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PROBLEMS

IN

DIFFERENTIAL CALCULUS

SUPPLEMENTARY TO A TREATISE ON DIFFERENTIAL CALCULUS

BY

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BC

PREFACE.

This collection of problems and examples is intended to accompany and to supplement my treatise on the Differential Calculus (Boston, Ginn & Company). The chapters correspond to the chapters in that book and the references are to its sections.

A slightly fuller treatment of definite integrals and of development in series than that in the original text is indicated and illustrated, and brief explanations and suggestions are given when they seem to me to be needed.

W. E. Byerly.

Cambridge, Mass.,
Sept. 20, 1895.

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PROBLEMS

IN

DIFFERENTIAL CALCULUS.

CHAPTER I.

INTRODUCTION.

1. If a body is rotating in any way about an axis the whole angle through which it turns in any period of time divided by the length of the period is called the *mean angular velocity* during the period.

The limit approached by the mean angular velocity during a period of arbitrary length as the length of the period is indefinitely decreased is called the *actual angular velocity* at the beginning of the period.

Define *uniform angular velocity.*

Angular velocity, whether mean or actual, is usually expressed in *radians per second.*

If ϕ is the number of radians through which a body turns in t seconds, and ω is the actual angular velocity at any instant, show that if $\phi = f(t)$

$$\omega = \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{\Delta \phi}{\Delta t} \right].$$

2. If a particle is moving in any way the whole change in its velocity during a given period of time divided by the

length of the period is called the *mean acceleration* of the particle during the period.

The limit approached by the mean acceleration during a period of arbitrary length as the length of the period is indefinitely decreased is called the *actual acceleration* at the beginning of the period.

If velocity is expressed in feet per second, acceleration, whether mean or actual, is expressed in *feet per second per second.*

Define *uniform acceleration.*

If a is the acceleration and v the velocity of a particle, and $v = f(t)$, show that

$$a = \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{\Delta v}{\Delta t} \right].$$

3. Given $s = 16\,t^2$ where s is measured in feet and t in seconds, show that $a = 32$ feet per second per second, and consequently that the acceleration of a falling body is uniform.

4. Given $s = v_0 t - 16\,t^2$ as the distance ascended by a body thrown vertically upward with an initial velocity of v_0 feet per second, find the velocity at the end of t seconds, and the acceleration.

5. A cannon ball is fired upward with a muzzle velocity of 640 feet a second. Find (a) its velocity at the end of 10 seconds; (b) the number of seconds it will rise; (c) the distance it will ascend. Neglect the resistance of the air.

Ans. (a) 320 ft. per second; (b) 20 seconds; (c) 6400 ft.

6. A man 6 feet high walks directly away from a lamp-post 10 feet high with a uniform velocity of 4 miles an hour. How fast does the end of his shadow move along the pavement? *Ans.* 10 miles an hour.

7. Show that if $x = f(y)$ is the equation of a curve in rectangular coördinates, and τ is the angle the tangent at any point of the curve makes with the axis of X,

$$\operatorname{ctn} \tau = \underset{\Delta y \doteq 0}{\text{limit}} \left[\frac{\Delta x}{\Delta y}\right].$$

Find the angle the tangent to the parabola $y^2 = 2\,mx$ at the point $\left(\frac{m}{2}, m\right)$ makes with the axis of X.

Ans. $\operatorname{ctn} \tau = 1$, $\tau = 45°$.

CHAPTER II.

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

Find the derivatives with respect to x of the following functions:—

1. $3a^2x^2 - 4ax + b$. *Ans.* $6a^2x - 4a$.

2. $6ax^4 + 3a^2x^5 + 2a^3x^6$. *Ans.* $24ax^3 + 15a^2x^4 + 12a^3x^5$.

3. $x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$. *Ans.* $\dfrac{3x+1}{2\sqrt{x}}$.

4. $(1+x^2)^{\frac{5}{4}}$. *Ans.* $\frac{5}{2}x(1+x^2)^{\frac{1}{4}}$.

5. $\dfrac{x+4}{x+1}$. *Ans.* $\dfrac{-3}{(x+1)^2}$.

6. $\dfrac{a+bx+cx^2}{x}$. *Ans.* $c - \dfrac{a}{x^2}$.

7. $\dfrac{a^2-b^2}{a^2-x^2}$. *Ans.* $(a^2-b^2)\dfrac{2x}{(a^2-x^2)^2}$.

8. $\dfrac{1}{1-2x^2}$. *Ans.* $\dfrac{4x}{(1-2x^2)^2}$.

9. $(a+x)\sqrt{a-x}$. *Ans.* $\dfrac{a-3x}{2\sqrt{a-x}}$.

10. $\dfrac{2x^4}{a^2-x^2}$. *Ans.* $\dfrac{4x^3(2a^2-x^2)}{(a^2-x^2)^2}$.

11. $\dfrac{x}{\sqrt{1+x^2}}$. *Ans.* $(1+x^2)^{-\frac{3}{2}}$.

12. $\dfrac{x^3}{a^2 - x^2}$. *Ans.* $\dfrac{x^2(3a^2 - x^2)}{(a^2 - x^2)^2}$.

13. $\sqrt{ax} + \sqrt{c^2x^3}$. *Ans.* $\dfrac{\sqrt{a} + 3cx}{2\sqrt{x}}$.

14. $\dfrac{x}{\sqrt{a^2 - x^2}}$. *Ans.* $\dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.

15. $\dfrac{\sqrt{a^2 - x^2}}{a^2x}$. *Ans.* $-\dfrac{1}{x^2\sqrt{a^2 - x^2}}$.

16. $(a^3 + x^3)(b^2 + 3x^2)$. *Ans.* $3(5x^3 + b^2x + 2a^3)x$.

17. $\dfrac{x^2 - 2a^2}{x - a}$. *Ans.* $1 + \dfrac{a^2}{(x - a)^2}$.

18. $\dfrac{x^3}{(1 - x^2)^3}$. *Ans.* $\dfrac{3x^2}{(1 - x^2)^5}$.

19. $\dfrac{x^3}{(1 - x^2)^{\frac{3}{2}}}$. *Ans.* $\dfrac{3x^2}{(1 - x^2)^{\frac{5}{2}}}$.

20. $\dfrac{1 - x}{\sqrt{1 + x^2}}$. *Ans.* $-\dfrac{1 + x}{(1 + x^2)^{\frac{3}{2}}}$.

21. $\dfrac{x\sqrt{(a + x)}}{\sqrt{a} - \sqrt{(a - x)}}$. *Ans.* $\dfrac{\sqrt{a}}{2\sqrt{(a + x)}} - \dfrac{x}{\sqrt{a^2 - x^2}}$.

CHAPTER III.

APPLICATIONS.

1. If the equation of a curve is given in the form $x=f(y)$ show that the formulas in Dif. Cal., Art. 27, can be written:

$$\operatorname{ctn}\tau = D_y x \tag{1}$$

$$\tan\nu = -D_y x; \tag{2}$$

and that the formulas in Dif. Cal., Art. 28, can be written:

$$y-y_0=\frac{1}{[D_y x]_{y=y_0}}(x-x_0) \tag{3}$$

$$y-y_0=-[D_y x]_{y=y_0}(x-x_0) \tag{4}$$

$$t_x = yD_y x \tag{5}$$

$$n_x=\frac{y}{D_y x} \tag{6}$$

$$t=y\sqrt{1+(D_y x)^2} \tag{7}$$

$$n=y(D_y x)^{-1}\sqrt{1+(D_y x)^2} \tag{8}$$

2. Find the subtangent and subnormal in the parabola $x^2=2my$.

Ans. $t_x=\frac{x}{2}$; $n_x=\frac{x^3}{2m^2}$.

3. Find the subtangent and subnormal in the ellipse and the hyperbola.

Ans. $t_x=\mp\frac{a^2y^2}{b^2x}$; $n_x=\mp\frac{b^2x}{a^2}$.

4. Given the formula

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta},$$

obtain a formula for the angle at which two given curves, $y = f(x)$ and $y = F(x)$, intersect.

Ans. If (x_0, y_0) is a point of intersection, and θ is the angle required, $\tan\theta = \dfrac{[D_xF(x)]_{x=x_0} - [D_xf(x)]_{x=x_0}}{1 + [D_xF(x)]_{x=x_0}\,[D_xf(x)]_{x=x_0}}$.

5. At what angle do the curves $y^2 = 4\,ax$ and $x^2 = 4\,ay$ intersect?

Ans. $90°$ at the point $(0,0)$; and $\tan^{-1}\frac{3}{4}$ at the point $(4\,a, 4\,a)$.

6. Find the angle at which the circle $x^2 + y^2 + 2\,x = 7$ crosses the parabola $y^2 = 4\,x$. *Ans.* $90°$.

7. Show that the curves $y^2 = 2\,mx + m^2$ and $y^2 = -2\,nx + n^2$ cross at right angles.

8. Show that the curves $x^2 - y^2 = a^2$ and $2\,xy = b^2$ cross at right angles.

9. Show that the curves $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$ and $\dfrac{x^2}{b^2} - \dfrac{y^2}{c^2 - b^2} = 1$ cross at right angles.

10. Find the equation of the tangent at the point $(1, 1)$ to the curve $y^4 = 3\,x - 2\,x^2$. *Ans.* $x + 4\,y = 5$.

11. Prove the following results without the aid of differentiation by reducing the fractions to their simplest terms before trying to find the limiting values: —

$$\underset{x \doteq 0}{\text{limit}}\left[\frac{\sin x}{\tan x}\right] = 1;$$

$$\underset{x \doteq 0}{\text{limit}}\left[\frac{1 - \cos x}{\sin x}\right] = 0;$$

$$\underset{x \doteq 90°}{\text{limit}}\left[\frac{\sec x}{\tan x}\right] = 1;$$

$$\underset{x \doteq 90^\circ}{\text{limit}}\left[\sec x - \tan x\right] = 0;$$

$$\underset{x \doteq 0}{\text{limit}}\left[\frac{1-\cos x}{\sin^2 x}\right] = \tfrac{1}{2};$$

$$\underset{x \doteq 0}{\text{limit}}\left[\frac{\tan x - \sin x}{1-\cos x}\right] = 0;$$

$$\underset{x \doteq 0}{\text{limit}}\left[\frac{\tan x - \sin x}{\sin^3 x}\right] = \tfrac{1}{2};$$

$$\underset{x \doteq 1}{\text{limit}}\left[\frac{x-1}{x^2-1}\right] = \tfrac{1}{2};$$

81 + 81 − 63 − ~~81~~

$$\underset{x \doteq 3}{\text{limit}}\left[\frac{x^4+3x^3-7x^2-27x-18}{x^4-3x^3-7x^2+27x-18}\right] = 10.$$

81 − 81 63 + 81

12. Show by any method that

$$\underset{x \doteq 1}{\text{limit}}\left[\frac{\sin^2(x-1)}{1-\cos(x-1)} \cdot \frac{x^{\frac{3}{2}}-1+(x-1)^{\frac{3}{2}}}{(x^2-1)^{\frac{3}{2}}-x+1}\right] = -3.$$

13. A cylindrical tin tomato can is to be made which shall have a given capacity. Find what should be the ratio of the height to the radius of the base that the smallest possible amount of tin shall be required. *Ans.* $h = 2r$.

14. What should be the ratio in the last question that the can may be as inexpensive as possible when the top and bottom are cut out of rectangular sheets of tin so that in making either top or bottom a regular hexagon circumscribing the required circle is used up?

Ans. $h = \dfrac{4\sqrt{3}}{\pi} r$; or the diameter of the base is to the height nearly in the ratio of 10 to 11.

15. What are the most economical proportions for a cylindrical tin cup?

Ans. $h=r$, if no allowance is made for waste;

$h=\frac{2\sqrt{3}}{\pi}r=\frac{11}{10}r$ approximately, if the waste is allowed for.

16. A copper water tank in the form of a rectangular parallelopiped is to be made. If its length is to be a times its breadth, how high should it be that for a given capacity it should cost as little as possible?

Ans. Height should be to breadth as $a : a+1$ if the tank is open; as $2a : a+1$ if the tank has a lid.

17. What are the most economical proportions for an open cylindrical water tank if it is to be made out of iron boiler plates, and the cost of the sides per square foot is two-thirds the cost of the bottom per square foot? *Ans.* $h=\frac{3}{2}r$.

18. The cost per hour of running a certain steamboat is proportional to the cube of its velocity in still water. At what speed should it be run to make a trip up stream against a four-mile current most economically?

Ans. 6 miles an hour.

19. An open box is to be made from a sheet of pasteboard 12 inches square by cutting equal squares from the four corners and bending up the sides. What are the dimensions of the largest box that can be made?

Ans. 8 in. by 8 in. by 2 in.

20. A house with a rectangular ground plan of given length and breadth is to be built. If shingles and clapboards are equally expensive, and garret room as valuable as room below the eaves, what should be the inclination of the roof to the horizon that the cost of outside walls and roof may be as little as possible? *Ans.* 30°.

21. A ship is 41 miles due north of a second ship. The first sails south at the rate of 8 miles an hour, the second east at the rate of 10 miles an hour. How rapidly are they approaching each other? How long will they continue to approach? *Ans.* For 2 hours.

22. The strength of a beam of rectangular cross-section, if supported at the ends and loaded in the middle, is proportional to the product of the breadth of the cross-section by the square of its depth. Find the dimensions of the cross-section of the strongest beam that can be cut from a log 12 inches in diameter. *Ans.* About 6.9 in. by 9.8 in.

