PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXV.

NOVEMBER, 1893, TO NOVEMBER, 1894.

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1894.

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VOL. XXV.

THIRTIETH SESSION, 1893–94.

November 9th, 1893.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

SPECIAL MEETING.

The President stated that, in accordance with a notice sent out to all members capable of attending, the meeting had been made a "special" one, for the purpose of considering the following resolution, which would be submitted by the Council, viz., "That the London Mathematical Society be incorporated as a Limited Liability Company, under Section 23 of the Companies' Act, 1867, and that the Council be empowered to take the necessary steps to carry this resolution into effect."

The adoption of the above resolution was moved by Mr. Basset, seconded by Dr. Larmor, and further supported by Mr. S. Roberts and the Chairman. The resolution was carried unanimously, and the meeting then became the

ANNUAL GENERAL MEETING.

The President informed the members of the recent decease of Mr. W. S. B. Woolhouse, F.R.A.S., and gave the following short sketch of his life and work:—

Since our last meeting we have lost by death one who had been a member of our Society for over twenty-five years, viz., Westley

Stoker Barker Woolhouse, who has just passed away at the age of eighty-four. In laying before the Society a very slender sketch of his life and achievements, I must express the obligations I am under to his grandson, Mr. Rea, and our Secretary, Mr. Tucker, for information without which I could not have ventured to make even the brief tribute of respect to his memory which I propose to make. I regret that the time and opportunities at my disposal have not been such as to enable me to do something more to testify to our appreciation of one who was in many respects remarkable.

Mr. Woolhouse was born on May 6th, 1809. As a boy, though fond of practical jokes, he was regarded as particularly dull, his stupidity at figures prompting a mode of education which was at one time much in vogue, viz., blows on the head with a ruler. This mode of stimulating the brain, though not approved of in our own times, seems to have been effective, for he suddenly developed a most pronounced mathematical faculty, which came to light by the discovery of some difficult integrations chalked on the shop shutters in the lower part of the town of North Shields, where he resided. These having been traced to young Woolhouse, the interest of the learned of the town was excited, and his progress became assured.

The communication of problems to the *Newcastle Magazine*, and the *Lady's and Gentleman's Diary*, followed, and at thirteen he obtained a prize from the latter for higher mathematics, over the heads of several mathematicians of repute. At nineteen, without, it is said, ever having seen any treatise on the subject, he published a work on geometry of two dimensions. He also, about the same time, published some interesting investigations in dynamics.

These indications of his powers led to the adoption of a scientific career, and, after beginning as a computer for the *Nautical Almanack*, he rose to the post of Deputy Superintendent. The appendices to the *Almanack* bear witness to the part he played in constructing the formulæ employed in its calculation. Thus, in the *Almanack* for 1835, there are to be found (Appendix, 1–39) "New Tables for computing the occultations of Jupiter's satellites by Jupiter, the transits of the satellites and their shadows over the disc of the planet, and the positions of the satellites with respect to Jupiter at any time," and a paper "On the Computation of an Ephemeris of a Comet from its Elements" (Appendix 40–48). The *Almanack* for 1836 has an Appendix by him (pp. 53–148) "On Eclipses," and that for 1837 one "On the Determination of the Longitude from an observed Solar Eclipse or Occultation" (Appendix, pp. 172–183).

Subsequently he became the actuary of the International Loan Fund. His actuarial work on the tables employed by life assurance offices was of first-rate importance; but perhaps "his most remarkable feat was the solution of a problem in probabilities in connexion with the Ten Hours Bill. The question was how far the factory girls had to run in a day, when attending the 'mules' and trotting backwards and forwards to tie the threads which were constantly breaking. Mr. Woolhouse was engaged by Lord Ashley to go down to Manchester and obtain the necessary data. He performed the journey, obtained the data, solved the problem, wrote his report, and sent it off by the same evening's post. Mr. Woolhouse's calculation showed that the thread girl ran upwards of thirty miles each working day." Some remarks of his on the problem are to be found in the *Lady's and Gentleman's Diary* for 1859, pp. 91–95.

The *Philosophical Magazine* contains several papers from his pen. There was one in 1836, "On the Theory of Gradients on Railways" (*Phil. Mag.*, VIII., 243–6), and another, in the same year, "On the Theory of Vanishing Fractions" (*Phil. Mag.*, VIII., 293–400; IX., 18–26, 209–212). One in 1860, "On the Deposit of Submarine Cables" (*Phil. Mag.*, XIX., 345–361), led to a letter from the late Astronomer Royal, highly complimenting him on having "completely mastered a rather difficult investigation." And a paper in the same magazine in 1861, "On the Rev. T. P. Kirkman's Problem respecting certain Triadic Arrangements of Fifteen Symbols" (*Phil. Mag.*, XXII., 510–515), shows that he must be numbered among those who have attacked the "Girls' School Problem."*

The Royal Society's catalogue of scientific writings refers also to a paper by him in the *Mathematical Miscellany* of 1838 (Vol. I., pp. 336–343), "On the Theory of Exponential and Imaginary Quantities."

To our own *Proceedings* he communicated but one paper, which was "On General Numerical Solution," Vol. II., p. 75, 1868.

In a list of his writings before 1880, which has been furnished me

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* In the *Lady's and Gentleman's Diary* for 1862 (pp. 84–88), and again in that for 1863 (pp. 79–90), Woolhouse fully discusses, in very small type, this question, which was first proposed, in the *Diary* for 1850, by the Rev. T. P. Kirkman. Of the previous solutions he says: "They are irregular in construction, and do not in their present state suggest any system of derivation." He proposes, in his remarks, to give "a complete and systematic solution, with a few observations." Further, in the *Diary* for 1865 (pp. 94, 95), he gives a short note on Combinations, in connexion with the *Phil. Mag.* article on Triads.

by Mr. Tucker, I find reference to the following works, besides those of which I have already spoken:—

"On the Application of Algebraic Analysis to Geometry," Lond., 1831, 8vo;

"On Musical Intervals," 1835, 8vo;

"Tables of Continental Lineal and Square Measures," 1836, 8vo;

"On the Mortality in the Indian Army," 1839, 8vo;

"Elements of the Differential Calculus," 1852, 8vo;

"Measures, Weights, and Monies of all Nations," 1856, 12mo;

"Memoirs of the Early Life of W. S. B. Woolhouse," North Shields, 8vo.

I have reason to believe that the list which I have given from the materials at my disposal is far from exhaustive,* and that other important papers might be referred to. Among his minor works may be mentioned the editing of the Almanacks of the Stationers' Company for half a century, and an edition of Tredgold "On the Steam Engine." He was a frequent contributor to the mathematical columns of the *Educational Times.*

Some allusion should be made in conclusion to his character, which was one of unblemished simplicity. He was, says his grandson, "a perfect child in business affairs, always too ready to please, and too willing to be led."

He next gave a brief account of Prof. Klein's mathematical work, in connexion with the fourth award of the De Morgan Medal (made at the June meeting of the Council) to that gentleman. As Prof. Klein was unable to be present, the medal, at his instance, was given in charge to Prof. Greenhill and Dr. Forsyth; these gentlemen made suitable replies to the President's address.†

The Treasurer then read his Report. Its reception was moved by Dr. Forsyth, seconded by Mr. S. Roberts, and supported by Mr. Basset (who expressed the hope that, with a succession of favourable reports and strict economy, the time was not far distant when the Society would have rooms of its own with suitable accommodation for its accumulating library), and carried unanimously.

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* The theorem now known as Holditch's Theorem (Williamson, *Int. Calc.*, third edition, p. 206) was first proposed in the *Diary* for 1858 (Prize Quest. 1928). Mr. Woolhouse gives a *general* solution (*cf.* Williamson, *l.c.*) on p. 96, and, in the *Diary* for 1859, he gives (pp. 89–95) some "general theorems in further extension of Quest. 1928."

† This address is given, *in extenso*, in *Nature*, November 23rd, 1893 (p. 80).

At the request of the Chairman, the Rev. T. R. Terry consented to act as Auditor.

From the Report of the Secretaries, it appeared that the number of the members during the session had been 221, but was now reduced to 216, in consequence of two deaths, two withdrawals, and one removal. The number of compounders is 98.

The Society had to regret the loss, by death, of Mr. Hari Dâs Sâstri, M.A., who was elected a member December 11th, 1890; and of Mr. Westley Stoker Barker Woolhouse, F.R.A.S., who was elected a member December 12th, 1867.

The following communications had been made or received :—

Collaboration in Mathematics (Valedictory Address): Prof. Greenhill.
Note on the Equation $y^2 = x(x^4-1)$: Prof. W. Burnside.
Some Properties of Homogeneous Isobaric Functions: Prof. E. B. Elliott.
On certain General Limitations affecting Hyper-Magic Squares: Mr. S. Roberts.
On a Theorem in Differentiation, and its Application to Spherical Harmonics: Dr. Hobson.
Notes on Determinants: Mr. J. E. Campbell.
On the Evaluation of a certain Surface-Integral, and its Application to the Expansion, in Series, of the Potential of Ellipsoids: Dr. Hobson.
On the Application of the Sylvester-Clifford Graphs to Ordinary Binary Quantics (second part): Mr. A. B. Kempe.
On the Vibrations of an Elastic Circular Ring: Mr. Love.
Note on Secondary Tucker-Circles: Mr. J. Griffiths.
On a Group of Triangles inscribed in a given Triangle *ABC*, whose Sides are Parallel to Connectors of any Point *P* with *A*, *B*, *C*: Mr. R. Tucker.
On the Thirty Cubes that can be constructed with Six differently Coloured Squares: Major P. A. MacMahon.
Note on the Stability of a Thin Elastic Rod: Mr. Love.
A Geometrical Note: Mr. R. Tucker.
The Dioptrics of Gratings: Dr. J. Larmor.
On a Threefold Symmetry in the Elements of Heine's Series: Prof. L. J. Rogers.
Note on the Centres of Similitude of a Triangle of Constant Form *inscribed* in a given Triangle: Mr. J. Griffiths.
On a Problem of Conformal Representation: Prof. W. Burnside.
On the Collapse of Boiler Flues: Mr. Love.
On some Formulæ of Codazzi and Weingarten in relation to the Application of Surfaces to each other: Prof. Cayley.
On Complex Primes formed with the Fifth Roots of Unity: Prof. Tanner.
The Singularities of the Optical Wave-Surface, Electric Stability, and Magnetic Rotatory Polarization: Dr. J. Larmor.
On the Linear Transformations between Two Quadrics: Dr. H. Taber.
Note on some Properties of Gauche Cubics: Mr. T. R. Lee.
Pseudo-Elliptic Integrals, and their Dynamical Applications: Prof. Greenhill.

Complex Integers derived from $\theta^3 - 2 = 0$, and on the Algebraical Integers derived from an Irreducible Cubic Equation: Prof. G. B. Mathews.
Note on the Centres of Similitude of a Triangle of Constant Form *circumscribed* to a given Triangle: Mr. J. Griffiths.
A Note on Triangular Numbers: Mr. R. W. D. Christie.
The Harmonics of a Ring: Mr. W. D. Niven.
Toroidal Functions: Mr. Basset.
On the Expansion of certain Infinite Products (2): Prof. L. J. Rogers.
A Theorem for Bicircular Quartic Curves and for Cyclides analogous to Ivory's Theorem for Conics and Conicoids: Mr. A. L. Dixon.
On Maps and the Problem of the Four Colours: Prince C. de Polignac.
On Fermat's Proof that Primes of the Form $4n+1$ can be broken up into the Sum of Two Squares: Mr. S. Roberts.
On Cauchy's Condensation Test for the Convergency of Series: Dr. M. J. M. Hill.

The same Journals had been subscribed for as in the preceding session. An additional exchange of *Proceedings* had been made with the Mathematical Society of Amsterdam.

As it is some years since the list of exchanges has been printed in the *Proceedings*, it is here given as it stands at the present time.

Addresses of Societies and Persons, not Members, who receive the "Proceedings of the London Mathematical Society."

1. The Royal Society.
2. The Royal Society of Edinburgh.
3. The Royal Irish Academy.
4. The Library of Trinity College, Dublin.
5. The Cambridge Philosophical Society.
6. The Philosophical Society of Manchester.
7. The Institute of Actuaries.
8. The Library of University College, *Gower Street.*
9. The Principal Librarian of the British Museum.
10. The University Library, *Cambridge.*
11. The Bodleian Library, *Oxford.*
12. The Faculty of Advocates, Edinburgh, *Advocates' Library.*
13. The Librarian, Mason Science College, *Birmingham.*
14. The Edinburgh Mathematical Society, Edinburgh.
15. The Editor of *Nature.*
16. The Canadian Institute, *Toronto.*
17. The Smithsonian Institute, *Washington, D.C., U.S.A.*
18. The United States Naval Observatory, *Washington, D.C., U.S.A.*
19. The Connecticut Academy, *New-Haven, Conn., U.S.A.*
20. The Editors of the "American Journal of Mathematics," *Johns Hopkins University, Baltimore, Md., U.S.A.*

21. The Editors of "The Annals of Mathematics," *Leander McCormick Observatory, University of Virginia, U.S.A.*
22. L'Institut National de France, *Paris.*
23. La Société Mathématique, 7 *Rue des Grands-Augustins, Paris.*
24. La Société Philomathique, 7 *Rue des Grands-Augustins, Paris.*
25. [M. le Général Commandant] l'Ecole Polytechnique, *Paris.*
26. La Société des Sciences physiques et naturelles, *Bordeaux.*
27. Bibliothèque Universitaire de médecine et des sciences alliées, *St. Michel, Toulouse.*
28. L'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique, *Palais des Académies, Bruxelles.*
29. La Société Hollandaise (par l'entremise du Bureau scientifique central Néerlandais), *Haarlem.*
30. M. le Prof. Bierens de Haan, Rédacteur de "Nieuen Archiv," *Leiden.*
31. The Editors of the "Annales de l'Ecole Polytechnique à Delft."
32. The Mathematical Society of Amsterdam.
33. Reale Istituto Lombardo di Scienze e Lettere, *Milan.*
34. Reale Accademia dei Lincei, *Palazzo delle Scienze, Lungara* 10, *Roma.*
35. Reale Accademia di Scienze, Lettere, ed Arti, *Modena.*
36. Reale "Accademia delle Scienze fisiche e matematiche," *Napoli.*
37. Reale Istituto Veneto di Scienze, Lettere, ed Arti, *Venezia.*
38. Circolo Matematico di Palermo.
39. M. le Prof. F. Gomes Teixeira, *Coimbra.*
40. La Société Mathématique, [Cabinet de Mécanique, Université,] *Odessa.*
41. Akademie der Wissenschaften, *Berlin.*
42. The Editor of the "Journal für die reine und angewandte Mathematik (Crelle)."
43. The Authors of the "Jahrbuch über die Fortschritte der Mathematik," *Berlin.*
44. Die Königliche Gesellschaft der Wissenschaften, *Göttingen.*
45. [Dem Herrn Bibliothekar von der] Universität (an der Physischen und Medicinischen Gesellschaft), *Erlangen.*
46. "Beiblätter zu den Annalen der Physik und Chemie," *Leipzig.*
47. Die Königliche Sächsische Gesellschaft, *Leipzig.*
48. Die Naturforschende Gesellschaft, *Zürich.*
49. The Editors of the "Prace matematyczno fizycne" of Warsaw.

The meeting next proceeded to the election of the new Council. Mr. Jenkins having read the rules bearing on the election, the President nominated, as Scrutators, Prof. Hudson and the Rev. T. R. Terry. These gentlemen, having examined the balloting lists, declared the following gentlemen duly elected:—

Mr. A. B. Kempe, F.R.S., President; Mr. A. B. Basset, F.R.S., Prof. Elliott, F.R.S., Prof. Greenhill, F.R.S., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Messrs. M. Jenkins and R. Tucker, Honorary Secretaries. Other Members of the Council: Lieut.-Col. J. R. Campbell, F.G.S., Lieut.-Col. A. J. Cunningham, R.E., Dr. A. R.

Forsyth, F.R.S., Dr. J. W. L. Glaisher, F.R.S., Dr. M. J. M. Hill, Dr. E. W. Hobson, F.R.S., Mr. A. E. H. Love, Major MacMahon, R.A., F.R.S., Mr. J. J. Walker, F.R.S.

The following communications were made:—

A Mechanical Solution of the Problem of Tethering a Horse to the Circumference of a Circular Field, so as to Graze over an *n*th part of it: Prof. L. J. Rogers. (The solution turned on a property of the cycloid.)

The Stability of certain Vortex Motions: Mr. A. E. H. Love.

Cyclotomic Quartics: Prof. G. B. Mathews.

On the Application of Elliptic Functions to the Curve of Intersection of Two Quadrics: Mr. J. E. Campbell.

Notes on the Theory of Groups of Finite Order: Prof. W. Burnside.

Prof. Hudson showed, and explained, some mechanical constructions by his son, R. W. Hudson, for the Parabola, Hyperbola, Cubical Parabola, and Semi-Cubical Parabola.

Messrs. Hill, Basset, Greenhill, Walker, and the President, took part in the discussions which followed the reading of the papers.

The following presents were received:—

Cabinet Likeness of Mr. G. Heppel, for the Album, from Mr. Heppel.

Mukhopadhyay, A.—"Elementary Treatise on the Geometry of Conics," 8vo; London, 1893.

"Proceedings of the Royal Society," Vol. LIV., No. 327.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stück 9; Leipzig, 1893.

"Journal of the Institute of Actuaries," Vol. XXXI., Part 1, No. 171; October, 1893.

"Institute of Actuaries," List of Members to September, 1893.

"Proceedings of the Royal Irish Academy," Vol. II., Nos. 4, 5, N. S., May, August, 1893; Dublin.

Issaly, Mons. l'Abbé.—"Optique Géométrique," 5e Mémoire, Extrait des Mémoires de la Société des Sciences physiques et naturelles de Bordeaux, T. IV., 4e Série.

"Bulletin of the New York Mathematical Society," Vol. III., No. 1; October, 1893.

"Rendiconti del Circolo Matematico di Palermo," Tomo VII., Fasc. 3, 4, and 5.

"Atti della Reale Accademia dei Lincei," 1893, Serie 5a, Rendiconti, Vol. II., Fasc. 7 and 8, 2e Sem.; Roma, 1893.

"Educational Times," November, 1893.

"Annals of Mathematics," Vol. VII., Nos. 5 and 6; October, 1893; University of Virginia.

"Indian Engineering," Vol. XIV., Nos. 13, 14, 15, 16.

Notes on the Theory of Groups of Finite Order. By Prof. W. Burnside. Received November 2nd, 1893. Read November 9th, 1893.

I. *On the proof of Sylow's Theorem.*

The one definite fact known with regard to the structure of a finite group, on whose order no restriction is placed, may be summed up in the following theorem, due to Herr Sylow.

If p^a is the highest power of a prime p which divides the order of a finite group, the group contains a single set of conjugate sub-groups of order p^a; and the number of such conjugate sub-groups is congruent with unity, modulus p.

Herr Sylow's proof of this theorem is given in a paper in Vol. v. of the *Math. Annalen.* He first shows that the group necessarily contains a sub-group of order p^a, this part of the proof being founded on the theorem, due to Cauchy, that if the order of a group is divisible by a prime p, the group contains an operation of order p. Cauchy's proof of this theorem, which is also given with slight modifications by M. Jordan in his *Traité des Substitutions*, involves the representation of the given group as a group of substitutions performed on a certain number of symbols. Herr Netto, in his "Substitutionentheorie," gives an independent proof of the first part of Sylow's theorem, depending again on the representation of the group as a permutation-group. Herr Sylow's proof of the second part of his theorem, which is reproduced by Herr Netto, depends on the conception of the transitivity of a permutation-group. Now, since every group of finite order can be represented as a permutation-group, there can be no objection to the validity of the proofs above referred to; but from the point of view of right method they leave something to be desired.

A theorem so fundamental in the theory, which also from its statement is independent of the infinite variety of forms in which a group can be presented, should not depend for its proof on the properties of a very special mode of representation. The proof that follows will be found to be entirely independent of such extraneous considerations.

For the sake of completeness, the three following well-known theorems are stated as lemmas.

Lemma I.—The order of a sub-group is a factor of the order of the main-group of which it is a sub-group.

Lemma II.—The operations of a group which are permutable with a given operation form a sub-group.

Lemma III.—The operations of a group which are permutable with every operation of the group (*i.e.* the self-conjugate operations) form a self-conjugate sub-group.

Every group contains one self-conjugate operation, namely, the identical operation. Suppose first that this is the only self-conjugate operation that the group contains, and divide up all the operations of the group into conjugate sets. If N is the order of the group, and n the order of the sub-group formed of those operations which are permutable with a given operation S, then S forms one of a set of $\frac{N}{n}$ conjugate operations. Now the total number of operations is N, and hence

$$N = 1 + \frac{N}{n} + \frac{N}{n'} + \frac{N}{n''} + \dots .$$

Each term on the right-hand side is an integer, and therefore one or more of the numbers n, n', n'' must be divisible by p^{α}; that is, the given group has a sub-group whose order is divisible by p^{α}.

If next the group contains more than one self-conjugate operation, these will form a self-conjugate sub-group Γ. Form the group G', to which the given group G is merihedrically isomorphous, so that to the identical operation of G' there corresponds the self-conjugate sub-group Γ of G. If G' contains no self-conjugate operation except the identical one, it must, by the previous case, contain a sub-group whose order is divisible by the highest power of p entering in the order of G', and G will then contain a corresponding sub-group whose order is divisible by p^{α}.

If, on the other hand, G' contains other self-conjugate operations, the same process may be repeated.

Hence, finally, in any case, the original group contains a sub-group whose order is divisible by p^{α}. The same reasoning shows that this sub-group must itself contain a sub-group whose order is divisible by p^{α}, and, continuing in this way, we must at last arrive at a sub-group whose order is p^{α}.

For the proof of the second part of the theorem it is convenient to point out that a group whose order is the power of a prime must contain self-conjugate operations; for otherwise we should have an equation of the form

$$N = 1 + \frac{N}{n} + \frac{N}{n'} + \dots,$$

where N, n, n' are all powers of p, while n, n', &c., are less than N, which is clearly impossible.

Let now H be a sub-group of G of order p^{α} whose existence has just been proved. If H is self-conjugate within a more extensive sub-group I, the latter will contain no operations whose orders are powers of p except those of H; for otherwise I would contain a sub-group whose order would be a higher power of p than p^{α}, which is impossible.

It follows that the only operations, whose orders are powers of p, that transform H into itself are the operations of H.

Let S be a self-conjugate operation of H, and suppose, if possible, that S is also contained in a sub-group H', conjugate to H.

The group H' will be transformed with itself by those operations of H which are common to H and H'; and these form a sub-group of order p^{γ} $(\gamma < \alpha)$.

Hence, when H' is transformed by *all* the operations of H, a set of $p^{\alpha-\gamma}$ different groups will result, and, since S is a self-conjugate operation of H, each of these groups contains S. If this set does not exhaust the groups conjugate to H which contain S, let H_1' be another. When H_1' is transformed by all the operations of H, a set of $p^{\alpha-\gamma_1}$ groups will result, which are all different from each other and from the preceding, while they all contain S. This process may be continued till the groups conjugate to H and containing S are exhausted, and the number of groups contained in each set so obtained is a power of p.

Hence the number of sub-groups conjugate to H that contain S is a multiple of p.

If now H'' is one of the conjugate set that does not contain S, neither will any of the series

$$H'', \; S^{-1}H''S, \; S^{-2}H''S^2, \; \dots .$$

Exactly as before, it may be shown that the number of different sub-groups in this series is a power of p; and that all those of the conjugate sub-groups which do not contain S may be arranged in similar series. Hence the number of sub-groups conjugate to H

which do not contain S is also a multiple of p; and therefore the total number of sub-groups in the conjugate set, including H, is congruent to unity (mod p).

Finally, let h be, if possible, a sub-group of order p^{α}, not contained in the previous conjugate set.

If s is a self-conjugate operation of h, the previous reasoning shows that the number of groups in the previous self-conjugate set containing s, and also the number not containing s, are both multiples of p; and this is obviously absurd.

Hence all the sub-groups of order p^{α} form a *single* conjugate set.

Whether the following deduction from Sylow's theorem, which can be very easily proved by the above methods, has ever been noticed before in its general form the writer cannot say. Netto states it for the case of the symmetric group.

Cor.—Every sub-group of order p^{β} $(\beta<\alpha)$ of the main group is contained as a sub-group in one at least of the conjugate set of sub-groups of order p^{α}.

Let K be such a sub-group of order p^{β}, and let J be the greatest sub-group that contains K self-conjugately.

If the order of J is not divisible by a higher power of p than p^{β}, the above reasoning shows that the only operations whose orders are powers of p that transform K into itself are its own operations. But, this being so, the preceding method shows that the number of sub-groups in the conjugate set of which K forms part is congruent to unity (mod p). This, however, is in direct contradiction to the supposition that p^{β} is the highest power of p that divides the order of J. Hence there must be an operation whose order is a power of p which, not being contained in K, transforms K into itself, and, combining this with K, a new group is formed of order p^{γ} $(\gamma>\beta)$ of which K is a sub-group. This process may be continued till we arrive at a group of order p^{α}; and the corollary is thus proved.

II. *On the Possibility of Simple Groups whose Orders are the Products of Four Primes.*

It is well known that there is no simple group whose order is the product of two primes.

In a memoir in Vol. xl. of the *Mathematischen Annalen*, Herr O. Hölder has shown that there is no simple group whose order is the product of three primes. As leading up naturally to the discussion of the case of the product of four primes, an independent proof of this theorem is here given, which suggests the point of view taken in the more complicated case.

If p, q, r are primes in descending order of magnitude, there are four cases to consider, according as the order is of the form p^3, p^2q, pq^2, or pqr. In the first case, the order being the power of a prime, it is known that there is no corresponding simple group.

In the second case, since q cannot be congruent to unity, mod p, the sub-group of order p^2 contained in the main group is self-conjugate, and therefore the group must be composite.

In the third case, q being a prime less than p, neither q nor q^2 can be congruent to unity, mod p, unless $p = 3$, $q = 2$; and it is known that there is no simple group of order 12. For all other values of p and q, the sub-group of order p is self-conjugate, and the group composite.

In the last case, unless qr is congruent to unity, mod p, the group evidently cannot be simple. If this congruence is satisfied, the group must contain $(p-1)\,qr$ different operations of order p, leaving over qr operations. Now the group cannot be simple unless there are at least p conjugate sub-groups of order q, and therefore $p\,(q-1)$ different operations of order q. But $p\,(q-1) > qr$, so that this is impossible. Hence the group must be composite.

When the order of the group is the product of four primes, there are eight cases to deal with, according as the order is of the form p^4, p^3q, p^2q^2, pq^3, p^2qr, pq^2r, pqr^2, or $pqrs$, where p, q, r, and s are primes in descending order.

It will be shown that the only simple group that occurs is the icosahedral group corresponding to the particular values $p = 5$, $q = 3$, $r = 2$ of the seventh case.

Cases 1, 2, 3. Orders p^4, p^3q, p^2q^2.—The reasoning just given for the case of three primes shows that in none of these cases can the group be simple.

Case 4. Order pq^3.—Since neither q nor q^2 can be congruent to unity (mod p), the sub-group of order p must be either self-conjugate or one of q^3 conjugate sub-groups. In the latter case, the group contains $(p-1)\,q^3$ different operations of order p, and hence the remaining q^3 form a self-conjugate sub-group.

Case 5. Order p^2qr.—This and the following cases require rather more detailed treatment.

In each case the group is assumed to be simple, and it is then shown that Sylow's theorem leads to relations between the numbers of operations of different orders which it is impossible to satisfy.

Consider first in this case the sub-groups of order r. If there were either p or q of them, the group would be composite, since its order is not a factor either of $p\,!$ or of $q\,!$

If there are p^2q of them, there are $p^2q\,(r-1)$ different operations of order r.

If there are p^2 of them, each is self-conjugate within a group of order qr, which is therefore cyclical, and hence there are

$$p^2(q-1)(r-1)+p^2(r-1) = p^2q\,(r-1)$$

operations of orders r and qr.

Similarly, if there are pq conjugate sub-groups of order r, it may be shown that there are $p^2q\,(r-1)$ different operations of orders r and pr.

Hence in any case there are $p^2q\,(r-1)$ operations whose orders are r or a multiple of r, leaving only p^2q operations for the rest of the group.

Consider next the sub-groups of orders q. If the group is simple there must be p^2r, p^2, or pr of them. The above reasoning shows that there cannot be p^2r. If there were p^2, there would be $p^2\,(q-1)$ different operations of order q, leaving only p^2 for the rest of the group, so that the sub-group of order p^2 would be self-conjugate.

If, finally, there were pr conjugate sub-groups, order q, each would be self-conjugate within a group of order pq, which is therefore cyclical. There would then be

$$pr\,(p-1)(q-1)+pr\,(q-1) = p^2r\,(q-1)$$

operations orders q and pq. But this is impossible, since

$$p^2q < p^2r\,(q-1).$$

Hence the group cannot be simple.

Case 6. Order pq^2r.—Consider first the conjugate sub-groups of order p. There must be q^2r or qr of them. In the former case, there are $q^2r\,(p-1)$ different operations of order p. In the latter case, unless $p\equiv 1 \pmod{q}$, each is self-conjugate in a cyclical sub-group of order pq, and there are $q^2r\,(p-1)$ operations of orders p and pq.

But, on the other hand, $p\equiv 1 \pmod{q}$ is inconsistent with

$$qr\equiv 1 \pmod{p}.$$

For these two congruences give

$$p-1 = q\,(\lambda p - r),$$

where λ is an undetermined positive integer.

Now, in the case considered, pq^2r must be a factor of $p!$, and therefore

$$q \ngtr \tfrac{1}{2}(p-1).$$

Hence $$\lambda p - r \nless (\lambda - \tfrac{1}{2})\,p \nless \tfrac{1}{2}p,$$

and $q \ngtr 2$, which is impossible.

There are therefore always $q^2r\,(p-1)$ operations whose orders are p, or a multiple of p.

Consider now the conjugate sub-groups of order r. There are either pq^2, pq, q^2, or p of them.

Now $pq^2\,(r-1)$ and $pq\,(r-1)$ are both necessarily greater than q^2r, so that there cannot be pq^2 or pq such sub-groups.

If there were q^2, the sub-group order q^2 would evidently be self-conjugate.

If there were p, each would be self-conjugate within a sub-group of order q^2r, and there would be at least

$$p\,(q-1)(r-1) + p\,(r-1) = pq\,(r-1)$$

operations of orders r and qr; and this, as before, is impossible, since

$$pq\,(r-1) > q^2r.$$

Hence the group cannot be simple.

Case 7. Order pqr^2. — There are qr^2 or qr conjugate groups of order p. In the former case, there are $qr^2\,(p-1)$ different operations of order p, leaving only qr^2. Hence at once there cannot be more than p conjugate groups order q, and p conjugate groups order r^2. But this structure of the group involves the congruences

$$qr^2 \equiv 1 \pmod{p},$$

$$p \equiv 1 \pmod{q},$$

$$p \equiv 1 \pmod{r};$$

from which $$p-1 = qr\,(\lambda p - r),$$

where λ is an indeterminate positive integer, at once follows.

Now $$r \ngtr \tfrac{1}{2}(p-1),$$

since in the case considered pqr^2 must be a factor of $p!$; so that the above congruences are not consistent.

Suppose next that there are qr conjugate sub-groups order p. If $p \not\equiv 1 \pmod{r}$, each is self-conjugate in a cyclical sub-group of order pr; so that there are $qr^2 (p-1)$ different operations of orders p and pr. Also, since $p \not\equiv 1 \pmod{r}$, there are pq conjugate sub-groups of order r^2, involving certainly not less than $pq\,(r-1)$ different operations of order r; and, since $pq\,(r-1) > qr^2$, this is impossible.

If, on the other hand, $p \equiv 1 \pmod{r}$, then, since $qr \equiv 1 \pmod{p}$, $p \not\equiv 1 \pmod{q}$, and there are either pr^2 or pr conjugate sub-groups of order q. There cannot be pr^2, for then the number of operations of orders p and q would exceed the order of the group.

If $q \not\equiv 1 \pmod{r}$, the pr sub-groups of order q are self-conjugate within cyclical sub-groups of order qr, and the number of operations of orders p, q, and qr would exceed again the order of the group.

Hence, finally, the only admissible case is given by the congruences

$$qr \equiv 1 \pmod{p}, \quad p \equiv 1 \pmod{r},$$

$$pr \equiv 1 \pmod{q}, \quad q \equiv 1 \pmod{r}.$$

Now $$qr \equiv 1 \pmod{p}, \; pr \equiv 1 \pmod{q},$$

give $$\lambda pq + 1 = (p+q)\,r,$$

where λ is indeterminate.

Since $pq > pr$ or qr, $\lambda = 1$.

Hence $pq+1$ or $p\,(q-1)+p-1+2$ is divisible by r, and, since $p-1$ and $q-1$ are both divisible by r, it follows that r is 2.

The equation between p and q may then be written

$$(p-2)(q-2) = 3,$$

and hence $$p = 5, \quad q = 3.$$

The only possible case in which the group can be simple is therefore that of order $5.3.2^2$.

Case 8. Order $pqrs$.—If all the operations are of prime order, the group must be composite. For on this supposition there must be pqr conjugate sub-groups of order s, since, if a sub-group of order s were self-conjugate within, say, a sub-group of order rs, there would be operations of order rs. So also there must be at least pq conjugate sub-groups of order r, and at least p sub-groups of order q. But this gives

$$pqr\,(s-1) + pq\,(r-1) + p\,(q-1) = pqrs - p$$

operations of orders q, r, and s; and therefore the remaining operations form a self-conjugate sub-group. If the group is simple, there must therefore be operations of composite order.

Suppose first that the group contains an operation whose order is a multiple of p, say pr. There must be qs conjugate cyclical sub-groups of order pr, and all the operations (except identity) of each of these must be different. For if the qs groups of order p contained in the cyclical sub-groups of order pr are not all different, there must be either q or s different ones, and the group could be expressed as a substitution-group of q or s symbols, which is impossible; and the same holds of the qs groups of order r.

Hence there are $qs\,(pr-1)$ different operations of orders pr, p, and r; leaving only qs operations over. Now there must be at least p conjugate sub-groups of order q; but, since

$$p\,(q-1) > qs,$$

this is impossible. Hence the group is composite if it contains operations of order pr; and it is evident that the same holds for pq and ps.

Suppose next that there is an operation of order rs. There must then be either pq or p conjugate cyclical sub-groups order rs.

If there are p of them each is self-conjugate within a sub-group of order qrs, which must be cyclical, and all the operations of each of the p cyclical groups of order qrs must be distinct, or else these groups would have a common sub-group which would be self-conjugate. This gives $p\,(qrs-1)$ different operations, and hence the sub-group order p must be self-conjugate.

If, on the other hand, there are pq conjugate cyclical sub-groups of order rs, the sets of pq contained sub-groups of orders r and s may or may not be all distinct.

If, first, there are pq sub-groups of both the orders r and s, there are

$$pq\,(rs-1)$$

different operations of orders rs, r, and s. There cannot be less than p conjugate sub-groups order q, giving at least $p\,(q-1)$ different operations order q. Hence in this case the group must be composite.

If, secondly, there are pq sub-groups of order s, and only p sub-groups of order r, the latter are necessarily self-conjugate within p cyclical sub-groups of order qr, and then there are

$$pq\,(r-1)(s-1)+pq\,(s-1)+p\,(q-1)(r-1)+p\,(r-1)=pq\,(rs-1)$$

different operations of orders rs, s, qr and r. This leaves only pq operations, and leads again at once to the conclusion that the group is composite.

The case in which there are pq sub-groups order r, and only p sub-groups order s, may be treated in exactly the same way.

If, lastly, there are only p sub-groups of order r, and p sub-groups of order s, each sub-group of order r must be self-conjugate within at least one cyclical sub-group of order qr, and each sub-group of order s must be self-conjugate within at least one cyclical sub-group of order qs; so that there are at least p cyclical sub-groups of each of the orders qr and qs. There are therefore at least

$$pq(r-1)(s-1)+p(q-1)(r-1)+p(q-1)(s-1)+p(r-1)+p(s-1)$$
$$=pq(rs-1)$$

different operations of orders rs, qr, qs, r, and s; and as before the group is composite.

Hence, finally, if the group has an operation of order rs, it cannot be simple.

The supposition that the group has an operation of order qs leads to the same result, and the course of reasoning by which this is established is so closely similar to that given in the first preceding case that it may, perhaps, be omitted.

Finally, if the group has no operation of composite order in which s enters as a factor, there must be $pqr(s-1)$ different operations of order s. Also there must be either p conjugate cyclical sub-groups of order qr whose operations are necessarily all distinct, or else ps conjugate cyclical sub-groups of order qr whose operations of this order are all distinct. In the former case, the sub-group of order p would be self-conjugate; and in the latter, the number of operations of orders s and qr would exceed the order of the group, which is impossible.

Hence, in any case whatever, a group whose order is the product of four different primes is necessarily composite.

On the Stability of certain Vortex Motions. By A. E. H. Love. Read November 9th, 1893.

I. *Introductory.*

The chief question here considered is that of the stability or instability of the state of steady motion of a liquid, generally known as "Kirchhoff's Elliptic Vortex". It has, in fact, been shown by Kirchhoff that a motion is possible in which the liquid within a certain elliptic cylinder is in uniform vortex motion, and the liquid without the same cylinder is in irrotational motion, provided the cylinder rotates with an angular velocity depending on the spin, or "molecular rotation", and on the eccentricity of the ellipse. The first complete investigation of this state of motion was given by Dr. M. J. M. Hill* in 1884. He proved that both the components of the velocity and the pressure are continuous in crossing the surface separating the fluid moving rotationally from that moving irrotationally. Hill, at the same time, proved that Kirchhoff's vortex is one of a series of possible steady motions of an elliptic cylindrical vortex inside a confocal rigid envelope, viz., it is that particular case for which the envelope goes off to an infinite distance, and he gave the relation which must connect the angular velocity of the principal axes of the two ellipses, the lengths of these axes, and the spin, in order that such a steady motion may be possible. These new steady motions, in which the fluid in irrotational motion is bounded externally by a confocal of the internal boundary, I propose to call "Hill's vortices".

In the following, I first investigate the steady motion of Kirchhoff's vortex. One reason for the introduction of this somewhat ancient matter is that I wish to emphasize a particular point. It appears, in fact, that, alike in this case and in all the remaining cases of steady motion and small oscillations here investigated, the condition of continuity of pressure across the surface of the vortex reduces to an identity when the stream-functions are adjusted to satisfy the conditions of continuity of tangential and normal velocity, and the condition that the surface of the vortex always contains the same particles. This is only known as a general theorem in the case of steady motion referred to fixed axes, *i.e.*, where the velocities at any point of space are independent of the time; but the results obtained suggest that it is always true for steady motion referred to axes moving in a uniform manner, and for small oscillations about such steady motions. We can throw this suggestion into another form by saying that in the investigation of the stability of any steady vortex motion it is only necessary to satisfy the kinematical conditions, and the dynamical conditions will be identically satisfied. This suggestion, if valid, obviously very much increases the confidence with which we can attack any problem concerning the stability of vortex motion.

The stability of Kirchhoff's vortex is investigated on the supposition that the spin ζ of any element in the disturbed motion is the

* "On the Motion of Fluid part of which is moving rotationally and part irrotationally," *Phil. Trans. R. S.*, 1884.

same as in the undisturbed motion. In any normal mode of oscillation it turns out to be convenient to estimate the departure from the undisturbed boundary by the ratio of the normal displacement of a point on the bounding surface to the central perpendicular on the tangent at that point. This ratio is proportional to the quantity denoted hereafter by $\delta\xi/h_0^2$, and it appears that in any normal mode of oscillation this quantity is a simple harmonic function of the eccentric angle and of the time. In the different normal modes there are 1, 2, ... m, ... wave-lengths to the circumference. The frequency $n/2\pi$ of any mode is given directly by an equation of the form

$$\frac{n^2}{\zeta^2} = \left(\frac{2mab}{(a+b)^2} - 1\right)^2 - \left(\frac{a-b}{a+b}\right)^{2m},$$

where a, b are the principal semi-axes of the undisturbed ellipse. It is easy to show that (with the exception of the case $m = 2$) all the values of n are real, provided $3b > a$. When $m = 2$, however, n vanishes. The question of stability for displacements having a component for which $m = 2$ is therefore undecided by this investigation, but it is shown that for displacements having no such component, the motion is stable when $3b > a$.

To see what happens when $m = 2$, it is necessary to examine the meaning of the question of stability with constant spin. The problem may be stated thus :—At a certain instant there exists in an infinite fluid a vortex with given uniform spin whose surface is very nearly identical with a *particular* elliptic cylinder; in the subsequent motion will the surface be always nearly identical with this cylinder, supposed to rotate with a certain angular velocity? Now the case $m = 2$ corresponds to elliptic displacement, that is to say, the normal section of the surface of the vortex is initially a slightly different ellipse from the particular ellipse considered. The motion that ensues is steady motion, and the new ellipse rotates with the angular velocity required by its eccentricity. It will be proved later that no other motion of a vortex with an elliptic boundary which remains elliptic is possible in an infinite fluid, the spin being assumed uniform, and there being no slip at the surface of the vortex. In other words, if ever there exists an elliptic vortex with uniform spin, it is a Kirchhoff's elliptic vortex. The case $m = 2$ ought therefore to be excluded from the discussion of the stability of the vortex, and our conclusion is that if there exists at any instant a vortex with uniform spin whose surface is very nearly elliptic, it proceeds to oscillate about the elliptic form, provided its major axis is less than three times its minor axis.

Partly to throw further light on this question, I proceed to investigate the steady motion and small oscillations of a Hill's vortex, *i.e.*, of an elliptic cylindrical vortex inside a confocal rigid envelope which rotates with the angular velocity necessary for steady motion. Here any departure from the undisturbed form gives a state which is not one of steady motion. In the case of elliptic displacements of the surface of the vortex the motion is stable, and the frequency has a finite value which tends to zero as the external boundary goes off to infinity. The general frequency-equation is obtained, and it is verified that when the elliptic vortex degenerates into a vortex sheet whose section is the line of foci of the envelope the motion is unstable.

II. *Steady Motion of Kirchhoff's Vortex.*

1. The mode of steady motion known as Kirchhoff's elliptic vortex is such that all the liquid within a certain elliptic cylinder is in uniform vortex motion, and all the liquid outside the cylinder is moving irrotationally, the cylinder itself rotating uniformly without change of shape.

Let the equation of the cylinder, referred to its principal axes, be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\text{.............................(1)},$$

and let the conjugate functions ξ, η be defined by the equation

$$x+\iota y = c\cosh(\xi+\iota\eta) \quad\text{.........................(2)},$$

where $c^2=a^2-b^2$, so that $2c$ is the distance between the foci. Suppose ζ is the spin, and ω the angular velocity with which the axes of x and y rotate; then the known condition of steady motion is

$$\omega = 2\zeta ab/(a+b)^2 \quad\text{.........................(3)},$$

and the stream-functions inside and outside the cylinder are respectively

$$\psi' = -\zeta(bx^2+ay^2)/(a+b) \text{ inside},$$

and

$$\psi = -\zeta ab\xi - \tfrac{1}{2}\zeta ab e^{-2\xi}\cos 2\eta \text{ outside}.$$

These satisfy the conditions (*a*) that the surface (1) always contains the same particles, (*b*) that the tangential and normal components of the velocity are continuous in crossing this surface, (*c*) that the pressure is continuous in crossing this surface, (*d*) that the velocity vanishes at ∞, (*e*) that the circulation in any circuit surrounding the vortex once is twice the surface integral of vortex strength over the cross-section of the cylinder.

It is noteworthy that these conditions are not independent. In fact, when the functions have been determined to satisfy the remaining conditions, the condition (*c*) of continuity of pressure is identically satisfied. As the investigation of this condition will be useful to us afterwards in forming the pressure equation of the disturbed motion, we shall proceed to discuss the steady motion.

2. Consider a stream-function ψ', given by the equation

$$\psi' = \tfrac{1}{2}(\alpha x^2 + \beta y^2) \quad \dots\dots\dots\dots(4),$$

where

$$\alpha + \beta = -2\zeta \quad \dots\dots\dots\dots(5);$$

and suppose that this is the stream-function of a rotational motion with uniform spin ζ inside the surface

$$x^2/a^2 + y^2/b^2 = 1,$$

which rotates (without change of form) with an angular velocity ω.

(*a*) The condition that the surface always contains the same particles is that

$$\frac{x}{a^2}(\beta+\omega)\,y - \frac{y}{b^2}(\alpha+\omega)\,x = 0$$

when

$$x^2/a^2 + y^2/b^2 = 1.$$

Hence

$$\omega = -\frac{\alpha a^2 - \beta b^2}{a^2 - b^2} \quad \dots\dots\dots\dots(6).$$

(*b*) To make the velocities continuous in crossing the surface, we must write

$$\psi' = \tfrac{1}{4}c^2\left[(\alpha\cosh^2\xi + \beta\sinh^2\xi) + (\alpha\cosh^2\xi - \beta\sinh^2\xi)\cos 2\eta\right],$$

and take for the stream-function of the irrotational motion outside

$$\psi = A\xi + Be^{-2\xi}\cos 2\eta\,;$$

then these have to give the same values of $\partial\psi/\partial\xi$ and $\partial\psi/\partial\eta$ for a particular value ξ_0 of ξ, which is such that

$$c\cosh\xi_0 = a, \quad \text{and} \quad c\sinh\xi_0 = b.$$

We thus obtain the equations

$$A = \tfrac{1}{2}(\alpha+\beta)\,ab = -\zeta ab \quad \dots\dots\dots\dots(7),$$

$$B\frac{a-b}{a+b} = \tfrac{1}{4}(\beta-\alpha)\,ab = \tfrac{1}{4}(\alpha a^2 - \beta b^2) \quad \dots\dots\dots\dots(8),$$

and from the second of these, combined with (5), it follows that

$$\alpha = -2\zeta\frac{b}{a+b}, \quad \beta = -2\zeta\frac{a}{a+b} \quad \text{..................(9)},$$

and, on substituting these in (6), we obtain the condition (3).

The condition (d) is satisfied identically by the form assumed for ψ, and the condition (e) is satisfied in consequence of (7). It thus appears that all the constants are determined by the conditions (a), (b), (d) without reference to (c) or (e), and if the motion is possible the pressure equation must be identically satisfied.

(c) The equations of motion are

$$\frac{\partial u}{\partial t} - \omega v + (u+\omega y)\frac{\partial u}{\partial x} + (v-\omega x)\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + \omega u + (u+\omega y)\frac{\partial v}{\partial x} + (v-\omega x)\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

where ρ is the density, and p the pressure at any point, and u, v are the component velocities parallel to the axes of x and y.

At a point within the vortex these equations can be integrated, and we find

$$\text{const.} - \frac{p}{\rho} = \tfrac{1}{2}x^2\left\{\alpha\omega - (\alpha+\omega)\beta\right\} + \tfrac{1}{2}y^2\left\{\beta\omega - (\beta+\omega)\alpha\right\} \quad \text{... (10)},$$

and, in virtue of the equations giving α, β, and ω, this becomes

$$\text{const.} - \frac{p}{\rho} = u\omega y - v\omega x.$$

At a point without the vortex the same equations give us

$$\text{const.} - \frac{p}{\rho} = u\omega y - v\omega x + \tfrac{1}{2}(u^2+v^2) \quad \text{..................(11)}.$$

The pressure is continuous in crossing the surface if $\frac{1}{2}(u^2+v^2)$ is constant at the surface, and this is the case, for

$$\tfrac{1}{2}(u^2+v^2) = 2\zeta^2\frac{a^2b^2}{(a+b)^2}$$

at the surface.

The steady motion is thus shown to satisfy all the conditions.

III. *Stability of Kirchhoff's Vortex.*

3. Suppose now that the steady motion is slightly disturbed; the stream-functions of the disturbed motion, inside and outside a cylindrical surface, differing very little from

$$x^2/a^2+y^2/b^2 = 1,$$

must both of them satisfy Laplace's equation. Let the stream-function of the motion inside the vortex be $\psi'+\delta\psi'$, and that outside $\psi+\delta\psi$; then we have to take

$$\left.\begin{aligned} \delta\psi' &= \Sigma\left[A_m \cosh m\xi \cos m\eta + B_m \sinh m\xi \sin m\eta\right] \\ \delta\psi &= \Sigma\left[A'_m \cosh m\xi_0 e^{-m(\xi-\xi_0)} \cos m\eta + B'_m \sinh m\xi_0 e^{-m(\xi-\xi_0)} \sin m\eta\right] \end{aligned}\right\} \quad \text{...............(12)},$$

where A_m, ... are small constants, and the summations refer to integral values of m. Also we have to suppose that the disturbed surface is given by the equation

$$F = \xi-(\xi_0+\delta\xi) = 0 \text{..........................(13)},$$

where $\delta\xi$ is a small function of η, but not of ξ, and $\delta\xi$, A_m, B_m, A'_m, B'_m are functions of the time.

The conditions to be satisfied are that at the disturbed surface the component velocities and the pressure are continuous, and that this surface always contains the same particles. Just as in the steady motion it will appear that when the remaining conditions are satisfied the pressure condition becomes an identity.

4. To make the velocities continuous in crossing the surface we have to make

$$\frac{\partial(\psi'+\delta\psi')}{\partial\xi} = \frac{\partial(\psi+\delta\psi)}{\partial\xi}, \quad \text{and} \quad \frac{\partial(\psi'+\delta\psi')}{\partial\eta} = \frac{\partial(\psi+\delta\psi)}{\partial\eta},$$

when $\xi = \xi_0+\delta\xi$.

Taking the second of these first, and observing that, when $\xi = \xi_0+\delta\xi$ and the square of $\delta\xi$ is neglected,

$$\frac{\partial\psi}{\partial\eta} = \frac{\partial\psi'}{\partial\eta} = \zeta ab\,\frac{a-b}{a+b}\,(1-2\delta\xi)\sin 2\eta,$$

we see that we must have

$$A'_m = A_m \quad \text{and} \quad B'_m = B_m.$$

In what follows we shall therefore omit the accents on A'_m and B'_m.

Again, we have

$$\frac{\partial \psi'}{\partial \xi} = -\zeta (a-b) \left[(a+b) \cosh \xi \sinh \xi - (a-b) \cosh \xi \sinh \xi \cos 2\eta \right]$$

$$= -\zeta ab \left(1 - \frac{a-b}{a+b} \cos 2\eta\right) - \zeta (a^2+b^2) \left(1 - \frac{a-b}{a+b} \cos 2\eta\right) \delta\xi,$$

when $\xi = \xi_0 + \delta\xi$.

In like manner

$$\frac{\partial \psi}{\partial \xi} = -\zeta ab \left(1 - e^{-2\xi} \cos 2\eta\right)$$

$$= -\zeta ab \left(1 - \frac{a-b}{a+b} \cos 2\eta\right) - 2\zeta ab \frac{a-b}{a+b} \cos 2\eta \, \delta\xi,$$

when $\xi = \xi_0 + \delta\xi$.

Hence
$$\frac{\partial \psi'}{\partial \xi} - \frac{\partial \psi}{\partial \xi} = -\zeta \left[(a^2+b^2) - (a^2-b^2) \cos 2\eta \right] \delta\xi$$

$$= -2\zeta \left(a^2 \sin^2 \eta + b^2 \cos^2 \eta\right) \delta\xi$$

$$= -\frac{2\zeta}{h_0^2} \delta\xi,$$

where h_0^2 is the value of $\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2$, when $\xi = \xi_0$.

Thus the condition of continuity of tangential velocity gives us

$$-\frac{2\zeta}{h_0^2} \delta\xi + \Sigma m \left[A_m e^{m\xi_0} \cos m\eta + B_m e^{m\xi_0} \sin m\eta \right] = 0 \quad \text{......(14)},$$

products of $\delta\xi$ with A_m and B_m being neglected.

5. The condition that the surface

$$F = \xi - (\xi_0 + \delta\xi) = 0$$

always contains the same particles is

$$\frac{\partial F}{\partial t} + (u + \omega y) \frac{\partial F}{\partial x} + (v - \omega x) \frac{\partial F}{\partial y} = 0,$$

or

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \frac{\partial}{\partial y} \left\{\psi + \delta\psi + \tfrac{1}{2}\omega (x^2+y^2)\right\} - \frac{\partial F}{\partial y} \frac{\partial}{\partial x} \left\{\psi + \delta\psi + \tfrac{1}{2}\omega (x^2+y^2)\right\} = 0,$$

or
$$-\frac{\partial \delta\xi}{\partial t}+h^2\frac{\partial\left[F,\ \psi+\delta\psi+\frac{1}{2}\omega(x^2+y^2)\right]}{\partial(\xi,\ \eta)}=0 \text{ ....(15)},$$

where h^2 denotes the Jacobian $\dfrac{\partial(\xi,\ \eta)}{\partial(x,\ y)}$. This condition has to be satisfied when $\xi=\xi_0+\delta\xi$.

Now, we have

$$\frac{\partial(F,\ \psi)}{\partial(\xi,\ \eta)}=\frac{\partial\psi}{\partial\eta}+\frac{\partial\delta\xi}{\partial\eta}\frac{\partial\psi}{\partial\xi}$$

$$=\zeta ab\frac{a-b}{a+b}(1-2\delta\xi)\sin 2\eta-\frac{\partial\delta\xi}{\partial\eta}\zeta ab\left(1-\frac{a-b}{a+b}\cos 2\eta\right),$$

when $\xi=\xi_0+\delta\xi$, and squares and products of small quantities are rejected.

Also

$$\frac{\partial(F,\ \delta\psi)}{\partial(\xi,\ \eta)}=\frac{\partial\delta\psi}{\partial\eta}=\Sigma m\left[B_m\sinh m\xi_0\cos m\eta-A_m\cosh m\xi_0\sin m\eta\right],$$

when $\xi=\xi_0+\delta\xi$, and squares and products of small quantities are rejected.

And thirdly, $$\frac{\partial\left[F,\ \frac{1}{2}\omega(x^2+y^2)\right]}{\partial(\xi,\ \eta)}$$

$$=\zeta\frac{ab}{(a+b)^2}c^2\left\{\left[-\cosh^2\xi+\sinh^2\xi\right]\sin 2\eta+\frac{\partial\delta\xi}{\partial\eta}2\cosh\xi\sinh\xi\right\}$$

$$=-\zeta ab\frac{a-b}{a+b}\sin 2\eta+2\zeta\frac{a^2b^2}{(a+b)^2}\frac{\partial\delta\xi}{\partial\eta},$$

when $\xi=\xi_0+\delta\xi$, and squares and products of small quantities are rejected.

Hence, adding and dividing by h^2, equation (15) may be written

$$-\frac{1}{h_0^2}\frac{\partial\delta\xi}{\partial t}-2\zeta ab\frac{a-b}{a+b}\sin 2\eta\,\delta\xi-\zeta\frac{ab}{(a+b)^2}\left[a^2+b^2-(a^2-b^2)\cos 2\eta\right]\frac{\partial\delta\xi}{\partial\eta}$$
$$+\Sigma m\left[B_m\sinh m\xi_0\cos m\eta-A_m\cosh m\xi_0\sin m\eta\right]=0.$$

Remembering that $\qquad h_0^2=a^2\sin^2\eta+b^2\cos^2\eta,$

we see that this may be written

$$\frac{\partial}{\partial t}\left(\frac{\delta\xi}{h_0^2}\right)+\frac{2\zeta ab}{(a+b)^2}\frac{\partial}{\partial\eta}\left(\frac{\partial\xi}{h_0^2}\right)$$
$$-\Sigma m\left[B_m\sinh m\xi_0\cos m\eta-A_m\cosh m\xi_0\sin m\eta\right]=0 \text{......(16)}.$$

The quantity $\delta\xi/h_0^2$ which here occurs is $ab.\delta\nu/p_1$, where $\delta\nu$ is the normal displacement of a point on the boundary, and p_1 is the central perpendicular on the tangent at the point.

6. To form the pressure equation, we suppose that the component velocities and the pressure in the steady motion are u, v, p, and in the disturbed motion $u+u'$, $v+v'$, $p+p'$. The complete equations of motion, referred to the moving axes, are

$$\frac{\partial}{\partial t}(u+u')-\omega(v+v')+(u+u'+\omega y)\frac{\partial}{\partial x}(u+u')+(v+v'-\omega x)\frac{\partial}{\partial y}(u+u') = -\frac{1}{\rho}\frac{\partial}{\partial x}(p+p'),$$

$$\frac{\partial}{\partial t}(v+v')+\omega(u+u')+(u+u'+\omega y)\frac{\partial}{\partial x}(v+v')+(v+v'-\omega x)\frac{\partial}{\partial y}(v+v') = -\frac{1}{\rho}\frac{\partial}{\partial y}(p+p').$$

In these equations we may omit products of small quantities u', v', and terms independent of small quantities u', v', p', so that we find

$$\frac{\partial u'}{\partial t}-\omega v'+(u+\omega y)\frac{\partial u'}{\partial x}+u'\frac{\partial u}{\partial x}+(v-\omega x)\frac{\partial u'}{\partial y}+v'\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p'}{\partial x},$$

$$\frac{\partial v'}{\partial t}+\omega u'+(u+\omega y)\frac{\partial v'}{\partial x}+u'\frac{\partial v}{\partial x}+(v-\omega x)\frac{\partial v'}{\partial y}+v'\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p'}{\partial y}.$$

Now u' and v' are derivable from a velocity potential which, inside the vortex, is the conjugate function to $\delta\psi'$, and, outside, is the conjugate function to $\delta\psi$.

If we denote the velocity potentials by $\delta\phi'$ and $\delta\phi$, then it is easy to show that the form of the pressure equation inside is

$$\text{const.}-\frac{p+p'}{\rho} = u\omega y - v\omega x + \frac{\partial\delta\phi'}{\partial t}+(u+\omega y)\,u'+(v-\omega x)\,v'+2\zeta\delta\psi',$$

and the corresponding equation outside is

$$\text{const.}-\frac{p+p'}{\rho} = \frac{\partial\delta\phi}{\partial t}+(u+u')\,\omega y-(v+v')\,\omega x+\tfrac{1}{2}(u^2+v^2)+uu'+vv'.$$

The condition that the pressure is continuous in crossing the surface is therefore that

$$\frac{\partial\delta\phi'}{\partial t}-\frac{\partial\delta\phi}{\partial t}-\tfrac{1}{2}(u^2+v^2)+2\zeta\,\delta\psi'=\text{const.}\quad\ldots\ldots\ldots\ldots(17),$$

when $\xi=\xi_0+\delta\xi$.

Now

$$\delta\phi'=\Sigma\left[B_m\cosh m\xi\cos m\eta-A_m\sinh m\xi\sin m\eta\right],$$

$$\delta\phi=\Sigma\left[-B_m\sinh m\xi_0 e^{-m(\xi-\xi_0)}\cos m\eta+A_m\cosh m\xi_0 e^{-m(\xi-\xi_0)}\sin m\eta\right],$$

so that (denoting differentiation with respect to the time by a dot)

$$\frac{\partial\delta\phi'}{\partial t}-\frac{\partial\delta\phi}{\partial t}=\Sigma\left[\dot{B}_m e^{m\xi_0}\cos m\eta-\dot{A}_m e^{m\xi_0}\sin m\eta\right],$$

when $\xi=\xi_0+\delta\xi$, and products of small quantities are rejected.

Also

$$-\tfrac{1}{2}(u^2+v^2)=-2\zeta^2\frac{a^2b^2}{(a+b)^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$$

$$=-2\zeta^2\frac{a^2b^2}{(a+b)^2}\left[1+\frac{2\delta\xi}{ab\,h_0^2}\right],$$

when $\xi=\xi_0+\delta\xi$, and squares of small quantities are rejected.

Hence the pressure equation becomes

$$\Sigma\left[\dot{B}_m e^{m\xi_0}\cos m\eta-\dot{A}_m e^{m\xi_0}\sin m\eta\right]-\frac{4\zeta^2ab}{(a+b)^2}\frac{\delta\xi}{h_0^2}$$

$$+2\zeta\Sigma\left[A_m\cosh m\xi_0\cos m\eta+B_m\sinh m\xi_0\sin m\eta\right]=0\ldots(18).$$

7. To deduce the conditions of stability, we notice in the first place that the conditions (14), (16), and (18) contain $\delta\xi$ only in the expression $\delta\xi/h_0^2$, and it is therefore convenient to take

$$\delta\xi/h_0^2=\Sigma(\alpha_m\cos m\eta+\beta_m\sin m\eta),$$

where α_m and β_m are small functions of the time; the condition (14) gives us

$$\alpha_m=\frac{mA_m e^{m\xi_0}}{2\zeta},\quad \beta_m=\frac{mB_m e^{m\xi_0}}{2\zeta}.$$

Since A_m and B_m refer to a normal mode of oscillation, this result describes the character of the waves.

The condition (16) now becomes

$$\Sigma\left[\frac{(\dot{A}_m \cos m\eta + \dot{B}_m \sin m\eta)\, e^{m\xi_0}}{2\zeta} + \frac{mab}{(a+b)^2}\,(B_m \cos m\eta - A_m \sin m\eta)\, e^{m\xi_0}\right.$$

$$\left. - B_m \sinh m\xi_0 \cos m\eta + A_m \cosh m\xi_0 \sin m\eta\right] = 0 \quad\text{.........(19).}$$

Also the condition (18) becomes

$$\Sigma\left[(\dot{B}_m \cos m\eta - \dot{A}_m \sin m\eta)\, e^{m\xi_0} - \frac{2\zeta mab}{(a+b)^2}\,(A_m \cos m\eta + B_m \sin m\eta)\, e^{m\xi_0}\right.$$

$$\left. + 2\zeta\,(A_m \cosh m\xi_0 \cos m\eta + B_m \sinh m\xi_0 \sin m\eta)\right] = 0 \quad\text{......(20).}$$

Each of the equations last written gives us

$$\dot{A}_m + \left[\frac{2\zeta mab}{(a+b)^2} - 2\zeta e^{-m\xi_0} \sinh m\xi_0\right] B_m = 0,$$

$$\dot{B}_m - \left[\frac{2\zeta mab}{(a+b)^2} - 2\zeta e^{-m\xi_0} \cosh m\xi_0\right] A_m = 0.$$

If now we suppose that A_m and B_m, as functions of t, are proportional to $e^{-\iota nt}$, the equation to find n is

$$n^2 = \zeta^2 \left[1 + \left(\frac{a-b}{a+b}\right)^m - \frac{2mab}{(a+b)^2}\right]\left[1 - \left(\frac{a-b}{a+b}\right)^m - \frac{2mab}{(a+b)^2}\right] \quad\text{...(21),}$$

and the condition of stability is that all the values of n^2 are positive.

Now the value of n^2 may be written

$$n^2 = \zeta^2 \left[\left(\frac{2ab}{(a+b)^2}\, m - 1\right)^2 - \left(\frac{a-b}{a+b}\right)^{2m}\right],$$

in which the negative term diminishes when m increases, and the positive term increases provided

$$m > \frac{a^2+b^2+ab}{2ab}.$$

When $m = 1$, we find

$$n^2 = 4a^2b^2\zeta^2/(a+b)^4 = \omega^2 \text{ is positive.}$$

When $m = 2$, we find $\qquad n^2 = 0.$

When $m = 3$, we find

$$n^2 = 4a^2b^2\zeta^2\,(a-3b)(b-3a)/(a+b)^6,$$

and this is positive if $a < 3b$. Also when $3b > a > b$, it is easy to see that

$$3 > \frac{a^2 + b^2 + ab}{2ab},$$

so that, with the same condition, n^2 will be certainly positive for greater values of m.

Hence for all values of m except $m = 2$, n^2 is positive if

$$a < 3b \qquad \text{.................................} (22),$$

i.e., if the eccentricity of the ellipse is $< \frac{2}{3}\sqrt{2}$.

The case $m = 2$ corresponds to displacements in which the boundary remains elliptic. For any displacement which has no component of the second order, the motion is stable provided $a < 3b$.

When a small change in the character of the motion is made in such a way that the boundary remains elliptic, the new ellipse moves without change of form with an angular velocity given by Kirchhoff's equation (3). The new principal axes therefore rotate relatively to the principal axes of the undisturbed boundary with a small angular velocity, and the disturbed motion tends in a very long time to become finitely different from the undisturbed motion.

When a small general disturbance takes place we can regard it as resolved into a series of co-existent small motions superposed upon the undisturbed motion. All but one of these are oscillatory in character, provided $a < 3b$, and the remaining one is secular. It is better therefore to regard the motion as consisting of a series of co-existent small oscillations executed about a state of steady motion in a Kirchhoff's elliptic vortex slightly different from the undisturbed state.

It is worth while to remark that the motion is stable for displacements corresponding to $m = 1$. If the vortex cylinder is slightly displaced in any direction without change of form, it tends to oscillate about its original position as a mean. The period of these oscillations is the same as the period of rotation of the vortex.

Further, it is noteworthy that the ellipse may be as nearly circular as we please, and in the limit it rotates with an angular velocity $\frac{1}{2}\zeta$. In the limiting condition this may be regarded as an oscillation about the circular form, and the period is π/ω or $2\pi/\zeta$. This is in accordance with the known period for the oscillation of the second order (elliptic deformation) of a circular vortex column. (See Basset's *Hydrodynamics*, Vol. II., p. 40.)

We notice also that with given constant spin we can have a series of elliptic forms of the same area, each corresponding to a possible state of steady motion of the liquid. The eccentricity of the elliptic boundary may be as small as we please, and all the forms are stable unless the eccentricity exceeds $\frac{2}{3}\sqrt{2}$. When the eccentricity has this value, the motion corresponding to $m = 3$ becomes unstable, and it is therefore probable that there exists another series of forms for the boundaries of steady vortices with constant spin, of which one is an ellipse with this eccentricity, and the others are not ellipses.

8. We conclude this part of our investigation by showing that an elliptic vortex with uniform spin cannot move so as to remain elliptic unless it retains its shape and rotates with the angular velocity given by Kirchhoff's condition (3).

Since we have already shown that if the motion is steady, the principal axes must rotate with the said angular velocity, we have merely to show further that the ellipse cannot change its form and remain elliptic.

We take as the most general stream-function ψ' within an elliptic boundary for which the spin is uniform,

$$\psi' = \tfrac{1}{2}(\alpha x^2 + \beta y^2 + 2\gamma xy),$$

and we suppose the axes of x and y to be the principal axes of the elliptic boundary

$$x^2/a^2 + y^2/b^2 - 1 = 0$$

of the vortex, a and b being functions of the time, and further we suppose these axes to rotate with some angular velocity ω which may be a function of the time.

The condition that the surface of the vortex always contains the same particles is that

$$-\left[\frac{\dot{a}}{a}\frac{x^2}{a^2} + \frac{\dot{b}}{b}\frac{y^2}{b^2}\right] + (\gamma x + \beta y + \omega y)\frac{x}{a^2} - (\gamma y + \alpha x + \omega x)\frac{y}{b^2} = 0,$$

when $$x^2/a^2 + y^2/b^2 = 1.$$

This condition requires that

$$\gamma = \frac{\dot{a}}{a} = -\frac{\dot{b}}{b}, \quad \omega = -\frac{a^2\alpha - b^2\beta}{a^2 - b^2}.$$

Now, taking conjugate functions, as in § 1, we may write

$$\psi' = \tfrac{1}{4}c^2\left[\alpha\cosh^2\xi+\beta\sinh^2\xi+(\alpha\cosh^2\xi-\beta\sinh^2\xi)\cos 2\eta \right.$$
$$\left. +2\gamma\sinh\xi\cosh\xi\sin 2\eta\right],$$

and the form of ψ outside is

$$\psi = A\xi+Be^{-2\xi}\cos 2\eta+Ce^{-2\xi}\sin 2\eta.$$

To make $\dfrac{\partial\psi}{\partial\eta}$ and $\dfrac{\partial\psi}{\partial\xi}$ continuous at the surface, we must have

$$A = \tfrac{1}{2}(\alpha+\beta)\,ab,$$

$$B\,\frac{a-b}{a+b} = \tfrac{1}{4}(a^2\alpha-b^2\beta) = -\tfrac{1}{4}(\alpha-\beta)\,ab,$$

$$C\,\frac{a-b}{a+b} = \tfrac{1}{2}\gamma ab = -\tfrac{1}{2}\gamma\,(a^2+b^2).$$

These equations are incompatible, unless $\gamma = 0$, in which case $\dot{a}$ and $\dot{b}$ vanish, and the motion is steady. As C is now zero, the conditions become identical with those in § 2, and the vortex is a Kirchhoff's elliptic vortex.

IV. *Hill's Vortex.*

9. We now come to the investigation of the steady motion of Hill's vortex. Suppose that within the elliptic cylinder, whose equation, referred to axes of x, y rotating with angular velocity ω, is

$$x^2/a^2+y^2/b^2 = 1,$$

there is fluid in rotational motion with uniform spin ζ, and that between this surface and a rigid confocal envelope, whose equation is

$$x^2/a'^2+y^2/b'^2 = 1,$$

there is fluid in irrotational motion.

We take for the stream-function within the vortex

$$\psi' = \tfrac{1}{2}(\alpha x^2+\beta y^2),$$

where $$\alpha+\beta = -2\zeta \quad\text{.............................(23)};$$

then, just as in § 2, the surface

$$x^2/a^2+y^2/b^2 = 1$$

always contains the same particles, provided

$$\omega = -\frac{a^2\alpha-b^2\beta}{a^2-b^2} \quad\text{.........................(24).}$$

Taking conjugate functions as before, we write

$$\psi' = \tfrac{1}{4}c^2\left[(\alpha\cosh^2\xi+\beta\sinh^2\xi)+(\alpha\cosh^2\xi-\beta\sinh^2\xi)\cos 2\eta\right],$$

and the stream-function of the irrotational motion between the two confocals is of the form

$$\psi = A\xi + Be^{-2\xi}\cos 2\eta + Ce^{2\xi}\cos 2\eta \quad \text{..............(25).}$$

Suppose the value $\xi=\xi_0$ corresponds to the surface of the vortex, and the value $\xi=\xi_1$ corresponds to the rigid confocal envelope. The condition that the surface $\xi=\xi_1$ always contains the same particles is that

$$\psi+\tfrac{1}{2}\omega(x^2+y^2) = \text{const.}$$

when $\xi=\xi_1$. This condition gives us

$$Be^{-2\xi_1}+Ce^{2\xi_1}+\tfrac{1}{4}\omega c^2 = 0,$$

or

$$B\frac{a'-b'}{a'+b'}+C\frac{a'+b'}{a'-b'}+\tfrac{1}{4}\omega c^2 = 0 \quad \text{..................(26).}$$

The conditions of continuity of velocity are that

$$\frac{\partial\psi}{\partial\xi}=\frac{\partial\psi'}{\partial\xi}, \text{ and } \frac{\partial\psi}{\partial\eta}=\frac{\partial\psi'}{\partial\eta},$$

when $\xi=\xi_0$. These conditions give us

$$-B\frac{a-b}{a+b}+C\frac{a+b}{a-b}=\tfrac{1}{4}(\alpha-\beta)ab,$$

$$B\frac{a-b}{a+b}+C\frac{a+b}{a-b}=\tfrac{1}{4}(\alpha a^2-\beta b^2);$$

from which

$$\left.\begin{aligned} B &= \tfrac{1}{8}(a+b)(\alpha a+b\beta) \\ C &= \tfrac{1}{8}(a-b)(\alpha a-b\beta) \end{aligned}\right\} \quad \text{......................(27).}$$

Hence, substituting in (26), and using (24) to eliminate ω, we have

$$\alpha a\left[\frac{(a-b)(a'+b')}{a'-b'}+\frac{(a+b)(a'-b')}{a'+b'}-2a\right]$$
$$=\beta b\left[\frac{(a-b)(a'+b')}{a'-b'}-\frac{(a+b)(a'-b')}{a'+b'}-2b\right].$$

This equation, with (23) and (24), gives us

$$\frac{\alpha}{b\left[\dfrac{(a-b)(a'+b')}{a'-b'}-\dfrac{(a+b)(a'-b')}{a'+b'}-2b\right]}$$

$$=\frac{\beta}{a\left[\dfrac{(a-b)(a'+b')}{a'-b'}+\dfrac{(a+b)(a'-b')}{a'+b'}-2a\right]}$$

$$=-\frac{\zeta}{(a'^2+b'^2)-(a^2+b^2)}=-\frac{\omega\,(a'+b')^2\,(a+b)^2}{ab\left[(a'+b')^4-(a+b)^4\right]}\;\ldots\ldots\ldots(28),$$

and the angular velocity with which the envelope must rotate is

$$\omega=\frac{ab\zeta\left[(a'+b')^4-(a+b)^4\right]}{(a+b)^2(a'+b')^2\left[(a'^2+b'^2)-(a^2+b^2)\right]}\;\ldots\ldots\ldots(29),$$

which reduces to $2\zeta ab/(a+b)^2$, when a' and b' are infinite and equal.

Now, just as in Kirchhoff's vortex, the pressure equations inside and outside are

$$\text{const.}-\frac{p}{\rho}=\tfrac{1}{2}x^2\left[\alpha\omega-(\alpha+\omega)\,\beta\right]+\tfrac{1}{2}y^2\left[\beta\omega-(\beta+\omega)\,\alpha\right]\quad\text{inside,}$$

and $$\text{const.}-\frac{p}{\rho}=\tfrac{1}{2}\,(u^2+v^2)+u\omega y-v\omega x\quad\text{outside.}$$

Since we have already made u and v continuous in crossing the surface, the pressure will be continuous if

$$\tfrac{1}{2}x^2\left[\alpha\omega+(\alpha+\omega)\,\beta+\alpha^2\right]+\tfrac{1}{2}y^2\left[\beta\omega+(\beta+\omega)\,\alpha+\beta^2\right]$$

is constant when $$x^2/a^2+y^2/b^2=1.$$

This will be the case if

$$a^2\,(\alpha+\omega)(\alpha+\beta)=b^2\,(\beta+\omega)(\alpha+\beta),$$

i.e., if $$\omega=-(a^2\alpha-b^2\beta)/(a^2-b^2),$$

a condition previously obtained.

It follows that, as in the cases previously investigated, the pressure condition becomes an identity when the kinematical conditions are satisfied.

10. To investigate small oscillations about the state of steady motion just discussed, we have to suppose that the disturbed surface of the vortex is

$$F = \xi - (\xi_0 + \delta\xi) = 0,$$

where $\delta\xi$ is a function of η and t. The stream-functions inside and outside will be $\psi' + \delta\psi'$ and $\psi + \delta\psi$, where ψ' and ψ have been already found, and

$$\left.\begin{aligned} \delta\psi' &= \Sigma\left[A_m \cosh m\xi \cos m\eta + B_m \sinh m\xi \sin m\eta\right] \\ \delta\psi &= \Sigma\left[A'_m \cosh m\xi_0 \frac{\sinh m(\xi_1 - \xi)}{\sinh m(\xi_1 - \xi_0)} \cos m\eta \right. \\ &\qquad \left. + B'_m \sinh m\xi_0 \frac{\sinh m(\xi_1 - \xi)}{\sinh m(\xi_1 - \xi_0)} \sin m\eta\right] \end{aligned}\right\} \ldots (30).$$

We have to satisfy the conditions that $\partial(\psi + \delta\psi)/\partial\xi$ and $\partial(\psi + \delta\psi)/\delta\eta$ are continuous in crossing the surface

$$\xi = \xi_0 + \delta\xi,$$

the condition that this surface always contains the same particles, and the pressure condition.

Now $$\frac{\partial\psi'}{\partial\eta} = -\tfrac{1}{2}c^2 \sin 2\eta\,(\alpha \cosh^2\xi - \beta \sinh^2\xi)$$

$$= -\tfrac{1}{2}(\alpha a^2 - \beta b^2) \sin 2\eta - (\alpha - \beta)\, ab \sin 2\eta\, \delta\xi,$$

when $\xi = \xi_0 + \delta\xi$.

Also

$$\frac{\partial\psi}{\partial\eta} = -2 \sin 2\eta\,(Be^{-2\xi} + Ce^{2\xi})$$

$$= -2\left[B\frac{a-b}{a+b} + C\frac{a+b}{a-b}\right] \sin 2\eta - 4\left[-B\frac{a-b}{a+b} + C\frac{a+b}{a-b}\right] \sin 2\eta\, \delta\xi,$$

when $\xi = \xi_0 + \delta\xi$.

