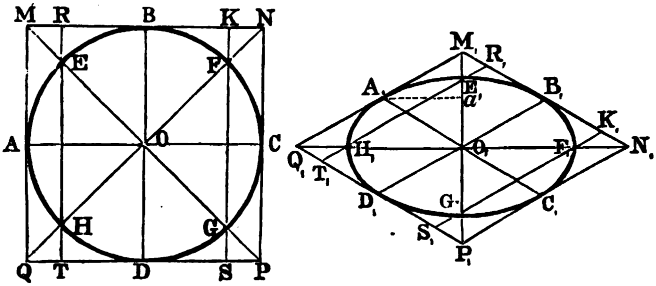

Fig. 163.

Hence the minor axis is in length equal to the diameter of the circle multiplied by $\frac{1}{\sqrt{3}}$ (the cosine of the angle of inclination). But although the above facts are sufficient for calculation, the circle is seldom or never projected from these data. It is generally circumscribed by a square. This square is projected isometrically as a rhombus. Then, by the aid of the corresponding diagonals and diameters in the two figures, points are determined in the required ellipse; or, in other words, the latter is inscribed in the rhombus.

Referring to the figure, let ABCD be the circle to be projected. Draw MNPQ, its circumscribing square, with its diagonals and diameters. These will divide the circle into 8 equal parts. Through EH and FG draw two lines parallel to the sides of the square, as RT and KS. Next project the square isometrically, as at $M_1N_1P_1Q_1$. (The isometric scale must, of course, be employed to obtain the lengths of the sides of the rhombus). Join M_1P_1 and N_1Q_1. These lines represent the two diagonals of the square. Bisect the sides of the rhombus, and join A_1C_1 and B_1D_1. These represent the two diameters of the circle. Hence $A_1B_1C_1D_1$ are four points in the required ellipse. Then mark off on P_1Q_1 two distances, P_1S_1 and Q_1T_1, whose lengths are equal to PS and QT isometrically reduced according to scale. Draw through T_1 and S_1 lines parallel to M_1Q_1 and N_1P_1, to intersect the diagonals in $E_1F_1G_1$ and H_1. These four points are the representations of EFG and H in the original drawing, and hence they are points in the ellipse. Draw the curve in by hand.

The lines A_1C_1 and B_1D_1 are called the *Isometrical Diameters* of the circle.

With regard to the relative numerical magnitudes of the principal lines in this ellipse, it is interesting to note that the major axis F_1H_1 is equal to the diameter of the circle (as previously explained). Further, as the angle $E_1H_1O_1$ is one of 30°, E_1O_1 is half of H_1E_1, and hence $H_1O_1 : E_1O_1$ as $\sqrt{3} : 1$ (Euclid i. 47). This is the relation, therefore, of the major axis to the minor.

From A_1 draw A_1a perpendicular to O_1M_1. Then from one of the properties of the ellipse O_1E_1 is a mean proportional between O_1M_1 and Oa; but O_1M_1 is equal to O_1A_1 ($O_1A_1M_1$ being an equilateral triangle). Hence $(O_1E_1)^2 = Oa.OA_1$. But O_1a is half of O_1M_1 or half of O_1A_1. Therefore $(O_1E_1)^2 = \frac{1}{2}(O_1A_1)^2$, or $\frac{O_1E_1}{1} = \frac{O_1A_1}{\sqrt{2}}$ and $O_1E_1 : O_1A_1 :: 1 : \sqrt{2}$.

The three main lines, O_1E_1, O_1A_1 and O_1H_1, are thus proved to be in magnitude in the ratio $\sqrt{3} : \sqrt{2} : 1$.

The area of the ellipse is to the area of the circle, of which it is the projection, as $\sqrt{3} : 1$, for they are proportional to the circumscribing rectangles, which in the case of the circle is a square 1 by 1 (1 being the diameter), and in that of the ellipse it it is 1 by $\frac{1}{\sqrt{3}}$ (see above).

PROBLEM CCXLVIII.

To determine the isometrical projections of a cylinder.

This problem is given to show the student a very direct way of projecting a circle, which is merely a modification of the principles just discussed. Let it be assumed that the cylinder is to be of 1″ diameter and 2″ long. Draw ab, 1″ long, and on it describe a semicircle aIb. Draw oC perpendicular to ab, and bisect the right angles. At G and F draw CD and CE, tangents to the semicircle, to meet ab produced in D and E, and oI in C. At D and E set out two lines, making 30° with ab and meeting on oI. From F and G draw Ff and Gg perpendicular to ab, to meet EK and DK in f and g. At a set out a parallel to

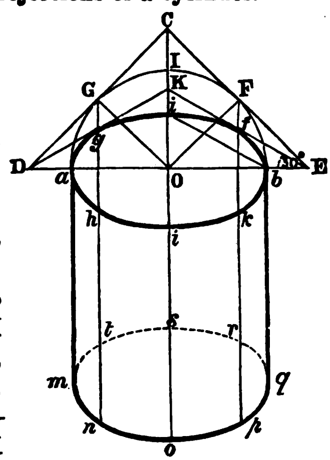

Fig. 164.

DK meeting OK in *i*. Then *a*, *b*, *f*, *g*, and *i* are points in the semi-ellipse which is the isometric projection of the semicircle, and the other half can be deduced from this one.

Instead of using the same construction for the other ellipse, a series of generatrices of the cylindrical surface can be drawn through the points determined in the upper base, and the length which represents 2″ isometrically projected can be taken off each line; the extremities thus giving points in the second ellipse.

It should be noticed that, in the above construction, DCE is one-half the circumscribing square; or, in other words, the semicircle and the square are constructed about DE into a vertical plane. The solution of the problem is the folding of these figures into an isometric plane.

In Plate LXXV. some examples are given of the application of isometrical projection to the representation of solids having curved surfaces.

Fig. 1 represents a cube with a frustum of a cone projecting at right angles from one of its faces.

The points of the circle inscribed in the rhombus A′B′C′D′ are obtained from the square A′B′CD in the usual way; but the student must, if he adopts this plan, be careful to remember that the ellipse he obtains is not the projection of the circle in this square, but of a larger one. But as the rhombus is already drawn the method shown is not objectionable, as the result is the same as if he had deduced the curve from the original circle. This ellipse being the projection of one base, the second one can be obtained thus:—In the true square, and concentric with it, draw a circle whose radius is equal to that of the required base. Circumscribe this circle by a parallel square, MNPQ. Determine the projection of this square upon the face A′B′C′D′, as at *mnpq*, and regard each of the corners as new regulating points from which lines *m*M′, *n*N′, etc., can be drawn parallel to the appropriate isometric axis, and from which the length of the frustum can be cut off. This determines the projection of the square on the new isometric plane, in which that of the required circle can be inscribed. The two remaining lines are tangents to both ellipses.

Fig. 2 is the isometric projection of two equal semi-cylinders whose axes are at right angles and in one plane. The construction needs no detailed description. The intersection of the solids forms two curved lines. One of these curves being perpendicular to the isometric plane of projection, it is shown as a straight line, whilst the other curve, being in a plane parallel to it, is projected in its true form.

The intersection points of the latter are formed by tracing horizontal lines *of the same height* upon the cylinders, until they meet. Thus, the line through *b* meets the line through *b′* in G, and that through *d′* in N.

In fig. 3 the plan, elevation, and profile view is given of a solid, which may be described as follows:—Its plan is a circle of 3″ diameter, its front elevation is a square of 3″ side, and its side elevation is an isosceles triangle, whose base and altitude are each 3″ long.

A model to illustrate its form may be made thus:—Obtain a cylindrical piece of wood 3″ in diameter and 3″ long. Then cut the solid through the diameter of one end, so that the plane of the cut may meet the extremity of a diameter on the other end, perpendicular to the first one. Repeat this cut on the opposite side, and the result will be a block which will pass through and yet fill a circular hole, a square hole, or a triangular one, according as to which part is parallel to the plane of the aperture.

Fig. 3 gives the orthographic projections of such a block, the circle AEBF being the plan, the square ABDC the front elevation, and the triangle EFC′ the profile view.

In fig. 4 this block is isometrically projected without the use of the scale, *i.e.*, all the dimensions on it are real, and not, as they theoretically should be, isometrically reduced.

The ellipse A′F′B′E′ is obtained in the usual manner. At A′ and B′ the two lines A′C′ and B′D′ are made 3″ long, or equal to AC and BD in the front elevation, and then C′ being joined to D′, the projection of the upper edge is determined. The points E′ and F′ give the positions where the two sloping planes are tangential to the base circle.

To determine points in the curves which are the projections of the two elliptical faces, proceed as follows:—

Divide the original circle into 16 equal parts, and upon the triangular elevation deduce by projectors from the division marks the heights of certain points on the two semi-ellipses. Thus, from I and VII we learn that the heights of the points of the elliptical edges vertically over these positions are equal to pC'', and at II and VI to qa''.

Through all the points of division draw parallels to the sides of the circumscribing square, and show on the isometric rhombus the projections of these parallels, thus finding the points on the ellipse corresponding to those on the base circle. At these points raise lines parallel to A′C′, and cut off the proper length on each, as described above, and draw the curves through the points obtained.

In Plate LXXVI. an excellent example of the application of isometric projection is given. In a double-framed floor the parts are arranged in planes mutually perpendicular. Hence an isometrical projection of such a construction is valuable, as giving a very clear conception of the relation of one part to another, and as providing a means whereby the workman can obtain all the principal dimensions. It is inserted in this work more to render the student familiar with the practical application of the subject he is studying than to add another geometrical exercise to those already given.

EXERCISES.

1. Draw an isometric projection of a brick; dimensions, 9″ × 4½″ × 2½″. Scale, ¼. Use Scale.*

2. Draw an isometric scale of $\frac{1}{60}$ to show feet long enough to measure 35 feet.

3. Required the isometric projection of a ladder 50 ft. long, and of the uniform width of 3 ft. Scale, 10 ft. to 1 in. Rungs and sides to be represented each by a single line. Three rungs only to be shown dividing the length of the ladder exactly. (Royal School of Mines, 1875.)

4. A circular mass of 2 ft. diameter and 4 in. thick has a hole through its centre 8″ square. Draw its isometric projection. Scale ⅛. Use Scale. (May Exam. 1872.)

5. Draw an isometric projection of an octahedron of 3″ edge—(*i.e.*, so that its three axes are so projected).

6. Determine the form of the curved figure whose isometrical projection is a circle.

7. Draw the isometric projection of a box without a lid. The height of the box is 1·5″, the base is 4″ × 2·5″, and the bottom and sides are ·25″ thick. It is to be projected in such a way as to show the interior of the box. (May Exam. 1877.)

* Where the words "Use Scale" are inserted after an exercise, the student is to understand that real dimensions are not to be directly taken on the isometric lines, but an isometric scale is to be used.

8. Two cubes, edge 1·5", rest on the h. p. 2" apart, with a vertical face in one parallel to a vertical face in the other. They are spanned by a semicircular arch springing from and covering exactly the highest horizontal faces of the cubes. Draw an isometric projection of the whole. (Royal School of Mines, 1876.)

9. Make an isometric projection which will explain the method of attaching the bridging and ceiling joists to the binder in a double floor. Scale ¼. (Cooper's Hill Exam. 1875.)

10. State and show by a figure what is the proportion between a real line and its isometric projection. (May Exam. 1875.)