23. A lamp-post stands on the edge of the sidewalk 10 feet from the street crossing. The houses on the opposite side of the street are 60 feet away. A man crosses the street, walking on the crossing at the rate of 4 miles an hour toward the corner near the lamp-post. How fast is his shadow moving on the walls of the opposite houses when he is 20 feet from the corner toward which he is going? When he is 5 feet from the corner? What is the acceleration of his shadow's velocity at each of the times referred to?

Ans. (*a*) 6 miles an hour; (*b*) 96 miles an hour; (*c*) $5\frac{61}{375}$ ft. per sec. per. sec.; (*d*) $501\frac{29}{375}$ ft. per sec. per sec.

24. A ladder 25 feet long rests against a house. A man takes hold of the bottom of the ladder and walks off with it with a uniform velocity of 2 feet a second. How fast is the top of the ladder descending when the bottom is 7 feet from the house? At what rate is its velocity increasing?

Ans. (*a*) $\frac{7}{12}$ ft. per sec.; (*b*) $\frac{2500}{13824}$ ft. per sec. per sec.

25. A vessel is anchored in 3 fathoms of water and the cable passes over a sheave in the bowsprit which is 6 feet above the water. If the cable is hauled in at the rate of a foot a second how fast is the vessel moving through the water

when there is 5 fathoms of cable out? What is the acceleration of the vessel's velocity?

Ans. (*a*) $\frac{5}{3}$ ft. per sec.; (*b*) $\frac{8}{81}$ ft. per sec. per sec.

26. A man standing on a wharf is drawing in the painter of a boat at the rate of 4 feet a second. If his hands are 6 feet above the bow of the boat how fast is the boat moving when it is 8 feet from the wharf? What is the acceleration of its velocity?

Ans. (*a*) 5 ft. per sec.; (*b*) $\frac{9}{8}$ ft. per sec. per sec.

27. A man is walking across a bridge at the rate of 4 miles an hour, when a boat passes under the bridge immediately below him rowing 8 miles an hour. The bridge is 20 feet above the boat. How rapidly are the boat and the pedestrian separating 5 minutes after the boat has passed under the bridge? *Ans.* 8.9 miles an hour.

28. If V is the volume of the solid generated by the revolution about the axis X of the plane figure bounded by a given curve $y=f(x)$, a given ordinate, and a second variable ordinate, prove that $D_xV=\pi y^2$.

29. Assuming that the pressure of a liquid on a submerged horizontal area is equal to the weight of a cylindrical column of the liquid having the area in question for its base and reaching to the surface of the liquid, and that a liquid presses equally in all directions; show that $D_xP=wxy$, if P is the pressure of a liquid on a submerged vertical plane surface bounded by a given horizontal ordinate at the distance x_0 below the surface, a variable horizontal ordinate at the distance x below the surface, the vertical axis of X, and a curve $y=f(x)$, and w is the weight of a cubic unit of the liquid.

30. Find the volume of a sphere, regarding it as generated by the revolution of a semicircle about its diameter.

Ans. $\frac{4}{3}\pi a^3$.

31. Find the volume of the solid generated by revolving an ellipse about its major axis. *Ans.* $\frac{4}{3}\pi ab^2$.

32. Find the number of cubic inches in a football whose long diameter is 16 inches and short diameter 8 inches.

Ans. $\frac{512\pi}{3}$ cu. in.

33. Find the volume of the solid generated by revolving the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ about the axis of X. *Ans.* $\frac{32\pi a^3}{105}$.

34. The area bounded by the parabola $y^2=4x$, the ordinate corresponding to $x=4$, and the ordinate corresponding to $x=16$ is revolved about the axis of X. Find the volume generated. *Ans.* 480π.

35. Find the pressure sustained by a rectangular floodgate 10 feet broad and 20 feet deep, the upper edge of which is at the surface of the water. Given that a cubic foot of water weighs about 62 pounds. *Ans.* 62 tons.

36. Find the pressure on the lower half of the gate.

Ans. $46\frac{1}{2}$ tons.

37. A vertical masonry dam in the form of a trapezoid is 200 feet long at the surface of the water, 150 feet long at the bottom, and is 60 feet high. What pressure must it withstand? *Ans.* 9300 tons.

38. A board in the shape of a parabola is immersed in water. Its vertex is in the surface and its axis is vertical. Find the pressure on one side of it down to a horizontal line x feet below the surface. *Ans.* $\frac{2}{5}w\sqrt{2mx^5}$.

39. A rectangular floodgate a feet deep and b feet broad, with its upper edge at the surface, is to be braced along a horizontal line. How far down must the brace be put that the gate may not tend to turn about it? *Ans.* $\frac{2}{3}a$ ft.

40. The subnormal of a curve is of constant length, m, and the curve passes through the origin. Find its equation.

Ans. $y^2 = 2\,mx$.

41. The subnormal of a curve is of constant length and is equal to 4. The curve passes through the point (1, 4). Find its equation.

Ans. $y^2 = 8\,x + 8$.

42. The subtangent at any point of a curve is equal to $\frac{1}{m}$ of the square of the ordinate of the point. Find the equation of the curve if it passes through the origin. *Ans.* $y^2 = 2\,mx$.

43. The tangent at any point of a curve is $y - y_0 = \frac{m}{y_0}(x - x_0)$, and the curve passes through the point $\left(\frac{m}{2}, m\right)$. Find its equation.

Ans. $y^2 = 2\,mx$.

44. The slope of the tangent at any point of a curve is equal to $\frac{1}{m}$ of the abscissa of the point. Find the equation of the curve.

Ans. $x^2 = 2\,my + C$.

45. A man is walking up a hemispherical hill at the rate of 2 miles an hour. What is his horizontal velocity in terms of his vertical distance above the base of the hill?

Ans. $v = \frac{2\,y}{a}$ if a is the radius of the hill.

46. The sun is just setting in the west as a horse is running on an elliptical race-track at the rate of m miles an hour. The major axis of the track lies north and south. How fast is the horse's shadow moving along a fence parallel to the axis of the track?

Ans. $v = \frac{may}{\sqrt{b^4 + (a^2 - b^2)\,y^2}}$.

47. A body moves with constant acceleration g. Given its initial velocity, show that if s is the distance it moves in t seconds, and v_0 is its initial velocity, $s = v_0 t + \frac{1}{2}\,gt^2$.

48. A rifle ball is fired through a three-inch plank, the resistance of which causes an unknown constant retardation of its velocity. Its velocity on entering the plank is 1000 feet a second, and on leaving the plank is 500 feet a second. How long does it take the ball to traverse the plank?

Ans. $\frac{1}{3000}$ sec.

49. When the brakes are put on a train, its velocity suffers a constant retardation. It is found that when a certain train is running 30 miles an hour the brakes will bring it to a dead stop in 2 minutes. If the train is to stop at a station, at what distance from the station should the engineer whistle "down brakes"? *Ans.* Half a mile.

CHAPTER IV.

TRANSCENDENTAL FUNCTIONS.

Find the derivatives with respect to x of the following functions: —

1. $e^x(x^3-3x^2+6x-6)$. *Ans.* $e^x x^3$.

2. $\dfrac{x-1}{x+1}e^x$. *Ans.* $\dfrac{x^2+1}{(x+1)^2}e^x$.

3. $\dfrac{a^x-a^{-x}}{a^x+a^{-x}}$. *Ans.* $\dfrac{4\log a}{(a^x+a^{-x})^2}$.

4. $(\sqrt{x})^{\sqrt{x}}$. *Ans.* $\dfrac{1+\log\sqrt{x}}{2\sqrt{x}}(\sqrt{x})^{\sqrt{x}}$.

5. $(\sin x)^x$. *Ans.* $(\log\sin x+x\operatorname{ctn}x)(\sin x)^x$.

6. $x^m\log x-\dfrac{x^m}{m}$. *Ans.* $mx^{m-1}\log x$.

7. $\log\dfrac{x}{\sqrt{1+x^2}}$. *Ans.* $\dfrac{1}{x(1+x^2)}$.

8. $\dfrac{1}{2\sqrt{2}}\log\dfrac{\sqrt{2+2x^2}-x}{\sqrt{2+2x^2}+x}+\log(x+\sqrt{1+x^2})$.

   Ans. $\dfrac{\sqrt{1+x^2}}{2+x^2}$.

9. $x-\sin x\cos x$. *Ans.* $2\sin^2 x$.

10. $\dfrac{\sin x}{a-b\cos x}$. *Ans.* $\dfrac{a\cos x-b}{(a-b\cos x)^2}$.

11. $x + \log \cos\left(\frac{\pi}{4} - x\right)$. *Ans.* $\frac{2}{1 + \tan x}$.

12. $2x \sin x + (2 - x^2) \cos x$. *Ans.* $x^2 \sin x$.

13. $\sin^{-1} \frac{1-x}{1+x}$. *Ans.* $\frac{-1}{(1+x)\sqrt{x}}$.

14. $\sin^{-1} \sqrt{\frac{x^2 - a^2}{b^2 - a^2}}$. *Ans.* $\frac{x}{\sqrt{(x^2 - a^2)(b^2 - x^2)}}$.

15. $\log \tan x - \frac{\operatorname{ctn} 2x}{\sin 2x}$. *Ans.* $\frac{4}{\sin^3 2x}$.

16. $\tan^{-1} \frac{x\sqrt{3}}{x+2}$. *Ans.* $\frac{\sqrt{3}}{2(x^2 + x + 1)}$.

17. $\tan^{-1} \frac{x}{a} + \log \sqrt{\frac{x-a}{x+a}}$. *Ans.* $\frac{2ax^2}{x^4 - a^4}$.

18. $\cos^{-1} \frac{b + a \cos x}{a + b \cos x}$. *Ans.* $\frac{\sqrt{a^2 - b^2}}{a + b \cos x}$.

19. $2 \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right)$. *Ans.* $\frac{\sqrt{a^2 - b^2}}{a + b \cos x}$.

20. $\log \sqrt{x + \sin x}$. *Ans.* $\frac{\cos^2 \frac{x}{2}}{x + \sin x}$.

21. Prove that the curves

$$y = \frac{\pi}{6} - \sin^{-1} \frac{x}{2} \quad \text{and} \quad y = \sqrt{3}\,(e^{x-1} - 1)$$

intersect at right angles at the point (1, 0).

22. Find the subtangent and subnormal to the curve $y = ce^{\frac{x}{a}}$. *Ans.* $t_x = a$; $n_x = \frac{y^2}{a}$.

23. Find the equation of the tangent to the curve $y = a \log x$ at the point (1, 0). *Ans.* $y = a(x - 1)$.

24. Find the equation of the tangent to the curve

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2,$$

and show that at the point (a, b) it is the same, whatever the value of n.

25. Find $\underset{x \doteq 1}{\text{limit}} \left[\dfrac{x\sqrt{3x - 2x^4} - x^{\frac{6}{5}}}{1 - x^{\frac{2}{3}}}\right]$. *Ans.* 4.05.

26. Find $\underset{x \doteq 0}{\text{limit}} \left[\dfrac{x \log(1+x)}{1 - \cos x}\right]$. *Ans.* 2.

27. Find $\underset{x \doteq 0}{\text{limit}} \left[\dfrac{e^x - 1}{x}\right]$. *Ans.* 1.

28. Find $\underset{x \doteq a}{\text{limit}} \left[\dfrac{a^n - x^n}{\log a - \log x}\right]$. *Ans.* na^n.

29. Find $\underset{x \doteq 0}{\text{limit}} \left[\dfrac{xe^{2x} - e^{-2x} - x + 1}{e^{2x} - 1}\right]$. *Ans.* 1.

30. Find $\underset{x \doteq \frac{\pi}{2}}{\text{limit}} \left[\dfrac{a^{\sin x} - a}{\log \sin x}\right]$. *Ans.* $a \log a$.

31. Find $\underset{x \doteq 0}{\text{limit}} \left[\dfrac{\sin x - x\cos x}{x - \sin x}\right]$. *Ans.* 2.

32. Find $\underset{x \doteq 0}{\text{limit}} \left[\dfrac{\tan x - \sin x}{x^3}\right]$. *Ans.* $\frac{1}{2}$.

33. Find $\underset{x \doteq \frac{\pi}{2}}{\text{limit}} \left[\dfrac{x \sin x - \frac{\pi}{2}}{\cos x}\right]$. *Ans.* -1.

34. The illumination of a plane surface by a luminous point being directly as the cosine of the angle of incidence of the rays and inversely as the square of the distance of the luminous point from the surface, find the height at which a gas-burner must be placed to best illuminate a point on the floor at the distance a feet from the wall. *Ans.* $\frac{1}{2} a \sqrt{2}$.

35. A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of copper. If the width of the strip is a, find the radius of the cross-section when the carrying capacity of the gutter is a maximum.

Ans. $\frac{a}{\pi}$.

36. How far down must the helm be put that the boat may come about as quickly as possible?

Ans. The tiller must make with the keel of the boat an angle whose tangent is $\sqrt{2}$.

37. A can-buoy in the form of a double cone is to be made from two equal circular iron plates. If the radius of each plate is a, find the radius of the base of the cone when the buoy is as large as possible. *Ans.* $a\sqrt{\frac{2}{3}}$.

38. At what point on the line joining the centres of two spheres must a light be placed to illuminate the largest possible amount of spherical surface?

Ans. The square of the ratio of the segments of the line must be equal to the cube of the ratio of the radii.

39. A wall 27 feet high is 64 feet from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside of the wall. *Ans.* 125 ft.

40. The captain of a man-of-war saw, one dark night, a privateersman crossing his path at right angles and at a distance ahead of c miles. The privateersman was making a miles an hour, while the man-of-war could make only b miles in the same time. The captain's only hope was to cross the track of the privateersman at as short a distance as possible under his stern, and to disable him by one or two well directed shots; so the ship's lights were put out and her course altered so as to effect this. Show that the man-of-war crossed the privateersman's track $\frac{c}{b}\sqrt{a^2-b^2}$ miles astern of the latter.