The values of $\partial\psi'/\partial\eta$ and $\partial\psi/\partial\eta$ are identical, and it follows that we must have

$$\partial\delta\psi'/\partial\eta = \partial\delta\psi/\partial\eta,$$

and hence that $\quad A_m = A'_m$, and $B_m = B'_m$.

We shall therefore, in what follows, suppress the accents on A'_m and B'_m.

D 2

Again $\dfrac{\partial \psi'}{\partial \xi} = \tfrac{1}{2}c^2 \sinh \xi \cosh \xi \left[(\alpha+\beta)+(\alpha-\beta)\cos 2\eta\right]$

$$= \tfrac{1}{2}\left[(\alpha+\beta)+(\alpha-\beta)\cos 2\eta\right]\left[ab+(a^2+b^2)\,\delta\xi\right],$$

when $\xi = \xi_0+\delta\xi$.

Also

$$\frac{\partial \psi}{\partial \xi} = A+(2Ce^{2\xi}-2Be^{-2\xi})\cos 2\eta$$

$$= A+2\left[C\frac{a+b}{a-b}-B\frac{a-b}{a+b}\right]\cos 2\eta+4\left[C\frac{a+b}{a-b}+B\frac{a-b}{a+b}\right]\cos 2\eta\,\delta\xi,$$

when $\xi = \xi_0+\delta\xi$.

Hence

$$\frac{\partial \psi'}{\partial \xi}-\frac{\partial \psi}{\partial \xi} = \tfrac{1}{2}\delta\xi\left\{(\alpha+\beta)(a^2+b^2)+\cos 2\eta\left[(\alpha-\beta)(a^2+b^2)-2(\alpha a^2-\beta b^2)\right]\right\}$$

$$= \tfrac{1}{2}(\alpha+\beta)\,\delta\xi\left[(a^2+b^2)-(a^2-b^2)\cos 2\eta\right]$$

$$= -2\zeta\frac{\delta\xi}{h_0^2},$$

where, as before, $h_0^{-2} = a^2\sin^2\eta+b^2\cos^2\eta$.

The condition that

$$\partial(\psi'+\delta\psi')/\partial\xi = \partial(\psi+\delta\psi)/\partial\xi,$$

when $\xi = \xi_0+\delta\xi$, now becomes

$$-2\zeta\frac{\delta\xi}{h_0^2}+\Sigma m\left[A_m\left\{\sinh m\xi_0+\cosh m\xi_0\coth m(\xi_1-\xi_0)\right\}\cos m\eta\right.$$
$$\left.+B_m\left\{\cosh m\xi_0+\sinh m\xi_0\coth m(\xi_1-\xi_0)\right\}\sin m\eta\right] = 0 \qquad \text{.........(31).}$$

11. The condition that the surface

$$F = \xi-(\xi_0+\delta\xi) = 0$$

always contains the same particles, is (as in § 5)

$$-\frac{\partial\delta\xi}{\partial t}+h^2\frac{\partial\left[F,\ \psi'+\delta\psi'+\tfrac{1}{2}\omega(x^2+y^2)\right]}{\partial(\xi,\eta)} = 0,$$

when $\xi = \xi_0+\delta\xi$.

Now we have

$$\frac{\partial(F, \psi')}{\partial(\xi, \eta)} = \frac{\partial\psi'}{\partial\eta} + \frac{\partial\psi'}{\partial\xi}\frac{\partial\delta\xi}{\partial\eta}$$

$$= -\tfrac{1}{2}(\alpha a^2 - \beta b^2)\sin 2\eta - ab(\alpha-\beta)\sin 2\eta\,\delta\xi$$

$$+\tfrac{1}{2}ab\left[(\alpha+\beta)+(\alpha-\beta)\cos 2\eta\right]\frac{\partial\delta\xi}{\partial\eta},$$

when $\xi = \xi_0 + \delta\xi$, and squares of small quantities are rejected.

Again $\frac{\partial(F, \delta\psi')}{\partial(\xi, \eta)} = \frac{\partial\delta\psi'}{\partial\eta}$ under the same conditions.

Also $\frac{\partial\{F, \frac{1}{2}\omega(x^2+y^2)\}}{\partial(\xi, \eta)} = -\tfrac{1}{2}\omega(a^2-b^2)\sin 2\eta + \omega ab\frac{\partial\delta\xi}{\partial\eta}$

under the same conditions.

Hence the condition that the surface $\xi = \xi_0 + \delta\xi$ always contains the same particles becomes

$$-\frac{1}{h^2}\frac{\partial\delta\xi}{\partial t} + ab\frac{\partial}{\partial\eta}\left[\left\{\tfrac{1}{2}(\alpha+\beta)+\omega+\tfrac{1}{2}(\alpha-\beta)\cos 2\eta\right\}\delta\xi\right] + \frac{\partial\delta\psi'}{\partial\eta} = 0.$$

Now $\tfrac{1}{2}(\alpha+\beta)+\omega = \tfrac{1}{2}(\alpha+\beta) - \frac{a^2\alpha - b^2\beta}{a^2-b^2} = -\tfrac{1}{2}(\alpha-\beta)\frac{(a^2+b^2)}{a^2-b^2}$.

Thus the above equation becomes

$$\frac{\partial}{\partial t}\left(\frac{\delta\xi}{h_0^2}\right) + \frac{ab(\alpha-\beta)}{a^2-b^2}\frac{\partial}{\partial\eta}\left(\frac{\delta\xi}{h_0^2}\right) \dots\dots$$

$$+\Sigma m\left[A_m\cosh m\xi_0\sin m\eta - B_m\sinh m\xi_0\cos m\eta\right] = 0 \quad \dots(32).$$

12. Again, just as in the case of Kirchhoff's vortex, the pressure equations inside and outside the vortex can be written :—

inside $\text{const.} - \frac{p+p'}{\rho} = \tfrac{1}{2}\{\alpha\omega - (\alpha+\omega)\beta\}x^2 + \tfrac{1}{2}\{\beta\omega - (\beta+\omega)\alpha\}y^2$

$$+\frac{\partial\delta\phi'}{\partial t} + (u+\omega y)u' + (v-\omega x)v' + 2\zeta\delta\psi',$$

outside $\text{const.} - \frac{p+p'}{\rho} = u\omega y - v\omega x + \tfrac{1}{2}(u^2+v^2)$

$$+\frac{\partial\delta\phi}{\partial t} + uu' + vv' + u'\omega y - v'\omega x.$$

As we have already made the velocities continuous at the surface, the condition of continuity of pressure becomes

$$\frac{\partial\delta\phi'}{\partial t}-\frac{\partial\delta\phi}{\partial t}+2\zeta\delta\psi'-\tfrac{1}{2}(\alpha+\beta)\left[(\alpha+\omega)x^2+(\beta+\omega)y^2\right]=\text{const.},$$

when $\xi=\xi_0+\delta\xi$.

The last term is $\tfrac{1}{2}(\alpha+\beta)\dfrac{\alpha-\beta}{a^2-b^2}a^2b^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)$,

and this is $\qquad \tfrac{1}{2}(\alpha+\beta)\dfrac{(\alpha-\beta)\,a^2b^2}{a^2-b^2}\left(1-\dfrac{2\delta\xi}{abh_0^2}\right)$,

Also, we have

$$\delta\phi'=\Sigma\left[B_m\cosh m\xi\cos m\eta-A_m\sinh m\xi\sin m\eta\right],$$

$$\delta\phi=\Sigma\left[-B_m\sinh m\xi_0\frac{\cosh m(\xi_1-\xi)}{\sinh m(\xi_1-\xi_0)}\cos m\eta\right.$$
$$\left.+A_m\cosh m\xi_0\frac{\cosh m(\xi_1-\xi)}{\sinh m(\xi_1-\xi_0)}\sin m\eta\right].$$

Thus the pressure equation becomes

$$\Sigma\left[\dot{B}_m\left\{\cosh m\xi_0+\sinh m\xi_0\coth m(\xi_1-\xi_0)\right\}\cos m\eta\right.$$
$$\left.-\dot{A}_m\left\{\sinh m\xi_0+\cosh m\xi_0\coth m(\xi_1-\xi_0)\right\}\sin m\eta\right]$$
$$+2\zeta\Sigma\left[A_m\cosh m\xi_0\cos m\eta+B_m\sinh m\xi_0\sin m\eta\right]$$
$$+2\zeta\frac{(\alpha-\beta)\,ab}{a^2-b^2}\frac{\delta\xi}{h_0^2}=0 \quad\ldots\ldots(33).$$

13. To discuss the stability of the vortex, we notice first that the equation (31) gives us

$$\frac{\delta\xi}{h_0^2}=\frac{1}{2\zeta}\Sigma m\left[A_m\frac{\cosh m\xi_1}{\sinh m(\xi_1-\xi_0)}\cos m\eta+B_m\frac{\sinh m\xi_1}{\sinh m(\xi_1-\xi_0)}\sin m\eta\right].$$

Hence equation (32) becomes

$$\Sigma\left[\frac{\dot{A}_m}{2\zeta}\frac{\cosh m\xi_1}{\sinh m(\xi_1-\xi_0)}\cos m\eta+\frac{\dot{B}_m}{2\zeta}\frac{\sinh m\xi_1}{\sinh m(\xi_1-\xi_0)}\sin m\eta\right]$$
$$+\frac{(\alpha-\beta)\,ab}{a^2-b^2}\Sigma m\left[\frac{B_m}{2\zeta}\frac{\sinh m\xi_1}{\sinh m(\xi_1-\xi_0)}\cos m\eta-\frac{A_m}{2\zeta}\frac{\cosh m\xi_1}{\sinh m(\xi_1-\xi_0)}\sin m\eta\right]$$
$$+\Sigma\left[A_m\cosh m\xi_0\sin m\eta-B_m\sinh m\xi_0\cos m\eta\right]=0.$$

This gives us

$$\frac{\dot{A}_m}{2\zeta}+B_m\left[\frac{mab\,(\alpha-\beta)}{2\zeta\,(a^2-b^2)}\tanh m\xi_1-\frac{\sinh m\xi_0\sinh m\,(\xi_1-\xi_0)}{\cosh m\xi_1}\right]=0,$$

$$\frac{\dot{B}_m}{2\zeta}-A_m\left[\frac{mab\,(\alpha-\beta)}{2\zeta\,(a^2-b^2)}\coth m\xi_1-\frac{\cosh m\xi_0\sinh m\,(\xi_1-\xi_0)}{\sinh m\xi_1}\right]=0,$$

and it is easy to see that the pressure equation (33) gives rise to the same two equations.

Now supposing that A_m and B_m are proportional to e^{int}, we find, for the frequency of the mth harmonic component vibration,

$$\frac{n^2}{\zeta^2}=\left[\frac{m\,(\alpha-\beta)\,ab}{\zeta\,(a^2-b^2)}-\frac{2\sinh m\xi_0\sinh m\,(\xi_1-\xi_0)}{\sinh m\xi_1}\right]$$

$$\times\left[\frac{m\,(\alpha-\beta)\,ab}{\zeta\,(a^2-b^2)}-\frac{2\cosh m\xi_0\sinh m\,(\xi_1-\xi_0)}{\cosh m\xi_1}\right]\ldots(34).$$

It is easy to verify that this reduces to equation (21) when ξ_1 is made infinite.

14. We shall consider the cases $m=1$ and $m=2$. First taking $m=1$, we have

$$\frac{n^2}{\zeta^2}=\left[\frac{ab\,(\alpha-\beta)}{\zeta\,(a^2-b^2)}-\frac{\cosh\xi_1-\cosh\,(\xi_1-2\xi_0)}{\sinh\xi_1}\right]$$

$$\times\left[\frac{ab\,(\alpha-\beta)}{\zeta\,(a^2-b^2)}-\frac{\sinh\xi_1+\sinh\,(\xi_1-2\xi_0)}{\cosh\xi_1}\right]\ldots(35).$$

Now we have, by (28),

$$\frac{ab\,(\alpha-\beta)}{\zeta\,(a^2-b^2)}=\frac{ab\left[(a'+b')^2-(a+b)^2\right]^2}{(a+b)^2\,(a'+b')^2\left[(a'^2+b'^2)-(a^2+b^2)\right]}$$

$$=\frac{\cosh\xi_0\sinh\xi_0\,(e^{2\xi_1}-e^{2\xi_0})^2}{e^{2\xi_1}e^{2\xi_0}\,(\cosh 2\xi_1-\cosh 2\xi_0)}.$$

Also

$$\frac{\cosh\xi_1-\cosh\,(\xi_1-2\xi_0)}{\sinh\xi_1}=\frac{a'}{b'}\,\frac{2b^2}{c^2}+\frac{2ab}{c^2}=\frac{2\sinh\xi_0}{\sinh\xi_1}\sinh\,(\xi_1-\xi_0).$$

The first factor on the right-hand side of (35) can now be reduced to

$$-\frac{\cosh\xi_1\sinh^2\xi_0\,(e^{2\xi_1}-e^{2\xi_0})^2}{e^{2\xi_1}e^{2\xi_0}\,(\cosh 2\xi_1-\cosh 2\xi_0)\sinh\xi_1}.$$

In like manner the second factor can be reduced to

$$-\frac{\sinh \xi_1 \cosh^2 \xi_0 \,(e^{2\xi_1}-e^{2\xi_0})^2}{e^{2\xi_1} e^{2\xi_0} (\cosh 2\xi_1 - \cosh 2\xi_0) \cosh \xi_1}.$$

Hence the frequency for modes corresponding to $m = 1$ is given by

$$\frac{n^2}{\zeta^2} = \frac{a^2b^2\left[(a'+b')^2-(a+b)^2\right]^4}{(a+b)^4(a'+b')^4\left[(a'^2+b'^2)-(a^2+b^2)\right]^2} \quad \text{.........(36).}$$

This agrees with a result found in § 7, when $a' = b' = \infty$.

Next, taking $m = 2$, we can find the equation

$$\frac{n^2}{\zeta^2} = \left[\frac{2ab\,(\alpha-\beta)}{\zeta c^2} - \frac{4ab\,(a^2+b^2)}{c^4}\left(1-\frac{ab}{a'b'}\,\frac{a'^2+b'^2}{a^2+b^2}\right)\right]$$
$$\times\left[\frac{2ab\,(\alpha-\beta)}{\zeta c^2} - \frac{4ab\,(a^2+b^2)}{c^4}\left(\frac{a'b'}{ab}\,\frac{a^2+b^2}{a'^2+b'^2}-1\right)\right],$$

and, by using analysis similar to that used in the case $m = 1$, we can transform this into

$$\frac{n^2}{\zeta^2} = \frac{4a^2b^2\left[(ab+a'b')(a^2+b^2)-ab\,(a'^2+b'^2)\right]\left[(a'+b')^2-(a+b)^2\right]^4}{(a+b)^4(a'+b')^4\,a'b'\,(a'^2+b'^2)\left[(a'^2+b'^2)-(a^2+b^2)\right]^2}$$
$$\text{............(37).}$$

It follows that the motion is stable for this kind of displacement, and the frequency tends to zero when $a' = b' = \infty$.

15. Although the steady motion has been proved to be stable for displacements corresponding to $m = 1$ and $m = 2$, yet a comparison of the general frequency-equation (34) with that (21) which holds in the case of Kirchhoff's vortex, shows that for very elongated forms of section, the motion must be unstable for displacements in which there are more than two wave-lengths to the circumference, and it is in fact not difficult to verify the occurrence of instability in the limiting case when the boundary of the vortex shrinks to the line of foci, so that the vortex itself reduces to a special vortex-sheet. When the boundary of the vortex is such that b is very small compared with a, it is necessary to suppose that ζ becomes infinite in such a way that ζb is finite. The strength of the vortex sheet, at any point distant x from the centre, is then $2\zeta y$, or $2\zeta b\sqrt{(1-x^2/a^2)}$. In

equation (34), we must multiply up by ζ^2, and obtain

$$n^2 = \left[m\frac{ab\,(\alpha-\beta)}{a^2-b^2} - \zeta\frac{2\sinh m\xi_0 \sinh m\,(\xi_1-\xi_0)}{\sinh m\xi_1}\right]$$

$$\times\left[m\frac{ab\,(\alpha-\beta)}{a^2-b^2} - \zeta\frac{2\cosh m\xi_0 \sinh m\,(\xi_1-\xi_0)}{\cosh m\xi_1}\right] \ldots (38).$$

The term

$$m\frac{ab\,(\alpha-\beta)}{a^2-b^2} = m\zeta ab\frac{\left[(a'+b')^2-(a+b)^2\right]^2}{(a+b)^2\,(a'+b')^2\left[(a'^2+b'^2)-(a^2+b^2)\right]},$$

and this is finite under the conditions contemplated. The term

$$\zeta\frac{2\sinh m\xi_0 \sinh m\,(\xi_1-\xi_0)}{\sinh m\xi_1}$$

is also finite, but the term

$$\zeta\frac{2\cosh m\xi_0 \sinh m\,(\xi_1-\xi_0)}{\cosh m\xi_1}$$

is ultimately infinite. Thus the second factor of (38) is ultimately infinite and negative. We must therefore investigate the sign of the first factor, when ξ_0 vanishes, and $\zeta\xi_0$ has a finite limit. Now it is not very difficult to show that the finite limits of the two terms of the first factor of (38) destroy each other. In fact these limits are respectively

$$m\zeta ab\frac{(e^{2\xi_1}-1)^2}{a^2e^{2\xi_1}\,(\cosh 2\xi_1-1)} \quad\text{and}\quad -2m\zeta\frac{b}{a}.$$

It follows that the first factor of (38) is small of the order $\zeta\xi_0^2$, and it is necessary to expand it as far as ξ_0^2 in order to determine its sign.

Now the term

$$m\frac{ab\,(\alpha-\beta)}{a^2-b^2} = m\zeta\frac{(e^{2\xi_1}-e^{2\xi_0})^2\sinh\xi_0\cosh\xi_0}{e^{2\xi_0}e^{2\xi_1}\,(\cosh 2\xi_1-\cosh 2\xi_0)}$$

$$= 2m\zeta\sinh\xi_0\cosh\xi_0\frac{e^{2\xi_1}-e^{2\xi_0}}{e^{2\xi_1}e^{2\xi_0}-1}$$

$$= 2m\zeta\xi_0\,(1-2\xi_0\coth\xi_1) + \text{terms in } \xi_0^3.$$

Again, the term

$$2\zeta\frac{\sinh m\xi_0\sinh m\,(\xi_1-\xi_0)}{\sinh m\xi_1} = \zeta\frac{\cosh m\xi_1-\cosh m\,(\xi_1-2\xi_0)}{\sinh m\xi_1}$$

$$= 2m\zeta\xi_0\,(1-m\xi_0\coth m\xi_1) + \text{terms in } \xi_0^3.$$

Also the second factor of (38) is ultimately

$$-2\zeta \tanh m\xi_1.$$

Hence (38) becomes ultimately

$$n^2 = 4\zeta^2\xi_0^2 m \left[2 \coth \xi_1 \tanh m\xi_1 - m \right].$$

Of the expression in the square brackets, the first term can never be greater than 2 coth ξ_1, but the second term increases indefinitely as larger values of m are taken, and thus there are always values of m for which the vortex sheet becomes unstable.

Thursday, December 14th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

The following gentlemen were elected members:—Arthur Berry, M.A., Fellow of King's College, Cambridge; J. H. Hooker, M.A., late Scholar of Queens' College, Cambridge; F. H. Jackson, M.A., Mathematical Master, Cowbridge School, S. Wales; A. H. Leahy, M.A., late Fellow of Pembroke College, Cambridge; and C. Morgan, M.A., Naval Instructor, Royal Naval College, Greenwich.

The Auditor (the Rev. T. R. Terry) having made his Report, a vote of thanks to him, for the trouble he had taken, was moved by Mr. Basset, seconded by the President, and carried unanimously.

The adoption of the Treasurer's Report was then moved by Lt.-Col. Campbell, seconded by Major MacMahon, and carried.

Mr. Basset read a paper on "The Stability of a Deformed Elastic Wire." Mr. Dallas gave an account of his paper, entitled "The Linear Automorphic Transformations of certain Quantics." Dr. Hobson gave a brief outline of a paper on "Bessel's Functions and Relations connecting them with Spherical and Hyperspherical Harmonics."

The following communications were taken as read:—

A Theorem of Liouville's: Prof. Mathews.

Note on non-Euclidian Geometry: Mr. H. F. Baker.

Note on an Identity in Elliptic Functions: Prof. L. J. Rogers.

Note on a Variable Seven-Points Circle analogous to the Brocard Circle of a Plane Triangle: Mr. J. Griffiths.

The following presents were received :—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stück 10; Leipzig, 1893.

D'Ocagne, Mons. M.—"Sur la Construction des Cubiques Cuspidales par Points et Tangentes" (extrait des "Nouvelles Annales de Mathématiques"); "Sur une Classe de Transformations dans le Triangle, et notamment sur certaine Transformation Quadratique Irrationnelle" (extrait des "Nouvelles Annales de Mathématiques"); "Remarque sur la Déformation des Surfaces de Révolution" (extrait de la "Bulletin de la Société de France"); "Sur la Sommation d'une certaine Classe de Séries"; "Sur une Méthode Nomographique, applicable à des équations pouvant contenir jusqu'à dix variables" (Comptes rendus).

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVII., Livraison 3; Harlem, 1893.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XI., No. 5; Coimbra, 1893.

"The Physical Society of London—Proceedings," Vol. XII., Pt. 2; October, 1893.

"Berichte über die Verhandlungen der Königl. Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1893, 4, 5, 6.

"Kansas University Quarterly," Vol. II., No. 2; October, 1893.

"Bulletin des Sciences Mathématiques," Tome XVII., 1893, Aout et Septembre; Paris.

"Bulletin of the New York Mathematical Society," Vol. III., No. 2; New York, 1893.

"Transactions of the Canadian Institute," Vol. III., Pt. 2, No. 6; Toronto, 1893; and the fifth "Annual Report," 1892-3.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 9, 2 Sem.; Roma, 1893.

"Journal of the Japan College of Science," Vol. VI., Part 3; Tokyo, 1893.

"Journal für die reine und angewandte Mathematik," Bd. CXII., Heft 4; Berlin.

"Acta Mathematica," XVII., 3, 4; Stockholm, 1893.

"Annales de la Faculté des Sciences de Toulouse," Tome VII., Fasc. 3; Paris, 1893.

"Educational Times," December, 1893.

"Memorie della Regia Accademia di Scienze, Lettere, ed Arti in Modena," Serie 2, Vol. VIII.; Modena, 1892.

Cayley, A.—"Mathematical Papers," Vol. VI.

"Indian Engineering," Vol. XIV., Nos. 17-21.

"American Journal of Mathematics," Vol. XV., No. 4.

Note on an Identity in Elliptic Functions. *By* Prof. L. J. ROGERS.
Received December 8th, 1893. Read December 14th, 1893.

1. In the *Fund. Nov.*, see Vol. I., p. 336, of his collected works, Jacobi establishes the identity,

$$\operatorname{sn} a \operatorname{sn} b + \operatorname{sn} x \operatorname{sn}(x+a+b) - \operatorname{sn}(x+a)\operatorname{sn}(x+b)$$
$$= k^2 \operatorname{sn} a \operatorname{sn} b \operatorname{sn} x \operatorname{sn}(x+a+b)\operatorname{sn}(x+a)\operatorname{sn}(x+b) \quad\ldots(1).$$

By putting $\qquad x = u - \frac{m}{2}, \quad a = \frac{m-l}{2}, \quad b = \frac{m+l}{2},$

this identity becomes

$$\operatorname{sn}\left(u-\frac{m}{2}\right)\operatorname{sn}\left(u+\frac{m}{2}\right) - \operatorname{sn}\left(u-\frac{l}{2}\right)\operatorname{sn}\left(u+\frac{l}{2}\right) + \operatorname{sn}\frac{m-l}{2}\operatorname{sn}\frac{m+l}{2}$$
$$= k^2 \operatorname{sn}\left(u-\frac{m}{2}\right)\operatorname{sn}\left(u+\frac{m}{2}\right)\operatorname{sn}\frac{u-l}{2}\operatorname{sn}\left(u+\frac{l}{2}\right)\operatorname{sn}\frac{m-l}{2}\operatorname{sn}\frac{m+l}{2}$$
$$\ldots\ldots\ldots\ldots\ldots(2).$$

It will be found convenient to write U_r for

$$\operatorname{sn}\left(u-\frac{rl}{2}\right)\operatorname{sn}\left(u+\frac{rl}{2}\right),$$

and $\operatorname{sn} rl!$ for $\qquad \operatorname{sn} l \operatorname{sn} 2l \ldots \operatorname{sn} rl,$

when r is a positive integer.

By (2), putting $m = 3l$, we get

$$k^2 U_1 U_3 \operatorname{sn} l \operatorname{sn} 2l = U_3 - U_1 + \operatorname{sn} l \operatorname{sn} 2l,$$

or, as we may write it, so as to correspond more exactly to formula to be obtained subsequently,

$$k^2 U_1 U_3 \operatorname{sn} 3l! = U_3 \operatorname{sn} 3l - \frac{\operatorname{sn} 3l}{\operatorname{sn} l} U_1 \operatorname{sn} l + \operatorname{sn} 3l! \quad\ldots\ldots\ldots(3).$$

Multiplying by U_5, we get on the right-hand side the products $U_3 U$ and $U_1 U_5$, which, by (1), may be replaced by linear functions of U_1 U_3, and U_5.

By actual calculation it will be found, after replacing

$$\frac{\operatorname{sn} 3l}{\operatorname{sn} l} - 1 + k^2 \operatorname{sn} l \operatorname{sn} 2l \operatorname{sn} 3l \operatorname{sn} 4l \quad \text{by} \quad \frac{\operatorname{sn} 4l}{\operatorname{sn} 2l},$$

to which it may be reduced by (1), we get

$$k^4 U_1 U_3 U_5 \,\mathrm{sn}\, 5l! = U_5 \,\mathrm{sn}\, 5l - \frac{\mathrm{sn}\, 5l}{\mathrm{sn}\, l} U_3 \,\mathrm{sn}\, 3l + \frac{\mathrm{sn}\, 5l \,\mathrm{sn}\, 4l}{\mathrm{sn}\, l \,\mathrm{sn}\, 2l} U_1 \,\mathrm{sn}\, l \quad \ldots(4).$$

It will be observed that in (3) and (4) the successive coefficients of expressions of the form $U_r \,\mathrm{sn}\, rl$ bear a similarity to binomial coefficients, each factor in the latter being replaced by a corresponding elliptic sine.

The object of the present paper is to establish a general formula for the product $U_1 U_3 U_5 \ldots$ to n factors.

It is easily seen, by continued repetition of Jacobi's formula, that we may reduce the product $U_1 U_3 U_5 \ldots$ to a linear expression in the U's, which may contain a term independent of u, as in (3), or not, as in (4).

Thus we may assume that

$$k^{2n} U_1 U_3 U_5 \ldots U_{2n+1} \,\mathrm{sn}\,(2n+1)\,l!$$

$$= A_0 U_{2n+1} \,\mathrm{sn}\,(2n+1)\,l - A_1 U_{2n-1} \,\mathrm{sn}\,(2n-1)\,l + \ldots + L_{2n+1} \ldots\ldots(5),$$

where L_{2n+1} is independent of u.

We see then, by (3) and (4), that

$$L_3 = \mathrm{sn}\, 3l!, \quad \text{and} \quad L_5 = 0.$$

By changing u into $u+K'i$, $\mathrm{sn}\, u$ becomes $1 \div k \,\mathrm{sn}\, u$, so that (5) becomes

$$\frac{\mathrm{sn}\,(2n+1)\,l!}{U_1 U_3 U_5 \ldots} = \frac{A_0 \,\mathrm{sn}\,(2n+1)\,l}{U_{2n+1}} - \frac{A_1 \,\mathrm{sn}\,(2n-1)\,l}{U_{2n-1}} + \ldots + k^2 L_{2n-1} \ldots(6).$$

By making $u = \frac{l}{2}, \frac{3l}{2} \ldots$ in succession, we may obtain the values of all the coefficients $A_0, A_1, \ldots$.

These coefficients may best be evaluated by considering the corresponding algebraic identity

$$\frac{(2n+1)!}{\left(x^2 - \frac{1}{4}\right)\left(x^2 - \frac{3^2}{4}\right)\ldots} = A_0 \frac{2n+1}{x^2 - \left(\frac{2n+1}{2}\right)^2} - A_1 \frac{2n-1}{x^2 - \left(\frac{2n-1}{2}\right)^2} + \ldots + C.$$

Let $$x = \frac{2r+1}{2};$$

then $(2n+1)!/r(r+1)(r-1)(r+2)\ldots$

$$\ldots 1 \,.\, 2r(2r+1)(-1)(2r+2)(-2)(2r+3)\ldots(-n+r)(r+n+1)$$
$$= (-1)^{n-r}A_{n-r};$$

therefore $$A_{n-r} = \frac{(2n+1)!}{(n-r)!\,(n+r+1)!},$$

so that the A's in (5) and (6) are fully established as corresponding to binomial coefficients, as we observed in (3) and (4), and

$$A_{n-r} = \frac{\operatorname{sn}(2n+1)!}{\operatorname{sn}(n-r)\,l!\,\operatorname{sn}(n+r+1)\,l!} \quad \ldots\ldots\ldots\ldots\ldots(7).$$

It remains now to find L_{2n+1}.

Multiplying (5) by $k^2U_{2n+3}\operatorname{sn}(2n+2)\,l\operatorname{sn}(2n+3)\,l$,

and remarking that, by (1),

$$k^2U_{2n+1}U_{2n+3} = \frac{U_{2n+3}-U_{2n+1}+\operatorname{sn}l\operatorname{sn}(2n+2)\,l}{\operatorname{sn}l\operatorname{sn}(2n+2)\,l}, \text{ \&c. } \ldots,$$

we see that the coefficient of U_{2n+3} in the new product is similar to the numerical expression

$$(2n+2)(2n+3)\left\{\frac{2n+1}{(2n+2)1} - \binom{2n+1}{1}\frac{2n-1}{(2n+1)2} + \binom{2n+1}{2}\frac{2n-3}{2n\,.\,3} - \ldots + k^2L_{2n+1}\right\}.$$

But we know, by (7), that this is $(2n+3)$; therefore

$$k^2L_{2n+1}(2n+2)(2n+3)$$
$$= (2n+3) - \binom{2n+3}{1}(2n+1) + \binom{2n+3}{2}(2n-1) - \ldots \quad \ldots(8),$$

where each numerical factor has to be replaced by the corresponding elliptic sine.

Similarly calculating the absolute term L_{2n+5} in

$$k^{2n+4}U_1U_3\ldots U_{2n+5}\operatorname{sn}(2n+5)\,l!,$$

by multiplying $\quad k^{2n+2}U_1U_3\ldots U_{2n+3}\operatorname{sn}(2n+3)\,l!$

by $\quad k^2U_{2n+5}\operatorname{sn}(2n+4)\,l\operatorname{sn}(2n+5)\,l,$

and noticing that by Jacobi's formula the absolute term, in any expression such as $k^2 U_{2n+5} U_{2r+1}$, is unity, we see that, in the corresponding algebraic form, we get

$$L_{2n+5} = (2n+4)(2n+5)\left\{(2n+3) - \binom{2n+3}{1}(2n+1) + \ldots\right\} \quad \ldots(9).$$

Hence, by (8),

$$L_{2n+5} = k^2 \operatorname{sn}(2n+2)\, l \operatorname{sn}(2n+3)\, l \operatorname{sn}(2n+4)\, l \operatorname{sn}(2n+5)\, l \,.\, L_{2n+1} \quad \ldots\ldots\ldots(10).$$

From the known values of L_3 and L_5, we see then that

$$L_{4m+1} = 0,$$

while $$L_{4m+3} = k^{2m} \operatorname{sn}(4m+3)\, l! \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

We have therefore completely determined a formula

$$k^{2n+1} U_1 U_3 \ldots U_{2n+1} \operatorname{sn}(2n+1)\, l!$$

$$= U_{2n+1} \operatorname{sn}(2n+1)\, l - \frac{\operatorname{sn}(2n+1)\, l}{\operatorname{sn} l} U_{2n-1} \operatorname{sn}(2n-1)\, l + \ldots + L_{2n+1} \quad \ldots\ldots\ldots\ldots(12),$$

where $$L_{2n+1} = 0 \text{ if } n \text{ is even},$$

and $$L_{2n+1} = k^{n-1} \operatorname{sn}(2n+1)\, l! \text{ if } n \text{ is odd},$$

a formula which gives an easily remembered equivalent for the product of $2n$ elliptic sines whose arguments are in arithmetic progression.

By changing u into $u+K'i$, we moreover deduce a similar expression for the reciprocal of such a product.

So, too, by the various transformations in elliptic functions, we may deduce corresponding formulae for tn-functions, cn-functions, dn-functions, sines, tangents, and cosines.

2. A similar formula may be derived from Jacobi's formula by writing

$$x = u+l, \quad a = -m-l, \quad b = -u-m,$$

whence $$k^2 U_m \operatorname{sn}(u+l) \operatorname{sn}(m+l) \operatorname{sn}(m-l) \operatorname{sn} 2m$$

$$= \operatorname{sn}(m+l) \operatorname{sn}(u+m) + \operatorname{sn}(m-l) \operatorname{sn}(u-m) - \operatorname{sn} 2m \operatorname{sn}(u+l) \ldots(1).$$

Thus $$k^2 \operatorname{sn} u \,.\, U_2 \operatorname{sn} l \operatorname{sn} 2l = \operatorname{sn}(u+l) - \frac{\operatorname{sn} 2l}{\operatorname{sn} l} \operatorname{sn} u + \operatorname{sn}(u-l) \quad \ldots(3).$$

By multiplying by U_4 and reducing such products as $U_4 \operatorname{sn}(u+rl)$ by (1), we get a linear expression in $\operatorname{sn}(u+2l)$, $\operatorname{sn}(u+l)$... for the product $k^4 \operatorname{sn} u \,.\, U_2 U_4 \operatorname{sn} 4l!$, there being no absolute term.

Similarly $\qquad k^{2n} \operatorname{sn} u \,.\, U_2 U_4 \dots U_{2n} \operatorname{sn} 2nl!$

may be reduced to the form

$$A_0 \operatorname{sn}(u+nl) - A_1 \operatorname{sn}\{u+(n-1)l\} + \dots - A_1 \operatorname{sn}\{u-(n-1)l\} + A_0 \operatorname{sn}(u-nl),$$

when the A's are independent of u.

By changing u into $u+K'i$, as in the last section, we can evaluate the coefficients, which will be found to correspond to those in the expansion of $(1-x)^{2n}$; thus

$$k^{2n} \operatorname{sn} u \,.\, U_2 U_4 \dots U_{2n} \operatorname{sn} 2nl!$$

$$= \operatorname{sn}(u+nl) - \frac{\operatorname{sn} 2nl}{\operatorname{sn} l}\{\operatorname{sn} u+(n-1)l\} + \frac{\operatorname{sn} 2nl \operatorname{sn}(2n-1)l}{\operatorname{sn} l \operatorname{sn} 2l} \operatorname{sn}\{u+(n-2)l\} - \dots \quad \dots\dots\dots(3).$$

This formula will also, by the various transformations in elliptic functions, lead to others, giving like relations connecting tn-functions, &c.

Thus k may be changed into k', and sn into tn throughout, and further we may put $k = 1$ and write tan for sn.

If, again, in this trigonometrical identity, we put

$$e^{-u} = q^{\frac{1}{2}}, \quad \text{and} \quad n = \infty,$$

we get the formula connecting the q-series and the q-product form for

$$\operatorname{dn} \frac{2K}{\pi} u \operatorname{tn} \frac{2K}{\pi} u.$$

On Bessel's Functions, and Relations connecting them with Hyper-Spherical and Spherical Harmonics. By E. W. Hobson, Sc.D. Received and read December 14th, 1893.

The Bessel's functions $J_m(r)$ of positive integral order m make their appearance in the product $\frac{\cos}{\sin} m\theta . J_m(r)$, which is a particular integral of the differential equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0,$$

where $$x = r\cos\theta, \quad y = r\sin\theta.$$

The Bessel's functions $J_{m+\frac{1}{2}}(r)$ of order half an odd integer (known sometimes as spherical functions) make their appearance in the product

$$\frac{1}{\sqrt{r}} J_{m+\frac{1}{2}}(r) . Y_m(\theta, \phi),$$

which satisfies the equation $\nabla^2 V + V = 0$,

Y_m denoting a surface harmonic of order m. There is, however, another mode in which both kinds of functions may be considered to arise; it appears that, if we consider the equation in p variables corresponding to

$$\nabla^2 V + V = 0,$$

the function $\frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$ plays the same part in relation to this equation that $J_0(r)$ does in relation to the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0,$$

and thus that $\frac{J_m(r)}{r^m}$ may be considered to be the Bessel's function of zero order when there are $2m+2$ variables, and also that $\frac{J_{m-\frac{1}{2}}(r)}{r^{m+\frac{1}{2}}}$ will be the Bessel's function of zero order when $2m+3$ is the number of variables. In the present paper, various properties of the functions are developed from this point of view, the method having the advantage of dealing with both classes of functions at once. A considerable number of relations connecting the functions of different orders, both amongst themselves and with the corresponding hyper-spherical harmonics, are obtained, many of which are believed to be new. Many of these theorems arise from a comparison of different ways of expressing the same solution of one of the equations

$$\nabla^2 V = 0, \quad \nabla^2 V + V = 0,$$

the number of variables being unrestricted. Expressions are obtained for the zonal and tesseral harmonics as definite integrals involving Bessel's functions.

A Theorem concerning a certain Differential Operator.

1. In a paper* "On a Theorem in Differentiation and its Application to Spherical Harmonics," I proved a theorem which may be stated thus:—If $f_n(x_1, x_2, \dots x_p)$ denote a rational integral function of degree n of the p variables $x_1, x_2, \dots x_p$, then

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right)\phi(r)$$

$$= \left\{2^n \frac{d^n\phi}{d(r^2)^n} + \frac{2^{n-1}}{2}\frac{d^{n-1}\phi}{d(r^2)^{n-1}}\nabla_p^2 + \frac{2^{n-2}}{2.4}\frac{d^{n-2}\phi}{d(r^2)^{n-2}}\nabla_p + \dots\right\} f_n(x_1, x_2, \dots x_p) \quad \dots\dots\dots\dots\dots(1),$$

where

$$r^2 = x_1^2 + x_2^2 + \dots + x_p^2,$$

and

$$\nabla_p^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2}.$$

Now suppose that f_n satisfies the differential equation

$$\nabla_p^2 f_n = 0,$$

so that f_n is a spherical or hyper-spherical harmonic; in the above theorem the series on the right-hand side reduces to its first term, and we have, denoting by $S_n(x_1, x_2, \dots x_p)$ such a value of f_n,

$$S_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right)\phi(r) = 2^n \frac{d^n\phi(r)}{d(r^2)^n} S_n(x_1, x_2, \dots x_p) \quad \dots(2).$$

In this paper, I shall make use of the theorem (2), and I give here two examples of its application to the differential equations of physics.

* See *Proc. Lond. Math. Soc.*, Vol. XXIV., p. 67.

(*a*) It is well known that the real part of $\frac{1}{r}e^{\iota\kappa(at-r)}$ represents the potential due to a simple source of vibrations in a gas, the expression

$$V=\frac{1}{r}e^{\iota\kappa(at\mp r)}$$

satisfying the differential equation

$$\frac{\partial^2 V}{\partial t^2}=a^2\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right);$$

it follows from the linear character of the equation that

$$S_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\left[\frac{1}{r}e^{\iota\kappa(a_t\mp r)}\right],$$

where $S_n(x, y, z)$ is a solid harmonic of degree n, also satisfies the differential equation; applying the theorem (1), we see that the function

$$S_n(x,y,z)\frac{d^n}{d(r^2)^n}\left[\frac{1}{r}e^{\iota\kappa(at\mp r)}\right],$$

and therefore also $S_n(x, y, z)\,e^{\iota\kappa at}\dfrac{J_{n+\frac{1}{2}}(\kappa r)}{(\kappa r)^{n+\frac{1}{2}}}$,

satisfies the differential equation. It has been remarked by Lord Rayleigh* that the potential of a multiple source, which is of the form

$$\frac{\partial^n}{\partial h_1\partial h_2\ldots\partial h_n}\frac{e^{\iota\kappa(at-r)}}{r},$$

does not in general contain a spherical harmonic $S_n(x, y, z)$ as a factor, as it does in the case $\kappa=0$, of the gravitation potential; the reason of this is that $\dfrac{\partial^n}{\partial h_1\partial h_2\ldots\partial h_n}$ differs from some operator of the form

$$S_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

by a multiple of
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

and thus
$$S_n(x,y,z)\,e^{\iota\kappa at}\frac{J_{n+\frac{1}{2}}(\kappa r)}{(\kappa r)^{n+\frac{1}{2}}}$$

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* *Theory of Sound*, Vol. II., p. 216.

is the potential due to a combination of sources of degrees $n, n-2$, &c., whereas, in the case $\kappa = 0$, we have, since

$$\nabla^2 \frac{1}{r} = 0,$$

$$\frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n} \frac{1}{r} = S_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{1}{r} = \frac{S_n(x, y, z)}{r^{2n+1}},$$

omitting numerical factors.

(*b*) It is well known that the equation

$$\frac{\partial V}{\partial t} = \kappa\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

is satisfied by $V = \int_0^t \frac{1}{\{2\sqrt{\pi\kappa(t-\lambda)}\}^3} f(\lambda)\, e^{-r^2/[4\kappa(t-\lambda)]}\, d\lambda$;

in fact this is the temperature at a point of an infinite solid of conductivity κ, due to a source of intensity $f(t)$, commencing at time $t = 0$. We see that

$$S_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V$$

satisfies the differential equation; thus the expression

$$S_n(x, y, z) \int_0^t \frac{1}{(t-\lambda)^{n+\frac{3}{2}}} f(\lambda)\, e^{-r^2/[4\kappa(t-\lambda)]}\, d\lambda$$

satisfies the differential equation; putting

$$\alpha^2 = \frac{r^2}{4\kappa(t-\lambda)},$$

we see that the function

$$\frac{S_n(x, y, z)}{r^{2n+1}} \int_{r/2\sqrt{\kappa t}}^{\infty} \alpha^{2n} \cdot e^{-\alpha^2} \cdot f\left(t - \frac{r^2}{4\kappa\alpha^2}\right) d\alpha$$

satisfies the differential equation, S_n denoting any solid harmonic. The particular case $n = 1$ gives the class of solutions which I have applied, in my paper* on "Synthetic Solutions in the Conduction of Heat," to certain problems of conduction.

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* See *Proc. Lond. Math. Soc.*, Vol. XIX., p. 289.

Bessel's Functions of rank p.

2. The equation $\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2}\right)V=0$ ....(3),

or
$$\nabla_p^2 V = 0,$$

has
$$\frac{p(p+1)\ldots(p+n-1)}{n!}-\frac{p(p+1)\ldots(p+n-3)}{(n-2)!},$$

or
$$(2n+p-2)\,\frac{(p-1)p(p+1)\ldots(p+n-3)}{n!},$$

distinct solutions which are rational integral functions of $x_1, x_2, \ldots x_p$ of degree n; we shall denote such a solution by $S_n(x_1, x_2, \ldots x_p)$.

Consider the equation $\nabla_p^2 V + V = 0$ ....(4).

Suppose $x_1, x_2, \ldots x_p$ to be expressed in terms of the usual hyper-polar system of variables $r, \theta_1, \theta_2, \ldots \theta_{p-1}$, and suppose V to be a function of r only; the equation (3) reduces in this case to

$$\frac{d^2V}{dr^2}+\frac{p-1}{r}\frac{dV}{dr}+V=0.$$

Now, in Bessel's equation of order m,

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\left(1-\frac{m^2}{r^2}\right)u=0,$$

put
$$u = r^m v;$$

then we have
$$\frac{d^2v}{dr^2}+\frac{2m+1}{r}\frac{dv}{dr}+v=0;$$

thus we see that (4) is satisfied by

$$V=\frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \quad\text{or}\quad \frac{Y_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}},$$

where $J_{\frac{1}{2}p-1}$, $Y_{\frac{1}{2}p-1}$ are the two Bessel's functions of order $\frac{1}{2}p-1$.

For simplicity we shall for the most part consider the function J only; many of the theorems will apply equally to Y.

The solutions of (4) which contain r only I shall call the Bessel's functions of zero order and rank p; thus the ordinary Bessel's functions $J_0(r)$, $Y_0(r)$ are of rank 2. These solutions of (4) may be denoted by

$$J_0(p, r), \quad Y_0(p, r),$$

where $$J_0(p, r) = \frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}, \quad Y_0(p, r) = \frac{Y_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \quad\text{............(5)};$$

we thus have

$$J_0(2, r) = J_0(r), \quad J_0(3, r) = \frac{J_{\frac{1}{2}}(r)}{r^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}}\,\frac{\sin r}{r},$$

$$J_0(2m, r) = \frac{J_{m-1}(r)}{r^{m-1}}, \quad J_0(2m+1, r) = \frac{J_{m-\frac{1}{2}}(r)}{r^{m-\frac{1}{2}}};$$

it thus appears that Bessel's functions of even rank are expressible in terms of the ordinary Bessel's functions of integral order, and that those of odd rank are expressible in terms of the ordinary functions of order half an odd integer.

3. In order to obtain unsymmetrical solutions of (3), the theorem (2) may be applied; thus

$$S_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x}\right) J_0(p, r) = 2^n S_n(x_1, x_2, \dots x_p)\,\frac{d^p}{d(r^2)^p} J_0(p, r);$$

now $$\frac{d^n}{d(r^2)^n} J_0(p, r) = \frac{d^n}{d(r^2)^n}\,\frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} = \frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}},$$

leaving out a numerical factor, as the form only of the result is required.

We see therefore that (4) is satisfied by

$$V = S_n(x_1, x_2, \dots x_p)\,\frac{J_{n+\frac{1}{2}p-1}}{r^{n+\frac{1}{2}p-1}},$$

that is, a solution of (4) is obtained by multiplying a solution S_n of (3) by $\frac{J_{n-1+\frac{1}{2}p}(r)}{r^{n-1+\frac{1}{2}p}}$. The cases $p = 2$, $p = 3$ of this theorem are well-known; thus, when $p = 2$, we have

$$(x \pm iy)^n \frac{J_n(r)}{r^n}, \quad\text{or}\quad {\cos \atop \sin} n\phi \,.\, J_n(r),$$

as a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + V = 0,$$

where $$x = r\cos\phi, \quad y = \rho\sin\phi;$$

again, when $p = 3$, we have

$$S_n(x, y, z)\frac{J_{n+\frac{1}{2}}(r)}{r^{n+\frac{1}{2}}},$$

as a solution of the equation

$$\nabla^2 V + V = 0.$$

4. We have considered only the case in which S_n is a rational algebraical solution of (3); it may, however, be shown that, if S_n is any solution of (3) of degree n in the p variables, $S_n . \frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}}$ is a solution of (4).

In (4), put $V = S_n u$; the equation then becomes, on the assumption that u is a function of r only,

$$\frac{2}{r}\frac{\partial u}{\partial r}\left(x_1\frac{\partial S_n}{\partial x_1} + x_2\frac{\partial S_n}{\partial x_2} + \ldots + x_p\frac{\partial S_n}{\partial x_p}\right) + S_n\nabla_p^2 u + S_n u = 0;$$

or, using the theorem

$$x_1\frac{\partial S_n}{\partial x_1} + \ldots + x_p\frac{\partial S_n}{\partial x_p} = nS_n,$$

this becomes
$$\frac{d^2u}{dr^2} + \frac{p+2n-1}{r}\frac{du}{dr} + u = 0,$$

of which the solution is

$$u = A\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}} + B\frac{Y_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}};$$

thus, whatever the nature of a function S_n of the n^{th} degree may be, which is a solution of (2), if it be multiplied by $\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}}$, we obtain a solution of (3). It will be observed that n may be negative, so that $S_{-n}\frac{J_{-n+\frac{1}{2}p-1}(r)}{r^{-n+\frac{1}{2}p-1}}$ satisfies (3), S_{-n} denoting a harmonic of negative degree.

5. In a paper* on "Systems of Spherical Harmonics," I have given a table of some spherical harmonics of degree zero; any one of these,

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* See *Proc.*, Vol. XXII., p. 435.

when multiplied by $\frac{\sin r}{\sqrt{r}}$, or $\frac{\cos r}{\sqrt{r}}$, is a solution of the equation

$$\nabla^2 V + V = 0;$$

we obtain, for example, as solutions of this equation

$$\frac{x}{r+z}\,\frac{\sin r}{\sqrt{r}},\quad \frac{x}{r+z}\,\frac{\cos r}{\sqrt{r}},\quad \frac{(r\pm z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\,{\cos \atop \sin}\, m\phi \,.\, \frac{1}{\sqrt{r}}\,{\sin \atop \cos}\, r,$$

$$\frac{1}{\sqrt{r}}\log\sqrt{\frac{r-z}{r+z}}\,{\sin \atop \cos}\, r,\quad \frac{1}{\sqrt{r}}\tan^{-1}\frac{y}{x}\,{\sin \atop \cos}\, r.$$

The most general harmonic of degree n is

$$r^{2n+1}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}f\left(\frac{x\pm \iota y}{r+z}\right)\right\};$$

we obtain therefore, as solutions of

$$\nabla^2 V + V = 0,$$

the expressions
$$r^{n+\frac{1}{2}}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}f\left(\frac{x\pm \iota y}{r+z}\right)\right\}J_{n+\frac{1}{2}}(r),$$

$$r^{n+\frac{1}{2}}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}f\left(\frac{x\pm \iota y}{r+z}\right)\right\}Y_{n+\frac{1}{2}}(r),$$

where f denotes any function.

Zonal and Tesseral Hyper-Spherical Harmonics.

6. The systems of zonal and tesseral harmonics for p variables have been discussed by various writers; as, however, some of their properties are needed below, I give what appears to me to be a simple method of investigating their forms.

The potential equation (3) is satisfied by

$$V = \frac{1}{\left\{(x_1-a_1)^2+(x_2-a_2)^2+\ldots+(x_p-a_p)^2\right\}^{\frac{1}{2}p-1}},$$

when $p > 2$; and by

$$V = -\tfrac{1}{2}\log_e\left\{(x_1-a_1)^2+(x_2-a_2)^2\right\},$$

when $p = 2$.

Denoting the zonal harmonic of rank p by $P_n(p, \cos\theta)$, we have, as in the case $p = 3$,

$$\left.\begin{aligned}\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p-1}} &= \Sigma h^n P_n(p, \cos\theta)\\ -\tfrac{1}{2}\log_e(1-2h\cos\theta+h^2) &= \Sigma h^n P_n(2, \cos\theta)\end{aligned}\right\} \quad \text{............(6)}.$$

Also, as in the ordinary case $p = 3$, we find, if $p \geqq 3$,

$$P_n(p, \cos\theta) = \frac{(-1)^n r^{2n+1}}{\Pi(n)} \frac{\partial^n}{\partial x_p^n} \frac{1}{(x_1^2+x_2^2+\ldots+x_p^2)^{\frac{1}{2}p-1}},$$

where $$x_p/r = \cos\theta = \mu.$$

Performing the differentiation by means of the theorem (1), we have at once

$$P_n(p, \mu) = 2^n \frac{\Pi(n+\frac{1}{2}p-2)}{\Pi(n)\,\Pi(\frac{1}{2}p-2)}$$

$$\times\left\{\mu^n - \frac{n(n-1)}{2.2n+p-4}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n+p-4)(2n+p-6)}\mu^{n-4} - \ldots\right\}$$

$$\text{..................(7)}.$$

In the case $p = 2$, we have

$$P_n(2, \cos\theta) = \frac{1}{n}\cos n\theta;$$

thus $$P_n(2, \mu)$$

$$= 2^{n-1}\left\{\mu^n - \frac{n(n-1)}{2.2n-2}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.2n-2.2n-4}\mu^{n-4} - \ldots\right\};$$

thus the factor of $P_n(2, \mu)$ in the bracket agrees with the series factor in (7).

7. The potential equation (3) may be written

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \ldots + \frac{\partial^2 V}{\partial x_{p-1}^2} + \left(\frac{\partial}{\partial x_p}\right)^2 V = 0,$$

so that when $\frac{\partial}{\partial x_p}$ is treated as a quantity, this equation is of the form (4) with $p-1$ variables; we see therefore that

$$V = S_n(x_1, x_2 \ldots x_{p-1}) \frac{J_{n+\frac{1}{2}(p-3)}\left(\sqrt{x_1^2+x_2^2+\ldots+x_{p-1}^2}\,\frac{\partial}{\partial x_p}\right)}{\left(\sqrt{x_1^2+x_2^2+\ldots+x_{p-1}^2}\,\frac{\partial}{\partial x_p}\right)^{n+\frac{1}{2}(p-3)}} \chi(x_p),$$

where $\chi(x_p)$ is any function of x_p, and a is any integer, satisfies the above equation. Let $\chi(x_p) = x_p^{n-a}$, and suppose $a \leq n$; we thus have, as a solution of (3),

$$S_a(x_1, x_2, \dots x_{p-1})\left\{x_p^{n-a} - (x_1^2 + x_2^2 + \dots + x_{p-1}^2)\frac{(n-a)(n-a-1)}{2.2a+p-1}x_p^{n-a-2} + \dots\right\},$$

where S_a denotes any solid hyper-harmonic of rank $p-1$. S_a may itself be expressed as the product of a harmonic of rank $p-2$, and a series; thus, proceeding in this way, we obtain, as a solution of (3), an expression of the form of the product

$$L(p, n, a, x_p)\, L(p-1, a, \beta, x_{p-1}) \dots L(3, \kappa, \lambda, x_3)(x_1 \pm \iota x_2)^\lambda \dots\dots (8),$$

where $L(p-r, \eta, \theta, x_{p-r})$ denotes the series

$$x_{p-r}^{\eta-\theta} - \frac{(\eta-\theta)(\eta-\theta-1)}{2.2\theta+p-r-1}x_{p-r}^{\eta-\theta-2}(x_1^2 + x_2^2 + \dots + x_{p-r-1}^2)$$

$$+ \frac{(\eta-\theta)(\eta-\theta-1)(\eta-\theta-2)(\eta-\theta-3)}{2.4.2\theta+p-r-1.2\theta+p-r+1}x_{p-r}^{\eta-\theta-4}(x_1^2 + x_2^2 + \dots + x_{p-r-1}^2)^2 - \dots$$

If different integral values are assigned to $a, \beta, \dots \kappa, \lambda$, such that

$$n \geq a \geq \beta \dots \geq \kappa \geq \lambda,$$

the form (8) expresses various hyper-spherical harmonics of degree n. It is easy to show that (8) gives a complete set of harmonics of degree n; the $p-1$ quantities $n-a, a-\beta, \dots \kappa-\lambda, \lambda$, are capable of all positive integral values (including zero) which are such that their sum is n; corresponding to any choice of these numbers we have two harmonics given by (8), except when $\lambda = 0$, in which case only one harmonic is given; the number of solutions in which $\lambda = 0$ is the number of ways in which the $p-2$ numbers $n-a, a-\beta \dots \kappa$ may be chosen so that their sum is n; it follows that the formula (8) represents

$$2\frac{(p-1)p(p+1)\dots(p+n-2)}{n!} - \frac{(p-2)(p-1)\dots(p+n-3)}{n!}$$

distinct harmonics; this number is equal to

$$(p+2n-2)\frac{(p-1)p(p+1)\dots(p+n-3)}{n!},$$

which, as we have seen in Section 2, is the number of independent harmonics of degree n and rank p. It has thus been shown that all the hyper-spherical harmonics of degree n are included in (8); these correspond to the system of zonal, tesseral, and sectorial harmonics in the ordinary case $p = 3$.

Expansion of an Exponential Function.

9. The differential equation (4) is satisfied by

$$V = e^{\iota x_p} = e^{\iota r \cos\theta};$$

it follows that $e^{\iota r\cos\theta}$ can be expanded in the form

$$\sum_0^{\infty} a_n r^n P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}},$$

since Bessel's functions of the second kind are infinite when $r = 0$, and therefore cannot occur. We have therefore

$$e^{\iota r\cos\theta} = \sum_0^{\infty} a_n P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}},$$

where a_n are constants to be determined; substituting the value for $P_n(p, \cos\theta)$ given by (7), and equating the coefficients of $r^n\cos^n\theta$ on both sides of the equation, we obtain

$$\frac{\iota^n}{\Pi(n)} = a_n\frac{1}{2^{n+\frac{1}{2}p-1}\,\Pi(n+\frac{1}{2}p-1)}\,2^n\frac{\Pi(n+\frac{1}{2}p-2)}{\Pi(n)\,\Pi(\frac{1}{2}p-2)};$$

thus $$a_n = \iota^n \,.\, 2^{\frac{1}{2}p-1}(n+\tfrac{1}{2}p-1)\,\Pi(\tfrac{1}{2}p-2),$$

whence we obtain the result

$$e^{\iota r\cos\theta} = 2^{\frac{1}{2}p-1}\,\Pi(\tfrac{1}{2}p-2)\sum_{n=0}^{n=\infty}\iota^n(n+\tfrac{1}{2}p-1)\,P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \quad \ldots(9).$$

Two cases of (9) are well known; putting $p = 3$, we have

$$e^{\iota r\cos\theta} = \sqrt{2\pi}\sum_0^{\infty}\iota^n(n+\tfrac{1}{2})\,P_n(\cos\theta)\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}};$$

again, putting $p = 2$, and taking account of the exceptional form of $P_n(2, \cos\theta)$, we have

$$e^{\iota r\cos\theta} = \Sigma\,\iota^n\cos n\theta\,.\,J_n(r).$$

Addition Theorems for Bessel's Functions.

10. Since $J_0(p, r)$ satisfies the equation (4), it follows that

$$J_0\left(p, \sqrt{r^2+r'^2-2rr'\cos\theta}\right)$$

satisfies the same equation, for

$$r^2+r'^2-2rr'\cos\theta = x_1^2+x_2^2+\ldots+(x_p-r')^2.$$

We see that if $\quad J_0\left(p, \sqrt{r^2+r'^2-2rr'\cos\theta}\right)$

is expanded in a series of the functions $P_n(p, \cos\theta)$, the coefficients must contain $\dfrac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$ and $\dfrac{J_{n+\frac{1}{2}p-1}(r')}{r'^{\frac{1}{2}p-1}}$ as factors; thus

$$J_0\left(p, \sqrt{r^2+r'^2-2rr'\cos\theta}\right)$$

$$= J_0(p, r)\, J_0(p, r') + \sum_{n=1} \beta_n \frac{J_{n+\frac{1}{2}p-1}(r)\, J_{n+\frac{1}{2}p-1}(r')}{(rr')^{\frac{1}{2}p-1}} P_n(p, \cos\theta),$$

where β_n is a constant to be determined. Write

$$R = \sqrt{r^2+r'^2-2rr'\cos\theta},$$

and differentiate both sides of the equation n times with respect to $\cos\theta$, remembering that

$$d(R^2) = -2rr'd(\cos\theta);$$

thus

$$(-2rr')^n \frac{d^n}{d(R^2)^n} \frac{J_{\frac{1}{2}p-1}(R)}{R^{\frac{1}{2}p-1}} = \beta_n \frac{J_{n+\frac{1}{2}p-1}(r)\, J_{n+\frac{1}{2}p-1}(r')}{(rr')^{\frac{1}{2}p-1}} \frac{d^n P_n(p, \cos\theta)}{d(\cos\theta)^n} + \ldots;$$

divide both sides of the equation by r'^n, and then put $r'=0$; we get

$$(-2r)^n \frac{d^n}{d(r^2)^n} \frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$$

$$= \beta_n \frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \frac{1}{2^{n+\frac{1}{2}p-1}\,\Pi(n+\frac{1}{2}p-1)} \frac{2^n\Pi(n+\frac{1}{2}p-2)}{\Pi(\frac{1}{2}p-2)}.$$

Now $\quad (-2r)^n \dfrac{d^n}{d(r^2)^n} \dfrac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} = \dfrac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}};$

hence we find $\quad \beta_n = 2^{\frac{1}{2}p-1}(n+\frac{1}{2}p-1)\,\Pi(\frac{1}{2}p-2);$

we thus obtain the theorem

$$\frac{J_{\frac{1}{2}p-1}(R)}{R^{\frac{1}{2}p-1}} = \frac{1}{(rr')^{\frac{1}{2}p-1}} \{J_{\frac{1}{2}p-1}(r)\, J_{\frac{1}{2}p-1}(r')$$

$$+ 2^{\frac{1}{2}p-1}\Pi(\tfrac{1}{2}p-2) \sum_1^\infty (n+\tfrac{1}{2}p-1)\, J_{n+\frac{1}{2}p-1}(r)\, J_{n+\frac{1}{2}p-1}(r')\, P_n(p, \cos\theta)\} \quad \ldots\ldots\ldots\ldots\ldots(10).$$

This theorem has been proved by other methods by Gegenbauer and by Sonnine.

If we put, in (10), $p = 2$, we obtain C. Neumann's addition theorem

$$J_0(R) = J_0(r)\, J_0(r') + 2\sum_1^\infty J_n(r)\, J_n(r') \cos n\theta.$$

Putting $p = 3$, we obtain the addition theorem for the spherical functions,

$$\frac{J_{\frac{1}{2}}(R)}{R^{\frac{1}{2}}}$$

$$= \frac{1}{(rr')^{\frac{1}{2}}} \{J_{\frac{1}{2}}(r)\, J_{\frac{1}{2}}(r') + 2^{\frac{1}{2}}\Pi(-\tfrac{1}{2}) \sum (n+\tfrac{1}{2})\, J_{n+\frac{1}{2}}(r)\, J_{n+\frac{1}{2}}(r')\, P_n(\cos\theta)\},$$

or

$$\frac{\sin R}{R}$$

$$= \frac{\sin r}{r}\frac{\sin r'}{r'} + \sum_1^\infty (2n+1)(4\rho\rho')^n \frac{d^n}{d(\rho^2)^n}\frac{\sin r}{r} \cdot \frac{d^n}{d(\rho'^2)^n}\frac{\sin r'}{r'} . P_n(\cos\theta).$$

It will be observed that the formula (10) is a general addition theorem for the functions $\frac{J_m(R)}{R^m}$, $\frac{J_{m+\frac{1}{2}}(R)}{R^{m+\frac{1}{2}}}$.

The Evaluation of a Surface Integral.

11. It can be shown by means of the differential equation (2), in the same way as in the case $p = 3$, that, if S_m, S_n are hyper-spherical harmonics of degrees m and n,

$$\int S_m S_n \, d\omega = 0,$$

provided m and n are unequal, the integration being taken over the surface of a sphere of unit radius, the centre being at the origin. We shall have occasion to use the value of

$$\int P_n(p, \cos\theta)\, S_n \, d\omega,$$

which I proceed to calculate. Using the system of hyper-polar coordinates, given by

$$x_1 = r\sin\theta\sin\phi_1\sin\phi_2\ldots\sin\phi_{p-2},$$

$$x_2 = r\sin\theta\sin\phi_1\sin\phi_2\ldots\cos\phi_{p-2},$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$x_{p-1} = r\sin\theta\cos\phi_1,$$

$$x_p\ = r\cos\theta,$$

we find, for an elementary volume, the expression

$$r^{p-1}\sin^{p-2}\theta\sin^{p-3}\phi_1\ldots\sin\phi_{p-3}\,dr\,d\theta\,d\phi_1\,d\phi_2\ldots d\phi_{p-2},$$

and thus $d\omega = \sin^{p-2}\theta\sin^{p-3}\phi_1\ldots\sin\phi_{p-3}\,d\theta\,d\phi_1\,d\phi_2\ldots d\phi_{p-2}$.

Using the equation

$$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}(p-2)}} = \sum_{n=0} P_n(p, \cos\theta)\,h^n,$$

we find

$$(p-2)\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}} = \Sigma\,(2n+p-2)\,P_n(p, \cos\theta)\,h^n;$$

multiplying both sides of this equation by S_n, and integrating over the surface of the sphere of unit radius, we have

$$\int P_n(p, \cos\theta)\,S_n\,d\omega = \frac{p-2}{2n+p-2}\int\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}}S_n\,d\omega,$$

where, on the right-hand side, the expression has its limiting value when $h=1$; we therefore have

$$\int P_n(p, \cos\theta)\,S_n\,d\omega = \frac{p-2}{2n+p-2}S_n(1)\int\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}}d\omega,$$

where $S_n(1)$ is the value of S_n at the pole of P_n.

Further, $$\int\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}}d\omega = \frac{1}{p-2}\int\Sigma h^n(2n+p-2)\,P_n\,d\omega$$

$$= \int d\omega = \frac{2\pi^{\frac{1}{2}p}}{\Pi\left(\frac{p-2}{2}\right)};$$

thus we obtain the theorem

$$\int P_n(p, \cos\theta)\,S_n\,d\omega = \frac{p-2}{2n+p-2}\,\frac{2\pi^{\frac{1}{2}p}}{\Pi\left(\frac{p-2}{2}\right)}S_n(1) \quad \ldots\ldots(11).$$

A particular case of this theorem is

$$\int \{P_n(p, \cos\theta)\}^2 (1-\mu^2)^{\frac{1}{2}(p-3)} d\mu$$

$$= \frac{p-2}{2n+p-2} \sqrt{\pi} \frac{\Pi(p+n-3)\,\Pi\left(\frac{p-3}{2}\right)}{\Pi(n)\,\Pi\left(\frac{p-2}{2}\right)\Pi(p-3)} \quad \text{.........(12).}$$

12. Next, let us evaluate $\int e^{a_1x_1+a_2x_2+\dots+a_px_p} S_n\, d\omega$ over the sphere of unit radius; this integral is, by changing the variables, reducible to an integral $\int e^{\beta\cos\theta} S_n\, d\omega$, where

$$\beta^2 = a_1^2 + a_2^2 + \dots + a_n^2.$$

Substitute for $e^{\beta\cos\theta}$ its value given by putting $-\iota\beta$ for r in the expansion (9); then, remembering that

$$\int S_n P_m(p, \cos\theta)\, d\omega = 0,$$

except when $m = n$, we have

$$\int e^{\beta\cos\theta} S_n\, d\omega = 2^{\frac{1}{2}p-1}\,\Pi(\tfrac{1}{2}p-2)\,\iota^n\,(n+\tfrac{1}{2}p-1)\,\frac{J_{n+\frac{1}{2}p-1}(-\iota\beta)}{(-\iota\beta)^{\frac{1}{2}p-1}} \int S_n P_n\, d\omega$$

$$= \frac{1}{2^n}\,\frac{\Pi(\frac{1}{2}p-2)(n+\frac{1}{2}p-1)}{\Pi(n+\frac{1}{2}p-1)} \int S_n P_n\, d\omega$$

$$\times \beta^n \left\{1 + \frac{\beta^2}{2.2n+p} + \frac{\beta^4}{2.4.2n+p.2n+p+2} + \dots\right\}.$$

If $f(x_1, x_2, \dots x_p)$ be a function which is finite and continuous throughout the volume of the hyper-sphere, we have

$$f(x_1, x_2, \dots x_p) = e^{x_1.\partial/\partial\xi_1 + x_2.\partial/\partial\xi_2 + \dots + x_p.\partial/\partial\xi_p} f(\xi_1, \xi_2, \dots \xi_p),$$

where $\xi_1, \xi_2, \dots \xi_p$ are put equal to zero after the operation is performed. Let

$$a_1 = \frac{\partial}{\partial\xi_1}, \quad a_2 = \frac{\partial}{\partial\xi_2}, \dots a_p = \frac{\partial}{\partial\xi_p};$$

we then have

$$\int S_n . f(x_1, x_2, \dots x_p)\, d\omega = \frac{1}{2^n}\,\frac{\Pi(\frac{1}{2}p-2)}{\Pi(n+\frac{1}{2}p-2)}\,\frac{p-2}{2n+p-2}\,\frac{2\pi^{\frac{1}{2}p}}{\Pi\left(\frac{p-2}{2}\right)}$$

$$\times S_n\left(\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \dots \frac{\partial}{\partial\xi_p}\right)\left(1 + \frac{\nabla^2}{2.2n+p} + \frac{\nabla^4}{2.4.2n+p.2n+p+2}\right)$$

$$f(\xi_1, \xi_2, \dots \xi_p) \quad \text{...(13),}$$

where
$$\nabla^2 = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \dots + \frac{\partial^2}{\partial \xi_p^2},$$

and where $\xi_1, \xi_2, \dots \xi_p$ are put equal to zero after the operations on the right-hand side are performed.

13. A large number of integrals are included as particular cases of (13); an important one for our purpose is the case in which

$$S_n = P_n(p, \cos\theta), \quad f = (x_1^2 + x_2^2 + \dots + x_{p-1}^2)^k;$$

we then have, supposing n even, keeping the only term which does not vanish,

$$\int P_n(p, \cos\theta) \sin^{2k}\theta \, d\omega$$

$$= \frac{2\pi^{\frac{1}{2}p}}{2^{2k}} (-1)^{\frac{1}{2}n} \frac{\Pi\left(\frac{p+n}{2} - 2\right)}{\Pi\left(\frac{p+n}{2} + k - 1\right) \Pi\left(\frac{n}{2}\right) \Pi\left(\frac{p}{2} - 2\right) \Pi\left(k - \frac{n}{2}\right)}$$

$$\times \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_{p-1}^2}\right)^k (x_1^2 + x_2^2 + \dots + x_{p-1}^2)^k.$$

It can easily be shown that

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_{p-1}^2}\right)^k (x_1^2 + x_2^2 + \dots + x_{p-1}^2)^k = \frac{4^k \Pi(k) \Pi\left(k + \frac{p-3}{2}\right)}{\Pi\left(\frac{p-3}{2}\right)};$$

hence the value of the surface integral is

$$(-1)^{\frac{1}{2}n} 2\pi^{\frac{1}{2}p} \frac{\Pi\left(\frac{p+n}{2} - 2\right) \Pi(k) \Pi\left(k + \frac{p-3}{2}\right)}{\Pi\left(\frac{p+n}{2} + k - 1\right) \Pi\left(\frac{n}{2}\right) \Pi\left(\frac{p}{2} - 2\right) \Pi\left(k - \frac{n}{2}\right) \Pi\left(\frac{p-3}{2}\right)};$$

or
$$\int_{-1}^{1} P_n(p, \mu)(1 - \mu^2)^{k + \ (p-3)} \, d\mu$$

$$= (-1)^{\frac{1}{2}n} \pi^{\frac{1}{2}} \frac{\Pi\left(\frac{p+n}{2} - 2\right) \Pi(k) \Pi\left(k + \frac{p-3}{2}\right)}{\Pi\left(\frac{p+n}{2} + k - 1\right) \Pi\left(\frac{n}{2}\right) \Pi\left(\frac{p}{2} - 2\right) \Pi\left(k - \frac{n}{2}\right)} \dots (14),$$

where n is an even integer.

In the particular case $p = 3$, we have

$$\int_{-1}^{1} P_n(\mu)(1-\mu^2)^k \, d\mu$$

$$= (-1)^{\frac{1}{2}n}\pi^{\frac{1}{2}} \frac{\Pi\left(\frac{n-1}{2}\right)\Pi(k)\,\Pi(k)}{\Pi\left(\frac{n+1}{2}+k\right)\Pi\left(\frac{n}{2}\right)\sqrt{\pi}\,\Pi\left(k-\frac{n}{2}\right)}$$

$$= (-1)^{\frac{1}{2}n} \frac{\Pi(k)\,\Pi(k)}{\Pi\left(\frac{n}{2}\right)\Pi\left(k-\frac{n}{2}\right)} \cdot \frac{2^{2k+1}\,\Pi(n)\,\Pi\left(\frac{n}{2}+k\right)}{\Pi(n+2k+1)\,\Pi\left(\frac{n}{2}\right)}$$

$$= (-1)^{\frac{1}{2}n} 2^{2k+1} \frac{\Pi(n)\,\Pi\left(k+\frac{n}{2}\right)\Pi(k)\,\Pi(k)}{\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{n}{2}\right)\Pi\left(k-\frac{n}{2}\right)\Pi(n+2k+1)} \quad \ldots(15).$$

The particular case (15) agrees with the value obtained by Mr. W. D. Niven.*

Expansions in Zonal Hyper-Harmonics.

14. Since $\qquad e^{\pm x_p} J_0\left(p-1, \sqrt{x_1^2+x_2^2+\ldots+x_{p-1}^2}\right)$,

or $\qquad e^{\pm r\cos\theta} J_0(p-1, r\sin\theta)$,

satisfies the equation $\qquad \nabla_p^2 V = 0$,

it is clear that this function must be capable of being exhibited in a series of zonal hyper-harmonics of rank p of positive integral degrees; thus

$$e^{r\cos\theta} \frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}} = \sum_{n=0}^{n=\infty} a_n r^n P_n(p, \cos\theta);$$

putting $\theta = 0$, we have

$$e^r \frac{1}{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)} = \sum_{n=0}^{n=\infty} a_n r^n P_n(p, 1);$$

* See *Phil. Trans.* for 1879, p. 385.

thus
$$a_n = \frac{1}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)\Pi(n)\,P_n(p,1)}$$
$$= \frac{\Pi(p-3)}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)\Pi(p+n-3)};$$

we therefore obtain the theorem

$$e^{r\cos\theta}J_{\frac{1}{2}(p-3)}(r\sin\theta) = \frac{\Pi(p-3)(r\sin\theta)^{\frac{1}{2}(p-3)}}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}\sum_{n=0}^{\infty}\frac{r^n}{\Pi(p+n-3)}P_n(p,\cos\theta) \quad \text{............(16)};$$

on changing r into $-r$, we have

$$e^{-r\cos\theta}J_{\frac{1}{2}(p-3)}(r\sin\theta) = \frac{\Pi(p-3)(r\sin\theta)^{\frac{1}{2}(p-3)}}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}\sum\frac{(-1)^n r^n}{\Pi(p+n-3)}P_n(p,\cos\theta) \quad \text{............(17)}.$$

In the particular case $p = 3$, we have

$$e^{r\cos\theta}J_0(r\sin\theta) = \sum_{n=0}^{n=\infty}\frac{r^n}{n!}P_n(\cos\theta) \quad \text{...............(18)},$$

$$e^{-r\cos\theta}J_0(r\sin\theta) = \sum_{n=0}^{n=\infty}\frac{(-1)^n r^n}{n!}P_n(\cos\theta) \quad \text{............(19)}.$$

On multiplying the series (18), (19) together, we have

$$\{J_0(r\sin\theta)\}^2 = \left\{1+\frac{r^2}{2!}P_2(\cos\theta)+\frac{r^4}{4!}P_4(\cos\theta)+\dots\right\}^2$$
$$-\left\{rP_1(\cos\theta)+\frac{r^3}{3!}P_3(\cos\theta)+\dots\right\}^2.$$

Relations connecting Bessel's Functions of Different Orders.

15. The equation $\nabla_p^2 V + V = 0$ .... (4)

is satisfied by $J_0\left(p-1, \sqrt{x_1^2+x_2^2+\dots+x_{p-1}^2}\right)$,

which is independent of x_p; it is therefore clear that $J_0(p-1, r\sin\theta)$ can be exhibited in a series of the functions

$$P_n(p,\cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}};$$

thus $$\frac{J_{\frac12(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac12(p-3)}} = \sum_{n=0}^{\infty} \beta_n P_n(p, \cos\theta)\frac{J_{n+\frac12p-1}(r)}{r^{\frac12p-1}},$$

where β_n denotes constants which must be determined.