11. Draw the isometric projection of a cube with another equal cube attached to each of the faces of the first one. (May Exam. 1878.)

12. A block 9" by 4·5" by 3" has a semicircular groove through its entire length cutting half through its thickness, and with the centre of the groove coinciding with the long centre line of a large face. Draw an isometrical projection of it. Scale ½. Use Scale. (Royal School of Mines, 1877.)

CHAPTER XVII.

ON THE SOLUTION OF THE SPHERICAL TRIANGLE.

ALL the plane sections of the sphere are circles, and if they contain the centre they are *great* circles.

The figure formed upon a spherical surface by the intercepted arcs of three great circles which meet, and whose planes have only one point (the centre) in common, is called a ***spherical triangle.**** This is illustrated in fig. 165, the triangle ABC being generated by arcs of the three great circles AM, CL, and BN.

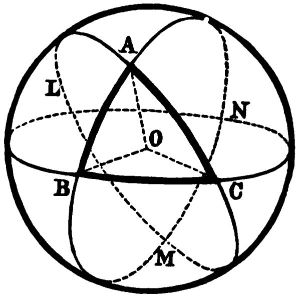

Fig. 165.

The three section planes which generate the great circles form at the centre of the sphere a *trihedral angle* similar to that which exists at the corner of a triangular pyramid, and their intersections are known as the *edges* of this angle. Consequently, a spherical triangle always subtends a companion trihedral angle at the centre of the sphere.

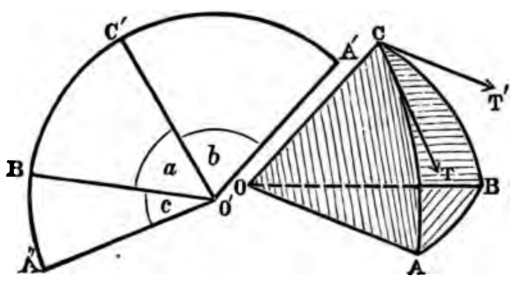

Fig. 166.

A simple model wherewith to illustrate the principles involved in the application of Practical Geometry to the solution of the Spherical Triangle can be readily prepared by first cutting a piece of cardboard into the form of a sector of a circle (A″O′A′, fig. 166), and then folding it along two radii, so selected that the free straight edges shall meet. Then the curves AB, BC, and CA will represent the *sides* of the

* Only great circles of the sphere are concerned in the generation of spherical triangles.

spherical triangle, whilst the pyramid-like corner at O will illustrate the corresponding trihedral angle.

The arcs or sides of the figure are not estimated in terms of linear measurement, but by the plane angles they subtend at the centre between the adjacent edges of the trihedral angle. Thus the side AC would be expressed by the number of degrees in the angle AOC.* Hence these angles are called the *sides* of the triangle.

The *angles* of a spherical triangle, *i.e.*, those included between the arcs forming it, are measured by the number of degrees in the plane angles included between the tangents at their points of intersection. Thus the angle at C is that which exists between CT and CT′ (these lines being the tangents to CB and CA at C).

But it is evident that CT and CT′ being perpendicular to the edge CO (for CO, a radius, is normal to all tangents to the spherical surface at C), this is equivalent to determining the dihedral angle between the contiguous faces COA and COB. And thus it appears that the solution of the spherical triangle is identical with that of its trihedral angle. Therefore, in all problems connected with this subject, the data are given and determined in respect to the trihedral angle, although spoken of as appertaining to the corresponding spherical triangle.

System of Notation.—Remembering these facts, it is easy to understand that if A, B, C denote the angular points of the spherical triangle, and *a*, *b*, *c* the curves forming its relatively opposite sides, that the former letters will refer to the angles between the faces of the trihedral angle, and the latter to the plane angles between its edges.

In finished drawings, it is usual to distinguish the *given* data by small arcs described in the angles in *firm* lines, whilst those which were to be determined are furnished with similar arcs *dotted*.

Properties of the Spherical Triangle:—

1. Any two sides must together be greater than the third side. It is necessary to observe this fact in making the model. (Fig. 166.)

2. The sum of the three sides must not exceed four right angles.

3. The three angles together must exceed two right angles, and be less than six.

4. Each angle must be less than two right angles.

* The linear measurement of the arc is connected with this angle by the equation $x = \frac{\theta^\circ}{180}\pi r$ where θ = the angle and r = radius.

The poles of a great circle are the points in which a straight line, perpendicular to its plane and passing through its centre, meets the spherical surface. Every great circle, therefore, has *two* poles.

If one pole of each of three great circles which generate a spherical triangle be so selected *that they all lie towards the same parts*, the great circles passing through these poles will give rise to a second spherical triangle, called, with reference to the first one, its "*Polar Triangle*," the original figure being known as the "Primitive."

Thus, in fig. 167, A′, B′, and C′ are poles of the circles generating the triangle ABC. Then A′B′C′, a second triangle, consisting of arcs of great circles passing through these three points is the polar of ABC, and conversely ABC is the polar of A′B′C′. These two triangles have an important property in common.

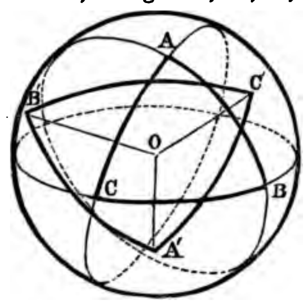

Fig. 167.

*The sides of the one are the supplements of the angles of the other.**

Thus, for instance, if *a*, *b*, and *c* be 40°, 60°, and 50° in a primitive triangle, the angles of its "polar" will be $A' = 180^{\circ} - 40^{\circ}$, or 140°; $B' = 180^{\circ} - 60^{\circ}$, or 120°; and $C' = 180^{\circ} - 50^{\circ}$, or 130°.

Hence, the one figure is sometimes described as supplemental to the other.

It follows, therefore, that the two trihedral angles subtended at the centre of the sphere possess the same relation, and hence are mutually polar, the one to the other.

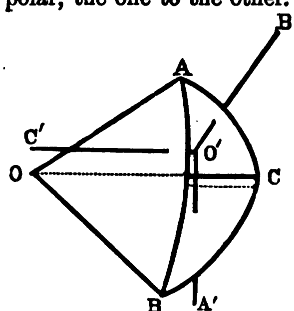

Fig. 168.

To illustrate this, arrange a simple model, as in fig. 168, and conceive a point O′, not contained by either of the three sides AOB, BOC, AOC. Imagine three straight lines to pass through this point, each of them perpendicular to one of the sides, like A′O′, B′O′, and C′O. The solid angle thus formed at O′ would be polar to that formed at O, for these three lines would be parallel to those diameters of the great circles which generate the angular points of the corresponding spherical triangle.

* For proof of this property, reference should be made to a companion volume *of this series* on Pure Mathematics, by E. Atkins.

As there are three sides and angles to the figure, any three of which being given the others can be determined, it follows that there are six cases for solution, which may be tabulated thus :—

1. Given the three sides, or from *a*, *b*, and *c*, to determine A, B, and C.
2. Given two sides and the included angle, or from *a*, *b*, C, to determine A, B, and *c*.
3. Given two sides and the angle opposite to one of them, or from *a*, *b*, A, to determine B, C, and *c*.
4. Given one side and the two adjacent angles, or from *b*, A, C, to determine B, *a*, and *c*.
5. Given one side, its opposite angle and an adjacent angle, or from *b*, B, A, to determine *a*, C, and *c*.
6. Given the three angles, or from A, B, and C, to determine *a*, *b*, and *c*.

Of these six cases, the first five are capable of direct solution conveniently, but the other one can best be solved indirectly by the aid of its polar triangle, as will be explained.

The student is strongly recommended to use a cardboard model, as previously described, whilst studying each of the cases which follow.

Problem CCXLIX.

Case 1. **Given the three sides.** (Plate LXXVII., figs. 1 and 2.)

1st Example (fig. 1).—Let the three given sides be $a=27°$, $b=35°$, $c=32°$, the sum total being less than two right angles.

Commence by setting out the three given angles in succession as at O, and consider the drawing to represent the sides developed in the plane of one of them (*b*) in the same manner as the sides of the pyramid are treated in Plate LII., fig. 2. Next, conceive *c* and *a* to turn upon the lines OA′ and OC′* towards each other, until their free edges meet. Then any two points, as *e′* and *e*, which, being one upon each of these free edges, are equidistant from O, must coincide when *a* and *b* are in their correct position, and be contained by their common intersection. And as they must move in vertical planes, perpendicular to the lines OA′ and OC′, draw through them *e′p*, *ep*, to represent the traces of these planes. Then *p* must be the projection on the plane of the side *b*, of a point in the intersection of *a* and *c*, and *op* is the plan of an edge of the trihedral angle. At *p* draw *pp″*, *pp′*, perpendicular to *e′p* and *ep* respectively. With *g* as centre, radius

* Dashes are added to the letters to prevent confusion as regards A and C, the angles to be determined.

ge', describe an arc to meet *pp″* in *p″*. Join *gp″*. Then C is the angle included between the faces *a* and *b*, for *p″gp* is a section of these two planes taken at right angles to both and folded about *e'p* into the plane of *b*.

(The aid of the model to show this is most advantageous.)

The angle A is determined in a similar manner.

The third angle B could be obtained by the construction described in Problem CXIV., Chapter VII. But it is generally solved as follows:—Draw any straight line perpendicular to *op* as *mn*, and consider it as the trace, on the plane of *b*, of a section plane perpendicular to the faces *a* and *c*, *i.e.*, to their intersection. At *m* and *n* set out *mt* and *nt*, making right angles with OB″ and OB′ respectively. Then these two lines represent on the developed sides *a* and *c*, the traces of the section plane previously adverted to. Hence, a triangle made up of *mn*, *mt*, and *nt*, will give between *mt* and *nt* the required angle between the two faces. This is most conveniently drawn by taking *m* and *n* in turn as centres, and describing arcs through t_1 and *t* to meet in *t″*.

Note that *ot* and ot_1 are necessarily equal, and that *t″* must fall upon the intersection *op*.

Thus A, B, and C are the three elements of the spherical triangle which were to be determined, as they are the angles between the faces of the subtended trihedral angle, as previously explained. The three projections of the edges are, in all cases, drawn in slightly firmer lines.

2nd Example.—In the case drawn, in fig. 2 of the same plate, the sum of the three given sides exceeds 180°. It is highly desirable to study an instance of this kind, as this problem cannot readily be solved through the medium of the polar triangle. The solution is identical with that just described, but the results give for A and C angles greater than 90.° A reference to a paper model will explain this readily. Note that *mn* is drawn perpendicular to *op* produced, and *mt* and nt_1 to OB″ and OB′ respectively, also produced. If the sides *a* and *c* were at right angles, their intersection would give an edge perpendicular to the plane of *b* (see a model).

Problem CCL.

Case 2. **Given two sides and the included angle.** (Plate LXXVII., figs. 3 and 4.)

