MACDONALD

PRINCIPLES

OF

PLANE GEOMETRY

BY

J. W. MACDONALD

PRINCIPAL OF THE STONEHAM (MASS.) HIGH SCHOOL

Let him know a thing because he has found it out for himself, and not because you have told him. — J. J. ROUSSEAU

Boston

ALLYN AND BACON

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University Press:
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PREFACE.

THE most appropriate preface for a book of this kind would be an exposition of the principles of psychology pertaining to the development and training of the reasoning and linguistic faculties. As such a preface, however, would be more pretentious than the book proper, and as these principles have been so well expounded by many eminent writers, I must content myself by merely urging studious investigation upon all teachers who are ambitious to practise the best methods. If teachers will do this, of one thing I am confident, — they will grant that the purpose of this book is right in theory, even if in practice certain difficulties may seem to them insurmountable. As a help, however, to teachers who may wish it, I have thought it advisable to publish as a companion to this book a monograph on teaching geometry,* illustrating actual class-work, and showing in detail how some of the most difficult topics may be mastered, — not, let me add, as a

* Geometry in the Secondary School. Willard Small, Publisher, 24 Franklin Street, Boston, Mass.

dogmatic declaration of *the* method of development, but only as a suggestive illustration of *a* method. The earnest teacher, who thoroughly understands the subject he is teaching, and who has a purpose clear in his mind, will make his own, and for him the best method.

It cannot be said of this Geometry, as I have heard it said of others, that it is designed to aid inefficient teachers. The teacher who does not thoroughly understand elementary geometry, — who is not sharp to detect inaccuracies in definitions and arguments, — who, in short, is dependent on the written text for what he teaches, — should not undertake to use this book.

I have thought it best to publish the books in an inexpensive form, so that, where the free text-book system exists, it would be as economical to purchase them new each year as to transmit a more expensive volume from pupil to pupil, with much distasteful accumulation. They will be furnished in a more durable form, if desired.

In conclusion, I wish to thank the publishers and the proof-reader, by whose suggestions and watchfulness the text has been much improved and saved from numerous errors.

J. W. M.

STONEHAM, August 1, 1889.

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c. How is a surface bounded?

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a. What is a straight line or right line?

b. What is a curved line?

c. What is a broken line?

Illustration. 

d. What is a mixed line?

Illustration. 

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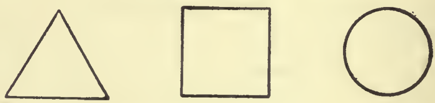
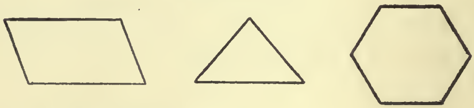


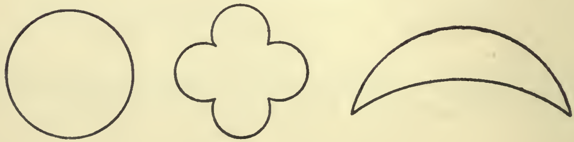
b. A rectilinear figure?

Illustrations.



c. A curvilinear figure?

Illustrations.



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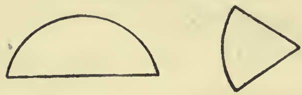
Illustrations.



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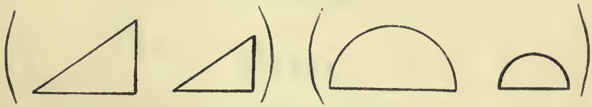
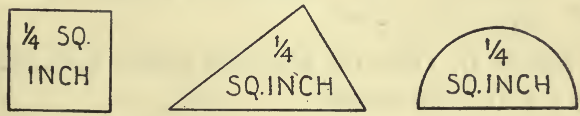
Illustrations.



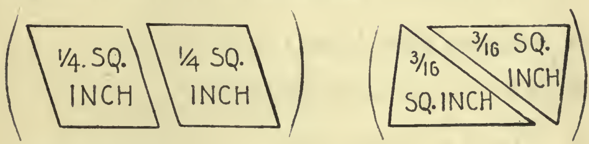
f. Equivalent figures?

Illustrations.



g. Equal figures?

Illustrations.



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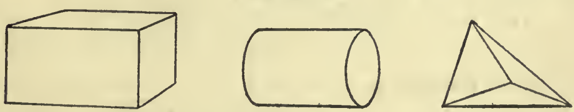
Illustrations.



14. What is magnitude?

a. Of a line?

b. Of a surface?

c. Of a solid?

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a. Of a line?

b. Of a surface?

c. Of a solid?

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LOCUS.

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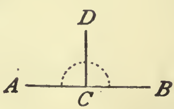
Illustration.



b. An acute angle?

Illustration.

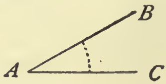


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Illustration. 

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Illustration. 

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e. Subtracting unequals.
f. Multiplying equals.
g. Dividing equals.
h. The whole and a part.
i. The whole and all the parts.

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a. Right angles?
b. Straight angles?
c. Complementary angles?
d. Supplementary angles?

28. What is a postulate?

29. What must be granted as to the following:

a. Drawing a straight line?

b. Prolonging a line?

c. Drawing a circumference?

d. Dividing lines, angles, etc.?

30. What is a proposition?

a. A problem?

b. A theorem?

c. A corollary?

d. A scholium?

31. Of what does every theorem consist?

IV. Mathematical Symbols and Abbreviations.

32. *a.* $+$, plus.

b. $-$, minus.

c. $\times$, multiplied by.

d. $\div$, divided by.

e. $=$, equal to.

f. ≏, equivalent to.

g. $>$, greater than.

h. $<$, less than.

i. :, : :, :, signs of proportion.

j. $\angle$, angle; $\angle$s, angles.

k. $\triangle$, triangle; $\triangle$s, triangles.

l. $\square$, square; [ⓢ], squares.

m. ▱, parallelogram; ▱ˢ, parallelograms.

n. ○, circle; Ⓢ, circles.

o. ⌒, arc; ⌒ˢ, arcs.

BOOK I.

Proposition I. A Theorem.

33. If one straight line meets another so as to form two adjacent angles, the sum of these angles is equal to two right angles; that is, the angles are supplements of each other.

Corollary I. Any number of angles in the same plane, formed about a given point on one side of a straight line, are equivalent to two right angles.

Corollary II. The sum of all the angles in the same plane, formed about a given point, is equal to four right angles.

Proposition II. A Theorem.

34. Conversely, if two angles whose sum equals two right angles are placed adjacent to each other, their exterior sides will form one straight line.

Proposition III. A Theorem.

35. If two straight lines intersect each other, the vertical angles are equal.

See page 5, §§ 24 and 27 *d*.

Proposition IV. A Theorem.

36. At any point in a straight line there can be but one perpendicular on either side.

Proposition V. A Theorem.

37. From any point outside of a straight line there can be but one perpendicular to the line.

Proposition VI. A Theorem.

38. Two lines in the same plane perpendicular to a third line are parallel to each other.

Proposition VII. A Theorem.

39. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

Proposition VIII. A Theorem.