Multiply both sides of the equation by $P_n(p, \cos\theta)(1-\cos^2\theta)^{\frac12(p-3)}$, and integrate with respect to $\cos\theta$ between the limits ± 1; we have then, in virtue of (12),

$$\beta_n \frac{p-2}{2n+p-2}\sqrt{\pi}\,\frac{\Pi(p+n-3)\,\Pi\left(\frac{p-3}{2}\right)}{\Pi(n)\,\Pi\left(\frac{p-2}{2}\right)\Pi(p-3)}\,\frac{J_{n+\frac12p-1}(r)}{r^{\frac12p-1}}$$

$$= \int_{-1}^{1}\frac{J_{\frac12(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac12(p-3)}}P_n(p, \cos\theta)(1-\cos^2\theta)^{\frac12(p-3)}\,d(\cos\theta);$$

equating the coefficients of r^n on both sides of this equation, we see that β_n is zero when n is odd, and that, when n is even,

$$\beta_n\frac{p-2}{2n+p-2}\,\frac{\sqrt{\pi}\,\Pi(p+n-3)\,\Pi\left(\frac{p-3}{2}\right)}{\Pi(n)\,\Pi\left(\frac{p-2}{2}\right)\Pi(p-3)}\,\frac{1}{2^{n+\frac12p-1}\,\Pi(n+\frac12p-1)}$$

$$= (-1)^{\frac12n}\frac{1}{2^{n+\frac12(p-3)}\,\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p+n-3}{2}\right)}$$

$$\times\int_{-1}^{1}P_n(p, \cos\theta)(1-\cos^2\theta)^{\frac12(p-3)+\frac12n}\,d(\cos\theta)$$

$$= \frac{\sqrt{\pi}}{2^{n+\frac12(p-3)}}\,\frac{\Pi\left(\frac{p+n}{2}-2\right)}{\Pi\left(n+\frac{p-2}{2}\right)\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p-4}{2}\right)}, \text{ by (14);}$$

hence $$\beta_n = \sqrt{2}\,\frac{\left(n+\frac{p-2}{2}\right)\Pi(n)\,\Pi(p-3)\,\Pi\left(\frac{p+n}{2}-2\right)}{\Pi\left(\frac{p-3}{2}\right)\Pi\left(\frac{n}{2}\right)\Pi(p+n-3)};$$

on changing n into $2n$, we have

$$\frac{J_{\frac12(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac12(p-3)}} = \sqrt{2}\,\frac{\Pi(p-3)}{r^{\frac12p-1}}\sum_{n=0}^{\infty}\frac{\left(2n+\frac{p-2}{2}\right)\Pi(2n)\,\Pi\left(n+\frac{p-4}{2}\right)}{\Pi\left(\frac{p-3}{2}\right)\Pi(n)\,\Pi(2n+p-3)}$$

$$\times P_{2n}(p, \cos\theta)\,J_{2n+\frac12p-1}(r) \quad\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

In (20), put $p = 3$; we obtain

$$J_0(r\sin\theta) = \sqrt{\frac{2}{r}} \sum_0^\infty \frac{(2n+\frac{1}{2})\,\Pi(n-\frac{1}{2})}{\Pi(n)} P_{2n}(\cos\theta)\, J_{2n+\frac{1}{2}}(r)$$

$$= \sqrt{\frac{2\pi}{r}} \sum \frac{(2n+\frac{1}{2})(2n)!}{2^{2n+1}\, n!\, n!} P_{2n}(\cos\theta)\, J_{2n+\frac{1}{2}}(r) \quad\text{......(21).}$$

Again, put $p = 4$; we have then

$$\frac{J_{\frac{1}{2}}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}}} = \frac{2\sqrt{2}}{\pi}\,\frac{1}{r} \sum P_{2n}(4, \cos\theta)\, J_{2n+1}(r).$$

In the theorem (20), put $\theta = \frac{\pi}{2}$; we then have

$$J_{\frac{1}{2}(p-3)}(r) = \sqrt{\frac{2}{r}}\, \frac{\Pi(p-3)}{\Pi\left(\frac{p}{2}-2\right)\Pi\left(\frac{p-3}{2}\right)} \sum_{n=0}^{n=\infty} (-1)^n$$

$$\times \frac{\left(2n+\frac{p-2}{2}\right)\Pi(2n)\,\Pi\left(n+\frac{p}{2}-2\right)\Pi\left(n+\frac{p}{2}-2\right)}{\Pi(n)\,\Pi(n)\,\Pi(2n+p-3)} J_{2n+\frac{1}{2}p-1}(r) \text{...(22);}$$

this expresses a Bessel's function of integral order in a series of Bessel's functions of order half an odd integer, and conversely.

A particular case of (22) is when $p = 3$;

$$J_0(r) = \sqrt{\frac{2}{\pi r}} \sum_{n=0}^{n=\infty} (-1)^n \left\{\frac{\Pi(n-\frac{1}{2})}{\Pi(n)}\right\}^2 (2n+\tfrac{1}{2})\, J_{2n+\frac{1}{2}}(r) \text{......(23).}$$

Again, when $p = 4$, we have

$$\sin r = 2\sum_{n=0}^{n=\infty} (-1)^n J_{2n+1}(r) \quad\text{.................(24).}$$

16. It is interesting to obtain an expansion corresponding to (23), by another method; on comparing the results obtained by the two methods, the evaluation of certain definite integrals is obtained.

We have

$$\int_0^\infty J_0(u)\,\frac{e^{\iota(u+r)}}{u+r}\,du = \int_0^\infty\int_0^\infty e^{(\iota-t)(u+r)} J_0(u)\,du\,dt;$$

on carrying out the integration with respect to u, the right hand becomes, by a known theorem, equal to

$$\int_0^\infty \frac{e^{(\iota-t)r}}{\sqrt{1+(t-\iota)^2}}\,dt, \quad\text{or}\quad e^{\iota r}\int_0^\infty \frac{e^{-tr}}{\sqrt{t^2-2t\iota}}\,dt;$$

putting $\iota - t = \iota t'$,

we have $$\int_0^\infty J_0(u)\,\frac{e^{\iota(u+r)}}{u+r}\,du = -\int_1^{\iota\infty} \frac{e^{\iota t' r}}{\sqrt{t'^2-1}}\,dt';$$

the integral $\int_1^{\iota\infty} \frac{e^{\iota t' r}}{\sqrt{t'^2-1}}\,dt'$ is one of a class of integrals which represent Bessel's functions, and has been considered by Hankel,* and other writers. Using a result obtained by Hankel, we have

$$\int_0^\infty J_0(u)\,\frac{e^{\iota(u+r)}}{u+r}\,du = -\frac{1}{2}\,Y_0(r) + \frac{\iota\pi}{2}\,J_0(r);$$

on equating the real and imaginary parts on both sides of this equation, we obtain the formulæ

$$J_0(r) = \frac{2}{\pi}\int_0^\infty \frac{\sin(u+r)}{u+r}\,J_0(u)\,du \quad\text{.................(25)},$$

$$Y_0(r) = -2\int_0^\infty \frac{\cos(u+r)}{u+r}\,J_0(u)\,du \quad\text{..............(26)}.$$

These formulæ were first obtained, by a different method, by Sonnine.

On substituting in (25), the value of $\frac{\sin(u+r)}{u+r}$, given by the addition formula of Section 10, we have

$$J_0(r) = \frac{\sin r}{r}\,\frac{2}{\pi}\int_0^\infty J_0(u)\,\frac{\sin u}{u}\,du$$

$$+\sum_{n=1}^{n=\infty}(-1)^n(2n+1)\,\frac{2}{\pi}\,(-2r)^n\,\frac{d^n}{d(r^2)^n}\,\frac{\sin r}{r}$$

$$\times\int_0^\infty J_0(u)(-2u)^n\,\frac{d^n}{d(u^2)^n}\,\frac{\sin u}{u}\,du,$$

which may be written

$$J_0(r) = \frac{1}{\sqrt{r}}\,J_0(r)\int_0^\infty J_0(u)\,\frac{J_{\frac{1}{2}}(u)}{\sqrt{u}}\,du$$

$$+\sum_{n=1}^{n=\infty}(-1)^n(2n+1)\,\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}\int_0^\infty J_0(u)\,\frac{J_{n+\frac{1}{2}}(u)}{\sqrt{u}}\,du.$$

---

* See *Mathematische Annalen*, Vol. I.

On comparing this expansion with (23), we see that

$$\int_0^\infty \frac{J_0(u)\,J_{n+\frac{1}{2}}(u)}{\sqrt{u}}\,du = 0,$$

when n is odd; and when n is even, writing $2n$ for n, we have

$$\int_0^\infty \frac{J_0(u)\,J_{2n+\frac{1}{2}}(u)}{\sqrt{u}}\,du = \sqrt{\frac{2}{\pi}}\,\frac{2n+\frac{1}{2}}{2n+1}\left\{\frac{\Pi\left(n-\frac{1}{2}\right)}{\Pi(n)}\right\}^2 \quad\ldots\ldots\ldots(27).$$

It is clear that, by using the addition theorem for $\frac{\cos(u+r)}{u+r}$, the equation (26) could be applied to obtain a development of $Y_0(r)$ in Bessel's functions of the second kind, and of orders equal to half an odd integer.

Definite Integral Relations between Bessel's Functions.

17. Since $\quad r\sin\theta = (x_1^2+x_2^2+\ldots+x_{p-1}^2)^{\frac{1}{2}},$

we see that $J_0(p-1, r\sin\theta)$ satisfies the equation

$$\nabla_p^2 V + V = 0,$$

being a solution which is independent of x_p; it follows that the mean value of $J_0(p-1, r\sin\theta)$ taken over the sphere of radius r is a solution of

$$\nabla_p^2 V + V = 0,$$

which is such that it depends only on r, and is therefore, except for a constant factor, equal to $J_0(p, r)$, as it is clear that the Bessel's function of the second kind cannot be involved.

We have $\quad \int_0^{\frac{1}{2}\pi} J_0(p-1, r\sin\theta)\sin^{p-2}\theta\,d\theta$

$$= \int_0^{\frac{1}{2}\pi} \frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}}\sin^{p-2}\theta\,d\theta$$

$$= \frac{1}{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)}\int_0^{\frac{1}{2}\pi}\left\{1-\frac{r^2\sin^2\theta}{2\,.\,p-1}+\frac{r^4\sin^4\theta}{2.4.p-1.p+1}-\ldots\right\}\sin^{p-2}\theta\,d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1}{2}(p-1)}\,\Pi\left(\frac{p-2}{2}\right)}\left\{1-\frac{r^2}{2\,.\,p}+\frac{r^4}{2\,.\,4\,.\,p\,(p+2)}-\ldots\right\}$$

$$= \sqrt{\frac{\pi}{2}}\,\frac{J_{\frac{1}{2}(p-2)}(r)}{r^{\frac{1}{2}(p-2)}} = \sqrt{\frac{\pi}{2}}\,J_0(p, r);$$

it follows that

$$\sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}\pi} J_{\frac{1}{2}(p-3)}(r\sin\theta)\sin^{\frac{1}{2}(p-1)}\theta\, d\theta = \frac{J_{\frac{1}{2}p-1}(r)}{\sqrt{r}} \quad \text{.........(28).}$$

Putting $p = 2n+3$, we have

$$\sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}\pi} J_n(r\sin\theta)\sin^{n+1}\theta\, d\theta = \frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}} \quad \text{............(29).}$$

Again, putting $p = 2n+2$, we have

$$\sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(r\sin\theta)\sin^{n+\frac{1}{2}}\theta = \frac{J_n(r)}{\sqrt{r}} \text{...............(30).}$$

A particular case of (29) is

$$\int_0^{\frac{1}{2}\pi} J_0(r\sin\theta)\sin\theta\, d\theta = \frac{\sin r}{r} \quad \text{..................(31);}$$

thus we have relations connecting Bessel's functions of orders differing by $\frac{1}{2}$.

The mode of verification of (28) shows that the relation holds even when p is not restricted to be a positive integer.

Expressions for Zonal and Tesseral Harmonics as Definite Integrals involving Bessel's Functions.

18. Let $$\rho^2 = x_1^2 + x_2^2 + \ldots + x_{p-1}^2;$$

then we have the well known theorem

$$\frac{1}{r} = \frac{1}{(\rho^2 + x_p^2)^{\frac{1}{2}}} = \int_0^\infty e^{-\lambda x_p} J_0(\lambda\rho)\, d\lambda.$$

Differentiating both sides of this equation m times with respect to ρ^2, we have

$$\frac{1}{r^{2m+1}} = \frac{1}{(\rho^2 + x_p^2)^{m+\frac{1}{2}}} = \frac{2^m \Pi(m)}{\Pi(2m)}\int_0^\infty e^{-\lambda x_p} \lambda^m \frac{J_m(\lambda\rho)}{\rho^m}\, d\lambda \text{......(32).}$$

In order to find a corresponding expression for the even powers of $\frac{1}{r}$, we have

$$\frac{1}{r^2} = \frac{1}{x_p^2 + \rho^2} = \int_0^\infty e^{-\lambda x_p} \frac{\sin\lambda\rho}{\rho}\, d\lambda$$

$$= \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda x_p} \lambda \frac{J_{\frac{1}{2}}(\lambda\rho)}{(\lambda\rho)^{\frac{1}{2}}}\, d\lambda;$$

on differentiation m times with respect to ρ^2, we have

$$\frac{1}{r^{2m+2}} = \frac{1}{(x_p^2+\rho^2)^{m+1}} = \frac{2^{m+\frac{1}{2}}\,\Pi\left(m+\frac{1}{2}\right)}{\Pi\,(2m+1)}\int_0^\infty e^{-\lambda x_p}\,\lambda^{2m+1}\,\frac{J_{m+\frac{1}{2}}(\lambda\rho)}{(\lambda\rho)^{m+\frac{1}{2}}}\,d\lambda \quad \ldots(33);$$

both the equations (32), (33) are included in the formula

$$\frac{1}{r^n} = \frac{1}{(x_p^2+\rho^2)^{\frac{1}{2}n}} = \frac{2^{\frac{1}{2}(n-1)}\,\Pi\left(\frac{n-1}{2}\right)}{\Pi\,(n-1)}\int_0^\infty e^{-\lambda x_p}\,\lambda^{\frac{1}{2}(n-1)}\,\frac{J_{\frac{1}{2}(n-1)}(\lambda\rho)}{\rho^{\frac{1}{2}(n-1)}}\,d\lambda \quad \ldots(34);$$

the most important case of this theorem is obtained by putting

$$n = p-2,$$

in which case we have

$$\frac{1}{r^{p-2}} = \frac{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)}{\Pi\,(p-3)}\int_0^\infty e^{-\lambda x_p}\,\lambda^{p-3}\,J_0\,(p-1,\,\lambda\rho)\,d\lambda \quad \ldots\ldots\ldots(35).$$

From the equation (35), we find, by differentiating n times with respect to x_p,

$$\frac{\partial^n}{\partial x_p^n}\frac{1}{r^{p-2}} = \frac{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)}{\Pi\,(p-3)}\int_0^\infty (-\lambda)^n\,e^{-\lambda x_p}\,\lambda^{p-3}\,J_0\,(p-1,\,\lambda\rho)\,d\lambda\,;$$

hence

$$\frac{P_n\,(p,\,\mu)}{r^{n+p-2}} = \frac{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)}{\Pi\,(p-3)\,\Pi\,(n)}\int_0^\infty \lambda^{n+p-3}\,e^{-\lambda x_p}\,J_0\,(p-1,\lambda\rho)\,d\lambda \quad \ldots(36);$$

this is the expression for the zonal hyper-harmonic of degree $-(n+p-2)$, in terms of Bessel's functions. In particular we have

$$\frac{P_n\,(\mu)}{r^{n+1}} = \frac{1}{\Pi\,(n)}\int_0^\infty \lambda^n e^{-\lambda z}\,J_0\,(\lambda\rho)\,d\lambda \quad \ldots\ldots\ldots\ldots\ldots(37).$$

19. For the ordinary system of zonal and tesseral harmonics of rank 3, we have from the equation

$$\frac{1}{r} = \frac{1}{(z^2+\xi\eta)^{\frac{1}{2}}} = \int_0^\infty e^{-\lambda z}\,J_0\,(\lambda\sqrt{\xi\eta})\,d\lambda,$$

where $$\xi = x + \iota y, \quad \eta = x - \iota y,$$

$$\frac{\partial^m}{\partial \xi^m}\frac{\partial^{n-m}}{\partial z^{n-m}}\frac{1}{r} = (-1)^{n-m}\int_0^\infty \lambda^{n-m} e^{-\lambda z}\frac{\partial^m}{\partial \xi^m} J_0\left(\lambda\sqrt{\xi\eta}\right) d\lambda$$

$$= (-1)^n \frac{1}{2^m} e^{-m\iota\phi}\int_0^\infty \lambda^n e^{-\lambda z} J_m(\lambda\rho)\, d\lambda.$$

Now $$\frac{\partial^m}{\partial \xi^m}\frac{\partial^{n-m}}{\partial z^{n-m}}\frac{1}{r} = (-1)^n \frac{e^{-m\iota\phi}}{r^{n+1}}\frac{(n-m)!}{2^m} P_n^m(\mu);$$

hence we have

$$\frac{P_n^m(\mu)}{r^{n+1}} = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda z} J_m(\lambda\rho)\, d\lambda \quad\text{.........(38)},$$

which gives an expression for the tesseral harmonic $\frac{P_n^m(\mu)}{r^{n+1}}\cos m\phi$ as a definite integral.

We have, putting $r = 1$,

$$P_n^m(\mu) = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda\cos\theta} J_m(\lambda\sin\theta)\, d\lambda \quad\text{.........(39)},$$

$$P_n(\mu) = \frac{1}{n!}\int_0^\infty \lambda^n e^{-\lambda\cos\theta} J_0(\lambda\sin\theta)\, d\lambda \quad\text{.........(40)}.$$

Some potential problems may be solved either in terms of the system of zonal and tesseral harmonics, or in terms of Bessel's functions; the formulæ of the present section afford the means of passing from one form of solution to the other.

20. It might be expected that, corresponding to (36) and (37), there should exist expressions for the positive harmonics $r^n P_n(p, \mu)$, $r^n P_n(\mu)$, as definite integrals involving Bessel's functions. If, in (16), we write λr for r, we see, by Cauchy's theorem, that

$$r^n P_n(p, \cos\theta)\frac{\Pi(p-3)}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)\Pi(n+p-3)}$$

$$= \frac{1}{2\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}}\frac{J_{\frac{1}{2}(p-3)}(\lambda r\sin\theta)}{(\lambda r\sin\theta)^{\frac{1}{2}(p-3)}}\, d\lambda,$$

or $$r^n P_n(p, \cos\theta)$$

$$= \frac{2^{\frac{1}{2}(p-5)}\Pi\left(\frac{p-3}{2}\right)\Pi(n+p-3)}{\Pi(p-3)\,\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}}\frac{J_{\frac{1}{2}(p-3)}(\lambda r\sin\theta)}{(\lambda r\sin\theta)^{\frac{1}{2}(p-3)}}\, d\lambda \quad\text{...(41)},$$

where the integral is taken along a complex path represented by a closed curve round the origin $\lambda = 0$.

In particular, we have

$$r^n P_n(\cos\theta) = \frac{n!}{2\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}} J_0(\lambda r\sin\theta)\,d\lambda \quad \ldots\ldots\ldots\ldots(42).$$

It may be shown that

$$r^n P_n^m(\cos\theta) = \frac{(n-m)!}{2\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}} J_m(\lambda r\sin\theta)\,d\lambda \ldots\ldots\ldots(43).$$

The expressions (42), (43) correspond exactly to (37) and (38), the only difference being that in the latter the integrals are taken along a real path, and in the former along a complex path.

Expressions for the Zonal and Tesseral Harmonics of the Second Kind in Terms of Bessel's Functions.

21. Let us evaluate the definite integral

$$\int_0^\infty e^{-\lambda z} Y_0(\lambda\rho)\,d\lambda.$$

Substituting for $Y_0(\lambda\rho)$, the value

$$\int_0^\infty \cos(\lambda\rho\cosh u)\,du,$$

we have

$$\int_0^\infty e^{-\lambda z} Y_0(\lambda\rho)\,d\lambda = \int_0^\infty\int_0^\infty e^{-\lambda z}\cos(\lambda\rho\cosh u)\,du\,d\lambda$$

$$= \int_0^\infty \frac{z\,d(2u)}{2z^2+\rho^2+\rho^2\cosh 2u}$$

$$= \frac{1}{2\sqrt{z^2+\rho^2}}\left[\cosh^{-1}\frac{(2z^2+\rho^2)\cosh 2u+\rho^2}{(2z^2+\rho^2)+\rho^2\cosh 2u}\right]_0^\infty$$

$$= \frac{1}{2\sqrt{z^2+\rho^2}}\left[\cosh^{-1}\frac{2z^2+\rho^2}{\rho^2}-\cosh^{-1}1\right]$$

$$= \frac{1}{2\sqrt{z^2+\rho^2}}\log_e\left(\frac{2z^2+\rho^2}{\rho^2}+\frac{2z\sqrt{z^2+\rho^2}}{\rho^2}\right)$$

$$= \frac{1}{\sqrt{z^2+\rho^2}}\log\frac{z+\sqrt{z^2+\rho^2}}{\rho};$$

thus we have the theorem

$$\frac{1}{r}\log_e\sqrt{\frac{1+\mu}{1-\mu}} = \int_0^\infty e^{-\lambda z}\, Y_0(\lambda\rho)\, d\lambda \qquad\text{.................(44),}$$

which corresponds to the known theorem

$$\frac{1}{r} = \int_0^\infty e^{-\lambda z}\, J_0(\lambda\rho)\, d\lambda.$$

From (44), we obtain, by differentiation n times with respect to r, the formula

$$\frac{Q_n(\mu)}{r^{n+1}} = \frac{1}{n!}\int_0^\infty \lambda^n e^{-\lambda z}\, Y_0(\lambda\rho)\, d\lambda \qquad\text{.................(45),}$$

where $Q_n(\mu)$ is the zonal harmonic of the second kind. As in the case of the harmonics of the first kind, we find

$$\frac{Q_n^m(\mu)}{r^{n+1}} = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda z}\, Y_m(\lambda\rho)\, d\lambda \qquad\text{...........(46);}$$

thus the tesseral harmonic $\frac{Q_n^m(\mu)}{r^{n+1}}\cos m\phi$ is expressed as a definite integral involving the elements $e^{-\lambda z}\, Y_m(\lambda\rho)\cos m\phi$.

Note on a Variable Seven-points Circle, analogous to the Brocard Circle of a Plane Triangle. By JOHN GRIFFITHS, M.A.

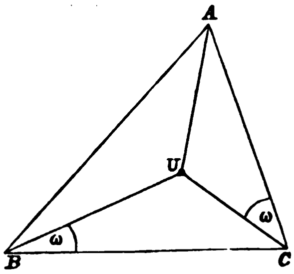
Received December 13th, 1893. Read December 14th, 1893.

The object of this note is to show that a seven-points circle can be constructed from a variable point U taken on one of three given circles connected with a triangle ABC.

1. On the side BC of a triangle ABC describe a circular arc BUC touching AC in C, and let U be any point on this arc. This con-

struction gives

$$\angle UBC = \angle UCA = \omega, \quad \text{and} \quad \angle BUC = \pi - C.$$

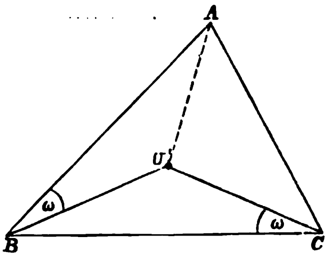


[If $\angle CUA$ be denoted by $\pi-\theta$, then

$$\cot\omega = \cot\theta + \cot B + \cot C.]$$

2. Let U' be the isogonal point of U; *i.e.*, let U' lie on the circular arc described on BC and touching AB in B, so that

$$\angle U'BA = \angle U'CB = \omega, \quad \text{and} \quad \angle BU'C = \pi - B.$$



[$\angle BU'A = \pi - \theta$,

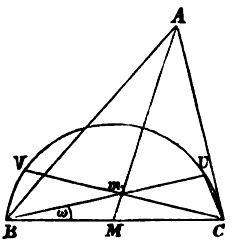
and $\cot\omega = \cot\theta + \cot B + \cot C$,

as before.

If $\theta = A$, then ω is the Brocard angle of ABC.]

This gives U' as a point dependent on U.

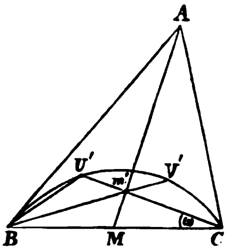
3. Let BU intersect the median AM of ΔABC in m; produce the join of C and m to meet the circle $CUVB$ in V; then V is a second point dependent on U.



[The points U, V form two homographic divisions on the circle $CUVB$, in which C and B are corresponding points. Hence the envelope of UV is a conic (a hyperbola), having double contact with the circle, along the homographic axis AM.]

4. Similarly, let CU' intersect the median AM in m'; produce the

join of B and m' to meet the circle $BU'V'C$ in V'; then V' is a third point dependent on U.

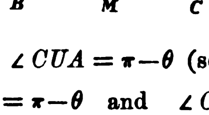


5. Let CV and BV' intersect in K. This gives K as a fourth point dependent on U.

6. Let the perpendicular to BC at M, the mid-point of BC, meet the side CA or CA produced in N; on CN describe a circular arc CON similar to the circular arc CUA, *i.e.*, such that

$$\angle CON = \angle CUA,$$

and let this arc CON meet the perpendicular MN in O; then O is a fifth point dependent on U.



If $$\angle CUA = \pi - \theta \text{ (see 1)},$$

then $$\angle CON = \pi - \theta \quad\text{and}\quad \angle COM = \theta,$$

where $$\cot\omega = \cot\theta + \cot B + \cot C.$$

If $$\theta = A,$$

U will be the positive Brocard point, and O will be the centre of the circumcircle ABC.

7. Lastly, let BU and CU' intersect in A'; then A' is a sixth point dependent on U.

Theorem I.

The seven points U, U', V, V', O, K, A' as defined above all lie on the circumference of a circle whose diameter is OK.

If U be the positive Brocard point, the circle in question is, in fact, the Brocard circle.

By similar constructions to the above with regard to the two other sides CA, AB of ABC, it is clear that we have in all three systems of variable seven-points circles, each of which has properties analogous to those of the Brocard circle of ABC.

Theorem II.

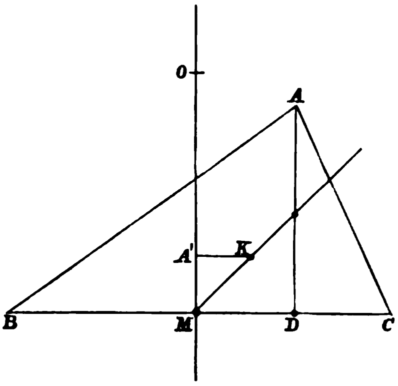
1. The centre of the circle through O, K, U, U', V, V', A' describes a hyperbola as U moves along the arc BUC.

2. The circle OKU is cut orthogonally by a fixed circle having its centre on the side BC.

3. Hence the envelope of the circle OKU is a bicircular quartic.

Theorem III.

As U moves along the fixed circle BUC, the points O and A' describe the right line MO perpendicular to BC at its mid-point M, while K moves on the line through M which bisects the perpendicular AD from A on BC.



If U be the positive Brocard point of ABC, then K is the symmedian point of ABC.

Theorem IV.

The angle subtended at O by UU' is equal to twice the angle UBC; *i.e.*, to 2ω. See 1.

Theorem V.

If from O we draw OA', OB', OC' perpendicular to the sides BC, CA, AB, meeting the circle OKU in A', B', C', then the triangle $A'B'C'$ is inversely similar to ABC.

As U moves on the fixed circular arc BUC, the points B' and C' each describe a hyperbola. [They can also be directly derived from the primary point U, since B' lies on CU and C' on BU'.]

Theorem VI.

Let $U'A$ meet the circle OKU again in B''; then the triangle $A'B''C'$ is similar to the pedal triangle of U with respect to ABC.

The above are some of the principal results which I have arrived at with regard to the variable seven-points circle in question. They have been deduced partly by elementary geometry, and partly by the use of isogonal coordinates, which I have employed in previous notes on the triangle.

The isogonal coordinates x, y, z of a point P are connected with its trilinear coordinates by the relations

$$\frac{x}{a} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{a\,(a\alpha + b\beta + c\gamma)}{\Sigma a\beta\gamma},$$

which give $$\Sigma ax = \Sigma ayz.$$

It thus follows that the equation of a circle in this system of coordinates is expressed by a linear relation

$$\lambda x + \mu y + \nu z = \delta.$$

If we take U as expressed by its isogonal coordinates

$$x = \frac{\sin C}{\sin B}, \quad y = \frac{\sin\theta}{\sin(\theta + B)}, \quad z = \frac{\sin(\theta + C)}{\sin\theta},$$

I have found that the equation of the above seven-points circle OKU

is $x\dfrac{\sin B\sin C}{\sin A}\sin(2\theta-A)+y\sin\theta\sin(\theta+B)+z\sin\theta\sin(\theta+C)$

$$=2\sin^2\theta\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right)=2\sin^2\theta\frac{\sin B\sin C}{\sin A}\cot\omega,$$

where $\cot\omega=\cot\theta+\cot B+\cot C$. (See 1.)

By writing this equation in the form

$$\xi\cos 2\theta+\eta\sin 2\theta+\zeta=0,$$

where $\xi=0$, $\eta=0$, and $\zeta=0$ denote circles, it is seen at once that the envelope of the seven-points circle is the bicircular quartic expressed by

$$\xi^2+\eta^2=\zeta^2.$$

Since the points U', V, V', O, K, A' are all dependent on U, there is no particular difficulty in expressing their coordinates as functions of those of U.

If U be given by

$$x=\frac{\sin C}{\sin B},\quad y=\frac{\sin\theta}{\sin(\theta+B)},\quad z=\frac{\sin(\theta+C)}{\sin\theta},$$

the coordinates of the six points in question are:—

$$U';\ X=\frac{1}{x},\quad Y=\frac{1}{y},\quad Z=\frac{1}{z}.$$

$$V;\ X=x,\quad Y=xz,\quad Z=\frac{1}{z}+\frac{\sin(B-C)}{\sin B}.$$

$$V';\ X=\frac{1}{x},\quad Y=y+\frac{\sin(C-B)}{\sin C},\quad Z=\frac{1}{xy}.$$

$$O;\ \frac{X}{\cos\theta}=\frac{-Y}{\cos(\theta+C)}=\frac{-Z}{\cos(\theta+B)}=\frac{2\sin\theta}{\sin(2\theta-A)}.$$

$$K;\ \frac{X}{\sin\theta}=\frac{Y}{\sin(\theta+C)}=\frac{Z}{\sin(\theta+B)}=k.$$

where $$k = 2(\sin\theta \sin A + \sin B \sin C \cos\theta)$$

$$\div \left\{2\sin(\theta+B)\sin(\theta+C)\sin A + \sin B \sin C \sin(2\theta - A)\right\}.$$

$$A';\ \frac{X}{xy} = \frac{Y}{x^2} = \frac{Z}{yz} = \frac{xy\sin A + x^2 \sin B + yz\sin C}{xy\,(xz\sin A + yz\sin B + x^2\sin C)},$$

or $$\frac{X}{\sin B\sin C\sin\theta} = \frac{Y}{\sin^2 C\sin(\theta+B)} = \frac{Z}{\sin^2 B\sin(\theta+C)}.$$

If $\theta = A$, we have the known point

$$\frac{X}{abc} = \frac{Y}{c^3} = \frac{Z}{b^3},$$

which lies on the Brocard circle.

As I have stated above, the coordinates X, Y, Z must, in all cases, satisfy the relation $$\Sigma(X - YZ)\sin A = 0.$$

[*Appendix.*

At the suggestion of Mr. Tucker I append some additional details with regard to the isogonal coordinates of the seven points U, U', V, V', O, K, A', in order to verify the fact that they are concyclic.

1. If we take the coordinates of the primary point U to be

$$x = \frac{\sin C}{\sin B},\quad y = \frac{\sin\theta}{\sin(\theta+B)},\quad z = \frac{\sin(\theta+C)}{\sin\theta},$$

the condition that this point shall lie on the circle expressed by the equation

$$X\frac{\sin B\sin C}{\sin A}\frac{\sin(2\theta - A)}{\sin^2\theta} + Y\frac{\sin(\theta+B)}{\sin\theta} + Z\frac{\sin(\theta+C)}{\sin\theta}$$
$$= 2\left(1 + \frac{\sin B\sin C}{\sin A}\cot\theta\right)$$

is $$\sin^2 C\sin(2\theta - A) + \sin A\sin^2\theta + \sin A\sin^2(\theta+C)$$
$$= 2\sin A\sin^2\theta + \sin B\sin C\sin 2\theta.$$

There is no difficulty in proving that this is a trigonometrical identity.

2. Since

$$z^2-\frac{1}{y^2}=\frac{\sin^2(\theta+C)-\sin^2(\theta+B)}{\sin^2\theta}=\frac{\sin(B-C)\sin(2\theta-A)}{\sin^2\theta},$$

it appears that the equation of the circle in question can also be written in the form

$$\left(z^2-\frac{1}{y^2}\right)(X-x)\frac{\sin B\sin C}{\sin A\sin(B-C)}+\frac{1}{y}(Y-y)+z(Z-z)=0.$$

This circle will pass through $U'\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ if

$$\left(z^2-\frac{1}{y^2}\right)\left(\frac{1}{x}-x\right)\frac{\sin B\sin C}{\sin A\sin(B-C)}+\frac{1}{y}\left(\frac{1}{y}-y\right)+z\left(\frac{1}{z}-z\right)=0,$$

i.e., if $\left(\frac{1}{x}-x\right)\sin B\sin C=\sin A\sin(B-C),$

or $\sin^2 B-\sin^2 C=\sin A\sin(B-C).$

3. The coordinates of V are

$$X=x,\quad Y=xz,\quad Z=\frac{1}{z}+\frac{\sin(B-C)}{\sin B};$$

V will therefore lie on the circle, if

$$\frac{1}{y}(xz-y)+z\left(\frac{1}{z}-z+\frac{\sin(B-C)}{\sin B}\right)=0,$$

or $$\frac{x}{y}-z+\frac{\sin(B-C)}{\sin B}=0.$$

Now, since

$$x=\frac{\sin C}{\sin B},\quad y=\frac{\sin\theta}{\sin(\theta+B)},\quad z=\frac{\sin(\theta+C)}{\sin\theta},$$

we have

$$\frac{1}{y\sin B}-\frac{z}{\sin C}=\cot B+\cot\theta-(\cot C+\cot\theta)=\frac{\sin(C-B)}{\sin B\sin C},$$

i.e., $$\frac{x}{y}-z+\frac{\sin(B-C)}{\sin B}=0.$$

4. The coordinates of V' are

$$X=\frac{1}{x},\quad Y=y+\frac{\sin(C-B)}{\sin C},\quad Z=\frac{1}{xy}.$$

Therefore V' will lie on the circle, if

$$\left(z^2-\frac{1}{y^2}\right)\left(\frac{1}{x}-x\right)\frac{\sin B\sin C}{\sin A\sin(B-C)}+\frac{1}{y}\frac{\sin(C-B)}{\sin C}+z\left(\frac{1}{xy}-z\right)=0,$$

or $$\left(z^2-\frac{1}{y^2}\right)+\frac{1}{y}\frac{\sin(C-B)}{\sin C}+z\left(\frac{1}{xy}-z\right)=0,$$

i.e., if $$\frac{x}{y}-z+\frac{\sin(B-C)}{\sin B}=0.$$

This is an equation satisfied by the coordinates of U, as I have just proved.

5. The coordinates of O are

$$\frac{X}{\cos\theta}=\frac{Y}{-\cos(\theta+C)}=\frac{Z}{-\cos(\theta+B)}=\frac{2\sin\theta}{\sin(2\theta-A)}.$$

This point will lie on the circle expressed by

$$X\frac{\sin B\sin C}{\sin A}\frac{\sin(2\theta-A)}{\sin^2\theta}+Y\frac{\sin(\theta+B)}{\sin\theta}+Z\frac{\sin(\theta+C)}{\sin\theta}$$
$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

if $$2\frac{\sin B\sin C}{\sin A}\cot\theta-2\frac{\sin(\theta+B)\cos(\theta+C)+\sin(\theta+C)\cos(\theta+B)}{\sin(2\theta-A)}$$
$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

i.e., if $\sin(\theta+B)\cos(\theta+C)+\sin(\theta+C)\cos(\theta+B)+\sin(2\theta-A)=0,$

or $$\sin(2\theta+B+C)+\sin(2\theta-A)=0,$$
$$\sin(2\theta-A+\pi)+\sin(2\theta-A)=0.$$

6. The coordinates of K are

$$X=k\sin\theta,\quad Y=k\sin(\theta+C),\quad Z=k\sin(\theta+B),$$

where $$\Sigma X\sin A=\Sigma YZ\sin A,$$

or $$k=\frac{\sin\theta\sin A+\sin(\theta+C)\sin B+\sin(\theta+B)\sin C}{\sin(\theta+B)\sin(\theta+C)\sin A+\sin(\theta+B)\sin\theta\sin B+\sin(\theta+C)\sin\theta\sin C}$$

$$=\frac{2(\sin A\sin\theta+\sin B\sin C\cos\theta)}{\sin B\sin C\sin(2\theta-A)+2\sin A\sin(\theta+B)\sin(\theta+C)}.$$

K will therefore lie on the circle expressed by the equation

$$X\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin^2\theta}+Y\frac{\sin(\theta+B)}{\sin\theta}+Z\frac{\sin(\theta+C)}{\sin\theta}$$
$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

if $$k\left\{\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin\theta}+2\,\frac{\sin(\theta+B)\sin(\theta+C)}{\sin\theta}\right\}$$
$$=2\left(1+\frac{\sin B\sin C}{\sin A}\cot\theta\right),$$

or $$k=\frac{2(\sin A\sin\theta+\sin B\sin C\cos\theta)}{\sin B\sin C\sin(2\theta-A)+2\sin A\sin(\theta+B)\sin(\theta+C)}.$$

7. The coordinates of A' are

$X=k\sin B\sin C\sin\theta,\ Y=k\sin^2 C\sin(\theta+B),\ Z=k\sin^2 B\sin(\theta+C).$

Now, since $$\Sigma X\sin A=\Sigma YZ\sin A,$$

we have $$k=\frac{N}{D},$$

where $$N=2(\sin A\sin\theta+\sin B\sin C\cos\theta),$$

$$D=\sin A\sin B\sin C\sin(\theta+B)\sin(\theta+C)$$
$$+\left\{\sin^2 B\sin(\theta+C)+\sin^2 C\sin(\theta+B)\right\}\sin\theta;$$

A' will therefore lie on the circle expressed by

$$X\frac{\sin B\sin C}{\sin A}\,\frac{\sin(2\theta-A)}{\sin^2\theta}+Y\frac{\sin(\theta+B)}{\sin\theta}+Z\frac{\sin(\theta+C)}{\sin\theta}$$
$$=2\left(1+\frac{\sin B\sin\theta\cot\theta}{\sin A}\right),$$

if $$k\left\{\sin^2 B\sin^2 C\sin(2\theta-A)+\sin^2 C\sin^2(\theta+B)\sin A\right.$$
$$\left.+\sin^2 B\sin^2(\theta+C)\sin A\right\}$$
$$=2(\sin A\sin\theta+\sin B\sin C\cos\theta)=N.$$

It thus appears that the circle will pass through A', if

$$\sin^2 B\sin^2 C\sin(2\theta-A)+\left\{\sin^2 C\sin^2(\theta+B)+\sin^2 B\sin^2(\theta+C)\right\}\sin A$$
$$=D=\sin A\sin B\sin C\sin(\theta+B)\sin(\theta+C)$$
$$+\left\{\sin^2 B\sin(\theta+C)+\sin^2 C\sin(\theta+B)\right\}\sin\theta.$$

There is no especial difficulty in proving that this is a trigonometrical identity.

8. It may be noticed here that, since the above note was communicated to the Society, I have found that the circle $UU'V\ldots$ passes through an eighth point dependent on U; viz., the intersection of the circles ACU, ABU'.

The coordinates of this point, W, say, are

$$x = \frac{\sin 2\theta}{\sin(2\theta - A)}, \quad y = \frac{\sin\theta}{\sin(\theta + B)}, \quad z = \frac{\sin\theta}{\sin(\theta + C)}.$$

As θ varies, or the circle $UU'V\ldots$ moves, the locus of W is a circular cubic having A for a double point.

The equation of this curve is

$$\frac{1}{y\sin B} - \frac{1}{z\sin C} = \cot B - \cot C,$$

which proves that it passes through B, C, and the foot of the perpendicular from A upon BC, and has an asymptote parallel to the median AM.]

On a Theorem of Liouville's. By Mr. G. B. Mathews.

Read December 14th, 1893.

In the first of the series of papers "Sur quelques Formules Générales qui peuvent être Utiles dans la Théorie des Nombres" [*Journ. de Math.*, (2) iii. (1858), p. 143], Liouville has stated without proof the following remarkable proposition:—

Let $2m$, the double of any odd integer, be expressed in all possible ways as the sum of two odd numbers, a and b, where the decompositions $2m = a+b$ and $2m = b+a$ are considered distinct, unless $a = b = m$; let α denote any divisor of a, and β any divisor of b, and let $f(x)$ be any even function of x, that is, such that

$$f(-x) = f(x).$$

Then, if μ denotes any divisor of m,

$$\Sigma\{f(\alpha-\beta) - f(\alpha+\beta)\} = \Sigma\mu\{f(0) - f(2\mu)\},$$

where the summation on the left applies to all pairs of divisors α and

β which can be derived from each of the partitions of $2m$, and the summation on the right applies to all the divisors of m.

A little consideration will show that the theorem must involve the following:—

The number of pairs of *unequal* associated divisors (α, β) for which the sum of α and β has any prescribed value is the same as that of the pairs for which the difference of α and β has the same prescribed value.

Moreover, when this second proposition is proved, it will be easy to infer the truth of the first.

To show the meaning of the theorems, and the way in which they are connected, suppose $m = 7$; then the partitions, and the values of α and β in each case, are

$1+13$,	$\alpha = 1$,	$\beta = 1, 13$,
$3+11$,	$1, 3$,	$1, 11$,
$5+9$,	$1, 5$,	$1, 3, 9$,
$7+7$,	$1, 7$,	$1, 7$,
$9+5$,	$1, 3, 9$,	$1, 5$,

and so on, the last partition being $13+1$.

The values of $\alpha+\beta$ are

$$2, \quad 4, \quad 6, \quad 8, \quad 10, \quad 12, \quad 14,$$

occurring respectively

$$7, \quad 4, \quad 2, \quad 4, \quad 2, \quad 2, \quad 7$$

times; and the values of $|\alpha-\beta|$ are

$$0, \quad 2, \quad 4, \quad 6, \quad 8, \quad 10, \quad 12,$$

occurring respectively

$$8, \quad 6, \quad 4, \quad 2, \quad 4, \quad 2, \quad 2$$

times. Thus, for instance, $|\alpha-\beta| = 4$, for the combinations (5, 1), (5, 9), (9, 5), (1, 5), and for no others. In this particular case, then,

$$\Sigma\{f(\alpha-\beta)-f(\alpha+\beta)\} = 8f(0)-f(2)-7f(14),$$

which agrees with $\Sigma\mu\{f(0)-f(2\mu)\}$, since the values of μ are 1 and 7.

The values of α, β in any associated pair are both odd, and their sum is therefore even. It is easy to see that the number of pairs

(α, β) for which $\alpha+\beta$ has a prescribed value $2t$ is equal to the number of positive integral solutions of the diophantine equations

$$\left.\begin{array}{r} x_1+(2t-1)\,y_1 = 2m \\ 3x_2+(2t-3)\,y_2 = 2m \\ 5x_3+(2t-5)\,y_3 = 2m \\ \vdots \qquad \vdots \qquad \vdots \\ (2t-1)\,x_t+y_t = 2m \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

In the same way, the number of pairs (α, β) for which $|\,\alpha-\beta\,| = 2t$ is *double* the number of positive integral solutions of

$$\left.\begin{array}{r} \xi_1+(2t+1)\,\eta_1 = 2m \\ 3\xi_2+(2t+3)\,\eta_2 = 2m \\ 5\xi_3+(2t+5)\,\eta_3 = 2m \\ \vdots \qquad \vdots \qquad \vdots \\ (2h-1)\,\xi_h+(2t+2h-1)\,\eta_h = 2m \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(B),$$

where, for a reason which will sufficiently appear as we proceed, the series of equations (B) is supposed to continue indefinitely, although it is clear that after a certain point, depending on the value of t, the equations cease to have positive solutions.

For convenience the r^{th} equation of the set (A) or (B) will be referred to as (A_r) or (B_r) respectively.

The equation (A_1) may be written

$$(x_1-2y_1)+(2t+1)\,y_1 = 2m\,;$$

hence, if there is a positive solution of the equation, such that $x_1 > 2y_1$, the values

$$\xi_1 = x_1-2y_1, \quad \eta_1 = y_1$$

give a positive solution of (B_1). Conversely, writing (B_1) in the form

$$(\xi_1+2\eta_1)+(2t-1)\,\eta_1 = 2m,$$

we see that from every positive solution of (B_1) may be derived a solution of (A_1) by putting

$$x_1 = \xi_1+2\eta_1, \quad y_1 = \eta_1,$$

and, moreover, this gives $\quad x_1-2y_1 = \xi_1,$

so that $x_1 > 2y_1$.

In the same way, every positive integral solution of (A_r) for which

$x_r > 2y_r$ is associated with a solution of (B_r); and, conversely, from every solution of (B_r) may be derived a solution of (A_r) for which $x_r > 2y_r$.

Again, the equation (A_1) may be written

$$(2t-1)(y_1-2x_1)+(4t-1)\,x_1 = 2m\,;$$

consequently, if there is a solution for which $y_1 > 2x_1$, we obtain a solution of (B_t) by putting

$$\xi_t = y_1-2x_1, \quad \eta_t = x_1,$$

and, conversely, from every positive solution of (B_t), we derive a solution of (A_1) by putting

$$x_1 = \eta_t,$$

$$y_1 = \xi_t+2\eta_t,$$

and this is a solution for which $y_1 > 2x_1$.

In the same way, every positive solution of (A_r) for which $y_r > 2x_r$ is associated with a solution of (B_{t+1-r}) by means of the relations

$$\xi_{t+1-r} = y_r-2x_r,$$

$$\eta_{t+1-r} = x_r\,;$$

and, conversely, from every positive solution of (B_{t+1-r}) may be derived a solution of (A_r) by putting

$$x_r = \eta_{t+1-r},$$

$$y_r = \xi_{t+1-r}+2\eta_{t+1-r},$$

and this is a solution for which $y_r > 2x_r$.

It will be observed that the conditions $x_r > 2y_r$ and $y_r > 2x_r$ are mutually exclusive; so that, on the whole, each positive solution of the equations (B_1), (B_2), ... (B_t) is associated with two distinct solutions of the equations (A).

Now the equation (B_{t+1}) is

$$(2t+1)\,\xi_{t+1}+(4t+1)\,\eta_{t+1} = 2m,$$

which may be written

$$(2\xi_{t+1}+3\eta_{t+1})+(2t-1)(\xi_{t+1}+2\eta_{t+1}) = 2m\,;$$

hence we obtain a solution of (A_1) by putting

$$x_1' = 2\xi_{t+1} + 3\eta_{t+1},$$

$$y_1' = \xi_{t+1} + 2\eta_{t+1}.$$

This gives

$$x_1' - 2y_1' = -\eta_{t+1},$$

$$2x_1' - 3y_1' = \xi_{t+1},$$

so that the solution is one for which

$$2y_1' > x_1' > \tfrac{3}{2}y_1';$$

and, therefore, distinct from those already considered.

In the same way, every positive solution of (B_{t+r}), where $r < t+1$, is associated with a solution of (A_r), say (x_r', y_r'), for which

$$2y_r' > x_r' > \tfrac{3}{2}y_r';$$

and, conversely, every solution of (A_r) for which these conditions of inequality are satisfied leads to a positive solution of (B_{t+r}) in the form

$$\xi_{t+r} = 2x_r' - 3y_r',$$

$$\eta_{t+r} = 2y_r' - x_r'.$$

Again, the equation (A_t) may be written

$$(4t-1)(2y_1 - 3x_1) + (6t-1)(2x_1 - y_1) = 2m,$$

so that, if there is a solution (x_r', y_r') for which

$$2x_1' > y_1' > \tfrac{3}{2}x_1',$$

we obtain a positive solution of (B_{2t}) by putting

$$\xi_{2t} = 2y_1' - 3x_1',$$

$$\eta_{2t} = 2x_1' - y_1';$$

and, conversely, every positive solution of (B_{2t}) leads to a corresponding solution of (A_1) for which

$$x_1' = \xi_{2t} - 2\eta_{2t},$$

$$y_1' = 2\xi_{2t} + 3\eta_{2t},$$

and

$$2x_1' > y_1' > \tfrac{3}{2}x_1';$$

and, in the same way, every positive solution of (B_{2t+1-r}) is associated with a solution of (A_r) for which

$$2x_r' > y_r' > \tfrac{3}{2}x_r',$$

by means of the relations

$$\xi_{2t+1-r} = 2y'_r - 3x'_r, \quad \eta_{2t+1-r} = 2x'_r - y'_r,$$

and conversely.

It is now easy to see that if the equations (B) are considered in successive groups, each containing t equations, then, if (ξ, η) is any positive solution of the r^{th} equation of the i^{th} group, we obtain a corresponding solution of (A_r) by putting

$$x_r^{(i)} = i\xi + (i+1)\,\eta,$$
$$y_r^{(i)} = (i-1)\,\xi + i\eta\,;$$

in fact, this gives

$$\begin{aligned}(2r-1)\,x_r^{(i)} + (2t-2r+1)\,y_r^{(i)} &= \{(2r-1)\,i + (2t-2r+1)(i-1)\}\,\xi \\ &\quad + \{(2r-1)(i-1) + (2t-2r+1)\,i\}\,\eta \\ &= \{2\,(i-1)\,t + 2r - 1\}\,\xi + \{2it + 2r - 1\}\,\eta \\ &= 2m\,;\end{aligned}$$

because (ξ, η) is a solution of (B_k), where

$$k = (i-1)\,t + r.$$

It will be found that

$$\xi = ix_r^{(i)} - (i+1)\,y_r^{(i)},$$
$$\eta = iy_r^{(i)} - (i-1)\,x_r^{(i)},$$

so that the solution of (A_r) is one for which

$$\frac{i}{i-1}\,y_r^{(i)} > x_r^{(i)} > \frac{i+1}{i}\,y_r^{(i)}\,;$$

and, conversely, from every such solution may be derived a corresponding solution of (B_k).

Similarly, to every solution $(x_r^{(i)}, y_r^{(i)})$ for which

$$\frac{i}{i-1}\,x^{(i)} > y_r^{(i)} > \frac{i+1}{i}\,x_r^{(i)}$$

corresponds a solution of (B_l), where

$$l = it + 1 - r,$$

which is given by
$$\xi = iy_r^{(i)} - (i+1)\, x_r^{(i)},$$
$$\eta = ix_r^{(i)} - (i-1)\, y_r^{(i)}.$$

Each solution (ξ, η) belonging to the i^{th} group of equations (B) is therefore associated with two solutions of the equations (A), while every solution of an equation (A) which satisfies the inequalities last written is associated with one, and only one, equation (B) of the group. Moreover, if we consider the inequalities

$$x > 2y, \quad 2y > x > \tfrac{3}{2}y, \quad \tfrac{3}{2}y > x > \tfrac{4}{3}y, \; \ldots \; \frac{i}{i-1}\, y > x > \frac{i+1}{i}\, y, \; \ldots,$$

$$y > 2x, \quad 2x > y > \tfrac{3}{2}x, \quad \tfrac{3}{2}x > y > \tfrac{4}{3}x, \; \ldots \; \frac{i}{i-1}\, x > y > \frac{i+1}{i}\, x, \; \ldots,$$

where i assumes all positive integral values, it is clear that any two positive integers x, y must satisfy one, and only one, of these conditions, except when
$$x = y,$$
or
$$x = i+1, \quad y = i,$$
or
$$x = i, \quad y = i+1.$$

In the case we are considering, it is easy to see that, since m is odd, x and y must be both odd or both even if (x, y) is a solution of an equation (A_r); consequently, the only exceptional case to be considered is when $x = y$.

This gives, for each of the equations (A),
$$2tx = 2m, \quad \text{or} \quad tx = m;$$
therefore t is a divisor of m, say μ, and
$$x = y = \mu',$$
where
$$\mu\mu' = m.$$

If, then, μ is any divisor of m, there will be exactly μ combinations (α, β) for which
$$\alpha + \beta = 2\mu,$$
namely, $(1, 2\mu-1), \; (3, 2\mu-3), \; \ldots \; (2\mu-1, 1),$

not associated with corresponding combinations (α', β') for which
$$|\, \alpha' - \beta' \,| = 2\mu.$$

These pairs are connected with the partitions

$$\mu' + (2\mu - 1)\,\mu', \quad 3\mu' + (2\mu - 3)\,\mu', \;\ldots\; (2\mu - 1)\,\mu' + \mu'$$

(where $\mu' = m/\mu$).

Consequently, the sum of the uncompensated terms $f(\alpha + \beta)$ is

$$\Sigma \mu f(2\mu).$$

The number of terms for which $\alpha - \beta = 0$ is very easily found. If $\alpha = \beta$, each must be a divisor of m; suppose

$$\alpha = \beta = \mu.$$

Then the combination (μ, μ) arises from each of the partitions

$$\mu + (2m - \mu), \quad 3\mu + (2m - 3\mu), \;\ldots\; (2m - \mu) + \mu,$$

and no others. The number of these partitions is

$$\frac{m}{\mu} = \mu', \text{ say};$$

therefore the number of times $\alpha - \beta = 0$ is $\Sigma\mu'$, or, which is the same thing, $\Sigma\mu$.

In the triple sum $\quad \Sigma \{f(\alpha - \beta) - f(\alpha + \beta)\}$,

every term $f(\alpha - \beta)$ in which $\alpha - \beta$ is not zero is cancelled by a corresponding term $f(\alpha' + \beta')$, and, therefore, finally,

$$\Sigma \{f(\alpha - \beta) - f(\alpha + \beta)\} = f(0)\,\Sigma\mu - \Sigma\mu f(2\mu)$$
$$= \Sigma\mu \{f(0) - f(2\mu)\},$$

which is Liouville's theorem.

Thursday, January 11th, 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

Mr. A. E. Daniels, B.A., late Scholar of Peterhouse, Cambridge, Mathematical Master of Nottingham High School, was elected a member. Messrs. H. M. Macdonald and C. Morgan were admitted into the Society.

The President communicated to the meeting the intelligence which had just reached him of the death, on the 10th January, of Dr. Heinrich Rudolf Hertz, who was elected an honorary foreign member of the Society on April 14th, 1892.*

The following communications were made:—

The Types of Wave-Motion in Canals: Mr. H. M. Macdonald.

On Green's Function for a System of non-Intersecting Spheres: Prof. W. Burnside.

The following presents were made to the Library:—

"University of Japan—Calendar for 1892–3"; Tokyo, 1893.

"Beiblätter zu den Annalen der Physik und Chemie," 1893, Bd. xvii., St. 11; Leipzig, 1893.

"Nieuw Archief voor Wiskunde," 2e Reeks, Deel i., 1; Amsterdam, 1894.

"Proceedings of the Royal Society," Vol. liv., No. 328.

"Nyt Tidsskrift for Mathematik," A. 4e Aargang, Nr. 4–6.

"Nyt Tidsskrift for Mathematik," B. 4e Aargang, Nr. 3; Copenhagen, 1893.

"Bulletin of the New York Mathematical Society," Vol. iii., No. 3; December, 1893.

"Proceedings of the Cambridge Philosophical Society," Vol. viii., Pt. 2; 1894.

"Revue Semestrielle des Publications Mathématiques," Tome ii., 1ère Partie; Amsterdam, 1894.

"Wiskundige Opgaven met de Oplossingen," Zesde Deel, 2de Stuk; Amsterdam, 1893.

"Bulletin de la Société Mathématique de France," Tome xxi., Nos. 7, 8; Paris.

"Bulletin des Sciences Mathématiques," Tome xvii., October, 1893; Paris.

Byerly, W. E.—"Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics," 8vo; Boston, 1893.

"L'Intermédiaire des Mathématiciens," Tome i., No. 1; Paris, 1894.

"Annali di Matematica," Serie 2, Tomo xxi., Fasc. 4; Milano.

"Educational Times," January, 1894.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. ii., Fasc. 10, 11, 2 Sem.; Roma, 1893.

"Annals of Mathematics," Vol. viii., No. 1.

"Indian Engineering," Vol. xiv., Nos. 22–5.

"Transactions of the Royal Irish Academy," Vol. xxx., Pts. 5–10.

"Proceedings of the Royal Irish Academy," Vol. iii., No. 1; Dublin, 1893.

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* *Cf. Nature*, January 18th, 1894 (p. 265), for a Notice by D. E. J.

On Green's Function for a System of non-Intersecting Spheres. *By* Prof. W. Burnside. Received January 8th, 1894. Read January 11th, 1894.

1. *Introduction.*

In the application of the method of images to the problem of two electrified spheres, series are obtained of the form

$$\frac{1}{OP}-\frac{c_1}{O_1P}+\frac{c_2}{O_2P}-\dots,$$

where O_1, O_2, &c., are the images of the original point O, backwards and forwards, in the spheres, and $c_1, c_2 \dots$ are certain constants depending on the radii of the spheres and the position of the points O. The point P is the variable point, whose coordinates are the independent variables of which the above series is a function.

Green's function for two spheres and the potential at any external point due to the charged spheres are both expressible by one or more series of the above form. Now, though for purposes of actual calculation these series are expressed in a sufficiently convenient form, they give, in the form written, no information as to the nature of the analytical function of x, y, z represented by them. A very simple modification of the separate terms of the series brings this latter point into a clear light. It may, in fact, be shown by elementary geometry that, if P_n is the point derived from P by taking in the inverse order the set of inversions through which O_n proceeds from O, then

$$\frac{c_n}{O_nP}=\frac{\mu_n^{\frac{1}{2}}}{OP_n},$$

where μ_n is the ratio of the infinitesimal linear elements at P_n and P. The series is thus expressible in the form

$$\sum_n (-1)^n \frac{\mu_n^{\frac{1}{2}}}{OP_n},$$

and the quasi-automorphic character of the function represented, for the group of point-transformations generated by inversions at the two spheres, is almost immediately evident.

It is the object of the present paper to lead up to this result, starting directly from a consideration of the group arising from any finite number of non-intersecting spheres. The convergency of the chief series involved, namely, $\sum_i \mu_i^4$, is not proved unconditionally; but it is shown that, whatever be the number of spheres, this series will certainly be convergent when certain inequalities between their radii and distances apart are satisfied. The groups of transformations dealt with are the analogues for three dimensions of those treated in the author's paper "On a Class of Automorphic Functions" (*Proc. Lond. Math. Soc.*, Vol. XXIII.); and many details in the treatment of the two are practically identical. The attempt has, however, been made to ensure that this paper shall be complete in itself.

2. *Notation.*

From n given spheres, and a given point, a series of points may be formed by successive inversions. The points thus proceeding from a given point P may be conveniently denoted by a suffix notation $P_{123\ldots}$, this symbol denoting the point derived from P by inverting first in sphere 1, then in sphere 2, and so on. It may be assumed that in the suffix no symbol occurs twice running, since two successive inversions in a sphere produce no change.

Where there is no risk of confusion, any point of the series will be represented by P_i, the suffix i being an abbreviation for any possible combination of the symbols 1, 2, 3, ... n.

The coordinates of each point in the series are rational functions of the coordinates of P; but, since when two points are inverses of each other the coordinates of either are rational functions of those of the other, it follows that the coordinates of all the points in the series are rational functions of those of any one of them, such as P_i.

The n rational reversible operations, by means of which the coordinates of the successive points in the series are expressed in terms of those of P, namely, the inversions at the n spheres, may be replaced by an inversion at a single sphere, say sphere 1, and $n-1$ pairs of inversions taken at the spheres 1 and 2, 1 and 3, ... 1 and n.

The new operations thus introduced, each consisting of a pair of inversions, are evidently also rational and reversible; and they also form a group, in the sense that the result of performing successively two operations of the series is always equivalent to some other operation of the same series. This latter statement is also clearly true of the original series of inversions, but it is not true of the series of operations each consisting of an odd number of inversions. If P_i be now used to denote any point proceeding from P by the operations of the group, which arises from the $n-1$ pairs of inversions, the original series of points will be denoted by the symbols P_i and P_{1i}, where i is replaced in succession by every possible combination of an even number of the original suffixes.

Since the corresponding infinitesimal figures each of which is the inverse of the other with respect to a sphere are similar, the same is true of the two corresponding infinitesimal figures one of which is derived from the other by any operation of the group now introduced. The symbol μ_i will be used to denote the ratio of an infinitesimal element of length in the neighbourhood of the point P_i to the corresponding element in the neighbourhood of P, or, in other words, the linear magnification at P_i. For a single inversion the linear magnification is evidently a rational function of the coordinates of either of the two inverse points considered; and therefore μ_i is a rational function of x_i, y_i, z_i the coordinates of P_i.

3. *Quasi-Automorphic Functions.*

Consider now the series

$$\sum_i \mu_i^m f(x_i, y_i, z_i) = F(x, y, z),$$

where the summation is extended to all the operations of the group, suffix 0 corresponding to the identical operation, so that x_0, y_0, z_0 are x, y, z, and μ_0 is unity. When the group is finite, and therefore also the number of terms in the series, the latter will be a one-valued function of the position of P, so long as $f(x, y, z)$ is a one-valued function of x, y, z, and m is integral. The latter condition may be dispensed with, if it is agreed that μ_i^m shall represent the real positive m^{th} power of μ_i (which is itself necessarily positive) whatever m may be. If the group is of infinite order, the convergency of the series must be considered. This will be dealt with later, and is for the present assumed.

If S_k is that operation of the group that changes P into P_k, so that

$$S_k x = x_k, \quad S_k y = y_k, \quad S_k z = z_k,$$

then
$$S_k F(x, y, z) = F(x_k, y_k, z_k).$$

Now the totality of the points P_i are unaltered by the operation S_k; and therefore the particular point P_i must by this operation be changed into some other one of the series. Moreover, no two can be changed into the same point, for from

$$S_k P_i = S_k P_{i'},$$

in consequence of the reversibility of the operation S_k,

$$P_i = P_{i'},$$

would follow.

Hence $$S_k P_i = P_j,$$

where, when i takes every possible value once, so also does j.

Now $$\mu_i^2 = \frac{dx_i^2 + dy_i^2 + dz_i^2}{dx^2 + dy^2 + dz^2},$$

and hence $$(S_k \mu_i)^2 = \frac{dx_j^2 + dy_j^2 + dz_j^2}{dx_k^2 + dy_k^2 + dz_k^2} = \frac{\mu_j^2}{\mu_k^2},$$

or $$S_k \mu_i = \frac{\mu_j}{\mu_k};$$

therefore $$S_k \sum_i \mu_i^m f(x_i, y_i, z_i) = \sum_j \frac{\mu_j^m}{\mu_k^m} f(x_j, y_j, z_j),$$

where, on the right-hand, j takes every possible value once.

Hence, finally, $$F(x_k, y_k, z_k) = \mu_k^{-m} F(x, y, z),$$

so that the function $F(x, y, z)$ is quasi-automorphic with regard to the group of operations considered.

Finite groups of operations of the kind here considered are exhaustively dealt with in a memoir by Goursat (*Ann. de l'Ecole Nor. Sup.*, t. 6). When two or more of the spheres intersect at angles which are incommensurable with two right angles, an infinite number of points P_i lie in the neighbourhood of any given one, and the series above considered cannot be convergent. The case in which each of the spheres is external to all the others is the one which it is proposed here to deal with.

In this case, if P is external to all the spheres, the points P_i are all within the spheres; and by a process which need not here be given in detail, as it is precisely analogous to the first convergency proof given in Poincaré's memoir on Fuchsian functions (*Acta Mathematica*, t. 1), it is easily shown that the series

$$\sum_i \mu_i^m$$

is absolutely convergent if m is not less than 3.

When the distances apart of the centres of the spheres are sufficiently great in comparison with their radii, the second convergency proof given in the author's paper "On a Class of Automorphic Functions" (*Proc. Lond. Math. Soc.*, Vol. XXIII.) may be modified as follows, to show that the above series is convergent so long as m is positive.

Let a and b be the radii of two spheres, and d the distance apart of their centres. Then it may easily be shown by elementary geometry that the linear magnification resulting from an inversion first at the sphere whose radius is a, and then at the sphere whose radius is b, is always less than

$$\left(\frac{b}{d-a}\right)^2.$$

If, now, λ is the greatest value of this fraction for any pair of the n given circles, the linear magnification after any operation compounded of r of the original operations is not greater than λ^r. Also corresponding to each operation compounded of r of the original operations there are $2(n-1)-1$, *i.e.*, $2n-3$, compounded of $r+1$. Hence the series

$$\sum_i \mu_i^m$$

is certainly convergent if $\sum_r \left[(2n-3)\,\lambda^m\right]^r$

is convergent. For instance, with three equal spheres whose centres form an equilateral triangle, the series for $m=\frac{1}{2}$ is certainly convergent if the diameters of the spheres are less than half the distances between the centres.

The proof just given shows that

$$\sum_i \mu_i^{\frac{1}{2}}$$

is certainly convergent for the group arising from a system of spheres each of which is external to all the others, provided that certain inequalities hold between the radii and the distances apart of the centres. It is series of this form, viz., $m=\frac{1}{2}$, that occur in the physical problems referred to in the title of the present paper; and, with regard to the limitations imposed by the inequalities mentioned above, it is to be noticed that these inequalities have been shown to be sufficient to ensure the convergence of the series, and therefore to justify the application of the method to the corresponding cases. But they are clearly not necessary conditions, as a consideration of the proof itself will show.

The convergency of the series $\sum_i \mu_i^m$ carries with it that of the series

$$\sum_i \mu_i^m f(x_i, y_i, z_i),$$

so long as $f(x, y, z)$ does not become infinite at any one of the series of points P_i. For, if this is the case, $f(x_i, y_i, z_i)$ will have a maximum value M independent of i, and then

$$\sum_i \mu_i^m f(x_i, y_i, z_i) < M \sum_i \mu_i^m.$$

4. *Physical Application.*

If (x', y', z') is the inverse point of (x, y, z) in a given sphere, and if $f(x, y, z)$ is a solution of Laplace's equation, so also is $\mu'^{\frac{1}{2}} f(x', y', z')$, and the two solutions are numerically equal at the surface of the sphere. This is a known theorem; a proof of it is given in a paper by Mr. W. D. Niven (*Proc. Lond. Math. Soc.*, Vol. VIII., p. 66). If, now, (x'', y'', z'') is the inverse point of (x', y', z') in another sphere, the linear magnification in passing from (x', y', z') to (x'', y'', z'') is μ''/μ'. Hence, since $f(x', y', z')$ is a solution of Laplace's equation when x', y', z' are the variables, so also by the above theorem is

$$\frac{\mu''^{\frac{1}{2}}}{\mu'^{\frac{1}{2}}} f(x'', y'', z'').$$

But, if $F(x', y', z')$ is a solution when x', y', z' are the variables, $\mu'^{\frac{1}{2}} F(x', y', z')$ is a solution with x, y, z as variables. Hence, finally, $\mu''^{\frac{1}{2}} f(x'', y'', z'')$ is a solution when x, y, z are the variables. It follows that $f(x, y, z)$ and $\mu'^{\frac{1}{2}} f(x', y', z')$ are simultaneously solutions of Laplace's equation when x', y', z' are derived from x, y, z by any number of inversions, even or odd.

Let, now, accented symbols be used to denote points derived from (x, y, z) by an odd number of inversions, and suppose the set of spheres such that $\sum_i \mu_i^{\frac{1}{2}}$ is a uniformly convergent series. Then, if $f(x, y, z)$ is a solution of Laplace's equation, whose only infinities are external to all the spheres, the function $F(x, y, z)$ defined by the series

$$\sum_i \mu_i^{\frac{1}{2}} f(x_i, y_i, z_i) - \sum_i \mu_i'^{\frac{1}{2}} f(x_i', y_i', z_i')$$

has the following properties :—

(i.) It satisfies Laplace's equation.

(ii.) Its only infinities in the space external to all the spheres coincide with those of $f(x, y, z)$.

(iii.) It vanishes at the surface of each sphere.

That the function satisfies Laplace's equation is evident, since it is defined by a uniformly convergent series, each of whose terms satisfies the equation.

If $f(x, y, z)$ becomes infinite at the point A, $f(x_i, y_i, z_i)$ becomes infinite at the point derived from A by the inverse of the operation which leads from P to P_i; and since, by supposition, A is external to all the spheres, $f(x_i, y_i, z_i)$ can therefore only be infinite at points within some one of the spheres.

The quantities μ_i have a superior limit in the space external to the spheres. It follows that the only infinities of the series outside the spheres are those of $f(x, y, z)$.

Finally, corresponding to each term $\mu_i^{\frac{1}{2}} f(x_i, y_i, z_i)$ of the first sum, there is a single term $\mu_i'^{\frac{1}{2}} f(x_i', y_i', z_i')$ of the second, such that x_i', y_i', z_i' is the inverse of x_i, y_i, z_i at any specified one of the system of spheres, and at the surface of this sphere the two terms are numerically equal. Hence at the surface of this sphere the terms of the two sums can be taken in pairs which destroy each other.

If, now, $$\mu' f(x', y', z') = f_1(x, y, z),$$

and if the operation which changes (x, y, z) into (x_i, y_i, z_i) transforms (x', y', z') into (x_i', y_i', z_i'), then

$$\frac{\mu_i'}{\mu_i^{\frac{1}{2}}} f(x_i', y_i', z_i') = f_1(x_i, y_i, z_i);$$

and therefore $$\Sigma \mu_i'^{\frac{1}{2}} f(x_i', y_i', z_i') = \Sigma \mu_i^{\frac{1}{2}} f_1(x, y, z).$$

Hence the function $F(x, y, z)$ can be expressed in the form

$$\sum_i \mu \ \{f(x_i, y_i, z_i) - f_1(x_i, y_i, z_i)\},$$

and it is therefore such that it is reproduced, except as to a factor depending on the transformation, when (x, y, z) is transformed by *any one of the operations* of the group.

For the case of two spheres, which has already been treated from the point of view of calculation in a great variety of ways, the necessary formulæ for representing explicitly the result just obtained may be conveniently written as follows.

Taking
$$x^2+y^2+z^2+2x\coth a+1 = 0,$$
$$x^2+y^2+z^2-2x\coth \beta+1 = 0$$
as the equations to the two spheres, the system of points proceeding from (x, y, z) by an even number of inversions all lie in a plane through the axis of x, and, when ρ is used for $\sqrt{y^2+z^2}$, are given by

$$x_n = \frac{(x^2+\rho^2+1)\sinh n\theta\cosh n\theta+x\cosh 2n\theta}{\sinh^2 n\theta\left(\{x+\coth n\theta\}^2+\rho^2\right)},$$

$$\rho_n = \frac{\rho}{\sinh^2 n\theta\left(\{x+\coth n\theta\}^2+\rho^2\right)},$$

$$\mu_n = \frac{\rho_n}{\rho},$$

where
$$\theta = a+\beta.$$

Waves in Canals. *By* H. M. Macdonald. Read January 11th, 1894. Received February 9th, 1894.

It has been usual to assume that the velocity potential of the fluid motion which consists of a train of progressive waves propagated along a canal of uniform cross-section can be represented by an expression of the form $f(y, z)\cos(mx-nt)$, the notation being the same as in Basset's *Hydrodynamics*, Vol. II., Art. 392. The wave motion which has a velocity potential of this form must be such that the crests of the waves are always in planes perpendicular to the length of the canal, the particles of fluid describing ellipses whose planes are perpendicular to the cross-section. In what follows it is proposed to investigate in what cases it is possible to propagate a train of such waves of any given wave length along a canal whose sides are planes equally inclined to the vertical.

1. Taking the origin at the lowest point of any cross-section of the canal, the coordinate planes being as above, ϕ the velocity potential is assumed to be of the form

$$\phi = f(y, z)\cos(mx - nt) \qquad (1);$$

then

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - m^2 f = 0 \qquad (2)$$

throughout the fluid,

$$l\frac{\partial f}{\partial z} - f = 0 \qquad (3)$$

at the free surface, where $z = h$ the depth of the canal, and $l = g/n^2$.

The solution of (2) may be written

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)f_0 + \frac{\sin\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)}{\sqrt{\frac{\partial^2}{\partial z^2} - m^2}} f_0',$$

where f_0 is the value of f, and f_0' of $\frac{\partial f}{\partial y}$, when $y = 0$.

It is clear that the two parts of this solution must satisfy (3) independently, (3) being true for all values of y, when $z = h$, and therefore correspond to different systems of waves. The first part $\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)f_0$ corresponds to waves produced by disturbances symmetrical with respect to the plane of symmetry of the canal, and such that the particles of fluid originally in that plane always remain in it. The other part $\frac{\sin\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)}{\sqrt{\frac{\partial^2}{\partial z^2} - m^2}} f_0'$ corresponds to waves produced by certain asymmetrical disturbances which are such that the particles of fluid originally in the plane of symmetry of the canal oscillate in straight lines perpendicular to it. The waves discussed by Green, *Cambridge Phil. Trans.*, 1838, belong to the first class, and for these only will the expression $\sqrt{\frac{gA}{b}}$ for the velocity of propagation of long waves there found be true; the velocity of propagation of long waves of the second class would be indefinitely great, that is, very long waves of this type could not be generated.

2. When the canal is formed by two planes equally inclined to the vertical plane, the conditions to be satisfied at the fixed boundary are

$$\frac{\partial f}{\partial y}\cos\alpha \pm \frac{\partial f}{\partial y}\sin\alpha = 0 \quad\quad (4),$$

when $$y\cos\alpha \pm z\sin\alpha = 0,$$

2α being the angle which the sides of the canal make with one another. Considering waves of the first class

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right) f_0 \quad\quad (5),$$

putting $$z = r\cos\theta,\quad y = r\sin\theta,$$

f satisfies $$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} - m^2 f = 0 \quad\quad (2)',$$

subject to $$\left.\begin{array}{l}\dfrac{\partial f}{\partial \theta} = 0 \\ \theta = \pm\alpha\end{array}\right\} \quad\quad (4)'.$$
when

Therefore $$f = \Sigma A_\mu J_\mu (miz) \cos\mu\theta,$$

where $$\mu = s\pi/\alpha,$$

s being a positive integer; hence

$$f_0 = \Sigma A_\mu J_\mu (miz),$$

and $$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\Sigma A_\mu J_\mu (miz).$$

Let $\pi/\alpha = 2k$ an even integer, then

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\int_0^\pi \cosh(mz\sin\chi)\sum_0^\infty B_s\cos 2ks\chi\, d\chi;$$

writing

$$F(\chi) \equiv \{ml\sin\chi\sinh(mh\sin\chi) - \cosh(mh\sin\chi)\}\sum_0^\infty B_s\cos 2ks\chi,$$

to satisfy (3), it is necessary that

$$\int_0^\pi \left(l\frac{\partial}{\partial z} - 1\right)\cosh(mz\sin\chi)\cosh(my\cos\chi)\sum_0^\infty B_s\cos 2ks\chi\, d\chi$$

should vanish for all values of y, when $z = h$, that is, observing that

$$\left(l\frac{\partial}{\partial z} - 1\right)\cosh(mz\sin\chi) = ml\sin\chi\sinh(mh\sin\chi) - \cosh(mh\sin\chi),$$

when $z = h$,
$$\int_0^\pi F(\chi)\,d\chi = 0,$$
$$\int_0^\pi F(\chi)\cos^2\chi\,d\chi = 0,\quad \int_0^\pi F(\chi)\cos^4\chi\,d\chi = 0, \text{ \&c.};$$
or
$$\int_0^\pi F(\chi)\,d\chi = 0,\quad \int_0^\pi F(\chi)\cos 2\chi\,d\chi = 0,$$
$$\int_0^\pi F(\chi)\cos 4\chi\,d\chi = 0, \text{ \&c.}$$

Now $F(\chi)$ can be expressed as a series involving cosines of even multiples of χ only; therefore $F(\chi) \equiv 0$ for all values of χ between 0 and π. That this may be true, and all values of m (*i.e.*, every wave length) be possible, $\sum_0^\infty B_s \cos 2ks\chi$ must vanish for all values of χ between 0 and π, except such as make
$$ml \sin\chi \sinh(mh\sin\chi) - \cosh(mh\sin\chi)$$
vanish, m and l remaining the same; if χ_1 is such a critical value of χ, the only other possible one is $\pi - \chi_1$, for $\sin\chi$ must be the same for all such values. Let
$$\sum_0^\infty B_s \cos 2ks\chi = G(\chi),$$
where $G(\chi)$ vanishes, except where $\chi = \chi_1$ or $\pi - \chi_1$, then
$$\frac{\pi B_s}{2} = \int_0^\pi G(\chi)\cos 2ks\chi\,d\chi = \cos 2ks\chi_1 \int_0^\pi G(\chi)\,d\chi;$$
therefore
$$G(\chi) = C(1 + 2\cos 2k\chi_1 \cos 2k\chi + \ldots + 2\cos 2ks\chi_1 \cos 2ks\chi + \ldots),$$
$$2G(\chi) = \mathrm{Lt}_{\zeta=1} C\left\{\frac{1-\zeta^2}{1-2\zeta\cos 2k(\chi+\chi_1)+\zeta^2} + \frac{1-\zeta^2}{1-2\zeta\cos 2k(\chi-\chi_1)+\zeta^2}\right\}.$$

Substituting this expression for $\sum_0^\infty B_s \cos 2ks\chi$ in the expression for f, and performing the integration, it becomes
$$f = C'\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\cosh(mz\sin\chi_1);$$
now,
$$\cosh(mz\sin\chi_1) = J_0(miz) + 2J_2(miz)\cos 2\chi_1 + 2J_4(miz)\cos 4\chi_1 + \text{\&c.};$$
and therefore χ_1 must be such that $\cos 2p\chi_1$ vanishes, p being an

integer, except when $p = 2ks$; this condition can only be satisfied when $k = 1$ or $k = 2$, and then $2k\chi_1 = \pi$.