41. The formula for the *range* of a projectile when the resistance of the air is neglected is $range = \frac{2V^2}{g} \sin\phi \cos\phi$; where V is the initial velocity, g the force of gravity, and ϕ the angle which the initial direction of the projectile makes with the horizontal. Find the angle of elevation at which a cannon must be discharged in order that the ball may go as far as possible, the resistance of the air being neglected.

Ans. 45°.

42. The formula for the range of a projectile on an inclined plane is $R = \frac{2V^2}{g\cos^2\beta} \cos a \sin(a - \beta)$, where V is the velocity of projection in feet, a the inclination of the gun to the horizontal, and β the inclination of the plane to the horizontal. A hillside makes an angle of 30° with the horizontal. Find the direction in which a rifle ball must be fired to reach the highest point possible on the hillside. Show that if the muzzle-velocity of the ball is 500 feet a second the maximum range is nearly a mile.

CHAPTER V.

INTEGRATION.

Find the value of the following integrals: —

1. $\int_x (3\sqrt{x} - 5\sqrt{x^3})$. *Ans.* $2(\sqrt{x^3} - \sqrt{x^5})$.

2. $\int_x \frac{1}{\sqrt{x}}$. *Ans.* $2\sqrt{x}$.

3. $\int_x \left(\frac{1}{x^2} + \frac{2}{x^3}\right)$. *Ans.* $-\frac{1}{x} - \frac{1}{x^2}$.

4. $\int_x (1 + 3x)^3$. *Ans.* $\frac{1}{12}(1 + 3x)^4$.

5. $\int_x \sqrt{(a - bx)^3}$. *Ans.* $-\frac{2}{5b}\sqrt{(a - bx)^5}$.

6. $\int_x \frac{x^{n-1}}{a + bx^n}$. *Ans.* $\frac{1}{nb}\log(a + bx^n)$.

Suggestion. When the expression to be integrated can be broken up into two factors the first of which is equal to any constant multiplied by the derivative of the second or by the derivative of a quantity of which the second is a simple function, it is usually worth while to try to simplify by substituting y for this quantity. Thus in Example 6 let $y = a + bx^n$.

7. $\int_x \frac{x^2}{a^3 + x^3}$. *Ans.* $\frac{1}{3}\log(a^3 + x^3)$.

8. $\int_x \frac{x^2}{\sqrt{a^3 + x^3}}$. *Ans.* $\frac{2}{3}\sqrt{a^3 + x^3}$.

9. $\int_x \frac{(\log x)^2}{x}$. *Ans.* $\frac{1}{3}(\log x)^3$.

10. $\int_x \sin x \cos x$. *Ans.* $\frac{1}{2}\sin^2 x$.

11. $\int_x x^2 \sin(x^3)$. *Ans.* $-\frac{1}{3}\cos(x^3)$.

12. $\int_x \frac{\sin x}{a + b\cos x}$. *Ans.* $-\frac{1}{b}\log(a + b\cos x)$.

13. $\int_x e^{2\sin x}\cos x$. *Ans.* $\frac{1}{2}e^{2\sin x}$.

14. $\int_x \frac{x}{1+x^4}$. Let $y = x^2$. *Ans.* $\frac{1}{2}\tan^{-1}(x^2)$.

15. $\int_x \frac{1}{x^2+4x+5}$. *Ans.* $\tan^{-1}(x+2)$.

Suggestion. When the denominator contains a mixed quadratic in x it often pays to complete the square by adding and subtracting a suitable constant, and then to let y equal the quantity whose square is then involved. Thus

$$\frac{1}{x^2+4x+5} = \frac{1}{(x+2)^2+1}.$$

Let $y = x + 2$.

16. $\int_x \frac{1}{2x^2 - 2x + 1} = \frac{1}{2}\int_x \frac{1}{x^2 - x + \frac{1}{4} + \frac{1}{4}}$.

Ans. $\tan^{-1}(2x - 1)$.

17. $\int_x \frac{1}{\sqrt{3x - x^2 - 2}}$. *Ans.* $\sin^{-1}(2x - 3)$.

18. $\int_x \frac{1}{x^2 + 4x}$. *Ans.* $\frac{1}{4}\log\frac{x}{x+4}$.

19. $\int_x \sin^3 x$. *Ans.* $\frac{1}{3}\cos^3 x - \cos x$.

Suggestion. $\sin^3 x = (1 - \cos^2 x) \sin x$. Let $y = \cos x$. *v.* Example 6.

20. $\int_x \cos^5 x$. *Ans.* $\frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x$.

21. $\int_x \sin^3 x \cos^3 x$. *Ans.* $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x$.

22. $\int_x \frac{\sin^3 x}{\cos^2 x}$. *Ans.* $\sec x + \cos x$.

23. $\int_x xe^x$. *Ans.* $(x-1)e^x$.

Suggestion. Integrate by parts.

24. $\int_x xe^{ax}$. *Ans.* $\left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$.

25. $\int_x x\cos x$. *Ans.* $x\sin x + \cos x$.

Definite Integrals.

26. Show that the area bounded by the curve $y = f(x)$, the axis of X, and the ordinates corresponding to $x = x_0$ and $x = x_1$ is

$$A = \left[\int_x f(x)\right]_{x=x_1} - \left[\int_x f(x)\right]_{x=x_0}.$$

27. If the curve $y = f(x)$ lies on the positive side of the axis of X and y has neither a maximum nor a minimum value between the points (x_0, y_0) and (x_1, y_1) show that if a set of $n-1$ equidistant ordinates are erected between y_0 and y_1 and with these as altitudes a set of inscribed rectangles and a set of circumscribed rectangles are constructed, the areas of the two sets will differ by the difference between the last rectangle of one set and the first rectangle of the other set; that is, by the absolute value of $(y_1 - y_0)\Delta x$ if Δx is $\frac{x_1 - x_0}{n}$. Hence, show that the area of either set will approach the

area of the curve as its limit as n is indefinitely increased. To formulate,

$$A = \underset{\Delta x \doteq 0}{\text{limit}} \Big[f(x_0)\,\Delta x + f(x_0 + \Delta x)\,\Delta x + f(x_0 + 2\,\Delta x)\,\Delta x$$

$$+ f(x_0 + 3\,\Delta x)\,\Delta x + \cdots\cdots + f(x_1 - \Delta x)\,\Delta x \Big]$$

$$= \underset{\Delta x \doteq 0}{\text{limit}} \sum_{x=x_0}^{x=x_1} f(x)\,\Delta x = \Big[\int_x f(x) \Big]_{x=x_1} - \Big[\int_x f(x) \Big]_{x=x_0}$$

28. Show that the formulas of Example 27 hold good even when y passes through maximum or minimum values between (x_0, y_0) and (x_1, y_1).

29. Write out and compute the value of $\sum\limits_{x=1}^{x=5} x\,\Delta x$ when $\Delta x = 1$; when $\Delta x = 0.5$; when $\Delta x = 0.1$; when $\Delta x = 0.01$; and show that the first differs from $\underset{\Delta x \doteq 0}{\text{limit}} \sum\limits_{x=1}^{x=5} x\,\Delta x$ by less than 4, the second by less than 2, the third by less than 0.4, and the last by less than 0.04. What is the value of $\underset{\Delta x \doteq 0}{\text{limit}} \sum\limits_{x=1}^{x=5} x\,\Delta x$? *Ans.* 12.

30. Write out and compute the value of $\sum\limits_{x=0}^{x=1} \dfrac{\Delta x}{1 + x^2}$ for $\Delta x = 0.2$; for $\Delta x = 0.1$; for $\Delta x = 0.05$. What is the value of $\underset{\Delta x \doteq 0}{\text{limit}} \sum\limits_{x=0}^{x=1} \dfrac{\Delta x}{1 + x^2}$? *Ans.* $\frac{1}{4}\pi$.

31. Write out and compute the value of $\sum\limits_{x=0}^{x=\frac{1}{2}} \dfrac{\Delta x}{\sqrt{1 - x^2}}$ for $\Delta x = 0.1$; for $\Delta x = 0.025$. What is the value of

$$\underset{\Delta x \doteq 0}{\text{limit}} \sum_{x=0}^{x=\frac{1}{2}} \frac{\Delta x}{\sqrt{1 - x^2}}?$$

Ans. $\frac{1}{6}\pi$.

32. $\underset{\Delta x \doteq 0}{\text{limit}} \sum_{x=x_0}^{x=x_1} f(x)\,\Delta x$ is written $\int_{x_0}^{x_1} f(x)\,dx$ and is called the definite integral of $f(x)\,dx$ from x_0 to x_1. It can always be represented graphically by the area bounded by the curve $y=f(x)$, the axis of X, and the ordinates y_0 and y_1. Hence

$$\int_{x_0}^{x_1} f(x)\,dx = \left[\int_x f(x)\right]_{x=x_1} - \left[\int_x f(x)\right]_{x=x_0}.$$

33. Show that the volume generated by revolving about the axis of X, the plane area bounded by the curve $y=f(x)$, the axis of X, and the ordinates y_0 and y_1 is $V=\int_{x_0}^{x_1} \pi y^2 dx$.

34. The area included between the curves $y^2=4\,ax$ and $x^2=4\,ay$ is revolved about the axis of X; find the volume generated. *Ans.* $\frac{96}{5}\pi a^3$.

35. Find the volume of the ring generated by revolving the circle $x^2+(y-b)^2=a^2$ about the axis of X. *Ans.* $2\,\pi^2 a^2 b$.

36. A woodcutter starts to fell a tree 4 feet in diameter, and cuts half way through. One face of the cut is horizontal, and the other face is inclined to the horizontal at an angle of 45°. Find the volume of wood cut out. *Ans.* $5\frac{1}{3}$ cu. ft.

37. A Rugby foot-ball is 16 inches long, and a plane section containing a seam of the cover is an ellipse 8 inches broad. Find (a) the volume of the ball assuming that the leather is so stiff that every plane cross-section is a square; (b) the volume of the ball on the assumption that the leather is so soft that every plane cross-section is a circle.

Ans. (a) $341\frac{1}{3}$ cu. in.; (b) $\frac{512\,\pi}{3}$ cu. in.

38. Find the volume of an ellipsoid. *Ans.* $\frac{4}{3}\ \pi abc.$

39. Obtain a formula for the pressure of a liquid on a vertical plane surface bounded by the curve $y = f(x)$, the axis of X, and the ordinates y_0 and y_1. Given that the axis of X is vertical, the origin at the distance a below the surface of the liquid, and that the weight of a cubic unit of the liquid is w.

Ans. $P = w \int_{x_0}^{x_1} (a + x)\, y dx.$

40. One end of an unfinished water-main 2 feet in diameter is closed by a temporary bulkhead and the water is let in from the reservoir. Find the pressure on the bulkhead if its centre is 40 feet below the surface of the water in the reservoir. *Ans.* $2480\,\pi$ lbs.

41. A vase has two opposite faces which are equal ellipses with their major axes vertical. Find the whole pressure on one of the faces when the vase is full of water. Call the semi-axes of the ellipse a and b, and the weight of a cubic inch of water w. *Ans.* $\pi a^2 bw.$

42. Show that the abscissa of the centre of gravity of the plane area bounded by two curves $y = f(x)$ and $y' = F(x)$, and the ordinates corresponding to x_0 and x_1 is

$$\bar{x} = \frac{\int_{x_0}^{x_1} (y - y')\, x dx}{\int_{x_0}^{x_1} (y - y')\, dx} = \frac{\int_{x_0}^{x_1} (y - y')\, x dx}{A}.$$

43. Find the centre of gravity of a semicircle.

Ans. $\bar{x} = \frac{4\,a}{3\,\pi}.$

44. Find the centre of gravity of a half-ellipse.

Ans. $\bar{x} = \frac{4a}{3\pi}$.

45. Find the centre of gravity of a parabolic segment.

Ans. $\bar{x} = \frac{3}{5} x_1$.

46. Find the centre of gravity of a triangle.

Ans. The intersection of the medians.

47. Find the centre of gravity of a hemisphere.

Ans. $\bar{x} = \frac{3}{8} a$.

48. Find the centre of gravity of a half-spheroid.

Ans. $\bar{x} = \frac{3}{8} a$.

49. Find the centre of gravity of a cone of revolution.

Ans. $\bar{x} = \frac{1}{4} h$.

50. Show that the abscissa of the centre of fluid-pressure on the plane area described in Example 39 is

$$\bar{\bar{x}} = \frac{\int_{x_0}^{x_1} (a+x) xy dx}{\int_{x_0}^{x_1} (a+x)\, y dx}.$$

51. Find the depth of the centre of pressure in the dam described in Example 37, Chapter III. *Ans.* 39 ft.

52. Find the centre of pressure on an immersed circle whose centre is b below the surface of the water.

Ans. $\bar{\bar{x}} = \frac{a^2}{4b}$.

53. One end of a cylindrical aqueduct 6 feet in diameter which is half full of water is closed by a water-tight bulkhead held in place by a brace. How far below the centre of the

bulkhead should the brace be put? What pressure must it be able to withstand? *Ans.* $\bar{\bar{x}} = \frac{9}{16}\pi$ ft.; $P = 1116$ lbs.

54. Show that in finding the limit of a sum of terms each of which approaches zero, while at the same time the number of terms increases in such a way that the limit in question is finite, any term may be replaced by a term taken at pleasure, provided that the limit of the ratio of the replacing term to the term replaced is unity. *v.* Dif. Cal., Art. 161.

55. Show by the aid of Example 54 that the formulas obtained in Examples 33, 39, 42, and 50 are rigorously accurate.

56. Show that in finding the limit of the ratio of two quantities which are approaching zero together either may be replaced by any other, provided that the limit of the ratio of the replacing quantity to the one replaced is unity. *v.* Dif. Cal., Art. 160.