40. Angles having the sides of the one parallel to the sides of the other are either equals or supplements.

SCHOLIUM. If both pairs of parallel sides extend in the same direction from the vertices, or both in opposite directions, the angles are equal; but if one pair extends in the same direction and the other pair in opposite directions, the angles are supplements.

Proposition IX. A Theorem.

41. Angles having the sides of the one perpendicular to the sides of the other are either equals or supplements.

SCHOLIUM. How can it be determined whether the angles are equals or supplements?

V. Angles having Special Names.

42. Let two straight lines be intersected by a third:

a. What are the exterior angles?

b. What are the interior angles?

c. What are alternate exterior angles?

d. What are alternate interior angles?

e. What are external angles on the same side (of the intersecting line)?

f. What are internal angles on the same side?

g. What are opposite external-internal angles?

Proposition X. A Theorem.

43. If two parallel straight lines are intersected by a third:

I. Alternate interior or exterior angles will be equal.

II. Opposite external-internal angles will be equal.

III. Interior or exterior angles on the same side will be supplements.

See Proposition VIII.

Proposition XI. A Theorem.

44. Two straight lines intersected by a third line will be parallel:

I. If alternate interior or exterior angles are equal.

II. If opposite external-internal angles are equal.

III. If interior or exterior angles on the same side are supplements.

Proposition XII. A Theorem.

45. Two lines parallel to a third are parallel to each other. *See* Proposition VII.

Proposition XIII. A Theorem.

46. The sum of two lines drawn from any point to the extremities of a line is greater than the sum of any two lines similarly drawn from an included point.

Proposition XIV. A Theorem.

47. The shortest distance from any point to a given straight line is a perpendicular to that line.

Proposition XV. A Theorem.

48. Two oblique lines extending from any point in a perpendicular to points in the base line equally distant from the foot of the perpendicular are equal.

Proposition XVI. A Theorem.

49. Of two oblique lines extending from any point in a perpendicular to points in the base line unequally distant from the foot of the perpendicular, the one extending to the farther point will be the longer.

Corollary. There can be but two equal oblique lines drawn from any point in a perpendicular to the base line.

Proposition XVII. A Theorem.

50. If two oblique lines drawn from any point in a perpendicular to the base line are equal, they extend to points equally distant from the foot of the perpendicular.

Proposition XVIII. A Theorem.

51. If a perpendicular be erected at the middle point of a straight line:

I. Any point in the perpendicular will be equally distant from the extremities of the line.

II. Any point out of the perpendicular will be unequally distant from the extremities of the line.

Corollary I. Conversely, all points equally distant from the extremities of a line are in the perpendicular at its middle point.

Corollary II. The perpendicular at the middle point of a line will cut the longer of two lines joining a point with its extremities.

Proposition XIX. A Problem.

52. To erect a perpendicular at the middle of a line.

See page 4, §§ 17 and 18.

Proposition XX. A Problem.

53. To bisect a given line.

Proposition XXI. A Problem.

54. To erect a perpendicular at any point in a straight line.

Proposition XXII. A Problem.

55. From a point outside of a straight line to draw a perpendicular to the line.

Proposition XXIII. A Theorem.

56. If an angle be bisected by a straight line, every point in the bisector is equally distant from the sides.

Proposition XXIV. A Theorem.

57. Conversely, every included point equally distant from the sides of an angle is in the bisector of the angle.

SCHOLIUM. What is the locus of a point equally distant from the sides of an angle?

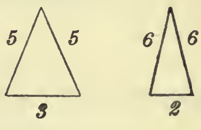
VI. TRIANGLES.

58. What is a triangle?

a. An equilateral triangle?

b. An isosceles triangle?

Illustrations.



c. A scalene triangle?

Illustrations.

d. A right-angled, or right triangle?

e. An obtuse-angled, or obtuse triangle?

f. An equiangular triangle?

59. What is the hypotenuse of a right triangle?

a. What are the other sides called?

60. When are triangles equal? Equivalent?

a. What are homologous sides and angles of equal triangles?

61. What is the base of a triangle?

a. What the altitude?

b. What are its medians?

Proposition XXV. A Theorem.

62. The sum of two sides of a triangle is greater than the third, and the difference is less.

Proposition XXVI. A Theorem.

63. If two triangles have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, the other homologous parts are also equal, and the triangles are equal.

Proposition XXVII. A Theorem.

64. If two triangles have two angles and the included side of one equal to two angles and the included side of the other, each to each, the other homologous parts are equal, and the triangles are equal.

Proposition XXVIII. A Theorem.

65. If two triangles have the three sides of one equal to the three sides of the other, each to each, the triangles are equal.

Proposition XXIX. A Problem.

66. Construct an equilateral triangle having sides equal each to a given line.

Proposition XXX. A Problem.

67. Construct a triangle having sides equal to the sides of a given triangle.

Proposition XXXI. A Theorem.

68. If two triangles have two sides of the one respectively equal to two sides of the other and the included angles unequal, the third side of the one having the greater angle will be longer than the third side of the other.

SCHOLIUM. Three different cases may arise; prove each.

Proposition XXXII. A Theorem.

69. Conversely, if two triangles have two sides of the one respectively equal to two sides of the other and the third sides unequal, the angle opposite the longer third side will be greater than the angle opposite the shorter.

Proposition XXXIII. A Theorem.

70. The sum of the angles of a triangle is equal to two right angles.

COROLLARY. In a right triangle the two acute angles are complements of each other.

SCHOLIUM. The exterior angle formed by prolonging one of the sides of a triangle is equal to the sum of the two opposite interior angles.

Proposition XXXIV. A Theorem.

71. In an isosceles triangle the angles opposite the equal sides are equal.

COROLLARY. An equilateral triangle is also equiangular.

Proposition XXXV. A Theorem.

72. If two angles of a triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.

Corollary. An equiangular triangle is also equilateral.

Proposition XXXVI. A Theorem.

73. If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter side.

Proposition XXXVII. A Theorem.

74. If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the lesser.

Proposition XXXVIII. A Theorem.

75. Two right triangles are equal if the hypotenuse and one side of the one are equal to the hypotenuse and one side of the other.

Proposition XXXIX. A Theorem.

76. Lines bisecting the angles of a triangle meet at a point which is equally distant from the side of the triangle.

See Propositions XXIII. and XXIV.

Proposition XL. A Theorem.

77. Perpendiculars bisecting the sides of a triangle meet at a point equally distant from the vertices of the angles.

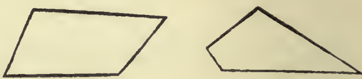
See Proposition XVIII.

VII. QUADRILATERALS.

78. What is a quadrilateral or quadrangle?

a. A trapezium?

Illustrations.



b. A trapezoid?

Illustrations.

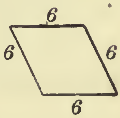
c. A parallelogram?

d. A rectangle?

e. A square?

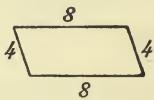
f. A rhombus?

Illustration.



g. A rhomboid?

Illustration.