Of these two cases $k = 1$ or $\pi/a = 2$ does not belong to a canal, and the only cases for which all wave lengths are possible is $k = 2$ or $\pi/a = 4$. In this case

$$f = C' \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\cosh\left(\frac{mz}{\sqrt{2}}\right),$$

that is, $$\phi = C' \cosh\frac{my}{\sqrt{2}}\cosh\frac{mz}{\sqrt{2}}\cos(mx - nt).$$

The wave velocity of propagation is found from the equation

$$\frac{ml}{\sqrt{2}}\sinh\left(\frac{mh}{\sqrt{2}}\right) - \cosh\left(\frac{mh}{\sqrt{2}}\right) = 0,$$

whence $$V^2 = \frac{g\lambda}{2\pi\sqrt{2}}\tan\left(\frac{2\pi h}{\lambda\sqrt{2}}\right),*$$

where λ is the wave length.

This result was obtained by Kelland, *Edin. Trans.*, Vol. xv., but it should be mentioned that the analysis from which it was obtained as a particular case is faulty.

A velocity potential can be found for π/a any even integer greater than four by giving a sufficient number of proper critical values to the series $\sum_0^\infty B_s \cos 2ks\chi$ to satisfy the conditions at the fixed boundaries, and this leads to conditions which m must satisfy.

If $\pi/a = 6$,

$$\phi = C'\left(\cosh mz + 2\cosh\frac{mz}{2}\cosh\frac{my\sqrt{3}}{2}\right)\cos(mx - nt),$$

where m must satisfy

$$m\coth mh = 2m\coth\frac{mh}{2}.$$

If $\pi/a = 8$,

$$\phi = C'\left\{\cosh\left(mz\sin\frac{\pi}{8}\right)\cosh\left(my\cos\frac{\pi}{8}\right) + \cosh\left(mz\sin\frac{3\pi}{8}\right)\cosh\left(my\cos\frac{3\pi}{8}\right)\right\}\cos(mx - nt),$$

* This expression for V^2 differs by a numerical factor from that given by Greenhill, *Amer. Jour. of Math.*, Vol. IX., and by Basset, *Hydrodynamics*, Vol. II., Art. 392; but it will be observed that in their notation $n^2 = g/2l$, not g/l.

where m must satisfy

$$m\coth\left(mh\sin\frac{\pi}{8}\right)\bigg/\sin\frac{\pi}{8}=m\coth\left(mh\sin\frac{3\pi}{8}\right)\bigg/\sin\frac{3\pi}{8}.$$

If $\pi/a=10$,

$$\phi=C'\left\{\cosh mz+2\cosh\left(mz\sin\frac{\pi}{10}\right)\cosh\left(my\cos\frac{\pi}{10}\right)\right.$$
$$\left.+2\cosh\left(mz\sin\frac{3\pi}{10}\right)\cosh\left(my\cos\frac{3\pi}{10}\right)\right\}\cos(mx-nt),$$

where m must satisfy

$$m\coth mh=m\coth\left(mh\sin\frac{3\pi}{10}\right)\bigg/\sin\frac{3\pi}{10}=m\coth\left(mh\sin\frac{\pi}{10}\right)\bigg/\sin\frac{\pi}{10},$$

and so on for any value of π/a an even integer.

It will be observed that the values of m which satisfy these conditions are complex quantities; hence when π/a is an even integer greater than four a train of waves with their crests in planes perpendicular to the length of the canal is impossible for any wave length. It also follows that a motion whose velocity potential is of the form $f(yz)\,e^{-p(x-ct)}$ is impossible.

When $\pi/a=4$, and λ is very great, $V^2=gh/2$, agreeing with Green's result.

3. Let $\pi/a=2k+1$ an odd integer, then

$$f=\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2}-m^2}\right)\int_0^\pi\left\{\cosh(mz\sin\chi)\sum_0^\infty B_s\cos(2k+1)\,2s\chi\right.$$
$$\left.+\sinh(mz\sin\chi)\sum_0^\infty B'_{2s+1}\sin(2s+1)(2k+1)\,\chi\right\}d\chi;$$

writing

$$F(\chi)\equiv\{ml\sin\chi\sinh(mh\sin\chi)-\cosh(mh\sin\chi)\}\sum_0^\infty B_s\cos 2s(2k+1)\chi$$
$$+\{ml\sin\chi\cosh(mh\sin\chi)-\sinh(mh\sin\chi)\}\sum_0^\infty B'_{2s+1}\sin(2s+1)(2k+1)\chi,$$

that (3) may be satisfied it is necessary that

$$\int_0^\pi F(\chi)\,d\chi=0,\quad \int_0^\pi F(\chi)\cos 2\chi\,d\chi=0,\quad \&c.;$$

hence, as in the foregoing, $F(\chi)$ vanishes for all values of χ between 0 and π.

Therefore, that every wave length should be possible,

$$\sum_0^\infty B_s\cos 2s(2k+1)\,\chi\quad\text{and}\quad\sum_0^\infty B'_{2s+1}\sin(2s+1)(2k+1)\,\chi$$

must vanish for all values of χ, except such as make

$$L\{ml\sin\chi\sinh(mh\sin\chi)-\cosh(mh\sin\chi)\}$$
$$+L'\{ml\sin\chi\cosh(mh\sin\chi)-\sinh(mh\sin\chi)\}$$

vanish. Let χ_1, χ_2, &c., be these critical values, then, remembering that

$$\cosh(mz\sin\chi) = J_0(miz)+2J_2(miz)\cos 2\chi+\&c.,$$
$$\sinh(mz\sin\chi) = 2J_1(miz)\sin\chi+2J_3(miz)\sin 3\chi+\&c.,$$

the conditions to be satisfied at the fixed boundaries require that

$$L_1\cos 2\chi_1+L_2\cos 2\chi_2+\&c. = 0,$$
$$L_1\cos 2(2k+2)\chi_1+L_2\cos 2(2k+2)\chi_2+\&c. = 0, \ \&c.,$$
$$L_1\cos 4\chi_1+L_2\cos 4\chi_2+\&c. = 0, \ \&c.;$$

from these it follows that

$$e^{2(2k+1)\chi_1\iota} = e^{2(2k+1)\chi_2\iota} = \&c.;$$

also $\quad L_1'\sin\chi_1+L_2'\sin\chi_2+\&c. = 0, \ \&c.,$

whence $\quad e^{(2k+1)\chi_1\iota} = e^{(2k+1)\chi_2\iota} = \&c.$

Hence, if χ_1 is a critical value, there are $2k$ other critical values given by

$$\chi_1+\frac{2\pi}{2k+1},\quad \chi_1+\frac{4\pi}{2k+1},\ \ldots\ \chi_1+\frac{4k\pi}{2k+1};$$

substituting these in the sets

$$L_1\cos 2\chi_1+L_2\cos 2\chi_2+\ldots = 0,$$
$$L_1\cos 4\chi_1+L_2\cos 4\chi_2+\ldots = 0,$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$L_1\cos 4k\chi_1+L_2\cos 4k\chi_2+\ldots = 0,$$

it follows that $\quad L_1 = L_2 = \&c.,$

so $\quad L_1' = L_2' = \&c.$

Therefore

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2}-m^2}\right)\left\{L\left[\cosh(mz\sin\chi_1)\right.\right.$$
$$\left.+\cosh\left\{mz\sin\left(\chi_1+\frac{2\pi}{2k+1}\right)\right\}+\&c.\right]$$
$$\left.+L'\left[\sinh(mz\sin\chi_1)+\&c.\right]\right\}\quad\ldots(\mathrm{A});$$

in order that all wave lengths may be possible, these critical values must be such that they give only two different values to $\sin\chi$, as there is only one constant, viz., L'/L, at our disposal; this will be so when $k=1$, *i.e.*, $\pi/a=3$, and only then, and χ_1 is then determined by

$$\pi-\chi_1=\chi_1+2\pi/3,$$

whence
$$\chi_1=\frac{\pi}{6}.$$

Therefore the only case for which all wave lengths are possible, π/a being an odd integer, is $\pi/a=3$, when

$$f=\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2}-m^2}\right)\Big[L\Big\{\cosh\left(mz\sin\frac{\pi}{6}\right)+\cosh\left(mz\sin\frac{5\pi}{6}\right)+\cosh\left(mz\sin\frac{3\pi}{2}\right)\Big\}$$
$$+L'\Big\{\sinh\left(mz\sin\frac{\pi}{6}\right)+\sinh\left(mz\sin\frac{5\pi}{6}\right)+\sinh\left(mz\sin\frac{3\pi}{2}\right)\Big\}\Big],$$

which may be written

$$f=A\left[\cosh m(z+\beta)+2\cosh m\left(\frac{z}{2}-\beta\right)\cosh\frac{my\sqrt{3}}{2}\right].$$

In this case l is given by

$$ml=\coth m(h+\beta)=2\coth m\left(\frac{h}{2}-\beta\right),$$

whence
$$m^2l^2-3ml\coth\frac{3mh}{2}+2=0.$$

Hence

$$\phi=A\Big[ml\cosh m(z-h)+\sinh m(z-h)$$
$$+2\cosh\frac{my\sqrt{3}}{2}\Big\{ml\cosh m\left(\frac{z}{2}+h\right)-\sinh m\left(\frac{z}{2}+h\right)\Big\}\Big]\cos(mx-nt),$$

the velocity of propagation being given by

$$g/V^2=\frac{3m}{2}\coth\frac{3mh}{2}\left\{1\pm\sqrt{1-\frac{8}{9}\tanh^2\frac{3mh}{2}}\right\}.$$

The form of the velocity potential in this, the only possible case for

π/a an odd integer, is somewhat complicated, but it may be verified by the case of long waves when it leads to Green's result $V^2 = gh/2$. The other solution $V^2 = \infty$ means that no long wave corresponding to the lower sign in the above expression for the velocity of propagation would be generated.

If the upper sign is taken in the expression for g/V^2, ml is always greater than 2, and the free surface of the wave is given by

$$\zeta = A\left\{1+2\sqrt{\frac{m^2l^2-1}{m^2l^2-4}}\cosh\left(\frac{my\sqrt{3}}{2}\right)\right\}\sin(mx-nt),$$

the cross-section of the wave at a crest being a catenary with its lowest point in the middle of the canal. If the lower sign is taken, ml is always less than 1, and the free surface is given by

$$\zeta = A\left\{1-2\sqrt{\frac{1-m^2l^2}{4-m^2l^2}}\cosh\frac{my\sqrt{3}}{2}\right\}\sin(mx-nt),$$

the cross-section of the wave at a crest being a catenary with its highest point in the middle of the canal. When the wave length λ is great, that form must be taken for which $ml > 2$, and when λ is small that for which $ml < 1$.

The above expressions for the velocity potential and velocity of propagation in the case $\pi/a = 3$ lead to Stokes' result for waves with their crests perpendicular to a beach sloping to the horizon at an angle of $\pi/6$. [In this case $ml > 2$.]

In the *Amer. Jour. of Math.*, Vol. IX., Greenhill has tried to obtain the solution of the above case by modifying the solution for standing waves across the same canal; the expression for the velocity potential so obtained can be got from the above equation (A) by putting

$$\chi_1 = \pi/12,$$

but it then contains one undetermined constant less than the number necessary to give a solution for every wave length. In the same paper the propagation of a bore along the canal is investigated, assuming

$$\phi = F\,.\cosh(mx-nt);$$

this expression seems objectionable on the ground that the displacement given by it could become large, and the theory is only applicable to waves where the displacement is always small.

The expression (A) gives a velocity potential for every case π/a an odd integer which satisfies the conditions at the fixed boundary, but the free surface condition in every case but $\pi/a = 3$ will lead to conditions which m must satisfy; *e.g.*, if $\pi/a = 5$, the velocity potential is

$$\phi = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\Big[L\Big\{\cosh mz + 2\cosh\left(mz\sin\frac{\pi}{10}\right) + 2\cosh\left(mz\sin\frac{3\pi}{10}\right)\Big\} + L'\Big\{\sinh mz + 2\sinh\left(mz\sin\frac{\pi}{10}\right) - 2\sinh\left(mz\sin\frac{3\pi}{10}\right)\Big\}\Big]\cos(mx - nt),$$

χ_1 being chosen so that m has to satisfy the least possible number of conditions; m is given by the equation

$$\{\epsilon\epsilon'(\coth mh\epsilon + \coth mh\epsilon') + \epsilon\coth mh\epsilon - \epsilon'\coth mh\epsilon'\}(\epsilon\coth mh\epsilon - \epsilon'\coth mh\epsilon') = (\epsilon + \epsilon')^2,$$

where $\epsilon = -1 + \sin\frac{\pi}{10}, \quad \epsilon' = \sin\frac{3\pi}{10} + 1;$

the roots of this equation are complex quantities. Similarly, the other cases π/a an odd integer may be investigated.

4. The solution $$f = \frac{\sin\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)}{\sqrt{\frac{\partial^2}{\partial z^2} - m^2}} f_0$$

gives a wave motion which is possible for all wave lengths when $\pi/a = 4$, and the velocity potential then is

$$\phi = A\sinh\frac{my}{\sqrt{2}}\sinh\frac{mz}{\sqrt{2}}\cos(mx - nt);$$

the wave velocity of propagation is given by

$$V^2 = \frac{g\lambda}{2\pi\sqrt{2}}\coth\frac{2\pi h}{\lambda\sqrt{2}},$$

and, if λ is small, $$V^2 = \frac{g\lambda}{2\pi\sqrt{2}}.$$

It can be shown, by an analysis similar to the preceding, that standing waves across a canal of triangular cross-section are only possible in the same two cases, the solutions for which were given by Kirchhoff, *Gesam. Abhand.*, Vol. II.

From the above investigation, it appears that, if a wave with a plane front is set up in a canal of triangular cross-section, it will be propagated without change of form in two cases only, viz., when the angle which the sides make with one another is either 90° or 120°; in all other cases the wave front will not remain plane for any great distance along the canal. It follows from what precedes that there is no angle which forms the limit between stability and instability, as stated in Basset's *Hydrodynamics*, Vol. II., Art. 394; a wave motion of some kind must be possible for any angle. Green's investigation above referred to requires that it should be possible to expand the velocity potential in powers of the y, z coordinates, and that powers higher than the squares of these should be negligible; this will be true when the front of the wave is approximately plane, so that the results there arrived at would be true in the case of a wave whose front is initially plane for some distance along the canal.

Thursday, February 8th, 1894.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Miss Edith Lees, Mr. F. W. Hill, M.A., City of London School, and Major Hippisley, R.E., were elected members of the Society. Miss Lees was admitted into the Society.

The President announced the death of Mr. William Racster, M.A., for many years a colleague, at Woolwich, of Prof. Sylvester. He was elected a member October 16th, 1865, and died December 30th, 1893. Also of Mr. William Paice, M.A., for twenty-two years an Assistant Master in University College School. He was a Life Governor of University College; a sub-Examiner in Mathematics, for five years, of the London University; an Assistant Examiner in Magnetism at South Kensington; and the author of a small work entitled *Energy and Motion*. On the death of the Rev. W. Stainton Moses, his colleague at the school, which event took place on the 5th September, 1892, he succeeded that gentleman as the editor of the

Spiritualist journal *Light.* He was elected a member April 11th, 1872, and died January 24th, 1894.

At the request of Lord Kelvin, P.R.S., and by the permission of the Council, Mr. J. J. Walker exhibited and described Lord Kelvin's model of his Tetrakaidekahedron.

This was a model (for the making of which Lord Kelvin acknowledges his obligation to Prof. Crum Brown, D.Sc., M.D., F.R.S., Prof. of Chemistry in the University of Edinburgh) of the form named "orthoidal" by the author; viz., it is a form derived by homogeneous strain from the "orthic," a surface bounded by eight regular hexagons and four squares, first described in the *Acta Mathematica*, Vol. XI., "The Division of Space with Minimum Partitional Area," a paper reproduced in the *Phil. Mag.* for 1887 (second half-year).

Lord Kelvin's own account of the surface will be found in Vol. LV., *Royal Society's Proceedings* (pp. 1-16).*

Votes of thanks were passed to Mr. Walker and to Lord Kelvin.

A conversation ensued in which Messrs. S. Roberts, Forsyth, MacMahon, Cunningham, Elliott, and the President took part.

Abstracts were communicated of the following papers:—

On a Class of Groups defined by Congruences: Prof. W. Burnside.

Some Properties of the Uninodal Quartic and Quintic having a Triple Point: Mr. W. R. W. Roberts.

A cabinet likeness of Prof. Mathews was presented by him to the Album.

The following presents were made to the Library:—

Bodhanundānath Swami.—"Kalyāna Manjushā"; Calcutta, 1893.

"Nautical Almanac for 1897."

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VIII., No. 1; Manchester, 1893-4.

"Bulletin des Sciences Mathématiques," 2ème Série, Tome XVII., November and December, 1893; Paris.

"Proceedings of the Royal Society," Vol. LIV., No. 329.

"Report of the Superintendent of the U.S. Naval Observatory," to June, 1893; Washington.

"Journal of the Institute of Actuaries," No. 172, January, 1894.

"Papers read before the Mathematical and Physical Society of the University of Toronto during 1891-2," Toronto, 1892.

"Proceedings of the Physical Society of London," Vol. XII., Pt. 3; December, 1893.

"Bulletin of the New York Mathematical Society," Vol. III., No. 4.

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* *Cf.* also *Nature* for March 8th and 15th, pp. 445-8, 469-71.

"Bulletin de la Société Mathématique de France," Tome xxi., No. 9; Paris.

"Bulletin de la Société Mathématique de France," Table des 20 Premiers Volumes; Paris, 1894.

Gram, J. P.—"Essai sur la Restitution du Calcul de Léonard de Pise sur l'équation $x^3+2x^2+10x=20$," pamphlet.

Gram, J. P.—"Rapport sur quelques Calculs entrepris par M. Bertelsen et concernant les Nombres Premiers," 4to pamphlet.

Zeuthen, H. G.—"Note sur l'Histoire des Mathématiques," pamphlet.

Barrett, T. S.—"Magic Squares," second edition, 8vo; Berkhamsted, 1894.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xvii., St. 12, 1893; Bd. xviii., St. 1, 1894; Leipzig.

"Atti della Reale Accademia dei Lincei," Serie 5, Rendiconti, Vol. iii., Fasc. 1, 1 Sem.; Vol. ii., Fasc. 12, 2 Sem.; Roma, 1894.

"Indian Engineering," Vol. xiv., Nos. 26, 27; Vol. xv., Nos. 1 and 2.

"Educational Times," February, 1894.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. vii., Fasc. 8–12; Napoli, 1894.

"Annales de la Faculté des Sciences de Toulouse," Tome vii., Année 1893, 4ème Fasc.; Paris.

On a Class of Groups defined by Congruences. By Prof. W. Burnside. Received February 7th, 1894. Read February 8th, 1894.

1. *Introductory.*

Most of the groups of finite order which occur in connexion with problems of higher analysis can be defined by means of congruences. This is true, for example, of the group of the modular equation, and of the groups on which the division of the periods of the hyper-elliptic functions depends. In his standard treatise (*Traité des Substitutions et des Equations Algébriques*) M. Camille Jordan has investigated at length the more important properties of the general linear group, defined by sets of congruences of the form

$$\left.\begin{aligned} x_1' &\equiv a_1x_1+b_1x_2+\ldots+c_1x_n \\ x_2' &\equiv a_2x_1+b_2x_2+\ldots+c_2x_n \\ \ldots & \quad\ \ldots \quad\ \ldots \quad\ \ldots \\ x_n' &\equiv a_nx_1+b_nx_2+\ldots+c_nx_n \end{aligned}\right\} \pmod{p},$$

where the coefficients are ordinary integers.

The group of the modular equation, which is isomorphous with the general linear group when the number of variables is two, has formed the subject of a large number of memoirs; but it was first exhaustively analysed in a paper by Herr J. Gierster (*Math. Ann.*, Vol. XVIII.), in which the order and type of all possible sub-groups contained in the modular group for a prime transformation are completely determined. Though the congruences defining the groups dealt with in these investigations involve only real coefficients, both the authors mentioned find it of great advantage to introduce in their discussions the imaginaries which Galois* first used in analysis.

If these imaginaries are introduced in the congruences defining the groups, a new class of groups arises, altogether distinct from those defined by congruences whose coefficients are real integers; that is to say, the simple groups occurring in the composition-series (*Reihe der Zusammensetzung*) of these new groups are new simple groups. In the present paper some of the more important properties of the fractional linear group to a prime modulus, *i.e.*, the group defined by

$$z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta} \text{ (mod } p \text{, prime)},$$

when α, β, γ, δ are any rational functions of the roots of an irreducible congruence of the n^{th} degree (mod p), are investigated. It is shown that in this way new simple groups of orders $2^n(2^{2n}-1)$ and $\frac{1}{2}p^n(p^{2n}-1)$, p an odd prime and n any integer, are defined; the latter being in many respects closely analogous to the group of the modular equation. The orders of the separate operations of the groups and their distribution in conjugate sets are determined, and the order and type of some of the simpler sub-groups. For the case of $p=2$, a complete discussion is given of all possible types of sub-group; to carry this out, for p an odd prime, would probably necessitate the separate treatment of each value of n.

In two memoirs in Liouville's *Journal*, 1860–1, M. E. Mathieu has shown the existence of the triply-transitive group, called G in this paper, of which the simple group of order $\frac{1}{2}p^n(p^{2n}-1)$ is a self-conjugate sub-group. These memoirs deal, however, in the main, with the formation of functions which are unaltered by the operations of transitive groups, and the nature and properties of the groups themselves are not entered upon.

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* *Cf.* Liouville's *Journal*, 1846, p. 381. Galois' papers have also been printed separately in a German translation by J. Springer, Berlin, 1889. *Cf.*, also, Serret, *Cours d'Algèbre Sup.*, Vol. II., p. 179, and Jordan's work mentioned above, p. 14.

In a subsequent paper the author hopes to deal with the simple groups that arise in connexion with systems of congruences involving n variables when the coefficients are imaginaries.

2. *Definition and Order of the Groups.*

Let x be a primitive root of the congruence

$$x^{p^n-1}-1 \equiv 0 \pmod{p},$$

so that x satisfies an irreducible congruence of the the form

$$x^n+a_1x^{n-1}+\ldots+a_n \equiv 0 \pmod{p},$$

where $a_1, \ldots a_n$ are real integers.

Then the p^n quantities

$$a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A),$$

where $a_1, a_2, \ldots a_n$ may have any of the values $0, 1, 2, \ldots p-1$, are all incongruent, and it is known that in a proper order they are the same as the series of quantities

$$0, x, x^2, \ldots x^{p^n-1}.$$

If, now, $\alpha, \beta, \gamma, \delta$ are any four of these quantities, such that

$$\alpha\delta-\beta\gamma \not\equiv 0,$$

the system of congruences $\quad z' \equiv \dfrac{\alpha z+\beta}{\gamma z+\delta}$

form a group; for the result of combining any two congruences of this form is a third congruence of the same form.

The congruences

$$z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta} \quad \text{and} \quad z' = \frac{m\alpha z+m\beta}{m\gamma z+m\delta}$$

are identical, and it may therefore be assumed that the determinant

$$\alpha\delta-\beta\gamma$$

of the substitution is either unity or a determinate quadratic non-residue, which may conveniently be taken as x.

Since $\quad z' \equiv \dfrac{\alpha z+\beta}{\gamma z+\delta} \quad$ and $\quad z' \equiv \dfrac{-\alpha z-\beta}{-\gamma z-\delta}$

are not distinct substitutions, the order of the group will be one half of the sum of the number of distinct solutions of the two separate congruences

$$\alpha\delta-\beta\gamma \equiv 1$$

and

$$\alpha\delta-\beta\gamma \equiv x.$$

In the first of these congruences, if α is zero, δ may have any one of the p^n possible values, and β and γ satisfy

$$\beta\gamma \equiv -1.$$

This congruence has clearly p^n-1 distinct solutions, and hence the number of solutions when α is zero is $p^n\,(p^n-1)$.

If α is different from zero, and δ such that

$$\alpha\delta \equiv 1,$$

then β, γ have $2p^n-1$ sets of values; while, if

$$\alpha\delta \not\equiv 1,$$

β, γ have p^n-1 sets of values, as in the first case.

Hence for each finite value of α there are

$$2p^n-1+(p^n-1)^2=p^{2n}$$

solutions of the congruence.

The total number of solutions is therefore

$$p^n\,(p^n-1)+p^{2n}\,(p^n-1)\equiv p^n\,(p^{2n}-1).$$

It is easy to show that the second congruence

$$\alpha\delta-\beta\gamma \equiv x$$

has an equal number of solutions, and therefore

$$p^n\,(p^{2n}-1)$$

is the order of the group.

The order of the group may also be simply determined as follows. The substitutions of the group permute the p^n+1 symbols consisting of the set (A) with ∞ among themselves. If $x_1, x_2, x_3, x_1', x_2', x_3'$ are any six of these symbols, the substitution

$$\frac{z'-x_1'}{z'-x_2'}\,\frac{x_3'-x_2'}{x_3'-x_1'} \equiv \frac{z-x_1}{z-x_2}\,\frac{x_3-x_2}{x_3-x_1}$$

is evidently a substitution of the group, and it replaces the three symbols x_1, x_2, x_3 by x_1', x_2', x_3' respectively. The group is therefore *triply-transitive* in the p^n+1 symbols, and its order is therefore divisible by $(p^n+1)p^n(p^n-1)$; while, since it clearly contains no substitution, except identity, which keeps more than two symbols fixed, the order must be equal to this number.

3. *The Generating Operations.*

It will now be shown that the group can be generated by the combination and repetition of the three substitutions

$$z' \equiv \frac{-1}{z}, \quad z' \equiv z+1, \quad z' \equiv xz,$$

which may be conveniently represented by the symbols T, S, X.

If S be transformed by X^i, the resulting substitution $X^{-i}SX^i$ is given by the congruence $z' \equiv z+x^i$.

Let now Σ or $$z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta}$$

be any substitution of the group; then i may be so chosen that $\Sigma X^{-i}SX^i$ or

$$z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta}+x^i$$

is of the form $$z' \equiv \frac{\beta'}{\gamma z+\delta}.$$

It follows that $\Sigma X^{-i}SX^iT$ is given by

$$z' \equiv \frac{\gamma z+\delta}{-\beta'},$$

and j may then be so chosen that $\Sigma X^{-i}SX_iTX^{-j}SX^j$ or

$$z' \equiv \frac{\gamma z+\delta}{-\beta'}+x^j$$

is the same as $$z' \equiv \frac{\gamma z}{-\beta'} \equiv x^k z.$$

Hence any substitution of the group can be expressed in the form

$$X^{\alpha}SX^{\beta}TX^{\gamma}SX^{-\gamma},$$

and the three substitutions T, S, X are therefore, as stated above, generating operations of the group.

It has been seen that one half of the substitutions of the group have unity, or a quadratic residue, for their determinant. These evidently form a self-conjugate sub-group, and it may be shown that the generating substitutions of this sub-group are T, S, and X^2;

indeed, the previous proof will hold, as it stands, so soon as it is shown that

$$z' \equiv z + x'$$

can be formed from these operations, whatever x' may be. Now, every operation of this form in which x' is a quadratic residue may be formed by transforming S by the powers of X^2; and, by combining these operations, every operation of the above form in which x' is the sum of any number of quadratic residues may be formed. But among these quantities non-residues must occur, for, if unity be added in turn to every quadratic residue, the sums are all different, and unity does not occur among them.

When $p = 2$, there are no quadratic non-residues, for the two solutions of the congruence

$$x^2 \equiv x'^2 \pmod 2$$

are congruent with each other, and therefore every one of the 2^n quantities (A) is in this case a quadratic residue.

In calculating the order of the group in this case the congruence

$$\alpha\delta - \beta\gamma \equiv x \pmod 2$$

does not occur. On the other hand, $\alpha, \beta, \gamma, \delta$ and $-\alpha, -\beta, -\gamma, -\delta$ are not now different solutions of the congruence

$$\alpha\delta - \beta\gamma \equiv 1 \pmod 2,$$

so that the order of the group is the total number of solutions of this congruence, viz., $2^n(2^{2n}-1)$. In this case also there is evidently no self-conjugate sub-group of index 2, corresponding to the one just referred to when p is an odd prime.

It will be convenient, to avoid repetition, to deal with the case $p = 2$ by itself, after considering the general case of p any odd prime. In what follows the triply-transitive group of order $p^n(p^{2n}-1)$ will be referred to as the group G, and the sub-group of order $\frac{1}{2}p^n(p^{2n}-1)$ as the group H. It will also be a useful abbreviation to speak of the substitution defined by the congruence

$$z' \equiv \frac{\alpha z + \beta}{\gamma z + \delta}$$

as the substitution $\dfrac{\alpha z + \beta}{\gamma z + \delta}$.

4. *On the Orders of the Operations of G, and their Distribution in Conjugate Sets.*

When G is regarded as a triply-transitive group in p^n+1 symbols, its operations either change all the symbols, all but one or all but two. Those that keep either one or two symbols fixed are necessarily regular, as otherwise their powers would keep more than two symbols fixed. On the other hand, some of those that change all the symbols may be such that their squares keep two symbols fixed; otherwise they also must be regular. Since the group is triply-transitive, there must occur among the substitutions conjugate to a substitution which contains a given transposition substitutions containing any other chosen transposition. If now the substitution $\frac{\alpha z+\beta}{\gamma z+\delta}$ transposes 0 and ∞, then $\alpha\equiv\delta\equiv 0$, and the substitution is therefore necessarily of order 2. Hence irregular substitutions, such as those suggested, cannot occur, and all the substitutions of G are regular.

The sub-groups which keep each one of the p^n+1 symbols successively unchanged are all conjugate, and that which keeps ∞ unchanged may be taken as their type.

This is clearly the group of order $p^n(p^n-1)$ which is generated by

$$z'\equiv z+1 \quad \text{and} \quad z'\equiv xz.$$

It is evident that this group contains the group of order p^n,

$$z'\equiv z,\quad z'\equiv z+1,\quad z'\equiv z+x,\ \dots\ z'\equiv z+x^{p^n-1},$$

self-conjugately. Hence this is the type of the single conjugate set of groups of order p^n, which according to Sylow's theorem is contained in G; and their number is p^n+1. This sub-group is such that all its operations, except identity, are of order p, and are permutable with each other.

Since $X^{-i}SX^i$ is the substitution $z+x^i$, the p^n-1 operations of order p form a single conjugate set within the sub-group which keeps ∞ unchanged, and therefore the $(p^n+1)(p^n-1)$ operations of order p contained in the group G form a single conjugate set. The remaining operations of the sub-group keeping ∞ unchanged (which all keep one other symbol fixed) consist of operations conjugate to xz and its powers, and therefore m can always be chosen so that any operation of the group which keeps two symbols unchanged is conjugate to $x^m z$. Since the first power of x which is congruent to unity is the $(p^n-1)^{\text{th}}$, the order of xz or X is p^n-1; and therefore the order of every operation which keeps two symbols fixed is a submultiple of p^n-1, while every such operation is a power of an operation of order p^n-1.

Since the operations which change all the symbols are regular, their orders must be sub-multiples of p^n+1; and, in close analogy with the operations which keep two symbols fixed, it may be shown that among the operations changing all the symbols there are operations of order p^n+1, while every operation changing all the symbols is the power of an operation of order p^n+1.

Thus, if the substitution

$$z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta}$$

is thrown into the form

$$\frac{z'-\mu}{z'-\nu} \equiv \lambda\frac{z-\mu}{z-\nu},$$

λ is given by
$$\lambda^2+\left(2-\frac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}\right)\lambda+1 \equiv 0.$$

Now the coefficient of λ in this congruence can take all possible values, for, when

$$\alpha\delta-\beta\gamma \equiv 1,$$

$\dfrac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}$ may be any quadratic residue, and, when

$$\alpha\delta-\beta\gamma \equiv x,$$

it may be any quadratic non-residue.

Now the congruence $\lambda^2-x^i\lambda+1 \equiv 0$

is reducible when j can be found such that

$$x^i \equiv x^j+x^{-j},$$

and, if this congruence can be satisfied at all, it can only be satisfied in one way, for

$$x^j+x^{-j} \equiv x^k+x^{-k}$$

gives at once
$$x^j \equiv x^k \text{ or } x^{-k}.$$

For all other values of x^i the congruence is irreducible. But

$$\prod_{0}^{p^n-1}(\lambda^2-x^i\lambda+1)$$

$$=\lambda^{p^n}\left(\lambda+\frac{1}{\lambda}\right)\prod_{1}^{p^n-1}\left(\lambda+\frac{1}{\lambda}-x^i\right)$$

$$=\lambda^{p^n}\left(\lambda+\frac{1}{\lambda}\right)\left[\left(\lambda+\frac{1}{\lambda}\right)^{p^n-1}-1\right]$$

$$\equiv\lambda^{p^n}\left(\lambda^{p^n}+\frac{1}{\lambda^{p^n}}-\lambda-\frac{1}{\lambda}\right)$$

$$\equiv(\lambda^{p^n+1}-1)(\lambda^{p^n-1}-1).$$

Those quadratic congruences which are reducible give the quadratic factors of $\lambda^{p^n-1}-1$, and the factors $\lambda-1$ and $\lambda+1$ of $\lambda^{p^n+1}-1$; and therefore the irreducible quadratic congruences give the remaining factors of $\lambda^{p^n+1}-1$.

Now, since
$$\lambda^{p^{2n}-1}-1\equiv 0$$
has primitive roots,
$$\lambda^{p^n+1}-1\equiv 0$$
must also have primitive roots.

Hence among the quantities λ, defined by the irreducible congruences
$$\lambda^2-x^i\lambda+1\equiv 0,$$
there must be some for which the first power of λ which is congruent to unity is the $(p^n+1)^{\text{th}}$. There are therefore operations $\dfrac{az+\beta}{\gamma z+\delta}$, whose order is p^n+1, since the order of the operation is equal to the index to which the corresponding λ belongs.

If $\dfrac{az+\beta}{\gamma z+\delta}$ is transformed into $\dfrac{a'z+\beta'}{\gamma'z+\delta'}$ by any substitution of determinant unity, it is well known that

$$a'+\delta'\equiv a+\delta \quad\text{and}\quad a'\delta'-\beta'\gamma'\equiv a\delta-\beta\gamma,$$

and this result is still true when the transforming substitution has determinant x, if the transformed substitution $\dfrac{a'z+\beta'}{\gamma'z+\delta'}$ be brought into its standard form so as to have unity or x for its determinant.

It may, however, be further shown that for the group G all the substitutions for which $\alpha+\delta$ and $\alpha\delta-\beta\gamma$ have given values form a single conjugate set. Thus, if $\frac{Az+B}{Cz+D}$ be the transforming substitution, α', β', γ', δ' are given by

$$A\alpha+B\gamma \equiv \alpha'A+\beta'C, \quad A\beta+B\delta \equiv \alpha'B+\beta'D,$$

$$C\alpha+D\gamma \equiv \gamma'A+\delta'C, \quad C\beta+D\delta \equiv \gamma'B+\delta'D,$$

and, since here $AD-BC$ may be either 1 or x, it is clear that, except when $\alpha\equiv\delta\equiv 1$ and $\beta\equiv\gamma\equiv 0$, α' and β' may be chosen arbitrarily, and therefore that every substitution $\frac{\alpha'z+\beta'}{\gamma'z+\delta'}$, for which

$$\alpha'+\delta'\equiv\alpha+\delta \quad \text{and} \quad \alpha'\delta'-\beta'\gamma'\equiv\alpha\delta-\beta\gamma,$$

is conjugate to $\frac{\alpha z+\beta}{\gamma z+\delta}$.

Now, the multiplier λ of a substitution Σ of order p^n+1 has been shown to satisfy an irreducible quadratic congruence of the form

$$\lambda^2-x^i\lambda+1\equiv 0,$$

and the multipliers λ^2, λ^3, &c., of its successive powers satisfy similar congruences. If λ^r satisfied the same congruence as λ^s, then

$$\lambda^{r+s}\equiv 1;$$

and hence, since λ is a primitive root of the congruence

$$\lambda^{p^n+1}-1\equiv 0,$$

$$r+s=p^n+1.$$

The multipliers of successive powers of Σ, with the exception of the $\frac{1}{2}(p^n+1)^{\text{th}}$, therefore satisfy $\frac{p^n-1}{2}$ different irreducible quadratic congruences of the above form. But this is the total number of such congruences, and therefore among the powers of Σ all possible values of $\frac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}$, for substitutions whose orders are submultiples of p^n+1, occur. Combining this with the previous result, it follows that every operation changing all the symbols is conjugate with a power of an operation of order p^n+1, and is therefore itself a power of an operation of order p^n+1.

The operations $\frac{\alpha z+\beta}{\gamma z+\delta}$ and $\frac{-\alpha z-\beta}{-\gamma z-\delta}$ being identical, the result just proved may be stated in the form that all operations for which

$(\alpha+\delta)^2$ and $\alpha\delta-\beta\gamma$ have given values constitute a single conjugate set of operations.

Since $(\alpha+\delta)^2$ may be zero, or any one of the $\frac{p^n-1}{2}$ quadratic residues, while $\alpha\delta-\beta\gamma$ may be either 1 or x, there must be p^n+1 conjugate sets of operations, exclusive of identity.

This discussion of the orders of the operations of G, and their distribution into conjugate sets, may be applied to simplify considerably the corresponding investigation for the group H.

5. *On the Distribution of the Operations of H in Conjugate Sets.*

The cyclical substitutions of p^n+1 and p^n-1 symbols that G contains are odd substitutions, *i.e.*, substitutions that are equivalent to an odd number of transpositions. Hence the self-conjugate subgroup H of index 2 consists of the even substitutions of G.

It follows at once that all the substitutions of order p contained in G belong to H, and that operations of G of orders $\frac{p^n+1}{\mu}$ and $\frac{p^n-1}{\nu}$ will belong to H when μ and ν are even.

Hence the operations of H which keep no symbols fixed have for their orders sub-multiples of $\frac{p^n+1}{2}$, and every such operation is the power of an operation of order $\frac{p^n+1}{2}$; while the operations which keep two symbols fixed are powers of operations of order $\frac{p^n-1}{2}$.

It is not, however, now the case that all the operations for which $(\alpha+\delta)^2$ is the same form a single conjugate set. For, if the substitution $z+1$ is transformed into $z+x^i$ by $\frac{\alpha z+\beta}{\gamma z+\delta}$,

$$\alpha \equiv \delta x^i \quad \text{and} \quad \gamma \equiv 0;$$

but
$$\alpha\delta \equiv 1;$$

and therefore x^i must be a quadratic residue.

The operations of order p therefore fall into two conjugate sets, each containing $\frac{1}{2}(p^{2n}-1)$.

If, now, $\frac{\alpha z+\beta}{\gamma z+\delta}$ be any substitution whose order is a sub-multiple of

$\frac{p^n+1}{2}$, and if this be transformed into $\frac{\alpha' z+\beta'}{\gamma' z+\delta'}$ by $\frac{Az+B}{Cz+D}$, by combining the first two of the equations for the transformed coefficients already given with

$$AD-BC \equiv 1,$$

it is easily shown that

$$\beta' \left[C^2\beta+CD(\delta-\alpha)-D^2\gamma\right] \equiv \alpha'^2-(\alpha+\delta)\alpha'+1.$$

Now, $\beta+\frac{D}{C}(\delta-\alpha)-\frac{D^2}{C^2}\gamma$ cannot vanish for any value of $\frac{D}{C}$, for, when equated to zero, it would give the fixed elements of the operation, and no elements remain fixed; and, when the p^n different possible values of $\frac{D}{C}$ are successively substituted for it, this expression will take $\frac{p^n+1}{2}$ different values, since the congruence

$$\beta+\frac{D}{C}(\delta-\alpha)-\frac{D^2}{C^2}\gamma \equiv \beta+x'(\delta-\alpha)-x'^2\gamma$$

will always have two and only two roots, and for one value of x' these will be equal. Moreover, there are only $\frac{p^n-1}{2}$ quadratic residues, so that the expression in question can, by suitably choosing $\frac{D}{C}$, be made either a residue or a non-residue.

It follows that, when all possible values are given to C and D,

$$C^2\beta+CD(\delta-\alpha)-D^2\gamma$$

can take every possible value except zero.

Hence, when α' is chosen arbitrarily, β' may have every possible value except zero. But since

$$\alpha'\delta'-\beta'\gamma' \equiv 1 \quad \text{and} \quad \alpha'+\delta' \equiv x',$$

where

$$\lambda^2-x'\lambda+1 \equiv 0$$

is irreducible, the value $\beta = 0$ is in any case inadmissible.

It follows from this discussion that all the operations which keep no symbols fixed, and for which $(\alpha+\delta)^2$ has the same value, form a single conjugate set.

A closely similar discussion of the congruence connecting α', β', C, D leads to the same result for operations whose orders are sub-multiples of $\frac{p^n-1}{2}$.

There are, therefore, two conjugate sets of substitutions for which

$$(\alpha+\delta)^2 = 4,$$

and one set of every other possible value. The total number of conjugate sets, exclusive of identity, is $\frac{p^n+3}{2}$.

6. *Proof that H is a Simple Group.*

The proof which Herr Weber gives in his *Elliptische Functionen und Algebraische Zahlen* that the group of the modular equation is simple may be applied directly, with suitable modifications, to show that the analogous group H is simple. The following proof of this property, founded on the discussion just given of the distribution of the operations in conjugate sets, is, however, considerably shorter.

Let K be a self-conjugate sub-group of H, and suppose first that K contains an operation of order p. It must then contain the whole of one of the two conjugate sets of operations of order p, and therefore the whole of both sets, since by a previous remark a sub-group which contains all the operations $z+x^i$ where x^i is a quadratic residue must also contain those where x^i is a non-residue. The group K therefore contains the two operations $\frac{\alpha z+\beta}{\gamma z+2-\alpha}$ and $z+x^i$, where α, γ and x^i may be chosen arbitrarily. The result of combining these two is

$$\frac{(\alpha+\gamma x^i)\,z+\beta+(2-\alpha)\,x^i}{\gamma z+2-\alpha},$$

and the sum of the first and last coefficients in this substitution, namely, $2+\gamma x^i$, may be made anything whatever.

Hence, in this case, K coincides with H.

Suppose now that K contains a substitution keeping two symbols fixed, say $\frac{x^i z}{x^{-i}}$. It will then contain $\frac{\alpha z+\beta}{\gamma z+\delta}$, where

$$\alpha+\delta = x^i+x^{-i}.$$

These combined give $$\frac{\alpha x^i z+\beta x^{-i}}{\gamma x^i z+\delta x^{-i}},$$

and, if now $\alpha \equiv x^{-i},\quad \delta \equiv x^i,\quad \gamma \equiv 0,\quad \beta \not\equiv 0,$

this substitution is $$z+\beta x^{-i},$$

which is of order p. Hence, again, K coincides with H.

Suppose, lastly, that K contains an operation, displacing all the symbols, for which

$$a+\delta \equiv x^i;$$

then K contains $\dfrac{x^iz+1}{-z}$ and $\dfrac{1}{-z+x^i}$, and these combined give $z-2x^i$, an operation of order p. Hence, once again, K coincides with H.

It follows, therefore, that, since H contains no self-conjugate sub-group different from itself, it is a simple group.

7. *On H_n regarded as a Sub-Group of H_{rn}.*

If a suffix be now used to denote the degree of the irreducible congruence on which the coefficients of the substitutions of G or H depend, it is immediately clear that H_N will contain H_n, if p^n-1 is a factor of p^N-1. For, if

$$p^N-1 = \lambda\,(p^n-1),$$

and if y is a primitive root of the congruence

$$y^{p^N-1}-1 \equiv 0,$$

then y^λ is a primitive root of

$$x^{p^n-1}-1 \equiv 0,$$

and the group H_n is derived from

$$z' \equiv \frac{-1}{z}, \quad z' \equiv z+1, \quad z' \equiv \frac{y^\lambda z}{y^{-\lambda}}.$$

The sub-group of H_N with which H_n is permutable may be determined as follows. If $\dfrac{az+\beta}{\gamma z+\delta}$ a substitution of H_n is transformed into $\dfrac{a'z+\beta'}{\gamma' z+\delta'}$ by any substitution $\dfrac{Az+B}{Cz+D}$ of H_N, then

$$\begin{aligned} a' &\equiv ADa-AC\beta+BD\gamma-BC\delta, \\ \beta' &\equiv -ABa+A^2\beta-B^2\gamma+AB\delta, \\ \gamma' &\equiv CDa-C^2\beta+D^2\gamma-CD\delta, \\ \delta' &\equiv -BCa+AC\beta-BD\gamma+AD\delta. \end{aligned}$$

Now $\dfrac{Az+B}{Cz+D}$, where $AD-BC \equiv 1$, is permutable with H_n, if, when these congruences are applied to the three generating substitutions of H_n, the values of α', β', γ', δ' obtained are all powers of y^λ.

The respective values of α, β, γ, δ are

$$\begin{array}{cccc} 0, & -1, & 1, & 0, \\ 1, & 1, & 0, & 1, \\ y^\lambda, & 0, & 0, & y^{-\lambda}; \end{array}$$

and therefore $AC+BD$, A^2+B^2, C^2+D^2, $1-AC$, A^2, C^2, $ADy^\lambda-BCy^{-\lambda}$, $AB(y^\lambda-y^{-\lambda})$, $CD(y^\lambda-y^{-\lambda})$, and $ADy^{-\lambda}-BCy^\lambda$ are powers of y^λ. Hence, since the sum of any number of powers of y^λ is again a power of y^λ, it follows that A^2, B^2, C^2, D^2, AB, AC, AD, BC, BD, CD must all of them be powers of y^λ. If, then, λ be odd, A, B, C, D must themselves be powers of y^λ, and the group H_n is only permutable with itself; if, however, λ be even, the coefficients may also be of the form $y^{(m+\frac{1}{2})\lambda}$. Now, if $\frac{\alpha z+\beta}{\gamma z+\delta}$ is any substitution of H_n, $\frac{\alpha y^{\frac{1}{2}\lambda}z+\beta y^{\frac{1}{2}\lambda}}{\gamma y^{-\frac{1}{2}\lambda}z+\delta y^{-\frac{1}{2}\lambda}}$ is a substitution of the second kind which transforms H_n into itself, and in this way each such substitution is obtained once and once only. It follows that, when λ is even, the order of the group with which H_n is permutable is twice the order of H_n. This group is evidently G_n.

Now p^n-1 will only divide p^N-1, when n divides N and λ is odd or even according as $\frac{N}{n}$ is odd or even. The group H_n is therefore one of a conjugate set of sub-groups of H_{ns} whose number is

$$p^{n(s-1)}\,(p^{2n(s-1)}+p^{2n(s-2)}+\ldots+p^{2n}+1),$$

or one half of this number according as s is odd or even. In particular H_1 the group of the modular equation is always contained as a sub-group in H_n.

Consider now the case of $s=2$, and H_n as a sub-group of H_{2n}. Since p^n+1 is not a factor of $p^{2n}+1$, none of the operations of H_n displace all the symbols of H_{2n}; and therefore any operation of H_n of the order $\frac{p^n+1}{2}$ occurs as the p^n-1 power of some operation of H_{2n} of order $\frac{p^{2n}-1}{2}$ which keeps two symbols fixed. But the cyclical sub-groups of H_{2n} of order $\frac{p^{2n}-1}{2}$ are all conjugate to the sub-group arising from $\frac{yz}{y^{-1}}$; and therefore H_n must contain operations of order $\frac{p^n+1}{2}$ which are conjugate, within H_{2n}, to $\frac{y^{p^n-1}z}{y^{-p^n+1}}$. It follows that among the groups conjugate to H_n, within H_{2n}, which are all isomorphous with H_n, there must be one at least containing the operation $\frac{y^{p^n-1}z}{y^{-p^n+1}}$.

A group H'_n, contained in H_{2n}, which is isomorphous with H_n, and which contains the operation $\frac{y^{p^n-1}z}{y^{-p^n+1}}$, consists of all operations of the form

$$\frac{\alpha' z+\beta'}{\gamma' z+\delta'}, \quad (\alpha'\delta'-\beta'\gamma' \equiv 1),$$

for which α', β', γ', δ' are given in form by

$$\alpha' \equiv r+s\,(y^{p^n-1}-y^{-p^n+1}), \qquad \beta' \equiv u+v\,(y^{p^n-1}-y^{-p^n+1}),$$

$$\delta' \equiv r-s\,(y^{p^n-1}-y^{-p^n+1}), \qquad \gamma' \equiv -u+v\,(y^{p^n-1}-y^{-p^n+1}),$$

where r, s, u, v are powers of y^{p^n+1}.

That these operations actually form a group may be verified at once by forming the operation which is compounded of any two operations of the above form, when it will be found that the coefficients in the resulting operation are again of the same form. That the group thus defined may be obtained by transforming H_n by an operation $\frac{Az+B}{Cz+D}$ may be proved as follows.

If the formulæ given at the foot of the last page but one for α', β', γ', δ' are equivalent to the above-written forms, the quantities r, u, $s\,(y^{p^n-1}-y^{-p^n+1})$ and $v\,(y^{p^n-1}-y^{-p^n-1})$ must be given by

$$r \equiv \alpha+\delta,$$

$$u \equiv (CD+AB)(\delta-\alpha)+(A^2+C^2)\,\beta-(B^2+D^2)\,\gamma,$$

$$s\,(y^{p^n-1}-y^{-p^n+1}) \equiv (AD+BC)(\alpha-\delta)-2AC\beta+2BD\gamma,$$

$$v\,(y^{p^n-1}-y^{-p^n+1}) \equiv (AB-CD)(\delta-\alpha)+(A^2-C^2)\,\beta+(D^2-B^2)\,\gamma.$$

It has therefore to be shown that an operation $\frac{Az+B}{Cz+D}$ can be found, such that, when r, u, s, v are given powers of x satisfying

$$r^2+u^2-(s^2+v^2)(y^{p^n-1}-y^{-p^n+1})^2 \equiv 1,$$

the above congruences determine α, β, γ, δ as powers of x or y^{p^n+1}.

Now it has been shown above that, since y^{p^n-1} is a primitive root of the congruence

$$\lambda^{p^n+1}-1 \equiv 0,$$

it must satisfy an irreducible quadratic congruence

$$x'^2 - x^i x' + 1 \equiv 0,$$

where

$$x = y^{p^n+1}$$

is a primitive root of

$$\lambda^{p^n-1} - 1 \equiv 0.$$

It follows that $y^{p^n-1} + y^{-p^n+1}$ is a power of x, and therefore so also is $(y^{p^n-1} - y^{-p^n+1})^2$, which will be called ξ^2. On the other hand, ξ is clearly not expressible rationally in terms of x.

If, now, m and n are any powers of x, the expression

$$m + n\xi$$

includes, with zero, p^{2n} incongruous values; and therefore every integral power of y can be expressed in this form.

Since ξ^2 is a power of x, so also is $(m+n\xi)(m-n\xi)$, and $m+n\xi$, $m-n\xi$ are therefore either both even or both odd powers of y.

Suppose then that

$$A^2 \equiv m+n\xi, \qquad B^2 = m'+n'\xi,$$

$$C^2 \equiv m-n\xi, \qquad D^2 = m'-n'\xi,$$

where $m+n\xi$, $m'+n'\xi$ are odd powers of y, and $m^2-n^2\xi^2$, $m'^2-n'^2\xi^2$ are odd powers of x. They can obviously be chosen so in a variety of ways satisfying

$$A^2D^2 + B^2C^2 - 2ABCD \equiv (AD-BC)^2 \equiv 1.$$

From these forms it at once follows that A^2+C^2 and B^2+D^2 are powers of x, while A^2-C^2, B^2-D^2, AC and BD are, each of them, powers of x multiplied by ξ.

Also $\qquad AD+BC = A^2D^2 - B^2C^2 \equiv 2\,(m'n - mn')\,\xi,$

and $\qquad (CD-AB)(CD+AB) \equiv -2\,(m'n+mn')\,\xi,$

while $\quad (CD-AB)(AD-BC) = AC\,(B^2+D^2) - BD\,(A^2+C^2)$

is also a power of x multiplied by ξ; and therefore $CD-AB$ is a power of x multiplied by ξ, while $CD+AB$ is a power of x. The values assumed for A, B, C, D satisfy therefore all the conditions given at the beginning of this investigation, and the group whose operations are of the form

$$\frac{(r+s\xi)\,z+u+v\xi}{(-u+v\xi)\,z+r-s\xi}$$

is obtained from the transformation of H_n by $\dfrac{Az+B}{Cz+D}$.

The operation $\dfrac{Az+B}{Cz+D}$ does not belong to H_{2n}, since A, B, C, D are not rationally expressible in terms of y; it will, however, evidently be an operation of the group H_{4n}; and the group H'_n is therefore conjugate with H_n within H_{4n}, but not within H_{2n}. It necessarily follows that H_{2n} contains at least two different conjugate sets of sub-groups, each isomorphous with H_n.

The values

$$r \equiv \tfrac{1}{2}\,(y^{p^n-1}+y^{-p^n+1}), \quad s \equiv \tfrac{1}{2}, \quad u \equiv v \equiv 0,$$

of which the first has been shown to be a power of x, give the operation $\dfrac{y^{p^n-1}z}{y^{-p^n+1}}$ contained in H'_n.

This transformed form of the group may be used to bring out the analogy between the cyclical sub-groups of orders $\tfrac{1}{2}(p^n-1)$ and $\tfrac{1}{2}(p^n+1)$. Thus in the original form of the group a typical cyclical sub-group of order $\tfrac{1}{2}(p^n-1)$ is that arising from $\dfrac{xz}{x^{-1}}$. This keeps the symbols 0, ∞ unchanged, and can therefore only be transformed into itself by operations which either keep 0, ∞ unchanged, or by operations which interchange them. The former are the operations of the sub-group itself, and the latter are the $\tfrac{1}{2}(p^n-1)$ operations of order 2 of the form $\dfrac{-x^i}{x^{-i}z}$ contained in H_n. Each of the latter transforms any operation of the cyclical sub-group into its own inverse; and the $\tfrac{1}{2}(p^n-1)$ operations of order 2, taken with the operations of the cyclical sub-group, form a sub-group of dihedral type of order p^n-1.

In the transformed group H'_n a typical cyclical sub-group of order $\tfrac{1}{2}(p^n+1)$ is that arising from $\dfrac{y^{p^n-1}z}{y^{-p^n+1}}$. Considered as an operation in the group H_{2n}, this keeps the symbols 0, ∞ unchanged, and therefore is only transformed into itself by the operations $\frac{y^i z}{y^{-i}}$ and $\frac{-y^i}{y^{-i} z}$ of H_{2n}. Those of the former type which belong to H_n are the operations of the cyclical sub-group itself, while those of the latter type are the operations included under the form

$$\frac{u+v\,(y^{p^n-1}-y^{-p^n+1})}{\{-u+v\,(y^{p^n-1}-y^{-p^n+1})\}\,z},$$

where $$u^2-v^2\,(y^{p^n-1}-y^{-p^n+1})^2 \equiv 1.$$

This congruence has for its solutions

$$u = \pm\tfrac{1}{2}\,(y^{m\,(p^n-1)}+y^{-m\,(p^n-1)}),$$

$$v = \pm\tfrac{1}{2}\,\frac{y^{m\,(p^n-1)}-y^{-m\,(p^n-1)}}{y^{p^n-1}-y^{-p^n+1}},$$

$$m = 0,\ 1,\ \dots\ p^n$$

and these correspond to the $\frac{1}{2}\,(p^n+1)$ operations

$$\frac{y^{m\,(p^n-1)}}{-y^{-m\,(p^n-1)}\,z},\quad \left[m = 0,\ 1,\ \dots\ \tfrac{1}{2}\,(p^n-1)\right].$$

Finally, these $\frac{1}{2}\,(p^n+1)$ operations of order 2, taken with the cyclical sub-group arising from $\frac{y^{p^n-1}z}{y^{-p^n+1}}$, give a dihedral group of order p^n+1.

The sub-groups of tetrahedral type, and those of octahedral and icosahedral types for the cases of $p^n \equiv \pm 1$ (mod 8 and mod 5, respectively), the existence of which Herr Gierster demonstrates in his memoir for the case $n = 1$, may also be shown to exist in the general case.

The complete discussion which is given in the following paragraph of all possible sub-groups for the case $p = 2$ indicates the lines on which a similar discussion may be carried out for the case of p an odd prime; and suggests that the types of sub-group which have been shown to exist, including those mentioned in the last sentence probably exhaust all types that actually exist.

8. *On the Group G, when $p = 2$.*

The necessary modifications of the foregoing theorems with respect to the omitted case of $p = 2$ are now very readily made, and it seems hardly necessary to repeat proofs which are almost identical with those already given.

As was shown in § 3, when $n = 2$, there are no quadratic non-residues, and therefore all the operations of the group G of order $2^n(2^{2n}-1)$ may be brought to the standard form in which the determinant is unity. Considered as a permutation group of 2^n+1 symbols, the operations which displace all the symbols are all powers of operations of order 2^n+1, those which keep one symbol fixed are all of order 2, and those which keep two symbols fixed are powers of operations of order 2^n-1.

The operations of order 2 form a single conjugate set, as also do all the operations for which $(a+\delta)^2$ has a given value; but here $(a+\delta)^2$, including the value zero which gives the operations of order 2, may have any one of 2^n values, and there are, therefore, 2^n different conjugate sets of operations, exclusive of identity.

The proof that H, for p an odd prime, is a simple group will apply exactly to show that G is simple, when $p = 2$.

It may also be shown, exactly as in the corresponding case for H, that corresponding to each cyclical sub-group of order 2^n+1 or 2^n-1 there is a sub-group of dihedral type of order $2(2^n+1)$ or $2(2^n-1)$ containing the cyclical group as a self-conjugate sub-group, and that no cyclical sub-group is contained self-conjugately in any sub-group of higher order than these dihedral groups. As regards sub-groups of tetrahedral, octahedral, and icosahedral types, there can clearly be none of octahedral type, since the groups contain no operations of order 4. If n is odd, 5 divides neither 2^n+1 nor 2^n-1, and hence, for an odd n, G cannot contain an icosahedral sub-group. If, however, n is even, G_n contains G_2 as a sub-group, and this, being of order 60, is necessarily an icosahedral group. Since the icosahedral group contains tetrahedral sub-groups, G_n, when n is even, will have sub-groups of tetrahedral type. Finally, when n is odd, 2^n+1 is divisible by 3 and not 2^n-1, and, since a sub-group of order 4 cannot be transformed into itself by an operation changing all the symbols, a tetrahedral sub-group, which must contain a group of order 4 self-conjugately, cannot exist in this case.

It will now be shown that the sub-groups already enunciated, together with the sub-group that keeps one symbol fixed, and *its* sub-groups, and the sub-group of type $G_{n'}$, where n' is a factor of n, exhaust all existing types. This will be proved by a modification and extension of the process used by Herr Gierster in his often referred to memoir in discussing the corresponding question for the modular group.

Let Γ be any sub-group of G_n of order $2^m g' = g$, where g' is a factor of $2^{2m}-1$, and let Σ be an operation of odd order p_1 contained in Γ, there being no operations in Γ of higher order than p_1. Then the sub-group arising from Σ is permutable within Γ, either with itself or with a dihedral group of order $2p_1$, and it therefore forms one of a set of either $\frac{g}{p_1}$ or $\frac{g}{2p_1}$ conjugate sub-groups. No two of these sub-groups contain a common operation, for, if they did, all their operations would be common. Hence, omitting identity from each such sub-group, they contain in all $\frac{(p_1-1)\,g}{p_1}$ or $\frac{(p_1-1)\,g}{2p_1}$ different operations. Let, now, Σ' be an operation of odd order p_2 contained among the remaining operations, there being no remaining operation of a higher order than p_2. Then, as before, the set of sub-groups conjugate with the cyclical sub-group arising from Σ' contain either $\frac{(p_2-1)\,g}{p_2}$ or $\frac{(p_2-1)\,g}{2p_2}$ different operations of odd order, and no one of these can coincide either with another of the same set or with one of the previous set. If this process is continued till the operations of odd order are exhausted, there remain only operations of order 2. Any one S of these is permutable with a sub-group of order 2^m, and therefore forms one of a set of $\frac{g}{2^m}$ or g' conjugate operations. If among these g' operations there occurs none of the group of order 2^m with which S is permutable, then each operation, except identity, of this group will give rise to a similar set, no two sets containing a common operation, and the number of operations of order 2 contained in Γ will be $(2^m-1)\,g'$. It is necessary therefore to determine in what cases the sub-group of order 2^m contains operations conjugate within Γ.

The general type of group of order 2^m contained in G is the group arising from the m permutable operations of order 2,

$$z+x^{a_1},\ z+x^{a_2},\ \ldots\ z+x^{a_m},$$

where $a_1, a_2, \ldots a_m$ are any m chosen integers from $1, 2, \ldots 2^n-1$.

No operation of this sub-group can be transformed into another, except by the operations of the sub-group arising from

$$z+1 \text{ and } xz.$$

Hence Γ must contain the operation $x^\nu z$, or an operation conjugate to it within the sub-group which keeps ∞ fixed, if the sub-group of order 2^m contains conjugate operations. Now, the $\frac{2^n-1}{\nu}$ powers of $x^\nu z$ transform $z+x^{n_1}$ into a set of $\frac{2^n-1}{\nu}$ conjugate operations of the same form $z+x^i$, and these must all be contained in the sub-group of order 2^m, as otherwise 2^m would not be the highest power of 2 dividing g.

Hence, if a is the symbol that any sub-group of Γ of order 2^m keeps fixed, and if $\frac{2^n-1}{\nu}$ is the order of the highest operation of odd order contained both in Γ and in the sub-group keeping a fixed, the operations of the sub-group of order 2^m will be conjugate in sets of $\frac{2^n-1}{\nu}$. This involves that $\frac{2^n-1}{\nu}$ is a factor of 2^m-1, and is also one of the numbers $p_1, p_2, \ldots$. If, then,

$$\frac{2^n-1}{\nu} = p_r,$$

the number of operations of order 2 contained in Γ is $\frac{(2^m-1)\,g'}{p_r}$.

Adding together the numbers of different operations thus obtained, including the identical operation, there results

$$g = 1+\Sigma\frac{(p_1-1)\,g}{s_1 p_1}+\frac{(2^m-1)\,g}{2^m p_r},$$

where each s is either 1 or 2.

Hence
$$g = \frac{1}{1-\Sigma\frac{p_1-1}{s_1 p_1}-\frac{2^m-1}{2^m p_r}}.$$

If m is zero, so that Γ contains no operations of order 2, each s must be unity, and the relation becomes

$$g = \frac{1}{1-\Sigma\frac{p_1-1}{p_1}}.$$

Now g is a positive integer, and each p is an odd number. Hence in this case there can be only one term under the sign of summation, and

$$g = p_1.$$

It follows that the only sub-groups not containing operations of order 2 are the cyclical sub-groups.

Again
$$\frac{p_1 - 1}{s_1 p_1} \geqslant \tfrac{1}{3},$$

so that, when m is not zero, there can be at most two terms under the sign of summation; and, if there are two terms, $s_1 = s_2 = 2$; while, if there is one term only, s_1 may be either 1 or 2.

With one term only, if $s_1 = 1$,

$$\frac{1}{g} = \frac{1}{p_1} - \frac{2^m - 1}{2^m p_r},$$

where p_r is either unity or p_1.

If $p_r = p_1$, $g = 2^m p_1$, and Γ is a sub-group of the sub-group that keeps one symbol fixed.

If $p_r = 1$, $\dfrac{2^m p_1}{2^m - (2^m - 1)\, p_1}$ must be an integer, and this can only be so when $p_1 = 1$. Γ is then a sub-group of order 2^m.

With one term only, and $s_1 = 2$,

$$\frac{1}{g} = \frac{1}{2} + \frac{1}{2p_1} - \frac{2^m - 1}{2^m p_r}.$$

If $p_r = p_1$,
$$\frac{1}{g} = \frac{2^{m-1} p_1 - (2^{m-1} - 1)}{2^m p_1},$$

and, since p_1 is a factor of $2^m - 1$, g can only be an integer when $p_1 = 1$; leading again to a sub-group of order 2^m.

If $p_r = 1$, m must be unity, and Γ is a dihedral group of order $2p_1$.

When there are two terms under the sign of summation, both s_1 and s_2 must be 2 that g may be a positive integer. Hence

$$\frac{1}{g} = \frac{1}{2p_1} + \frac{1}{2p_2} - \frac{2^m - 1}{2^m p_r},$$

where p_r is either 1, p_1 or p_2.

Since p_1, p_2 are odd integers, g cannot be positive if p_r is unity, and therefore it may be taken as p_1. Then

$$\frac{1}{g}=\frac{2^{m-1}p_1-(2^{m-1}-1)\,p_2}{2^m p_1 p_2}.$$

If q is the G.C.M. of p_1 and p_2, so that

$$p_1 = qp_1', \quad p_2 = qp_2',$$

where p_1', p_2' are relatively prime, then

$$\frac{1}{g}=\frac{2^{m-1}p_1'-(2^{m-1}-1)\,p_2'}{2^m q p_1' p_2'}.$$

The numerator of this fraction is prime relatively to $2^m p_1' p_2'$, and hence, if g is an integer,

$$2^{m-1}p_1'-(2^{m-1}-1)\,p_2' = q',$$

a factor of q.

The general values of p_1' and p_2' which satisfy this equation are given by

$$p_1' = q'+k\,(2^{m-1}-1),$$
$$p_2' = q'+k\,2^{m-1};$$

and therefore

$$p_1 = qq'+kq\,(2^{m-1}-1),$$
$$p_2 = qq'+kq\,2^{m-1}.$$

Now p_1 is a factor of 2^m-1, so that kq cannot be greater than 2. Also q is odd and is therefore unity, as also must therefore be q'. Hence

$$p_1 = 1+k\,(2^{m-1}-1),$$
$$p_2 = 1+k\,2^{m-1},$$

where k can only be 1 or 2, and must be 2 since p_1 is odd. Then

$$p_1 = 2^m-1, \quad p_2 = 2^m+1.$$

Also, since p_1 is a factor of 2^n-1, m must be a factor of n. Hence this last possible case leads to the sub-groups of type $G_{n'}$, where n' is a factor of n.

9. *On certain Special Cases of the Groups G and H.*

There are three values of p^n for which the corresponding groups are already known. When

$$p^n = 2^2,$$

G is a simple group of order 60, and must therefore be a form of the icosahedral group. Thus a new and very simple specification is obtained for this group, as consisting of all substitutions of the form

$$\left.\begin{aligned} z' &\equiv \frac{\alpha z+\beta}{\gamma z+\delta} \\ \alpha\delta-\beta\gamma &\equiv 1 \end{aligned}\right\} \pmod 2,$$

where the coefficients are either 0, 1, x, or x^2; x being such that

$$x^2+x+1 \equiv 0 \pmod 2.$$

When $$p^n = 2^3,$$

G is a simple group of order 504. This group was first discovered by Dr. Cole (*American Journal of Mathematics*, Vol. xv.).

When $$p^n = 3^2,$$

H is a simple group of order 360, and must therefore be a form of the alternating group of 6 symbols. G is a triply-transitive group of 10 symbols, order 720, containing H self-conjugately.

It is curious to notice that, as is well known, the symmetric group of 6 symbols, order 720, which also contains the alternating group self-conjugately, can be expressed as a doubly-transitive group of 10 symbols; so that there are two distinct transitive groups of 10 symbols, one doubly and the other triply transitive, both of order 720, and both containing the same doubly-transitive simple group of order 360 as a self-conjugate sub-group.

10. *On a Property of certain Transitive Groups.*

The sub-group of G which keeps one symbol fixed is doubly transitive in p^n symbols while its order is $p^n(p^n-1)$. Now the order of a doubly-transitive group in m symbols is necessarily divisible by $m(m-1)$, and it may be shown that, when it is equal to this number, m is the power of a prime, and moreover that, as has been seen to be the case with G, the operations of the sub-group of order m must all be permutable with each other. Thus, assuming the existence of a doubly-transitive group in m symbols of order $m(m-1)$, its operations must displace all the symbols or all but one; and therefore the $m-2$ symbols into which a given symbol is changed, by the operations of a sub-group keeping one symbol fixed, are all different. Hence among the operations of the m sub-groups, each of which keeps one symbol unchanged, there must be $m-2$ operations which change a given symbol a into another given symbol

b; for each sub-group contains one such operation except those that keep a and b respectively unchanged.

Now the group contains $m-1$ operations changing a into b, and hence among the $m-1$ operations which change all the symbols there is one, and only one, which changes a into b. It follows, therefore, that these $m-1$ operations with identity form a sub-group of order m, since the product of any two of them necessarily changes all the symbols. This sub-group of order m is evidently self-conjugate.

If, now, P is any particular operation of this sub-group of order m, and if A is any operation of the sub-group that keeps a fixed, the operations

$$A^{-1}PA$$

are clearly all different, for the symbols into which they change a are all different. Hence the $m-1$ operations which change all the symbols form a single conjugate set within the main group, and they are therefore all of the same order. Since with identity these operations form a group, their common order must be a prime, and hence finally m must be the power of a prime.

Since P is one of a set of $m-1$ conjugate operations, the operations permutable with P form a sub-group whose order is

$$m(m-1) \div (m-1),$$

i.e., m. Hence every operation of the sub-group of order m is permutable with every other, and the sub-group is therefore Abelian.

The type of doubly-transitive group of order $p^n(p^n-1)$ which appears as that sub-group of G which keeps one symbol fixed is not, however, the only possible type. Thus Dr. Cole in his analysis of the transitive groups of 9 letters (*Bulletin of the New York Mathematical Society*, July, 1893) has shown that there are two such doubly-transitive groups of order 82. All possible types may be obtained by the following considerations. The sub-group of order p^n is generated by n permutable operations $P_1, P_2, \ldots P_n$ of order p. Let Σ be any operation of the sub-group K that keeps one symbol fixed, and let

$$\Sigma^{-1}P_1\Sigma = P_1^{a_1}P_2^{a_2}\ldots P_n^{a_n},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\Sigma^{-1}P_n\Sigma = P_1^{n_1}P_2^{n_2}\ldots P_n^{n_n},$$

so that

$$\Sigma^{-1}P_1^xP_2^y\ldots P_n^z\Sigma$$

$$= P_1^{a_1x+b_1y+\ldots+n_1z}P_2^{a_2x+b_2y+\ldots+n_2z}\ldots P_n^{a_nx+b_ny+\ldots+n_nz}.$$

Since no operation of the sub-group K is permutable with any operation of the sub-group of order p^n, it follows that, if the p^n-1 operations of this sub-group other than identity be transformed by all the operations of the sub-group K in succession, a permutation group is obtained which is holohedrically isomorphous with K. The operations of this sub-group are, as the last equation shows, defined by congruences of the form

$$\left.\begin{array}{l} x' \equiv a_1x+b_1y+\ldots+n_1z \\ y' \equiv a_2x+b_2y+\ldots+n_2z \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ z' \equiv a_nx+b_ny+\ldots+n_nz \end{array}\right\} \pmod{p};$$

and therefore the problem of determining all possible forms of the sub-group K is equivalent to that of finding all the sub-groups of the general homogeneous linear group in n variables which are of order p^n-1, and all of whose operations displace all the p^n-1 symbols in terms of which the group can be expressed transitively.

Thursday, March 8th, 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

Mr. Adam Brand, M.A., Fellow of Pembroke College, Cambridge, was elected a member. Mr. F. W. Hill, M.A., and Major Hippisley, R.E., were admitted into the Society.

The following communications were made:—

Groups of Points on Curves: Mr. F. S. Macaulay.

On a Simple Contrivance for Compounding Elliptic Motions: Mr. G. H. Bryan.*

On the Buckling and Wrinkling of Plating supported on a Framework under the Influence of Oblique Stresses: Mr. G. H. Bryan.

On the Motion of Paired Vortices with a Common Axis: Mr. A. E. H. Love.

On the Existence of a Root of a Rational Integral Equation: Prof. E. B. Elliott.

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* For an account of this contrivance, see *Nature*, March 22nd, 1894, p. 498.

The following presents to the Library were received :—

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," 38er Jahrgang, Hefte 3, 4 ; Zurich, 1893.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 2 ; Leipzig, 1894.

"Proceedings of the Royal Society," Vol. LIV., Nos. 330, 331.

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"Proceedings of the Royal Society of Edinburgh," Vol. XIX., Session 1891–2.

"Nyt Tidsskrift for Mathematik," A. Fjerde Aargang, Nos. 7, 8 ; B. Fjerde Aargang, No. 4 ; Copenhagen, 1893.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," Bd. III., Heft 4.

D'Ocagne, M.—"Abaque général de la Trigonométrie Sphérique," pamphlet.

"Berichte über die Verhandlungen der Koniglich sächsischen Gesellschaft der Wissenschaften zu Leipzig, Math. Phys. Classe," 1893, 7–9.

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"Bulletin des Sciences Mathématiques," Tome XVIII., Janvier, 1894 ; Paris.

Macfarlane, A.—"On the Definitions of the Trigonometric Functions," 8vo ; Boston.

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"Rendiconti del Circolo Matematico di Palermo," Tomo VII., Fasc. 6 ; November, December, 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. III., Fasc. 2, 3 ; Roma.

"Journal de l'Ecole Polytechnique," 63ème Cahier ; Paris, 1893.

"Educational Times," March, 1894.

"Journal für die reine und angewandte Mathematik," Bd. CXIII., Heft 1 ; Berlin, 1894.

"Annals of Mathematics," Vol. VIII., No. 2 ; University of Virginia.

"Indian Engineering," Vol. XV., Nos. 3–6.

"American Journal of Mathematics," Vol. XVI., No. 1 ; Baltimore.

On the Buckling and Wrinkling of Plating when supported on Parallel Ribs or on a Rectangular Framework. By G. H. Bryan. Read and received March 8th, 1894.

Introduction.

1. In a communication made to the Society three years ago,* I discussed the kind of buckling which arises when a rectangular sheet of plating is subjected to tractions applied perpendicularly to its edges in its plane, if these tractions are sufficiently great to make the plane form unstable. In the present paper I shall consider certain cases in which a rectangular plate or a strip of plating is made to buckle by tractions applied in an oblique direction instead of perpendicular to its edges.

If we were to suppose the plate to be clamped or supported round its margin, there would be two boundary conditions to be satisfied at each edge, and the solution of the problem would become rather too complicated to lead to any interesting results.

This difficulty will be obviated if, instead of a *single* strip or rectangle, we consider an infinite sheet of plating supported on equidistant parallel ribs, which divide it into strips or which are also crossed at right angles by a second set of supporting strips thus dividing the plate into rectangles.

