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THE

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Physics Mech.

THE PRINCIPLES OF MECHANICS

PRESENTED IN A NEW FORM

BY

HEINRICH HERTZ

LATE PROFESSOR OF PHYSICS IN THE UNIVERSITY OF BONN

WITH AN INTRODUCTION BY

H. VON HELMHOLTZ

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EDITOR'S PREFACE

THE volume now published is Heinrich Hertz's last work. To it he devoted the last three years of his life. The general features were settled and the greater part of the book written within about a year; the remaining two years were spent in working up the details. At the end of this time the author regarded the first part of the book as quite finished, and the second half as practically finished. He had arranged to work once more through the second half. But soon his plans became only hopes, and his hopes were doomed to disappointment. Death was soon to claim him in the prime of his power. Shortly before he died he forwarded to the publishers the greater part of the manuscript. At the same time he sent for me and asked me to edit the book, in case he should not be able to see it through the press.

From first to last I have done this with the greatest care, seeking especially to give a faithful rendering of the sense of the original. I have also endeavoured, as far as possible, to retain the form; but to do this in all cases, without due reference to the contents and connection, would have been contrary to the author's wish. Hence I have slightly changed the form in places where, after careful study of the book, I felt convinced that the author would himself have made such changes. I have not thought it necessary to specify where these changes occur, inasmuch as none of them affects the sense. In order to guard against this I have carefully studied all the rough notes and earlier manuscripts of the work. Several of the first drafts had been carefully written out, and some of them are fuller than the manuscript as finally prepared for the press. With regard to two paragraphs of the work I have found it impossible to satisfy myself, the author's intention as to the final form having remained doubtful to me. I have marked these two paragraphs and have thought it best to leave them entirely unaltered.

After sending off the manuscript the author had noted certain corrections in a second copy; all these have been included before printing off. I have completed the references to earlier paragraphs of the book (of which few were given in the second part, and scarcely any in the last chapter), and have drawn up an index to the definitions and notation.

P. LENARD.

TRANSLATORS' NOTE

HERTZ'S *Principles of Mechanics* forms the third (and last) volume of his collected works, as edited by Dr. Philipp Lenard. English translations of the first and second volumes (*Miscellaneous Papers* and *Electric Waves*) have already been published.

The translation of the first two volumes was comparatively easy; the third has proved to be a more difficult undertaking. If it has been brought to a satisfactory conclusion this will be largely due to Professor Lenard, through whose hands the proof-sheets have passed. He has again, notwithstanding the pressure of other work, been good enough to advise and assist us from time to time, and we tender to him our warmest thanks.

We also desire to thank the publishers and printers for the extreme consideration shown by them while the book was being prepared for the press.

D. E. J.
J. T. W.

September 1899.

PREFACE BY H. VON HELMHOLTZ

On the 1st of January 1894 Heinrich Hertz died. All who regard human progress as consisting in the broadest possible development of the intellectual faculties, and in the victory of the intellect over natural passions as well as over the forces of nature, must have heard with the deepest sorrow of the death of this highly favoured genius. Endowed with the rarest gifts of intellect and of character, he reaped during his lifetime (alas, so short!) a bounteous harvest which many of the most gifted investigators of the present century have tried in vain to gather. In old classical times it would have been said that he had fallen a victim to the envy of the gods. Here nature and fate appeared to have favoured in an exceptional manner the development of a human intellect embracing all that was requisite for the solution of the most difficult problems of science,—an intellect capable of the greatest acuteness and clearness in logical thought, as well as of the closest attention in observing apparently insignificant phenomena. The uninitiated readily pass these by without heeding them; but to the practised eye they point the way by which we can penetrate into the secrets of nature.

Heinrich Hertz seemed to be predestined to open up to mankind many of the secrets which nature has hitherto concealed from us; but all these hopes were frustrated by the malignant disease which, creeping slowly but surely on, robbed us of this precious life and of the achievements which it promised.

To me this has been a deep sorrow; for amongst all my pupils I have ever regarded Hertz as the one who had penetrated furthest into my own circle of scientific thought, and it was to him that I looked with the greatest confidence for the further development and extension of my work.

Heinrich Rudolf Hertz was born on 22nd February 1857, in Hamburg, and was the eldest son of Dr. Hertz, who was then a barrister and subsequently became senator. Up to the time of his confirmation he was a pupil in one of the municipal primary schools (*Bürgerschulen*). After a year's preparation at home he entered the High School of his native town, the *Johanneum*; here he remained until 1875, when he received his certificate of matriculation. As a boy he won the appreciation of his parents and teachers by his high moral character. Already his pursuits showed his natural inclinations. While still attending school he worked of his own accord at the bench and lathe, on Sundays he attended the Trade School to practise geometrical drawing, and with the simplest appliances he constructed serviceable optical and mechanical instruments.

At the end of his school course he had to decide on his career, and chose that of an engineer. The modesty which in later years was such a characteristic feature of his nature, seems to have made him doubtful of his talent for theoretical science. He liked mechanical work, and felt surer of success in connection with it, because he already knew well enough what it meant and what it required. Perhaps, too, he was influenced by the tone prevailing in his native town and tending towards a practical life. It is in young men of unusual capacity that one most frequently observes this sort of timid modesty. They have a clear conception of the difficulties which have to be overcome before attaining the high ideal set before their minds; their strength must be tried by some practical test before they can secure the self-reliance requisite for their difficult task. And even in later years men of great ability are the less content with their own achievements the higher their capacity and ideals. The most gifted attain the highest and truest success because they are most keenly alive to the presence of imperfection and most unwearied in removing it.

For fully two years Heinrich Hertz remained in this state of doubt. Then, in the autumn of 1877, he decided upon an academic career; for as he grew in knowledge he grew in the conviction that only in scientific work could he find enduring satisfaction. In the autumn of 1878 he came to Berlin, and it was as an university student there, in the physical laboratory under my control, that I first made his acquaintance. Even while he was going through the elementary course of practical work, I saw that I had here to deal with a pupil of quite unusual talent; and when, towards the end of the summer semester, it fell to me to propound to the students a subject of physical research for a prize, I chose one in electromagnetics, in the belief that Hertz would feel an interest in it, and would attack it, as he did, with success.

In Germany at that time the laws of electromagnetics were deduced by most physicists from the hypothesis of W. Weber, who sought to trace back electric and magnetic phenomena to a modification of Newton's assumption of direct forces acting at a distance and in a straight line. With increasing distance these forces diminish in accordance with the same laws as those assigned by Newton to the force of gravitation, and held by Coulomb to apply to the action between pairs of electrified particles. The force was directly proportional to the product of the two quantities of electricity, and inversely proportional to the square of their distance apart; like quantities produced repulsion, unlike quantities attraction. Furthermore, in Weber's hypothesis it was assumed that this force was propagated through infinite space instantaneously, and with infinite velocity. The only difference between the views of W. Weber and of Coulomb consisted in this—that Weber assumed that the magnitude of the force between the two quantities of electricity might be affected by the velocity with which the two quantities approached towards or receded from one another, and also by the acceleration of such velocity.

Side by side with Weber's theory there existed a number of others, all of which had this in common—that they regarded the magnitude of the force expressed by Coulomb's law as being modified by the influence of some component of the velocity of the electrical quantities in motion. Such theories were advanced by F. E. Neumann, by his son C. Neumann, by Riemann, Grassmann, and subsequently by Clausius. Magnetised molecules were regarded as the axes of circular electric currents, in accordance with an analogy between their external effects previously discovered by Ampère.

This plentiful crop of hypotheses had become very unmanageable, and in dealing with them it was necessary to go through complicated calculations, resolutions of forces into their components in various directions, and so on. So at that time the domain of electromagnetics had become a pathless wilderness. Observed facts and deductions from exceedingly doubtful theories were inextricably mixed up together. With the object of clearing up this confusion I had set myself the task of surveying the region of electromagnetics, and of working out the distinctive consequences of the various theories, in order, wherever that was possible, to decide between them by suitable experiments.

I arrived at the following general result. The phenomena which completely closed currents produce by their circulation through continuous and closed metallic circuits, and which have this common property, that while they flow there is no considerable variation in the electric charges accumulated upon the various parts of the conductor,—all these phenomena can be equally well deduced from any of the above-mentioned hypotheses. The deductions which follow from them agree with Ampère's laws of electromagnetic action, with the laws discovered by Faraday and Lenz, and also with the laws of induced electric currents as generalised by F. E. Neumann. On the other hand, the deductions which follow from them in the case of conducting circuits which are not completely closed are essentially different. The accordance between the various theories and the facts which have been observed in the case of completely closed circuits is easily intelligible when we consider that closed currents of any desired strength can be maintained as long as we please—at any rate long enough to allow the forces exerted by them to exhibit plainly their effects; and that on this account the actual effects of such currents and their laws are well known and have been carefully investigated. Thus any divergence between any newly-advanced theory and any one of the known facts in this well-trodden region would soon attract attention and be used to disprove the theory.

But at the open ends of unclosed conductors between which insulating masses are interposed, every motion of electricity along the length of the conductor immediately causes an accumulation of electric charges; these are due to the surging of the electricity, which cannot force its way through the insulator, against the ends of the conductor. Between the electricity accumulated at the end and the electricity of the same kind which surges against it there is a force of repulsion; and an exceedingly short time suffices for this force to attain such magnitude that it completely checks the flow of the electricity. The surging then ceases; and after an instant of rest there follows a resurging of the accumulated electricity in the opposite direction.

To every one who was initiated into these matters it was then apparent that a complete understanding of the theory of electromagnetic phenomena could only be attained by a thorough investigation of the processes which occur during these very rapid surgings of unclosed currents. W. Weber had endeavoured to remove or lessen certain difficulties in his electromagnetic hypothesis by suggesting that electricity might possess a certain degree of inertia, such as ponderable matter exhibits. In the opening and closing of every electric current effects are produced which simulate the appearance of such electric inertia. These, however, arise from what is called electromagnetic induction, *i.e.* from a mutual action of neighbouring conductors upon each other, according to laws which have been well known since Faraday's time. True inertia should be proportional only to the mass of the electricity in motion, and independent of the position of the conductor. If anything of the kind existed we ought to be able to detect it by a retardation in electric oscillations, such as are produced by the sudden break of an electric current in metallic wires. In this manner it should be possible to find an upper limit to the magnitude of this electric inertia; and so I was led to propound the problem of carrying out experiments on the magnitude of extra-currents. Extra-currents in double-wound spirals, the currents traversing the branches in opposite directions, were suggested in the statement of the problem as being apparently best adapted for these experiments. Heinrich Hertz's first research of importance consisted in solving this problem. In it he gives a definite answer to the question propounded, and shows that of the extra-current in a double-wound spiral $\frac{1}{30}$ to $\frac{1}{20}$ at most could be ascribed to the effect of an inertia of electricity. The prize was awarded to him for this investigation.

But Hertz did not confine himself to the experiments which had been suggested. For he recognised that although the effects of induction are very much weaker in wires which are stretched out straight, they can be much more accurately calculated than in spirals of many turns; for in the latter he could not measure with accuracy the geometrical relations. Hence he used for further experiments a conductor consisting of two rectangles of straight wire; he now found that the extra-current due to inertia could at most not exceed $\frac{1}{250}$ of the magnitude of the induction current.

Investigations on the effect of centrifugal force in a rapidly rotating plate upon the motion of electricity passing through it, led him to find a still lower value to the upper limit of the inertia of electricity.

These experiments clearly impressed upon his mind the exceeding mobility of electricity, and pointed out to him the way towards his most important discoveries.

Meanwhile in England the ideas introduced by Faraday as to the nature of electricity were extending. These ideas, expressed as they were in abstract language difficult of comprehension, made but slow progress until they found in Clerk Maxwell a fitting interpreter. In explaining electrical phenomena Faraday was bent upon excluding all preconceived notions involving assumptions as to the existence of phenomena or substances which could not be directly perceived. Especially did he reject, as did Newton at the beginning of his career, the hypothesis of the existence of action-at-a-distance. What the older theories assumed seemed to him inconceivable—that direct actions could go on between bodies separated in space without any change taking place in the intervening medium. So he first sought for indications of changes in media lying between electrified bodies or between magnetic bodies. He succeeded in detecting magnetism or diamagnetism in nearly all the bodies which up to that time had been regarded as non-magnetic. He also showed that good insulators undergo a change when exposed to the action of electric force; this he denoted as the "dielectric polarisation of insulators."

It could not be denied that the attraction between two electrically charged bodies or between two magnet poles in the direction of their lines of force was considerably increased by introducing between them dielectrically or magnetically polarised media. On the other hand there was a repulsion across the lines of force. After these discoveries men were bound to recognise that a part of the magnetic and electric action was produced by the polarisation of the intervening medium; another part might still remain, and this might be due to action-at-a-distance.

Faraday and Maxwell inclined towards the simpler view that there was no action-at-a-distance; this hypothesis, which involved a complete upsetting of the conceptions hitherto current, was thrown into mathematical form and developed by Maxwell. According to it the seat of the changes which produce electrical phenomena must be sought only in the insulators; the polarisation and depolarisation of these are the real causes of the electrical disturbances which apparently take place in conductors. There were no longer any closed currents; for the accumulation of electric charges at the ends of a conductor, and the simultaneous dielectric polarisation of the medium between them, represented an equivalent electric motion in the intervening dielectric, thus completing the gap in the circuit.

Faraday had a very sure and profound insight into geometrical and mechanical questions; and he had already recognised that the distribution of electric action in space according to these new views must exactly agree with that found according to the older theory.

By the aid of mathematical analysis Maxwell confirmed this, and extended it into a complete theory of electromagnetics. For my own part, I fully recognised the force of the facts discovered by Faraday, and began to investigate the question whether actions-at-a-distance did really exist, and whether they must be taken into account. For I felt that scientific prudence required one to keep an open mind at first in such a complicated matter, and that the doubt might point the way to decisive experiments.

This was the state of the question at the time when Heinrich Hertz attacked it after completing the investigation which we have described.

It was an essential postulate of Maxwell's theory that the polarisation and depolarisation of an insulator should produce in its neighbourhood the same electromagnetic effects as a galvanic current in a conductor. It seemed to me that this should be capable of demonstration, and that it would constitute a problem of sufficient importance for one of the great prizes of the Berlin Academy.

In the Introduction to his interesting book, *Untersuchungen über die Austreitung der elektrischen Kraft*,[1] Hertz has described how his own discoveries grew out of the seeds thus

[1] [*Electric Waves.* London, Macmillan, 1893.]

sown by his contemporaries, and has done this in such an admirably clear manner that it is impossible for any one else to improve upon it or add anything of importance. His Introduction is of exceeding value as a perfectly frank and full account of one of the most important and suggestive discoveries. It is a pity that we do not possess more documents of this kind on the inner psychological history of science. We owe the author a debt of gratitude for allowing us to penetrate into the inmost working of his thoughts, and for recording even his temporary mistakes.

Something may, however, be added as to the consequences which follow from his discoveries.

The views which Hertz subsequently proved to be correct had been propounded, as we have already said, by Faraday and Maxwell before him as being possible, and even highly probable; but as yet they had not been actually verified. Hertz supplied the demonstration. The phenomena which guided him into the path of success were exceedingly insignificant, and could only have attracted the attention of an observer who was unusually acute, and able to see immediately the full importance of an unexpected phenomenon which others had passed by. It would have been a hopeless task to render visible by means of a galvanometer, or by any other experimental method in use at that time, the rapid oscillations of currents having a period as short as one ten-thousandth or even only a millionth of a second. For all finite forces require a certain time to produce finite velocities and to displace bodies of any weight, even when they are as light as the magnetic needles of our galvanometers usually are. But electric sparks can become visible between the ends of a conductor even when the potential at its ends only rises for a millionth of a second high enough to cause sparking across a minute air-gap. Through his earlier investigations Hertz was thoroughly familiar with the regularity and enormous velocity of these rapid electric oscillations; and when he essayed in this way to discover and render visible the most transient electric disturbances, success was not long in coming. He very soon discovered what were the conditions under which he could produce in unclosed conductors oscillations of sufficient regularity. He proceeded to examine their behaviour under the most varied circumstances, and thus determined the laws of their development. He next succeeded in measuring their wave-length in air and their velocity. In the whole investigation one scarcely knows which to admire most, his experimental skill or the acuteness of his reasoning, so happily are the two combined.

By these investigations Hertz has enriched physics with new and most interesting views respecting natural phenomena. There can no longer be any doubt that light-waves consist of electric vibrations in the all-pervading ether, and that the latter possesses the properties of an insulator and a magnetic medium. Electric oscillations in the ether occupy an intermediate position between the exceedingly rapid oscillations of light and the comparatively slow disturbances which are produced by a tuning-fork when thrown into vibration; but as regards their rate of propagation, the transverse nature of their vibrations, the consequent possibility of polarising them, their refraction and reflection, it can be shown that in all these respects they correspond completely to light and to heat-rays. The electric waves only lack the power of affecting the eye, as do also the dark heat-rays, whose frequency of oscillation is not high enough for this.

Here we have two great natural agencies—on the one hand light, which is so full of mystery and affects us in so many ways, and on the other hand electricity, which is equally mysterious, and perhaps even more varied in its manifestations: to have furnished a complete demonstration that these two are most closely connected together is to have achieved a great feat. From the standpoint of theoretical science it is perhaps even more important to be able to understand how apparent actions-at-a-distance really consist in a propagation of an action from one layer of an intervening medium to the next. Gravitation still remains an unsolved puzzle; as yet a satisfactory explanation of it has not been forthcoming, and we are still compelled to treat it as a pure action-at-a-distance.

Amongst scientific men Heinrich Hertz has secured enduring fame by his researches. But not through his work alone will his memory live; none of those who knew him can ever forget his uniform modesty, his warm recognition of the labours of others, or his genuine gratitude towards his teachers. To him it was enough to seek after truth; and this he did with all zeal and devotion, and without the slightest trace of self-seeking. Even when he had some right to claim discoveries as his own he preferred to remain quietly in the background. But although naturally quiet, he could be merry enough amongst his friends, and could enliven social intercourse by many an apt remark. He never made an enemy, although he knew how to judge slovenly work, and to appraise at its true value any pretentious claim to scientific recognition.

His career may be briefly sketched as follows. In the year 1880 he was appointed Demonstrator in the Physical Laboratory of the Berlin University. In 1883 he was induced by the Prussian Education Department (*Kultusministerium*) to go to Kiel with a view to his promotion to the office of Privat-docent there. In Easter of 1885 he was called to Karlsruhe as ordinary Professor of Physics at the Technical School. Here he made his most important discoveries, and it was during his stay at Karlsruhe that he married Miss Elizabeth Doll, the daughter of one of his colleagues. Two years later he received a call to the University of Bonn as ordinary Professor of Physics, and removed thither in Easter 1889.

Few as the remaining years of his life unfortunately were, they brought him ample proof that his work was recognised and honoured by his contemporaries. In the year 1888 he was awarded the Matteucci Medal of the Italian Scientific Society, in 1889 the La Caze Prize of the Paris Academy of Sciences and the Baumgartner Prize of the Imperial Academy of Vienna, in 1890 the Rumford Medal of the Royal Society, and in 1891 the Bressa Prize of the Turin Royal Academy. He was elected a corresponding member of the Academies of Berlin, Munich, Vienna, Göttingen, Rome, Turin, and Bologna, and of many other learned societies; and the Prussian Government awarded him the Order of the Crown.

He was not long spared to enjoy these honours. A painful abscess began to develop, and in November 1892 the disease became threatening. An operation performed at that time appeared to relieve the pain for a while. Hertz was able to carry on his lectures, but only with great effort, up to the 7th of December 1893. On New Year's day of 1894 death released him from his sufferings.

In the present treatise on the Principles of Mechanics, the last memorial of his labours here below, we again see how strong was his inclination to view scientific principles from the most general standpoint. In it he has endeavoured to give a consistent representation of a complete and connected system of mechanics, and to deduce all the separate special laws of this science from a single fundamental law which, logically considered, can, of course, only be regarded as a plausible hypothesis. In doing this he has reverted to the oldest theoretical conceptions, which may also be regarded as the simplest and most natural; and he propounds the question whether these do not suffice to enable us to deduce, by consistent and rigid methods of proof, all the recently discovered general principles of mechanics, even such as have only made their appearance as inductive generalisations.

The first scientific development of mechanics arose out of investigations on the equilibrium and motion of solid bodies which were directly connected with one another; we have examples of these in the simple mechanics, the lever, pulleys, inclined planes, etc. The law of virtual velocities is the earliest general solution of all the problems which thus arise. Later on Galileo developed the conception of inertia and of the accelerating action of force, although he represented this as consisting of a series of impulses. Newton first conceived the idea of action-at-a-distance, and showed how to determine it by the principle of equal action and reaction. It is well known that Newton, as well as his contemporaries, at first only accepted the idea of direct action-at-a-distance with the greatest reluctance.

From that time onwards Newton's idea and definition of force served as a basis for the further development of mechanics. Gradually men learned how to handle problems in which conservative forces were combined with fixed connections; of these the most general solution is given by d'Alembert's Principle. The chief general propositions in mechanics (such as the law of the motion of the centre of gravity, the law of areas for rotating systems, the principle of the conservation of *vis viva*, the principle of least action) have all been developed from the assumption of Newton's attributes of constant, and therefore conservative, forces of attraction between material points, and of the existence of fixed connections between them. They were originally discovered and proved only under these assumptions. Subsequently it was discovered by observation that the propositions thus deduced could claim a much more general validity in nature than that which followed from the mode in which they were demonstrated. Hence it was concluded that certain general characteristics of Newton's conservative forces of attraction were common to all the forces of nature; but no proof was forthcoming that this generalisation could be deduced from any common basis. Hertz has now endeavoured to furnish mechanics with such a fundamental conception from which all the laws of mechanics which have been recognised as of general validity can be deduced in a perfectly logical manner. He has done this with great acuteness, making use in an admirable manner of new and peculiar generalised kinematical ideas. He has chosen as his starting-point that of the oldest mechanical theories, namely, the conception that all mechanical processes go on as if the connections between the various parts which act upon each other were fixed. Of course he is obliged to make the further hypothesis that there are a large number of imperceptible masses with invisible motions, in order to explain the existence of forces between bodies which are not in direct contact with each other. Unfortunately he has not given examples illustrating the manner in which he supposed such hypothetical mechanism to act; to explain even the simplest cases of physical forces on these lines will clearly require much scientific insight and imaginative power. In this direction Hertz seems to have relied chiefly on the introduction of cyclical systems with invisible motions.

English physicists—*e.g.* Lord Kelvin, in his theory of vortex-atoms, and Maxwell, in his hypothesis of systems of cells with rotating contents, on which he bases his attempt at a mechanical explanation of electromagnetic processes—have evidently derived a fuller satisfaction from such explanations than from the simple representation of physical facts and laws in the most general form, as given in systems of differential equations. For my own part, I must admit that I have adhered to the latter mode of representation and have felt safer in so doing; yet I have no essential objections to raise against a method which has been adopted by three physicists of such eminence.

It is true that great difficulties have yet to be overcome before we can succeed in explaining the varied phenomena of physics in accordance with the system developed by Hertz. But in every respect his presentation of the *Principles of Mechanics* is a book which must be of the greatest interest to every reader who can appreciate a logical system of dynamics developed with the greatest ingenuity and in the most perfect mathematical form. In the future this book may prove of great heuristic value as a guide to the discovery of new and general characteristics of natural forces.

AUTHOR'S PREFACE

All physicists agree that the problem of physics consists in tracing the phenomena of nature back to the simple laws of mechanics. But there is not the same agreement as to what these simple laws are. To most physicists they are simply Newton's laws of motion. But in reality these latter laws only obtain their inner significance and their physical meaning through the tacit assumption that the forces of which they speak are of a simple nature and possess simple properties. But we have here no certainty as to what is simple and permissible, and what is not: it is just here that we no longer find any general agreement. Hence there arise actual differences of opinion as to whether this or that assumption is in accordance with the usual system of mechanics, or not. It is in the treatment of new problems that we recognise the existence of such open questions as a real bar to progress. So, for example, it is premature to attempt to base the equations of motion of the ether upon the laws of mechanics until we have obtained a perfect agreement as to what is understood by this name.

The problem which I have endeavoured to solve in the present investigation is the following:—To fill up the existing gaps and to give a complete and definite presentation of the laws of mechanics which shall be consistent with the state of our present knowledge, being neither too restricted nor too extensive in relation to the scope of this knowledge. The presentation must not be too restricted: there must be no natural motion which it does not embrace. On the other hand it must not be too extensive: it must admit of no motion whose occurrence in nature is excluded by the state of our present knowledge. Whether the presentation here given as the solution of this problem is the only possible one, or whether there are other and perhaps better possible ones, remains open. But that the presentation given is in every respect a possible one, I prove by developing its consequences, and showing that when fully unfolded it is capable of embracing the whole content of ordinary mechanics, so far as the latter relates only to the actual forces and connections of nature, and is not regarded as a field for mathematical exercises.

In the process of this development a theoretical discussion has grown into a treatise which contains a complete survey of all the more important general propositions in dynamics, and which may serve as a systematic text-book of this science. For several reasons it is not well suited for use as a first introduction; but for these very reasons it is the better suited to guide those who have already a fair mastery of mechanics as usually taught. It may lead them to a vantage-ground from which they can more clearly perceive the physical meaning of mechanical principles, how they are related to each other, and how far they hold good; from which the ideas of force and the other fundamental ideas of mechanics appear stripped of the last remnant of obscurity.

In his papers on the principle of least action and on cyclical systems,[1] von Helmholtz has already treated in an indirect manner the problem which is investigated in this book, and has given one possible solution of it. In the first set of papers he propounds and maintains the thesis that a system of mechanics which regards as of universal validity, not only Newton's laws, but also the special assumptions involved (in addition to these laws) in Hamilton's Principle,

[1] H. von Helmholtz, "Über die physikalische Bedeutung des Prinzips der kleinsten Wirkung," *Journal für die reine und angewandte Mathematik*, **100**, pp. 137-166, 213-222, 1887; "Prinzipien der Statik monocyklischer Systeme," *ibid.* **97**, pp. 111-140, 317-336, 1884.

would yet be able to embrace all the processes of nature. In the second set of papers the meaning and importance of concealed motions is for the first time treated in a general way. Both in its broad features and in its details my own investigation owes much to the above-mentioned papers: the chapter on cyclical systems is taken almost directly from them. Apart from matters of form, my own solution differs from that of von Helmholtz chiefly in two respects. Firstly, I endeavour from the start to keep the elements of mechanics free from that which von Helmholtz only removes by subsequent restriction from the mechanics previously developed. Secondly, in a certain sense I eliminate less from mechanics, inasmuch as I do not rely upon Hamilton's Principle or any other integral principle. The reasons for this and the consequences which arise from it are made clear in the book itself.

In his important paper on the physical applications of dynamics, J. J. Thomson[1] pursues a train of thought similar to that contained in von Helmholtz's papers. Here again the author develops the consequences of a system of dynamics based upon Newton's laws of motion and also upon other special assumptions which are not explicitly stated. I might have derived assistance from this paper as well; but as a matter of fact my own investigation had made considerable progress by the time I became familiar with it. I may say the same of the mathematical papers of Beltrami[2] and Lipschitz,[3] although these are of much older date. Still I found these very suggestive, as also the more recent presentation of their investigations which Darboux[4] has given with

[1] J. J. Thomson, "On some Applications of Dynamical Principles to Physical Phenomena," *Philosophical Transactions*, **176**, II., pp. 307-342, 1885.

[2] Beltrami, "Sulla teoria generale dei parametri differenziali," *Memorie della Reale Accademia di Bologna*, 25 Febbrajo 1869.

[3] R. Lipschitz, "Untersuchungen eines Problems der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist," *Journal für die reine und angewandte Mathematik*, **74**, pp. 116-149, 1872. "Bemerkungen zu dem Princip des kleinsten Zwanges," *ibid.* **82**, pp. 316-342, 1877.

[4] G. Darboux, *Leçons sur la théorie générale des surfaces*, livre v., chapitres vi. vii. viii., Paris, 1889.

additions of his own. I may have missed many mathematical papers which I could and should have consulted. In a general way I owe very much to Mach's splendid book on the *Development of Mechanics*.[1] I have naturally consulted the better-known text-books of general mechanics, and especially Thomson and Tait's comprehensive treatise.[2] The notes of a course of lectures on analytical dynamics by Borchardt, which I took down in the winter of 1878-79, have proved useful. These are the sources upon which I have drawn; in the text I shall only give such references as are requisite. As to the details I have nothing to bring forward which is new or which could not have been gleaned from many books. What I hope is new, and to this alone I attach value, is the arrangement and collocation of the whole—the logical or philosophical aspect of the matter. According as it marks an advance in this direction or not, my work will attain or fail of its object.

[1] E. Mach, *Die Mechanik in ihrer Entwickelung historisch-kritisch dargestellt*, Leipzig, 1883 (of this there is an English translation by T. J. M'Cormack, *The Science of Mechanics*, Chicago, 1893).

[2] Thomson and Tait, *Natural Philosophy*.

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INTRODUCTION

THE most direct, and in a sense the most important, problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation. As a basis for the solution of this problem we always make use of our knowledge of events which have already occurred, obtained by chance observation or by prearranged experiment. In endeavouring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought. Experience teaches us that the requirement can be satisfied, and hence that such a conformity does in fact exist. When from our accumulated previous experience we have once succeeded in deducing images of the desired nature, we can then in a short time develop by means of them, as by means of models, the consequences which in the external world only arise in a comparatively long time, or as the result of our own interposition. We are thus enabled to be in advance of the facts, and to decide as to present affairs in accordance with the insight so obtained. The images which we here speak of are our conceptions of things. With the things themselves they are in conformity in *one* important respect, namely, in satisfying the above-mentioned requirement. For our purpose it is not necessary that they should be in conformity with the things in any other respect whatever. As a matter of fact, we do not know, nor have we any means of knowing, whether our conceptions of things are in conformity with them in any other than this *one* fundamental respect.

The images which we may form of things are not determined without ambiguity by the requirement that the consequents of the images must be the images of the consequents. Various images of the same objects are possible, and these images may differ in various respects. We should at once denote as inadmissible all images which implicitly contradict the laws of our thought. Hence we postulate in the first place that all our images shall be logically permissible—or, briefly, that they shall be permissible. We shall denote as incorrect any permissible images, if their essential relations contradict the relations of external things, *i.e.* if they do not satisfy our first fundamental requirement. Hence we postulate in the second place that our images shall be correct. But two permissible and correct images of the same external objects may yet differ in respect of appropriateness. Of two images of the same object that is the more appropriate which pictures more of the essential relations of the object,—the one which we may call the more distinct. Of two images of equal distinctness the more appropriate is the one which contains, in addition to the essential characteristics, the smaller number of superfluous or empty relations,—the simpler of the two. Empty relations cannot be altogether avoided: they enter into the images because they are simply images,—images produced by our mind and necessarily affected by the characteristics of its mode of portrayal.

The postulates already mentioned are those which we assign to the images themselves: to a scientific representation of the images we assign different postulates. We require of this that it should lead us to a clear conception of what properties are to be ascribed to the images for the sake of permissibility, what for correctness, and what for appropriatenesss. Only thus can we attain the possibility of modifying and improving our images. What is ascribed to the images for the sake of appropriateness is contained in the notations, definitions, abbreviations, and, in short, all that we can arbitrarily add or take away. What enters into the images for the sake of correctness is contained in the results of experience, from which the images are built up. What enters into the images, in order that they may be permissible, is given by the nature of our mind. To the question whether an image is permissible or not, we can without ambiguity answer yes or no; and our decision will hold good for all time. And equally without ambiguity we can decide whether an image is correct or not; but only according to the state of our present experience, and permitting an appeal to later and riper experience. But we cannot decide without ambiguity whether an image is appropriate or not; as to this differences of opinion may arise. One image may be more suitable for one purpose, another for another; only by gradually testing many images can we finally succeed in obtaining the most appropriate.

Those are, in my opinion, the standpoints from which we must estimate the value of physical theories and the value of the representations of physical theories. They are the standpoints from which we shall here consider the representations which have been given of the Principles of Mechanics. We must first explain clearly what we denote by this name.

Strictly speaking, what was originally termed in mechanics a principle was such a statement as could not be traced back to other propositions in mechanics, but was regarded as a direct result obtained from other sources of knowledge. In the course of historical development it inevitably came to pass that propositions, which at one time and under special circumstances were rightly denoted as principles, wrongly retained these names. Since Lagrange's time it has frequently been remarked that the principles of the centre of gravity and of areas are in reality only propositions of a general nature. But we can with equal justice say that other so-called principles cannot bear this name, but must descend to the rank of propositions or corollaries, when the representation of mechanics becomes based upon one or more of the others. Thus the idea of a mechanical principle has not been kept sharply defined. We shall therefore retain for such propositions, when mentioning them separately, their customary names. But these separate concrete propositions are not what we shall have in mind when we speak simply and generally of the principles of mechanics: by this will be meant any selection from amongst such and similar propositions, which satisfies the requirement that the whole of mechanics can be developed from it by purely deductive reasoning without any further appeal to experience. In this sense the fundamental ideas of mechanics, together with the principles connecting them, represent the simplest image which physics can produce of things in the sensible world and the processes which occur in it. By varying the choice of the propositions which we take as fundamental, we can give various representations of the principles of mechanics. Hence we can thus obtain various images of things; and these images we can test and compare with each other in respect of permissibility, correctness, and appropriateness.

I

The customary representation of mechanics gives us a first image. By this we mean the representation, varying in detail but identical in essence, contained in almost all text-books which deal with the whole of mechanics, and in almost all courses of lectures which cover the whole of this science. This is the path by which the great army of students travel and are inducted into the mysteries of mechanics. It closely follows the course of historical development and the sequence of discoveries. Its principal stages are distinguished by the names of Archimedes, Galileo, Newton, Lagrange. The conceptions upon which this representation is based are the ideas of space, time, force, and mass. In it force is introduced as the cause of motion, existing before motion and independently of it. Space and force first appear by themselves, and their relations are treated of in statics. Kinematics, or the science of pure motion, confines itself to connecting the two ideas of space and time. Galileo's conception of inertia furnishes a connection between space, time, and mass alone. Not until Newton's Laws of Motion do the four fundamental ideas become connected with each other. These laws contain the seed of future developments; but they do not furnish any general expression for the influence of rigid spacial connections. Here d'Alembert's principle extends the general results of statics to the case of motion, and closes the series of independent fundamental statements which cannot be deduced from each other. From here on everything is deductive inference. In fact the above-mentioned ideas and laws are not only necessary but sufficient for the development of the whole of mechanics from them as a necessary consequence of thought; and all other so-called principles can be regarded as propositions and corollaries deduced by special assumptions. Hence the above ideas and laws give us, in the sense in which we have used the words, a first system of principles of mechanics, and at the same time the first general image of the natural motions of material bodies.

Now, at first sight, any doubt as to the logical permissibility of this image may seem very far-fetched. It seems almost inconceivable that we should find logical imperfections in a system which has been thoroughly and repeatedly considered by many of the ablest intellects. But before we abandon the investigation on this account, we should do well to inquire whether the system has always given satisfaction to these able intellects. It is really wonderful how easy it is to attach to the fundamental laws considerations which are quite in accordance with the usual modes of expression in mechanics, and which yet are an undoubted hindrance to clear thinking. Let us endeavour to give an example of this. We swing in a circle a stone tied to a string, and in so doing we are conscious of exerting a force upon the stone. This force constantly deflects the stone from its straight path. If we vary the force, the mass of the stone, and the length of the string, we find that the actual motion of the stone is always in accordance with Newton's second law. But now the third law requires an opposing force to the force exerted by the hand upon the stone. With regard to this opposing force the usual explanation is that the stone reacts upon the hand in consequence of centrifugal force, and that this centrifugal force is in fact exactly equal and opposite to that which we exert. Now is this mode of expression permissible? Is what we call centrifugal force anything else than the inertia of the stone? Can we, without destroying the clearness of our conceptions, take the effect of inertia twice into account,—firstly as mass, secondly as force? In our laws of motion, force was a cause of motion, and was present *before* the motion. Can we, without confusing our ideas, suddenly begin to speak of forces which arise through motion, which are a consequence of motion? Can we behave as if we had already asserted anything about forces of this new kind in our laws, as if by calling them forces we could invest them with the properties of forces? These questions must clearly be answered in the negative. The only possible explanation is that, properly speaking, centrifugal force is not a force at all. Its name, like the name *vis viva*, is accepted as a historic tradition; it is convenient to retain it, although we should rather apologise for its retention than endeavour to justify it. But, what now becomes of the demands of the third law, which requires a force exerted by the inert stone upon the hand, and which can only be satisfied by an actual force, not a mere name?

I do not regard these as artificial difficulties wantonly raised: they are objections which press for an answer. Is not their origin to be traced back to the fundamental laws? The force spoken of in the definition and in the first two laws acts upon a body in one definite direction. The sense of the third law is that forces always connect two bodies, and are directed from the first to the second as well as from the second to the first. It seems to me that the conception of force assumed and created in us by the third law on the one hand, and the first two laws on the other hand, are slightly different. This slight difference may be enough to produce the logical obscurity of which the consequences are manifest in the above example. It is not necessary to discuss further examples. We can appeal to general observations as evidence in support of the above-mentioned doubt.

As such, in the first place, I would mention the experience that it is exceedingly difficult to expound to thoughtful hearers the very introduction to mechanics without being occasionally embarrassed, without feeling tempted now and again to apologise, without wishing to get as quickly as possible over the rudiments, and on to examples which speak for themselves. I fancy that Newton himself must have felt this embarrassment when he gave the rather forced definition of mass as being the product of volume and density. I fancy that Thomson and Tait must also have felt it when they remarked that this is really more a definition of density than of mass, and nevertheless contented themselves with it as the only definition of mass. Lagrange, too, must have felt this embarrassment and the wish to get on at all costs; for he briefly introduces his Mechanics with the explanation that a force is a cause which imparts "or tends to impart" motion to a body; and he must certainly have felt the logical difficulty of such a definition. I find further evidence in the demonstrations of the elementary propositions of statics, such as the law of the parallelogram of forces, of virtual velocities, etc. Of such propositions we have numerous proofs given by eminent mathematicians. These claim to be rigid proofs; but, according to the opinion of other distinguished mathematicians, they in no way satisfy this claim. In a logically complete science, such as pure mathematics, such a difference of opinion is utterly inconceivable.

Weighty evidence seems to be furnished by the statements which one hears with wearisome frequency, that the nature of force is still a mystery, that one of the chief problems of physics is the investigation of the nature of force, and so on. In the same way electricians are continually attacked as to the nature of electricity. Now, why is it that people never in this way ask what is the nature of gold, or what is the nature of velocity? Is the nature of gold better known to us than that of electricity, or the nature of velocity better than that of force? Can we by our conceptions, by our words, completely represent the nature of any thing? Certainly not. I fancy the difference must lie in this. With the terms "velocity" and "gold" we connect a large number of relations to other terms; and between all these relations we find no contradictions which offend us. We are therefore satisfied and ask no further questions. But we have accumulated around the terms "force" and "electricity" more relations than can be completely reconciled amongst themselves. We have an obscure feeling of this and want to have things cleared up. Our confused wish finds expression in the confused question as to the nature of force and electricity. But the answer which we want is not really an answer to this question. It is not by finding out more and fresh relations and connections that it can be answered; but by removing the contradictions existing between those already known, and thus perhaps by reducing their number. When these painful contradictions are removed, the question as to the nature of force will not have been answered; but our minds, no longer vexed, will cease to ask illegitimate questions.

I have thrown such strong doubts upon the permissibility of this image that it might appear to be my intention to contest, and finally to deny, its permissibility. But my intention and conviction do not go so far as this. Even if the logical uncertainties, which have made us solicitous as to our fundamental ideas, do actually exist, they certainly have not prevented a single one of the numerous triumphs which mechanics has won in its applications. Hence, they cannot consist of contradictions between the essential characteristics of our image, nor, therefore, of contradictions between those relations of mechanics which correspond to the relations of things. They must rather lie in the unessential characteristics which we have ourselves arbitrarily worked into the essential content given by nature. If so, these dilemmas can be avoided. Perhaps our objections do not relate to the content of the image devised, but only to the form in which the content is represented. It is not going too far to say that this representation has never attained scientific completeness; it still fails to distinguish thoroughly and sharply between the elements in the image which arise from the necessities of thought, from experience, and from arbitrary choice. This is also the opinion of distinguished physicists who have thought over and discussed [1] these questions, although it cannot be said that all of them are in agreement.[2] This opinion also finds confirmation in the increasing care with which the logical analysis of the elements is carried out in the more recent text-books of mechanics.[3] We are con-

[1] See E. Mach, *The Science of Mechanics*, p. 244. See also in *Nature* (**48**, pp. 62, 101, 117, 126 and 166, 1893; and *Proc. Phys. Soc.* **12**, p. 289, 1893) a discussion on the foundations of dynamics introduced by Prof. Oliver Lodge and carried on in the Physical Society of London.

[2] See Thomson and Tait, *Natural Philosophy*, § 205 *et seq.*

[3] See E. Budde, *Allgemeine Mechanik der Punkte und starren Systeme*, p. 111-138

vinced, as are the authors of these text-books and the physicists referred to, that the existing defects are only defects in form; and that all indistinctness and uncertainty can be avoided by suitable arrangement of definitions and notations, and by due care in the mode of expression. In this sense we admit, as every-one does, the permissibility of the content of mechanics. But the dignity and importance of the subject demand, not simply that we should readily take for granted its logical clearness, but that we should endeavour to show it by a representation so perfect that there should no longer be any possibility of doubting it.

Upon the correctness of the image under consideration we can pronounce judgment more easily and with greater certainty of general assent. No one will deny that within the whole range of our experience up to the present the correctness is perfect; that all those characteristics of our image, which claim to represent observable relations of things, do really and correctly correspond to them. Our assurance, of course, is restricted to the range of previous experience: as far as future experience is concerned, there will yet be occasion to return to the question of correctness. To many this will seem to be excessive and absurd caution: to many physicists it appears simply inconceivable that any further experience whatever should find anything to alter in the firm foundations of mechanics. Nevertheless, that which is derived from experience can again be annulled by experience. This over-favourable opinion of the fundamental laws must obviously arise from the fact that the elements of experience are to a certain extent hidden in them and blended with the unalterable elements which are necessary consequences of our thought. Thus the logical indefiniteness of the representation, which we have just censured, has one advantage. It gives the foundations an appearance of immutability; and perhaps it was wise to introduce it in the beginnings of the science and to allow it to remain for a while. The correctness of the image in all cases was carefully provided for by making the reservation that, if need be, facts derived from experience should determine definitions or *vice versa*. In a perfect science such groping, such an appearance of certainty, is inadmissible.

(Berlin: 1890). The representation there given shows at the same time how great are the difficulties encountered in avoiding discrepancies in the use of the elements.

Mature knowledge regards logical clearness as of prime importance: only logically clear images does it test as to correctness; only correct images does it compare as to appropriateness. By pressure of circumstances the process is often reversed. Images are found to be suitable for a certain purpose; are next tested as to their correctness; and only in the last place purged of implied contradictions.

If there is any truth in what we have just stated, it seems only natural that the system of mechanics under consideration should prove most appropriate in its applications to those simple phenomena for which it was first devised, *i.e.* especially to the action of gravity and the problems of practical mechanics. But we should not be content with this. We should remember that we are not here representing the needs of daily life or the standpoint of past times; we are considering the whole range of present physical knowledge, and are, moreover, speaking of appropriateness in the special sense defined in the beginning of this introduction. Hence we are at once bound to ask,—Is this image perfectly distinct? Does it contain all the characteristics which our present knowledge enables us to distinguish in natural motions? Our answer is a decided—No. All the motions of which the fundamental laws admit, and which are treated of in mechanics as mathematical exercises, do not occur in nature. Of natural motions, forces, and fixed connections, we can predicate more than the accepted fundamental laws do. Since the middle of this century we have been firmly convinced that no forces actually exist in nature which would involve a violation of the principle of the conservation of energy. The conviction is much older that only such forces exist as can be represented as a sum of mutual actions between infinitely small elements of matter. Again, these elementary forces are not free. We can assert as a property which they are generally admitted to possess, that they are independent of absolute time and place. Other properties are disputed. Whether the elementary forces can only consist of attractions and repulsions along the line connecting the acting masses; whether their magnitude is determined only by the distance or whether it is also affected by the absolute or relative velocity;

whether the latter alone comes into consideration, or the acceleration or still higher differential coefficients as well—all these properties have been sometimes presumed, at other times questioned. Although there is such difference of opinion as to the precise properties which are to be attributed to the elementary forces, there is a general agreement that more of such general properties can be assigned, and can from existing observations be deduced, than are contained in the fundamental laws. We are convinced that the elementary forces must, so to speak, be of a simple nature. And what here holds for the forces, can be equally asserted of the fixed connections of bodies which are represented mathematically by equations of condition between the coordinates and whose effect is determined by d'Alembert's principle. It is mathematically possible to write down any finite or differential equation between coordinates and to require that it shall be satisfied; but it is not always possible to specify a natural, physical connection corresponding to such an equation: we often feel, indeed sometimes are convinced, that such a connection is by the nature of things excluded. And yet, how are we to restrict the permissible equations of condition? Where is the limiting line between them and the conceivable ones? To consider only finite equations of condition, as has often been done, is to go too far; for differential equations which are not integrable can actually occur as equations of condition in natural problems.

In short, then, so far as the forces, as well as the fixed relations, are concerned, our system of principles embraces all the natural motions; but it also includes very many motions which are not natural. A system which excludes the latter, or even a part of them, would picture more of the actual relations of things to each other, and would therefore in this sense be more appropriate. We are next bound to inquire as to the appropriateness of our image in a second direction. Is our image simple? Is it sparing in unessential characteristics—ones added by ourselves, permissibly and yet arbitrarily, to the essential and natural ones? In answering this question our thoughts again turn to the idea of force. It cannot be denied that in very many cases the forces which are used in mechanics for treating physical problems are simply sleeping partners, which keep out of the business altogether when actual facts have to be represented. In the simple relations with which mechanics originally dealt, this is not the case. The weight of a stone and the force exerted by the arm seem to be as real and as readily and directly perceptible as the motions which they produce. But it is otherwise when we turn to the motions of the stars. Here the forces have never been the objects of direct perception; all our previous experiences relate only to the apparent position of the stars. Nor do we expect in future to perceive the forces. The future experiences which we anticipate again relate only to the position of these luminous points in the heavens. It is only in the deduction of future experiences from the past that the forces of gravitation enter as transitory aids in the calculation, and then disappear from consideration. Precisely the same is true of the discussion of molecular forces, of chemical actions, and of many electric and magnetic actions. And if after more mature experience we return to the simple forces, whose existence we never doubted, we learn that these forces which we had perceived with convincing certainty, were after all not real. More mature mechanics tells us that what we believed to be simply the tendency of a body towards the earth, is not really such: it is the result, imagined only as a single force, of an inconceivable number of actual forces which attract the atoms of the body towards all the atoms of the universe. Here again the actual forces have never been the objects of previous experience; nor do we expect to come across them in future experiences. Only during the process of deducing future experiences from the past do they glide quietly in and out. But even if the forces have only been introduced by ourselves into nature, we should not on that account regard their introduction as inappropriate. We have felt sure from the beginning that unessential relations could not be altogether avoided in our images. All that we can ask is that these relations should, as far as possible, be restricted, and that a wise discretion should be observed in their use. But has physics always been sparing in the use of such relations? Has it not rather been compelled to fill the world to overflowing with forces of the most various kinds—with forces which never appeared in the phenomena, even with forces which only came into action in exceptional cases? We see a piece of iron resting upon a table, and we accordingly imagine that no causes of motion —no forces—are there present. Physics, which is based upon the mechanics considered here and necessarily determined by this basis, teaches us otherwise. Through the force of gravitation every atom of the iron is attracted by every other atom in the universe. But every atom of the iron is magnetic, and is thus connected by fresh forces with every other magnetic atom in the universe. Again, bodies in the universe contain electricity in motion, and this latter exerts further complicated forces which attract every atom of the iron. In so far as the parts of the iron themselves contain electricity, we have fresh forces to take into consideration; and in addition to these again various kinds of molecular forces. Some of these forces are not small: if only a part of these forces were effective, this part would suffice to tear the iron to pieces. But, in fact, all the forces are so adjusted amongst each other that the effect of the whole lot is zero; that in spite of a thousand existing causes of motion, no motion takes place; that the iron remains at rest. Now if we place these conceptions before unprejudiced persons, who will believe us? Whom shall we convince that we are speaking of actual things, not images of a riotous imagination? And it is for us to reflect whether we have really depicted the state of rest of the iron and its particles in a simple manner. Whether complications can be entirely avoided is questionable; but there can be no question that a system of mechanics which does avoid or exclude them is simpler, and in this sense more appropriate, than the one here considered; for this latter not only permits such conceptions, but directly obtrudes them upon us.

Let us now collect together as briefly as possible the doubts which have occurred to us in considering the customary mode of representing the principles of mechanics. As far as the form is concerned, we consider that the logical value of the separate statements is not defined with sufficient clearness. As far as the facts are concerned, it appears to us that the motions considered in mechanics do not exactly coincide with the natural motions under consideration. Many properties of the natural motions are not attended to in mechanics; many relations which are considered in mechanics are probably absent in nature. Even if these objections are acknowledged to be well founded, they should not lead us to imagine that the customary representation of mechanics is on that account either bound to or likely to lose its value and its privileged position; but they sufficiently justify us in looking out for other representations less liable to censure in these respects, and more closely conformable to the things which have to be represented.

II

There is a second image of mechanical processes which is of much more recent origin than the first. Its development from, and side by side with, the latter is closely connected with advances which physical science has made during the past few decades. Up to the middle of this century its ultimate aim was apparently to explain natural phenomena by tracing them back to innumerable actions-at-a-distance between the atoms of matter. This mode of conception corresponded completely to what we have spoken of as the first system of mechanical principles: each of the two was conditioned by the other. Now, towards the end of the century, physics has shown a preference for a different mode of thought. Influenced by the overpowering impression made by the discovery of the principle of the conservation of energy, it likes to treat the phenomena which occur in its domain as transformations of energy into new forms, and to regard as its ultimate aim the tracing back of the phenomena to the laws of the transformation of energy. This mode of treatment can also be applied from the beginning to the elementary phenomena of motion. There thus arises a new and different representation of mechanics, in which from the start the idea of force retires in favour of the idea of energy. It is this new image of the elementary processes of motion which we shall denote as the second; and to it we shall now devote our attention. In discussing the first image we had the advantage of being able to assume that it stood out plainly before the eyes of all physicists. With the second image this is not the case. It has never yet been portrayed in all its details. So far as I know, there is no text-book of mechanics which from the start teaches the subject from the standpoint of energy, and introduces the idea of energy before the idea of force. Perhaps there has never yet been a lecture on mechanics prepared according to this plan. But to the founders of the theory of energy it was evident that such a plan was possible; the remark has often been made that in this way the idea of force with its attendant difficulties could be avoided; and in special scientific applications chains of reasoning frequently occur which belong entirely to this mode of thought. Hence we can very well sketch the rough outlines of the image; we can give the general plan according to which such a representation of mechanics must be arranged. We here start, as in the case of the first image, from four independent fundamental ideas; and the relations of these to each other will form the contents of mechanics. Two of them—space and time—have a mathematical character; the other two—mass and energy—are introduced as physical entities which are present in given quantity, and cannot be destroyed or increased. In addition to explaining these matters, it will, of course, also be necessary to indicate clearly by what concrete experiences we ultimately establish the presence of mass and energy. We here assume this to be possible and to be done. It is obvious that the amount of energy connected with given masses depends upon the state of these masses. But it is as a general experience that we must first lay down that the energy present can always be split up into two parts, of which the one is determined solely by the relative positions of the masses, while the other depends upon their absolute velocities. The first part is defined as potential energy, the second as kinetic energy. The form of the dependence of kinetic energy upon the velocity of the moving bodies is in all cases the same, and is known. The form of the dependence of potential energy upon the position of the bodies cannot be generally stated; it rather constitutes the special nature and characteristic peculiarity of the masses under consideration. It is the problem of physics to ascertain from previous experience this form for the bodies which surround us in nature. Up to this point there come essentially into consideration only three elements—space, mass, energy, considered in relation to each other. In order to settle the relations of all the four fundamental ideas, and thereby the course in time of the phenomena, we make use of one of the integral principles of ordinary mechanics which involve in their statement the idea of energy. It is not of much importance which of these we select; we can and shall choose Hamilton's principle. We thus lay down as the sole fundamental law of mechanics, in accordance with experience, the proposition that every system of natural bodies moves just as if it were assigned the problem of attaining given positions in given times, and in such a manner that the average over the whole time of the difference between kinetic and potential energy shall be as small as possible. Although this law may not be simple in form, it nevertheless represents without ambiguity the transformations of energy, and enables us to predetermine completely the course of actual phenomena for the future. In stating this new law we lay down the last of the indispensable foundations of mechanics. All that we can further add are only mathematical deductions and certain simplifications of notation which, although expedient, are not necessary. Among these latter is the idea of force, which does not enter into the foundations. Its introduction is expedient when we are considering not only masses which are connected with constant quantities of energy, but also masses which give up energy to other masses or receive it from them. Still, it is not by any new experience that it is introduced, but by a definition which can be formed in more than one way. And accordingly the properties of the force so defined are not to be ascertained by experience, but are to be deduced from the definition and the fundamental laws. Even the confirmation of these properties by experience is superfluous, unless we doubt the correctness of the whole system. Hence the idea of force as such cannot in this system involve any logical difficulties: nor can it come in question in estimating the correctness of the system; it can only increase or diminish its appropriateness.

Somewhat after the manner indicated would the principles of mechanics have to be arranged in order to adapt them to the conception of energy. The question now is, whether this second image is preferable to the first. Let us therefore consider its advantages and disadvantages.

It will be best for us here to consider first the question of appropriateness, since it is in this respect that the improvement is most obvious. For, to begin with, our second image of natural motions is decidedly more distinct: it shows more of their peculiarities than the first does. When we wish to deduce Hamilton's principle from the general foundations of mechanics we have to add to the latter certain assumptions as to the acting forces and the character of contingent fixed connections. These assumptions are of the most general nature, but they indicate a corresponding number of important limitations of the motions represented by the principle. And, conversely, we can deduce from the principle a whole series of relations, especially of mutual relations between every kind of possible force, which are wanting in the principles of the first image; in the second image they are present, and likewise occur, which is the important point, in nature. To prove this is the object of the papers published by von Helmholtz under the title, *Ueber die physikalische Bedeutung des Prinzips der kleinsten Wirkung.* It would be more correct to say that the fact which has to be proved forms the discovery which is demonstrated and communicated in that paper. For it is truly a discovery to find that from such general assumptions, conclusions so distinct, so weighty, and so just can be drawn. We may then appeal to that paper for confirmation of our statement; and, inasmuch as it represents the furthest advance of physics at the present time, we may spare ourselves the question whether it be possible to conform yet more closely to nature, say by limiting the permissible forms of potential energy. We shall simply emphasise this, that in respect of simplicity as well, our present image avoids the stumbling-blocks which endangered the appropriateness of the first. For if we ask ourselves the real reasons why physics at the present time prefers to express itself in terms of energy, our answer will be, Because in this way it best avoids talking about things of which it knows very little, and which do not at all affect the essential statements under consideration. We have already had occasion to remark that in tracing back phenomena to force we are compelled to turn our attention continually to atoms and molecules. It is true that we are now convinced that ponderable matter consists of atoms; and we have definite notions of the magnitude of these atoms and of their motions in certain cases. But the form of the atoms, their connection, their motion in most cases—all these are entirely hidden from us; their number is in all cases immeasurably great. So that although our conception of atoms is in itself an important and interesting object for further investigation, it is in no wise specially fit to serve as a known and secure foundation for mathematical theories. To an investigator like Gustav Kirchhoff, who was accustomed to rigid reasoning, it almost gave pain to see atoms and their vibrations wilfully stuck in the middle of a theoretical deduction. The arbitrarily assumed properties of the atoms may not affect the final result. The result may be correct. Nevertheless the details of the deduction are in great part presumably false; the deduction is only in appearance a proof. The earlier mode of thought in physics scarcely allowed any choice or any way of escape. Herein lies the advantage of the conception of energy and of our second image of mechanics: that in the hypotheses of the problems there only enter characteristics which are directly accessible to experience, parameters, or arbitrary coordinates of the bodies under consideration; that the examination proceeds with the aid of these characteristics in a finite and complete form; and that the final result can again be directly translated into tangible experience. Beyond energy itself in its few forms, no auxiliary constructions enter into consideration. Our statements can be limited to the known peculiarities of the system of bodies under consideration, and we need not conceal our ignorance of the details by arbitrary and ineffectual hypotheses. All the steps in the deduction, as well as the final result, can be defended as correct and significant. These are the merits which have endeared this method to present-day physics. They are peculiar to our second image of mechanics: in the sense in which we have used the words they are to be regarded as advantages in respect of simplicity, and hence of appropriateness.