79. What is the diagonal of a quadrilateral?

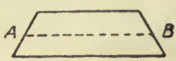
80. What are the upper and lower bases of a quadrilateral?

81. What is the altitude of a parallelogram or trapezoid?

82. What are the bases of a trapezoid?

a. What is its median?

Illustration.



Proposition XLI. A Theorem.

83. The opposite sides and angles of a parallelogram are equal.

COROLLARY I. The diagonal divides a parallelogram into equal triangles.

COROLLARY II. The parts of parallel lines cut off between parallel lines are equal.

Proposition XLII. A Theorem.

84. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Proposition XLIII. A Theorem.

85. If two sides of a quadrilateral are equal and parallel, the other two sides are also equal and parallel, and the figure is a parallelogram.

Proposition XLIV. A Theorem.

86. The diagonals of a parallelogram bisect each other.

Proposition XLV. A Theorem.

87. Two parallelograms are equal if they have two sides and the included angle of one equal to two sides and the included angle of the other, each to each.

Proposition XLVI. A Theorem.

88. Parallel lines are everywhere equally distant.

VIII. Polygons of more than Four Sides.

89. What is a polygon?

a. A pentagon?

b. A hexagon?

c. A heptagon?

d. An octagon?

e. A nonagon?

f. A decagon?

g. An undecagon?

h. A duodecagon?

90. What are salient angles of a polygon?

91. What are re-entrant angles?

92. What is an equilateral polygon?

93. What is an equiangular polygon?

94. What is a concave polygon?

95. When are two polygons mutually equiangular?

96. When are two polygons mutually equilateral?

97. What are homologous sides or angles?

98. What are equal polygons?

99. When is a polygon symmetrical with reference to any dividing line?

100. What is an axis of symmetry?

101. What is a centre of symmetry?

Proposition XLVII. A Theorem.

102. Two equal polygons may be divided into the same number of equal triangles.

Proposition XLVIII. A Theorem.

103. The sum of the interior angles of a polygon is equal to as many right angles as twice a number two less than the number of its sides.

SCHOLIUM. To how many right angles is the sum of the angles of figures from pentagons to duodecagons equal? If equiangular, how large is each angle?

Proposition XLIX. A Theorem.

104. If each side of a polygon be produced in order, the sum of the exterior angles equals four right angles.

OPTIONAL PROPOSITIONS.

Proposition L. A Theorem.

105. I. In a regular polygon having an odd number of sides, a line joining the vertex of an angle with the middle point of the opposite side is an axis of symmetry.

II. In a regular polygon having an even number of sides, a line joining the vertices of opposite angles, or the middle points of opposite sides, is an axis of symmetry.

Proposition LI. A Problem.

106. Draw parallel lines a given distance apart.

Proposition LII. A Theorem.

107. Any number of parallel lines equally distant from each other intercept equal parts on any transverse line crossing them.

SUPPLEMENTARY PROPOSITIONS.

1. What is the supplement to an angle of 35°? The complement?

2. If three or more angles be formed at the same point on the same side of a straight line, any one of them will be a supplement to the sum of all the others.

3. If two adjacent supplementary angles be bisected, the bisectors will form a right angle.

4. A line bisecting one of two vertical angles will, if continued, bisect the other.

5. The sum of any two sides of a triangle is greater than the sum of any two lines drawn from any point in the triangle to the extremities of the third side.

6. An equiangular triangle is also equilateral.

7. The bisector of the vertical angle of an isosceles triangle, if continued to the base, is an axis of symmetry.

8. In a right triangle the two acute angles are complements of each other.

9. In a right triangle, if one of the acute angles is of 30°, the side opposite is one half the hypotenuse.

10. Find the locus of a point equally distant from two points.

11. The opposite angles of a parallelogram are equal.

12. In a parallelogram, the angles adjacent to any one side are supplements.

13. All the angles of a parallelogram are equal to four right angles.

14. The diagonals of a rectangle are equal.

15. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

16. If a diagonal divides a quadrilateral into two equal triangles, the quadrilateral is a parallelogram.

17. Through a given point to draw a parallel to a given line.

18. If several parallel lines intercept equal parts on any transverse line, they are an equal distance apart.

19. If a line drawn through a triangle parallel to one of the sides bisects one of the other sides, it will bisect both of them.

20. A line bisecting two sides of a triangle is parallel to the third.

21. A line connecting the middle points of two sides of a triangle is half the length of the third.

22. The medians of a triangle intersect one another at the same point, which is distant from the vertex of each angle two thirds the length of its median.

23. The median of a trapezoid is parallel to the bases and equally distant from them.

24. The median of a trapezoid is equal to half the sum of the bases.

BOOK II.

THE PRINCIPLES OF PROPORTION AND THE THEORY OF LIMITS.

I. Ratio and Proportion.

108. What is a ratio?

a. How expressed? Example. $a:b$, or $\frac{a}{b}$.

b. What names are given the terms?

109. What is a proportion?

a. How expressed? Ex. $a:b::c:d$, or $\frac{a}{b}=\frac{c}{d}$.

b. What names are given the terms?

110. What is a fourth proportional? A mean proportional? A third proportional?

111. Given a proportion $a:b::c:d$, what is changing it:

a. By inversion? Ex. $b:a::d:c$.

b. By alternation? Ex. $a:c::b:d$, or $d:b::c:a$.

c. By composition? Ex. $a+b:b::c+d:d$.

d. By division? Ex. $a-b:b::c-d:d$.

112. What common divisor has 8 and 34? Their ratio? What common divisor has 3.6 and 54? Their ratio? What common divisor has 5 and $\sqrt{40}$? Their ratio?

113. In each of the following problems how long a line will exactly divide both the given lines?

a. Lines 10 and 25 inches, respectively. Their ratio?

b. Lines $8\frac{1}{2}$ and $15\frac{1}{4}$ inches, respectively. Their ratio?

c. Lines 6 and $\sqrt{8}$ inches, respectively. Their ratio?

114. What are commensurable quantities? Incommensurable quantities?

115. How can a common measure and the ratio of two lines be found?

116. What are equimultiples of two quantities?

Proposition I. A Theorem.

117. If four quantities are in proportion, the product of the extremes equals the product of the means.

COROLLARY. A mean proportional is equal to the square root of the product of the other two terms.

Proposition II. A Theorem.

118. If two sets of proportional quantities have a ratio in each equal, the other ratios will be in proportion.

COROLLARY. If the antecedents or consequents are the same in both, the other terms are in proportion.

Proposition III. A Theorem.

119. If the product of two quantities equals the product of two other quantities, either two may be made the means and the other two the extremes of a proportion.

Proposition IV. A Theorem.

120. If four quantities are in proportion, they will be in proportion by inversion.

Proposition V. A Theorem.

121. If four quantities are in proportion, they will be in proportion by alternation.

Proposition VI. A Theorem.

122. If four quantities are in proportion, they will be in proportion by composition.

Proposition VII. A Theorem.

123. If four quantities are in proportion, they will be in proportion by division.

Proposition VIII. A Theorem.

124. If four quantities are in proportion, they will be in proportion by composition and division.

Proposition IX. A Theorem.