1st Example.—Let *a* and *b* (fig. 3) be the given sides, and C the given angle between them.* Set out the two given sides in one plane, as at *a* and *b*. On OB″ take any point *e′*, and draw *e′g* perpendicular to it, and produce indefinitely. At *g* make p_1gp'' equal to the given angle C. With *g* as centre, radius equal to *ge′*, describe an arc to meet *gp″* in *p″*. Through *p″* draw *p″p*, perpendicular to *e′g*, to meet it in *p*. Then *op*, from reasons explained in the previous problem, is the projection of the free edge of the face *a*, when in its proper position relative to *b*. Hence it represents the intersection of *a* with the unknown side *c*. Through *p* draw *pe*, indefinite in length, but perpendicular to OA′, and make *pp′* (at right angles to *pe*) equal to *pp″*. Join *fp′*. Then A is the angle between the faces *a* and *c*. With *f* as centre, radius *fp′*, describe an arc to meet *pf* in *e*. Then OB′ drawn through *e* gives the developed edge of the side *c*. The angle B is determined as in the last case.

2nd Example.—A second example is shown in fig. 4, where the given angle C is 90°. The drawing will explain itself, as no new principle is introduced in its solution; it is only a special instance of the case previously described.

Problem CCLI.

Case 3. **Given two sides and the angle opposite to one of them.** (Plate LXXVIII., figs. 1 and 2.)

Let the given data be $a = 27°$, $b = 44°$, $A = 36°$.

This case is analogous to that known in the solution of the plane triangle as the *ambiguous* one. For, from the given data, two values of the angle C may result. The case of the plane triangle is easily illustrated; thus:—Let AB, AC (fig. 169), represent the lengths of two given sides, and let θ be the angle at B. Produce BD indefinitely, and with A as centre, AC as radius, describe an arc to meet AD in C′ and C″. Then either of the triangles AC′B, or AC″B, satisfies the given conditions. Hence the ambiguity. If the arc described were merely to touch

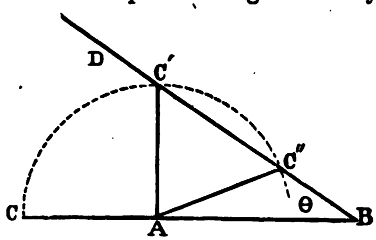

Fig. 169.

* C is here taken greater than 90°. The student, if he feels that this difficulty is troublesome at first, may refer to fig. 1, and assume that *a*, *b*, and C in that drawing constitute the given data, as the letters of reference are alike in both examples. Hence the description applies to either.

AD, there would be one solution only, the angle at C being a right angle. And if it did not meet AD at all, the case would be impossible. All these considerations equally apply to the sides of the trihedral angle. Whilst the plane of the side C is fixed in its position as regards *b*, if the face *a* be rotated upon the edge OC′ towards *c*, they may meet twice during the rotation, or once only (and in that case they would be mutually perpendicular), or not at all. This should be tried with a paper model. We see, therefore, that under the first condition C may have two distinct magnitudes.

The case illustrated in Plate LXXVIII., from the data assumed to be given, admits of two solutions. One of these is shown in fig. 1, and the other in fig. 2 (*a*, *b*, and A being the same in both). Having developed the given sides *a* and *b*, as in previous problems, assume any point *f′* in OA′, and draw *f′g* perpendicular to it to meet OC′ in *g*. At *f′* make the given angle A, and at *g* erect a perpendicular to *f′g* to meet *f′s* in *s*. Then *gf′s* represents a section of the given side *a* and of the unknown side *c*, taken at right angles to their intersection, and "constructed" into the plane of *b*. At *g* draw *gi* perpendicular to OC′, and meeting OB″ in *e′*. Consider this line as the trace of a vertical plane, in which *e′* moves when the face *a* is rotated on OC′. With *g* as centre, *gs* as radius, describe an arc to meet OC′ in *s′*. Then *s′* is a new elevation of *s* on the vertical plane in which *e′* moves, and *is′* (by joining *s′* to *i*) must therefore be the trace of the side *c* on this same vertical plane.

With *g* as centre, radius *ge′*, describe an arc to meet *is′* in *p′*, and note that it also meets the same line in *q″*. This arc is the elevation of the path of *e′*, whilst the face *a* is turning on OC′, and *p′* and *q″* give two points, one in each of the two intersection lines where that face meets *c*. Thus arise the two solutions.

In the first example *p′* is chosen, and from it *p′p* is drawn perpendicular to *e′i*, and meeting it in *p*. Then *op* is the projection of the unknown edge of the trihedral angle. The remainder of the construction is similar to that adopted in Case 2.

Example 2 (fig. 2).—Here the solution is given, using *q″* as the point in the edge. The construction is the same as before. Note that in discovering the angle B the section is turned towards O to save space. The arcs, having *m* and *n* as centres, are therefore drawn in the opposite directions to those in the previous cases.

Had the arc *p′q″* been tangential at a point K″ to *is′*, there would have been only one solution, and from K″ the remainder of the construction would have to be completed as before. Then the angle B would be a right angle.

Problem CCLII.

Case 4. **Given one side and the two adjacent angles.** (Plate LXXVIII., fig. 3.)

Assume $b = 30°$ as the given side, and $A = 60°$, $C = 50°$, as the given angles. Consider the side *b* to be horizontal as usual. Take two points, *f* and *g'*, one upon each of the edges OC', OA', and through them draw perpendiculars. Then set out the given angles as shown. Next, consider *g's'* and *fp'* as vertical traces of the sides *a* and *c* upon auxiliary vertical planes perpendicular to them. Hence, deduce the plans of two horizontal lines of the same height, one upon each plane. Thus, *q'* and *r'* may be taken as the vertical projections of two such horizontals (the dotted parallels ensuring the equality of heights). Their plans *qv* and *rv* meet in *v*, and thus give the projection on the plane of *b* of a point contained by both *a* and *c*, and hence in their intersection. The remainder of the construction from this point is exactly the same as in the previous cases.

Problem CCLIII.

Case 5. **Given two angles, and the side opposite to one of them.** (Plate LXXVIII., fig. 4.)

Let $b = 35°$, $A = 50°$, $B = 67°$, be the data given. Assume the horizontal plane to be that of the side C (the one between the two given angles), and draw upon it any straight line, OA', to represent the edge between the faces *b* and *c*. Set out A'OC', an angle of 35°, and consider it to represent the side *b*, developed in the plane of *c*. Take any vertical plane, *eq* perpendicular to OA', and determine *p'* (as in other cases) the elevation of a point in the intersection of faces *b* and *a*, and thence deduce *op* the plan of that intersection. The plane of side *a*, therefore, contains this line, and makes an angle B with the plane of side *c*, *i.e.*, with the plane of projection. Take *p'* to represent the vertex of a right cone, having its base on the h. p., the angle of its generatrices being B°.

In the drawing, *p'qp* is the elevation of one-half of this cone, and the circle *gqr* is its plan. Then any plane tangential to this cone will be inclined B° to the h. p., and will contain p'. Hence OB′ is the h. t. of a plane of this kind, which contains both o and p. It is therefore the plane of the side a, and OA′ is the second edge of the side c. The determination of a and C then depends on principles taught in previous problems.

The student will notice that through o a second trace could have been drawn to touch the circle on the opposite side; but it would be inadmissible to solve the problem, for although its plane would be inclined B° to the plane of c, the angle between the two sides would be $180° - B°$, which is not the datum given.

From this fact it is clear that, if the angle B were given greater than 90°, the cone must have an angle at its base equal to $180° - B°$, and the edge OB′ would then be drawn on the opposite side, as previously referred to.

Frequently it is considered advisable to solve Case 5 through the medium of its polar triangle.

It then resolves into the third case, thus:—Let b, B, and A be the given data. Then, as previously explained, $\pi - b = B'$, $\pi - B = b'$, and $\pi - A = a'$ of the polar triangle, so that we have given two sides and the angle opposite to one of them. If, then, this triangle be solved, the result will give A′, C′, and c', and from these the required elements of the primitive can be obtained; thus:—

$$\pi - A' = a, \quad \pi - C' = c, \text{ and } \pi - c' = C.$$

But the third case has been shown to admit, under certain conditions, of two values for the angles to be determined. Hence this would involve doubt in the solution of Case 5, which is not ambiguous.

To clear this, the student will find that one only of the results is allowable from the properties of the spherical triangle previously described. For instance, one result may make a, b, and c together greater than 2π. If so, the other solves the problem.

But this difficulty does not arise when Case 5 is worked *directly* as in the plate.

Problem CCLIV.

Case 6. Given the three angles, A, B, C.

The direct method of solving the triangle under these conditions is extremely complicated. It is best, therefore, to use the polar triangle always in this case. It resolves into the first case, thus: The supplements of the three given angles become the given sides of the polar. Thence the supplements of the three angles found become the sides required of the primitive.

Problem CCLV.

Let *abc* (fig. 170) be the plan of an angle, whose plane is inclined 30°, *ab* being horizontal; required to determine the plan of a line, *bd*, meeting the given ones in *b*, so that the angles *abd*, *dbc*, and *abc*, shall be equal.

This problem is introduced as an exercise upon the principles involved in the cases of the spherical triangle. Take any vertical plane XY perpendicular to the horizontal line *ab*, and draw the ver-

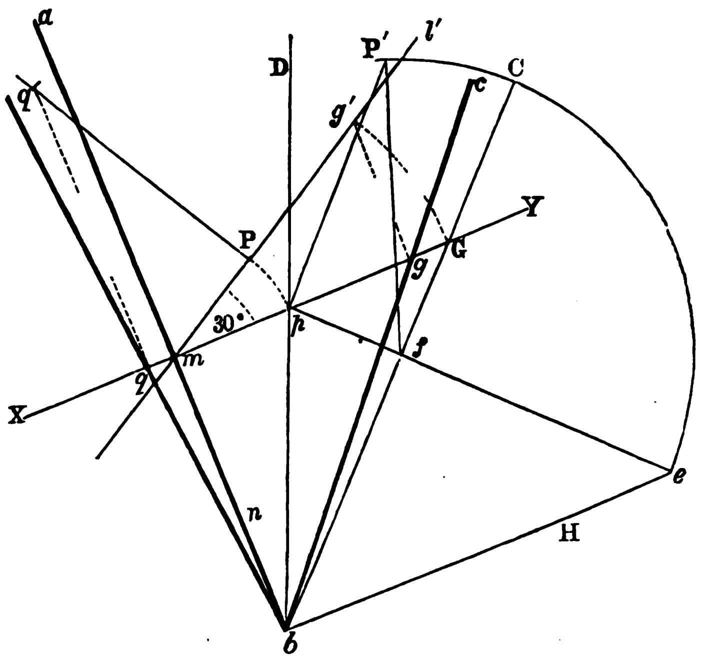

Fig. 170.

tical trace *l′m* of a plane inclined 30°. Then *g′* is the elevation of *g*, a point in the inclined line *bc*. Construct *g′* into the horizontal plane as at G, and draw *b*C through G. Then *ab*C is the plan of the angle *abc* when it is horizontal. Bisect this angle as at *b*D, and consider that line to be the plan of another which meets the first two in *b*, and makes angles equal to *ab*C with each of them. It is the plan of this line which is required in the problem when the angle *abc* is inclined. Hence, the whole arrangement has to be rebatted into the first position. To do this, the height of some point in *b*D, above the plane of *abc*, must first be obtained.

