THE PRINCIPLES

OF

ELLIPTIC AND HYPERBOLIC ANALYSIS

BY

ALEXANDER MACFARLANE, M.A., D.Sc., LL.D.

Fellow of the Royal Society of Edinburgh; Professor of Physics in the University of Texas

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[Abstract read before the Mathematical Congress at Chicago, August 24, 1893.*]

In several papers recently published, entitled "Principles of the Algebra of Physics," "The Imaginary of Algebra," and "The Fundamental Theorems of Analysis generalized for Space," I have considered the principles of vector analysis; and also the principles of versor analysis, the versor being circular, logarithmic, or equilateral-hyperbolic. In the present paper, I propose to consider the versor part of space analysis more fully, and to extend the investigation to elliptic and hyperbolic versors. The order of the investigation is as follows: The fundamental theorem of trigonometry is investigated for the sphere, the ellipsoid of revolution, and the general ellipsoid; then for the equilateral hyperboloid of two sheets, the equilateral hyperboloid of one sheet, and the general hyperboloid. Subsequently, the principles arrived at are applied to find the complete form of other theorems in spherical trigonometry, and to deduce the generalized theorems for the ellipsoid and the hyperboloid. At the end, the analogues of the rotation theorem are deduced.

FUNDAMENTAL THEOREM FOR THE SPHERE.

Let α^A and β^B denote any two spherical versors; their planes will intersect in the axis which is perpendicular to α and β, and

* Jan. 8, 1894. I have rewritten and extended the original paper so as to include the trigonometry of the general ellipsoid and hyperboloid. At the time of reading the paper, I had discovered how to make this extension, but had not had time to work it out.

which we denote by $\overline{\alpha\beta}$. Let OPA (Fig. 1) represent α^A, and OAQ represent β^B; then OPQ, the third side of the spherical triangle, represents the product $\alpha^A\beta^B$.

To prove that

$$\alpha^A\beta^B = \cos A\cos B - \sin A\sin B\cos\alpha\beta$$
$$+\{\cos B\sin A\cdot\alpha + \cos A\sin B\cdot\beta - \sin A\sin B\sin\alpha\beta\cdot\overline{\alpha\beta}\}^{\frac{\pi}{2}}.$$

The first part of this proposition, namely, that

$$\cos\alpha^A\beta^B = \cos A\cos B - \sin A\sin B\cos\alpha\beta,$$

is equivalent to the well-known fundamental theorem of Spherical Trigonometry; the only difference is, that $\alpha\beta$ denotes, not the angle included by the sides, but the angle between the planes; or, to speak more accurately, the angle between the axes α and β. It is more difficult to prove the complementary proposition, namely, that

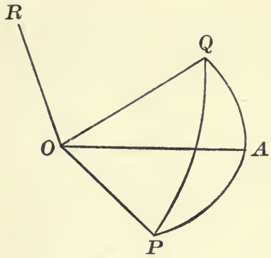

FIG. 1.

$$\operatorname{Sin}\alpha^A\beta^B = \cos B\sin A\cdot\alpha + \cos A\sin B\cdot\beta$$
$$- \sin A\sin B\sin\alpha\beta\cdot\overline{\alpha\beta},$$

for it is necessary to prove, not only that the magnitude of the right-hand member is equal to $\sqrt{1-\cos^2\alpha^A\beta^B}$, but also that its direction coincides with the axis normal to the plane of OPQ. At page 7 of "*Fundamental Theorems,*" I have proved the above statement as regards the magnitude, but I was then unable to give a general proof as regards the axis. Now, however, I am able to supply a general proof, and it will be found of the highest importance in the further development of the analysis.

In Fig. 1, OP is the initial line of α^A, and OQ the terminal line of β^B; let OR be drawn equal to

$$\cos B\sin A\cdot\alpha + \cos A\sin B\cdot\beta - \sin A\sin B\sin\alpha\beta\cdot\overline{\alpha\beta};$$

it is required to prove that OR is perpendicular to OP and to OQ.

Now, $$OP = \alpha^{-A}\,\overline{\alpha\beta} = (\cos A - \sin A\cdot\alpha^{\frac{\pi}{2}})\cdot\overline{\alpha\beta}$$
$$= \cos A\cdot\overline{\alpha\beta} - \sin A\cdot\alpha^{\frac{\pi}{2}}\,\overline{\alpha\beta}.$$

Similarly, $OQ = \beta^B \overline{\alpha\beta} \;= (\cos B + \sin B \cdot \beta^{\frac{\pi}{2}}) \cdot \overline{\alpha\beta}$

$$= \cos B \cdot \overline{\alpha\beta} + \sin B \cdot \beta^{\frac{\pi}{2}} \, \overline{\alpha\beta}.$$

By $\alpha^{\frac{\pi}{2}} \overline{\alpha\beta}$ is meant the axis which is perpendicular to α and β, after it is rotated by a quadrant round α. In Fig. 2, let OA and OB represent α and β, any two axes drawn from O, then $\overline{\alpha\beta}$ is drawn from O upwards, normal to the plane of the paper. Hence $\alpha^{\frac{\pi}{2}} \overline{\alpha\beta}$ is OL, which is of unit length, and drawn in the plane of the paper, perpendicular to α. It is required to find the components of OL along α and β. Draw LN parallel to β, and LM parallel to α. Now OM or NL is $-\dfrac{1}{\sin \alpha\beta} \cdot \beta$, and ON is $\dfrac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \alpha$; hence,

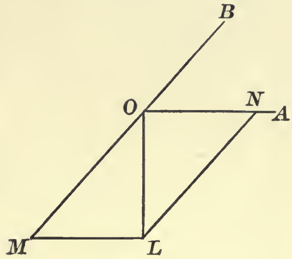

FIG. 2.

$$\alpha^{\frac{\pi}{2}} \overline{\alpha\beta} = \frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \alpha - \frac{1}{\sin \alpha\beta} \cdot \beta.$$

Similarly, $\beta^{\frac{\pi}{2}} \overline{\alpha\beta} = -\beta^{\frac{\pi}{2}} \overline{\beta\alpha} = -\dfrac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \beta + \dfrac{1}{\sin \alpha\beta} \cdot \alpha.$

Consequently, the three lines expressed in terms of the axes α, β, and $\overline{\alpha\beta}$, are

$$OR = \quad \cos B \sin A \;\cdot \alpha + \cos A \sin B \cdot \beta - \sin A \sin B \sin \alpha\beta \cdot \overline{\alpha\beta};$$

$$OP = -\sin A \frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \alpha + \sin A \frac{1}{\sin \alpha\beta} \cdot \beta + \cos A \cdot \overline{\alpha\beta};$$

$$OQ = \quad \sin B \frac{1}{\sin \alpha\beta} \cdot \alpha - \sin B \frac{\cos \alpha\beta}{\sin \alpha\beta} \cdot \beta + \cos B \cdot \overline{\alpha\beta}.$$

Hence $\cos(OR)(OP)$

$$= -\cos B \sin^2 A \left(\frac{\cos \alpha\beta}{\sin \alpha\beta} - \frac{\cos \alpha\beta}{\sin \alpha\beta} \right)$$

$$- \cos A \sin A \sin B \left(\frac{\cos^2 \alpha\beta}{\sin \alpha\beta} - \frac{1}{\sin \alpha\beta} + \sin \alpha\beta \right)$$

$$= 0.$$

Similarly, it may be shown that $\cos(OR)(OQ) = 0$; hence OR has the direction of the normal to the plane of OPQ.

To find the general expression for a spherical versor, when reference is made to a principal axis.

Let OA represent the principal axis (Fig. 3), and let it be denoted by α. Any versor OPA, which passes through the principal axis, may be denoted by β^u, where β denotes a unit axis perpendicular to α. Similarly, OAQ, another versor passing through the principal axis, may be denoted by γ^v, where γ denotes a unit axis perpendicular to α. The product versor OPQ is circular, but it will not, in general, pass through OA; let it be denoted by ξ^θ. Now

$$\begin{aligned}\xi^\theta &= \beta^u \gamma^v \\ &= \cos u \cos v - \sin u \sin v \cos \beta\gamma \\ &\quad + \{\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma - \sin u \sin v \sin \beta\gamma \cdot \overline{\beta\gamma}\}^{\frac{\pi}{2}}.\end{aligned}$$

We observe that the directed sine may be broken up into two components, namely, $\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma$, which is perpendicular to the principal axis, and $-\sin u \sin v \sin \beta\gamma \cdot \overline{\beta\gamma}$, which has the direction of the negative of the principal axis, for $\overline{\beta\gamma} = \alpha$.

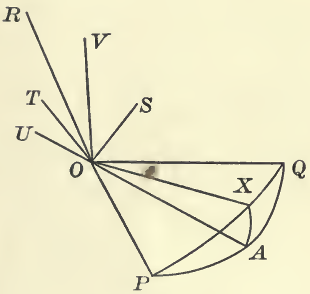

FIG. 3.

Draw OS to represent the first component $\cos v \sin u \cdot \beta$, OT to represent the second component $\cos u \sin v \cdot \gamma$, and OU to represent the third component $-\cos u \cos v \sin \beta\gamma \cdot \alpha$. Draw OV, the resultant of the first two, and OR, the resultant of all three. The plane of OA and OV passes through OR, which is normal to the plane OPQ; hence these planes cut at right angles in a line OX; and the angle between OA and OX is equal to that between OV and OR, for OV is perpendicular to OA, and OR to OX. Let ϕ denote the angle AOX, then

$$\cos \phi = \frac{\sqrt{\cos^2 v \sin^2 u + \cos^2 u \sin^2 v + 2 \cos u \cos v \sin u \sin v \cos \beta\gamma}}{\sqrt{1 - (\cos u \cos v - \sin u \sin v \cos \beta\gamma)^2}},$$

and

$$\sin \phi = \frac{\sin u \sin v \sin \beta\gamma}{\sqrt{1 - (\cos u \cos v - \sin u \sin v \cos \beta\gamma)^2}}.$$

Figure 4 represents a section through the plane of OA and OV. Let XM be drawn from X perpendicular to OA; it is equal in magnitude to $\sin\phi$; and OM is equal in magnitude to $\cos\phi$.

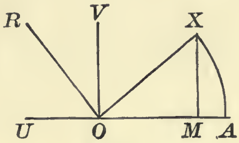

FIG. 4.

Hence the axis ξ has the form

$$\cos\phi\cdot\epsilon - \sin\phi\cdot\alpha,$$

where ϵ denotes a unit axis perpendicular to α. And

$$\xi^\theta = \cos\theta + \sin\theta(\cos\phi\cdot\epsilon - \sin\phi\cdot\alpha)^{\frac{\pi}{2}}$$

is determined by the equations,

$$\cos\theta = \cos u\cos v - \sin u\sin v\cos\beta\gamma, \tag{1}$$

$$\sin\theta\sin\phi = \sin u\sin v\sin\beta\gamma, \tag{2}$$

$$\sin\theta\cos\phi\cdot\epsilon = \cos v\sin u\cdot\beta + \cos u\sin v\cdot\gamma. \tag{3}$$

The unit axis ϵ may be expressed in terms of two axes β and γ, which are at right angles to one another and to α, and the angle which ϵ makes with β. Hence the more general expression for any spherical versor is

$$\xi^\theta = \cos\theta + \sin\theta\{\cos\phi(\cos\psi\cdot\beta + \sin\psi\cdot\gamma) - \sin\phi\cdot\alpha\}^{\frac{\pi}{2}}.$$

We observe that the line OX is the principal axis of the product versor POQ.

To find the product of two spherical versors of the general form given above.

The two factor versors may be expressed by

$$\xi^u = \cos u + \sin u(\cos\phi\cdot\beta - \sin\phi\cdot\alpha)^{\frac{\pi}{2}},$$

and $$\eta^v = \cos v + \sin v\,(\cos\phi'\cdot\gamma - \sin\phi'\cdot\alpha)^{\frac{\pi}{2}},$$

where β and γ denote any unit axes perpendicular to α. The product has the form

$$\zeta^w = \cos w + \sin w(\cos\phi''\cdot\gamma - \sin\phi''\cdot\alpha)^{\frac{\pi}{2}}.$$

Since $$\xi^u\eta^v = \cos u\cos v - \sin u\sin v\cos\xi\eta$$

$$+\{\cos v\sin u\cdot\xi + \cos u\sin v\cdot\eta - \sin u\sin v\sin\xi\eta\cdot\overline{\xi\eta}\}^{\frac{\pi}{2}},$$

and $\cos\xi\eta = \cos\phi\cos\phi'\cos\beta\gamma + \sin\phi\sin\phi'$,

and $$\text{Sin}\,\xi\eta = \cos\phi\cos\phi'\sin\beta\gamma\cdot\overline{\beta\gamma} - (\cos\phi\sin\phi'\cdot\overline{\beta\alpha} + \cos\phi'\sin\phi\cdot\overline{\alpha\gamma}),$$

therefore $$\cos w = \cos u\cos v - \sin u\sin v(\cos\phi\cos\phi'\cos\beta\gamma + \sin\phi\sin\phi'), \quad (1)$$

$$\sin w\sin\phi'' = \cos u\sin v\sin\phi' + \cos v\sin u\sin\phi + \sin u\sin v\cos\phi\cos\phi'\sin\beta\gamma, \quad (2)$$

$$\sin w\cos\phi''\cdot\epsilon = \cos u\sin v\cos\phi'\cdot\gamma + \cos v\sin u\cos\phi\cdot\beta + \sin u\sin v(\cos\phi\sin\phi'\cdot\overline{\beta\alpha} + \cos\phi'\sin\phi\cdot\overline{\alpha\gamma}). \quad (3)$$

From equation (1) we obtain w, then from (2) we obtain ϕ'', and finally from (3) we obtain ϵ.

When the factor versors are restricted to one plane, the axes coincide; that is, $\eta = \xi$. The above formula then becomes

$$\xi^{\theta+\theta'} = \cos\theta\cos\theta' - \sin\theta\sin\theta' + (\cos\theta\sin\theta' + \cos\theta'\sin\theta)\{\cos\phi\cdot\beta - \sin\phi\cdot\alpha\}^{\frac{\pi}{2}},$$

which is the fundamental theorem for trigonometry in any plane.

When the axes are coplanar with the initial line, we have γ identical with β, but ϕ', in general, different from ϕ. The theorem then becomes

$$\begin{aligned}\xi^{\theta}\eta^{\theta'} = {} & \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi'-\phi) \\ & + \{(\cos\theta\sin\theta'\cos\phi' + \cos\theta'\sin\theta\cos\phi)\cdot\beta \\ & + \sin\theta\sin\theta'\sin(\phi'-\phi)\cdot\overline{\beta\alpha} \\ & - (\cos\theta\sin\theta'\sin\phi' + \cos\theta'\sin\theta\cos\phi)\cdot\alpha\}^{\frac{\pi}{2}}.\end{aligned}$$

If, in addition, the middle term of the sine vanishes, the axis of the product will also be in the same plane with the other axes and the initial line.

To prove that the sum of the squares of the three components of the product of two general spherical versors is unity.