125. Equimultiples of two quantities are proportional to the quantities themselves.

Corollary. Any equimultiples of the antecedents are proportional to any equimultiples of the consequents.

Proposition X. A Theorem.

126. If four quantities are in proportion, their like powers or like roots are in proportion.

Proposition XI. A Theorem.

127. In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to its consequent.

Corollary. The sum of any number of the antecedents is to the sum of their consequents as any one antecedent is to its consequent.

Proposition XII. A Theorem.

128. If two or more proportions be multiplied together, term by term, the products are in proportion.

II. The Theory of Limits.

129. What is a variable? A constant?

a. An increasing variable?

b. A decreasing variable?

130. Suppose a point x moving on the line AB in such a

way that it goes one half the distance from A to B the first second, one half the remaining distance the next second, one half the remaining distance the third, and so on indefinitely:

a. What two varying distances does it produce?

b. What distance is the distance Ax approaching, and when will it reach it?

c. What is the distance xB approaching, and when will it reach it?

131. Reduce the fraction $\frac{1}{3}$ to a decimal:

a. How is the value of the decimal affected by each division, and what is it approaching?

b. How is the difference between $\frac{1}{3}$ and the decimal affected by each division, and what is it approaching?

132. What is the limit to a variable?

a. A superior limit?

b. An inferior limit?

133. How near may a variable be conceived to approach its limit?

134. Suppose a right triangle to be continually changing by the shortening of one of its legs:

a. What lines would be variables? Their limits?

b. What angles would be variables? Their limits?

c. How would its area be affected? Its limit?

135. Why could not the diminishing leg become zero?

136. Suppose a regular polygon, as a square or equilateral triangle, to be inscribed in a circle (see Book III. § 159), and that by bisecting the arcs and drawing chords it be changed to a regular inscribed polygon of double the number of sides, four times the number of sides, and so on indefinitely:

a. What variables result?

b. What are their limits?

137. Sometimes the variable does not indefinitely approach limits, as, for example, suppose the process in § 136 reversed.

Proposition XIII. A Theorem.

138. If two variables as they indefinitely approach their limits have any constant ratio, their limits have the same ratio.

Corollary. If two variables as they indefinitely approach their limits are constantly equal, their limits are equal.

Scholium. In the above corollary, the variables have the constant ratio 1, as have also their limits.

Proposition XIV. A Theorem.

139. If the product of two variables as they indefinitely approach their limits is constantly equal to a third variable, the products of their limits will equal the limits of the third.

Scholium. Sometimes the product of an increasing and a decreasing variable is a constant.

See Book IV., Proposition XVII.

Proposition XV. A Theorem.

140. If several parallel lines are crossed by an oblique line, the segments of the oblique line are proportional to the distances between the parallels.

Case I. When the parallels are an equal distance apart.

Case II. When the parallels are unequal distances apart.

a. When the distances between them are commensurable.

b. When these distances are incommensurable.

Corollary. The corresponding segments of two oblique transversals are in proportion.

Proposition XVI. A Theorem.

141. If one or more parallel lines be drawn through a triangle parallel to one side, the other two sides will be divided proportionally.

Corollary. The intersected sides are to each other as any two corresponding segments.

See Book II., Proposition VI.

Proposition XVII. A Theorem.

142. If a straight line divide the sides of a triangle proportionally, it is parallel to the third side.

Proposition XVIII. A Problem.

143. To divide a given line into parts proportional to given lines, or given parts of a given line.

Proposition XIX. A Problem.

144. To find a fourth proportional to three given lines.

Proposition XX. A Problem.

145. To find a third proportional to two given lines.

BOOK III.

I. The Circle.

146. What is a circle?

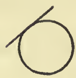
147. What is the circumference?

148. What is the radius?

149. What is a chord? Illustration. 

150. What is a diameter?

151. What is a secant? Illustration. 

152. What is a tangent? Illustration. 

153. What is an arc?

154. What is a segment? Illustration. 

155. What is a sector? Illustration. 

a. A quadrant? Illustration. 

156. When do circles touch each other internally? When externally?

157. When is an angle inscribed in a circle? Ill. 

158. When is an angle inscribed in a segment? Ill. 

159. When is a polygon inscribed in a circle? Ill. 

a. What is the relation of the circle to the polygon?

160. When is a circle inscribed in a polygon? Ill. 

a. What is the relation of the polygon to the circle?

161. What are concentric circles? Illustration. 

162. When will circles be equal?

163. What is an angle at the centre?

Proposition I. A Theorem.

164. The diameter of a circle is an axis of symmetry.

Corollary. The diameter bisects the circle and its circumference.

Proposition II. A Theorem.

165. A straight line cannot intersect the circumference of a circle at more than two points.

See Book I., Proposition XVI., Corollary.

Proposition III. A Theorem.

166. The diameter is longer than any other chord.

Proposition IV. A Theorem.

167. In the same circle, or in equal circles, equal arcs are subtended by equal chords; and conversely, if the chords are equal, the arcs also are equal.

Proposition V. A Theorem.

168. In the same circle, or in equal circles, of two unequal arcs each less than a semicircumference, the greater arc is subtended by the longer chord; and conversely, of two unequal chords, the longer subtends the greater arc.

See Book I., Propositions XXXI. and XXXII.

Proposition VI. A Theorem.

169. A radius drawn perpendicular to a chord bisects both the chord and its arc.

SCHOLIUM. A line drawn perpendicular to the middle point of a chord is a radius and bisects the arc.

Proposition VII. A Theorem.

170. Through three given points not in the same straight line one circumference can be drawn, and but one.

Proposition VIII. Problems.

171. I. Given three points not in a straight line, to draw a circumference through them.

See Book I., Propositions XVIII. and XIX.

II. Given a circumference, to find its centre.

Proposition IX. A Theorem.

172. In the same circle, or in equal circles, equal chords are equally distant from the centre; and conversely, chords equally distant from the centre are equal.

See Book I., Proposition XXXVIII.

Proposition X. A Theorem.

173. In the same circle, or in equal circles, of two unequal chords the shorter is the farther from the centre.

See Book I., Proposition XIV.

Proposition XI. A Theorem.

174. Conversely, of two chords unequally distant from the centre, the farther one will be the shorter.

Proposition XII. A Theorem.

175. A tangent is perpendicular to the radius drawn to the point of tangency.

See Book I., Proposition XIV.

COROLLARY. Conversely, a straight line perpendicular to a radius at its termination in the circumference is a tangent to the circle.

Proposition XIII. A Theorem.

176. If from a point outside of a circle two tangents to the circle be drawn, and also a straight line to the centre of the circle:

I. The tangents will be equal.

II. The line drawn to the centre bisects the angle formed by the tangents.

See Book I., Proposition XXIV.

Proposition XIV. A Theorem.

177. Parallels intercept on a circumference equal arcs.

CASE I. When both parallels are tangents.

CASE II. When one is a tangent and the other a secant.

CASE III. When both are secants.

Proposition XV. A Theorem.