57. Show that the limit of the ratio of the length of an arc to the length of its chord is unity. *v.* Dif. Cal., Art. 165.

58. Show that the length of the arc of the curve $y = f(x)$ from x_0 to x_1 is $s = \int_{x_0}^{x_1} \sqrt{1 + (D_x y)^2} \,.\, dx.$

59. Show that the area of the surface generated by revolving the portion of the curve $y = f(x)$ between x_0 and x_1 about the axis of X is $S = 2\pi \int_{x_0}^{x_1} y\sqrt{1 + (D_x y)^2} \,.\, dx.$

60. Find the surface of a sphere; of a paraboloid of revolution. *Ans.* $4\pi a^2$; $\frac{2\pi}{3m}\left[m^3 - (m^2 + 2mx_1)^{\frac{3}{2}}\right]$.

61. Show that the abscissa of the centre of gravity of an arc is

$$\bar{x} = \frac{1}{s}\int_{x_0}^{x_1} x\sqrt{1+(D_x y)^2}\,.\,dx.$$

62. Find the centre of gravity of a semi-circumference.

Ans. $\bar{x} = \dfrac{2a}{\pi}$.

63. Show that the abscissa of the centre of gravity of a surface of revolution is $\bar{x} = \dfrac{2\pi}{S}\displaystyle\int_{x_0}^{x_1} xy\sqrt{1+(D_x y)^2}\,.\,dx.$

64. Find the centre of gravity of the surface of a hemisphere; of a zone. *Ans.* $\bar{x} = \dfrac{a}{2}$; $\bar{x} = \dfrac{x_0 + x_1}{2}$.

65. Write out the sum of terms whose limit is

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}.$$

Compute its approximate value for $\Delta x = 0.1$. About how accurate is this approximation? Compute $\displaystyle\int_0^1 \sqrt{1-x^4}\,.\,dx$ to the nearest tenth.

CHAPTER VI.

CURVATURE.

1. Find the evolute of $2xy = a^2$.

Ans. $(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

2. Find the evolute of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

3. Find the evolute of

$$x = a(\cos\theta + \theta\sin\theta)$$
$$y = a(\sin\theta - \theta\cos\theta).$$

Ans. $x^2 + y^2 = a^2$.

4. Find the evolute of

$$x = a\cos^3\theta$$
$$y = a\sin^3\theta.$$

Ans. $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

CHAPTER VII.

THE CYCLOID.

1. Obtain the equations of the *involute of the circle* by regarding it as generated by unwinding a string from the circle.

Ans. $x = a(\cos\theta + \theta\sin\theta)$
$y = a(\sin\theta - \theta\cos\theta)$.

2. If a circle rolls on a straight line in its plane, the curve traced by a point of the circle not in its circumference is called a trochoid. Find the equations of a trochoid in terms of a the radius of the circle, b the distance of the tracing point from the centre of the circle, and θ the angle through which the circle has rotated since the tracing point was in its lowest position.

Ans. $x = a\theta - b\sin\theta$
$y = a - b\cos\theta$.

3. Show that the normal to the trochoid always passes through the lowest point of the rolling circle.

4. Obtain the polar equation of the cardioid from its definition by the aid of Elementary Geometry.

5. Prove geometrically that the hypocycloid is a straight line when the radius of the rolling circle is one-half the radius of the fixed circle.

CHAPTER VIII.

PROBLEMS IN MECHANICS.

If a force acts on a particle in the direction of its motion the mass of the particle multiplied by the acceleration of its velocity is proportional to the force.

The force which acting on a mass of one pound produces an acceleration of one foot per second per second is taken as the unit of force, and is called a *poundal.*

The force exerted by gravity on a mass of M pounds is Mg poundals.

The force exerted by gravity on a pound of mass, that is a force of g poundals, is sometimes spoken of as a *pound of force*, and a poundal is approximately equal to the force exerted by gravity on a mass of half an ounce.

1. A 12-pound weight rests on a smooth, horizontal table; a string fastened to it passes over a smooth pulley in the edge of the table, and is attached to a 4-pound weight which hangs down. Find the motion of the system. What is the tension of the string when the system is moving?

Ans. The system moves with a constant acceleration of $\frac{1}{4}g$ feet per sec. per sec. The tension is equal to $3g$ poundals, *i.e.* 3 lbs.

2. Solve Example 1 if the weights both hang vertically from a smooth pulley.

Ans. Acceleration is equal to $\frac{1}{2}g$. Tension is equal to $6g$ poundals, or 6 lbs.

3. Solve Examples 1 and 2 if the masses are M and m pounds respectively.

$$\textit{Ans.}\quad (1)\ \text{Accel.} = \frac{mg}{M+m};\quad \text{Ten.} = \frac{mM}{M+m}\ \text{lbs.}$$

$$(2)\ \text{Accel.} = \frac{M-m}{M+m}g;\quad \text{Ten.} = \frac{2\,mM}{M+m}\ \text{lbs.}$$

4. A pail of water containing an 8-pound stone (sp. gr. 4), and weighing in all 30 pounds, is fastened to a rope which passes over a smooth pulley and is then attached to a 50-pound weight. Find the pressure of the stone on the bottom of the pail when the system is in motion.

What is the pressure if the weight is only 18 pounds instead of 50? *Ans.* $7\frac{1}{2}$ lbs.; $4\frac{1}{2}$ lbs.

5. Suppose the stone in Example 4 replaced by a 4-pound cork-float (sp. gr. $\frac{1}{2}$), and 4 pounds more of water poured into the pail. If the cork is held under water by a string fastened to the bottom of the pail, find the tension of the string in the two cases considered in Example 4.

Ans. Ten. $=5$ lbs.; Ten. $=3$ lbs.

6. A railway train running at the rate of 30 miles an hour strikes a snowdrift and is brought to a standstill after going 200 yards. Assuming that the drift offers a constant resistance to the passage of the train, find how long the train keeps in motion. *Ans.* $27\frac{3}{11}$ sec.

7. A bicycle rider finds that the brake will barely stop his machine on a hill which has an inclination of 15°. Riding one day on a level road at the rate of 10 miles an hour he suddenly discovers an obstruction 15 feet ahead and puts the brake on hard. Will he stop in time? *Ans.* Yes.

8. From an express-train which weighs 100 tons, going at full speed, the rear car, which weighs 20 tons, is slipped at a

distance of half a mile from a way-station; prove that if the car comes to rest just at the station the rest of the train will then be 3300 feet beyond the station, provided that the pull exerted by the engine remains constant.

9. A vessel of mass 200,000 is moving with a velocity of 1 foot per second when its way is suddenly checked by a rope, one end of which is attached to the vessel and the other to a post on a pier behind the vessel. Under a tension of p pounds the rope is stretched $\frac{2p}{2000+p}$ feet beyond its ordinary length. How far does the vessel move after the rope is taut? Show that unless the rope can stand a strain of about 7500 pounds it will break.

Ans. The vessel goes 1.58 ft.

10. Show that $D_t^2 x = D_t v = v D_x v$, and hence that $mvD_x v = f$ is a form of the fundamental equation for rectilinear motion.

11. The original pneumatic gun used by Lieutenant Zalinski for throwing dynamite shells consisted of a tube 50 feet long and 4 inches in diameter connected with a reservoir containing compressed air giving a pressure of 500 pounds to the square inch. The volume of the reservoir was 12 times that of the tube, and the weight of the projectile was 80 pounds. Find the extreme range, neglecting the resistance of the air, and assuming that the pressure of a gas is inversely as its volume.

Given: Range $= \frac{V^2}{g} \cdot \quad g = 32.$ *Ans.* About 2500 yds.

12. Prove that a body on the surface of the moon must be thrown up towards the earth with an initial velocity of about 2275 metres per second in order that it may just reach the point between the moon and the earth where the attraction of both of these heavenly bodies is the same.

Data: —

Mass of moon $= \frac{1}{81} \times$ [mass of earth (m)]
Radius of moon $= \frac{3}{11} \times$ [radius of earth (r)]
Distance between centres of moon and earth $= 60\,r$.
$r = \dfrac{20{,}000{,}000}{\pi}$. $g = 9.809$ metres per sec. per sec.

13. A body of mass m, with an initial velocity v_0, moves in a resisting medium which resists with a force proportional to the velocity and equal to k poundals for a velocity of 1 foot a second. Show that

$$v = v_0 - \frac{k}{m}x = v_0 e^{-\frac{kt}{m}}; \; t = \frac{m}{k}\log\frac{v_0}{v_0 - \frac{k}{m}x} = \frac{m}{k}\log\frac{v_0}{v}.$$

14. A ship whose mass is 200,000 pounds is moving with a velocity of 2 feet a second, and is resisted by the water with a force proportional to the velocity and equal to 200 poundals when the velocity is 1 foot a second. How far will the ship go before its velocity is reduced to 1 foot a second? How long before this reduction of velocity is accomplished?

Ans. 1000 ft.; 11 min. 33 sec.

15. A body of mass m, with an initial velocity v_0, moves in a medium which resists with a force proportional to the square of the velocity and equal to k poundals for a velocity of 1 foot a second. Show that

$$x = \frac{m}{k}\log\frac{v_0}{v}; \; t = \frac{m}{kv_0}\left(e^{\frac{kx}{m}} - 1\right) = \frac{m}{k}\left(\frac{1}{v} - \frac{1}{v_0}\right).$$

16. The rear car of a train moving at the rate of 30 miles an hour on a straight level track, breaks loose 2000 feet short of a certain station at which the train makes a stop. The car weighs 40,000 pounds, and the air resists its motion with a force proportional to the square of the velocity and equal to 10 poundals when the velocity is 1 foot per second. Show that the car will reach the station with a velocity of

about 27 feet a second, and that if the train stops at the station more than 14 seconds there will be a collision.

17. A body acted on by a constant force g moves in a medium which resists with a force proportional to the square of the velocity, and which is equal to k poundals for a velocity of 1 foot per second. Show that

$$x = \frac{m}{2k} \log \frac{mg - kv_0^2}{mg - kv^2};$$

$$2t = \sqrt{\frac{m}{kg}} \log \left[\frac{\sqrt{\frac{mg}{k}} + v}{\sqrt{\frac{mg}{k}} - v} \cdot \frac{\sqrt{\frac{mg}{k}} - v_0}{\sqrt{\frac{mg}{k}} + v_0} \right] \text{ if } g > 0;$$

and $t = \sqrt{\frac{m}{-kg}} \left[\tan^{-1} v_0 \sqrt{\frac{k}{-mg}} - \tan^{-1} v \sqrt{\frac{k}{-mg}} \right]$ if $g < 0$.

18. A man and a parachute weigh 150 pounds. How large must the parachute be that the man may trust himself to it at any height if 25 feet a second is a safe velocity with which to reach the ground? Given that the resistance of the air is as the square of the velocity and is equal to 2 pounds per square foot of opposing surface for a velocity of 30 feet a second. *Ans.* About 12 ft. in diameter.

19. A particle is projected vertically upward in a medium in which the resistance is equal to kv^2; if V be the velocity of projection and V_1 the particle's velocity when it again reaches the starting point show that $V_1^2 = \frac{mgV^2}{mg + kV^2}$.

20. A toboggan slide of constant slope is a quarter of a mile long and has a fall of 200 feet. Assuming that the coefficient of friction is $\frac{3}{100}$, that the resistance of the air is proportional to the square of the velocity and is equal to 2

pounds per square foot of opposing surface for a velocity of 30 feet a second, that a loaded toboggan weighs 300 pounds and presents a surface of 3 square feet to the resistance of the air; find the velocity acquired during the descent and the time required to reach the bottom.

Find the limit of the velocity that could be acquired by a toboggan under the given conditions, supposing the hill of infinite length.

Ans. (*a*) 68 ft. a second; (*b*) 30 seconds; (*c*) 74 ft. a second.

In problems on elastic strings assume that the force exerted by a stretched string is proportional to the difference between its stretched length and its natural length.

21. One end of an elastic string of inconsiderable mass and of natural length a is fastened at a point on the surface of a smooth table. A particle of mass m is attached to the other end of the string and is drawn back till the string is stretched by the amount b, and is then released. Find the time of a complete oscillation if a force of P pounds would stretch the string to double its original length.

Ans. $T = 4\sqrt{\dfrac{ma}{Pg}}\left(\dfrac{a}{b} + \dfrac{\pi}{2}\right)$.

22. If in Example 21 the mass of the particle is 2 pounds, the natural length of the string is 2 feet, the amplitude of an oscillation is 6 feet, and the observed time of an oscillation is 2 seconds, find what force would be required to double the length of the string. *Ans.* 6.37 pounds.

23. Can the apparatus described in Example 22 be made to beat half-seconds? What would be the amplitude of the required oscillation? *Ans.* About 24 feet.

24. A heavy particle is attached by an extensible string to a fixed point, from which the particle is allowed to fall freely; when the particle is in its lowest position the string is of

twice its natural length. Prove that the weight which would stretch the string to twice its natural length is four times the weight of the particle, and show that the time during which the string is stretched is $2\sqrt{\frac{a}{g}}\tan^{-1}\sqrt{2}$ seconds.

25. A cylindrical spar buoy (specific gravity $\frac{1}{2}$) is anchored so that it is just submerged at high water. If the cable should break at high tide show that the spar would jump entirely out of the water.

26. A number of iron weights are attached to one end of a long, round, wooden spar so that when left to itself the spar floats vertically in water. A ten-kilogram weight having become accidentally detached, the spar is seen to oscillate with a period of 4 seconds. The radius of the spar is 10 centimetres; find the sum of the weights of the spar and attached iron. Through what distance does the spar oscillate? *Ans.* (*a*) About 125 kilograms; (*b*) 0.64 metres.