For shortness, let $x = \cos\theta$, $y = \sin\theta\cos\phi$, $z = \sin\theta\sin\phi$; $x' = \cos\theta'$, $y' = \sin\theta'\cos\phi'$, $z' = \sin\theta'\sin\phi'$. Then

$$\begin{aligned}\cos^2\theta'' &= (xx' - yy'\cos\beta\gamma - zz')^2\\ &= x^2x'^2 + y^2y'^2\cos^2\beta\gamma + z^2z'^2 - 2xx'yy'\cos\beta\gamma - 2xx'zz'\\ &\qquad + 2yy'zz'\cos\beta\gamma,\\ (\sin\theta''\cos\phi''\cdot\epsilon)^2 &= \{xy'\cdot\gamma + x'y\cdot\beta + yz'\cdot\overline{\beta\alpha} - zy'\cdot\overline{\gamma\alpha}\}^2\\ &= x^2y'^2 + x'^2y^2 + y^2z'^2 + z^2y'^2 + 2xx'yy'\cos\beta\gamma + 2xyy'z'\cos\gamma\overline{\beta\alpha}\\ &\qquad - 2yz'x'y'\cos\beta\overline{\gamma\alpha} - 2yzy'z'\cos\overline{\beta\alpha}\cdot\overline{\gamma\alpha},\\ (\sin\theta''\sin\phi'')^2 &= \{xz' + x'z + yy'\sin\beta\gamma\}^2\\ &= x^2z'^2 + z^2x'^2 + y^2y'^2\sin^2\beta\gamma + 2xx'zz' + 2xyy'z'\sin\beta\gamma\\ &\qquad + 2x'yy'z\sin\beta\gamma.\end{aligned}$$

The sum of the square terms is $(x^2+y^2+z^2)(x'^2+y'^2+z'^2)$, that is, 1; and the sum of the product terms reduces to

$$\begin{aligned}&2yy'zz'(\cos\beta\gamma - \cos\overline{\beta\alpha}\cdot\overline{\gamma\alpha}) + 2xyy'z'(\cos\gamma\overline{\beta\alpha} + \sin\beta\gamma)\\ &\qquad - 2yz'x'y'(\cos\beta\overline{\gamma\alpha} - \sin\beta\gamma).\end{aligned}$$

Now, β and γ both being perpendicular to α, $\cos\beta\gamma = \cos\overline{\beta\alpha}\cdot\overline{\gamma\alpha}$, and $\sin\beta\gamma = -\cos\gamma\overline{\beta\alpha} = \cos\beta\overline{\gamma\alpha}$. Hence the sum of the product terms vanishes.

FUNDAMENTAL THEOREM FOR THE ELLIPSOID OF REVOLUTION.

Imagine a circle APB (Fig. 5) to be projected on the plane of AQB, by means of lines drawn from the points of the circle, perpendicular to the plane, as PQ from P; the projection of the circle is an ellipse, having the initial line for semi-major axis. Let λ denote the axis of the circle, and β that of the plane; all lines perpendicular to the initial line are in the projected figure, diminished by the ratio $\cos\lambda\beta$, while all lines parallel to the initial line remain unaltered. Any area A in the circle will be changed into $A\cos\lambda\beta$ in the ellipse; and this is true whatever the form of the area. For shortness, $\cos\lambda\beta$ will be denoted by k.

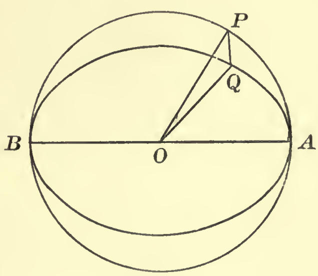

FIG. 5.

The projecting lines, instead of being drawn perpendicular to the plane of projection, may be drawn perpendicular to the plane of the circle; the ratio of projection then becomes $\sec\lambda\beta$, which may likewise be denoted by k, but k is then always greater than unity. The figure obtained is an ellipse, having the initial line for semi-minor axis. By the revolution of the former ellipse round the initial line we obtain a prolate ellipsoid; by the revolution of the latter, an oblate ellipsoid.

The Fundamental Equation of Elliptic Trigonometry.

The elliptic versor is expressed by $\frac{1}{OA}OP$ (Fig. 6), and

$$\frac{1}{OA}OP = \frac{OM}{OA} + \frac{1}{OA}MP.$$

The problem is, to find the correct analytical expressions for these three terms. If by u we denote the ratio of twice the area of the sector AOP to the square on OA, then,

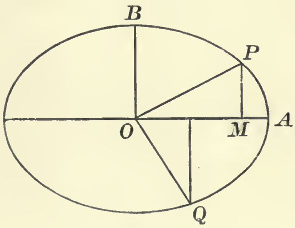

Fig. 6.

$$\frac{OM}{OA} = \cos\frac{u}{k} \text{ and } \frac{MP}{OA} = k\sin\frac{u}{k}.$$

Hence, if β denote a unit axis normal to the plane of the ellipse, the equation may be written

$$(k\beta)^u = \cos\frac{u}{k} + \sin\frac{u}{k}\cdot(k\beta)^{\frac{\pi}{2}}.$$

But we observe that it is much simpler to define u as the ratio of twice the area of AOP to the rectangle formed by OA and OB, the semi-axes; for then we have

$$(k\beta)^u = \cos u + \sin u\cdot(k\beta)^{\frac{\pi}{2}}.$$

We attach the k to the axis rather than to the ratio, because in forming a product of versors it does not enter as an ordinary multiplier. When the elliptic sector does not start from the principal axis, the element u must still be taken as the ratio of twice the area of the sector to the rectangle formed by the axes. The index $\frac{\pi}{2}$ is due to the rectangular nature of the components; it expresses the circular versor between OA and MP. When oblique components are used, the index is then w, the angle of the obliquity. This is proved in *Fundamental Theorems*, page 10.

To find the product of two elliptic versors which are in one plane passing through the principal axis.

Let the two versors be represented by OQA and OAP (Fig. 6); then their product is represented by OQP. Let β denote a unit axis normal to the plane; the former versor may be denoted by $(k\beta)^u$, and the latter by $(k\beta)^v$. Then

$$\begin{aligned}(k\beta)^u(k\beta)^v &= \{\cos u + \sin u\cdot(k\beta)^{\frac{\pi}{2}}\}\{\cos v + \sin v\cdot(k\beta)^{\frac{\pi}{2}}\}\\ &= \cos u\cos v + \cos u\sin v\cdot(k\beta)^{\frac{\pi}{2}} + \cos v\sin u\cdot(k\beta)^{\frac{\pi}{2}}\\ &\qquad + \sin u\sin v\cdot(k\beta)^{\frac{\pi}{2}}(k\beta)^{\frac{\pi}{2}}.\end{aligned}$$

Now
$$\begin{aligned}(k\beta)^u(k\beta)^v &= (k\beta)^{u+v}\\ &= \cos(u+v) + \sin(u+v)\cdot(k\beta)^{\frac{\pi}{2}}\\ &= \cos u\cos v - \sin u\sin v\\ &\qquad + (\cos u\sin v + \cos v\sin u)\cdot(k\beta)^{\frac{\pi}{2}}.\end{aligned}$$

Hence $(k\beta)^{\frac{\pi}{2}}(k\beta)^{\frac{\pi}{2}} = \beta^{\pi} = -1$. From this we infer that k is such a multiplier that it does not affect the terms of the cosine.

To find the product of two elliptic versors which intersect in the principal axis of the ellipsoid of revolution.

Let $\frac{1}{OP}OA$ and $\frac{1}{OA}OQ$ (Fig. 7) represent the two versors; their axes are β and γ, respectively, each being perpendicular to α, the direction of the principal axis OA. Let u denote the ratio of twice the area of OPA to the rectangle formed by the semi-axes of its ellipse, and v the ratio of twice the area of OAQ to the rectangle formed by the semi-axes of its ellipse. The versors are denoted by $(k\beta)^u$ and $(k\gamma)^v$. Now

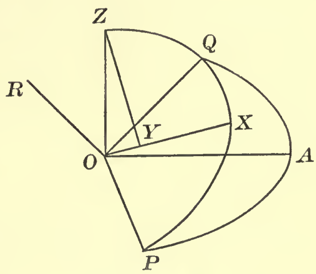

FIG. 7.

$$(k\beta)^u = \cos u + \sin u\cdot(k\beta)^{\frac{\pi}{2}},$$
and
$$(k\gamma)^v = \cos v + \sin v\cdot(k\gamma)^{\frac{\pi}{2}},$$
therefore
$$\begin{aligned}(k\beta)^u(k\gamma)^v &= \cos u\cos v + \cos v\sin u\cdot(k\beta)^{\frac{\pi}{2}}\\ &\qquad + \cos u\sin v\cdot(k\gamma)^{\frac{\pi}{2}} + \sin u\sin v\cdot(k\beta)^{\frac{\pi}{2}}(k\gamma)^{\frac{\pi}{2}}.\end{aligned}$$

By means of the principle that the first power of k is k, we see that the second and third terms contribute

$$k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma)$$

to the Sine component. It remains to determine the meaning of the fourth term, that is, the values of the coefficients x and y in the equation

$$(k\beta)^{\frac{\pi}{2}}(k\gamma)^{\frac{\pi}{2}} = x \cos \beta\gamma + y \sin \beta\gamma \cdot \overline{\beta\gamma}^{\frac{\pi}{2}}.$$

From the form of the product of two coplanar versors (page 9), it appears that x is -1; the value of y appears to be either $-k^2$ or -1.

On the former hypothesis the directed sine OR would be

$$k \cos v \sin u \cdot \beta + k \cos u \sin v \cdot \gamma - k^2 \sin u \sin v \sin \beta\gamma \cdot \alpha.$$

Now $$OP = \cos u \cdot \alpha - k \sin u \cdot \beta^{\frac{\pi}{2}} \overline{\beta\gamma},$$

and $$OQ = \cos v \cdot \alpha + k \sin v \cdot \gamma^{\frac{\pi}{2}} \overline{\beta\gamma};$$

consequently $$\cos(OR)(OP) = -k^2 \cos v \sin^2 u \left(\frac{\cos \beta\gamma}{\sin \beta\gamma} - \frac{\cos \beta\gamma}{\sin \beta\gamma}\right) - k^2 \cos u \sin u \sin v \left(\frac{\cos^2 \beta\gamma}{\sin^2 \beta\gamma} - \frac{1}{\sin \beta\gamma} + \sin \beta\gamma\right),$$

which vanishes, as before (page 3). Similarly $\cos(OR)(OQ) = 0$. Hence the above expression gives the direction of the normal to the plane of the product versor. But suppose that $\frac{1}{OP} OA$ and $\frac{1}{OA} OQ$ are quadrantal elliptic versors, then $\cos u = \cos v = 0$, and $\sin u = \sin v = 1$; consequently the cosine of the product would then be $-\cos \beta\gamma$ and the sine of the product $-k^2 \sin\beta\gamma \cdot \alpha^{\frac{\pi}{2}}$. But it is evident that in this case the product versor is circular, namely, $-(\cos \beta\gamma + \sin \beta\gamma \cdot \alpha^{\frac{\pi}{2}})$. Hence it appears that k^2 cannot enter as a factor of the third term of the Sine.

On the other hypothesis the directed sine is

$$k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma) - \sin u \sin v \sin \beta\gamma \cdot \alpha.$$

This expression satisfies the test of becoming circular under the conditions mentioned; but its direction is not normal to the plane of the product versor. How then, is its direction related to that plane? It will be found that it has the direction of the conjugate axis to the plane. Draw OV (Fig. 8), to represent $k(\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma)$, the component perpendicular to the principal axis OA, and OU' in the direction opposite to the

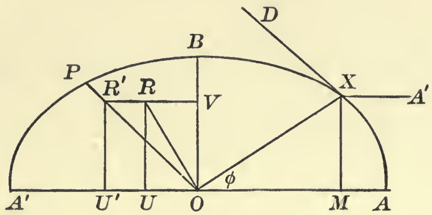

FIG. 8.

principal axis to represent $-\sin u \sin v \sin \beta\gamma$, also OU to represent the same quantity multiplied by k^2; and draw OR' and OR, the two resultants. The plane through OA and OV will cut the ellipsoid in a principal ellipse AXB, and as it passes through the normal OR it will cut the plane of the product ellipse at right angles; let OX denote the line of intersection. Draw XA' parallel to OA and XD the tangent at X, and let θ denote the circular versor between AO and OX. Now

$$\tan\theta = \frac{MX}{OM} = \frac{OU}{OV}$$

$$= \frac{k \sin u \sin v \sin \beta\gamma}{\sqrt{\cos^2 v \sin^2 u + \cos^2 u \sin^2 v + 2 \cos u \cos v \sin u \sin v \cos \beta\gamma}};$$

but $\tan A'XD = -k^2 \cot\text{an}\,\theta$

$$= -\frac{k\sqrt{\cos^2 v \sin^2 u + \cos^2 u \sin^2 v + 2 \cos u \cos v \sin u \sin v \cos \beta\gamma}}{\sin u \sin v \sin \beta\gamma}$$

$$= \text{cotan}\, VOR' = \tan AOR'.$$

Thus the direction of OR' is that of the conjugate axis of the plane of the product versor.

Let ϕ denote the ratio of twice the area of AOX to the square of OA; it is equal to the angle which OX made with OA before the contraction. The direction of the axis was then $\cos\phi$ along OB, and $\sin\phi$ along OA'; by the contraction, $\cos\phi$ has been changed into $k\cos\phi$; hence the axis of the ellipsoid, along the direction of OR', is $k\cos\phi\cdot\epsilon-\sin\phi\cdot\alpha$, where ϵ denotes a unit axis in the direction of OB.

The magnitude of the product versor is determined by the cosine function,

$$\cos u\cos v-\sin u\sin v\cos\beta\gamma.$$

Suppose that an elliptic sector OXZ (Fig. 7), having the area of the third side of the ellipsoidal triangle, starts from the semi-major axis OX, and let OY and OZ be the rectangular projections of the bounding radius vector OZ. As the small ellipse OPQ is derived from a principal ellipse by diminishing all lines parallel to OX in the ratio of OX to OA, that is, in the ratio of $\sqrt{\cos^2\phi+k^2\sin^2\phi}$ to 1, while the transverse lines remain unaltered; the ratio of OY to OX is equal to the corresponding ratio in the principal ellipse; hence the ratio of OY to OX is equal to $\cos u\cos v-\sin u\sin v\cos\beta\gamma$.