178. I. If two circles cut each other, the line joining their centres will be perpendicular to their common chord.

II. If two circles touch each other externally or internally, the line joining their centres will be perpendicular to their common tangent.

Proposition XVI. A Theorem.

179. In the same circle, or in equal circles, radii forming equal angles at the centre intercept equal arcs on the circumference; and conversely, if the arcs intercepted are equal, the angles at the centre are equal.

Proposition XVII. A Theorem.

180. In the same circle, or in equal circles, angles at the centre are to each other as their arcs.

CASE I. When they are commensurable.

CASE II. When they are incommensurable.

SCHOLIUM I. The angle may be measured by the arc; why?

SCHOLIUM II. Explain the division of the arc into degrees, etc.

SCHOLIUM III. What arc measures a right angle? An acute angle? An obtuse angle?

Proposition XVIII. A Theorem.

181. An inscribed angle is measured by half the intercepted arc.

CASE I. When one of the chords forming the angle is the diameter.

See Book I., Proposition X.

CASE II. When the chords are on opposite sides of the centre.

CASE III. When both chords are on the same side of the centre.

COROLLARY I. All angles inscribed in the same segment are equal.

COROLLARY II. All angles inscribed in a semicircle are right angles.

COROLLARY III. An angle inscribed in a segment smaller than a semicircle is obtuse; one inscribed in a segment greater than a semicircle is acute.

Proposition XIX. A Theorem.

182. The angle formed by two chords which intersect each other is measured by half the sum of the included arcs.

Proposition XX. A Theorem.

183. The angle formed by two secants is measured by half the difference of the included arcs.

Proposition XXI. A Theorem.

184. The angle formed by a tangent and a chord is measured by half the intercepted arc.

Proposition XXII. A Theorem.

185. An angle formed by two tangents is measured by half the difference of the intercepted arcs.

NOTE. In circles that are not equal, radii forming equal angles at the centre intercept arcs whose absolute length is not the same, but they contain the same number of degrees, and may be called homologous. This can be easily shown by means of concentric circles. Hence, in any circles, (*a*) radii forming equal angles at the centre, (*b*) equal inscribed angles, (*c*) equal angles formed by intersecting chords, by secants, or by tangents, intercept homologous arcs; and, conversely, if the arcs intercepted are homologous, the angles are equal. It is on this principle that Supplementary Propositions 13, 14, and 15 of this Book depend.

Proposition XXIII. A Problem

186. To erect a perpendicular at the end of a given line.

See Book III., Proposition XVIII., Corollary II.

Proposition XXIV. A Problem.

187. At a point on a line, to construct an angle equal to a given angle.

See Book III., Proposition XVII., Scholium I., and Proposition IV.

Proposition XXV. A Problem.

188. To bisect a given arc.

See Book III., Proposition VI.

Proposition XXVI. A Problem.

189. To bisect a given angle.

Proposition XXVII. A Problem.

190. Through a given point to draw a line parallel to a given line.

See Book I., Proposition XI.

Proposition XXVIII. A Problem.

191. Two angles of a triangle being given, to find the third.

Proposition XXIX. A Problem.

192. To construct a triangle when two of its sides and the angle included by them are given.

Proposition XXX. A Problem.

193. Given a side and two angles, to construct the triangle.

Proposition XXXI. A Problem.

194. Given two sides of a triangle and the angle opposite one of them, to construct the triangle.

Proposition XXXII. A Problem.

195. Given the three sides, to construct the triangle.

See Book I., Proposition XXX.

Proposition XXXIII. A Problem.

196. To construct a parallelogram when its adjacent sides and their included angle are given.

Proposition XXXIV. A Problem.

197. From a given point to draw a tangent to a given circle.

See Book III., Proposition XII., and Proposition XVIII., Corollary II.

Proposition XXXV. A Problem.

198. At a given point in the circumference of a circle to draw a tangent.

See Book III., Proposition XXIII.

Proposition XXXVI. A Problem.

199. In a given triangle to inscribe a circle.

See Book I., Proposition XXXIX.

Proposition XXXVII. A Problem.

200. The chord being given, to construct a circle such that any angle inscribed in one of the segments will be equal to a given angle.

See Book III., Proposition XXI., Proposition XVIII., Corollary I., and Proposition XII.

Proposition XXXVIII. A Problem.

201. Two arcs or two angles being given, to find their common measure.

Proposition XXXIX. A Theorem. (*Optional.*)

202. The side of an inscribed equilateral triangle and the radius perpendicular to it bisect each other.

SUPPLEMENTARY PROPOSITIONS.

1. From any point in a circle the shortest distance to the circumference will be on the radius passing through the point.

2. From any point in a circle the farthest distance to the circumference will be on the line passing through the centre.

3. If a circle is touched internally by another circle having half the diameter, any chord of the larger circle drawn from the point of contact is bisected by the circumference of the smaller circle.

4. The shortest chord that can be drawn through any point in a circle is the one drawn at right angles to the radius passing through the point.

5. The opposite angles of an inscribed quadrilateral are supplements of each other.

6. If the opposite angles of a quadrilateral are supplements of each other, a circumference can be circumscribed about it.

7. The sides of an inscribed equilateral triangle are half the length of the sides of a similar circumscribed triangle.

See Book I., Supplementary Propositions 19, 20, and 21.

8. If two circles intersect each other, the distance between their centres is less than the sum and greater than the difference of their radii.

9. The sum of the opposite arcs intercepted by two chords crossing each other at right angles equals a semi-circumference.

10. If two equal circles intersect each other, parallel secants passing through the points of intersection cut off reciprocally equal arcs and segments.

11. If two equal intersecting circles are cut by two secants passing through the points of intersection, chords subtending the exterior arcs intercepted by these secants will be parallel.

Case I. When the secants do not cross.

Case II. When the secants cross each other in one of the circles.

12. If two equal circles touch each other, two secants passing through the point of contact, will intercept equal arcs; and the chords subtending these arcs will be parallel.

13. If two unequal circles intersect each other, two parallels passing through the points of intersection and terminated by the exterior arcs, will be equal.

See Note, page 35.

14. If two unequal intersecting circles are cut by secants passing through the points of intersection, chords subtending the exterior arcs intercepted are parallel.

Case I. When the secants do not cross.

Case II. When the secants cross each other in one of the circles.

See Note, page 35.

15. If two unequal circles touch each other, two secants passing through the point of contact will intercept homologous arcs, and the chords subtending these arcs will be parallel.

16. Construct an angle of 60°; of 120°; of 30°; of 15°.

17. Construct an angle of 45°. Divide it into three equal angles.

18. Divide a right angle into three equal angles.

19. Find a point equidistant from three given points.

20. Find a point equidistant from two given points, and a given distance from a third given point.

21. Construct a perpendicular from the vertex of one angle of a triangle to the opposite side.

22. Divide a line into three equal parts.

See Book I., Supplementary Proposition 18.