27. If the point of application of a constant force of f pounds moves a distance of x feet in the direction of the force, the force is said to do $f.x$ *foot-pounds of work*. If the force is variable the work done is equal to $\int_0^x f.dx$ *foot-pounds.*

A ten-pound meteor falls from a great distance to the earth. Find the work done on it by the earth's attraction.

Ans. About 211,200,000 foot-pounds.

28. The effectiveness of an engine depends upon the amount of useful work it can do per minute, and is expressed in terms of the *Horse-Power*. An engine that can do 33,000 foot-pounds of work in a minute is an engine of one Horse-Power.

A certain marine engine has a 6-foot cylinder; the shaft makes 30 revolutions a minute; the average steam pressure is 22 pounds per square inch; the area of the piston is 1800 square inches. How much work does the steam do during one stroke of the piston? During one revolution of the shaft? How much work per minute is done by the steam? If one-fourth of the work is wasted in overcoming the internal resistances of the engine, what is its effective Horse-Power?

Ans. (*a*) 237,600 foot-pounds; (*b*) 475,200 foot-pounds; (*c*) 14,256,000 foot-pounds; (*d*) 324 H. P.

29. The diameter of the piston of an engine is 80 inches; the length of stroke is 10 feet; the piston makes 11 strokes a minute; and the mean pressure of the steam is 25 pounds per square inch. One-fourth the work done by the steam is wasted. What is the effective Horse-Power of the engine?

Ans. 314.16 H. P.

30. The cylinder of a steam engine has an internal diameter of 3 feet; the length of stroke is 11 feet; the piston makes 6 strokes a minute. One-fourth of the work done by the steam is wasted. What must be the average steam pressure that the effective Horse-Power of the engine may be 150? *Ans.* About 98 pounds per square inch.

31. What work is expended in drawing a train weighing 100 tons a distance of a mile on a level track if the retarding forces of friction and the resistance of the air amount to 8 pounds per ton? *Ans.* About 4,224,000 foot-pounds.

32. A train weighing 100 tons is dragged a mile up an incline where the rise is one foot in 50. The resistance of friction is 9 pounds per ton. How much work is done?

Ans. 25,872,000 foot-pounds.

33. Find the Horse-Power of a locomotive which can pull a train weighing 100 tons on an up-grade of 1 in 50 with a velocity of 30 miles an hour if the resistance of friction amounts to 8 pounds a ton. *Ans.* 384 H. P.

34. A train weighing 100 tons is drawn by an engine of 150 Horse-Power. The resistances amount to 11 pounds a ton. Find the maximum speed of which the train is capable on a level track. *Ans.* About 51 miles an hour.

35. A wheelman rides at the rate of 10 miles an hour against a ten-mile breeze. What Horse-Power does he exert, if the surface he offers to the wind is 4 square feet and the law of the resistance of the wind is that stated in Example 18? *Ans.* About $\frac{1}{5}$ H. P.

36. A fan with a rectangular blade a feet long and b feet broad rotates with uniform angular velocity of ω radians per second, and is resisted by the air with a force proportional to the square of the velocity and equal to k pounds per foot of opposing surface for a velocity of one foot a second. Find the work done by the fan in a second. *Ans.* $\frac{1}{4}kab^4\omega^3$ foot-pounds.

37. A four-bladed fan each blade of which is 6 feet long and one foot broad makes 120 revolutions a minute. Required the Horse-Power of the electric motor which drives it. Given that the resistance of the air is equal to a force of 2 pounds per square foot of opposing surface for a velocity of 30 feet a second. *Ans.* 0.048 H. P.

38. If the fan in Example 37 is driven by a weight of 20 pounds fastened to a string which is wound round a drum attached to the axis of the fan, what is the diameter of the drum? *Ans.* $2\frac{1}{2}$ inches.

39. Show that if a mass of m pounds moves from rest and attains a velocity of v feet a second under the action of a constant force the work done by the force is $\frac{1}{2}mv^2$ foot-poundals.

40. Show that the result of Example 39 is true even when the force is variable.

$\frac{1}{2}mv^2$ is called the *energy* of the moving particle and is g times the work in foot-pounds done on the particle in giving it its velocity. It is, moreover, the work in foot-poundals that the particle can do against resistance before it is reduced to rest.

41. A particle of mass m moves with uniform angular velocity ω in a circle of radius a. What is its energy?

Ans. $ma^2\dfrac{\omega^2}{2}$.

42. A set of particles rigidly connected with one another are revolving with angular velocity ω about an axis. Show that the energy of the system is $\dfrac{\omega^2}{2}\left(\sum mr^2\right)$.

43. A cylinder of radius a and length b is rotating with angular velocity ω about its axis. Assuming that it is homogeneous and that the mass of a cubic foot is ρ pounds; find its energy.

Ans. $\frac{1}{2}Ma^2\dfrac{\omega^2}{2}$, where M is the mass of the cylinder.

44. A homogeneous sphere of mass M is rotating with angular velocity ω about a diameter. Find its energy.

Ans. $\frac{2}{5}Ma^2\dfrac{\omega^2}{2}$.

45. A fly-wheel is making 30 revolutions a minute. The inner diameter of the rim is 18 feet, the outer diameter 20 feet, and the rim is 6 inches broad. Assuming that the wheel is made of iron which weighs 400 pounds to the cubic foot and that the mass of the wheel is mainly in its rim, show that the wheel can do about 5,000,000 foot-poundals of work if it is disconnected with the engine and allowed to run till it stops.

CHAPTER IX.

DEVELOPMENT IN SERIES.

The student will find it convenient to know or temporarily to take for granted the facts briefly stated in the following Syllabus on the Theory of Series. For the best treatment of the subject in the English language he is referred to Chrystal's Algebra.

SYLLABUS.

Series whose terms are constants.

1. Let $u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ be a series of constant terms.

If the sum of the terms up to and including u_n approaches a limit as n increases indefinitely the series is *convergent*, and this limit is called the *sum of the series* and is taken as the value of the series.

For example: the Geometrical series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is convergent and its sum is 2.

If the sum of the terms up to and including u_n increases indefinitely as n is increased the series is *divergent.*

For example: $1 + 2 + 3 + 4 + \cdots$ is divergent.

If the sum of the terms up to and including u_n neither approaches a limit nor increases indefinitely as n increases indefinitely the series is *oscillating.*

For example: $\frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 + \cdots$ is oscillating. Obviously a series cannot oscillate if its terms are all of the same sign.

Oscillating and divergent series are to be avoided in mathematical work.

2. If a series of positive terms is convergent, or if a series containing negative terms would be convergent if all its terms were positive it is said to be *absolutely convergent.*

A series which is convergent but not absolutely convergent is said to be *semi-convergent.*

An absolutely convergent series may be broken up into parts in any way, or the order of its terms may be changed at pleasure, without affecting the sum of the series. Hence any absolutely convergent series is convergent even if some of its terms are negative.

3. If the terms of an absolutely convergent series are multiplied by any quantity, or by any set of quantities no one of which exceeds in absolute value some given quantity, the resulting series will be absolutely convergent.

Tests of convergence.

4. If the terms of a series are all positive and as n is increased $\frac{u_{n+1}}{u_n}$ eventually becomes and remains less than something less than 1, the series is convergent. If $\frac{u_{n+1}}{u_n}$ eventually becomes and remains greater than 1, the series is divergent.*

5. If the terms of a series are all positive and as n is increased $n \log \frac{u_n}{u_{n+1}}$ eventually becomes and remains greater than something greater than 1, the series is convergent. If $n \log \frac{u_n}{u_{n+1}}$ eventually becomes and remains less than 1, the series is divergent.

* If $\frac{u_{n+1}}{u_n}$ eventually becomes and remains less than 1 but approaches 1 as its limit as n is indefinitely increased no conclusion can be drawn as to the convergence of the series.

6. If the terms of a series are all positive and as n is increased $n\left[\frac{u_n}{u_{n+1}}-1\right]$ eventually becomes and remains greater than something greater than 1 the series is convergent. If $n\left[\frac{u_n}{u_{n+1}}-1\right]$ eventually becomes and remains less than 1 the series is divergent.

7. If the terms of a series are alternately positive and negative as n increases and the absolute value of $\frac{u_{n+1}}{u_n}$ eventually becomes and remains less than 1, while u_n approaches zero as a limit, the series is convergent.*

8. If in any series u_n fails to approach zero as a limit as n is indefinitely increased, the series is divergent or oscillating.

Series whose terms are functions of a variable, x.

9. If $u_0, u_1, u_2, \cdots u_n$, etc., are finite, continuous, single-valued functions of x throughout the interval from $x=a$ to $x=b$, the series $u_0+u_1+u_2+\cdots+u_n+\cdots$ if convergent for all values of x between a and b will itself be a finite, single-valued function of x throughout the interval in question, but will not necessarily be continuous. Let us represent this function of x by $F(x)$.

Then for values of x between a and b

$$\int_a^x u_0dx+\int_a^x u_1dx+\int_a^x u_2dx+\cdots+\int_a^x u_ndx+\cdots$$

* If the terms of a series are alternately positive and negative and the series is convergent, the sum of the terms up to and including u_n differs from the sum of the series by less than the absolute value of u_{n+1}.

will *usually* * be convergent and be equal to $\int_a^x F(x)dx$, and $D_x u_0 + D_x u_1 + D_x u_2 + \cdots + D_x u_n + \cdots$ if convergent will *usually* * be equal to $D_x F(x)$.

10. If the series whose terms are the greatest absolute values through which the corresponding terms of

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots$$

pass between $x = a$ and $x = b$ is convergent

$$\int_a^x u_0 dx + \int_a^x u_1 dx + \int_a^x u_2 dx + \cdots + \int_a^x u_n dx + \cdots$$

is *always* convergent for all values of x between a and b and is equal to $\int_a^x F(x)\,dx$; and if the series whose terms are the greatest absolute values through which the corresponding terms of $D_x u_0 + D_x u_1 + D_x u_2 + \cdots + D_x u_n + \cdots$ pass between $x = a$ and $x = b$ is convergent $D_x u_0 + D_x u_1 + D_x u_2 + \cdots + D_x u_n + \cdots$ is for all values of x between a and b *always* equal to $D_x F(x)$.

11. If the terms of a series which is absolutely convergent for all values of x between a and b are multiplied by a function of x which is finite, continuous, and single-valued, throughout the interval from $x = a$ to $x = b$, or by a set of such functions no one of which for any value of x between a and b exceeds some given quantity in absolute value, the resulting series is absolutely convergent for values of x between a and b.

12. A series of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ is called a *Power Series.* It is absolutely convergent, and is a finite, continuous, single-valued function of x, within the

* It should be said that convergent series which cannot safely be integrated term by term are exceedingly rare, so rare, indeed, that it is only within a few years that the possibility of their existence has been recognized.

interval between but not necessarily including the extreme values of x for which it is convergent. Within this same interval its term by term integral and its term by term derivative are absolutely convergent; are finite, continuous, single-valued functions of x; and are equal respectively to the integral and to the derivative of the function represented by the series. For the extreme values of x for which the series is convergent its term by term integral is convergent and is equal to the integral of the function represented by the series; and its term by term derivative, if convergent, is equal to the derivative of the function represented by the series.*

Examples.

1. Are the following series convergent or divergent?

(*a*) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\cdots.$

(*b*) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\cdots.$

(*c*) $1+\dfrac{1}{2^p}+\dfrac{1}{3^p}+\dfrac{1}{4^p}+\dfrac{1}{5^p}+\cdots\cdots.$

(*d*) $1+\dfrac{1}{2}+\dfrac{1}{3^2}+\dfrac{1}{4^3}+\dfrac{1}{5^4}+\cdots\cdots.$

(*e*) $1+\dfrac{1^2}{2!}+\dfrac{2^3}{3!}+\dfrac{3^4}{4!}+\dfrac{4^5}{5!}+\cdots\cdots.$

(*f*) $1+\dfrac{1}{2^p}+\dfrac{2}{3^p}+\dfrac{3}{4^p}+\dfrac{4}{5^p}+\cdots\cdots.$

(*g*) $\dfrac{1}{2}+\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{1+\sqrt{3}}+\dfrac{1}{1+\sqrt{4}}+\cdots\cdots.$

Ans. (*a*) div.; (*b*) semi-conv.; (*c*) conv. if $p>1$, div. if $p=1$, div. if $p<1$; (*d*) conv.; (*e*) div.; (*f*) conv. if $p>2$, div. if $p=2$, div. if $p<2$; (*g*) div.

* If at either end of an interval within which a power series is convergent the series fails to converge, its term by term integral if convergent is still equal to the integral of the function represented within the interval by the series.