Unfortunately we begin to be uncertain as to the value of our system when we test its correctness and its logical permissibility. The question of correctness at once gives rise to legitimate doubts. Hamilton's principle can be deduced from the accepted foundations of Newtonian mechanics; but this does not by any means guarantee an accordance with nature. We have to remember that this deduction only follows if certain assumptions hold good; and also that our system claims not only to describe certain natural motions correctly, but to embrace all natural motions. We must therefore investigate whether these special assumptions which are made in addition to Newton's laws are universally true; and a single example from nature to the contrary would invalidate the correctness of our system as such, although it would not disturb in the least the validity of Hamilton's principle as a general proposition. The doubt is not so much whether our system includes the whole manifold[1] of forces, as whether it embraces the whole manifold of rigid connections which may arise between the bodies of nature. The application of Hamilton's principle to a material system does not exclude the existence of fixed connections between the chosen coordinates. But at any rate it requires that these connections be mathematically expressible by finite equations between the coordinates: it does not permit the occurrence of connections which can only be represented by differential equations. But nature itself does not appear to entirely exclude connections of this kind. They arise, for example, when bodies of three dimensions roll on one another without slipping. By such a connection, examples of which frequently occur, the position of the two bodies with respect to each other is only limited by the condition that they must always have one point of their surfaces common; but the freedom of motion of the bodies is further diminished by a degree. From the connection, then, there can be deduced more equations between the changes of the coordinates than between the coordinates themselves; hence there must amongst these equations be at least one non-integrable differential equation. Now Hamilton's principle cannot be applied to such a case; or, to speak more correctly, the application, which is mathematically possible, leads to results which are physically false. Let us restrict our consideration to the case of a sphere rolling without slipping upon a horizontal plane under the influence of its inertia alone. It is not difficult to see, without calculation, what motions the sphere can actually execute. We can also see what motions would correspond to Hamilton's principle; these would have to take place

[1] [*Mannigfaltigkeit* is thus rendered throughout.—*Tr.*]

in such a way that with constant *vis viva* the sphere would attain given positions in the shortest possible time. We can thus convince ourselves, without calculation, that the two kinds of motions exhibit very different characteristics. If we choose any initial and final positions of the sphere, it is clear that there is always one definite motion from the one to the other for which the time of motion, *i.e.* the Hamilton's integral, is a minimum. But, as a matter of fact, a natural motion from every position to every other is not possible without the co-operation of forces, even if the choice of the initial velocity is perfectly free. And even if we choose the initial and final positions so that a natural free motion between the two is possible, this will nevertheless not be the one which corresponds to a minimum of time. For certain initial and final positions the difference can be very striking. In this case a sphere moving in accordance with the principle would decidedly have the appearance of a living thing, steering its course consciously towards a given goal, while a sphere following the law of nature would give the impression of an inanimate mass spinning steadily towards it. It would be of no use to replace Hamilton's principle by the principle of least action or by any other integral principle, for there is but a slight difference of meaning between all these principles, and in the respect here considered they are quite equivalent. Only in one way can we defend the system and preserve it from the charge of incorrectness. We must decline to admit that rigid connections of the kind referred to do actually and strictly occur in nature. We must show that all so-called rolling without slipping is really rolling with a little slipping, and is therefore a case of friction. We have to rest our case upon this—that generally friction between surfaces is one of the processes which we have not yet been able to trace back to clearly understood causes; that the forces which come into play have only been ascertained quite empirically; and hence that the whole problem is one of those which we cannot at present handle without making use of force and the roundabout methods of ordinary mechanics. This defence is not quite convincing. For rolling without slipping does not contradict either the principle of energy or any other generally accepted law of physics. The process is one which is so nearly realised in the visible world that even integration machines are constructed on the assumption that it strictly takes place. We have scarcely any right, then, to exclude its occurrence as impossible, at any rate from the mechanics of unknown systems, such as the atoms or the parts of the ether. But even if we admit that the connections in question are only approximately realised in nature, the failure of Hamilton's principle still creates difficulties in these cases. We are bound to require of every fundamental law of our mechanical system, that when applied to approximately correct relations it should always lead to approximately correct results, not to results which are entirely false. For otherwise, since all the rigid connections which we draw from nature and introduce into the calculations correspond only approximately to the actual relations, we should get into a state of hopeless uncertainty as to which admitted of the application of the law and which not. And yet we do not wish to abandon entirely the defence which we have proposed. We should prefer to admit that the doubt is one which affects the appropriateness of the system, not its correctness, so that the disadvantages which arise from it may be outweighed by other advantages.

The real difficulties first meet us when we try to arrange the elements of the system in strict accordance with the requirements of logical permissibility. In introducing the idea of energy we cannot proceed in the usual way, starting with force, and proceeding from this to force-functions, to potential energy, and to energy in general. Such an arrangement would belong to the first representation of mechanics. Without assuming any previous consideration of mechanics, we have to specify by what simple, direct experiences we propose to define the presence of a store of energy, and the determination of its amount. In what precedes we have only assumed, not shown, that such a determination is possible. At the present time many distinguished physicists tend so much to attribute to energy the properties of a substance as to assume that every smallest portion of it is associated at every instant with a given place in space, and that through all the changes of place and all the transformations of the energy into new forms it retains its identity. These physicists must have the conviction that definitions of the required kind can be found; and it is therefore permissible to assume that such definitions can be given. But when we try to throw them into a concrete form, satisfactory to ourselves and likely to command general acceptance, we become perplexed. This mode of conception as a whole does not yet seem to have arrived at a satisfactory and conclusive result. At the very beginning there arises a special difficulty, from the circumstance that energy, which is alleged to resemble a substance, occurs in two such totally dissimilar forms as kinetic and potential energy. Kinetic energy itself does not really require any new fundamental determination, for it can be deduced from the ideas of velocity and mass; on the other hand potential energy, which does require to be settled independently, does not lend itself at all well to any definition which ascribes to it the properties of a substance. The amount of a substance is necessarily a positive quantity; but we never hesitate in assuming the potential energy contained in a system to be negative. When the amount of a substance is represented by an analytical expression, an additive constant in the expression has the same importance as the rest; but in the expression for the potential energy of a system an additive constant never has any meaning. Lastly, the amount of any substance contained in a physical system can only depend upon the state of the system itself; but the amount of potential energy contained in given matter depends upon the presence of distant masses which perhaps have never had any influence upon the system. If the universe, and therefore the number of such distant masses, is infinite, then the amount of many forms of potential energy contained in even finite quantities of matter is infinitely great. All these are difficulties which must be removed or avoided by the desired definition of energy. We do not assert that such a definition is impossible, but as yet we cannot say that it has been framed. The most prudent thing to do will be to regard it for the present as an open question, whether the system can be developed in logically unexceptionable form.

It may be worth while discussing here whether there is any justification for another objection which might be raised as to the permissibility of this second system. In order that an image of certain external things may in our sense be permissible, not only must its characteristics be consistent amongst themselves, but they must not contradict the characteristics of other images already established in our knowledge. On the strength of this it may be said to be inconceivable that Hamilton's principle, or any similar proposition, should really play the part of a fundamental law of mechanics, and be a fundamental law of nature. For the first thing that is to be expected of a fundamental law is simplicity and plainness, whereas Hamilton's principle, when we come to look into it, proves to be an exceedingly complicated statement. Not only does it make the present motion dependent upon consequences which can only exhibit themselves in the future, thereby attributing intentions to inanimate nature; but, what is much worse, it attributes to nature intentions which are void of meaning. For the integral, whose minimum is required by Hamilton's principle, has no simple physical meaning; and for nature it is an unintelligible aim to make a mathematical expression a minimum, or to bring its variation to zero. The usual answer, which physics nowadays keeps ready for such attacks, is that these considerations are based upon metaphysical assumptions; that physics has renounced these, and no longer recognises it as its duty to meet the demands of metaphysics. It no longer attaches weight to the reasons which used to be urged from the metaphysical side in favour of principles which indicate design in nature, and thus it cannot lend ear to objections of a metaphysical character against these same principles. If we had to decide upon such a matter we should not think it unfair to place ourselves rather on the side of the attack than of the defence. A doubt which makes an impression on our mind cannot be removed by calling it metaphysical; every thoughtful mind as such has needs which scientific men are accustomed to denote as metaphysical. Moreover, in the case in question, as indeed in all others, it is possible to show what are the sound and just sources of our needs. It is true we cannot *a priori* demand from nature simplicity, nor can we judge what in her opinion is simple. But with regard to images of our own creation we can lay down requirements. We are justified in deciding that if our images are well adapted to the things, the actual relations of the things must be represented by simple relations between the images.

And if the actual relations between the things can only be represented by complicated relations, which are not even intelligible to an unprepared mind, we decide that these images are not sufficiently well adapted to the things. Hence our requirement of simplicity does not apply to nature, but to the images thereof which we fashion; and our repugnance to a complicated statement as a fundamental law only expresses the conviction that, if the contents of the statement are correct and comprehensive, it can be stated in a simpler form by a more suitable choice of the fundamental conceptions. The same conviction finds expression in the desire we feel to penetrate from the external acquaintance with such a law to the deeper and real meaning which we are convinced it possesses. If this conception is correct, the objection brought forward does really justify a doubt as to the system; but it does not apply so much to its permissibility as to its appropriateness, and comes under consideration in deciding as to the latter. However, we need not return to the consideration of this.

If we once more glance over the merits which we were able to claim for this second image, we come to the conclusion that as a whole it is not quite satisfactory. Although the whole tendency of recent physics moves us to place the idea of energy in the foreground, and to use it as the corner-stone of our structure, it yet remains doubtful whether in so doing we can avoid the harshness and ruggedness which were so disagreeable in the first image. In fact I have discussed this second mode of representation at some length, not in order to urge its adoption, but rather to show why, after due trial, I have felt obliged to abandon it.

III

A third arrangement of the principles of mechanics is that which will be explained at length in this book. Its principal characteristics will be at once stated, so that it may be criticised in the same way as the other two. It differs from them in this important respect, that it only starts with three independent fundamental conceptions, namely, those of time, space, and mass. The problem which it has to solve is to represent the natural relations between these three, and between these three alone. The difficulties have hitherto been met with in connection with a fourth idea, such as the idea of force or of energy; this, as an independent fundamental conception, is here avoided. G. Kirchhoff has already made the remark in his *Text-book of Mechanics* that three independent conceptions are necessary and sufficient for the development of mechanics. Of course the deficiency in the manifold which thus results in the fundamental conceptions necessarily requires some complement. In our representation we endeavour to fill up the gap which occurs by the use of an hypothesis, which is not stated here for the first time; but it is not usual to introduce it in the very elements of mechanics. The nature of the hypothesis may be explained as follows.

If we try to understand the motions of bodies around us, and to refer them to simple and clear rules, paying attention only to what can be directly observed, our attempt will in general fail. We soon become aware that the totality of things visible and tangible do not form an universe conformable to law, in which the same results always follow from the same conditions. We become convinced that the manifold of the actual universe must be greater than the manifold of the universe which is directly revealed to us by our senses. If we wish to obtain an image of the universe which shall be well-rounded, complete, and conformable to law, we have to presuppose, behind the things which we see, other, invisible things—to imagine confederates concealed beyond the limits of our senses. These deep-lying influences we recognised in the first two representations; we imagined them to be entities of a special and peculiar kind, and so, in order to represent them in our image, we created the ideas of force and energy. But another way lies open to us. We may admit that there is a hidden something at work, and yet deny that this something belongs to a special category. We are free to assume that this hidden something is nought else than motion and mass again,—motion and mass which differ from the visible ones not in themselves but in relation to us and to our usual means of perception. Now this mode of conception is just our hypothesis. We assume that it is possible to conjoin with the visible masses of the universe other masses obeying the same laws, and of such a kind that the whole thereby becomes intelligible and conformable to law. We assume this to be possible everywhere and in all cases, and that there are no causes whatever of the phenomena other than those hereby admitted. What we are accustomed to denote as force and as energy now become nothing more than an action of mass and motion, but not necessarily of mass and motion recognisable by our coarse senses. Such explanations of force from processes of motion are usually called dynamical; and we have every reason for saying that physics at the present day regards such explanations with great favour. The forces connected with heat have been traced back with certainty to the concealed motions of tangible masses. Through Maxwell's labours the supposition that electro-magnetic forces are due to the motion of concealed masses has become almost a conviction. Lord Kelvin gives a prominent place to dynamical explanations of force; in his theory of vortex atoms he has endeavoured to present an image of the universe in accordance with this conception. In his investigation of cyclical systems von Helmholtz has treated the most important form of concealed motion fully, and in a manner that admits of general application; through him "concealed mass" and "concealed motion" have become current as technical expressions in German.[1] But if this hypothesis is capable of gradually eliminating the mysterious forces from mechanics, it can also entirely prevent their entering into mechanics. And if its use for the former purpose is in accordance with present tendencies of physics, the same must hold good of its use for the latter purpose. This is the leading thought from which we start. By following it out we arrive at the third image, the general outlines of which will now be sketched.

We first introduce the three independent fundamental ideas of time, space, and mass as objects of experience; and we specify the concrete sensible experiences by which time, mass, and space are to be determined. With regard to the masses we stipulate that, in addition to the masses recognisable by the senses, concealed masses can by hypothesis be

[1] [*Verborgene Masse; verborgene Bewegung.*]

introduced. We next bring together the relations which always obtain between these concrete experiences, and which we have to retain as the essential relations between the fundamental ideas. To begin with, we naturally connect the fundamental ideas in pairs. Relations between space and time alone form the subject of kinematics. There exists no connection between mass and time alone. Experience teaches us that between mass and space there exists a series of important relations. For we find certain purely spacial connections between the masses of nature: from the very beginning onwards through all time, and therefore independently of time, certain positions and certain changes of position are prescribed and associated as possible for these masses, and all others as impossible. Respecting these connections we can also assert generally that they only apply to the relative position of the masses amongst themselves; and further that they satisfy certain conditions of continuity, which find their mathematical expression in the fact that the connections themselves can always be represented by homogeneous linear equations between the first differentials of the magnitudes by which the positions of the masses are denoted. To investigate in detail the connections of definite material systems is not the business of mechanics, but of experimental physics: the distinguishing characteristics which differentiate the various material systems of nature from each other are, according to our conception, simply and solely the connections of their masses. Up to this point we have only considered the connections of the fundamental ideas in pairs: we now address ourselves to mechanics in the stricter sense, in which all three have to be considered together. We find that their general connection, in accordance with experience, can be epitomised in a single fundamental law, which exhibits a close analogy with the usual law of inertia. In accordance with the mode of expression which we shall use, it can be represented by the statement:—Every natural motion of an independent material system consists herein, that the system follows with uniform velocity one of its straightest paths. Of course this statement only becomes intelligible when we have given the necessary explanation of the mathematical mode of expression used; but the sense of the law can also be expressed in the usual language of mechanics. The law condenses into one single statement the usual law of inertia and Gauss's Principle of Least Constraint. It therefore asserts that if the connections of the system could be momentarily destroyed, its masses would become dispersed, moving in straight lines with uniform velocity; but that as this is impossible, they tend as nearly as possible to such a motion. In our image this fundamental law is the first proposition derived from experience in mechanics proper: it is also the last. From it, together with the admitted hypothesis of concealed masses and the normal connections, we can derive all the rest of mechanics by purely deductive reasoning. Around it we group the remaining general principles, according to their relations to it and to each other, as corollaries or as partial statements. We endeavour to show that the contents of mechanics, when arranged in this way, do not become less rich or manifold than its contents when it starts with four fundamental conceptions; at any rate not less rich or manifold than is required for the representation of nature. We soon find it convenient to introduce into our system the idea of force. However, it is not as something independent of us and apart from us that force now makes its appearance, but as a mathematical aid whose properties are entirely in our power. It cannot, therefore, in itself have anything mysterious to us. Thus according to our fundamental law, whenever two bodies belong to the same system, the motion of the one is determined by that of the other. The idea of force now comes in as follows. For assignable reasons we find it convenient to divide the determination of the one motion by the other into two steps. We thus say that the motion of the first body determines a force, and that this force then determines the motion of the second body. In this way force can with equal justice be regarded as being always a cause of motion, and at the same time a consequence of motion. Strictly speaking, it is a middle term conceived only between two motions. According to this conception the general properties of force must clearly follow as a necessary consequence of thought from the fundamental law; and if in possible experiences we see these properties confirmed, we can in no sense feel surprised, unless we are sceptical as to our fundamental law. Precisely the same is true of the idea of energy and of any other aids that may be introduced.

What has hitherto been stated relates to the physical content of the image, and nothing further need be said with regard to this; but it will be convenient to give here a brief explanation of the special mathematical form in which it will be represented. The physical content is quite independent of the mathematical form, and as the content differs from what is customary, it is perhaps not quite judicious to present it in a form which is itself unusual. But the form as well as the content only differ slightly from such as are familiar; and moreover they are so suited that they mutually assist one another. The essential characteristic of the terminology used consists in this, that instead of always starting from single points, it from the beginning conceives and considers whole systems of points. Every one is familiar with the expressions "position of a system of points," and "motion of a system of points." There is nothing unnatural in continuing this mode of expression, and denoting the aggregate of the positions traversed by a system in motion as its path. Every smallest part of this path is then a path-element. Of two path-elements one can be a part of the other: they then differ in magnitude and only in magnitude. But two path-elements which start from the same position may belong to different paths. In this case neither of the two forms part of the other: they differ in other respects than that of magnitude, and thus we say that they have different directions. It is true that these statements do not suffice to determine without ambiguity the characteristics of "magnitude" and "direction" for the motion of a system. But we can complete our definitions geometrically or analytically so that their consequences shall neither contradict themselves nor the statements we have made; and so that the magnitudes thus defined in the geometry of the system shall exactly correspond to the magnitudes which are denoted by the same names in the geometry of the point,—with which, indeed, they always coincide when the system is reduced to a point. Having determined the characteristics of magnitude and direction, we next call the path of a system straight if all its elements have the same direction, and curved if the direction of the elements changes from position to position. As in the geometry of the point, we measure curvature by the rate of variation of the direction with position. From these definitions we at once get a whole series of relations; and the number of these increases as soon as the freedom of motion of the system under consideration is limited by its connections. Certain classes of paths which are distinguished among the possible ones by peculiar simple properties then claim special attention. Of these the most important are those paths which at each of their positions have the least possible curvature: these we shall denote as the straightest paths of the system. These are the paths which are referred to in the fundamental law, and which have already been mentioned in stating it. Another important type consists of those paths which form the shortest connection between any two of their positions: these we shall denote as the shortest paths of the system. Under certain conditions the ideas of straightest and shortest paths coincide. The relation is perfectly familiar in connection with the theory of curved surfaces; nevertheless it does not hold good in general and under all circumstances. The compilation and arrangement of all the relations which arise here belong to the geometry of systems of points. The development of this geometry has a peculiar mathematical attraction; but we only pursue it as far as is required for the immediate purpose of applying it to physics. A system of n points presents a $3n$-manifold of motion,—although this may be reduced to any arbitrary number by the connections of the system. Hence there arise many analogies with the geometry of space of many dimensions; and these in part extend so far that the same propositions and notations can apply to both. But we must note that these analogies are only formal, and that, although they occasionally have an unusual appearance, our considerations refer without exception to concrete images of space as perceived by our senses. Hence all our statements represent possible experiences; if necessary, they could be confirmed by direct experiments, viz. by measurements made with models. Thus we need not fear the objection that in building up a science dependent upon experience, we have gone outside the world of experience. On the other hand, we are bound to answer the question how a new, unusual, and comprehensive mode of expression justifies itself; and what advantages we expect from using it. In answering this question we specify as the first advantage that it enables us to render the most general and comprehensive statements with great simplicity and brevity. In fact, propositions relating to whole systems do not require more words or more ideas than are usually employed in referring to a single point. Here the mechanics of a material system no longer appears as the expansion and complication of the mechanics of a single point; the latter, indeed, does not need independent investigation, or it only appears occasionally as a simplification and a special case. If it is urged that this simplicity is only artificial, we reply that in no other way can simple relations be secured than by artificial and well-considered adaptation of our ideas to the relations which have to be represented. But in this objection there may be involved the imputation that the mode of expression is not only artificial, but far-fetched and unnatural. To this we reply that there may be some justification for regarding the consideration of whole systems as being more natural and obvious than the consideration of single points. For, in reality, the material particle is simply an abstraction, whereas the material system is presented directly to us. All actual experience is obtained directly from systems; and it is only by processes of reasoning that we deduce conclusions as to possible experiences with single points. As a second merit, although not a very important one, we specify the advantage of the form in which our mathematical mode of expression enables us to state the fundamental law. Without this we should have to split it up into Newton's first law and Gauss's principle of least constraint. Both of these together would represent accurately the same facts; but in addition to these facts they would by implication contain something more, and this something more would be too much. In the first place they suggest the conception, which is foreign to our system of mechanics, that the connections of the material system might be destroyed; whereas we have denoted them as being permanent and indestructible throughout. In the second place we cannot, in using Gauss's principle, avoid suggesting the idea that we are not only stating a fact, but also the cause of this fact. We cannot assert that nature always keeps a certain quantity, which we call constraint, as small as possible, without suggesting that this quantity signifies something which is for nature itself a constraint,—an uncomfortable feeling. We cannot assert that nature acts like a judicious calculator reducing his observations, without suggesting that deliberate intention underlies the action. There is undoubtedly a special charm in such suggestions; and Gauss felt a natural delight in giving prominence to it in his beautiful discovery, which is of fundamental importance in our mechanics. Still, it must be confessed that the charm is that of mystery; we do not really believe that we can solve the enigma of the world by such half-suppressed allusions. Our own fundamental law entirely avoids any such suggestions. It exactly follows the form of the customary law of inertia, and like this it simply states a bare fact without any pretence of establishing it. And as it thereby becomes plain and unvarnished, in the same degree does it become more honest and truthful. Perhaps I am prejudiced in favour of the slight modification which I have made in Gauss's principle, and see in it advantages which will not be manifest to others. But I feel sure of general assent when I state as the third advantage of our method, that it throws a bright light upon Hamilton's method of treating mechanical problems by the aid of characteristic functions. During the sixty years since its discovery this mode of treatment has been well appreciated and much praised; but it has been regarded and treated more as a new branch of mechanics, and as if its growth and development had to proceed in its own way and independently of the usual mechanics. In our form of the mathematical representation, Hamilton's method, instead of having the character of a side branch, appears as the direct, natural, and, if one may so say, self-evident continuation of the elementary statements in all cases to which it is applicable. Further, our mode of representation gives prominence to this: that Hamilton's mode of treatment is not based, as is usually assumed, on the special physical foundations of mechanics; but that it is fundamentally a purely geometrical method, which can be established and developed quite independently of mechanics, and which has no closer connection with mechanics than any other of the geometrical methods employed in it. It has long since been remarked by mathematicians that Hamilton's method contains purely geometrical truths, and that a peculiar mode of expression, suitable to it, is required in order to express these clearly. But this fact has only come to light in a somewhat perplexing form, namely, in the analogies between ordinary mechanics and the geometry of space of many dimensions, which have been discovered by following out Hamilton's thoughts. Our mode of expression gives a simple and intelligible explanation of these analogies. It allows us to take advantage of them, and at the same time it avoids the unnatural admixture of supra-sensible abstractions with a branch of physics.

We have now sketched the content and form of our third image as far as can be done without trenching upon the contents of the book; far enough to enable us to submit it to criticism in respect of its permissibility, its correctness, and its appropriateness. I think that as far as logical permissibility is concerned it will be found to satisfy the most rigid requirements, and I trust that others will be of the same opinion. This merit of the representation I consider to be of the greatest importance, indeed of unique importance. Whether the image is more appropriate than another; whether it is capable of including all future experience; even whether it only embraces all present experience, all this I regard almost as nothing compared with the question whether it is in itself conclusive, pure and free from contradiction. For I have not attempted this task because mechanics has shown signs of inappropriateness in its applications, nor because it in any way conflicts with experience, but solely in order to rid myself of the oppressive feeling that to me its elements were not free from things obscure and unintelligible. What I have sought is not the only image of mechanics, nor yet the best image; I have only sought to find an intelligible image and to show by an example that this is possible and what it must look like. We cannot attain to perfection in any direction; and I must confess that, in spite of the pains I have taken with it, the image is not so convincingly clear but that in some points it may be exposed to doubt or may require defence. And yet it seems to me that of objections of a general nature there is only a single one which is so pertinent that it is worth while to anticipate and remove it. It relates to the nature of the rigid connections which we assume to exist between the masses, and which are absolutely indispensable in our system. Many physicists will at first be of opinion that by means of these connections forces are introduced into the elements of mechanics, and are introduced in a way which is secret, and therefore not permissible. For, they will assert, rigid connections are not conceivable without forces; they cannot come into existence except by the action of forces. To this we reply—Your assertion is correct for the mode of thought of ordinary mechanics, but it is not correct independently of this mode of thought; it does not carry conviction to a mind which considers the facts without prejudice and as if for the first time. Suppose we find in any way that the distance between two material particles remains constant at all times and under all circumstances. We can express this fact without making use of any other conceptions than those of space; and the value of the fact stated, as a fact, for the purpose of foreseeing future experience and for all other purposes, will be independent of any explanation of it which we may or may not possess. In no case will the value of the fact be increased, or our understanding of it improved, by putting it in the form—"Between these masses there acts a force which keeps them at a constant distance from one another," or "Between them there acts a force which makes it impossible for their distance to alter from its fixed value." But it will be urged that this latter explanation, although apparently only a ludicrous circumlocution, is nevertheless correct. For all the connections of the actual world are only approximately rigid; and the appearance of rigidity is only produced by the action of the elastic forces which continually annul the small deviations from the position of rest. To this we reply as follows:—With regard to rigid connections which are only approximately realised, our mechanics will naturally only state as a fact that they are approximately satisfied; and for the purpose of this statement the idea of force is not required. If we wish to proceed to a second approximation and to take into consideration the deviations, and with them the elastic forces, we shall make use of a dynamical explanation for these as for all forces. In seeking the actual rigid connections we shall perhaps have to descend to the world of atoms. But such considerations are out of place here; they do not affect the question whether it is logically permissible to treat of fixed connections as independent of forces and precedent to them. All that I wished to show was that this question must be answered in the affirmative, and this I believe I have done. This being so, we can deduce the properties and behaviour of the forces from the nature of the fixed connections without being guilty of a *petitio principii*. Other objections of a similar kind are possible, but I believe they can be removed in much the same way.

By way of giving expression to my desire to prove the logical purity of the system in all its details, I have thrown the representation into the older synthetic form. For this purpose the form used has the merit of compelling us to specify beforehand, definitely even if monotonously, the logical value which every important statement is intended to have. This makes it impossible to use the convenient reservations and ambiguities into which we are enticed by the wealth of combinations in ordinary speech. But the most important advantage of the form chosen is that it is always based upon what has already been proved, never upon what is to be proved later on: thus we are always sure of the whole chain if we sufficiently test each link as we proceed. In this respect I have endeavoured to carry out fully the obligations imposed by this mode of representation. At the same time it is obvious that the form by itself is no guarantee against error or oversight; and I hope that any chance defects will not be the more harshly criticised on account of the somewhat presumptuous mode of presentation. I trust that any such defects will be capable of improvement and will not affect any important point. Now and again, in order to avoid excessive prolixity, I have consciously abandoned to some extent the rigid strictness which this mode of representation properly requires. Before proceeding to mechanics proper, as dependent upon physical experience, I have naturally discussed those relations which follow simply and necessarily from the definitions adopted and from mathematics; the connection of these latter with experience, if any, is of a different nature from that of the former. Moreover, there is no reason why the reader should not begin with the second book. The matter with which he is already familiar and the clear analogy with the dynamics of a particle will enable him easily to guess the purport of the propositions in the first book. If he admits the appropriateness of the mode of expression used, he can at any time return to the first book to convince himself of its permissibility.

We next turn to the second essential requirement which our image must satisfy. In the first place there is no doubt that the system correctly represents a very large number of natural motions. But this does not go far enough; the system must include all natural motions without exception. I think that this, too, can be asserted of it; at any rate in the sense that no definite phenomena can at present be mentioned which would be inconsistent with the system. We must of course admit that we cannot extend a rigid examination to all phenomena. Hence the system goes a little beyond the results of assured experience; it therefore has the character of a hypothesis which is accepted tentatively and awaits sudden refutation by a single example or gradual confirmation by a large number of examples. There are in especial two places in which we go beyond assured experience: firstly, in our limitation of the possible connections; secondly, in the dynamical explanation of force. What right have we to assert that all natural connections can be expressed by linear differential equations of the first order? With us this assumption is not a matter of secondary importance which we might do without. Our system stands or falls with it; for it raises the question whether our fundamental law is applicable to connections of the most general kind. And yet connections of a more general kind are not only conceivable, but they are permitted in ordinary mechanics without hesitation. There nothing prevents us from investigating the motion of a point where its path is only limited by the supposition that it makes a given angle with a given plane, or that its radius of curvature is always proportional to another given length. These are conditions which are not permissible in our system. But why are we certain that they are debarred by the nature of things? We might reply that these and similar connections cannot be realised by any practical mechanism; and in this respect we might appeal to the great authority of Helmholtz's name. But in every example possibilities might be overlooked; and ever so many examples would not suffice to substantiate the general assertion. It seems to me that the reason for our conviction should more properly be stated as follows. All connections of a system which are not embraced within the limits of our mechanics, indicate in one sense or another a discontinuous succession of its possible motions. But as a matter of fact it is an experience of the most general kind that nature exhibits continuity in infinitesimals everywhere and in every sense: an experience which has crystallised into firm conviction in the old proposition—*Natura non facit saltus.* In the text I have therefore laid stress upon this: that the permissible connections are defined solely by their continuity; and that their property of being represented by equations of a definite form is only deduced from this. We cannot attain to actual certainty in this way. For this old proposition is indefinite, and we cannot be sure how far it applies—how far it is the result of actual experience, and how far the result of arbitrary assumption. Thus the most conscientious plan is to admit that our assumption as to the permissible connections is of the nature of a tentatively accepted hypothesis. The same may be said with respect to the dynamical explanation of force. We may indeed prove that certain classes of concealed motions produce forces which, like actions-at-a-distance in nature, can be represented to any desired degree of approximation as differential coefficients of force-functions. It can be shown that the form of these force-functions may be of a very general nature; and in fact we do not deduce any restrictions for them. But on the other hand it remains for us to prove that any and every form of the force-functions can be realised; and hence it remains an open question whether such a mode of explanation may not fail to account for some one of the forms occurring in nature. Here again we can only bide our time so as to see whether our assumption is refuted, or whether it acquires greater and greater probability by the absence of any such refutation. We may regard it as a good omen that many distinguished physicists tend more and more to favour the hypothesis. I may mention Lord Kelvin's theory of vortex-atoms: this presents to us an image of the material universe which is in complete accord with the principles of our mechanics. And yet our mechanics in no wise demands such great simplicity and limitation of assumptions as Lord Kelvin has imposed upon himself. We need not abandon our fundamental propositions if we were to assume that the vortices revolved about rigid or flexible, but inextensible, nuclei; and instead of assuming simply incompressibility we might subject the all-pervading medium to much more complicated conditions, the most general form of which would be a matter for further investigation. Thus there appears to be no reason why the hypothesis admitted in our mechanics should not suffice to explain the phenomena.

We must, however, make one reservation. In the text we take the natural precaution of expressly limiting the range of our mechanics to inanimate nature; how far its laws extend beyond this we leave as quite an open question. As a matter of fact we cannot assert that the internal processes of life follow the same laws as the motions of inanimate bodies; nor can we assert that they follow different laws. According to appearance and general opinion there seems to be a fundamental difference. And the same feeling which impels us to exclude from the mechanics of the inanimate world as foreign every indication of an intention, of a sensation, of pleasure and pain,—this same feeling makes us unwilling to deprive our image of the animate world of these richer and more varied conceptions. Our fundamental law, although it may suffice for representing the motion of inanimate matter, appears (at any rate that is one's first and natural impression) too simple and narrow to account for even the lowest processes of life. It seems to me that this is not a disadvantage, but rather an advantage of our law. For while it allows us to survey the whole domain of mechanics, it shows us what are the limits of this domain. By giving us only bare facts, without attributing to them any appearance of necessity, it enables us to recognise that everything might be quite different. Perhaps such considerations will be regarded as out of place here. It is not usual to treat of them in the elements of the customary representation of mechanics. But there the complete vagueness of the forces introduced leaves room for free play. There is a tacit stipulation that, if need be, later on a contrast between the forces of animate and inanimate nature may be established. In our representation the outlines of the image are from the first so sharply delineated, that any subsequent perception of such an important division becomes almost impossible. We are therefore bound to refer to this matter at once, or to ignore it altogether.

As to the appropriateness of our third image we need not say much. In respect of distinctness and simplicity, as the contents of the book will show, we may assign to it about the same position as to the second image; and the merits to which we drew attention in the latter are also present here. But the permissible possibilities are somewhat more extensive than in the second image. For we pointed out that in the latter certain rigid connections were wanting; by our fundamental assumptions these are not excluded. And this extension is in accordance with nature, and is therefore a merit; nor does it prevent us from deducing the general properties of natural forces, in which lay the significance of the second image. The simplicity of this image, as of the second, is very apparent when we consider their physical applications. Here, too, we can confine our consideration to any characteristics of the material system which are accessible to observation. From their past changes we can deduce future ones by applying our fundamental law, without any necessity for knowing the positions of all the separate masses of the system, or for concealing our ignorance by arbitrary, ineffectual, and probably false hypotheses. But as compared with the second image, our third one exhibits simplicity also in adapting its conceptions so closely to nature that the essential relations of nature are represented by simple relations between the ideas. This is seen not only in the fundamental law, but also in its numerous general corollaries which correspond to the so-called principles of mechanics. Of course it must be admitted that this simplicity only obtains when we are dealing with systems which are completely known, and that it disappears as soon as concealed masses come in. But even in these cases the reason of the complication is perfectly obvious. The loss of simplicity is not due to nature, but to our imperfect knowledge of nature. The complications which arise are not simply a possible, but a necessary result of our special assumptions. It must also be admitted that the co-operation of concealed masses, which is the remote and special case from the standpoint of our mechanics, is the commonest case in the problems which occur in daily life and in the arts. Hence it will be well to point out again that we have only spoken of appropriateness in a special sense—in the sense of a mind which endeavours to embrace objectively the whole of our physical knowledge without considering the accidental position of man in nature, and to set forth this knowledge in a simple manner. The appropriateness of which we have spoken has no reference to practical applications or the needs of mankind. In respect of these latter it is scarcely possible that the usual representation of mechanics, which has been devised expressly for them, can ever be replaced by a more appropriate system. Our representation of mechanics bears towards the customary one somewhat the same relation that the systematic grammar of a language bears to a grammar devised for the purpose of enabling learners to become acquainted as quickly as possible with what they will require in daily life. The requirements of the two are very different, and they must differ widely in their arrangement if each is to be properly adapted to its purpose.

In conclusion, let us glance once more at the three images of mechanics which we have brought forward, and let us try to make a final and conclusive comparison between them. After what we have already said, we may leave the second image out of consideration. We shall put the first and third images on an equality with respect to permissibility, by assuming that the first image has been thrown into a form completely satisfactory from the logical point of view. This we have already assumed to be possible. We shall also put both images on an equality with respect to appropriateness, by assuming that the first image has been rendered complete by suitable additions, and that the advantages of both in different directions are of equal value. We shall then have as our sole criterion the correctness of the images: this is determined by the things themselves and does not depend on our arbitrary choice. And here it is important to observe that only one or the other of the two images can be correct: they cannot both at the same time be correct. For if we try to express as briefly as possible the essential relations of the two representations, we come to this. The first image assumes as the final constant elements in nature the relative accelerations of the masses with reference to each other: from these it incidentally deduces approximate, but only approximate, fixed relations between their positions. The third image assumes as the strictly invariable elements of nature fixed relations between the positions: from these it deduces when the phenomena require it approximately, but only approximately, invariable relative accelerations between the masses. Now, if we could perceive natural motions with sufficient accuracy, we should at once know whether in them the relative acceleration, or the relative relations of position, or both, are only approximately invariable. We should then know which of our two assumptions is false; or whether both are false; for they cannot both be simultaneously correct. The greater simplicity is on the side of the third image. What at first induces us to decide in favour of the first is the fact that in actions-at-a-distance we can actually exhibit relative accelerations which, up to the limits of our observation, appear to be invariable; whereas all fixed connections between the positions of tangible bodies are soon and easily perceived by our senses to be only approximately constant. But the situation changes in favour of the third image as soon as a more refined knowledge shows us that the assumption of invariable distance-forces only yields a first approximation to the truth; a case which has already arisen in the sphere of electric and magnetic forces. And the balance of evidence will be entirely in favour of the third image when a second approximation to the truth can be attained by tracing back the supposed actions-at-a-distance to motions in an all-pervading medium whose smallest parts are subjected to rigid connections; a case which also seems to be nearly realised in the same sphere. This is the field in which the decisive battle between these different fundamental assumptions of mechanics must be fought out. But in order to arrive at such a decision it is first necessary to consider thoroughly the existing possibilities in all directions. To develop them in one special direction is the object of this treatise,—an object which must necessarily be attained even if we are still far from a possible decision, and even if the decision should finally prove unfavourable to the image here developed.

BOOK I

GEOMETRY AND KINEMATICS OF MATERIAL SYSTEMS

1. **Prefatory Note.** The subject-matter of the first book is completely independent of experience. All the assertions made are *a priori* judgments in Kant's sense. They are based upon the laws of the internal intuition of, and upon the logical forms followed by, the person who makes the assertions; with his external experience they have no other connection than these intuitions and forms may have.

CHAPTER I

TIME, SPACE, AND MASS

2. **Explanation.** The time of the first book is the time of our internal intuition. It is therefore a quantity such that the variations of the other quantities under consideration may be regarded as dependent upon its variation; whereas in itself it is always an independent variable.

The space of the first book is space as we conceive it. It is therefore the space of Euclid's geometry, with all the properties which this geometry ascribes to it. It is immaterial to us whether these properties are regarded as being given by the laws of our internal intuition, or as consequences of thought which necessarily follow from arbitrary definitions.

The mass of the first book will be introduced by a definition.

3. **Definition 1.** A material particle is a characteristic by which we associate without ambiguity a given point in

space at a given time with a given point in space at any other time.

Every material particle is invariable and indestructible. The points in space which are denoted at two different times by the same material particle, coincide when the times coincide. Rightly understood, the definition implies this.

4. **Definition 2.** The number of material particles in any space, compared with the number of material particles in some chosen space at a fixed time, is called the mass contained in the first space.

We may and shall consider the number of material particles in the space chosen for comparison to be infinitely great. The mass of the separate material particles will therefore, by the definition, be infinitely small. The mass in any given space may therefore have any rational or irrational value.

5. **Definition 3.** A finite or infinitely small mass, conceived as being contained in an infinitely small space, is called a material point.

A material point therefore consists of any number of material particles connected with each other. This number is always to be infinitely great: this we attain by supposing the material particles to be of a higher order of infinitesimals than those material points which are regarded as being of infinitely small mass. The masses of material points, and in especial the masses of infinitely small material points, may therefore bear to one another any rational or irrational ratio.

6. **Definition 4.** A number of material points considered simultaneously is called a system of material points, or briefly a system. The sum of the masses of the separate points is, by § 4, the mass of the system.

Hence a finite system consists of a finite number of finite material points, or of an infinite number of infinitely small material points, or of both. It is always permissible to regard a system of material points as being composed of an infinite number of material particles.

7. **Observation 1.** In what follows we shall always treat a finite system as consisting of a finite number of finite material points. But as we assign no upper limit to their number, and

no lower limit to their mass, our general statements will also include as a special case that in which the system contains an infinite number of infinitely small material points. We need not enter into the details required for the analytical treatment of this case.

8. **Observation 2.** A material point can be regarded as a special case and as the simplest example of a system of material points.

CHAPTER II

POSITIONS AND DISPLACEMENTS OF POINTS AND SYSTEMS

Position

9. **Definition 1.** The point of space which is indicated by a given particle at a given time is called the position of the particle at that time. The position of a material point is the common position of its particles.

10. **Definition 2.** The aggregate of the positions which all the points of a system simultaneously occupy is called the position of the system.

11. **Definition 3.** Any given position of a material point in infinite space is called a geometrically conceivable, or for shortness a conceivable, position of the point. The aggregate of any conceivable positions whatsoever of the points of a system is called a conceivable position of the system.

At any time two particles may differ as to their position, two material points as to their mass and their position, and two systems of material points as to the number, the mass, and the positions of their points. But, in accordance with the definitions which we have already given of them, particles, material points, and systems of material points cannot differ in any other respect.

12. **Analytical Representation of the Position (*a*) of a Point.**—The position of a material point can be represented analytically by means of its three rectangular coordinates referred to a set of fixed axes. These coordinates will always be denoted by x_1, x_2, x_3. Every conceivable position of the point

implies a singly-determined value-system of these coördinates, and conversely every arbitrarily chosen value-system of the coordinates implies a singly-determined conceivable position of the point.

The position of a point can also be represented by any r quantities $p_1 \dots p_\rho \dots p_r$ whatsoever, provided we agree to associate continuously a given value-system of these quantities with a given position of the point, and conversely. The rectangular coordinates are then functions of these quantities, and conversely. The quantities p_ρ are called the general coordinates of the point. If $r > 3$, then for geometrical reasons $r - 3$ equations must exist between the quantities p_ρ which enable us to determine these quantities as functions of three independent quantities, for instance, x_1, x_2, x_3. However, we shall exclude dependence of the coordinates on one another on account of purely geometrical relations, and consequently it must always be understood that $r \overline{\gtrless} 3$. If $r < 3$, then all conceivable positions of the point cannot be represented by means of p_ρ, but only a portion of these positions. The positions not expressed in terms of p_ρ will be considered as being *eo ipso* excluded from discussion whilst we are using the coordinates p_ρ.

13. **Analytical Representation** (*b*) **of a System.**—The position of a system of n material points can be analytically represented by means of the $3n$ rectangular coordinates of the points of the system. These coordinates will be denoted by $x_1, x_2 \dots x_\nu \dots x_{3n}$, so that x_1, x_2, x_3 are the coordinates of the first point, $x_{3\mu-2}$, $x_{3\mu-1} x_{3\mu}$ the coordinates of the μ^{th} point. We shall call these $3n$ coordinates x_ν, the rectangular coordinates of the system. Every conceivable position of the system implies a singly-determined value-system of its rectangular coordinates, and conversely every arbitrarily chosen value-system of x_ν a singly-determined conceivable position of the system.

We may also consider the system as determined by means of any r quantities $p_1 \dots p_\rho \dots p_r$ whatever, as long as we agree to associate continuously a given value-system of these coordinates with a given position of the system, and conversely. The rectangular coordinates are therefore functions of these quantities, and conversely. The quantities p_ρ are called the general coordinates of the system. If $r > 3n$, then for geometrical reasons $r-3n$ equations must exist between p_ρ. However, we shall assume that no geometrical relations exist between p_ρ, and consequently $r \leqq 3n$. If $r < 3n$, then all conceivable positions of the system cannot be expressed in terms of p_ρ, but only a portion of them. Those positions not expressed in terms of p_ρ will be considered as being *eo ipso* excluded from consideration when we are using the general coordinates p_ρ.

Configuration and Absolute Position

14. **Definition 1.** The aggregate of the relative positions of the points of a system is called the configuration of the system.

The configuration of the system and the absolute position of the configuration in space determine together the position of the system.

15. **Definition 2.** By a coordinate of configuration we mean any coordinate of the system whose value cannot change without the configuration of the system changing.

Whether a given coordinate is a coordinate of configuration or not does not depend on the choice of the remaining coordinates of the system.

16. **Definition 3.** By a coordinate of absolute position we mean any coordinate of the system through whose change the configuration of the system cannot be altered so long as the remaining coordinates of the system do not alter.

Whether a given coordinate is a coordinate of absolute position or not depends therefore on the choice of the remaining coordinates.

Corollaries

17. 1. A coordinate cannot be at one and the same time both a coordinate of configuration and a coordinate of absolute position. On the other hand, a given coordinate can and in general will be neither a coordinate of configuration nor a coordinate of absolute position.

18. 2. So long as $n>3$, in every position $3n$ independent coordinates can be chosen in various ways, so that there are as

many as $3n - 6$ coordinates of configuration amongst them, but in no way so as to include more than $3n - 6$ such coordinates.

For, let us choose from the coordinates the three distances of any three points of the system from each other, and the $3(n - 3)$ distances of the remaining points from them; then we have $(3n - 6)$ coordinates of configuration; and any $(3n - 6)$ different functions of these distances give $(3n - 6)$ coordinates of configuration of the system. Fewer coordinates of configuration can exist; for example, if we take the $3n$ rectangular coordinates, none exist. But there cannot be more than $(3n - 6)$ coordinates of configuration amongst independent coordinates; for if, amongst the given coordinates of a system, there were more than $(3n - 6)$ coordinates of configuration, then the latter could be expressed in terms of these $(3n - 6)$ distances, and consequently would not be independent of one another.

19. 3. So long as $n > 3$, $3n$ independent coordinates for all conceivable positions of a system can be chosen in various ways so that there are amongst them as many as 6, but not more than 6 coordinates of absolute position.

For, let us choose the coordinates in such a manner that there are amongst them $(3n - 6)$ coordinates of configuration, and take with them any 6 coordinates, say 6 of the rectangular coordinates of the system; then the last are *eo ipso* coordinates of absolute position, for no change in them changes the configuration so long as the rest are fixed. Fewer than 6 may exist; none exist, for instance, when we use the rectangular coordinates of the system. More than 6 cannot exist. For did more than 6 exist, then, for a particular choice of the coordinates, all conceivable configurations of the system would be determined by the remaining fewer than $3n - 6$ coordinates; and consequently there would not be left $(3n - 6)$ coordinates of configuration independent of one another for the system, which would be contrary to § 18.

20. 4. If $3n$ independent coordinates of a system of n points are so chosen that there are amongst them $(3n - 6)$ coordinates of configuration, then the remaining 6 are necessarily coordinates of absolute position. And if these $3n$ coordinates are so chosen that 6 of them are coordinates of absolute

position, then the remaining $(3n - 6)$ are necessarily coordinates of configuration.

For, if there were amongst the latter even one which could be changed without altering the configuration of the system, then the absolute position of the configuration would be determined by more than 6 independent coordinates, which is impossible.

21. 5. Any quantity can be used as a coordinate of absolute position, provided its change alters the position of the system, and provided it is not itself a coordinate of configuration. Any 6 quantities which satisfy these conditions and are independent of one another, can be taken as coordinates of absolute position, and become coordinates of absolute position by the fact that no other quantities are associated with them as coordinates unless they have the properties of coordinates of configuration.

Finite Displacements

(*a*) *Of Points*

22. **Definition 1.** The passage of a material point from an initial to a final position, without regard to the time or manner of the passage, is called a displacement of the point from the initial position to the final one.

The displacement of a point is completely determined by its initial and final position. It is also completely known when we are given its initial position, its direction, and its magnitude.

23. **Observation 1.** The magnitude of the displacement of a point is equal to the distance of its final position from its initial one. Let the quantities x_ν be the rectangular coordinates of its initial position, and x_ν' the rectangular coordinates of its final position, then the magnitude s' of the displacement is the positive root of the equation

$$s'^2 = \sum_1^3{}_\nu (x_\nu' - x_\nu)^2.$$

24. **Observation 2.** The direction of a displacement is the direction of a straight line which is drawn from the initial

position of the point to the final one. Let s', x_ν, x_ν' have the same meaning as before, and let x_ν°, x_ν'', s'' be the coordinates of the initial and final positions, and the length of a second displacement, then the angle $\widehat{s's''}$ between the two displacements is given by the equation

$$s's'' \cos \widehat{s's''} = \sum_1^3 {}_\nu (x_\nu' - x_\nu)(x_\nu'' - x_\nu^\circ) \qquad \text{(i).}$$

For, if we consider a triangle whose sides are equal in length to s' and s'', and the included angle equal to $\widehat{s's''}$, we obtain the equation

$$s'^2 + s''^2 - 2s's'' \cos \widehat{s's''} = \sum_1^3 {}_\nu [(x_\nu' - x_\nu) - (x_\nu'' - x_\nu^\circ)]^2 \quad \text{(ii),}$$

from which, together with § 23, equation (i) follows.

25. **Definition 2.** Two displacements of a point are said to be identical when they have the same initial and final positions; two displacements of a point are said to be equal when they have the same magnitude and direction; they are said to be parallel when they have the same direction.

26. **Note.** Let $x_1, x_2 \ldots x_k$ denote the k rectangular coordinates of a point in space of k dimensions, $x_1', x_2' \ldots x_k'$ the coordinates of a second point; then the additional statement that the distance between the two points is the positive root of the equation

$$s'^2 = \sum_1^k {}_\nu (x_\nu' - x_\nu)^2$$

extends the whole of the following investigation, as well as the whole of mechanics, to space of k dimensions, without necessitating anything but a change in the wording. No use will be made of this remark, but the investigation will refer, as stated at the beginning, simply to the space of Euclidian geometry.

(*b*) *Of Systems*

27. **Definition.** The passage of a system of material points from an initial position to a final one without regard to the time

or manner of the passage is called a displacement of the system from the initial to the final position.

The displacement of a system is completely known when we know its initial and final positions. It is also completely known when its initial position, and what are termed its direction and magnitude, are given.

28. **Notation.** It will be convenient to call the positive root of the arithmetic mean of the squares of a series of quantities their quadratic mean value.

29. **Definition a.** The magnitude of the displacement of a system is the quadratic mean value of the magnitudes of the displacements of all its particles.

The magnitude of the displacement which a system undergoes in moving from one position to another is called the distance between the two positions. The magnitude of a displacement is also called its length.

30. **Note.** The distance between two positions of a system is defined independently of the form of its analytical representation, and in particular is independent of the choice of the coordinates of the system.

31. **Problem.** To express the distance between two positions of a system in terms of its rectangular coordinates.

Let there be n material points in the system. Let x_ν be the value of one of the rectangular coordinates of the system before the displacement, and x_ν' the value of the same after the displacement. The coordinate x_ν is at the same time a coordinate of one of the points of the system: let the mass of this point be m_ν, ν ranges from 1 to $3n$, but all the m_ν's are not unequal, since for every μ from 1 to $3n$

$$m_{3\mu-2} = m_{3\mu-1} = m_{3\mu}.$$

If now η be the number of particles in the unit of mass, the mass m_ν contains $m_\nu\eta$ particles, and the whole mass m of the system $m.\eta$. Consequently, with this notation, the quadratic mean value s' of the displacements of all particles is the positive root of the equation

$$ms'^2 = \sum_1^{3n} {}_\nu m_\nu \, (x_\nu' - x_\nu)^2 \qquad \text{(i)},$$

and this root is therefore the required distance. Finally

$$m = \tfrac{1}{3} \sum_1^{3n} {}_\nu m_\nu \qquad \text{(ii)}.$$

32. **Proposition.** The distance between two positions of a system is always smaller than the sum of the distances of the two positions from a third.

Let the quantities x_ν', x_ν'', x_ν''' be the rectangular coordinates of the positions 1, 2, 3; let s_{12}, s_{13}, s_{23} be their distances from each other. For shortness write

$$\sqrt{\frac{m_\nu}{m}}(x_\nu''' - x_\nu') = a_\nu \qquad \sqrt{\frac{m_\nu}{m}}(x_\nu''' - x_\nu'') = b_\nu,$$

then $s_{13}^{\ 2} = \sum_1^{3n} {}_\nu . a_\nu^2 \quad s_{23}^{\ 2} = \sum_1^{3n} {}_\nu b_\nu^2 \quad s_{12}^{\ 2} = \sum_1^{3n} {}_\nu (a_\nu - b_\nu)^2$. If then it were possible that $s_{12} > s_{13} + s_{23}$ we should get on squaring $s_{12}^{\ 2} - s_{13}^{\ 2} - s_{23}^{\ 2} > 2 s_{13} . s_{23}$, and on squaring again,

$$4 \, s_{13}^{\ 2} s_{23}^{\ 2} - (s_{12}^{\ 2} - s_{13}^{\ 2} - s_{23}^{\ 2})^2 < 0.$$

This is, however, impossible, for, on substituting the values of s given as above, the left-hand side becomes

$$4 \sum_1^{3n} {}_\nu \sum_1^{3n} {}_\mu (a_\nu b_\mu - a_\mu b_\nu)^2,$$

which, being a sum of squares, is necessarily positive. Therefore our assumption was unwarranted, and consequently

$$s_{12} \leqq s_{13} + s_{23}.$$

33. **Corollary.** It is therefore always possible to construct a plane triangle whose three sides are equal to the three distances of any three positions of a system from each other.

34. **Definition b.** The difference in direction between two displacements of a system from the same initial position is the included angle of a plane triangle which has the lengths of the

two displacements as sides, and whose base is the distance between their final positions.

The difference in direction between two displacements is also called the angle between them, or their inclination towards one another.

35. **Note 1.** The inclination towards one another of two displacements with the same initial position is in all cases a singly-determined real angle, smaller than π.

For the triangle which determines that angle can always (§ 32) be drawn.

36. **Note 2.** The difference in direction between two displacements is defined independently of the form of the analytical representation, and in particular is independent of the choice of the coordinates used.

37. **Problem.** To express the angle between two displacements from the same initial position in terms of the rectangular coordinates of the initial and final positions.

Let the quantities x_ν be the coordinates of the common initial positions, x_ν' and x_ν'' the coordinates of the two final positions, s' and s'' the lengths of the two displacements, $\widehat{s's''}$ the included angle. By consideration of the plane triangle whose three sides are the three distances between the three positions, we obtain

$$2ms's'' \cos \widehat{s's''} = \sum_1^{3n}{}_\nu m_\nu(x_\nu' - x_\nu)^2 + \sum_1^{3n}{}_\nu m_\nu(x_\nu'' - x_\nu)^2$$
$$- \sum_1^{3n}{}_\nu m_\nu[(x_\nu'' - x_\nu) - (x_\nu - x_\nu)]^2,$$

and therefore

$$ms's'' \cos \widehat{s's''} = \sum_1^{3n}{}_\nu m_\nu(x_\nu'' - x_\nu)(x_\nu' - x_\nu) \qquad \text{(i)},$$

in which equation we consider s' and s'' expressed as in § 31 (i) in terms of the rectangular coordinates.

38. **Proposition.** Two displacements of a system from the same initial position have the angle between them equal to zero when the displacements of the individual points of both

the systems are parallel and correspondingly proportional, and conversely.

For, if the displacements of all points are parallel and proportional, then for all values of ν

$$x_\nu'' - x_\nu = \epsilon(x_\nu' - x_\nu),$$

where ϵ is the same constant factor for all values of ν. The right-hand side of equation § 37 (i) becomes therefore $m\epsilon s'^2$. Moreover $s'' = \epsilon s'$; thus by this equation $\cos \widehat{s's''} = 1$, and since $\widehat{s's''}$ is an interior angle of a triangle $\widehat{s's''} = 0$ (§ 35).

Conversely when $\widehat{s's''} = 0$, $\cos \widehat{s's''} = 1$, and then the equation 37 (i) squared gives when the values of s' and s'' are substituted

$$0 = [\sum_1^{3n}{}_\nu m_\nu(x_\nu'' - x_\nu)(x_\nu' - x_\nu)]^2 - \sum_1^{3n}{}_\nu m_\nu(x_\nu'' - x_\nu)^2 . \sum_1^{3n}{}_\nu m_\nu(x_\nu' - x_\nu)^2$$

$$= \sum_1^{3n}{}_\nu \sum_1^{3n}{}_\mu m_\nu m_\mu[(x_\nu'' - x_\nu)(x_\mu' - x_\mu) - (x_\mu'' - x_\mu)(x_\nu' - x_\nu)]^2,$$

and this is only possible when for every value of μ and ν

$$\frac{x_\mu'' - x_\mu}{x_\nu'' - x_\nu} = \frac{x_\mu' - x_\mu}{x_\nu' - x_\nu};$$

wherefore the converse is proved.

39. **Corollary 1.** If two displacements from the same initial position have their inclinations to a third displacement from the same initial position zero, then their inclination to one another is zero.

All displacements whose inclinations to any given displacement are zero, have consequently their inclinations to each other zero. The common property of all such displacements is called their direction.

40. **Corollary 2.** When two displacements of a system have the same direction, they are equally inclined to a third displacement.

Thus all displacements from the same initial position, and having the same direction, make the same angle with all displacements which have another common direction. This angle is called the angle between the two directions or the inclination of the two directions.

41. **Definition.** Two displacements of a system are said to be identical when the displacements of the points of the two systems are identical. Two displacements are equal when the displacements of the individual points are equal and two displacements are parallel when the displacements of the individual points in both are parallel and correspondingly proportional.

42. **Corollary.** Two displacements of a system from different initial positions are parallel when each of them has the same direction as a displacement which passes through its initial position and is equal to the other displacement, and conversely.

43. **Additional Note.** By the difference in direction between two displacements of a system from different initial positions we mean the angle between either of them, and a parallel displacement to the other from its own initial position.

44. **Problem.** To express the angle between any two displacements of a system in terms of the rectangular coordinates of their four end positions.

Let s' and s'' be the magnitudes of the two displacements, and $\widehat{s's''}$ the angle between them. Let x_ν and x_ν' be the coordinates of the initial and final positions of the first, x_ν° and x_ν'', the coordinates of the initial and final positions of the second displacement. A displacement whose initial coordinates are x_ν, and whose final coordinates have the value $x_\nu + x_\nu'' - x_\nu^\circ$, has the same initial position as the first, and is equal to the second. Hence it makes with the first the required angle, for which we obtain the equation

$$ms's'' \cos \widehat{s's''} = \sum_1^{3n} {}_\nu m_\nu (x_\nu' - x_\nu)(x_\nu'' - x_\nu^\circ).$$

The same value is obtained if we choose a displacement through the initial position of the second, and equal to the first, and then find the angle between this and the second.

Our definition in § 43 was thus unique, and therefore permissible.

45. **Definition.** Two displacements of a system are said to be perpendicular to one another when the angle between them is a right angle.

46. **Corollary 1.** The necessary and sufficient analytical condition that two displacements should be perpendicular to one another is the equation

$$\sum_{1}^{3n}{}_\nu m_\nu(x_\nu' - x_\nu)(x_\nu'' - x_\nu^\circ) = 0,$$

in which use is made of the notation of § 44.

47. **Corollary 2.** In a system of n points there is from a given position a $(3n-1)$ manifold of displacements, and therefore a $(3n-2)$ manifold of directions conceivable which are perpendicular to a given direction.

48. **Definition.** The component of a displacement in a given direction is a displacement whose direction is the given direction, and whose magnitude is equal to the orthogonal projection of the magnitude of the given displacement upon the given direction.

Thus, if the magnitude of the given displacement is s, and it makes with the given direction the angle ω, then its component in this direction is equal to $s \cos \omega$.

The magnitude of the component in a given direction will be simply termed the component in that direction.

Composition of Displacements

49. **Note.** Let there be given to a system several displacements, which are equal to given displacements, and which are so related to one another that the final position of the preceding displacement is the initial position of the succeeding one, then the final position attained is independent of the succession of displacements.

Since this is true for the displacements which the individual points suffer, it must also be true for the system.

50. **Definition 1.** A displacement which carries the

system into the same final position as a succession of displacements, which are equal to given displacements, is called the sum of these given displacements.

51. **Definition 2.** The difference between a chosen displacement and another is a displacement whose sum, together with the latter one, gives the former.

52. **Corollary** (to § 49). The addition and subtraction of displacements is subject to the rules of algebraic addition and subtraction.

CHAPTER III

INFINITELY SMALL DISPLACEMENTS AND PATHS OF A SYSTEM OF MATERIAL POINTS

53. **Prefatory Note.** From here on we shall no longer deal with single material points by themselves, but shall regard their investigation as being included in that of systems. Hence what follows must be understood as referring always to displacements of systems, even when this is not expressly stated.

Infinitely Small Displacements

54. **Explanation.** A displacement is said to be infinitely small when its length is infinitely small.

The position of the infinitely small displacement is a position to which the bounding points of the displacement lie indefinitely near.

An infinitely small displacement is determined in magnitude and direction when we know its position, and the infinitely small changes which the coordinates of the system undergo owing to the displacement.

55. **Problem 1a.** To express the length ds of an infinitely small displacement in terms of the changes dx_ν of the $3n$ rectangular coordinates of the system.