Set out at *b* the line *b*H, making the angle *cb*H equal to *ab*C, and treat it as the developed side of a trihedral angle. Thence obtain the height of *p*, as in the case of the spherical triangle, thus :—Draw *pe* perpendicular to *b*C, and meeting it in *f*. With *f* as centre, *fe* as radius, describe the arc *e*P′ to meet *p*P′, a perpendicular to *pe*. Then *p*P′ is the height of point *p* above the plane of *ab*C. The remainder of the construction is elementary. The perpendicular to the plane is shown at P*q*′ in elevation, and equal to P′*p*. Thence *q* is obtained, and *ab*, the required third line, is drawn through that point. This problem could be varied by assuming *ab* and *ac* *both* to be inclined. In that case, the plane containing them must be found, the remainder of the construction being as above.

EXERCISES.

Solve the following spherical triangles :—

1.	$a = 40°$,	$b = 50°$,	$c = 36°$.	
2.	$A = 146°$,	$B = 140°$,	$C = 125°$.	
3.	$a = 75°$,	$b = 45°$,	$C = 40°$.	
4.	$C = 30°$,	$b = 36°$,	$A = 110°$.	
5.	$a = 30°$,	$b = 90°$,	$C = 100°$.	
6.	$A = 48°$,	$B = 58°$,	$a = 60°$.	
7.	$a = 52°$,	$b = 37°$,	$C = 90°$.	(Honours, May Exam. 1870.)
8.	$b = 39°$,	$C = 60°$,	$B = 70°$.	

9. Draw the plan of a triangle ABC from the following conditions :—AB=2″, BC=3·5″, AC=3·2″, the heights of A, B, and C above the paper to be ·75″, ·35″, and 1″ respectively. Consider the plan to be that of the base of a pyramid, whose edge CD makes angles of 35° with AC, and 40° with BC, and is 3″ long. Complete the plan of the solid.

10. Taking the same conditions as in the previous problem, regarding the shape and position of the triangle ABC, complete the plan of the solid when the sides BCD and ACD are inclined 55° and 50° to ABC, the edge CD being 3″ long.

CHAPTER XVIII.

ON HORIZONTAL PROJECTION.

I. EXPLANATION OF PRINCIPLES.

THE actual position of a point or line is said to be determined when its projections are known, upon two planes mutually perpendicular. If the projectors employed be assumed at right angles with the planes, the system is said to be *orthographic*, if otherwise (the true angles between the projectors and the planes being constant, as described in the chapter on Shadows), it is *orthogonal*. But each of these methods necessitates the use of *two* projections. In Chapter XVI. a device is described, which, under certain restricted conditions, enables us to dispense with *one* of these.

The same object, however, may be attained by another system of procedure: A single projection on *one* assumed plane may be made to serve the purpose of indicating the relations of points, lines, and planes in space, by marking to a given linear scale the actual lengths of the projectors used in the determination of the *one* projection. Thus, the position of a point may be indicated by its *plan* only, its distance above or below the horizontal plane of projection being given by a figure bearing reference to a certain unit of length arranged beforehand. If that unit were ·1″, then a point marked 15 would be understood to be vertically over the plan, and 1·5″ (15 units) above it.

Similarly, all the elements of inclination, length, etc., of a straight line may be deduced from a single projection only, if the extremities or any two points in it are indexed as described.

This principle receives frequent application in the construction of drawings of fortification and topographical plans, where the intersections of the lines and surfaces concerned are exceedingly remote, or where they themselves are nearly horizontal.

Further, as separate elevations are thus dispensed with, the work is rendered more compact, and problems, which would otherwise involve complex constructions, are, by the adaptation of this method, much simplified.

As the single plane upon which the drawings are made is assumed in all cases to be horizontal, the system has been termed *Horizontal Projection.*

II. NOTES AND DEFINITIONS.

1. The horizontal plane upon which the figured plans are made is called "the plane of reference."

2. This plane may be assumed at any convenient level, depending entirely upon the nature of the heights of the points concerned. Frequently, in fact, in the same drawing advantage is taken of this principle, and several planes of reference are assumed, at different levels, care being taken to remember their relation throughout, so as not to affect the ultimate result.

If it be taken at level zero, then it becomes the ordinary h. p., and a point indixed 5 (unit = ·1″) would be recognised as the plan of one, ½″ above the reference plane.

3. Negative indices (those having a minus sign before them) are adopted to denote distances BELOW the level of zero. Thus point *b*, fig. 174, is marked – 5, and it is understood to be five-tenths below the plane of reference.

4. If a plane of reference be at level 25, a point indexed 30 must be understood to represent one whose height above that plane is 30 – 25, or 5 units. Hence, a second point indexed 10 would indicate a position 25 – 10, or 15 units *below* the plane.

5. In all cases the plane of reference should be understood to be at level zero, unless otherwise stated.

6. From the preceding paragraphs, it follows that a trace of a line is that point in it which has the same index as the reference plane.

7. If a point is not indexed, it must be understood as the projection of an indefinite vertical line. To be definite, the two indices of the extremities would be marked on the plan.

8. If a line be horizontal, it follows that it will be indexed in two points with the same figure.

9. Any straight line may be assumed as the XY, or ground line, of a vertical plane, upon which an elevation of a point or line can be determined in the usual manner, the heights being taken as the differences between the level of the plane of reference and those of the points concerned. Hence any straight line not indexed must be understood to represent the XY of a vertical plane.

10. Parallel lines have their indexed plans parallel, the figures

indicating an equal amount of rise or fall for the same lengths in plan. Thus, in fig. 171, the lines are parallel (28 – 16 being equal to 13 – 1, and *ab* being equal to *cd*).

11. The method of indicating an oblique plane may be described thus:—

Any two horizontal lines contained by the plane being selected, their indexed plans are sufficient to indicate both the direction and the amount of its slope.

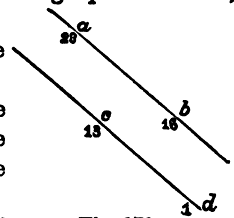

Fig. 171.

Referring to fig. 172, *fh* and *pq*, the plans of two of the horizontals of the plane *v'fh*, being figured as shown—*fh* the h. t. being marked zero, and *pq* 5 units as determined from the elevation—it is evident that the slope of the plane must be at right angles to these lines, and that the inclination θ can be ascertained by making an elevation, as at *tk"*, on a v. p. perpendicular to these horizontals. Further, as horizontals on the same plane, whose heights increase or decrease by equal amounts, will be projected equidistant from one another, it follows that, any straight line perpendicular to their plans being assumed, scale of heights can be recorded along it to indicate the absolute rise of the plane. A plane is said to be determined when this divided line is known. It is usual to arrange it in the form of an ordinary plain scale, *i.e.*, a double line is used (one darker than the other),* and the divisions are set off along it and figured, as shown in fig. 172. This enables one to distinguish an indexed line from an oblique plane. The scale is termed the "*Scale of Slope*" of the plane.

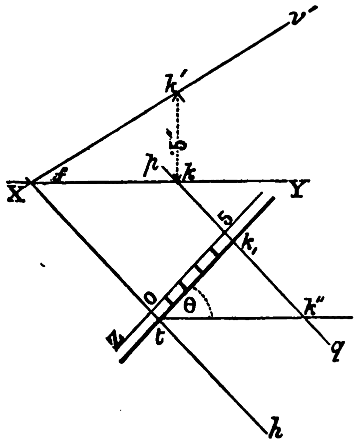

Fig. 172.

12. Parallel planes have their scales of slope parallel, and are similarly divided and indexed, *i.e.*, equal differences of height are indexed equally distant on the scale.

13. To "make an elevation" of an oblique plane, is to discover its

* The dark line should be on the left hand of a person when ascending the plane.

trace upon a v. p. parallel to the scale of slope, as shown in fig. 172. The scale of slope itself is often used as the XY.

14. If a line be perpendicular to an oblique plane, its indexed plan is parallel to the scale of slope of the plane.

15. The unit adopted in all cases is $\frac{1}{10}$ of an inch.

Problem CCLVI.

Given the indexed plan, *ab*, fig. 173, of a straight line, to determine its real length, trace, and inclination.

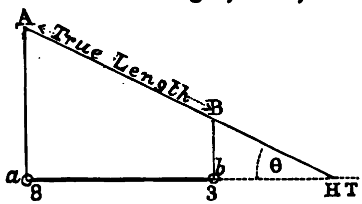

Fig. 173.

Make an elevation of the line, using *ab* as XY, the heights of A and B being ·8″ and ·3″ respectively. Then AB produced meets *ab*, also produced, in the h. t. required. The inclination is θ.

Problem CCLVII.

Given the indexed plan of a straight line (fig. 174), to determine (1) a point in it having a given index, and (2) the index of a given point.

In fig. 174, the given line is indexed 11 at *a* and − 5 at *b*. An elevation being determined, as AB (the height of B being − 5, or 5 units below *ab*), let it be required to index point *c*. Find *c′* the elevation of *c*. Then the index of *c* is evidently the number of units contained by *c′c*, its projector. Conversely, let it be required to determine a point *c* in *ab*, whose index shall be 6. Having found the elevation of AB as before, mark off on *a*A 6 units measuring from *a*. Draw C*c′* parallel to *ab*, then *c′* is the elevation, and *c* the plan of the required point. This problem is of constant recurrence in future solutions.

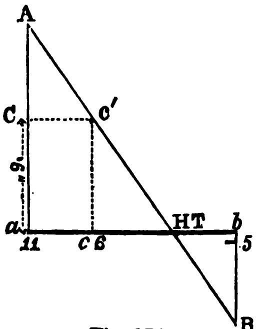

Fig. 174.

PROBLEM CCLVIII.

Given the indexed plans of two straight lines, to determine whether they meet.

If their plans be parallel, it is evident they cannot meet; if not, produce the plans till they intersect. Then determine by the last problem whether the point of intersection has the same index on elevations of each of the two given lines. If so, they do meet at the height determined.

PROBLEM CCLIX.

Through a given point *c*, to draw a line parallel to a given line *ab*. (Fig. 175.)

The projecting planes of parallel lines are themselves parallel, hence their intersections with the h. p. are also parallel. *The projections, therefore, of parallel lines are parallel.* Through *c* draw *cd* indefinite and parallel to *ab*. To index *d* add to that of *c* the difference of *a* and *b*, for parallel lines are equally inclined. Then as $19 - 7 = 12$, add 12 to -4. Then the index of *d* is 8.

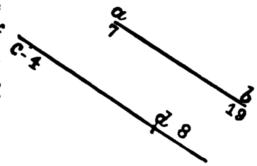

Fig. 175.

PROBLEM CCLX.

Given the scale of slope of a plane to determine its inclination, and through a given point to draw a second plane parallel to the given one. (Fig. 176.)

Make an elevation of the plane, using the given scale of slope as XY. As the indices of the given plane are 10 and 15, assume that the h. p. is at a level 10, and at 15 raise *ba* perpendicular to the scale, and 5 $(15 - 10)$ units high. Join *ac*. Then θ is the angle required.

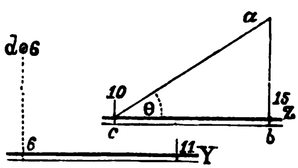

Fig. 176.

If d_6 (fig. 176) be the point given through which a plane is to pass, parallel to Z, draw the required scale of slope parallel to that of Z, and set off a distance equal to $10 - 15$ along this scale, and index it $6 - 11$. The principle of this construction has already been explained.

Problem CCLXI.

To determine the scale of slope of the plane containing three given points. (Fig. 177.)