Let w denote the ratio of twice the sector OPQ to the rectangle formed by OX and the minor semi-axis of the ellipse OPQ; this ratio is equal to the ratio of twice the corresponding circular sector to the square of OA. By the corresponding circular sector is meant that circular sector from which the elliptic sector was formed by contraction along the two axes. Also, let ξ denote the elliptic axis, $\cos\phi\cdot k\epsilon-\sin\phi\cdot\alpha$. The product versor then takes the form

$$\xi^w=\cos w+\sin w(\cos\phi\cdot k\epsilon-\sin\phi\cdot\alpha)^{\frac{\pi}{2}},$$

the quantities w, ϕ, and ϵ being determined by

$$\cos w=\cos u\cos v-\sin u\sin v\cos\beta\gamma, \tag{1}$$

$$\sin\phi=\frac{\sin u\sin v\sin\beta\gamma}{\sqrt{1-\cos^2 w}}, \tag{2}$$

$$\epsilon=\frac{\cos v\sin u\cdot\beta+\cos u\sin v\cdot\gamma}{\sin w\cos\phi}. \tag{3}$$

Consequently we have for the elliptic axis OP,

$$\xi=\frac{k(\cos v\sin u\cdot\beta+\cos u\sin v\cdot\gamma)-\sin u\sin v\sin\beta\gamma\cdot\alpha}{\sqrt{1-\cos^2 w}}.$$

The locus of the poles of the several elliptic areas is the original ellipsoid.

To find the product of two ellipsoidal versors of the above general form.

The two factor versors are expressed by

$$\xi^u = \cos u + \sin u (\cos\phi \cdot k\beta - \sin\phi \cdot \alpha)^{\frac{\pi}{2}},$$

and $$\eta^v = \cos v + \sin v (\cos\phi' \cdot k\gamma - \sin\phi' \cdot \alpha)^{\frac{\pi}{2}};$$

it is required to show that their product has the form

$$\zeta^w = \cos w + \sin w (\cos\phi'' \cdot k\epsilon - \sin\phi'' \cdot \alpha)^{\frac{\pi}{2}}.$$

We have

$$\begin{aligned}\xi^u\eta^v &= (\cos u + \sin u \cdot \xi^{\frac{\pi}{2}})(\cos v + \sin v \cdot \eta^{\frac{\pi}{2}}) \\ &= \cos u \cos v - \sin u \sin v \cos \xi\eta \\ &\qquad + \{\cos u \sin v \cdot \eta + \cos v \sin u \cdot \xi - \sin u \sin v \operatorname{Sin} \xi\eta\}^{\frac{\pi}{2}}.\end{aligned}$$

The problem is reduced to finding the value of $\cos\xi\eta$ and $\operatorname{Sin}\xi\eta$. Now $\xi\eta$ means the elliptic versor between the elliptic axes

$$\cos\phi \cdot k\beta - \sin\phi \cdot \alpha \quad \text{and} \quad \cos\phi' \cdot k\gamma - \sin\phi' \cdot \alpha.$$

To find them, we apply the following principle:

Restore the elliptic axes to their spherical originals, find the versor between these unit axes according to the ordinary rule, and reduce its axes back to the ellipsoidal form. Applied to the above, the rule means: suppose $k = 1$, form the cosine and the directed Sine, and introduce k as a multiplier of those components of the directed Sine which are perpendicular to α. Hence

$$\cos\xi\eta = \cos\phi \cos\phi' \cos\beta\gamma + \sin\phi \sin\phi',$$

and $$\begin{aligned}\operatorname{Sin}\xi\eta = {} & \cos\phi \cos\phi' \sin\beta\gamma \cdot \alpha \\ & - k(\cos\phi \sin\phi' \cdot \overline{\beta\alpha} + \sin\phi \cos\phi' \cdot \overline{\alpha\gamma}).\end{aligned}$$

If we express $\operatorname{Sin}\xi\eta$ as $\sin\xi\eta \cdot \overline{\xi\eta}$, what must $\overline{\xi\eta}$ now mean? Its length is not unity, nor is it normal to the plane of ξ and η. It means

$$\frac{\cos\phi \cos\phi' \sin\beta\gamma \cdot \alpha - k(\cos\phi \sin\phi' \cdot \overline{\beta\alpha} + \sin\phi \cos\phi' \cdot \overline{\alpha\gamma})}{\sqrt{1 - \cos^2\xi\eta}};$$

that is, the elliptic axis conjugate to the plane of ξ and η.

Hence

$$\cos w = \cos u \cos v - \sin u \sin v (\cos\phi \cos\phi' \cos\beta\gamma + \sin\phi \sin\phi'), \quad (1)$$

$$\begin{aligned}\sin w \sin\phi'' &= \cos u \sin v \sin\phi' + \cos v \sin u \sin\phi \\ &\quad + \sin u \sin v \cos\phi \cos\phi' \sin\beta\gamma, \quad (2)\end{aligned}$$

$$\begin{aligned}\sin w \sin\phi'' \cdot \epsilon &= \cos u \sin v \cos\phi' \cdot \gamma + \cos v \sin u \cos\phi \cdot \beta \\ &\quad + \sin u \sin v (\cos\phi \sin\phi' \cdot \overline{\beta\alpha} + \cos\phi' \sin\phi \cdot \overline{\alpha\gamma}). \quad (3)\end{aligned}$$

FUNDAMENTAL THEOREM FOR THE GENERAL ELLIPSOID.

To find the product of two ellipsoidal versors whose axes have the same directions as the minor axes of the ellipsoid.

In the general ellipsoid there are three principal axes mutually rectangular; in Fig. 9 they are represented by OA, OB, OC. We shall suppose the greatest semi-axis to be taken as the initial line, but either of the others might be chosen. Let unit axes along OA, OB, and OC be denoted by α, β, γ, respectively; let k' denote the ratio of OB to OA, and k that of OC to OA. A versor POA in the plane COA is expressed by $(k\beta)^u$, while a versor AOQ in the plane of AOB is expressed by $(k'\gamma)^v$; u denoting the ratio of twice POA to the rectangle COA, and v that of twice AOQ to the rectangle AOB.

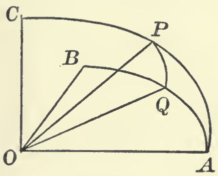

Fig. 9.

$$\begin{aligned}\text{Now } (k\beta)^u (k'\gamma)^v &= \{\cos u + \sin u \cdot (k\beta)\}^{\frac{\pi}{2}} \{\cos v + \sin v \cdot (k'\gamma)\}^{\frac{\pi}{2}} \\ &= \cos u \cos v + \cos v \sin u \cdot (k\beta)^{\frac{\pi}{2}} \\ &\quad + \cos u \sin v \cdot (k'\gamma)^{\frac{\pi}{2}} + \sin u \sin v \cdot (k\beta)^{\frac{\pi}{2}} (k'\gamma)^{\frac{\pi}{2}}.\end{aligned}$$

The fourth term, as it involves two axes which are at right angles, can contribute nothing to the cosine; the cosine is $\cos u \cos v$. The second and third terms contribute $k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma$ to the directed Sine; while the fourth contributes either $-kk' \sin u \sin v \cdot \alpha$ or $-\sin u \sin v \cdot \alpha$.

It may be shown, in the same manner as before (page 2), that

$$k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma - kk' \sin u \sin v \cdot \alpha$$

is perpendicular to both OP and OQ, hence has the direction of the normal to their plane; and, by the principle stated at page 13, it is seen that

$$k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma - \operatorname{Sin} u \sin v \cdot \alpha$$

is the axis conjugate to the plane of POQ.

Let a plane pass through the principal axis and the perpendicular component $k \cos v \sin u \cdot \beta + k' \cos u \sin v \cdot \gamma$; as it passes through the normal to the plane POQ it must cut that plane at right angles, and OX, the line of intersection, is the principal axis of the ellipse PQ. Let ϕ denote the elliptic ratio of AOX, and ψ the angle between β and $\cos v \sin u \cdot \beta + \cos u \sin v \cdot \gamma$, and w the ratio of twice the elliptic versor POQ to the rectangle of the semi-axes of its ellipse; then the product versor takes the form

$$\xi^w = \cos w + \sin w \{\cos \phi (k \cos \psi \cdot \beta + k' \sin \psi \cdot \gamma) - \sin \phi \cdot \alpha\}^{\frac{\pi}{2}}.$$

For
$$\cos w = \cos u \cos v, \tag{1}$$
$$\sin w \sin \phi = \sin u \sin v, \tag{2}$$
$$\sin w \cos \phi \cos \psi = \cos v \sin u, \tag{3}$$
$$\sin w \cos \phi \sin \psi = \cos u \sin v. \tag{4}$$

To find the product of two ellipsoidal versors of the above form.

Let the one versor be ξ^u, where

$$\xi = \cos \phi (k \cos \psi \cdot \beta + k' \sin \psi \cdot \gamma) - \sin \phi \cdot \alpha,$$

and let the other be η^v, where

$$\eta = \cos \phi' (k \cos \psi' \cdot \beta + k' \sin \psi' \cdot \gamma) - \sin \phi' \cdot \alpha;$$

it is required to show that $\xi^u \eta^v$ has the form ζ^w, where

$$\zeta = \cos \phi'' (k \cos \psi'' \cdot \beta + k' \sin \psi \cdot \gamma) - \sin \phi'' \cdot \alpha.$$

Since
$$\xi^u \eta^v = \cos u \cos v - \sin u \sin v \cos \xi\eta$$
$$+ \{\cos v \sin u \cdot \xi + \cos u \sin v \cdot \eta - \sin u \sin v \operatorname{Sin} \xi\eta\}^{\frac{\pi}{2}},$$

the problem reduces to finding $\cos \xi\eta$ and $\operatorname{Sin} \xi\eta$. By $\xi\eta$ is meant the elliptic angle between the elliptic axes ξ and η; the ratio of the sector $\xi\eta$ to the rectangle of its ellipse is the same as the ratio of the sector of the primitives of ξ and η to 1. Hence the cosine is obtained by supposing k and k' to be one, and the Sine is obtained by the same method, and then reducing by k the component having the axis β, and by k' the component having the axis γ. We obtain

$$\cos\xi\eta = \cos\phi\cos\phi'\cos(\psi-\psi') + \sin\phi\sin\phi',$$

and $$\begin{aligned}\operatorname{Sin}\xi\eta = &\cos\phi\cos\phi'\sin(\psi-\psi')\cdot\alpha \\ &+ k'(\cos\phi\cos\psi\sin\phi' - \cos\phi'\cos\psi'\sin\phi)\cdot\gamma \\ &- k(\cos\phi\sin\psi\sin\phi' - \cos\phi'\sin\psi'\sin\phi)\cdot\beta.\end{aligned}$$

Hence $\cos w$

$$= \cos u\cos v - \sin u\sin v\{\cos\phi\cos\phi'\cos(\psi-\psi') + \sin\phi\sin\phi'\}, \quad (1)$$

$$\begin{aligned}\sin w\cos\phi''\cos\psi'' \\ = &\cos u\sin v\cos\phi'\cos\psi' + \cos v\sin u\cos\phi\cos\psi \\ &+ \sin u\sin v(\cos\phi\sin\psi\sin\phi' - \cos\phi'\sin\psi'\sin\phi), \quad (2)\end{aligned}$$

$$\begin{aligned}\sin w\cos\phi''\sin\psi'' \\ = &\cos u\sin v\cos\phi'\sin\psi' + \cos v\sin u\cos\phi\sin\psi \\ &- \sin u\sin v(\cos\phi\cos\psi\sin\phi' - \cos\phi'\cos\psi'\sin\phi), \quad (3)\end{aligned}$$

$$\begin{aligned}\sin w\sin\phi'' = &\cos u\sin v\sin\phi' + \cos v\sin u\sin\phi \\ &- \sin u\sin v\cos\phi\cos\phi'\sin(\psi-\psi'). \quad (4)\end{aligned}$$

The elliptic axis is given in magnitude and direction by $\dfrac{\operatorname{Sin}\xi\eta}{\sqrt{1-\cos^2\xi\eta}}$. The locus of these axes is an ellipsoid derived from the original ellipsoid by interchanging the ratios k and k'.

FUNDAMENTAL THEOREM FOR THE EQUILATERAL HYPERBOLOID OF TWO SHEETS.

In order to distinguish readily the equilateral from the general hyperbola, it is desirable to have a single term for the equilateral hyperbola. The term *excircle,* with the corresponding adjective *excircular,* have been introduced by Mr. Hayward, in his "Algebra of Coplanar Vectors." These terms are brief and suggestive, for the equilateral hyperbola is the analogue of the circle. If we consider the sphere, we find that its hyperbolic analogue consists of three sheets. Two of these are similar, the one being merely the negative of the other with respect to the centre, and are classed together as the equilateral hyperboloid of two sheets; the third is called the equilateral hyperboloid of one sheet. For brevity we propose to call these the *exsphere of two sheets,* and the *exsphere of one sheet,* the two together being called the *exsphere.* In treating of the exsphere of two sheets, we shall generally consider the positive sheet.

To find the expression for an exspherical versor, the plane of which passes through the principal axis.

Let OA (Fig. 10) be the principal axis of an equilateral hyperboloid of two sheets, QAP a section through OA, AOP the sector of a versor in that plane, and PM perpendicular to OA. The versor is denoted by $\frac{1}{OA}OP$, or $(OA)(OP)$, if OA is of unit length. Now

$$\frac{1}{OA}OP = \frac{1}{OA}(OM + MP)$$
$$= \frac{OM}{OA} + \frac{1}{OA}MP.$$

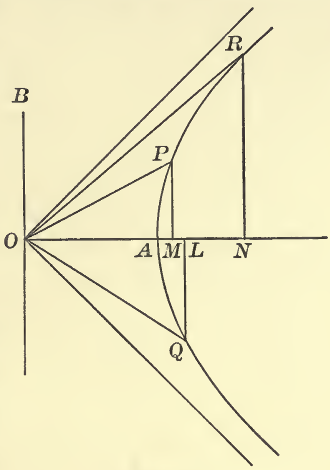

FIG. 10.

The problem is to find the proper analytical expression for this equation. Let β denote a unit axis normal to the plane of QAP, and u the ratio of twice the area of the sector AOP to the square of OA, or rather to the area of the rectangle AOB, and let i denote $\sqrt{-1}$. The above equation, if the starting line is indifferent, is expressed by

$$\beta^{iu} = \cos iu + \sin iu \cdot \beta^{\frac{\pi}{2}}$$
$$= \cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}}.$$

We observe that $\cosh u = \frac{OM}{OA}$, and $\sinh u = \frac{MP}{OA}$, and that $\beta^{\frac{\pi}{2}}$ expresses the circular versor between OA and MP. What is the geometrical meaning of the i? It expresses the fact that $\cosh u$ and $\sinh u$ are related, not by the condition

$$\cosh^2 u + \sinh^2 u = 1,$$

but by the condition $\cosh^2 u - \sinh^2 u = 1$.

With this notation, we can deduce readily from any spherical theorem the corresponding exspherical theorem.

A plausible hypothesis is that the i before $\sinh u$ may be considered as an index $\frac{\pi}{2}$ to be given to the axis β, making

$$\beta^{iu} = \cosh u + \sinh u \cdot \beta^{\pi};$$

but this would leave out entirely the axis of the plane, for the equation would reduce to

$$\beta^{iu} = \cosh u - \sinh u.$$

The quantity here denoted by i is the scalar $\sqrt{-1}$, while the index $\frac{\pi}{2}$ expresses the vector $\sqrt{-1}$.

The series for e^{iu} is wholly scalar; but the series for $e^{iu\cdot\beta^{\frac{\pi}{2}}}$ breaks up into a scalar and a vector part.