2. When are the following series absolutely convergent? When are they divergent?

(*a*) $1+x+x^2+x^3+x^4+\cdots\cdots.$

(*b*) $1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+\cdots\cdots.$

(*c*) $1+2^px+3^px^2+4^px^3+5^px^4+\cdots\cdots.$

(*d*) $\frac{3}{2}x+\frac{5}{5}x^2+\frac{7}{10}x^3+\cdots\cdots+\frac{2n+1}{n^2+1}x^n+\cdots\cdots.$

(*e*) $\frac{x}{1+x^3}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x^5}+\cdots\cdots.$

(*f*) $1+\frac{x}{1}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\cdots\cdots.$

(*g*) $1+\frac{x}{1}+\frac{1}{2}\frac{x^2}{3}+\frac{1.3}{2.4}\frac{x^3}{5}+\frac{1.3.5}{2.4.6}\frac{x^4}{7}+\cdots\cdots.$

(*h*) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\cdots.$

(*i*) $ax+\frac{a(a+1)}{2\,!}x^2+\frac{a(a+1)(a+2)}{3\,!}x^3+\cdots\cdots. \qquad a>0.$

Ans. (*a*) abs. conv. if $x^2<1$, div. if $x^2=1$, div. if $x^2>1$; (*b*) abs. conv. if $x^2<1$, div. if $x^2=1$, div. if $x^2>1$; (*c*) abs. conv. if $x^2<1$, div. if $x^2>1$, abs. conv. if $x^2=1$ and $p<-1$, div. if $x^2=1$ and $p>-1$, div. if $x=1$ and $p=-1$, semi-conv. if $x=-1$ and $p=-1$; (*d*) abs. conv. if $x^2<1$, div. if $x^2>1$, div. if $x=1$, semi-conv. if $x=-1$; (*e*) abs. conv. if $x^2<1$, div. if $x^2>1$, div. if $x=1$, meaningless if $x=-1$; (*f*) abs. conv. for all values of x; (*g*) abs. conv. if $x^2<1$, abs. conv. if $x^2=1$, div. if $x^2>1$; (*h*) abs. conv. if $x^2<1$, semi-conv. if $x=1$, div. if $x=-1$, div. if $x^2>1$; (*i*) abs. conv. if $x^2<1$, div. if $x^2>1$, div. if $x^2=1$ and $a=1$ or $a>1$, abs. conv. if $x^2=1$ and $a<1$.

3. Obtain the following developments and find when you can for what values of x the series are convergent.

(*a*) $\frac{1}{2}(e^x+e^{-x})=1+\frac{x^2}{2\,!}+\frac{x^4}{4\,!}+\frac{x^6}{6\,!}+\cdots\cdots.$

(*b*) $\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\cdots.$

(*c*) $\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\cdots.$

(*d*) $\log(x + \sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \cdots\cdots.$

(*e*) $\sin^{-1}\frac{2x}{1+x^2} = 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\cdots\right].$

(*f*) $\cos^{-1}\frac{x^2-1}{x^2+1} = 2\left[\frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots\cdots\right].$

(*g*) $\cos^2 x = \frac{1}{2}\left[2 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots\cdots\right].$

(*h*) $\cos^3 x = 1 - \frac{3^2+3}{4.2!}x^2 + \frac{3^4+3}{4.4!}x^4 - \frac{3^6+3}{4.6!}x^6 + \cdots\cdots.$

(*i*) $\frac{1}{2}(e^x + e^{-x})\cos x = 1 - \frac{2^2x^4}{4!} + \frac{2^4x^8}{8!} - \frac{2^6x^{12}}{12!} + \cdots\cdots.$

(*j*) $\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\frac{x^4}{4} + \frac{2.4}{3.5}\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\frac{x^8}{8} + \cdots\cdots.$

(*k*) $(1+x)^x = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \cdots\cdots.$

(*l*) $e^{x\sin x} = 1 + x^2 + \frac{1}{3}x^4 + \frac{x^6}{120} + \cdots\cdots.$

(*m*) $\frac{1}{1+x+x^2+x^3} = 1 - x + x^4 - x^5 + x^8 - x^9 + x^{12} - x^{13} + \cdots\cdots.$

(*n*) $\frac{\log(1+x)}{1+x} = \frac{1}{1}x - \left(\frac{1}{1} + \frac{1}{2}\right)x^2 + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)x^3 - \cdots\cdots.$

(*o*) $\frac{\log(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 - \frac{2.4.6}{3.5.7}x^7 + \cdots\cdots.$

(*p*) $\frac{1}{x\sqrt{x^2-1}} = \frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^{-\frac{1}{2}}$

$$= \frac{1}{x^2} + \frac{1}{2}\frac{1}{x^4} + \frac{1.3}{2.4}\frac{1}{x^6} + \frac{1.3.5}{2.4.6}\frac{1}{x^8} + \cdots\cdots.$$

(q) $\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2.3}\,\frac{1}{x^3} - \frac{1.3}{2.4.5}\,\frac{1}{x^5} - \frac{1.3.5}{2.4.6.7}\,\frac{1}{x^7} - \cdots\cdots.$

(r) $\log\left[\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right]$

$$= \frac{1}{x} - \frac{1}{2.3}\,\frac{1}{x^3} + \frac{1.3}{2.4.5}\,\frac{1}{x^5} - \frac{1.3.5}{2.4.6.7}\,\frac{1}{x^7} + \cdots\cdots.$$

Ans. (a) abs. conv. for all values of x; (b) abs. conv. for all values of x; (c) abs. conv. if $x^2 < 1$, div. if $x^2 = 1$ or $x^2 > 1$; (d) abs. conv. if $x^2 < 1$ or $x^2 = 1$, div. if $x^2 > 1$; (e) and (f) abs. conv. if $x^2 < 1$, semi-conv. if $x^2 = 1$, div. if $x^2 > 1$; (g), (h), and (i) abs. conv. for all values of x; (j) abs. conv. if $x^2 < 1$ or $x^2 = 1$, div. if $x^2 > 1$; (m), (n), and (o) abs. conv. if $x^2 < 1$, div. if $x^2 = 1$ or $x^2 > 1$; (p) abs. conv. if $x^2 > 1$, div. if $x^2 = 1$ or $x^2 < 1$; (q) and (r) abs. conv. if $x^2 > 1$ or $x^2 = 1$, div. if $x^2 < 1$.

4. Obtain the following equations and find for what values of x they are true.

(a) $$\int_0^x \frac{dx}{\sqrt{1-x^4}} = \frac{x}{1} + \frac{1}{2}\,\frac{x^5}{5} + \frac{1.3}{2.4}\,\frac{x^9}{9} + \frac{1.3.5}{2.4.6}\,\frac{x^{13}}{13} + \cdots\cdots.$$

(b) $$\int_0^x \sqrt{\frac{x}{1-x^4}}\,dx = \frac{2}{3}\,x^{\frac{3}{2}} + \frac{1}{2}\cdot\frac{2}{11}\,x^{\frac{11}{2}} + \frac{1.3}{2.4}\,\frac{2}{19}\,x^{\frac{19}{2}} + \cdots\cdots.$$

(c) $$\int_0^x \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\left[x^2 + \frac{1}{2}\,\frac{x^6}{3} + \frac{1.3}{2.4}\,\frac{x^{10}}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^{14}}{7} + \cdots\cdots\right]$$
$$= \tfrac{1}{2}\sin^{-1}(x^2).$$

(d) $$\int_a^x \frac{dx}{\sqrt{1+x^4}} = \frac{x}{1} - \frac{1}{2}\,\frac{x^5}{5} + \frac{1.3}{2.4}\,\frac{x^9}{9} - \frac{1.3.5}{2.4.6}\,\frac{x^{13}}{13} + \cdots\cdots$$
$$- \left[\frac{a}{1} - \frac{1}{2}\,\frac{a^5}{5} + \frac{1.3}{2.4}\,\frac{a^9}{9} - \frac{1.3.5}{2.4.6}\,\frac{a^{13}}{13} + \cdots\cdots\right]$$
if $a = 0$ or $0 < a < 1$.

(e) $\displaystyle\int_a^x \frac{dx}{\sqrt{1+x^4}} = -\frac{1}{x} + \frac{1}{2.5}\frac{1}{x^5} - \frac{1.3}{2.4.9}\frac{1}{x^9} + \frac{1.3.5}{2.4.6.13}\frac{1}{x^{13}} - \cdots\cdots$

$\displaystyle + \frac{1}{a} - \frac{1}{2.5}\frac{1}{a^5} + \frac{1.3}{2.4.9}\frac{1}{a^9} - \frac{1.3.5}{2.4.6.13}\frac{1}{a^{13}} + \cdots\cdots$

if $a=1$ or $a>1$.

(f) $\displaystyle\int_0^x e^{-x^2}dx = x - \frac{x^3}{1.3} + \frac{x^5}{2\,!\,5} - \frac{x^7}{3\,!\,7} + \frac{x^9}{4\,!\,9} - \cdots\cdots.$

(g) $\displaystyle\int_0^x \cos(x^2)\,dx = x - \frac{x^5}{2\,!\,5} + \frac{x^9}{4\,!\,9} - \frac{x^{13}}{6\,!\,13} + \cdots\cdots.$

(h) $\displaystyle\int_0^x \frac{\sin x}{x}\,dx = x - \frac{x^3}{3\,!\,3} + \frac{x^5}{5\,!\,5} - \frac{x^7}{7\,!\,7} + \cdots\cdots.$

(i) $\displaystyle\int_0^x x^{\frac{1}{2}} e^{-x}\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}\frac{x^{\frac{5}{2}}}{1} + \frac{2}{7}\frac{x^{\frac{7}{2}}}{2\,!} - \frac{2}{9}\frac{x^{\frac{9}{2}}}{3\,!} + \cdots\cdots.$

(j) $\displaystyle\int_0^x \frac{\log(1+x)}{x}\,dx = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots\cdots.$

Ans. (a) from $x=-1$ to $x=1$; (b) from $x=0$ to $x=1$; (c) and (d) from $x=-1$ to $x=1$; (e) from $x=1$ to $x=\infty$; (f) (g) and (h) for all values of x; (i) for all positive values of x; (j) from $x=-1$ to $x=1$.

5. Compute the value of π to four decimal places from the following series:

$$\frac{\pi}{6} = \sin^{-1}\frac{1}{2} = \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{1.3}{2.4.5.2^5} + \frac{1.3.5}{2.4.6.7.2^7} + \cdots\cdots.$$

$$\frac{\pi}{4} = \sin^{-1}\frac{\sqrt{2}}{2} = \sqrt{2}\left[\frac{1}{2} + \frac{1}{2.3.2^2} + \frac{1.3}{2.4.5.2^3} + \frac{1.3.5}{2.4.6.7.2^4} + \cdots\cdots\right].$$

$$\frac{\pi}{6} = \tan^{-1}\frac{\sqrt{3}}{3} = \sqrt{3}\left[\frac{1}{3} - \frac{1}{3.3^2} + \frac{1}{5.3^3} - \frac{1}{7.3^4} + \cdots\cdots\right].$$

6. Find the values of the following definite integrals correct to three decimal places:

(a) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^4}}.$

(b) $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^4}}.$

(c) $\displaystyle\int_2^{10} \frac{dx}{\sqrt{1+x^4}}.$

(d) $\displaystyle\int_0^{2} e^{-x^2} dx.$

(e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx.$

(f) $\displaystyle\int_0^{1} \frac{\log(1+x)}{x}\, dx.$

(g) $\displaystyle\int_0^{\frac{1}{5}} (1+x)^x\, dx.$

(h) $\displaystyle\int_5^{10} \sec^{-1} x \,.\, dx.$

Ans. (*a*) 0.503; (*b*) 0.247; (*c*) 0.397; (*d*) 0.882; (*e*) 1.371; (*f*) 0.822; (*g*) 0.202; (*h*) 7.159.

CHAPTER X.

INFINITESIMALS.

In all the Calculus problems hitherto considered we have had to deal either with the limit approached by the ratio of two variables as both approached zero (a *derivative*), the variables being so related that a change in one necessitated a change in the other; or with the limit approached by the sum of a set of variables as they all approached zero (a *definite integral*), the variables being so related that they were obliged to change simultaneously, and their number increasing as they separately diminished, but in such wise that the sum was not increasing indefinitely.

We have seen that in solving such problems the work can often be greatly simplified by the aid of the theorems stated in Articles 160 and 161.

The practical application of these theorems is much facilitated by the use of the somewhat artificial but exceedingly convenient conception of infinitesimals of different orders.

It must be kept in mind, however, that just as no *fixed* magnitude however large is infinite, so no *fixed* magnitude however small is infinitesimal. Both infinites and infinitesimals are by definition *variables,* the former capable of indefinite increase, the latter of indefinite diminution.

An infinite may be capable of very small values or an infinitesimal of enormously great ones, but in dealing with infinites we are usually concerned with what is happening to functions of them as the infinites in question are increasing beyond all bounds; as in dealing with infinitesimals we are usually concerned with what is happening to functions of them as the infinitesimals in question are diminishing indefinitely. Indeed, in problems where infinitesimals naturally enter we invariably need to find the limit of the ratio of two of them as they approach zero together, or the limit of the sum of a set of them as they all approach zero.

1. Show by the principles of this chapter that the formulas given in Examples 33, 39, 42, and 50 in Chapter V are rigorously accurate.

2. Show that

$$\underset{\Delta\phi \doteq 0}{\text{limit}} \sum_{\phi=0}^{\phi=\frac{\pi}{2}} 2\,a \cos\phi \,.\, \Delta\phi$$

$$= \underset{\Delta\phi \doteq 0}{\text{limit}} \sum_{\phi=0}^{\phi=\frac{\pi}{2}} 2\,a \cos(\phi+\Delta\phi)\,\Delta\phi$$

$$= \underset{\Delta\phi \doteq 0}{\text{limit}} \sum_{\phi=0}^{\phi=\frac{\pi}{2}} 2\,a \cos\phi \sin\Delta\phi$$

$$= \underset{\Delta\phi \doteq 0}{\text{limit}} \sum_{\phi=0}^{\phi=\frac{\pi}{2}} 2\,a \cos(\phi+\Delta\phi) \sin\Delta\phi$$

$$= \underset{\Delta\phi \doteq 0}{\text{limit}} \sum_{\phi=0}^{\phi=\frac{\pi}{2}} 2\,a \cos\phi \tan\Delta\phi$$

$$= \underset{\Delta\phi \doteq 0}{\text{limit}} \sum_{\phi=0}^{\phi=\frac{\pi}{2}} 2\,a \cos(\phi+\Delta\phi) \tan\Delta\phi$$

$$= 2\,a\,;$$

and draw and explain a figure for each case.