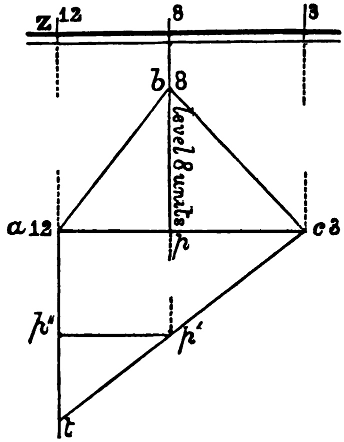

Fig. 177.

Join the two given points which have the highest and lowest indices as a_{12} and c_3. On this line ac determine a point having the same index as the remaining one b_8, as at p. Then it is evident that bp is a level or horizontal of the required plane. Next, set out a scale perpendicular to bp, as at Z, and through a or c draw a second horizontal, which will be sufficient to determine the necessary indices.

If two of the points were at the same level, the scale of slope would obviously be perpendicular to the line joining them, as that line must be a horizontal of the plane.

Problem CCLXII.

To determine the distance between two given parallel planes. (Fig. 178.)

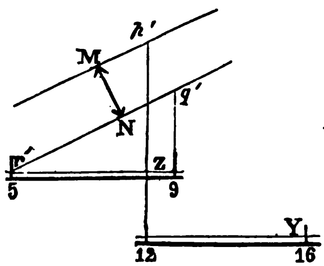

Fig. 178.

Let Y, Z (fig. 178) be the two planes given. Make an elevation of Z (working to level 5) as $r'q'$. Then, retaining the scale of plane Z for ground line, make an elevation of plane Y, remembering that p' on the projector through 12 must be only 7 units (12 − 5) in height. Then a parallel to $r'q'$ through p' will give the trace of plane Y, and MN will determine the distance required.

Problem CCLXIII.

To determine the angle contained by two given straight lines which meet. (Fig. 179.)

Let *a*, *b*, *c* be the indexed plans of the two lines. Determine a point on *ab* having the same index, as *c*. Join *pc*. Next, by means of elevations of *pb* and *bc*, find their true lengths, as at *p*B″ and *c*B′. Then a triangle whose sides are equal to the three lines *pc*, *p*B″, *c*B′ respectively, will determine the required angle, as at B. If one of the given lines were horizontal, it would be necessary to find the true length of the other one and of that joining the extremities, as, for instance, if *pc* were one of those *given* with *bp*, B*pc* would then be the required angle.

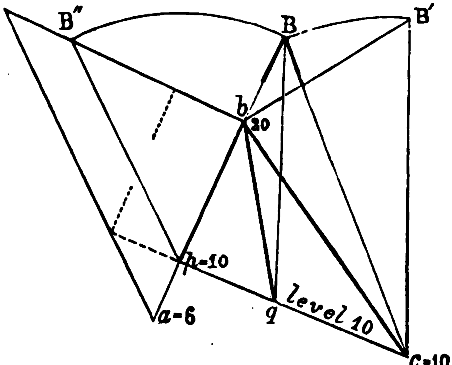

Fig. 179.

Problem CCLXIV.

To bisect the angle between two given straight lines. (Fig. 179.)

First, determine the included angle by the construction described in the previous problem. Then bisect the true angle by a line meeting the level line used. This is shown in fig. 179 as B*q*. Join *q* to *b*. It is evident, then, that *bq* is the projection of the line B*q*, as the point *q* does not move during the construction of the plane of *pbc*, *pc* being the horizontal upon which the motion hinges.

Problem CCLXV.

To determine the intersection of two given planes. (Fig. 180.)

Case 1. *When the scales of slope are parallel.* (Fig. 180.) When this is the case, as it is clear that the horizontals of both planes must be parallel, the required intersection must be a horizontal line. Referring to the figure, *v′f* and *l′m* are traces of the given planes on an elevation plane, perpendicular to their horizontals, the level of the plane of reference being taken at 10 units. Then *l′*, the point of intersection, gives the elevation of the required line in which the planes meet, and *ab* is its plan. The index of the line is obtained by adding the height of *l′*, above XY, to the assumed level 10, and should be marked as at *a* and *b* (13·7). The XY is taken in the figure parallel to the scales of slope for perspicuity. Of course, either of these scales could have been used instead.

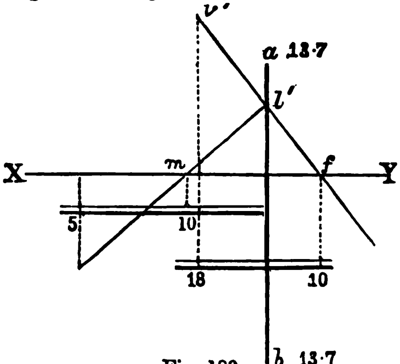

Fig. 180.

Case 2. *When the scales of slope are not parallel.* (Fig. 181.) Let X and Z (fig. 181) be the two given planes. Draw the plans

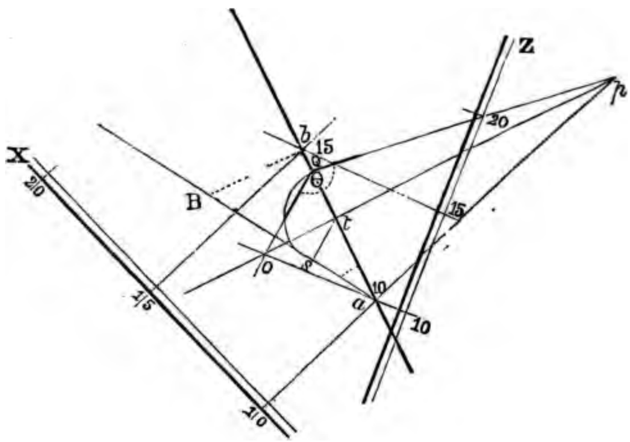

Fig. 181.

of a horizontal in each plane having the same index. Then it is evident that the intersection of these two horizontals must be in that of the two planes, as it is a point common to both. Thus the levels *10, 10,* in X and Z in the figure, meet in *a*. Then a second point being similarly determined, as at *b* (using another level), *ab* is the required intersection. Its indices are of course at *a* and *b*, those of the levels used.*

The line *a*B is an elevation of *ab*, using the plan as the XY of the v. p. (B*b* being 16 – 10 units in length). Then the angle at *a* is the inclination of the intersection determined. †

CASE 3. *When the planes are nearly parallel.* (Fig. 182.) When the planes are nearly parallel, the horizontals meet at such remote points that the construction becomes very inconvenient in practice. To obviate this difficulty, the following plan may be adopted :— Assume any third plane, and find its intersection with each of the two given ones. These intersection lines will meet in a point common to the three planes, and it follows therefore that this point must be in the required intersection of the two given. If, then, another assumed plane be taken, and thence by the same construction another point be found, the line joining the one to the other will be that required. In the figure, M and N are the given planes; Q is a plane assumed at pleasure, and its intersection with M is shown at *st*, and that with N at *cd*. Then *a* (the meeting point of these two intersections) is obviously in planes M, N, and Q. It is therefore (being common to M and N) a point in their intersection.

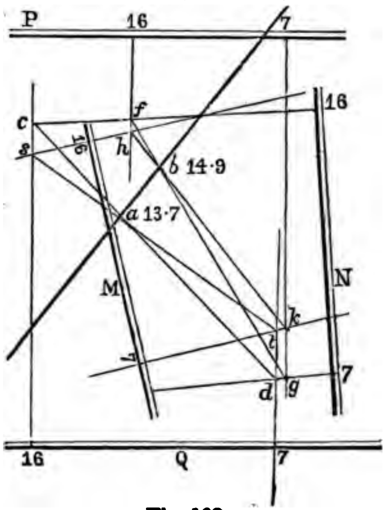

Fig. 182.

Similarly, the lines *fg* and *hk* are the intersections of the second assumed plane P with those given, and *b* is therefore another point in the meeting line of M and N. Hence, a straight line through *a* and *b* gives the plan of the intersection required. The indices of *a* and *b* are best obtained by referring to the scale of one of the planes containing them.

* The student will notice that the two planes might be given with any indices. He would then have to consider which would suit the purpose best. They are assumed alike on both planes in the figure for convenience of space.

† The remainder of the lines shown on the figure refer to the determination *of the dihedral angle* between the planes. (See next problem.)

Problem CCLXVI.

To determine the dihedral angle between two planes given by their scales of slope. (Fig. 181.)

The principle involved in the solution shown is exactly the same as that described in Problem CXIV., Chapter VII., Solid Geometry. A section of the two planes is taken perpendicular to their intersection, and the resulting triangle formed by the two lines of section on the given planes and the third line upon the plane of reference is constructed into the latter, the details of the drawing being as follows:—*op* is drawn perpendicular to *ab*. This is the trace of the section plane upon the plane of reference. It intersects that plane (on the levels 10) in *o* and *p.*; *st* is the distance of the intersection of the two planes from *t.*, and hence gives the radius by which the triangle described above can be "constructed." Again, *o*Q*p* is the true form of the triangular section; hence the angle at Q is the one desired.

Problem CCLXVII.

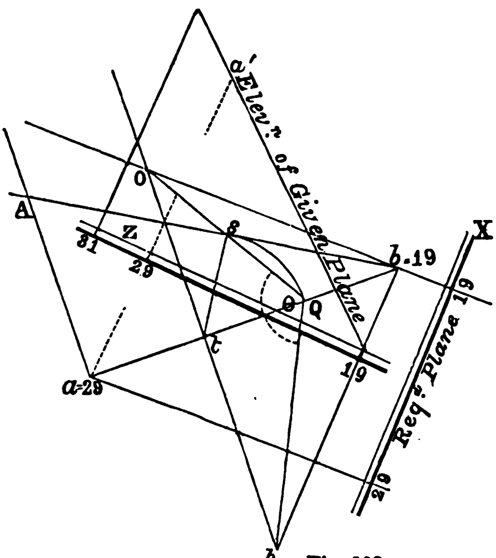

Fig. 183.

Through a given line in a given plane, to draw a second plane making a given angle with the first. (Fig. 183.)

This is a deduction from the previous problem, as the given line is necessarily the intersection of the given plane with the one required. The details of the construction may be described as follows:—Z being the given plane, and $a_{29}b_{19}$ the given line,* the

* The student should notice that *ab* is in the plane Z, for if an elevation be made of line and plane using the scale of slope as XY, then they are seen to coincide. This is shown in the figure.

line *op* is first drawn perpendicular to *ab* until it meets a horizontal 19 of the given plane Z in *p*. An elevation of *ab* is then made as at A*b*, working to level 19; thus, A*a* = 29 − 19 units, and *st* is drawn through *t* perpendicular to this elevation. Then point *s* is "constructed" by means of the arc *s*Q into the plane of reference, and the given angle with *p*Q is set out at Q, and as Q*o* meets *po* in *o*, that point must be in the level 19 of the plane required. Further, as b_{19} is also in the same level, *o* is joined to *b*, and the required scale of slope is taken perpendicular to it, a second index being taken from a_{29}.

Problem CCLXVIII.

To determine the intersection of a straight line with a plane. (Figs. 184, 185.)

1st Solution. (Fig. 184.)

Make an elevation of the plane and line, using the scale of slope as XY. Thus P6 is the trace of the plane (level 6), and *a′b′* is the projection of the line. Then *c′c* is the point of intersection required, as it is in the plane and line too.