2. Let the supporting ribs be the systems of straight lines

$$x = ma \quad (m = -\infty \text{ to } m = +\infty),$$

$$y = nb \quad (n = -\infty \text{ to } n = +\infty).$$

Let the stresses in the plane of the plate be specified by the thrusts T_1, T_2, and the shear M (each estimated per unit of length measured in the surface of the plate). Then, if these are sufficiently great to make equilibrium in the plane form critical, the plate will still be in equilibrium when displaced into a certain form differing infinitely little from a plane, and the normal displacement w in this form will be given by the differential equation

$$\beta\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + T_1\frac{\partial^2 w}{\partial x^2} + 2M\frac{\partial^2 w}{\partial x \partial y} + T_2\frac{\partial^2 w}{\partial y^2} = 0 \; \ldots \; (1),$$

* *Proceedings*, Vol. XXII., p. 54.

where the various letters have the same meaning as in my previous paper. In the notation adopted by Love,*

$$\beta = C, \quad T_1 = \mathfrak{P}_1, \quad T_2 = \mathfrak{P}_2.$$

In what follows it will be convenient to use the letter C instead of β to denote the cylindrical rigidity of the surface.

The differential equation (1) can be established in the way indicated by Love, or by applying the variational equation

$$\delta V - \delta W = 0$$

to the expressions found in my previous paper (*cf.* the method there adopted in § 12). Hence it admits of being verified *without* the use of the not altogether satisfactory " energy test."

3. The cases which I propose to consider in this paper are those where T_1, T_2, M are the components of a simple thrust acting in some direction not necessarily perpendicular to the edges of the plate. If this thrust is P, and its direction makes an angle α with the axis of x, then

$$T_1 = P\cos^2\alpha, \quad T_2 = P\sin^2\alpha, \quad M = P\sin\alpha\cos\alpha \quad \text{.........(2)}.$$

Hence the differential equation (1) becomes

$$C\nabla^4 w + P\left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)^2 w = 0 \quad \text{...............(3)}.$$

Stability of a Plate supported on Parallel Ribs.

First Solution.

4. Consider in the first place a plate supported only along the lines

$$y = nb,$$

and divided into infinitely long strips by these lines.

Assume the differential equation (3) to have a solution of the form

$$w = A\cos(px + qy) \quad \text{...........................(4)};$$

then we have, on substituting in it,

$$C(p^2+q^2)^2 - P(p\cos\alpha + q\sin\alpha)^2 = 0 \quad \text{...............(5)}.$$

Putting

$$P/C = 4k^2 \quad \text{..................................(6)},$$

* *Theory of Elasticity*, Vol. II., p. 305.

this gives either $(p^2+q^2)-2k\,(p\cos\alpha+q\sin\alpha) = 0$..................(7),

or $(p^2+q^2)+2k\,(p\cos\alpha+q\sin\alpha) = 0$..................(8).

For a given value of p, either of these alternatives gives two values of q, and, by combining the corresponding expressions for w together, we are able to satisfy the required conditions at the ribs.

Taking the first equation

$$p^2+q^2-2k\,(p\cos\alpha+q\sin\alpha) = 0 \quad \text{..................(7),}$$

let q_1, q_2 be its roots when regarded as a quadratic in q. Then the differential equation (3) is satisfied by

$$\mathrm{w} = A\,\{\cos(px+q_1y)+\cos(px+q_2y)\}$$

$$= 2A\sin\{px+\tfrac{1}{2}(q_1+q_2)\,y\}\sin\tfrac{1}{2}(q_1-q_2)\,y \quad \text{............(9).}$$

The condition to be satisfied at the ribs is

$$\mathrm{w} = 0,$$

when $y = nb$;

therefore $\tfrac{1}{2}(q_1-q_2)\,b$ must be a multiple of π;

therefore $q_1-q_2 = 2m\pi/b$, where m is an integer ....(10).

Now (7) may be written

$$(q-k\sin\alpha)^2+(p-k\cos\alpha)^2 = k^2;$$

therefore $q = k\sin\alpha \pm \sqrt{\{k^2-(p-k\cos\alpha)^2\}}$ ....(11)

therefore $q_1-q_2 = 2\sqrt{\{k^2-(p-k\cos\alpha)^2\}}$.

Therefore the condition of critical equilibrium requires that

$$\frac{m^2\pi^2}{b^2} = k^2-(p-k\cos\alpha)^2 \quad \text{....................(12).}$$

Now m is any integer, and p may have any value whatever, for $2\pi/p$ is the wave length of the corrugations set up when the plate buckles, and this is entirely at our disposal. To find the greatest thrust P consistent with stability, we must choose m, p so as to make k a minimum.

Evidently we must take $m = 1$, and the least value of k^2 is then $= \pi^2/b^2$, and occurs when

$$p-k\cos\alpha = 0.$$

Therefore the plane form is unstable if

$$k^2 > \frac{\pi^2}{b^2};$$

that is, if

$$\frac{P}{C} > \frac{4\pi^2}{b^2} \quad\text{..........................(13)};$$

and therefore the greatest thrust consistent with stability for any displacement of the assumed type is

$$P = \frac{4\pi^2 C}{b^2} \quad\text{..........................(13a)}.$$

We observe that the limiting value of the thrust does not depend on the direction in which it is applied.

5. The most interesting part of the problem, however, consists in determining the nature of the corrugations produced when buckling takes place. To find p we have, from above,

$$p = k\cos a = \frac{\pi\cos a}{b} \quad\text{......................(14)}.$$

Also, from (7) or (11),

$$q_1 + q_2 = 2k\sin a = \frac{2\pi\sin a}{b},$$

and we have seen that

$$q_1 - q_2 = \frac{2\pi}{b}.$$

Hence the equation of the displaced surface (9) takes the form

$$w = A\sin\left\{\frac{\pi}{b}(x\cos a + y\sin a)\right\}\sin\frac{\pi y}{b} \quad\text{...........(15)}.$$

If we regard one side of the plate as its upper side, the corrugations may be said to consist of alternate elevations and depressions, separated by the lines

$$x\cos a + y\sin a = 0,\ b,\ 2b,\ \ldots.$$

These lines are perpendicular to the direction of the thrust, and their distance apart is equal to the breadth of the strip. Hence the corrugations divide the plate into a series of rhombi.

When $a = 0$ or the thrust is parallel to the axis of x, the corrugations divide the plate into squares, and the condition of critical equilibrium agrees with that found in § 5 of my previous paper for the case when $a = \infty$.

Another Form of Solution.

6. If we had taken the two values of q given by the second equation (8), we should have arrived at the same result, for this equation only differs in the sign of k, and this difference would disappear when we came to make k^2 a minimum.

But, by taking two values of q, one given by (7) and the other by (8), we obtain a different solution leading to different conditions of instability.

For, assuming the displacement to be

$$w = 2A \sin\left\{px + \tfrac{1}{2}(q_1+q_2)\, y\right\} \sin \tfrac{1}{2}(q_1-q_2)\, y \quad \text{.........(16)},$$

where

$$\left.\begin{aligned} p^2+q_1^2-2k\,(p\cos\alpha+q_1\sin\alpha) &= 0 \\ p^2+q_2^2+2k\,(p\cos\alpha+q_2\sin\alpha) &= 0 \end{aligned}\right\} \quad \text{...............(17)},$$

the conditions at the ribs require, as before, that

$$q_1-q_2 = 2m\pi/b \quad \text{...........................(18)},$$

where it is easy to see that m will have to be taken to be unity, as in the preceding case.

Put $\qquad c = \pi/b,$

$$\left.\begin{aligned} &\text{so that} && q_1-q_2 = 2c \\ &\text{and let} && q_1+q_2 = 2f \end{aligned}\right\} \quad \text{...........................(19)}.$$

Then $\qquad q_1 = f+c, \quad q_2 = f-c \quad \text{........................(20)},$

and equations (17) become

$$\left.\begin{aligned} p^2+(f+c)^2-2k\left\{p\cos\alpha+(f+c)\sin\alpha\right\} &= 0 \\ p^2+(f-c)^2+2k\left\{p\cos\alpha+(f-c)\sin\alpha\right\} &= 0 \end{aligned}\right\} \quad \text{.........(21)}.$$

By addition, $\qquad p^2+f^2+c^2-2kc\sin\alpha = 0,$

and, by subtraction, $fc-k\,(p\cos\alpha+f\sin\alpha) = 0,$

Eliminating f, we have

$$2kc\sin\alpha = c^2+p^2\left\{1+\left(\frac{k\cos\alpha}{c+k\sin\alpha}\right)^2\right\} \quad \text{............(22)}.$$

We must now take p such as to make k a minimum, and evidently this requires that

$$\left.\begin{aligned} & && p = 0 \\ &\text{and} && \therefore\ f = 0 \end{aligned}\right\} \quad \text{.................................(23)}.$$

Therefore the condition of critical equilibrium becomes

$$2k \sin a = c = \pi/b,$$

whence $$P = 4k^2 C = \frac{\pi^2 C}{b^2 \sin^2 a} \quad \text{.......................(24).}$$

7. The corresponding expression for the displacement is

$$w = 2A \sin \frac{\pi y}{b} \quad \text{.........................(25),}$$

showing that the displaced form is cylindrical, the corrugations being divided by the ribs. The condition of critical equilibrium could be much more simply worked out by *assuming* the displacement to have the form just found; but such an assumption would leave certain possible solutions of the differential equation (3) unexamined.

General Results for a Plate supported on Parallel Ribs.

8. Having found two independent criteria of critical equilibrium corresponding to the two different assumed types of displacement, and given respectively by

$$P = \frac{4\pi^2 C}{b^2} \quad \text{..............................(13),}$$

$$P = \frac{\pi^2 C}{b^2 \sin^2 a} \quad \text{..........................(24),}$$

we must select that one which makes P least.

Hence the first or the second is the proper condition according as

$$\sin a < \text{ or } > \tfrac{1}{2}.$$

Therefore, if the direction of the thrust makes an angle of less than 30° with the ribs, wrinkles will appear on the plate as soon as instability sets in, and these wrinkles will run perpendicular to the thrust. But, if the thrust makes an angle of more than 30° with the ribs, the plate will buckle into simple corrugations running parallel to the ribs.

Statement of the Method of Solution for a Single Strip in any State of Stress.

9. Although the above seems to be the only case in which the differential equation and boundary conditions lead to simple results, the form of the solution may be briefly stated for the more general

case when the stress-components in the plane of the plate do not reduce to a single thrust, and the plating consists of a single strip clamped or supported in any manner at its two parallel edges $y = 0$ and $y = b$.

If we assume a displacement of the form

$$w = A \cos (px + qy + \epsilon),$$

we have, on substituting in (1), and writing $C = \beta$,

$$C (p^2 + q^2)^2 - (T_1 p^2 + 2Mpq + T_2 q^2) = 0 \quad \text{............(26)}.$$

For any assumed value of p this is to be regarded as a biquadratic in q, and, if its roots are q_1, q_2, q_3, q_4, the general expression for the displacement is

$$\begin{aligned} w = \ & A_1 \cos (px + q_1 y + \epsilon_1) + A_2 \cos (px + q_2 y + \epsilon_2) \\ & + A_3 \cos (px + q_3 y + \epsilon_3) + A_4 \cos (px + q_4 y + \epsilon_4) \quad \text{......(27)}. \end{aligned}$$

There are two boundary conditions at $x = 0$, and two at $x = b$. Since these have to be satisfied for all values of x, we have to equate coefficients of $\cos px$ and $\sin px$ separately to zero in each. We are thus able to eliminate the four A's and the four ϵ's, and obtain a transcendental equation connecting p with C, T_1, T_2, M. We must then make C a maximum by the variation of p (for C a maximum corresponds to k a minimum in §§ 4, 6). The corresponding value of p determines the nature of the buckling or wrinkling, and the value of C determines the condition of critical equilibrium.

The simple form assumed by the solutions found in §§ 4–8 is due to the biquadratic splitting into two quadratics, and to the fact that for a plate supported on a series of ribs the expression for w can only contain two terms, so that only two of the four values of q occur in any solution. The truth of the last statement is easy enough to see in a general kind of way, and it could be proved rigorously if thought worth while.

Stability of a Plane Plate supported on a Rectangular Framework.

10. Let us now endeavour to apply the methods developed in §§ 4, 5 to the stability of a plate supported on two sets of parallel ribs which cross each other at right angles, and which lie along the lines

$$x = ma,$$

$$y = nb.$$

Assuming a displacement of the form

$$w = A \cos (px+qy),$$

p, q are connected, as before, by one or other of the quadratic equations (7, 8), which may be written

$$(p-k\cos a)^2+(q-k\sin a)^2 = k^2 \quad\text{.................}(28),$$

$$(p+k\cos a)^2+(q+k\sin a)^2 = k^2 \quad\text{.................}(29).$$

Introducing a new quantity γ, we see that (28) will be satisfied whatever be the value of γ, provided that

$$\left.\begin{aligned} p &= k(\cos a \pm \cos\gamma), \\ q &= k(\sin a \pm \sin\gamma), \end{aligned}\right\}$$

and that (29) will be satisfied if

$$\left.\begin{aligned} p &= -k(\cos a \pm \cos\gamma), \\ q &= -k(\sin a \pm \sin\gamma), \end{aligned}\right\}$$

where the double signs in p, q are independent of one another.

Let us first try to satisfy the boundary conditions by using only the first solution given by (28). Write

$$\left.\begin{aligned} p_1 &= k(\cos a+\cos\gamma), & p_2 &= k(\cos a-\cos\gamma) \\ q_1 &= k(\sin a+\sin\gamma), & q_2 &= k(\sin a-\sin\gamma) \end{aligned}\right\}\text{.........}(30);$$

then the differential equation will be satisfied by taking

$$\begin{aligned} w = A\,\{&\cos(p_1x+q_1y)-\cos(p_1x+q_2y) \\ &-\cos(p_2x+q_1y)+\cos(p_2x+q_2y)\} \\ = \;& 2A\sin\{p_1x+\tfrac{1}{2}(q_1+q_2)\,y\}\sin\tfrac{1}{2}(q_2-q_1)\,y \\ &-2A\sin\{p_2x+\tfrac{1}{2}(q_1+q_2)\,y\}\sin\tfrac{1}{2}(q_2-q_1)\,y \\ = & -4A\sin\tfrac{1}{2}(q_1-q_2)y\sin\tfrac{1}{2}(p_1-p_2)x\cos\tfrac{1}{2}\{(p_1+p_2)x+(q_1+q_2)\,y\} \end{aligned}$$

$$\text{..............}(31).$$

Therefore w vanishes at the ribs, provided that

$$p_1-p_2 = \frac{2r\pi}{a}, \quad q_1-q_2 = \frac{2s\pi}{b} \quad\text{.................}(32),$$

where r, s are any integers.

Now, $\quad p_1-p_2 = 2k\cos\gamma, \quad q_1-q_2 = 2k\sin\gamma.$

Therefore the conditions of critical equilibrium give

$$k\cos\gamma = \frac{r\pi}{a},\quad k\sin\gamma = \frac{s\pi}{b} \quad\text{.................(33)};$$

and
$$\therefore\quad k^2 = \frac{r^2\pi^2}{a^2} + \frac{s^2\pi^2}{b^2}.$$

Therefore k is least when $r = 1$, $s = 1$, zero values of r or s being precluded, because they would make w vanish everywhere.

Therefore equilibrium will be critical for the assumed type of displacement, if

$$k^2 = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

that is, if
$$\frac{P}{C} = 4\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{4\pi^2}{h^2} \quad\text{.................(34)},$$

where h is the altitude of the two triangles into which any rectangle is divided by its diagonal.

11. Since $p_1+p_2 = 2k\cos\alpha$, $q_1+q_2 = 2k\sin\alpha$,

the equation of the displaced surface (31) takes the form

$$\mathrm{w} = 4A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\cos k(x\cos\alpha + y\sin\alpha)$$

$$= 4A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\cos\frac{\pi\,(x\cos\alpha + y\sin\alpha)}{h} \quad\text{...........(35)}.$$

Hence, as in the case of an infinite strip, the wrinkles are perpendicular to the direction of the thrust, but, of course, they vanish at the four edges of each rectangle. Also the lines crossing the plate which separate the alternate elevations and depressions are at a distance h apart. The distance h is equal to the distance between the parallel diagonals of successive rectangles.

Other Conditions of Stability for a Rectangular Framework.

12. Since a series of parallel ribs may be regarded as the limit of a rectangular framework when one of the sides of each rectangle becomes infinitely long, we might naturally expect to find in the general case alternative conditions of critical equilibrium analogous with that of §§ 6, 7.

But on trial it will be found impossible to satisfy the differential equation and boundary conditions by any simple solution similar to that given by (31), and involving *both* the quadratic factors (28) and

(29). I have tried assuming various simple expressions for the displacement, and, after examining them at considerable length, find that they all either lead to incompatible conditions or to solutions identical with that found above. It will be sufficient for our present purpose to state that such is the case, and that the investigation leads to no interesting results such as would warrant its being treated in detail.

There is, however, an important difference between the two problems, which is sufficient to account for the present discrepancy. In an infinite strip the displacement may be non-periodic along its length, and this is what we actually find in § 6, while in a series of finite rectangles the displacement is necessarily periodic.

Summary.

15. The present investigations are chiefly interesting as forming an addition to the small and curious class of problems in which the question of stability arises in connexion with the theory of elasticity. A rough and ready illustration of the results here arrived at may be readily obtained by causing a strip of paper to wrinkle either by wetting it or otherwise. By suitably straining the paper it is easy to vary the direction in which the wrinkles cross its breadth.

Another illustration is afforded when a flat leaden house-top is exposed to strong sunshine, which causes the lead to swell up and assume an undulating form. Moreover, certain structures in which plating attached to parallel ribs is used in the construction of ships have been known to buckle, and considerable damage has resulted. This danger is now effectually obviated by attaching a flange to the plating.

14. A more elaborate treatment of the possibility of alternative conditions of stability in rectangularly supported plating under the distributions of thrust considered in the present paper might perhaps be given. But it is doubtful whether the results arrived at would be of sufficient importance to justify a fuller investigation than that given above, for they would certainly be very complicated, and such problems soon lose interest when their results do not admit of simple interpretation.

Some Properties of the Uninodal Quartic. *By* W. R. WESTROPP ROBERTS, M.A. Read February 8th, 1894. Received April 10th, 1894.

1. I propose in this paper to investigate some properties of the uninodal quartic which are, perhaps, more readily discovered and perceived by the application of Abelian integrals and functions to this curve than by other methods.

The uninodal quartic belongs to that great and important class of curves the coordinates of which can be expressed in terms of Abelian functions. To this class belongs the cubic whose coordinates can be expressed in terms of elliptic integrals.

Let A, B, and C be three binary forms in x, y, A being a quadratic, B a cubic, and C a quartic; it is obvious then that the equation

$$Az^2 - 2Bz + C = 0$$

represents a uninodal quartic, having its node, which we shall denote by O, at the point x, y; z being a line which passes through the points in which the four lines whose equation is $C = 0$ meet the curve.

We shall now express the coordinates of a point x_1, y_1, z_1 on the curve in terms of a parameter θ. Let us seek the points in which the line $x = \theta y$ meets the curve.

Introducing this value of x into the equation of the curve, we find, after dividing by y^2,

$$\bar{A}z^2 - 2\bar{B}zy + \bar{C}y^2 = 0,$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ are respectively quadratic, cubic, and quartic functions of θ; in fact, $\bar{A}$ is what A becomes when we put $x = \theta$ and $y = 1$, and, similarly, $\bar{B}$ and $\bar{C}$ are what B and C become for the same values of x and y.

Solving the above quadratic, it is plain we may write

$$\rho x_1 = \bar{A}\theta,$$

$$\rho y_1 = \bar{A},$$

$$\rho z_1 = \bar{B} \pm \sqrt{R},$$

where $$R \equiv B^2 - AC.$$

If we now multiply each of the above equations by y^2, and substitute for θ its value $\frac{x}{y}$, it is easy to see we may write

$$\rho_1 x_1 = Ax, \qquad \rho_1 x_2 = Ax,$$
$$\rho_1 y_1 = Ay, \qquad \rho_1 y_2 = Ay,$$
$$\rho_1 z_1 = B + \sqrt{R}, \quad \rho_1 z_2 = B - \sqrt{R},$$

equations which determine the two points in which the line through the node whose equation is $x = \theta y$ meets the curve. I call such points *corresponding points*.

2. A line drawn through O, the node, meets the curve in corresponding points, and it is clear from geometrical considerations that these points will coincide when the line touches the curve, and, since the class of the curve is 10, it follows that six tangents can be drawn from O to the curve. There are, consequently, six points which coincide with their corresponding points. The same conclusion is arrived at by equating the values of z_1 and z_2 in the last article, when we find

$$B + \sqrt{R} = B - \sqrt{R},$$

or

$$R = 0.$$

We shall call the roots of $R = 0$, $a_1, a_2, a_3, a_4, a_5, a_6$, and sometimes write it in the form

$$R \equiv D_1 D_2 D_3,$$

D_1, D_2, D_3 being three quadratics whose roots are a_1, a_2; a_3, a_4; a_5, a_6.

The six points of contact of tangents from O to the curve, we shall refer to as the R points.

3. Let

$$z = lx + my$$

be the equation of any line, and let us seek the equation which determines the parameters of the four points of section.

Substituting for z, its value $lx + my$, in the equation of the curve, we obtain, as the equation of the four lines joining the four points of section to O,

$$A\,(lx+my)^2 - 2B\,(lx+my) + C = 0\,;$$

consequently the four θ's, which we shall call $\theta_1, \theta_2, \theta_3, \theta_4$, of the points of section are found from the equation

$$\phi\,(\theta) \equiv \bar{A}\,(l\theta+m)^2 - 2\bar{B}\,(l\theta+m) + \bar{C} = 0.$$

If we now seek the change in θ consequent on changes dl and dm in l and m respectively, we obtain, by differentiating the above equation,

$$\frac{d\phi(\theta)}{d\theta}\,d\theta+2\{\bar{A}(l\theta+m)-\bar{B}\}(\theta dl+dm)=0.$$

Now, $$\{\bar{A}(l\theta_1+m)-\bar{B}\}^2=\bar{R}_1,$$

and $$\frac{d\phi(\theta_1)}{d\theta_1}=M\phi'(\theta_1),$$

where M is a function of l and m; hence we find easily

$$\frac{d\theta_1}{\sqrt{R_1}}+\frac{2(\theta_1 dl+dm)}{M\phi'(\theta_1)}=0,$$

$$\frac{d\theta_2}{\sqrt{R_2}}+\frac{2(\theta_2 dl+dm)}{M\phi'(\theta_2)}=0,$$

$$\frac{d\theta_3}{\sqrt{R_3}}+\frac{2(\theta_3 dl+dm)}{M\phi'(\theta_3)}=0,$$

$$\frac{d\theta_4}{\sqrt{R_4}}+\frac{2(\theta_4 dl+dm)}{M\phi'(\theta_4)}=0.$$

Now, by the theory of partial fractions,

$$\Sigma\frac{1}{\phi'(\theta_1)}=0,\quad \Sigma\frac{\theta_1}{\phi'(\theta_1)}=0,$$

and $$\Sigma\frac{\theta_1^2}{\phi'(\theta_1)}=0,$$

Σ denoting summation from θ_1 to θ_4; consequently, if we add these equations, we obtain

$$\Sigma\frac{d\theta}{\sqrt{R}}=0,$$

and, if we multiply each equation by the corresponding value of θ, and add, we obtain

$$\Sigma\frac{\theta d\theta}{\sqrt{R}}=0;$$

consequently, if we write

$$a_1=\int^{\theta_1}\frac{d\theta}{\sqrt{R}},\quad a_2=\int^{\theta_2}\frac{d\theta}{\sqrt{R}},\quad \&c.,$$

$$b=\int^{\theta_1}\frac{\theta d\theta}{\sqrt{R}},\quad b_2=\int^{\theta_2}\frac{\theta d\theta}{\sqrt{R}},\quad \&c.$$

We infer, by integrating the two preceding equations, that any line meets the curve in four points, such that

$$\begin{cases} \Sigma a = c_1, \\ \Sigma b = c_2, \end{cases}$$

c_1 and c_2 being constants.

4. We shall now determine the values of c_1 and c_2. We know that a double point on a curve has *two* parameters, one corresponding to each branch of the curve passing through the double point. The parameters of O are obviously the roots of $A = 0$, which we shall call n_1 and n_2. The constants c_1 and c_2 being the same for all lines, their values will remain unaltered if we suppose the line to be one of the tangents drawn through O to the curve, and touching it in the point whose parameter is a_1, say; we have then

$$c_1 = \int^{n_1} \frac{d\theta}{\sqrt{R}} + \int^{n_2} \frac{d\theta}{\sqrt{R}} + 2\int^{a_1} \frac{d\theta}{\sqrt{R}},$$

$$c_2 = \int^{n_1} \frac{\theta\, d\theta}{\sqrt{R}} + \int^{n_2} \frac{\theta\, d\theta}{\sqrt{R}} + 2\int^{a_1} \frac{\theta\, d\theta}{\sqrt{R}};$$

now, if we call

$$\int^{n_1} \frac{d\theta}{\sqrt{R}} + \int^{n_2} \frac{d\theta}{\sqrt{R}} = N_0,$$

$$\int^{n_1} \frac{\theta\, d\theta}{\sqrt{R}} + \int^{n_2} \frac{\theta\, d\theta}{\sqrt{R}} = N_1,$$

$$\int^{a_1} \frac{d\theta}{\sqrt{R}} = I_0(a_1),$$

$$\int^{a_1} \frac{\theta\, d\theta}{\sqrt{R}} = I_1(a_1),$$

we find

$$c_1 = N_1 + 2I_0(a_1),$$

$$c_2 = N_2 + 2I_1(a_1).$$

We know, however, from the theory of Abelian integrals that, if we add to the right-hand sides of the following transcendental equations

$$\Sigma a = N_1 + 2I_0(a_1),$$

$$\Sigma b = N_2 + 2I_1(a_1),$$

the terms

$$2\{I_0(a_2) - I_0(a_1)\},$$

and

$$2\{I_1(a_2) - I_1(a_1)\},$$

respectively, the algebraic relations connecting the superior limits of the integrals are unaltered.

Hence we might equally write

$$c_1 = N_1 + 2I_0(\alpha_2),$$

$$c_2 = N_2 + 2I_1(\alpha_2),$$

and, consequently, it is indifferent what root of $R = 0$ we may choose.

Now, let us take the inferior limit of each integral equal to a roots α of $R = 0$, and write

$$a = \int_\alpha^\theta \frac{d\theta}{\sqrt{R}}, \quad b = \int_\alpha^\theta \frac{\theta\, d\theta}{\sqrt{R}},$$

and

$$N_0 = \int_\alpha^{x_1} \frac{d\theta}{\sqrt{R}} + \int_\alpha^{x_2} \frac{d\theta}{\sqrt{R}},$$

$$N_1 = \int_\alpha^{x_1} \frac{\theta\, d\theta}{\sqrt{R}} + \int_\alpha^{x_2} \frac{\theta\, d\theta}{\sqrt{R}},$$

$$I_0(\alpha_r) = \int_\alpha^{\alpha_r} \frac{d\theta}{\sqrt{R}}, \quad I_1(\alpha_r) = \int_\alpha^{\alpha_r} \frac{\theta\, d\theta}{\sqrt{R}};$$

a and b might be called the arguments of the point whose parameter is θ.

We arrive then at the following theorem that any line meets the curve in four points, so that

$$\Sigma a = N_0,$$

$$\Sigma b = N_1.$$

If the line pass through the node meeting the curve in corresponding points x_1 and x_2, we have, obviously,

$$a_1 + a_2 = 0,$$

$$b_1 + b_2 = 0.$$

5. We shall now discuss *pairs* of points. Any pair of points the parameters of which are θ_1 and θ_2 may be determined in the following manner.

If we write
$$u = \int_a^{\theta_1} \frac{d\theta}{\sqrt{R}} + \int_a^{\theta_2} \frac{d\theta}{\sqrt{R}},$$
$$v = \int_a^{\theta_1} \frac{\theta d\theta}{\sqrt{R}} + \int_a^{\theta_2} \frac{\theta d\theta}{\sqrt{R}},$$
the pair of points whose parameters are θ_1 and θ_2 is completely determined if u and v are known, except when
$$u = 0, \quad v = 0,$$
in which case the line joining them passes through the node.

We might call u and v the *arguments* of the *pair of points*, without, I think, any valid objection, just as we speak of the argument of a point on a cubic curve, the coordinates of which can be expressed in terms of *elliptic* functions.

Let, now, X_1 and X_2 be any pair of points, the arguments of which are u and v, and let the line joining X_1 and X_2 meet the curve in a pair of points Y_1 and Y_2. Then, if we call the arguments of the pair Y_1 and Y_2, Lu and Lv, respectively, we have
$$u + Lu = N_1,$$
$$v + Lv = N_2,$$
therefore
$$Lu = N_1 - u,$$
$$Lv = N_2 - v.$$

If now the lines joining X_1 and X_2 to the node meet the curve in Z_1 and Z_2, we shall have, if we call Cu and Cv the arguments of Z_1 and Z_2,
$$u + Cu = 0 \quad \text{or} \quad Cu = -u,$$
$$v + Cv = 0 \quad \text{or} \quad Cv = -v.$$

Now, the line joining the last pair meets the curve in another pair given by
$$LCu \quad \text{and} \quad LCv;$$
let us now denote this pair by the symbol τu and τv, so that
$$\tau u = LCu = N_1 - Cu = N_1 + u,$$
$$\tau v = LCv = N_2 - Cv = N_2 + v;$$
again, if we denote by σu and σv the arguments of the pair of points
$$CLu, \quad CLv,$$

we shall have
$$\sigma u = CLu = -Lu = u - N_0,$$
$$\sigma v = CLv = -Lv = v - N_1.$$

We have also
$$\tau^2 u = u + 2N_1, \quad \sigma^2 u = u - 2N_0,$$
$$\tau^2 v = v + 2N_2, \quad \sigma^2 v = v - 2N_1,$$
$$\tau\sigma u = \sigma\tau u = u;$$

and, by adding the values of τu and σu, we find
$$\tau u + \sigma u = 2u,$$
$$\tau v + \sigma v = 2v.$$

We shall presently make use of these equations.

6. We shall now seek the relations connecting the points of section of a curve of the m^{th} degree having O for a point of the $m-1$ order. We know, from the theory of curves, that, if a curve have a point of the k^{th} order, the point being given in position, this is equivalent to $\frac{1}{2}k(k+1)$ conditions, and further the point of the k^{th} order counts as $\frac{1}{2}k(k-1)$ double points.

Now, a curve of the m^{th} degree requires $\frac{1}{2}m(m+3)$ conditions to determine it completely, and if it have O as a point of the $m-1^{\text{th}}$ order, this counts as $\frac{1}{2}m(m-1)$ conditions, and the curve will require $2m$ conditions to determine it completely, or, what is the same thing, it will be completely determined if made to pass through $2m$ points, or m pairs of points. We shall call such curves O curves, that is to say, an O curve is a curve of the m^{th} order having O as a point of the $m-1^{\text{th}}$ order.

The deficiency of such curves is obviously zero.

7. The equation of an O curve of the m^{th} degree is easily seen to be of the form
$$az - b = 0,$$
where a and b are binary forms of the $m-1^{\text{th}}$ and m^{th} degree in x, y, respectively.

By reasoning precisely similar to that which we employed in investigating the transcendental relations connecting the θ's of the points of section of a line with the curve, we find that an O curve of the m^{th} degree meets the quartic in $2(m+1)$ points, such that
$$\Sigma a = c_1,$$
$$\Sigma b = c_2.$$

Now, an O curve may be replaced by $m-1$ lines through O, and a line not passing through O; hence it is easy to conclude that

$$\Sigma a = N_0,$$

$$\Sigma b = N_1.$$

We now proceed to state a theory of residuation analogous to Sylvester's theory of residuation for the cubic.

If two systems of *pairs of points* α, β make up the complete intersection of an O curve of the m^{th} degree with the quartic, I call one of these systems the residual of the other.

Since through the system α an infinity of O curves can be drawn, it follows that the system α has an infinite number of residuals.

Again, in conformity with Sylvester's nomenclature, I call two systems of pairs of points β_1 and β_2 *coresidual*, if both are the residuals of the same system α.

Let us now denote the arguments of the several pairs of points in the system α by u_1, v_1; u_2, v_2, and let

$$\alpha = \Sigma u,$$

$$\alpha' = \Sigma v,$$

and, in like manner, let β denote the sum of the u arguments, and β' the sum of the v arguments of the several pairs of points in the system β; then, if α and β are residuals, we must have

$$\alpha + \beta = N_0,$$

$$\alpha' + \beta' = N_1.$$

8. If two systems β_1 and β_2 are coresidual, any system γ which is a residual of β_1 will be a residual of β_2.

We have

$$\begin{cases} \alpha + \beta_1 = N_0, \\ \alpha + \beta_1' = N_1, \end{cases} \quad \begin{cases} \alpha + \beta_2 = N_0, \\ \alpha' + \beta_2' = N_0, \end{cases} \quad \begin{cases} \gamma + \beta_1 = N_0, \\ \gamma' + \beta_1' = N_1; \end{cases}$$

hence

$$\gamma + \beta_2 = N_0,$$

$$\gamma' + \beta_2' = N_1,$$

which proves the theorem.

We see, then, that the conditions that two systems β_1 and β_2 should be coresidual are the following:—

$$\begin{cases} \beta_1 = \beta_2, \\ \beta_1' = \beta_2', \end{cases}$$

from which we infer that two pairs of points which are coresidual must coincide.

If we took a system α of an even number of points $2m$, and through α described an O curve of the degree m, which will be completely determined by the passage through the system α, this curve would meet the quartic in a pair of points u_1, v_1, the residual of the system α, and if through the pair u_1, v_1 we draw a line to meet the curve in u_2, v_2, this pair would be the coresidual of the system α.

Now, instead of proceeding from the system α to the residual u_1, v_1 by one stage, and to the coresidual u_2, v_2, by two stages, we might employ any *odd* number of stages in the first case, and any *even* number in the second.

We shall now prove that the pair of points we ultimately arrive at after any odd number of stages is in all cases the same, namely, the pair u_1, v_1, and that the pair arrived at after any even number of stages is always the pair u_2, v_2.

We have
$$\begin{cases} u_1 + \alpha = N_0, \\ v_1 + \alpha' = N_1, \end{cases}$$

two equations which determine the pair of points u_1, v_1, the residual pair.

If, now, through the $2m$ points of the system α we describe an O curve of the degree $m+p$, meeting the quartic in a residual system β of $2p+2$ points, and through the system β we describe an O curve of the degree $p+q$ meeting the quartic in a system γ of $2q$ points, and finally through the $2q$ points of the γ system we describe an O curve of the degree q, this curve will meet the quartic in a pair of points identical with the pair u_1, v_1.

For
$$\begin{cases} \alpha + \beta = N_0, \\ \alpha' + \beta' = N_1, \end{cases} \qquad \begin{cases} \beta + \gamma = N_0, \\ \beta' + \gamma' = N_1. \end{cases}$$

Consequently,
$$\begin{cases} \gamma + u_1 = N_0, \\ \gamma' + v_1 = N_1. \end{cases}$$

It is obvious then, without any further proof, that the pair of points arrived at by any odd number of stages is the pair u_1, v_1, and by similar reasoning the pair arrived at by any even number of stages is the pair u_2, v_2, where

$$\begin{cases} u_2 = a, \\ v_2 = a', \end{cases}$$

two equations determining the pair coresidual to the system a.

9. We now discuss some special forms of the equation of the curve.

Suppose the equation given in the form

$$Az^2 - 2Bz + C = 0;$$

we find, if we put $z = z_1 + f$,

f being a line through the node,

$$Az_1^2 - 2(B - Af) z_1 + C - 2Bf + Af^2 = 0,$$

or

$$Az_1^2 - 2B_1 z_1 + C_1 = 0,$$

where

$$B_1 \equiv B - Af,$$

$$C_1 \equiv C - 2Bf + Af^2,$$

Now, it is clear we can, as we have two constants at our disposal, determine them so that B_1 and C_1 may have a common *quadratic* factor, which we shall call D.

We may write then

$$B_1 = DF,$$

$$C_1 = DH,$$

F being of the first degree, and H of the second in x, y.

The equation which gives the R points must remain unaltered, and consequently we must have

$$B^2 - AC = B_1^2 - AC_1 = D^2F^2 - ADH = D(DF^2 - AH);$$

hence D is a quadratic factor of R, and the equation of the curve could be written

$$Az^2 - 2DFz + DH = 0.$$

10. The equation of the curve being given in the form

$$Az^2 - 2Bz + C = 0,$$

let us now see how we are to reduce it to the form

$$Az^2 - 2DFz + DH = 0.$$

Let us suppose the sextic $B^2 - AC$ resolved into three quadratics, so that

$$R \equiv D_1 D_2 D_3;$$

then, if we write $$z = z_1 + f,$$

we have, as before,

$$Az_1^2 - 2(B - Af)z_1 + C - 2Bf + Af^2 = 0.$$

We may now write $$B - Af \equiv D_1 F_1 \quad \text{.......................(1)},$$

$$C - 2Bf + Af^2 \equiv D_1 H_1 \quad \text{.......................(2)},$$

D_1 being a known quadratic factor of $R = 0$. Substituting successively, in the identical equation (1), the roots of $D_1 = 0$, we completely determine f; the expression $B - Af$ will then be found divisible by D_1, and the remaining factor will be F_1; in the same way,

$$C - 2Bf + Af^2$$

will be found divisible by D_1, and the remaining factor is H_1.

Hence, to any pair of roots of $R = 0$, corresponds one, and but one, form of the equation of the curve in which B_1 and C_1 have two roots in common.

And, as there are fifteen ways of arranging the six R points in pairs, there are fifteen ways in which the curve can be reduced to the form

$$Az^2 - 2D_1E_1z + D_1H_1 = 0.$$

In this form z is the line joining two R points.

11. The curve being given by

$$Az^2 - 2Bz + C = 0,$$

by transforming z by the equation

$$z = z_1 + f,$$

we have, as before,

$$B_1 = B - Af, \qquad C_1 = C - 2Bf + Af^2;$$

having two constants in f, we could determine f so that

$$C_1 \equiv AQ,$$

Q being a quadratic, and it is to be remarked that this can be effected in only *one* way.

The equation of the curve is now

$$Az^2 - 2Bz + AQ = 0,$$

and z is the line joining the points where the tangents at O meet the curve again (I call them the A points); the line joining the A points meets the curve in the points given by the equation $Q = 0$ (I call them the Q points)—a pair of points connected in a remarkable manner with the geometry of the quartic.

12. We must now determine the arguments of the A points, and those of the Q points.

The node is easily seen to correspond to each of the A points; hence we conclude that the arguments of the A points are $-N_0$, $-N_1$, and consequently those of the Q points are $2N_0$, $2N_1$.

The arguments of the six R points are evidently $I_0(a_1)$, $I_1(a_1)$; $I_0(a_2)$, $I_1(a_2)$; and the inferior limit of each integral being some one root of $R = 0$, as in Art. 4.

We are now in a position to determine the residual and coresidual of any four of the R points. Let u, v be the arguments of the residual pair of points of the points $I_0(a_1)$, $I_1(a_1)$; $I_0(a_2)$, $I_1(a_2)$; $I_0(a_3)$, $I_1(a_3)$; $I_0(a_4)$, $I_1(a_4)$; u and v are then found from the equations

$$u + I_0(a_1) + I_0(a_2) + I_0(a_3) + I_0(a_4) = N_0,$$

$$v + I_1(a_1) + I_1(a_2) + I_1(a_3) + I_1(a_4) = N_1.$$

But we know from Jacobi's theory that we may write

$$I_0(a_1) + I_0(a_2) + I_0(a_3) = I_0(a_4) + I_0(a_5) + I_0(a_6),$$

$$I_1(a_1) + I_1(a_2) + I_1(a_3) = I_1(a_4) + I_1(a_5) + I_1(a_6);$$

hence, easily,

$$u + I_0(a_5) + I_0(a_6) = N_0,$$

$$v + I_1(a_5) + I_1(a_6) = N_1,$$

from which we infer that, if an O conic be drawn through any four of the R points, it meets the curve in a pair of points which is the residual pair of the remaining two R points.

13. Any conic drawn through the Q points and the node meets the curve in four points whose corresponding points lie on a line.

This theorem is easily proved as follows: let u_1, v_1; u_2, v_2 be the arguments of two pairs of points in which the O conic through the Q points meets the curve. We have then

$$\begin{cases} u_1+u_2+Q=N_0, \\ v_1+v_2+Q'=N_1. \end{cases}$$

Now $$Q=2N_0, \quad \text{and} \quad Q'=2N_1.$$

Consequently $$\begin{cases} -u_1-u_2=N_0, \\ -v_1-v_2=N_1, \end{cases}$$

which proves the theorem, the converse of which is also true.

14. Given the curve, to determine the Q points.

By the aid of the theorem of the last article we can find the Q points by drawing any line meeting the curve in four points. By means of the ruler alone we determine their four corresponding points, and through these latter points we describe an O conic which meets the curve in the Q points.

Since we now know the Q points, we can draw the tangents at the node O. The line joining the Q points meets the curve in the A points, the lines joining which to O are the tangents at the node.

15. If any line be drawn through one of the Q points, Q_1, meeting the curve in three other points whose arguments are a_1, b_1; a_2, b_2; a_3, b_3, then their three corresponding points lie on a line which passes through the other Q point, Q_2.

We have $$\begin{cases} a_1+a_2+a_3+Q_1=N_0, \\ b_1+b_2+b_3+Q_1'=N_1, \end{cases}$$

and, since $$Q_1+Q_2=2N_0,$$
$$Q_1'+Q_2'=2N_1,$$

we find $$\begin{cases} -a_1-a_2-a_3+Q_2=N_0, \\ -b_1-b_2-b_3+Q_2'=N_1, \end{cases}$$

which proves the theorem.

On account of the importance of this theorem we give another proof.

Let the factors of Q be q_1 and q_2, so that

$$Q \equiv q_1 q_2,$$

and let us seek the equation which determines the points in which a line $z = \lambda q_1$, drawn through one of the Q points, meets the curve. Substituting this value of z in the equation

$$Az^2 - 2Bz + AQ = 0,$$

we find $$A\lambda^2 q_1^2 - 2B\lambda q_1 + Aq_1 q_2 = 0,$$

or, dividing by λq_1, $$A(\lambda q_1 + \lambda^{-1} q_2) - 2B = 0.$$

Now this is the same equation we should find to determine the points in which a line

$$z = \lambda^{-1} q_2$$

meets the curve; hence it follows that the lines

$$z - \lambda q_1 = 0 \quad \text{and} \quad z - \lambda^{-1} q_2 = 0$$

meet the curve in points which correspond.

Now these two lines pass one through one of the Q points, and the other through the remaining Q point, and are obviously connected by a 1, 1 relation, and the locus of their intersection is the conic whose equation is

$$z^2 = Q,$$

which we shall call the Q conic.

16. The six R points lie on the Q conic. For, if a line be drawn through the node meeting the curve in two corresponding points X_1, X_2, the lines Q_1X_1, Q_2X_2 intersect on the Q conic; consequently the Q conic must meet the curve in points which coincide with their corresponding points or the points of contact of the tangents from the node to the curve.

We now show how to construct geometrically the Q conic, and consequently to determine geometrically the six R points.

We have shown already how to determine the Q points; consequently, if we draw any three lines through one of the Q points, and through the other Q point the three lines which correspond to them, we determine by their intersections three points, which, with the Q points, enable us to determine the Q conic completely.

$\therefore$ The R points are consequently found by describing the Q conic as above indicated.

17. To draw a tangent to the curve at a given point P.

Let the arguments of P be a, b. Join P to Q_1 and Q_2 by lines PQ_1, PQ_2, meeting the curve in pairs of points whose arguments are respectively u_1, v_1; u_2, v_2.

We have then
$$\left.\begin{aligned} a+Q_1+u_1 &= N_0 \\ b+Q_1'+v_1 &= N_1 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\left.\begin{aligned} a+Q_2+u_2 &= N_0 \\ b+Q_2'+v_2 &= N_1 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from which we find, by the addition of the first equations in (1) and (2),
$$2a+Q_1+Q_2+u_1+u_2 = 2N_0,$$

or, since
$$Q_1+Q_2 = Q = 2N_1,$$
$$2a+u_1+u_2 = 0;$$

similarly,
$$2b+v_1+v_2 = 0,$$

or
$$\begin{cases} -u_1-u_2+(N_0-2a) = N_0, \\ -v_1-v_2+(N_1-2b) = N_1, \end{cases}$$

the signification of which is that the O conic through the pairs corresponding to u_1, v_1; u_2, v_2 passes through the residual of the pair of consecutive points at P.

Hence we have the following construction for drawing the tangent at P.

Through P draw PQ_1 and PQ_2, meeting the curve in points A_1, B_1 and A_2, B_2, respectively. Through the corresponding points of A_1, B_1; A_2, B_2 describe an O conic, meeting the curve in a pair of points X, Y.

The line joining XY touches the curve at P.

18. To draw the tangents at the Q points.

It is clear that the construction given in the last article becomes illusory when the point at which the tangent is to be drawn is one or other of the Q points.

It remains then to show how to draw the tangents at these points.

We know from Article 15 that, if a line be drawn through Q_1 meeting the curve in three points A_1, B_1, C_1, their three corresponding points A_2, B_2, C_2 lie on a line which passes through Q_2.

The line joining Q_2 to the corresponding point of Q_1 consequently meets the curve in two points such that the line joining their corresponding points touches the curve at Q_1.

The tangent at Q_2 is, of course, drawn by a similar construction.

19. If a line be drawn through one of the points of contact of tangents from the node to the curve, the three pairs of points in which the tangents at the points of section meet the curve lie on a conic which passes through the node.

Let a_1, b_1; a_2, b_2; a_3, b_3 be the arguments of the three points in which the line drawn through the R point meets the curve; also let $I_0(a_1)$, $I_1(a_1)$ be the arguments of this point. We have then

$$I_0(a_1)+a_1+a_2+a_3 = N_0,$$

$$I_1(a_1)+b_1+b_2+b_3 = N_1,$$

and, if u_1, v_1; u_2, v_2; u_3, v_3 be the arguments of the pairs of points in which the tangents meet the curve again, we have, for the determination of u_1 and v_1, the following equations:

$$2a_1+u_1 = N_0,$$

$$2b_1+v_1 = N_1,$$

and similar equations for u_2, v_2; u_3, v_3.

From these equations we readily find

$$u_1+u_2+u_3 = N_0,$$

$$v_1+v_2+v_3 = N_1,$$

equations which show that the pairs u_1, v_1; u_2, v_2; u_3, v_3 lie on an O conic.

20. If a line be drawn through one of the Q points Q_1 to touch the curve in a point P_1, then the tangent at P_2, the point corresponding to P_1, will pass through Q_2, and the correspondence between the lines Q_1P_1 and Q_2P_2 will be of the kind pointed out in Article 15.

Now eight tangents can be drawn from each of the Q points to the curve, and to each tangent from Q_1, such as Q_1P_1, corresponds a tangent from Q_2, namely, Q_2P_2, so that the anharmonic ratio of any four of the tangents from Q_1 is equal to that of the four corresponding tangents through Q_2.

The eight points of intersection, then, of the tangents from Q_1 with the *corresponding* tangents from Q_2 lie on a conic which passes through Q_1 and Q_2.

Again, remembering that the anharmonic ratio of four points 1, 2, 3, 4 is unaltered by writing them in the order 2, 3, 4, 1, or 3, 4, 1, 2, or 4, 1, 2, 3, it is easy to conclude that the sixty-four points of intersection of the first set of tangents with the second set lie by fours on sixteen conics which pass through Q_1 and Q_2.

If we now suppose the curve circular, and the Q points the imaginary circular points at infinity, it follows that in this case the sixty-four foci arising from the intersection of the eight tangents from Q_1 with the eight from Q_2 can be arranged in fours on sixteen circles.

21. If a, b be the arguments of a point of inflexion, and c, d those of the point in which the inflexion tangent meets the curve, we have

$$3a+c = N_0,$$

$$3b+d = N_1.$$

We shall now show that, if an O cubic be drawn through two points of inflexion on the curve, and having the same points as points of inflexion with the same inflexional tangents, it will meet the curve again in two points whose corresponding points and the points in which the inflexion tangents of the quartic meet the curve again lie on a line.

Let a_1, b_1; c_1, d_1 be the arguments of a point of inflexion and those of the point in which the inflexion tangent meets the curve, a_2, b_2; c_2, d_2 the arguments of a second point of inflexion and those of the point in which the second inflexion tangent meets the curve; we have then

$$3a_1+c_1 = N_0, \qquad 3a_2+c_2 = N_0,$$

$$3b_1+d_1 = N_1, \qquad 3b_2+d_2 = N_1.$$

Also, for the points in which the cubic meets the curve, if u, v be the arguments of the remaining pair of points in which it intersects the quartic,

$$3a_1+3a_2+u = N_0,$$

$$3b_1+3b_2+v = N_1;$$

consequently,

$$-u+c_1+c_2 = N_0,$$

$$-v+d_1+d_2 = N_1,$$

which proves the theorem.

The problem of finding the inflexions on the curve is evidently identical with that of finding the conics which can be drawn through the node and the Q points to have contact of the second order with the curve.

I am obliged to omit further discussion of the inflexions at present, owing to the length and complexity of the equations which determine their positions.

22. We now proceed to the discussion of the double tangents.

If u and v be the arguments of a pair of points of contact of a double tangent, we have, for the determination of u and v, the equations

$$2u = N_0,$$
$$2v = N_1.$$

There being but four independent periods of the form $2I_0(a)$, and four corresponding periods of the form $2I_1(a)$, there will be consequently sixteen solutions of the above equations. We shall, however, find it more convenient to work with the six pairs of integrals of the form $I_0(a)$, $I_1(a)$, one pair of which will consist of integrals in which the inferior limit equals the superior, as in all cases we suppose the inferior limit some one value of a.

It will not now be difficult to see that the sixteen values of u and v may be written as coming under one or other of the following forms :

$$\left.\begin{aligned} u &= \frac{N_0}{2} \\ v &= \frac{N_1}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

or

$$\left.\begin{aligned} u &= \frac{N_0}{2} + I_0(a_r) + I_0(a_s) \\ v &= \frac{N_1}{2} + I_1(a_r) + I_1(a_s) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

there being fifteen forms included in the forms of u and v given in (2) ; a_r and a_s being any two of the six roots of $R = 0$; these fifteen added to the values given in (1) make up the sixteen values of u and v.

23. We now arrange the sixteen bitangents in eight pairs which have a special relation to a pair of the R points, say the pair whose arguments are

$$I_0(a_1) + I_0(a_2), \qquad I_1(a_1) + I_1(a_2).$$

Let us write

$$\text{I.} \qquad\qquad\qquad\qquad \text{II.}$$

$$u-\frac{N_0}{2}=\begin{cases} 1.\ I_0(a_1)+I_0(a_3), \\ 2.\ I_0(a_1)+I_0(a_4), \\ 3.\ I_0(a_1)+I_0(a_5), \\ 4.\ I_0(a_1)+I_0(a_6), \\ 5.\ I_0(a_3)+I_0(a_4), \\ 6.\ I_0(a_3)+I_0(a_5), \\ 7.\ I_0(a_3)+I_0(a_6), \\ 8.\ \quad 0, \end{cases} \qquad u-\frac{N_0}{2}=\begin{cases} 1.\ I_0(a_2)+I_0(a_3), \\ 2.\ I_0(a_2)+I_0(a_4), \\ 3.\ I_0(a_2)+I_0(a_5), \\ 4.\ I_0(a_2)+I_0(a_6), \\ 5.\ I_0(a_6)+I_0(a_3), \\ 6.\ I_0(a_5)+I_0(a_4), \\ 7.\ I_0(a_4)+I_0(a_5), \\ 8.\ I_0(a_1)+I_0(a_2), \end{cases}$$

and similar values for $v-\frac{N_1}{2}$.

Now take a pair of values on the same horizontal line from each of the columns I. and II., and suppose we commence with the first pair; write

$$u_1-\frac{N_0}{2}=I_0(a_1)+I_0(a_3),$$

$$u_1'-\frac{N_0}{2}=I_0(a_2)+I_0(a_3),$$

and similar values for v_1 and v_1'; then we find, by adding these equations,

$$u_1+u_1'-N_0=I_0(a_1)+I_0(a_2)+2I_0(a_3),$$

and, since $2I_0(a_3)$ is a period, we may neglect this term and write

$$u_1+u_1'+\{I_0(a_1)+I_0(a_2)\}=N_0,$$

$$v_1+v_1'+\{I_1(a_1)+I_1(a_2)\}=N_1.$$

Consequently an O conic passes through the points R_1 and R_2, and through the points of contact of two bitangents. It follows, then, that through each pair of R points R_r and R_s, eight conics can be drawn passing through the points of contact of two bitangents and the node. If we call a conic passing through the node and through the points R_r and R_s and the points of contact of two bitangents a conic belonging to the R_{rs} system of conics, it is clear there are fifteen such systems, and that through the points of contact of any one bitangent can be drawn fifteen conics, one from each system, each conic passing through two R points and the points of contact of another bitangent and the node.

24. We shall now examine more closely the nature of the conics of these systems. Let us write the equation of the curve in the form discussed in Article 10, viz.,

$$Az^2-2DFz+DH=0,$$

and let us suppose the lines through the node whose equation is $D=0$ touch the curve at the points R_1 and R_2.

We know that there are two values of k for which $D+kA$ becomes a perfect square; let us call these values k and k', and write

$$D+kA \equiv f^2,$$

$$D+k'A \equiv f'^2.$$

If we now multiply the above equation of the curve by k, and replace kA by its value f^2-D, we find

$$z^2(f^2-D)-2zkDF+kDH=0,$$

which we can write in the form

$$(fz-D)^2 = D\left\{z^2-2z(f-kF)+D-kH\right\},$$

showing that the curve is the envelope of the conic

$$\left\{z^2-2z(f-kF)+D-kH\right\}-2(\rho-1)z(f-kF)+D(\rho-1)^2=0,$$

ρ being a variable parameter, or

$$z^2-2z(\rho f-kF)+D\rho^2-kH=0;$$

and this conic touches the curve in four points where it is met by the conic

$$zf=\rho D.$$

The four points of contact of the enveloping conic consequently lie on a conic which passes through the points R_1 and R_2, and touches the known line whose equation is $f=0$ at the node.

Now, four conics of the system

$$z^2-2z(\rho f-kF)+D\rho^2-kH=0$$

reduce to a pair of right lines, for the discriminant of this conic involves ρ in the fourth degree.

It is patent that, when an enveloping conic breaks up into a pair of right lines, the four points of contact lie two on each line, and each line is consequently a bitangent, the four points of contact lying on the conic

$$zf=\rho D,$$

where ρ is now determined by the condition that the discriminant of the enveloping conic vanishes. There are obviously four other conics corresponding to the line whose equation is $f' = 0$.

Consequently the eight conics of the system R_{12} consist of four which touch $f = 0$ at the node and four which touch $f' = 0$ at the same point.

25. Let us now denote by

$$D_{rs} = 0$$

the equation of the tangents at the points R_r and R_s, which, of course, pass through the node, and let us write

$$f_{rs}^2 \equiv k_{rs} A + D_{rs},$$

$$f_{rs}'^2 \equiv k_{rs}' A + D_{rs},$$

and call F_{rs} and H_{rs} the corresponding forms of F and H. We can construct by elementary geometry all the lines

$$f_{rs} \text{ and } f_{rs}';$$

consequently we can determine geometrically the four points of contact of any enveloping conic by describing through an arbitrary point and the points R_r and R_s a conic touching the line f_{rs} at the node, for, by what we have already proved, this conic meets the curve in the desired points.

26. We are now in a position to determine geometrically the bitangents of the quartic.

Let us call the systems of conics which pass through the points R_r and R_s and touch the line

$$f_{rs} = 0$$

at the node the system (rs).

Now, from Article 23, a conic from each of the systems R_{12}, R_{13}, R_{14}, R_{15} will pass through the points of contact of a bitangent, and, as the eight conics of the system R_{12} are each of them conics of the system (12), it follows that the problem of determining the bitangents is reduced to that of finding the pair of points common to conics, one from each of the systems (12), (13), (14), and (15). I give Mr. Russell's solution of this latter problem, as being more elegant than my own. Take a particular conic of the system (12) which will pass through R_1, R_2 and touch f_{12} at the node O.

A variable conic of the system (13) will intersect this conic in a chord which passes through a point on OR_3 which we shall call 123. This point 123, of course, varies with the conic of the system (12). Similarly, conics of the system (14) intersect the particular conic of the system (12) in chords which pass through a point 124, and conics of the system (15) intersect the same conics in chords passing through a point 125. If the points 123, 124, 125 are collinear, the problem is solved, and this line intersects the particular conic of the system (12) in the required points. If the three points are not collinear, we proceed as follows.

As the conic of the system (12) varies, the points 123, 124, and 125 determine three homographic systems on the fixed lines OR_3, OR_4, and OR_5, and the three coincide at O. Hence the line joining 123 to 124 passes through a fixed point, and, in like manner, the line joining 123 to 125 passes through a fixed point. The line joining these two centres of perspective is the desired line of collinearity. It is obvious that three particular conics of the system 12 are necessary in order to determine the homographic systems.

These may be taken as follows:—

(i.) The lines R_1R_2, f_{12}.

(ii.) The lines OR_1, OR_2.

(iii.) Any conic of the system (12).

Having found the line of collinearity, it is necessary to determine the points on it. This is easily done. It reduces to simultaneous harmonic division of two segments on the same line.

There are other interesting theorems in connexion with the points of contact of the bitangents, but, as I have already exceeded the contemplated length of this paper, I shall defer their communication to another occasion.

On the Existence of a Root of a Rational Integral Equation.
By E. B. ELLIOTT. Read and received March 8th, 1894.

1. It may be that a proof, not depending on the theory of functions of a complex variable, of the theorem that every rational integral algebraic equation has a root is still a desideratum. It is at all events worth while to examine whether such proofs as have been given are or can be made sound. I have recently been studying two simple apparent proofs depending on the theory of elimination, one by the late Professor Clifford (*Math. Papers*, p. 20; *Camb. Proceedings*, Vol. II.), and one by Mr. J. C. Malet (*Transactions of the Royal Irish Academy*, Vol. XXVI.), and find both to be wanting in completeness. I have also, and I hope successfully, endeavoured to construct a proof of the same character free from the corresponding defects.*

Clifford's method is to show that a quadratic divisor x^2+px+q of

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$$

can be found if a root q exists of a certain equation of degree $\frac{n(n-1)}{2}$, the result of eliminating p between $M=0$ and $N=0$, where $Mx+N$ is the remainder when the n-ic is divided by the quadratic, the corresponding p being the common root of the equations $M=0$, $N=0$, with that value inserted for q in them. Now, n being m times even, *i.e.*, of the form $2^m \times$ an odd number, $\frac{n(n-1)}{2}$ is only $m-1$ times even. Thus the argument is that, if every equation of degree $m-1$ times even has a root, every equation of degree m times even has a pair of roots. Now every equation of odd degree with real coefficients has a root. Hence every equation with real coefficients of degree once even has a root. It is then concluded by induction that every equation whatever has a root.

The successs of the induction is considerably interfered with by the question of imaginary roots and coefficients. There is, however, a far more fundamental objection to the validity of the method. Everything depends on the uniqueness of the value of p found as corresponding to a known q. There is no reason to assume that, when the q-eliminant of M and N vanishes, the G.C.M. of M and N is linear. What is proved as a basis for mathematical induction is at

* *Cf.* Gordan, *Math. Ann.*, x., pp. 573, &c., for a proof in which the essential argument is similar. [*June*, 1894.]

most that an equation of degree $n = 2^m \times$ odd number has a root, if one of degree $2^{m-1} \times$ odd number has one, and if every equation of degree less than $n-1$ has.

The idea that the vanishing of the eliminant of u and v is sufficient to ensure that $u = 0$, $v = 0$ have a common root is one altogether subsequent to and dependent upon that of the fact that an equation has roots. The eliminant approached without previous idea of roots is, as will be seen later, the criterion only for a common factor of unknown degree of u and v. To assume that such a common factor implies a common root or roots is to assume the theorem of which a proof is desired.

2. Mr. Malet's argument is almost identical, though different in analytical form. His method is practically to show that p is determined if an equation of degree $\frac{n(n-1)}{2}$ can be solved, and that x^2, where x is a root, then follows as the common root of two equations. His induction proceeds exactly as Clifford's. His tacit assumption which needs justification is that of the determinateness of a common root of two equations when p is known, just as Clifford's was that of the determinateness of p when q is known. He does not ignore the question of the imaginary.

In the following articles I do not endeavour to perfect either of the two proofs criticised in the form in which it stands, but find it convenient to adopt a somewhat different (and in one respect more cumbrous) analysis leading to the same essential argument.

3. Since the ordinary theory of the order of eliminants is based on the assumption that an equation of the n^{th} degree has n roots, it is in the first place necessary to have a clear idea of what is necessitated by the vanishing of the dialytic determinant of two forms when we are not entitled to make any such assumption.

For simplicity's sake, I write down only

$$\begin{vmatrix} a & b & c & d & e & & \\ & a & b & c & d & e & \\ & & a & b & c & d & e \\ a' & b' & c' & d' & & & \\ & a' & b' & c' & d' & & \\ & & a' & b' & c' & d' & \\ & & & a' & b' & c' & d' \end{vmatrix},$$

the dialytic determinant of the quartic and cubic

$$ax^4+bx^3+cx^2+dx+e,$$

$$a'x^3+b'x^2+c'x+d',$$

the argument for this case applying generally.

It is convenient to consider the determinant from Euler's rather than Sylvester's point of view. Its vanishing necessitates that y_1, y_2, y_3, y_4 and z_1, z_2, z_3, not all zero, exist, such that

$$\begin{aligned}
az_1 &= a'y_1,\\
bz_1+az_2 &= b'y_1+a'y_2,\\
cz_1+bz_2+az_3 &= c'y_1+b'y_2+a'y_3,\\
dz_1+cz_2+bz_3 &= d'y_1+c'y_2+b'y_3+a'y_4,\\
ez_1+dz_2+cz_3 &= d'y_2+c'y_3+b'y_4,\\
ez_2+dz_3 &= d'y_3+c'y_4,\\
ez_3 &= d'y_4,
\end{aligned}$$

i.e., that an auxiliary cubic $y_1x^3+\dots$ and quadratic $z_1x^2+\dots$ exist, such that

$$(z_1x^2+z_2x+z_3)(ax^4+bx^3+cx^2+dx+e)$$

$$\equiv(y_1x^3+y_2x^2+y_3x+y_4)(a'x^3+b'x^2+c'x+d');$$

and this implies that

$$ax^4+bx^3+cx^2+dx+e,$$

$$a'x^3+b'x^2+c'x+d'$$

have a common measure of the first or some higher degree in x.

Thus in all cases the vanishing of the dialytic determinant of two binary forms expresses that those forms have a common factor.

In case the two forms can be expressed as products of linear factors, the product of all differences between roots of the one and roots of the other, made integral by the smallest adequate powers of a and a' as factors, is, by the usual theory, of the same order in the coefficients as the dialytic determinant, and expresses by its vanishing the same property. The two are then in such a case identical.

4. Now consider the dialytic determinant

$$\Delta \equiv \left| \begin{array}{l} a_0\rho^n,\ a_1\rho^{n-1},\ \ldots\ a_{n-1}\rho,\ a_n \\ \qquad a_0\rho^n,\ \ldots\ a_{n-2}\rho^2,\ a_{n-1}\rho,\ a_n \\ \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\ \qquad\qquad\qquad a_0\rho^n,\ a_1\rho^{n-1},\ a_2\rho^{n-2},\ \ldots\ a_n \\ a_n,\ a_{n-1}\rho,\ \ldots\ a_1\rho^{n-1},\ a_0\rho^n \\ \qquad a_n,\ \ldots\ a_2\rho^{n-2},\ a_1\rho^{n-1},\ a_0\rho^n \\ \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\ \qquad\qquad\qquad a_n,\ a_{n-1}\rho,\ a_{n-2}\rho^2,\ \ldots\ a_0\rho^n \end{array} \right|$$

of
$$a_0\rho^n \cdot z^n + a_1\rho^{n-1} \cdot z^{n-1} + \ldots + a_n$$
and
$$a_n \cdot z^n + a_{n-1}\rho \cdot z^{n-1} + \ldots + a_0\rho^n.$$

The condition $\Delta = 0$, if it can be satisfied by a value of ρ, will necessitate that these two forms with that value of ρ inserted in them have a common factor involving z.

Δ is of degree $2n^2$ in ρ. To examine its form, let us for a moment take for $a_0, a_1, \ldots a_n$ the coefficients $a_0', a_1', \ldots a_n'$ in an equation

$$f(x) \equiv a_0'x^n + a_1'x^{n-1} + \ldots + a_n' \equiv a_0'(x-x_1)(x-x_2)\ldots(x-x_n)$$
$$= 0,$$

which has been so formed as to have n roots. This imposes no relation on $a_0', a_1', \ldots a_n'$. Otherwise $a_0', x_1, x_2, \ldots x_n$, of which they are functions, must be connected by a relation, whereas they may be taken absolutely independent of each other. Thus the form of Δ is not affected by the substitutions.*

---

* [This must not be misunderstood. To say that no *relation* connects $a_0', a_1', \ldots a_n'$ is not to say that there is no *restriction* upon the values of those letters. That there is no *restriction* is what we are about to prove.

The distinction may be illustrated by reference to other theories. Thus, for instance, no *relation* connects the coefficients in $ax^2 + 2bx + c = 0$ when its roots, supposed to exist, are real. Otherwise three perfectly arbitrary real quantities, a and x_1, x_2 the two roots, are connected by a relation. But there is a *restriction* on the values of a, b, c; viz., their values must be such that $ac - b^2$ is negative. Any function of a, b, c and other letters, which we may call ρ, σ, τ, ..., will have its algebraical form in all the letters perfectly independent of any such *restriction* on the ranges of values to which we may consider them open, though a *relation* in them might make the form special. The question in the text is one of algebraical form in certain letters, and not of arithmetical form when numbers are substituted for those letters.]

Now, when two equations have numbers of roots indicated by their degrees, their dialytic determinant is the product of all the differences between a root of one and a root of the other, made integral by a product of the lowest adequate powers of their leading coefficients.

The dialytic determinant Δ' of

$$a'_0\rho^n z^n + a'_1\rho^{n-1}z^{n-1} + \dots + a'_n$$

and
$$a'_n z^n + a'_{n-1}\rho z^{n-1} + \dots + a'_0\rho^n$$

is then equivalent, but perhaps for a numerical and sign multiplier, to

$$(a'_0\rho^n)^n\, a'^n_n\, \Pi\left(\frac{x_r}{\rho} - \frac{\rho}{x_s}\right),$$

r and s having separately given them all values from 1 to n inclusive, *i.e.*, to

$$a'^{2n}_0\, \Pi\,(x_r x_s - \rho^2).$$

Now, in this product, a factor $x_r^2 - \rho^2$ in which $r = s$ occurs once, but a factor in which r and s are different occurs twice, once as $x_r x_s - \rho^2$, and once as $x_s x_r - \rho^2$. Thus Δ' is equivalent to

$$a'^n_0\, \Pi\,(x_r^2 - \rho^2)\,\{a'^{n-1}_0\, \Pi\,(x_r x_s - \rho^2)\}^2,$$

i.e., to
$$f(\rho)\, f(-\rho) \times \text{perfect square},$$

the squared function being of degree $\dfrac{n(n-1)}{2}$ in ρ^2, with coefficients which, by the theory of symmetric functions, are rational and integral in $a'_0, a'_1, a'_2, \dots a'_n$.

This form is preserved when for $a'_0, a'_1, \dots a'_n$ are written $a_0, a_1, \dots a_n$, as above explained. Thus a factor of Δ is a rational integral function in ρ^2 of degree $\dfrac{n(n-1)}{2}$, its coefficients being rational integral functions of $a_0, a_1, \dots a_n$.

Consequently, if a certain rational integral equation of degree $\dfrac{n(n-1)}{2}$ in ρ^2 has a root, the two forms

$$a_0\rho^n \cdot z^n + a_1\rho^{n-1} \cdot z^{n-1} + \dots + a_n,$$

$$a_n \cdot z^n + a_{n-1}\rho \cdot z^{n-1} + \dots + a_0\rho^n,$$

with that value of ρ^2 substituted in them, have a common factor.

5. Now, every equation of odd degree with real coefficients has certainly a real root. We proceed to consider an equation,

$$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0,$$

with real coefficients whose degree n is twice an odd number, $= 2(2m+1)$, say. For this $\frac{n(n-1)}{2} = (2m+1)(4m+1)$ is odd.

By the above a real ρ^2, and consequently a real or purely imaginary ρ, exists, which makes

$$a_0\rho^n z^n + a_1\rho^{n-1}z^{n-1} + \ldots + a_n,$$

$$a_n z^n + a_{n-1}\rho z^{n-1} + \ldots + a_0\rho^n,$$

have a common factor. The two cases must be regarded separately.

Firstly, if ρ^2 be positive, and so ρ real, the two expressions in z have real coefficients. Their G.C.M. (proved to exist) has then real coefficients, as the ordinary process for finding it necessitates. Call it $P(z)$, and let $Q(z)$ be the complementary factor of $a_0\rho^n z^n + \ldots$. Then, writing x for ρz, the equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$$

is equivalent to $$P\left(\frac{x}{\rho}\right) Q\left(\frac{x}{\rho}\right) = 0,$$

or, say, to $$P'(x)\, Q'(x) = 0,$$

the coefficients in P' and Q' being all real.

Secondly, if ρ^2 be negative, and so ρ a pure imaginary $r\sqrt{-1}$, the two forms in z may be written, multiplying the second by $(\sqrt{-1})^n$, *i.e.*, by -1, and putting ζ for $z\sqrt{-1}$,

$$a_0r^n\zeta^n + a_1r^{n-1}\zeta^{n-1} + a_2r^{n-2}\zeta^{n-2} + \ldots + a_n,$$

$$a_n\zeta^n - a_{n-1}r\zeta^{n-1} + a_{n-2}r^2\zeta^{n-2} - \ldots + a_0r^n.$$

These two expressions in ζ with real coefficients have a common factor which can be found by the G.C.M. process. Call it $P(\zeta)$ and let $Q(\zeta)$ be the complementary factor of the first form. The coefficients in these are found as real quantities. Thus, writing $\frac{x}{r}$ for ζ, the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

is the same as $$P\left(\frac{x}{r}\right) Q\left(\frac{x}{r}\right) = 0,$$

or, say, as $$P'(x)\, Q'(x) = 0,$$

where the coefficients in the factors are real.

6. Two cases again arise. Either P' may be of degree n, *i.e.*, be the whole form, and Q' a constant, or P' may be of degree between 1 and $n-1$ inclusive.

The former case would mean that the two equations

$$a_0\rho^n z^n + a_1\rho^{n-1}z^{n-1} + \dots + a_n = 0,$$

$$a_n z^n + a_{n-1}\rho z^{n-1} + \dots + a_0\rho^n = 0,$$

or the two equations

$$a_0 r^n \zeta^n + a_1 r^{n-1}\zeta^{n-1} + \dots + a_n = 0,$$

$$a_n \zeta^n - a_{n-1} r\zeta^{n-1} + \dots + a_0 r^n = 0,$$

as the case may be, are identical. This being so, each of the first pair would be

$$a_0\rho^n(z^n+1) + a_1\rho^{n-1}(z^{n-1}+z) + \dots + 2a_{\frac{1}{2}n}z^{\frac{1}{2}n} = 0,$$

which is an equation of degree $\frac{n}{2}$, *i.e.* odd degree, in $z+\frac{1}{z}$; or else each of the second pair would be

$$a_0 r^n(\zeta^n+1) + a_1 r^{n-1}\zeta(\zeta^{n-2}-1) + a_2 r^{n-2}\zeta^2(\zeta^{n-4}+1) + a_3 r^{n-3}\zeta^3(\zeta^{n-6}-1)$$
$$+ \dots + a_{\frac{1}{2}n-1}\zeta^{\frac{1}{2}n-1}(\zeta^2+1),$$

which vanishes when $$\zeta = \pm\sqrt{-1},$$

since n, $n-4$, $n-8$, ... 2 are twice odd numbers, and $n-2$, $n-6$, $n-10$, ... 4 are twice even numbers. In this case our equation of degree n has the roots $\pm r\sqrt{-1}$. In the former case, a real value of $z+\frac{1}{z}$ is given by an equation of odd degree $\frac{1}{2}n$, and consequently two values, real or of the form $\alpha+\beta\sqrt{-1}$, of z, *i.e.*, two roots ρz, real or of that imaginary form, of our equation in x.

In the more general case, P' and Q' have complementary degrees both between 1 and $n-1$ inclusive. Now, these degrees cannot both be divisible by 4. Otherwise their sum n would be, as by supposition it is not. Either then one of the two degrees must be odd, that of Q', say, in which case $Q' = 0$ has a real root, or one at least, that of Q, say again, must be twice an odd number less than the odd number $\frac{1}{2}n$.

Thus an equation $$a_0 x^n + \ldots + a_n = 0,$$

whose coefficients are real and whose degree is twice an odd number $2m+1$, has certainly a root, real or of the form $\alpha+\beta\sqrt{-1}$, if every equation $Q' = 0$ whose coefficients are real and whose degree is twice an odd number less than $2m+1$ has. Now a quadratic, whose degree is twice the smallest odd number, can be solved, its two roots being real or of the form $\alpha+\beta\sqrt{-1}$. Thus induction establishes that every equation with real coefficients whose degree is twice an odd number has a root, real or of the form $\alpha+\beta\sqrt{-1}$.

7. The next step in the argument is to prove that every equation of odd degree whose coefficients are of the form $a+b\sqrt{-1}$ has a root, real or of that form.

If $$f(x) + \sqrt{-1}\,\phi(x) = 0$$

be such an equation, where the coefficients of $f(x)$ and $\phi(x)$ are real, then $$\{f(x) + \sqrt{-1}\,\phi(x)\}\,\{f(x) - \sqrt{-1}\,\phi(x)\} = 0$$

is an equation of degree twice an odd number with real coefficients. It has then a root, real or of the form $\alpha+\beta\sqrt{-1}$. This root must make one of the two factors vanish, for the product of two non-vanishing quantities cannot vanish even though both be of the form $A+B\sqrt{-1}$.

If the root be real and make

$$f(x) - \sqrt{-1}\,\phi(x) = 0,$$

it must make $f(x) = 0$ and $\phi(x) = 0$ separately, and so also be a root of $$f(x) + \sqrt{-1}\,\phi(x) = 0.$$

If, on the other hand, it be of the form $\alpha+\beta\sqrt{-1}$, there must also be a conjugate root $\alpha-\beta\sqrt{-1}$; and if

$$f(x) - \sqrt{-1}\,\phi(x)$$

be the factor which $\alpha+\beta\sqrt{-1}$ makes vanish, then $\alpha-\beta\sqrt{-1}$ satisfies

$$f(x)+\sqrt{-1}\,\phi(x)=0,$$

for, if $$A+B\sqrt{-1}-\sqrt{-1}\,(C+D\sqrt{-1})=0,$$

then $$A-B\sqrt{-1}+\sqrt{-1}\,(C-D\sqrt{-1})=0.$$

Thus every equation of odd degree with coefficients of the form $a+b\sqrt{-1}$ has a root, real or of the form $\alpha+\beta\sqrt{-1}$.

8. Now, let it be assumed that every equation whose degree is $m-1$ or fewer times even, *i.e.*, contains the factor 2, if at all, not more than $m-1$ times, and whose coefficients are real or of the form $a+b\sqrt{-1}$ has a root, real or of the form $\alpha+\beta\sqrt{-1}$.

Consider the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0 \quad\ldots\ldots\ldots\ldots(1),$$

in which $a_0, a_1, \ldots a_n$ are real, or of the form $a+b\sqrt{-1}$, and in which

$$n=2^m\,(2p+1),$$

i.e., is m times even.

By § 4 and our assumption, a value of ρ^2, and consequently two values of ρ, real or of the form $\alpha+\beta\sqrt{-1}$, given as a root of an equation of degree

$$\frac{n\,(n-1)}{2}=2^{m-1}\,(2p+1)\,\{2^m\,(2p+1)-1\},$$

which is only $m-1$ times even, exists, the substitution of which in

$$a_0\rho^n z^n+a_1\rho^{n-1}z^{n-1}+\ldots+a_n,$$

$$a_nz^n+a_{n-1}\rho z^{n-1}+\ldots+a_0\rho^n,$$

makes them have a common factor. This common factor, found by the ordinary G.C.M. process, will have coefficients real or of the form $a+b\sqrt{-1}$.