3. Prove by strict reasoning that the area bounded by a curve and two radii vectores is given by the formula

$$A = \tfrac{1}{2}\int_{\phi_0}^{\phi_1} r^2 d\phi.$$

4. Find the area of a Cardioid $r = 2a(1 - \cos\phi)$ by the aid of the formula given in Example 3. *Ans.* $A = 6\pi a^2$.

5. If ϵ is the angle that the tangent at any point of a curve makes with the radius vector drawn to the point, prove rigorously that $\sin\epsilon = \frac{r}{D_\phi s}$, $\cos\epsilon = \frac{1}{D_r s}$, and $\tan\epsilon = \frac{r}{D_\phi r}$.

6. Prove the formula $s = \int_{\phi_0}^{\phi_1} \sqrt{r^2 + (D_\phi r)^2} \,.\, d\phi$ for the length of the arc of a curve.

Geometrical Problems.

7. Two circles have their centres at A and B on a given curve and pass through a fixed point C not on the curve; show that the limiting position of their chord of intersection as B is made to move up toward coincidence with A is the perpendicular from C upon the tangent at A.

8. Circles of the same radius R are described cutting a straight line ABC in AB and AC respectively; prove that if C is made to approach B the limiting position of the line joining the centres of the circles makes with ABC an angle whose sine is $\frac{AB}{2R}$.

9. AP and AQ are chords of a given circle; show that the limiting position of the point of intersection of circles described on AP and AQ as diameters, as Q is made to approach P, is the foot of the perpendicular from A upon the tangent to the given circle at P.

10. In a right triangle the base is made to vary while the altitude is kept constant; show that the limit of the ratio of the increment of the base to the increment of the hypothenuse as both approach zero is equal to the ratio of the hypothenuse of the triangle to its base.

11. Show that the ratio of the two parts into which any arc is divided by a perpendicular at the middle point of its chord approaches the limit unity as the arc is indefinitely diminished.

12. Two straight lines AB and $A'B'$ cut off equal areas from a given oval curve; show that the limiting position of their point of intersection as $A'B'$ is made to approach coincidence with AB is the middle point of AB.

13. A line moves so that the product of its distances from two fixed points A and B is constant; show that the lines joining A and B with the limiting position approached by the point of intersection of two positions of the moving line, as the second position is made to approach coincidence with the first, make equal angles with the moving line.

14. Show that the sides of the minimum triangle that can be circumscribed about an oval curve are bisected by their points of contact.

15. A circle is revolved about an external axis in its own plane; show that the surface of the ring generated is equal to the length of the circumference of the circle multiplied by the length of the circumference described by its centre; and that the volume of the ring is equal to the area of the circle multiplied by the length of the circumference described by its centre.

Suggestion. — Draw a line parallel to the axis and at the same distance from the centre.

16. Extend 15 to the case of the ring generated by any oval curve which is symmetrical with respect to a diameter parallel to the external axis of revolution.

17. The Cardioid is the locus of the points obtained by drawing chords of a circle through a fixed point on the cir-

cumference and extending each by an amount equal to the diameter of the circle. Find the area of the Cardioid.

Ans. Six times the area of the circle.

18. A string which just surrounds an oval curve is cut and one end is securely fastened to the curve. The string is then held by the other end and being kept tightly stretched is unwrapped, carried completely round and wrapped on again the other way. Show that the distance traveled by the end of the string is equal to one and a half times the circumference of a circle of which the string is the radius.

19. A straight rod of given length moves about in a plane; show that the area it sweeps over is equal to the length of the rod multiplied by the distance its middle point moves in a direction always at right angles with the rod.

20. If a plane curve rolls on another curve in its own plane, show that the rolling curve can be brought from any of its positions into another infinitely near the first by a rotation about its point of contact through an infinitesimal angle followed by a displacement in which all of its points describe equal parallel straight lines whose length is infinitesimal of higher order than that of the angle.

Apply this conception of rolling to the study of the Cycloid.

21. A circle rolls from one end to the other of a curved line equal in length to the circumference of the circle, and then rolls back again on the other side of the curve; prove that, if the curvature of the curve be throughout less than that of the circle, the area contained within the closed curve traced by the point of the circle which was first in contact with the fixed curve is six times the area of the circle, and its perimeter is eight times the diameter of the circle.

CHAPTER XI.

DIFFERENTIALS.

Differentials of the First Order.

Note that if y is a function of x, dx and Δx are identical; Δy and dy are functions of x and Δx, but dy is generally much simpler than Δy; $\frac{\Delta y}{\Delta x}$ is a function of x and Δx; $\frac{dy}{dx}$ is a function of x only; $\Delta y - dy$ is a function of x and Δx, and the limit of its ratio to $(\Delta x)^2$ is finite, *i.e.* it is an infinitesimal of the second order if Δx or dx is the principal infinitesimal.

When a *derivative* is written as the ratio of two differentials the denominator is always the differential of the independent variable, and may be written either as an increment or as a differential.

1. If $y = x^3$ find

$$\Delta y,\ dy,\ \frac{\Delta y}{\Delta x},\ \frac{\Delta y}{dx},\ \frac{dy}{\Delta x},\ \frac{dy}{dx},\ \Delta y - dy.$$

2. What are the numerical values of the seven quantities called for in Example 1, (*a*) if $x = 4$ and $dx = 10$; (*b*) if $x = 4$ and $dx = 1$; (*c*) if $x = 4$ and $dx = 0.1$; (*d*) if $x = 4$ and $dx = 0.01$?

Ans. (*a*) 2680, 480, 268, 268, 48, 48, 2200; (*b*) 61, 48, 61, 61, 48, 48, 13; (*c*) 4.921, 4.8, 49.21, 49.21, 48, 48, 0.121; (*d*) 0.481201, 0.48, 48.1201, 48.1201, 48, 48, 0.001201.

3. If $y = \sin x$ find the values of $\frac{dy}{dx}$, dy, and Δy when $x = \frac{\pi}{3}$ and $dx = \frac{\pi}{6}$. *Ans.* $\frac{dy}{dx} = \frac{1}{2}$; $dy = \frac{\pi}{12}$; $\Delta y = 1 - \frac{1}{2}\sqrt{3}$.

4. If $y = \tan^{-1} x$ find dy and Δy.

Ans. $dy = \dfrac{\Delta x}{1 + x^2}$; $\Delta y = \tan^{-1} \dfrac{\Delta x}{1 + x^2 + x\Delta x}$.

5. If $y = \tan x$ find dy and Δy.

Ans. $dy = \sec^2 x \,.\, \Delta x$; $\quad \Delta y = \dfrac{\sec^2 x \tan(\Delta x)}{1 - \tan x \tan(\Delta x)}$.

6. If $y = f(x)$ is the equation of a curve, draw lines representing Δx, Δy, and dy, supposing x the independent variable.

If y is the independent variable, draw a line representing dx and show that in this case it is not equal to Δx.

Prove geometrically from your figure that

$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}.$$

Differentials of Different Orders.

7. If y is a function of x, we have seen that dy is a new function of x and dx which of course may be differentiated with respect to x by our ordinary formulas.

If $y = x^3$ find the values of dy, d^2y, and d^3y, supposing (a) that dx may be taken at pleasure and is consequently independent of x; (b) that dx involves x.

Ans. (a) $dy = 3x^2dx$, $d^2y = 6\,xdx^2$, $d^3y = 6\,dx^3$; (b) $dy = 3x^2dx$, $d^2x = 6\,xdx^2 + 3\,x^2d^2x$, $d^3y = 6\,dx^3 + 18\,xdxd^2x + 3\,x^2d^3x$.

8. If $y = \sin x$ find dy, d^2y, and d^3y on the same two hypotheses as in Example 7.

Ans. (a) $dy = \cos x\, dx$, $d^2y = -\sin x\, dx^2$, $d^3y = -\cos x\, dx^3$; (b) $dy = \cos x\, dx$, $d^2y = -\sin x\, dx^2 + \cos x\, d^2x$, $d^3y = -\cos x\, dx^3 - 3 \sin x\, dxd^2x + \cos x\, d^3x$.

9. Show geometrically that in finding $D_x{}^2y$ we naturally give x two successive increments, and that if these increments are equal $\Delta \dfrac{dy}{dx} = \dfrac{\Delta dy}{\Delta x}$, but not otherwise.

CHAPTER XII.

PARTIAL DERIVATIVES.

A *partial derivative* of a function is purely formal. It is taken on a hypothesis which may or may not be actually possible, namely, that only one of the variables changes its value. If in a concrete problem one of the variables may conceivably change its value without thereby compelling some of the others to change, the partial derivative with respect to it has a geometrical or physical interpretation, but not otherwise.

The total differential of a function of several variables like the differential of a function of a single variable is arbitrarily defined. If $y = f(x)$ $dy = D_x y \,.\, \Delta x$. If $u = f(x, y)$ $du = D_x u \,.\, \Delta x + D_y u \,.\, \Delta y$.

Just as dy is usually the simplest function of x and Δx that can replace Δy in problems involving the limit of a ratio or the limit of a sum, so du is usually the simplest function of x, y, Δx, and Δy that can replace Δu in like problems. It is generally very much simpler than Δu.

1. Form Δu and du when $u = x^4 y^3$, and show that they differ by an infinitesimal of higher order than the first if Δx and Δy are of the first order.

2. Form Δu and du if $u = \tan^{-1}\dfrac{y}{x}$, and show that the limit of their ratio as Δx and Δy approach zero is one.

$$\textit{Ans.}\quad \Delta u = \tan^{-1}\left[\frac{x\Delta y - y\Delta x}{x^2 + y^2 + x\Delta x + y\Delta y}\right].$$

$$du = \frac{x\Delta y - y\Delta x}{x^2 + y^2}.$$

3. Show that if $u = \log(x^2 + y^2)$

$$\Delta u = \log\left[1 + \frac{2\,x\Delta x + 2\,y\Delta y + (\Delta x)^2 + (\Delta y)^2}{x^2 + y^2}\right]$$

and $$du = \frac{2\,x\Delta x + 2\,y\Delta y}{x^2 + y^2}.$$

4. Show that if $u = F(y, z)$ and y and z are functions of x so that u is indirectly a function of x, $du = D_y u\,.\,dy + D_z u\,.\,dz$ when du, dy and dz are not increments, but are differentials as defined in Chapter XI.

5. Verify the formula $D_x D_y u = D_y D_x u$ when

$$(a)\quad u = \sin\frac{y}{x};$$

$$(b)\quad u = \sin^{-1}\frac{y}{x};$$

$$(c)\quad u = \frac{xy}{2\,x + z};$$

$$(d)\quad u = \log\left(x \tan^{-1}\sqrt{x^2 + y^2}\right).$$

6. Show that $V = e^{ay}\cos ax$ satisfies the equation $D_x^2 V + D_y^2 V = 0$, as do also $V = \log(x^2 + y^2)$ and $V = \tan^{-1}\dfrac{y}{x}$.

7. Show that $y = f(x + at) + F(x - at)$ satisfies the equation $D_t^2 y = a^2 D_x^2 y$.

8. Show that

$$V = \frac{1}{\sqrt{x^2 + y^2 + z^2}},$$

$$V = \frac{1}{\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}},$$

and $$V = \frac{xz}{x^2 + y^2}$$

satisfy the equation $D_x^2 V + D_y^2 V + D_z^2 V = 0$.

9. If $f(x, y) = 0$ and $F(x, z) = 0$, show that

$$D_x F(x, z) \,.\, D_y\, f(x, y) \frac{dy}{dz} = D_x f(x, y)\, D_z F(x, z).$$

10. If $f(x, y, z) = 0$, show that $D_y x + D_z y + D_x z + 1 = 0$.

CHAPTER XIII.

CHANGE OF VARIABLE.

Ordinary Derivatives.

1. Show that if $y = F(x)$ and $x = f(t)$

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}; \qquad \frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{dt^2}\dfrac{dx}{dt} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3}.$$

2. Show that

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}; \qquad \frac{d^2y}{dx^2} = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}.$$

3. Change the independent variable from θ to x in the equation $\dfrac{d^2z}{d\theta^2} + \operatorname{ctn}\theta\dfrac{dz}{d\theta} + m(m+1)z = 0$; given $x = \cos\theta$.

Ans. $(1 - x^2)\dfrac{d^2z}{dx^2} - 2x\dfrac{dz}{dx} + m(m+1)z = 0.$

4. Change the independent variable from x to θ in the equation $(1 - x^2)^2\dfrac{d^2y}{dx^2} + y = 0$; given $x = \sin\theta$.