23. Given the radius and two points in the circumference, to construct the circle.

24. A chord and a point in the circumference given, to construct the circle.

25. To lay off on a given circumference an arc of 180°; of 90°; of 60°; of 30°; of 120°.

26. The base, the altitude, and one of the angles at the base given, to construct the triangle.

27. Given one side, the diagonal, and the included angle, to construct a parallelogram.

28. In a given circle to inscribe an equilateral triangle.

29. About a given circle to circumscribe an equilateral triangle.

30. The radius is two thirds the altitude of an inscribed, and one third the altitude of a circumscribed equilateral triangle.

31. Find in a given circumference two points such that tangents passing through them will meet at an angle of 30°.

32. Find in a given circumference two points such that two tangents passing through them will meet at an angle of 90°.

33. Given the perimeter and altitude of a triangle, and the point on the perimeter where the perpendicular from the opposite angle, which equals the altitude, would fall, to construct the triangle.

34. To construct a triangle, the base, altitude, and angle at the vertex being given.

See Book III., Proposition XXXVII.

35. To construct a triangle, the base, angle at the vertex, and median connecting them being given.

36. From a given point draw tangents to a given circle; connect these tangents by a line drawn tangent to the smaller intercepted arc; a triangle will be formed, the sum of whose sides will be constant at whatever point on the arc the connecting tangent be drawn.

See Book III., Proposition XIII.

37. If, with the conditions as given in 36, lines be drawn from the centre of the circle to the extremities of the connecting tangent, the angle at the centre will remain constant through all positions of the tangent.

38. To construct a right triangle, when given:

a. Hypotenuse and one side.

b. Hypotenuse and altitude on the hypotenuse.

c. One side and altitude on the hypotenuse.

39. To construct a scalene triangle, when given:

a. The perimeter and angles.

b. One side, an adjacent angle, and sum of the other sides.

c. The sum of two sides and the angles.

d. The angles and the radius of an inscribed circle.

e. An angle, its bisector, and the altitude from the given angle.

40. To construct a rectangle, when given:

a. One side and the angle formed by the diagonals.

b. The perimeter and a diagonal.

41. To construct a rhombus, when given:

a. One side and the radius of the inscribed circle.

b. One angle and the radius of the inscribed circle.

42. To construct a rhomboid, when given:

a. One side and the two diagonals.

b. The base, the altitude, and one angle.

43. To construct a trapezoid, when given:

a. The bases, the altitude, and one angle.

b. One base, the adjacent angles, and one side.

c. One base, the adjacent angles, and the median.

BOOK IV.

I. Similar Polygons.

203. When are polygons similar? *

204. What are their homologous parts?

205. What is meant by their ratio of similitude?

Proposition I. A Theorem.

206. Two mutually equiangular triangles are similar.

See Book II., Proposition XVI.

Corollary. Triangles having two angles mutually equal, or an angle in each equal and the sides including it in proportion, are similar.

Proposition II. A Theorem.

207. If triangles have their sides taken in order in proportion, they are similar.

Proposition III. A Problem.

208. The ratio of the homologous sides being equal to the ratio of two given lines, to construct a triangle similar to a given triangle.

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* Form what proportions you can from two similar triangles; from two similar quadrilaterals; from two similar pentagons.

Proposition IV. A Theorem.

209. Two triangles whose sides are parallel or perpendicular are similar.

Proposition V. A Theorem.

210. Two similar polygons may be divided into the same number of triangles, similar each to each.

Proposition VI. A Theorem.

211. If two polygons can be divided into the same number of triangles, similar each to each, and similarly placed, the two polygons are similar.

Proposition VII. A Problem.

212. A polygon being given, on a line corresponding to one of its sides, to construct a similar polygon.

Proposition VIII. A Theorem.

213. The perimeters of two similar polygons have the same ratio as any two homologous sides.

Proposition IX. A Theorem.

214. Any number of straight lines intersecting at a common point intercept proportional segments on two parallels.

CASE I. When the parallels are on the same side of the common point.

CASE II. When they are on opposite sides.

NOTE. This principle may be used in finding the sides in Proposition VII.

Proposition X. A Theorem.

215. Conversely, all non-parallel lines intercepting proportional segments on two parallel lines intersect at a common point.

II. Division of Lines.

216. What is dividing a line internally?

Example. 

a. What are the segments?

217. What is dividing a line externally?

Example. 

a. What are the segments?

SPECIAL PROBLEMS.

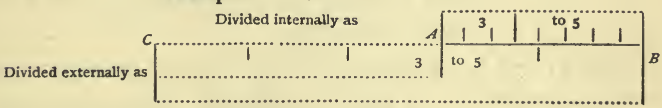
218. *a.* To divide a line internally in the ratio of 2 : 3; of 3 : 5; of 2 : 7; etc.

See Proposition IX., or Book II., Proposition XVIII.

b. To divide a line externally in the ratio of 2 : 3; of 3 : 5; of 2 : 7; etc.

219. What is dividing a line harmonically?

Example.



SPECIAL PROBLEMS.

220. To divide a given line harmonically in the ratio of 3 : 4; of 3 : 5; of 2 : 7; etc.

Proposition XI. A Theorem.

221. A line bisecting an angle of a triangle divides the opposite side into segments proportional to the adjacent sides including the angle.

Proposition XII. A Theorem.

222. A line bisecting an exterior angle of a triangle divides the opposite side externally into segments proportional to the other two sides.

Proposition XIII. A Theorem.

223. Lines bisecting adjacent interior and exterior angles of a triangle divide the opposite side harmonically.

Proposition XIV. A Problem.

224. To divide a line harmonically.

Proposition XV. A Theorem.

225. In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse:

I. The right triangle is divided into two triangles similar to itself and to each other.

II. The perpendicular is a mean proportional between the segments of the hypotenuse.

III. Each side of the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.

Corollary I. The squares of the sides of the right angle are proportional to the adjacent segments of the hypotenuse.

Corollary II. The square of the hypotenuse of a right triangle is equivalent to the sum of the squares of the other two sides.

Proposition XVI. A Problem.

226. To find a mean proportional between two given lines.

Proposition XVII. A Theorem.

227. The products of the two segments of all chords drawn through any fixed point in a circle are constant.*

Proposition XVIII. A Theorem.

228. From a point without a circle, in whatever direction a secant is drawn, the product of the whole secant by its external segment is constant.

Proposition XIX. A Theorem.

229. If from a point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and its external segment.

How can this be proved by the Theory of Limits?

III. Extreme and Mean Ratio.

230. What is dividing a line in extreme and mean ratio?

Examples.

Internally.

$AB : AC :: AC : CB.$

Externally.

$CB : CA :: CA : AB.$

* Find a line proportional to three given lines by this principle.

Proposition XX. A Problem.

231. To divide a given line in extreme and mean ratio.

See Proposition XIX., and Book II., Proposition VII.

Proposition XXI. A Theorem.

232. In any triangle, the product of any two sides is equal to the product of the perpendicular to the third side from the opposite angle by the diameter of the circumscribed circle.