In specifying an exspherical versor, it is necessary to give not only the ratio and the perpendicular axis of the plane, but also the principal axis of the versor. This is the reason why the spherical versor has to be treated with reference to a principal axis, in order to obtain theorems which can be translated into theorems for the exspherical versor.

To find the product of two coplanar exspherical versors, when the common plane passes through the principal axis.

Suppose the versors shifted without change of area until the line of meeting coincides with the principal axis. Let QOA (Fig. 10) be denoted by β^{iu}, and AOP by β^{iv}, expressions which are independent of the shifting. Then

$$\beta^{iu} = \cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}},$$

$$\beta^{iv} = \cosh v + i \sinh v \cdot \beta^{\frac{\pi}{2}};$$

therefore
$$\begin{aligned}\beta^{iu}\beta^{iv} &= (\cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}})(\cosh v + \sinh v \cdot \beta^{\frac{\pi}{2}}) \\ &= \cosh u \cosh v + i(\cosh u \sinh v + \cosh v \sinh u) \cdot \beta^{\frac{\pi}{2}} \\ &\quad + i^2 \sinh u \sinh v \cdot \beta^{\pi};\end{aligned}$$

but $i^2 = -$, and $\beta^{\pi} = -$; hence

$$\begin{aligned}\beta^{iu}\beta^{iv} &= \cosh u \cosh v + \sinh u \sinh v \\ &\quad + i(\cosh u \sinh v + \cosh v \sinh u) \cdot \beta^{\frac{\pi}{2}}.\end{aligned}$$

Hence $\beta^{iu}\beta^{iv} = \beta^{i(u+v)}$.

Suppose that the sector QOP is shifted without change of area till it starts from OA, and becomes AOR. Then

$$\frac{ON}{OA} = \cosh u \cosh v + \sinh u \sinh v,$$

and
$$\frac{NR}{OA} = \cosh u \sinh v + \cosh v \sinh u.$$

To find the product of two diplanar exspherical versors when the plane of each passes through the principal axis.

Let the two versors POA and AOQ (Fig. 11) be denoted by β^{iu} and γ^{iv}, the axes β and γ being each perpendicular to the principal axis α. Then

$$\begin{aligned}\beta^{iu}\gamma^{iv} &= (\cos iu + \sin iu \cdot \beta^{\frac{\pi}{2}})(\cos iv + \sin iv \cdot \gamma^{\frac{\pi}{2}}) \\ &= \cos iu \cos iv - \sin iu \sin iv \cos \beta\gamma \\ &\quad + \{\cos iv \sin iu \cdot \beta + \cos iu \sin iv \cdot \gamma - \sin iu \sin iv \sin \beta\gamma \cdot \alpha\}^{\frac{\pi}{2}}.\end{aligned}$$

But $\cos iu = \cosh u$, and $\sin iu = i \sinh u$, therefore,

$$\begin{aligned}\beta^{iu}\gamma^{iv} &= \cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma, \\ &+ i\{\cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma - i \sinh u \sinh v \sin \beta\gamma \cdot \alpha\}^{\frac{\pi}{2}}.\end{aligned}$$

Hence $\cosh \beta^{iu}\gamma^{iv} = \cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma$

and
$$\operatorname{Sinh} \beta^{iu}\gamma^{iv} = \cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma - i \sinh u \sinh v \sin \beta\gamma \cdot \alpha.$$

By expanding, it may be shown that

$$(\cosh \beta^{iu}\gamma^{iv})^2 - (\operatorname{Sinh} \beta^{iu}\gamma^{iv})^2 = 1,$$

or
$$(\cos \beta^{iu}\gamma^{iu})^2 + (\operatorname{Sin} \beta^{iu}\gamma^{iv})^2 = 1.$$

The function Sinh is the same as Sin, only an i has been dropped from all the terms of the latter. The product versor is also represented by a sector of an excircle of unit semi-axis.

The first and second components of the excircular Sine are perpendicular to the principal axis; hence their resultant,

$$\cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma,$$

is also perpendicular to the principal axis. Let it be represented by OV (Fig. 11). The difficulty consists in finding the true direction of the third component, $-i \sinh u \sinh v \sin \beta\gamma \cdot \alpha$. At page 53 of *The Imaginary of Algebra,* I suggested the following construction:

With V as centre, and radius equal to $\sinh u \sinh v \sin \beta\gamma$, describe a circle in the plane of OA and OV, and draw OS or OS' a tangent to this circle.

But another hypothesis presents itself; namely, to make the same construction as in the case of the sphere.

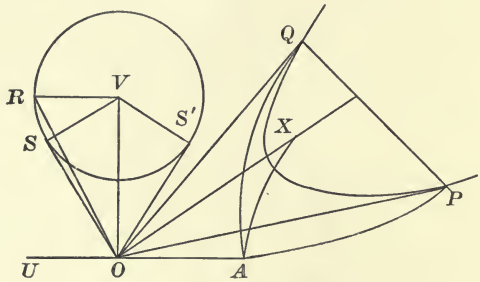

FIG. 11.

Draw OU opposite to OA, and equal to $\sinh u \sinh v \sin \beta\gamma$; and find OR, the resultant of OV and OU. We shall show that OR satisfies the condition of being normal to the plane POQ, while OS or OS' does not.

The reasoning at page 2 applies to give the expression for the vectors OP and OQ. Hence the expressions for the three vectors OR, OP, OQ, are

$$OR = \cosh v \sinh u \cdot \beta + \cosh u \sinh v \cdot \gamma - \sinh u \sinh v \sin \beta\gamma \cdot \overline{\beta\gamma},$$

$$OP = -\sinh u \frac{\cos \beta\gamma}{\sin \beta\gamma} \cdot \beta + \sinh u \frac{1}{\sin \beta\gamma} \cdot \gamma + \cosh u \cdot \overline{\beta\gamma},$$

$$OQ = -\sinh v \frac{1}{\sin \beta\gamma} \cdot \beta - \sinh v \frac{\cos \beta\gamma}{\sin \beta\gamma} \cdot \gamma + \cosh v \cdot \overline{\beta\gamma}.$$

It follows, as there, that

$$\cos(OR)(OP) = 0, \quad \text{and} \quad \cos(OR)(OQ) = 0.$$

Hence OR is normal to the plane POQ, and OS is not.

The function of the i before the third component of the Sine is to indicate that the magnitude of the Sine is not $\sqrt{OV^2 + VR^2}$ but $\sqrt{OV^2 - VR^2}$. This gives

$$\sinh \beta^{iu}\gamma^{iv}$$
$$= \sqrt{\{\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v + 2 \cosh u \cosh v \sinh u \sinh v \cos \beta\gamma - \sinh^2 u \sinh^2 v \sin^2 \beta\gamma\}}$$
$$= \sqrt{(\cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma)^2 - 1}.$$

The expression $\dfrac{OR}{\sqrt{OV^2 - VR^2}}$ gives the excircular axis both in magnitude and direction. The plane of OA and OV cuts the exsphere in an excircle, and as it passes through the normal OR, it must cut the plane POQ at right angles. Let OX be the line of intersection (Fig. 12). Draw XM perpendicular to OA;

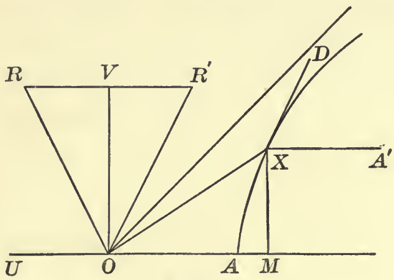

FIG. 12.

draw XD a tangent to the excircle at X, and XA' parallel to OA, and OR' the reflection of OR with respect to OV. Let ϕ denote the excircular angle of AOX; that is, the ratio of twice the area of AOX to the square of OA.

As OR is normal to the plane POQ, it is perpendicular to OX; but OV is perpendicular to OA; therefore the angle AOX is equal to the angle VOR. Also as the angle AOR' is the complement of $R'OV$ and $A'XD$ the complement of AOX, the line OR' is parallel to the tangent XD.

Hence $\cosh \phi = \dfrac{OM}{OA} = \dfrac{OV}{\sqrt{OV^2 - VR^2}} =$

$$\sqrt{\frac{\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v + 2 \cosh u \cosh v \sinh u \sinh v \cos \beta\gamma}{(\cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma)^2 - 1}}$$

and $\qquad \sinh \phi = \dfrac{MX}{OA} = \dfrac{VR}{\sqrt{OV^2 - VR^2}}$

$$= \frac{\sinh u \sinh v \sin \beta\gamma}{\sqrt{(\cosh u \cosh v + \sinh u \sinh v \cos \beta\gamma)^2 - 1}}.$$

The above analysis shows that the product versor of POQ may be specified by three elements: *first,* ϵ a unit axis drawn perpendicular to OA in the plane of OA and the normal to the plane of POQ; *second,* ϕ the excircular angle of AOX determined by OA and OX drawn at right angles to the normal in the plane of OA and the normal; *third,* w the versor of a unit excircle determined by the conditions of passing through the points P and Q and having its vertex on the line OX.

When u and v are equal, half of the line joining PQ is the sinh of half of the versor of the product. Let y denote the sinh of each of the factor versors, then it is easy to see from geometrical considerations (v. *The Imaginary of Algebra,* page 53), that

$$\sinh\frac{w}{2}=\frac{1}{\sqrt{2}}y\sqrt{1+\cos\beta\gamma},$$

therefore
$$\cosh\frac{w}{2}=\frac{1}{\sqrt{2}}\sqrt{2+y^2(1+\cos\beta\gamma)}.$$

But it is also evident that the distance from O to the midpoint of PQ is

$$\sqrt{\frac{y^2(1-\cos\beta\gamma)+2(y^2+1)}{y^2(1+\cos\beta\gamma)+2}}.$$

The excess of this distance over $\cosh\frac{w}{2}$ gives the distance by which the axis has been displaced along OX.

Hence the product versor may be expressed by an excircular axis and an excircular versor as ξ^{iw}, where

$$\xi=\cosh\phi\cdot\epsilon-i\sinh\phi\cdot\alpha.$$

To determine these quantities, we have, as in the case of the sphere, the three equations

$$\cosh w=\cosh u\cosh v+\sinh u\sinh v\cos\beta\gamma, \qquad (1)$$

$$\sinh w\cosh\phi=\sinh u\sinh v\sin\beta\gamma, \qquad (2)$$

$$\sinh w\sinh\phi\cdot\epsilon=\cosh v\sinh u\cdot\beta+\cosh u\sinh v\cdot\gamma. \qquad (3)$$

The axis ϵ may be expressed in terms of two axes β and γ forming with α a set of mutually rectangular axes, and the angle ψ which it makes with β; so that for the excircular axis we have

$$\xi=\cosh\phi(\cos\psi\cdot\beta+\sin\psi\cdot\gamma)-i\sinh\phi\cdot\alpha.$$

In the above investigation it is assumed that the magnitude of the perpendicular component of the Sine is necessarily greater than the component parallel to the principal axis. This means that

$$\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v + 2\cosh u \cosh v \sinh u \sinh v \cos \beta\gamma$$

is necessarily greater than $\sinh^2 u \sinh^2 v \sin^2 \beta\gamma$.

Let $\sin \beta\gamma = 1$; then $\cos \beta\gamma = 0$; and we have to compare

$$\cosh^2 v \sinh^2 u + \cosh^2 u \sinh^2 v \quad \text{with} \quad \sinh^2 u \sinh^2 v.$$

Now each term on the left is greater than the term on the right; therefore their sum must be greater, for each term is the square of a real quantity. Next let $\sin \beta\gamma = 0$; then $\cos \beta\gamma = 1$; the former term becomes a complete square while the latter is 0; hence the former must always be greater than the latter.

To find the product of two exspherical versors of the general kind.

The two versors are expressed by

$$\xi^{iu} = \cosh u + i \sinh u (\cosh \phi \cdot \beta - i \sinh \phi \cdot \alpha)^{\frac{\pi}{2}},$$

and $$\eta^{iv} = \cosh v + i \sinh v (\cosh \phi' \cdot \gamma - i \sinh \phi' \cdot \alpha)^{\frac{\pi}{2}};$$

it is required to show that their product has the form

$$\zeta^{iw} = \cosh w + i \sinh w (\cosh \phi'' \cdot \epsilon - i \sinh \phi'' \cdot \alpha)^{\frac{\pi}{2}}.$$

We have $\xi^{iu} = \cosh u + i \sinh u \cdot \xi^{\frac{\pi}{2}}$

and $$\eta^{iv} = \cosh v + i \sinh v \cdot \eta^{\frac{\pi}{2}},$$

therefore

$$\xi^{iu}\eta^{iv} = \cosh u \cosh v + \sinh u \sinh v \cos \xi\eta$$
$$+ i\{\cosh u \sinh v \cdot \eta + \cosh v \sinh u \cdot \xi - i \sinh u \sinh v \sin \xi\eta \cdot \overline{\xi\eta}\}^{\frac{\pi}{2}}.$$

It remains to determine $\cos \xi\eta$ and $\operatorname{Sin} \xi\eta$.

Since $$\xi = \cosh \phi \cdot \beta - i \sinh \phi \cdot \alpha,$$

and $$\eta = \cosh \phi' \cdot \gamma - i \sinh \phi' \cdot \alpha,$$

and as we have seen that the i is merely scalar, and does not affect the direction, we conclude that

$$\cos\xi\eta = \cosh\phi\cosh\phi'\cos\beta\gamma - \sinh\phi\sinh\phi',$$

$$\begin{aligned}\operatorname{Sin}\xi\eta = {} & \cosh\phi\cosh\phi'\sin\beta\gamma\cdot\alpha \\ & - i(\cosh\phi\sinh\phi'\cdot\overline{\beta\alpha} + \cosh\phi'\sinh\phi\cdot\overline{\alpha\gamma}).\end{aligned}$$

Substituting these values of $\cos\xi\eta$ and $\operatorname{Sin}\xi\eta$, we obtain

$$\begin{aligned}\cosh w = {} & \cosh u\cosh v \\ & + \sinh u\sinh v(\cosh\phi\cosh\phi'\cos\beta\gamma - \sinh\phi\sinh\phi'),\end{aligned} \quad (1)$$

$$\begin{aligned}\sinh w\sinh\phi'' = {} & \cosh u\sinh v\sinh\phi' + \cosh v\sinh u\sinh\phi \\ & + \sinh u\sinh v\cosh\phi\cosh\phi'\sin\beta\gamma,\end{aligned} \quad (2)$$

$$\begin{aligned}\sinh w\cosh\phi''\cdot\epsilon = {} & \cosh u\sinh v\cosh\phi'\cdot\gamma + \cosh v\sinh u\cosh\phi\cdot\beta \\ & - \sinh u\sinh v(\cosh\phi\sinh\phi'\cdot\overline{\beta\alpha} + \cosh\phi'\sinh\phi\cdot\overline{\alpha\gamma}).\end{aligned} \quad (3)$$

Let us consider, more minutely, the above equations

$$\cos\xi\eta = \cosh\phi\cosh\phi'\cos\beta\gamma - \sinh\phi\sinh\phi',$$

and
$$\begin{aligned}\operatorname{Sin}\xi\eta = {} & \cosh\phi\cosh\phi'\sin\beta\gamma\cdot\alpha \\ & - i(\cosh\phi\sinh\phi'\cdot\overline{\beta\alpha} + \cosh\phi'\sinh\phi\cdot\overline{\alpha\gamma}).\end{aligned}$$

If we square these functions, we find

$$\begin{aligned}(\cos\xi\eta)^2 = {} & \cosh^2\phi\cosh^2\phi'\cos^2\beta\gamma + \sinh^2\phi\sinh^2\phi' \\ & - 2\cosh\phi\cosh\phi'\sinh\phi\sinh\phi'\cos\beta\gamma,\end{aligned}$$

$$\begin{aligned}(\operatorname{Sin}\xi\eta)^2 = {} & \cosh^2\phi\cosh^2\phi'\sin^2\beta\gamma - \cosh^2\phi\sinh^2\phi' - \cosh^2\phi'\sinh^2\phi \\ & - 2\cosh\phi\cosh\phi'\sinh\phi\sinh\phi'\cos\overline{\beta\alpha}\ \overline{\alpha\gamma};\end{aligned}$$

but $\cos\overline{\beta\alpha}\ \overline{\alpha\gamma} = -\cos\beta\gamma$, and $\cosh^2 = 1 + \sinh^2$, therefore,

$$(\cos\xi\eta)^2 + (\operatorname{Sin}\xi\eta)^2 = 1.$$

As the symbol i does not affect the geometrical composition, $\operatorname{Sin}\xi\eta$ must be normal to the plane of ξ and η; hence, if we analyze it into $\sin\xi\eta\cdot\overline{\xi\eta}$, we must have $\sin\xi\eta = \sqrt{1-(\cos\xi\eta)^2}$, and $\overline{\xi\eta} = \dfrac{\operatorname{Sin}\xi\eta}{\sqrt{1-(\cos\xi\eta)^2}}$.