If this G.C.M. be of degree n, the two expressions are identical, but for a factor free from z, with

$$a_0\rho^n\,(z^n+1)+a_1\rho^{n-1}\,(z^{n-1}+z)+a_2\rho^{n-2}\,(z^{n-2}+z^2)+\ldots,$$

and this equated to zero is an equation of degree $\frac{1}{2}n$, *i.e.*, $m-1$ times even only, in $z+\dfrac{1}{z}$, and is accordingly by our present assumption satisfied by a value of $z+\frac{1}{z}$, real or of the form $a+\beta\sqrt{-1}$, and consequently by two values of z real or of that form. In other words, the equation (1) is satisfied by two values ρz of x, real or of the form $a+\beta\sqrt{-1}$.

On the other hand, if the G.C.M. be of degree less than n, call it $P(z)$. Then the left-hand side of equation (1) has the factor $P\left(\frac{x}{\rho}\right)$, or, say $P'(x)$. Let $Q'(x)$ be the complementary factor. We have thus (1) resolved into

$$P'(x)\ Q'(x) = 0,$$

the coefficients in P' and Q' being real or of the form $a+b\sqrt{-1}$.

Now the degrees of P' and Q' cannot both be divisible by 2^{m+1}, for their sum n is only divisible by 2^m. One or the other of them, the degree of Q', say, must then be m times even at most.

On the assumption therefore that every equation of degree $m-1$ or fewer times even, and that every equation of degree m times even and less than n, has a root, it is proved that any equation of degree n, which is m times even, has a root.

Take now 2^m the smallest number which is m times even. An equation of this degree has, by the same argument, a root if one of degree $2^{m-1}(2^m-1)$, which is $m-1$ times even, has a root, and if every equation $Q'(x) = 0$ of degree less than 2^m has. On our assumption this is the case, no such degree being so many as m times even. We thus proceed to degrees $2^m.3$, $2^m.5$, ... $2^m(2p+1)$, ..., so that the following general statement is accurate.

"If every equation with coefficients real or of the form $a+b\sqrt{-1}$, whose degree is $m-1$ or fewer times even, has a root, real or of the form $a+\beta\sqrt{-1}$, then every equation with such coefficients and of degree m times even has such a root."

In this m may be any positive integer, unity included. Now, the last article has established the existence of a root for odd degrees, *i.e.*, for the case $m = 1$. It follows, then, successively for the cases $m = 2, 3, 4, \ldots$, *i.e.*, for equations of degrees once, twice, three times, and generally any number of times, even.

[*April* 17*th*, 1894.

As some doubt has been thrown on the argument of § 4, I proceed to show otherwise that

$$a_0\rho^n \cdot z^n + a_1\rho^{n-1} \cdot z^{n-1} + \ldots + a_n \equiv F(\rho z, 1)$$

and
$$a_n z^n + a_{n-1}\rho \cdot z^{n-1} + \ldots + a_0\rho^n \equiv F(\rho, z)$$

can be made to have a common factor if ρ^2 can be determined so as to satisfy an equation of degree $\frac{1}{2}n(n-1)$.

As in the applications required n is always even, I, for simplicity, confine myself to this case.

The two will have a common factor if, and only if, their sum and difference have.

Now, the difference of $F(\rho z, 1)$ and $F(\rho, z)$ is divisible by z^2-1, n being even. The complete condition that a factor of z^2-1 be the common factor in question is

$$F(\rho, 1)\, F(-\rho, 1) = 0,$$

i.e., is $(a_0\rho^n + a_2\rho^{n-2} + \ldots + a_n)^2 - (a_1\rho^{n-1} + a_3\rho^{n-3} + \ldots + a_{n-1}\rho)^2 = 0$,

of which the left-hand side is a function of ρ^2.

Also the sum of $F(\rho z, 1)$ and $F(\rho, z)$ may be written

$$(a_0\rho^n + a_n)(z^2+1)^{\frac{1}{2}n} + B\,(z^2+1)^{\frac{1}{2}n-1} z + \ldots + Kz^{\frac{1}{2}n},$$

and the difference, after the removal of the factor z^2-1, may be written
$$(a_0\rho^n - a_n)(z^2+1)^{\frac{1}{2}n-1} + B'\,(z^2+1)^{\frac{1}{2}n-2} z + \ldots + K'z^{\frac{1}{2}n-1},$$

where $B, \ldots K, B', \ldots K'$ are of degree n in ρ. Thus the remaining factor of the criterion required is the dialytic determinant of

$$(a_0\rho^n + a_n)\, t^{\frac{1}{2}n} + Bt^{\frac{1}{2}n-1} + \ldots + K$$

and
$$(a_0\rho^n - a_n)\, t^{\frac{1}{2}n-1} + B't^{\frac{1}{2}n-2} + \ldots + K',$$

whose degree in ρ is at most $n\left(\frac{n}{2} + \frac{n}{2} - 1\right)$, *i.e.*, $n(n-1)$, and will be seen below to be exactly this number. Call this dialytic determinant δ.

Accordingly the complete condition expressed by $\Delta = 0$ of § 4, whose degree is $2n^2$ in ρ, is expressed also by the alternatives

$$F(\rho, 1)\, F(-\rho, 1) = 0, \quad \delta = 0,$$

whose degrees are $2n$, and at most $n(n-1)$, respectively.

Moreover, Δ is a function of ρ^2, for the result of changing ρ into $-\rho$ in it is to alter the signs of alternate columns, and then the signs of alternate rows, *i.e.*, is to multiply it by $(-1)^{2n}$, *i.e.*, not to alter it. Also $F(\rho, 1)\,F(-\rho, 1)$ is a function of ρ^2. Consequently so is δ.

Now the degree of δ in ρ, seen above to be not greater than $n(n-1)$, cannot be less than $n(n-1)$. This will certainly be true, $a_0, a_1, a_2, \ldots a_n$ being unrestricted, if it is true even when particular restrictions are imposed on them, for a function of any degree in ρ cannot have that degree raised by supposing its coefficients made special. Now, when $a_0, a_1, a_2, \ldots a_n$ are so chosen that the equation

$$F(x, 1) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

has n roots, as may certainly be done by taking it to be

$$a_0(x-x_1)(x-x_2)\ldots(x-x_n) = 0,$$

there are $\frac{1}{2}n(n-1)$ values of ρ^2 which must satisfy $\delta = 0$, namely, the $\frac{1}{2}n(n-1)$ products of two and two of $x_1, x_2, \ldots x_n$. [For instance, the value $\sqrt{x_1x_2}$ of ρ makes $F(\rho z, 1)$ and $F(\rho, z)$ have the common factors $z-\sqrt{\dfrac{x_1}{x_2}}$, $z-\sqrt{\dfrac{x_2}{x_1}}$.] [These $\frac{1}{2}n(n-1)$ products with the squares $x_1^2, x_2^2, \ldots x_n^2$, *i.e.*, the values of ρ^2 which make

$$F(\rho, 1)\,F(-\rho, 1) = 0,$$

make up all the solutions of the equation in ρ^2, $\Delta = 0$.]

The condition $\delta = 0$ is then, in the general case, no less than in the one which has at present to be taken as special, one of degree exactly $\frac{1}{2}n(n-1)$ in ρ^2.

δ is, of course, the square root of the quotient $\Delta/F(\rho, 1)\,F(-\rho, 1)$.

It is unfortunate for the simplicity of the argument of this paper that the property of such a determinant as Δ, that, after division by its obvious factors,

$$F(\rho, 1) \equiv a_0\rho^n + a_1\rho^{n-1} + a_2\rho^{n-2} + \ldots + a_n$$

and $F(-\rho, 1)$, it leaves a perfect square as quotient, is one which direct algebraic methods have as far as I know not yet supplied. For low values of n the proof is, of course, easy, but I have not yet succeeded in giving a general form to such proofs as I have obtained and had given me by my friends.]

On the Motion of Paired Vortices with a Common Axis.

By A. E. H. Love. Read and received March 8th, 1894.

1. The investigation in this paper was undertaken with the view of throwing some further light on a problem in the motion of vortex-rings which was first considered by Helmholtz in his original memoir on vortex-motion.* He found that two vortex-rings having the same axis and circulations in the same direction travel in the same direction parallel to the axis; "the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately."† It is extremely difficult to obtain a more detailed account of the motion here described. We are ignorant of the condition that the motion may be periodic, and we can only make guesses at the length of the period when the unknown condition is satisfied. Yet in applications of the vortex-atom theory to problems of radiation and chemical combination, it is conceivable that this period and the type of motion may play an important part. I propose here to imitate some of the circumstances of the problem by considering the case where there are present in an infinite fluid two pairs of cylindrical vortices of indefinitely small section, the circulations about the two vortices of each pair being equal and of opposite sign, the circulations about the four vortices being equal in absolute magnitude, and the line of symmetry for one pair coinciding with that for the other. A single pair of this kind moves parallel to the axis of symmetry with constant velocity. Two pairs with circulations in the proper directions influence each other's motions in a manner analogous to that exhibited by thin rings. I find a condition that the motion may be periodic, the length of the period, and the form of the curve described by one vortex of one pair relative to the homologous vortex of the other pair.

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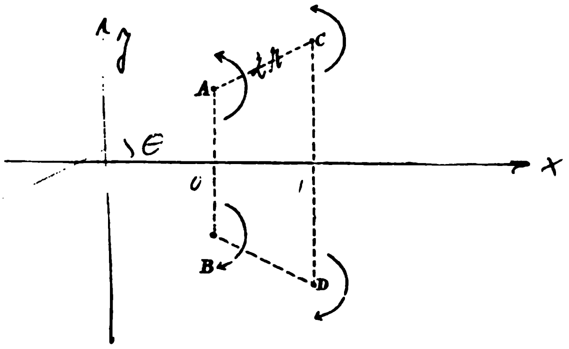
* "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle-Borchardt*, LV., 1858.

† Tait's translation of Helmholtz's memoir, *Phil. Mag.* (4), XXXIII., 1867.

2. Let the axis of symmetry be chosen as the axis of x, and let the coordinates of the vortices at time t be

$$(x_0, y_0),\ (x_0, -y_0),\ (x_1, y_1),\ (x_1, -y_1);$$

then the two with suffix 0 form a pair, and the two with suffix 1 also form a pair.



The figure shows a possible configuration of the system in which A, B are taken to be the pair with suffix 0.

Let m be the absolute value of the strength of either vortex (*i.e.*, half the circulation in any circuit surrounding it once, and not surrounding either of the others), and suppose the strengths of the four A, B, C, D to be m, $-m$, m, $-m$; then the stream function ψ which gives the motion at any point (x, y) is

$$\psi = \frac{m}{2\pi}\log\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}+\frac{m}{2\pi}\log\frac{(x-x_1)^2+(y+y_1)^2}{(x-x_1)^2+(y-y_1)^2} \quad \text{.....(1).}$$

From this expression we may form the components of velocity of the two vortices by leaving out the terms in $\partial\psi/\partial y$, and $-\partial\psi/\partial x$ which become infinite at (x_0, y_0) or (x_1, y_1).

We thus find

$$\left.\begin{aligned}\frac{dx_1}{dt} &= -\frac{m}{\pi}\,\frac{y_1-y_0}{(x_1-x_0)^2+(y_1-y_0)^2}+\frac{m}{\pi}\,\frac{y_1+y_0}{(x_1-x_0)^2+(y_1+y_0)^2}+\frac{m}{2\pi y_1}\\ \frac{dy_1}{dt} &= \frac{m}{\pi}\,\frac{x_1-x_0}{(x_1-x_0)^2+(y_1-y_0)^2}-\frac{m}{\pi}\,\frac{x_1-x_0}{(x_1-x_0)^2+(y_1+y_0)^2}\end{aligned}\right\} \quad \text{............(2),}$$

and dx_0/dt and dy_0/dt can be obtained from these by interchanging the suffixes 0, 1. The differential equations for x_0, y_0, x_1, y_1 as functions of t can clearly be put in the form

$$\frac{dx_0}{\frac{\partial\chi}{\partial y_0}} = \frac{dy_0}{-\frac{\partial\chi}{\partial x_0}} = \frac{dx_1}{\frac{\partial\chi}{\partial y_1}} = \frac{dy_1}{-\frac{\partial\chi}{\partial x_1}} = dt \quad \text{..................(3)},$$

where
$$\chi = \frac{m}{2\pi}\log y_1 y_0 \frac{(x_1-x_0)^2+(y_1+y_0)^2}{(x_1-x_0)^2+(y_1-y_0)^2} \quad \text{..................(4)}.$$

We find at once two integrals of the equations (3) in the form

$$\left.\begin{aligned} y_1+y_0 &= \text{const.} = 2c \\ y_1 y_0 \frac{(x_1-x_0)^2+(y_1+y_0)^2}{(x_1-x_0)^2+(y_1-y_0)^2} &= \text{const.} = a^2 \end{aligned}\right\} \quad \text{...............(5)}.$$

The first of these shows that the middle point of the line joining the two vortices A, C moves parallel to the axis x and at a distance c from it. It is easy to see by multiplying this equation by m that it represents the condition of constancy of momentum parallel to x of the fluid in the half-plane y positive, for this momentum is equal to that generated by impulsive pressure proportional to m per unit area applied at all points of half a barrier between the vortices A, B and at all points of half a barrier between C and D. The second of equations (5) is really χ = const., and it may in like manner be interpreted as representing the constancy of the energy of the fluid motion in the same half-plane. For, according to Helmholtz's formula, the energy of a plane vortex-motion, in terms of the density ρ, the spin ζ, and the stream function ψ, is

$$\rho \iint \psi \zeta \, dx\, dy,$$

the integral extending to all the points where ζ is different from zero. For isolated vortex-filaments this expression contains an infinite constant, and the finite part of it divided by ρ is the sum of the strengths of the vortices each multiplied by the part of ψ which is finite at the point occupied by the vortex.

3. The two equations (5) are sufficient to determine the paths of the vortices A and C relative to each other, for, if $2r$ is the distance between A and C at any time, and θ the angle the line AC makes with the axis x, they give rise to the polar equation

$$(r^2\cos^2\theta + c^2)(c^2 - r^2\sin^2\theta) = a^2 r^2 \quad \text{..................(6)},$$

which represents the curve described by either vortex relative to the middle point of the line joining them. In (x, y) coordinates with origin at this middle point the same equation is

$$x^2y^2+(a^2-c^2)\,x^2+(a^2+c^2)\,y^2-c^4=0 \quad\text{.................(7).}$$

The condition that the motion may be periodic is the condition that this equation represents a closed curve. This is the case if

$$a^2>c^2,$$

as is seen by writing (7) in the form

$$(x^2+a^2+c^2)(y^2+a^2-c^2)=a^4.$$

When this condition is satisfied one of the values of r^2 given by equation (6) is positive for every value of θ. This value of r^2 is

$$c^2\operatorname{cosec}^2\theta-\tfrac{1}{2}(a^2+c^2)\sec^2\theta\operatorname{cosec}^2\theta\,\{1-\sqrt{(1-\kappa^2\cos^2\theta)}\},$$

where $\kappa^2=4a^2c^2/(a^2+c^2)^2,$

and the square root is to be taken positive. On evaluating the indeterminate, which occurs when $\theta=0$, we find

$$r^2=c^4/(a^2-c^2),$$

as is given directly by (6). This value becomes infinite when $a^2=c^2$, and for values of a^2/c^2 which are less than unity the curve has no real points on the line $\theta=0$, but there exists a pair of parallel asymptotes, $y^2=c^2-a^2$, between which there are no real points on the curve. In every case there is a positive value of r^2 for $\theta=\frac{1}{2}\pi$, so that every system of two pairs such as we are considering will either at some future time have its four vortices A, B, C, D in a straight line perpendicular to the axis x, or may be regarded as having been at some past time in this state. We may therefore interpret the constants a and c, more particularly in terms of the distances between the vortices in a pair at the instant when each vortex of one pair is passing between the corresponding vortex of the other pair and the axis of symmetry, or, as it may be otherwise expressed, when one pair is passing through the other pair.

Let r_0 and r_1 be the distances of A and C from the axis x when A, B, C, D are in a straight line; then we have

$$r_0+r_1=2c,$$

$$r_1r_0\left(\frac{r_1+r_0}{r_1-r_0}\right)^2=a^2,$$

and the condition $a^2 > c^2$ is

$$6r_0r_1 - r - r_1 > 0.$$

Taking $r_1 > r_0$, this requires that $r_1 : r_0 < 3 + 2\sqrt{2}$. Thus the motion is periodic, if, at the instant when one pair passes through the other, the ratio of the breadths of the pairs is less than $3 + 2\sqrt{2}$. When the ratio has this precise value the smaller pair shoots ahead of the larger and widens, while the larger contracts, so that each is ultimately of the same breadth $2c$, and the distance between them is ultimately infinite. When the ratio in question is greater than $3 + 2\sqrt{2}$, the smaller shoots ahead and widens, and the latter falls behind and contracts, but in such a way that the former never attains so great a breadth as $2c$, nor the latter so small a breadth. When the ratio is less than $3 + 2\sqrt{2}$, the motion of the two pairs is similar to the motion described by Helmholtz for two rings on the same axis, and it is probable that there is for his case also a critical condition in which the rings, after one has passed through the other, ultimately separate to an infinite distance, and attain equal diameters.

4. The curves given by (7) when $a^2 > c^2$ always lie inside and near to the ellipses

$$\frac{x^2}{\frac{c^4}{a^2 - c^2}} + \frac{y^2}{\frac{c^4}{a^2 + c^2}} - 1 = 0,$$

and their longest and shortest diameters are the same as the major and minor axes of these ellipses. In the limiting case, when $a^2 = c^2$, it is clear that there are inflexions. In general it can be proved that the positions of the inflexions are given by the equation

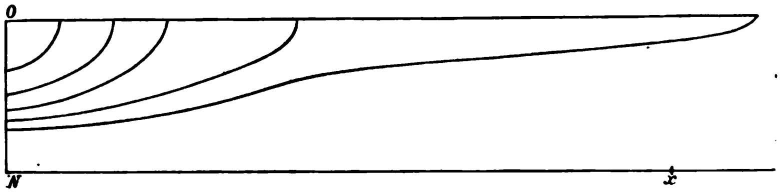
$$3y^4(a^2 + c^2) - 2y^2c^4 + c^4(a^2 - c^2) = 0.$$

All the roots are imaginary if $a^2 > \frac{1}{3} 2\sqrt{3}\, c^2$, but, if $c^2 < a^2 < \frac{1}{3} 2\sqrt{3}\, c^2$, they are all real. There are consequently no inflexions, unless the ratio of the major to the minor axis of the ellipse

$$x^2(a^2 - c^2) + y^2(a^2 + c^2) = c^4$$

exceeds $2 + \sqrt{3}$, so that there are no inflexions unless the relative path is very elongated.

At the end of the paper will be found a table giving the principal axes of the ellipses which nearly coincide with the relative paths of the vortices for some simple values of the ratio $r_1 : r_0$, and a drawing showing the character of these curves is given here. The line Nx in the figure is the axis of symmetry of the two vortex-pairs, and the point O is the middle point of the line joining the two vortices on one side of the axis. The curves show quadrants of the relative paths of these two vortices for the values 2, 3, 4, 5, $\frac{11}{3}$ of the ratio $r_1 : r_0$. It is noteworthy how very nearly circular these curves are for values of this ratio up to 2.



5. We have next to consider the dependence of the relative situation of the vortices upon the time. For this purpose we form an equation connecting $d\theta/dt$ with θ. By means of this equation the time can be expressed as a function of θ, and, in particular, the period of the motion, when it is periodic, can be deduced.

The distance between the vortices A and C being $2r$, and the angle AC makes with the axis x being θ, we have

$$4r^2\frac{d\theta}{dt} = (x_1-x_0)\frac{d(y_1-y_0)}{dt} - (y_1-y_0)\frac{d(x_1-x_0)}{dt}$$

$$= \frac{m}{2\pi}\left[\frac{(y_1-y_0)^2}{y_1y_0} + 4 - 4\frac{(x_1-x_0)^2}{(x_1-x_0)^2+(y_1+y_0)^2}\right]$$

$$= \frac{2m}{\pi}\left[1 + \frac{r^2\sin^2\theta\,(r^2\cos^2\theta+c^2)}{a^2r^2} - \frac{r^2\cos^2\theta}{r^2\cos^2\theta+c^2}\right],$$

or

$$\frac{d\theta}{dt} = \frac{m}{2\pi}\,\frac{(r^2\cos^2\theta+c^2)^2\sin^2\theta+a^2c^2}{a^2r^2\,(r^2\cos^2\theta+c^2)} \quad\text{.........................(8).}$$

Now, from the equation of the relative path

$$(r^2\cos^2\theta+c^2)(r^2\sin^2\theta-c^2)+a^2r^2=0,$$

we find, for the positive value of r^2,

$$c^2\operatorname{cosec}^2\theta - \tfrac{1}{2}(a^2+c^2)\sec^2\theta\operatorname{cosec}^2\theta\left\{1-\sqrt{(1-\kappa^2\cos^2\theta)}\right\},$$

where

$$\kappa^2 = 4a^2c^2/(a^2+c^2)^2 \quad\text{..........................(9),}$$

are the square root is to be taken positive. From this, after some reductions, we find

$$\frac{d\theta}{dt}=\frac{m}{\pi a^2}\,\frac{\sin^2\theta\cos^2\theta\sqrt{(1-\kappa^2\cos^2\theta)}}{\sqrt{(1-\kappa^2\cos^2\theta)}-(\kappa'\cos^2\theta+\sin^2\theta)},$$

where $$\kappa'^2=1-\kappa^2,$$

so that $$\kappa'=(a^2-c^2)/(a^2+c^2).$$

The equation determining the time as a function of θ is

$$t=\frac{\pi a^2}{m}\left[\int\frac{d\theta}{\sin^2\theta\cos^2\theta}-\int\frac{\kappa'\cos^2\theta+\sin^2\theta}{\sin^2\theta\cos^2\theta}\,\frac{d\theta}{\sqrt{(1-\kappa^2\cos^2\theta)}}\right]\ldots(10).$$

To find the period τ we may take for the limits of θ the values 0 and $\frac{1}{2}\pi$, and multiply the right-hand side by 4, and then, writing $\frac{1}{2}\pi-\phi$ for θ, we find

$$\tau=\frac{4\pi a^2}{m}\left[\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sin^2\phi\cos^2\phi}-\int_0^{\frac{1}{2}\pi}\frac{\kappa'\sin^2\phi+\cos^2\phi}{\sin^2\phi\cos^2\phi}\,\frac{d\phi}{\sqrt{(1-\kappa^2\sin^2\phi)}}\right].$$

Introducing elliptic functions of modulus κ and argument u, such that

$$\phi=\operatorname{am}u,\quad \sin\phi=\operatorname{sn}u,\ \ldots,$$

we obtain, from (10),

$$\frac{mt}{\pi a^2}=\int\frac{d\phi}{\sin^2\phi}-\int\frac{du}{\operatorname{sn}^2u}+\int\frac{d\phi}{\cos^2\phi}-\kappa'\int\frac{du}{\operatorname{cn}^2u},$$

or $$\frac{mt}{\pi a^2}=\left[-\frac{\operatorname{cn}u}{\operatorname{sn}u}-u+E\operatorname{am}u+\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}\right]$$

$$+\left[\frac{\operatorname{sn}u}{\operatorname{cn}u}-\kappa'\left(u-\frac{1}{\kappa'^2}E\operatorname{am}u+\frac{\operatorname{sn}u\operatorname{dn}u}{\kappa'^2\operatorname{cn}u}\right)\right],$$

where $$E\operatorname{am}u=\int\operatorname{dn}^2u\,du;$$

and hence, observing that the terms that become infinite at the limits cancel one another, we deduce

$$\frac{m\tau}{\pi a^2}=4\,\frac{1+\kappa'}{\kappa'}\,(E-K\kappa'),$$

where K is the real quarter-period of the elliptic functions with modulus κ, and E is the complete elliptic integral of the second kind given by

$$E = \int_0^K \mathrm{dn}^2 u \, du.$$

It is easily verified that $E - K\kappa'$ is always positive and less than unity, for it vanishes when $\kappa = 0$, and becomes equal to 1 when $\kappa = 1$, and further

$$\frac{d}{d\kappa}(E - K\kappa') = \frac{1-\kappa'}{\kappa\kappa'}(K - E),$$

so that the function constantly increases when $1 > \kappa > 0$.

Since $$a^2/c^2 = (1+\kappa')/(1-\kappa'),$$

the period τ is thus proved to be the positive quantity

$$\tau = \frac{4\pi c^2}{m} \frac{(1+\kappa')^2}{\kappa'(1-\kappa')}(E - K\kappa') \quad \text{..................(11)},$$

and this becomes infinite when $\kappa' = 0$ or $a^2 = c^2$, in accordance with the result of § 3, that when $a^2 = c^2$ the motion ceases to be periodic.

6. To get an idea of the way the period varies with the velocities and sizes of the vortex-pairs, we shall compare it with the time taken by a vortex-pair of strength m and breadth $2c$, when undisturbed, to travel a distance equal to its breadth. The velocity of such a pair is $m/2\pi c$, and consequently the time in question is $4\pi c^2/m$. Calling this τ', we have

$$\frac{\tau}{\tau'} = \frac{(1+\kappa')^2}{\kappa'(1-\kappa')}(E - K\kappa').$$

As in the discussion of the relative path, we can make an interpretation in terms of the radii (r_1 and r_0) of the pairs at the instant when one is passing through the other. When $\kappa = 0$ or $\kappa' = 1$ the ratio τ/τ' can be shown to vanish, and when $\kappa' = 0$ or $\kappa = 1$ it becomes infinite. The value for κ very small is small, and can be proved to be $\pi^2(r_1 - r_0)^2/m$, which is the same as the period of rotation round each other of two vortex-filaments of equal strength m at a distance $r_1 - r_0$. As κ increases, the quantity $E - K\kappa'$ increases very slowly, and the ratio τ/τ' does not become large until κ is very nearly unity. The arithmetical details of some particular cases are given in the table of § 8 below.

The period of the slowest oscillation of a thin ring of radius c is known to be $\frac{1}{3}\sqrt{3}\,\pi c/V$, where V is the velocity of translation.* The latter velocity depends on the ratio of the radius of the section to the radius of the aperture, and becomes logarithmically infinite when this ratio vanishes. For the application to radiation it is perhaps best to regard τ' as comparable with the period required by a particle of fluid circulating between the vortices of the pair of breadth $2c$ which is never quite close to either vortex of the pair or to the axis. The period τ, in which one pair goes through the other, is thus of the same order of magnitude as the period in which a particle of the fluid in irrotational motion circulates through either pair when the other is absent. From the nature of the case it is highly probable that a like statement holds for rings, *i.e.*, that the period of one ring going through another is comparable with the period in which a particle of fluid carried forward with the ring circulates through it, and we appear to be justified in concluding that the period of two thin rings passing through each other alternately is long compared with any period of oscillation of either ring about its circular form.

7. So far we have been concerned with the motion of either of two vortices on the same side of the axis relative to the middle point of the line joining them. For the motion of this middle point we need only remark that it describes a straight line $y = c$ parallel to the axis with a variable velocity. This velocity is given by the equation

$$\frac{d}{dt}\frac{1}{2}(x_1+x_0) = \frac{mc}{2\pi a^2}\,\frac{2c^2+r^2\cos 2\theta}{r^2},$$

which is easily deduced from equations (2). It is easy to prove that this has its greatest value when $\theta = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, and its least value when $\theta = 0$ or π. These values are respectively

$$\frac{mc}{2\pi a^2}\,\frac{2a^2+3c^2}{c^2} \quad\text{and}\quad \frac{mc}{2\pi a^2}\,\frac{2a^2-c^2}{c^2},$$

or

$$\frac{m}{2\pi c}\,\frac{5-\kappa'}{1+\kappa'} \quad\text{and}\quad \frac{m}{2\pi c}\,\frac{1+3\kappa'}{1+\kappa'}.$$

8. A table is appended showing arithmetical details of some particular cases. These are arranged so as to give simple values of r_1/r_0 lying between the two extreme values 0 and $3+2\sqrt{2}$ for which the motion is periodic. The first column gives the value assumed

* J. J. Thomson, "On the Motion of Vortex Rings," p. 35.

for $r_1 : r_0$, and the second gives the corresponding value of κ'. The number $(5-\kappa')/(1+\kappa')$ in the third column is the ratio of the greatest velocity of the middle point of the line joining the two vortices in the same half-plane to the velocity of a vortex-pair of strength m and breadth $2c$, and the number $(1+3\kappa')/(1+\kappa')$ in the fourth column is the ratio of the least velocity of the same middle point to the velocity of the same vortex pair. The number $c/\sqrt{(a^2-c^2)}$ in the fifth column is the ratio of the greatest semi-diameter of the relative path to the distance of the middle point from the axis of symmetry, and the number $c/\sqrt{(a^2+c^2)}$ in the sixth column is the ratio of the least semi-diameter of the relative path to the same distance. The number τ/τ' in the last column is the ratio of the period of one pair going through the other to the time occupied by a pair of strength m and breadth $2c$ in moving over a distance equal to its breadth.

$\frac{r_1}{r_0}$	κ'	$\frac{5-\kappa'}{1+\kappa'}$	$\frac{1+3\kappa'}{1+\kappa'}$	$\frac{c}{\sqrt{(a^2-c^2)}}$	$\frac{c}{\sqrt{(a^2+c^2)}}$	$\frac{\tau}{\tau'}$
1	1	2	2	0	0	0
2	$\frac{7}{9}$	2·375	1·875	·378...	·333...	·402...
3	$\frac{1}{2}$	3	1·666...	·707...	·5	1·195...
4	$\frac{7}{25}$	3·6875	1·4375	1·082...	·6	2·701...
5	$\frac{1}{9}$	4·4	1·2	2	·666...	8·538...
$\frac{17}{3}$	$\frac{1}{50}$	4·882...	1·039...	4·949	·7	45·11...
$3+2\sqrt{2}$	0	5	1	∞	·707...	∞

Pseudo-Elliptic Integrals and their Dynamical Applications.

By A. G. GREENHILL.

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When the elliptic integral of the third kind is expressible, as an exceptional case, by a logarithm, or by an inverse circular function, the integral is called a pseudo-elliptic integral; the first investigation of pseudo-elliptic integrals is to be found in two memoirs by Abel, "Sur l'intégration de la formule $\int \frac{\rho\, dx}{\sqrt{R}}$, R et ρ étant des fonctions entières," *Crelle*, t. I., 1826, *Œuvres Complètes*, t. I., p. 164; "Théorie des transcendantes elliptiques," *Œuvres Complètes*, t. II., p. 139.

1. In his memoirs, Abel seeks the form of the elliptic integral

$$\int \frac{x+k}{\sqrt{X}}\, dx,$$

where X is a quartic function of x, when it can be expressed in the form

$$A \log \frac{P+Q\sqrt{X}}{P-Q\sqrt{X}},$$

where P, Q are rational integral functions of x; and to effect this he shows that it is requisite to expand $\sqrt{X}$ in the form of a continued fraction, having first reduced X to the form

$$X = (x^2+ax+b)^2+px;$$

and when this continued fraction is periodic, and $\frac{P}{Q}$ is the value of its first period, so that

$$\sqrt{X} = \frac{P}{Q+}\,\frac{P}{Q+\dots},$$

then k can be chosen so as to make

$$\int \frac{x+k}{\sqrt{X}}\, dx = A \log \frac{P+Q\sqrt{X}}{P-Q\sqrt{X}};$$

or, changing the sign of x and X, when the integral is circular,

$$\int \frac{x-k}{\sqrt{(-X)}}\,dx = 2A\tan^{-1}\frac{Q}{P}\sqrt{(-X)}.$$

Abel shows further that we may put, without loss of generality,

$$P^2 - Q^2X = 1,$$

so that the circular integral can also be written

$$\int \frac{x-k}{\sqrt{(-X)}}\,dx = 2A\tan^{-1}\frac{Q}{P}\sqrt{(-X)} = 2A\cos^{-1}P$$

$$= 2A\sin^{-1}Q\sqrt{(-X)};$$

and therefore the first logarithmic or hyperbolic integral, by the use of the hyperbolic functions, direct and inverse, can be written, by analogy,

$$\int \frac{x+k}{\sqrt{X}}\,dx = 2A\tanh^{-1}\frac{Q}{P}\sqrt{X} = 2A\cosh^{-1}P = 2A\sinh^{-1}Q\sqrt{X}.$$

In the dynamical applications it is the circular form of the elliptic integral of the third kind which always makes its appearance; but, following Abel, it is generally simpler to work with the logarithmic or hyperbolic form, as imaginaries are thereby avoided; and we can always change immediately from the hyperbolic to the circular form by changing the signs of x and X.

Thus, denoting the integral by I, then, in the hyperbolic form,

$$e^{I/2A} = P + Q\sqrt{X},$$

and in the circular form,

$$e^{Ii/2A} = P + Q\sqrt{(-X)}.$$

The form of the quartic X, with the sign of x changed,

$$X = (x^2 - ax + b)^2 - px,$$

shows that the roots, x, of $X = 0$ are the squares, t^2, of the roots of a simpler quartic

$$T = t^4 - at^2 - \sqrt{p}\,t + b = 0.$$

Further, putting $$x = \frac{py}{4b},$$

$$\int \frac{x+k}{\sqrt{X}}\,dx = \int \frac{y+k'}{\sqrt{Y}}\,dz,$$

where $$Y = \{y^2+(n-1)\,y+m\}^2+4my,$$

and $$m = \frac{16b^3}{p^2},\quad n = 1+\frac{4ab}{p}.$$

The quantities employed by Abel, denoted by p_m, q_m, g_m, c_m, &c., are now simple rational functions of m and n; and Mr. G. B. Mathews points out that $-m$ and $-n$ are in fact the quantities denoted by x and y in Halphen's *Fonctions elliptiques*, t. I., p. 103.

2. The degree of P and Q can be reduced to one half of that given by Abel, when we know a factor, $x-a$, of X; and then the substitution

$$x-a = \frac{M}{s}$$

reduces the elliptic integral to the form

$$\int \frac{As+M}{s\sqrt{S}}\,ds,$$

where S is a cubic in s.

We choose, as the canonical form of this elliptic integral of the third kind, the form

$$I = \tfrac{1}{2}\int \frac{\rho s+\mu xy}{s\sqrt{S}}\,ds,$$

where $$S = 4s\,(s-x)^2+\{(y+1)\,s-xy\}^2,$$

when the integral is hyperbolic; but, if the integral is circular, we change the sign of s and S, and then put

$$S = 4s\,(s+x)^2-\{(y+1)\,s+xy\}^2,$$

so that the roots, s, of the cubic $S = 0$ are the squares, t^2, of the roots of the cubic

$$T = 2t^3-(y+1)\,t^2+2xt-xy = 0.$$

We shall now find that our x and y are the quantities employed by Halphen (*F. E.*, t. I., p. 103); and, introducing the Weierstrassian notation, by putting

$$s-x = \wp u-\wp v,$$

$$s = \wp u-\wp w,$$

in $$S = 4s\,(s-x)^2+\{(y+1)\,s-xy\}^2,$$

and making the coefficient of $(\wp u)^2$ vanish, by putting

$$12\wp v = (y+1)^2+4x,$$

$$12\wp w = (y+1)^2-8x,$$

then $$\wp'^2 u = S = 4s(s-x)^2+\{(y+1)s-xy\}^2,$$

$$\wp'' u = 2(s-x)^2+4s(s-x)+(y+1)\{(y+1)s-xy\};$$

and therefore $$\wp'^2 v = x^2, \quad \wp'' v = (y+1)x;$$

$$\wp'^2 w = x^2y^2, \quad \wp'' w = 2x^2-(y+1)xy.$$

Then $$\wp 2v = -2\wp v+\tfrac{1}{4}\left(\frac{\wp'' v}{\wp' v}\right)^2$$

$$= -\tfrac{1}{6}(y+1)^2-\tfrac{2}{3}x+\tfrac{1}{4}(y+1)^2$$

$$= \tfrac{1}{12}(y+1)^2-\tfrac{2}{3}x = \wp w,$$

so that $$w = 2v.$$

In the Weierstrassian notation the integral I, distinguished as $I(w)$, becomes

$$I(w) = \tfrac{1}{2}\int \frac{\rho s+\mu xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}\int \frac{\rho(\wp u-\wp w)+\mu\wp' w}{\wp u-\wp w}\,du,$$

and Abel's conditions that this is a pseudo-elliptic integral are equivalent to saying that w must be the aliquot part, one-μ^{th}, of a period of the elliptic integral

$$u = \int \frac{ds}{\sqrt{S}}.$$

The integral $I(v)$ may equally well be chosen as the canonical form, and

$$I(v) = \tfrac{1}{2}\int \frac{\rho(s-x)+\mu x}{(s-x)\sqrt{S}}\,ds,$$

and $I(v)$ has the advantage of starting at the beginning of the series of parameters v, $2v$, $3v$, ..., mv, ..., all relating to associated pseudo-elliptic integrals, when v is an aliquot part of a period.

The determination of ρ is rather complicated, and is reserved for *the present* (§ 9).

3. Writing $$s_m - x = \wp mv - \wp v,$$
then $$s_1 - x = 0, \quad S_1 = x^2;$$
$$s_2 - x = -x, \quad S_2 = x^2 y^2,$$
or $$s_2 = 0.$$

Thus, from the formulas below,

$$s_3 - x = -y, \qquad S_3 = (y-x-y^2)^2;$$

$$s_4 - x = -\frac{x(y-x)}{y^2}, \qquad S_4 = \frac{x^2\{x(y-x-y^2)-(y-x)^2\}^2}{y^6};$$

$$s_5 - x = -\frac{xy(y-x-y^2)}{(y-x)^2}, \qquad S_5 = \frac{x^2\{y^2(xy-x^2-y^2)-x(y-x-y^2)^2\}^2}{(y-x)^6};$$

$$s_6 - x = -\frac{(y-x)\{(y-x)x-y^2\}}{(y-x-y^2)^2},$$

$$S_6 = \quad \dots \quad \dots \quad \dots;$$

$$s_7 - x = \quad \dots \quad \dots, \qquad S_7 = \quad \dots \quad \dots \quad \dots;$$

and so on; and we change the sign of s and S for the circular form.

By comparison with Abel's q_m (*Œuvres complètes*, t. II., p. 158), calculated from the quartic

$$Y = \{y^2 + (n-1)y + m\}^2 + 4my,$$

in which $-m$ and $-n$ are afterwards replaced by Halphen's x and y, we shall find that $s_m - x$ is the same as Abel's $\frac{1}{2}q_{m-1}$.

Abel's recurring equation for q_m (*Œuvres*, t. II., p. 157),

$$q_m + q_{m-2} = \frac{\frac{1}{2}p^2}{q_{m-1}^2} + \frac{ap}{q_{m-1}},$$

is thus merely equivalent to the elliptic function formula

$$\wp(u+v) + \wp(u-v) = 2\wp v + \frac{\wp'^2 v}{(\wp u - \wp v)^2} + \frac{\wp'' v}{\wp u - \wp v},$$

with $$u = (m-1)v,$$

and the continued fraction expansion employed by Abel is not required in this method, at least not for the present.

Knowing $\wp v$, $\wp' v$, $\wp'' v$, and $\wp 2v$ in terms of x and y, as above, we can now determine $\wp 3v$, $\wp 4v$, ... $\wp mv$, by means of this formula; while $\wp' mv$ or $\sqrt{S_m}$ is determined by means of the formula

$$\wp(u-v) - \wp(u+v) = \frac{\wp' u \, \wp' v}{(\wp u - \wp v)^2},$$

or $$\wp' mv \, \wp' v = \{\wp(m-1)v - \wp(m+1)v\}(\wp mv - \wp v)^2,$$

or $$\sqrt{S_m}\sqrt{S_1} = (s_{m-1} - s_{m+1})(s_m - s_1)^2.$$

4. But, introducing Halphen's function $\psi_m(v)$, (*F. E.*, t. I., p. 96), defined by the relation

$$\psi_m(v) = \frac{\sigma(mv)}{(\sigma v)^{m^2}},$$

or $$\wp mv - \wp v = -\frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2},$$

or, more generally, $$\wp mu - \wp nv = -\frac{\psi_{m-n}\psi_{m+n}}{\psi_m^2 \psi_n^2};$$

then $$\wp' mv \, \wp' v = \frac{\psi_{2m}\psi_2}{\psi_{m-1}^2 \psi_{m+1}^2}\left(\frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2}\right)^2 = \frac{\psi_{2m}\psi_2}{\psi_m^4},$$

or, with $$\wp' v = -\psi_2,$$

$$\wp' mv = -\frac{\psi_{2m}}{\psi_m^4}.$$

Again, Halphen's function γ_m is defined by the relation (*F. E.*, t. I., p. 102)

$$\gamma_m = \psi_m \psi_2^{-\frac{1}{2}(m^2-1)};$$

so that, with $$\psi_2 = -\wp' v = x,$$

$$s_m + x, \quad \text{or} \quad s_m - s_1 = -x^{\frac{1}{2}}\frac{\gamma_{m-1}\gamma_{m+1}}{\gamma_m^2} = \tfrac{1}{2}q_{m-1};$$

$$s_m, \quad \text{or} \quad s_m - s_2 = -x^{\frac{1}{2}}\frac{\gamma_{m-2}\gamma_{m+2}}{\gamma_m^2};$$

and, generally, $$s_m - s_n = -x^{\frac{1}{2}}\frac{\gamma_{m-n}\gamma_{m+n}}{\gamma_m^2\gamma_n^2};$$

while $$\sqrt{S_m} = -x\frac{\gamma_{2m}}{\gamma_m^4}.$$

According to Halphen (*F. E.*, t. I., p. 103),

$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad \gamma_3 = x^4, \quad \gamma_4 = y, \quad \gamma_5 = y-x,$$
$$\gamma_6 = x^4(y-x-y^2),$$
$$\gamma_7 = (y-x)\,x-y^2,$$
$$\gamma_8 = y\,\{x(y-x-y^2)-(y-x)^2\},$$
$$\gamma_9 = x^4\,\{y^3(y-x-y^2)-(y-x)^3\},$$
$$\gamma_{10} = (y-x)\,\{y^2(xy-x^2-y^3)-x(y-x-y^2)^2\},$$
$$\gamma_{11} = (xy-x^2-y^3)(y-x)^2-xy\,(y-x-y^2)^2,$$
&c.

Since $$\wp mv-\wp nv = (\wp mv-\wp v)-(\wp nv-\wp v),$$

therefore $$\frac{\psi_{m-n}\psi_{m+n}}{\psi_m^2\psi_n^2} = \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2} - \frac{\psi_{n-1}\psi_{n+1}}{\psi_n^2},$$

or $$\frac{\gamma_{m-n}\gamma_{m+n}}{\gamma_m^2\gamma_n^2} = \frac{\gamma_{m-1}\gamma_{m+1}}{\gamma_m^2} - \frac{\gamma_{n-1}\gamma_{n+1}}{\gamma_n^2},$$

or $$\gamma_{m-n}\gamma_{m+n} = \gamma_{m-1}\gamma_{m+1}\gamma_n^2 - \gamma_{n-1}\gamma_{n+1}\gamma_m^2;$$

and therefore also

$$\gamma_{2n+1} = \gamma_{n+2}\gamma_n^3 - \gamma_{n-1}\gamma_{n+1}^3,$$

$$\gamma_{2n} = \gamma_n\,(\gamma_{n+2}\gamma_{n-1}^2-\gamma_{n-2}\gamma_{n+1}^2),$$

recurring formulas by means of which Halphen calculated γ_n; and from which we deduce the values of s_m and $\sqrt{S_m}$.

5. The elliptic integral of the third kind is pseudo-elliptic, that is, it can be expressed by a logarithm or an inverse circular or hyperbolic function, when the parameter v is an aliquot part, one-μ^{th} suppose, of a period; and then

$$\wp(\mu-1)\,v = \wp v,$$

$$\wp(\mu-m)\,v = \wp mv;$$

expressed in the preceding notation by

$$s_{\mu-1}+x = 0, \quad \gamma_\mu = 0, \quad \text{or} \quad q_{\mu-2} = 0 \text{ (Abel)};$$

or, more generally, by

$$s_{\mu-m} = s_m, \quad q_{\mu-m-1} = q_{m-1} \text{ (Abel)}.$$

This condition, expressed by Halphen's γ function, is

$$\frac{\gamma_{\mu-m-1}\gamma_{\mu-m+1}}{\gamma_{\mu-m}^2} = \frac{\gamma_{m-1}\gamma_{m+1}}{\gamma_m^2},$$

or

$$\frac{\gamma_{\mu-m-1}}{\gamma_{m+1}}\frac{\gamma_{\mu-m+1}}{\gamma_{m-1}} = \left(\frac{\gamma_{\mu-m}}{\gamma_m}\right)^2,$$

so that we may put

$$\frac{\gamma_{\mu-m}}{\gamma_m} = \lambda^{\mu-2m}, \quad \text{or} \quad \lambda^m.$$

6. Treating x and y as the coordinates of a point on the curve

$$\gamma_\mu = 0,$$

our first object is to determine x and y in terms of a parameter z, or p, or c; and this is easily effected for certain simple values of μ.

Thus, for instance,

$\mu = 3$, $$x = 0;$$

$\mu = 4$, $$y = 0;$$

$\mu = 5$, $$y-x = 0, \quad y = x;$$

$\mu = 6$, $$y-x-y^2 = 0,$$

$$y = z, \quad x = z(1-z);$$

$\mu = 7$, $$(y-x)x-y^2 = 0,$$

$$y = z(1-z), \quad x = z(1-z)^2;$$

$\mu = 8$, $$x(y-x-y^2)-(y-x)^2 = 0,$$

$$y = z\frac{1-2z}{1-z}, \quad x = z(1-2z);$$

$\mu = 9$, $$y^2(y-x-y^2)-(y-x)^2 = 0,$$

$$y = p^2(1-p), \quad x = p^2(1-p)(1-p+p^2);$$

$\mu = 10$, $$y^2(xy-x^2-y^3)-x(y-x-y^2)^2 = 0,$$

$$y = \frac{-c(1+c)}{(2+c)(1-c-c^2)}, \quad x = \frac{-c(1+c)}{(2+c)(1-c-c^2)^2};$$

$\mu = 11$, $(xy - x^2 - y^3)(y - x)^2 - xy(y - x - y^2)^2$,

so that, if $x = y(1 - z)$, $y = z\left(1 - \frac{z}{p}\right)$,

then $2z = 1 + \sqrt{P}$,

where $P = 1 - 4p^2 + 4p^3$;

$\mu = 12$, $\frac{\gamma_8}{\gamma_4} = \left(\frac{\gamma_7}{\gamma_5}\right)^2$,

$$y = -c(1+c)(1+c+c^2), \quad x = -c(1+c)(1+c+c^2)\frac{2+2c+c^2}{2+c},$$

or $$y = -\frac{(p-1)(2p-1)(3p^2-3p+1)}{p^3},$$

$$x = -\frac{(p-1)(2p-1)(3p^2-3p+1)(2p^2-2p+1)}{p^4}.$$

7. Putting $s = \wp u - \wp v$,

we notice that $x = \wp 2v - \wp v$,

$$y = \wp 3v - \wp v,$$

$$z = \frac{\wp 3v - \wp 2v}{\wp 3v - \wp v},$$

$$p = \frac{\wp 2v - \wp v}{\wp 5v - \wp},$$

$$\frac{x}{p} = \wp 5v - \wp v,$$

$$c = \frac{\wp 5v - \wp v}{\wp 3v - \wp v}\frac{\wp 3v - \wp 2v}{\wp 2v - \wp 5v}.$$

Also $$\frac{1 + \frac{c}{p}}{1+c)^2} = \frac{\wp 8v - \wp v}{\wp 4v - \wp v}.$$

In this method, originated by Abel, we determine the values of the series of functions of v, $2v$, $3v$, $4v$, ..., aliquot parts of a period, in terms of a single parameter, z, p, t, or c; and thence the value of the modulus can be inferred; this is a reversal of the ordinary procedure, in which the modulus is supposed given; and the degree of the equations requiring solution, of the nature of those given by Halphen (*F. E.*, t. III., Chaps. I. and II.), is thereby considerably reduced.

8. By means of various transformations of the curve $\gamma_\mu = 0$, it is possible to reduce the degree of the curve to a considerable extent; and when the curve $\gamma_\mu = 0$ is unicursal, it is possible to reduce the degree in one of the coordinates to unity; and to degree two, when the curve $\gamma_\mu = 0$ has a deficiency (*genre*, *Geschlecht*) of unity.

The transformations employed in the preceding cases were

$$y - x = yz, \quad \text{or} \quad x = y(1-z),$$

$$z - y = \frac{z^2}{p}, \quad \text{or} \quad y = z\left(1 - \frac{z}{p}\right),$$

$$z = c(p-1), \quad p = \frac{q+c}{q-1}, \quad \text{\&c.}$$

With these transformations, we find

$$\gamma_3 = x^{\frac{1}{2}},$$

$$\gamma_4 = y,$$

$$\gamma_5 = yz,$$

$$\gamma_6 = x^{\frac{1}{2}} \frac{yz^2}{p},$$

$$\gamma_7 = -\frac{y^2z^2(p-1)}{p} = -\frac{y^2z^3}{cp},$$

$$\gamma_8 = -\frac{y^3z^3(z+p-1)}{p} = -\frac{y^3z^3(1+c)}{cp},$$

$$\gamma_9 = -x^{\frac{1}{2}}y^3z^3\frac{z+p(p-1)}{p^2} = -x^{\frac{1}{2}}y^3z^3\frac{(p-1)(p+c)}{p^2},$$

$$\gamma_{10} = -y^4z^4\frac{(p-1)\{(1-c-c^2)p+2c+c^2\}}{p^2},$$

$$\gamma_{11} = -y^5z^5\frac{z-z^2+p^2(p-1)}{p^3} = -y^5z^5\frac{(p-1)(p^2-c^2p+c+c^2)}{p^3},$$

$$\gamma_{12} = -x^{\frac{1}{2}}y^6z^6\frac{pz+(p-1)(2p-1)}{p^3} = -x^{\frac{1}{2}}y^6z^6\frac{(p-1)\{(2+c)p-1\}}{p^3},$$

$$\gamma_{13} = y^7z^7\frac{(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}}{p^4},$$

$$\gamma_{14} = y^8z^8\frac{(p-1)^3\{(1+c-2c^2-c^3)p^2+(2c+3c^2)p+c^2+c^3\}}{p^5},$$

$$\gamma_{15} = \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

9. Having determined x and y, and thence $\wp v$, $\wp 2v$, ... as functions of a single parameter, c, suppose, where x and y are the coordinates of a point on Halphen's curve

$$\gamma_\mu = 0,$$

so that v is a μ^{th} aliquot part of a period, say

$$v = \frac{2r\omega_1}{\mu},$$

we must now proceed to the determination of ρ in the elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{\rho(s-x)+\mu x}{(s-x)\sqrt{S}}\,ds,$$

or, in the Weierstrassian form,

$$I(v) = \tfrac{1}{2}\int \frac{\rho(\wp u-\wp v)-\mu\wp' v}{\wp u-\wp v}\,du$$

$$= \tfrac{1}{2}(\rho-2\mu\zeta v)\,u + \tfrac{1}{2}\log\frac{\sigma(v+u)}{\sigma(v-u)}$$

or $$I(v) = \tfrac{1}{2}\int \frac{\rho(\wp u-\wp v)-\mu\wp' u+\mu(\wp' u-\wp' v)}{\wp u-\wp v}\,du$$

$$= \tfrac{1}{2}\rho u - \log(\wp u-\wp v)^{\frac{1}{2}\mu} - \mu u\zeta v + \mu\log\frac{\sigma(v+u)}{\sigma v\,\sigma u}.$$

Then, adding $\mu-1$ of these integrals to $I(1-\mu)v$, which is the same as $I(v)$, we find

$$\mu I(v) = \mu u\left\{\tfrac{1}{2}\rho - (\mu-1)\zeta v - \zeta(1-\mu)v\right\} - \mu\log(\wp u-\wp v)^{\frac{1}{2}\mu}$$

$$+\mu\log\frac{\sigma(v+u)^{\mu-1}\sigma(v-pv+u)}{(\sigma v)^{\mu}(\sigma u)^{\mu}},$$

and $$\frac{\sigma(v+u)^{\mu-1}\sigma(v-\mu v+u)}{(\sigma v)^{\mu}(\sigma u)^{\mu}} = \Omega,$$

a rational integral function of $\wp u$ and $\wp' u$, and therefore of the form

$$\Omega = A + B\wp' u,$$

where A and B are rational integral functions of $\wp u$.

Thus $I(v) = \{\tfrac{1}{2}\rho - (\mu-1)\zeta v - \zeta(1-\mu)v\}u + \log\dfrac{A+B\wp' u}{(\wp u - \wp v)^{\frac{1}{2}\mu}}$,

and for this to be pseudo-elliptic, so that

$$I(v) = \log\frac{A+B\wp' u}{(\wp u-\wp v)^{\frac{1}{2}\mu}},$$

we must have

$$\tfrac{1}{2}\rho = (\mu-1)\zeta v + \zeta(1-\mu)v = -\frac{1}{(\mu-1)}\frac{d}{dv}\log\frac{\sigma(\mu-1)v}{(\sigma v)(\mu-1)^2}$$

$$= -\frac{1}{\mu-1}\frac{d}{dv}\log\psi_{\mu-1}(v) = -\frac{1}{\mu-1}\frac{\psi'_{\mu-1}}{\psi_{\mu-1}}.$$

Now, since

$$\psi_{\mu-1} = \frac{\psi_{\mu-1}\psi_{\mu-3}}{\psi^2_{\mu-2}}\left(\frac{\psi_{\mu-2}\psi_{\mu-4}}{\psi^2_{\mu-3}}\right)^2\left(\frac{\psi_{\mu-3}\psi_{\mu-5}}{\psi^2_{\mu-4}}\right)^3\cdots\left(\frac{\psi_3\psi_1}{\psi^2_2}\right)^{\mu-3}(\psi_2)^{\mu-2}$$

$$= (s_{\mu-2}-s_1)(s_{\mu-3}-s_1)^2(s_{\mu-4}-s_1)^3\cdots(s_2-s_1)^{\mu-3}(-\wp' v)^{\mu-2};$$

therefore, differentiating logarithmically,

$$\tfrac{1}{2}(\mu-1)\rho = -\sum_{r=2}^{r=\mu-2}(\mu-r-1)\frac{r\wp' rv - \wp' v}{\wp rv - \wp v} - (\mu-2)\frac{\wp'' v}{\wp' v},$$

or $\tfrac{1}{2}(\mu-1)\rho\wp' v = \sum_{r=2}^{r=\mu-2}(\mu-r-1)r(\wp rv-\wp v)\{\wp(r+1)v - \wp(r-1)v\}$

$$+\Sigma(\mu-r-1)(\wp rv-\wp v)\{\wp(r+1)v+\wp(r-1)v-2\wp v\}$$

$$-\Sigma(\mu-r-1)\wp'' v - (\mu-2)\wp'' v,$$

by means of the formulas of § 3.

Making use of Abel's notation of

$$\tfrac{1}{2}q_{m-1} \text{ for } s_m - x \text{ or } \wp mv - \wp v,$$

then
$$\begin{aligned}\tfrac{1}{2}(\mu-1)\rho\wp' v =\ & \tfrac{1}{4}\Sigma(\mu-r-1)rq_{r-1}(q_r-q_{r-2})\\ &+\tfrac{1}{4}\Sigma(\mu-r-1)q_{r-1}(q_r+q_{r-2})\\ &-\tfrac{1}{2}(\mu-1)(\mu-2)\wp'' v\\ =\ & \tfrac{1}{4}\Sigma(\mu-r-1)(r+1)q_rq_{r-1}\\ &-\tfrac{1}{4}\Sigma(\mu-r-1)(r-1)q_{r-1}q_{r-2}\\ &-\tfrac{1}{2}(\mu-1)(\mu-2)\wp'' v\\ =\ & \tfrac{1}{4}(\mu-1)(q_{\mu-2}q_{\mu-3}+q_{\mu-3}q_{\mu-4}+\ldots+q_2q_1)\\ &-\tfrac{1}{2}(\mu-1)(\mu-2)\wp'' v,\end{aligned}$$

or
$$\rho\wp' v = \tfrac{1}{2}(q_{\mu-2}q_{\mu-3}+q_{\mu-3}q_{\mu-4}+\ldots+q_2q_1) - (\mu-2)\wp'' v,$$

in which
$$q_{\mu-2} = 2\{\wp(\mu-1)v - \wp v\} = 0.$$

If we compare this expression for ρ with that given by Abel for the corresponding quantity k in his treatment, we shall find that we must put

$$\wp' v = \tfrac{1}{4}p, \quad \wp'' v = \tfrac{1}{4}ap,$$

and then

$$\rho = \mu\,(2k-a).$$

Since

$$\tfrac{1}{2}q_{r-1}q_r = 2\,(s_r-x)(s_{r+1}-x),$$

$$\tfrac{1}{2}q_{r-2}q_{r-1}+\tfrac{1}{2}q_{r-1}q_r = \frac{2\wp'^2 v}{s_r-x}\,2\wp'' v,$$

and

$$\rho\,\wp' v = \sum_{r=2}^{r=\mu-2} \frac{\wp'^2 v}{s_r-x} - \wp'' v,$$

or

$$\rho = x\Sigma\frac{1}{s_r-x} - y - 1.$$

With the circular form of the integral,

$$\rho = x\Sigma\frac{1}{s_r+x} + y + 1.$$

This expression for ρ is more convenient for purposes of calculation than the one derived by logarithmic differentiation from the determinant form of $\psi_{\mu-1}(v)$, namely,

$$\psi_{\mu-1}(v) = \frac{\sigma\,(\mu-1)\,v}{(\sigma v)^{(\mu-1)^2}}$$

$$= \frac{(-1)^{\mu}}{\{1!\,2!\,3!\ldots(\mu-2)!\}^2}\begin{vmatrix} \wp' v, & \wp'' v, & \ldots & \wp^{(\mu-2)} v \\ \wp'' v, & \wp''' v, & \ldots & \wp^{(\mu-1)} v \\ \ldots & \ldots & \ldots & \ldots \\ \wp^{(\mu-2)} v, & \wp^{(\mu-1)} v, & \ldots & \wp^{(2\mu-5)} v \end{vmatrix}$$

(Schwarz, *Formeln und Lehrsätze*, § 15; Halphen, *Fonctions elliptiques*, t. I., p. 223).

10. Returning to the pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{\rho\,(s-x)+\mu x}{(s-x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}\int \frac{\rho\,(\wp u-\wp v)-\mu\,\wp' v}{\wp u-\wp v}\,du$$

$$= \log\frac{A+B\wp' u}{(\wp u-\wp v)^{\frac{1}{2}\mu}},$$

we notice that a change of sign of u changes the sign of the integral, so that

$$\log \frac{A - B\wp' u}{(\wp u - \wp v)^{\frac{1}{2}\mu}} = -\log \frac{A + B\wp' u}{(\wp u - \wp v)^{\frac{1}{2}\mu}},$$

or

$$A^2 - B^2 \wp'^2 u = (\wp u - \wp v)^{\mu};$$

so that we may write

$$I(v) = \cosh^{-1} \frac{A}{(\wp u - \wp v)^{\frac{1}{2}\mu}} = \sinh^{-1} \frac{B\wp' u}{(\wp u - \wp v)^{\frac{1}{2}\mu}},$$

or

$$I(v) = \cosh^{-1} \frac{A}{(s - x)^{\frac{1}{2}\mu}} = \sinh^{-1} \frac{B\sqrt{S}}{(s - x)^{\frac{1}{2}\mu}},$$

which may also be expressed as

$$(s - x)^{\frac{1}{2}\mu} e^{I(v)} = A + B\sqrt{S},$$

$$(s - x)^{\frac{1}{2}\mu} e^{-I(v)} = A - B\sqrt{S}.$$

11. Knowing the factors of S, say

$$S = 4(s - e_1)(s - e_2)(s - e_3),$$

a change in the parameter v from the form $\frac{2r\omega_1}{\mu}$ to $\omega_3 + \frac{2r\omega_1}{\mu}$ is made by putting

$$s - e_3 = \frac{(e_1 - e_3)(e_2 - e_3)}{s' - e_3},$$

$$s - e_1 = \frac{-(e_1 - e_3)(s' - e_2)}{s' - e_3},$$

$$s - e_2 = \frac{-(e_2 - e_3)(s' - e_1)}{s' - e_3},$$

$$S = \frac{(e_1 - e_3)^2 (e_2 - e_3)^2 S'}{(s' - e_3)^4}.$$

In this way we find that the result is given by

$$I\left(\omega_3 + \frac{2r\omega_1}{\mu}\right) = \cosh^{-1} \frac{A'\sqrt{(s' - e_3)}}{(s' - s_1')^{\frac{1}{2}\mu}} = \sinh^{-1} \frac{B'\sqrt{(s' - e_1 \,.\, s' - e_2)}}{(s' - s_1')^{\frac{1}{2}\mu}},$$

in which the accents may afterwards be omitted; and, similarly, we can put

$$I\left(\omega_2 + \frac{2r\omega_1}{\mu}\right) = \cosh^{-1} \frac{A\sqrt{(s - e_2)}}{(s - s_1)^{\frac{1}{2}\mu}} = \sinh^{-1} \frac{B\sqrt{(s - e_1 - s - e_3)}}{(s' - s_1')^{\frac{1}{2}\mu}}.$$

12. To obtain the pseudo-elliptic integrals of the circular form, corresponding to the parameters

$$v = \frac{2r\omega_3}{\mu}, \quad \text{or} \quad \omega_1 + \frac{2r\omega_3}{\mu}, \quad \text{or} \quad \omega_2 + \frac{2r\omega_3}{\mu},$$

change the signs of s and S, so that

$$S = 4s\,(s+x)^2 - \{(y+1)\,s+xy\}^2;$$

and now, putting $$s+x = \wp u - \wp v,$$

where $$v = \frac{2r\omega_3}{\mu},$$

then $$I(v) = \tfrac{1}{2}\int \frac{\rho\,(s+x)-\mu xy}{(s+x)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{A}{2\,(s+x)^{\frac{1}{2}\mu}} = \sin^{-1}\frac{B\sqrt{S}}{2\,(s+x)^{\frac{1}{2}\mu}},$$

or $$2\,(s+x)^{\frac{1}{2}\mu}\,e^{iI(v)} = A+iB\sqrt{S};$$

and the results corresponding to the parameters

$$\omega_1 + \frac{2r\omega_3}{\mu} \quad \text{or} \quad \omega_2 + \frac{2r\omega_3}{\mu}$$

are obtained by linear substitutions of the preceding form.

Now $$A^2+B^2S = 4\,(s+x)^{\mu},$$

so that, if μ is odd, A is of degree $\frac{1}{2}(\mu-1)$, and B is of degree $\frac{1}{2}(\mu-3)$ in s, S being of the third degree.

A verification by differentiation shows that the coefficient P of the leading term in A is ρ, and of the leading term in B is 1; so that

$$A = Ps^{\frac{1}{2}(\mu-1)} + Qs^{\frac{1}{2}(\mu-3)} + Rs^{\frac{1}{2}(\mu-5)} + \dots,$$

$$B = \quad s^{\frac{1}{2}(\mu-3)} + Cs^{\frac{1}{2}(\mu-5)} + Ds^{\frac{1}{2}(\mu-7)} + \dots;$$

and, $P = \rho$ being determined, the remaining coefficients $Q, R, \dots, C, \dots$ are readily determined by a verification; and now

$$2\,(s+x)^{\frac{1}{2}\mu}\,e^{iI(v)} = Ps^{\frac{1}{2}(\mu-1)} + Qs^{\frac{1}{2}(\mu-3)} + Rs^{\frac{1}{2}(\mu-5)} + \dots$$
$$\dots + i\left\{s^{\frac{1}{2}(\mu-3)} + Cs^{\frac{1}{2}(\mu-5)} + \dots\right\}\sqrt{S}.$$

If μ is even, then A is of degree $\frac{1}{2}\mu$, and B of degree $\frac{1}{2}\mu-2$; and therefore

$$A = s^{\frac{1}{2}\mu} + Qs^{\frac{1}{2}\mu-1} + Rs^{\frac{1}{2}\mu-2} + Ss^{\frac{1}{2}\mu-3} + \dots,$$

$$B = \qquad Ps^{\frac{1}{2}\mu-2} + Cs^{\frac{1}{2}\mu-3} + \dots.$$

13. We proceed now to illustrate the preceding theory by the results for the simplest numerical values of μ; it will be noticed that, when μ is even, the results include those for the case of $\frac{1}{2}\mu$, and in addition that the resolution of the cubic S is also effected; for

$$S_{\frac{1}{2}\mu} = x\frac{\gamma_\mu}{\gamma_{\frac{1}{2}\mu}^4} = 0,$$

so that $s - s_{\frac{1}{2}\mu}$ is a factor of S; and the other factors of S are inferred by the solution of a quadratic. This resolution of the cubic S appears essential in the dynamical applications.

Passing over the cases of $\mu = 1$ and $\mu = 2$, which have no signification in the theory, we begin with

$$\mu = 3.$$

Here $x = 0$, and the integral I, as written above in § 2, assumes an illusory form; but, writing it

$$I = \tfrac{1}{2}\int \frac{s-3x}{s\sqrt{S}}\,ds,$$

where $$S = 4s^3 + (s-x)^2,$$

then $$I = \log\frac{\sqrt{S}-s+x}{2s^{\frac{3}{2}}}$$

$$= -\cosh^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} = -\sinh^{-1}\frac{s-x}{2s^{\frac{3}{2}}},$$

or $$2s^{\frac{3}{2}}e^{-I} = \sqrt{S}+s-x,$$

$$2s^{\frac{3}{2}}e^{I} = \sqrt{S}-s+x.$$

In the circular form, corresponding to parameter $v = \frac{2}{3}\omega_3$,

$$I = \tfrac{1}{2}\int \frac{s+3x}{s\sqrt{S}}\,ds,$$

where $$S = 4s^3 - (s+x)^2,$$

and $$I = \cos^{-1}\frac{s+x}{2s^{\frac{3}{2}}} = \sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}},$$

or $$2s^{\frac{3}{2}}e^{iI} = s+x+i\sqrt{S}.$$

The integrals considered in Chap. XXVI. of Legendre's *Fonctions elliptiques*, t. I., are pseudo-elliptic integrals of this class $\mu = 3$.

Putting $s = \wp u - \wp v,$

then $S = \wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3,$

provided that $12\wp v = -1,$

and then $12 g_2 = 1 + 24x,$

$$216 g_3 = 1 + 36x + 216x^2,$$

$$\Delta = -x^3(1+27x).$$

Thus the relation between this x and Klein's parameter τ, employed in his "Modular Equation of the Third Order" (*Proc. Lond. Math. Soc.*, IX., p. 123),

$$J : J-1 : 1 = (\tau-1)(9\tau-1)^3 : (27\tau^2-18\tau-1)^2 : -64\tau,$$

is expressed by $\tau = \frac{1+27x}{27x}, \quad \tau - 1 = \frac{1}{27x}.$

$$\mu = 4.$$

14. Here $y = 0$, and, working with the integral in the circular form, we find

$$I = \tfrac{1}{2}\int \frac{-s+x}{(s+x)\sqrt{S}}\,ds,$$

where $S = 4s(s+x)^2 - s^2,$

and $I = \cos^{-1}\frac{\sqrt{\{4(s+x)^2 - s\}}}{2(s+x)} = \sin^{-1}\frac{\sqrt{s}}{2(s+x)},$

or $2(s+x)\,e^{iI} = \sqrt{\{4(s+x)^2 - s\}} + i\sqrt{s}.$

Now, putting $s + x = \wp u - \wp v,$

and $S = \wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3,$

then $12\wp v = -1 + 8x,$

$$12 g_2 = 1 - 16x + 16x^2,$$

$$216 g_3 = (1-8x)(1-16x-8x^2),$$

$$\Delta = x^4(1-16x).$$

This x is connected with Klein's parameter τ, employed in his "Modular Equation of the Fourth Order" (*Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (\tau^2+14\tau+1)^3 : (\tau^3-33\tau^2-33\tau+1)^2 : 108\tau(1-\tau)^4,$$

by the relation $\tau = 1 - 16x.$

In Gierster's "Modular Equation of the Fourth Order" (*Math. Ann.*, XIV., p. 541),

$$J : J-1 : 1 = (4\tau^2-8\tau+1)^3 : (\tau-1)^2(8\tau^2-16\tau-1)^2 : 27\tau(\tau-2),$$

the connecting relation is $\tau = \dfrac{1}{8x}$.

If we put $s = \tfrac{1}{4}s'$,

and $x = -\tfrac{1}{4}(c+c^2)$,

then

$$16S = s'(s'-c-c^2)^2-s'^2$$
$$= s'-(1+c)^2\,.\,s'-c^2\,.\,s'$$
$$= S', \text{ suppose};$$

so that, dropping the accents, we may write

$$I = \tfrac{1}{2}\int \frac{(c+c^2-s)-2(c+c^2)}{(c+c^2-s)\sqrt{S}}\,ds$$
$$= \sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s} = \cos^{-1}\frac{\sqrt{s}}{c+c^2-s},$$

and $S = (s-s_\alpha)(s-s_\beta)(s-s_\gamma)$,

where, arranged in descending order if c is positive,

$$s_\alpha = (1+c)^2, \quad s_\beta = c^2, \quad s_\gamma = 0.$$

The six roots of Gierster's "Modular Equation of the Fourth Order" thus form the group

$$-\frac{1}{2c+2c^2}, \quad \frac{(1+2c)^2}{2c+2c^2}, \quad \frac{\{1\pm\sqrt{(1+2c)}\}^4}{\pm 8(1+c)\sqrt{(1+2c)}}, \quad \frac{\{1\pm i\sqrt{(1+2c)}\}^4}{\pm 8i(1+c)\sqrt{(1+2c)}}.$$

The value $s = c+c^2$ is intermediate to s_α and s_β, and the parameter corresponding to I must be taken as $\omega_1+\tfrac{1}{2}\omega_3$, the value $s = -c-c^2$ corresponding to the parameter $\tfrac{1}{2}\omega_3$; and we find

$$I(\tfrac{1}{2}\omega_3) = \tfrac{1}{2}\int \frac{(1+2c)(c+c^2+s)-2(1+2c)(c+c^2)}{(c+c^2+s)\sqrt{S}}\,ds$$
$$= \sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2+s} = \cos^{-1}\frac{(1+2c)\sqrt{s}}{c+c^2+s};$$

and thus

$$(c+c^2-s)\,e^{iI(\omega_1+\frac{1}{2}\omega_3)} = i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}+\sqrt{s},$$
$$(c+c^2+s)\,e^{iI(\frac{1}{2}\omega_3)} = i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}+(1+2c)\sqrt{s},$$

The signs are chosen so as to be suitable for the dynamical applications, in which s lies between s_β and s_γ, or between c^2 and 0.

The factors of S being known, the preceding results can be readily translated into the notation employed by Jacobi; thus

$$\kappa^2 = \frac{s_\beta - s_\gamma}{s_\alpha - s_\gamma} = \frac{c^2}{(1+c)^2}, \qquad \kappa'^2 = \frac{s_\alpha - s_\beta}{s_\alpha - s_\gamma} = \frac{1+2c}{(1+c)^2},$$

$$\text{cn}^2\,\tfrac{1}{2}\text{K}'i = \frac{-c-c^2-s_\alpha}{-c-c^2-s_\gamma} = \frac{-c-c^2-(1+c)^2}{-c-c^2} = \frac{1+2c}{c};$$

$$\text{cn}^2\,(\tfrac{1}{2}\text{K}', \kappa') = \frac{c}{1+2c}, \qquad \text{sn}^2\,(\tfrac{1}{2}\text{K}', \kappa') = \frac{1+c}{1+2c}.$$

$$\mu = 5.$$

15. The relation to be satisfied is

$$\gamma_5 = 0,$$

or

$$y = x;$$

so that, working with the circular form of the integral,

$$S = 4s\,(s+x)^2 - \{(1+x)\,s + x^2\}\,;$$

and the values $\quad s = -x \;$ and $\; 0$

may be taken to correspond to the parameters

$$\tfrac{2}{5}\omega_3 \text{ and } \tfrac{4}{5}\omega_3.$$

To calculate ρ for the parameter

$$v = \tfrac{2}{5}\omega_3,$$

we have

$$\rho\wp' v = \tfrac{1}{2}q_1 q_2 - 3\wp'' v,$$

where

$$\wp' v = -x, \quad q_1 = q_2 = 2x;$$

so that

$$\rho = x+3;$$

and thus

$$I\,(\tfrac{2}{5}\omega_3) = \tfrac{1}{2}\int \frac{(x+3)(s+x)-5x}{(s+x)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(x+3)\,s^2+Qs+R}{2\,(s+x)^{\frac{5}{2}}} = \sin^{-1}\frac{(s+C)\sqrt{S}}{2\,(s+x)^{\frac{5}{2}}},$$

and the condition

$$\{(x+3)\,s^2+Qs+R\}^2 + (s+C)^2\,S = 4\,(s+x)^5$$

gives, by equating coefficients,

$$C = -1+x, \quad Q = -1+4x+2x^2, \quad R = x^2+x^3.$$

If we put
$$s+x=t,$$
$$I(\tfrac{2}{5}\omega_3)=\tfrac{1}{2}\int\frac{(3+x)\,t-5x}{t\sqrt{T}}\,dt,$$
where
$$T=4\,(t-x)\,t^2-\{(1+x)\,t-x\}^2$$
$$=4t\,(t-x)^2-\{(1-x)\,t-x\}^2;$$
and then
$$I=\cos^{-1}\frac{(3+x)\,t^2-(1+2x)\,t^2+x}{2t^{\frac{5}{2}}}=\sin^{-1}\frac{(t-1)\sqrt{T}}{2t^{\frac{5}{2}}},$$
or
$$2t^{\frac{5}{2}}\,e^{iI}=(3+x)\,t^2-(1+2x)\,t+x+i\,(t-1)\sqrt{T}.$$

Corresponding to the parameter $\frac{4}{5}\omega_3$, we have
$$s=0\quad\text{or}\quad t-x=0;$$
and the corresponding pseudo-elliptic integral is
$$I(\tfrac{4}{5}\omega_3)=\tfrac{1}{2}\int\frac{(1-3x)(t-x)-5x^2}{(t-x)\sqrt{T}}\,dt$$
$$=\cos^{-1}\frac{(1-3x)\,t^2-(2x-4x^2-x^3)\,t+x^2-x^3}{2\,(t-x)^{\frac{5}{2}}}$$
$$=\sin^{-1}\frac{(t-x-x^2)\sqrt{T}}{2\,(t-x)^{\frac{5}{2}}},$$

This last integral can be deduced from the first integral with respect to s, by putting
$$s=\frac{t}{x^2},$$
and by writing $-1/x$ in place of x.