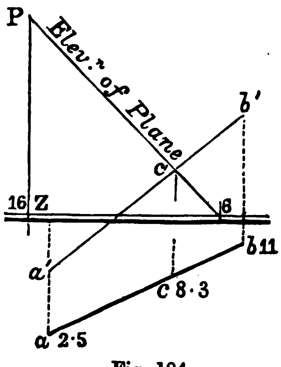

Fig. 184.

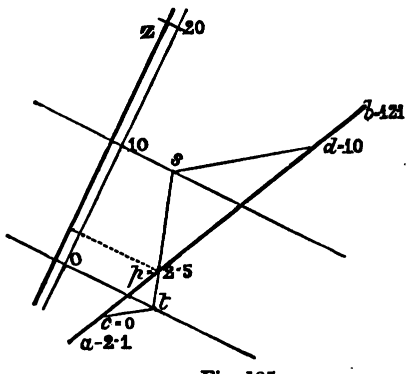

Fig. 185.

2nd Solution.

Another method of solving this problem is given in fig. 185, where Z is the given plane and $a_{-2\cdot1}b_{12\cdot1}$ the given line. Two convenient points are selected in the line, such as $c = 0$ and $d = 10$. Two horizontals of the plane are drawn next, which have the same indices as *c* and *d*. Then *d* being joined to any point in the horizontal, having the same index as *s*, *ct* is drawn parallel to *ds*, to meet in *t* the horizontal having the same level as *c*. Then *s* is joined to *t*, and this line discovers by its intersection with *ab* the point *p*, which is the desired trace of the given line on the given plane. Its index can be obtained from an elevation of the line or plane.

The principle upon which this solution depends is that a second plane is really assumed, and two of its horizontals having indices similar to those of the given plane are drawn. These are *ds* and *ct*. Hence *st* is the intersection of the given plane with the assumed one. Therefore every point in *st* is *in* both planes. But point *p*. is not only in *st* but also in line *ab*. Therefore *p* is the intersection of *a.b* with the plane Z.

Problem CCLXIX.

Through a given point to draw a line perpendicular to a given plane. (Fig. 186.)

Let Z be the given plane and b_{13} the given point. As perpendiculars to planes are always projected perpendicular to the traces of the planes, it will be evident that the plan of the required line will be perpendicular to the horizontals of the given plane, or, in other words, parallel to the scale of slope. Draw, then, *ab* parallel to Z, and consider it as the indefinite plan of the line required. Make an elevation of the plane, using the scale of slope as XY, and also an elevation *b′* of the given point. Through *b′* draw the elevation of the line, and index the plane at *c* from the height of *c′* above the plane of reference (level 5).

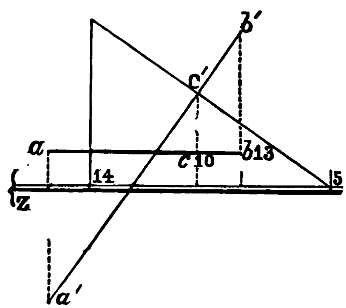

Fig. 186.

Problem CCLXX.

Through a given point to draw a plane perpendicular to a given line. (Fig. 187.)

This is the converse of the preceding problem. Make an elevation of the line and point, using the plan as XY and working to level of *a* (– 3). In the figure *ab′* is the elevation of the line, and *c′* that of the point. Then *c′e′h* are the traces of the required plane on the planes of projection, and *e′h* is a horizontal of it taken at the level of the plane of reference (– 3). A second index is obtainable from point *c′c*, which is contained by the plane.

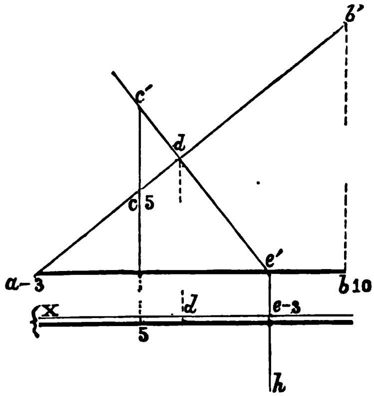

Fig. 187.

Problem CCLXXI.

To determine the angle between a given straight line and a given plane. (Fig. 188.)

The principles adopted and fully explained in Problem CXV. are

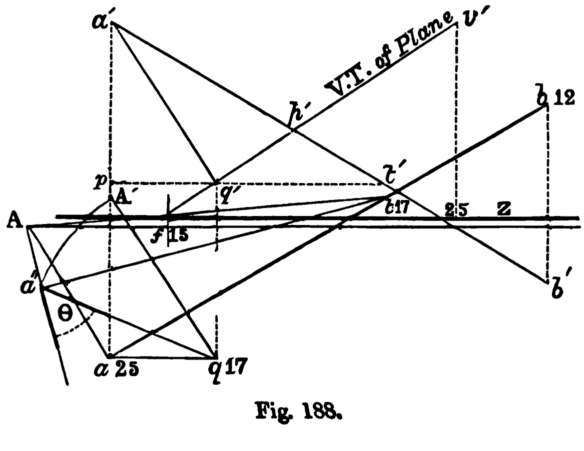

Fig. 188.

those used here, and the only necessary explanation is the adaptation of the methods of horizontal projection to the solution shown in the figure. The "Scale of Slope" gives the plane Z, and $a_{25}b_{12}$ is the plan of the line. An elevation of the plane is shown at $v'f$ (working to level 15). At a', $a'q'$ is drawn perpendicular to $v'f$, and through a, aq is made parallel to the scale of slope. Then $A'q$ is a perpendicular to the given plane passing through a point in the given line, and $q'q$ is the intersection.

A point tt' is next found on $a'b',ab$, at the same level as $q'q$ (17), and the angle between $a'q',aq$ and $a'p',ap$ is determined by folding the triangle $a'q't',aqt$ about $q't,qt$ into a horizontal plane of the same level as that line (17).

The method of working this part of the problem depends upon finding the real lengths of the two inclined lines concerned, as at $A'q$ and $At.$, and then constructing a triangle on qt, whose other two sides are equal to these real lengths just determined.

Then $qa''t$ is the angle between the given line and a perpendicular to the plane. Hence its complement θ is the angle required, as explained in Problem CXV.

Problem CCLXXII.

Through a given point in a given plane to draw a line inclined at a given angle, and contained by the plane. (Fig. 189.)

Let Z be the given plane and b_{18} the point in it. Make an elevation of the plane on any level, as 10 for instance. Find b', the elevation of b, and set out $b'A'$, making the given angle θ with the scale of slope. Then cA' gives the length of the plan of any line which, being inclined θ, has its extremities at levels 18 and that of the reference plane (in this case 10). Draw the horizontal 10 of the given plane, and with b as centre cut it with radius equal to cA', as at a. Then $ab.$ is the required projection.

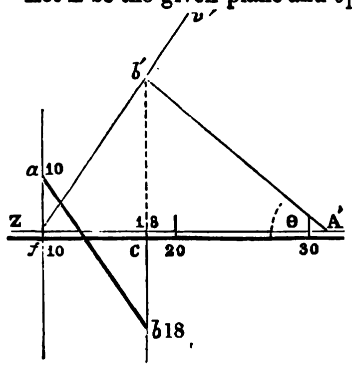

Fig. 189.

Problem CCLXXIII.

To determine a plane inclined θ, and containing a given line. (Fig. 190.)

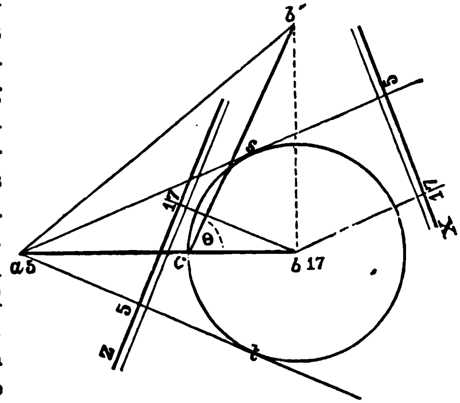

Fig. 190.

Let $a_5 b_{17}$ be the given line. Make an elevation of it, as *ab'*, using the plan as XY, and working to level of *a* (5). At any point *b'* set out the elevation of a semi-cone, having θ for its base angle. With *b* as centre, radius *bc*., describe the circle which is the trace of this cone on the plane of reference. Then any plane tangential to the curved surface will necessarily be inclined $\theta°$, and if it contains the given line, it will also contain the trace of that line. Now *a* is the trace of *ab* on the reference plane. Hence *a t*, drawn through *a* and touching the circle, must be a horizontal of a plane fulfilling the given conditions, and its scale of slope can be drawn at once as at *Z*, the necessary second horizontal (17) being obtained from *b*.

There are two planes which solve the problem. Both are shown in the figure.

Problem CCLXXIV.

Through a given point to draw a straight line inclined $\theta°$ and parallel to a given plane.

It is considered unnecessary to illustrate this problem, as it is easily solved by two others which precede it. First place a line inclined $\theta°$ in the given plane (Problem CCLXXII). Then through the given point draw a line parallel to the one determined (CCLIX.) This latter satisfies the conditions.

PROBLEM CCLXXV.

Through a given point to draw a plane inclined at a given angle and perpendicular to a given plane. (Fig. 191.)

Let Z (fig. 191) be the given plane and a_{29} the given point. Make an elevation of the plane and point, and through a' draw the vertical projection of a cone, whose base angle is $\theta°$, and of a line $a't'$ perpendicular to the given plane. Determine the traces of the cone and perpendicular on the plane of reference. These are the circle (centre a) and t_{20}. A line tangential to the circle, and passing through t, is a horizontal 20 of a plane X satisfying the conditions. There could be a second plane, which is not shown in the figure.

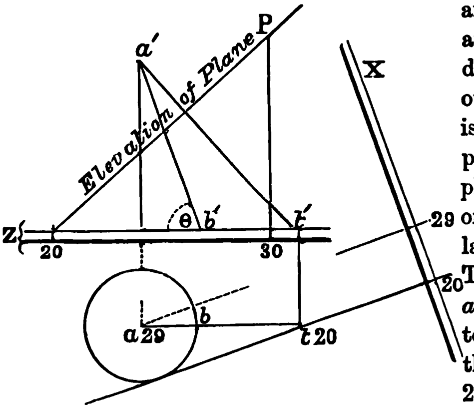

Fig. 191.

PROBLEM CCLXXVI.

Through a given line to draw a plane perpendicular to a given plane. (Fig. 192.)

Let Z be the given plane and *ab* the given line. From any two points in *ab* set out perpendiculars to the given plane. The traces of these two lines on any plane of reference will be in a horizontal of the required plane, the index of it being, of course, that of the reference plane. In the figure $a'c'$, ac and $a'd'$,

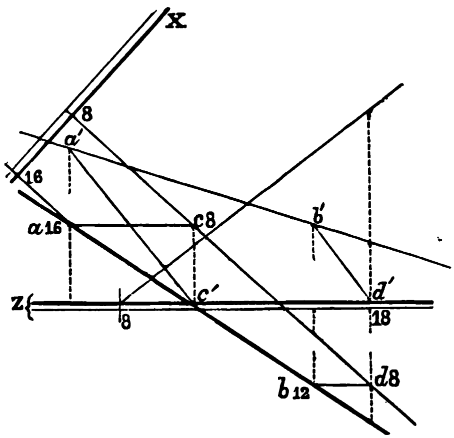

Fig. 192.

ad are the two perpendiculars to plane Z, and c_8, d_8 are their traces on the plane (level 8) to which the problem is worked. Then *cd* is a horizontal 8 of the plane X, which is the one required.