Consider the special case, when $\gamma = \beta$. Then

$$\cos\xi\eta = \cosh\phi\cosh\phi' - \sinh\phi\sinh\phi',$$

and $\operatorname{Sin}\xi\eta = -i(\cosh\phi\sinh\phi' - \cosh\phi'\sinh\phi)\,\overline{\beta\alpha}.$

Hence $\xi\eta$ becomes an excircular versor. Consider next the special case where γ is perpendicular to β. Then

$$\cos\xi\eta = -\sinh\phi\sinh\phi',$$

and $\operatorname{Sin}\xi\eta = \cosh\phi\cosh\phi'\cdot\alpha + i(\cosh\phi\sinh\phi'\cdot\gamma + \cosh\phi'\sinh\phi\cdot\beta)$.

It appears that the locus of the poles of all the axes is the equilateral hyperboloid of one sheet. (v. page 27.)

FUNDAMENTAL THEOREM FOR THE EQUILATERAL HYPERBOLOID OF ONE SHEET.

To find the product of a circular and an excircular versor, when they have a common plane.

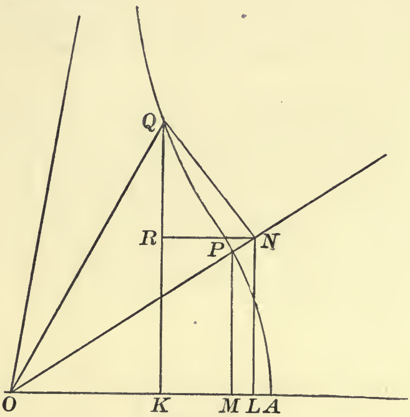

FIG. 13.

Let AOP represent a circular, and POQ an excircular, versor (Fig. 13); and let them be denoted by β^u and β^{iv}. We have

$$\begin{aligned}\beta^u\beta^{iv} = \beta^{u+iv} &= (\cos u + \sin u\cdot\beta^{\frac{\pi}{2}})(\cosh v + i\sinh v\cdot\beta^{\frac{\pi}{2}}) \\ &= \cos u\cosh v - i\sin u\sinh v \\ &\qquad + (\cosh v\sin u + i\cos u\sinh v)\cdot\beta^{\frac{\pi}{2}}.\end{aligned}$$

What is the meaning of the i which occurs in these scalar functions? Is the magnitude of the cosine

$$\sqrt{(\cos u\cosh v)^2 - (\sin u\sinh v)^2},$$

or is it $\qquad \cos u\cosh v - \sin u\sinh v$?

At page 48 of *Definitions of the Trigonometric Functions,* I show that

$$\cos(u+iv)=\frac{OK}{OA}, \quad \text{and} \quad \sin(u+iv)=\frac{KQ}{OA},$$

and that the ordinary proof for the cosine and the sine of the sum of two angles gives

$$\frac{OK}{OA}=\frac{OM}{OA}\frac{ON}{OP}-\frac{MP}{OA}\frac{NQ}{OP};$$

that is, $\cos(u+iv)=\cos u\cosh v-\sin u\sinh v,$

and

$$\frac{KQ}{OA}=\frac{MP}{OA}\frac{ON}{OP}+\frac{OM}{OA}\frac{NQ}{OP};$$

that is, $\sin(u+iv)=\sin u\cosh v+\cos u\sinh v.$

What, then, is the function of the i? It shows that if you form the two squares, taking account of it, their sum will be equal to unity. Also, in forming the products of versors, it must be taken into account. When it is preserved, the rules for circular versors apply without change to excircular versors.

Here we have the true geometric meaning of a *bi-versor,* and consequently of a *bi-quaternion;* for the latter is only the former multiplied by a line.

As a special case, let $u=\frac{\pi}{2}$; we then have

$$\beta^{\frac{\pi}{2}+iv}=-i\sinh v+\cosh v\cdot\beta^{\frac{\pi}{2}};$$

this versor evidently refers to the conjugate hyperbola.

Again, let $u=\pi$; we have

$$\beta^{\pi+iv}=-(\cosh v+i\sinh v\cdot\beta^{\frac{\pi}{2}}),$$

which refers to the opposite hyperbola.

In the following table, the related excircular versors are placed in the same line with their circular analogues, and the diagram (Fig. 14) shows the related versors graphically.

Circular.		Excircular.		
β^{u}	$= \cos u + \sin u \cdot \beta^{\frac{\pi}{2}}$	β^{iu}	$= \cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}}$	AOP_1
$\beta^{\frac{\pi}{2}-u}$	$= \sin u + \cos u \cdot \beta^{\frac{\pi}{2}}$	$\beta^{\frac{\pi}{2}-iu}$	$= i \sinh u + \cosh u \cdot \beta^{\frac{\pi}{2}}$	AOP_2
$\beta^{\frac{\pi}{2}+u}$	$= - \sin u + \cos u \cdot \beta^{\frac{\pi}{2}}$	$\beta^{\frac{\pi}{2}+iu}$	$= - i \sinh u + \cosh u \cdot \beta^{\frac{\pi}{2}}$	AOP_3
$\beta^{\pi-u}$	$= - \cos u + \sin u \cdot \beta^{\frac{\pi}{2}}$	$\beta^{\pi-iu}$	$= - \cosh u + i \sinh u \cdot \beta^{\frac{\pi}{2}}$	AOP_4
$\beta^{\pi+u}$	$= - \cos u - \sin u \cdot \beta^{\frac{\pi}{2}}$	$\beta^{\pi+iu}$	$= - \cosh u - i \sinh u \cdot \beta^{\frac{\pi}{2}}$	AOP_5
$\beta^{-\frac{\pi}{2}-u}$	$= - \sin u - \cos u \cdot \beta^{\frac{\pi}{2}}$	$\beta^{-\frac{\pi}{2}-iu}$	$= - i \sinh u - \cosh u \cdot \beta^{\frac{\pi}{2}}$	AOP_6
$\beta^{-\frac{\pi}{2}+u}$	$= \sin u - \cos u \cdot \beta^{\frac{\pi}{2}}$	$\beta^{-\frac{\pi}{2}+iu}$	$= i \sinh u - \cosh u \cdot \beta^{\frac{\pi}{2}}$	AOP_7
β^{-u}	$= \cos u - \sin u \cdot \beta^{\frac{\pi}{2}}$	β^{-iu}	$= \cosh u - i \sinh u \cdot \beta^{\frac{\pi}{2}}$	AOP_8

It is evident that AOP_2 is the complement, AOP_4 the supplement, and AOP_8 the reciprocal, of AOP_1. It is not the circular

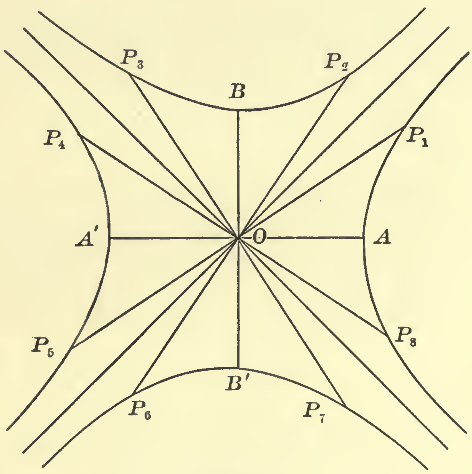

Fig. 14.

term of the complex exponent which is affected by the $\sqrt{-1}$, but the excircular term. Thus space analysis throws a new light upon the periodicity of the hyperbolic functions.

To find the product of two versors of the equilateral hyperboloid of one sheet, when each passes through the principal axis of the hyperboloid.

Let P be a point on the excircle of one sheet (Fig. 15), OP its radius; draw OB equal to OA, in the plane of OA and OP; AB is joined by a quadrant of a circle, and BOP by a sector of an excircle. Let u denote the ratio of twice the area of the sector POB to the square of OA; $\frac{\pi}{2}$ is the ratio of twice the area of BOA to the square of OA. Hence if β is a unit axis perpendicular to OB and OA, the expression for the versor POA is $\beta^{\frac{\pi}{2}+iu}$. Similarly, the expression for the versor AOQ is $\gamma^{\frac{\pi}{2}+iv}$.

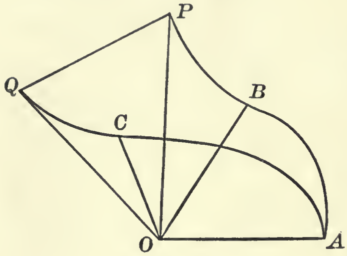

FIG. 15.

$$\text{Now } \beta^{\frac{\pi}{2}+iu}\gamma^{\frac{\pi}{2}+iv} = (-i\sinh u + \cosh u\cdot\beta^{\frac{\pi}{2}})(-i\sinh v + \cosh v\cdot\gamma^{\frac{\pi}{2}})$$
$$= -(\sinh u \sinh v + \cosh u \cosh v \cos\beta\gamma)$$
$$-\{i(\cosh u \sinh v\cdot\beta + \cosh v \sinh u\cdot\gamma) + \cosh u \cosh v \sin\beta\gamma\cdot\alpha\}^{\frac{\pi}{2}}.$$

Now the magnitude of $\cosh u \sinh v\cdot\beta + \cosh v \sinh u\cdot\gamma$ may be greater or less than $\cosh u \cosh v \sin\beta\gamma$. If it is greater, then the directed sine may be thrown into the form

$$-i\{(\cosh u \sinh v\cdot\beta + \cosh v \sinh u\cdot\gamma) - i\cosh u \cosh v \sin\beta\gamma\cdot\alpha\},$$

consequently, the ratio is excircular, and the axis excircular; hence the product takes the form

$$-\xi^{iw}, \text{ where } \xi = \cosh\phi\cdot\epsilon - i\sinh\phi\cdot\alpha.$$

But if $\cosh u \cosh v \sin\beta\gamma$ is the greater, the directed sine takes the form

$$-\{\cosh u \cosh v \sin\beta\gamma\cdot\alpha + i(\cosh u \sinh v\cdot\beta + \cosh v \sinh u\cdot\gamma)\}.$$

The ratio of the product is circular, but the axis is excircular. Let w denote the ratio; the axis has the form $\cosh\phi\cdot\alpha - i\sinh\phi\cdot\epsilon$, so that the product is of the form

$$-\xi^{w} = -\cos w - \sin w(\cosh\phi\cdot\alpha - i\sinh\phi\cdot\epsilon)^{\frac{\pi}{2}}.$$

In the former case, the locus of the poles of the axes is the exsphere of one sheet; in the latter, the opposite sheet of the exsphere of two sheets.

To find the product of two general versors of the equilateral hyperboloid of one sheet.

The one versor may be represented by

$$-\{x+(iy\cdot\beta+z\cdot\alpha)^{\frac{\pi}{2}}\},$$

where $x^2-y^2+z^2=1$, and β is perpendicular to α. Similarly, the other versor may be represented by

$$-\{x'+(iy'\cdot\gamma+z'\cdot\alpha)^{\frac{\pi}{2}}\},$$

where $x'^2-y'^2+z'^2=1$, and γ is perpendicular to α.

The cosine of the product is

$$xx'+yy'\cos\beta\gamma-zz',$$

and the Sine of the product is

$$i(xy'\cdot\gamma+x'y\cdot\beta)+(xz'+x'z+yy'\sin\beta\gamma)\cdot\alpha.$$

As before, if $(xy')^2+(x'y)^2+2xx'yy'\cos\beta\gamma$ is greater than $(xz'+x'z+yy'\sin\beta\gamma)^2$, the ratio of the product is excircular; but if less, it is circular. In the former case the axis is an axis of the exsphere of one sheet, in the latter it is an axis of the exsphere of two sheets.

To find the product of two versors which pass through the principal axis, when the one belongs to the exsphere of two sheets, the other to the exsphere of one sheet.

Let the former versor be denoted by β^{iu}, and the latter by $\gamma^{\frac{\pi}{2}+iv}$. Then

$$\begin{aligned}{}^{u}\gamma^{\frac{\pi}{2}+iv} &= (\cosh u+i\sinh u\cdot\beta^{\frac{\pi}{2}})(-i\sinh v+\cosh v\cdot\gamma^{\frac{\pi}{2}})\\ &= -i(\cosh u\sinh v+\sinh u\cosh v\cos\beta\gamma)\end{aligned}$$

$$+\{\cosh u\cosh v\cdot\gamma+\sinh u\sinh v\cdot\beta-i\sinh u\cosh v\sin\beta\gamma\cdot\alpha\}^{\frac{\pi}{2}}.$$

As the magnitude of $\cosh u\cosh v\cdot\gamma+\sinh u\sinh v\cdot\beta$ is by reasoning similar to that at page 23 seen to be greater than $\sinh u\cosh v\sin\beta\gamma$, we see that the axis is excircular; and the i before the scalar term shows that the ratio is excircular. From comparison of the table, page 27, we see that the product versor has the form

$$\xi^{\frac{\pi}{2}+iw}, \text{ where } \xi = \cosh\phi\cdot\epsilon - i\sinh\phi\cdot\alpha,$$

the equations being

$$\sinh w = \cosh u \sinh v + \sinh u \cosh v \cos\beta\gamma, \qquad (1)$$

$$\cosh w \sinh\phi = \cosh u \sinh v + \sinh u \cosh v \cos\beta\gamma, \qquad (2)$$

$$\cosh w \cosh\phi\cdot\epsilon = \cosh u \cosh v\cdot\gamma + \sinh u \sinh v\cdot\beta. \qquad (3)$$

FUNDAMENTAL THEOREM FOR THE HYPERBOLOID.