If in equation § 31 (i) we substitute dx_ν for $x_\nu' - x_\nu$, we obtain

$$mds^2 = \sum_1^{3n} {}_\nu\, m_\nu dx_\nu^2.$$

56. **Problem 1b.** To express the angle $\widehat{ss'}$ between two infinitely small displacements ds and ds' in terms of the changes dx_ν and dx_ν' in the $3n$ rectangular coordinates of the system.

If in the equation § 44 we substitute dx_ν for $x_\nu' - x_\nu$ and dx_ν' for $x_\nu'' - x_\nu^\circ$ we obtain

$$m ds ds' \cos \widehat{ss'} = \sum_1^{3n}{}_\nu m_\nu dx_\nu dx_\nu'.$$

This expression holds whether both displacements have the same position or not.

57. **Problem 2a.** To express the length ds of an infinitely small displacement in terms of the changes dp_ρ of the r general coordinates p_ρ of the system.

The rectangular coordinates x_ν are functions of the p_ρ's, and moreover of the p_ρ's alone, since they are completely determined by these, and since displacements of the system which are not expressible in terms of the changes of p_ρ are excluded from consideration (§ 13).

Putting now for shortness

$$\frac{\partial x_\nu}{\partial p_\rho} = a_{\nu\rho} \qquad \text{(i)},$$

we get $3n$ equations of the form

$$dx_\nu = \sum_1^{r}{}_\rho a_{\nu\rho} dp_\rho \qquad \text{(ii)},$$

where $a_{\nu\rho}$ are functions of the position, and can therefore be expressed as functions of p_ρ. Substituting these values in equation § 55, and putting for shortness

$$m a_{\rho\sigma} = \sum_1^{3n}{}_\nu m_\nu a_{\nu\rho} a_{\nu\sigma} \qquad \text{(iii)},$$

we get as the solution of the problem

$$ds^2 = \sum_1^{r}{}_\rho \sum_1^{r}{}_\sigma a_{\rho\sigma} dp_\rho dp_\sigma \qquad \text{(iv)}.$$

58. **Problem 2b.** To express the angle $\widehat{ss'}$ between two infinitely small displacements of lengths ds and ds' and having the same position in terms of the changes dp_ρ and dp_ρ' in the r general coordinates p_ρ of the system.

We form the values of dx_ν' by means of § 57 (ii), and substitute these and the values of dx_ν in equation § 56. Remembering that for both displacements the values of the coordinates, and therefore of the quantities $a_{\rho\sigma}$, are equal, we obtain

$$ds ds' \cos \widehat{ss'} = \sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\sigma} dp_\rho dp_\sigma'.$$

Properties of $\mathbf{a}_{\rho\sigma}$ and $a_{\rho\sigma}$. Introduction of $b_{\rho\sigma}$.

59. 1. For all values of ρ, σ, τ (cf. § 57 (i)),

$$\frac{\partial \mathbf{a}_{\rho\sigma}}{\partial p_\tau} = \frac{\partial \mathbf{a}_{\rho\tau}}{\partial p_\sigma}.$$

60. 2. For all values of ρ and σ (cf. § 57 (iii)),

$$a_{\rho\sigma} = a_{\sigma\rho}.$$

61. 3. The number of the quantities $\mathbf{a}_{\rho\sigma}$ is equal to $3nr$; the number of the quantities $a_{\rho\sigma}$ different from one another is $\frac{1}{2} r(r+1)$.

62. 4. For all values of ρ

$$a_{\rho\rho} > 0.$$

For all values of ρ and σ

$$a_{\rho\rho} a_{\sigma\sigma} - a^2_{\rho\sigma} > 0.$$

For the right-hand side of the equation § 57 (iv.), on account of its derivation from the equation § 55, is a necessarily positive quantity, whatever may be the values of dp_ρ. For this the foregoing inequalities are necessary conditions.

63. 5. For all values of ρ, σ, τ, the following equation holds,

$$\sum_1^{3n}{}_\nu m_\nu \mathbf{a}_{\nu\sigma} \left(\frac{\partial \mathbf{a}_{\nu\rho}}{\partial p_\tau} + \frac{\partial \mathbf{a}_{\nu\tau}}{\partial p_\rho} \right) = m \left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} + \frac{\partial a_{\tau\sigma}}{\partial p_\rho} - \frac{\partial a_{\rho\tau}}{\partial p_\sigma} \right).$$

In order to prove this equation we must substitute on the right-hand side the values of $a_{\rho\sigma}$ given in § 57 (iii), and make use of the properties of $\mathbf{a}_{\rho\sigma}$ given in § 59.

64. 6. Let the determinant formed by the r^2 quantities

$a_{\rho\sigma}$ be Δ. The factor of $a_{\rho\sigma}$ in Δ, divided by Δ will always be denoted by $b_{\rho\sigma}$. Thus we have as a definition

$$b_{\rho\sigma} = \frac{1}{\Delta}\frac{\partial\Delta}{\partial a_{\rho\sigma}}.$$

For all values of ρ and σ then

$$b_{\rho\sigma} = b_{\sigma\rho}.$$

The number of quantities $b_{\rho\sigma}$ different from one another is equal to $\frac{1}{2}r(r+1)$.

65. 7. The value of the expression

$$\sum_1^r{}^{\rho}\, a_{\rho\iota} b_{\rho\chi}$$

is equal to unity so long as $\iota = \chi$; its value is zero if ι and χ are different.

For if $\iota = \chi$, the expression $\sum_1^r{}^{\rho}\, a_{\rho\iota} b_{\rho\chi}\Delta$ represents the determinant Δ itself. If, however, ι and χ are different, it represents the determinant which results from Δ when the row $a_{\rho\chi}$ is replaced by the row $a_{\rho\iota}$. In this determinant two rows are equal, and consequently its value is zero.

66. 8. For all values of ι and χ we have the two equations

$$\sum_1^r{}^{\rho}\sum_1^r{}^{\sigma}\, b_{\rho\sigma} a_{\rho\iota} a_{\sigma\chi} = a_{\iota\chi},$$

$$\sum_1^r{}^{\rho}\sum_1^r{}^{\sigma}\, a_{\rho\sigma} b_{\rho\iota} b_{\sigma\chi} = b_{\iota\chi}.$$

For if we form by means of § 65 the value of the expression $\sum_1^r{}^{\rho}\, b_{\rho\sigma} a_{\rho\iota}$, or $\sum_1^r{}^{\rho}\, a_{\rho\sigma} b_{\rho\iota}$ for all values of σ from 1 to r, and then multiply the resulting equations one by one with $a_{\sigma\chi}$ or $b_{\sigma\chi}$ respectively, and add, the equations follow.

67. 9. Definite changes of the quantities $a_{\rho\sigma}$ involve definite changes of the quantities $b_{\rho\sigma}$. Let us denote by $\delta a_{\rho\sigma}$

and $\delta b_{\rho\sigma}$ any variation of $a_{\rho\sigma}$ and the resulting variation in $b_{\rho\sigma}$, then the following equations hold,

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\iota} a_{\sigma\chi} \delta b_{\rho\sigma} = - \delta a_{\iota\chi}$$

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\iota} b_{\sigma\chi} \delta a_{\rho\sigma} = - \delta b_{\iota\chi}.$$

If we vary the equations § 66 and make use of the results of § 65 the equations follow.

68. 10. If we vary in $a_{\rho\sigma}$ and $b_{\rho\sigma}$ only one definite coordinate, p_τ, on which they depend, then for every value of τ

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\iota} a_{\sigma\chi} \frac{\partial b_{\rho\sigma}}{\partial p_\tau} = - \frac{\partial a_{\iota\chi}}{\partial p_\tau},$$

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\iota} b_{\sigma\chi} \frac{\partial a_{\rho\sigma}}{\partial p_\tau} = - \frac{\partial b_{\iota\chi}}{\partial p_\tau}.$$

Displacements in the Direction of the Coordinates

69. **Definition 1.** A displacement in the direction of a definite coordinate is an infinitely small displacement in which only this one coordinate is changed without the remaining ones changing.

The direction of all the displacements in the direction of the same coordinate from the same position is the same; it is called the direction of the coordinate in that position.

70. **Note.** The direction of a coordinate depends on the choice of the remaining coordinates in use.

71. **Definition 2.** The reduced component of an infinitely small displacement in the direction of a given coordinate is the component of the displacement in the direction of the coordinate (§§ 48, 69) divided by the ratio of the change of the coordinate to a displacement in its own direction.

The reduced component in the direction of a coordinate is called for shortness the component along the coordinate.

Thus we speak of the component of a given displacement in a given direction; we cannot speak of the reduced component in a given direction, but only of the reduced component of an infinitely small displacement in the direction of a coordinate.

72. **Problem 1a.** To express the inclination $\widehat{sx_\nu}$ of the displacement ds to the rectangular coordinate x_ν in terms of the $3n$ increments dx_ν.

In equation § 56, put the dx_ν's equal to zero for all values of ν, except the given one to which the problem refers. Then the direction of ds' is, by § 69, that of x_ν, and the angle $\widehat{ss'}$ becomes the required angle. Moreover, since by § 55 $mds'^2 = m_\nu dx_\nu'^2$, we get as solution

$$ds \cos \widehat{sx_\nu} = \sqrt{\frac{m_\nu}{m}} dx_\nu,$$

where for ds its value in terms of dx_ν is to be substituted.

73. **Problem 1b.** To express the components $\overline{dx_\nu}$ of the displacement ds along the rectangular coordinates x_ν in terms of the changes dx_ν of the coordinates.

Put $\widehat{sx_\nu} = 0$ in the foregoing proposition; then we get the displacement ds in the direction of the coordinate x_ν, and we observe that the ratio of the change of the coordinate to a displacement in its own direction is equal to dx_ν/ds, or to $\sqrt{m/m_\nu}$. The left-hand side of equation § 72 represents immediately the component of ds in the direction of x_ν; then if we divide the equation by $\sqrt{m/m_\nu}$ we obtain (§ 71) as the solution of the problem

$$\overline{dx_\nu} = \frac{m_\nu}{m} dx_\nu.$$

74. **Problem 1c.** To express the changes dx_ν of the rectangular coordinates in a displacement in terms of the reduced components of the displacement along these coordinates.

The solution of the foregoing problem gives immediately

$$dx_\nu = \frac{m}{m_\nu} \overline{dx_\nu}.$$

75. **Problem 2a.** To express the inclination $\widehat{sp_\rho}$ of the displacement ds to the general coordinate p_ρ in terms of the r increments dp_ρ.

Put in equation § 58 dp_ρ' zero for all values of ρ except the chosen one to which the problem refers. The direction of ds' is then by § 69 that of p_ρ, and the angle $\widehat{ss'}$ is the required angle. Since at the same time by § 57 $ds'^2 = a_{\rho\rho} dp'_\rho{}^2$, we obtain as solution of the problem

$$\sqrt{a_{\rho\rho}}\, ds \cos \widehat{sp_\rho} = \sum_1^r {}_\sigma\, a_{\rho\sigma} dp_\sigma,$$

where for ds its value in terms of dp_σ must be substituted.

76. **Observation 1.** If in the foregoing expression we put all the dp_σ's equal to zero with the exception of a given one, say dp_σ, the direction of ds becomes the direction of this coordinate p_σ and the angle $\widehat{sp_\rho}$ becomes the angle $\widehat{p_\sigma p_\rho}$ which the coordinate p_σ makes with the coordinate p_ρ. Since at the same time $ds^2 = a_{\sigma\sigma} dp_\sigma{}^2$, we obtain for this angle

$$\cos \widehat{p_\sigma p_\rho} = \frac{a_{\rho\sigma}}{\sqrt{a_{\rho\rho} a_{\sigma\sigma}}},$$

and this angle is always, by § 62, a real angle.

77. **Observation 2.** The coordinates p_ρ are called orthogonal when each of them is in every position perpendicular to the remaining ones. The necessary and sufficient condition is (§ 76) that $a_{\rho\sigma}$ should vanish whenever ρ and σ are different. For example, rectangular coordinates are orthogonal coordinates.

78. **Problem 2b.** To express the components $d\bar{p}_\rho$ of the displacement ds along the coordinates p_ρ in terms of the increments dp_ρ of these coordinates in the displacement.

In equation § 75 put $\widehat{sp_\rho}$ equal to zero, and we get the displacement ds given by this equation in the direction of p_ρ; every dp_σ is zero except dp_ρ, and the equation thus becomes $\sqrt{a_{\rho\rho}}\, ds = a_{\rho\rho} dp_\rho$. The ratio of the change of p_ρ to a displacement in its own direction is thus $1/\sqrt{a_{\rho\rho}}$. If we remember that according to § 48 $ds \cos \widehat{sp_\rho}$ is the component of ds in the direction p_ρ and pay attention to definition § 71, we see that the left-hand side of the equation § 75 represents the reduced component along p_ρ and we obtain the expression,

$$d\bar{p}_\rho = \sqrt{a_{\rho\rho}}\, ds \cos \widehat{sp_\rho} \qquad \text{(i)},$$

and thus

$$d\bar{p}_\rho = \sum_1^r {}_\sigma\, a_{\rho\sigma} dp_\sigma \qquad \text{(ii)}.$$

79. **Problem 2c.** To express the increments dp_ρ of the coordinates, owing to a displacement ds, in terms of the components $d\bar{p}_\rho$ of the displacement along the coordinates p_ρ.

Using the equation § 78 (ii), along with the notation of § 64, we get immediately

$$dp_\rho = \sum_1^r {}_\sigma\, b_{\rho\sigma} d\bar{p}_\sigma.$$

80. **Problem 3a.** To express the components $d\bar{p}_\rho$ of a displacement along the general coordinates p_ρ in terms of the components $d\bar{x}_\nu$ of the displacement along the rectangular coordinates of the system.

We obtain successively by use of §§ 78, 57 (iii), 57 (ii), and 74,

$$d\bar{p}_\rho = \sum_1^r {}_\sigma\, a_{\rho\sigma} dp_\sigma = \sum_1^r {}_\sigma \sum_1^{3n} {}_\nu \frac{m_\nu}{m} \alpha_{\nu\rho} \alpha_{\nu\sigma} dp_\sigma$$

$$= \sum_1^{3n} {}_\nu \frac{m_\nu}{m} \alpha_{\nu\rho} dx_\nu = \sum_1^{3n} {}_\nu\, \alpha_{\nu\rho} d\bar{x}_\nu.$$

81. **Problem 3b.** To express the components $d\bar{x}_\nu$ of a displacement along the rectangular coordinates x_ν in terms of the components $d\bar{p}_\rho$ of the displacement along the general coordinates p_ρ of the system.

We obtain successively by means of §§ 73, 57 (ii), and 79,

$$d\bar{x}_\nu = \frac{m_\nu}{m} dx_\nu = \frac{m_\nu}{m} \sum_1^r {}_\sigma\, \alpha_{\nu\sigma} dp_\sigma$$

$$= \frac{m_\nu}{m} \sum_1^r {}_\sigma\, \alpha_{\nu\sigma} \sum_1^r {}_\rho\, b_{\rho\sigma} d\bar{p}_\rho,$$

thus, writing for shortness

$$\frac{m_\nu}{m}\sum_1^r{}_\sigma a_{\nu\sigma} b_{\rho\sigma} = \beta_{\nu\rho} \qquad \text{(i)},$$

we obtain

$$d\bar{x}_\nu = \sum_1^r{}_\rho \beta_{\nu\rho} d\bar{p}_\rho \qquad \text{(ii)}.$$

82. **Problem 4.** To express the length of an infinitely small displacement in terms of its reduced components along the coordinates of the system.

If we employ the general coordinates p_ρ, we obtain by successive use of §§ 78 (ii) and 79 with the equation § 57 (iv)

$$\begin{aligned} ds^2 &= \sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\sigma} dp_\rho dp_\sigma \\ &= \sum_1^r{}_\rho dp_\rho d\bar{p}_\rho = \sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} d\bar{p}_\rho d\bar{p}_\sigma . \end{aligned}$$

83. If we employ rectangular coordinates these equations take the form

$$\begin{aligned} ds^2 &= \sum_1^{3n}{}_\nu \frac{m_\nu}{m} dx_\nu^2 \\ &= \sum_1^{3n}{}_\nu dx_\nu d\bar{x}_\nu = \sum_1^{3n}{}_\nu \frac{m}{m_\nu} d\bar{x}_\nu^2 . \end{aligned}$$

84. **Problem 5a.** To express the angle between two infinitely small displacements from any position in terms of the reduced components of both displacements along the rectangular coordinates.

By successive use of § 73 and § 74 in the equation § 56 we obtain the forms

$$\begin{aligned} ds\,ds' \cos \widehat{ss'} &= \sum_1^{3n}{}_\nu \frac{m_\nu}{m} dx_\nu dx_\nu' \\ &= \sum_1^{3n}{}_\nu dx_\nu d\bar{x}_\nu' = \sum_1^{3n}{}_\nu d\bar{x}_\nu dx_\nu' = \sum_1^{3n}{}_\nu \frac{m}{m_\nu} d\bar{x}_\nu d\bar{x}_\nu' . \end{aligned}$$

In these we must substitute for ds and ds' their values in terms of $d\bar{x}_\nu$ given in § 83.

85. **Problem 5b.** To express the angle between two

infinitely small displacements from the same position in terms of the components of the two displacements along the general coordinates p_ρ.

By successive use of §§ 78 and 79 in the equation § 58 we obtain the forms

$$dsds' \cos \widehat{ss'} = \sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\sigma} dp_\rho dp_\sigma'$$

$$= \sum_1^r{}_\rho dp_\rho d\bar{p}_\rho' = \sum_1^r{}_\rho d\bar{p}_\rho dp_\rho' = \sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} d\bar{p}_\rho d\bar{p}_\sigma'.$$

Here again we must substitute for ds and ds' their values in terms of $d\bar{p}_\rho$ given in § 82.

86. **Problem 6.** To express the angle between two infinitely small displacements in terms of the angles which both make with the coordinates of the system.

Divide the last of equations § 85 by $dsds'$ and remember that by § 78 (i)

$$\sqrt{a_{\rho\rho}} \cos \widehat{sp_\rho} = \frac{d\bar{p}_\rho}{ds}, \quad \sqrt{a_{\rho\rho}} \cos \widehat{s'p_\rho} = \frac{d\bar{p}_\rho'}{ds},$$

we then obtain

$$\cos \widehat{ss'} = \sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} \sqrt{a_{\rho\rho} a_{\sigma\sigma}} \cos \widehat{sp_\rho} \cos \widehat{s'p_\sigma}.$$

87. When we employ rectangular coordinates the foregoing equation takes the form

$$\cos \widehat{ss'} = \sum_1^{3n}{}_\nu \cos \widehat{sx_\nu} \cos \widehat{s'x_\nu}.$$

It is to be noticed that the equation § 86 assumes the same position for the two displacements, whereas the equation of § 87 is free from this assumption.

88. **Proposition.** The r angles which any direction in a definite position makes with the r general coordinates are connected by the equation

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} \sqrt{a_{\rho\rho} a_{\sigma\sigma}} \cos \widehat{sp_\rho} \cos \widehat{sp_\sigma} = 1\,;$$

for this equation follows when in § 86 the directions of ds and ds' are made to coincide.

89. **Corollary.** In particular the $3n$ angles which any displacement of the system makes with the rectangular coordinates of the system satisfy the equation

$$\sum_{1}^{3n}{}_{\nu} \cos^2 \widehat{sx_\nu} = 1.$$

Use of Partial Differential Coefficients

90. **Notation.** The length ds of an infinitely small displacement is determined by the values of the coordinates p_ρ of its position and their changes dp_ρ. If we change one of these constituent elements, whilst the rest remain constant, the resulting partial differential of ds will be denoted by $\partial_p ds$.

If we consider, as we may, the coordinates p_ρ and the components $\overline{dp}_\rho$ along them as the independent constituent elements of ds, then the resulting partial differential of ds will be denoted by $\partial_q ds$.

Other partial differentials of ds are of course possible, but it is not necessary for our purpose to specify them. The symbol ∂ds, which is usually used for them, will be retained, and will be more particularly defined on each occasion in words.

91. **Note 1.** The components of a displacement along the coordinates can be expressed as partial differential coefficients of the length of the displacement. Thus, by differentiating the equation § 57 (iv), and making use of § 78, we get

$$\overline{dp}_\rho = \tfrac{1}{2}\frac{\partial_p ds^2}{\partial dp_\rho} = ds\frac{\partial_p ds}{\partial dp_\rho}.$$

92. **Note 2.** The inclination of an infinitely small displacement to the coordinate p_ρ can be expressed by means of the partial differential coefficients of its length. Thus, using §§ 91 and 78,

$$\sqrt{a_{\rho\rho}} \cos \widehat{sp_\rho} = \frac{\partial_p ds}{\partial dp_\rho}.$$

93. **Observation.** In particular, if in §§ 91 and 92 we use rectangular coordinates we obtain

$$d\bar{x}_\nu = ds\frac{\partial ds}{\partial dx_\nu} \qquad \text{(i)},$$

$$\sqrt{\frac{m_\nu}{m}} \cos \widehat{sx_\nu} = \frac{\partial ds}{\partial dx_\nu} \qquad \text{(ii)},$$

where the meaning of the partial differentials is clear from what precedes.

94. **Note 3.** The changes which the coordinates p_ρ suffer in an infinitely small displacement can be expressed as partial differential coefficients of the length of the displacement. Thus, using the equations in §§ 82 and 79,

$$d\bar{p}_\rho = \tfrac{1}{2}\frac{\partial_q ds^2}{\partial dp_\rho} = ds\frac{\partial_q ds}{\partial dp_\rho}.$$

95. **Note 4.** For all values of the index τ the following relation exists between the partial differential coefficients of ds—

$$\frac{\partial_p ds}{\partial p_\tau} = -\frac{\partial_q ds}{\partial p_\tau} \qquad \text{(i)}.$$

For

$$\frac{\partial_p ds}{\partial p_\tau} = \frac{1}{2ds}\sum_1^r{}_\rho\sum_1^r{}_\sigma\frac{\partial a_{\rho\sigma}}{\partial p_\tau}dp_\rho dp_\sigma,$$

and

$$\frac{\partial_q ds}{\partial p_\tau} = \frac{1}{2ds}\sum^r{}_\rho\sum_1^r{}_\sigma\frac{\partial b_{\rho\sigma}}{\partial p_\tau}d\bar{p}_\rho d\bar{p}_\sigma.$$

If we put in the first form for dp_ρ and dp_σ their values in terms of $d\bar{p}_\rho$ and $d\bar{p}_\sigma$ given in § 79, and make use of the relations in § 68 and the second form, the proof follows. In a similar manner we may proceed with the second form.

96. **Proposition.** If the position of an infinitely small displacement suffers two such changes, whereby the first time the components along the coordinates, and the second time the changes of the coordinates retain their original value, then the changes in the length of the displacement in both cases are equal, but of opposite signs.

For in the second case $\delta dp_\rho = 0$, whilst the coordinates p_ρ suffer the changes δp_ρ, and thus the change in the length of the displacement is given by

$$\delta_p ds = \sum_1^r{}_\tau \frac{\partial_p ds}{\partial p_\tau} \delta p_\tau \qquad \text{(i).}$$

In the first case $\delta d\bar{p}_\rho = 0$, whilst the coordinates suffer the same changes δp_ρ so that

$$\delta_q ds = \sum_1^r{}_\tau \frac{\partial_q ds}{\partial p_\tau} \delta p_\tau \qquad \text{(ii).}$$

From the equations (i) and (ii) and the equation § 95 (i) we get

$$\delta_p ds = -\delta_q ds.$$

Paths of Systems

Explanations

97. The aggregate of positions which a system occupies in its passage from one position to another is called a path of the system.

A path may also be considered as the aggregate of displacements which a system undergoes in its passage from one position to another.

98. A portion of the path which is limited by two infinitely near positions is called an element of the path. Such an element is an infinitely small displacement; it has both length and direction.

99. The direction of the path of a system in a given position is the direction of one of the elements of the path infinitely near that position.

The length of the path of a system between two of its positions is the sum of the lengths of the elements of the path between these positions.

100. **Analytical Representation.**—The path of a system is represented analytically when the coordinates of its positions

are given as functions of any one chosen variable. With every position of the path is a value of the variable associated. One of the coordinates themselves may serve as independent variable. It is frequently convenient to choose as independent variable the length of the path, measured from a given position of the path. The differential coefficients with regard to this chosen variable, and therefore with regard to the length of path, will be denoted in the manner of Lagrange by accents.

101. **Definition 1.** The path of a system is said to be straight when it has the same direction in all its positions.

102. **Corollary.** If a system describes a straight path, then its individual points describe straight lines, whose lengths measured from their starting-point are always proportional to one another (§ 38).

103. **Definition 2.** The path of a system is said to be curved when the direction of the path changes as we pass from one position to another. The rate of change of the direction with regard to the length of the path is called the curvature of the path.

The curvature of the path is therefore the limiting value of the ratio of the angle between two neighbouring elements to their distance.

104. **Observation.** The value of the curvature is therefore defined independently of the form of the analytical representation; hence, in particular, it is independent of the choice of the coordinates of the system.

105. **Problem 1.** To express the curvature c of the path in terms of the changes of the angles which the path makes with the rectangular coordinates of the system.

Let $d\epsilon$ be the angle between the direction of the path at the beginning and end of the path-element ds. Then by definition (§ 103)

$$c = \frac{d\epsilon}{ds}.$$

Let, further, $\cos \widehat{sx_\nu}$ be the cosine of the angle which the path makes with x_ν at the beginning of ds; and let

$\cos \hat{sx_\nu} + d \cos \hat{sx_\nu}$ be the value of the same quantity at the end of ds. Then, by equation § 87,

$$\cos (d\epsilon) = \sum_1^{3n}{}_\nu \cos \hat{sx_\nu} (\cos \hat{sx_\nu} + d \cos \hat{sx_\nu}).$$

Further, by equation 89,

$$\sum_1^{3n}{}_\nu \cos^2 \hat{sx_\nu} = 1,$$

and

$$\sum_1^{3n}{}_\nu (\cos \hat{sx_\nu} + d \cos \hat{sx_\nu})^2 = 1.$$

If, then, we subtract twice the first equation from the sum of the last two we obtain

$$2 - 2 \cos (d\epsilon) = d\epsilon^2 = \sum_1^{3n}{}_\nu (d \cos \hat{sx_\nu})^2,$$

and on dividing by ds^2

$$c^2 = \sum_1^{3n}{}_\nu \left(\frac{d \cos \hat{sx_\nu}}{ds} \right)^2.$$

106. **Problem 2.** To express the curvature of the path in terms of the changes of the rectangular coordinates of the system with respect to the length of the path.

From § 72 we have (§ 100)

$$\cos \hat{sx_\nu} = \sqrt{\frac{m_\nu}{m}} x_\nu',$$

and

$$(\cos \hat{sx})' = \sqrt{\frac{m_\nu}{m}} x_\nu''.$$

Hence by § 105 the solution of the problem is

$$mc^2 = \sum_1^{3n}{}_\nu m_\nu x_\nu''^2.$$

107. **Problem 3.** To express the curvature of the path in terms of the changes in the rectangular coordinates, themselves considered as functions of any variable τ.

According to the rules of the differential calculus

$$x_\nu'' = \frac{d}{ds}\left(\frac{dx_\nu}{d\tau}\cdot\frac{d\tau}{ds}\right) = \left(\frac{d\tau}{ds}\right)^3 \left\{ \frac{ds}{d\tau}\cdot\frac{d^2x_\nu}{d\tau^2} - \frac{dx_\nu}{d\tau}\cdot\frac{d^2s}{d\tau^2} \right\}.$$

If we substitute this expression in c^2 and remember (§ 55) that

$$m\left(\frac{ds}{d\tau}\right)^2 = \sum_1^{3n}{}_\nu\, m_\nu\left(\frac{dx_\nu}{d\tau}\right)^2 \qquad \text{(i)},$$

and

$$m\frac{ds}{d\tau}\cdot\frac{d^2s}{d\tau^2} = \sum_1^{3n}{}_\nu\, m_\nu\,\frac{dx_\nu}{d\tau}\cdot\frac{d^2x_\nu}{d\tau^2} \qquad \text{(ii)},$$

we obtain

$$m\left(\frac{ds}{d\tau}\right)^4 c^2 = \sum_1^{3n}{}_\nu\, m_\nu\left(\frac{d^2x_\nu}{d\tau^2}\right)^2 - m\left(\frac{d^2s}{d\tau^2}\right)^2,$$

where for $ds/d\tau$ and $d^2s/d\tau^2$ their values determined by the foregoing equations are to be substituted.

108. **Problem 4.** To express the curvature of the path in terms of the changes in the general coordinates p_ρ of the system with regard to the length of the path.

Substitute in the expression § 106 instead of the rectangular coordinates, p_ρ, supposing x_ν'' expressed in terms of p_ρ' and p_ρ''.

Thus, by § 57 (ii),

$$x_\nu' = \sum_1^r{}_\rho\, a_{\nu\rho}p_\rho',$$

and hence

$$x_\nu'' = \sum_1^r{}_\rho\,(a_{\nu\rho}p_\rho'' + a_{\nu\rho}'p_\rho');$$

therefore

$$x_\nu''^2 = \sum_1^r{}_\rho\sum_1^r{}_\sigma\,(a_{\nu\rho}a_{\nu\sigma}p_\rho''p_\sigma'' + 2a_{\nu\rho}'a_{\nu\sigma}p_\rho'p_\sigma'' + a_{\nu\rho}'a_{\nu\sigma}'p_\nu'p_\sigma').$$

If we form these equations for all values of ν and multiply each of them by $\frac{m_\nu}{m}$ and then add, the left-hand side becomes c^2. The summation on the right with regard to ν can be obtained by aid of the quantities $a_{\rho\sigma}$ for the first two terms. For the first term we get immediately by § 57 (iii) $a_{\rho\sigma}$. For the coefficient of p_σ'' in the second term we have

$$2\sum_1^r{}_\rho p_\rho'\sum_1^{3n}{}_\nu \frac{m_\nu}{m} a_{\nu\sigma} a_{\nu\rho}' = 2\sum_1^r{}_\rho\sum_1^r{}_\tau p_\rho' p_\tau' \sum_1^{3n}{}_\nu \frac{m_\nu}{m} a_{\nu\sigma}\frac{da_{\nu\rho}}{dp_\tau}$$

$$= \sum_1^r{}_\rho\sum_1^r{}_\tau p_\rho' p_\tau' \sum_1^{3n}{}_\nu \frac{m_\nu}{m} a_{\nu\sigma}\left(\frac{\partial a_{\rho\nu}}{\partial p_\tau} + \frac{\partial a_{\nu\tau}}{\partial p_\rho}\right)$$

$$= \sum_1^r{}_\rho\sum_1^r{}_\tau p_\rho' p_\tau' \left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} + \frac{\partial a_{\tau\sigma}}{\partial p_\rho} - \frac{\partial a_{\rho\tau}}{\partial p_\sigma}\right) \qquad \text{(by § 63)}$$

$$= \sum_1^r{}_\rho\sum_1^r{}_\tau p_\rho' p_\tau' \left(2\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{\partial a_{\rho\tau}}{\partial p_\sigma}\right).$$

In the transition from the second to the third form, and from the fourth to the fifth, use is made of the fact that, when $F(\rho,\sigma)$ is any expression involving ρ and σ, then

$$\sum_1^r{}_\rho\sum_1^r{}_\sigma F(\rho,\sigma) \equiv \sum_1^r{}_\rho\sum_1^r{}_\sigma F(\sigma,\rho).$$

The coefficient of the third term cannot be expressed in terms of $a_{\rho\sigma}$. In order to make the connection with the rectangular coordinates disappear from the final result, let us put

$$a_{\rho\sigma\lambda\mu} = \sum_1^{3n}{}_\nu \frac{m_\nu}{m}\frac{\partial a_{\nu\sigma}}{\partial p_\lambda}\cdot\frac{\partial a_{\nu\rho}}{\partial p_\mu}.$$

Then we obtain

$$c^2 = \sum_1^r{}_\rho\sum_1^r{}_\sigma \left\{ a_{\rho\sigma} p_\rho'' p_\sigma'' + \sum_1^r{}_\tau\left(2\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{\partial a_{\rho\tau}}{\partial p_\sigma}\right) p_\rho' p_\tau' p_\sigma'' \right.$$

$$\left. + \sum_1^r{}_\lambda\sum_1^r{}_\mu a_{\rho\sigma\lambda\mu} p_\rho' p_\sigma' p_\lambda' p_\mu' \right\}.$$

In these results the values of $a_{\rho\sigma}$ are given by means of § 57 as functions of p_ρ; the quantities $a_{\rho\sigma\lambda\mu}$ are to be regarded as newly introduced functions of the same quantities. The number of these newly introduced functions is equal to

$$\tfrac{1}{4}r^2(r+1)^2.$$

CHAPTER IV

POSSIBLE AND IMPOSSIBLE DISPLACEMENTS. MATERIAL SYSTEMS

Explanations

109. There exists a connection between a series of material points when from a knowledge of some of the components of the displacements of those points we are able to state something as to the remaining components.

110. When connections exist between the points of a system, some of the conceivable displacements of the system are excluded from consideration, namely, those displacements of the system whose occurrence would contradict the statements above referred to. Conversely, every statement that some of the conceivable displacements of the system are excluded from consideration, implies a connection between the points of the system. The connections between the points of a system are completely given when for every conceivable displacement of the system it is known whether it is, or is not, excluded from our consideration.

111. Those displacements which are not excluded from our consideration are called possible ones, the others impossible ones. Possible displacements are also called virtual. They are always called possible displacements when as a narrower idea they are contrasted with conceivable displacements; they are only called virtual when as a broader idea they are contrasted with a narrower one, *e.g.* the case of actual displacements.

112. Possible paths are those paths which are composed of possible displacements. Possible positions are all those positions which can be reached *via* possible paths.

113. Thus all positions of possible paths are possible positions. But it is not to be understood that any conceivable path whatever through possible positions is also a possible path. On the contrary, a displacement between infinitely neighbouring possible positions may be an impossible displacement.

114. Between two possible positions there is always one possible path. For if from any one actual position even a single possible path can be drawn to each of the two positions, then these two paths must together form one possible path between the two positions; if no possible path could be drawn to one of the two, then would this position not be a possible position.

115. **Definition 1.** A connection of a system is said to be a continuous one when it is not inconsistent with the three following assumptions:—

1. That the knowledge of all possible finite displacements should be included in the knowledge of all possible infinitely small displacements.

2. That every possible infinitely small displacement can be traversed in a straight, continuous path.

3. That every infinitely small displacement, which is possible from a given position, is also possible from any infinitely neighbouring position, except for variations of the order of the distance between the positions or of a higher order.

116. **Corollary.** When only continuous connections exist in a system, the sum of any possible infinitely small displacements whatever from the same position is itself a possible displacement from the same position. (Superposition of infinitely small displacements.)

For, according to § 115 (3), the individual displacements may be performed successively, and consequently, by § 115 (2), the direct displacement from the initial position to the final one is itself a possible displacement.

117. **Definition 2.** A connection of a system is said to

be an internal one when it only affects the mutual position of the points of the system.

118. **Corollary.** When in a system only internal connections exist, every displacement of the system which does not alter the configuration is a possible displacement, and conversely.

119. **Definition 3.** A connection of a system is said to be normal (*gesetzmässiger*) when it exists independently of the time.

A normal connection is therefore implied in the statement that of the conceivable displacements of a system some are possible, others not, and this at all times or independently of the time.

120. **Observation.** So long as we treat solely of the geometry of systems, the difference between normal and abnormal connections does not appear, for in this case our investigations are not affected by the time. If the connections of a system are different at two different times, then for the present we must consider that we are dealing with two different systems. It will practically amount to the same thing if we assume that in this first book all the connections are normal.

121. **Definition 1.** A system of material points which is subject to no other than continuous connections is called a material system.

122. **Definition 2.** A material system which is subject to no other than internal and normal connections is called a free system.

123. **Definition 3.** A material system between whose possible positions all conceivable continuous motions are also possible motions is called a holonomous system.

The term means that such a system obeys integral (ὅλος) laws (νόμος), whereas material systems in general obey only differential conditions. (*Cf.* § 132 *infra.*)

Analytical Representation

124. **Note.** A system of material points satisfies the conditions of a material system when the differentials of its

rectangular coordinates are subject to no other conditions than a series of homogeneous linear equations whose coefficients are continuous functions of possible values of the coordinates.

For the first kind of continuity which Definition 115 requires must be presupposed, when mention is made of the differentials of the coordinates of the system; the other two kinds of continuity are satisfied by the restriction of the differentials employed.

125. **Converse.** If a system of material points satisfies the conditions of a material system, then the differentials of its rectangular coordinates are subject to no other limitations than to a series of homogeneous linear equations, whose coefficients are continuous functions of possible values of the coordinates.

To prove this let us take a possible position of the system, and the possible displacements from it. For a given displacement the $3n$ increments dx_ν may be supposed to have to one another the ratios

$$\epsilon_{11} : \epsilon_{12} \ . \ . \ . : \epsilon_{13n}.$$

If now we consider du_1 as any infinitely small quantity whatever, then by means of the set of equations

$$dx_\nu = \epsilon_{1\nu} du_1$$

a set of possible displacements is given. Now either all possible displacements are contained in these, or this is not the case. If not, then we must take a second displacement, which cannot be represented in this form, and for this the $3n$ increments dx_ν may bear to one another the ratios

$$\epsilon_{21} : \epsilon_{22} \ . \ . \ . : \epsilon_{23n}.$$

Then taking any second infinitely small quantity du_2, by means of the set of equations

$$dx_\nu = \epsilon_{1\nu} du_1 + \epsilon_{2\nu} du_2,$$

by § 116 a more general set of possible displacements is given.

Now either all possible displacements are contained in this set, or not. If not, we must choose another such quantity du_3, and continue the process until, on account of the exhaustion of all possible displacements, it is not possible to continue it further. Its continuance becomes impossible when we have taken $3n$ such quantities du_λ; and then the expression

$$dx_\nu = \sum_1^{3n}{}_\lambda\, \epsilon_{\lambda\nu} du_\lambda \qquad \text{(i)}$$

represents all possible displacements of the system, when all conceivable displacements are possible ones, and thus when no connections exist between the points of a system. In general the process must come to an end earlier, and all possible displacements may therefore be expressed by equations of condition of the form

$$dx_\nu = \sum_1^{l}{}_\lambda\, \epsilon_{\lambda\nu} du_\lambda,$$

where under all circumstances

$$l \overline{\overline{<}} 3n.$$

In order that this form may be satisfied by arbitrarily chosen values of dx_ν, it is sufficient that the dx_ν's should satisfy the $3n-l$ linear homogeneous equations which result from the elimination of du_λ from the equations (i). The quantities $\epsilon_{\lambda\nu}$ must, according to § 115 (3), be continuous functions of the position. However (by § 124), the increments dx_ν are not to be subject to further limitations than these.

126. **Observation.** The number and the content of the equations which we obtain between dx_ν by the foregoing process, are independent of the particular choice of the displacements.

For if we take other displacements and express dx_ν in terms of other quantities dv_λ, then we can substitute the values of dx_ν in the equations which we have already obtained by elimination. If these are not identically satisfied then the quantities dv_λ would not be independent, which would be contrary to the assumption under which they are chosen. Thus these equations are identically satisfied, and cannot therefore be different from the equations or linear combinations of the equations which were obtained by elimination of the quantities dv_λ in terms of which the increments dx_ν are expressed. The number of equations obtained by means of du_λ can not be greater than the number obtained by means of dv_λ: neither can it be less; for then the converse process would show that the quantities du_λ would not be independent of one another.

127. **Corollary 1.** The connection of a material system can be completely expressed analytically by stating a single possible position of the system and a set of homogeneous linear equations between the differentials of its rectangular coordinates.

For relations between these differentials cannot by § 125 be given in any other manner than by such a set of equations. This does not exclude the existence of finite equations between the coordinates. However, all such finite equations can be completely replaced by means of a single possible position, and just as many homogeneous linear equations between the differentials. These last, however, must not be inconsistent with the given differential equations; they must either reduce to them, or must be associated with them in a complete representation.

128. **Notation.** The equations which represent the connection of a material system, in terms of its rectangular coordinates, will in future always be expressed in the following form

$$\sum_{1}^{3n} {}^{\nu}\, x_{\iota\nu} dx_\nu = 0.$$

It is to be understood that i such equations exist, and that the ι's have values from 1 to i. The quantities $x_{\iota\nu}$ are to be considered continuous functions of x_ν.

129. **Corollary 2.** The connection of a material system whose positions are expressed in terms of general coordinates can also be completely expressed analytically by stating a single possible position and a set of homogeneous linear equations between the differentials of the coordinates.

Using the general coordinates p_ρ whose number r is less than $3n$, a connection between the points of the system is *ipso facto* in existence. First suppose the connection to be completely expressed by the rectangular coordinates according to § 128. In the corresponding differential equations let the values of dx_ν be substituted in terms of dp_ρ by means of equation § 57 (ii). The resulting linear homogeneous equations can be so arranged that $3n - r$ of them are identically satisfied in consequence of the $3n-r$ equations which express the fact that the $3n$ quantities x_ν are functions of the r quantities p_ρ. The remaining $k=i-3n+r$ equations between dp_ρ give completely all the equations between dx_ν, and therefore (§ 127), with a knowledge of one possible position, are sufficient to describe completely the connection of the system.

130. **Notation.** The equations which express the connection of a material system in the general coordinates p_ρ, will in future always be written in the form

$$\sum_1^r {}_\rho \, p_{\chi\rho} dp_\rho = 0.$$

They will be taken as k in number, and all values from 1 to k are to be given to χ. The quantities $p_{\chi\rho}$ are to be considered continuous functions of p_ρ.

131. **Observation.** The equations § 128 or § 130 are called the differential equations, or the equations of condition of the system.

132. **Proposition.** When from the differential equations of a material system an equal number of finite equations between the coordinates of the system can be deduced, the system is a holonomous system (§ 123).

For the coordinates of every possible position must satisfy the finite equations. The differences between the coordinates of two neighbouring positions satisfy consequently an equal number of homogeneous linear differential equations, and since these must not be inconsistent with the equal number of the differential equations of the system, they must satisfy these also. The displacement between any two possible positions is consequently a possible displacement, whence the assertion follows.

133. **Converse.** If a material system is holonomous, then its differential equations admit an equal number of finite or integral equations between the coordinates.

For let us take from the r coordinates of the system, between whose differentials the k equations exist, any, say the first $r-k$ as independent variables, and pass from any initial position of the system along different possible paths to a position for which the independent coordinates have given values. Now if with a continually changing path one arrived at continuously changing values of the remaining coordinates, consequently at different positions, these positions would be possible positions, and therefore the displacements between them would by hypothesis be possible displacements. There would then be a value-system of the differentials, different from zero, which would satisfy the differential equations, even when the first $r-k$ are put zero. This is not possible, for the equations are homogeneous and linear. Thus we must always arrive at the same values not only of the first $r-k$, but also of the remaining coordinates. The latter are consequently definite functions of the former. The k finite equations which express this are, since they cannot be inconsistent with the differential equations, integral equations of these latter.

Freedom of Motion

134. **Definition.** The number of infinitely small changes of the coordinates of a system that can be taken arbitrarily is called the number of free motions of the system, or the degree of freedom of its motion.

135. **Note 1.** The number of free motions of a system is equal to the number of its coordinates, diminished by the number of the differential equations of the system.

136. **Note 2.** The number of free motions of a material system is independent of the choice of the coordinates.

In the notation of §§ 128-130 the number of degrees of freedom is equal to $r-k$, or, by § 129, to $3n-i$, and is therefore always the same number, whatever numbers r and k may represent.

137. **Note 3.** The number of degrees of freedom of a system does not change with the position of the system.

For the connection being a continuous one, the number of degrees of freedom cannot differ by a finite quantity in neighbouring positions; thus, since a continuous change in this number is excluded, it does not change in finitely distant positions.

138. **Note 4.** The proof of the set of equations in § 125 furnishes a solution of the problem—To find, but not without trial, the number of degrees of freedom of a completely known material system. The number l of the auxiliary quantities du_λ found according to the method of that proof is the required number.

It is known that the possible positions of the system can be represented by means of r general coordinates p_ρ, and so in that proof these coordinates can be used instead of x_ν.

139. **Definition.** A coordinate of a material system whose changes can take place independently of the changes of the remainder of the coordinates is called a free coordinate of the system.

140. **Corollary.** A free coordinate does not appear in the differential equations of its system, and conversely every coordinate which does not appear in the differential equations of the system is a free coordinate.

141. **Observation 1.** Whether a given coordinate is a free coordinate or not depends on the choice of the remaining coordinates simultaneously employed.

For if a certain coordinate does not appear in the differential equations of the system, and we choose instead of one of those coordinates which do appear in the differential equations, a function of this and the first one as coordinate, then the first one loses its property of being a free coordinate, a property which it possessed until then.

142. **Observation 2.** In a free system every coordinate of absolute position is a free coordinate. See §§ 118 and 122.

143. **Proposition.** When the possible positions of a material system can be represented by means of coordinates which are all free, then the system is holonomous (§ 123).

For every displacement of the system between possible positions is expressed in terms of a value-system of the differentials of the free coordinates; every such value-system is possible since it is subject to no conditions, and therefore every displacement between possible positions is a possible displacement.

144. **Converse.** In a holonomous system all possible positions can be expressed in terms of free coordinates.

If a holonomous system has r coordinates, between which k differential equations exist, then k of the coordinates can be expressed as functions of the remaining $(r-k)$. (See § 133.) Hence these $r-k$ arbitrarily chosen coordinates determine completely the position of the system, and can by omission of the remaining coordinates be taken as free coordinates of the system. Also any $(r-k)$ functions of the original r coordinates may serve a similar purpose.

145. **Observation 1.** The number of free coordinates of a holonomous system is equal to the number of its degrees of freedom.

146. **Observation 2.** If the number of coordinates of a material system is equal to the number of its degrees of freedom, then all the coordinates are free coordinates, and the system is holonomous.

For should even a single differential equation between the coordinates exist, then the number of coordinates of the system would be greater than the number of degrees of freedom. The number of coordinates can not be less than the number of degrees of freedom.

147. **Observation 3.** The possible positions of a system, which is not holonomous, can not be fully represented by means of free coordinates alone.

For the opposite of this statement would be contrary to § 143.

Displacements Perpendicular to Possible Displacements

148. **Proposition.** If the r components $d\bar{p}_\rho$ of a displacement ds of a system along the coordinates p_ρ can be expressed by means of k quantities γ_χ in the form

$$d\bar{p}_\rho = \sum_1^k {}_\chi \cdot p_{\chi\rho}\gamma_\chi,$$

where the $p_{\chi\rho}$'s are taken from the equations of condition of the system (§ 130), then the displacement is perpendicular to every possible displacement of the system from the same position.

Let ds' be the length of any possible displacement from the same position, and let dp_ρ' denote the changes of the coordinates owing to this displacement. If now we multiply the equations of the series, each with dp_ρ' and add them, then using equations § 85 and § 130

$$\sum_1^r{}_\rho\, d\bar{p}_\rho dp_\rho' = ds ds' \cos \widehat{ss'} = \sum_1^k{}_\chi\, \gamma_\chi \sum_1^r{}_\rho\, p_{\chi\rho} dp_\rho' = 0 ;$$

thus $\cos \widehat{ss'} = 0$; and $\widehat{ss'} = 90°$ as was to be proved.

149. **Additional Note.** The r components $d\bar{p}_\rho$ of a displacement ds along the coordinates p_ρ are singly determined when we know k of them, and know also that the displacement is perpendicular to every possible displacement of the system.

Let dp_ρ' be again the changes of p_ρ for any possible displacement. By means of the k equations of condition we can represent k of them as homogeneous linear functions of the remaining $(r-k)$, and then substitute these values in the equation

$$\sum_1^r{}_\rho\, d\bar{p}_\rho dp_\rho' = 0.$$

The dp_ρ''s appearing in this equation are now completely arbitrary, and thus the coefficient of each one of them must vanish. This gives $(r-k)$ homogeneous linear equations between $d\bar{p}_\rho$ which permit us to express $(r-k)$ of them as single-valued linear functions of the remaining k.

150. **Converse.** If a conceivable displacement is perpendicular to every possible displacement of a system, then its r components $d\bar{p}_\rho$ along p_ρ can always, by suitable choice of the k quantities γ_χ, be expressed in the form

$$d\bar{p}_\rho = \sum_1^k{}_\chi\, p_{\chi\rho}\gamma_\chi.$$

For if we determine the γ_χ's by means of k of these equations and calculate by means of these values all the components, we must obtain the given values of $d\bar{p}_\rho$. For the displacement so obtained is by § 148 perpendicular to all possible displacements, and has with the given displacement k components common. It has thus by § 149 all the r components along p_ρ common with the same.

CHAPTER V

SPECIAL PATHS OF MATERIAL SYSTEMS

1. Straightest Paths

151. **Definition 1.** An element of a path of a material system is said to be straighter than any other when it has a smaller curvature.

152. **Definition 2.** The straightest element is defined as a possible element, which is straighter than all other possible ones which have the same position and direction.

153. **Definition 3.** A path, all of whose elements are straightest elements, is called a straightest path.

154. **Analytical Representation.** All elements of a path of which one straightest element is the straightest, have the same position and direction; hence the values of their coordinates, and the first differentials of these coordinates with regard to the independent variables, are equal. The curvature, however, is determined not by means of these values alone, but also by means of the second differential coefficients of the coordinates. By the values of these the elements are distinguished, and for the straightest element the second differential coefficients must be such functions of the coordinates and of their first differential coefficients as make the curvature a minimum.

The equations which express this condition must be satisfied for all positions of a straightest path, and they are thus the differential equations of such a path.

155. **Problem 1.** To express the differential equations of the straightest paths of a material system in terms of the rectangular coordinates of the system.

Let us choose as independent variable the length of the path. Since only possible paths are to be considered, the $3n$ quantities x_ν' according to §§ 128 and 100 are subject to i equations of the form

$$\sum_1^{3n}{}_\nu\, x_{\iota\nu} x_\nu' = 0 \qquad \text{(i).}$$

Thus the $3n$ quantities x_ν'' are subject to i equations of the form

$$\sum_1^{3n}{}_\nu\, x_{\iota\nu} x_\nu'' + \sum_1^{3n}{}_\nu \sum_1^{3n}{}_\mu \frac{\partial x_{1\nu}}{\partial x_\mu} x_\nu' x_\mu' = 0 \qquad \text{(ii),}$$

which follow from (i) by differentiation.

With the condition that these equations are not to be violated, the quantities x_ν'' will be determined so as to make the curvature c (§ 106), or what is the same thing, the value of $\frac{1}{2}c^2$, viz.,

$$\tfrac{1}{2}\sum_1^{3n}{}_\nu \frac{m_\nu}{m} x_\nu''^2 \qquad \text{(iii),}$$

a minimum.

According to the rules of the differential calculus, we proceed as follows :—

Multiply each of the equations (ii) by a factor to be determined later, which for the ι^{th} equation we may denote by Ξ_ι; add the partial differential coefficients on the left-hand side of the resulting equations arranged according to each of the quantities x_ν'' to the partial differential coefficients of (iii) (the quantity which is to be made a minimum) arranged according to the same quantities; then finally put the result equal to zero, and we get $3n$ equations of the form

$$\frac{m_\nu}{m} x_\nu'' + \sum_1^{i}{}_\iota\, x_{\iota\nu} \Xi_\iota = 0 \qquad \text{(iv);}$$

which, together with the i equations (ii), give $3n+i$ linear but not homogeneous equations to determine the $3n+i$ quantities x_ν'' and Ξ_ι; and from these the values of these quantities can be found, and consequently the value of the least curvature. The satisfying of the equations (iv) at all positions of a possible path is thus a necessary condition that the path should be a straightest one, and the equations (iv) are therefore the required differential equations.

156. **Observation 1.** The equations (iv) are moreover the sufficient conditions for the occurrence of a minimum. For the second differential coefficients $\frac{\partial^2 c^2}{\partial x_\nu''\partial x_\mu''}$ vanish whenever ν and μ are different, and are necessarily positive when ν and μ are equal. The value of the curvature thus admits no other special value.

The satisfying of equations (iv) for all positions of a possible path is thus the sufficient condition for a straightest path.

157. **Observation 2.** By use of § 72 the equations (iv) can be written in the form

$$\sqrt{\frac{m_\nu}{m}}\frac{d}{ds}\left(\cos \widehat{sx_\nu}\right) = -\sum_1^i {}_\iota x_{\iota\nu}\Xi_\iota.$$

The equations (iv) therefore determine how the direction of a path must change from position to position in order that it may remain a straightest path; and moreover every single equation determines how the inclination of the path to a given rectangular coordinate changes.

158. **Problem 2.** To express the differential equations of the straightest paths of a material system in terms of the general coordinates of the system.

Choose again as independent variable the length of the path. The coordinates p_ρ and their differential coefficients p_ρ' satisfy (§ 130) the k equations

$$\sum_1^r {}_\rho p_{\chi\rho}p_\rho' = 0 \qquad \text{(i)},$$

thus the quantities p_ρ'' satisfy the equations

$$\sum_1^r {}_\rho p_{\chi\rho}p_\rho'' + \sum_1^r {}_\rho \sum_1^r {}_\sigma \frac{\partial p_{\chi\rho}}{\partial p_\sigma}p_\rho' p_\sigma' = 0 \qquad \text{(ii)}.$$

From all values of p_ρ'' which satisfy these equations those are to be determined which make the value of c or $\frac{1}{2}c^2$, that is, the right-hand side of the equation § 108 (iii), a minimum. We proceed according to the rules of the differential calculus, as in § 155, and take Π_χ for the factor, with which we multiply the χ^{th} of equations (ii), and we obtain the necessary conditions for the minimum as r equations of the form

$$\sum_1^r{}_\sigma\, a_{\rho\sigma}p_\sigma'' + \sum_1^r{}_\sigma\sum_1^r{}_\tau\left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho}\right)p_\sigma' p_\tau' + \sum_1{}_\chi\, p_{\chi\rho}\Pi_\chi = 0 \quad \text{(iii)},$$

where to ρ in each equation a definite value from 1 to r has to be given. These make together with the equations (ii) $(r+k)$ linear but not homogeneous equations for the $(r+k)$ quantities p_ρ'' and Π_χ, by which these quantities, and thus the least curvature, can be found by § 108. The satisfying of equations (iii) for all positions of a possible path is the necessary condition that the path should be a straightest path.

159. **Observation 1.** The satisfying of equations (iii) is also the sufficient condition for a minimum, and thus for a straightest path. For the result of § 108 is only a transformation of § 106 for the curvature, and like § 156 this value in § 158 only admits one special value, which is a minimum.

160. **Observation 2.** By § 75 we have

$$\sqrt{a_{\rho\rho}}\cos \widehat{sp_\rho} = \sum_1^r{}_\sigma\, a_{\rho\sigma}p_\sigma',$$

and therefore

$$\frac{d}{ds}\left(\sqrt{a_{\rho\rho}}\cos \widehat{sp_\rho}\right) = \sum_1^r{}_\sigma\, a_{\rho\sigma}p_\sigma'' + \sum_1^r{}_\sigma\sum_1^r{}_\tau\frac{\partial a_{\rho\sigma}}{\partial p_\tau}p_\sigma' p_\tau'.$$

Thus the equations 158 (iii) can be written in the form

$$\frac{d}{ds}\left(\sqrt{a_{\rho\rho}}\cos \widehat{sp_\rho}\right) = \frac{1}{2}\sum_1^r{}_\sigma\sum_1^r{}_\tau\frac{\partial a_{\sigma\tau}}{\partial p_\rho}p_\sigma' p_\tau' - \sum_1^k{}_\chi\, p_{\chi\rho}\Pi_\chi.$$

The equations (158) (iii) determine thus how the direction of the path must change from position to position in order that it may remain a straightest path; and moreover every single equation determines how the inclination to a given coordinate p_ρ changes.

161. **Proposition.** From a given position in a given direction there is always one and only one straightest path possible.

For when a position and a direction in it are given, the equations 155 (iv) and 158 (iv) always give definite, and moreover unique, values for the change of direction; thus by means of the given quantities the initial position, the direction at the next element of the path, and therefore at the successive positions right to the final position, are singly determined.

162. **Corollary.** It is in general not possible to draw a straightest path from any position of a given system to any other position.

For since the manifold of possible displacements from a position is equal to the number of free motions of the system, the manifold of possible directions in any position and therefore the manifold of straightest paths from it is smaller by unity. The manifold of positions which are to be reached by straightest paths from a given position is thus again equal to the number of free motions. But the manifold of possible positions may be equal to the number of coordinates used, and is therefore in general greater than the former.

163. **Note 1.** In order to be able to express all straightest paths of a material system whose positions are denoted in terms of p_ρ, by equations between p_ρ it is not necessary to know any $3n$ functions whatever which fully determine the position of the separate points of the system as functions of p_ρ. It is sufficient that, together with the equations of condition of the system in terms of p_ρ, the $\frac{1}{2}\, r\,(r+1)$ functions $a_{\rho\sigma}$ of p_ρ should be known.

For the differential equations of the straightest paths can be explicitly written down when together with the $p_{\chi\rho}$'s the $a_{\rho\sigma}$'s are given as functions of p_ρ.

164. **Note 2.** In order to be able to express the straightest paths of a material system whose positions are denoted in terms of p_ρ by equations between p_ρ, it is sufficient to know, together with the equations of condition between p_ρ, the length of every possible infinitely small displacement as a function of these coordinates p_ρ and their changes.

For if ds is the expression for this length in the desired form, then

$$a_{\rho\sigma} = \frac{\partial^2 ds^2}{\partial dp_\rho \partial dp_\sigma}.$$

165. **Note 3.** In order to know the value of the curvature itself in any position of a straightest path, it is not sufficient to know the $\frac{1}{2}r(r+1)$ functions $a_{\rho\sigma}$. We require in addition the $\frac{1}{4}r^2(r+1)^2$ functions $a_{\rho\sigma\lambda\mu}$ (§ 108).

The knowledge of the position of all the separate points as functions of p_ρ is not necessary for the determination of the curvature itself.

2. Shortest and Geodesic Paths

166. **Definition 1.** The shortest path of a material system between two of its positions is a possible path between these positions, whose length is less than the length of any of the other infinitely neighbouring paths between the same positions.

167. **Note 1.** The definition does not exclude the possibility, which may actually arise, of there being more than one shortest path between the two positions. The shortest of these shortest paths is the absolutely shortest path. It is at the same time the shortest path which is at all possible between the two positions.

168. **Note 2.** Between any two possible positions of a material system there is always at least one shortest path possible.

For possible paths always exist between the two positions (§ 114), and consequently there is amongst them an absolutely shortest path which is shorter than the neighbouring ones,—such as, according to §§ 121, 115, it must possess,—and is consequently a shortest path.