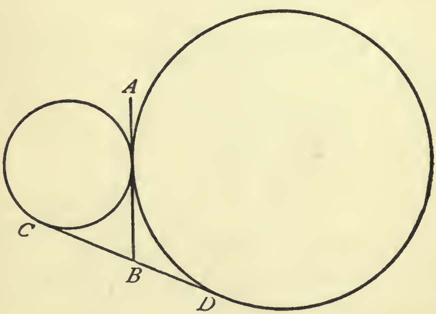
Proposition XXII. A Theorem.

233. If an angle of a triangle be bisected by a line terminating in the opposite side, the product of the segments of this side plus the square of the bisector equals the product of the other two sides.

Proposition XXIII. A Theorem.

234. Homologous altitudes of similar triangles are proportional to any two homologous sides.

SUPPLEMENTARY PROPOSITIONS.



1. The chord AB bisects the common tangent CD.

2. The common tangent CD is a mean proportional between the diameters of the circles.

3. If two circles intersect each other, the common chord produced bisects the common tangent.

4. If the common chord of two intersecting circles be produced, tangents drawn from any point in it to the circles are equal.

5. To inscribe a square in a given triangle.

6. To inscribe a square in a semicircle.

7. To inscribe in a given triangle a rectangle similar to a given rectangle.

8. To circumscribe about a circle a triangle similar to a given triangle.

9. To construct a circle whose circumference will be tangent to a given line and pass through two given points.

10. To construct a circle whose circumference will be tangent to two given lines and pass through one given point.

BOOK V.

MEASUREMENT AND COMPARISON OF RECTILINEAR FIGURES.

I. Area.

235. What is area?

a. How measured?

Proposition I. A Theorem.

236. The area of a rectangle is equal to the product of its base and altitude.

Case I. When the base and altitude are commensurable.

Case II. When they are incommensurable.

Proposition II. A Theorem.

237. The area of a parallelogram is equal to the product of its base and altitude.

Corollary. Parallelograms having equal bases and altitudes are equivalent.

Proposition III. A Theorem.

238. Parallelograms are to each other as the products of their respective bases and altitudes.

Corollary. Parallelograms having equal altitudes are to each other as their bases; those having equal bases are to each other as their altitudes.

Proposition IV. A Theorem.

239. The area of a triangle is equal to half the product of its base and altitude.

Corollary. Triangles having the same base and altitude are equivalent.

Proposition V. A Theorem.

240. Triangles are to each other as the products of their respective bases and altitudes.

Corollary. Triangles having the same altitudes are to each other as their bases, and those having the same bases are to each other as their altitudes.

Proposition VI. A Theorem.

241. The area of a trapezoid is equal to the product of its altitude by half the sum of its bases, or by its median.

Proposition VII. A Theorem.

242. The areas of two triangles having an angle in each equal are to each other as the products of the sides including the equal angle.

Proposition VIII. A Theorem.

243. The square described on the sum of two lines is equal to the sum of their squares plus two rectangles contained by the lines.

SCHOLIUM. Compare $(a + b)^2 = a^2 + 2ab + b^2$.

Proposition IX. A Theorem.

244. The square described on the difference of two lines is equal to the sum of their squares minus two rectangles contained by the lines.

SCHOLIUM. Compare $(a - b)^2 = a^2 - 2ab + b^2$.

Proposition X. A Theorem.

245. The rectangle contained by the sum and difference of two lines is equal to the difference of their squares.

SCHOLIUM. Compare $(a + b)(a - b) = a^2 - b^2$.

Proposition XI. A Theorem.

246. The square described on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

COROLLARY. The square described on either side forming the right angle is equal to the square of the hypotenuse minus the square of the other side.

Proposition XII. A Problem.

247. To construct a square equal to the sum of two given squares.

Proposition XIII. A Problem.

248. To construct a square equal to the difference of two given squares.

Proposition XIV. A Problem.

249. To construct a square equal to the sum of any given number of given squares.

Proposition XV. A Problem.

250. I. To construct a square equivalent to a given rectangle.

See Book IV., Proposition XVI.

II. Equivalent to a given triangle.

Proposition XVI. A Problem.

251. The sum of the base and altitude given, to construct a parallelogram equivalent to a given square.

Proposition XVII. A Problem.

252. The difference between the base and altitude given, to construct a parallelogram equivalent to a given square.

Proposition XVIII. A Theorem.

253. The areas of similar triangles are to each other as the squares of their homologous sides.

See Proposition VII.

Proposition XIX. A Theorem.

254. The areas of any similar polygons are proportional to the squares of their homologous sides.

Proposition XX. A Problem.

255. I. To construct a polygon similar to two given polygons but equal to their sum.

See Proposition XII.

II. Equal to their difference.

Proposition XXI. A Problem.

256. I. To construct a triangle equivalent to a given polygon.

II. To construct a square equivalent to a given polygon of five or more sides.

Proposition XXII. A Problem.

257. To construct a square having a given ratio to a given square.

Proposition XXIII. A Problem.

258. In a given ratio between their areas, to construct a polygon similar to a given polygon.

Proposition XXIV. A Problem.

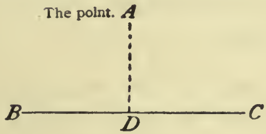
259. To construct a polygon similar to one given polygon but equivalent to another.

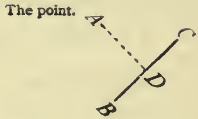
See Proposition XXI.

II. Projection.

260. What is the projection of a point on a line?

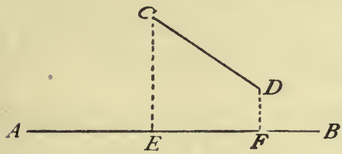
Illustrations.

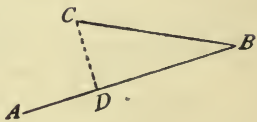




261. What is the projection of a line on another line?

Illustrations.





Proposition XXV. A Theorem.

262. In a triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the rectangle formed by one of these sides and the projection of the other on it.

Proposition XXVI. A Theorem.

263. In a triangle, the square of the side opposite an obtuse angle is equal to the sum of the squares of the other two sides plus twice the rectangle formed by one of these sides and the projection of the other on it.

Proposition XXVII. A Theorem.

264. In a triangle, if a median line is drawn from the vertex of any angle:

I. The sum of the squares of the sides including the angle is equal to twice the square of the median plus twice the square of half the side it bisects.

II. The difference of the squares of the two sides including the angle is equal to twice the rectangle formed by the third side and the projection of the median on it.

Corollary I. In any quadrilateral (not a parallelogram) the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.

Corollary II. In a parallelogram the sum of the squares of the four sides is equal to the sum of the squares of the diagonals.

SPECIAL PROBLEMS.

1. Express the altitude of an equilateral triangle in terms of its sides.

Suggestion. Let a be the length of one side, and x the altitude.

2. Express the area of an equilateral triangle in terms of its sides.

Suggestion. Let A represent the area.

3. Express the altitude of any triangle in terms of its sides.

Suggestion. Let a, b, and c represent the lengths of the different sides.

4. Express the area of a triangle in terms of its sides.

5. Express a median of a triangle in terms of its sides.

6. Express the bisector of an angle of a triangle in terms of its sides.

7. Express the radius of a circle circumscribed about a triangle in terms of the sides of the triangle.

BOOK VI.