PROBLEM CCLXXVII.

Through a given point to draw a straight line to meet two given straight lines which are not parallel and do not meet.

Two solutions of this problem were illustrated and described in Problem CXXXI., Chapter VII., and the principle adopted in this case is identical with one of the methods shown there. Two planes are determined, each containing the given point, and *one* of the given lines. The intersection of these two planes is the line required. The plane Z contains $e._{13}$ (the given point) and *ab*, whilst the plane Y contains $e._{13}$ and *cd*. These are determined by Problem CCLXI. Their intersection is *gh*, which meets the lines in *g* and *f*. The indices of the two latter points can be obtained either from the planes or from elevations of *ab* and *cd*.

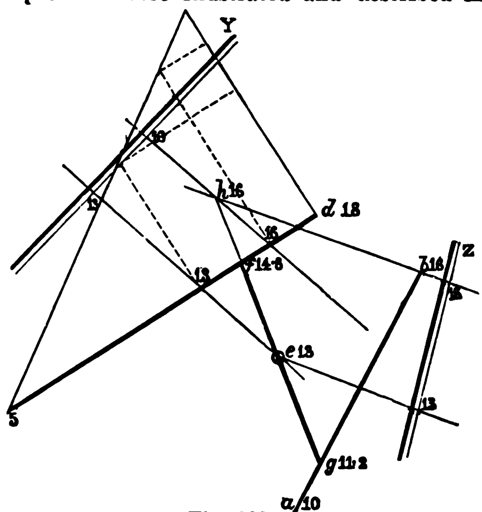

Fig. 193.

PROBLEM CCLXXVIII.

Through a given line to draw a plane parallel to another given line. (Fig. 194.)

This problem is illustrated in connection with the next one (fig. 194), where it is assumed that a plane is to be determined containing *cd*, and parallel to *ab*. The principle of solution adopted consists in setting out from any point in *cd* a line parallel to *ab*. This is shown at *dk*, which is equal to *ab*, and indexed similarly, *i.e.*, with the same run and rise. Then the plane X is determined which contains *cd* and *k*, by Problem CCLXI. This is the one required, as it is parallel to *ab*, for it contains a line parallel to it.

Problem CCLXXIX.

To draw a straight line perpendicular to two given straight lines which are neither parallel nor which meet. (Fig. 194.)

This is the same question as is discussed in Problem CXXXIII., and the method of solution is identical with that there employed. Further, as the limits and possibilities of the solution have been fully described, the student is referred to that case for their explanation.

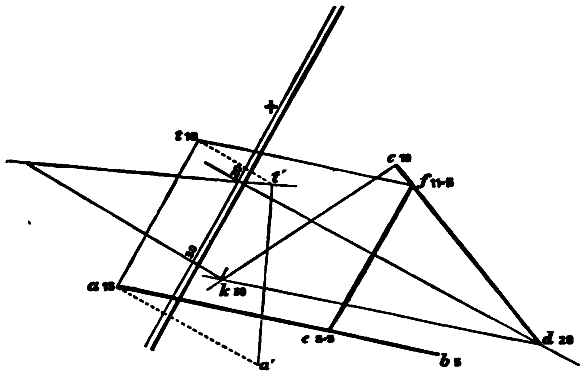

Fig. 194.

In the figure 194, the plane X containing *cd* and parallel to *ab* is first determined. Then from *a* a perpendicular to this plane is found, as $a't'$, at. The intersection of the perpendicular and plane is next discovered at t_{18}. Then $a't$, at is assumed to move parallel to itself, keeping one extremity in *ab* until it intersects *cd* in *f*. Then *ef* is the line required. Its indices are obtainable from elevations of *ab* and *cd*.

Problem CCLXXX.

Given two straight lines as the axes of two cylinders of equal diameter which touch each other. To determine the radius and complete the plans of the solids. (Plate LXXIX., fig. 2.)

This is an application of the previous problem, for the perpendicular to the two axes is the line which passes through the point of contact of the two cylinders, and which, being bisected, determines their radii.

The common perpendicular is $p_{18\cdot3}$, $q_{27\cdot3}$, determined exactly as described in the last problem; pq' gives the real length of this line, and half of it, pt', the radius of each cylinder. Further, the plan of t', indexed 22·8, is the projection of the point of contact. The elliptical ends of the cylinders in plan are obtained from elevations, using their respective axes as ground lines.

Problem CCLXXXI.

To determine a plane tangential to a given sphere at a given point on its surface. (Fig. 195.)

Let c_{15} be the centre of the given sphere, and a the plan of the given point. Join ca, and consider the plan circle as the elevation of the sphere, using ca as ground line. Project point a upon it at a'. Then a line $a't'$, drawn tangent to the circle at a', may be considered as the vertical trace of the required tangent plane. The scale of slope must, therefore, be parallel to ca, and two of its horizontals can be indexed, one from t', which gives the level 15, and the other from a, which gives 19·3.

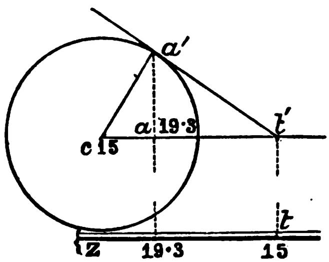

Fig. 195.

Problem CCLXXXII.

Given the indexed plan of a vertical cone resting on the h. p., also that of a point without the cone. Through the latter to determine a line inclined 20°, and tangential to the cone. (Fig. 196.)

Let c_{15} be the given cone, and a_6 the given point. Make an elevation of the axis of the cone and of the point, as at cc' and a', on a vertical plane containing them. Join $c'a'$ and produce to meet ca in f. Then $c'a'$ is the elevation of a line joining the apex of the cone to the given point, and is necessarily in the tangent plane. Hence t must be in the trace of it, upon the plane of the base which in the figure is the plane of reference. Draw *fg* (to touch the circle) as the level zero of a plane passing through *a′a* and tangent to the given cone. The problem then resolves into placing a line inclined 20° in this plane to pass through *a′a*. To effect this, set out *a′*B′, making 20° with the ground line. Then B′*a* is the length of the plan of a line inclined 20° between levels 6 and zero. Take, therefore, *a* as centre and *a*B′ as radius, and draw an arc to meet *fg* in *b*. Then *ab* is the plan of the required line. To determine the exact touching point of the line and cone, the simplest method is to show the contact line of plane and cone as *cg*. Then the intersection of *cg* with *ab* gives *t*, the desired point. Its index can be obtained from the plane, or by referring back to *a′*B′.

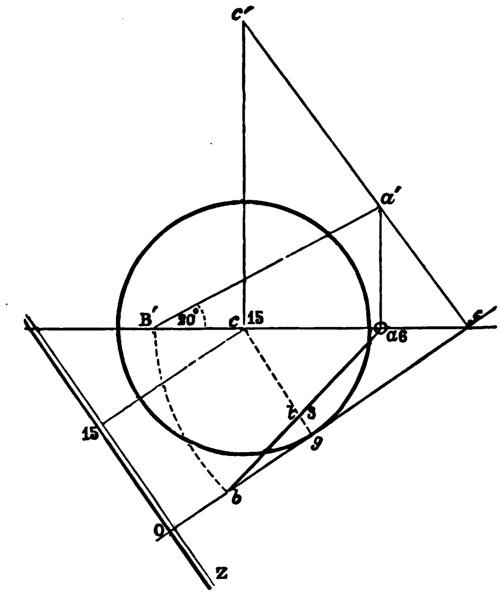

Fig. 196.

Problem CCLXXXIII.

To determine a tangent plane to an oblique cone to pass through a given point C. (Plate LXXX., fig. 1.)

Having prepared the projections of cone and point, as in the Plate, proceed as follows :—

1. Join the apex of the cone to the given point.

2. Produce this line, and discover its intersection with the plane of the base.

3. From the intersection point draw a line tangential to the base circle, and determine the exact point of contact.

4. Join this point to the apex of the cone. The line so formed will be the contact line of the required plane and cone.

5. Determine the scale of slope of the plane containing the contact line and the given point.

Referring to the figure, the details of the above construction are shown as follows :—

The vertex a_{31} is joined to the given point c_{29}, and produced indefinitely. Then an elevation of the axis being drawn (level 16) as at $a'b$, the elevation of the plane of the base is taken through b at right angles to this line. To discover the intersection of ac with its plane, an elevation of the line is made on the same v. p. as at $a'c'$, and this line is produced to meet the trace at m'. Project $m_{23 \cdot 4}$ from m'. The next proceeding would be to draw a tangent to the ellipse from m. But a better plan, so as to ensure accuracy, is to construct the plane of the base together with the circle, and point M into the reference plane. This is done by drawing a semicircle on fg to represent half the base, and by rebatting point $m'm$ in the usual way, as at M. Then MK drawn tangential to the circle discovers K, the point in the base where the contact line has its extremity. From K we get $k_{28 \cdot 5}$. Join ak. Then the scale of slope of the plane required is found (Problem CCLXI.) as passing through ak and c. It should be noticed that a second plane could be determined by drawing a second tangent from M, but only one is shown in the figure.

Problem CCLXXXIV.

To determine a plane tangential to an oblique cylinder, and parallel to a given line CD. (Plate LXXX., fig. 2.)

This is a somewhat similar problem to the preceding. Having drawn the cylinder axis ab and the given line cd, the various steps of the solution are as follows :—

1. At any point in the axis a line is drawn parallel to the given one.

Thus b_{30}, f_{15} is parallel to d_{40}, c_{25}.

2. This line is produced until it intersects the plane of the base in e_7. The elevation of this point in the figure is at e' in the trace of the base plane (perpendicular to the elevation of the axis). It should

be noticed that the working level being taken at 15, the elevation f' occurs on *ab*. The plan of e' is at e_7.

3. The point $e'e$ is then joined to a, and the base plane, together with the circle and this line, are constructed into the plane of reference. The constructed point $e'e$ is shown at E. Therefore Ea represents the line $e'a'$, ea. Now, this line is really the intersection with the plane of the base of a plane containing the axis and parallel to the given line. But the two tangent planes (for there are two, one on each side of the cylinder) being parallel to the same given line, must be parallel to that just described. Hence their intersections with the plane of the base must be parallel to $a'e'$, ae, and, further, must touch the base circle.

4. Hence SR is drawn parallel to aE to represent one of these intersection lines, and the point S is discovered, from which s_{22} is obtained, as one extremity of the line in which the cylinder is touched by the required tangent plane.

5. Then, as this line is necessarily parallel to the axis, *st* can be drawn at once, being indexed with the same run and rise as *ab*.

6. It is then necessary to determine a third point not in *st* before the plane can be found. Now, it is known that the line SR is in the plane required. Therefore its plan *sr* must be obtained by rotation of the base plane, and r_{14}* will then complete the data by which the scale of slope Z may be determined.

This problem would be required if the separation lines of light and shade on the cylinder were to be obtained, the rays producing it being parallel to *cd*.

Problem CCLXXXV.

To determine a tangent plane to a given cone and sphere. (Fig. 197.)

Let the circle a_{10} represent the cone, and $c._7$ the sphere.