169. **Note 3.** A shortest path between two positions is at the same time a shortest path between any two of its intermediate positions. Every portion of a shortest path is itself a shortest path.

170. **Note 4.** The length of a shortest path differs only by an infinitely small quantity of a higher order from the lengths of all neighbouring paths between the same end posi-

tions. By infinitely small quantities of the first order are meant the lengths of the displacements necessary to pass from a neighbouring path to the shortest path.

171. **Definition 2.** A geodesic path of a material system is any path whose length between any two of its positions differs only by an infinitely small quantity of a higher order from the lengths of any of the infinitely neighbouring paths whatever between the same positions.

172. **Note 1.** Every shortest path between any two positions is a geodesic path.

Thus the definition § 171 does not involve anything in the nature of an inconsistency, for there are paths which satisfy this definition.

173. **Note 2.** There is always at least one geodesic path possible between any two possible positions of a material system (§§ 168, 172).

174. **Note 3.** A geodesic path is not necessarily at the same time a shortest path between any two of its positions.

It cannot be concluded from the definition that every geodesic path is also a shortest path, and simple examples show that there are in fact geodesic paths which are not also shortest paths between their end positions. Such examples may be taken from the geometry of the single material point, that is, from ordinary geometry, and thus be assumed known.

175. **Note 4.** When between two positions there is only one geodesic path, then this is also a shortest path, and moreover the absolutely shortest path between the two positions.

For the opposite would by §§ 168 and 172 be contrary to the hypothesis.

176. **Note 5.** A geodesic path is always a shortest path between any two sufficiently neighbouring but still finitely distant positions on it.

There may be between any two positions of the geodesic path under consideration a number of other geodesic paths. The absolutely shortest path between the two positions must coincide with one of these paths (§ 172). If we now make the positions approach one another along the geodesic path considered, then the length of this path as well as the length of the absolutely shortest path tends to zero, whilst the remaining geodesic paths remain finite. At least, from a certain finite distance of the positions onwards the geodesic path, along which the two positions approach each other, must coincide with the absolutely shortest path.

177. **Analytical Representation.**—In order that a path may be a geodesic path, it is necessary and sufficient that the integral of the path-elements, (§ 99) viz.,

$$\int ds,$$

taken between any two positions of the path should not vary when any continuous variations are given to the coordinates of the positions of the path, it being only supposed (1) that these variations should vanish at the limits of the integral, and (2) that after the variation the coordinates and their differentials should satisfy the equations of condition of the system. The necessary and sufficient conditions for this are a set of differential equations, which the coordinates of the path, considered as functions of any single variable, must satisfy, and which are consequently the differential equations of the geodesic paths.

178. That these differential equations should be satisfied for all points of a possible path is also by § 172 the necessary condition that the path should be a shortest path, and hence these equations are also the differential equations of the shortest paths. The vanishing of the variation of the integral is, however, not also a sufficient condition that the path should be a shortest path between its bounding positions. It is further necessary that for every admissible variation of the coordinates the second variation of the integral should have an essentially positive value. For sufficiently near positions of a path, which satisfies the differential equations, this condition is always satisfied by § 176 of itself.

179. **Problem 1.** To express the differential equations of the geodesic paths of a material system in terms of its rectangular coordinates.

The $3n$ rectangular coordinates x_ν which are regarded as functions of any variable, must both before and after the variation satisfy (§ 128) the i equations

$$\sum_1^{3n}{}^{\nu} x_{\iota\nu} dx_\nu = 0 \qquad \text{(i).}$$

The $3n$ variations δx_ν are therefore associated with the i equations which result from these after variation, viz.,

$$\sum_1^{3n}{}^{\nu} x_{\iota\nu} d\delta x_\nu + \sum_1^{3n}{}^{\mu} \sum_1^{3n}{}^{\nu} \frac{\partial x_{\iota\nu}}{\partial x_\mu} \delta x_\mu dx_\nu = 0 \qquad \text{(ii).}$$

As the length ds of an element of the path does not depend on x_ν, but only on dx_ν, then its variation is

$$\delta ds = \sum_1^{3n}{}^{\nu} \frac{\partial ds}{\partial dx_\nu} \delta dx_\nu = \sum_1^{3n}{}^{\nu} \frac{\partial ds}{\partial dx_\nu} d\delta x_\nu \qquad \text{(iii).}$$

This being understood,

$$\delta \int ds = \int \delta ds$$

must be made zero. According to the rules of the Calculus of Variations, we multiply each of the equations (ii) by a function of x_ν to be determined later, which for the ιth equation will be denoted by ξ_ι, and add the sum of the left-hand sides of the resulting equations, which sum is equal to zero, to the varied element of the integral. By partial integration we get rid of the differentials of the variations; finally we put the coefficients of each one of the arbitrary functions δx_ν equal to zero. We thus obtain $3n$ differential equations of the form

$$d\left(\frac{\partial ds}{\partial dx_\nu}\right) + \sum_1^{i}{}^{\iota} x_{\iota\nu} d\xi_\iota - \sum_1^{i}{}^{\iota} \sum_1^{3n}{}^{\mu} \left(\frac{\partial x_{\iota\mu}}{\partial x_\nu} - \frac{\partial x_{\iota\nu}}{\partial x_\mu}\right) \xi_\iota dx_\mu = 0 \qquad \text{(iv),}$$

which, together with the i equations (i), give $(3n+i)$ equations for the $(3n+i)$ functions x_ν and ξ_ι. These differential equations are necessary conditions for the vanishing of the variation of the integral; every geodesic path thus satisfies them, and consequently they represent the required solution.

180. **Observation 1.** The differential equations 179 (iv) are moreover the sufficient conditions that the path which satisfies them should be a geodesic path. For if these equations are satisfied, then the variation of the integral $\int ds$ becomes

equal to the series which results from partial integration under the integral sign; it thus becomes with the usual notation, the upper limit being denoted by 1 and the lower by 0,

$$\delta\int ds = \sum_1^{3n}{}_\nu \left[\left(\frac{\partial ds}{\partial dx_\nu} + \sum_1^{i}{}_\iota x_{\iota\nu}\xi_\iota\right)\delta x_\nu\right]_0^1.$$

If we make the variations δx_ν for any two positions of the path vanish, then the variation of the integral between these positions as limiting positions vanishes, and therefore the required sufficient analytical condition for a geodesic path is by § 177 satisfied.

181. **Observation 2.** Let us take the current length of the path as independent variable, then by use of §§ 55, 100, the equations 179 (iv), after division by ds, take the form

$$\frac{m_\nu}{m}x''_\nu + \sum_1^{i}{}_\iota x_{\iota\nu}\xi_\iota' - \sum_1^{i}{}_\iota\sum_1^{3n}{}_\mu\left(\frac{\partial x_{\iota\mu}}{\partial x_\nu} - \frac{\partial x_{\iota\nu}}{\partial x_\mu}\right)\xi_\iota x_\mu' = 0 \qquad \text{(i);}$$

which, together with the i equations resulting from differentiating 179 (i), viz.,

$$\sum_1^{3n}{}_\nu x_{\iota\nu}x''_\nu + \sum_1^{3n}{}_\nu\sum_1^{3n}{}_\mu\frac{\partial x_{\iota\nu}}{\partial x_\mu}x_\nu' x_\mu' = 0 \qquad \text{(ii),}$$

furnish $(3n+i)$ unhomogeneous, linear equations for the $(3n+i)$ quantities x_ν'' and ξ_ι', and thus permit these quantities to be expressed as single-valued functions of the quantities x_ν, x_ν', ξ_ι.

182. **Observation 3.** By use of § 72 the equations 181 (i) can be put in the form

$$\sqrt{\frac{m_\nu}{m}}\frac{d}{ds}(\cos \widehat{sx_\nu}) = -\sum_1^{i}{}_\iota x_{\iota\nu}\xi_\iota' + \sum_1^{i}{}_\iota\sum_1^{3n}{}_\mu\left(\frac{\partial x_{\iota\mu}}{\partial x_\nu} - \frac{\partial x_{\iota\nu}}{\partial x_{\iota\mu}}\right)\xi_\iota x_\mu'.$$

The equations 181 (i) thus express how the direction of the path must continually change from a given initial value in order that it may remain a geodesic path; and moreover every single equation expresses how the inclination to a given rectangular coordinate changes.

183. **Problem 2.** To express the differential equations of

the geodesic paths of a material system in terms of the general coordinates p_ρ.

The r coordinates p_ρ of the system are connected by the k equations

$$\sum_1^r {}_\rho p_{\chi\rho} dp_\rho = 0 \qquad \text{(i)},$$

and thus the r variations by the equations

$$\sum_1^r {}_\rho p_{\chi\rho} d\delta p_\rho + \sum_1^r {}_\rho \sum_1^r {}_\sigma \frac{\partial p_{\chi\rho}}{\partial p_\sigma} \delta p_\sigma dp_\rho = 0 \qquad \text{(ii)}.$$

Now the length ds of an infinitely small displacement depends not only on the differentials dp_ρ, but also on the values of p_ρ themselves, and thus

$$\delta ds = \sum_1^r {}_\rho \frac{\partial ds}{\partial dp_\rho} d\delta p_\rho + \sum_1^r {}_\rho \frac{\partial ds}{\partial p_\rho} \delta p_\rho.$$

This understood,

$$\delta \int ds = \int \delta ds \text{ must be made zero} \qquad \text{(iii)}.$$

Then we proceed according to the rules of the Calculus of Variations as in § 179, and denoting the factor of the χth equation by π_χ we obtain the r differential equations

$$d\left(\frac{\partial ds}{\partial p_\rho}\right) - \frac{\partial ds}{\partial p_\rho} + \sum_1^k {}_\chi p_{\chi\rho} d\pi_\chi - \sum_1^k {}_\chi \sum_1^r {}_\sigma \left(\frac{\partial p_{\chi\sigma}}{\partial p_\rho} - \frac{\partial p_{\chi\rho}}{\partial p_\sigma}\right) \pi_\chi dp_\sigma = 0 \text{ (iv)},$$

which, together with the equations (i), give $(r+k)$ differential equations for the $(r+k)$ quantities p_ρ and π_χ as functions of the independent variables. These equations are the necessary conditions for the vanishing of the variation, and thus are satisfied in all positions of a geodesic path; they accordingly contain the solution of the problem.

184. **Observation 1.** The differential equations 183 (iv) are moreover the sufficient conditions that the path which satisfies them should be a geodesic path. For if these equations are satisfied, then the variation of the length of the path becomes (*cf.* § 180)

$$\delta \int ds = \sum_1^r {}_\rho \left[\left(\frac{\partial ds}{\partial dp_\rho} + \sum_1^k {}_\chi p_{\chi\rho} \pi_\chi\right) \delta p_\rho\right]_0^1.$$

If we make the variations δp_ρ of any two positions of the path vanish, then the variation of the integral between these positions as limits also vanishes, and therefore the required analytical condition for a geodesic path is satisfied (§ 177).

185. **Observation 2.** If we choose the length of the path as independent variable and divide the equations 183 (iv) by *ds*, and for *ds* substitute its value given by § 57 (iv) in terms of p_ρ and dp_ρ, we obtain the equations of the geodesic paths in the form of the r equations

$$\sum_1^r{}_\sigma a_{\rho\sigma}p_\sigma'' + \sum_1^r{}_\sigma\sum_1^r{}_\tau\left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho}\right)p_\sigma' p_\tau' + \sum_1^k{}_\chi\, p_{\chi\rho}\pi_\chi' - \sum_1^k{}_\chi\sum_1^r{}_\sigma\left(\frac{\partial p_{\chi\sigma}}{\partial p_\rho} - \frac{\partial p_{\chi\rho}}{\partial p_\sigma}\right)\pi_\chi p_\sigma' = 0 \quad \text{(i)};$$

which, together with the k equations obtained from § 183 (i),

$$\sum_1^r{}_\rho\, p_{\rho\chi}p_\rho'' + \sum_1^r{}_\rho\sum_1^r{}_\sigma\frac{\partial p_{\chi\rho}}{\partial p_\sigma}p_\rho' p_\sigma' = 0 \quad \text{(ii)},$$

give $(r+k)$ unhomogeneous, linear equations for the $(r+k)$ quantities p_ρ'' and π_χ', and enable us to express these quantities as single-valued functions of p_ρ, p_ρ', and π_χ.

186. **Observation 3.** When by use of the length of the path as independent variable we consider the equation § 92, we obtain the equations 185 (i) in the form

$$\frac{d}{ds}\left(\sqrt{a_{\rho\rho}}\cos \widehat{sp_\rho}\right) = \frac{1}{2}\sum_1^r{}_\sigma\sum_1^r{}_\tau\frac{\partial a_{\sigma\tau}}{\partial p_\rho}p_\sigma' p_\tau' - \sum_1^k{}_\chi\, p_{\chi\rho}\pi_\chi' + \sum_1^r{}_\chi\sum_1^r{}_\sigma\left(\frac{\partial p_{\chi\sigma}}{\partial p_\rho} - \frac{\partial p_{\chi\rho}}{\partial p_\sigma}\right)\pi_\chi p_\sigma'.$$

Thus these equations again express how the direction of the path must change in order that the path may constantly remain geodesic; and moreover every single equation expresses how the inclination to a chosen coordinate p_ρ changes.

187. **Note 1.** A geodesic path is not completely known if we know the length and direction of one of its elements, but from a given position in a given direction there is in general an infinite series of geodesic paths possible.

When the quantities p_ρ, p_ρ' and the k quantities π_χ are given us for one position of the path, then they are (§ 185) also singly determined for the next element, and the continuation of the path is only possible in a single given manner. The knowledge of the direction of the path at that given position, however, only furnishes us with the quantities p_ρ and p_ρ', and this is not sufficient for the determination of the path, but admits, when particular conditions do not prevent, an infinity of the kth order of geodesic paths.

188. **Note 2.** When the differential equations of the system permit of no integral, consequently in the general case, $2r-k$ of the $2r$ quantities p_ρ and p_ρ' which determine a position and the direction at it, can be arbitrarily chosen, viz., the r quantities p_ρ and $r-k$ of the quantities p_ρ'. These $2r-k$ arbitrary values, together with the k arbitrary values of π_χ in that position, may be regarded as the $2r$ arbitrary constants which, together with the differential equations § 185 (i), determine a geodesic path, and must therefore exist in the integrals of these equations, for by § 173 it must be possible to connect every possible position of the system with every other by means of a geodesic path. For if the differential equations of the system furnish no finite relation between p_ρ, then every conceivable value-system of these quantities is a possible value-system; an arbitrary initial and final position are thus determined by means of these $2r$ arbitrary values of the coordinates.

189. **Note 3.** For every integral, which the differential equations of the material system admit, the number of the constants which determine uniquely a geodesic path diminishes by two.

For if from the equations of condition of the system l finite equations between p_ρ can be derived, then only $r-l$ of the r coordinates p_ρ can be arbitrarily chosen, and consequently of the $2r$ quantities p_ρ and p_ρ' which determine a position and a direction at it only $2r-l-k$. Further in this case the differential equations by multiplication by proper factors and by addition can be brought into such a form that l of them immediately give integrable equations, viz., those equations which are got by differentiation of the l finite relations. In each of these equations, one of which we may typify by the index λ, we get

$$\frac{\partial p_{\lambda\sigma}}{\partial p_\rho} - \frac{\partial p_{\lambda\rho}}{\partial p_\sigma} = 0.$$

Thus the corresponding quantities π_λ vanish from the equations 185 (i), and all the quantities p_ρ'' and π_χ' are singly determined in terms of the $k-l$ values of the remaining π_χ. On the whole, therefore, we have still $2r-2l$ arbitrary quantities; two have disappeared for every finite equation.

Finally, these $2r-2l$ arbitrary constants are always sufficient to connect every possible position of the system with every other by means of a geodesic path. If then l finite equations exist between p_ρ, it is sufficient to traverse the path in such a manner that two of its positions should each have $r-l$ coordinates common with the given positions; the coincidence of the remaining will then ensue of itself.

3. Relations between Straightest and Geodesic Paths

190. **Proposition.** In a holonomous system every geodesic path is a straightest path, and conversely.

To prove this let us use rectangular coordinates. Then if the system is holonomous, such a form may be given to the i equations of condition by multiplication by proper factors and addition in a proper order as to make every one of them directly integrable, namely, that form in which the left-hand side of each of them coincides with the exact differential of one of the i integrals of the equations. For every value-system of ι, μ, ν, then,

$$\frac{\partial x_{\iota\mu}}{\partial x_\nu} - \frac{\partial x_{\iota\nu}}{\partial x_\mu} = 0 \qquad \text{(i)},$$

and the differential equations of the geodesic paths by 181 (i) now become

$$\frac{m_\nu}{m} x_\nu'' + \sum_1^i {}_\iota\, x_{\iota\nu} \xi_\iota' = 0 \qquad \text{(ii)}.$$

These equations differ only in notation from the equations of the straightest paths (§ 155 (iv)), viz.,

$$\frac{m_\nu}{m} x_\nu'' + \sum_1^i {}_\iota\, x_{1\nu} \Xi_\iota = 0 \qquad \text{(iii)},$$

as neither ξ_ι nor Ξ_ι appear in the remaining equations to be satisfied. Every possible path, which after a proper determination of ξ_ι satisfies the first of these equations, also satisfies the second when Ξ_ι is made equal to ξ_ι', and every solution of the second is also a solution of the first. The satisfying of the equations (ii) and (iii) is moreover a sufficient condition that the path should be a geodesic one or a straightest one.

191. **Corollary 1.** In a holonomous system only one geodesic path is possible from a possible position in a possible direction (§ 161).

192. **Corollary 2.** In a holonomous system there is always at the least one straightest path between any two possible positions (§ 173).

193. **Proposition.** If in a material system every geodesic path is also a straightest path, then the system is holonomous.

For from every possible position there is only one straightest path in a given direction by § 161, and consequently by hypothesis only one geodesic path. Moreover, it is possible by § 173 to reach every possible position by one of these paths. Thus the number of degrees of freedom of the system is equal to the number of its independent coordinates, and consequently by § 146 the system is holonomous.

194. **Corollary.** In a system which is not holonomous a geodesic path is not in general a straightest path.

This follows from the fact that in any direction there is only one straightest path, whereas many geodesic paths are possible (§ 161, 187).

195. **Note.** In a system which is not holonomous a straightest path is not in general a geodesic path.

The assertion is proved if examples of systems are given in which the straightest paths are not amongst the geodesic ones. Let us choose for simplicity a system in which there exists only a single unintegrable equation of condition between the r coordinates p_ρ of the system, and let this be

$$\sum_1^r {}_\rho p_{1\rho} p_\rho' = 0 \qquad \text{(i).}$$

Let us now assume that every straightest path is also geodesic. Then for all possible systems of values of p_ρ and p_ρ' at least one system of values of p_ρ'' can be obtained so as to satisfy simultaneously the equations 158 (iv) and 185 (i). Then the equations obtained by subtraction of these equations in pairs, viz.,

$$p_{1\rho}\left(\Pi_1 - \pi_1'\right) + \pi_1 \sum_1^r {}_\sigma \left(\frac{\partial p_{1\sigma}}{\partial p_\rho} - \frac{\partial p_{1\rho}}{\partial p_\sigma}\right) p_\sigma' = 0,$$

are to be satisfied for all possible values of p_ρ and p_ρ'. But these are r equations for the single quantity $(\Pi_1 - \pi_1')/\pi_1$, and they are only consistent with one another when for all pairs of values of ρ and τ

$$\frac{1}{p_{1\rho}} \sum_1^r {}_\sigma \left(\frac{\partial p_{1\sigma}}{\partial p_\rho} - \frac{\partial p_{1\rho}}{\partial p_\sigma}\right) p_\sigma' = \frac{1}{p_{1\tau}} \sum_1^r {}_\sigma \left(\frac{\partial p_{1\sigma}}{\partial p_\tau} - \frac{\partial p_{1\tau}}{\partial p_\sigma}\right) p_\sigma'.$$

Let us now substitute in $(r-1)$ of these equations independent of one another, by aid of equation (i), one of the quantities p_ρ' in terms of the remaining, then the ratios between the last are now entirely arbitrary quantities. The coefficient of each of these quantities must consequently vanish. We thus obtain as a necessary consequence of our assumption $(r-1)^2$ equations between the r functions $p_{1\rho}$ and their r^2 first partial differential coefficients. In particular cases these equations can be satisfied, for they are satisfied when the equation (i) is integrable. But in general we have no right to make the functions $p_{1\rho}$ subject to even a single condition, and thus in general our assumption is unwarranted. Hence the statement is proved.

196. **Summary (190–195).** In holonomous systems the ideas of straightest and geodesic paths are completely identical as regards their content: in systems which are not holonomous neither of these ideas includes the other, but both have in general a completely different content.

CHAPTER VI

ON THE STRAIGHTEST DISTANCE IN HOLONOMOUS SYSTEMS

Prefatory Notes

197. This chapter is confined to holonomous systems alone, and by a system simply, is meant a holonomous one. It will therefore be assumed that the coordinates p_ρ of the system are all free coordinates. The number of these coordinates is equal to the number of degrees of freedom of the system, and is thus quite unarbitrary; we shall always denote them by r.

198. Straightest and geodesic paths in this chapter are the same (§ 196), and the common differential equations of these paths can be written in the form of the r equations

$$d\left(\sqrt{a_{\rho\rho}}\cos\widehat{sp_\rho}\right)=\frac{\partial ds}{\partial p_\rho},$$

which are obtained from § 186 or § 160, when we remember that for the chosen coordinates all the quantities $p_{\chi\rho}$ are zero.

199. As a consequence of this, we obtain from § 184 for the variation of the length of a path which satisfies the foregoing differential equations, that is for the length of a geodesic path,

$$\delta\!\int ds=\sum_1^r{}_\rho\left[\frac{\partial ds}{\partial dp_\rho}\delta p_\rho\right]_0^1,$$

or using § 92,

$$\delta\!\int ds=\sum_1^r{}_\rho\left[\sqrt{a_{\rho\rho}}\cos\widehat{sp_\rho}\,\delta p_\rho\right]_0^1,$$

where the quantities δp_ρ denote the variations of the coordinates of the final position, and $\cos \widehat{sp_\rho}$ the direction cosines of the final elements of the geodesic path under consideration.

1. Surfaces of Positions

200. **Definition.** By a surface of positions is meant, in general, a continuously connected aggregate of positions. In particular, however, here by surface will be understood an aggregate of possible positions of a holonomous system which is characterised by the fact that the coordinates of the positions which belong to it satisfy a single finite equation between them.

The aggregate of the positions which simultaneously belong to two or more surfaces we define as the intersection of these surfaces.

201. **Observation 1.** Through every position of a surface an infinite manifold of paths can be drawn, all of whose positions belong to the surface. We say of these paths that they belong to or lie on the surface; we employ the same expressions for the elements of the paths and for infinitely small displacements.

202. **Observation 2.** A path which does not lie on a surface has in general a finite number of positions common with it.

For the path is analytically expressed by means of $(r-1)$ equations between the coordinates of its positions, the surface by means of a single equation. By supposition the former equations are independent of the latter. Therefore in all they give r equations for the r coordinates of the common positions, which equations in general permit of none or a finite number of real solutions.

203. **Observation 3.** From any position of a surface a manifold of the $(r-1)$th order of infinitely small displacements is possible on the surface.

For of the r independent changes of the coordinates which characterise the displacement, $(r-1)$ can be arbitrarily chosen; the rth is then determined from the fact that the displacement lies along the given surface.

204. **Proposition 1.** It is always possible to determine one, and in general only one, direction which is perpendicular to the $(r-1)$ different infinitely small displacements of a system from the same position (§ 197).

Let $d_\tau p_\rho$ be the change of the coordinate p_ρ for the τth of the $(r-1)$ displacements; let δp_ρ be the change of the coordinate p_ρ for a second displacement. The necessary and sufficient condition that the latter should be perpendicular to the former is that $(r-1)$ equations of the form (§ 58)

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\sigma} d_\tau p_\rho \delta p_\sigma = 0$$

should be satisfied. These, however, give $(r-1)$ unhomogeneous, linear equations for the $(r-1)$ ratios of δp_ρ to one another; they can thus always be satisfied, and in general only satisfied, by a single value-system of these ratios. In exceptional cases indeterminateness may arise; this may happen, for instance, when any three of the $(r-1)$ displacements are so chosen that every displacement which is perpendicular to two of them is also perpendicular to the third.

205. **Proposition 2.** If a direction is perpendicular to $(r-1)$ different displacements which lie on a surface in a given position, then it is perpendicular to every displacement which lies on the surface in that position.

The displacements, which lie on a surface in a given position, are characterised by the fact that the corresponding dp_ρ's satisfy a single homogeneous, linear relation between them, namely, the equation which is obtained by differentiation of the equation of the surface. If now the $(r-1)$ value-systems of $d_\tau p_\rho$ satisfy that equation, then so do also the quantities given by

$$dp_\rho = \sum_1^{r-1}{}_\tau \lambda_\tau d_\tau p_\rho,$$

where λ_τ denote arbitrary factors. Thus the dp_ρ's belong to any displacement on the surface, and moreover every displacement on the surface can be expressed in this form since it contains an arbitrary manifold of the $(r-1)$th order. By hypothesis now (§ 204)

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\sigma} d_\tau p_\rho \delta p_\sigma = 0 ;$$

by multiplying these equations by λ_τ and adding we get

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\rho\sigma} dp_\rho \delta p_\sigma = 0,$$

which is the required proof (§ 58).

206. **Definition.** A displacement from a position of a surface is said to be perpendicular to the surface when it is perpendicular to every displacement which lies on the surface in the same position.

207. **Corollary 1.** In every position of a surface there is always one, and in general only one, direction which is perpendicular to the surface.

208. **Corollary 2.** In every position of a surface it is always possible to draw one, and in general only one, straightest path perpendicular to the surface.

209. **Definition 1.** By a series of surfaces we mean an aggregate of surfaces whose equations (§ 200) differ y n the value of the contained constant.

210. **Notation.** Every series of surfaces can be analytically expressed by an equation of the form

$$R = \text{constant},$$

which is obtained by the solution of the equation of one of the surfaces in terms of the variable constant; and in which the right-hand side denotes the possible values of this constant, whilst the left is a function of the coordinates p_ρ. To every surface of this series there corresponds a definite value of the constant, that is a definite value of the function R. Those surfaces, for which the value of the function R only differs by an infinitely small quantity, are called neighbouring surfaces.

211. **Definition 2.** An orthogonal trajectory of a series of surfaces is a path which cuts the series orthogonally, *i.e.* which is perpendicular to every surface of the series in the common positions (§ 202).

212. **Proposition.** In order that a path may be an orthogonal trajectory of the series

$$R = \text{constant} \qquad \text{(i)}$$

it is necessary and sufficient that it should satisfy in each of its positions r equations of the form

$$\sqrt{a_{\rho\rho}} \cos \widehat{sp_\rho} = f \frac{\partial R}{\partial p_\rho} \qquad \text{(ii)},$$

where the quantities $\widehat{sp_\rho}$ denote the inclinations of the path to the coordinates p_ρ, and f is a quantity identical for all the r equations, but which changes with a change of p_ρ.

Draw from the position under consideration an infinitely small displacement whose length is $\delta\sigma$, and denote the resulting changes of p_ρ and R by δp_ρ and δR, and let this displacement make an angle $\widehat{s\sigma}$ with the path considered; then multiply the equations (ii) each with the corresponding δp_ρ and add; we thus obtain (§§ 78 (i) and 85)

$$\delta\sigma \cos \widehat{s\sigma} = \sum_1^r {}_\rho f \frac{\partial R}{\partial p_\rho} \delta p_\rho = f \delta R \qquad \text{(iii)}.$$

If now the displacement $\delta\sigma$ lies on a surface of the series (i), namely, that surface which has the position under consideration common with the path, then $\delta R = 0$, and thus $\widehat{s\sigma} = 90°$. The direction of the path is therefore perpendicular to the surface which it intersects (§ 206), and the equations (ii) are consequently the sufficient conditions that this should happen at every position. They are, moreover, the necessary conditions, since, apart from exceptional cases, at every position there is only one direction which satisfies the given requirement.

213. The orthogonal distance between two neighbouring surfaces of the series in any position is equal to

$$f dR.$$

For, let the displacement $\delta\sigma$ of the foregoing article coincide in direction and length with the portion of the orthogonal trajectory which lies between the two surfaces; then $\delta\sigma$ coincides with the distance under consideration, and the angle

$\overset{\wedge}{s\sigma}$ is equal to zero, and thus the proof follows from § 122 (iii).

214. The function f which enters into the equations of the orthogonal trajectory is a root of the equation

$$\frac{1}{f^2} = \sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} \frac{\partial \mathrm{R}}{\partial p_\rho} \frac{\partial \mathrm{R}}{\partial p_\sigma}.$$

For this equation follows when we substitute the value of the r direction cosines from § 212 (ii) in the equation § 88, which they must satisfy. The root to be chosen depends on whether we consider the direction of the trajectory positive along increasing or decreasing values of R.

2. Straightest Distance

215. **Definition.** By the straightest distance between two positions of a holonomous system is meant the length of one of the straightest paths connecting them.

216. **Observation.** Two positions may have more than one straightest distance. Amongst them are the lengths of the shortest paths between both positions, consequently, too, the length of the absolutely shortest path. When mention is made of the shortest distance between two positions as of a quantity determined without ambiguity, then the last is meant.

217. **Analytical Representation.** The straightest distance between two positions can be expressed as a function of the coordinates of these positions. That position which is regarded as the initial position will be denoted by 0, and its coordinates by $p_{\rho 0}$; whilst that position which is regarded as the final position will be denoted by 1, and its coordinates by $p_{\rho 1}$, so that the direction of the straightest path is positive from 0 to 1. The straightest distance for all value-systems of $p_{\rho 0}$ and $p_{\rho 1}$ is then a definite function of these $2r$ quantities. The analytical expression for the straightest distance, in terms of these variables, will be denoted by S, and for shortness this will be termed the straightest distance of the system.

218. **Observation 1.** The function S is in general a many-valued function of its independent variables. Of the branches of this function one and only one vanishes with the vanishing of the difference between $p_{\rho 0}$ and $p_{\rho 1}$. It is to this branch (§ 216) that we shall refer whenever we say that S is a given single-valued function.

219. **Observation 2.** The function S is symmetrical with regard to $p_{\rho 1}$ and $p_{\rho 0}$ in the sense that it does not change its value when for all values of ρ these quantities are interchanged.

For this interchange only implies an interchange of the final and initial position.

220. **Note.** When the straightest distance of a system is given in terms of any free coordinates, then all the straightest paths of the system are given in terms of these same coordinates, without its being necessary to know in what manner the position of the separate material points of the system depends on these coordinates.

For the straightest distance between any two infinitely near positions of the system is at the same time the length of the infinitely small displacement between them; but if this latter can be expressed in terms of the chosen coordinates, then the statement follows by § 163.

221. **Problem.** To obtain from the straightest distance of a system the expression for the length of its infinitely small displacements.

In S substitute for $p_{\rho 0}$, p_ρ, and for $p_{\rho 1}$, $p_\rho + \delta p_\rho$, and suppose δp_ρ to become very small. We already know (§ 57 (iv)) that the distance between the two positions is expressed as the quadratic root of a homogeneous quadratic function of δp_ρ. S itself cannot thus be expressed in a series of ascending powers of dp_ρ, but S^2 can, and in this expansion the quadratic terms must be the first which do not vanish. If, then, we denote by a bar that in the function under consideration $p_{\rho 0} = p_{\rho 1} = p_\rho$, we obtain for the distance between the two points, and therefore for the magnitude of the displacement, the expression

$$ds^2 = \tfrac{1}{2} \sum_1^r {}_\rho \sum_1^r {}_\sigma \left(\overline{\frac{\partial^2 S^2}{\partial p_{\rho 1} \partial p_{\sigma 1}}} \right) dp_\rho dp_\sigma,$$

and the function $a_{\rho\sigma}$ becomes

$$a_{\rho\sigma} = \tfrac{1}{2}\left(\frac{\overline{\partial^2 S^2}}{\partial p_{\rho 1}\partial p_{\sigma 1}}\right).$$

We might equally correctly have

$$a_{\rho\sigma} = \tfrac{1}{2}\left(\frac{\overline{\partial^2 S^2}}{\partial p_{\rho 0}\partial p_{\sigma 0}}\right).$$

These values of $a_{\rho\sigma}$ can be employed to obtain indirectly, from the function S, the straightest paths, but the following propositions enable us to determine them in a more direct way.

222. **Proposition.** A surface, all of whose positions have equal straightest distances from a fixed position, is cut orthogonally by all straightest paths through this fixed position.

Let $p_{\rho 0}$ be the coordinates of the fixed position and $p_{\rho 1}$ the coordinates of a position of the surface. Let us pass from the latter to another position of the surface for which $p_{\rho 1}$ has changed by $dp_{\rho 1}$. In this the straightest distance from the fixed position has, by hypothesis, not changed; but by § 199 it has changed by $\sum_1^r {}_\rho \sqrt{a_{\rho\rho 1}} \cos \widehat{sp_{\rho 1}} dp_{\rho 1}$, where $\widehat{sp_{\rho 1}}$ denotes the angle which the straightest path at 1 makes with the direction of p_ρ. Thus then

$$\sum_1^r {}_\rho \sqrt{a_{\rho\rho 1}} \cos \widehat{sp_{\rho 1}} dp_{\rho 1} = 0,$$

and this equation expresses that the shortest path is perpendicular to the displacement of $dp_{\rho 1}$ (§§ 85 and 78 (i)). Since this holds for any displacement which lies on the surface at 1, the proposition follows (§ 206).

223. **Corollary 1.** The straightest paths which pass through a fixed position are the orthogonal trajectories of a series of surfaces which satisfy the condition that all the positions of each one of them have the same distance from this fixed position.

224. **Corollary 2.** All the straightest paths which pass through the fixed position 0 satisfy the r equations

$$\sqrt{a_{\rho\rho 1}} \cos \widehat{sp_{\rho 1}} = \frac{\partial S}{\partial p_{\rho 1}} \qquad \text{(i)},$$

where $p_{\rho 1}$ are to be considered the coordinates of the variable position of the path, and $\cos \widehat{sp}_{\rho 1}$ the direction cosine of the path in this position.

For the equations (i) are the equations of the orthogonal trajectories of a series of surfaces which are represented by the equation

$$S = \text{constant} \qquad \text{(ii)}.$$

For if S were any function of the variable coordinates $p_{\rho 1}$, then by § 212 the equations of the orthogonal trajectories would be

$$\sqrt{a_{\rho\rho 1}} \cos \widehat{sp}_{\rho 1} = f\frac{\partial S}{\partial p_{\rho 1}}, \qquad \text{(iii)},$$

and the perpendicular distance between two neighbouring surfaces would be equal to $f d$S. On account of the special nature (§§ 217, 222) of our function S, however, this distance is equal to dS itself, and consequently

$$f = 1 \qquad \text{(iv)},$$

and the general equations (iii) take the particular form (i).

225. **Observation 1.** The equations 224 (i), which are differential equations of the first order, can also be regarded as the equations of straightest paths in a finite form, if we regard $p_{\rho 0}$ as variable and the $2r$ quantities $p_{\rho 1}$ and $\widehat{sp}_{\rho 1}$ as constants.

For let us determine from these equations a series of positions 0 in such a manner that with fixed values of $p_{\rho 1}$, the values of $\widehat{sp}_{\rho 1}$ do not change, then we obtain positions 0 such that the straightest paths drawn from them towards the position 1 have in this position 1 a fixed direction. Since now only one straightest path having this property is possible, all the positions 0 so obtained must be on this one path; their aggregate forms this path and this last is expressed by the equations § 224 (i).

226. **Observation 2.** In the proof of § 222 we might equally well have made 1 the fixed and 0 the variable position. In place of the equations § 224 (i) we should then have obtained the equations

$$\sqrt{a_{\rho\rho 0}} \cos \widehat{sp}_{\rho 0} = -\frac{\partial \mathrm{S}}{\partial p_{\rho 0}} \qquad \text{(i)}.$$

The difference in the sign of the right-hand side results from the fact that the direction from the fixed position is now negative (§ 217). Like the equations § 224 (i), the equations § 226 (i) also represent straightest paths. They are the differential equations of the first order of all straightest paths which pass through the fixed position $p_{\rho 1}$, and at the same time the finite equations of a definite path which passes through the position $p_{\rho 0}$, and there makes with the coordinates the angles $\widehat{sp}_{\rho 0}$.

227. **Corollary 3.** The straightest distance S of a system satisfies, as a function of $p_{\rho 0}$, the partial differential equation of the first order

$$\sum_1^r {}_\rho \sum_1^r {}_\sigma\, b_{\rho\sigma 0} \frac{\partial \mathrm{S}}{\partial p_{\rho 0}} \frac{\partial \mathrm{S}}{\partial p_{\sigma 0}} = 1 \qquad \text{(i)},$$

and as a function of $p_{\rho 1}$ the partial differential equation of the first order

$$\sum_1^r {}_\rho \sum_1^r {}_\sigma\, b_{\rho\sigma 1} \frac{\partial \mathrm{S}}{\partial p_{\rho 1}} \frac{\partial \mathrm{S}}{\partial p_{\sigma 1}} = 1 \qquad \text{(ii)}.$$

For both equations follow from § 214 and § 224 (iv); they are also immediately found when we substitute in § 88 the direction cosines of a straightest path expressed by means of S from § 224 (i) or 226 (i), which the angles of any inclination to the coordinates satisfy.

228. **Proposition.** If we erect at all positions of any surface straightest paths perpendicular to the surface, and cut off from each equal lengths, then the surface so obtained is cut orthogonally by each of these straightest paths.

Let the positions of the original surface be denoted by 0, and of the new surface by 1. Let $\widehat{sp}_{\rho 0}$ and $\widehat{sp}_{\rho 1}$ denote the angles which a chosen straightest path makes with the coordinates at the first and second surface respectively. If we proceed from this straightest path to any neighbouring one, then the length of the path changes (§ 199) by

$$\sum_1^r {}_\rho \sqrt{a_{\rho\rho 1}} \cos \overset{\wedge}{sp_{\rho 1}} dp_{\rho 1} - \sum_1^r {}_\rho \sqrt{a_{\rho\rho 0}} \cos \overset{\wedge}{sp_{\rho 0}} dp_{\rho 0},$$

where $dp_{\rho 1}$ and $dp_{\rho 0}$ denote the changes of p_ρ in the positions 1 and 0. But by construction this change is zero, and also by construction

$$\sum_1^r {}_\rho \sqrt{a_{\rho\rho 0}} \cos sp_{\rho 0} dp_\rho = 0,$$

for every path is perpendicular to the original surface.

Thus then also

$$\sum_1^r {}_\rho \sqrt{a_{\rho\rho 1}} \cos \overset{\wedge}{sp_{\rho 1}} dp_{\rho 1} = 0\ ;$$

and since $dp_{\rho 1}$ denotes any displacement on the surface in the position 1, the conclusion follows.

229. **Corollary 1.** The orthogonal trajectories of any series of surfaces, each of which in all its positions has the same perpendicular straightest distance from its neighbouring ones, are straightest paths.

230. **Corollary 2.** If R is a function of the r coordinates p_ρ of such a nature that the equation

$$\text{R} = \text{constant} \qquad \text{(i)}$$

represents a series of surfaces each of which has in all its positions the same perpendicular straightest distance dR from its neighbours, then the equations

$$\sqrt{a_{\rho\rho}} \cos \overset{\wedge}{sp_\rho} = \frac{\partial \text{R}}{\partial p_\rho} \qquad \text{(ii)}$$

are the equations of the orthogonal trajectories, and consequently the equations of the straightest paths. And, moreover, these equations are differential equations of the first order for these paths.

For if R were any function whatever of p_ρ, then the equations 212 (ii) would represent the orthogonal trajectories of the series (i), and the perpendicular distance between two neighbouring surfaces would, by § 213, be equal to fdR. According to our particular hypothesis, however, this distance is constant and equal to $d\mathrm{R}$, consequently $f=1$, and thus the equations 212 (ii) reduce to the above-mentioned ones.

231. **Corollary 3.** If the equation

$$\mathrm{R} = \text{constant}$$

represents a series of surfaces of such a nature that each of them in all its positions has the same straightest orthogonal distance $d\mathrm{R}$ from its neighbours, then the function R satisfies the partial differential equation

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\sigma\rho} \frac{\partial \mathrm{R}}{\partial p_\rho} \frac{\partial \mathrm{R}}{\partial p'_\sigma} = 1\,;$$

for this equation follows from § 214 and § 230. It is also immediately found when we substitute the direction cosines of a straightest path, given by § 230 (ii), in the equation § 88, which the angles of every inclination to the coordinates satisfy.

232. **Proposition 1.** **(Converse of § 231.)** If the function R satisfies the partial differential equation

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} \frac{\partial \mathrm{R}}{\partial p_\rho} \frac{\partial \mathrm{R}}{\partial p_\sigma} = 1,$$

then the equation

$$\mathrm{R} = \text{constant}$$

represents a series of surfaces of such a nature that each of them in all its positions has the same orthogonal straightest distance from its neighbours, and, moreover, this distance is measured by the change of R.

For if R were any function, then the orthogonal trajectories of the series would be given by equations of the form § 212 (ii), and the orthogonal distance between two neighbouring surfaces would in every position be $fd\mathrm{R}$. But by our special hypothesis as to the nature of R, $f=1$ (§ 214), and thus the proposition is true.

233. **Proposition 2.** If the function R of p_ρ is any solution of the partial differential equation

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} \frac{\partial \mathrm{R}}{\partial p_\rho} \frac{\partial \mathrm{R}}{\partial p_\sigma} = 1 \qquad \text{(i)},$$

then the equations

$$\sqrt{a_{\rho\rho}} \cos \widehat{sp_\rho} = \frac{\partial R}{\partial p_\rho} \qquad \text{(ii)}$$

are the equations of straightest paths. And, moreover, they are differential equations of the first order of the straightest paths represented by them.

This follows immediately from § 230 and § 232.

234. **Observation.** Although every path which is represented by the equations 233 (ii) is a straightest path, yet in general every straightest path cannot conversely be represented in this form. The manifold of straightest paths, which are contained in the given form, depends rather on the manifold which the function R as a solution of the differential equation possesses, that is on the number of its arbitrary constants.

In particular, however, if R is a complete solution, *i.e.* if R contains r arbitrary constants a_0, a_1 . . . a_{r-1}, the first of which is the additive constant necessarily present, then all straightest paths of the system may be expressed in the form § 233 (ii). For the right-hand sides of these r equations (of which only $r-1$ are independent of one another) contain then $(r-1)$ constants which are sufficient to furnish an arbitrarily chosen direction of the path represented at an arbitrary position in terms of $(r-1)$ independent direction cosines. But if we can arbitrarily choose one position of the path represented, and its direction at this position, then we can represent all straightest paths.

235. **Proposition 3.** (Jacobi's Proposition.) Let R denote a complete solution of the differential equation

$$\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma} \frac{\partial R}{\partial p_\rho} \frac{\partial R}{\partial p_\sigma} = 1 \qquad \text{(i)},$$

and let its arbitrary constants, with the exception of the additive one, be a_1, a_2 . . . a_{r-1}. Then the $(r-1)$ equations

$$\frac{\partial R}{\partial a_\tau} = \beta_\tau \qquad \text{(ii)},$$

where the β_τ's are $(r-1)$ new arbitrary constants, give the equations of the straightest paths of the system in a finite form.

As proof we show that the paths which are represented by the equations (ii) are orthogonal trajectories of the series

$$R = \text{constant} \qquad \text{(iii)};$$

whence the proof follows by §§ 232 and 229.

In order now firstly to find the direction of the path represented, we differentiate the equations (ii) each in its own direction, *i.e.* we form these equations for two positions of the path at distance ds, in which p_ρ differs from its next value by dp_ρ, then we subtract and divide by ds. We thus obtain $(r-1)$ equations of the form

$$\sum_1^r {}_\sigma \frac{\partial^2 R}{\partial p_\sigma \partial a_\tau} \frac{\partial p_\sigma}{\partial s} = 0\,;$$

or, when we substitute in these by §§ 79 and 78 the direction cosines of the element of the path under consideration,

$$\sum^r {}_\rho \sqrt{a_{\rho\rho}} \cos \widehat{sp_\rho} \sum_1^r {}_\sigma b_{\rho\sigma} \frac{\partial^2 R}{\partial p_\sigma \partial a_\tau} = 0 \qquad \text{(iv)};$$

which equations give $(r-1)$ unhomogeneous, linear equations for the $(r-1)$ ratios of the direction cosines to one another.

Secondly, we notice that the equation (i) holds for all values of the constants a_τ; we can thus differentiate them with regard to these quantities, and we then obtain $(r-1)$ equations, which may be written in the form

$$\sum_1^r {}_\rho \frac{\partial R}{\partial p_\rho} \sum_1^r {}_\sigma b_{\rho\sigma} \frac{\partial^2 R}{\partial p_\sigma \partial a_\tau} = 0 \qquad \text{(v)},$$

and which express relations which the partial differential coefficients of R must satisfy as a consequence of our particular hypotheses with regard to this function.

If now the equations (ii) represent a definite path for the values of a_τ and β_τ under consideration, then from the equations (iv) must be obtained singly-determined values for the ratios of the direction cosines to one of them. But these same single values for the ratios of the quantities $\dfrac{\partial R}{\partial p_\rho}$ to one of them must be given by the equations (v). Thus if f is a factor which still remains to be determined, then

$$\sqrt{a_{\rho\rho}} \cos \widehat{sp_\rho} = f\frac{\partial \mathrm{R}}{\partial p_\rho}.$$

Thus by § 212 the path under consideration is the orthogonal trajectory of the series (iii), as was to be proved. The factor f is found equal to unity.

The hypothesis that the $(r-1)$ equations (ii) represent, for definite values of a_τ and β_τ, a definite path, would not be correct if these equations were not independent of one another. In that case the arbitrary constants would not be independent of one another, and the solution would not be, as was supposed, a complete one.

236. **Problem.** From any complete solution R of the differential equations 235 (i) to obtain the straightest distance S of the system.

By S is again to be understood the straightest distance between two positions 0 and 1 with the coordinates $p_{\rho 0}$ and $p_{\rho 1}$. In the $(r-1)$ equations, § 235 (ii), we substitute for p_ρ in the first place $p_{\rho 0}$, and in the second $p_{\rho 1}$. From the resulting $(2r-2)$ equations we eliminate β_τ and express a_τ as functions of $p_{\rho 0}$ and $p_{\rho 1}$. These functions are symmetrical with regard to $p_{\rho 0}$ and $p_{\rho 1}$, and give those values which a_τ must have in order that the paths defined by them may pass through the definite positions 0 and 1.

We have then, in the first place, for any position 1, by §§ 224 (i) and 233 (ii),

$$\frac{\partial \mathrm{S}}{\partial p_{\rho 1}} = \left(\frac{\partial \mathrm{R}}{\partial p_\rho}\right)_1;$$

and secondly, for any position 0, by §§ 226 (i) and 233 (i),

$$\frac{\partial \mathrm{S}}{\partial p_{\rho 0}} = -\left(\frac{\partial \mathrm{R}}{\partial p_\rho}\right)_0.$$

We substitute in the right-hand side of these equations the values of a_τ in terms of $p_{\rho 0}$ and $p_{\rho 1}$, and put p_ρ in the first equal to $p_{\rho 1}$, and in the second equal to $p_{\rho 0}$; we then obtain the first differential coefficients of S with regard to all the independent variables expressed as functions of these variables. S can then be found by a single integration.

CHAPTER VII

KINEMATICS

1. Vector Quantities with regard to a System

237. **Definition.** A vector quantity with regard to a system is any quantity which bears a relation to the system, and which has the same kind of mathematical manifold as a conceivable displacement of the system.

238. **Note 1.** A displacement of a system is itself a vector quantity with regard to the system. Every product of a displacement of the system with any scalar quantity whatever is a vector quantity with regard to the system.

239. **Note 2.** Every vector quantity with regard to a system can be represented geometrically by a conceivable displacement of the system. The direction of the displacement representing it is called the direction of the vector quantity. The measure of the representation can and will always be so chosen that the displacement representing it is indefinitely small. Every vector with regard to a system which changes with the position of the system can then be represented as an infinitely small displacement of the system from the position to which its instantaneous value belongs.

240. **Note 3.** A vector quantity with regard to a single material point is a vector in the ordinary sense of the word. Every vector with regard to a point can be represented by a geometrical displacement of the point; in particular, by an infinitely small displacement from its actual position.

241. **Note 4.** By components and reduced components

of a vector are meant those vectors of the same kind which are represented by the components and reduced components of that infinitely small displacement which represents the original vector (§§ 48, 71).

The reduced component of a definite vector in the direction of a coordinate p_ρ is called for short the component of the vector along p_ρ, or the vector along the coordinate p_ρ.

When no misunderstanding can arise, the magnitude of such a component is simply called a component or reduced component.

242. **Problem 1a.** To deduce the components k_ρ of a vector along the general coordinates p_ρ from the components h_ν along the $3n$ rectangular coordinates.

Let $\overline{dx}_\nu$ be the components along x_ν of that displacement which represents the vector quantity, and let $\overline{dp}_\rho$ be the components of the same displacement along p_ρ, then $\overline{dp}_\rho$ is given in terms of $\overline{dx}_\nu$ in § 80. But k_ρ and h_ν are respectively proportional to $\overline{dp}_\rho$ and $\overline{dx}_\rho$; consequently

$$k_\rho = \sum_1^{3n}{}_\nu\, a_{\nu\rho} h_\nu = \sum_1^{3n}{}_\nu \frac{\partial x_\nu}{\partial p_\rho} h_\nu.$$

243. **Problem 1b.** To deduce the components h_ν of a vector along rectangular coordinates, from the components k_ρ of the vector along p_ρ.

The equations § 242 give only r equations for the $3n$ quantities h_ν, from which the latter consequently cannot be found. In fact the problem is in general indeterminate. For all conceivable positions and displacements of a system cannot be expressed in terms of p_ρ but only a part of them, amongst which are the possible displacements.

The proposition can thus only be solved in the case when the given vector is parallel to a displacement which can be expressed in terms of p_ρ and its changes. In this case, by § 81,

$$h_\nu = \sum_1^{r}{}_\rho\, \beta_{\nu\rho} k_\rho.$$

244. **Problem 2a.** To determine the magnitude h of a vector, from its components h_ν along rectangular coordinates.

Using § 83 we obtain

$$h^2 = \sum_1^{3n} {}_\nu \frac{m}{m_\nu} h_\nu^{\,2}.$$

245. **Problem 2b.** To determine the magnitude k of a vector in terms of its components k_ρ along the general coordinates p_ρ.

The problem is again, as in § 243, in general indeterminate.

A solution is only possible when the vector in question is parallel to a displacement which can be represented in terms of p_ρ, and then, by § 82,

$$k^2 = \sum_1^{r} {}_\rho \sum_1^{r} {}_\sigma b_{\rho\sigma} k_\rho k_\sigma.$$

246. **Problem 3a.** To find the components of a vector in the direction of any displacement ds from its components h_ν along x_ν.

If ds' denote the length and $d\bar{x}_\nu'$ the reduced components of the displacement by which we represent the vector, then the component of this displacement in the direction of ds is, by §§ 48 and 84,

$$ds' \cos \widehat{ss'} = \frac{1}{ds} \sum_1^{3n} {}_\nu \, d\bar{x}_\nu d\bar{x}_\nu'.$$

If we multiply this equation by the ratio of the magnitude of the vector to the length of the displacement by which it is represented, we obtain on the left-hand side the required component and on the right-hand side h_ν instead of $d\bar{x}_\nu'$; we thus get as a solution of the problem the required quantity equal to

$$\sum_1^{3n} {}_\nu \, h_\nu \frac{d\bar{x}_\nu}{ds},$$

or, by § 72, equal to

$$\sum_1^{3n} {}_\nu \sqrt{\frac{m}{m_\nu}} \, h_\nu \cos \widehat{sx_\nu}.$$

247. **Problem 3b.** To find the components of a vector in the direction of any displacement ds, expressed in terms of p_ρ, from the components k_ρ along p_ρ.

If we employ the same method as in the previous problem we obtain by §§ 48 and 85 the required quantity equal to

$$\sum_1^r {}_\rho k_\rho \frac{dp_\rho}{ds},$$

or, by §§ 78 and 89, equal to

$$\sum_1^r {}_\rho \sum_1^r {}_\sigma b_{\rho\sigma} k_\rho \sqrt{a_{\sigma\sigma}} \cos \widehat{sp_\sigma}.$$

248. **Observation.** Thus although in general all components of a vector are not determined by means of the quantities k_ρ, yet the components of the vector are determined by means of these quantities in all such directions as can be expressed in terms of p_ρ, and consequently in every possible direction.

249. **Proposition 1.** In order that the vector, whose components along p_ρ are the quantities k_ρ, may be perpendicular to a displacement for which p_ρ suffer the changes dp_ρ, it is necessary and sufficient that the equation

$$\sum_1^r {}_\rho k_\rho dp_\rho = 0$$

should be satisfied.

This follows from § 85 when we consider k_ρ proportional to $\overline{dp_\rho}'$.

250. **Proposition 2.** In order that the vector, whose components along p_ρ are k_ρ, should be perpendicular to every possible displacement of the system, it is necessary and sufficient that the r quantities k_ρ can be expressed in the form

$$k_\rho = \sum_1^k {}_\chi p_{\chi\rho} \cdot \gamma_\chi,$$

where $p_{\chi\rho}$ occur in the equations of condition of the system (§ 130) and γ_χ are quantities to be determined as we please.

This follows from §§ 148 and 150 when we consider k_ρ expressed by means of $\overline{dp_\rho}$.

251. **Note 1.** Vectors with regard to one and the same

system can be compounded and resolved like the conceivable displacements of the system.

Consequently, the compounding of the vectors of the same system follows the rules of algebraic addition.

252. **Note 2.** Vectors with regard to different systems are to be considered quantities of a different nature; they can neither be compounded nor added.

253. **Note 3.** A vector quantity with regard to a given system may be considered as a vector quantity of any more extended system of which the original forms a part.

254. **Problem 1.** The same vector quantities may, at one time, be considered as vector quantities with regard to a partial system, and, at another time, as vector quantities with regard to the complete system. From the components h_ν along the rectangular coordinates x_ν in the first case, the components h_ν' along the corresponding coordinates x_ν' in the second can be determined.

Let m be the mass of the partial system, m' the mass of the complete system. The coordinates x_ν of the partial system are at the same time coordinates of the complete system, only for clearness they are denoted as such by x_ν'. If now the partial system suffers any displacement, which is, of course, at the same time a displacement of the complete system, then $dx_\nu' = dx_\nu$ for the common coordinates, whereas $dx_\nu' = 0$ for the remaining ones. Now, by § 73 $m'd\bar{x}_\nu' = m_\nu dx_\nu'$ and $md\bar{x}_\nu = m_\nu dx_\nu$, consequently $m'd\bar{x}_\nu' = md\bar{x}_\nu$. In the case of a vector which is represented by means of this displacement, the component along x_ν is proportional to $d\bar{x}_\nu$, and that along x_ν' to $d\bar{x}_\nu'$. Thus we obtain

$$m'h_\nu' = mh_\nu$$

for every ν which the systems have in common, whereas for the remainder

$$h_\nu' = 0.$$

255. **Problem 2.** The same vector quantities may, at one time, be considered as vector quantities with regard to a partial system, and, at another time, as vector quantities with regard to the complete system. To determine, in the former

case, the components k_ρ' along the coordinates p_ρ' in terms of the components k_ρ along the coordinates p_ρ.

Let m be again the mass of the partial system, m' that of the complete. We assume that the coordinates p_ρ of the partial system are also coordinates of the complete system, only for clearness in the latter case they will be denoted by p_ρ'. Of the coordinates p_ρ' which are not common to the two systems we assume that they are not coordinates of the partial system. With these assumptions, an analogous consideration to the foregoing (§ 254) gives

$$m'k_\rho' = mk_\rho$$

for the common coordinates, whereas for the remainder

$$k_\rho' = 0.$$

But without the assumptions named the problem is indeterminate.

2. Motion of Systems

Explanations

256. (1) The passage of a system of material points from an initial position to a final one, considered with reference to the time and manner of the passage, is called a motion of the system from the initial to the final position (cf. § 27).

Consequently, in any definite motion the system describes a definite path, and moreover it describes definite lengths in definite times.

257. (2) Every motion of a system along a conceivable path is called a conceivable motion of the system (§ 11).

258. (3) Every motion of a system along a possible path is called a possible motion of the system (§ 112).

259. (4) Kinematics, or the theory of pure motion, treats of the conceivable and possible motions of systems.

So long as we deal only with normal systems (§§ 119, 120), kinematical investigations almost coincide with those of geometry. But when an abnormal system is investigated and the time appears in the equations of condition of the system, then kinematics possesses greater generality than geometry. However, it is not necessary to enter into purely kinematical investigations here; we may then be satisfied with the discussion of a series of fundamental ideas.

260. **Analytical Representation.** The motion of a system is analytically represented when in the representation of the path described, the time t is taken as independent variable, or, what is the same thing, when the coordinates of the position of the system are given as functions of the time.

Following Newton, the differential coefficients of all quantities with regard to the time will be denoted by dots.

Velocity

261. **Definition 1.** The instantaneous rate of motion of a system is called its velocity.

The velocity is determined by the change which the position of the system suffers in an infinitely small time, and by the time itself. It is measured by the ratio of these quantities which is independent of their absolute value.

By the condition of a system we shall mean its position and velocity.

262. **Corollary.** The velocity of a system may be regarded as a vector quantity with regard to the system. The direction of the velocity is then the direction of the instantaneous path-element; the magnitude of the velocity is equal to the differential coefficient of the length of path traversed with regard to the time.

The magnitude of the velocity is also called the velocity of the system along its path, or, when misunderstanding cannot arise, the velocity simply.

263. **Definition 2.** A motion of a system in which the velocity does not change its magnitude is called a uniform motion.

264. **Observation.** A straight motion of a system is motion in a straight path. In this motion the velocity does not change its direction.

265. **Problem 1.** To express the magnitude of the

velocity, its components and reduced components in the direction of the rectangular coordinates, in terms of the rates of change of these coordinates.

The magnitude v of the velocity is given by the positive root of the equation,

$$mv^2 = m\left(\frac{ds}{dt}\right)^2 = \sum_1^{3n}{}_\nu m_\nu \dot{x}_\nu^2.$$

Thus, then (§ 241), the components of the velocity in the direction of x_ν are equal to

$$\sqrt{\frac{m_\nu}{m}}\dot{x}_\nu,$$

and the reduced components in the same direction, or the components along x_ν, to

$$\frac{m_\nu}{m}\dot{x}_\nu.$$

266. **Observation.** The magnitude of the velocity of a system is the quadratic mean value of the magnitudes of the velocities of all its particles.