I. Regular Polygons.

265. What is a regular polygon?

a. The apothem?

Proposition I. A Theorem.

266. A circle may be circumscribed about and one inscribed within a regular polygon.

Corollary. The radius drawn to the vertex of an angle of a regular inscribed polygon bisects the angle.

Proposition II. A Theorem.

267. An equilateral polygon inscribed in a circle is regular.

Proposition III. A Theorem.

268. If a circumference be divided into equal arcs:

I. The chords subtending these arcs form a regular polygon.

II. The tangents drawn at the points of division form a regular polygon.

See Book III., Proposition XIII.

Proposition IV. A Problem.

269. I. Given a regular inscribed polygon, to construct one having double the number of sides.

II. Given a regular circumscribed polygon, to construct one having double the number of sides.

Proposition V. A Problem.

270. In a given circle to construct a square.

Proposition VI. A Theorem.

271. The side of a regular inscribed hexagon is equal to the radius of the circumscribed circle.

Proposition VII. A Problem.

272. To inscribe a regular hexagon in a given circle.

Proposition VIII. A Problem.

273. To inscribe a regular decagon in a given circle.

See Book IV., Proposition XX.

SCHOLIUM. Inscribe a regular pentagon.

Proposition IX. A Problem.

274. To inscribe a regular pentedecagon in a given circle.

Proposition X. A Theorem.

275. Two regular polygons of the same number of sides are similar.

Proposition XI. A Theorem.

276. The perimeters of two regular polygons of the same number of sides are to each other:

I. As their sides.

II. As the radii of circumscribed circles.

III. As the radii of inscribed circles.

Corollary. If in two circles all possible regular polygons be drawn, the perimeters of those in one circle will have to the perimeters of the similar ones in the other circle a constant ratio.

Proposition XII. A Theorem.

277. The circumferences of circles are to each other as their radii or their diameters.

See Book II., §§ 120–128.

Corollary. The ratio of circumferences to their radii or to their diameters is constant.

Scholium I. The constant ratio of the circumference to the diameter is represented by the Greek letter π, and it will be hereafter one of our objects to ascertain its numerical value.

Scholium II. Let $2R$ represent the diameter; the circumference will be $2\pi R$.

Proposition XIII. A Theorem.

278. The areas of two regular polygons of the same number of sides are to each other:

I. As the squares of their sides.

II. As the squares of the radii of circumscribed circles.

III. As the squares of the radii of inscribed circles.

Proposition XIV. A Theorem.

279. The areas of circles are to each other as the squares of their radii or of their diameters.

Corollary. The areas of similar sectors or segments are to each other as the squares of the radii or of the diameters.

Proposition XV. A Theorem.

280. I. The difference between the perimeters of regular inscribed and circumscribed polygons of the same number of sides is indefinitely diminished as the sides are indefinitely multiplied.

II. The difference between their areas is indefinitely diminished as the sides are indefinitely multiplied.

Proposition XVI. A Theorem.

281. The area of a regular polygon is equal to half the product of its perimeter by its apothem.

Proposition XVII. A Theorem.

282. The area of a circle is equal to half the product of the circumference by the radius.

Scholium. If $2 \pi R$ (see Proposition XII., Scholium II.) represents the circumference, what will be the area of the circle?

Corollary. The area of a sector is equal to half the product of its arc and the radius.

Proposition XVIII. A Problem.

283. Given the radius and a chord, to compute the chord of half the arc subtended.

SCHOLIUM. This principle can be used, when the side of a regular inscribed polygon is known, to find the side, and therefore the perimeter, of a regular polygon of double the number of sides.

Proposition XIX. Problems.

284. I. To find the ratio between the perimeter of a regular inscribed hexagon and the diameter of the circle.

II. Between the perimeter of a regular inscribed duodecagon and the diameter of the circumscribed circle.

Proposition XX. A Problem.

285. To compute the numerical value of π.

OPTIONAL PROPOSITIONS.

Proposition XXI. A Problem.

286. The perimeters of a regular inscribed and a similar circumscribed polygon being known, to compute the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.

Proposition XXII. A Problem.

287. To compute the numerical value of π from the preceding problem.

SPECIAL PROBLEMS.

Proposition XXIII. A Problem.

288. I. Express the side of an inscribed equilateral triangle in terms of the radius.

II. The same of a regular inscribed hexagon.

III. The same of a regular inscribed duodecagon.

NOTE. Continue this as far as desirable.

SCHOLIUM. What would the perimeters be in each case?

COROLLARY. Express the areas of the above in terms of the radius.

Proposition XXIV. Problems.

289. I. Express the side of an inscribed square in terms of the radius.

II. The same of a regular inscribed octagon.

NOTE. Continue as far as desirable.

SCHOLIUM. What would the perimeters be in each case?

COROLLARY. Express their areas in terms of the radius.

Proposition XXV. Problems.

290. I. Express the side (and perimeter) of a regular inscribed decagon in terms of the radius.

See Proposition VIII.

II. Express the side of a regular inscribed pentagon in terms of the radius.

COROLLARY. Express their areas in terms of the radius.

Proposition XXVI. A Problem.

291. Compute the numerical value of π from one of the above problems.

SUPPLEMENTARY PROBLEMS.

Geometrical Construction of Algebraic Equations.

Note. In these problems, the first letters of the alphabet express known, or given lines; in performing operations with them, the following points should be kept in mind:

1. The product of two lines, or the square of a line is a surface.
2. The product of three lines is a solid.
3. A surface divided by a line, is a line.
4. The square root of the product of two lines is a line.

The letter x represents the element to be constructed and may be a line, surface, or solid as the case requires.

Problem I. Construct $x = a + b$.

Problem II. Construct $x = a - b$.

Problem III. Construct $x = a\,b$.

Problem IV. Construct $x = a\,b\,c$.

Problem V. Construct $x = \frac{a\,b}{c}$.

See Book II., Proposition XIX.

Problem VI. Construct $x = \frac{a^2}{b}$.

See Book II., Proposition XX.

Problem VII. Construct $x = \sqrt{a^2 + b^2}$.

See Book V., Proposition XI.

Problem VIII. Construct $x = \sqrt{a^2 - b^2}$.

Problem IX. Construct $x = \sqrt{a\,b}$.

See Book IV., Proposition XVI.

Problem X. Construct $x = \sqrt{a^2 - a\,b}$.

See Book IV., Proposition XIX.

PROBLEM XI. Construct $x = a \pm \sqrt{a^2 - b^2}$.

SUG. Construct a line equal to a, and at one extremity construct a perpendicular equal to b. With the remote end of b as a centre and a radius equal to a, draw an arc cutting a and a prolonged.

PROBLEM XII. Form the equation for the larger segment of a line a divided in extreme and mean ratio.

See Book IV., Proposition XX.

PROBLEM XIII. Form the equation for the side of a square inscribed in a triangle whose base and altitude are given.

PROBLEM XIV. Given the radii and the distance between the centres of two unequal circles, form the equation for the distance to the point where their common tangents will meet.