Join a to c, and make an elevation of both as in the figure. Over the sphere draw a cone, having the same base angle as the given one,

* Notice that as SR is parallel to aE, *sr* is drawn parallel to *ae*, and r is indexed 14 as e being (15 − 7) lower than a, r is made 8 lower than s_{22}.

and proceed to find the level zero as in Problem CCXI. Further explanation is unnecessary.

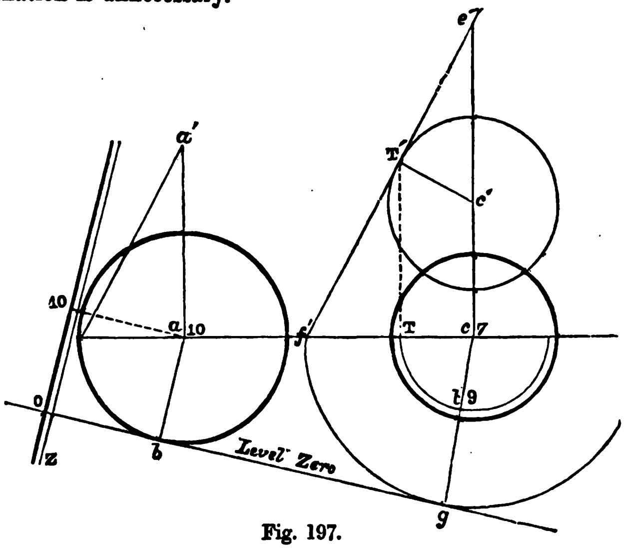

Fig. 197.

PROBLEM CCLXXXVI.

To determine a plane tangential to a given ellipsoid, whose major axis is horizontal, and to pass through a given point upon its surface. (Fig. 198.)

Let the ellipse (fig. 198) be the plan of the given ellipsoid, its centre at level 7. Also, let $e._{11}$ be the given point on its surface.

A tangent plane to a curved surface contains all the lines which are themselves tangent to that surface, and which pass through the point of contact.

From the above definition it follows that, if two tangent lines can be found which shall touch the ellipsoid at the given point, the plane containing these lines will be the one required. Make, therefore, a section $e'f'$ through e perpendicular to the major axis, and construct it into the plane of reference. In the figure, the solution is worked to the level of $o._7$. Hence $e'if'$ is one half of the "constructed" section, and e'' represents e upon that section. Next, draw $e''h$ to touch the semicircle in e'', and produce it until it meets $e'f'$ produced in h_7. This line represents one tangent through the given point, and *h* is its trace on the plane of reference.

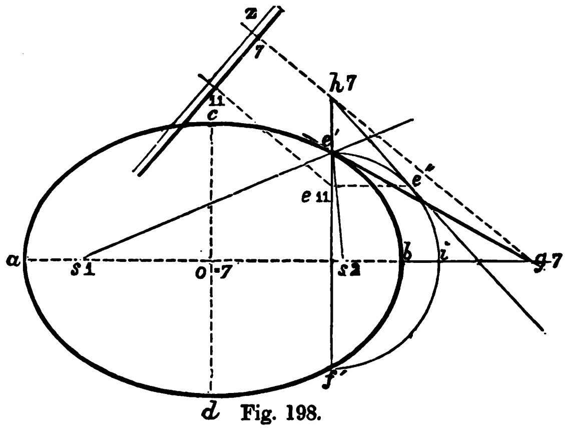

Fig. 198.

Again, conceive a plane section of the solid containing the major axis and the given point. This section would be in true shape an ellipse similar to *acbd*. Imagine this section constructed on the line *ab* into the reference plane. Then the section would coincide with *acbd*, and point *e* would be at *e'*.

Conceive also that a tangent to this ellipse at *e'e* in the plane of section has been constructed with it. Then *e'g* would represent that tangent on the plane of reference. But during the "construction" *g* is stationary, being on the major axis produced. Hence its index is 7; and as it is also in the required tangent plane, a horizontal 7 of that plane is discoverable by joining *g* to 7, and the scale of slope can be set out at once, the second index being obtained from e_{11}, the given point.

Problem CCLXXXVII.

Given the indexed plans of the axis of a cone and sphere. Assuming that the two surfaces are in contact, required to determine the projection of the cone. (Plate LXXIX., fig. 1.)

Let a_{24}, b_6 (see figure) be the plan of the axis of the cone, and the circle centre c_{36} that of the sphere.

The problem is solved by the following construction:—

1. The plane containing the given line and point is first determined.

This plane would make a section of both solids, which would contain the point of contact; such section consisting of an isosceles triangle tangential to a circle.

2. The plane just determined is "constructed" into the plane of reference, together with the given line and point.

3. The isosceles triangle is then completed in such a way that the given axis may be its centre line, and one of its sides be tangent to the circle. This determines the diameter of the base.

4. An elevation of the whole cone can then be made on a vertical plane containing the axis, and thence the required plan can be deduced.

The details of the construction in the figure are as follows:—The points b_6 and c_{36} being joined, d_{24} is determined, and ad becomes a level 24 of the plane containing the three points. Then xy is taken through a, and used as the intersection line of a plane of elevation upon which its trace $b'c'$ is drawn. Points b and c are then constructed into the reference plane at B and C, and a circle equal to the original circle is drawn with C as centre. Then through B, a tangent to this circle gives one side of the isosceles triangle, whilst a right angle through A to meet it completes one half of it, and discovers the length of the radius of the base in aE, and the constructed point of contact in T.

Again, using ab as ground line, an elevation ba' is made of the axis, and a line MN, perpendicular to $a'b$, becomes the trace of the plane of the base, and a'M and a'N being made equal to aE, the ellipse is projected from it, thus completing the plan. Point t_{31} is obtained directly from T, by referring to the plane of a, b, and c.

Problem CCLXXXVIII.

Given the indexed plans of a cylinder, and of the apex and axis of a cone. Assuming that the two surfaces are in contact, required to complete the plan of the cone. (Plate LXXXI.)

Let the cylinder $a._{10}$, b_{20}, and the line c_4, d_{60}, be the given data, c being the apex of the cone.

The principle of the solution may be described thus:—

1. It is known that the two surfaces, if in contact, have a common tangent plane. Therefore, through c_4 a plane is first determined tangential to the given cylinder.

2. The line of contact of the cone with this plane is its intersection with another plane perpendicular to it, containing the given line *ab*. But c_4 is necessarily in this intersection. Hence a perpendicular to the tangent plane passing through d_{60} gives a second point *t*, by which this contact line can be finished.

3. The contact lines on the two surfaces meet in i_{20}, which is therefore the point of contact of cone and cylinder.

4. Elevations of (i.) the axis *cd*, (ii.) of the plane of the base perpendicular to *cd*, and (iii.) of the contact line *ct*, are then made on a v. p. containing *cd*, and the intersection of the contact line with the base plane is determined at *s*. S is therefore in the circular base.

5. The point *s* being in the circumference, *ds* is a radius, and its true length being discovered, one is able to complete the elliptical plan from the elevation.

Turning to the details, the tangent plane described in par. 1 is found thus:—Through c_4 a line is drawn parallel to the axis, and by means of elevations, taking *ab* as ground line, the intersection of this line with the plane of the base of the cylinder is determined at *e′.e.* Note that the level used is 10 units, and that *ab′* and *c′e′* are the elevations of the axis and line, *mn* being the trace of the base plane. Point *e′e* and the base circle are then constructed about the horizontal, through *a*, into the reference plane, giving *y*F′*z* and E. Through E a tangent is drawn to the semicircle to meet it in F′, which gives the point in the base circle where the tangent plane touches it. Then $f_{15 \cdot 5}$ is projected from F′, and the entire contact line *fg* is drawn parallel to *ab* with the same run and rise. The tangent plane Z is shown by its "scale of slope" as containing points *f*, *g*, and *c*.

The scale of slope is then used as XY for an elevation of the plane and of point *d* (working to level 15·5), and from *d′*, *d′t′* is drawn perpendicular to the trace of the plane, and meeting it in *t′*; and $t_{47 \cdot 5}$ is deduced from *t′* on *dt*, drawn parallel to the "scale of slope." Then c_4 is joined to $t_{47 \cdot 5}$ to give the line of contact of cone and plane. These are the details of the proceeding described in par. 2. (See Problem CCLXIX.)

Next, *cd* is taken as ground line, and elevations of *cd*, of the base plane, and of *ct*, the line of contact, are made; but as the height of *d″* would be so great if taken at 60, the whole has been supposed to be lowered 32 units. This was done to enable the lines to be included in the plate. Hence D is taken at (60 − 32) units high, *c″* at 4 − 32, or − 28 units below, and *t″* at 47·5 − 32, or 15·5 units. Then *c″t″* is an elevation of the contact line, and s' is that of its intersection with the base plane from which $s_{50\cdot5}$ is obtained upon *ct* produced. The length of the radius of the base is $d''s$, which is measured from D to p' and q', and thence the elliptical projection of the base of the cone is determined.

EXERCISES.

1. Draw an isosceles triangle, the base 2″, and the equal sides 3·5″; index the apex as 30, and the corners of the base as 10 and 17. Determine the true shape of the figure, of which this is the projection.

2. Taking the data of the previous question, draw a line through either corner to meet the opposite side at a right angle.

3. Taking the data of the first question, determine the projection of the circle containing the three angular points.

4. Draw an irregular quadrilateral figure, no side less than 2·5″ long, and index the corners as 5, 12, 20, and 17 units. Determine the sphere passing through the four points.

5. Draw the scale of slope of a plane where two horizontals, ·6″ apart, are indexed 30 and 40. In this plane place two lines meeting at level 27, and inclined 27° and 50° respectively. Then determine the angle between these lines.

6. Taking the same data as in Plate LXXIX., fig. 1, assume the given line to be the axis of a cylinder touching the sphere. Determine the projections.

7. Draw the indexed plans of three unequal circles at different levels, and determine a tangent plane to them, passing *over* all three.

8. Draw an equilateral triangle of 3″ side, and index the corners at 7, 12, and 18. With these points as centres, describe circles of 1·5″, 1″, and ·6″ radius respectively. Consider the circles as plans of spheres, and determine the centre of a fourth sphere of ·75″ radius, resting upon and touching the three given ones.

9. Assume two planes by their scales of slope, and draw a line through any point parallel to both planes.

10. Determine a tangent plane to the cone and sphere in Plate LXXIX., fig. 1, not passing through the contact point of the two surfaces.

12. Draw a square of 2″ side, and index three of its angular points as 6, 20, 13. Then determine the index of the fourth corner (all being in one plane), and also the plan of a point 3″ from each of the four.

12. Taking the sphere and line in Plate LXXIX. as data, determine the plan of the tetrahedron which circumscribes the sphere, and has one edge parallel to the given line.

13. Referring to figure 193, find the shortest line which, passing through e_{12}, shall meet the planes Y and Z, and be bisected in that point.

14. Draw a straight line 3″ long, and index its extremities at 5 and 27. Consider this as the axis of a sphere, and determine the plan of the great circle whose plane is perpendicular to the line.

15. Show by their scales of slope two planes inclined 47° and 60°, and perpendicular to each other.

16. Draw two straight lines, *ab* and *cd*, each 4·5″ long, bisecting each other at right angles. Figure them as follows:—$a=6$, $b=26$, $c=10$, $d=60$. Consider the two lines to be the axes of two cones in contact, and having their apices at *a* and *c*, the base diameter of the lower one to be 3·5″. Determine the plans of both, and of the point of contact.