267. **Problem 2.** To express the magnitude of the velocity, its components and reduced components along the general coordinates p_ρ, in terms of the rates of change $\dot{p}_\rho$ of these coordinates.

By transformation of § 265 by means of § 57 we obtain the magnitude of the velocity as the positive root of the equation

$$v^2 = \sum_1^r{}_\rho \sum_1^r{}_\sigma a_{\sigma\rho}\dot{p}_\rho\dot{p}_\sigma.$$

Thence, by § 241, the components in direction of p_ρ are equal to

$$\frac{1}{\sqrt{a_{\rho\rho}}}\sum_1{}_\sigma a_{\rho\sigma}\dot{p}_\sigma,$$

and the reduced components in the same direction, or the components along p_ρ, to

$$\sum_1^r{}_\sigma a_{\rho\sigma}\dot{p}_\sigma.$$

Momentum

268. **Definition.** The product of the mass of a system into its velocity is called the quantity of motion, or momentum, of the system.

The momentum of the system is thus a vector quantity with regard to the system. The component of the momentum along any coordinate will usually be simply called the momentum of the system along this coordinate (§ 241).

269. **Notation.** The momenta of a system along the general coordinates p_ρ will always be denoted by q_ρ.

270. **Problem 1.** To express the momenta q_ρ of a system along p_ρ in terms of the rates of change of these coordinates.

From §§ 268 and 267 we obtain

$$q_\rho = m\sum_1^r {}_\sigma a_{\rho\sigma}\dot{p}_\sigma.$$

271. **Problem 2.** To express the rates of change of the general coordinates p_ρ in terms of the momenta of the system along these coordinates.

From the foregoing equation we obtain

$$\dot{p}_\rho = \frac{1}{m}\sum_1^r {}_\sigma b_{\rho\sigma}q_\sigma.$$

272. **Observation.** The velocity and the quantity of motion of a system are vectors with regard to the system of such a nature that they are always parallel to possible displacements of the system (§§ 243, 245).

Acceleration

273. **Definition.** The instantaneous rate of change of the velocity of a system is called its acceleration.

The acceleration is determined by the change which the velocity suffers in an infinitely small time and by the time itself; it is measured by the ratio of these two quantities which is independent of their absolute value.

274. **Corollary.** The acceleration of a system may be

regarded as a vector quantity with regard to the system. We take from the actual position of the system two displacements, of which the one represents the actual velocity, the other the velocity at the next instant; then the difference of these gives a new displacement, whose direction is the direction of the acceleration, whilst the magnitude of the acceleration is equal to the ratio of the length of this new displacement to the differential of the time.

275. **Problem 1.** To express the magnitude f of the acceleration and its components along the rectangular coordinates in terms of the differential coefficients of these coordinates with regard to the time.

The components of the velocity along x_ν, now, and after the time dt, are (§ 265)

$$\frac{m_\nu}{m}\dot{x}_\nu \quad \text{and} \quad \frac{m_\nu}{m}\dot{x}_\nu + \frac{m_\nu}{m}\ddot{x}_\nu dt,$$

the components of their difference are thus $\frac{m_\nu}{m}\ddot{x}_\nu dt$; the ratio of these to the time dt gives the components of the acceleration along x_ν equal to

$$\frac{m_\nu}{m}\ddot{x}_\nu,$$

whence by § 244 the magnitude of the acceleration is the positive root of the equation

$$mf^2 = \sum_1^{3n}{}_\nu\, m_\nu \ddot{x}_\nu^{\,2}.$$

276. **Observation.** The magnitude of the acceleration of a material system is the quadratic mean value of the magnitudes of the accelerations of its particles.

277. **Problem 2.** To express the components f_ρ of the acceleration of a system along the general coordinates p_ρ, in terms of the differential coefficients of these with regard to the time.

By § 242,

$$f_\rho = \sum_1^{3n}{}_\nu \frac{m_\nu}{m} \alpha_{\nu\rho} \ddot{x}_\nu,$$

and in this is to be substituted, as in § 108,

$$\ddot{x}_\nu = \sum_1^r{}_\sigma a_{\nu\sigma}\ddot{p}_\sigma + \sum_1^r{}_\sigma \sum_1^r{}_\tau \frac{\partial a_{\nu\sigma}}{\partial p_\tau}\dot{p}_\sigma \dot{p}_\tau.$$

Thus proceeding as in § 108 we obtain

$$f_\rho = \sum_1^r{}_\sigma a_{\rho\sigma}\ddot{p}_\sigma + \sum_1^r{}_\sigma \sum_1^r{}_\tau \left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho}\right)\dot{p}_\sigma \dot{p}_\tau.$$

278. **Observation 1.** The components of the acceleration are thus in general linear functions of the second differential coefficients of the coordinates, quadratic functions of the first differential coefficients, and implicit functions of the coordinates themselves.

279. **Observation 2.** The acceleration of a system is not necessarily parallel to a possible displacement of a system, nor even to a displacement which can be expressed by the coordinates p_ρ.

The components f_ρ do not therefore in general suffice to determine the magnitude of the acceleration nor even its components along all the rectangular coordinates (§§ 243, 245). On the other hand the quantities f_ρ are sufficient to determine the components of the acceleration in the direction of every one of the possible motions of the system (§ 248).

280. **Problem 3.** To find the component of the acceleration in the direction of the path.

The direction cosines of the path are by § 72 equal to $\sqrt{\frac{m_\nu}{m}}\frac{dx_\nu}{ds}$; and thus by § 265 to $\sqrt{\frac{m_\nu}{m}}\frac{\dot{x}_\nu}{v}$. Thence follows by § 246, with the help of § 275 for the tangential component f_t,

$$f_t = \sum_1^{3n}{}_\nu \frac{m_\nu}{m}\frac{\dot{x}_\nu \ddot{x}_\nu}{v} = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \ddot{s},$$

where s is the current length of the path.

281. **Note.** If we resolve the acceleration of a system into two components, of which one is in the direction of the path and the other is perpendicular to the path, then the magnitude of the latter is equal to the product of the curvature of the path into the square of the velocity of the system in the path.

If, in equation § 107 (iii), we take the time t as independent variable, we obtain

$$mv^4c^2 = \sum_1^{3n}{}_\nu m_\nu \ddot{x}_\nu^{\,2} - m\ddot{s}^2\,;$$

thus by use of §§ 275 and 280

$$v^4c^2 = f^2 - f_t^2.$$

If now we call the second, the radial or centrifugal component of the acceleration f_r, then $f^2 = f_t^2 + f_r^2$, for f_r and f_t are perpendicular to one another; consequently

$$f_r = cv^2,$$

as was to be proved.

Energy

282. **Definition.** The energy of a system is half the product of its mass into the square of the magnitude of its velocity.

283. **Problem 1.** To express the energy E of a system in terms of the rates of change of its rectangular coordinates.

By § 265

$$\mathrm{E} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\sum_1^{3n}{}_\nu m_\nu \dot{x}_\nu^{\,2}.$$

284. **Corollary 1.** The energy of a system is the sum of the energies of its particles.

285. **Corollary 2.** If several systems together form a greater system, then the energy of the latter is the sum of the energies of the former.

286. **Problem 2.** To express the energy of a system in terms of the rates of change of the general coordinates of the system and the momenta along these coordinates.

Using §§ 267, 270, 271, we obtain successively

$$\mathrm{E} = \tfrac{1}{2}m\sum_1^{r}{}_\rho \sum_1^{r}{}_\sigma a_{\rho\sigma}\dot{p}_\rho \dot{p}_\sigma \qquad \text{(i)}$$

$$= \tfrac{1}{2}\sum_1^{r}{}_\rho q_\rho \dot{p}_\rho \qquad \text{(ii)}$$

$$= \tfrac{1}{2}m \sum_{1}^{r}{}_{\rho} \sum_{1}^{r}{}_{\sigma} b_{\rho\sigma} q_\rho q_\rho. \qquad \text{(iii).}$$

287. **Observation (on §§ 261-286).** The velocity, momentum, acceleration, and energy of a system are defined independently of their analytical representation, and, in particular, independently of the choice of the coordinates of the system.

Use of Partial Differential Coefficients

288. **Notation** (*cf.* § 90). The partial differential of the energy E will be denoted by $\partial_p E$ only when we consider the coordinates p_ρ and their rates of change $\dot{p}_\rho$ as the independent variable elements of the energy (§ 286 (i)).

The partial differential of the energy E will be denoted by $\partial_q E$ only when we consider the coordinates p_ρ and the momenta q_ρ along these coordinates as the independent variable elements of the energy (§ 286 (iii)).

Either of these assumptions excludes the other. When misunderstanding cannot arise, any partial differential of E will be denoted as usual by ∂E, *e.g.* the first or the second of those mentioned above, or any third kind.

289. **Note 1.** The momenta q_ρ of a system along the coordinates p_ρ may be expressed as partial differential coefficients of the energy of the system with regard to the rates of change of the coordinates.

For, by equation § 286 (i) and § 270 (cf. § 91),

$$q_\rho = \frac{\partial_p E}{\partial \dot{p}_\rho}.$$

290. **Note 2.** The rates of change $\dot{p}_\rho$ of the coordinates p_ρ of a system may be expressed as partial differential coefficients of the energy of the system with regard to the momenta.

For, by equation § 286 (iii), and § 271 (cf. § 94),

$$\dot{p}_\rho = \frac{\partial_q E}{\partial q_\rho}.$$

291. **Note 3.** The components f_ρ of the acceleration of a system along the coordinates p_ρ can be expressed as partial differential coefficients of the energy.

For, by equation 286 (i), firstly,

$$\frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho} = m \sum_1^r {}_\sigma\, a_{\rho\sigma} \dot{p}_\sigma,$$

thus

$$\frac{d}{dt}\left(\frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho}\right) = m \sum_1^r {}_\sigma\, a_{\rho\sigma} \ddot{p}_\sigma + m \sum_1^r {}_\sigma \sum_1^r {}_\tau \frac{\partial a_{\rho\sigma}}{\partial p_\tau} \dot{p}_\sigma \dot{p}_\tau\,;$$

and, secondly, by the same equation,

$$\frac{\partial_p \mathrm{E}}{\partial p} = \tfrac{1}{2} m \sum_1^r {}_\sigma \sum_1^r {}_\tau \frac{\partial a_{\sigma\tau}}{\partial p_\rho} \dot{p}_\sigma \dot{p}_\tau.$$

By subtracting the second equation from the first and comparing with § 227,

$$m f_\rho = \frac{d}{dt}\left(\frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho}\right) - \frac{\partial_p \mathrm{E}}{\partial p_\rho} \qquad \text{(i)},$$

for which may be written (cf. § 289)

$$m f_\rho = \dot{q}_\rho - \frac{\partial_p \mathrm{E}}{\partial p_\rho}.$$

292. **Note 4.** If we change one coordinate p_τ of a system twice by the same infinitely small amount, whereby the first time we let the rates of change of the coordinates, the second time the momenta along these coordinates, retain their original values, then the energy of the system in the two cases suffers an equal and opposite change.

For, if the equation § 95 (i) is multiplied by *mds* and divided by dt^2, we get

$$\frac{\partial_p \mathrm{E}}{\partial p_\tau} = - \frac{\partial_q \mathrm{E}}{\partial p_\tau},$$

which proves the statement.

293. **Proposition.** If the position of a system suffers twice the same infinitely small displacements whereby the first time the rates of change of the coordinates, and the second time the momenta along the coordinates, retain their original values then the energy of the system in the two cases suffers an equal and opposite change.

For the change of the energy is in the first case

$$\delta_p \mathrm{E} = \sum_1^r{}_\tau \frac{\partial_p \mathrm{E}}{\partial p_\tau} \delta p_\tau$$

and in the second

$$\delta_q \mathrm{E} = \sum_1^r{}_\tau \frac{\partial_q \mathrm{E}}{\partial p_\tau} \delta p_\tau \,;$$

thus then, by § 292,

$$\delta_p \mathrm{E} = -\,\delta_q \mathrm{E}.$$

294. **Corollary.** The components of the acceleration of a system along its coordinates p_ρ can also (by §§ 291 (ii) and 292) be expressed in the form

$$mf_\rho = \dot{q}_\rho + \frac{\partial_q \mathrm{E}}{\partial p_\rho}.$$

Concluding Note on the First Book

295. As has already been stated in the prefatory note (§ 1), no appeal is made to experience in the investigations of this book. Consequently, if in the sequel we again meet with the results here obtained, we shall know that they are not obtained from experience but from the given laws of our intuition and thought, combined with a series of arbitrary statements.

It is true that the formation of the ideas and the development of their relations has only been performed with a view to possible experiences; it is thus none the less true that experience alone must decide on the value or worthlessness of our investigations. But the correctness or incorrectness of these investigations can be neither confirmed nor contradicted by any possible future experiences.

BOOK II

MECHANICS OF MATERIAL SYSTEMS

296. **Prefatory Note.** In this second book we shall understand times, spaces, and masses to be symbols for objects of external experience; symbols whose properties, however, are consistent with the properties that we have previously assigned to these quantities either by definition or as being forms of our internal intuition. Our statements concerning the relations between times, spaces, and masses must therefore satisfy henceforth not only the demands of thought, but must also be in accordance with possible, and, in particular, future experiences. These statements are based, therefore, not only on the laws of our intuition and thought, but in addition on experience. The part depending on the latter, in so far as it is not already contained in the fundamental ideas, will be comprised in a single general statement which we shall take for our Fundamental Law. No further appeal is made to experience. The question of the correctness of our statements is thus coincident with the question of the correctness or general validity of that single statement.

CHAPTER I

TIME, SPACE, AND MASS

297. Time, space, and mass in themselves are in no sense capable of being made the subjects of our experience, but only definite times, space-quantities, and masses. Any definite time, space-quantity, or mass may form the result of a definite experience. We make, that is to say, these conceptions symbols for objects of external experience in that we settle by what sensible perceptions we intend to determine definite

times, space-quantities, or masses. The relations which we state as existing between times, spaces, and masses, must then in future be looked upon as relations between these sensible perceptions.

298. **Rule 1.** We determine the duration of time by means of a chronometer, from the number of beats of its pendulum. The unit of duration is settled by arbitrary convention. To specify any given instant, we use the time that has elapsed between it and a certain instant determined by a further arbitrary convention.

This rule contains nothing empirical which can prevent us from considering time as an always independent and never dependent quantity which varies continuously from one value to another. The rule is also determinate and unique, except for the uncertainties which we always fail to eliminate from our experience, both past and future.

299. **Rule 2.** We determine space-relations according to the methods of practical geometry by means of a scale. The unit of length is settled by arbitrary convention. A given point in space is specified by its relative position with regard to a system of coordinates fixed with reference to the fixed stars and determined by convention.

We know by experience that we are never led into contradictions when we apply all the results of Euclidean geometry to space-relations determined in this manner. The rule is also determinate and unique, except for the uncertainties which we always fail to eliminate from our actual experience, both past and future.

300. **Rule 3.** The mass of bodies that we can handle is determined by weighing. The unit of mass is the mass of some body settled by arbitrary convention.

The mass of a tangible body as determined by this rule possesses the properties attributed to the ideally defined mass (§ 4). That is to say, it can be conceived as split up into any number of equal parts, each of which is indestructible and unchangeable and capable of being employed as a mark to refer, without ambiguity, a point of space at one time to a point of space at any other time (§ 3). The rule is also determinate and unique as regards bodies which we can handle, apart from the uncertainties which we cannot eliminate from our actual experience, either past or future.

301. **Addition to Rule 3.** We admit the presumption that in addition to the bodies which we can handle there are other bodies which we can neither handle, move, nor place in the balance, and to which Rule 3 has no application. The mass of such bodies can only be determined by hypothesis.

In such hypothesis we are at liberty to endow these masses only with those properties which are consistent with the properties of the ideally defined mass.

302. **Observation 1.** The three foregoing rules are not new definitions of the quantities time, space, and mass, which have been completely defined previously. They present rather the laws of transformation by means of which we translate external experience, *i.e.* concrete sensations and perceptions, into the symbolic language of the images of them which we form (*vide* Introduction), and by which conversely the necessary consequents of this image are again referred to the domain of possible sensible perceptions. Thus, only through these three rules can the symbols time, space, and mass become parts of our images of external objects. Again, only by these three rules are they subjected to further demands than are necessitated by our thought.

303. **Observation 2.** The indeterminateness which our rules involve and which we have acknowledged, does not arise from the indeterminateness of our images, nor of our laws of transformation, but from the indeterminateness of the external experience which has to be transformed. By this we mean that there is no actual method which, with the aid of our senses, determines time more accurately than can be done by the help of the best chronometer; nor position than when it is referred to a system of coordinates fixed with regard to the fixed stars; nor mass than when determined by the best balance.

304. **Observation 3.** There is, nevertheless, some apparent warrant for the question whether our three rules furnish true or absolute measures of time, space, and mass, and this

question must in all probability be answered in the negative, inasmuch as our rules are obviously in part fortuitous and arbitrary. In truth, however, this question needs no discussion here, not affecting the correctness of our statements, even if we attached to the question a definite meaning and answered it in the negative. It is sufficient that our rules determine such measures as enable us to express without ambiguity the results of past and future experiences. Should we agree to use other measures, then the form of our statements would suffer corresponding changes, but in such a manner that the experiences, both past and future, expressed thereby, would remain the same.

Material Systems

305. **Explanation.** By a material system is henceforth understood a system of concrete masses, whose properties are not inconsistent with the properties of the ideally defined material system (§ 121). Thus in a natural material system some positions and displacements are possible, others impossible; and the aggregate of possible positions and displacements satisfies the conditions of continuity (§ 121). In a natural free system the connections are independent of the position of the system relative to all masses not included in it, as well as of the time (§ 122).

306. **Note thereupon.** We know from experience that there is an actual content corresponding to the conceptions so defined.

For, firstly, experience teaches us that there are connections, and moreover continuous connections, between the masses of nature. There are thus material systems in the sense of § 305. We may even assert that other than continuous connections are not found in nature, and that, consequently, every natural system of material points is a material system.

Secondly, experience teaches us that the connections of a material system may be independent of its position relative to other systems, and of its absolute position. We may even assert that this independence always appears, so long as a material system is sufficiently distant in space from all other systems. Thus, there are systems which have only internal connections, and we possess also a general method for recognising and constructing such systems.

Thirdly and finally, experience teaches us that absolute time has no effect on the behaviour of natural systems which are only subject to internal connections. Every such natural system is thus subject only to normal (§ 119) connections and is therefore a free system. There are thus free systems in the sense of § 305, and we can construct free systems and recognise them as such independently of the statements which we shall have to make again concerning free systems.

307. **Observation.** The normal connections of free systems form those very properties which exist independently of the time. It is the problem of experimental physics to separate those finite groups of masses which can exist independently as free systems, from the infinite world of phenomena, and to deduce from those phenomena which occur in time and in connection with other systems those properties which are unaffected by time.

CHAPTER II

THE FUNDAMENTAL LAW

308. We consider the problem of mechanics to be to deduce from the properties of a material system which are independent of the time those phenomena which take place in time and the properties which depend on the time. For the solution of this problem we lay down the following, and only the following, fundamental law, inferred from experience.

309. **Fundamental Law.** Every free system persists in its state of rest or of uniform motion in a straightest path.

Systema omne liberum perseverare in statu suo quiescendi vel movendi uniformiter in directissimam.

310. **Note 1.** The fundamental law is so worded that its statement has reference only to free systems. But since a portion of a free system can be an unfree (*unfreies*) system, results may be deduced from the fundamental law which have reference to unfree systems.

311. **Note 2.** The aggregate of inferences with regard to a free system and its unfree portions which may be drawn from the fundamental law forms the content of mechanics. Our mechanics does not recognise other causes of motion than those which arise from the law. The knowledge of the fundamental law is, according to our view of it, not only necessary for the solution of the problem of mechanics, but also sufficient for this purpose, and this is an essential part of our assertion.

312. **Note 3. (Definition.)** Every motion of a free material system, or of its parts, which is consistent with the

fundamental law, we call a natural motion of the system in contradistinction to its conceivable and possible motions (§§ 257, 258).

Thus mechanics treats of the natural motions of free material systems and their parts.

313. **Note 4.** We consider a phenomenon of the material world to be mechanically and thereby physically explained when we have proved it a necessary consequence of the fundamental law and of those properties of material systems which are independent of the time.

314. **Note 5.** The complete explanation of the phenomena of the material world would therefore comprise: (1) their mechanical or physical explanation; (2) an explanation of the fundamental law; (3) the explanation of those properties of the material world which are independent of time. The second and third of these explanations we, however, regard as beyond the domain of physics.

Validity of the Fundamental Law

315. We consider the law to be the probable outcome of most general experience. More strictly, the law is stated as a hypothesis or assumption, which comprises many experiences, which is not contradicted by any experience, but which asserts more than can be proved by definite experience at the present time. For, as regards their relation to the fundamental law, the material systems of nature can be divided into three classes.

316. 1. The first class comprises those systems of bodies or parts of such systems which satisfy the conditions of a free system, as can be immediately seen from experience, and to which the fundamental law applies directly. Such are, for example, rigid bodies moving in free space or perfect fluids moving in closed vessels.

The fundamental law is deduced from experiences on such material systems. With regard to this first class it merely represents an experiential fact.

317. 2. The second class comprises those systems of bodies which do not immediately conform to the assumptions of

the fundamental law, or which do not at first sight obey the law, but which can be adapted to the assumptions or can be made to obey the law when, and in fact only when, to direct sensible experience certain definite hypotheses as to the nature of this experience are adjoined.

(*a*) Amongst these are included, firstly, those systems which do not seem to satisfy the condition of continuity in particular positions; *i.e.* those systems in which impulses, in the widest meaning of the term, occur. In this case it is sufficient to use the exceedingly probable hypothesis that all discontinuities are only apparent and vanish when we succeed in taking into consideration sufficiently small space- and time-quantities.

(*b*) Secondly, there are included amongst them those systems in which actions-at-a-distance, the forces due to heat and other causes of motion, not always fully understood, are in operation. When we bring to rest the tangible bodies of such systems, they do not remain in this state, but on being set free enter into a state of motion again. Thus, apparently, they do not obey the law. In this case it is highly probable that the tangible bodies are not the only masses, nor their visible motions the only motions of these systems, but that when we have reduced the visible motions of the tangible bodies to rest, other concealed motions still exist in the systems which are communicated to the tangible bodies again when we set them free. It appears that assumptions can always be made with regard to these concealed motions such that the complete systems obey the fundamental law.

As regards the second class of natural systems the law bears the character of a hypothesis which is in part highly probable, in part fairly probable, but which, as far as we can see, is always permissible.

318. 3. The third class of systems of bodies comprises those systems whose motions cannot be represented directly as necessary consequences of the law, and for which no definite hypotheses can be adduced to make them conformable to it. Amongst these are included, for instance, all systems which contain organic or living beings. We know, however, so little of all the systems included under this head, that it cannot be regarded as proved that such hypotheses are impossible,

and that the phenomena in these systems contradict the fundamental law.

Thus, then, with regard to the third class of systems of bodies the fundamental law has the character of a permissible hypothesis.

319. **Observation.** If we may assume that there is no free system in nature which is not conformable to the law, then we may consider any system whatever as such a free system, or as part of such a free system; so that, on this assumption, there is in nature no system whose motions cannot be determined by means of its connections and the fundamental law.

Limitation of the Fundamental Law

320. In a system of bodies which conforms to the fundamental law there is neither any new motion nor any cause of new motion, but only the continuance of the previous motion in a given simple manner. One can scarcely help denoting such a material system as an inanimate or lifeless one. If we were to extend the law to the whole of nature, as the most general free system, and to say—"The whole of nature pursues with uniform velocity a straightest path,"—we should offend against a feeling which is sound and natural. It is therefore prudent to limit the probable validity of the law to inanimate systems. This amounts to the statement that the law, applied to a system of the third class (§ 318), forms an improbable hypothesis.

321. No attention is, however, paid to this consideration, nor is it necessary, seeing that the law gives a permissible hypothesis if not a probable one. If it could be proved that living systems contradicted the hypothesis, then they would separate themselves from mechanics. In that case, but only in that case, our mechanics would require supplementing with reference to those unfree systems which, although themselves lifeless, are nevertheless parts of such free systems as contain living beings.

As far as we know, such a supplement could be formed, namely, from the experience that animate systems never produce any different results on inanimate ones than those which can also be produced by an inanimate system. Thus it is possible to substitute for any animate system an inanimate one; this may replace the former in any particular problem under consideration, and its specification is requisite in order that we may reduce the given problem to a purely mechanical one.

322. **Observation.** In the usual presentation of mechanics such a reservation is omitted as superfluous and it is assumed that the fundamental laws include animate as well as inanimate nature. And, indeed, in that presentation it is permissible, because we give the freest play to the forms of the forces which there enter into the fundamental laws, and reserve to ourselves an opportunity of explaining, later and outside of mechanics, whether the forces of animate and inanimate nature are different, and what properties may distinguish the one from the other. In our presentation of the subject greater prudence is necessary, since a considerable number of experiences which primarily relate to inanimate nature only are already included in the principle itself, and the possibility of a later narrowing of the limits is much lessened.

Analysis of the Fundamental Law

323. The form in which we have stated the law purposely assimilates itself to the statement of Newton's First Law. However, this statement comprises three others independent of one another, namely, the following:—

1. Of the possible paths of a free system its straightest paths are the only one which it pursues.

2. Different free systems describe in identical times lengths of their paths proportional to each other.

3. Time, as measured by a chronometer (§ 298), increases proportionally to the length of the path of any one of the free moving systems.

The first two statements alone contain facts of a general nature derived from experience. The third only justifies our arbitrary rule for the measure of time, and only includes the particular experience that in certain respects a chronometer behaves as a free system, although, strictly speaking, it is not such.

Method of applying the Fundamental Law

324. When a given question with regard to the motion of a material system is asked, then one of the three following cases must necessarily arise:—

1. The question may be stated in such a manner that the fundamental law itself provides a definite answer. In this case, the problem is a definite mechanical one, and the application of the fundamental law gives its solution.

325. 2. The question may be stated in such a manner that the fundamental law itself does not directly furnish a definite reply, but one or more assumptions may be joined with the question by means of which the definite application of the law is rendered possible.

If only one such assumption is possible and we assume that the problem is a mechanical one, this assumption must also be an appropriate one; the problem can thus be considered as a definite mechanical one, and the application of the assumption and the fundamental law gives the solution.

If several assumptions are possible and we assume that the problem is a mechanical one, one of these assumptions must be appropriate; the problem may then be considered as an indeterminate mechanical one, and the application of the fundamental law to the different possible assumptions gives the possible solutions.

326. 3. The question may be stated in such wise that the fundamental law is insufficient for the solution and that no assumption may be joined to it such as to render the application of the law possible. In this case the question must contain assumptions contradicting the fundamental law or the properties of the system to which it relates; the proposition stated cannot then be considered a mechanical problem.

Approximate Application of the Fundamental Law

327. **Note.** When equations result from the given equations of condition of a system and the fundamental law, which have strictly the form of equations of condition, then for the determination of the motion of the system it is indifferent whether we consider the original equations alone, or instead of them the derived equations, as a representation of the connections of the system.

For if we omit from the series of original equations of condition all those which may be obtained analytically from the remainder and from the derived equations of condition, then only possible displacements, although in general not all the displacements which were possible according to the original equations, satisfy those of the original equations which are left and the derived equations. A path which was a straightest path under the original more general manifold will be one also *a fortiori* under the present more limited manifold. And since the natural paths must be included under this more limited manifold, the natural paths are the straightest amongst those which are possible by the present equations of condition. Thus the proof follows.

328. **Corollary 1.** If we know from experience that a system actually satisfies given equations of condition, then in applying the fundamental law it is quite indifferent whether these connections are original ones, *i.e.* whether they do not admit of a further physical explanation (§ 313), or whether they are connections which may be represented as necessary consequences of other connections and of the fundamental law, and which consequently admit of a mechanical explanation.

329. **Corollary 2.** If we know from experience that given equations of condition of a material system are only approximately but not completely satisfied, then it is still permissible to leave those equations of condition as an approximate representation of a true connection, and by applying the fundamental law to them to obtain approximate statements concerning the motion of the system, although it is quite certain that these approximate equations of condition do not

represent an original, continuous and normal connection, but can only be regarded as the approximate result of unknown connections and the fundamental law.

330. **Observation.** Every practical application of our mechanics is founded upon the foregoing corollary. For in all connections between sensible masses which physics discovers and mechanics uses, a sufficiently close investigation shows that they have only approximate validity, and therefore can only be derived connections. We are compelled to seek the ultimate connections in the world of atoms, and they are unknown to us. But even if they were known to us we could not apply them to practical purposes, but should have to proceed as we now do. For the complete control over any problem always requires that the number of variables should be extremely small, whereas a return to the connections amongst the atoms would require the introduction of an immense number of variables.

However, the fact that we may employ the fundamental law in the manner we do, is not to be regarded as a new experience in addition to the law, but is, as we have seen, a necessary consequence of the law itself.

CHAPTER III

MOTION OF FREE SYSTEMS

General Properties of the Motion

1. Determinateness of the Motion

331. **Proposition.** A natural motion of a free system is singly determined when the position and velocity of the system at any given time are known.

For the path of the system is singly determined (§ 161) by its position and the direction of its velocity; the constant velocity of the system in its path is given by the magnitude of the velocity at the initial time.

332. **Corollary 1.** The future and past conditions of a free system for all times are singly determined by its present condition (§ 261).

333. **Corollary 2.** If it were possible to reverse the velocity of a system in any position (a thing which would in no wise contravene the equations of condition of the system), then the system would pass through the positions of its former motion in reverse order.

334. **Note 1.** In a free holonomous system (§ 123) there is always a natural motion which carries the system in a given time from an arbitrarily given initial position to an arbitrarily given final one.

For a natural path is always possible between the two positions (§ 192). Any velocity is permissible in this path, and therefore such an one as makes the system traverse the given distance in the given time.

335. **Observation.** The foregoing note still holds when instead of the time of the transference the velocity of the system in its path or its energy is given.

336. **Note 2.** A free system which is not holonomous cannot be carried from every possible initial position to every possible final one by a natural motion (§ 162).

337. **Proposition.** A natural motion of a free holonomous system is determined by specifying two positions of the system at two given times.

For by these data the path of the system and its velocity in the path are determined.

338. **Observation 1.** The determination of a natural motion by means of two positions between which it takes place is in general not unique; it is unique so long as the distance between the two positions does not exceed a certain finite quantity and the length of the path described is of the order of this distance (cf. §§ 167, 172, 190, 176).

339. **Observation 2.** A natural motion of a free holonomous system, apart from the absolute value of the time, is also determined by two positions of the system and either the duration of the transference, or the velocity of the system in its path, or the energy of the system.

2. Conservation of Energy

340. **Proposition.** The energy of a free system in any motion does not change with the time.

For the energy (§ 282) is determined by the mass of the system, which is invariable, and the velocity in its path, which is also invariable.

341. **Observation 1.** Of the three partial statements into which the fundamental law can be subdivided (§ 323), only the second and third are needed for the proof of the proposition. We might also make the third unnecessary and render the proposition independent of any given method of measuring time by stating it in the form :—

The ratio of the energies of any two free systems in any motion does not change with the time.

342. **Observation 2.** The law of the conservation of energy is a necessary consequence of the fundamental law. Conversely, from the law of the conservation of energy the second partial statement (§ 323) of that law follows, but not the first, and consequently not the entire law. There might be free systems conceivable, for which the law of the conservation of energy held, and which nevertheless did not move in straightest paths. It is conceivable, for instance, that the law of the conservation of energy might also hold good for animated systems although these might not be embraced in our mechanics. Conversely, natural systems might be conceived which only moved in straightest paths, and for which nevertheless the law of the conservation of energy might not hold good.

343. **Observation 3.** Lately the opinion has been repeatedly expressed that the energy of a moving system is associated with a definite place and is propagated from place to place. On this account energy, in this respect as well as in respect to its indestructibility, has been compared with matter. This conception of energy is obviously very different from that implied in our method of treatment. We have no stronger reason for saying that the seat of the energy of a moving system is where the system is, than for saying that the seat of the velocity of a moving body is where the body is. But naturally this last mode of expression is never used.

3. Least Acceleration

344. **Proposition.** A free system moves in such a manner that the magnitude of its acceleration at any instant is the smallest which is consistent with the instantaneous position, the instantaneous velocity and the connections of the system.

For the square of the magnitude of the acceleration is by §§ 280, 281, equal to

$$v^4c^2 + \dot{v}^2.$$

Now for the natural motion $\dot{v} = o$; v has a value given by the instantaneous velocity, and c has the least value which is consistent with the given direction of motion and the connections of the system. Hence the expression itself must take the smallest value consistent with the given circumstances.

345. **Observation 1.** The property of the natural motion stated in the above proposition determines this motion uniquely, and therefore the proposition can completely replace the fundamental law.

For if the expression $v^4c^2 + \dot{v}^2$ is to become a minimum, firstly $\dot{v}$ must be zero, and consequently the system must traverse its path with constant velocity; secondly either v must be zero, in which case the system is at rest, or c must have the smallest value possible for the direction of the path, in which case the path is a straightest path.

346. **Observation 2.** Proposition 344 might be regarded as a preferable form of statement of the fundamental law, inasmuch as it condenses the law into a single indivisible statement, not only externally into one sentence. The chosen form, however, has the advantage of making its meaning clearer and more unmistakable.

4. Shortest Path

347. **Proposition.** The natural path of a free holonomous system between any two sufficiently near positions is shorter than any other possible path between the two positions.

For in a holonomous system a straightest path between any two sufficiently near positions is also a shortest one (§§ 190, 176).

348. **Observation 1.** If the restriction to sufficiently near positions is removed, then it can no longer be asserted that the natural path is shorter than all other paths, nor even that it is shorter than all neighbouring paths. However, the assertion contained in the foregoing proposition, that the variation of the length of the path vanishes in a transference to any neighbouring possible path, always holds (§§ 190, 171).

349. **Observation 2.** The foregoing proposition corresponds to the Principle of Least Action in the form given to it by Jacobi. If for the moment we take m_ν to be the mass, ds_ν the path-length described by the ν^{th} of the n points of the system in a given element of time, then the proposition asserts that the variation of the integral

$$\int ds = \frac{1}{\sqrt{m}} \int \sqrt{\sum_1^n {}_\nu m_\nu ds_\nu^2}$$

vanishes in the natural motion of the system, and this is Jacobi's form of that principle.

350. **Observation 3.** In order to establish more strictly the relation between the proposition of § 347 and Jacobi's Law, it is necessary to make the following statement:—According to the usual conception of mechanics the proposition contains a particular case of Jacobi's Law, viz., the case where no forces act.

Conversely, according to our conception, the assumptions of the complete Law of Jacobi are to be considered as less general. According to this conception Jacobi's Law is an adaptation of the proposition to particular relations and a modification of it to the assumptions in them.

351. **Observation 4.** The law of the conservation of energy is not postulated by the proposition of § 347, nor is the latter deduced from the law; they are quite independent of one another. In conjunction with the law of energy the proposition may completely replace the fundamental law, but only for holonomous systems. If the proposition were applied to other systems, it would certainly determine definite motions; but these motions would contradict the fundamental law (§ 194) and would consequently furnish false solutions of the stated mechanical problem.

5. Shortest Time

352. **Proposition.** The natural motion of a free holonomous system carries the system in a shorter time from a given initial position to a sufficiently near final one, than could be

done with any other possible motion, with the same constant value of the energy.

For if the energy, and consequently the velocity in the path, are the same for all the motions compared, then the duration of the motion is proportional to the length of the path. Consequently it is the smallest for the shortest path, that is for the natural path.

353. **Observation.** If the restriction to sufficiently near positions is removed, then the time of the motion is no longer necessarily a minimum, but it still retains the property of always being equal for the natural path and for all its infinitely near possible paths (see § 348).

354. **Corollary 1.** For the natural motion of a free holonomous system between given sufficiently near final positions, the time-integral of the energy is always smaller than for any other possible motion performed with the same constant value of the energy.

For the time-integral is equal to the product of the given constant value of the energy and the duration of the transference.

355. **Observation 1.** The proposition of § 352, particularly in the form of § 354, contains Maupertuis' Principle of Least Action. If it is desired to establish more strictly its relation to this principle, we must express ourselves in the manner done in § 350.

356. **Observation 2.** The corollary § 354, and also the proposition § 352, assume for the motions compared with one another the constancy of the energy with the time. With the assumption that the natural motion is included in those compared, they are sufficient for its determination, and could replace the fundamental law, but only in the case of holonomous systems. Their assumptions applied to other systems would lead to false mechanical solutions.

357. **Corollary 2.** A free holonomous system is carried from its initial position in a given time through a greater straightest distance by its natural motion than by any other possible motion which takes place with the same constant value of the energy as the natural motion.

6. Least Time-Integral of the Energy

358. **Proposition.** The time-integral of the energy in the transference of a free holonomous system from a given initial position to a sufficiently near final one is smaller for the natural motion than for any other possible motion by which the system may pass from the given initial position to the final one in an equal time.

For firstly, if we compare only motions in one and the same path, of length S, then the time-integral attains its minimum value for that one in which the velocity v is constant. For since the sum of the quantities vdt has the given value S, then the sum of the quantities v^2dt will attain its smallest value only when all the v's are equal. But if the velocity is constant, then the time-integral of the energy is equal to $\frac{1}{2}mS^2/T$, where T is the duration of the transference. Since T is given, the time-integral of the energy for different paths of the system varies as the square of the length of the path; hence the first quantity, like the last, has its minimum value for the natural path.

359. **Observation 1.** If the limitation to sufficiently near positions is removed, then the time-integral of the energy will no longer necessarily be a minimum, but its variation, nevertheless, always vanishes in the transference to any other of the motions considered (*cf.* § 348).

360. **Observation 2.** The foregoing proposition corresponds to Hamilton's Principle. If it is desired to establish more closely its connection with this principle, we must use the mode of expression of § 300.

361. **Observation 3.** The proposition § 358 and the corollary § 354 agree in this, that amongst certain classes of possible motions they distinguish the natural motion by one and the same characteristic, viz., the minimum value of the time-integral of the energy. They differ essentially from one another in this, that they consider entirely different kinds of possible motions.

362. **Observation 4.** The law of the conservation of energy is a necessary consequence of the proposition in § 358;

and this proposition, employed as a principle, can therefore completely replace the fundamental law, but still only in its application to holonomous systems. If the restriction to holonomous systems is removed, then the proposition determines definite motions of the material systems; but these in general contradict the fundamental law, and are, therefore, mechanically considered, false solutions of the stated problem.

363. **Retrospect to §§ 347-362.** If we employ the properties of the natural motion stated in the propositions 347, 352, 354, and 358 as principles for the complete or partial determination of this motion, then we make the changes now entering into the condition of the system dependent on such peculiarities of the motion as can only appear in the future, and which often seem in human affairs as objects worth striving for. This circumstance has occasionally led physicists and philosophers to perceive in the laws of mechanics the expression of a conscious intention as to future aims, combined with a certain foresight as to the most suitable means for attaining them. Such a conception is, however, neither necessary nor permissible.

364. That such a conception of these principles is not necessary is shown by the fact that the properties of the natural motion which seem to indicate an intention can be recognised as the necessary consequences of a law in which one finds no expression of any intention as to the future.

365. That this conception of the principles is inadmissible is seen from the fact that the properties of the natural motion which appear to denote an outlook to future issues are not found in all natural motions. Had nature the design of aiming at a shortest path, a least expenditure of energy and a shortest time, it would be impossible to understand why there could be systems in which this design, although attainable, should still be regularly missed by nature.

366. If one wishes to recognise in the fact that a system always chooses a straightest path-element amongst all possible ones the expression of a definite intention, then this is allowable; the expression of a definite intention is then already seen in the fact that a natural system always chooses out

of all possible motions no arbitrary one, but always one which is determinable beforehand and is marked by particular characteristics.

Analytical Representations. Differential Equations of Motion

367. **Explanation.** By the differential equations of motion of a system we understand a set of differential equations in which the time is the independent variable, the coordinates of the system the dependent variables; and which, together with an initial position and initial velocity, uniquely determine the motion of the system (§ 331).

368. **Problem 1.** To express the differential equations of the motion of a free system in terms of its rectangular coordinates.

In § 155 (iv) we have found the differential equations of the straightest paths of the system in terms of the rectangular coordinates. In these equations we introduce the time t as independent variable instead of the length of the path. By the fundamental law, $ds/dt = v$ is independent of t, and consequently also of s. Thus we have

$$\dot{x}_\nu = x'_\nu \,.\, v, \quad \ddot{x}_\nu = x''_\nu \,.\, v^2.$$

We then multiply the equations 155 (iv) by mv^2 and put for shortness X_ι instead of $mv^2\Xi_\iota$. We thus obtain as solution of the problem the $3n$ equations

$$m_\nu \ddot{x}_\nu + \sum_1^i {}_\iota x_{\iota\nu} X_\iota = 0 \qquad \text{(i)},$$

which with the i equations (*cf.* § 155 ii.)

$$\sum_1^{3n} {}_\nu x_{\iota\nu} \ddot{x}_\nu + \sum_1^{3n} {}_\nu \sum_1^{3n} {}_\mu \frac{\partial x_{\iota\nu}}{\partial x_\mu} \dot{x}_\nu \dot{x}_\mu = 0 \qquad \text{(ii)}$$

determine the $3n$ quantities $\ddot{x}_\nu$ and X_ι as single-valued functions of x_ν and $\dot{x}_\nu$.

369. **Observation 1.** The equations of motion of the free system in the form of § 368 are usually known as Lagrange's equations of the first form.

370. **Observation 2.** Every single equation of § 368 (i) gives us, after having first determined the quantities X_ι, the component of the acceleration of the system along one of the rectangular coordinates of the system.

371. **Problem 2.** To express the differential equations of motion of a free system in terms of its general coordinates p_ρ.

The differential equations of the straightest paths in terms of p_ρ are given in § 158 (iv). In these we introduce the time as independent variable instead of the length of the path; and we again note that according to the fundamental law

$$\dot{p}_\rho = p'_\rho v, \quad \ddot{p}_\rho = p''_\rho v^2.$$

We consequently multiply the equations § 158 (iv) by mv^2, and putting P_χ for $mv^2\Pi_\chi$ we obtain as solution of the problem the r equations

$$m\left\{\sum_1^r{}_\sigma a_{\rho\sigma}\ddot{p}_\sigma + \sum_1^r{}_\sigma\sum_1^r{}_\tau\left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho}\right)\dot{p}_\sigma\dot{p}_\tau\right\} + \sum_1^k{}_\chi p_{\chi\rho}P_\chi = 0 \quad \text{(i)},$$

which with the k equations (*cf.* § 158 (ii))

$$\sum_1^r{}_\rho p_{\chi\rho}\ddot{p}_\rho + \sum_1^r{}_\rho\sum_1^r{}_\sigma\frac{\partial p_{\chi\rho}}{\partial p_\sigma}\dot{p}_\rho\dot{p}_\sigma = 0 \quad \text{(ii)}$$

determine the $r+k$ quantities $\ddot{p}_\rho$ and P_χ as single-valued functions of p_ρ and $\dot{p}_\rho$.

372. **Observation.** If we make use of the notation of § 277 we can write the equations of motion of § 371 (i) in the form

$$mf_\rho + \sum_1^k{}_\chi p_{\chi\rho}P_\chi = 0.$$

If we assume that the quantities P_χ have been determined, each of these equations gives us the component of the acceleration along a given coordinate p_ρ expressed as a function of the instantaneous position and velocity of the system.

373. **Corollary 1.** If we express by using the notation of § 291 (i) the components of the acceleration in terms of the energy, then the equations of motion of a free system take the form

$$\frac{d}{dt}\left(\frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho}\right) - \frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho} + \sum_1^k{}_\chi p_{\chi\rho}\mathrm{P}_\chi = 0.$$

374. **Observation 1.** The differential equations of motion in this form are called also the generalised Lagrangian equations of motion or Lagrange's equations of the second form (*cf.* § 369).

375. **Observation 2.** If the coordinate p_ρ is a free coordinate, then it does not appear in the equations of condition of the system, and the quantities $p_{\chi\rho}$ are consequently all equal to zero. The equation of motion corresponding to p_ρ then becomes

$$\frac{d}{dt}\left(\frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho}\right) - \frac{\partial_p \mathrm{E}}{\partial p_\rho} = 0.$$

In a holonomous system all the equations of motion can be expressed in this simple form (§ 144).

376. **Corollary 2.** The equations of motion of a free holonomous system expressed in any r free coordinates p_ρ of the system can be written in the form of the $2r$ equations

$$q_\rho = \frac{\partial_p \mathrm{E}}{\partial \dot{p}_\rho} \qquad \text{(i)}$$

$$\dot{q}_\rho = \frac{\partial_p \mathrm{E}}{\partial p_\rho} \qquad \text{(ii).}$$

Of these the former contain only definitions; but the latter contain experiential facts. One can thus regard the equations of motion in this form as $2r$ differential equations of the first order for the $2r$ quantities p_ρ and q_ρ. These equations, together with the $2r$ initial values of the quantities, determine them for all times.

377. **Observation 1.** The equations 376 (i) and (ii) one might correctly term Poisson's form of the equations of motion.

378. **Observation 2.** From the equations § 376 there follow two reciprocal relations, analytically expressed by the equations

[from (ii)]
$$\frac{\partial_p \dot{q}_\rho}{\partial p_\sigma} = \frac{\partial_p \dot{q}_\sigma}{\partial p_\rho} \qquad \text{(i)}$$

[from (i) and (ii)]
$$\frac{\partial_p q_\rho}{\partial \dot{p}_\sigma} = \frac{\partial_p \dot{q}_\sigma}{\partial \dot{p}_\rho} \qquad \text{(ii)},$$

and which possess a simple physical meaning. Both relations contain elements of experience and would not hold for every possible motion of the system. Hence they may, under certain conditions, be utilised for testing the fundamental law. A third analogous relation, deduced solely from § 376 (i), would only be a consequence of our definitions.

379. **Corollary 3.** The equations of motion of a free holonomous system in terms of any r free coordinates p_ρ of the system can be written in the form of the $2r$ equations (§§ 290, 289, 292, 375)

$$\dot{p}_\rho = \frac{\partial_q E}{\partial q_\rho} \qquad \text{(i)}$$

$$\dot{q}_\rho = -\frac{\partial_q E}{\partial p_\rho} \qquad \text{(ii)}.$$

Of these the former contain only definitions; but the latter contain experiential facts. In this form also the equations of motion appear as $2r$ differential equations of the first order for the $2r$ quantities p_ρ and q_ρ. These equations, together with the $2r$ initial values of the quantities, determine them for all times.

380. **Observation 1.** The foregoing equations are usually known as the Hamiltonian form of the equations of motion for a free system.

381. **Observation 2.** Two reciprocal relations follow from the equations § 379, which are analytically expressed by the equations

$$\frac{\partial_q \dot{q}_\rho}{\partial p_\sigma} = \frac{\partial_q \dot{q}_\sigma}{\partial p_\rho} \qquad \text{(i)}$$

$$\frac{\partial_q \dot{p}_\rho}{\partial q_\sigma} = -\frac{\partial_q \dot{q}_\sigma}{\partial q_\rho} \qquad \text{(ii)},$$

and which possess a simple physical meaning. Both relations contain elements of experience and distinguish the natural motion from other possible motions. Hence they may conversely, under certain conditions, be utilised for testing the fundamental law. A third analogous relation, deduced solely from § 379 (i), would only be the consequence of our definitions, and would therefore have no mechanical significance.

It is necessary to insist on the fact that the equations 378 (i) and 381 (i) represent different statements, and not the same statements in a different form.

Internal Constraint of Systems

382. **Proposition.** A system of material points between which no connections exist, persists in its condition of rest or uniform motion in a straight path.

For in such a system the straight path is also the straightest.

383. **Corollary 1.** A free material point persists in its condition of rest or uniform motion in a straight path (Galileo's Law of Inertia or Newton's First Law).

384. **Corollary 2.** The acceleration of a system of material points between which no connections exist is zero. The connections between the points of a material system can thus be regarded as the cause owing to which the acceleration differs in general from zero.

385. **Definition.** The change in the acceleration caused by all the connections of a material system is called the constraint which the connections impose on the system; this change is also called for shortness the internal constraint, or, still shorter, the constraint of the system.

The constraint is measured by the difference between the actual acceleration of the system and the acceleration of that natural motion which would result on removal of all the equations of condition of the system; it is equal to the former diminished by the latter.

386. **Corollary 1.** The internal constraint of a system is, like the acceleration, a vector quantity with regard to the system.

387. **Corollary 2.** In a free system the internal con-

straint is equal to the acceleration of the system: it is here in fact only another mode of regarding the acceleration (§ 382).

388. **Proposition 1.** The magnitude of the constraint is at every instant smaller for the natural motion of a free system than for any other possible motion which coincides with it in position and velocity at the particular instant considered.

For this statement is by § 387 only different in form from proposition § 344.

389. **Corollary.** Any connection which is added to the connections of the system already in existence increases the constraint of the system. The removal of any connection changes the natural motion in such a manner that the constraint is diminished.

390. **Observation 1.** The foregoing theorem corresponds to Gauss's Principle of Least Constraint. In order to present clearly its connection with this principle we should have to use the same mode of expression as in § 350.

391. **Observation 2.** Gauss's Principle and the Law of Inertia (§ 383) may together replace completely the fundamental principle, and that for all systems.

For they together are equivalent to the proposition § 344.

392. **Proposition 2.** The direction of the constraint in the natural motion of a free system is constantly perpendicular to every possible or virtual (§ 111) displacement of the system from its instantaneous position.

For the components of the constraint in a free system along the coordinates p_ρ are by § 387 equal to f_ρ, and may thus be written in the form

$$-\frac{1}{m}\sum_{1}^{k}{}_{\chi}p_{\chi\rho}P_{\chi}.$$

Thus by § 250 they are perpendicular to every possible displacement of the system.

393. **Symbolical Expression.** If we denote by δp_ρ the changes of the coordinates p_ρ for any possible or virtual displacement of the system, then the equation

$$\sum_1^r {}_\rho f_\rho \delta p_\rho = 0 \qquad \text{(i)}$$

furnishes a symbolical expression of the foregoing proposition. For the equation replaces the proposition by § 249, and it is symbolical, since it stands as a symbol for an infinite number of equations.

If we use rectangular coordinates and denote by δx_ν the change of x_ν for any possible or virtual displacement, then the equation takes the form

$$\sum_1^{3n} {}_\nu m_\nu \ddot{x}_\nu \delta x_\nu = 0 \qquad \text{(ii).}$$

394. **Observation 1.** The foregoing proposition, § 392, corresponds to d'Alembert's Principle; the equations 393 (i) and (ii) correspond to the usual expression of that principle. In order to establish clearly the relation between that principle and the proposition we should have to use the same mode of expression as in § 350.

395. **Observation 2.** From the condition that the constraint is perpendicular to every virtual displacement of the system we get by § 250 the equations of motion of the free system in the form § 372. Consequently d'Alembert's Principle can by itself replace the fundamental law, and that for all systems. Our fundamental law has over d'Alembert's Principle the advantage of a simpler and clearer meaning.

396. **Corollary 1.** In a free system the acceleration is always perpendicular to any possible displacement of the system from its instantaneous position.

397. **Corollary 2.** In the motion of a free system the acceleration is always perpendicular to the direction of the actual instantaneous motion.

398. **Corollary 3.** In the motion of a free system the component of the acceleration in any direction of a possible motion is always zero.

399. **Corollary 4.** The component of the acceleration of a free system in the direction of any free coordinate is always equal to zero.

400. **Proposition.** A free system moves in such a manner that the components of the acceleration in the direction of any coordinate of absolute position always remain zero, whatever is the internal connection between the points of the system.

For whatever is the connection of the system, every coordinate of its absolute position is a free coordinate (§ 142).

401. **Corollary.** If we choose the coordinates of a free system in all other respects arbitrarily, but so that there are amongst them six coordinates of absolute position (§ 19), we can without knowledge of the connection of the system, or without complete knowledge of it, write down immediately six differential equations of the motion of the system.

402. **Particular Selection of Coordinates.** The following choice of coordinates of absolute position is permissible for every system.

We denote by

$$\alpha_1, \alpha_2, \alpha_3,$$

the arithmetic mean value of those rectangular coordinates of all particles which are respectively parallel to $x_1 x_2 x_3$. The quantities $\alpha_1 \alpha_2 \alpha_3$ we consider as rectangular coordinates of a point of mean position, which we call the centre of gravity of the system. Through the centre of gravity we draw three straight lines parallel to the three coordinate axes. Through these three straight lines and all the particles we draw planes and denote by

$$\omega_1, \omega_2, \omega_3,$$

the arithmetic mean value of the inclinations of all the planes drawn through these straight lines to any one of them. The six quantities α and ω are variable quantities independent of each other, whose change necessarily causes a change in the position of the system, and which are not determined by the configuration alone. We can consequently make these six quantities coordinates of absolute position (§ 21), and we make them coordinates of absolute position so long as we introduce only coordinates of configuration for the remaining coordinates.

If we give α and ω any changes whilst we fix the remaining coordinates, the system moves as a rigid body.

We obtain then from purely geometrical considerations for the changes of the rectangular coordinates, when we allow the index ν to pass from 1 to n (§ 13),

$$\begin{cases} dx_{3\nu} = da_1 + (x_{3\nu-1} - a_2)d\omega_3 - (x_{3\nu-2} - a_3)d\omega_2 \\ dx_{3\nu-1} = da_2 + (x_{3\nu-2} - a_3)d\omega_1 - (x_{3\nu} - a_1)d\omega_3 \\ dx_{3\nu-2} = da_3 + (x_{3\nu} - a_1)d\omega_2 - (x_{3\nu-1} - a_2)d\omega_1 \end{cases} \quad \text{(i)}.$$

From this we can obtain, when we consider the x_ν's as functions of all the coordinates, the values of the partial differential coefficients of the x_ν's with respect to a and ω; thus, for instance,

$$\frac{\partial x_{3\nu}}{\partial a_1} = 1, \quad \frac{\partial x_{3\nu}}{\partial a_2} = 0, \quad \frac{\partial x_{3\nu}}{\partial a_3} = 0 \qquad \text{(ii)},$$

$$\frac{\partial x_{3\nu}}{\partial \omega_1} = 0, \quad \frac{\partial x_{3\nu}}{\partial \omega_2} = -(x_{3\nu-2} - a_3), \quad \frac{\partial x_{3\nu}}{\partial \omega_3} = x_{3\nu-1} - a_2 \quad \text{(iii)}.$$

403. **Corollary 1.** As a consequence of the remark that the accelerations of the system along the coordinates a_1, a_2, a_3 must vanish (§ 400), we get the three equations

$$\sum_1^n {}_\nu m_\nu \ddot{x}_{3\nu} = 0, \quad \sum_1^n {}_\nu m_\nu \ddot{x}_{3\nu-1} = 0, \quad \sum_1^n {}_\nu m_\nu \ddot{x}_{3\nu-2} = 0.$$

For by § 242 and § 275 the acceleration along the coordinate a_1 of the centre of gravity is equal to

$$\sum_1^{3n} {}_\nu \frac{\partial x_\nu}{\partial a_1} \cdot \frac{m_\nu}{m} \ddot{x}_\nu,$$

therefore by § 402 (ii) equal to

$$\sum_1^n {}_\nu \frac{m_\nu}{m} \ddot{x}_{3\nu},$$

and similar expressions hold for the accelerations along a_2 and a_3.

404. **Observation.** The three equations § 403 can be immediately integrated twice and then express that the centre of gravity of a free system moves uniformly and in a straight line. This is known as the Principle of the Centre of Gravity.

405. **Corollary 2.** From the fact that the accelerations

of the system along the coordinates ω_1, ω_2, ω_3 must vanish (§ 400), we get the three equations

$$\sum_1^n {}_\nu m_\nu(x_{3\nu-2}\ddot{x}_{3\nu-1} - x_{3\nu-1}\ddot{x}_{3\nu-2}) = 0$$

$$\sum_1^n {}_\nu m_\nu(x_{3\nu}\ddot{x}_{3\nu-2} - x_{3\nu-2}\ddot{x}_{3\nu}) = 0$$

$$\sum_1^n {}_\nu m_\nu(x_{3\nu-1}\ddot{x}_{3\nu} - x_{3\nu}\ddot{x}_{3\nu-1}) = 0.$$

For by § 242 and § 275 the acceleration along ω_1 is equal to

$$\sum_1^{3n} {}_\nu \frac{\partial x_\nu}{\partial \omega_1} \cdot \frac{m_\nu}{m}\ddot{x}_\nu,$$

thus by § 402 (iii) equal to

$$\sum_1^n {}_\nu \frac{m_\nu}{m}\left\{(x_{3\nu-2} - a_3)\ddot{x}_{3\nu-1} - (x_{3\nu-1} - a_2)\ddot{x}_{3\nu-2}\right\},$$

then by using § 403 equal to

$$\sum_1^n {}_\nu (x_{3\nu-2}\ddot{x}_{3\nu-1} - x_{3\nu-1}x_{3\nu-2});$$

and corresponding values hold for the accelerations along ω_2 and ω_3.

406. **Observation.** These three equations contain the so-called Principle of Areas. These equations can be immediately integrated once, and then give the differential equations of the first order

$$\sum_1^n {}_\nu m_\nu(x_{3\nu-2}\dot{x}_{3\nu-1} - x_{3\nu-1}\dot{x}_{3\nu-2}) = \text{const},$$

$$\sum_1^n {}_\nu m_\nu(x_{3\nu}\dot{x}_{3\nu-2} - x_{3\nu-2}\dot{x}_{3\nu}) = \text{const},$$

$$\sum_1^n {}_\nu m_\nu(x_{3\nu-1}\dot{x}_{3\nu} - x_{3\nu}\dot{x}_{3\nu-1}) = \text{const}.$$

These admit of the following geometrical interpretation which suggests the name :—

Draw to each particle of the system from the origin a radius vector; then the sum of the projections of the areas described by these radii on each of the three coordinate planes increases uniformly with the time.

407. **Observation 1 (on §§ 402-406).** We have introduced the Principles of the Centre of Gravity and of Areas as particular cases of the general proposition § 400. We should not have been right in this, if we regarded, as is sometimes done, the essential features of these principles as lying in the fact that they furnish integrals of the equations of motion. One reason why this view seems to us inadmissible, is that the result derived from the Principle of Areas can only be called an integral in a figurative sense. We rather consider the essential features of the principles as lying in the fact that they furnish properties which are of general validity and can be stated quite independently of the particular connection of the system.