The theorems for the hyperboloid are obtained from the theorems for the exsphere in the same manner as the theorems for the ellipsoid are deduced from those for the sphere.

Two general versors for the hyperboloid of two sheets are expressed by ξ^{iu} and η^{iv}, where

$$\xi = \cosh\phi\ (\cos\psi\cdot k\beta + \sin\psi\cdot k'\gamma) - i\sinh\phi\cdot\alpha,$$

and $$\eta = \cosh\phi'(\cos\psi'\cdot k\beta + \sin\psi'\cdot k'\gamma) - i\sinh\phi'\cdot\alpha.$$

Now $$\xi^{iu}\eta^{iv} = (\cosh u + i\sinh u\cdot\xi^{\frac{\pi}{2}})(\cosh v + i\sinh v\cdot\eta^{\frac{\pi}{2}})$$

$$= \cosh u\cosh v + \sinh u \sinh v \cos\xi\eta$$

$$+ \{i(\cosh v \sinh u\cdot\xi + \cosh u \sinh v\,.\,\eta) + \sinh u \sinh v \operatorname{Sin}\xi\eta\}^{\frac{\pi}{2}}.$$

The problem is reduced to finding the versor $\xi\eta$. We apply the same principle as that employed in finding the versor between two elliptic axes (page 13), namely: Restore the axes to their excircular primitives, find the versor between these excircular axes (page 23), and change its axis according to the ratios of the contraction of the hyperboloid. This gives

$$\cos\xi\eta = \cosh\phi\cosh\phi'\{\cos(\psi-\psi')\} - \sinh\phi\sinh\phi',$$

$$\begin{aligned}\operatorname{Sin}\xi\eta = {} & \cosh\phi\cosh\phi'\sin(\psi-\psi')\cdot\alpha \\ & - i(\cosh\phi\sinh\phi'\sin\psi - \cosh\phi'\sinh\phi\sin\psi')\cdot k\beta \\ & + i(\cosh\phi\sinh\phi'\cos\psi - \cosh\phi'\sinh\phi\cos\psi')\cdot k'\gamma.\end{aligned}$$

In this manner, each theorem proved for the exsphere may be generalized for the hyperboloid.

DE MOIVRE'S THEOREM.

To find any integral power of a versor.

Let n denote any integral number. For the general spherical versor we have $(\xi^u)^n = \xi^{nu}$, because the axes of the factor versors are all the same. Hence

$$\cos nu + \sin nu \cdot \xi^{\frac{\pi}{2}}$$

$$= (\cos u + \sin u \cdot \xi^{\frac{\pi}{2}})^n$$

$$= \cos^n u + n \cos^{n-1} u \sin u \cdot \xi^{\frac{\pi}{2}} + \frac{n(n-1)}{2!} \cos^{n-2} u \sin^2 u \cdot \xi^{\pi} +,$$

from which it follows that

$$\cos nu = \cos^n u - \frac{n(n-1)}{2!} \cos^{n-2} u \sin^2 u +,$$

and $\sin nu = n \cos^{n-1} u \sin u - \dfrac{n(n-1)(n-2)}{3!} \cos^{n-3} u \sin^3 u +.$

Similarly for the exspherical versor $(\xi^{iu})^n$, as the axes are all the same $(\xi^{iu})^n = \xi^{inu}$, and

$$\cosh nu + i \sinh nu \cdot \xi^{\frac{\pi}{2}} = (\cosh u + i \sinh u \cdot \xi^{\frac{\pi}{2}})^n$$

$$= \cosh^n u + ni \cosh^{n-1} u \sinh u \cdot \xi^{\frac{\pi}{2}} + \frac{n(n-1)}{2!} i^2 \cosh^{n-2} u \sinh^2 u \cdot \xi^{\pi} +;$$

therefore

$$\cosh nu = \cosh^n u + \frac{n(n-1)}{2!} \cosh^{n-2} u \sinh^2 u +,$$

and $\sinh nu = n \cosh^{n-1} u \sinh u + \dfrac{n(n-1)(n-2)}{3!} \cosh^{n-3} u \sinh^3 u +.$

The only difference in the case of the general ellipsoidal versor is that u is measured elliptically and ξ is an ellipsoidal axis. So for the general hyperboloidal versor, u is measured hyperbolically and ξ is a hyperboloidal axis.

To find any integral root of a versor.

Consider first the case of an ellipsoidal versor. If u is defined as the ratio of twice the sector to the rectangle formed by the semi-axes, it cannot be greater than 2π. Then $(\xi^u)^{\frac{1}{n}}$ is unambiguously equal to $\xi^{\frac{u}{n}}$. Hence

$$\cos\frac{u}{n}+\sin\frac{u}{n}\cdot\xi^{\frac{\pi}{2}}=(\cos u+\sin u\cdot\xi^{\frac{\pi}{2}})^{\frac{1}{n}}.$$

If $\cos u$ is not less than $\sin u$, then

$$\cos\frac{u}{n}+\sin\frac{u}{n}\cdot\xi^{\frac{\pi}{2}}=(\cos u)^{\frac{1}{n}}\{1+\tan u\cdot\xi^{\frac{\pi}{2}}\}^{\frac{1}{n}}$$

$$=(\cos u)^{\frac{1}{n}}\left\{1+\frac{1}{n}\tan u\cdot\xi^{\frac{\pi}{2}}+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2\,!}\tan^2 u\cdot\xi^{\pi}+\right\};$$

therefore

$$\cos\frac{u}{n}=(\cos u)^{\frac{1}{n}}\left\{1+\frac{(n-1)}{n^2\,2\,!}\tan^2 u\right.$$
$$\left.-\frac{(n-1)(2n-1)(3n-1)}{n^4\,4\,!}\tan^4 u+\right\},$$

and $$\sin\frac{u}{n}=(\cos u)^{\frac{1}{n}}\left\{\frac{1}{n}\tan u-\frac{(n-1)(2n-1)}{n^3\,3\,!}\tan^3 u+\right\}.$$

But if $\sin u$ is not less than $\cos u$, we have the complementary series

$$\xi^{\frac{u}{n}}=(\sin u)^{\frac{1}{n}}\xi^{\frac{\pi}{2n}}\{1+\cot u\cdot\xi^{-\frac{\pi}{2}}\}^{\frac{1}{n}}.$$

Consider next the case of a hyperboloidal versor. A versor for the hyperboloid of two sheets is denoted by ξ^{iu}. Now

$$(\xi^{iu})^{\frac{1}{n}}=\xi^{\frac{iu}{n}}=\{\cosh u+i\sinh u\cdot\xi^{\frac{\pi}{2}}\}^{\frac{1}{n}}$$
$$=(\cosh u)^{\frac{1}{n}}\{1+i\tanh u\cdot\xi^{\frac{\pi}{2}}\}^{\frac{1}{n}},$$

for $\cosh u$ is always greater than $\sinh u$; therefore

$$\cosh\frac{u}{n}=(\cosh u)^{\frac{1}{n}}\left\{1-\frac{n-1}{n^2\,2\,!}\tanh^2 u\right.$$
$$\left.-\frac{(n-1)(2n-1)(3n-1)}{n^4\,4\,!}\tanh^2 u+\cdots\right\},$$

and $\sinh\frac{u}{n} = (\cosh u)^n \left\{\frac{1}{n}\tanh u - \frac{(n-1)(2n-1)}{n^3\, 3\,!}\tanh^3 u + \right\}.$

But a versor for the hyperboloid of one sheet is expressed by $\xi^{\frac{\pi}{2}+iu}$. Now

$$\left(\xi^{\frac{\pi}{2}+iu}\right)^{\frac{1}{n}} = \xi^{\frac{\pi}{2n}+\frac{iu}{n}} = \left\{-i\sinh u + \cosh u\cdot\xi^{\frac{\pi}{2}}\right\}^{\frac{1}{n}}$$
$$= (\cosh u)^{\frac{1}{n}}\xi^{\frac{\pi}{2n}}\left\{1 - i\tanh u\cdot\xi^{-\frac{\pi}{2}}\right\}^{\frac{1}{n}},$$

which is expanded as before.

POLAR THEOREM.

To deduce in the trigonometry of the sphere the polar theorem corresponding to the fundamental theorem.

The cosine theorem, which is the fundamental theorem of spherical trigonometry, expresses the side of a spherical triangle in terms of the opposite sides and their included angle. In treatises on spherical trigonometry, it is shown how to deduce from the cosine theorem a polar or supplemental theorem which expresses an angle in terms of the other two angles and the opposite side. It is our object to find the polar theorem corresponding to the complete fundamental theorem.

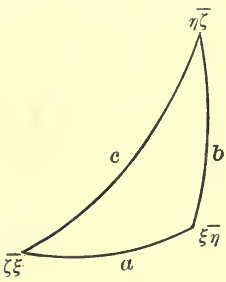

FIG. 16.

Let the versors of the three sides of the spherical triangle (Fig. 16), taken the same way round, be denoted by ξ^a, η^b, ζ^c, where ξ, η, ζ are unit axes, and a, b, c denote the ratio of twice the area of the sector to the area of the rectangle formed by the semi-axes of its circle (which, in this case, is simply the square of the radius). The angles included by the sides are usually denominated A, B, C, respectively, but what it is necessary to consider in view of further generalization is the angles between the planes, or rather the versors between the axes. These in accordance with our notation are denoted by $\eta\zeta$, $\zeta\xi$, and $\xi\eta$ respectively; the axes of these versors, which are also of unit length, are denoted by $\overline{\eta\zeta}$, $\overline{\zeta\xi}$, and $\overline{\xi\eta}$, respectively, and they correspond to the poles of the corners of the triangle as indicated by the figure.

The fundamental theorem is

$$\xi^a\eta^b = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{\cos b \sin a \cdot \xi + \cos a \sin b \cdot \eta - \sin a \sin b \sin \xi\eta \cdot \overline{\xi\eta}\}^{\frac{\pi}{2}};$$

but as ζ^c is taken in the opposite direction, we have

$$\zeta^c = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{-\cos b \sin a \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \sin \xi\eta \cdot \overline{\xi\eta}\}^{\frac{\pi}{2}}.$$

The polar theorem is obtained by changing each side into the supplement of the corresponding angle and the angle into the supplement of the corresponding side. Hence

$$\cos(\pi - \xi\eta) = \cos(\pi - \eta\zeta)\cos(\pi - \zeta\xi)$$
$$-\sin(\pi - \eta\zeta)\sin(\pi - \zeta\xi)\cos(\pi - c);$$

that is, $\cos \xi\eta = -\cos \eta\zeta \cos \zeta\xi - \sin \eta\zeta \sin \zeta\xi \cos c.$

When A, B, C, are used to denote the external angles between the sides, the above equation is written

$$\cos C = -\cos A \cos B - \sin A \sin B \cos c.$$

Apply the same rule of change to the Sine part, and we obtain

$$\mathrm{Sin}(\pi - \xi\eta) = -\cos(\pi - \zeta\xi)\,\mathrm{Sin}(\pi - \eta\zeta) - \cos(\pi - \eta\zeta)\,\mathrm{Sin}(\pi - \zeta\xi)$$
$$+ \sin(\pi - \eta\zeta)\sin(\pi - \zeta\xi)\sin c \cdot \zeta;$$

that is, $\mathrm{Sin}\,\xi\eta = \cos\zeta\xi\,\mathrm{Sin}\,\eta\zeta + \cos\eta\zeta\,\mathrm{Sin}\,\zeta\xi + \sin\eta\zeta \sin\zeta\xi \sin c \cdot \zeta.$

To deduce the polar theorem for the ellipsoid.

Let ξ^a, η^b, ζ^c denote the three versors of the original ellipsoidal triangle taken the same way round; then the corresponding versors of the polar triangle are $\eta\zeta$, $\zeta\xi$, and $\xi\eta$. The third versor of the original triangle is given in terms of the other two by the theorem

$$\zeta^c = \cos a \cos b - \sin a \sin b \cos \xi\eta$$
$$+ \{-\cos b \sin a \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b\,\mathrm{Sin}\,\xi\eta\}^{\frac{\pi}{2}}.$$

The third versor of the polar triangle is obtained in terms of the other two by changing each versor into the supplement of its corresponding versor; hence

$$\cos\xi\eta = -\cos\eta\zeta\cos\zeta\xi - \sin\eta\zeta\sin\zeta\xi\cos c,$$

and $\mathrm{Sin}\,\xi\eta = \cos\zeta\xi\,\mathrm{Sin}\,\eta\zeta + \cos\eta\zeta\,\mathrm{Sin}\,\zeta\xi + \sin\eta\zeta\sin\zeta\xi\,\mathrm{Sin}\,\zeta^c$.

In form it is the same as for the sphere; the only difference is in the expressions for the ellipsoidal axes ξ, η, ζ, and the manner of deducing the cosine and Sine of the versor between two such axes. (See page 13.) The polar ellipsoid is not identical with the original ellipsoid; the ratios of the two minor axes are interchanged.

To deduce the polar theorem for the exsphere of two sheets.

Let ξ^{ia}, η^{ib}, ζ^{ic} denote the versors for the three sides of a triangle of the exsphere of two sheets, taken in the same order round. The axes ξ, η, ζ have their poles on the exsphere of two sheets (page 23); it is required to deduce the theorem for that polar triangle. For the original triangle, we have

$$\zeta^{ic} = \cos ia\cos ib - \sin ia\sin ib\cos\xi\eta$$
$$+\{-\cos ib\sin ia\cdot\xi - \cos ia\sin ib\cdot\eta + \sin ia\sin ib\,\mathrm{Sin}\,\xi\eta\}^{\frac{\pi}{2}}.$$

By changing each versor into the supplement of the corresponding versor, we obtain

$$\xi\eta = -\cos\eta\zeta\cos\zeta\xi - \sin\eta\zeta\sin\zeta\xi\cosh c$$
$$+\{\cos\zeta\xi\,\mathrm{Sin}\,\eta\zeta + \cos\eta\zeta\,\mathrm{Sin}\,\zeta\xi + i\sin\eta\zeta\sin\zeta\xi\sinh c\cdot\zeta\}^{\frac{\pi}{2}}.$$

The above cosine equation has a marked resemblance to the fundamental equation of non-euclidean geometry (see Dr. Günther's *Hyperbelfunctionen*, pages 306 and 322). It is true that $\eta\zeta$ and $\zeta\xi$ are not simple circular versors, but the functions are *cos* and *sin* in a generalized sense. I venture the opinion that non-euclidean geometry is nothing but trigonometry on the exsphere; and that the so-called elliptic and hyperbolic geometries are identical with the ellipsoidal and hyperboloidal trigonometry developed in this paper.

To deduce the general polar theorem for the exsphere.