Ans. $\cos^2\theta\dfrac{d^2y}{d\theta^2} + \sin\theta\cos\theta\dfrac{dy}{d\theta} + y = 0.$

5. Show that

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + \cos^2x \,.\, y = 0$$

reduces to

$$\frac{d^2y}{dz^2} + y = 0$$

if

$$z = \sin x.$$

6. Show that

$$(a^2-x^2)\frac{d^2y}{dx^2}-\frac{a^2}{x}\frac{dy}{dx}+\frac{x^2}{a}=0$$

reduces to
$$\frac{d^2y}{dz^2}+\frac{1}{a}=0$$

if
$$x^2+z^2=a^2.$$

7. Show that if

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0 \quad \text{and} \quad z=\frac{1}{x}$$

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+y=0.$$

8. Show that if

$$\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-y\left(\frac{dy}{dx}\right)^8=0$$

$$\frac{d^2x}{dy^2}+\frac{dx}{dy}+y=0.$$

9. Show that if $x=r\cos\phi$ and $y=r\sin\phi$

$$\frac{dy}{dx}=\frac{\sin\phi\,\dfrac{dr}{d\phi}+r\cos\phi}{\cos\phi\,\dfrac{dr}{d\phi}-r\sin\phi}=\frac{\sin\phi+r\cos\phi\,\dfrac{d\phi}{dr}}{\cos\phi-r\sin\phi\,\dfrac{d\phi}{dr}};$$

$$\frac{d^2y}{dx^2}=\frac{r^2+2\left(\dfrac{dr}{d\phi}\right)^2-r\,\dfrac{d^2r}{d\phi^2}}{\left(\cos\phi\,\dfrac{dr}{d\phi}-r\sin\phi\right)^3}=\frac{r\,\dfrac{d^2\phi}{dr^2}+r^2\left(\dfrac{d\phi}{dr}\right)^3+2\,\dfrac{d\phi}{dr}}{\left(\cos\phi-r\sin\phi\,\dfrac{d\phi}{dr}\right)^3};$$

$$x\frac{dx}{dt}+y\frac{dy}{dt}=r\frac{dr}{dt};$$

$$-y\frac{dx}{dt}+x\frac{dy}{dt}=r^2\frac{d\phi}{dt}.$$

Partial Derivatives.

10. Show that if $x = r\cos\phi$ and $y = r\sin\phi$, whence $r = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}\dfrac{y}{x}$;

$$D_r x = \cos\phi = \frac{x}{\sqrt{x^2+y^2}}; \quad D_r y = \sin\phi = \frac{y}{\sqrt{x^2+y^2}};$$

$$D_\phi x = -r\sin\phi = -y; \quad D_\phi y = r\cos\phi = x;$$

$$D_x r = \frac{x}{\sqrt{x^2+y^2}} = \cos\phi; \quad D_y r = \frac{y}{\sqrt{x^2+y^2}} = \sin\phi;$$

$$D_x\phi = \frac{-y}{x^2+y^2} = -\frac{\sin\phi}{r}; \quad D_y\phi = \frac{x}{x^2+y^2} = \frac{\cos\phi}{r};$$

$$D_x V = \cos\phi\, D_r V - \frac{\sin\phi}{r} D_\phi V;$$

$$D_y V = \sin\phi\, D_r V + \frac{\cos\phi}{r} D_\phi V;$$

$$D_r V = \frac{1}{\sqrt{x^2+y^2}}(x D_x V + y D_y V);$$

$$D_\phi V = -y D_x V + x D_y V.$$

11. Transform $D_x^2 V + D_y^2 V = 0$ to polar coördinates.

Ans. $D_r^2 V + \dfrac{1}{r} D_r V + \dfrac{1}{r^2} D_\phi^2 V = 0.$

12. Show that if $x = r\cos\theta$, $y = r\sin\theta\cos\phi$, and $z = r\sin\theta\sin\phi$, whence $r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\dfrac{\sqrt{y^2+z^2}}{x}$, and $\phi = \tan^{-1}\dfrac{z}{y}$;

$$D_r x = \cos\theta = \frac{x}{\sqrt{x^2+y^2+z^2}};$$

$$D_r y = \sin\theta\cos\phi = \frac{y}{\sqrt{x^2+y^2+z^2}};$$

$$D_r z = \sin\theta \sin\phi = \frac{z}{\sqrt{x^2+y^2+z^2}};$$

$$D_\theta x = -r\sin\theta = -\sqrt{y^2+z^2};$$

$$D_\theta y = r\cos\theta\cos\phi = \frac{xy}{\sqrt{y^2+z^2}};$$

$$D_\theta z = r\cos\theta\sin\phi = \frac{xz}{\sqrt{y^2+z^2}};$$

$$D_\phi x = 0;$$

$$D_\phi y = -r\sin\theta\sin\phi = -z;$$

$$D_\phi z = r\sin\theta\cos\phi = y;$$

$$D_x r = \frac{x}{\sqrt{x^2+y^2+z^2}} = \cos\theta;$$

$$D_x \theta = -\frac{\sqrt{y^2+z^2}}{x^2+y^2+z^2} = -\frac{\sin\theta}{r};$$

$$D_x \phi = 0;$$

$$D_y r = \frac{y}{\sqrt{x^2+y^2+z^2}} = \sin\theta\cos\phi;$$

$$D_y \theta = \frac{xy}{(x^2+y^2+z^2)\sqrt{y^2+z^2}} = \frac{\cos\theta\cos\phi}{r};$$

$$D_y \phi = -\frac{z}{y^2+z^2} = -\frac{\sin\phi}{r\sin\theta};$$

$$D_z r = \frac{z}{\sqrt{x^2+y^2+z^2}} = \sin\theta\sin\phi;$$

$$D_z \theta = \frac{xz}{(x^2+y^2+z^2)\sqrt{y^2+z^2}} = \frac{\cos\theta\sin\phi}{r};$$

$$D_z \phi = \frac{y}{y^2+z^2} = \frac{\cos\phi}{r\sin\theta};$$

$$D_x V = \cos\theta\, D_r V - \frac{\sin\theta}{r} D_\theta V;$$

$$D_y V = \sin\theta \cos\phi\, D_r V + \frac{\cos\theta \cos\phi}{r} D_\theta V - \frac{\sin\phi}{r \sin\phi} D_\phi V;$$

$$D_z V = \sin\theta \sin\phi\, D_r V + \frac{\cos\theta \sin\phi}{r} D_\theta V + \frac{\cos\phi}{r \sin\theta} D_\phi V;$$

$$D_r V = \frac{1}{\sqrt{x^2+y^2+z^2}}(x D_x V + y D_y V + z D_z V);$$

$$D_\theta V = \frac{1}{\sqrt{y^2+z^2}}[-(x^2+y^2) D_x V + xy D_y V + xz D_z V];$$

$$D_\phi V = -z D_y V + y D_z V;$$

$$(D_x V)^2 + (D_y V)^2 + (D_z V)^2 = (D_r V)^2 + \left(\frac{1}{r} D_\theta V\right)^2 + \left(\frac{1}{r \sin\theta} D_\phi V\right)^2.$$

13. Transform

$$D_x^2 V + D_y^2 V + D_z^2 V = 0$$

to polar coördinates.

Ans. $D_r^2 V + \frac{2}{r} D_r V + \frac{1}{r^2} D_\theta^2 V + \operatorname{ctn}\theta\, D_\theta V + \frac{1}{r^2 \sin^2\theta} D_\phi^2 V = 0,$

or $\frac{1}{r^2}[r D_r^2 (rV)] + \frac{1}{\sin\theta} D_\theta (\sin\theta\, D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V = 0.$

CHAPTER XIV.

TANGENT LINES AND PLANES IN SPACE.

1. If $z = f(x, y)$ is the equation of a surface and P (coördinates x, y, z) and Q (coördinates $x + \Delta x$, $y + \Delta y$, $z + \Delta z$) are two points on the surface, show that $D_x z$ is the tangent of the angle which the tangent line at P to the intersection with the surface of a plane through P parallel to the plane of XZ makes with the axis of X; that $D_y z$ is the tangent of the angle which the tangent line at P to the intersection with the surface of a plane through P parallel to the plane of YZ makes with the axis of Y; and that dz is equal to the length of the portion of a line through Q parallel to the axis of Z, intercepted between the tangent plane at P and a plane through P parallel to the plane of XY.

2. Find the equation of the tangent plane at the point (x_0, y_0, z_0) of the surface $z = \phi(x, y)$, regarding it as the limiting position approached by a plane through (x_0, y_0, z_0), $(x_0 + \Delta x, y_0, z_0 + \Delta_x z)$, and $(x_0, y_0 + \Delta y, z_0 + \Delta_y z)$ as Δx and Δy approach zero.

Ans. $z - z_0 = (x - x_0) D_{x_0} z + (y - y_0) D_{y_0} z.$

3. Show that if the equation of the surface is $f(x, y, z) = 0$ the answer in Example 2 can be written

$$(x - x_0) D_{x_0} f + (y - y_0) D_{y_0} f + (z - z_0) D_{z_0} f = 0.$$

4. Show that the condition that the surfaces $f(x, y, z) = 0$ and $F(x, y, z) = 0$ shall cut orthogonally is

$$(D_x f)(D_x F) + (D_y f)(D_y F) + (D_z f)(D_z F) = 0$$

if (x, y, z) is a point of intersection of the surfaces.

5. Show that if $\lambda > b > \mu > c > \nu$

the ellipsoid $$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} = 1,$$

the ruled hyperboloid

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} = 1,$$

and the biparted hyperboloid

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} = 1,$$

cut one another orthogonally no matter what the values of λ, μ, and ν.

6. Show that if $b > \mu > c > \nu$

the sphere $$x^2 + y^2 + z^2 = \lambda^2,$$

the cone $$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} = 0,$$

and the cone $$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} = 0,$$

cut orthogonally.

7. Show that the anchor ring

$$(x^2 + y^2 + z^2 + a^2)^2 = 4\,a^2\lambda^2\,(x^2 + y^2),$$

the sphere $$x^2 + y^2 + z^2 - 2\,a\mu z = a^2,$$

and the plane $$y = \nu x,$$

cut orthogonally, no matter what the values of λ, μ, and ν.

CHAPTER XV.

DEVELOPMENT OF A FUNCTION OF SEVERAL VARIABLES.

1. Develop $(x+h)^4+(x+h)(y+k)^3$ by Taylor's Theorem.

Ans. $x^4y^3+(4x^3+y^3)h+3xy^2k+6x^2h^2+3y^2hk+3xyk^2$
$$+4xh^3+3xyhk^2+xk^3+h^4+hk^3.$$

2. Show that $$(x+h)^p(y+k)^q$$
$$=x^py^q+(px^{p-1}y^qh+qx^py^{q-1}k)$$
$$+\frac{1}{2!}[p(p-1)x^{p-2}y^qh^2+2pqx^{p-1}y^{q-1}hk+q(q-1)x^py^{p-2}k^2]+\cdots\cdots.$$

3. Show that $$\sin x\sin y$$
$$=\frac{1}{2!}2xy-\frac{1}{4!}xy(4x^2+4y^2)+\frac{1}{6!}xy(6x^4+20x^2y^2+6y^4)+\cdots\cdots.$$

4. Show that $$e^x\sin y$$
$$=y+\frac{1}{2!}2xy+\frac{1}{3!}(3x^2y-y^3)+\frac{1}{4!}(4x^3y-4xy^3)+\cdots\cdots.$$

5. Show by developing $f(x+\Delta x,\ y+\Delta y)$ by Taylor's Theorem that if $u=f(x,y)$, $\Delta u-du$ is infinitesimal of higher order than the first if Δx and Δy are infinitesimal of the first order.

6. Show by direct differentiation that $xD_xu+yD_yu=nu$ in the case of the following functions:—

(*a*) $u=(x+y)(\log x-\log y)$.

(*b*) $u=xe^{\frac{y}{x}}-ye^{\frac{x}{y}}$.

(*c*) $u=\sin^{-1}\dfrac{x+y}{\sqrt{x^2+y^2}}$.

7. If $f(x,y,z)$ is homogeneous, show that every tangent plane to $f(x,y,z)=0$ passes through the origin.

CHAPTER XVI.

MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.

1. A triangular space is to be diminished by fencing off the corners, each fence being an arc of a circle and having the nearest corner as centre. Show how to leave the greatest possible central space with a given length of fence.

Ans. The fences must all have the same radius.

2. Find the point so situated that the sum of its distances from the three vertices of an acute-angled triangle is a minimum.

Ans. The lines joining the point with the vertices make with one another angles of 120°.

3. What is the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$?

Ans. $V=\frac{8\,abc}{3\sqrt{3}}$.

4. A torpedo in the form of a cylinder with equal conical ends is to be made out of boiler plates, and is to just float when loaded. What must be its proportions that it may carry the greatest weight of dynamite?

Ans. The length of the torpedo must be three times the length of the cylindrical portion, and the diameter of the torpedo must be $\sqrt{5}$ multiplied by the length of the cylindrical portion.

5. What are the proportions of the roomiest wall-tent that can be made of a given amount of canvas?

Ans. Length and breadth equal. Height of eaves and height above the eaves equal. Inclination of roof 30°.

CHAPTER XVII.

THEORY OF PLANE CURVES.

1. Find the points of inflection of the curves

$$(a)\quad x=y^3+3y^2;$$
$$(b)\quad y=x^2\log(1-x);$$
$$(c)\quad y=e^{x^{\frac{1}{3}}};$$
$$(d)\quad y=\frac{e^x-e^{-x}}{2};$$
$$(e)\quad y=xe^x.$$

Ans. (a) $(2,-1)$; (b) $(0,0)$; (c) $(8, e^2)$; (d) $(0,0)$; (e) $\left(-2,-\frac{2}{e^2}\right)$.

2. Show that the cubic $y=\frac{a^2x}{x^2+a^2}$ has three points of inflection, and that they all lie on the line $x=4y$.

3. Show that the Cissoid $y^2=\frac{x^3}{2a-x}$ has a cusp at the origin.

4. Show that the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ has cusps at the points where it crosses the axes.

5. Show that the curve $x^4-2ay^3-3a^2y^2-2a^2x^2+a^4=0$ has nodes at $(a,0)$, $(-a,0)$ and $(0,-a)$. Find the tangents at these points.

6. Show that the locus of the centres of rectangular hyperbolas having contact of the third order with the parabola $y^2=4ax$ is the equal parabola $y^2+4a(x+2a)=0$.

7. A series of circles have their centres on an equilateral hyperbola and pass through its centre. Show that their envelope is a lemniscate.

8. Show that the equation of the normal to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ may be written in the form $y \cos \phi - x \sin \phi = a \cos 2\phi$. Find the evolute of the curve.

Ans. $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.