408. **Observation 2 (on §§ 402-406).** In deducing the Principles of the Centre of Gravity and of Areas as special cases of § 400 we have not made use of all the properties which the definitions assigned to a and ω. In fact, we might have been able to deduce these principles by using other coordinates, for instance, all coordinates which are in the same direction as a and ω without being identical with them. Of course, with any choice of coordinates, we should not obtain in every case six equations which would furnish a new physical meaning, or which would be quite independent of the equations § 403 and § 405; but they would always be those equations which result from the equations § 403 and § 405 by transformation to the chosen coordinates. But the proposition § 400 gives for all these different forms a common expression and physical meaning.

Holonomous Systems

409. **Note.** If the straightest distance (§ 217) is known for a holonomous system, then the equations of the straightest paths can be expressed in a finite form (§ 225). These paths, moreover, are the natural paths of the system, so long as it is

free; and all motions by which they could be traversed with a constant velocity, are natural motions of the system. The equations of motion of a free holonomous system can thus be expressed in a finite form.

410. **Problem.** To express the equations of motion of a free holonomous system by means of its straightest distance.

As before, let S be the straightest distance of the system, considered as a function of the free coordinates $p_{\rho 0}$ and $p_{\rho 1}$ of its initial and final positions. Let t_0 be the time at which the system passes through the initial position, and t_1 the time at which it passes through the final position. Then $t_1 - t_0$ is the duration of the motion, and thus

$$v = \frac{S}{t_1 - t_0} \qquad \text{(i)}$$

gives the constant velocity of the system in its path; its energy is given by

$$E = \tfrac{1}{2} m \frac{S^2}{(t_1 - t_0)^2} \qquad \text{(ii)},$$

and its momenta $q_{\rho 0}$ and $q_{\rho 1}$ at the times t_0 and t_1 by

$$\left.\begin{aligned} q_{\rho 0} &= m \frac{S}{t_1 - t_0} \sqrt{a_{\rho\rho 0}} \cos \widehat{sp_{\rho 0}} \\ q_{\rho 1} &= m \frac{S}{t_1 - t_0} \sqrt{a_{\rho\rho 1}} \cos \widehat{sp_{\rho 1}} \end{aligned}\right\} \qquad \text{(iii)}.$$

For the equations of the straightest paths we find two forms in the equations § 224 (i) and § 226 (i). If we multiply these by $m/S(t_1 - t_0)$, or, what is the same thing (ii), by $\sqrt{2mE}$, we obtain the four following sets of r equations—

$$q_{\rho 1} = \tfrac{1}{2} \frac{m}{t_1 - t_2} \frac{\partial S^2}{\partial p_{\rho 1}} \qquad \text{(iv)}$$

$$q_{\rho 0} = -\tfrac{1}{2} \frac{m}{t_1 - t_0} \frac{\partial S^2}{\partial p_{\rho 0}} \qquad \text{(v)}$$

$$q_{\rho 1} = \sqrt{2mE} \frac{\partial S}{\partial p_{\rho 1}} \qquad \text{(vi)}$$

$$q_{\rho 0} = -\sqrt{2mE} \frac{\partial S}{\partial p_{\rho 0}} \qquad \text{(vii)}.$$

Thus our problem is solved in a variety of ways.

For if we consider t_1 as the variable time, and consequently $p_{\rho 1}$ as the coordinates of the position changing with this time, the r equations (v) determine these r coordinates as finite functions of t_1, and the equations (vii) give us the same result if we associate with them the relation between E and t_1, *i.e.* the equation (ii). The $2r$ quantities $p_{\rho 0}$ and $q_{\rho 0}$ behave here as $2r$ arbitrary constants. From similar considerations the equations (iv), or (vi) and (ii), give us the equations of motion of the system; these are now in the form of differential equations of the first order, in which the r quantities $p_{\rho 0}$ behave as r arbitrary constants.

Or, if we consider, as is equally permissible, the time t_0 as the variable time, and thus the position 0 as the variable position, the equations (iv), or (vi) and (ii), give us the equations of motion in a finite form, with the time t_0 as independent, the quantities $p_{\rho 0}$ as dependent variables, and the quantities $p_{\rho 1}$ and $q_{\rho 1}$ as $2r$ arbitrary constants. Thus, again, the equations (v), or (vii) and (ii), give the equations of motion in the form of differential equations of the first order, in which $p_{\rho 1}$ behave as r arbitrary constants.

411. **Corollary 1.** If we put

$$\sqrt{2\mathrm{E}m}\,.\,\mathrm{S} = \mathrm{V} \qquad \text{(i)},$$

and consider V as a function of $p_{\rho 0}$, $p_{\rho 1}$ and E, then the natural motions of the system can be expressed in the form

$$q_{\rho 1} = \frac{\partial \mathrm{V}}{\partial p_{\rho 1}} \qquad \text{(ii)}$$

$$q_{\rho 0} = -\frac{\partial \mathrm{V}}{\partial p_{\rho 0}} \qquad \text{(iii)}$$

$$t_1 - t_0 = \frac{\partial \mathrm{V}}{\partial \mathrm{E}} \qquad \text{(iv)}.$$

For the equations (ii) and (iii) coincide with the equations § 410 (vi) and (vii), and the equation (iv) follows from the equation (i) and § 410 (ii).

412. **Observation.** The function V here introduced is Hamilton's Characteristic Function of the System; Hamilton

denotes it by the same symbol. Such a function, therefore, only exists for holonomous systems. Its mechanical meaning is this. Suppose that the system moves with given energy from a given initial to a given final position: then the characteristic function gives twice the value of that time-integral of the energy which results, considered as a function of that energy and of the coordinates of the initial and final positions.

For by equations 411 (i) and 410 (ii)

$$V = 2E(t_1 - t_0)$$

in value, but in form only when, on the right-hand side, we regard the duration of the motion $t_1 - t_0$ expressed as a function of E, $p_{\rho 1}$ and $p_{\rho 0}$.

413. **Proposition.** The characteristic function V of a free holonomous system satisfies the following two partial differential equations of the first order—

$$\frac{1}{2m}\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma 1} \frac{\partial V}{\partial p_{\rho 1}} \frac{\partial V}{\partial p_{\sigma 1}} = E$$

$$\frac{1}{2m}\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma 0} \frac{\partial V}{\partial p_{\rho 0}} \frac{\partial V}{\partial p_{\sigma 0}} = E.$$

For they are obtained by multiplying the equations § 227 for the straightest distance by $2mE$, and using equation § 411 (i).

414. **Corollary 2.** If we put

$$\frac{mS^2}{2(t_1 - t_0)} = P \qquad \text{(i)},$$

and regard P as a function of $p_{\rho 0}$, $p_{\rho 1}$, t_0 and t_1, the equations

$$q_{\rho 1} = \frac{\partial P}{\partial p_{\rho 1}} \qquad \text{(ii)}$$

$$q_{\rho 0} = -\frac{\partial P}{\partial P_{\rho 0}} \qquad \text{(iii)}$$

represent the natural motions of the system. The energy E of the system can be immediately obtained from P by means of the equations

$$\mathrm{E} = -\frac{\partial \mathrm{P}}{\partial t_1} = \frac{\partial \mathrm{P}}{\partial t_0} \qquad \text{(iv).}$$

For the equations (ii) and (iii) coincide with the equations § 410 (iv) and (v), and the equations (iv) follow from (i) and § 410 (ii).

415. **Observation.** The function P, here introduced, is Hamilton's Principal Function of the System; it is called by Hamilton S. Such a function exists only for holonomous systems. Its mechanical meaning is this. Suppose that the system moves in a given time from a given initial to a given final position: then the Principal Function gives that value of the time-integral of the energy which results, considered as a function of that time and of the initial and final values of the coordinates.

For by equations § 414 (i) and 410 (ii)

$$\mathrm{P} = \mathrm{E}(t_1 - t_0)$$

as regards its value, but as regards it form only when we regard E, on the right-hand side, as a function of $p_{\rho 1}$, $p_{\rho 0}$, t_1 and t_0.

416. **Proposition.** The principal function of a holonomous system satisfies the two following partial differential equations of the first order—

$$\frac{1}{2m}\sum_1^r{}_\rho\sum_1^r{}_\sigma b_{\rho\sigma 1}\frac{\partial \mathrm{P}}{\partial p_{\rho 1}}\frac{\partial \mathrm{P}}{\partial p_{\sigma 1}} = -\frac{\partial \mathrm{P}}{\partial t_1}$$

$$\frac{1}{2m}\sum_1^r{}_\rho\sum_1^r{}_\sigma b_{\rho\sigma 0}\frac{\partial \mathrm{P}}{\partial p_{\rho 0}}\frac{\partial \mathrm{P}}{\partial p_{\sigma 0}} = \frac{\partial \mathrm{P}}{\partial t_0}.$$

For these are obtained when the equations § 227 are multiplied by (§ 410 (ii))

$$\frac{m\mathrm{S}^2}{2(t_1 - t_0)^2} = \mathrm{E}$$

and the relations § 414 (i) and (iv) made use of.

417. **Observation on §§ 411-416.** Starting from the differential equations § 227, we were able to consider in §§ 232-236 functions which were related to the straightest distance and capable of replacing it in all respects analytically, but without having the same simple geometrical meaning. In

just the same way, starting from the differential equations §§ 413, 416, we can arrive at functions which are related to the characteristic and principal functions and analytically serve the same purpose, or even offer advantages over these; but their physical significance, on account of the mathematical complications, becomes more and more obscure. Such functions would be suitably denoted as Jacobi's Principal Functions and Characteristic Functions.

It appears, moreover, that even in the characteristic and principal functions it is only the simple idea of the straightest distance which appears, and this, too, somewhat indistinctly; so that the introduction of these two functions together and in addition to the straightest distance would have but little significance if all the systems to be considered were always, as here, completely known and free.

Dynamical Models

418. **Definition.** A material system is said to be a dynamical model of a second system when the connections of the first can be expressed by such coordinates as to satisfy the following conditions:—

(1) That the number of coordinates of the first system is equal to the number of the second.

(2) That with a suitable arrangement of the coordinates for both systems the same equations of condition exist.

(3) That by this arrangement of the coordinates the expression for the magnitude of a displacement agrees in both systems.

Any two of the coordinates so related to one another in the two systems are called corresponding coordinates. Corresponding positions, displacements, etc., are those positions, displacements, etc., in the two systems which involve similar values of the corresponding coordinates and their changes.

419. **Corollary 1.** If one system is a model of a second, then, conversely, the second is also a model of the first. If two systems are models of a third system, then each of these systems is also a model of the other. The model of the model of a system is also a model of the original system.

All systems which are models of one another are said to be dynamically similar.

420. **Corollary 2.** The property which one system possesses of being a model of another, is independent of the choice of the coordinates of one or the other system, although it is only clearly exhibited by a particular choice of coordinates.

421. **Corollary 3.** A system is not completely determined by the fact that it is a model of a given system. An infinite number of systems, quite different physically, can be models of one and the same system. Any given system is a model of an infinite number of totally different systems.

For the coordinates of the masses of the two systems which are models of one another can be quite different in number and can be totally different functions of the corresponding coordinates.

422. **Corollary 4.** The models of holonomous systems are themselves holonomous. The models of non-holonomous systems are themselves non-holonomous.

423. **Observation.** In order that a holonomous system may be a model of another, it is sufficient that both should have such free coordinates that the expression for the magnitude of the displacements of both systems should be the same.

424. **Proposition.** If two systems, each of which is a model of the other, have corresponding conditions at a definite time, then they have corresponding conditions at all times.

For by the equations of condition of a system, the expression for the magnitude of the displacement (§ 164) and the initial values of the coordinates and their change (§ 332), the course of these coordinates is determined for all times,—this being true whatever function of these coordinates the position of the masses of the system is.

425. **Corollary 1.** In order to determine beforehand the course of the natural motion of a material system, it is sufficient to have a model of that system. The model may be much simpler than the system whose motion it represents.

426. **Corollary 2.** If the same quantities are corresponding coordinates of a number of material systems which are models of one another, and if these corresponding coordinates alone are accessible to observation, then, so far as this limited observation is concerned, all these systems are not different from one another; they appear as like systems, however different in reality they may be in the number and the connection of their material points.

Thus it is impossible, from observation alone of the natural motions of a free system, *i.e.* without direct determination of its masses (§ 300), to obtain any wider knowledge of the connection of the system than that one could specify a model of the system.

427. **Observation 1.** If we admit generally and without limitation that hypothetical masses (§ 301) can exist in nature in addition to those which can be directly determined by the balance, then it is impossible to carry our knowledge of the connections of natural systems further than is involved in specifying models of the actual systems. We can then, in fact, have no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone,—that the one set of systems are models of the other.

428. **Observation 2.** The relation of a dynamical model to the system of which it is regarded as the model, is precisely the same as the relation of the images which our mind forms of things to the things themselves. For if we regard the condition of the model as the representation of the condition of the system, then the consequents of this representation, which according to the laws of this representation must appear, are also the representation of the consequents which must proceed from the original object according to the laws of this original object. The agreement between mind and nature may therefore be likened to the agreement between two systems which are models of one another, and we can even account for this agreement by assuming that the mind is capable of making actual dynamical models of things, and of working with them.

CHAPTER IV

MOTION OF UNFREE SYSTEMS

429. **Prefatory Note 1.** Every unfree system we conceive to be a portion of a more extended free system; from our point of view there are no unfree systems for which this assumption does not obtain. If, however, we wish to emphasise this relation, we shall denote the unfree system as a partial system, and the free system of which it forms a part, as the complete system.

430. **Prefatory Note 2.** When a part of a free system is considered an unfree system it is assumed that the rest of the system is more or less unknown, so that an immediate application of the fundamental law is impossible. This deficiency of knowledge must in some way be met by special data. Such data can be given in various ways. As it is not our purpose to take every possible form for these data, we shall only consider two forms which, in previous developments of mechanics, have obtained special significance.

In the first form the motion of the unfree system is denoted as guided; whilst in the second we say that the motion is affected by forces.

I. Guided Unfree System

431. **Definition.** A guided motion of an unfree system is any motion which the system performs while the other masses of the complete system perform a determinate and

prescribed motion. A system whose motion is guided is called a guided system.

432. **Additional Note 1.** A possible motion of a guided system is such a motion as is not inconsistent with the connection of the complete system and the prescribed motion of the other masses.

433. **Additional Note 2.** A natural motion of a guided system is such a motion as forms, with the prescribed motion of the remaining masses, a natural motion of the complete system.

434. **Problem.** To represent analytically the possible motions of a guided system.

Let the r quantities p_ρ be the general coordinates of the partial system considered, and the $\mathfrak{r}$ quantities $\mathfrak{p}_\rho$ be any coordinates whatever of the remaining masses of the complete system. The $r+\mathfrak{r}$ quantities p_ρ and $\mathfrak{p}_\rho$ are then general coordinates of the complete system, and its connections are expressible by a series of equations, say h in number, of the form

$$\sum_1^r{}_\rho\, p_{\chi\rho}\dot{p}_\rho + \sum_1^{\mathfrak{r}}{}_\rho\, \mathfrak{p}_{\chi\rho}\dot{\mathfrak{p}}_\rho = 0 \qquad \text{(i)},$$

where $p_{\chi\rho}$ and also $\mathfrak{p}_{\chi\rho}$ may be functions both of p_ρ and $\mathfrak{p}_\rho$. If now the motion of the masses whose coordinates are $\mathfrak{p}_\rho$ are determined, then the $\mathfrak{p}_\rho$'s are given functions of the time. The equations (i) are in part identically satisfied by these functions; in part they take, on substitution of these, the form of the r equations

$$\sum_1^r{}_\rho\, p_{\chi\rho}\dot{p}_\rho + p_{\chi t} = 0 \qquad \text{(ii)},$$

or

$$\sum_1^r{}_\rho\, p_{\chi\rho}dp_\rho + p_{\chi t}dt = 0 \qquad \text{(iii)},$$

which are called the equations of condition of the guided system, and in which $p_{\chi\rho}$ and $p_{\chi t}$ are now functions of p_ρ and the time t alone. All possible motions of the guided system satisfy these equations, and all motions which satisfy them are possible motions.

435. **Observation 1.** If the guided system is holonomous, then the differential equations (ii) and (iii) for it can be replaced by the same number of finite equations between the r coordinates of the system and the time t. The possible positions of a guided holonomous system can be expressed by coordinates which are subject to no other conditions than this, that a number of them are given functions of the time.

436. **Observation 2.** Thus the equations of condition of a guided system contain in general the time, and therefore the guided system, considered in itself, would be inconsistent with the requirements of normality (§ 119). Conversely, we now consider every system whose equations of condition in the ordinary language of mechanics contain the time explicitly, and which in our mode of expression is apparently abnormal, as a guided system, *i.e.* as a system which with other unknown masses satisfies the conditions of normality. If this assumption is permissible, then by it the problem is reduced to a determinate mechanical problem (§ 325). But if, owing to any particular form of the equations of condition, this assumption is not permissible, then these equations of condition already involve a contradiction to the fundamental law or its assumptions, and no questions asked concerning the system would be mechanical problems (§ 326).

437. **Observation 3.** The fundamental law is not directly applicable to a guided system. For the idea of straightest paths is only defined for normal connections (§ 120); and the internal connections of the guided system are abnormal. Some other characteristics must therefore be sought by which the natural motions of a guided system may be distinguished from the greater manifold of possible motions.

438. **Proposition 1.** A guided system, just like a free system, moves in such a manner that the magnitude of its acceleration is always smaller for the actual motion than for any other motion which satisfies the equations of condition and which, at the moment under consideration, coincides in position and velocity with the actual motion.

For the square of the magnitude of the acceleration of the complete system is equal to the sum of the corresponding quantities for the partial system and the remaining system, these quantities being multiplied by the masses of their systems and divided by the mass of the complete system. This sum, by § 344, is to be a minimum; the second member of the summation is supposed to be already determined and such a function of the time as allows the sum to be a minimum (§ 436); this minimum is then only obtained when the first member is made a minimum.

439. **Proposition 2.** A guided holonomous system, just like a free one, moves in such a manner that the time-integral of the energy in a motion between sufficiently near positions is smaller for the actual motion than for any other motion which satisfies the equations of condition, and which carries the system in the same time from the given initial to the final position.

For the time-integral of the energy for the complete system is equal to the sum of the corresponding quantities for the partial system and the remaining system. This sum is, by § 358, to be a minimum; the second member of the summation is supposed to be already determined and to be such as admits of a minimum sum; this minimum is then only obtained when the first member is made a minimum.

440. **Observation 1.** The two preceding propositions contain the adaptation of articles 344 and 358 to the special assumptions of this chapter. In the ordinary language of mechanics their contents could be put into the following form:—The Law of Least Acceleration and Hamilton's Principle still hold even where the equations of condition of a system contain the time explicitly.

441. **Observation 2.** The laws of energy, of the shortest path, and of the least time (§§ 340, 347, 352) can not be directly adapted in a similar manner to the assumptions of a guided system. In the ordinary language of mechanics this statement can be put in the following form:—The Principles of Energy and Least Action lose their validity when the equations of condition contain the time explicitly.

442. **Problem.** To obtain the differential equations of motion of a guided system.

Let, as before, m be the mass, p_ρ the coordinates, and f_ρ the accelerations along p_ρ for the guided system. Further, let $\mathfrak{m}$ be the mass and $\mathfrak{p}_\rho$ any coordinates of the remaining material points of the complete system. Thus p_ρ and $\mathfrak{p}_\rho$ may be taken as coordinates of the complete system. The components of the acceleration along these coordinates may be denoted for the complete system by f'_ρ and $\mathfrak{f}'_\rho$. Then the motion of the complete system is singly determined by its h equations of condition of the form § 434 (i), and by $r+\mathfrak{r}$ equation of motion of the form (§ 372)

$$(m+\mathfrak{m})f'_\rho + \sum_1^h{}_\chi p_{\chi\rho}\mathrm{P}_\chi = 0 \qquad \text{(i)}$$

$$(m+\mathfrak{m})\mathfrak{f}'_\rho + \sum_1^h{}_\chi \mathfrak{p}_{\chi\rho}\mathrm{P}_\chi = 0 \qquad \text{(ii)}.$$

Now by hypothesis we have to regard the quantities $\mathfrak{p}_\rho$ as such determined functions of the time as identically satisfy the equations (ii), and through whose substitution the h equations of condition of the complete system are transformed into the k equations of condition (§ 434 (ii)) of the constrained system. Further, by § 255 we have

$$(m+\mathfrak{m})f'_\rho = mf_\rho \qquad \text{(iii)}.$$

Thus we obtain as equations to be considered the r equations of motion

$$mf_\rho + \sum_1^k{}_\chi p_{\chi\rho}\mathrm{P}_\chi = 0 \qquad \text{(iv)},$$

and the k equations of condition

$$\sum_1^r{}_\rho p_{\chi\rho}\dot{p}_\rho + p_{\chi t} = 0 \qquad \text{(v)}.$$

These $(r+k)$ equations do not now contain any reference to the unknown masses of the complete system; and as they are sufficient for the unique determination of the $r+k$ quantities $\ddot{p}_\rho$ and P_χ, they contain the solution of the stated problem.

443. **Corollary 1.** The differential equations of motion of a guided system have the same form as those of a free system.

In the ordinary language of mechanics we may say that the validity of this form does not depend on whether the equations of condition contain the time or not. The equations of motion of a guided system will therefore admit of exactly the same transformations as those of a free system (368 *et seq.*); but of course those forms which assume that all the coordinates are free will lose their applicability.

444. **Corollary 2.** A natural motion of a guided system is singly determined by a knowledge of the position and velocity of the system at any given time (*cf.* § 331).

445. **Note.** In a guided, as in a free system, the constraint is equal to the acceleration of the system.

For if all the equations of condition of a guided system are removed, then the material points of the system will be free points and the acceleration of the natural motion of the system will be zero (§ 385).

446. **Proposition 1.** The magnitude of the constraint at any instant in a guided system, as in a free one, is smaller for the natural motion than for any other possible motion which, at the moment considered, coincides with it in position and velocity.

The proof follows from §§ 445 and 448.

447. **Proposition 2.** In the natural motion of a guided system, as in that of a free system, the direction of the constraint is always perpendicular to every possible or virtual displacement of the system from its instantaneous position.

This follows from §§ 445 and 442 as in § 392.

448. **Observation.** The two foregoing propositions contain the adaptation of propositions 388 and 392 to the particular case of guided systems. In the usual language of mechanics they might be expressed in the following form:—Gauss's Principle of Least Constraint and d'Alembert's Principle retain their validity even when the equations of condition contain the time explicitly.

449. **Note.** When the coordinates $\mathfrak{p}_\rho$ of the complete system which appear together with p_ρ in the equations § 434 (i) are not functions of the time, but are constant, then the equations of condition of the guided system take the form

$$\sum_1^r{}_\rho p_{\chi\rho}\dot{p}_\rho = 0,$$

where the $p_{\chi\rho}$'s do not contain the time. The guided system appears in this case as a normal one, but it does not necessarily cease to be an unfree system. For $p_{\chi\rho}$ may be functions of the absolute position, whilst in the equations of condition of a free system they are independent of the absolute position.

In such guided, but nevertheless normal systems, the idea of the straightest path retains its applicability. It also follows that the fundamental law is immediately applicable to such systems; and all the propositions which have been proved for the motion of a free system also hold good for such systems, excepting only those which refer to absolute position, *i.e.* excepting only proposition 400 and its corollaries.

II. Systems acted on by Forces

450. **Definition.** Two material systems are said to be directly coupled (*gekoppelt*) when one or more coordinates of the one are always equal to one or more coordinates of the other. Two systems will be simply said to be coupled when their coordinates can be so chosen that the systems become directly coupled. Coupled systems which are not directly coupled are said to be indirectly coupled.

451. **Corollary 1.** The coupling of two systems is a relation between them which is independent of our choice, and in particular independent of the choice of coordinates. But whether an existing coupling is direct or indirect does depend on the choice of coordinates, and is thus a question for our arbitrary determination.

452. **Corollary 2.** Every coupling which exists between two systems can be made direct by a proper choice of coordinates. When the contrary is not definitely expressed, we shall hereafter assume that this has been done. The coordinates of the coupled systems which are always equal we shall denote as their common coordinates.

453. **Corollary 3.** Each of two coupled systems is

necessarily an unfree system; but both together, or with other systems with which they are coupled, form a free system. When the contrary is not expressly stated it will be assumed in what follows that there is no coupling with more systems, so that the two coupled systems together form a free system.

454. **Analytical Representation.** Let p_ρ be the coordinates of the one, $\mathfrak{p}_\rho$ of the other system; then a coupling between the two systems is expressed by the fact that for one or more pairs of values of ρ and σ, p_ρ and $\mathfrak{p}_\sigma$ are always equal. We can, however, without restricting the generality, so arrange the indices that congruent coordinates in both systems have the same index. The systems are then coupled when for one or more values of ρ

$$\mathfrak{p}_\rho - p_\rho = 0 \qquad \text{(i)}$$

continually. From this equation the equations

$$\dot{\mathfrak{p}}_\rho - \dot{p}_\rho = 0 \qquad \text{(ii)}$$

or

$$d\mathfrak{p}_\rho - dp_\rho = 0 \qquad \text{(iii)}$$

immediately follow.

455. **Definition.** By a force we understand the independently conceived effect which one of two coupled systems, as a consequence of the fundamental law, exerts upon the motion of the other.

456. **Corollary.** To every force there is necessarily always a counterforce (*Gegenkraft*).

For the conception of the effect which the system, referred to in the definition as the second, produces upon the first, is by the definition itself also a force. Force and counterforce are reciprocal in the sense that we are free to consider either of them as the force or the counterforce.

457. **Problem.** To obtain an expression for the effect which one of two coupled systems produces upon the motion of the other.

Let m be the mass, and the r quantities p_ρ the coordinates of the first system; and let the k equations

$$\sum_1^r \rho\, p_{\chi\rho}\dot{p}_\rho = 0 \qquad \text{(i)}$$

be its equations of condition. Let $\mathfrak{m}$ be the mass and the $\mathfrak{r}$ quantities $\mathfrak{p}_\rho$ the coordinates of the second system; and let the $\mathfrak{k}$ equations

$$\sum_1^r \rho\, \mathfrak{p}_{\chi\rho}\dot{\mathfrak{p}}_\rho = 0 \qquad \text{(ii)}$$

be its equations of condition. Between the two there may further be for one or more, say h, values of ρ, equations of coupling of the form

$$\dot{\mathfrak{p}}_\rho - \dot{p}_\rho = 0 \qquad \text{(iii).}$$

Let us now consider the motion of the first system under the action of the second, and regard it as a guided system. So long as the p_ρ's do not appear in the equations (iii) the accelerations along them are given by the equations (442)

$$mf_\rho + \sum_1^k \chi\, p_{\chi\rho}\mathrm{P}_\chi = 0 \qquad \text{(iv);}$$

but for those p_ρ's which appear in (iii) we must take into consideration these equations as well, and consequently multiply the coefficient of $\dot{p}_\rho$ in them, namely -1, by an undetermined factor which may be called P_ρ, and add the product to the left-hand side; thus, then, for these—

$$mf_\rho + \sum_1^k \chi\, p_{\chi\rho}\mathrm{P}_\chi - \mathrm{P}_\rho = 0 \qquad \text{(v).}$$

The appearance of the h quantities P_ρ in the equations of motion increases the number of unknowns in them by h, and for the determination of these h quantities the number of equations of condition is also increased by the h equations (iii), in which we must regard the $\dot{\mathfrak{p}}_\rho$'s as given explicit functions of the time. But if we assume that the quantities P_ρ are not unknown, but are given immediately as functions of the time, then the h equations (iii) and any knowledge of $\dot{\mathfrak{p}}_\rho$ and of the second system are unnecessary; the $k+r$ equations (i), (iv) and (v) are again sufficient for the unique determination of the $k+r$ unknowns P_χ and $\ddot{p}_\rho$. The h factors P_ρ consequently represent completely the effect of the second system on the first, and their aggregate can be regarded as an analytical expression for this effect, as is required by the problem.

458. **Theorem 1.** If we wish to represent in a symmetrical manner the effect of the first system on the second, we must write the equations of coupling in the form

$$\dot{p}_\rho - \dot{\mathfrak{p}}_\rho = 0 \qquad \text{(i)},$$

and for the $\mathfrak{p}_\rho$'s which do not appear in (i) we have the following equations of motion

$$\mathfrak{m}\mathfrak{f}_\rho + \sum_1^{\mathfrak{k}} {}_\chi \mathfrak{p}_{\chi\rho} \mathfrak{P}_\chi = 0 \qquad \text{(ii)},$$

while for the remaining $\mathfrak{p}_\rho$'s they take the form

$$\mathfrak{m}\mathfrak{f}_\rho + \sum_1^{\mathfrak{k}} {}_\chi \mathfrak{p}_{\chi\rho} \mathfrak{P}_\chi - \mathfrak{P}_\rho = 0 \qquad \text{(iii)};$$

where by the $\mathfrak{P}_\rho$'s are understood the undetermined multipliers of equations (i). The aggregate of the $\mathfrak{P}_\rho$'s gives us an expression for the effect which the first at any instant has on the motion of the second.

459. **Theorem 2.** Thus we can write all equations of motion of the first system in the form

$$mf_\rho + \sum_1^{k} {}_\chi p_{\chi\rho} \mathrm{P}_\chi - \mathrm{P}_\rho = 0 \qquad \text{(i)},$$

and of the second in the form

$$\mathfrak{m}\mathfrak{f}_\rho + \sum_1^{\mathfrak{k}} {}_\chi \mathfrak{p}_{\chi\rho} \mathfrak{P}_\chi - \mathfrak{P}_\rho = 0 \qquad \text{(ii)},$$

when we decide (which is permissible, although arbitrary) that for all coordinates which are not coupled the quantities P_ρ and $\mathfrak{P}_\rho$ are to be zero. It is true that P_ρ and $\mathfrak{P}_\rho$ thereby lose their significance as a system of multipliers of the equations 457 (iii) and 458 (i); but they retain their significance as an expression for the effect which the one system has on the other.

460. **Analytical Representation of Force.** In accordance with the definition § 455 we may and shall decide that the aggregate of the quantities P_ρ, singly determined for all values of p_ρ by § 459, forms the analytical expression for the force

which the system $\mathfrak{p}_\rho$ exerts on the system p_ρ. In a similar manner the aggregate of the quantities $\mathfrak{P}_\rho$ forms the analytical expression for the force which the system p_ρ exerts on the system $\mathfrak{p}_\rho$. The individual quantities P_ρ or $\mathfrak{P}_\rho$ are called the components of the force along the corresponding coordinates p_ρ or $\mathfrak{p}_\rho$, or, for short, the forces along these coordinates.

By this determination we place ourselves in agreement with the existing notation of mechanics; and the necessity for securing such an agreement sufficiently justifies us in choosing this particular determination out of several permissible ones.

461. **Corollary 1.** The force which a system exerts on a second may be considered a vector quantity with regard to the second system: *i.e.* as a vector quantity whose components along the common coordinates are in general different from zero; whose components along the coordinates which are not common vanish; but whose components in such directions as cannot be expressed by changes of the coordinates used remain undetermined.

462. **Corollary 2.** The force which one system exerts on another may also be considered as a vector quantity with regard to the first system: *i.e.* as a vector quantity whose components along the common coordinates are in general different from zero; whose components along the coordinates which are not common vanish; but whose components in such directions as cannot be expressed by means of changes in the coordinates used remain undetermined.

463. **Observation.** Considered as vector quantities with regard to a system, every force contains components which depend on the choice of coordinates, *i.e.* on arbitrary convention. This arises from the fact that on the choice of coordinates depends the manifold of those motions of a system which we take into consideration; and in the direction of which we may therefore admit a possible effect.

464. **Note 1.** If a system is coupled successively with several other systems, and the same force is thereby exerted upon it by these systems, then its motion is the same, however much these other systems may differ amongst themselves.

We therefore speak (in accordance with definition § 455) of the motion of a system under the influence or action of a force simply, without mentioning the other system to which it is due, and without which it could not be conceived.

465. **Note 2.** If a system is coupled successively with several other systems, and the same motion results, then it may exert upon each of these other systems the same force, even though these systems may be entirely different from each other.

We therefore speak (in accordance with definition § 455) of the force which a moving system exerts simply, without mentioning the other system upon which this force is exerted, and without which it could not be conceived.

466. **Note 3.** Since all forces which are simply spoken of as such can be no other than those which are exerted by material systems on material systems in accordance with the fundamental law, all forces must as a matter of course have certain properties common. The sources of all such common properties are the properties of material systems and the fundamental law.

Action and Reaction

467. **Notation.** (1) The components of the force which the $\mathfrak{p}_\rho$ system exerts on the p_ρ one, considered as vector quantities with regard to the p_ρ system, have already been denoted in § 460 by P_ρ. If we regard this same force as a vector quantity with regard to the system $\mathfrak{p}_\rho$, then its components along $\mathfrak{p}_\rho$ will be denoted by $\mathfrak{P}_\rho'$. Thus for all common coordinates

$$\mathrm{P}_\rho = \mathfrak{P}_\rho'$$

identically.

(2) The components of the force which the p_ρ system exerts on the $\mathfrak{p}_\rho$ system, considered as vector quantities with regard to the $\mathfrak{p}_\rho$ system, have already been denoted in § 460 by $\mathfrak{P}_\rho$. If we regard this same force as a vector quantity with regard to the p_ρ system, then its components along p_ρ will be denoted by P_ρ'. Thus for all coordinates

$$\mathfrak{P}_\rho = \mathrm{P}_\rho'$$

identically.

The forces exerted on a system are thus denoted by unaccented letters, whilst the forces exerted by a system are denoted by accented letters, as long as we regard them as vector quantities with regard to the system itself.

468. **Proposition.** Force and counterforce are always equal and opposite. By this is meant that their components, along each of the coordinates used, are equal and opposite; and this is equally true whether we regard force and counterforce as vector quantities with regard to the one or the other system.

For we may regard the two coupled systems (§ 457) as a single free system. Its mass is $m+\mathfrak{m}$, and its coordinates are p_ρ and $\mathfrak{p}_\rho$. Its equations of condition are the equations 457 (i) and (ii) and the equations of coupling as in 457 (iii). If in addition we denote the multipliers of the equations (i) by P_χ°, those of equation (ii) by $\mathfrak{P}_\chi^\circ$, and those of the equation (iii) by P_ρ°, then the equations of motion of the total system take the form

$$m f_\rho + \sum_1^{k}{}_\chi\, p_{\chi\rho} P_\chi^\circ - P_\rho^\circ = 0 \qquad \text{(i)}$$

$$\mathfrak{m}\mathfrak{f}_\rho + \sum_1^{\mathfrak{k}}{}_\chi\, \mathfrak{p}_{\chi\rho} \mathfrak{P}_\chi^\circ + P_\rho^\circ = 0 \qquad \text{(ii)},$$

where, for the coordinates that do not appear in the equations of coupling, the P_ρ°'s are to be put zero.

But the motion represented by these equations is that which before was considered as the motion of the separate systems. We consequently obtain a possible solution for the above equations when we substitute for f_ρ and $\mathfrak{f}_\rho$ their former values, and make

$$P_\chi^\circ = P_\chi, \quad \mathfrak{P}_\chi^\circ = \mathfrak{P}_\chi \qquad \text{(iii)},$$

and in (i)

$$P_\rho^\circ = \quad P_r \qquad \text{(iv)},$$

and in (ii)

$$P_\rho^\circ = -\mathfrak{P}_\rho \qquad \text{(v)}.$$

Moreover, since the undetermined multipliers are singly determined by the equations (i) and (ii), this possible solution is at the same time the only possible solution. Therefore the equations (iv) and (v) necessarily hold; thus from them

$$\mathrm{P}_\rho = -\mathfrak{P}_\rho,$$

or, using the notation of § 467,

$$\mathrm{P}_\rho = -\mathrm{P}_\rho'$$
$$\mathfrak{P}_\rho = -\mathfrak{P}_\rho',$$

which proves the proposition.

469. **Observation 1.** The foregoing proposition corresponds to Newton's Third Law, and is also known as the Principle of Reaction. Nevertheless its content is not quite identical with that of Newton's Third Law. Their true relation is as follows:—

Newton's Law, as he intended it to be understood, contains our proposition completely; this is shown by the examples appended to his statement of the law.

But Newton's Law contains more. At least it is usually applied to actions-at-a-distance, *i.e.* to forces between bodies which have no common coordinates. But our mechanics does not recognise such actions. Thus in order to be able to adduce as a consequence of our proposition the fact that a planet attracts the sun with the same force that the sun attracts the planet, it is necessary that further data should be given as to the nature of the connection between the two bodies.

470. **Observation 2.** It is open to doubt, whether the extension of the application of the principle of reaction beyond what is contained in proposition 468 as to its form and content, can rightly be used as a fundamental principle of mechanics; or whether rather the actual and universally valid content of that principle has not been completely included in proposition 468.

As far as the form is concerned, it is manifest that the statement of the law is not quite clearly determined when applied to actions-at-a-distance. For when force and counterforce affect different bodies, it is not quite clear what is meant by opposite. For example, this is seen in the case of the mutual action between current-elements.

As far as the content is concerned, the application of the principle of reaction to actions-at-a-distance commonly found in mechanics manifestly represents an experiential fact, concerning the correctness of which in all cases people are beginning to be doubtful. For instance, in Electromagnetics we are almost convinced that the mutual action between moving magnets is not in all cases strictly subject to the principle.

Composition of Forces

471. **Proposition.** If a system is simultaneously coupled with several other systems, then the force which the aggregate of these systems exerts on the first is equal to the sum of the forces which the individual systems exert on it.

For let there be a system 1 of mass m and coordinates p_ρ, whose equations of condition are the k equations,

$$\sum_1^r {}_\rho p_{\chi\rho}\dot{p}_\rho = 0 \qquad \text{(i)},$$

and let this be simultaneously coupled with the systems 2, 3, etc., whose coordinates are $\mathfrak{p}_\rho''$, $\mathfrak{p}_\rho'''$, etc.

First consider the systems 2, 3, etc., as separate systems. Then the equations of coupling for every common coordinate p_ρ are to be written in the form

$$\dot{\mathfrak{p}}_\rho'' - \dot{p}_\rho = 0 \qquad \text{(ii)}$$

$$\dot{\mathfrak{p}}_\rho''' - \dot{p}_\rho = 0, \text{ etc.} \qquad \text{(iii)}.$$

If now we treat the system made up of 1, 2, 3, etc., as a free system, and denote the multipliers of the equations (i) by P_χ, those of (ii) by P_ρ'' and of (iii) by P_ρ''', etc., then we obtain the equations of motion of the system 1 in the form

$$mf_\rho + \sum^k {}_\chi p_{\chi\rho}P_\chi - P_\rho'' - P_\rho''' - \text{etc.} = 0 \qquad \text{(iv)},$$

where all the quantities P_ρ'', P_ρ''', etc., as well as P_χ are singly-determined quantities. P_ρ'', P_ρ''', etc., represent the components of the forces which the systems 2, 3, etc., respectively exert on the system 1.

Secondly, if we regard the systems 2, 3, etc., as forming one system, then for the quantities $\mathfrak{p}_\rho''$, $\mathfrak{p}_\rho'''$, etc., which by equations (ii), (iii), etc., are equal, one single coordinate $\mathfrak{p}_\rho$ can be used, and in place of those equations of coupling we have now for each common coordinate p_ρ the one equation

$$\dot{\mathfrak{p}}_\rho - \dot{p}_\rho = 0 \qquad \text{(v)}.$$

If P_ρ is its multiplier, and we denote by $P_\chi{}^\circ$ the multipliers of the equations (i) which correspond to the present system of equations of motion, then these take the form

$$mf_\rho + \sum_1^k{}_\chi p_{\chi\rho} P_\chi{}^\circ - P_\rho = 0 \qquad \text{(vi)}.$$

The P_ρ's represent the components of the total force exerted on the system 1.

Now this different mode of conception cannot alter the motion which ensues according to the fundamental law. Therefore a possible solution of the equations (vi) is obtained by using the former solution and putting

$$P_\chi{}^\circ = P_\chi \qquad \text{(vii)}$$

$$P_\rho = P_\rho{}'' + P_\rho{}''' + \ldots \qquad \text{(viii)}.$$

Moreover, since there is only one possible solution, the foregoing is the one, and the equation (viii) which contains our proposition must necessarily hold.

472. **Corollary 1.** Any number of forces exerted on a system, or by a system, can be regarded as a single force, namely, that force which, considered as a vector quantity with regard to the system, is equal to the sum of these forces.

When we represent a number of forces in this way, we say that we compound them. The result of the composition is called the resultant of the individual forces.

473. **Corollary 2.** Any force exerted on a system, or by a system, can be conceived as a sum of any number of forces, namely, of any number of forces the sum of which, regarded as vector quantities with regard to the system, is equal to that original force.

When we represent a force in this way, we say that we resolve it; the forces which result from such a resolution we call the components of the original force.

474. **Observation.** The geometrical components of a force

along the coordinates can at the same time be considered components in the sense of § 473.

475. **Definition.** A force which is exerted by a single material point, or on a single material point, is called an elementary force.

476. **Observation.** As a rule, elementary mechanics means by forces only elementary forces. By way of distinction, the more general forms of forces hitherto considered by us are denoted as Lagrangian forces. Similarly we might denote the elementary forces as Galilean or Newtonian forces.

477. **Corollary 1.** Every elementary force can be represented by the geometrical displacement of a point, and therefore by a straight line given in magnitude and direction.

For each elementary force is a vector quantity with regard to a single point.

478. **Corollary 2.** The composition of the elementary forces, which act at the same point, is performed according to the method of geometrical composition and resolution of straight lines.

In particular, two forces acting at the same point can be combined into a single force, which is represented in magnitude and direction by the diagonal of a parallelogram whose sides represent these forces in magnitude and direction (Parallelogram of Forces).

479. **Corollary 3.** Every Lagrangian force can be represented as a sum of elementary forces, and is therefore capable of being resolved into elementary forces.

For every displacement of a system can be conceived as a sum of displacements of its individual points.

480. **Corollary 4.** The components of a force along the rectangular coordinates of the system on which the force acts, or which exerts the force, can be directly conceived as elementary forces, which act on the individual material points of the system.

Motion under the Action of Forces

481. **Problem 1.** To determine the motion of a material system under the action of a given force.

The solution follows directly from § 457. Let the P_ρ's be the given components of the force acting along p_ρ, then one uses the r equations

$$mf_\rho + \sum_1^k \chi p_{\chi\rho} P_\chi = P_\rho$$

together with the k equations of condition of the system for the determination of the $r+k$ quantities $\ddot{p}_\rho$ and P_χ, and these equations are sufficient to determine them without ambiguity.

482. **Observation 1.** The equations of motion of a system acted on by forces have in rectangular coordinates the form of the $3n$ equations

$$m_\nu \ddot{x}_\nu + \sum_1^i \iota x_{\iota\nu} X_\iota = X_\nu,$$

where the X_ν's are the components of the force along x_ν, and for the rest the notation of § 368 is used.

483. **Observation 2.** If the coordinate p_ρ is a free coordinate, then the equation of motion corresponding to it takes the simple form

$$mf_\rho = P_\rho,$$

If in a holonomous system all the coordinates p_ρ are free, then all the equations of motion of the system take this form, and these r equations are sufficient to determine the r quantities $\ddot{p}_\rho$.

484. **Corollary.** The natural motion of a material system from a given instant onwards is singly determined by position and velocity of the system at that instant and the knowledge of the forces acting on the system for all times from that instant onwards (*cf.* §§ 331, 444).

485. **Proposition.** The acceleration which a number of forces simultaneously acting produce in a system is equal to the sum of the accelerations which each force acting alone would produce.

For the equations of motion § 481 are linear in f_ρ and P_χ. Thus if the value-systems $f_{\rho 1}$ $P_{\chi 1}$, $f_{\rho 2}$ $P_{\chi 2}$, etc., are the solutions for these equations for the forces $P_{\rho 1}$, $P_{\rho 2}$, etc., then the value-system $f_{\rho 1}+f_{\rho 2}+\ldots$, $P_{\chi 1}+P_{\chi 2}\ldots$ is the solution for the force $P_{\rho 1}+P_{\rho 2}+\ldots$

486. **Observation.** The content of the above proposition can also be rendered by the statement that any number of simultaneously acting forces are independent of one another with regard to the acceleration which they produce. This principle has been known and used since Galileo's time.

487. **Corollary.** The acceleration which the resultant of any number of forces produces in a system is equal to the sum of the accelerations which the components acting alone would produce on the system (§§ 472, 473).

488. **Proposition.** If a force, as a vector quantity, is perpendicular to every possible displacement of a material system, then it has no effect on the motion of the system—and conversely.

For if π is such a force, then its components π_ρ along p_ρ have the form (§ 250)

$$\pi_\rho = \sum_1^k{}_\chi p_{\chi\rho}\gamma_\chi.$$

If now this force be made to act on the system in addition to the force P, then the equations of motion can be written in the form

$$mf_\rho + \sum_1^k{}_\chi p_{\chi\rho}\left(P_\chi - \gamma_\chi\right) = P_\rho.$$

In the solution of these equations with regard to $\ddot{p}_\rho$ and P_χ the P_χ's alone are increased by γ_χ; the $\ddot{p}_\rho$'s, which alone determine the motion, remain unaltered.

Conversely—if the addition of the components π_ρ to the right-hand side of the equations § 481 does not alter f_ρ, but only P_χ, then π_ρ can be written in the form

$$\pi_\rho = \sum_1^k{}_\chi p_{\chi\rho}\gamma_\chi.$$

Thus the force π is perpendicular to every possible displacement of the system (§ 250).

489. **Observation.** The proposition states to what condition that part of a force, considered as a vector quantity, is subjected, which depends upon the choice of coordinates and therefore upon our will (§ 463). For this part must necessarily be such as to have no effect on the actual motion.

490. **Corollary.** Although the motion of a system can be obtained without ambiguity from a knowledge of the forces which act on the system, still the force which acts on a system can not be determined without ambiguity from the motion of the system.

491. **Problem 2.** To determine the force which a material system exerts in a given motion.

In accordance with § 467 we denote by P_ρ' the component of the force required along p_ρ; then by §§ 468 and 481 we get

$$P_\rho' = mf_\rho - \sum_1^k {}_\chi p_{\chi\rho} P_\chi.$$

In these equations the f_ρ's are to be considered as given, and must, moreover, satisfy the equations of condition. The quantities P_χ are likewise determined, when that system is given with which the one considered is coupled. But when only the motion of the p_ρ system is given, the P_χ's remain unknown. The force which a moving system exerts is thus not completely determined by the knowledge of the motion of the system alone, but contains an undetermined summation whose components have the form

$$\pi_\rho = \sum_1^k {}_\chi p_{\chi\rho} \gamma_\chi,$$

and which is therefore perpendicular to every possible displacement of the system.

492. **Observation.** Although all the components of the force exerted by a moving system are not singly determined by the motion of the system, yet the components in the direction of every possible displacement of the system are singly determined by its motion.

493. **Corollary.** The components of the force which a

moving system exerts in the direction of every free coordinate of the system are singly determined by the motion.

For if p_ρ is a free coordinate, then the $p_{\chi\rho}$'s vanish, and with them the undetermined series; thus the component of the force of the system along p_ρ can be written in the forms

$$\begin{aligned} P_\rho' &= -mf_\rho && (\S\ 491) \quad (i) \\ &= \frac{\partial_p E}{\partial p_\rho} - \frac{d}{dt}\left(\frac{\partial_p E}{\partial p_\rho}\right) && (\S\ 291)\ (i) \quad (ii) \\ &= \frac{\partial_p E}{\partial p_\rho} - \dot{q}_\rho && (\S\ 291)\ (ii) \quad (iii) \\ &= -\frac{\partial_q E}{\partial p_\rho} - \dot{q}_\rho && (\S\ 294) \quad (iv). \end{aligned}$$

Internal Constraint

494. **Proposition.** The acceleration of a system of material points between which no connections exist, takes place in the direction of the force which acts on the system, and its magnitude is equal to the magnitude of the force, divided by the mass of the system.

For when no connections exist between the n points of a material system, then for every one of the $3n$ rectangular coordinates of the system (§ 482)

$$\frac{m_\nu}{m}\ddot{x}_\nu = \frac{X_\nu}{m};$$

but the left-hand side of the equation represents the components of the acceleration of the system along x_ν (§ 275).

495. **Corollary.** The acceleration of a single material point takes place in the direction of the force acting on it, and its magnitude is equal to the magnitude of the force, divided by the mass of the point (Newton's Second Law).

496. **Observation.** If connections exist between the points of a material system on which a force acts, then the acceleration of the system differs in general from that given by proposition

§ 494. We may thus regard the connections of the system as the cause of this difference, and by § 385 we have to denote this difference as the internal constraint of the system.

497. **Problem.** To determine the internal constraint of a system which moves under the action of forces.

The actual component of the acceleration of the system along the general coordinate p_ρ is f_ρ; the component which would arise if the equations of condition did not exist is (§ 494) P_ρ/m; the difference of the two quantities, or

$$z_\rho = f_\rho - \frac{P_\rho}{m} \qquad \text{(i)}$$

is thus the component of the constraint along p_ρ.

The knowledge of the components themselves along p_ρ is in general insufficient for the determination of the magnitude of the constraint. If, however, we use rectangular coordinates, we obtain for the component along x_ν

$$z_\nu = \frac{1}{m}(m_\nu \ddot{x}_\nu - X_\nu) \qquad \text{(ii)},$$

and consequently for the magnitude z of the constraint the positive root of the equation (§ 244)

$$\begin{aligned} mz^2 &= \sum_1^{3n}{}_\nu \frac{1}{m_\nu}(m_\nu \ddot{x}_\nu - X_\nu)^2 \\ &= \sum_1^{3n}{}_\nu m_\nu \left(\ddot{x}_\nu - \frac{X_\nu}{m_\nu}\right)^2 \end{aligned} \qquad \text{(iii)}.$$

498. **Proposition 1.** The magnitude of the constraint of a material system under the action of forces is at every instant, as in a free system, smaller for the natural motion than for any other possible motion which coincides with it, at the moment considered, in position and velocity.

For the necessary and sufficient condition that, with given values of X_ν, the quantity $\frac{1}{2}mz^2$ should be a minimum, is that the $3n$ equations, obtained as in § 155, viz.,

$$m_\nu \ddot{x}_\nu - X_\nu + \sum_1^{i}{}_\iota x_{\iota\nu} X_\iota = 0,$$

should be satisfied; where X_ι denote the i undetermined multipliers which with the $3n$ quantities $\ddot{x}_\nu$ are to be singly determined from these $3n$ equations and the i equations of condition of the system. But the foregoing equations give the same values for $\ddot{x}_\nu$ and X_ν as the corresponding equations for the natural motion (§ 482).

499. **Observation.** The foregoing proposition contains a complete statement of Gauss's Principle of Least Constraint. We might regard proposition 388 as a particular case of it. But according to our general conception we prefer to regard that proposition as the general one, and to consider the foregoing as the application of it to particular and more complex relations.

500. **Proposition 2.** In the natural motion of a system under the action of a force the direction of the constraint, as in the natural motion of a free system, is always perpendicular to every possible or virtual displacement of the system from its instantaneous position.

For by §§ 497 (i) and 481 the components of the constraint along p_ρ may be written in the form

$$z_\rho = -\frac{1}{m}\sum_1^k {}_\chi p_{\chi\rho} P_\chi.$$

The constraint as a vector quantity is thus (§ 250) perpendicular to every possible displacement of the system.

501. **Symbolical Expression.** If we denote by δp_ρ the changes of the coordinates p_ρ for any possible displacement of the system, then we can express the foregoing proposition by the following symbolical equation (*cf.* § 393)—

$$\sum_1^r {}_\rho\left(f_\rho - \frac{P_\rho}{m}\right)\delta p_\rho = 0 \qquad \text{(i)},$$

which in rectangular coordinates takes the form

$$\sum_1^{3n} {}_\nu(m_\nu\ddot{x}_\nu - X_\nu)\delta x_\nu = 0 \qquad \text{(ii)}.$$

502. **Observation.** Proposition § 500 contains the complete Principle of d'Alembert, and the equations § 501 (i) and (ii) the usual expression for it. With regard to the relation between

proposition § 500 and proposition § 392 the same remark is to be made as in § 499.

503. **Corollary 1.** The component of the acceleration of a material system in the direction of any possible motion is equal to the component of the force acting in this direction, divided by the mass of the system.

For the component of the constraint vanishes in the direction of every possible motion.

504. **Corollary 2.** The component of the acceleration of a material system in the direction of its actual motion is equal to the component of the force acting in that direction, divided by the mass of the system.

505. **Corollary 3.** The component of the acceleration of a material system along any free coordinate of the system is equal to the component of the force acting in that direction, divided by the mass of the system.

506. **Proposition.** In the natural motion of a material system under the action of forces the component of the acceleration along every coordinate of absolute position is always equal to the component of the force acting in that direction, divided by the mass of the system; and this holds good whatever the internal connection of the system is.

507. **Corollary 1.** If we choose the coordinates of a system in any manner so that there are six coordinates of absolute position amongst them, then we can with a knowledge of the forces acting on the system,—yet without a knowledge of the internal connection of the system,—always obtain six of the equations of motion of the system.

508. **Corollary 2.** In particular, if we arrange the coordinates of absolute position as in § 402, and apply the proposition to the direction of the three coordinates a_1, a_2, a_3, then we get the three equations

$$\sum_1^n {}_\nu m_\nu \ddot{x}_{3\nu} = \sum_1^n {}_\nu X_{3\nu}$$

$$\sum_1^n {}_\nu m_\nu \ddot{x}_{3\nu-1} = \sum_1^n {}_\nu X_{3\nu-1}$$

$$\sum_1^n{}_\nu m_\nu \ddot{x}_{3\nu-2} = \sum_1^n{}_\nu X_{3\nu-2}.$$

These three equations, which admit of the interpretation that the centre of gravity moves as if the whole mass were condensed at the centre of gravity and all the elementary forces applied there, form the so-called extended Principle of the Centre of Gravity (*cf.* § 404).

509. **Corollary 3.** Applied to the direction of the three coordinates of absolute position ω_1, ω_2, ω_3, the proposition gives the three equations

$$\sum_1^n{}_\nu m_\nu(x_{3\nu-2}\ddot{x}_{3\nu-1} - x_{3\nu-1}\ddot{x}_{3\nu-2}) = \sum_1^n{}_\nu (x_{3\nu-2}X_{3\nu-1} - x_{3\nu-1}X_{3\nu-2})$$

$$\sum_1^n{}_\nu m_\nu(x_{3\nu}\ddot{x}_{3\nu-2} - x_{3\nu-2}\ddot{x}_{3\nu}) = \sum_1^n{}_\nu (x_{3\nu}X_{3\nu-2} - x_{3\nu-2}X_{3\nu})$$

$$\sum_1^n{}_\nu m_\nu(x_{3\nu-1}\ddot{x}_{3\nu} - x_{3\nu}\ddot{x}_{3\nu-1}) = \sum_1^n{}_\nu (x_{3\nu-1}X_{3\nu} - x_{3\nu}X_{3\nu-1}).$$

These three equations form the so-called extended Principle of Areas (*cf.* § 406).

Energy, Work

510. **Definition.** The increase in the energy of a system, conceived as a consequence of force exerted on the system, is called the work of that force.

The work which a force performs in a given time is measured by the increase of the energy of the system on which it acts, in that time.

Any decrease in the energy owing to the action of force we consider a negative increase. The work of a force may thus be positive or negative.

511. **Corollary.** When a force acting on a system does a certain amount of work, the counterforce exerted by the system always does an equal and opposite amount of work.

For the latter work is equal to the increase of the energy of that system, with which the one under consideration is coupled; the sum of the energies of the two systems is, however, constant.

512. **Proposition.** The work which a force does on a system whilst it traverses an element of its path is equal to the product of the length of the element and the component of the force in its direction.

For the increase dE of the energy in the time-element dt, in which the element ds is traversed, is (§ 283)

$$dE = m\, v\, \dot{v}\, dt = m\, \dot{v}\, ds.$$

By § 280 $\dot{v}$ is the component of the acceleration of the system in the direction of its path; thus by § 504 $m\dot{v}$ is the component of the force in that direction.

513. **Observation 1.** The work is also equal to the product of the magnitude of the force and the component of the element of the path in its direction.

514. **Observation 2.** If during the motion along the path-element ds the coordinates p_ρ suffer the changes dp_ρ, then the work done by the force is represented by the equation

$$dE = \sum_1^r {}_\rho P_\rho \delta p_\rho.$$

For the component of the force in the direction of the path-element is by § 247 equal to

$$\sum_1^r {}_\rho P_\rho \frac{dp_\rho}{ds}.$$

515. **Corollary 1.** The force acting on a system does positive or negative work, according as the angle which it makes with the velocity of the system is smaller or greater than a right angle. If the force is perpendicular to the direction of motion, it does no work.

516. **Corollary 2.** A force which acts on a system at rest, does no work.

Equilibrium, Statics

517. **Definition.** We say that two or more forces which act on the same system are in equilibrium when any one of them counteracts the effect of the others, *i.e.* when the system moves

under the action of both, or of all of them, as though none of them existed.

518. **Proposition.** Two or more forces are in equilibrium when their sum is perpendicular to every possible (virtual) displacement of the system from its instantaneous position, and conversely.

The proposition follows immediately from §§ 471 and 488.

519. **Symbolical Expression.** If we denote by P_ρ', P_ρ'', etc., the components of the respective forces along p_ρ, and by δp_ρ the changes of p_ρ for any possible displacement of the system, then the foregoing proposition can be expressed in the form of the symbolical equations

$$\sum_1^r {}_\rho (P_\rho' + P_\rho'' + \ldots)\delta p_\rho = 0.$$

Cf. §§ 393, 501.

520. **Observation.** The foregoing proposition contains the Principle of Virtual Velocities (displacements, momenta). and the equation § 519 the usual analytical form of it.

521. **Corollary 1.** If several forces acting on a system are in equilibrium, then the sum of the work done by the forces in any possible (virtual) displacement of the system from its instantaneous position is zero, and conversely (Principle of Virtual Work).

For if we write the equation § 519 in the form

$$\sum_1^r {}_\rho P_\rho'\delta p_\rho + \sum_1^r P_\rho''\delta p_\rho + \ldots = 0,$$

then the proof follows by § 514.

522. **Corollary 2.** If two or more forces preserve equilibrium in a system, then the sum of their components in the direction of any possible motion of the system is zero.

523. **Corollary 3.** If two or more forces preserve equilibrium in a system, then the sum of their components along every free coordinate of the system vanishes.

524. **Proposition.** If two or more forces preserve equilibrium in a system, then the sum of their components along

any coordinate of absolute position is zero, no matter what may be the internal connection of the system.