Let ξ^a, η^b, ζ^c denote the three sides of an exspherical triangle; the axes ξ, η, ζ are exspherical, but the ratios a, b, c may be circular or excircular, or be compounded of π or $\frac{\pi}{2}$ and an excircular ratio. For the original triangle, we have

$$\begin{aligned}\zeta^c &= \cos a \cos b - \sin a \sin b \cos \xi\eta \\ &\qquad + \{-\cos a \sin b \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \operatorname{Sin} \xi\eta\}^{\frac{\pi}{2}},\end{aligned}$$

and for the polar triangle,

$$\begin{aligned}\xi\eta &= -\cos \eta\zeta \cos \zeta\xi - \sin \eta\zeta \sin \zeta\xi \cos c \\ &\qquad + \{\cos \zeta\xi \operatorname{Sin} \eta\zeta + \cos \eta\zeta \operatorname{Sin} \zeta\xi + \sin \eta\zeta \sin \zeta\xi \operatorname{Sin} \zeta^c\}^{\frac{\pi}{2}}.\end{aligned}$$

Here the functions cos and sin are used in their most general meaning.

SINE THEOREM.

To prove that if ξ^a, η^b, ζ^c denote the three versors of a spherical triangle, then

$$\frac{\sin \eta\zeta}{\sin a} = \frac{\sin \zeta\xi}{\sin b} = \frac{\sin \xi\eta}{\sin c}.$$

We have $\cos c = \cos a \cos b - \sin a \sin b \cos \xi\eta$,

and $\qquad \sin c \cdot \zeta = -\cos b \sin a \cdot \xi - \cos a \sin b \cdot \eta + \sin a \sin b \sin \xi\eta \cdot \overline{\xi\eta}$.

By squaring the second equation, we obtain

$$\begin{aligned}\sin^2 c &= \cos^2 b \sin^2 a + \cos^2 a \sin^2 b + \sin^2 a \sin^2 b \sin^2 \xi\eta \\ &\qquad + 2 \cos a \cos b \sin a \sin b \cos \xi\eta;\end{aligned}$$

then, by substituting for $\cos \xi\eta$ from the first equation, and reducing, we obtain

$$\sin a \sin b \sin \xi\eta = \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}.$$

Hence
$$\frac{\sin \xi\eta}{\sin c} = \frac{\sin \eta\zeta}{\sin a} = \frac{\sin \zeta\xi}{\sin b}.$$

This theorem is also true for an ellipsoid of revolution, for then

$$\sin a \sin b \sin \xi\eta = k \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}.$$

To find the analogue for the exsphere of the sine theorem.

Let ξ, η, ζ denote exspherical axes, and a, b, c versors which may be circular, or excircular, or both combined. Then, with the general meaning of the sin and cos functions,

$$\sin a \sin b \sin \xi\eta = \sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}.$$

Hence $$\frac{\sin \xi\eta}{\sin c}=\frac{\sin \eta\zeta}{\sin a}=\frac{\sin \zeta\xi}{\sin b}.$$

We have seen that, if a and b are both simply excircular, it does not follow that c is (page 28).

SUM AND DIFFERENCE THEOREMS.

The reciprocal of a given versor.

By the reciprocal of a given versor is meant the versor of equal index but of opposite axis. Let ξ^u denote the given spherical versor; its reciprocal is $(-\xi)^u$. But it may be shown that $\xi^{-u}=(-\xi)^u$. For

$$\begin{aligned}\xi^{-u} &= \cos(-u)+\sin(-u)\cdot\xi^{\frac{\pi}{2}}\\ &= \cos u-\sin u\cdot\xi^{\frac{\pi}{2}}\\ &= \cos u+\sin u\cdot(-\xi)^{\frac{\pi}{2}}\\ &= (-\xi)^u.\end{aligned}$$

Similarly the reciprocal of an exspherical versor ξ^{iu} is $(-\xi)^{iu}$ or ξ^{-iu}, and

$$\xi^{-iu}=\cosh u-i\sinh u\cdot\xi^{\frac{\pi}{2}}.$$

The reciprocal of an ellipsoidal versor ξ^u is also ξ^{-u}, the only difference being that ξ is no longer a spherical, but an ellipsoidal axis. So for the hyperboloidal versor.

To find the analogues of the sum and difference theorems of plane trigonometry.

At page 45 of "The Imaginary of Algebra," I have shown how to generalize for the sphere the following well-known theorems in plane trigonometry, namely,

$$\cos(A+B)+\cos(A-B)=\quad 2\cos A\cos B,$$
$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B,$$

$$\sin(A+B)+\sin(A-B)=2\cos B\sin A,$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B,$$

and
$$\cos C+\cos D=\quad 2\cos\frac{C+D}{2}\cos\frac{C-D}{2},$$
$$\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2},$$
$$\sin C+\sin D=\quad 2\sin\frac{C+D}{2}\cos\frac{C-D}{2},$$
$$\sin C-\sin D=\quad 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}.$$

The generalized formulæ of the first set for the sphere are, using general axes ξ and η,

$$\cos\xi^A\eta^B+\cos\xi^A\eta^{-B}=\quad 2\cos A\cos B,$$
$$\cos\xi^A\eta^B-\cos\xi^A\eta^{-B}=-2\cos(\operatorname{Sin}\xi^A\operatorname{Sin}\eta^B),$$
$$\operatorname{Sin}\xi^A\eta^B+\operatorname{Sin}\xi^A\eta^{-B}=\quad 2\cos B\operatorname{Sin}\xi^A,$$
$$\operatorname{Sin}\xi^A\eta^B-\operatorname{Sin}\xi^A\eta^{-B}=\quad 2\{\cos A\operatorname{Sin}\eta^B-\operatorname{Sin}(\operatorname{Sin}\xi^A\operatorname{Sin}\eta^B)\}.$$

Corresponding to the latter set of four equations we have

$$\cos\zeta^C+\cos\omega^D=\quad 2\cos\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\cos(\omega^{-D}\zeta^C)^{\frac{1}{2}},$$
$$\cos\zeta^C-\cos\omega^D=-2\cos[\operatorname{Sin}\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\operatorname{Sin}(\omega^{-D}\zeta^C)^{\frac{1}{2}}],$$
$$\operatorname{Sin}\zeta^C+\operatorname{Sin}\omega^D=\quad 2\cos(\omega^{-D}\zeta^C)^{\frac{1}{2}}\operatorname{Sin}\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\},$$
$$\operatorname{Sin}\zeta^C-\operatorname{Sin}\omega^D=\quad 2\cos\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\operatorname{Sin}(\omega^{-D}\zeta^C)^{\frac{1}{2}}$$
$$-2\operatorname{Sin}\operatorname{Sin}\{\omega^D(\omega^{-D}\zeta^C)^{\frac{1}{2}}\}\operatorname{Sin}(\omega^{-D}\zeta^C)^{\frac{1}{2}}.$$

The corresponding theorems for the ellipsoid are the same, excepting that

$$\xi=\cos\phi\cdot k\beta-\sin\phi\cdot\alpha,\quad \eta=\cos\phi'\cdot k\gamma-\sin\phi'\cdot\alpha.$$

Consequently $\cos\xi\eta$ is the same as before, but

$$\operatorname{Sin}\xi\eta=\cos\phi\cos\phi'\sin\beta\gamma\cdot\alpha-k(\cos\phi\sin\phi'\cdot\overline{\beta\alpha}+\cos\phi'\sin\phi\cdot\overline{\alpha\gamma}).$$

For the general ellipsoid the only difference is in the expressions for ξ, η, and $\sin\xi\eta\cdot\overline{\xi\eta}$.

EXPONENTIAL THEOREM.

To find the exponential series for an ellipsoidal versor.

In the expression ξ^u for a spherical versor, the u and ξ are truly related as index to base, for $\log \xi^u = u \log \xi^1 = u \cdot \xi^{\frac{\pi}{2}}$, and therefore $\xi^u = e^{u \cdot \xi^{\frac{\pi}{2}}}$. Consequently

$$\xi^u = 1 - \frac{1}{2\,!} u^2 + \frac{1}{4\,!} u^4 -$$
$$+ \left\{ u - \frac{u^3}{3\,!} + \frac{u^5}{5\,!} - \right\} \cdot \xi^{\frac{\pi}{2}}.$$

In the case of the spherical versor, $\xi = \cos\phi \cdot \beta - \sin\phi \cdot \alpha$, or $\cos\phi(\cos\psi \cdot \beta + \sin\psi \cdot \gamma) - \sin\phi \cdot \alpha$, where α, β, γ are unit axes mutually rectangular.

The expansion for the ellipsoidal versor ξ^u differs only in the way in which u is measured, and in the expression for ξ, which is now $\cos\phi \cdot k\beta - \sin\phi \cdot \alpha$, or $\cos\phi(\cos\psi \cdot k\beta + \sin\psi \cdot k'\gamma) - \sin\phi \cdot \alpha$.

To find the exponential series for a hyperboloidal versor.

The expression for a versor on the exsphere of two sheets is ξ^{iu}. Now

$$\begin{aligned}
\xi^{iu} &= e^{iu \cdot \xi^{\frac{\pi}{2}}} \\
&= 1 + iu \cdot \xi^{\frac{\pi}{2}} + \frac{(iu)^2}{2\,!} \cdot \xi^{\pi} + \frac{(iu)^3}{3\,!} \cdot \xi^{3\frac{\pi}{2}} + \\
&= 1 + \frac{u^2}{2\,!} + \frac{u^4}{4\,!} + \\
&\quad + i \left\{ u + \frac{u^3}{3\,!} + \frac{u^5}{5\,!} + \quad \right\} \cdot \xi^{\frac{\pi}{2}}.
\end{aligned}$$

The expression for a mixed exspherical versor is ξ^{u+iv}. Now

$$\begin{aligned}
\xi^{u+iv} &= e^{(u+iv) \cdot \xi^{\frac{\pi}{2}}} \\
&= 1 + (u+iv) \cdot \xi^{\frac{\pi}{2}} + \frac{(u+iv)^2}{2\,!} \cdot \xi^{\pi} + \frac{(u+iv)^3}{3\,!} \cdot \xi^{3\frac{\pi}{2}} + \\
&= 1 - \frac{(u+iv)^2}{2\,!} + \frac{(u+iv)^4}{4\,!} - \\
&\quad + \left\{ u + iv - \frac{(u+iv)^3}{3\,!} + \quad \right\} \cdot \xi^{\frac{\pi}{2}}.
\end{aligned}$$

Both the cosine and the sine break up into two components, the one independent of i, and the other involving i. Here we have the sine and the cosine of the ordinary complex quantity.

As the ratio of a hyperboloidal versor may be circular or excircular, or both combined, the general versor may be expressed by ξ^a, where a is as general as stated. Then

$$\begin{aligned}\xi^a &= e^{a\cdot\xi^{\frac{\pi}{2}}}\\ &= 1-\frac{a^2}{2!}+\frac{a^4}{4!}-\\ &\qquad +\left\{a-\frac{a^3}{3!}+\frac{a^5}{5!}-\right\}\cdot\xi^{\frac{\pi}{2}}.\end{aligned}$$

To find the exponential series for the product of two ellipsoidal versors.

In the paper on *The Fundamental Theorems of Analysis Generalized for Space* I have shown that if ξ^u and η^v denote any two spherical versors, then

$$\begin{aligned}\xi^u\eta^v &= e^{u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}}}\\ &= 1+(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})+\frac{1}{2!}(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})^2+\frac{1}{3!}(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})^3+,\end{aligned}$$

where the powers of the binomial are expanded according to the binomial theorem, but subject to the special proviso that the order of the axes ξ, η must be preserved in all the axial terms. Thus

$$\begin{aligned}\xi^u\eta^v &= 1+u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}}\\ &\qquad +\frac{1}{2!}\{u^2\cdot\xi^{\pi}+2uv\cdot\xi^{\frac{\pi}{2}}\eta^{\frac{\pi}{2}}+v^2\cdot\eta^{\pi}\}\\ &\qquad +\frac{1}{3!}\{u^3\cdot\xi^{\frac{3\pi}{2}}+3u^2v\cdot\xi^{\pi}\eta^{\frac{\pi}{2}}+3uv^2\cdot\xi^{\frac{\pi}{2}}\eta^{\pi}+v^3\cdot\eta^{\frac{3\pi}{2}}\}\\ &\qquad +\text{etc.}\end{aligned}$$

$$\left.\begin{aligned}&= 1-\frac{1}{2!}\{u^2+2uv\cos\xi\eta+v^2\}\\ &\qquad +\frac{1}{4!}\{u^4+4u^3v\cos\xi\eta+6u^2v^2+4uv^3\cos\xi\eta+v^4\}\\ &\qquad -\text{etc.}\end{aligned}\right\}\qquad(1)$$

$$+\left\{u-\frac{1}{3!}(u^3+3uv^2)+\text{etc.}\right\}\cdot\xi^{\frac{\pi}{2}} \tag{2}$$

$$+\left\{v-\frac{1}{3!}(3u^2v+v^3)+\text{etc.}\right\}\cdot\eta^{\frac{\pi}{2}} \tag{3}$$

$$+\left\{-\frac{1}{2!}2uv+\frac{1}{4!}(4u^3v+4uv^3)-\text{etc.}\right\}\sin\xi\eta\cdot\overline{\xi\eta}^{\frac{\pi}{2}}. \tag{4}$$

In the case of the sphere

$$\xi=\cos\phi\cdot\beta-\sin\phi\cdot\alpha,$$

and $$\eta=\cos\phi'\cdot\gamma-\sin\phi'\cdot\alpha;$$

consequently $\cos\xi\eta=\cos\phi\cos\phi'\cos\beta\gamma+\sin\phi\sin\phi'$, and

$$\operatorname{Sin}\xi\eta=\cos\phi\cos\phi'\sin\beta\gamma\cdot\alpha-(\cos\phi\sin\phi'\cdot\overline{\beta\alpha}+\cos\phi'\sin\phi\cdot\overline{\alpha\gamma}).$$

For the ellipsoid of revolution the expansion is obtained by introducing ellipsoidal axes ξ and η; and the corresponding theorems for the hyperboloid are obtained by changing the axes and indices into hyperboloid axes and indices.

To find the exponential series for the product of two hyperboloidal versors.

Let ξ and η denote any two hyperboloidal axes, and u and v general hyperboloidal ratios (p. 40). Then the product is

$$\xi^u\eta^v=e^{u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}}}$$

$$=1+(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})+\frac{(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})^2}{2!}+\frac{(u\cdot\xi^{\frac{\pi}{2}}+v\cdot\eta^{\frac{\pi}{2}})^3}{3!}+\cdots$$

The form of the theorem is the same as before.

LOGARITHMIC VERSORS.