16. Calculating the invariants of T in the usual manner, we find
$$12g_2=1-12x+14x^2+12x^3+x^4,$$
$$216g_3=(1+x^2)(1-18x+74x^2+18x^3+x^4),$$
$$\Delta=x^5\,(1-11x-x^2);$$
or, as they may be written,
$$\frac{12g_2}{x^2}=\frac{1}{x^2}-\frac{12}{x}+14+12x+x^2,$$
$$\frac{216g_3}{x^3}=\left(\frac{1}{x}+x\right)\left(\frac{1}{x^2}-\frac{18}{x}+74+18x+x^2\right),$$
$$\frac{\Delta}{x^6}=\frac{1}{x}-11-x.$$

Comparing these results with Klein's "Modular Equation of the Fifth Order" (*Proc. Lond. Math. Soc.*, IX., p. 126; *Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (12g_2)^3 : (216g_3)^2 : 1728\Delta$$
$$= (\tau^2-10\tau+5)^3 : (\tau^2-22\tau+125)(\tau^2-4\tau-1)^2 : -1728\tau,$$

we find $$\tau = -\frac{1}{x}+11+x = -\frac{\Delta}{x^6}.$$

Also $$t = x^2$$

is a root of Halphen's equation (9), p. 5, *Fonctions elliptiques*, t. III.,

$$5t^6-12g_2 t^5+10\Delta t^3+\Delta^2 = 0,$$

or $t = 1$ is a root of this equation, if g_2 and Δ are replaced by

$$\frac{g_2}{x^2} \text{ and } \frac{\Delta}{x^6};$$

and then a root of Halphen's equation (4), p. 3, is

$$-\frac{1}{6}\left(\frac{1}{x}+x\right).$$

Also $$12\wp\tfrac{2}{5}\omega_3 = -\frac{1}{x}-6-x,$$

$$12\wp\tfrac{1}{5}\omega_3 = -\frac{1}{x}+6-x.$$

Mr. W. Burnside points out that, if one root τ_∞ of the equation

$$\frac{(\tau^2-10\tau+5)^3}{-1728\tau} = J$$

is given by $$\tau_\infty = \frac{125}{\tau_0},$$

where $$\tau_0 = x+11-\frac{1}{x},$$

then the remaining five roots are given by

$$\tau_r = \frac{(\epsilon^{-r}x^{\frac{1}{5}}+1-\epsilon^r x^{-\frac{1}{5}})^6}{x+11-\frac{1}{x}},$$

$$r = 0,\ 1,\ 2,\ 3,\ 4; \quad \epsilon = e^{\frac{2}{5}\pi i};$$

so that $x^{\frac{1}{5}}$ is the *ikosahedron irrationality* (Klein, *Math. Ann.*, XIV., p. 156; and *Lectures on the Ikosahedron*).

$$\mu = 6.$$

17. The relation $\qquad \gamma_6 = 0,$

or $\qquad y - x - y^2 = 0,$

gives $\qquad x = y - y^2;$

and $\qquad s_3 + x = y,$

or $\qquad s_3 = y^2;$

and this value of s_3 is a root of $S = 0$, where

$$S = 4s\,(s+y-y^2)^2 - \{(1+y)\,s + y^2 - y^3\}^2$$
$$= (s-y^2)\,\{4s^2 - (1-y)(1-5y)\,s + y^2\,(1-y)^2\}.$$

Calculating the invariants of S, we find that y is connected with τ in the "Modular Equation of the Sixth Order" (Gierster, *Math. Ann.*, XIV., p. 541; Klein and Fricke, *Modulfunctionen*, II., p. 61),

$$J : J-1 : 1 = 4\,(\tau+3)^3\,(\tau^3+9\tau^2+21\tau+3)^3$$
$$: (\tau^2+6\tau+6)^2\,(2\tau^4+24\tau^3+96\tau^2+126\tau-9)^2$$
$$: 27\tau\,(\tau+4)^3\,(2\tau+9)^2,$$

by the relation $\qquad \tau = \dfrac{1-9y}{2y}.$

According to Gierster, this τ is connected with the τ for $\mu = 3$, distinguished as τ_3, by the relation

$$\tau_3 = \frac{\tau\,(2\tau+9)^2}{-27\,(\tau+4)},$$

so that $\qquad x_3 = \dfrac{1}{27\,(\tau_3-1)} = -\dfrac{\tau+4}{4\,(\tau+3)^3} = -\dfrac{y^2-y^3}{(1-3y)^3}.$

This is easily inferred by a rearrangement of terms in S, when

$$S = 4s^3 - \{(1-3y)\,s - y^2\,(1-y\}^2,$$

and putting $\qquad s = \dfrac{s'}{m^6};$

then $\qquad m^6 S = 4s'^3 - \{(1-3y)\,ms' - m^3 y^2\,(1-y)\}^2 = S'$, suppose,

and $\qquad S' = 4s'^3 - (s'+x_3)^2.$

This is the same form as in the case of $\mu = 3$, if we take

$$m = \frac{1}{1-3y}, \quad x_3 = -m^3 y^2\,(1-y) = -\frac{y^2-y^3}{(1-3y)^3}.$$

18. Taking out the linear factor $s-y^2$ of S, the discriminant of the remaining quadratic factor is

$$(1-y)^2(1-5y)^2-16y^2(1-y)^2 = (1-y)^3(1-9y);$$

and the roots of the quadratic are rational functions of c, if we put

$$\frac{1-9y}{1-y} = \left(\frac{1+c}{1-c}\right)^2,$$

and then

$$y = \frac{c}{(1-2c)(c-2)},$$

$$r = -\frac{1}{c}-2-c = -\frac{(1+c)^2}{c},$$

$$r+4 = -\frac{(1-c)^2}{c},$$

$$2r+9 = -\frac{(1-2c)(2-c)}{c},$$

and

$$r_3 = -\frac{(1+c)^2(2-c)^2(1-2c)^2}{27(c-c^2)^2},$$

$$r_3-1 = -\frac{4(1-c+c^2)^3}{27(c-c^2)^2}.$$

The three roots of S are now

$$\frac{c^2}{(2-5c+2c^2)^2},\quad \frac{(1-c)^2}{(2-5c+2c^2)^2},\quad \frac{(c-c^2)^2}{(2-5c+2c^2)^2},$$

and

$$x = -\frac{2c(1-c)^2}{(2-5c+2c^2)^2}.$$

We may drop the denominator $(2-5c+2c^2)^2$, and write the corresponding pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int\frac{P\{2c(1-c)^2-s\}+Q}{\{2c(1-c)^2-s\}\sqrt{S}}\,ds,$$

where

$$S = s-(1-c)^2\,.\,s-c^2\,.\,s-(c-c^2)^2,$$

so that, arranged in descending order, if $c<\tfrac{1}{2}$,

$$s_\alpha = (1-c)^2,\quad s_\beta = c^2,\quad s_\gamma = (c-c^2)^2,$$

also

$$S = s^3-\{(1-c+c^2)\,s-(c-c^2)^2\}^2.$$

As $2c(1-c)^2$ now lies between s_α and s_β, the corresponding parameter

$$v = \omega_1+\tfrac{1}{3}\omega_3.$$

19. Taking the integral $I(\frac{2}{3}\omega_3)$ as derived from $\mu = 3$, and making the requisite substitutions,

$$(1)\quad I(\tfrac{2}{3}\omega_3) = \tfrac{1}{2}\int \frac{(1-c+c^2)\,s-3\,(c-c^2)^2}{s\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(1-c+c^2)\,s-(c-c^2)^2}{s^{\frac{3}{2}}} = \sin^{-1}\frac{\sqrt{S}}{s^{\frac{3}{2}}}.$$

By the substitution

$$s-s_\gamma = \frac{s_\alpha-s_\gamma\,.\,s_\beta-s_\gamma}{t-e_\gamma},$$

afterwards replacing t by s, we deduce

$$(2)\quad I(\tfrac{1}{3}\omega_3) = \tfrac{1}{2}\int \frac{(1+c)(2-c)\left\{s+2c-2c^2-3\,(c-c^2)\right\}}{(s+2c-2c^2)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(s-2+c-c^2)\sqrt{\{s-(c-c^2)^2\}}}{(s+2c-2c^2)^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{(1+c)(2-c)\sqrt{\{(1-c)^2-s\,.\,c^2-s\}}}{(s+2c-2c^2)^{\frac{3}{2}}}.$$

Again, by the substitution

$$s-s_\alpha = \frac{s_\alpha-s_\beta\,.\,s_\alpha-s_\gamma}{t-e_\alpha},$$

where t is afterwards replaced by s, we obtain

$$(3)\quad I(\omega_1+\tfrac{2}{3}\omega_3) = \tfrac{1}{2}\int \frac{(1+c)(1-2c)\left\{2c^2-2c^3-s-3\,(c^2-c^3)\right\}}{(2c^2-2c^3-s)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(s-c^2-c^3+2c^4)\sqrt{\{(1-c)^2-s\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{(1+c)(1-2c)\sqrt{\{c^2-s\,.\,s-(c-c^2)^2\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}};$$

and finally, in a similar manner,

$$(4)\quad I(\omega_1+\tfrac{1}{3}\omega_3) = \tfrac{1}{2}\int \frac{(2-c)(1-2c)\left\{2c\,(1-c)^2-s-3c\,(1-c)^2\right\}}{\left\{2c\,(1-c)^2-s\right\}\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{\left\{s-(1-c)^2\,(2-3c+2c^2)\right\}\sqrt{(c^2-s)}}{\left\{2c\,(1-c)^2-s\right\}^{\frac{3}{2}}}$$

$$= \cos^{-1}\frac{(2-c)(1-2c)\sqrt{\{(1-c)^2-s\,.\,s-(c-c^2)^2\}}}{\left\{2c\,(1-c)^2-s\right\}^{\frac{3}{2}}},$$

so that we infer in the original integral

$$P = (2-c)(1-2c), \quad Q = -3c\,(1-c)^2\,(2-c)(1-2c).$$

These four integrals can be expressed concisely as follows

$$(1)\quad s^{\frac{3}{2}}\,e^{iI(\frac{2}{3}\omega_3)} = (1-c+c^2)\,s-(c-c^2)^2+i\sqrt{S}$$

$$= (1-c+c^2)\,s-(c-c^2)^2$$

$$+i\sqrt{\{s-(1-c)^2\,.\,s-c^2\,.\,s-(c-c^2)^2\}},$$

$$(2)\quad (s+2c-2c^2)^{\frac{3}{2}}\,e^{iI(\frac{1}{3}\omega_3)} = (s-2+c-c^2)\sqrt{\{s-(c-c^2)^2\}}$$

$$+i\,(1+c)(2-c)\sqrt{\{(1-c)^2-s\,.\,c^2-s\}},$$

$$(3)\quad (2c^2-2c^3-s)^{\frac{3}{2}}\,e^{iI(\omega_1+\frac{2}{3}\omega_3)} = (s-c^2-c^3+2c^4)\sqrt{\{(1-c)^2-s\}}$$

$$-i\,(1+c)(1-2c)\sqrt{\{c^2-s\,.\,s-(c-c^2)^2\}},$$

$$(4)\quad \{2c\,(1-c)^2-s\}^{\frac{3}{2}}\,e^{iI(\omega_1+\frac{1}{3}\omega_3)} = -\{s-(1-c)^2\,(2-3c+2c^2)\}\sqrt{(c^2-s)}$$

$$-i\,(1-2c)(2-c)\sqrt{\{(1-c)^2-s\,.\,s-(c-c^2)^2\}}.$$

20. With

$$s = \wp u-\wp 2v,$$

where

$$3\wp 2v = -(1-c+c^2)^2,$$

the preceding values show that

$$3\wp\tfrac{2}{3}\omega_3 = -(1-c+c^2)^2,$$

$$3\wp\tfrac{1}{3}\omega_3 = -(1-c+c^2)^2-6c+6c^3,$$

$$3\wp\,(\omega_1+\tfrac{2}{3}\omega_3) = -(1-c+c^2)^2+6c^2-6c^3,$$

$$3\wp\,(\omega_1+\tfrac{1}{3}\omega_3) = -(1-c+c^2)+6c\,(1-c)^2.$$

Then

$$\wp\tfrac{1}{3}\omega_3+\wp\tfrac{2}{3}\omega_3+\wp\omega_3 = -1,$$

$$\wp\,(\omega_1+\tfrac{1}{3}\omega_3)+\wp\tfrac{2}{3}\omega_3+\wp\omega_2 = -(1-c)^4.$$

In the Jacobian notation

$$\kappa^2 = \frac{2c^3-c^4}{(1-c)^3\,(1+c)}, \quad \kappa'^2 = \frac{1-2c}{(1-c)^3\,(1+c)};$$

and then

$$\mathrm{cn}\,(\tfrac{2}{3}K', \kappa') = c, \quad \mathrm{sn}\,(\tfrac{2}{3}K', \kappa') = 1-c,$$

so that

$$\mathrm{sn}\,\tfrac{1}{3}K'+\mathrm{cn}\,\tfrac{2}{3}K' = 1,$$

a well known relation.

21. To find the values of s corresponding to the argument $\frac{2}{3}\omega_1$, we start with the equation

$$\wp 2u + 2\wp u = \tfrac{1}{4}\left(\frac{\wp'' u}{\wp' u}\right)^2,$$

and put
$$\wp 2u = \wp u;$$

then
$$3\wp u = \tfrac{1}{4}\left(\frac{\wp'' u}{\wp' u}\right)^2.$$

or, expressed in terms of s,

$$3s-(1-c+c^2)^2 = \frac{\left[3s^2-2(1-c+c^2)\{(1-c+c^2)s-(c-c^2)^2\}\right]^2}{4s^3-4\{(1-c+c^2)s-(c-c^2)^2\}^2},$$

$$12s^4-4(1-c+c^2)^2s^3-12s\{(1-c+c^2)s-(c-c^2)^2\}^2$$
$$+4(1-c+c^2)^2\{(1-c+c^2)^2s-(c-c^2)^2\}^2$$
$$=9s^4-12s^2(1-c+c^2)\{(1-c+c^2)s-(c-c^2)^2\}$$
$$+4(1-c-c^2)^2\{(1-c+c^2)s-(c-c^2)^2\}^2,$$

or $$3s^3-4(1-c+c^2)^2s^2+12(1-c+c^2)(c-c^2)^2s-12(c-c^2)^4=0.$$

This equation may be written as the difference of two cubes

$$\{(1-c+c^2)s-3(c-c^2)^2\}^3-\tfrac{1}{4}(1+c)^2(1-2c)^2(2-c)^2s^3=0,$$

so that $$\frac{3(c-c^2)^2}{s} = 1-c+c^2-\tfrac{1}{2}\sqrt[3]{2}\,(1+c)^{\frac{2}{3}}(1-2c)^{\frac{2}{3}}(2-c)^{\frac{2}{3}},$$

giving the value of s corresponding to $\frac{2}{3}\omega_1$; the values of s in which $\frac{1}{2}\sqrt[3]{2}$ is replaced by $\frac{1}{2}\omega\sqrt[3]{2}$ or $\frac{1}{2}\omega^2\sqrt[3]{2}$, where ω denotes an imaginary cube root of unity, correspond to the arguments

$$\tfrac{2}{3}(\omega_1 \pm \omega_3).$$

Mr. W. Burnside points out that these results are in agreement with those given in the *Math. Ann.*, XIV., p. 156, and in Klein and Fricke's *Modulfunctionen*, I., p. 630, if the tetrahedral forms ξ_3 and ξ_4 are given by

$$\xi_3 = -2x_1 = 1-c+c^2,$$

$$\xi_4^3 = x_2^3 = \tfrac{1}{4}(1+c)^2(2-c)^2(1-2c)^2 = \xi_3^3-\tfrac{27}{4}(c-c^2)^2.$$

22. With given r_3 in § 18, the three values of r_6 form the group

$$-\frac{(1+c)^2}{c},\quad \frac{(2-c)^2}{1-c},\quad \frac{(1-2c)^2}{c-c^2}.$$

But, with given J, the four roots of the tetrahedral equation or "Modular Equation of the Third Order" can be written (*Math. Ann.*, XIV., p. 155)

$$\tau_3 = \frac{1}{8\eta^3+1}, \text{ and } \frac{1}{9}\,\frac{(2\epsilon\eta+1)^4}{8\eta^3+1}, \quad \epsilon^3 = 1;$$

so that
$$8\eta^3 = -\frac{4(1-c+c^2)^3}{(1+c)^2(2-c)^2(1-2c)^2},$$

and thence the remaining nine roots of the Gierster's "Modular Equation of the Sixth Order" can be determined.

In the notation of the *Modulfunctionen*, I., pp. 684, 686,

$$x^4 = 2\frac{(1-c)^2}{c} = -2(\tau_6+4),$$

$$y^2 = -\frac{(2-c)(1-2c)}{c}.$$

The relation connecting τ_3 with $\tau = \tau_6$ is (Gierster, *Math. Ann.*, XIV., p. 540)

$$\tau_3 = \frac{\tau(\tau+4)^3}{-4(2\tau+9)};$$

and (*Math. Ann.*, XIV., p. 154)

$$\tau_3 = \frac{1}{4\kappa^2\kappa'^2}, \quad -\frac{(1-\kappa^2)^2}{4\kappa^2}, \quad -\frac{\kappa^4}{4(1-\kappa^2)},$$

for a given J; we thus obtain another grouping of the twelve roots of the "Modular Equation of the Sixth Order."

23. If we had worked with Abel's form of the integral

$$I = 6\int \frac{x+k}{\sqrt{\{(x^2+ax+b)^2+ex\}}}\,dx,$$

then Abel's condition
$$q_4 = 0,$$
is equivalent to
$$(e+2ab)(e+4ab) = 8b^3$$
(*Œuvres complètes*, I., p. 143); or, putting

$$\frac{16b^3}{e^2} = m, \quad 1+\frac{4ab}{e} = n,$$

this becomes
$$m = n(n+1);$$

and then, if $$n = -p^2,$$

$$X = \{x^2-(1+p^2)x-p^2(1-p^2)\}^2$$
$$-4p^2(1-p^2)x\{x^2-(1+p^2)x+p^2(1+p^2)\}^2-4p^2(x-p^2)^2$$
$$= X_1X_2,$$

where $$X_1 = x^2-(1+p)^2x+(p+p^2)^2,$$
$$X_2 = x^2-(1-p)^2x+(p-p^2)^2.$$

We now find by Abel's method that

$$6k = -1-3p^2,$$

so that, finally,

$$I = \int \frac{6x-1-3p^2}{\sqrt{(X_1X_2)}}\,dx$$
$$= 2\cosh^{-1}\frac{x^2-(2+p+p^2)x+(1+p)(1+p^2)}{2p^{\frac{1}{2}}(1-p^2)}\sqrt{X_2}$$
$$= 2\sinh^{-1}\frac{x^2-(2-p+p^2)x+(1-p)(1-p^2)}{2p^{\frac{1}{2}}(1-p^2)}\sqrt{X_1};$$

and, on comparing this with the former use of c in § 20, we find

$$\text{sn}\,\tfrac{1}{3}K = 1-c = \frac{1-p}{1+p}.$$

$$\mu = 7.$$

24. The relation $$\gamma_7 = 0,$$
or $$(y-x)x-y^3 = 0,$$
on putting $$y-x = yz,$$
is satisfied by $$y = z(1-z),\quad x = z(1-z)^2;$$
so that, taking the circular form of the integrals,

$$S = 4s\{s+z(1-z)^2\}^2-\{(1+z-z^2)s+z^2(1-z)^3\}^2.$$

Now, with the notation

$$s_m+x = \wp mv-\wp v,$$

where $$12\wp v = -4x-(y+1)^2,$$
then $$s_1 = -x = -z(1-z)^2,$$
$$s_2 = 0,$$
$$s_3 = y-x = z^2(1-z)^2,$$

which we may suppose to correspond to the parameters

$$\tfrac{2}{7}\omega_3,\quad \tfrac{4}{7}\omega_3,\quad \tfrac{6}{7}\omega_3;$$

and thus $$12\wp\tfrac{2}{7}\omega_3 = -1-6z+\ \ 9z^2-\ \ 2z^3-z^4,$$

$$12\wp\tfrac{4}{7}\omega_3 = -1+6z-15z^2+10z^3-z^4,$$

$$12\wp\tfrac{6}{7}\omega_3 = -1+6z-\ \ 3z^2-\ \ 2z^3-z^4;$$

so that $$G_1 = \wp\tfrac{2}{7}\omega_3+\wp\tfrac{4}{7}\omega_3+\wp\tfrac{6}{7}\omega_3 = -\tfrac{1}{4}(1-z+z^2)^2,$$

This expression is a root x of Halphen's equation (15), *F. E.*, III., p. 51,

$$x^8-21g_2x^6-2.3^2.7g_3x^5 \ldots - \frac{3^4.7}{2^8}g_2^4 = 0,$$

while $$y = -4x = -4G_1 = -7B_0^2$$

is a root of equation (10), p. 398, *Modulfunctionen*, II.

Also, in Halphen's notation, p. 52,

$$t = -(s_2-s_3)(s_3-s_1)(s_1-s_2) = z^4(1-z)^4,$$

and this is a root of his equation (59), p. 75,

$$7t^8-2^3.3^3g_3t^7-2.5.7.\Delta t^6-3^3.7.\Delta^2t^4-2.7.\Delta^3t^2-\Delta^4 = 0.$$

Calculating the invariants of the cubic S, we shall find

$$12g_2 = (1-z+z^2)(1-11z+30z^2-15z^3-10z^4+5z^5+z^6),$$

$$216g_3 = 1-18z+\ldots \qquad\qquad \ldots+6z^{11}+z^{12},$$

$$\Delta = z^7(1-z)^7(1-8z+5z^2+z^3).$$

25. The parameter τ employed in Klein's "Modular Equation of the Seventh Order" (*Proc. Lond. Math. Soc.*, IX., p. 125; *Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1$$

$$= (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3 : (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau,$$

is now readily seen to be connected with z by the relation

$$\tau = \frac{1-8z+5z^2+z^3}{z(1-z)};$$

and then $$\tau^2+13\tau+49 = \frac{(1-z+z^2)^3}{z^2(1-z)^2}.$$

This relation is unaltered if z is replaced by $\dfrac{1}{1-z}$, or by $\dfrac{z-1}{z}$, thus constituting a *group* for this cubic equation in z, corresponding to the parameters $\frac{2}{7}\omega_3$, $\frac{4}{7}\omega_3$, $\frac{6}{7}\omega_3$.

Referring to Klein's article in the *Math. Ann.*, XIV., p. 425, "Ueber Transformation siebenter Ordnung," and to the *Modulfunctionen*, t. I., Abschnitt III., Chaps. VI. and VII., we find that we can put

$$\lambda = z_2 = -z\,(z-1)^{\frac{2}{7}},$$

$$\mu = z_1 = -z^{\frac{5}{7}}(z-1)^{\frac{1}{7}},$$

$$\nu = z_4 = z^{\frac{4}{7}}(z-1)^{\frac{4}{7}};$$

thus satisfying the relation

$$\lambda^3\mu+\mu^3\nu+\nu^3\lambda = 0,$$

or

$$z_1^3 z_4+z_4^3 z_2+z_2^3 z_1 = 0.$$

Also

$$A_0\sqrt{-\Delta} = z^{\frac{1}{2}}(z-1)^{\frac{1}{2}},$$

$$A_1\sqrt{-\Delta} = -z^{\frac{13}{14}}(z-1)^{\frac{5}{14}},$$

$$A_2\sqrt{-\Delta} = -z^{\frac{5}{14}}(z-1)^{\frac{13}{14}},$$

$$A_4\sqrt{-\Delta} = z^{\frac{3}{14}}(z-1)^{\frac{5}{14}};$$

so that, as a verification,

$$\frac{A_1}{A_0} = -z^{\frac{3}{7}}(z-1)^{-\frac{1}{7}} = \frac{z_2}{z_4},$$

$$\frac{A_2}{A_0} = -z^{-\frac{1}{7}}(z-1)^{\frac{3}{7}} = \frac{z_4}{z_1},$$

$$\frac{A_4}{A_0} = z^{-\frac{2}{7}}(z-1)^{-\frac{1}{7}} = \frac{z_1}{z_2}.$$

Now, if for given J, one root of the "Modular Equation of the Seventh Order" is given by

$$\tau_\infty = -\frac{49z(z-1)}{z^3+5z^2-8z+1}$$

$$= -\frac{49}{5+z+\dfrac{1}{1-z}+\dfrac{z-1}{z}},$$

the seven remaining roots are expressed by

$$\tau_r = -\frac{\{1+\epsilon^{-r}z^{\frac{3}{7}}(z-1)^{-\frac{6}{7}}+\epsilon^{-2r}z^{-\frac{1}{7}}(z-1)^{\frac{2}{7}}+\epsilon^{-4r}z^{-\frac{2}{7}}(z-1)^{-\frac{3}{7}}\}^4}{5+z+\dfrac{1}{1-z}+\dfrac{z-1}{z}},$$

$$r = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6; \quad \epsilon = e^{\frac{2}{7}\pi i}.$$

Thus the irrationality $\qquad z^{\frac{3}{7}}(z-1)^{\frac{1}{7}}$

plays the same part here as the icosahedron irrationality $x^{\frac{1}{5}}$ in § 16 above.

We can put
$$z = -x-\frac{1}{x},$$

and then
$$\tau = \frac{x^7-1}{x(x^2+1)(x^3-1)}.$$

26. Taking the integral
$$I(v) = \tfrac{1}{2}\int \frac{\rho(s+x)-7x}{(s+x)\sqrt{S}}\,ds,$$

and calculating $\rho = P$ by the formula of § 9,

$$\rho x = \tfrac{1}{2}(q_4q_3+q_3q_2+q_2q_1)-5\wp'' v,$$

where
$$\tfrac{1}{2}q_1 = \tfrac{1}{2}q_4 = x = z(1-z)^2,$$
$$\tfrac{1}{2}q_2 = \tfrac{1}{2}q_3 = y = z(1-z),$$
$$\wp'' v = x(y+1) = z(1-z)^2(1+z-z^2),$$

then
$$\rho = -5+z+z^2,$$

and
$$I(v) = \cos^{-1}\frac{Ps^3+Qs^2+Rs+T}{2(s+x)^{\frac{7}{2}}}$$
$$= \sin^{-1}\frac{(s^2+Cs+D)\sqrt{S}}{2(s+x)^{\frac{7}{2}}},$$

or
$$2(s+x)^{\frac{7}{2}}e^{iI(v)} = Ps^3+Qs^2+Rs+T+i(s^2+Cs+D)\sqrt{S}.$$

Knowing ρ or P, we can find Q, R, T, C, D, by comparing coefficients in the relation

$$(Ps^3+Qs^2+Rs+T)^2+(s^2+Cs+D)^2S = 4(s+x)^7,$$

and also in the verification by differentiation; and thence

$$C = -3+4z-4z^2+2z^3,$$
$$D = (1-z)^2(1-2z+z^2-z^3),$$
$$Q = 5-17z+28z^2-15z^3-3z^4+3z^5,$$
$$R = -(1-z)^3(1-5z+11z^2-12z^3+0+3z^5),$$
$$T = z^2(1-z)^6(1-2z+z^2+z^3).$$

A similar procedure will serve for the pseudo-elliptic integrals,

$$I(2v) = \tfrac{1}{2}\int \frac{P_2 s+7xy}{s\sqrt{S}}\,ds$$

and
$$I(3v) = \tfrac{1}{2}\int \frac{P_3(s+x-y)+7\sqrt{(-S_3)}}{(s+x-y)\sqrt{S}}\,ds.$$

By putting
$$s+x = t,$$
$$I(v) = \tfrac{1}{2}\int \frac{Pt-7z(1-z)^2}{t\sqrt{T}}\,dt,$$

where
$$T = 4t(t+z^2-z^3)^2-\{(1-z+z^2)\,t-z(1-z)^2\}^2;$$

and putting
$$s+x-y = v,$$
$$I(3v) = \tfrac{1}{2}\int \frac{P_3 v+7z(1-z)}{v\sqrt{V}}\,dv,$$

where
$$V = 4v(v+z^2-z^3)^2-\{(1-3z+z^2)\,v-z^3(1-z)\}^2.$$

$$\mu = 8.$$

27. The relation
$$\gamma_8 = 0,$$
or
$$x(y-x-y^2)-(y-x)^2 = 0,$$
is found, on putting
$$x = y(1-z),$$
to be satisfied by
$$y = \frac{z-2z^2}{1-z},\quad x = z-2z^2.$$

Now
$$s_4+x = \frac{x(y-x)}{y^2} = z-z^2,$$
so that
$$s_4 = z^2,$$
is a root of
$$S = 4s(s+x)^2-\{(y+1)\,x+xy\}^2 = 0;$$
so that
$$S = (s-z^2)\left\{4s^2-\frac{(1-2z)^4}{(1-z)^2}\,s+\frac{z^2(1-2z)^4}{(1-z)^2}\right\}.$$

Calculating the invariants of this cubic S,

$$12g_2(1-z)^4 = \left\{(1-2z)^4+4z^2(1-z)^2\right\}^2-24z^2(1-z)^2(1-2z)^4,$$

$$216g_3(1-z)^6 = \left\{(1-2z)^4+4z^2(1-z)^2\right\}^3$$
$$-36z^2(1-z)^2(1-2z)^4\left\{(1-2z)^4+4z^2(1-z)^2\right\}$$
$$+216z^4(1-z)^4(1-2z)^4,$$

$$\Delta(1-z)^{12} = z^8(1-z)^8(1-2z)^4(1-8z+8z^2).$$

Gierster's "Modular Equation of the Eighth Order" is (*Math. Ann.*, XIV., p. 541)

$$J:J-1:1 = 4(\tau^4-8\tau^3+20\tau^2-16\tau+1)^3$$
$$:(\tau^2-4\tau+2)^2(2\tau^4-16\tau^3+40\tau^2-32\tau-1)^2:27\tau(\tau-4)(\tau-2)^2;$$

so that this τ, distinguished as τ_8, is connected with our z by

$$\tau=\tau_8=\frac{1}{2z(1-z)},\quad \tau-2=\frac{(1-2z)^2}{2z(1-z)},\quad \tau-4=\frac{1-8z+8z^2}{2z(1-z)}.$$

This τ_8 is connected with τ_4, employed in Gierster's "Modular Equation of the Fourth Order," by the relation

$$\tau_4 = -\tfrac{1}{2}\tau_8(\tau_8-4) = -\frac{1-8z+8z^2}{8z^2(1-z)^2},$$

so that the corresponding x, distinguished as x_4, is given by

$$x_4=\frac{1}{8\tau_4}=-\frac{1}{4\tau_8(\tau_8-4)}=-\frac{z^2(1-z)^2}{1-8z+8z^2}.$$

$$1+2c_4=\sqrt{(1-16x_4)}=\frac{(1-2z)^2}{\sqrt{(1-8z+8z^2)}}.$$

28. Starting with the integral

$$I=\tfrac{1}{2}\int\frac{\rho s-8x}{(s+x)\sqrt{S}}\,ds,$$

and calculating $\rho = P$, then, with $\mu = 8$,

$$-\rho x = \tfrac{1}{2}(q_5q_4+q_4q_3+q_3q_2+q_2q_1)-6\wp''c.$$

and

$$q_5=q_1=2(s_2+x)=2x,$$

$$q_4=q_2=2(s_3+x)=2y,$$

$$q_3 \quad =2(s_4+x)=2x\frac{y-x}{y^2};$$

so that
$$-\rho x = q_1 q_2 + q_2 q_3 - 6\wp'' r$$
$$= 4xy + 4x\frac{y-x}{y} - 6x(y+1)$$
$$= -4\frac{x^2}{y} - 2x(y+1),$$
$$\rho = 4\frac{x}{y} + 2y + 2$$
$$= 4(1-z) + 2\frac{1-2z^2}{1-z} = 2\frac{3-4z}{1-z}.$$

Replacing s by $\frac{t}{(1-z)^2}$, the pseudo-elliptic integral can be written
$$I = \tfrac{1}{2}\int \frac{(3-4z)\{t+z(1-z)^2(1-2z)\} - 4z(1-z)^3(1-2z)}{\{t+z(1-z)^2(1-2z)\}\sqrt{T}}\,dt,$$
where
$$T = 4t\{t+z(1-z)^2(1-2z)\}^2 - \{(1-2z^2)t + z^2(1-z)^2(1-2z)^2\}^2$$
$$= \{t - z^2(1-z)^2\}\{4t^2 - (1-2z)^4 t + z^2(1-z)^2(1-2z)^4\}.$$

29. Denoting the roots of this equation $T = 0$, in descending order, by
$$t_\alpha,\ t_\beta,\ t_\gamma,$$
$$t_\alpha = (1-2z)^2\left\{\frac{1+\sqrt{(1-8z+8z^2)}}{4}\right\}^2,$$
$$t_\beta = (1-2z)^2\left\{\frac{1-\sqrt{(1-8z+8z^2)}}{4}\right\}^2,$$
$$t_\gamma = z^2(1-z)^2.$$

Now we find that the integral
$$I = \cos^{-1}\frac{t+C}{2\{t+z(1-z)^2(1-2z)\}^{\frac{3}{2}}}$$
$$\sqrt{\{4t^2-(1-2z)^4 t + z^2(1-z)^2(1-2z)^4\}}$$
$$= \sin^{-1}\frac{(3-4z)t+Q}{2\{t+z(1-z)^2(1-2z)\}^{\frac{3}{2}}}\sqrt{\{t-z^2(1-z)^2\}},$$
where, as determined by an algebraical verification,
$$C = -(1-z)^2(1-2z+2z^2),$$
$$Q = -(1-z)^2(1-2z)^3.$$

This integral I being $I(v)$, where we may suppose $v = \frac{1}{4}\omega_3$, then $I(2v)$ is of the form

$$I(2v) = \tfrac{1}{2}\int \frac{Pt-4xy}{t\sqrt{T}}\,dt;$$

but, as
$$I(2v) = I(\tfrac{1}{2}\omega_3),$$

it falls under the case of $\mu = 4$, and the expression can be correspondingly simplified, at the same time affording interesting comparisons and verifications.

Since
$$I(3v) = I(\omega_3 - \tfrac{1}{4}\omega_3),$$

it can be derived from $I(v)$ by the substitution

$$t - t_3 = \frac{t_1 - t_3 \,.\, t_2 - t_3}{t' - t_3},$$

or
$$t - z^2(1-z)^2 = \frac{z^4(1-z)^4}{t' - z^2(1-z)^2};$$

and we thus find, dropping the accent of t,

$$I(\tfrac{3}{4}\omega_3) = \tfrac{1}{2}\int \frac{(1-4z)\{t-z^2(1-z)(1-2z)\} - 4z^3(1-z)(1-2z)}{\{t-z^2(1-z)(1-2z)\}\sqrt{T}}\,dt.$$

30. The roots of $T = 0$ can be expressed rationally in terms of a parameter c, by putting

$$z = \frac{c-2c^2}{1-2c^2},$$

and then
$$1 - 8z + 8z^2 = \frac{(1-4c+2c^2)^2}{(1-2c^2)^2},$$

$$t_\alpha = \tfrac{1}{4}\frac{(1-2c)^2(1-2c+2c^2)^2}{(1-2c^2)^4},$$

$$t_\beta = \frac{c^2(1-c)^2(1-2c+2c^2)^2}{(1-2c^2)^4},$$

$$t_\gamma = \frac{c^2(1-c)^2(1-2c)^2}{(1-2c^2)^4},$$

and
$$t_\alpha > t_\beta,$$

provided $\dfrac{1-4c+2c^2}{1-2c^2} = \dfrac{2(1-c)^2-1}{1-2c^2}$ is positive.

To ensure that $v = \frac{1}{4}\omega_3$, we must have

$$-z(1-z)^2(1-2z) - z^2(1-z)^2 = -z(1-z)^2,$$

and

$$z^2(1-z)(1-2z) - z^2(1-z)^2 = -z^2(1-z),$$

both negative, and therefore

$$z(1-z) \quad \text{or} \quad c(1-c)(1-2c) \text{ must be positive ;}$$

otherwise we should have

$$v = \omega_1 + \tfrac{1}{4}\omega_3,$$

and $I(v)$ would be suitable for the construction of an algebraical herpolhode.

According to Gierster (*Math. Ann.*, XIV., p. 540) the parameter τ_2 is connected with $\tau_8 = \tau$ by the relation

$$4\tau_2 = -\tau(\tau-4)(\tau-2)^2;$$

so that, expressed in terms of c,

$$4\tau_2 = -\frac{(1-2c^2)^2(1-4c+2c^2)^2(1-2c+2c^2)^2}{16c^4(1-c)^4(1-2c)^4};$$

and this τ_2 is therefore given by

$$\tau_2 = -\frac{(1-\kappa^2)^2}{4\kappa^2},$$

since, expressed in terms of c,

$$\kappa^2 = \frac{4c^4(1-c)^4}{(1-2c)^4}, \quad \kappa'^2 = \frac{(1-2c^2)(1-4c+2c^2)(1-2c+2c^2)^2}{(1-2c)^4}.$$

31. It is convenient to drop the denominator $(1-2c^2)^4$, by putting

$$t = \frac{s}{(1-2c^2)^4};$$

and now

$$I(\tfrac{1}{4}\omega_3) = \tfrac{1}{2}\int \frac{(1-2c^2)(3-4c+2c^2)\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\} - 4c(1-c)^3(1-2c)(1-2c+2c^2)(1-2c^2)}{\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\}\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{s-(1-c)^2(1-2c+2c^2-4c^3+4c^4)}{2\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\}^{\frac{1}{2}}}$$

$$\sqrt{\{4c-(1-2c)^2(1-2c+2c^2)^2 \cdot s - c^2(1-c)^2(1-2c+2c^2)^2\}}$$

$$= \cos^{-1}\frac{(1-2c^2)(3-4c+2c^2)\,s-(1-c)^2(1-2c+2c^2)^2}{2\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\}^{\frac{1}{2}}}$$

$$\sqrt{\{s-c^2(1-c)^2(1-2c)^2\}}.$$

$$I\left(\tfrac{3}{4}\omega_3\right) = \tfrac{1}{2}\int \frac{\begin{array}{r}(1-2c^2)(1-4c+6c^2)\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\} \\ -4c^3(1-c)(1-2c)^3(1-2c+2c^2)(1-2c^2)\end{array}}{\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\}\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{s-c^2(1-2c)^2(1-4c^3+4c^4)}{\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\}^2}\sqrt{(s-s_\alpha\,.\,s-s_\beta)}$$

$$= \cos^{-1}\frac{Ps+Q}{\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\}^{\frac{3}{2}}}\sqrt{(s-s_\gamma)}.$$

32. Working with Abel's form, involving the quartic X, we shall find that, with $\mu = 8$, we can split up the quartic X into two quadratics X_1 and X_2, of the form

$$X = \{x^2-(1-2c^2)\,x-c\,(1-c)^2(1-2c)\}^2-4c\,(1-c)^4(1-2c)\,x$$

$$= \{x^2-(1-2c^2)\,x+c\,(1-c)^2\}^2-4c\,(1-c)^4\,(x-c+c^2)^2,$$

$$X_1, X_2 = x^2-(1-2c^2)\,x+c\,(1-c)^2 \pm 2\,(1-c)\sqrt{(c-c^2)(x-c+c^2)};$$

and then $$8k = -1-4c+8c^2,$$

and $$I = \int \frac{8x-1-4c+8c^2}{\sqrt{X}}\,dx$$

is a pseudo-elliptic integral, and the result is

$$I = \cosh^{-1} A\,(x^3-Px^2+Qx-R)\sqrt{X_1}$$

$$= \sinh^{-1} A\,(x^3-Bx^2+Cx-D)\sqrt{X_2},$$

where
$$A^2 = 4c^3(1-c)^7(1-2c)^3\sqrt{(c-c^2)},$$

$$P+B = 2\,(3-2c-3c^2),$$

$$P-B = 2\,(1-c)\sqrt{(c-c^2)};$$

$$Q+C = 2\,(3-6c+2c^2-2c^3+4c^4),$$

$$Q-C = 2\,(1-c)(2-c-2c^2)\sqrt{(c-c^2)};$$

$$R+D = 2\,(1-c)^2(1-2c+c^2+c^3-2c^4),$$

$$R-D = 2\,(1-c)^2(1-c+c^2-2c^3)\sqrt{(c-c^2)},$$

and the irrationality $\sqrt{(c-c^2)}$ can be removed by putting

$$c = \frac{(1-a)^2}{2+2a^2},\quad 1-c = \frac{(1+a)^2}{2+2a^2}.$$

$$\mu = 9.$$

33. The relation to be satisfied is

$$\gamma_9 = 0,$$

or $$y^2(y-x-y^2)-(y-x)^2 = 0.$$

Put $$y-x = yz, \quad x = y(1-z),$$

then $$y(z-y)-z^2 = 0.$$

Again, put $$z-y = \frac{z^2}{p},$$

then $$z = p-p^2,$$
$$y = p^2(1-p),$$
$$x = p^2(1-p)(1-p+p^2).$$

Forming the invariants of the cubic

$$S = 4s(s+x)^2-\{(y+1)s+xy\}^2,$$

then with $$s = \wp u-\wp v, \quad 12\wp v = -(y+1)^2-4x,$$

$$12g_2 = 144\wp^2 v-24\wp'' v = \{(y+1)^2+4x\}^2-24(y+1)x,$$

$$\begin{aligned} 216g_3 &= 864\wp^3 v-216g_2\wp v-216\wp'^2 v \\ &= -\tfrac{1}{2}\{(y+1)^2+4x\}^3+\tfrac{3}{2}\{(y+1)^2+4x\} \\ &\qquad \times\left[\{(y+1)^2+4x\}^2-24(y+1)x\right]-216x^2 \\ &= \{(y+1)^2+4x\}^3-36\{(y+1)^2+4x\}(y+1)x-216x^2, \end{aligned}$$

$$\begin{aligned} 1728\Delta &= (12g_2)^3-(216g_3)^2, \\ \Delta &= x^3\{x(y+1)^2-y(y+1)^3-16x^2+18xy(y+1)-27xy^2\} \\ &= x^3\{(y+1)^2(x-y-y^2)-16x^2+18xy(y+1)-27xy^2\}, \end{aligned}$$

and with the above values of x and y, we find

$$\Delta = p^9(1-p)^9(1-p+p^2)^3(1-6p+3p^2+p^3),$$

Quoting the "Modular Equation of the Ninth Order," given by Gierster (*Math. Ann.*, XIV., p. 541),

$$\begin{aligned} J : J-1 : 1 &= (\tau-1)^3(9\tau^3-27\tau^2+27\tau-1)^3 \\ &: (27\tau^6-162\tau^5+405\tau^4-504\tau^3+297\tau^2-54\tau-1)^2 \\ &: -64\tau(\tau^2-3\tau+3), \end{aligned}$$

or by Kiepert (*Math. Ann.*, XXXII., p. 66),

$$J : J-1 : 1 = (\xi+3)^3(\xi^3+9\xi^2+27\xi+3)^3$$
$$: (\xi^6+18\xi^5+135\xi^4+504\xi^3+891\xi^2+486\xi-27)^2$$
$$: 1728\xi(\xi^2+9\xi+27);$$

and similar equations given by Joubert (*Sur les équations dans l théorie de la transformation des fonctions elliptiques*, Paris, 1876), we fin that the quantities p, r, and ξ are connected by the relation

$$-3r = \xi = \frac{1-6p+3p^2+p^3}{p(1-p)},$$

so that $$9(r^2-3r+3) = \xi^2+9\xi+27 = \frac{(1-p+p^2)^3}{p^2(1-p)^2}.$$

We can write $$\xi = \frac{1}{p} - \frac{1}{1-p} - 4 - p;$$

and the substitutions of $\dfrac{1}{1-p}$ and $-\dfrac{1-p}{p}$ for p leave ξ unchanged and thus we have the group of substitutions for this cubic in which may be taken to correspond to the parameters

$$\tfrac{2}{9}\omega_3, \quad \tfrac{4}{9}\omega_3, \quad \tfrac{8}{9}\omega_3.$$

If we put $$p = -q - \frac{1}{q},$$

then $$-3(\tau_9-1) = \xi+3 = \frac{(q^9-1)(q-1)}{(q^3+1)(q^3-1)}.$$

34. We may suppose $$v = \tfrac{2}{9}\omega_3,$$

and then
$$12\wp v = -(y+1)^2-4x$$
$$= -(1+p^2-p^3)^2-4p^2(1-p)(1-p+p^2)$$
$$= -1+0-6p^2+10p^3-9p^4+6p^5-p^6,$$
$$12\wp 2v = 12\wp v+12x$$
$$= -(y+1)^2+8x$$
$$= -1+0+6p^2-14p^3+15p^4-6p^5-p^6,$$
$$12\wp 3v = 12\wp v+12y$$
$$= -(y+1)^2-4x+12y$$
$$= -1+0+6p^2-2p^3-9p^4+6p^5-p^6,$$

$$12\wp 4v = -(y+1)^2 - 4x + 12p(1-p)(1-p+p^2)$$
$$= -1 + 12p - 30p^2 + 34p^3 - 21p^4 + 6p^5 - p^6,$$

$$12G_1 = 12(\wp v + \wp 2v + \wp 3v + \wp 4v)$$
$$= -4(y+1)^2 - 4x + 12y + 12p(1-p)(1-p+p^2),$$
$$3G_1 = -(1-p+p^2)^3.$$

In the pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{P(s+x)-9x}{(s+x)\sqrt{S}}\,ds,$$

$$-Px = \tfrac{1}{2}(q_1q_2+q_2q_3+q_3q_4+q_4q_5+q_5q_6) - 7\wp'' v$$
$$= q_1q_2 + q_2q_3 + \tfrac{1}{2}q_3^2 - 7\wp'' v,$$
$$\tfrac{1}{4}q_1 = x = p^3(1-p)(1-p+p^2),$$
$$\tfrac{1}{4}q_2 = y = p^2(1-p),$$
$$\tfrac{1}{4}q_3 = \frac{x(y-x)}{y^2} = p(1-p)(1-p+p^2),$$

so that

$$-Px = 4xy + 4x\frac{y-x}{y} + 2x^2\frac{(y-x)^2}{y^4} - 7x(y+1),$$
$$P = 3y + 3 + 4\frac{x}{y} - 2x\frac{(y-x)^2}{y^4}$$
$$= 3p^2 - 3p^3 + 3 + 4(1-p+p^2) - 2(1-p)(1-p+p^2)$$
$$= 5 + 3p^2 - p^3.$$

35. The result which is given in the *Proc. Lond. Math. Soc.*, XXIV., p. 10, is obtained by putting

$$p = \frac{1}{1-c}, \quad \text{and} \quad s+x = (1-c)^4 t;$$

and now
$$I(v) = \int \frac{Pt - 9c(1-c+c^2)}{t\sqrt{T}}\,dt,$$
where

$$T = 4t\{(1-c)^2t + c(1-c+c^2)\}^2 - \{(1-2c+c^2+c^3)t - c(1-c+c^2)\}^2,$$

and the result of the integration may be written

$$2(1-c)^3 t^{\frac{9}{2}} e^{iI} = Pt^4 + Qt^3 + Rt^2 + St + V + i(t^3 + Ct^2 + Dt + E)\sqrt{T},$$

where
$$P = 7-18c+15c^2-5c^3,$$
$$Q = -14+53c-76c^2+61c^3-25c^4+5c^5,$$
$$R = (1-c+c^2)(7-33c+38c^2-23c^3+6c^4-c^5),$$
$$S = -(1-c+c^2)^2(1-9c+6c^2-2c^3),$$
$$V = -c(1-c+c^2)^3,$$
$$C = -3(2-2c+c^2),$$
$$D = (1-c+c^2)(5-3c+c^2),$$
$$E = -(1-c+c^2)^2;$$
the work has been verified by Mr. T. I. Dewar.

These results were obtained originally by putting
$$q_3 = q_4$$
in Abel's formulas (*Œuvres complètes*, II., p. 162); and then it was found that we could write
$$X = \{x^2-(1-n)x+m\}^2+4mx,$$
and
$$n = -p^2+p^3, \quad m = -p^2(1-p)(1-p+p^2),$$
when X has the factor
$$x-(1-p)(1-p+p^2);$$
also
$$9k = -2-3p^2+4p^3,$$
and then the substitution
$$x-(1-p)(1-p+p^2) = \frac{p(1-p+p^2)}{t}$$
will lead to the preceding results.

$$\mu = 10.$$

36. The relation
$$\gamma_{10} = 0,$$
or
$$y^2(xy-x^2-y^3)-x(y-x-y^2)^2 = 0,$$
becomes, on putting $x = y(1-z)$,
$$y(z-z^2-y)-(1-z)(z-y)^2 = 0;$$
and this again, on putting $z-y = \dfrac{z^2}{p}$,
becomes
$$(p-z)(1-p)-z(1-z) = 0;$$

so that, putting $$z = (1+a)(1-p),$$

$$p = \frac{1-a^2}{1-a-a^2},$$

$$z = \frac{-a(1+a)}{1-a-a^2}, \quad 1-z = \frac{1}{1-a-a^2},$$

$$y = \frac{-a(1+a)}{(1-a)(1-a-a^2)},$$

$$x = \frac{-a(1+a)}{(1-a)(1-a-a^2)^2}.$$

Then $$s_5 + x = \frac{xy(y-x-y^2)}{(y-x)^2} = -\frac{a}{(1-a)^2(1-a-a^2)},$$

so that $$s_5 = \frac{a^2}{(1-a)^2(1-a-a^2)^2},$$

and $s - s_5$ is a factor of

$$S = 4s\left\{s - \frac{a(1+a)}{(1-a)(1-a-a^2)^2}\right\}$$
$$- \left\{\frac{1-3a-a^2+a^3}{(1-a)(1-a-a^2)}s + \frac{a^2(1+a)^2}{(1-a)^2(1-a-a^2)^3}\right\}^2.$$

Put $$s = \frac{t}{(1-a)^2(1-a-a^2)^2},$$

and $$(1-a)^6(1-a-a^2)^6 S = T$$

$$= 4t\{t - a(1-a^2)\}^2 - \{(1-3a-a^2+a^3)t + a^2(1+a)^2(1-a)\}^2$$

$$= (t-a^2)\{4t^2 - (1+a)^2(1-a)(1+a+3a^2-a^3)t + a^2(1+a)^4(1-a)^2\}$$

$$= 4(t-a^2)\left[t - \left(\frac{1-a^2}{4}\right)^2\left\{1-a \pm (1+a)\sqrt{\frac{1+4a-a^2}{1-a^2}}\right\}^2\right].$$

We can also write

$$T = 4t\{t - a^2(1-a^2)\}^2 - \{(1+a-a^2+a^3)t - a^2(1+a)^2(1-a)\}^2,$$

so that, on comparison with the case of $\mu = 5$, the x there, distinguished as x_5, is given in terms of a by

$$x_5 = -\frac{1+a}{a^2(1-a)},$$

Then $$r_5 = -\frac{1}{x_5} + 11 + x_5$$

$$= -\frac{(1+4a-a^2)(1-a-a^2)^2}{a^2(1-a^2)}.$$

37. Forming the invariants of T, and comparing the values with Gierster's "Modular Equation of the Tenth Order" (*Math. Ann.*, XIV., p. 542),

$$J : J-1 : (4\tau^6-40\tau^5+160\tau^4-320\tau^3+320\tau^2-130\tau+5)^3$$

$$: (\tau^2-4\tau+5)(\tau^2-3\tau+1)^2 (2\tau^2-6\tau+5)^2 (4\tau^4-28\tau^3+66\tau^2-52\tau-1)^2$$

$$: 27\tau (\tau-2)^5 (2\tau-5)^2,$$

we find
$$\tau_{10} = \tau = \frac{1+4a-a^2}{2a};$$

so that
$$\tau-2 = \frac{1-a^2}{2a},$$

$$2\tau-5 = \frac{1-a-a^2}{a},$$

$$\tau^2-4\tau+5 = \frac{(1+a^2)^2}{4a^2},$$

$$\tau^2-3\tau+1 = \frac{1-6a+2a^2+6a^3+a^4}{4a^2},$$

$$2\tau^2-6\tau+5 = \frac{1-6a+8a^2+6a^3+a^4}{2a^2},$$

and
$$\tau_5 = -\frac{\tau (2\tau-5)^2}{\tau-2}$$

$$= -\frac{(1-4a-a^2)(1-a-a^2)^2}{a^2 (1-a^2)},$$

as before.

It is convenient to put
$$a = \frac{1+c}{1-c};$$

and then
$$y = -\frac{(1-c^2)}{c (1+4c-c^2)},$$

$$x = \frac{(1+c)(1-c)^3}{c (1+4c-c^2)^2}.$$

Then, denoting the roots of $T = 0$ by t_α, t_β, t_γ, and putting

$$A = (1-a^2)(1+4a-a^2), \quad C = c^3+c^2-c,$$

$$t_\alpha = \left\{ \frac{(1+a)(1-a)^2+(1+a)\sqrt{A}}{4} \right\}^2 = 4\frac{(c^2+\sqrt{C})^2}{(1-c)^6},$$

$$t_\beta = a^2 = \left(\frac{1+c}{1-c}\right)^2,$$

$$t_\gamma = \left\{ \frac{(1+a)(1-a)^2-(1+a)\sqrt{A}}{4} \right\}^2 = 4\frac{(c^2-\sqrt{C})^2}{(1-c)^6}.$$

38. It is convenient to suppose t_β the middle root of $T = 0$, so that we may put

$$v = \omega_1 + \tfrac{1}{5}\omega_3 ;$$

and therefore we must suppose

$$(t_\beta - t_a)(t_\beta - t_\gamma)$$

$$= 4a^4 - (1+a)^2(1-a)(1+a+3a^2-a^3)\,a^2 + a^2(1+a)^4(1-a)^2$$

$$= -4a^6\left(\frac{1}{a} + 1 - a\right)$$

to be negative; that is,

$$\frac{1}{a} + 1 - a = \frac{1-4c-c^2}{1-c^2} \text{ must be positive,}$$

which is the case if $0 < c < 1$; but as C must be positive, if

$$\tfrac{1}{2}(\surd 5 - 1) < c < 1.$$

Putting
$$t = \frac{s}{(1-c)^6},$$

and denoting by s_1, s_2, s_3, s_4 the values of s corresponding to v, $2v$, $3v$, $4v$, then

$$s_1 = (1-c)^6 t_1 = (1-c)^6(1-a)^2(1-a-a^2)^2 s_1$$

$$= -(1-c)^6(1-a)^2(1-a-a^2)^2 x$$

$$= (1-c)^6(1-a)^2(1-a-a^2)^2\,\frac{a(1+a)}{(1-a)(1-a-a^2)^2}$$

$$= -4c(1+c)(1-c)^3,$$

and s_1 must be positive, and therefore $c > 1$ if positive, if s_1 is to lie between s_4 and s_3, in which case we can take

$$v = \omega_1 + \tfrac{1}{5}\omega_3.$$

Similarly
$$s_2 = 0,$$

$$s_3 = -8c(1+c)^2(1-c),$$

$$s_4 = -4c(1+c)^2(1-c)^2,$$

so that, to ensure making s_3 positive and s_4 negative, we had better suppose c positive and > 1.

39. We now investigate the pseudo-elliptic integrals

$$I(v) = \tfrac{1}{2}\int \frac{P_1\{s-4c(c+1)(c-1)^3\}-20c^2(c+1)(c-1)^3(c^2-4c-1)}{\{s-4c(c+1)(c-1)^3\}\sqrt{S}}\,ds,$$

$$I(2v) = \tfrac{1}{2}\int \frac{P_2 s-20c(c+1)^2(c-1)^4}{s\sqrt{S}}\,ds,$$

$$I(3v) = \tfrac{1}{2}\int \frac{P_3(s-s_3)-20c(c+1)^3(c-1)(c^2-4c-1)}{(s-s_3)\sqrt{S}}\,ds,$$

$$I(4v) = \tfrac{1}{2}\int \frac{P_4(s-s_4)-20c^2(c+1)^4(c-1)^2}{(s-s_4)\sqrt{S}},$$

where $$S = s-s_\alpha \,.\, s-s_\beta \,.\, s-s_\gamma,$$

and
$$s_\alpha = 4(c^2+\sqrt{C})^2,$$
$$s_\beta = (c+1)^2(c-1)^4,$$
$$s_\gamma = 4(c^2-\sqrt{C})^2,$$
$$C = c^3+c^2-c.$$

Then (§ 9) $$\frac{\rho_1}{(1-a)(1-a-a^2)}$$

$$= x\left\{\frac{2}{x}+\frac{2}{y}+\frac{2y^2}{x(y-x)}+\frac{(y-x)^2}{y(y-x-y^2)}\right\}+y+1$$

$$= 2+2-2z+2\frac{y}{z}+p+y+1$$

$$= 4\,\frac{2-3a-a^2+a^3}{(1-a)(1-a-a^2)},$$

$$P_1 = 4(2-3a-a^2+a^3).$$

With $$a = \frac{1+c}{1+c},\quad P_1 = 4\,\frac{3c^3-13c^2+c+1}{(c-1)^3},$$

but the denominator $(c-1)^3$ of P_1 must be omitted when employed in the last preceding form of the integral.

The values of P_3 and P_4 can be inferred from the integrals discussed under $\mu = 5$; thus

$$\tfrac{1}{4}P_3 = c(c+1)^2(1-3x) = c^3-c^2+7c-3,$$

$$\tfrac{1}{4}P_4 = c(c+1)^2\;(x+3) = 3c^3+7c^2+c+1.$$

The determination of P_2, P_3, ... may be simplified by noticing that if the sum of two parameters is equal to a half-period, say

$$v_2 + v_3 = \omega'_a,$$

then, in the general case,

$$P_2 \pm P_3 = \frac{\mu\sqrt{-S_2}}{s_2 - s_a} = \frac{\mu\sqrt{-S_3}}{s_3 - s_a};$$

this follows because the pseudo-elliptic integrals are transformed into each other by the substitution

$$s - s_a = \frac{s_a - s_\beta \,.\, s_a - s_\gamma}{s' - s_a}.$$

Thus, as examples, $\quad P_4 - P_1 = 20c^2,$

$$P_2 - P_3 = 20c,$$

so that $\quad \frac{1}{4}P_3 = c^3 - c^2 + 2c - 3.$

We now infer that

$$I(\tfrac{2}{5}\omega_3) = \tfrac{1}{2}\int \frac{(c^3 - c^2 + 7c - 3)\,s - 5c\,(c+1)^2\,(c-1)^4}{s\sqrt{(s - s_a \,.\, s - s_\beta \,.\, s - s_\gamma)}}\,ds$$

$$= \sin^{-1}\frac{s - 4\,(c-1)^4}{s^{\frac{1}{2}}}\sqrt{(s - s_a \,.\, s - s_\beta \,.\, s - s_\gamma)}$$

$$= \cos^{-1}\frac{(c^3 - c^2 + 7c - 3)\,s^2 + \dots}{s^{\frac{1}{2}}},$$

$$I(\tfrac{4}{5}\omega_3) = \tfrac{1}{2}\int \frac{(3c^3 + 7c^2 + c + 1)(s - s_4) - 5c^2\,(c+1)^4\,(c-1)^2}{(s - s_4)\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{s - 4c\,(c+1)^2\,(c^3 + c^2 + 3c - 1)}{(s - s_4)^{\frac{1}{2}}}\sqrt{S}$$

$$= \cos^{-1}\frac{(3c^3 + 7c^2 + c + 1)\,s^2 + \dots}{(s - s_4)^{\frac{1}{2}}};$$

and thence the remaining integrals

$$I(\tfrac{1}{5}\omega_3), \quad I(\tfrac{3}{5}\omega_3), \quad I\left(\omega_1 + \frac{1, 2, 3, 4}{5}\,\omega_3\right)$$

can be inferred by linear substitutions.

$$\mu = 11.$$

40. It was in working out this case in Abel's method that the clue was obtained by which Halphen's function γ_ρ and his x and y were found to be available for the theory of pseudo-elliptic integrals.

With the notation employed on p. 162, t. II., of Abel's *Œuvres complètes*, we put

$$q_9 = 0, \quad \text{or} \quad q_4 = q_5,$$

and then, with
$$c = 0,$$

$$\tfrac{1}{2}p^2 + apq_4 - q_3 q_4^2 - q_4^3 = 0.$$

We can replace a, b, and p by $\lambda(n-1)$, $\lambda^2 m$, and $4\lambda^3 m$; this is equivalent to taking $\dfrac{x}{\lambda}$ as independent variable; and now

$$q_3 = 2\lambda^2 \frac{m(m-n)}{n^2},$$

$$q_4 = 2\lambda^2 \frac{mn(n^2-m+n)}{(m-n)^2};$$

so that, dropping the common factor $8\lambda^6 m^3$,

$$1 + \frac{n(n-1)(n^2-m+n)}{(m-n)^2} - \frac{m(n^2-m+n)^2}{(m-n)^3} - \frac{mn^2(n^2-m+n)^3}{(m-n)^6} = 0$$

This becomes, on putting

$$m-n = nq,$$

$$1 + (n-1)\frac{n-q}{q^2} - (1+q)\frac{(n-q)^2}{q^3} - n(1+q)\frac{(n-q)^3}{q^3} = 0,$$

and again, on putting
$$n-q = \frac{q^2}{1+c},$$

$$1 + \left(q + \frac{q^2}{1+c} - 1\right)\frac{1}{1+c} - (1+q)\frac{q}{(1+c)^2}$$
$$- q\left(1 + \frac{q}{1+c}\right)(1+q)\frac{1}{(1+c)^3} = 0,$$

or
$$(1+c+q)\left\{c(1+c)^2 - q(1+q)\right\} = 0.$$

41. The factor $$1+c+q=0$$
makes $n=0$, corresponding to $\mu=4$; so that we take
$$q(1+q)=c(1+c)^2,$$
or
$$2q=-1+\sqrt{C},$$
where
$$C=1+4c+8c^2+4c^3.$$
Thence
$$m=c(1+c)(1+c+q),$$
$$n=c(1+c)+\frac{cq}{1+c}=q\frac{1+c+q}{1+c};$$
$$q_4=q_5=2c(1+c+q),$$
and, putting
$$\lambda=1,$$
$$X=\{x^2+(n-1)x+m\}^2+4mx$$
has a factor
$$x+c\frac{1+c+q}{1+c};$$
and we can throw X into the form
$$X=\left(x+c\frac{1+c+q}{1+c}\right)\left[x(x-1+c^2)^2+c\frac{1+c+q}{1+c}\{x+(1+c)^2\}^2\right].$$
Calculated according to Abel's formula, we find
$$11k=a+\frac{2q_1q_3+2q_2q_3+2q_3q_4+q_4^2}{p}$$
$$=-1+6c+5c^2+2\frac{1+3c}{1+c}q.$$

But now we reduce the integral to the form we have employed previously by the substitution
$$x+c\frac{1+c+q}{1+c}=\frac{M}{t},$$
and, taking
$$M=\frac{q}{c^4(1+c)^3},$$
we shall find that the integral assumes the form
$$I=\tfrac{1}{2}\int\frac{Pt+11c^4(1+c)^2}{t\sqrt{T}}\,dt,$$
where $T=4t\{t-c^2(1+c)q\}^2-\{(1+2c-c^3+2q+cq)t+c^4(1+c)^2\}^2$,
$$P=-(1+c)(1+5c-5c^2)+(2-5c)q.$$

Then $$I = \cos^{-1}\frac{Pt^5+Qt^4+Rt^3+St^2+Ut+V}{2t^{\frac{11}{2}}}$$

$$= \sin^{-1}\frac{t^4+Ct^3+Dt^2+Et+F}{2t^{\frac{11}{2}}}\sqrt{T},$$

or $2t^{\frac{11}{2}}e^{iI} = Pt^5+Qt^4+Rt^3+St^2+Ut+V+i(t^4+Ct^3+Dt^2+Et+F)\sqrt{T}$,

and, knowing P, the values of the coefficients $Q, R, \ldots C, D, \ldots$ can be calculated by an algebraical verification and by a differentiation.

42. To connect up these results with the "Modular Equation of the Eleventh Order," given by Kiepert in the *Math. Ann.*, XXXII., p. 93, and by Klein and Fricke in their *Modulfunctionen*, t. II., p. 437, it was necessary to form the discriminant of T; the algebraical labour was very great, and I am indebted to the Rev. J. Holme Pilkington, Rector of Framlingham, for a verification of this and other results; the result obtained is

$$\Delta = -c^{11}(1+c)^{11}\{(1+c)^2+(2+c)q\}^2\{(1+c)^3(1+3c)+(2+2c-c^2)q\}.$$

We must next form the expression for Kiepert's f (*Math. Ann.*, XXVI., p. 393) given by the formula

$$f^{-2} = (t_1-t_2)(t_2-t_4)(t_4-t_3)(t_3-t_5)(t_5-t_1),$$

where $$t_m = \wp mv - \wp v.$$

Since Halphen's equation

$$\gamma_{11} = 0$$

is equivalent to the relations

$$\lambda = \frac{\gamma_6}{\gamma_5},\quad \lambda^3 = \frac{\gamma_7}{\gamma_4},\quad \lambda^5 = \frac{\gamma_8}{\gamma_3},\quad \lambda^7 = \frac{\gamma_9}{\gamma_2},\quad \lambda^9 = \frac{\gamma_{10}}{\gamma_1};$$

therefore $$f^{-2} = x^{\frac{12}{2}}\frac{\gamma_1\gamma_3\gamma_2\gamma_6\gamma_1\gamma_7\gamma_3\gamma_8\gamma_4\gamma_6}{(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^4} = \frac{x^{\frac{12}{2}}\lambda^{10}}{(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^2};$$

and thence, or otherwise, we find that

$$f^{-2} = c^6q^4(1+c+q)^2 = -c^{10}(1+c)^{10}\{(1+c)^2+(2+c)q\}^2.$$

Next we find that, in Klein and Fricke's notation (*Modulfunctionen*, II., p. 442),

$$y_\infty = 11A^2 = \Delta f^2 = c(1+c)\{(1+c)^2(1+3c)+(2+2c-c^2)q\}.$$

43. The determination of Klein and Fricke's B^2 was not effected without the kind assistance of Dr. Robert Fricke, who pointed out that the requisite clue was afforded by his equation (*Math. Ann.*, XL., p. 478)

$$\tfrac{55}{12}B^2-\tfrac{22}{3}A^3=\sum_{r=1}^{r=5}\wp\frac{2r\omega_3}{11};$$

and thence we find

$$11B^2=-(1+c)^3(1+c-2c^2+c^3)+(2c+5c^2-2c^4)\,q.$$

The parameter τ employed by Klein and Fricke, II., p. 440, given by

$$\tau=\frac{A^3}{B^2},$$

which is connected with Kiepert's parameter η (*Math. Ann.*, XXXII., p. 96) by the relation $$\eta+8=\frac{1}{\tau},$$

now leads to the relation

$$\eta+8=\frac{1}{\tau}=\frac{B^2}{A^3}=\frac{-(1+c)^3(1+c-2c^2+c^3)+(2c+5c^2-2c^4)\,q}{c\,(1+c)\left\{(1+c)^2(1+3c)+(2+2c-c^2)\,q\right\}}.$$

Thence $$q=\frac{(1+c)^2\left\{(1+c-2c^2+c^3)\,\tau+c+3c^2\right\}}{(2c+5c^2-2c^4)\,\tau-c\,(1+c)(2+2c-c^2)},$$

$$1+q=\frac{(1+4c+9c^2-3c^3-2c^4+c^5+c^6)\,\tau+c\,(1+c)(-1+3c+8c^2+3c^3)}{(2c+5c^2-2c^4)\,\tau-c\,(1+c)(2+2c-c^2)}.$$

Multiplying these two equations together, putting

$$q\,(1+q)=c\,(1+c)^3,$$

and reducing, we shall find, finally,

$$\begin{aligned}(1+4c-9c^2-27c^3-13c^4+c^5)\,c^3\,(1+c)^3\\ -\tau\,(10+40c+31c^2-28c^3-9c^4+10c^5)\,c^3\,(1+c)^3\\ -\tau^2\,(1+4c+2c^2-5c^3-2c^4+c^5)^2=0.\end{aligned}$$

Putting $$F=1+4c+2c^2-5c^3-2c^4+c^5,$$

this quadratic equation for τ can be written

$$\left\{F-11c^2(1+c)^2\right\}c^3(1+c)^3-\tau\left\{10F+11c^2(1+c)^2\right\}c^3(1+c)^3-\tau^2F^2=0;$$

and putting $$H=\frac{F}{c^3\,(1+c)^3},$$

$$H^2+\frac{10H+11}{\tau}-\frac{H-11}{\tau^2}=0.$$

Solving this equation as a quadratic in H,

$$H = \frac{1-10\tau+\tau'}{2\tau^2},$$

where, as in Klein and Fricke's notation,

$$\tau'^2 = 1-20\tau+56\tau^2-44\tau^3.$$

Thus the required relation between c and τ is

$$\frac{1+4c+2c^2-5c^3-2c^4+c^5}{c^3(1+c)^2} = \frac{1-10\tau+\tau'}{2\tau^2},$$

or

$$= \tfrac{1}{2}(\eta^2+6\eta-16+W),$$

where

$$\eta+8 = \frac{1}{\tau},$$

$$W = \frac{\tau'}{\tau^2} = \sqrt{(\eta^4+12\eta^3-40\eta^2-940\eta-2912)}$$

$$= \sqrt{\{(\eta+8)(\eta^3+4\eta^2-72\eta-364)\}},$$

in Kiepert's notation (*Math. Ann.*, XXXII., p. 96).

Given τ or η, there is thus a quintic equation for c, corresponding to the five parameters

$$(2,\ 4,\ 6,\ 8,\ 10)\frac{\omega_2}{11};$$

and the group of this quintic is

$$c,\quad \frac{1+2c\pm\sqrt{C}}{2c^2},\quad \frac{-1-4c-2c^2\pm\sqrt{C}}{2(1+c)^2}.$$

The "Modular Equation of the Eleventh Order" is now, according to Klein and Fricke,

$$J : J-1 : 1 = (2^5.11.\tau^2-2^4.23.\tau+61-2^2.3.5.\tau')^3$$

$$: \{7.2^8.11^2.\tau^3-7.2^6.3.11.\tau^2+7.2^4.3.23.\tau-7.5.19$$

$$-2.3^2.\tau'(2^3.11\tau-37)\}^2$$

$$: 2^4.3^3.\tau\{2^3.11.\tau^2-3.7.\tau+1-\tau'(11\tau-1)\}^2.$$

These authors add in a foot-note the remark that the identification of their form of the modular equation with those given by Kiepert in the *Math. Ann.*, XXXII., p. 96, is easily carried out: "Die Ueberführung dieser beiden letzen Gleichungen in einander lässt sich in den That mühelos vollziehen"; but I must confess that I did not find this operation very easy.

44. If, however, we adopt Kiepert's notation, and put

$$\eta = \xi^2+4\xi+\frac{4}{\xi},$$

$$w^2 = (\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16),$$

$$W = (\xi+2-2\xi^{-2})\, w,$$

$$W^2 = (\eta+8)(\eta^3+4\eta^2-72\eta-364),$$

$$A = (\xi+2)(\xi^4+9\xi^3+26\xi^2+36\xi+16) = \xi^5+11\xi^4+44\xi^3+88\xi^2+88\xi+32,$$

$$C = A^2+2\xi^5 = \xi^{10}+11\,(2\xi^9+19\xi^8+104\xi^7+368\xi^6+886\xi^5+1472\xi^4$$
$$+1664\xi^3+1216\xi^2+512\xi)+1024,$$

then we may put

$$\frac{12g_2}{m^4} = \xi C+32-\xi\,(\xi+1)(\xi+4)\,Aw,$$

$$\frac{2\surd\Delta}{m^6} = 2\surd(2\xi)\,\{(\xi+1)(\xi+4)\,w-A\},$$

or $$\frac{12g_2}{M^4} = 61\eta^2+608\eta+1312-60W,$$

$$\frac{2\surd\Delta}{M^6} = (\eta^2-5\eta-16)\surd(\eta+8)+(\eta-3)\surd(\eta^3+4\eta^2-72\eta-364).$$

Then

$$\frac{m^4}{M^4} = \frac{61\eta^2+608\eta+1312-60W}{\xi C+32-\xi\,(\xi+1)(\xi+4)\,Aw},$$

$$\frac{m^6}{M^6} = \frac{(\eta^2-5\eta-16)\surd(\eta+8)+(\eta-3)\surd(\eta^3+4\eta^2-72\eta-364)}{2\surd(2\xi)\,\{(\xi+1)(\xi+4)\,w-A\}};$$

and therefore

$$\frac{m^2}{M^2} = \frac{(\eta^2-5\eta-16)\surd(\eta+8)+(\eta-3)\surd(\eta^3+4\eta-72\eta-364)}{61\eta^2+608\eta+1312-60W}$$
$$\times\frac{\xi C+32-\xi\,(\xi+1)(\xi+4)\,Aw}{2\surd(2\xi)\,\{(\xi+1)(\xi+4)\,w-A\}},$$

and, after algebraical reduction,

$$\frac{m^2}{M^2} = \frac{(\xi+5)\surd(\xi^3+4\xi^2+8\xi+4)+(\xi+3)\surd(\xi^3+8\xi^2+16\xi+16)}{2\surd(2\xi)}.$$

Memoirs on the "Transformation of the Eleventh Order," by Brioschi (*Annali di Math.*, XXI., Dec., 1893, also Dec., 1883), and by Klein (*Math. Ann.*, XV.), may be consulted for the employment of other associated parameters.

Substituting for $\frac{1}{r}$ its value in terms of ξ, namely,

$$\frac{1}{r} = \eta+8 = \frac{\xi^3+4\xi^2+8\xi+4}{\xi},$$

and putting $$H = 11(K+1),$$

then $$11(K+1)^2+\frac{10K+11}{r}-\frac{K}{r^2} = 0,$$

or $K(\xi^3+4\xi^2+8\xi+4)^2-(10K+11)(\xi^4+4\xi^3+8\xi^2+4\xi)-11(K+1)\xi^2 = 0$,

or $$K\xi^6+8K\xi^5+11(2K-1)\xi^4+(32K-44)\xi^3$$
$$-(11K^2+6K+99)\xi^2+(24K-44)\xi+16K = 0.$$

If we could discover a quadratic factor of this sextic equation, say of the form $$\xi^2+A\xi+4,$$
we should obtain the relation connecting our c with Kiepert's ξ.

45. The transformations we have usually employed, namely,

$$y-x = yz, \quad z-y = \frac{z^2}{p}, \quad z = c(p-1),$$

reduce the equation $$\gamma_{11} = 0$$

to $$p^3-c^2p+c+c^2 = 0,$$

and will lead to similar results.