525. **Observation.** Thus without a knowledge of the internal connection of a system, we can nevertheless always write down six necessary equations of condition for equilibrium. If we choose as coordinates of absolute position the six quantities $a_1a_2a_3$, $\omega_1\omega_2\omega_3$, which were introduced in § 402, then the foregoing proposition furnishes those six equations which correspond to the principles of the centre of gravity and of areas, and which Lagrange investigates in chapter iii. §§ 1 and 2 of the first part of the *Mécanique Analytique.*

526. **Note 1.** If two or more forces are in equilibrium when the system is in a given position and has a given velocity, then these forces are also in equilibrium in the same position, no matter what the velocity be.

For the condition of equilibrium does not contain the actual velocity of the system.

527. **Note 2.** If two or more forces are in equilibrium when the system is at rest, then the system continues in its state of rest. And conversely—if, notwithstanding the action of two or more forces, a system is at rest, then the forces on the system are in equilibrium.

528. **Corollary 1.** Two forces which, acting simultaneously on a system at rest, do not disturb the equilibrium of the system, have equal and opposite components in the direction of every possible motion of the system.

529. **Corollary 2.** Two forces which act successively on the same system at rest at the same time as other forces, and leave the system at rest, have the same components in the direction of every possible motion of the system.

530. **Observation.** From the last two corollaries the statical comparison of forces is deduced.

Machines and Internal Forces

531. **Definition.** A system whose masses are considered vanishingly small in comparison with the masses of the systems with which it is coupled, is called a machine.

A machine is thus completely represented, as to its effect on the motion of the other systems, by its equations of condition; the knowledge of the expression for the energy of the machine in terms of its coordinates is not necessary.

A machine is called simple when it has only one degree of freedom.

532. **Proposition.** So long as a machine moves with finite velocity the forces acting on the machine are continually in equilibrium.

For if these forces gave a component in the direction of any possible motion of the machine, then the component of the acceleration in this direction would be infinitely great on account of the vanishing mass (§ 504).

533. **Corollary.** There exists a series of homogeneous linear equations between the components of the forces acting on a machine along its coordinates, and their number is equal to the number of degrees of freedom of the machine. A simple machine is represented by a single homogeneous linear equation between the forces acting on its coordinates.

534. **Note 1.** If a machine is coupled as to all its coordinates with two or more material systems, then the mechanical connection produced between the latter can be analytically represented by a series of homogeneous linear differential equations between the coordinates of the connected systems. For in the equations of condition of the machine we can replace its coordinates by the equal coordinates of the connected systems.

Conversely, therefore, we can physically interpret any given analytical series of homogeneous linear differential equations between the coordinates of two or more systems as a mechanical connection of the kind which we denote as a coupling of these systems by means of the machine.

535. **Corollary.** If two or more systems are coupled by a machine, then the work done by each of the systems is equal and opposite to the work done by the other systems. Consequently no work is gained by coupling systems by a machine.

For the forces due to the systems preserve equilibrium in the machine, and thus the sum of the work done by them is zero.

536. **Note 2.** Any material system can in various ways be regarded as made up of two or more systems which

are coupled by machines. For if we divide up the masses of the system into several parts, and if p_ρ' are the coordinates of the first part, p_ρ'' of the second, etc., then we may consider those equations of condition of the complete system which only contain p_ρ', as equations of condition of the first partial system, those equations which only contain p_ρ'' as equations of condition of the second partial system, and so on; whilst those equations of condition of the complete system which contain $p_\rho', p_\rho'' \ldots$, may be regarded as the equations of the machine coupling the partial systems.

The forces which in this permissible though arbitrary conception are exerted on the partial systems by the machine coupling them will be denoted as internal forces of the system.

537. **Corollary 1.** Every such series of internal forces may replace a portion of the connection of the system. For if we set aside those equations of condition of the whole system which represent the machines between the partial systems, but retain the forces exerted by the machines, then the system moves as before.

538. **Corollary 2.** The whole connection of a system can be set aside and replaced by a series of elementary forces which act on the individual material parts of the system.

For we may regard the individual points as partial systems, and the whole system as the aggregate of the partial systems coupled by machines.

539. **Corollary 3.** The internal forces which entirely or partially replace the connection of a system are always in equilibrium when acting on the original system.

For by § 532 they preserve equilibrium in the machines which form parts of the original system.

540. **Observation.** This last consideration is the one by which, in the usual development of mechanics, the transition is made from the laws of equilibrium (the Principle of Virtual Velocities) to the laws of motion (d'Alembert's Principle).

Measurement of Forces

541. Our considerations give three independent methods of measuring directly those components of the forces which

affect phenomena. By the application of any one of these three methods the forces can be made objects of direct experience, *i.e.* symbols for determinate connections of sensible perceptions.

542. The first method determines the force from the masses and motions of the system by which it is exerted. Physically this method is known as the measurement of force according to its origin. It is, for instance, applied on the assumption that equally stretched springs, equal quantities of explosive powder, etc., *ceteris paribus*, exert equal forces.

543. The second method determines the force by means of the masses and motions of the system on which it acts. In physics this method is known as the dynamical measurement of force. It was, for instance, applied by Newton when he deduced the force acting on the planets from their motion.

544. The third method determines the force by reducing it to equilibrium with known forces. This method is known as the statical method. For example, all measures of forces by the balance depend upon this.

545. When these three methods are used for the determination of one and the same force, paying attention to the relations deduced by us, they must lead in all cases to the same result, provided the fundamental law, on which our considerations are based, actually comprises correctly all possible mechanical experience.

CHAPTER V

SYSTEMS WITH CONCEALED MASSES

I. Cyclical Motion

546. **Definition 1.** A free coordinate of a system is said to be cyclical when the length of an infinitesimal displacement of the system does not depend on the value of the coordinate, but only on its change.

547. **Observation 1.** Cyclical coordinates exist; for instance, a rectangular coordinate of the system, when free, satisfies the definition. Cyclical coordinates can always be introduced, when infinitesimal displacements of the system are possible which do not involve a change in the mass-distribution in space, but only a cyclical interchange of the masses; hence the name. Cyclical coordinates may, however, appear under other circumstances, as the example of rectangular coordinates shows.

548. **Observation 2.** The energy of a system does not depend on the value of its cyclical coordinates, but only on their time-rates of change.

549. **Definition 2.** A cyclical system is a material system whose energy approximates sufficiently near to a homogeneous quadratic function of the rates of change of its cyclical coordinates.

A cyclical system is monocyclic, dicyclic, etc., according as it possesses one, two, etc., cyclical coordinates.

In a cyclical system the non-cyclical coordinates are also known as the parameters of the system; the rates of change of the cyclical coordinates are also called the cyclical intensities.

550. **Observation 1.** The condition that must be approximately satisfied for cyclical systems cannot be rigorously satisfied except in the case when the system possesses only cyclical coordinates.

For if a quantity is a coordinate of a system, then its change must involve a displacement of at least one material point of the system; the energy of this point is consequently a quadratic function of the rate of change of that coordinate, and the same holds for the energy of the system. Strictly speaking, then, the energy of any system contains necessarily the rates of change of all quantities which are coordinates of the system, and consequently the energy of a cyclical system contains the rates of change of its parameters.

551. **Observation 2.** But this condition for the appearance of a cyclical system can be satisfied to any degree of approximation so long as the system possesses cyclical coordinates.

It is, for instance, satisfied in the case when the parts of the energy which contain the rates of change of the parameters vanish in comparison with the parts which depend on the cyclical intensities. This is always possible by taking the rates of change of the parameters sufficiently small, or the cyclical intensities sufficiently great. As to how small the former must be taken or how great the latter, in order that a given degree of approximation may be attained, depends on the particular values of the coefficients in the expression for the energy.

In what follows it will always be assumed that the condition for a cyclical system is satisfied to such a degree of approximation that we may regard it as absolutely satisfied.

552. **Notation.** We shall denote the cyclical coordinates of the system by $\mathfrak{p}_\rho$, their number by $\mathfrak{r}$, and the momenta along $\mathfrak{p}_\rho$ by $\mathfrak{q}_\rho$. The r non-cyclical coordinates may be denoted by p_ρ, and their momenta along p_ρ by q_ρ. Let the mass of the cyclical system be $\mathfrak{m}$.

Let the external forces which act on the system have P_ρ as their components along p_ρ, and $\mathfrak{P}_\rho$ as their components along

$\mathfrak{p}_\rho$. The forces which the system itself exerts then have components along p_ρ, likewise along $\mathfrak{p}_\rho$, which by § 467 are to be denoted by P'_ρ and $\mathfrak{P}'_\rho$ respectively.

553. **Corollary 1.** The energy $\mathfrak{E}$ of a cyclical system can be written in the form

$$\mathfrak{E} = \tfrac{1}{2}\mathfrak{m}\sum_1^{\mathfrak{r}}{}_\rho\sum_1^{\mathfrak{r}}{}_\sigma \mathfrak{a}_{\rho\sigma}\dot{\mathfrak{p}}_\rho\dot{\mathfrak{p}}_\sigma$$
$$= \frac{1}{2\mathfrak{m}}\sum_1^{\mathfrak{r}}{}_\rho\sum_1^{\mathfrak{r}}{}_\sigma \mathfrak{b}_{\rho\sigma}\mathfrak{q}_\rho\mathfrak{q}_\sigma,$$

where $\mathfrak{a}_{\rho\sigma}$ and $\mathfrak{b}_{\rho\sigma}$ are functions of p_ρ alone, but not (§ 548) of $\mathfrak{p}_\rho$, while in other respects they have the same properties and connection as $a_{\rho\sigma}$ and $b_{\rho\sigma}$ (§ 59 *et seq.*).

If we consider $\mathfrak{E}$ a function of p_ρ and $\dot{\mathfrak{p}}_\rho$, as the first form represents it, then its partial differential may be denoted by $\partial_{\mathfrak{p}}\mathfrak{E}$; but if we regard it as a function of p_ρ and $\mathfrak{q}_\rho$, as the second form represents it, its partial differential may be denoted by $\partial_{\mathfrak{q}}\mathfrak{E}$ (*cf.* § 288).

554. **Corollary 2.** For all values of ρ the following equations hold—

$$\frac{\partial_{\mathfrak{p}}\mathfrak{E}}{\partial \dot{p}_\rho} = q_\rho = 0 \qquad (\text{§ } 289)\ (\text{i})$$

$$\frac{\partial_{\mathfrak{q}}\mathfrak{E}}{\partial q_\rho} = \dot{p}_\rho = 0 \qquad (\text{§ } 290)\ (\text{ii})$$

$$\frac{\partial_{\mathfrak{p}}\mathfrak{E}}{\partial \mathfrak{p}_\rho} = 0 \qquad (\text{iii})$$

$$\frac{\partial_{\mathfrak{q}}\mathfrak{E}}{\partial \mathfrak{p}_\rho} = 0 \qquad (\text{iv}).$$

These equations contain the peculiar characteristics of cyclical systems, and from them are deduced their special properties.

The equation (ii) repeats the observation (§ 550) that a contradiction exists between the assumption that the form of the energy is strictly the one assumed, and that nevertheless the p_ρ's are quantities which change with the time. We have then, conformably with § 551, to take the equation to mean that when $\mathfrak{E}$ has very approximately the chosen form, the p_ρ's must be considered as quantities which change very slowly.

Forces and Force-Functions

555. **Problem 1.** To determine the force P_ρ' which the cyclical system exerts along its parameter p_ρ.

By equations § 493 (iii) and (iv) and § 554 (i) we obtain

$$P'_\rho = \frac{\partial_\mathfrak{p} \mathfrak{E}}{\partial p_\rho} = -\frac{\partial_\mathfrak{q} \mathfrak{E}}{\partial p_\rho} \qquad \text{(i)},$$

or, in a more extended form,

$$P_\rho' = \tfrac{1}{2}\mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{\mathfrak{r}}{}_\tau \frac{\partial \mathfrak{a}_{\sigma\tau}}{\partial p_\rho}\dot{\mathfrak{p}}_\sigma \dot{\mathfrak{p}}_\tau$$

$$= -\frac{1}{2\mathfrak{m}}\sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{\mathfrak{r}}{}_\tau \frac{\partial \mathfrak{b}_{\sigma\tau}}{\partial p_\rho}\mathfrak{q}_\sigma \mathfrak{q}_\tau \qquad \text{(ii)}.$$

556. **Corollary.** The forces of a cyclical system along its parameters are independent of the rates of change of these parameters.

It is always assumed that these rates of change do not exceed the values which permit us to treat the system as a cyclical one. Thus in Electromagnetics the attractions between magnets are independent of the velocity of their motion, but only so long as this velocity is considerably less than the velocity of light.

557. **Problem 2.** To determine the force $\mathfrak{P}_\rho'$ which the cyclical system exerts along its cyclical coordinate $\mathfrak{p}_\rho$.

By equations § 493 (iii) and § 554 (iii) we obtain

$$\mathfrak{P}_\rho' = -\dot{\mathfrak{q}}_\rho \qquad \text{(i)}.$$

When developed by § 270 we have

$$\mathfrak{q}_\rho = \mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \mathfrak{a}_{\rho\sigma}\dot{\mathfrak{p}}_\sigma \qquad \text{(ii)}$$

$$\mathfrak{P}_\rho' = -\mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \mathfrak{a}_{\rho\sigma}\ddot{\mathfrak{p}}_\sigma - \mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{r}{}_\tau \frac{\partial \mathfrak{a}_{\rho\sigma}}{\partial p_\tau}\dot{\mathfrak{p}}_\sigma \dot{p}_\tau \qquad \text{(iii)}.$$

558. **Corollary.** If an external force acts on a cyclical system, and if its components along $\mathfrak{p}_\rho$ are $\mathfrak{P}_\rho$, then the changes of the cyclical momenta are given by the equation.

$$\dot{\mathfrak{q}}_\rho = \mathfrak{P}_\rho.$$

559. **Proposition.** When no forces act on the cyclical coordinates of a cyclical system, then all the cyclical momenta of the system are constant with regard to time.

For if the $\mathfrak{P}_\rho$'s are zero, then the foregoing equations give on integration

$$\mathfrak{q}_\rho = \text{constant}.$$

560. **Definition.** A motion of a cyclical system in which its cyclical momenta remain constant is called an adiabatic motion; and when its cyclical intensities remain constant it is called an isocyclic motion.

The cyclical system itself is called adiabatic or isocyclic when it is constrained to perform only adiabatic or isocyclic motions.

561. **Observation 1.** The analytical condition for adiabatic motion is that for all values of ρ

$$\dot{\mathfrak{q}}_\rho = 0, \quad \mathfrak{q}_\rho = \text{constant}.$$

The analytical condition for isocyclic motion is that for all values of ρ

$$\ddot{\mathfrak{p}}_\rho = 0, \quad \dot{\mathfrak{p}}_\rho = \text{constant}.$$

562. **Observation 2.** The motion of a cyclical system is adiabatic so long as no forces act along the cyclical coordinates; it is isocyclic when it is coupled as to its cyclical coordinates with other systems which possess constant rates of change for the coupled coordinates. Thus, in order that a motion may be isocyclic, appropriate forces must act on the cyclical coordinates.

563. **Definition.** If the forces of a cyclical system along its parameters can be expressed as the partial differential coefficients with regard to the parameters of a function of these parameters and some constant quantities, then this function is called the force-function of the cyclical system.

564. **Proposition.** There exists a force-function both for adiabatic and for isocyclic motion.

From § 555 (iii) for adiabatic motion we get

$$\mathrm{P}_\rho' = -\frac{\partial}{\partial p_\rho}\sum_1^r{}_\sigma \sum_1^r{}_\tau \mathfrak{b}_{\sigma\tau}\frac{\mathfrak{q}_\sigma \mathfrak{q}_\tau}{2\mathfrak{m}} \qquad \text{(i)},$$

where the quantities $\mathfrak{q}_\sigma\mathfrak{q}_\tau/\mathfrak{m}$ are constants and the quantities $\mathfrak{b}_{\sigma\tau}$ functions of the parameters solely.

Similarly we get for isocyclic motion from § 555 (ii)

$$P_\rho' = \frac{\partial}{\partial p_\rho}\sum_1^r{}_\sigma \sum_1^r{}_\tau \mathfrak{a}_{\sigma\tau}\frac{\mathfrak{m}}{2}\dot{\mathfrak{p}}_\sigma\dot{\mathfrak{p}}_\tau \qquad \text{(ii)},$$

where the quantities $\mathfrak{m}\dot{\mathfrak{p}}_\sigma\dot{\mathfrak{p}}_\tau$ are constants and the quantities $\mathfrak{a}_{\sigma\tau}$ functions of the parameters solely.

565. **Observation.** We also distinguish the force-functions for adiabatic or isocyclic motions as adiabatic or isocyclic force-functions. There are other forms of motion of the system for which force-functions exist, but such a function does not exist for every given motion.

566. **Additional Note 1.** The force-function of an adiabatic system is equal to the decrease of the energy of the system, measured from some arbitrarily chosen initial condition. It is therefore equal to an arbitrary—*i.e.* not determined by definition—constant, diminished by the energy of the system.

567. **Additional Note 2.** The force-function of an isocyclic system is equal to the increase of the energy of the system measured from some arbitrarily chosen initial condition. It is therefore equal to the energy of the system diminished by an arbitrary constant.

Reciprocal Characteristics

568. **Proposition 1a.** If in an adiabatic system an increase of the parameter p_μ increases the component of the force along another parameter p_λ, then conversely an increase of p_λ increases the force along p_μ. Moreover, in an infinitesimally small increase, the quantitative relation between cause and effect is the same in both cases.

For in an adiabatic system we may regard the quantities p_ρ as sufficiently independent elements for determining P_ρ'; hence the equation § 564 (i), which holds for adiabatic systems, gives us

$$\frac{\partial P_\lambda'}{\partial p_\mu} = \frac{\partial P_\mu'}{\partial p_\lambda},$$

which proves the proposition.

569. **Proposition 1b.** If in an isocyclic system an increase of the parameter p_μ increases the component of the force along another parameter p_λ, then conversely an increase of p_λ increases the force along p_μ. Moreover, in an infinitesimal increase, the quantitative relation between cause and effect is the same in both cases.

For in an isocyclic system we may regard the quantities p_ρ as sufficiently independent elements for determining P_ρ': hence the equation § 564 (ii), which holds for isocyclic systems, gives us

$$\frac{\partial P_\lambda'}{\partial p_\mu} = \frac{\partial P_\mu'}{\partial p_\lambda},$$

which proves the proposition.

It is to be noted that this equation differs from the previous one in meaning although it is identical in form.

570. **Observation.** In order that the two foregoing propositions may admit of a physical application, it is sufficient that two parameters of the cyclical system and the forces along them should be accessible to direct observation.

571. **Proposition 2a.** If in a cyclical system an increase of the cyclical momentum $\mathfrak{q}_\mu$, with fixed values of the parameters, involves an increase of the force along the parameter p_λ, then the adiabatic increase of the parameter p_λ causes a decrease of the cyclical intensity $\dot{\mathfrak{p}}_\mu$, and conversely. Moreover, in an infinitesimal change the quantitative relation between cause and effect is the same in both cases.

For we have

$$P_\lambda' = -\frac{\partial_{\mathfrak{q}} \mathfrak{E}}{\partial p_\lambda} (555\ (\text{i})), \quad \dot{\mathfrak{p}}_\mu = \frac{\partial_{\mathfrak{q}} \mathfrak{E}}{\partial \mathfrak{q}_\mu} \quad (\S\ 290);$$

thus

$$\frac{\partial P_\lambda'}{\partial \mathfrak{q}_\mu} = -\frac{\partial \dot{\mathfrak{p}}_\mu}{\partial p_\lambda} \qquad (\text{i}),$$

and the proposition furnishes the correct interpretation of this equation.

572. **Corollary.** If in a monocyclic system an increase of the cyclical intensity $\dot{\mathfrak{p}}$, with fixed values of the parameters, involves an increase of the force along the parameter p_λ, then

the adiabatic increase of the parameter p_λ causes a decrease of the cyclical intensity $\dot{\mathfrak{p}}$, and conversely.

For in a monocyclic system increase of the cyclical intensity always goes hand in hand with increase of the cyclical momentum, the parameters remaining fixed. In fact, for a monocyclic system

$$\mathfrak{q} = \mathfrak{m}\mathfrak{a}\dot{\mathfrak{p}},$$

where $\mathfrak{a}$ is a necessarily positive (§ 62) function of the parameters of the system.

573. **Proposition 2b.** If in a cyclical system an increase of the cyclical intensity $\dot{\mathfrak{p}}_\mu$, the parameters remaining fixed, involves an increase of the force along the parameter p_λ, then the isocyclic increase of the parameter p_λ involves an increase of the cyclical momentum $\mathfrak{q}_\mu$, and conversely. Moreover, in an infinitesimal change the quantitative relation between cause and effect is the same in both cases.

For we have

$$P_\lambda' = \frac{\partial_{\mathfrak{p}} \mathfrak{E}}{\partial p_\lambda} \; (555(\text{i})), \quad \mathfrak{q}_\mu = \frac{\partial_{\mathfrak{p}} \mathfrak{E}}{\partial \dot{\mathfrak{p}}_\mu'} \; (289);$$

thus

$$\frac{\partial P_\lambda'}{\partial \dot{\mathfrak{p}}_\mu} = \frac{\partial \mathfrak{q}_\mu}{\partial p_\lambda} \qquad (\text{i}),$$

and the proposition expresses this equation in words.

574. **Corollary.** If in a monocyclic system an increase of the cyclical momentum $\mathfrak{q}$ involves an increase of the force along the parameter p_λ, the parameters remaining fixed, then the isocyclic increase of the parameter p_λ involves an increase of the cyclical momentum $\mathfrak{q}$, and conversely.

The reason is the same as in § 572.

575. **Observation.** The foregoing propositions **2a** and **2b** admit of a physical application when it is possible to determine a cyclical intensity and also the corresponding cyclical momentum directly, *i.e.* to determine it without a knowledge of the coefficients $\mathfrak{a}_{\rho\sigma}$. This may happen. For instance, in Electrostatics the differences of potential of conductors correspond to cyclical intensities, the charges of the conductors to the cyclical momenta, and both quantities can be directly determined independently of one another.

The corollaries require only the direct determination either of the cyclical intensity or cyclical momentum.

576. **Proposition 3a.** If in a cyclical system a force exerted on the cyclical coordinate $\mathfrak{p}_\mu$ involves an increase with the time of the force along the parameter p_λ, then an adiabatic increase of the parameter p_λ causes a decrease of the cyclical intensity $\dot{\mathfrak{p}}_\mu$ and conversely. Moreover, in an infinitesimal change the quantitative relation between cause and effect is the same in both cases.

For if we regard on the left-hand side of equation § 571 (i) the changes $\partial \mathrm{P}_\lambda{}'$ and $\partial \mathfrak{q}_\mu$ as happening in the time *dt*, and divide the differential coefficients in the numerator and denominator by this time *dt* and make use of equation § 558, where the change $\partial \mathfrak{q}_\mu$ is considered as the effect of the force $\mathfrak{P}_\mu$, then

$$\frac{\dot{\mathrm{P}}_\lambda{}'}{\mathfrak{P}_\mu} = -\frac{\partial \dot{\mathfrak{p}}_\mu}{\partial p_\lambda},$$

and the proposition expresses fully this equation in words.

577. **Proposition 3b.**[1] If in a cyclical system an increase of the cyclical intensity $\dot{\mathfrak{p}}_\mu$, the parameters remaining fixed, involves an increase of the force along the parameter p_λ, then an isocyclic increase of the parameter p_λ involves a decrease of the force of the system along the cyclical coordinate $\mathfrak{p}_\mu$, and conversely. Moreover, in an infinitesimal change, the quantitative relation between cause and effect is the same in both cases.

For if we regard in the right-hand side of the equation § 573 (i) the changes $\partial \mathfrak{q}_\mu$ and ∂p_λ as occurring in the time *dt*, we can put

$$\partial \mathfrak{q}_\mu = \frac{d}{dt}\partial \mathfrak{q}_\mu \,.\, dt = \partial \dot{\mathfrak{q}}_\mu dt = -\partial \mathfrak{P}_\mu{}' dt \qquad (557\ \text{(i)})$$

$$\partial p_\lambda = \frac{d}{dt}\partial p_\lambda dt = \partial \dot{p}_\lambda dt\,;$$

thus that equation becomes

$$\frac{\partial \mathrm{P}_\lambda{}'}{\partial \dot{\mathfrak{p}}_\mu} = -\frac{\partial \mathfrak{P}_\mu{}'}{\partial \dot{p}_\lambda},$$

and the proposition expresses this in words.

[1] Printed as in the original MSS.—Ed.

578. **Note.** The propositions **3a** and **3b** admit of a physical application when a cyclical intensity and also the corresponding cyclical force-component are accessible to direct observation. This happens, for instance, in Electromagnetics, and one can best illustrate the meanings of these theorems by translating them into the technical language of this branch of physics.

Energy and Work

579. **Proposition 1.** In the isocyclic motion of a cyclical system the work done on it through the coupling of its cyclical coordinates is always twice the work it does through the coupling of its parameters.

In the isocyclic motion $\ddot{\mathfrak{p}}_\rho$ is equal to zero for all values of ρ, and thus by § 514 and § 557 (iii) the work which the external forces acting on the cyclical coordinates perform in the unit of time is equal to

$$-\sum_1^{\mathfrak{r}}{}_\rho \mathfrak{P}_\rho' \dot{\mathfrak{p}}_\rho = \mathfrak{m}\sum_1^{\mathfrak{r}}{}_\rho \sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{r}{}_\tau \frac{\partial \mathfrak{a}_{\rho\sigma}}{\partial p_\tau} \dot{\mathfrak{p}}_\sigma \dot{p}_\tau \dot{\mathfrak{p}}_\rho.$$

But the work which the system performs through the forces along the parameters, calculated for unit time, is found equal to

$$\sum_1^{r}{}_\rho \mathrm{P}_\rho' \dot{p}_\rho = \tfrac{1}{2}\mathfrak{m}\sum_1^{r}{}_\rho \sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{\mathfrak{r}}{}_\tau \frac{\partial \mathfrak{a}_{\sigma\tau}}{\partial p_\rho} \dot{\mathfrak{p}}_\sigma \dot{\mathfrak{p}}_\tau \dot{p}_\rho$$

by the use of § 555 (ii).

The summations in both equations are identical except for the notation, and the value of the series in the first equation is therefore double that of the second.

580. **Corollary.** When an isocyclic system does work through the forces along its parameters, then the energy of the system increases at the same time, and by the same amount as the work done. When an isocyclic system has work done on it through the forces along its parameters, then the energy of the system decreases at the same time, and by the amount of the work done on it.

For the increase of the energy of the system is equal to the difference between the work done on it through its cyclic coordinates and the work it does through its parameters.

581. **Observation.** When an adiabatic system does work through the forces along its parameters, then the energy of the system decreases at the same time, and by the same amount as the work done. When an adiabatic system has work done on it by the forces along its parameters, then the energy of the system increases at the same time, and by the amount of work done.

For the work done on an adiabatic system through the cyclical coordinates is zero (§ 562).

582. **Proposition 2.** In an adiabatic displacement of a cyclical system the cyclical intensities always suffer changes in such a sense that the forces along the parameters caused by these changes do negative work.

Let the quantities p_ρ suffer the changes δp_ρ and the intensities $\dot{\mathfrak{p}}_\rho$ the changes $\delta\dot{\mathfrak{p}}_\rho$ from the displacement. If only the latter took place, then the forces P_ρ' would change by the amount (§ 555 (ii))

$$\delta P_\rho' = \mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{\mathfrak{r}}{}_\tau \frac{\partial \mathfrak{a}_{\sigma\tau}}{\partial p_\rho} \dot{\mathfrak{p}}_\sigma \delta\dot{\mathfrak{p}}_\tau,$$

and these $\delta P_\rho'$'s are what the proposition denotes as the forces caused by $\delta\dot{\mathfrak{p}}_\tau$. The work done by them is given by

$$\sum_1^{r}{}_\rho \delta P_\rho' \delta p_\rho = \mathfrak{m}\sum_1^{r}{}_\rho \sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{\mathfrak{r}}{}_\tau \frac{\partial \mathfrak{a}_{\sigma\tau}}{\partial p_\rho} \dot{\mathfrak{p}}_\sigma \delta\dot{\mathfrak{p}}_\tau \delta p_\rho$$
$$= \mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \sum_1^{\mathfrak{r}}{}_\tau \delta\mathfrak{a}_{\sigma\tau} \dot{\mathfrak{p}}_\sigma \delta\dot{\mathfrak{p}}_\tau,$$

and the proof requires that this work should be necessarily negative. But for the adiabatic motion

$$\mathfrak{q}_\tau = \mathfrak{m}\sum_1^{\mathfrak{r}}{}_\sigma \mathfrak{a}_{\rho\sigma} \dot{\mathfrak{p}}_\sigma = \text{const},$$

thus

$$\sum_1^{\mathfrak{r}}{}_\sigma \delta\mathfrak{a}_{\sigma\tau} \dot{\mathfrak{p}}_\sigma = -\sum_1^{\mathfrak{r}}{}_\sigma \mathfrak{a}_{\sigma\tau} \delta\dot{\mathfrak{p}}_\sigma.$$

If we form these equations for all values of τ, multiply them in succession by the corresponding $\mathfrak{m}\delta\dot{\mathfrak{p}}_\tau$ and add them, we obtain on the left-hand side the foregoing expression for the work done, and on the right-hand side a necessarily positive quantity (§ 62), which completes the proof.

583. **Corollary.** In an adiabatic displacement of a cyclical system the cyclical intensities always suffer changes in such a sense that the forces caused by these changes tend to stop the motion which produces them.

This is in fact only another form of the foregoing proposition. It corresponds to Lenz's Law in Electromagnetics.

584. **Note.** In any infinitesimally small motion of a monocyclic system, the work received through the cyclical coordinates of the system bears the same ratio to the energy of the system as twice the increase of the cyclical momentum of the system does to this momentum.

For the work $d\mathfrak{Q}$ done through the cyclical coordinate $\mathfrak{p}$ in the time dt is given by

$$d\mathfrak{Q} = \mathfrak{P}d\mathfrak{p} = \dot{\mathfrak{q}}d\mathfrak{v} = \dot{\mathfrak{q}}\mathfrak{p}dt = \dot{\mathfrak{p}}d\mathfrak{q},$$

while the energy $\mathfrak{E}$ may be written

$$\mathfrak{E} = \tfrac{1}{2}\mathfrak{q}\dot{\mathfrak{p}},$$

thus

$$\frac{d\mathfrak{Q}}{\mathfrak{E}} = 2\frac{d\mathfrak{q}}{\mathfrak{q}},$$

which proves the proposition.

585. **Corollary 1.** In any motion of a monocyclic system the expression

$$\frac{d\mathfrak{Q}}{\mathfrak{E}}$$

is the complete differential of a function of the parameters and cyclical intensity of the system. This function is

$$2 \log\frac{\mathfrak{q}}{\mathfrak{q}_0},$$

where $\mathfrak{q}_0$ denotes the cyclical momentum for an arbitrarily chosen initial position. This function is also called the entropy of the monocyclic system.

586. **Corollary 2.** The value of the integral

$$\int\frac{d\mathfrak{Q}}{\mathfrak{E}}$$

for any finite motion of a monocyclic system depends only on the condition of the system in its initial and final positions, and not on the condition at any intermediate position. The value of this integral is zero for every motion which carries the system back to its initial position.

For the value of this integral is equal to the difference between the entropy in the initial and final positions.

587. **Corollary 3.** In the adiabatic motion of a monocyclic system the entropy is constant. For in the adiabatic motion $\mathfrak{P}$, and consequently $d\mathfrak{Q}$, is equal to zero. Hence the adiabatic motion of a monocyclic system is called isentropic.

Time-Integral of the Energy

588. **Note 1.** If in the adiabatic motion of a cyclical system the cyclical coordinates $\mathfrak{p}_\rho$ change in a given finite time by $\bar{\mathfrak{p}}_\rho$, then the time-integral of the energy of the system for that time is equal to

$$\frac{1}{2}\sum_1^{\mathfrak{r}}{}_\rho\, \mathfrak{q}_\rho \bar{\mathfrak{p}}_\rho,$$

for the energy of the system can be written in the form (286 (ii))

$$\frac{1}{2}\sum_1^{\mathfrak{r}}{}_\rho\, \mathfrak{q}_\rho \dot{\mathfrak{p}}_\rho,$$

and for the adiabatic motion the $\mathfrak{q}_\rho$'s are constant.

589. **Note 2.** The variation of the time-integral of the energy of an adiabatic system when the motion of the system is varied depends—firstly, on the variation of the parameters during the whole time for which the integral is taken, and secondly, on the variations which the constant cyclical momenta of the system suffer.

590. **Notation.** We shall in what follows use the following notation:—δ will denote a variation by which the cyclical momenta suffer arbitrary variations,

$\delta_{\mathfrak{q}}$ a variation by which the cyclical momenta suffer no variations, and, finally, $\delta_{\mathfrak{p}}$ a variation by which the cyclical momenta suffer such variations that the initial and final values of the cyclical coordinates remain unaltered.

591. **Corollary.** From the notation we immediately get for all values of ρ

$$\delta_{\mathfrak{q}}\mathfrak{q}_{\rho} = 0, \quad \delta_{\mathfrak{p}}\bar{\mathfrak{p}}_{\rho} = 0,$$

and consequently by § 588 for any variations of the parameters

$$\delta_{\mathfrak{q}}\int \mathfrak{E}dt = \tfrac{1}{2}\sum_{1}^{\mathfrak{r}}{}_{\rho}\,\mathfrak{q}_{\rho}\delta_{\mathfrak{q}}\bar{\mathfrak{p}}_{\rho} \qquad \text{(i)}$$

$$\delta_{\mathfrak{p}}\int \mathfrak{E}dt = \tfrac{1}{2}\sum_{1}^{\mathfrak{r}}{}_{\rho}\,\bar{\mathfrak{p}}_{\rho}\delta_{\mathfrak{p}}\mathfrak{q}_{\rho} \qquad \text{(ii).}$$

592. **Observation.** In an adiabatic system it is always possible, and in general possible in only one way, to give the cyclical momenta such variations with any variation of the parameters that the initial and final values of the cyclical coordinates remain unaltered.

For from the general relation

$$\dot{\mathfrak{p}}_{\rho} = \frac{1}{\mathfrak{m}}\sum_{1}^{\mathfrak{r}}{}_{\sigma}\,\mathfrak{b}_{\rho\sigma}\mathfrak{q}_{\sigma}$$

it follows that in an adiabatic system, when the $\mathfrak{p}_{\rho}$'s change from the values $\mathfrak{p}_{\rho 0}$ to the values $\mathfrak{p}_{\rho 1}$,

$$\mathfrak{p}_{\rho 1} - \mathfrak{p}_{\rho 0} = \frac{1}{\mathfrak{m}}\sum_{1}^{\mathfrak{r}}{}_{\sigma}\,\mathfrak{q}_{\sigma}\int_{0}^{1}\mathfrak{b}_{\rho\sigma}dt\,;$$

thus in any variation of the parameters and cyclical momenta

$$\delta\mathfrak{p}_{\rho 1} - \delta\mathfrak{p}_{\rho 0} = \frac{1}{\mathfrak{m}}\sum_{1}^{\mathfrak{r}}{}_{\sigma}\,\mathfrak{q}_{\sigma}\delta\int_{0}^{1}\mathfrak{b}_{\rho\sigma}dt + \frac{1}{\mathfrak{m}}\sum_{1}^{\mathfrak{r}}{}_{\sigma}\,\delta\mathfrak{q}_{\sigma}\int_{0}^{1}\mathfrak{b}_{\rho\sigma}dt.$$

These equations form $\mathfrak{r}$ unhomogeneous linear equations for the $\mathfrak{r}$ quantities $\delta\mathfrak{q}_{\sigma}$, and thus admit of one, and in general only one, solution—in particular in the case when the variations on the left-hand side vanish.

Variations of the kind denoted by $\delta_{\mathfrak{p}}$ are thus always possible with any variation of the parameters.

593. **Proposition.** In equal and arbitrary variations of the parameters in a given time the variations of the time-integral of the energy in an adiabatic system are equal and opposite when in the first instance the cyclical momenta of the system are not varied, and in the second are varied in such a manner that the initial and final values of the cyclical coordinates remain unaltered.

For in any variation

$$\delta\int \mathfrak{E}dt = \delta_{\mathfrak{q}}\int \mathfrak{E}dt + \sum_1^{\mathfrak{r}}{}_\rho \int \frac{\partial_{\mathfrak{q}}\mathfrak{E}}{\partial \mathfrak{q}_\rho}\delta\mathfrak{q}_\rho dt$$
$$= \delta_{\mathfrak{q}}\int \mathfrak{E}dt + \sum_1^{\mathfrak{r}}{}_\rho \bar{\mathfrak{p}}_\rho \delta\mathfrak{q}_\rho ;$$

thus, in particular for a variation in which the initial and final values of $\mathfrak{p}_\rho$ remain unaltered,

$$\delta_{\mathfrak{p}}\int \mathfrak{E}dt = \delta_{\mathfrak{q}}\int \mathfrak{E}dt + \sum_1^{\mathfrak{r}}{}_\rho \bar{\mathfrak{p}}_\rho \delta_{\mathfrak{p}}\mathfrak{q}_\rho .$$

If twice the equation § 591 (ii) is subtracted from this, then

$$\delta_{\mathfrak{q}}\int \mathfrak{E}dt = -\delta_{\mathfrak{p}}\int \mathfrak{E}dt,$$

which proves the proposition.

With these we may compare the allied propositions § 96 and § 293.

II. Concealed Cyclical Motion

Explanations and Definitions

594. 1. We say that a system contains concealed masses when the position of all the masses of the system is not determined by means of those coordinates of the system which are accessible to observation, but only the position of a portion of them.

595. 2. Those masses whose position still remains unknown when the coordinates accessible to observation have been completely specified are called concealed masses, their

motions concealed motions, and their coordinates concealed coordinates. In contradistinction to these the remaining masses are called visible masses, their motions visible motions, and their coordinates visible coordinates.

596. 3. The problem which a system with concealed masses offers for the consideration of mechanics is the following:—To predetermine the motions of the visible masses of the system, or the changes of its visible coordinates, notwithstanding our ignorance of the position of the concealed masses.

597. 4. A system which contains concealed masses differs from a system without concealed masses only as regards our knowledge of the system. All the propositions hitherto made are therefore applicable to systems with concealed motions, if we understand by masses, coordinates, etc., all its masses, coordinates, etc. Thus alterations are only necessary when we restrict our propositions to the visible quantities. The problem can therefore be reduced to specifying what alterations our previous propositions must undergo, when by masses, coordinates, etc., we mean visible masses, coordinates, etc., only.

598. 5. It is evident that whether the problem is stated in the one form or the other, a solution cannot be obtained without some data as to the effect which the concealed masses produce on the motion of the visible masses. Such data are possible. A guided system, or a system under the action of forces, may be conceived as a system with concealed masses, if we consider either the unknown masses of the guiding system, or of the system producing the forces, as concealed. In general, however, in these cases it is possible also to ascertain physically the masses of the guiding system or of the system which exerts the forces, and it then rests with us to decide whether we regard them as concealed or not. But at present we are mainly interested in cases where a knowledge of the concealed masses cannot be obtained by physical observation.

599. 6. Continually recurrent motions, and therefore cyclical motions, are frequently concealed motions; for these, when existing alone, cause no change in the mass-distribution, nor therefore in the appearance of things. Thus to all appear-

ance the motion of a homogeneous fluid in a closed vessel is concealed; it is only rendered visible when its strictly cyclical character is destroyed by the introduction of dust or other such substances.

Conversely, concealed motions are almost always cyclical. For motions which do not recur continually must sooner or later produce a change in the mass-distribution, and therefore in the aspect of things, and thus become visible.

600. 7. Even cyclical motions cannot long retain their property of being concealed if we obtain means to affect the individual cyclical coordinates, and produce changes in the cyclical intensities. The manifold of our influence on the system is in this case as great as the actual manifold of the system, and we can argue from one to the other. The case is different, however, if any direct and arbitrary influence on the cyclical coordinates is permanently excluded. This may happen in adiabatic cyclical systems (§ 560), and in these we shall rather have to seek the motions which are concealed from our observation.

We therefore restrict our consideration of concealed motion in the first place to such cases. Our treatment, however, has the effect that even in these cases we treat the concealed motions as though they were visible, and only investigate subsequently which of our propositions are still applicable notwithstanding that they are now supposed to be concealed.

Conservative Systems

601. **Definition 1.** A material system which contains no other concealed masses than those which form adiabatic cyclical systems is called a conservative system.

The name is derived from a property of these systems which will appear later; at present it is sufficiently justified by its connection with the already established usage of mechanics.

602. **Observation.** Every conservative system may be regarded as consisting of two partial systems, of which one contains all the visible masses, the other all the concealed masses of the complete system. The coordinates of the

visible partial system, *i.e.* the visible coordinates of the complete system, are at the same time parameters of the concealed partial system.

We shall denote the mass of the visible partial system by m, its coordinates by p_ρ, and its momenta along p_ρ by q_ρ. The mass of the concealed partial system will be denoted by $\mathfrak{m}$, its coordinates by $\mathfrak{p}_\rho$, and its momenta along these coordinates by $\mathfrak{q}_\rho$.

603. **Definition 2.** By the force-function of a conservative system is meant the force-function of its concealed partial system (§ 563).

The force-function of a conservative system is thus in general given as a function of the visible coordinates and constant quantities, without any explicit statement of the connection between these constants and the momenta of the cyclical partial system. The form of this function is subject to no restriction by our considerations.

We shall denote the force-function of a conservative system by U.

604. **Note.** In order to fully determine the motion of the visible masses of a conservative system it is sufficient to know its force-function as a function of its visible coordinates, and this knowledge renders any further knowledge of the concealed masses of the system unnecessary.

For the forces which the concealed partial system exerts on the visible one can be completely obtained from the force-function in the given form, and these forces replace completely effect of the former on the latter (§ 457 *et seq.*).

605. **Definition 3.** That part of the energy of a conservative system which arises from the motion of its visible masses is called the kinetic energy of the whole system. In opposition thereto the energy of the concealed masses of the system is called the potential energy of the whole system.

Kinetic energy is also called *vis viva* (*lebendige Kraft*). According to another and older mode of expression this term denotes twice the kinetic energy.

606. **Notation.** We shall denote the kinetic energy by T. T is thus a homogeneous quadratic function of $\dot{p}_\rho$ or of q_ρ; the coefficients of this function are functions of p_ρ. We shall

denote the partial differential of T by $\partial_p T$ when we regard p_ρ and $\dot{p}_\rho$ as variables independent of one another, but by $\partial_q T$ when we regard p_ρ and q_ρ as variables independent of one another.

The energy of the concealed cyclical partial system, *i.e.* the potential energy of the whole system, may be denoted as previously (§ 553) by $\mathfrak{E}$.

607. **Observation.** The kinetic and the potential energy of a conservative system do not differ in their nature, but only in the voluntary standpoint of our conception, or the involuntary limitation of our knowledge of the masses of the system. That energy, which from one particular standpoint of our conception or knowledge is to be denoted as potential, is from a different standpoint of our conception or knowledge denoted as kinetic.

608. **Corollary 1.** The energy of a conservative system is equal to the sum of its kinetic and potential energies.

We shall denote the total energy of the conservative system by E, and we thus have

$$E = T + \mathfrak{E}.$$

609. **Corollary 2.** In a free conservative system the sum of the potential and kinetic energies is constant in time. As the kinetic energy increases the potential energy decreases, and conversely (§ 340).

610. **Corollary 3.** In a free conservative system the difference between the kinetic energy and the force-function is constant in time; the kinetic energy and the force-function increase and decrease simultaneously and by the same amount (§ 566).

611. **Definition 4.** We shall call the difference between the kinetic energy and the force-function of a conservative system the mathematical or analytical energy of the system.

We shall denote the mathematical energy by h. It differs from the energy of the system only by a constant which is independent of the time and the position of the system, but is in general unknown. In mathematical applications it may completely take the place of the energy, but it lacks the physical meaning which the latter possesses.

612. **Observation.** The definition is represented by the equation

$$T - U = h \quad \text{(i)},$$

or

$$U + h = T \quad \text{(ii)}.$$

If the conservative system is free, then the quantity h in this equation is a constant independent of the time, and the equation is then called the equation of energy for the conservative system.

From (ii) and § 608 we obtain the relation

$$U + h = E - \mathfrak{E} \quad \text{(iii)}.$$

613. **Definition 5.** The time-integral of the kinetic energy of a system, taken between two definite times as limits, is called the action or "expenditure of power" (*Kraftaufwand*) between the two times.

The action in the motion of a conservative system during a given time is thus represented by the integral

$$\int T dt$$

taken between the initial and final values of that time.

614. **Observation 1.** If ds denotes a path-element of the visible partial system, and v its velocity in its path, then the action can be represented in the form of the integral

$$\frac{1}{2} m \int v ds$$

taken between the positions in which the system is found at the beginning and end of the time considered.

615. **Observation 2.** The name "action" (*Wirkung*) for the integral in the text has often been condemned as unsuitable. It is not easy to see, however, why the term "expenditure of power," invented by Jacobi, is better; nor why the term (*action*) originally chosen by Maupertuis should be preferred. All these names suggest conceptions which have nothing to do with the objects they denote. It is difficult to see how the summation of the energies existing at different times could yield anything else than a quantity for calculation, and it is not only difficult, but impossible, to find a suitable name, of simple meaning, for the integral in the text.

The other terms and notations introduced in this chapter are also justified less by their essential suitability than by the necessity of employing as much as possible the existing terms of mechanics.

Differential Equations of Motion

616. **Problem.** To form the differential equations of motion of a conservative system.

The solution of the problem consists only in specifying the equations of motion for the visible partial system. The mass of this portion is m, its coordinates p_ρ; let the k equations

$$\sum_1^r{}_\rho p_{\chi\rho} dp_\rho = 0 \qquad \text{(i)}$$

be its equations of condition. Since the p_ρ's are at the same time the parameters of the concealed partial system, the components of the force which it exerts on the visible partial system are equal to $\partial \mathrm{U}/\partial p_\rho$ (§ 563). Let an additional force act on the visible partial system on account of a coupling with other visible systems and let P_ρ be its components. Then the equations of motion of the system are by § 481

$$mf_\rho + \sum_k^1{}_\chi p_{\chi\rho}\mathrm{P}_\chi = \frac{\partial \mathrm{U}}{\partial p_\rho} + \mathrm{P}_\rho \qquad \text{(ii)},$$

and these r equations, together with the k equations (i), are sufficient for the unique determination of the $r+k$ quantities $\ddot{p}_\rho$ and P_χ.

617. **Observation 1.** If the conservative system is free, then no external forces act on it, and the P_ρ's are zero; the equations of motion thus take the form

$$mf_\rho + \sum_1^k{}_\chi p_{\chi\rho}\mathrm{P}_\chi = \frac{\partial \mathrm{U}}{\partial p_\rho}.$$

618. **Observation 2.** In particular, if the coordinate p_ρ is a free coordinate of the visible partial system, then the equation of motion for the index $_\rho$ takes the form

$$mf_\rho = \frac{\partial \mathrm{U}}{\partial p_\rho},$$

since then all the $p_{\chi\rho}$'s vanish.

619. **Observation 3.** If we substitute in the equations § 616-618 for the accelerations along p_ρ their different expressions from § 291, we obtain for these equations a series of different forms corresponding to the forms which we obtained for a completely known system in § 368 *et seq.*

620. **Corollary 1.** If in a holonomous conservative system all the p_ρ's are free coordinates, and we put for short

$$T + U = L,$$

then the equations of motion of the system may be expressed in the form of the $2r$ equations

$$q_\rho = \frac{\partial_p L}{\partial \dot{p}_\rho} \qquad \text{(i)}$$

$$\dot{q}_\rho = \frac{\partial_p L}{\partial p_\rho} \qquad \text{(ii)},$$

which may be regarded as so many differential equations of the first order for the $2r$ quantities p_ρ and q_ρ, and which, with given initial values, singly determine the course of these quantities.

For if we substitute the value of L, develop the partial differential coefficients and remember that U does not contain $\dot{p}_\rho$, and thus that

$$\frac{\partial_p U}{\partial \dot{p}_\rho} = 0, \qquad \frac{\partial_p U}{\partial p_\rho} = \frac{\partial U}{\partial p_\rho},$$

then we recognise that the equations (i) coincide with the relation between q_ρ and $\dot{p}_\rho$ which follows from the definitions, but that the equations (ii) coincide with the equations of motion in the form § 618 (§§ 289, 291).

621. **Observation.** The function L, by whose use the differential equations of motion take the simple form of the equations § 620 (i) and (ii), has been called Lagrange's function. This function consequently exists only for a holonomous system, and it is here equal to the difference between the kinetic and potential energies, except for an arbitrary constant.

622. **Corollary 2.** If in a holonomous conservative system all the p_ρ's are free coordinates, and we put for short

$$T - U = H,$$

then the equations of motion can be expressed in the form of the $2r$ equations

$$\dot{p}_\rho = \frac{\partial_q H}{\partial q_\rho} \qquad \text{(i)}$$

$$\dot{q}_\rho = -\frac{\partial_q H}{\partial p_\rho} \qquad \text{(ii)},$$

which may be regarded as so many partial differential equations of the first order for which the $2r$ quantities p_ρ and q_ρ, and which with given initial values, singly determine the course of these quantities.

For if we substitute the value of H, and remember that U does not contain q_ρ, and consequently that

$$\frac{\partial_q U}{\partial q_\rho} = 0, \qquad \frac{\partial_q U}{\partial p_\rho} = \frac{\partial U}{\partial p_\rho},$$

we see that the equations (i) represent the relation between q_ρ and $\dot{p}_\rho$ resulting from the definitions; while the equations (ii) coincide with the equations of motion (§ 618) deduced from experience (§§ 290, 294).

623. **Observation.** The function H, through whose use the equations of motion take the simple form given in § 622 (i) and (ii), is known as Hamilton's function. This function therefore exists only for a holonomous system, and for such a system is equal to the sum of the potential and kinetic energies, except for an arbitrary constant; it is also equal to the total energy of the system, except for an arbitrary constant.

In general it is permissible to define Hamilton's function for a system with any, not necessarily cyclic, concealed motions, by the equations § 622 (i) and (ii), *i.e.* as a function of the visible p_ρ's and q_ρ's through whose use (assuming there is such a function) the equations of motion take that simple form. With this more general definition, Hamilton's function is not always equal to the sum of the kinetic and potential energies.

624. **Note.** From the equations § 620 and § 622 the same reciprocal properties can be obtained for a system with concealed cycles as were deduced in § 378 and § 381 for a completely known system. This is unnecessary, but it is implied in these relations that each of them is valid quite independently of whether the coordinates, momenta, etc., appearing in them are visible or concealed coordinates, momenta, etc.

Integral Propositions for Holonomous Systems

625. **Note 1.** The integral

$$\int_{t_0}^{t_1} (\mathrm{T} - \mathrm{U})dt$$

for the motion of a free holonomous system with concealed adiabatic cycles between sufficiently near positions 0 and 1 is smaller for the natural motion of the system than for any other possible motion by which both the visible and concealed coordinates pass in the same time from their initial to their final values.

For since $\mathrm{T} - \mathrm{U}$ is equal to the energy of the system, increased by a constant which is the same for all possible motions, the note is the same as proposition § 358 expressed by means of the notation just adopted.

626. **Observation 1.** If the restriction that the final positions should be sufficiently near is removed, then it can only be asserted that the variation of the integral vanishes in a transition to any one of the other motions considered. Using the notation of § 590 the statement takes the form that

$$\delta_{\mathrm{p}} \int_{t_0}^{t_1} (\mathrm{T} - \mathrm{U})dt = 0,$$

in a transition from the natural motion to any other possible motion, when the variations of the initial and final times as well as the initial and final values of the visible coordinates vanish (*cf.* § 359).

627. **Observation 2.** Note 1 distinguishes the natural motions from every other possible motion, and may therefore be used to determine the natural motion if it is actually possible to form the variation of Observation 1. But if, as is assumed, the cyclical coordinates are concealed, then the formation of variations of the form ∂_{p} is not possible, and the note, although still correct, becomes inapplicable.

628. **Proposition 1.** The integral

$$\int_{t_0}^{t_1} (T + U)dt$$

in the motion of a free holonomous system with concealed adiabatic cycles between sufficiently near positions of its visible masses is smaller for the natural motion than for any other possible motion which, in the same time and with the same momenta of the concealed cyclical motions, carries the visible coordinates from the given initial to the given final values.

The proof can be obtained by reference to Note 1, § 625. For this purpose we associate (as is possible by § 592) each one of the varied motions required by this proposition, with a second in which the visible coordinates undergo the same variation, but in which the cyclical momenta vary in such a manner that the initial and final values of the cyclical coordinates remain unaltered. We must denote, according to § 590, a variation by a transition to a motion of the first kind by δ_q, and a variation to the corresponding motion of the second kind by δ_p.

Now, firstly, since T depends on the visible coordinates alone,

$$\delta_q \int T dt = \delta_p \int T dt \qquad \text{(i).}$$

Secondly, since the duration of the motion is not varied, and $-U$ differs only by a constant from the energy of the cyclical motion (§ 566), we get by § 593

$$\delta_q \int U dt = -\delta_p \int U dt \qquad \text{(ii).}$$

Adding (i) and (ii) we get

$$\delta_q \int (T + U) dt = \delta_p \int (T - U) dt \qquad \text{(iii).}$$

Now by §§ 626, 625, the variation on the right-hand side has for the natural motion always a vanishing value, and for sufficiently near final positions a necessarily negative value, and therefore so has the variation on the left-hand side. Consequently the integral on the left has a minimum value for the natural motion between sufficiently near final positions; which proves the proposition.

629. **Observation 1.** If the restriction to sufficiently near positions is omitted, then it can only be asserted that the variation of the integral vanishes. The analytical expression for this statement is in our notation (in contradistinction to the statement of § 626)

$$\delta_q \int_{t_0}^{t_1} (T + U)dt = 0.$$

630. **Observation 2.** The property of the natural motion stated in the proposition distinguishes it without ambiguity from every other possible motion. The variation δ_q can be formed even though the cyclical motions are considered concealed; for its formation only requires that the constants appearing in the force-function should be left unvaried. The proposition can thus be used for the determination of the natural motion of conservative systems. Its validity is rigorously limited to holonomous systems.

631. **Observation 3.** The above proposition (§ 628) employed as in § 630 bears the name of Hamilton's Principle. Its physical meaning can in our opinion be no other than that of proposition § 358, from which we have deduced the principle. The principle represents the form which must be given to proposition § 358 in order that, notwithstanding our ignorance of the peculiarities of cyclical motion, it should remain applicable to the determination of the motion of the visible system.

632. **Note 2.** If we denote by ds a path-element of the visible masses of a free holonomous system which contains concealed adiabatic cycles, then the integral

$$\int_0^1 \frac{ds}{\sqrt{U + h}}$$

in a motion between sufficiently near positions 0 and 1 is smaller for the natural paths of the system than for any other possible paths by which the values both of the visible and cyclical coordinates pass from the given initial to the given final values. The quantity h is here to be considered a constant varying from one natural path to another, while for all the paths compared at any instant it is to be regarded as the same constant.

For if we introduce the time and make the arbitrary but permissible assumption that the system traverses the paths under consideration with a constant velocity, this being such that the constant h denotes the value of the analytical energy, then

$$T = U + h = \frac{1}{2} m \frac{ds^2}{dt^2} \qquad \text{(i)},$$

and thus the integral considered is equal to

$$\sqrt{\frac{2}{m}} \int_{t_0}^{t_1} dt.$$

The integral, except for the coefficient, is therefore equal to the duration of the motion. But this, by § 352 regarded as a consequence of § 347, is a minimum for a given value of the energy, *i.e.* of the constant h. Hence the content of this note is identical with that of proposition § 352, but is expressed by means of the notation since introduced.

633. **Observation 1.** If the restriction that the positions should be sufficiently near is omitted, then the vanishing of a variation only can be asserted: in our notation this statement is represented in the form

$$\delta_p \int_0^1 \frac{ds}{\sqrt{U + h}} = 0.$$

634. **Observation 2.** By means of the property stated in Note 2, the natural paths, which correspond to different values of the constant h, are uniquely distinguished from all other possible paths; and the proposition may be used for the determination of the natural paths of the system, if it is possible to form the variation δ_p. If, however, as is assumed, the peculiarities of the cyclical motion are concealed, then it is not possible to form this variation, and the note, although still correct, ceases to be applicable to the purpose in question.

635. **Proposition 2.**[1] In the motion of a free holonomous system which contains concealed adiabatic cycles, between two sufficiently near positions 0 and 1 of the visible masses, the integral

$$\int_0^1 \sqrt{U+h}\, ds$$

is smaller for the natural paths than for any other possible paths by which, with the same values of the concealed cyclical momenta and the constant h, the visible coordinates pass from the given initial values to the given final ones.

We again give the proof by a reference to the foregoing note (§ 632). For this purpose we introduce the time, and make the arbitrary but permissible assumption that the system traverses the paths considered with constant velocity, this being such that the constant h is equal to the mathematical energy. The integral can then be written in the form

$$\sqrt{\frac{2}{m}}\int_{t_0}^{t_1}(U+h)dt.$$

Further, we again associate, as is permissible by § 592, each one of the varied motions mentioned in the proposition with a second in which the visible coordinates undergo the same variation, and in which the constant h, and consequently the energy E, remains unaltered; the cyclical momenta must, however, vary in such a manner that the initial and final values of the cyclical coordinates retain their original values. A variation corresponding to the requirements of the proposition we shall again denote by $\delta_{\mathfrak{q}}$, and a variation corresponding to the second motion by $\delta_{\mathfrak{p}}$.