In the paper on *The Fundamental Theorems of Analysis Generalized for Space,* page 16, I have shown that when the index of α, in $e^{A\cdot\alpha^{\frac{\pi}{2}}}$, is generalized, we obtain the expression for the versor corresponding to a sector of a logarithmic spiral. Let w denote the general angle, and α_w^A the generalized versor; then

$$\begin{aligned}\alpha_w^A &= e^{A\cdot\alpha^w}\\ &= 1 + A\cdot\alpha^w + \frac{A^2\cdot\alpha^{2w}}{2!} + \frac{A^3\cdot\alpha^{3w}}{3!} + \frac{A^4\cdot\alpha^{4w}}{4!} + \\ &= 1 + A\cos w + \frac{A^2\cos 2w}{2!} + \frac{A^3\cos 3w}{3!} + \text{etc.}\\ &\quad + \left\{ A\sin w + \frac{A^2\sin 2w}{2!} + \frac{A^3\sin 3w}{3!} + \text{etc.} \right\}\cdot\alpha^{\frac{\pi}{2}}\\ &= e^{A\cos w}\, e^{A\sin w\,.\,\alpha^{\frac{\pi}{2}}}.\end{aligned}$$

It is there shown that w is the constant angle between the radius vector and the tangent, or rather that it is the constant difference between the circular versor from the principal axis to the tangent, and that from the principal axis to the radius vector. It is also shown that $A\sin w$ gives the ratio of twice the area of the corresponding circular sector to the square of the radius, while $A\cos w$ gives the logarithm of the ratio of the radius vector to the principal axis.

I have there called such a logarithmic versor, when multiplied by a length, a *quinternion*. In his *Synopsis der Hoheren Mathematik,* Mr. Hagen has pointed out that the proper classical word is *quinion.* A quaternion means a ratio of three elements multiplied by a length; therefore, a ratio involving an additional element when multiplied by a length, is a quinion.

In the paper on *The Imaginary of Algebra,* an excircular analogue is deduced, namely, $\alpha_{iw}^A = e^{A\alpha^{iw}}$, but there are in reality three, according to whether A or w, or both, are affected by the $\sqrt{-1}$.

To deduce the four forms of logarithmic versor.

First: circular-circular. Let ξ^u denote a general spherical versor, then

$$\begin{aligned}\xi_w^u &= e^{u\cdot\xi^w} = e^{u\cos w + u\sin w\cdot\xi^{\frac{\pi}{2}}}\\ &= 1 + u\xi^w + \frac{u^2}{2!}\xi^{2w} + \frac{u^3}{3!}\xi^{3w} + \text{etc.}\end{aligned}$$

Here w denotes the constant difference between the versor from the principal axis to the tangent and that from the principal axis to the radius vector.

Second: circular-excircular. Let iw denote the constant difference between the excircular versor from the principal axis to the tangent, and that from the principal axis to the radius vector; then

$$\xi^u_{iw} = e^{u \cdot \xi^{iw}} = e^{u \cosh w + iu \sinh w \cdot \xi^{\frac{\pi}{2}}}$$

$$= 1 + u \cdot \xi^{iw} + \frac{u^2}{2!} \cdot \xi^{2iw} + \frac{u^3}{3!} \cdot \xi^{3iw} +$$

$$= 1 + u \cosh w + \frac{u^2}{2!} \cosh 2w + \frac{u^3}{3!} \cosh 3w +$$

$$+ i \left\{ u \sinh w + \frac{u^2}{2!} \sinh 2w + \frac{u^3}{3!} \sinh 3w + \right\} \cdot \xi^{\frac{\pi}{2}}.$$

Third: excircular-circular. Let ξ^{iu} denote a general exspherical versor; it is equal to $e^{iu \cdot \xi^{\frac{\pi}{2}}}$, and here $\frac{\pi}{2}$ denotes the constant sum of the circular versors above mentioned. Let that constant sum be any other circular versor w. Then

$$\xi^{iu}_{w} = e^{iu \cdot \xi^{w}} = e^{iu \cos w + iu \sin w \cdot \xi^{\frac{\pi}{2}}}$$

$$= 1 + iu \cdot \xi^{w} + \frac{(iu)^2}{2!} \cdot \xi^{2w} + \frac{(iu)^3}{3!} \cdot \xi^{3w} + \text{etc.}$$

$$= 1 - \frac{u^2}{2!} \cdot \xi^{2w} + \frac{u^4}{4!} \cdot \xi^{4w} +$$

$$+ i \left\{ u \cdot \xi^{w} - \frac{u^3}{3!} \cdot \xi^{3w} + \right\}$$

$$= 1 - \frac{u^2}{2!} \cos 2w + \frac{u^4}{4!} \cos 4w - \text{etc.}$$

$$+ i \left\{ u \cos w - \frac{u^3}{3!} \cos 3w + \text{etc.} \right\}$$

$$+ \left\{ - \frac{u^2}{2!} \sin 2w + \frac{u^4}{4!} \sin 4w - \right\} \cdot \xi^{\frac{\pi}{2}}$$

$$+ i \left\{ u \sin w - \frac{u^3}{3!} \sin 3w + \text{etc.} \right\} \cdot \xi^{\frac{\pi}{2}}.$$

Here both the cosine and the sine consists of a real and an apparently imaginary part. The geometrical meaning has already been explained (page 25).

Fourth: excircular-excircular. Let iw denote the constant sum of the excircular versors mentioned in the second case. Then

$$\begin{aligned}
\xi_{iw}^{iu} &= e^{iu \cdot \xi^{iw}} = e^{iu \cosh w - u \sinh w \cdot \xi^{\frac{\pi}{2}}} \\
&= 1 + iu \cdot \xi^{iw} + \frac{(iu)^2}{2!} \cdot \xi^{2iw} + \frac{(iu)^3}{3!} \cdot \xi^{3iw} + \\
&= 1 - \frac{u^2}{2!}(\cosh 2w + i \sinh 2w \cdot \xi^{\frac{\pi}{2}}) + \\
&\quad + iu(\cosh w + i \sinh w \cdot \xi^{\frac{\pi}{2}}) - \\
&= 1 - \frac{u^2}{2!}\cosh 2w + \frac{u^4}{4!}\cosh 4w - \\
&\quad + i\left\{u \cosh w - \frac{u^3}{3!}\cosh 3w + \right\} \\
&\quad - \left\{u \sinh w - \frac{u^3}{3!}\sinh 3w + \right\} \cdot \xi^{\frac{\pi}{2}} \\
&\quad + i\left\{-\frac{u^2}{2!}\sinh 2w + \frac{u^4}{4!}\sinh 4w - \right\} \cdot \xi^{\frac{\pi}{2}}.
\end{aligned}$$

To find the product of two logarithmic versors of the most general kind.

Let ξ and η denote general axes, and u, w, v, t general ratiòs; that is, each may be a sum of a circular and an excircular ratio. Then ξ_w^u and η_t^v each denote a general logarithmic versor. Then

$$\begin{aligned}
\xi_w^u \eta_t^v &= e^{u \cdot \xi^w + v \cdot \eta^t} \\
&= 1 + (u \cdot \xi^w + v \cdot \eta^t) + \frac{(u \cdot \xi^w + v \cdot \eta^t)^2}{2!} + \frac{(u \cdot \xi^w + v \cdot \eta^t)^3}{3!} + \text{etc.}
\end{aligned}$$

The powers of the binomial are formed according to the same rule as before. (*Fundamental Theorems*, page 18.)

COMPOSITION OF ROTATIONS.

To find the resultant of two elliptic rotations round axes which pass through a common point.

Two circular rotations are compounded by the principle that the product of the half rotations is half of the resultant rotation.

Let any two circular rotations be denoted by ξ^u and η^v, and their resultant by $\xi^u \times \eta$; then

$$\xi^u \times \eta^v = (\xi^{\frac{u}{2}}\eta^{\frac{v}{2}})^2$$
$$= \left\{\cos\frac{u}{2}\cos\frac{v}{2} - \sin\frac{u}{2}\sin\frac{v}{2}\cos\xi\eta \right.$$
$$\left. + \left(\cos\frac{v}{2}\sin\frac{u}{2}\cdot\xi + \cos\frac{u}{2}\sin\frac{v}{2}\cdot\eta - \sin\frac{u}{2}\sin\frac{v}{2}\operatorname{Sin}\xi\eta\right)^{\frac{\pi}{2}}\right\}^2.$$

Let $x = \cos\frac{u}{2}\cos\frac{v}{2} - \sin\frac{u}{2}\sin\frac{v}{2}\cos\xi\eta$,

$$y = \sqrt{1-x^2},$$

$$\zeta = \frac{\cos\frac{v}{2}\sin\frac{u}{2}\cdot\xi + \cos\frac{u}{2}\sin\frac{v}{2}\cdot\eta - \sin\frac{u}{2}\sin\frac{v}{2}\operatorname{Sin}\xi\eta}{\sqrt{1-x^2}};$$

then $$\xi^u \times \eta^v = x^2 - y^2 + 2xy\cdot\zeta^{\frac{\pi}{2}}.$$

The elliptic generalization is obtained by generalizing the axes ξ and η and finding $\cos\xi\eta$ and $\operatorname{Sin}\xi\eta$, as at page 15.

To find the resultant of two hyperbolic rotations round axes which pass through a common point.

Let ξ^{iu} and η^{iv} denote two exspherical rotations which have a common principal axis; let their resultant be denoted by $\xi^{iu} \times \eta^{iv}$. By analogy we deduce that

$$\xi^{iu} \times \eta^{iv} = (\xi^{\frac{iu}{2}}\eta^{\frac{iv}{2}})^2$$
$$= \left\{\cosh\frac{u}{2}\cosh\frac{v}{2} + \sinh\frac{u}{2}\sinh\frac{v}{2}\cos\xi\eta \right.$$
$$\left. + i\left(\cosh\frac{v}{2}\sinh\frac{u}{2}\cdot\xi + \cosh\frac{u}{2}\sinh\frac{v}{2}\cdot\eta - i\sinh\frac{u}{2}\sinh\frac{v}{2}\operatorname{Sin}\xi\eta\right)^{\frac{\pi}{2}}\right\}^2.$$

Let $x = \cosh\frac{u}{2}\cosh\frac{v}{2} + \sinh\frac{u}{2}\sinh\frac{v}{2}\cos\xi\eta$.

$$y = \sqrt{x^2-1},$$

$$\zeta = \frac{\cosh\frac{v}{2}\sinh\frac{u}{2}\cdot\xi + \cosh\frac{u}{2}\sinh\frac{v}{2}\cdot\eta - i\sinh\frac{u}{2}\sinh\frac{v}{2}\operatorname{Sin}\xi\eta}{\sqrt{x^2-1}}.$$

Then $$\xi^{iu} \times \eta^{iv} = x^2 + y^2 + 2xy\cdot\zeta^{\frac{\pi}{2}}.$$

Suppose a fluid to move round the axis ξ, each particle describing a hyperbolic angle u, and then round the axis η by a hyperbolic angle v, the principal axes of the two motions coinciding; the resultant gives the angle, the plane, and the principal axes of the equivalent single motion of the same kind. The axis of that motion does not pass through the intersection of the axes of the components.

A more general result is obtained by supposing the ratios to be complex; the theorem is then expressed by the spherical theorem taken in a generalized sense, just as in ordinary algebra x may be positive or negative.

To find the effect of an elliptic rotation on a line.

The effect of a circular rotation ξ^u upon a unit axis ρ, is given by the equation

$$\xi^u\rho = \cos\xi\rho\cdot\xi + \sin u\ \mathrm{Sin}\,\xi\rho + \cos u\ \mathrm{Sin}(\mathrm{Sin}\,\xi\rho)\,\xi.$$

(*Principles of the Algebra of Physics,* page 100.)

It was shown by Cayley that the effect of ξ^u upon ρ is given by the Sine of the product $\xi^{-\frac{u}{2}}\rho^{\frac{\pi}{2}}\xi^{\frac{\pi}{2}}$. For by the expansion of

$$\left(\cos\frac{u}{2}-\sin\frac{u}{2}\cdot\xi^{\frac{\pi}{2}}\right)\rho^{\frac{\pi}{2}}\left(\cos\frac{u}{2}+\sin\frac{u}{2}\cdot\xi^{\frac{\pi}{2}}\right),$$

the directed sine is found to be

$$\cos^2\frac{u}{2}\cdot\rho + \sin^2\frac{u}{2}\cos\xi\rho\cdot\xi + \cos\frac{u}{2}\sin\frac{u}{2}\ \mathrm{Sin}\,\xi\rho - \sin^2\frac{u}{2}\ \mathrm{Sin}(\mathrm{Sin}\,\xi\rho)\,\xi.$$

But $\quad \cos^2\frac{u}{2}\cdot\rho = \cos^2\frac{u}{2}\cos\xi\rho\cdot\xi + \cos^2\frac{u}{2}\mathrm{Sin}(\mathrm{Sin}\,\xi\rho)\,\xi,$

therefore the directed sine is

$$\cos\xi\rho\cdot\xi + \sin u\ \mathrm{Sin}\,\xi\rho + \cos u\ \mathrm{Sin}(\mathrm{Sin}\,\xi\rho)\,\xi.$$

To generalize for an elliptic rotation we substitute the more general value of ξ and form $\cos\xi\rho$, $\mathrm{Sin}\,\xi\rho$, and $\mathrm{Sin}(\mathrm{Sin}\,\xi\rho)\,\xi$, according to the rules stated at page 15. For example, let

$$\xi = k\cos\phi\cdot\beta - \sin\phi\cdot\alpha,$$

$$\rho = \sin\theta\cdot\gamma + \cos\theta\cdot\alpha;$$

then

$$\cos \xi\rho = \cos\phi \sin\theta \cos\beta\gamma - \sin\phi\cos\theta,$$

$$\operatorname{Sin}\xi\rho = \cos\phi\sin\theta\sin\beta\gamma\cdot\alpha + k(\cos\phi\cos\theta\cdot\overline{\beta\alpha} - \sin\phi\sin\theta\cdot\overline{\alpha\gamma}).$$

To find the effect of a hyperbolic rotation on a line.

Consider the simplest exspherical analogue of the spherical theorem of the preceding article; it is

$$\xi^{iu}\rho = \cos\xi\rho\cdot\xi + i\sinh u \operatorname{Sin}\xi\rho + \cosh u \operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi.$$

But ξ is now an excircular axis of the form

$$\xi = \cosh\phi\cdot\beta - i\sinh\phi\cdot\alpha.$$

Let, as before, $\rho = \sin\theta\cdot\gamma + \cos\theta\cdot\alpha$;

then $\cos\xi\rho = \cosh\phi\sin\theta\cos\beta\gamma - i\sinh\phi\cos\theta,$

$$\operatorname{Sin}\xi\rho = \cosh\phi\sin\theta\sin\beta\gamma\cdot\alpha + \cosh\phi\cos\theta\cdot\overline{\beta\alpha} - i\sinh\phi\sin\theta\cdot\overline{\alpha\gamma},$$

$$\begin{aligned}\operatorname{Sin}(\operatorname{Sin}\xi\rho)\xi &= \cosh^2\phi\sin\theta\sin\beta\gamma\cdot\overline{\alpha\beta} + \cosh^2\phi\cos\theta\cdot\alpha - \sinh^2\phi\sin\theta\cdot\gamma \\ &\quad - i\cosh\phi\sinh\phi\cos\theta\cdot\beta - i\cosh\phi\sinh\phi\sin\theta\sin\overline{\alpha\gamma}\beta\cdot\alpha.\end{aligned}$$

The effect of a hyperbolic rotation is obtained by taking the more general value of ξ and applying the hyperbolic rules of multiplication.

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