Now, firstly, for any variations $\delta\mathfrak{q}_\rho$ of the cyclical momenta $\mathfrak{q}_\rho$ (§ 566)

$$\begin{aligned}\delta\int(U+h)\,dt &= \delta_{\mathfrak{q}}\int(U+h)\,dt + \sum_1^{\mathfrak{r}}{}_\rho\int\frac{\partial(U+h)}{\partial\mathfrak{q}_\rho}\delta\mathfrak{q}_\rho dt \\ &= \delta_{\mathfrak{q}}\int(U+h)dt - \tfrac{1}{2}\sum_1^{\mathfrak{r}}{}_\rho\,\bar{\mathfrak{p}}_\rho\delta\mathfrak{q}_\rho,\end{aligned}$$

thus in particular for a variation $\delta_{\mathfrak{p}}$

[1] Printed as in the original MSS.—ED.

$$\delta_{\mathfrak{p}}\int(\mathrm{U}+h)dt=\delta_{\mathfrak{q}}\int(\mathrm{U}+h)dt-\tfrac{1}{2}\sum_{1}^{\mathfrak{r}}{}_{\rho}\,\bar{\mathfrak{p}}_{\rho}\delta_{\mathfrak{p}}\mathfrak{q}_{\rho} \qquad \text{(i).}$$

Secondly, we obtain from the equation § 612 (iii), remembering the relation § 588 and the constancy of E,

$$\int(\mathrm{U}+h)dt=\mathrm{E}(t_1-t_0)-\tfrac{1}{2}\sum_{1}^{\mathfrak{r}}{}_{\rho}\,\bar{\mathfrak{p}}_{\rho}\mathfrak{q}_{\rho},$$

thus by a variation of the kind $\delta_{\mathfrak{p}}$

$$\delta_{\mathfrak{p}}\int(\mathrm{U}+h)dt=\mathrm{E}\delta_{\mathfrak{p}}(t_1-t_0)-\tfrac{1}{2}\sum_{1}^{\mathfrak{r}}{}_{\rho}\,\bar{\mathfrak{p}}_{\rho}\delta_{\mathfrak{p}}\mathfrak{q}_{\rho} \qquad \text{(ii).}$$

Subtraction of (i) and (ii) gives

$$\delta_{\mathfrak{q}}\int(\mathrm{U}+h)dt=\mathrm{E}\delta_{\mathfrak{p}}(t_1-t_0) \qquad \text{(iii)};$$

or when, by aid of § 632 (i), we again eliminate the time,

$$\delta_{\mathfrak{q}}\int_0^1\sqrt{\mathrm{U}+h}\,ds=\mathrm{E}\delta_{\mathfrak{p}}\int_0^1\frac{ds}{\sqrt{\mathrm{U}+h}} \qquad \text{(iv).}$$

The variation on the right has always, by § 632, for the natural motion a vanishing, and for sufficiently near positions a negative value; and hence, since E is necessarily positive, the same holds for the variation on the left. The integral on the left has thus, for the natural motion and for sufficiently near final positions, a minimum value, which proves the proposition.

636. **Observation 1.** If the restriction that the positions should be sufficiently near is removed, then it can only be asserted that the variation of the integral vanishes. The analytical expression of this statement is in our notation (in contradistinction to § 633)

$$\delta_{\mathfrak{q}}\int_0^1\sqrt{(\mathrm{U}+h)}\,ds=0.$$

637. **Observation 2.** For every value of the constant h the proposition distinguishes without ambiguity a natural path from all other possible paths. The property of natural paths which the proposition states, may therefore be used for

the determination of these paths; it can even be used if the cyclical motions are assumed to be concealed.

For the formation of the variation δ_q only requires that the constants appearing in the force-function should remain unaltered; the variation can thus be formed notwithstanding our ignorance of the peculiarities of cyclical motion.

638. **Observation 3.** Proposition 2, employed in the conception of the last observation, is Jacobi's form of the Principle of Least Action. For if, for the moment, we take m_ν to be the mass of the νth of the visible points of the system, ds_ν an element of the path of this point, then

$$mds^2 = \sum_1^n {}_\nu\, m_\nu ds_\nu^{\,2},$$

and thus the integral for which we establish a minimum value is, except for coefficient,

$$\int \sqrt{\mathrm{U}+h}\sqrt{\sum_1^n {}_\nu\, m_\nu ds_\nu^{\,2}},$$

which (again excepting a constant coefficient) is Jacobi's integral.

The physical meaning of Jacobi's Principle we conceive to be no other than that contained in propositions § 352 or § 347, from which it is deduced. It represents the form which we must give to that proposition in order that, notwithstanding our ignorance of the peculiarities of cyclical motions, it may be applicable to the determination of the motion of the visible system. The validity of Jacobi's Law is also confined to holonomous systems.

639. **Proposition 3.** In the motion of a free holonomous conservative system between sufficiently near positions, the time-integral of the kinetic energy is smaller for the natural motion than for any other possible motion which carries the system from the given initial to the given final values of the visible coordinates, and which is performed with the same given value of the mathematical energy which is constant with regard to the time.

For if we take h to be the given value of the mathematical energy, then for all the paths considered (§ 611)

$$\mathrm{T} - \mathrm{U} = h,$$

and thus the integral of which the proposition treats, viz.

$$\int_{t_0}^{t_1} \mathrm{T}dt,$$

is (except for a constant coefficient) the integral of which Proposition 2 treats; the present proposition is thus only another mode of expressing the content of that proposition.

Observations similar to Nos. 1 and 2 after Proposition 2 are also applicable here.

640. **Observation.** Proposition § 639 expresses the Principle of Least Action as originally stated by Maupertuis. This form is preferable to Jacobi's in that it can be expressed more simply, and therefore appears to contain a simple physical meaning. But it has the disadvantage that it contains the time unnecessarily, inasmuch as the actual statement only determines the path of the system and not the motion in it; this motion being rather determined only by the note which is added, viz. that only motions with constant energy will be considered.

Retrospect to §§ 625-640

641. 1. From our investigations we see that, for the natural motion of a free conservative system, each one of the integrals

$$\int (\mathrm{T}-\mathrm{U})dt, \qquad \int (\mathrm{T}+\mathrm{U})dt,$$

$$\int \frac{ds}{\sqrt{\mathrm{U}+h}}, \qquad \int \sqrt{\mathrm{U}+h}\,ds,$$

$$\int \mathrm{T}dt,$$

takes a special value under determined conditions. While the two upper integrals relate to the motion of the system, the others refer only to the path. The two integrals on the left relate to the case when all the coordinates of the system, even the cyclical ones, are considered, and when only those positions of the system are considered the same in which the latter coordinates as well as the former have the same values. The remaining integrals relate to the case when the cyclical coordinates are concealed, and when those positions of the system are considered the same in which the visible coordinates have the same values. The consideration of the last integral assumes the validity of the Principle of the Conservation of Energy; the consideration of the two upper ones allows the deduction of this principle; the two middle ones can be considered independently of this principle.

642. 2. The physical meaning of the two integrals on the left is extremely simple; the statements expressing them are immediate consequences of the fundamental law. The integrals on the right have lost their simple physical meaning; but the statement that they take special values for the natural motion always represents a form of the fundamental law, even though it be complicated and obscure. This has happened because the law has been adapted to complicated and obscure hypotheses. The statement which relates to the last integral has an illusory appearance of an independent and simple physical meaning.

Our method of proof was not chosen with a view to being as simple as possible, but to making the above relations stand out as clearly as possible.

643. 3. That Nature is not constituted so as to make any one of these integrals a minimum, is seen firstly from the fact that even in holonomous systems with a more extended motion a minimum does not always appear; and, secondly, from the fact that there are natural systems for which the minimum never appears, and for which the variation of these integrals never vanishes. An expression comprehending all the laws of natural motion cannot therefore be assigned to any of these integrals; and this justifies us in regarding the apparently simple meaning of the last integral as illusory.

Finite Equations of Motion for Holonomous Systems

644. **Note 1.** Let us denote by V′ the value of the integral

$$\sqrt{\frac{m}{2}}\int_0^1 \frac{ds}{\sqrt{U+h}}$$

taken for the natural path between two value-systems of all the coordinates of a free holonomous system with adiabatic cycles, regarded as a function of the initial and final values of these coordinates, *i.e.* of $p_{\rho 0}$, $p_{\rho 1}$, and $\mathfrak{p}_{\rho 0}$, $\mathfrak{p}_{\rho 1}$, and the quantity h; then the expression

$$V'\sqrt{\frac{2E}{m+\mathfrak{m}}}$$

represents the straightest distance of the system. The notation is the same as we have used previously in this chapter.

By § 632 V′ is equal to the duration of the natural motion between the given positions, for the mathematical energy h. If then S is the straightest distance between the two positions, we get

$$E = \tfrac{1}{2}(m+\mathfrak{m})\frac{S^2}{V'^2},$$

from which the proof follows.

645. **Corollary.** By means of the function V′ the natural paths of the system considered may be represented in a concise form.

For if ds denotes an element of the path of the visible partial system, and $d\mathfrak{s}$ a similar quantity for the cyclical partial system, and $d\sigma$ for the complete system, then

$$(m+\mathfrak{m})d\sigma^2 = m ds^2 + \mathfrak{m} d\mathfrak{s}^2 \qquad \text{(i)},$$

and therefore (§ 57) with the previous notation

$$d\sigma^2 = \sum_1^r{}_\rho \sum_1^r{}_\sigma \frac{m}{m+\mathfrak{m}} a_{\rho\sigma} dp_\rho dp_\sigma + \sum_1^r{}_\rho \sum_1^r{}_\sigma \frac{\mathfrak{m}}{m+\mathfrak{m}} \mathfrak{a}_{\rho\sigma} d\mathfrak{p}_\rho d\mathfrak{p}_\sigma \quad \text{(ii)}.$$

If $\widehat{\sigma p_\rho}$ and $\widehat{\sigma \mathfrak{p}_\rho}$ are the angles which the path of the complete system makes with the coordinates p_ρ and $\mathfrak{p}_\rho$ of that system, then the equations of the natural paths, after division of both sides by a constant factor, are obtained by §§ 224, 226 in the form

$$\sqrt{a_{\rho\rho1}} \cos \sigma, p_{\rho1} = \sqrt{\frac{2E}{m}} \frac{\partial V'}{\partial p_{\rho1}} \qquad \text{(iii)}$$

$$\sqrt{a_{\rho\rho0}} \cos \sigma, p_{\rho0} = -\sqrt{\frac{2E}{m}} \frac{\partial V'}{\partial p_{\rho0}} \qquad \text{(iv)}$$

$$\sqrt{\mathfrak{a}_{\rho\rho1}} \cos \sigma, \mathfrak{p}_{\rho1} = \sqrt{\frac{2E}{\mathfrak{m}}} \frac{\partial V'}{\partial \mathfrak{p}_{\rho1}} \qquad \text{(v)}$$

$$\sqrt{\mathfrak{a}_{\rho\rho0}} \cos \sigma, \mathfrak{p}_{\rho0} = -\sqrt{\frac{2E}{\mathfrak{m}}} \frac{\partial V'}{\partial \mathfrak{p}_{\rho0}} \qquad \text{(vi)},$$

and these equations admit of a dual interpretation, namely, either that they give the equations of the natural paths as differential equations of the first order or as equations of a finite form.

646. **Observation.** The foregoing equations (iii) to (vi) are correct in all cases, whether we regard the cyclical coordinates as visible or concealed. They cease, however, to be applicable if the latter be the case; for then the complete expression for V' is unknown and the equations cannot be developed.

647. **Problem 1.** To transform the foregoing equations of motion of a free holonomous system so that they remain applicable even when the cyclical motions of the system are concealed.

We denote by V the value of the integral

$$\sqrt{2m}\int_0^1 \sqrt{U + h}ds,$$

taken for the natural path between two value-systems of the visible coordinates. In the determination of this natural path we shall regard the cyclical momenta in the force-function as invariable constants; V will therefore be considered a function of the initial and final values alone of the visible coordinates and the constant h. By § 635 (iv), for the transition from one natural path to another with visible coordinates varied in any manner,

$$\delta_q \sqrt{2m}\int_0^1 \sqrt{U + h}ds = 2E\delta_p\sqrt{\frac{m}{2}}\int_0^1 \frac{ds}{\sqrt{U + h}} \qquad \text{(i)},$$

so that, in particular, in a transition from one natural path to any neighbouring natural path

$$\delta_q V = 2E\delta_p V' \qquad \text{(ii)},$$

therefore

$$\left.\begin{aligned}\frac{\partial V}{\partial p_{\rho 1}} &= 2E\frac{\partial V'}{\partial p_{\rho 1}}\\ \frac{\partial V}{\partial p_{\rho 0}} &= 2E\frac{\partial V'}{\partial p_{\rho 0}}\end{aligned}\right\} \qquad \text{(iii)}.$$

By the help of these equations we can eliminate the cyclical coordinates from the right-hand sides of the equations § 645 (iii) and (iv). Then for the left-hand sides we have to replace the angle $\widehat{\sigma p_\rho}$ by the angle $\widehat{sp_\rho}$. We then have, by § 645 (ii) (§ 75),

$$\left.\begin{aligned}\sqrt{\frac{m}{m+\mathfrak{m}}}\sqrt{a_{\rho\rho}}d\sigma\cos\sigma,p_\rho &= \sum_1^r{}_\sigma\frac{m}{m+\mathfrak{m}}a_{\rho\sigma}dp_\sigma\\ &= \frac{m}{m+\mathfrak{m}}\sqrt{a_{\rho\rho}}ds\cos s,p_\rho\end{aligned}\right\} \qquad \text{(iv)},$$

and further, from the equations

$$\left.\begin{aligned}U + h = T &= \tfrac{1}{2}m\frac{ds^2}{dt^2}\\ \text{and}\qquad E &= \tfrac{1}{2}(m+\mathfrak{m})\frac{d\sigma^2}{dt^2}\end{aligned}\right\} \qquad \text{(v)},$$

by division

$$d\sigma = \sqrt{\frac{m}{m+\mathfrak{m}}}\sqrt{\frac{E}{U+h}}ds \qquad \text{(vi)};$$

thus from (iv) and (vi)

$$\cos\sigma,p_\rho = \sqrt{\frac{U+h}{E}}\cos s,p_\rho \qquad \text{(vii)}.$$

If now we substitute the result (iii) on the right, and the result (vii) on the left of the equations to be transformed, we obtain the equations

$$\sqrt{a_{\rho\rho 1}}\cos s,p_{\rho 1} = \frac{1}{\sqrt{2m(U+h)_1}}\frac{\partial V}{\partial p_{\rho 1}},$$

$$\sqrt{a_{\rho\rho 0}}\cos s,p_{\rho 0} = -\frac{1}{\sqrt{2m(\mathrm{U}+h)_0}}\frac{\partial \mathrm{V}}{\partial p_{\rho 0}} \qquad \text{(viii)},$$

which are the required transformations. For they no longer contain any quantities which refer to the concealed partial system, and they admit the dual interpretation that they present the natural paths of the visible partial system as differential equations of the first order, or in a finite form.

648. **Observation 1.** The function V does not contain the time and gives only the natural paths of the system, but not its motion in these paths. But since the natural paths are traversed with constant velocities, and we have already assigned the interpretation of analytical energy to the constant h appearing in V, it is easy to introduce the time as an independent variable in the equations. In the first place, the connection of the time with the length of the path, previously regarded as the independent variable, is given by the equation

$$\frac{\partial \mathrm{V}}{\partial h} = \sqrt{\frac{m}{2}}\int_0^1 \frac{ds}{\sqrt{\mathrm{U}+h}} = t_1 - t_0 \qquad \text{(i)}.$$

Thus we obtain after multiplication of the equations § 647 (viii) by

$$\sqrt{2m(\mathrm{U}+h)} = \sqrt{2m\mathrm{T}} = m\frac{ds}{dt},$$

and using § 75 and § 270,

$$q_{\rho 1} = \frac{\partial \mathrm{V}}{\partial p_{\rho 1}} \qquad \text{(ii)}$$

$$q_{\rho 0} = -\frac{\partial \mathrm{V}}{\partial p_{\rho 0}} \qquad \text{(iii)}.$$

Finally, we obtain for the value of the function itself,

$$\mathrm{V} = 2\int_{t_0}^{t_1} \mathrm{T}dt \qquad \text{(iv)}.$$

In form these equations are much simpler than the equations of the foregoing problem, but the former have the advantage of containing one less independent variable.

649. **Observation 2.** The function V is the same function as Hamilton denoted by a similar symbol, and is known as the characteristic function of the conservative system. This statement agrees with that of § 412, for by the assumption made there that all coordinates were visible, the function here denoted by V is transformed into the function there denoted by the same symbol.

Finally, it appears that the characteristic function of a system, according to the now extended definition, is a quantity for calculation without any physical meaning. For, according as we treat greater or lesser parts of cyclical motions as concealed, we may write down different characteristic functions for the same system; and these serve the same purpose analytically although they possess different values for identical motions of the system.

650. **Proposition.** The characteristic function V of a conservative system satisfies the two partial differential equations of the first order

$$\frac{1}{2m}\sum_1^r{}_\rho\sum_1^r{}_\sigma b_{\rho\sigma 1}\frac{\partial \mathrm{V}}{\partial p_{\rho 1}}\frac{\partial \mathrm{V}}{\partial p_{\rho 1}} = (\mathrm{U}+h)_1$$

$$\frac{1}{2m}\sum_1^r{}_\rho\sum_1^r{}_\sigma b_{\rho\sigma 0}\frac{\partial \mathrm{V}}{\partial p_{\rho 0}}\frac{\partial \mathrm{V}}{\partial p_{\sigma 0}} = (\mathrm{U}+h)_0,$$

which correspond to the differential equations § 227 for the straightest distance.

For these equations are obtained by the substitution of the direction cosines from the equations § 647 (viii) in the equation § 88, which these direction cosines satisfy.

651. **Note 2.** If we denote by P′ the value of the integral

$$\int_{t_6}^{t_1} (\mathrm{T}-\mathrm{U})dt,$$

taken for the natural motion between two value-systems of all the coördinates of a free holonomous system with adiabatic cycles, and considered as a function of these values and the duration of the motion, then P′ differs from the principal function of the system (§ 415) only by the product of the duration of the motion and an (unknown) constant.

For T − U differs from the energy of the system only by an (unknown) constant.

652. **Corollary.** With the aid of the function P′ the natural motions of the system can be expressed in a concise form.

In fact the difference between P′ and the principal function defined in § 415 does not prevent the immediate application of the equations § 414 (ii) and (iii), so that we obtain as equations of motion

$$q_{\rho 1} = \frac{\partial \mathrm{P}'}{\partial p_{\rho 1}} \qquad \text{(i)}$$

$$q_{\rho 1} = -\frac{\partial \mathrm{P}'}{\partial p_{\rho 0}} \qquad \text{(ii)}$$

$$q_{\rho 1} = \frac{\partial \mathrm{P}'}{\partial \mathfrak{p}_{\rho 1}} \qquad \text{(iii)}$$

$$q_{\rho 9} = -\frac{\partial \mathrm{P}'}{\partial \mathfrak{p}_{\rho 0}} \qquad \text{(iv).}$$

On the other hand the equation § 414 (iv) requires a slight modification; we obtain instead of it

$$h = -\frac{\partial \mathrm{P}'}{\partial t_0} = \frac{\partial \mathrm{P}'}{\partial t_1}.$$

653. **Observation.** The foregoing equations (i) to (iv) are correct in every case, whether all the coordinates are accessible to observation or not; but they cease to be applicable when the cyclical motions of the system are considered concealed.

654. **Problem 2.** To transform the foregoing equations of motion of a free holonomous system, so that they remain applicable even when the cyclical motions of the system are concealed.

We denote by P the value of the integral

$$\int_{t_0}^{t_1} (\mathrm{T} + \mathrm{U})\,dt,$$

taken for the natural motion between the two value-systems of the visible coordinates existing at the times t_0 and t_1. In the determination of this natural motion the cyclical momenta contained in the constants of the force-function will be considered invariable, and P will thus be considered a function of the initial and final values alone of these coordinates and of the times t_0 and t_1.

Now, by § 628 (iii), for a transition from a natural motion to any neighbouring motion of equal duration, the equation

$$\delta_{\mathrm{q}}\int(\mathrm{T}+\mathrm{U})dt = \delta_{\mathrm{p}}\int(\mathrm{T}-\mathrm{U})dt$$

holds. If we apply this equation to the transition from a natural motion to a neighbouring natural motion of equal duration, we get

$$\delta_{\mathrm{q}}\mathrm{P} = \delta_{\mathrm{p}}\mathrm{P}',$$

thus

$$\frac{\partial \mathrm{P}}{\partial p_{\rho 0}} = \frac{\partial \mathrm{P}'}{\partial p_{\rho 0}}, \qquad \frac{\partial \mathrm{P}}{\partial p_{\rho 1}} = \frac{\partial \mathrm{P}'}{\partial p_{\rho 1}}.$$

By means of these equations we eliminate the concealed coordinates from the right of equations § 652. For the left, it is sufficient to remark that the momentum q_ρ of the whole system along p_ρ is also the momentum of the visible partial system along p_ρ, regarded as a coordinate of this partial system. We then obtain as equations of motion of the visible partial system

$$q_{\rho 1} = \quad \frac{\partial \mathrm{P}}{\partial p_{\rho 1}} \qquad \text{(i)},$$

$$q_{\rho 0} = -\frac{\partial \mathrm{P}}{\partial p_{\rho 0}} \qquad \text{(ii)},$$

which are the required transformations.

655. **Observation 1.** The function P here introduced is the function which Hamilton denoted by S, and is known as the principal function of the conservative system. This statement agrees with § 415, for by the assumption there made, that all the coordinates are visible, the present function P transforms into the function there denoted by the same symbol.

656. **Observation 2.** The value of the principal function for a definite transference is related to the characteristic function in a simple manner. For by a simple transformation we obtain

$$\int_{t_0}^{t_1}(\mathrm{T}+\mathrm{U})dt = \int_{t_0}^{t_1}(2\mathrm{U}+h)dt$$

$$= \sqrt{2m}\int_0^1 \sqrt{\mathrm{U}+h}ds - \sqrt{\frac{m}{2}}\int_0^1 \frac{hds}{\sqrt{\mathrm{U}+h}}.$$

Thus (§§ 647, 644)

$$\mathrm{P} = \mathrm{V} - h(t_1 - t_0) \qquad \text{(i)},$$

where we have to regard the quantity h introduced on the right-hand side, in V and in the second summation, as a function of $(t_1 - t_0)$, $p_{\rho 0}$ and $p_{\rho 1}$.

Conversely,

$$\mathrm{V} = \mathrm{P} + h(t_1 - t_0) \qquad \text{(ii)},$$

where, on the right-hand side in P and in the second summation, the quantity $(t_1 - t_0)$ is regarded as a function of h, $p_{\rho 0}$ and $p_{\rho 1}$.

657. **Observation 3.** The analytical energy h does not appear in the principal function. Still it can be indirectly deduced from it by means of the equations § 654 (i), (ii), § 286 (iii) and § 612 (i). It can also be directly expressed by means of P. For if we change on the right-hand side of § 656 (i) t_1 and t_0, but not $p_{\rho 1}$ and $p_{\rho 0}$, and denote by dh the change of h which necessarily results therefrom, we get

$$d\mathrm{P} = \frac{\partial \mathrm{V}}{\partial h}dh - hd(t_1 - t_0) - (t_1 - t_0)dh,$$

and thus, by § 648 (i),

$$d\mathrm{P} = -hd(t_1 - t_0),$$

from which follows

$$h = -\frac{\partial \mathrm{P}}{\partial t_1} = \frac{\partial \mathrm{P}}{\partial t_0}.$$

658. **Proposition.** The principal function P of a conservative system satisfies the two differential equations of the first order

$$\frac{1}{2m}\sum_1^r{}_\rho\sum_1^r{}_\sigma b_{\rho\sigma 1}\frac{\partial \mathrm{P}}{\partial p_{\rho 1}}\frac{\partial \mathrm{P}}{\partial p_{\sigma 1}} + \frac{\partial \mathrm{P}}{\partial t_1} = \mathrm{U}_1$$

$$\frac{1}{2m}\sum_1^r{}_\rho \sum_1^r{}_\sigma b_{\rho\sigma 0}\frac{\partial \mathrm{P}}{\partial p_{\rho 0}}\frac{\partial \mathrm{P}}{\partial p_{\sigma 0}} - \frac{\partial \mathrm{P}}{\partial t_0^1} = \mathrm{U}_0,$$

which correspond to the differential equations § 227 for the straightest distance.

For these equations are obtained when the analytical energy h is expressed in terms of the differential coefficients of P, the first time directly by means of § 657, and the second time indirectly by means of § 612 (i) and § 654 (i), (ii).

Retrospect to §§ 644-658

659. 1. In §§ 644-658 there are given four finite representations of the motion of a holonomous system with adiabatic cycles. In the first and third all the coordinates of the system were considered capable of being observed, and in the second and fourth the cyclical coordinates were treated as concealed. The first and third representation, which led to the characteristic function, essentially gave only the path of the system and corresponded to the Principle of Least Action. The second and fourth, which led to the principal function, gave the motion completely, and corresponded to Hamilton's Principle.

660. 2. All the four representations have the same simple physical sense, and in all of them the cause of the mathematical complexity is the same. The simple physical sense consists in the fact that the natural paths are always straightest paths, and in the purely geometrical connections of these paths with the straightest distance in holonomous systems. The cause of the mathematical complexity consists in this, that we did not always treat in the same manner all the essential elements for determining the motion, but eliminated some of them as concealed. We may also say that difference in the treatment consists in the fact that for some coordinates the initial and final values were the elements introduced, and for others the initial velocities. Our course of investigation was not adopted as being the simplest possible, but rather as putting this relation as clearly as possible.

661. 3. Further representations of the motion of a

holonomous system could be given by eliminating other coordinates, or by introducing for the visible coordinates as well, not their initial and final values, but other quantities as elements; or by proceeding from the partial differential equations, § 650 or § 658, in the same manner as is done for the straightest distance in §§ 232 *et seq.* Such representations may in particular cases have certain mathematical advantages, as Jacobi has shown in a comprehensive manner. But the further one proceeds in this direction the more is the physical meaning obscured under its mathematical form, and the more the functions used take the character of auxiliary constructions with which it is no longer possible to associate a physical meaning.

Non-Conservative Systems

Explanations and Notes

662. 1. If a material system contains only such concealed masses as are in adiabatic cyclical motion, then if the visible coordinates are under our free control it is possible at every instant to transform back the energy which has become the energy of the concealed masses, into the energy of the visible masses. The visible energy once residing in the system may therefore be permanently retained as visible energy.

It is on account of this property that we have called these systems conservative. For the same reason we denote the forces exerted by the concealed masses of such systems as conservative forces.

663. 2. On the other hand, those systems in which we cannot sufficiently control the visible coordinates so as to retransform the concealed energy at every instant into visible energy are called non-conservative, and the forces of their concealed masses non-conservative forces. Non-conservative systems in which the energy tends to change from the energy of the visible masses into that of the concealed masses, but not conversely, are called dissipative systems, and the forces due to their concealed masses dissipative forces.

664. 3. In general the systems and forces of nature are non-conservative if concealed masses come into consideration. This circumstance is a necessary consequence of the fact that conservative systems are exceptions, and even exceptions attained only more or less approximately (§ 550); so that for any natural system taken at random the probability of its being conservative is infinitely small. Again we know by experience that the systems and forces of nature are dissipative if concealed masses come into consideration. This circumstance is sufficiently explained by the hypothesis that in nature the number of concealed masses and of their degrees of freedom is infinitely great compared with the number of visible masses and their visible coordinates; so that for any motion taken at random the probability of the energy concentrating itself in a special direction from that large number of masses into this definite and small number is infinitely small.

665. 4. The difference between conservative and dissipative systems of forces does not lie in nature, but results simply from the voluntary restriction of our conception, or the involuntary limitation of our knowledge of natural systems. If all the masses of nature were considered visible, then the difference would cease to exist, and all the forces of nature could be regarded as conservative forces.

666. 5. Conservative forces appear in general as differential coefficients of force-functions, *i.e.* as such functions of the visible coordinates of the system as are independent of the time. The non-conservative forces depend in general on the first and higher differential coefficients of the visible coordinates with regard to the time. With any given analytical form of a force of either kind, the question may be raised whether this form is consistent with the assumptions of our mechanics, or the reverse.

667. To this question an answer cannot in general be given; in particular cases it is to be judged from the following considerations:—

(1) If it can be shown that there exists a normal continuous system which exerts forces of the given form, then it is proved that the given form satisfies the postulates of our mechanics.

(2) If it can be proved that the existence of such a system is impossible, then it is shown that the given form contradicts our mechanics.

(3) If it can be shown that there exists in nature any system which we know by experience to exert forces of the given form, then we consider it thereby proved that the given form is consistent with our mechanics.

If no one of the three cases happens, then the question must remain an open one. Should such a form of force be found as would be rejected by the second consideration, but permitted by the third, then the insufficiency of the hypothesis on which our mechanics reposes, and in consequence the insufficiency of our mechanics itself, would be proved.

CHAPTER VI

DISCONTINUITIES OF MOTION

Explanations and Notes

668. 1. All systems of material points to which the fundamental law in accordance with its assumptions is applicable must possess continuous connections. Hence the coefficients of all the equations of condition of such systems are throughout continuous functions of the position (§ 124). This, however, does not prevent these functions from changing very quickly near given positions, so that the equations have, in positions very near to one another, coefficients which differ by finite quantities.

669. 2. When the system considered passes through such a position of very rapid change, then a complete knowledge of its motion requires a complete knowledge of the equations of condition during the rapid change itself. Certain statements may, however, be made concerning the motion even when the form of the equations of condition of the system is given only before and after the place of its sudden change. If we limit ourselves to this class of statements, then it is analytically simpler to pay no attention to the special manner of the change, and to use the equations of condition as though their coefficients were discontinuous. In this case the system is regarded as discontinuous, owing to the voluntary limitation imposed by our mode of treatment.

670. 3. But it may happen that while our physical means permit us to completely investigate the connection of a system

in other respects, they are yet insufficient to investigate it at the places of very sudden change, although we are convinced, and indeed may physically prove, that even here this connection is continuous. If this happens we are compelled to represent the connection analytically as discontinuous, unless we renounce the possibility of a single representation of it. In this case the system must be regarded as discontinuous on account of the involuntary limitation of our knowledge of the system.

671. 4. Conversely, if the coefficients of the equations of condition of a system are directly given as discontinuous functions of the position, without a knowledge of how these functions are obtained, then we assume that one of the two cases previously mentioned happens. We regard the given equations only as an incomplete and approximate presentation of the true and continuous form. We therefore assume, from this very fact, that a complete determination of the motion of such a system is not required of us, but only the specification of those statements which can be made notwithstanding the incomplete knowledge of the system, with the supposition that even in the positions of discontinuity the unknown connection is in reality continuous.

672. 5. If a system passes through a point of very rapid change with a finite velocity, then its equations of condition undergo finite changes in a vanishing time. If during the whole change the system is in reality normal, as the fundamental law assumes, then, to all appearances, it ceases to be normal at the instant of its passing through that position, although this has not actually occurred. Hence if a system is given us analytically, and if its equations of condition are independent of the time but at a certain moment instantaneously take a new form, then we consider the equations of condition at this moment as only an approximate representation of another connection, unknown and perhaps more intricate, but at the same time not only continuous but also normal. Hence we assume again that a complete determination of the motion of the system is not required of us, but only a specification of those statements which, notwithstanding our ignorance, can be made by means of the fundamental law, with the supposi-

tion that even at the time of the discontinuity the true connection of the system is continuous and normal.

673. 6. When we regard all positions and times of discontinuity in the foregoing manner we have renounced the investigation of actually discontinuous systems. The fundamental law, too, would not be applicable to these. This restriction, however, does not imply a refusal to investigate any natural system whatever, for everything points to the conclusion, that there are in nature only apparent, and not actual, discontinuities. That the motion of systems through apparent positions of discontinuity is not completely determined by the fundamental law alone, corresponds entirely with the physical experience that the knowledge of a system before and after a position of discontinuity is not sufficient to determine completely the change of the motion during the passage through that position.

Impulsive Forces or Impulses

674. **Note.** If a system passes through a position of discontinuity, then its velocity undergoes a change of finite magnitude. The differential coefficients of the coordinates with regard to the time suddenly jump to new values.

For immediately before and after such a position these differential coefficients, and consequently the components of that velocity, must satisfy linear equations with finitely different coefficients.

675. **Corollary 1.** In a motion through a position of discontinuity the acceleration becomes infinitely great, but in such a manner that the time-integral of the acceleration taken for the time of the motion retains in general a finite value.

For this time-integral is the change of the velocity which in general is finite.

676. **Corollary 2.** If the equations of condition of one of two or more coupled systems are subject to discontinuity, then in the motion through this discontinuity the force acting between the systems becomes in general infinitely great, but in such a manner that the time-integral of the force, taken for the time of the motion, remains finite.

For in general the components of the acceleration of the discontinuous system along the common coordinates become infinite in the sense of Corollary 1. But since the coefficients of the equations of condition remain finite during the discontinuity, the force is of the order of the acceleration.

677. **Definition.** An impulsive force or impulse is the time-integral of the force exerted by one system on another during the motion through a position of discontinuity, taken for the duration of the motion through this position.

678. **Observation.** When all the systems considered have finite velocities, finite and infinitesimal, but not infinite impulses, may appear. In what follows we shall assume the impulses to be finite.

679. **Corollary 1.** To every impulse there is always a counter-impulse. It is the time-integral of the force which the system regarded as the second exerts on the first.

680. **Corollary 2.** An impulse is always exerted by, as well as exerted on, a system which suffers a discontinuity of motion; it is not conceivable without two such systems mutually acting on one another.

We may speak of impulses simply without expressly mentioning the systems which cause or suffer them, for exactly the same reasons as we thus speak of forces.

681. **Corollary 3.** An impulse may always be considered as a vector quantity with regard to that system which causes it, as well as with regard to that system on which it acts. Its components along the common coordinates are in general different from zero; its components along the coordinates which are not common are zero; its components in directions which cannot be expressed in terms of the coordinates used remain undetermined.

For this statement holds for the force of which the impulse is the time-integral.

682. **Notation.** If a system with the coordinates p_ρ suffers a discontinuity of motion, then we shall denote the components along p_ρ of the impulse which acts on the system, by J_ρ. But the components of the impulse which the system causes along p_ρ will be denoted by J_ρ'. For the second system whose co-

ordinates are denoted by $\mathfrak{p}_\rho$, the corresponding quantities will be denoted by $\mathfrak{J}_\rho$ and $\mathfrak{J}_\rho'$ respectively (*cf.* § 467). Thus, then,

$$J_\rho = \mathfrak{J}_\rho',$$
$$\mathfrak{J}_\rho = J_\rho',$$

identically.

683. **Proposition.** An impulse and its counter-impulse are always equal and opposite, *i.e.* their components along every coordinate are equal and opposite whether we consider these quantities as vector quantities with regard to the one system, or with regard to the other system.

For an impulse and its counter-impulse can also be regarded as is the time-integrals of force and counter-force (*cf.* § 468).

With the notation employed the proposition is given by the equation

$$J_\rho = -J_\rho'$$
$$\mathfrak{J}_\rho = -\mathfrak{J}_\rho'.$$

Composition of Impulses

684. **Proposition.** If a system is simultaneously coupled with other systems, then any impulse which the aggregate of these systems exerts is equal to the sum of the impulses exerted by the several systems.

For the proposition holds at every instant during the impulses for the acting forces (§ 471), and therefore also for their integrals, *i.e.* for the impulses.

685. **Corollary.** If impulses simultaneously act on the same system or are exerted by the same system, they can be compounded and resolved by the rules for the composition and resolution of vector quantities. We speak of the components of an impulse and of resultant impulses in the same sense as we speak of the components of forces and resultant forces (*cf.* §§ 472-474).

686. **Definition.** An impulse which is exerted by or on a single material point is called an elementary impulse.

687. **Corollary 1.** Every impulse which is exerted by or on a material system can be resolved into a series of elementary impulses (*cf.* § 479).

688. **Corollary 2.** The composition and resolution of elementary impulses are performed by means of the rules for the composition and resolution of geometrical quantities. (Parallelogram of impulses.) (*Cf.* § 478.)

Motion under the Action of Impulses

689. **Problem 1.** To determine the motion of a material system under the action of a given impulse.

The solution of the problem consists simply in stating the change which the velocity of the system suffers through the impulse. Let the system considered be the same as in § 481; let us denote by P_ρ the components of the infinite force which acts on the system during the impulse, then, by § 481, during this time,

$$mf_\rho + \sum_1^k {}_\chi p_{\chi\rho} P_\chi = P_\rho \qquad \text{(i).}$$

Multiply this equation by dt and integrate for the duration of the impulse. Since the values of the coordinates during this time are constant,

$$m\int f_\rho dt = q_{\rho 1} - q_{\rho 0} \qquad \text{(ii),}$$

where we denote quantities before the impulse by the index 0 and after by the index 1. We have further, by § 682,

$$\int P_\rho dt = J_\rho \qquad \text{(iii),}$$

and putting for short

$$\int P_\chi dt = J_\chi \qquad \text{(iv),}$$

we obtain r equations of the form

$$q_{\rho 1} - q_{\rho 0} + \sum_1^k {}_\chi p_{\chi\rho} J_\chi = J_\rho \qquad \text{(v).}$$

Since the velocity of the system before and after the impulse must satisfy the connections of the system, we obtain from the k equations of condition of the system, k equations of the form

$$\sum_1^r {}_\rho p_{\chi\rho} (\dot{p}_{\rho 1} - \dot{p}_{\rho 0}) = 0 \qquad \text{(vi)},$$

which, with the equations (v), may be regarded as $k+r$ unhomogeneous linear equations for the $k+r$ quantities $\dot{p}_{\rho 1} - \dot{p}_{\rho 0}$ and J_χ, or for the $k+r$ quantities $q_{\rho 1} - q_{\rho 0}$ and J_χ; and they singly determine these quantities, and therefore the change in the velocity of the system.

690. **Observation 1.** If the velocity of the system before the impulse is given, and thus the quantities $q_{\rho 0}$ and $\dot{p}_{\rho 0}$ known, then we may regard the r equations § 689 (v), together with the k equations § 689 (vi), or, what is the same thing, the k equations

$$\sum_1^r {}_\rho p_{\chi\rho} \dot{p}_{\rho 1} = 0,$$

as $r+k$ unhomogeneous linear equations for the $r+k$ quantities $\dot{p}_{\rho 1}$ and J_χ, which singly determine these quantities, and therefore the velocity of the system after the impulse.

691. **Observation 2.** If we use rectangular coordinates and denote the component of the impulse along x_ν by I_ν, then the equations of the impulse take the form of the $3n$ equations

$$m_\nu(\dot{x}_{\nu 1} - \dot{x}_{\nu 0}) + \sum_1^i {}_\iota x_{\iota\nu} I_\iota = I_\nu \qquad \text{(i)},$$

which, with the i equations deduced from the equations of condition, namely,

$$\sum_1^{3n} {}_\nu x_{\iota\nu}(\dot{x}_{\nu 1} - \dot{x}_{\nu 0}) = 0 \qquad \text{(ii)},$$

singly determine the $3n$ components $\dot{x}_{\nu 1} - \dot{x}_{\nu 0}$ of the change of the velocity and the i quantities I_ι.

692. **Observation 3.** If the coordinate p_ρ is a free coordinate, then the corresponding quantities $p_{\chi\rho}$ are zero, and the equation of impulse relative to p_ρ takes the simple form

$$q_{\rho 1} - q_{\rho 0} = J_\rho.$$

If in a holonomous system all the coordinates are free, then all the equations take this form, and the resulting r equations are sufficient to determine the r quantities $\dot{p}_{\rho 1} - \dot{p}_{\rho 0}$, which are known linear functions of the quantities $q_{\rho 1} - q_{\rho 0}$, immediately given by these equations.

693. **Corollary 1 (to § 689).** In order to impress suddenly on a system at rest a given possible velocity, it is sufficient to apply to the system an impulse in the given direction and equal in magnitude to the product of the given velocity and the mass of the system.

For if $q_{\rho 0} = 0$, and the given values of $\dot{p}_{\rho 1}$ satisfy the equations of condition, then the assumption

$$J_\chi = 0$$
$$J_\rho = q_{\rho 1}$$

satisfies the equations § 689 (v) and (vi).

694. **Corollary 2.** In order to bring a moving system suddenly to rest in its instantaneous position, it is sufficient to apply to the system an impulse opposite in direction and equal in magnitude to the product of the velocity of the system and its mass.

For if $q_{\rho 1} = 0$, and if the quantities $\dot{p}_{\rho 0}$ satisfy the equations of condition of the system, then the assumption

$$J_\chi = 0$$
$$J_\rho = -q_{\rho 0}$$

satisfies the equations § 689 (v) and (vi).

695. **Proposition.** The change of velocity which several impulses, acting simultaneously, produce in a system is the sum of the changes of velocity which the impulses, acting singly, would produce.

All impulses are considered as acting simultaneously which take place within a vanishing time, without regard to their succession in this time.

The theorem follows (*cf.* § 485) from the linear form of the equations § 689 (v) and (vi), and it can also be regarded as an immediate consequence of § 485.

696. **Observation.** The content of the foregoing pro-

position may also be expressed by the usual statement that several simultaneous impulses are quite independent as regards the velocity which they produce.

697. **Proposition.** If the direction of an impulse is perpendicular to every possible displacement of the system on which it acts, then the impulse produces no effect on the motion of the system. And conversely: If an impulse produces no effect on the motion of the system on which it acts, then it is perpendicular to every possible displacement of the system.

The proposition may be regarded as an immediate consequence of § 488, or it can be deduced from the equations § 689 (v) and (vi).

698. **Note.** Although the change of motion which an impulse produces can be singly determined when we know the impulse, yet the impulse cannot conversely be singly determined when we know the sudden change of motion which it has produced.

699. **Problem 2.** To determine the impulse which a material system exerts in a given sudden change of motion.

As in § 682 we denote the components of the impulse by J'_ρ, and by §§ 683 and 689 (v) these are

$$J'_\rho = -q_{\rho 1} + q_{\rho 0} - \sum_1^k {}_\chi p_{\chi\rho} J_\chi .$$

In this equation $q_{\rho 1}$ and $q_{\rho 0}$ are determined by the data of the problem, but the J_χ's are not so given unless the motion of the second system on which the impulse acts is also given. The solution of the problem is thus not determinate, but contains an undetermined summation which represents an impulse perpendicular to every possible displacement of the system.

700. **Observation 1.** Although all the components of the impulse which a system exerts in a sudden change of motion are not determined by the change of motion of the system, still all the components in the direction of a possible motion are determined by this change.

701. **Observation 2.** Although all the components of the

impulse which a system exerts in a sudden change of motion are not determined by the change of motion of the system, yet every component in the direction of a free coordinate is singly determined by this change.

702. **Observation 3.** If p_ρ is a free coordinate, then the impulse exerted in the direction of this coordinate can be written in the form

$$\begin{aligned} J'_\rho &= -q_{\rho 1} + q_{\rho 0} \\ &= -\left(\frac{\partial_p E}{\partial \dot{p}_\rho}\right)_1 + \left(\frac{\partial_p E}{\partial \dot{p}_\rho}\right)_0. \end{aligned}$$

Internal Constraint in an Impulse

703. **Note 1.** If an impulse acts on a system of material points between which no connections exist, it produces a change of velocity whose direction is that of the impulse, and whose magnitude is equal to the magnitude of the impulse divided by the mass of the system.

704. **Note 2.** If connections exist between the points of the system, then the change of velocity differs in general from that given in the foregoing remark. The connections of the system may thus be considered the causes of this difference.

705. **Definition.** By internal constraint, or constraint simply, in an impulse, we mean the alteration which all the connections of a system produce in the change of velocity of the system due to the impulse.

The constraint in an impulse is measured by the difference between the actual change of velocity and that change of velocity which would take place if all the equations of condition of the system were removed; it is equal to the former diminished by the latter.

706. **Corollary.** The constraint in an impulse is the time-integral of the internal constraint of the system taken for its whole duration.

707. **Problem.** To determine the constraint of a system in an impulse.

We shall denote the components of the constraint along p_ρ

by Z_ρ. If then we multiply the equation § 497 (i) by mdt and integrate for the duration of the impulse, we obtain

$$mZ_\rho = q_{\rho 1} - q_{\rho 0} - J_\rho \qquad \text{(i)}.$$

The components along any coordinates are not in general sufficient for the determination of the magnitude of the constraint. If, therefore, we use rectangular coordinates and denote the component of the constraint along x_ν by Z_ν, we obtain

$$mZ_\nu = m_\nu(\dot{x}_{\nu 1} - \dot{x}_{\nu 0}) - I_\nu \qquad \text{(ii)};$$

then the magnitude Z of the constraint is the positive root of the equation

$$mZ^2 = \sum_1^{3n} {}_\nu m_\nu \left(\dot{x}_{\nu 1} - \dot{x}_{\nu 0} - \frac{I_\nu}{m_\nu} \right)^2.$$

708. **Proposition 1.** The magnitude of the constraint in an impulse is smaller for the natural change of motion than it would be for any other possible change of motion.

For the necessary and sufficient condition (*cf.* §§ 155, 498) that with given values of I_ν the quantity $\frac{1}{2}mZ^2$ should be a minimum, is given by the $3n$ equations

$$m_\nu(\dot{x}_{\nu 1} - \dot{x}_{\nu 0}) - I_\nu + \sum_1^{i} {}_\iota x_{\iota\nu} I_\iota = 0,$$

where the quantities I_ι denote any undetermined multipliers, and these with the i equations

$$\sum_1^{3n} {}_\nu x_{\iota\nu}(\dot{x}_{\nu 1} - \dot{x}_{\nu 0}) = 0$$

singly determine the $3n + i$ quantities $\dot{x}_{\nu 1} - \dot{x}_{\nu 0}$ and I_ι. But since the equations coincide with the equations of motion (§ 691) of the system, they are satisfied by the natural changes of velocity, and only by these.

709. **Observation.** The foregoing theorem contains the adaptation of Gauss's Principle of Least Constraint to the particular case of impulses.

710. **Corollary.** If, owing to the connections of the system, the angle between an impulse and the change of velocity caused by it is not zero (§ 703), then this angle is as

small as possible, consistently with the connections of the system.

For, if we draw a plane triangle whose sides represent the magnitude of the impulse divided by the mass of the system, the magnitude of any possible change of velocity and the magnitude of the difference of these two quantities, that is to say, the constraint which corresponds to this change of velocity, then the angle ϵ included between the first two sides represents the angle between the impulse and the change of velocity (§ 34). Now a possible change of velocity in a given direction may take all values; but amongst all the changes of velocity in given directions, the natural one can only be that in which the constraint is perpendicular to the change of velocity (§ 708). If, then, we restrict ourselves to those changes of velocity which are subject to this consideration, all the triangles to be drawn are right-angled, and the hypothenuse is equal in all and is given. But the side opposite to the angle ϵ is smaller for the natural change of velocity than for any other (§ 708); therefore, for this change of velocity the angle ϵ itself is a minimum, which proves the proposition.

711. **Proposition 2.** The direction of the constraint in an impulse is perpendicular to every possible (virtual) displacement of the system from its instantaneous position.

For by §§ 707, 689, the components of the constraint can be represented in the form

$$-\frac{1}{m}\sum_{1}^{k}{}_{\chi}\, p_{\chi\rho}\mathrm{J}_{\chi}.$$

Thus (§ 250) the constraint as a vector quantity is perpendicular to every possible displacement of the system. The proposition may also be immediately deduced from § 500.

712. **Symbolical Expression.** If we denote by δp_ρ the changes of the coordinates p_ρ for every possible displacement of the system, then the foregoing proposition can be expressed in the form of the symbolical equation

$$\sum_{1}^{r}{}_{\rho}(q_{\rho 1} - q_{\rho 0} - \mathrm{J}_{\rho})\delta p_\rho = 0 \qquad \text{(i)},$$

which, for rectangular coordinates, takes the form

$$\sum_{1}^{3n}{}_{\nu}[m_\nu(\dot{x}_{\nu 1} - \dot{x}_{\nu 0}) - \mathrm{I}_\nu]\delta x_\nu = 0 \qquad \text{(ii)}$$

(*cf.* §§ 393, 501).

713. **Observation.** The foregoing proposition (§ 711) contains the adaptation of d'Alembert's Principle to the particular case of impulses, and the symbolical form § 712 is the usual expression for this adaptation.

714. **Corollary 1.** The component of the change of motion in the direction of every possible motion produced by an impulse is equal to the component of the impulse in that direction divided by the mass of the system.

715. **Corollary 2.** The component of the change of motion produced by an impulse in the direction of every free coordinate is equal to the component of the impulse along this coordinate divided by the mass of the system.

716. **Corollary 3.** The component of the velocity along every coordinate of absolute position changes by an amount which is equal to the component of the impulse acting in that direction divided by the mass of the system—whatever be the connections of the system.

717. **Observation.** Without any knowledge, or without a complete knowledge of the connection between the masses of a system, we can always find six equations for the motion of a system under the action of an impulse. If we choose as coordinates of absolute position the six quantities α_1, α_2, α_3, ω_1, ω_2, ω_3, introduced in § 402, then the six equations which we obtain represent the adaptation of the Principle of the Centre of Gravity and of Areas to the particular case of impulses.

Energy, Work

718. **Definition.** The increase of the energy of a system produced by an impulse acting on the system is called the work of the impulse.

Any decrease of the energy owing to an impulse is regarded as a negative increase. Thus the work of an impulse may be positive or negative.

719. **Corollary.** The work of an impulse is the time-integral of the work performed by that force whose time-integral is the impulse.

720. **Proposition.** The work of an impulse is equal to the product of the magnitude of the impulse and the component in its direction of the mean value of the initial and final velocities of the system.

For whatever may be the actual values of the force acting during the time of the impulse and the motion of the system during this time, the final motion, and consequently the work of the impulse, will be the same as though the force acted with a constant mean value in the direction of the impulse. Now, if we make this simple assumption, then, firstly, the magnitude of the force acting is equal to the magnitude of the impulse divided by its duration. Secondly, the velocity changes uniformly from its initial to its final value, and its mean value is the arithmetic mean of the initial and final values. The component of the portion of the path described during the impulse is, however, equal to the component of that mean value, multiplied by the time. Then, if we calculate by § 513 the work performed by the force during its time of application, *i.e.* the work of the impulse, the time drops out and the proposition follows.

721. **Observation.** With the notation hitherto used, the analytical expression for the proposition is the statement that the work of the impulse is equal to

$$\frac{1}{2}\sum_{1}^{r}{}_{\rho} J_{\rho}(\dot{p}_{\rho 1} + \dot{p}_{\rho 0}).$$

722. **Corollary 1.** The work of an impulse is equal to the product of the impulse and the component of the original velocity taken in its direction, increased by half the product of the magnitude of the impulse and the component in its direction of the change of velocity produced by it.

The analytical expression for this is that the work of the impulse is equal to

$$\sum_{1}^{r}{}_{\rho} J_{\rho}\dot{p}_{\rho 0} + \frac{1}{2}\sum_{1}^{r}{}_{\rho} J_{\rho}(\dot{p}_{\rho 1} - \dot{p}_{\rho 0}),$$

which coincides with § 721.

723. **Corollary 2.** The work of an impulse which sets in motion a system at rest is equal to half the product of the impulse, and the component in its direction of the velocity produced by it.

For if the quantities $\dot{p}_{\rho 0}$ are zero, then the work of the impulse is

$$\tfrac{1}{2}\sum_{1}^{r}{}_{\rho}\, \mathrm{J}_{\rho}\dot{p}_{\rho 1}.$$

724. **Proposition.** If a system at rest is set in motion by an impulse, then it moves in that direction in which the impulse performs the most work, *i.e.* in which it performs more work than it would if it were compelled to move in any other direction by additional connections. (The so-called Bertrand's Law.)

For if J is the magnitude of the impulse, v that of the velocity produced, and ϵ the angle between them, then for every original or additional connection we have by § 714

$$v = \frac{\mathrm{J}}{m}\cos\epsilon.$$

Thus the work of the impulse is by § 723 equal to

$$\tfrac{1}{2}\mathrm{J}v\cos\epsilon = \frac{\mathrm{J}^2}{2m}\cos^2\epsilon.$$

But the angle ϵ for the natural action of the impulse takes (§ 710) the smallest value consistent with the original connection, and consequently ϵ can only be increased by any additional connection, *i.e.* $\cos^2\epsilon$ decreased, which proves the proposition.

725. **Corollary.** The energy which an impulse on a system at rest produces in that system is greater the fewer the connections of the system. The greatest possible value of that energy, which, however, can only be attained by dropping all the connections, is equal to the square of the magnitude of the impulse divided by twice the mass of the system.

Impact of Two Systems

Explanations

726. 1. We say that two systems impinge when they behave as though they had been coupled for a very short time.

We assume this coupling to be direct by assuming (§ 452) a special choice of the coordinates of the two systems.

727. 2. We have to conceive such a temporary coupling as a permanent coupling of the two systems with a third unknown system which possesses the property that it in general has no effect on their motion, but that in the immediate neighbourhood of those positions in which certain coordinates of the one system are equal to certain coordinates of the other it constrains these coordinates to remain temporarily equal. We call such coordinates the common coordinates of the two systems.

728. 3. Before and after the impact the rates of change of the coordinates of each of the two systems are subject simply to the equations of condition of its own system. But during the impulse the rates of change of the common coordinates are also related by the equations of coupling. These rates of change, then, just like the coordinates themselves, must during the impulse have become respectively equal and must have remained so for a time. But the time in which this takes place we regard as vanishingly small, and what takes place during this time as quite unknown. We consider the systems only before and after the impulse, and expect that only such information with regard to the impact will be required as can be given without a knowledge of what takes place during the impact.

729. **Problem.** To determine the subsequent motion of two impinging systems from their motion before the impact, as far as is possible without a knowledge of what takes place during the impulse.

Let the quantities p_ρ be the r coordinates of the one system and $\mathfrak{p}_\rho$ the r coordinates of the other. Let the number of common coordinates be s. In the impact each of the systems suffers an impulse; let the components of the impulse on the first system be J_ρ and on the second $\mathfrak{J}_\rho$. Quantities before and after the impulse will be distinguished by the indices 0 and 1.

Now, in the first place, for all coordinates of the first system equations of the form § 689 (v) hold good, and for all coordinates of the second system corresponding equations. In the second place, the impulses which the two systems suffer stand in the relation of impulse and counter-impulse, and consequently for all common coordinates we have by §§ 682, 683,

$$J_\rho = -\mathfrak{J}_\rho,$$

and for all the coordinates of the two systems which are not common,

$$J_\rho = 0, \quad \mathfrak{J}_\rho = 0.$$

If now we combine the two relations we obtain for the s common coordinates, s equations of the form

$$q_{\rho 1} - q_{\rho 0} + \sum_1^k {}_\chi p_{\chi\rho} J_\chi = -\mathfrak{q}_{\rho 1} + \mathfrak{q}_{\rho 0} - \sum_1^{\mathfrak{k}} {}_\chi \mathfrak{p}_{\chi\rho} \mathfrak{J}_\chi \quad \text{(i)};$$

while for the $(r-s)+(\mathfrak{r}-s)$ coordinates which are not common, $r-s$ equations of the form

$$q_{\rho 1} - q_{\rho 0} + \sum_1^k {}_\chi p_{\chi\rho} J_\chi = 0 \qquad \text{(ii)}$$

and $(\mathfrak{r}-s)$ of the form

$$\mathfrak{q}_{\rho 1} - \mathfrak{q}_{\rho 0} + \sum_1^{\mathfrak{k}} {}_\chi \mathfrak{p}_{\chi\rho} \mathfrak{J}_\chi = 0 \qquad \text{(iii)}$$

are obtained. The equations (i), (ii), (iii), together with the $k+\mathfrak{k}$ equations of condition of the two systems, we may regard as equations for the quantities $\dot{p}_{\rho 1}$ and $\dot{\mathfrak{p}}_{\rho 1}$, which determine the motion of the system after the impulse, and for the quantities J_χ and $\mathfrak{J}_\chi$. We have thus altogether $r+\mathfrak{r}-s+k+\mathfrak{k}$ unhomogeneous linear equations which the $r+\mathfrak{r}+k+\mathfrak{k}$ unknowns must satisfy and which contain the requirements of the problem.

730. **Observation.** If the coordinates p_ρ and $\mathfrak{p}_\rho$ are free coordinates of their systems, then the equations of impact can be written in a simpler form. By paying attention to the common coordinates of the system, s equations of the form

$$q_{\rho 1} + \mathfrak{q}_{\rho 1} = q_{\rho 0} + \mathfrak{q}_{\rho 0} \qquad \text{(i)}$$

will be obtained; for the coordinates of the first system which are not common, $r-s$ equations of the form

$$q_{\rho 1} = q_{\rho 0} \qquad \text{(ii)};$$

and for the coordinates of the second system which are not common, $\mathfrak{r} - s$ equations of the form

$$\mathfrak{q}_{\rho 1} = \mathfrak{q}_{\rho 0} \qquad \text{(iii)};$$

these give $r + \mathfrak{r} - s$ equations to determine the $r + \mathfrak{r}$ unknowns $\dot{p}_{\rho 1}$ and $\dot{\mathfrak{p}}_{\rho 1}$.

731. **Corollary 1.** The motion of two systems after impact is not completely determined by their motion before impact and the general laws of mechanics, but its determination requires also a knowledge of further relations obtained from other sources. The number of these additional necessary relations is equal to the number of common coordinates during the impact.

732. **Corollary 2.** If in an impact it is possible to obtain, in addition to the relations deduced from the general laws of mechanics, as many linear equations for the components of the velocity after the impact as there are common coordinates, then the motion after the impact is singly determined by means of the previous motion.

733. **Observation.** The special relations which are necessary for the determination of the motion in an impact, and which do not spring from the general laws of mechanics, depend on the special nature of that system which causes the coupling and whose peculiarities are not known to us in detail. It is this concealed system which takes up the energy lost by the impinging systems, or which supplies the energy gained by the impinging systems. The first case occurs, for instance, in an inelastic impact where the immediate neighbourhood of the point of impact is to be regarded as the coupling system. The second case occurs in explosions. The detailed consideration of these special relations is, however, not a part of general mechanics.

Concluding Note on the Second Book

734. In this second book our object has not been to determine the necessary relations between the creations of our own mind, but rather to consider the experiential connections between

the objects of our external observation. It was therefore inevitable that our investigations should be founded not only on the laws of thought, but also on the results of previous experience. As the necessary contribution of experience, we thus took from our observation of nature the fundamental law.

735. At first it might have appeared that the fundamental law was far from sufficient to embrace the whole extent of facts which nature offers us and the representation of which is already contained in the ordinary system of mechanics. For while the fundamental law assumes continuous and normal connections, the common applications of mechanics bring us face to face with discontinuous and abnormal connections as well. And while the fundamental law expressly refers to free systems only, we are also compelled to investigate unfree systems. Even all the normal, continuous, and free systems of nature do not conform immediately to the law, but seem to be partly in contradiction to it. We saw, however, that we could also investigate abnormal and discontinuous systems if we regarded their abnormalities and discontinuities as only apparent; that we could also follow the motion of unfree systems if we conceived them as portions of free systems; that, finally, even systems apparently contradicting the fundamental law could be rendered conformable to it by admitting the possibility of concealed masses in them. Although we have associated with the fundamental law neither additional experiential facts nor arbitrary assumptions, yet we have been able to range over the whole domain covered by mechanics in general. Nor does our special hypothesis prevent us from understanding that mechanics could and must have been developed in the manner in which it actually has developed.

In conclusion, then, we may assert that the fundamental law is not only necessary but also sufficient to represent completely the part which experience plays in the general laws of mechanics.

INDEX TO DEFINITIONS AND NOTATIONS

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