In fact, the first two transformations lead to

$$z(1-z) = p^2-p^3,$$

so that $$2z = 1+\sqrt{P},$$

where $$P = 1-4p^2+4p^3,$$

and now the relations

$$p = 1+c, \quad z = -q,$$

are sufficient to identify these results with those which we have employed.

$$\mu = 12.$$

46. The relation $\gamma_{12} = 0$

is equivalent to $$\frac{\gamma_8}{\gamma_4} = \left(\frac{\gamma_7}{\gamma_5}\right)^2,$$

or $$(y-x)^2\{x(y-x-y^2)-(y-x)^2\}-\{(y-x)x-y^2\}^2 = 0.$$

Put $$y-x = yz,\quad z-y = \frac{z^2}{p},\quad z = (1+a)(1-p);$$

then $$p = \frac{1}{1-a},$$

$$z = \frac{-a(1+a)}{1-a},$$

$$y = \frac{-a(1+a)(1+a+a^2)}{1-a},$$

$$x = \frac{-a(1+a)(1+a+a^2)(1+a^2)}{(1-a)^2}.$$

Forming the expression for s_6,

$$s_6+x = \frac{(y-x)\{(y-x)x-y^2\}}{(y-x-y^2)^2}$$

$$= (p-z)(1-p)$$

$$= -\frac{a(1+a+a^2)}{(1-a)^2},$$

$$s_6 = \frac{a^2(1+a+a^2)^2}{(1-a)^2},$$

and then $s-s_6$ is a factor of

$$S = 4s\left\{s-\frac{a(1+a)(1+a+a^2)(1+a^2)}{(1-a)^2}\right\}^2$$
$$-\left\{\frac{1-2a-2a^2-2a^3-a^4}{1-a}s+\frac{a^2(1+a)^2(1+a+a^2)^2(1+a^2)}{(1-a)^3}\right\}^2.$$

Put $$s = \frac{t}{(1-a)^2},$$

and $$(1-a)^6 S = T = 4t\{t-a(1+a)(1+a+a^2)(1+a^2)\}^2$$
$$-\{(1-2a-2a^2-2a^3-a^4)t+a^2(1+a)^2(1+a+a^2)^2(1+a^2)\}^2$$
$$=\{t-a^2(1+a+a^2)^2\}\{4t^2-(1+a)^2(1+a^2)(1+2a+6a^2+2a^3+a^4)t$$
$$+a^2(1+a)^4(1+a+a^2)^2(1+a^2)^2\}.$$

47. The discriminant of the quadratic factor of T is

$$(1+a)^6(1-a)^2(1+a^2)^2(1+4a+a^2),$$

and the discriminant of the cubic T is

$$a^{12}(1+a)^6(1-a)^2(1+a^2)^2(1+a+a^2)^4(1+4a+a^2).$$

Gierster's "Modular Equation of the Twelfth Order" is (*Math. Ann.*, XIV., p. 542)

$$J:J-1:1=(\tau^2-6\tau+6)^3(\tau^6-18\tau^5+126\tau^4-432\tau^3+732\tau^2-504\tau+24)^3$$

$$:(\tau^4-12\tau^3+48\tau^2-72\tau+24)^2(\tau^8-24\tau^7+240\tau^6-1296\tau^5+4080\tau^4$$

$$-7488\tau^3+7416\tau^2-3024\tau-72)^2$$

$$:1728\tau(\tau-6)(\tau-2)^3(\tau-4)^3(\tau-3)^4;$$

so that this τ, or τ_{12}, is connected with a by the relation

$$\tau_{12}=\frac{1+4a+a^2}{a}=\frac{1}{a}+4+a,$$

$$\tau-2=\frac{(1+a)^2}{a},$$

$$\tau-3=\frac{1+a+a^2}{a},$$

$$\tau-4=\frac{1+a^2}{a},$$

$$\tau-6=\frac{(1-a)^2}{a},$$

$$\tau^2-6\tau+6=\frac{1+2a+2a^3+a^4}{a^2}.$$

Also (Gierster)

$$\tau_6=\tfrac{1}{2}\tau(\tau-6),$$

$$\tau_4=\frac{\tau(\tau-4)^3}{8(\tau-3)},$$

$$\tau_3=-\frac{\tau(\tau-6)(\tau-3)^4}{27(\tau-2)(\tau-4)},$$

$$\tau_2=\frac{\tau(\tau-6)(\tau-2)^3(\tau-4)^3}{-64(\tau-3)^3}.$$

48. Denoting the roots of T, in descending order, by t_α, t_β, t_γ, then

$$t_\alpha = (1+a)^2(1+a^2)^2\left\{\frac{1-a+(1+a)\sqrt{A}}{4}\right\}^2,$$

$$t_\beta = (1+a)^2(1+a^2)^2\left\{\frac{1-a-(1+a)\sqrt{A}}{4}\right\}^2,$$

$$t_\gamma = a^2(1+a+a^2)^2,$$

where
$$A = \frac{1+4a+a^2}{1+a^2}.$$

Denoting by t_1 the value of t corresponding to s_1 or v,

$$t_1 = -(1-a)^2 x = a(1+a)(1+a+a^2)(1+a^2),$$

and
$$t_1 - t_\gamma = a(1+a+a^2);$$

and this is positive, or t_1 lies between t_α and t_β, if a is positive; and now we can take
$$v = \omega_1 + \tfrac{1}{6}\omega_3.$$

49. The pseudo-elliptic integral corresponding to v is

$$I(v) = \tfrac{1}{2}\int \frac{P(t_1-t)+12a(1-a^4)(1+a+a^2)}{(t_1-t)\sqrt{T}}\,dt,$$

or
$$= \tfrac{1}{2}\int \frac{\rho(s_1-s)+12x}{(s_1-s)\sqrt{S}}\,ds,$$

where
$$\rho x = \tfrac{1}{2}(q_1q_2+q_2q_3+q_3q_4+q_4q_5+q_5q_6+\dots+q_8q_9)-10\wp''v$$
$$= q_1q_2+q_2q_3+q_3q_4+q_4q_5-10\wp''v,$$

and
$$q_1 = 2x,$$
$$q_2 = 2y,$$
$$q_3 = 2\frac{x(y-x)}{y^2},$$
$$q_4 = 2\frac{xy(y-x-y^2)}{(y-x)^2},$$
$$q_5 = 2\frac{(y-x)\{(y-x)x-y^2\}}{(y-x-y^2)^2},$$

$$\rho x = 4xy+4x\frac{y-x}{y}+4\frac{x^2(y-x-y^2)}{y(y-x)}+4xy\frac{(y-x)x-y^2}{(y-x)(y-x-y^2)}$$
$$-10x(y+1),$$

$$\rho = 4y+4-4\frac{x}{y}+4\frac{x}{y}\,\frac{y-x-y^2}{y-x}+4y\,\frac{(y-x)\,x-y^3}{(y-x)(y-x-y^2)}-10(y+1)$$

$$= -6y-6-4\frac{xy}{y-x}+4y\,\frac{(y-x)\,x-y^3}{(y-x)(y-x-y^2)}$$

$$= -6y-6-\frac{4y^3}{y-x-y^2}$$

$$= 6-2\,(1+a+a^2)(2+a)$$

$$= -2\,(5+3a+3a^2+a^3).$$

Now

$$I(v) = \tfrac{1}{2}\int\frac{(1-a)(5+3a+3a^2+a^3)(t_1-t)-6a\,(1-a^4)(1+a+a^2)}{(t_1-t)\sqrt{T}}\,dt$$

$$= \cos^{-1}\frac{t^2+Ct+D}{2\,(t_1-t)^3}\sqrt{\{4t^3-(1+a)^2(1+a^2)(1+2a+6a^2+2a^3+a^4)t + a^2\,(1+a)^4\,(1+a+a^2)^2\,(1+a^2)^2\}}$$

$$= \sin^{-1}\frac{Pt^2+Qt+R}{2\,(t_1-t)^3}\sqrt{\{t-a^2\,(1+a+a^2)^2\}},$$

where $P = (1-a)(5+3a+3a^2+a^3)$,

and thence the values of Q, R, C, D are inferred by a verification; we find

$$C = 3+0+5a^2+4a^3+6a^4+4a^5+2a^6,$$

$$D = (1+a^2)\,(1+a+a^2)^2\,(1+0+2a^2+0+2a^4+2a^5+a^6),$$

$$Q = (1-a)\,(1+a^2)\,(5+11a+25a^2+35a^3+34a^4+22a^5+10a^6+2a^7),$$

$$R = (1-a)\,(1+a)^2\,(1+a^2)^2\,(1+a+a^2)^2\,(1+0+2a^2+a^3).$$

The pseudo-elliptic integrals corresponding to

$2v$, or $\tfrac{1}{3}\omega_3$,

$3v$, or $\omega_1+\tfrac{1}{2}\omega_3$,

$4v$, or $\tfrac{2}{3}\omega_3$,

fall under the head of preceding cases; and the case of

$5v$, or $\omega_1+\tfrac{5}{6}\omega_3$,

can be constructed from $I(v)$ by the substitution

$$t-t_\gamma = \frac{t_\alpha-t_\gamma\,.\,t_\beta-t_\gamma}{t'-t_\gamma},$$

and then

$$P_5 = (1-a)\,(1+3a+3a^2+5a^3),\quad t_5 = a^2\,(1+a)\,(1+a+a^2)\,(1+a^2).$$

$$\mu = 13.$$

50. The relation $$\gamma_{13} = 0,$$

being equivalent to $$\frac{\gamma_8}{\gamma_5} = \left(\frac{\gamma_7}{\gamma}\right)^3,$$

or $$\gamma_8\gamma_6^3 - \gamma_5\gamma_7^3 = 0,$$

or $y\{x(y-x-y^3)-(y-x)^3\}\,x(y-x-y^3)^3-(y-x)\{(y-x)x-y^3\}^3=0$,

becomes, with the usual transformations,

$$\frac{\gamma_8}{\gamma_5} = -y^3z^3\frac{1+c}{cp},$$

$$\frac{\gamma_7}{\gamma_6} = -\frac{yz}{cx^{\frac{1}{3}}},$$

$$y^3z^3\frac{1+c}{cp} = \frac{y^3z^3}{c^3x},$$

or $$pyz = c^3(1+c)\,x = c^3(1+c)\,y\,(1-z),$$

$$pz = c^3(1+c)(1-z),$$

$$p\,(p-1) = c\,(1+c)(1+c-cp),$$

$$p^2-(1-c^2-c^3)\,p-c\,(1+c)^2 = 0,$$

$$2p = 1-c^2-c^3+\sqrt{C},$$

where $$C = 1+4c+6c^2+2c^3+c^4+2c^5+c^6$$
$$= (1+2c-c^2-c^3)^2+4c^2\,(1+c)^2.$$

Then $$z = c\,(p-1)$$
$$= \tfrac{1}{2}c\,(-1-c^2-c^3+\sqrt{C}),$$

$$1-z = \tfrac{1}{2}\,(2+c+c^3+c^4-c\sqrt{C}),$$

$$\frac{1-z}{1-p} = \frac{1+0-c^2-c^3+\sqrt{C}}{-2c\,(1+c)},$$

$$1-\frac{z}{p} = 1-c\frac{p-1}{p} = 1-c+\frac{c}{p}$$

$$= 1-c+\frac{-1+c^2+c^3+\sqrt{C}}{2\,(1+c)^2}$$

$$= \frac{1+2c-c^2-c^3+\sqrt{C}}{2(1+c)^2},$$

$$y = z\left(1-\frac{z}{p}\right)$$

$$= c^2\,\frac{1+3c+2c^4+c^5+(1-c-c^2)\sqrt{C}}{2(1+c)^2},$$

$$1+y = \frac{2+4c+3c^2+3c^3+2c^6+c^7+c^2(1-c-c^2)\sqrt{C}}{2(1+c)^2},$$

$$x = y(1-z)$$

$$= \frac{c^2}{2(1+c)}\{1+2c-2c^2+2c^3+6c^4+c^5+2c^7+c^8$$
$$+(1-2c-c^2+c^3-c^4-c^5)\sqrt{C}\}.$$

51. The next operation is to determine the relation connecting c with Klein's parameter τ, employed in his "Modular Equation of the Thirteenth Order" (*Proc. Lond. Math. Soc.*, IX., p. 126; *Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (\tau^2+5\tau+13)(\tau^4+7\tau^3+20\tau^2+19\tau+1)^3$$
$$: (\tau^2+6\tau+13)(\tau^6+10\tau^5+46\tau^4+108\tau^3+122\tau^2+38\tau-1)^2$$
$$: 1728\tau,$$

Kiepert's parameter L being connected with Klein's τ by the relation

$$\tau = L^2$$

(*Math. Ann.*, XXVI., p. 428).

After very great algebraical labour I have found finally that

$$\tau = \frac{1-c-4c^2-c^3}{c(1+c)} = \frac{1}{c}+\frac{1}{1+c}-c-3,$$

so that, considered as a cubic in c, the group of substitutions

$$c,\quad -\frac{1}{1+c},\quad -\frac{1+c}{c}$$

leaves τ unaltered.

Then $$\tau^2+5\tau+13=\frac{(1+c+c^2)^2}{c^2(1+c)^2},$$

$$\tau^2+6\tau+13=\frac{C}{c^2(1+c)^2};$$

and $$12g_2=m^4(\tau^2+5\tau+13)^{\frac{1}{2}}(\tau^4+7\tau^3+20\tau^2+19\tau+1),$$

$$216g_3=m^6(\tau^2+6\tau+13)^{\frac{1}{2}}(\tau^6+10\tau^5+46\tau^4+108\tau^3+122\tau^2+38\tau-1),$$

$$\Delta=m^{12}\tau.$$

52. If we choose the value

$$m^2=c^{\frac{1}{2}}(1+c)^{\frac{1}{2}}\sqrt{C},$$

then $$12g_2=C(1+c+c^2)(1+3c-4c^2-25c^3-23c^4+22c^5+40c^6+18c^7$$
$$+22c^8+40c^9+29c^{10}+9c^{11}+c^{12}),$$

$$216g_3=C^2(1+4c-3c^2-40c^3-65c^4+32c^5+235c^6\ldots$$
$$\ldots+264c^{15}+82c^{16}+14c^{17}+c^{18}),$$

$$\Delta=C^3c^{13}(1+c)^{13}(1-c-4c^2-c^3).$$

Kiepert's expression (*Math. Ann.*, XXVI., p. 427)

$$\tau=L^2=\Delta f^2$$

was employed for the determination of τ; the expression of Δ as a function of c and $\sqrt{C}$ was calculated, in the form

$$H+K\sqrt{C};$$

and then Kiepert's f was calculated from the relation

$$f^{-2}=(s_1-s_2)(s_2-s_4)(s_4-s_8)(s_8-s_{16})(s_{16}-s_{32})(s_{32}-s_{64})$$
$$=(s_1-s_2)(s_2-s_4)(s_4-s_5)(s_5-s_3)(s_3-s_6)(s_6-s_1)$$
$$=x^4\frac{\gamma_7^4\gamma_9^2}{\gamma_2^2\gamma_4^4\gamma_5^2\gamma_6^6}=x^4c^2(1+c)^2\left(1-\frac{1}{p}\right)^4(p+c)^2,$$

or $$f^{-2}=\sqrt{(S_1S_2S_3S_4S_5S_6)};$$

and then, if we find

$$f^{-2}=M+N\sqrt{C},$$

$$\tau=\frac{H+K\sqrt{C}}{M+N\sqrt{C}}.$$

It was thus found on rationalizing the denominator that the numerator and denominator differed by an irrational factor; and then

$$r = \frac{1-c-4c^2-c^3}{c(1+c)};$$

but the algebraical labour was very heavy, so that a more direct method probably exists.

The quadratic relation

$$y(1+cy)+c(1+c) = 0$$

changes the equation $1-c-4c^2-c^3 = 0$

into $$y^6+y^5-5y^4-4y^3+6y^2+3y-1 = 0,$$

and now $$y = x+\frac{1}{x}$$

changes this into $$\frac{x^{13}-1}{x-1} = 0,$$

thus showing the connection of these forms with the thirteenth roots of unity (Burnside and Panton, *Theory of Equations*, Ex. 15, p. 101).

53. Another long algebraical calculation will show that

$$-24(1+c)^4 \wp v = 2+8c+18c^2+38c^3+45c^4+22c^5+31c^6+96c^7+102c^8$$
$$+60c^9+38c^{10}+36c^{11}+24c^{12}+8c^{13}+c^{14}$$
$$+c^3\sqrt{C}\,(6+6c-19c^2-32c^3-18c^4-11c^5-18c^6-17c^7-7c^8-c^9),$$

$$-24(1+c)^4 \wp 2v = 2+8c+6c^2-22c^3-39c^4-14c^5-57c^6-180c^7$$
$$-174c^8-72c^9-58c^{10}-72c^{11}-36c^{12}-4c^{13}-c^{14}$$
$$+c^3\sqrt{C}\,(-6-6c+29c^2+52c^3+18c^4+13c^5+42c^6+31c^7+5c^8-c^9),$$

and so on.

But we shall find that, if

$$m^2 = -\frac{c^2(1+c)^2}{\sqrt{C}}\,\frac{p}{(p-z)(1-p)^2},$$

or $$2m^2\sqrt{C} = 6c^2+6c^3+c^4+2c^5+3c^6+c^7+(2+2c^2+c^4)\sqrt{C},$$

then
$$12m^2\wp v = \frac{6c^3(1+c)}{\sqrt{C}} + 1 \quad +3c^2+4c^3+c^4,$$

$$12m^2\wp 2v = \frac{6c^4(1+c)^2}{\sqrt{C}} + 1 \quad -3c^2-2c^3+c^4,$$

$$12m^2\wp 3v = -\frac{6c^4(1+c^3)}{\sqrt{C}} + 1 \quad -3c^2-2c^3+c^4,$$

$$12m^2\wp 4v = -\frac{6c(1+c)^4}{\sqrt{C}} + 1+6c+9c^2+4c^3+c^4,$$

$$12m^2\wp 5v = -\frac{6c^3(1+c)}{\sqrt{C}} + 1 \quad +3c^2+4c^3+c^4,$$

$$12m^2\wp 6v = \frac{6c(1+c)^4}{\sqrt{C}} + 1+6c+9c^2+4c^3+c^4;$$

so that, by addition,

$$12m^2 G_1 = 12m^2 \sum_{r=1}^{r=6} \wp\frac{2r\omega_3}{13}$$
$$= 6(1+c+c^2)^2.$$

54. If one root of Klein's "Modular Equation of the Thirteenth Order" for given J is written

$$\tau_\infty = \frac{13}{\tau} = \frac{13c(1+c)}{1-c-4c^2-c^3}$$
$$= -\frac{13}{4+c-\dfrac{1}{1+c}-\dfrac{1+c}{c}},$$

then, guided by the results of Klein's article, "Elliptische Functionen und Gleichungen fünften Grades" (*Math. Ann.*, XIV., pp. 145, 146), we should expect, by analogy with the cases of $\mu = 5$ and $\mu = 7$, that the remaining thirteen roots are expressible in the form

$$\tau_r = -\frac{\left(1+\epsilon^r\dfrac{A_1}{A_0}+\ldots+\epsilon^{6r}\dfrac{A_6}{A_0}\right)^2}{4+c-\dfrac{1}{1+c}-\dfrac{1+c}{c}},$$

$$r = 0, 1, 2, \ldots 12; \quad \epsilon = e^{\frac{2}{13}\pi i};$$

where the A's are expressions such that A^{13} is a rational function of c and $\sqrt{C}$.

$$\mu = 14.$$

55. The equation $$\gamma_{14} = 0$$

is equivalent to $$\frac{\gamma_9}{\gamma_5} = \left(\frac{\gamma_8}{\gamma_6}\right)^2,$$

or $$\gamma_6^2\gamma_9 - \gamma_5\gamma_8^2 = 0,$$

or $$x(y-x-y^2)^2\{y^3(y-x-y^2)-(y-x)^3\} - (y-x)\,y^3\{x(y-x-y^2)-(y-x)^2\}^2 = 0.$$

Putting $$y-x = yz,$$

$$z-y = \frac{z^2}{p},$$

$$z = c\,(p-1),$$

this equation reduces to the quadratic in p,

$$(1+c-2c^2-c^3)\,p^2+(2c+3c^2)\,p+c^2+c^3 = 0\,;$$

or, putting $$p = \frac{c+c^2}{q},$$

$$q^2+(2+3c)\,q+(1+c)(1+c-2c^2-c^3) = 0\,;$$

or, putting $$p = \frac{r+c}{r-1},$$

$$r^2-c^2r-c^3\,\frac{1+c}{1+2c} = 0.$$

Then $$2p = \frac{-2c-3c^2+c\sqrt{C}}{1+c-2c^2-c^3},$$

where $$C = c\,(1+2c)(4+5c+2c^2),$$

$$2z = c\,\frac{-2-4c+c^2+2c^3+c\sqrt{C}}{1+c-2c^2-c^3},$$

$$1-\frac{z}{p} = \frac{-3c-2c^2-\sqrt{C}}{2\,(1+c)},$$

$$y = c\,\frac{3c+6c^2-4c^3-8c^4-2c^5+(1+2c-2c^2-2c^3)\sqrt{C}}{2\,(1+c)(1+c-2c^2-c^3)},$$

$$2-2z = \frac{2+4c-3c^3-2c^4-c^2\sqrt{C}}{1+c-2c^2-c^3},$$

$$x = c\,\frac{3c+9c^2-3c^3-28c^4-17c^5+16c^6+21c^7+6c^8 + (1+3c-c^2-8c^3-3c^4+6c^5+3c^6)\sqrt{C}}{2\,(1+c-2c^2-c^3)^2}.$$

Also
$$s_7 = s_7 - s_2 = x\,\frac{\gamma_9\gamma_5}{\gamma_7^2\gamma_2^2} = x\,\frac{\gamma_5^2\gamma_8^2}{\gamma_7^2\gamma_2^2\gamma_6^2},$$

$$\sqrt{s_7} = x^{\frac{1}{2}}\,\frac{\gamma_5\gamma_8}{\gamma_7\gamma_2\gamma_6}$$

$$= \frac{(y-x)\,y\,\{x\,(y-x-y^2)-(y-x)^2\}}{\{x\,(y-x)-y^2\}\,(y-x-y^2)}$$

$$= \frac{(p-z)(1-p-z)}{1-p}$$

$$= (p-cp+c)(1+c)$$

$$= \tfrac{1}{2}c\,(1+c)\,\frac{c\,(1+c)(1-2c)+(1-c)\sqrt{C}}{1+c-2c^2-c^3},$$

and $s-s_7$ is a factor of

$$S = 4s\,(s+x)^2 - \{(y+1)\,s+xy\}^2.$$

The resolution of S into factors can now be effected, and the corresponding pseudo-elliptic integrals for parameters

$$v = \tfrac{1}{7}\omega_3 \quad\text{or}\quad \omega_1+\tfrac{1}{7}\omega_3,$$

and also for parameters $2v$, $3v$, $4v$, $5v$, $6v$, can be constructed; but the algebraical expressions involved will obviously be long and complicated.

$$\mu = 15.$$

56. The relation $\gamma_{15} = 0$

can be expressed by the elimination of λ between the equations

$$\lambda = \frac{\gamma_8}{\gamma_7} = y\,(1+c),$$

$$\lambda^3 = \frac{\gamma_9}{\gamma_6} = -y^3z\,\frac{(p-1)(p+c)}{p},$$

$$\lambda^5 = \frac{\gamma_{10}}{\gamma_5} = -y^5z^3\,\frac{(p-1)\{(1-c-c^2)\,p+2c+c^2\}}{p^2},$$

&c.,

employing the usual transformations.

We thus obtain the quadratic equation in p,

$$p^2 - c(c-1)(c^2+3c+3)\,p + c^2(c^2+3c+3) = 0;$$

so that $$2p = c(c-1)(c^2+3c+3) + c(c+1)\sqrt{C},$$

where $$C = (c^2-c-1)(c^2+3c+3).$$

Then $$z = \tfrac{1}{2}c\left\{c(c-1)(c^2+3c+3) - 2 + c(c+1)\sqrt{C}\right\}$$

$$= \tfrac{1}{2}c(c+1)(c^3+c^2-c-2+c\sqrt{C}),$$

$$1-z = -\tfrac{1}{2}\left\{c^5+2c^4-3c^2-2c-2+c^2(c+1)\sqrt{C}\right\},$$

$$\frac{c}{p} = \frac{(c-1)(c^2+3c+3)-(c+1)\sqrt{C}}{2(c^2+3c+3)},$$

$$1+\frac{c}{p} = \frac{(c+1)(c^2+3c+3-\sqrt{C})}{2(c^2+3c+3)},$$

$$\frac{z}{p} = c+1-1-\frac{c}{p}$$

$$= \frac{(c+1)(c^2+3c+3+\sqrt{C})}{2(c^2+3c+3)},$$

$$y = z\left(1-\frac{z}{p}\right)$$

$$= -\frac{c(c+1)}{2(c^2+3c+3)}\left\{(c^2+3c+3)(c^4-2c^3-c+1) + (c^4+2c^3-3c-1)\sqrt{C}\right\},$$

$$x = y(1-z)$$

$$= \frac{c(c^2+3c+3)(c^{10}+3c^9+0-10c^7-11c^6+3c^5+12c^4+8c^3+2c^2-c-1) + c(c+1)(c^9+4c^8+4c^7-6c^6-15c^5-8c^4+4c^3+6c^2+4c+1)\sqrt{C}}{2(c^2+3c+3)}.$$

57. Making use of these values for the calculation of $\wp v$, $\wp 2v$, $\wp 3v$, ..., and putting $$m^2 = 24(c^2+3c+3),$$

we find

$$m^2\wp v = -8(c^2+3c+3)\,x - 2(c^2+3c+3)(y+1)^2$$

$$= -(c^{14}+9+30+35-45-186-195+15+219+228+123+21-27-12+6)$$

$$-c(c+1)\sqrt{C}\,(c^{10}+7+16+6-34-54-21+15+21+21+6),$$

using the method of detached coefficients.

Then

$$\tfrac{1}{12}m^2(\wp 2v-\wp v) = 2(c^2+3c+3)\,x$$
$$= 0+1+6+12- \quad -41-60-12+53+62+29+\ 2-\ 6-3+0+c(c+1)\sqrt{C}(0+1+4+4-6-15-8+\ 4+6+4+1),$$

$$\tfrac{1}{12}m^2(\wp 3v-\wp v) = 2(c^2+3c+3)\,y$$
$$= 0+0+0+\ 0+\ 0+\ 0-\ 1-\ 4-\ 4+\ 6+15+\ 8-\ 3-3+0+c(c+1)\sqrt{C}(0+0+0+0+0+\ 0-1-\ 2-0+3+1),$$

$$\tfrac{1}{12}m^2(\wp 4v-\wp v) = 2(c^2+3c+3)\,z\,(1-z)$$
$$= 0+0-1-\ 7-19-18+22+77+80+24-26-35-21-6+0+c(c+1)\sqrt{C}(0+0-1-5-9-3+11+16+9+3+0),$$

$$\tfrac{1}{12}m^2(\wp 5v-\wp v) = 2(c^2+3c+3)\,\frac{z}{p}$$
$$= 0+0+0+\ 0+\ 0+\ 1+\ 4+\ 4-\ 6-16-11+\ 1+\ 6+5+2+c(c+1)\sqrt{C}(0+0+0+0+0+1+\ 2+\ 0-3-2-1),$$

$$\tfrac{1}{12}m^2(\wp 6v-\wp v) = 2(c^2+3c+3)(p-z)(1-p)$$
$$= 0+0+0+\ 1+\ 6+12-\ 1-40-55-\ 4+51+41+\ 6-3+0+c(c+1)\sqrt{C}(0+0+0+1+4+4-\ 6-14-5+6+3),$$

$$\tfrac{1}{12}m^2(\wp 7v-\wp v) = -2(c^2+3c+3)\,c\,(1+c)(1-z)\left(1-\frac{z}{p}\right)$$
$$= 0+0+0+\ 0-\ 1-\ 5-\ 8+\ 2+21+24+\ 8-\ 4-\ 6-3+0+c(c+1)\sqrt{C}(0+0+0+0-1-3-\ 2+\ 3+4+2+1),$$

and
$$\wp 8v-\wp v = \left(1+\frac{c}{p}\right)\frac{\wp 4v-\wp v}{(1+c)^2}$$

gives the same as $\wp 7v-\wp v$, so that
$$\wp 7v = \wp 8v,$$

a verification; and we are now able theoretically to determine Kiepert's parameter ξ_1 as a function of c and $\sqrt{C}$ (*Math. Ann.*, XXXII., p. 121.).

We thus find, using detached coefficients of descending powers of c, beginning with c^{14},

$$m^2 \wp v$$
$$= -1-9-30-35+45+186+195+6-198-228-123-21+27+12-6+c(c+1)\sqrt{C}(-1-7-16-6+34+54+21-15-21-21-6),$$

$$m^2 \wp 2v$$
$$= -1+3+42+73+111-66-417-450+18+474+471+183-9-24-6+c(c+1)\sqrt{C}(-1+5+32+42-38-126-75+33+51+27+6),$$

$$m^2 \wp 3v$$
$$= -1-9-30-35+45+186+183-42-246-156+57+75-9-24-6+c(1+c)\sqrt{C}(-1-7-16-6+34+54+9-39-21+15+6),$$

$$m^2 \wp 4v$$
$$= -1-9-42-119-183-30+459+930+762+60-423-393-153-24-6+c(1+c)\sqrt{C}(-1-7-28-66-74+18+153+177+87+15-6),$$

$$m^2 \wp 5v$$
$$= -1-12-42-38+102+279+177-156-324-231-111-42+27+54+18+c(1+c)\sqrt{C}(-1-4-10-15+4+63+90+15-72-69-24),$$

$$m^2 \wp 6v$$
$$= -1-9-30-23+117+330+183-474-858-276+489+471+99-24-6+c(1+c)\sqrt{C}(-1-7-16+6+82+102-51-183-81+51+30),$$

$$m^2 \wp 7v$$
$$= -1-9-30-35+33+126+99+30+54+60-27-69-45-24-6+c(1+c)\sqrt{C}(-1-7-16+6+22+18-3+21+27+3+6),$$

$$m^2 G_1$$
$$= -7-51-150-173+135+690+837+219-465-444-69+9-99-72-18+c(1+c)\sqrt{C}(-7-37-76-42+94+186+99-21-15+45+18).$$

By analogy with preceding results, this expression for G_1 ought to be capable of a simplification, but I have not been able to discover this.

$$\mu = 16.$$

58. The relation $$\gamma_{16} = 0,$$

being equivalent to $$\lambda^2 = \frac{\gamma_9}{\gamma_7},$$

$$\lambda^4 = \frac{\gamma_{10}}{\gamma_6},$$

$$\lambda^6 = \frac{\gamma_{11}}{\gamma_5},$$

&c.,

may be replaced by $$\frac{\gamma_{10}}{\gamma_6} - \left(\frac{\gamma_9}{\gamma_7}\right)^2 = 0,$$

or $$\frac{\gamma_9}{\gamma_7}\frac{\gamma_{11}}{\gamma_5} - \left(\frac{\gamma_{10}}{\gamma_6}\right)^2 = 0,$$

or $$\frac{\gamma_{11}}{\gamma_5} - \left(\frac{\gamma_9}{\gamma_7}\right)^3 = 0,$$

&c.;

and these relations, with the usual transformations, will lead to the cubic in p,

$$(1-2c-c^2)\,p^3+4cp^2+(c^2-c^3)\,p+c^3\,(1+c) = 0;$$

and, putting $$p = \frac{q+c}{q-1},$$

this becomes $$q^3-c\,(1+c)\,q^2-c^2q-c^3\,(1+c) = 0.$$

Put $$q = ca;$$

then $$c^2a^2-c\,(a^3-a^2-a-1)+1 = 0,$$

a quadratic for c in terms of a, the discriminant of the quadratic being

$$(a^3-a^2-a-1)^2-4a^2 = (a^4-1)(a^2-2a-1)$$

$$= A, \text{ suppose};$$

so that $$c = \frac{a^3-a^2-a-1+\sqrt{A}}{2a^2},$$

$$q = ca = \frac{a^3-a^2-a-1+\sqrt{A}}{2a},$$

$$p = \frac{q+c}{q-1} = \frac{(1+a)\,c}{q-1}$$

$$= \frac{(a+1)(a^2-2a-1)+\sqrt{A}}{2a\,(a^2-2a-1)},$$

$$z = \frac{(a^2-1)(a^2-2a-1)+(a^2-a-1)\sqrt{A}}{2a^3\,(a^2-2a-1)}, \text{ \&c.}$$

59. The expression of Gierster's parameter τ or τ_{16} as a function of a or c has still to be discovered; this can be effected by a return to the case of $\mu = 8$ (§ 26); and (Gierster, *Math. Ann.*, XIV., p. 541)

$$(\tau-2)^2 = -2\,(\tau_8-2) = \frac{(1-2z)^2}{z-z^2}.$$

Now, in § 26,

$$\frac{\wp\,\omega_3-\wp\,\frac{1}{4}\omega_3}{\wp\,\omega_3-\wp\,\frac{3}{4}\omega_3} = \frac{z^2\,(1-z)^2+z\,(1-z)^2\,(1-2z)}{z^3\,(1-z)^2-z^3\,(1-z)(1-2z)} = \left(\frac{1-z}{z}\right)^2,$$

and, expressed in the notation for $\mu = 16$, this is

$$= \frac{s_8-s_2}{s_8-s_6} = \frac{\gamma_{10}\,\gamma_6^2}{\gamma_{14}} = \left(\frac{\gamma_6\gamma_7}{\gamma_9}\right)^4 = \left(\frac{c}{q}\right)^4 = \frac{1}{a^4}$$

and therefore

$$\frac{1-z}{z} = \frac{1}{a^2}, \quad z = \frac{a^2}{a^2+1}, \quad 1-z = \frac{1}{a^2+1}, \quad 1-2z = -\frac{a^2-1}{a^2+1};$$

$$\tau-2 = \frac{a^2-1}{a},$$

$$\tau = \frac{a^2+2a-1}{a},$$

$$\tau-4 = \frac{a^2-2a-1}{a},$$

$$\tau^2-4\tau+8 = \frac{(a^2+1)^2}{a^2}.$$

Also (Gierster) $\quad \tau_2 = -\dfrac{\tau\,(\tau-4)(\tau^2-4\tau+8)(\tau-2)^4}{64};$

and, taking

$$\tau_2 = \frac{1}{4}\left(\frac{1}{\kappa} - \kappa\right)^2,$$

$$\left(\frac{1}{\kappa} - \kappa\right)^2 = \frac{(a^2+1)^2(a^2-1)^4(a^4-6a^2+1)}{16a^8},$$

$$\left(\frac{1}{\kappa} + \kappa\right)^2 = \frac{(a^8-4a^6-2a^4-4a^2+1)^2}{16a^8},$$

$$\kappa = \frac{a^8-4a^6-2a^4-4a^2+1-(a^2+1)(a^2-1)^2\sqrt{(a^4-6a^2+1)}}{8a^4},$$

$$\sqrt{\kappa} = \frac{(a^2-1)^2-(a^2+1)\sqrt{(a^4-6a^2+1)}}{4a^2},$$

$$\sqrt[4]{\kappa} = \frac{a^2+1-\sqrt{(a^4-6a^2+1)}}{2\sqrt{2}\,a},$$

$$\sqrt[8]{\kappa} = \frac{\sqrt{(a^2+2\sqrt{2}\,a+1)}-\sqrt{(a^2-2\sqrt{2}\,a+1)}}{2\sqrt[4]{2}\sqrt{a}},$$

or $2\sqrt[4]{2}\sqrt[8]{\kappa} = \sqrt{\left(a+2\sqrt{2}+\frac{1}{a}\right)}-\sqrt{\left(a-2\sqrt{2}+\frac{1}{a}\right)}$;

and now the pseudo-elliptic integrals

$$I\left(\frac{1,3,5,7}{8}\omega_2\right) \quad \text{or} \quad I\left(\omega_1+\frac{1,3,5,7}{8}\omega_2\right)$$

can be constructed by means of this parameter a.

$$\mu = 17.$$

60. The relation $\gamma_{17} = 0$

is equivalent to

$$\lambda = \frac{\gamma_9}{\gamma_8},\quad \lambda^3 = \frac{\gamma_{10}}{\gamma_7},\quad \lambda^5 = \frac{\gamma_{11}}{\gamma_6},\ \&c.,$$

so that, with the preceding transformations, we obtain

$$p^4-(4-c-3c^2-c^3)c^2p^3+(4+4c-9c^2-8c^3-2c^4)cp^2$$
$$-(1-4c-10c^2-5c^3-c^4)cp-c^2(1+c) = 0;$$

or, with $p = \frac{q+c}{q-1}$,

$$q^4-c(1+2c)q^3+c(1+c)(1-c+c^2)q^2-c(1+c)(1+c+c^2)q$$
$$-c^3(1+c)^2 = 0,$$

$$\{q^2-\tfrac{1}{2}c\,(1+2c)\,y+\tfrac{1}{2}m\}^2$$

$$=\{m+\tfrac{1}{4}c^2(1+2c)^2-c\,(1+c^3)\}\,q^2$$

$$-\{\tfrac{1}{2}mc\,(1+2c)+c\,(1+c)(1+c+c^3)\}\,q+\tfrac{1}{4}m^2+c^4(1+c)^2,$$

a perfect square, if

$$\{m^2+4c^4(1+c)^2\}\{m+\tfrac{1}{4}c^2(1+2c)^2-c\,(1+c^3)\}$$

$$-\{\tfrac{1}{2}mc\,(1+2c)+c\,(1+c)(1+c+c^3)\}^2=0,$$

or $m^3-m^2c\,(1+c^3)-mc^2(1+c)(1-c-2c^2+c^3+2c^4)$

$$-c^4(1+c)^2(1+2c+5c^2+c^3-2c^4+c^6)=0;$$

but this cubic appears irreducible.

$$\mu=18.$$

61. Here

$$\lambda=\frac{\gamma_{10}}{\gamma_8}=\quad yz\,\frac{c\,(p-1)\{(1-c-c^2)p+2c+c^2\}}{(1+c)\,p}\quad\ldots\ldots\ldots(1).$$

$$\lambda^2=\frac{\gamma_{11}}{\gamma_7}=\quad y^3z^3\,\frac{p^2-c^2p+c+c^3}{p^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\lambda^3=\frac{\gamma_{12}}{\gamma_6}=-y^5z^4\,\frac{(p-1)\{(2+c)p-1\}}{p^3}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\lambda^4=\frac{\gamma_{13}}{\gamma_5}=\quad y^6z^6\,\frac{(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}}{p^4}\ldots(4).$$

From (2) and (4),

$$(c^3+3c^2-3)\,p^3-(c^4+3c^3+6c^2+3c-3)\,p^2$$

$$+(2c^4+5c^3+5c^2+2c-1)\,p-c\,(c+1)^2=0.$$

Putting $\dfrac{p+c}{1+c}=\dfrac{t}{t-1},\quad \dfrac{p-1}{1+c}=\dfrac{1}{t-1},\quad p=\dfrac{t+c}{t-1},$

$$t^3-(2c^2-1)\,t^2+(c+1)(c^3-c^2+2c+1)\,t-2c^2(c+1)^2=0.$$

Put $t=(c+1)\,x$, and $c=y$ for the moment;

therefore $(1+y)\,x^3+(1-2y^2)\,x^2+(1+2y-y^2+y^3)\,x-2y^2=0.$

$$U_4=xy\,(x-y)^2,\quad U_3=x\,(x^2-y^2),\quad U_2=(x+y)^2-3y^2,\quad U_1=x.$$

Put $x-y=q,\quad x+y=r,\quad x=\frac{r+q}{2},\quad y=\frac{r-q}{2}$;

then $r^2(q+1)^2+2r(q^2+3q+1)-q^4-3q^2+2q=0$;

therefore $r=\frac{-q^2-3q-1+\sqrt{Q}}{(q+1)^2}$,

$$Q=q^6+2q^5+5q^4+10q^3+10q^2+4q+1,$$

$$\frac{t}{c+1}=x=\frac{q^3+q^2-2q-1+\sqrt{Q}}{2(q+1)^2},\quad y=\frac{-q^3-3q^2-4q-1+\sqrt{Q}}{2(q+1)^2}=c;$$

therefore

$$c+1=\frac{-q^3-q^2+0+1+\sqrt{Q}}{2(q+1)^2},\quad t=\frac{3q^4+7q^3+6q^2+q+0-q\sqrt{Q}}{2(q+1)^4},$$

$$t-1=\frac{q^4-q^3-6q^2-7q-2-q\sqrt{Q}}{2(q+1)^4},$$

$$p-1=\frac{c+1}{t-1}=(q+1)^2\frac{-q^3-q^2+0+1+\sqrt{Q}}{q^4-q^3-6q^2-7q-2-q\sqrt{Q}}$$

$$=\frac{(q+1)^2\{2(q+1)^3(q^3+3q^2-1)-2(q+1)^3\sqrt{Q}\}}{-4(q+1)^4(q^3-3q-1)},$$

$$p-1=\frac{-q^4-4q^3-3q^2+q+1+(q+1)\sqrt{Q}}{2(q^3-3q-1)},$$

$$p=\frac{-q^4-2q^3-3q^2-5q-1+(q+1)\sqrt{Q}}{2(q^3-3q-1)},$$

$$z=c(p-1)$$

$$=\frac{(-q^3-3q^2-4q-1+\sqrt{Q})(q+1)(-q^3-3q^2+0+1+\sqrt{Q})}{4(q+1)^3(q^3-3q-1)}$$

$$=\frac{q^5+3q^4+6q^3+5q^2-(q^2+2q)\sqrt{Q}}{2(q^3-3q-1)},$$

$$\frac{1}{p}=\frac{q^4+2q^3+3q^2+5q+1+(q+1)\sqrt{Q}}{2q},$$

$$\frac{z}{p}=\frac{\{q^4+3q^3+6q^2+5q-(q+2)\sqrt{Q}\}\{q^4+2q^3+3q^2+5q+1+(q+1)\sqrt{Q}\}}{4(q^3-3q-1)}$$

$$=\frac{2(q^3-3q-1)(q^3+q^2+0+1)+2(q^3-3q-1)\sqrt{Q}}{4(q^3-3q-1)}$$

$$=\frac{q^3+q^2+0+1+\sqrt{Q}}{2},$$

$$1-\frac{z}{p}=\frac{-q^3-q^2+0+1-\sqrt{Q}}{2},$$

$$y=z\left(1-\frac{z}{p}\right)$$

$$=q\,(q+1)\,\frac{5q^3+9q^2+6q+1-(2q+1)\sqrt{Q}}{2\,(q^3-3q-1)},$$

$$1-z=\frac{-q^5-3q^4-4q^3-5q^2-6q-2+(q^2+2q)\sqrt{Q}}{2\,(q^3-3q-1)},$$

$$x=y\,(1-z)$$

$$=(q+1)\,\frac{\begin{array}{r}-(1+7+23+51+80+90+70+36+10+1)\\ +\sqrt{Q}\,(1+6+15+19+15+6+1)\end{array}}{2(q^3-3q-1)^2}$$

$$=q\,(q+1)(q^2+q+1)\,\frac{\begin{array}{r}-(1+6+16+29+35+26+9+1)\\ +\sqrt{Q}\,(1+5+9+5+1)\end{array}}{2\,(q^3-3q-1)^2},$$

$$\frac{x}{p}=q\,(q+1)(q^2+q+1)\,\frac{-q^4+q^3+6q^2+7q+2-q\sqrt{Q}}{2\,(q^3-3q-1)},$$

$$z(1-z)=q\,(q+1)(q^2+q+1)\,\frac{\begin{array}{r}-q\,(q^5+4q^4+8q^3+14q^2+17q+7)\\ +\sqrt{Q}\,(q+2)(q^2+q+1)\end{array}}{2\,(q^3-3q-1)^2},$$

$$s_9+x=x^3\,\frac{\gamma_{10}\gamma_8}{\gamma_9^2}$$

$$=x^3\,\frac{y^4z^4\,\dfrac{(p-1)\{(1-c-c^2)p+2c+c^2\}}{p^2}\,y^3z^3\,\dfrac{1+c}{cp}}{x^3y^6z^6\,\dfrac{(p-1)^2(p+c)^2}{p^4}}$$

$$=\frac{yz}{p-1}\,\frac{p\,(1+c)\,\{(1-c-c^2)\,p+2c+c^2\}}{c\,(p+c)^2}$$

$$=(1+c)\,py\,\frac{(1-c-c^2)\,p+2c+c^2}{(p+c)^2}$$

$$=(1+c)\,z\,(p-z)\,\frac{(1-c-c^2)\,p+2c+c^2}{(p+c)^2}.$$

Put $$p = \frac{t+c}{t-1},\quad p-1 = \frac{1+c}{t-1};$$

therefore $$z = \frac{c+c^2}{t-1},\quad p+c = \frac{1+c}{t-1},$$

$$p-z = \frac{t-c^2}{t-1}.$$

$$s_9+x = \frac{(1+c)\,c\,(1+c)}{t-1}\,\frac{t-c^2}{t-1}\,\frac{\dfrac{(1+c)(t-c-c^2)}{t-1}}{\dfrac{(1+c)^2}{(t-1)^2}}$$

$$= c\,(1+c)\,\frac{(t-c^2)(t-c-c^2)}{t-1}.$$

But s_9 can be determined more rapidly by noticing that

$$s_9 = s_0 - s_2 = x^1\,\frac{\gamma_{11}\gamma_7}{\gamma_9^2} = x^1\,\frac{\gamma_{10}^2\gamma_7^2}{\gamma_9^2\gamma_8^2},$$

$$\sqrt{s_9} = x^{\frac{1}{2}}\,\frac{\gamma_{10}\gamma_7}{\gamma_9\gamma_8} = \frac{c\,(p-1)\{(1-c-c^2)\,p+2c+c^2\}}{(1+c)(p+c)}$$

$$= \frac{c\,(t-c-c^2)}{t\,(t-1)} = c^2\,(1+c)\left(\frac{1}{t}-\frac{1}{t-1}\right)+\frac{c}{t-1}.$$

62. Now, if $$t-1 = -y\,(q+1)^4\,(q^2-2q-1),$$

taking, from $\mu = 9$,

$$x = c^3\,(1+c)(1+c+c^2),\quad y = c^2+c^3,$$

$$S = 4s\,(s+x)^2 - \{(1+y)\,x+xy\}^2,$$

and writing $$s+x = t,$$

$$T = 4t^2\,(t-x) - \{(1+y)\,t-x\}^2$$

$$= 4t^2\,\{t-c^3\,(1+c)(1+c+c^2)\}$$

$$-\{(1+c^2+c^3)\,t-c^3\,(1+c)(1+c+c^2)\}^2,$$

this can be written

$$T = 4t\,\{t+c\,(1+c)(1+c+c^2)\}^2$$

$$-\{(1+4c+3c^2+c^3)\,t+c^3\,(1+c)(1+c+c^2)\}^2$$

$$= 4s\,(s+x)^2 - \{(1+y)\,s+xy\}^2;$$

again, if
$$t = m^2 s,$$
$$c(1+c)(1+c+c^2) = m^2 x,$$
$$1+4c+3c^2+c^3 = m(1+y),$$
$$c^2(1+c)(1+c+c^2) = m^2 xy;$$
therefore
$$my = c, \quad m = (1+c)^2;$$
$$x = \frac{c(1+c+c^2)}{(1+c)^4}, \quad y = \frac{c}{(1+c)^2}.$$

63. We have still to determine Gierster's parameter τ or τ_{18} as a function of q; this proved very laborious, but it was finally effected in the following manner, by means of Gierster's relations (*Math. Ann.*, XIV., p. 540).

Putting
$$\tau_{18}+2 = x,$$
thence $\tau = \tau_{18}$ is connected with τ_9 by the relation
$$\tau_9 = -\frac{\tau(\tau+3)^2}{3(\tau+2)},$$
$$-3(\tau_9-1) = \frac{x^3-2}{x} = x^2 - \frac{2}{x}.$$
Also
$$\tau_3-1 = (\tau_9-1)^3,$$
and referring to the case of $\mu = 6$ (§ 18),
$$-27(\tau_3-1) = 4\frac{(1-c+c^2)^3}{(c-c^2)^2},$$
so that x and c are connected by the relation
$$c-c^2 = \frac{2}{x^3}.$$
Now, in § 19,
$$\frac{\wp\frac{2}{3}\omega_3-\wp\frac{1}{3}\omega_3}{\wp\omega_3-\wp\frac{2}{3}\omega_3} = \frac{2c-2c^2}{(c-c^2)^2} = \frac{2}{c-c^2} = x^3;$$
so that, with the notation for $\mu = 18$,
$$x^3 = \frac{s_6-s_3}{s_9-s_6} = \frac{\gamma_9^3}{\gamma_{15}\gamma_3^2} = \frac{\gamma_9^3\gamma_7^3}{\gamma_{11}^3\gamma_3^3},$$
$$x = \frac{\gamma_9\gamma_7}{\gamma_{11}\gamma_3} = -\frac{(p-1)(p+c)}{p^2-c^2p+c+c^2},$$
$$-\frac{1}{x} = 1+\frac{1}{p-1}-\frac{c+c^2}{p+c} = 1+\frac{1}{p-1}-c+\frac{c}{t}.$$

But $$\frac{1}{p-1} = \frac{-q^3-3q^2+0+1-\sqrt{Q}}{2q(q+1)^2},$$

$$c = \frac{-q^3-3q^2-4q-1+\sqrt{Q}}{2(q+1)^2},$$

$$t = q\frac{3q^3+7q^2+6q+1-\sqrt{Q}}{2(q+1)^4},$$

$$\frac{c}{t} = \frac{-q^3-5q^2-6q-3+\sqrt{Q}}{4q(q+1)},$$

so that $$-\frac{1}{x} = \frac{q^3+q^2-2q-1-\sqrt{Q}}{4q(q+1)},$$

$$x = \frac{q^3+q^2-2q-1+\sqrt{Q}}{2q(q+1)},$$

$$\tau_{18} = \frac{q^3-3q^2-6q-1+\sqrt{Q}}{2q(q+1)}.$$

Thence, with the $p = p_9$ for $\mu = 9$ in § 33, we find

$$-3(\tau_9-1) = -1-p+\frac{1}{p}+\frac{1}{p-1}$$

$$= \frac{q^6+3q^5+3q^4+q^3+3q^2+3q+1+(q^3+0-3q-1)\sqrt{Q}}{2q^2(q+1)^2},$$

and this is satisfied by

$$p_9 = \frac{q^3+3q^2+2q+1+\sqrt{Q}}{2q(q+1)^2},$$

so that $$p_9-1 = \frac{c+1}{q}.$$

64. It will be found that Joubert's parameter x employed on p. 89 of his memoir, "*Sur les équations qui se rencontrent dans la théorie de la transformation des fonctions elliptiques*" (Paris, 1876), is connected with Gierster's $\tau = \tau_{18}$ by the relation

$$x = \frac{\tau}{\tau+2};$$

and then $$\tau_2 = \frac{1}{4\kappa^2\kappa'^2} = -\frac{64\{(x-1)^3+1\}}{(x-1)^3\{(x-1)^3-8\}};$$

while Joubert's parameter x on p. 91, giving

$$\tau_2 = -\frac{1}{4}\left(\frac{1}{\kappa}-\kappa\right)^2 = -\frac{64\{(x-1)^3+1\}}{(x-1)^9\{(x-1)^3-8\}},$$

is connected with Gierster's τ by the relation

$$x = \tau+3.$$

So also Joubert's parameter x on p. 103 is connected with Gierster's $\tau = \tau_{10}$ by the relation

$$x = \frac{\tau}{\tau-2},$$

and then
$$\tau_2 = \frac{1}{4\kappa^2\kappa'^2} = -\frac{64x}{(x-1)^5(x-5)}.$$

$$\mu = 19.$$

65. The relation
$$\gamma_{19} = 0$$

being replaced by the relations

$$\lambda = \frac{\gamma_{10}}{\gamma_9} = \frac{yz}{x^{\frac{1}{2}}}\,\frac{(1-c-c^2)p+2c+c^2}{p+c},$$

$$\lambda^3 = \frac{\gamma_{11}}{\gamma_8} = y^3z^3\,\frac{c(p-1)(p^2-c^2p+c+c^2)}{(1+c)p^2},$$

$$\lambda^5 = \frac{\gamma_{12}}{\gamma_7} = x^{\frac{1}{2}}y^4z^4\,\frac{(2+c)p-1}{p^2},$$

$$\lambda^7 = \frac{\gamma_{13}}{\gamma_6} = \frac{y^5z^5}{x^{\frac{1}{2}}}\,\frac{(p-1)^2\{(1+c-2c^2-c^3)p^2+(2c+3c^2)p+c^2+c^3\}}{p^3},$$

&c.,

the elimination of λ in any manner between these relations is found always to lead to the relation

$$\begin{aligned} p^5 &+ (-2+0+4c^2+5c^3-3c^4-4c^5-c^6)\,p^4 \\ &+ (1-4c-12c^2-c^3+18c^4+11c^5+2c^6)\,p^3 \\ &+ (0+3c+2c^2-15c^3-21c^4-7c^5-c^6)\,p^2 \\ &+ (0+0+3c^2+7c^3+3c^4-c^5+0)\,p \\ &+ c^3(1+c)^2 = 0. \end{aligned}$$

If we put $$p = \frac{c(1+c)}{r},$$

$$r^5 + (3+4c-c^2)\,r^4 + (3+2c-15c^2-21c^3-7c^4-c^5)\,r^3$$
$$+ (1+c)(1-4c-12c^2-c^3+18c^4+11c^5+2c^6)\,r^2$$
$$+ c\,(1+c)^2\,(-2+0+4c^2+5c^3-3c^4-4c^5-c^6)\,r$$
$$+ c^3\,(1+c)^3 = 0.$$

No factor or reduction of this quintic equation, in p or r, has been discovered so far, although it was hoped that a quadratic factor might be discovered, from the analogy between the cases of $\mu = 19$ and $\mu = 11$, worked out by Dr. Robert Fricke in the *Math. Ann.*, XL.

Thus, in the case of $\mu = 19$, he shows there that the "Modular Equation of the Nineteenth Order" may be expressed by the relations

$$12\,(A^2-B^2)(g_2, g_2')$$
$$= 2^5.3.19.A^6 - 2^4.211.A^4B^2 + 3.503A^2B^4 - 181B^6 \mp 60BE\,(5A^2-3B^2),$$
$$216\,(\quad)(g_3, g_3') = \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$2\sqrt[6]{\Delta},\; 2\sqrt[6]{\Delta'} = \frac{A^{\frac{1}{3}}}{(A^2-B^2)^{\frac{1}{2}}}\,(B^3-8A^2B \pm E),$$
$$\sqrt[6]{\Delta\Delta'} = \frac{19A^{\frac{19}{6}}}{(A^2-B^2)^{\frac{1}{2}}},$$

and putting $$\tau = \frac{A^2}{B^2}, \quad \tau' = \frac{E}{B^3},$$

$$\tau'^2 = -2^2.19\tau^3 + 2^6.\tau^2 - 2^4.\tau + 1;$$

also $$P = -2A^2 + \tfrac{3}{4}B^2,$$

where $$19P = \sum_{r=1}^{r=9} \wp\,\frac{2r\omega_3}{19}.$$

$$\mu = 20.$$

66. The relation $$\gamma_{20} = 0$$

being replaced, as before, by the relations

$$\lambda^2 = \frac{\gamma_{11}}{\gamma_9} = \frac{y^2z^2}{x^4}\,\frac{p^2-c^2p+c+c^2}{p\,(p+c)},$$

$$\lambda^4 = \frac{\gamma_{12}}{\gamma_8} = x^4y^3z^3\,\frac{c\,(p-1)\,\{(2+c)\,p-1)\}}{(1+c)\,p^2},$$

$$\lambda^6 = \frac{\gamma_{13}}{\gamma_7} = \qquad \&c.,$$

we obtain, by the elimination of λ,

$$\{1-c(p-1)\}(p+c)^2\{(2+c)p-1\}-(1+c)(p^2-c^2p+c+c^2)^2=0,$$

or, putting $$p=\frac{q+c}{q-1},$$

$$(q-1-c-c^2)(q+1+c)=(q^2-c^2q+c+c^2)^2.$$

If we put $$q^2-c^2q+c+c^2=r,$$

then this equation becomes

$$r-(1+c)^2=r^2,$$

or, with $$1+c=-a,$$

$$2r=1+\sqrt{(1+4a^2)};$$

and thus q, p, z, y, and x can all be expressed in terms of a single parameter a.

$$\mu=21.$$

67. Here $$\gamma_{21}=0,$$

or $$\lambda=\frac{\gamma_{11}}{\gamma_{10}}=yz\frac{p^2-c^2p+c+c^2}{p\{(1-c-c^2)p+2c+c^2\}},$$

$$\lambda^3=\frac{\gamma_{12}}{\gamma_9}=y^3z^3\frac{(2+c)p-1}{p(p+c)},$$

$$\lambda^5=\frac{\gamma_{13}}{\gamma_8}=-y^4z^4\frac{c(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}}{(1+c)p^3};$$

and, by the elimination of λ, we obtain

$$(p^2-c^2p+c+c^2)^3(p+c)-p^2\{(1-c-c^2)p+2c+c^2\}^3\{(2+c)p-1\}=0,$$

or else $$yz\frac{\{(2+c)p-1\}^2}{p^2(p+c)^2}$$

$$+\frac{c(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}(p^2-c^2p+c+c^2)}{(1+c)p^4\{(1-c-c^2)p+2c+c^2\}}=0,$$

each reducing to

$$c(1+c)p\{(1-c)p+c\}\{(2+c)p-1\}^2\{(1-c-c^2)p+2c+c^2\}$$
$$+(p+c)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}(p^2-c^2p+c+c^2)=0,$$

an equation of the sixth degree in p, but having a factor $p-1$; this case does not look promising (Kiepert, *Math. Ann.*, XXXII., p. 123).

$$\mu = 22.$$

68. Here $\lambda^2 = \dfrac{\gamma_{12}}{\gamma_{10}} = \quad x^4y^2z^2 \dfrac{(2+c)\,p-1}{p\{(1-c-c^2)\,p+2c+c^2\}},$

$$\lambda^4 = \frac{\gamma_{13}}{\gamma_9} = -\frac{y^4z^4}{x^4}\,\frac{(p-1)\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\}}{p^2\,(p+c)},$$

and therefore, eliminating λ,

$$x\frac{\{(2+c)\,p-1\}^2}{\{(1-c-c^2)\,p+2c+c^2\}^2}+\frac{(p-1)\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\}}{p+c}=0.$$

But $x = y\,(1-z) = z\,(1-z)\dfrac{p-z}{p} = c\,(p-1)(1+c-cp)\dfrac{p+c-cp}{p}$;

so that

$$c\,(1+c-cp)(p+c-cp)(p+c)\,\{(2+c)\,p-1\}^2$$
$$+p\,\{(1-c-c^2)\,p+2c+c^2\}^2\,\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\} = 0.$$

If we put $\qquad p = \dfrac{q+c}{q-1},$

this equation reduces to a quartic in q,

$$(1+c)^2\,q^4-c\,(2+5c+4c^2+2c^3)\,q^3-c\,(1+3c+c^2-2c^3-c^4-c^5)\,q^2$$
$$+c^3\,(1+c)(2+4c+c^2+c^3)\,q-c^5\,(1+c)^2 = 0.$$

Writing this equation

$$\{(1+c)^2\,q^2-\tfrac{1}{2}\,(2c+5c^2+4c^3+2c^4)\,q+\tfrac{1}{2}m\}^2$$
$$= \{m+\tfrac{1}{4}\,(2c+5c^2+4c^3+2c^4)+c\,(1+c)^2(1+3c+c^2-2c^3-c^4-c^5)\}\,q^2$$
$$-2\,\{(2c+5c^2+4c^3+2c^4)\,\tfrac{1}{2}m+c^3\,(1+c)^2(2+4c+c^2+c^3)\}\,q$$
$$+\tfrac{1}{4}m^2c^5\,(1+c)^4,$$

and making the right-hand side of the equation a perfect square, the reducing cubic for m has a root $-c^4\,(1+c)$; so that

$$\{2\,(1+c)^2\,q^2-(2c+5c^2+4c^3+2c^4)\,q-c^4\,(1+c)\}^2$$
$$= (4c+8c^2+4c^3+c^4)\,\{(1+2c)\,q-c^2\,(1+c)\}^2,$$

and the resolution of the quartic is effected.

I am indebted to Mr. St. Bodfan Griffiths, pupil of Professor G. B. Mathews, at University College, Bangor, N. Wales, for the solution of these cubic and quartic equations, and for a general verification of the algebraical reductions in this case of $\mu = 22$.

$$\mu = 25.$$

69. Then
$$\gamma_{25} = 0,$$

or
$$\frac{\gamma_{13}}{\gamma_{12}}\frac{\gamma_{15}}{\gamma_{10}} = \left(\frac{\gamma_{14}}{\gamma_{11}}\right)^2,$$

$$y^6 z^6 \frac{(p-1)^3 \{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\}\{p^2+(3c-2c^2-c^3)\,p+3c^2+3c^3+c^4\}}{p^4\{(2+c)\,p-1\}\{(1-c-c^2)\,p+2c+c^2\}}$$

$$= y^6 z^6 \frac{(p-1)^4\{(1+c-2c^2-c^3)\,p^2+(2c+3c^2)\,p+c^2+c^3\}^2}{p^4\,(p^2-c^2p+c+c^2)^2},$$

or
$$(p^2-c^2p+c+c^2)^2\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\}$$
$$\times\{p^2+(3c-2c^2-c^3)\,p+3c^2+3c^3+c^4\}$$
$$-(p-1)\{(2+c)\,p-1\}\{(1-c-c^2)\,p+2c+c^2\}$$
$$\times\{(1+c-2c^2-c^3)\,p^2+(2c+3c^2)\,p+c^2+c^3\}^2 = 0,$$

an equation of the eighth degree in p.

Simplifications probably exist, as this case of $\mu = 25$ is the highest order of modular equation treated by Gierster (*Math. Ann.*, XIV., p. 543).

Expressed in terms of the x of § 16, Gierster's τ_{25} is given by

$$\tau_{25} = x+1-x^{-1}.$$

Dynamical Applications of Pseudo-Elliptic Integrals to the Motion of a Top or Gyrostat.

70. With the notation explained in Routh's *Rigid Dynamics*, the principles of energy and of momentum lead to the two equations

$$\tfrac{1}{2}A\left(\frac{d\theta}{dt}\right)^2+\tfrac{1}{2}A\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = Wg\,(d-h\cos\theta) \quad\text{.........(1)},$$

$$A\sin^2\theta\,\frac{d\psi}{dt}+Cr\cos\theta = G \quad\text{........................(2)},$$

where r denotes the constant angular velocity of the top about its axis of figure OC, $\frac{d\psi}{dt}$ the angular velocity of the vertical plane through OC about the vertical Oz, θ the inclination of the axis OC to the upward drawn vertical Oz, W the weight of the top, and h the distance of its centre of gravity from O, C and A the moments of inertia about OC and any axis through O perpendicular to OC, while d, G represent arbitrary constants.

We also put
$$\frac{A}{Wh} = l = OP,$$
as in the simple pendulum, and call P the centre of oscillation, as in plane vibrations; and also put
$$g/l = n^2.$$

The elimination of $\frac{d\psi}{dt}$ between (1) and (2) leads to

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \left(\frac{G - Cr\cos\theta}{A}\right)^2$$
$$= 2n^2(\cos\theta - \cos\alpha)(\cos\theta - \cos\beta)(\cos\theta - \cosh\gamma)$$
$$= 2n^2\Theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

suppose, the inclination of the axis of the top oscillating between α and β, chosen such that $\alpha > \theta > \beta$, and therefore
$$-1 < \cos\alpha < \cos\theta < \cos\beta < 1 < \cosh\gamma.$$

71. The solution of equation (3) is expressed in Weierstrass's notation by
$$\wp u - \wp w = \lambda\cos\theta,$$
where
$$u = \omega_2 + mt, \quad\text{or}\quad \omega_3 + mt,$$
if we suppose, initially, $\theta = \beta$ or α when $t = 0$; the half period ω_2 or ω_3 being added to mt so as to make $\wp u$ oscillate between e_2 and e_3, and $\cos\theta$ between $\cos\beta$ nd $\cos\alpha$.

Denoting by a and b the values of u corresponding to
$$\cos\theta = -1 \quad\text{and}\quad \cos\theta = +1,$$
then
$$\wp u - \wp a = \lambda(1+\cos\theta), \quad \wp b - \wp u = \lambda(1-\cos\theta);$$
and, since
$$-1 < \cos\alpha < \cos\theta < \cos\beta < 1 < \cosh\gamma,$$
we must take
$$a = M\omega_3, \quad b = \omega_1 + N\omega_3,$$
where M and N may be considered as real positive proper fractions; and now
$$m^2\lambda^2\wp'^2 a = -\tfrac{1}{2}\left(\frac{G+Cr}{An}\right)^2, \quad m^2\lambda^2\wp'^2 b = -\tfrac{1}{2}\left(\frac{G-Cr}{An}\right)^2,$$
$$im\lambda\wp' a = \frac{G+Cr}{\sqrt{(2AWgh)}}, \quad im\lambda\wp' b = -\frac{G-Cr}{\sqrt{(2AWgh)}},$$

and, if $G-Cr$ is negative, we put

$$b = \omega_1 - N\omega_3.$$

From equation (2),

$$\frac{d\psi}{dt} = \frac{G-Cr\cos\theta}{A\sin^2\theta} = \tfrac{1}{2}\frac{G+Cr}{A}\frac{1}{1+\cos\theta}+\tfrac{1}{2}\frac{G-Cr}{A}\frac{1}{1-\cos\theta},$$

or

$$\psi = \psi_a + \psi_b,$$

where $\psi_a = \tfrac{1}{2}\dfrac{G+Cr}{A}\displaystyle\int_0 \frac{dt}{1+\cos\theta} = \frac{G+Cr}{2\sqrt{(2AWgh)}}\int_\theta \frac{\sin\theta\, d\theta}{(1+\cos\theta)\sqrt{\Theta}},$

$$\psi_b = \tfrac{1}{2}\frac{G-Cr}{A}\int_0 \frac{dt}{1-\cos\theta} = \frac{G-Cr}{2\sqrt{(2AWgh)}}\int_\theta \frac{\sin\theta\, d\theta}{(1-\cos\theta)\sqrt{\Theta}};$$

so that ψ is composed of two elliptic integrals of the third kind, ψ_a and ψ_b, having poles at the lowest and highest positions of the axis of the top, and parameters which we have denoted by a and b.

72. As we are concerned now with the application of the pseudo-elliptic integrals, we use the s, formerly employed, as independent variable, and put

$$s = \wp u - \wp v,$$

$$s - s_a = \lambda(1+\cos\theta), \quad s_b - s = \lambda(1-\cos\theta), \quad s_b - s_a = 2\lambda.$$

Employing the suffixes 1, 2, 3, instead of α, β, γ, then

$$s_3 - s_a = \lambda(1+\cos\alpha), \quad s_b - s_3 = \lambda(1-\cos\alpha);$$

$$s_2 - s_a = \lambda(1+\cos\beta), \quad s_b - s_2 = \lambda(1-\cos\beta);$$

$$s_1 - s_a = \lambda(\cosh\gamma + 1), \quad s_1 - s_b = \lambda(\cosh\gamma - 1);$$

$$S = \lambda^3\Theta,$$

$$\int_s^{s_a}\frac{ds}{\sqrt{S}} = \int_\theta^\pi \frac{\sin\theta\, d\theta}{\sqrt{\lambda}\sqrt{\Theta}} = nt\sqrt{\frac{2}{\lambda}};$$

$$\frac{(G+Cr)^2}{2AWgh} = -\frac{S_a}{\lambda^3}, \quad \frac{(G-Cr)^2}{2AWgh} = -\frac{S_b}{\lambda^3}, \quad \left(\frac{G+Cr}{G-Cr}\right)^2 = \frac{S_a}{S_b},$$

$$\psi_a = \tfrac{1}{2}\int_s^{s_a}\frac{\sqrt{(-S_a)}\,ds}{(s-s_a)\sqrt{S}}, \quad \psi_b = \tfrac{1}{2}\int_s^{s_b}\frac{\sqrt{(-S_b)}\,ds}{(s_b-s)\sqrt{S}};$$

but the negative sign must be taken with ψ_b if $G-Cr$ is negative.

In the Weierstrassian notation

$$i\psi_a = \tfrac{1}{2}\int\frac{\wp' a\, du}{\wp u - \wp a},$$

$$i\psi_b = \tfrac{1}{2}\int\frac{\wp' b\, du}{\wp b - \wp u},$$

73. In the pseudo-elliptic applications of order μ, M and N are proper fractions with denominator μ, so that we put

$$a = \frac{q\omega_3}{\mu}, \quad b = \omega_1 + \frac{q'\omega_3}{\mu},$$

$$I_a = \tfrac{1}{2}\int_s^{s_a} \frac{P_a(s-s_a) - \mu\sqrt{(-S_a)}}{(s-s_a)\sqrt{S}}\,ds,$$

$$I_b = \tfrac{1}{2}\int_s^{s_b} \frac{P_b(s_b-s) - \mu\sqrt{(-S_b)}}{(s_b-s)\sqrt{S}}\,ds;$$

and therefore

$$I_a = \tfrac{1}{2}P_a nt\sqrt{\frac{2}{\lambda}} - \mu\psi_a = \mu(p_a t - \psi_a),$$

$$I_b = \tfrac{1}{2}P_b nt\sqrt{\frac{2}{\lambda}} - \mu\psi_b = \mu(p_b t - \psi_b),$$

where
$$p_a = \frac{P_a}{\mu}\frac{n}{\sqrt{(2\lambda)}} = \frac{P_a}{\mu}\sqrt{\frac{g}{2l\lambda}},$$

$$p_b = \frac{P_b}{\mu}\frac{n}{\sqrt{(2\lambda)}} = \frac{P_b}{\mu}\sqrt{\frac{g}{2l\lambda}}.$$

Then, if $G - Cr$ is positive,

$$I_a + I_b = \mu(pt - \psi),$$

where
$$p = p_a + p_b,$$

and
$$e^{\mu(pt-\psi)i} = e^{iI_a} e^{iI_b},$$

which the preceding investigations have shown us can be expressed as an algebraical function of s or $\cos\theta$, in such a form that

$$(\sin\theta)^\mu e^{\mu(pt-\psi)i} = A\sqrt{(\cosh\gamma - \cos\theta \,.\, \cos\theta - \cos\alpha)} + iB\sqrt{(\cos\beta - \cos\theta)},$$

or
$$A\sqrt{(\cosh\gamma - \cos\theta)} + iB\sqrt{(\cos\beta - \cos\theta \,.\, \cos\theta - \cos\alpha)},$$

where A and B are rational integral functions of s or $\cos\theta$; and thus the curve described by a point, P suppose, on the axis of the top is determined.

If $G - Cr$, and therefore also ψ_a, is negative, we must put

$$I_b = \mu(p_b t + \psi_b),$$

and now $$I_b - I_a = \mu\,(qt+\psi),$$

where $$q = p_b - p_a;$$

and $$e^{\mu(qt+\psi)i} = e^{iI_b} \,.\, e^{-iI_a},$$

and the curve described by P is obtained as before.

Introducing Euler's coordinate angle ϕ, given by

$$\frac{d\phi}{dt} = r - \cos\theta\,\frac{d\psi}{dt} = \left(1-\frac{C}{A}\right)r + \tfrac{1}{2}\,\frac{G+Cr}{A}\,\frac{1}{1+\cos\theta} - \tfrac{1}{2}\,\frac{G-Cr}{A}\,\frac{1}{1-\cos\theta},$$

then $$\phi = \left(1-\frac{C}{A}\right)rt + \psi_a - \psi_b,$$

and ϕ is also pseudo-elliptic.

It will be noticed that a change of sign of N interchanges G and Cr, or ψ and $\phi - \left(1-\frac{C}{A}\right)rt$.

74. When $$b-a = \omega_1 \quad\text{or}\quad \omega_1+\omega_3;$$

that is, when $$q'-q = 0 \quad\text{or}\quad \mu,$$

there is a further simplification in the value of ϕ, as it can now be expressed in the form

$$\sin\theta\, e^{(\phi-pt)i} = C\sqrt{(\cosh\gamma-\cos\theta)} + iD\sqrt{(\cos\beta-\cos\theta\,.\,\cos\theta-\cos\alpha)};$$

or $$C\sqrt{(\cosh\gamma-\cos\theta\,.\,\cos\theta-\cos\alpha)} + iD\sqrt{(\cos\beta-\cos\theta)},$$

and ψ also receives a similar simplification when

$$b+a = \omega_1, \quad\text{or}\quad \omega_1+\omega_3;$$

these considerations are useful as a check upon the accuracy of the algebra in the formulas, which becomes very complicated and baffling.

In these cases the values of I_a and I_b are deducible, the one from the other, by the substitution

$$(\cosh\gamma-\cos\theta)(\cosh\gamma-\cos\theta') = (\cosh\gamma-\cos\beta)(\cosh\gamma-\cos\alpha),$$

$$(s_1-s)(s_1-s') = (s_1-s_2)(s_1-s_3);$$

or $$(\cos\beta-\cos\theta)(\cos\beta-\cos\theta') = (\cos\beta-\cos\alpha)(\cos\beta-\cosh\gamma),$$

$$(s_2-s)(s_2-s') = (s_2-s_3)(s_2-s_1);$$

the accent on θ or s being afterwards dropped.

When $$b \pm a = \omega_1,$$

$$\left(\frac{s_1 - s_b}{s_1 - s_a}\right)^2 = \frac{S_b}{S_a},$$

$$\left(\frac{\cosh\gamma - 1}{\cosh\gamma + 1}\right)^2 = \left(\frac{G - Cr}{G + Cr}\right)^2 = \frac{1-\cos\alpha}{1+\cos\alpha}\,\frac{1-\cos\beta}{1+\cos\beta}\,\frac{\cosh\gamma - 1}{\cosh\gamma + 1},$$

or $$\frac{\cosh\gamma - 1}{\cosh\gamma + 1} = \frac{1-\cos\alpha}{1+\cos\alpha}\,\frac{1-\cos\beta}{1+\cos\beta},$$

$$\tanh\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}\alpha\,\tan\tfrac{1}{2}\beta,$$

and $$\frac{d}{h} = \cosh\gamma = \frac{G}{Cr}, \text{ or } \frac{Cr}{G},$$

according as $G - Cr$ is positive or negative, or as

$$b - a \quad\text{or}\quad b + a = \omega_1.$$

When $$b \pm a = \omega_1 + \omega_3,$$

$$\left(\frac{s_b - s_2}{s_2 - s_a}\right)^2 = \frac{S_b}{S_a},$$

or $$\left(\frac{1-\cos\beta}{1+\cos\beta}\right)^2 = \left(\frac{G - Cr}{G + Cr}\right)^2 = \frac{1-\cos\alpha}{1+\cos\alpha}\,\frac{1-\cos\beta}{1+\cos\beta}\,\frac{\cosh\gamma - 1}{\cosh\gamma + 1},$$

so that $$\tan\tfrac{1}{2}\beta = \tan\tfrac{1}{2}\alpha\,\tanh\tfrac{1}{2}\gamma,$$

and $$\frac{d}{h} = \cos\beta = \frac{Gr}{C}, \text{ or } \frac{Cr}{G},$$

according as $G - Cr$ is negative or positive, or as

$$b - a \quad\text{or}\quad b + a = \omega_1 + \omega_3.$$

75. It is curious that when

$$b - a = \omega_1, \quad\text{or}\quad \omega_1 + \omega_3,$$

the arc described by a point P on the axis of the top is easily rectifiable.

For, denoting the length of the arc in the general case by σ, then

$$\left(\frac{d\sigma}{dt}\right)^2 = l^2\left(\frac{d\theta}{dt}\right)^2 + l^2\sin^2\theta\left(\frac{d\psi}{dt}\right)^2$$

$$= \frac{2Wgl^2}{A}(d - h\cos\theta) = 2gl\left(\frac{d}{h} - \cos\theta\right),$$

if $$l = A/Wh;$$

and $$\left(\frac{d\sigma}{dt}\right)^2 = \frac{2gl}{\lambda}(\wp c - \wp u),$$

if $$\wp c - \wp u = \lambda\left(\frac{d}{h} - \cos\theta\right).$$

But the formulas of elliptic functions prove that

$$c = b - a$$

is the value of u corresponding to

$$\cos\theta = \frac{d}{h},$$

the value $$e = b + a$$

corresponding to $$\cos\theta = \frac{d}{h} - \frac{G^2 - C^2r^2}{2A^2n^2}.$$

This follows because

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right)(1 - \cos^2\theta) - \left(\frac{G - Cr\cos\theta}{A}\right)^2$$

$$= 2n^2\left(\frac{d}{h} - \frac{G^2 - C^2r^2}{2A^2n^2} - \cos\theta\right)(1 - \cos^2\theta) - \left(\frac{Cr - G\cos\theta}{A}\right)^2;$$

and therefore a linear relation of the form

$$\alpha + \beta\wp v + \gamma\wp' v = 0$$

connects $\wp v$ and $\wp' v$, when $v = a, b, c, e$.

76. Thus σ is pseudo-elliptic when c or $b - a = \omega_1$ or $\omega_1 + \omega_3$.

When $$b - a = \omega_1,$$

$$\left(\frac{d\sigma}{dt}\right)^2 = 2gl(\cosh\gamma - \cos\theta),$$

and $$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cosh\gamma - \cos\theta)(\cos\beta - \cos\theta)(\cos\theta - \cos\alpha),$$

$$\frac{d\sigma}{d\theta} = \frac{l\sin\theta}{\sqrt{(\cos\beta - \cos\theta \,.\, \cos\theta - \cos\alpha)}},$$

$$\frac{\sigma}{l} = 2\tan^{-1}\sqrt{\frac{\cos\theta - \cos\alpha}{\cos\beta - \cos\theta}} = 2\sin^{-1}\sqrt{\frac{\cos\theta - \cos\alpha}{\cos\beta - \cos\alpha}}$$

$$= 2\cos^{-1}\sqrt{\frac{\cos\beta - \cos\theta}{\cos\beta - \cos\alpha}}.$$

The solution of equation (3), by means of Jacobi's elliptic functions, is

$$\cos\theta = \cos a \, \mathrm{cn}^2 mt + \cos\beta \, \mathrm{sn}^2 mt,$$

where
$$m^2 = \tfrac{1}{2}(\cosh\gamma - \cos a)\frac{g}{l};$$

so that
$$\sigma = 2l \, \mathrm{am}\, mt.$$

When
$$b - a = \omega_1 + \omega_3,$$

$$\frac{d\sigma}{d\theta} = \frac{l\sin\theta}{\sqrt{(\cosh\gamma - \cos\theta \,.\, \cos\theta - \cos a)}},$$

$$\frac{\sigma}{l} = 2\tan^{-1}\sqrt{\frac{\cos\theta - \cos a}{\cosh\gamma - \cos\theta}} = \&c.,$$

or
$$\sigma = 2l\tan^{-1}\frac{\kappa\,\mathrm{sn}\, mt}{\mathrm{dn}\, mt} = 2l\tan^{-1}\mathrm{cn}\,(\mathrm{K} - mt)$$

$$= 2\sin^{-1}\kappa\,\mathrm{sn}\, mt = 2\cos^{-1}\mathrm{dn}\, mt.$$

This last spherical curve has a series of cusps on the circle defined by $\theta = \beta$; and it is practically the most interesting case, as the top, if spun initially with its axis at an inclination β to the vertical, will proceed to describe this curve if the angular velocity r is of appropriate magnitude; we shall therefore illustrate in general the pseudo-elliptic applications with reference to this case.

77. Let us begin with the application of the pseudo-elliptic integrals corresponding to
$$\mu = 4.$$

Then we take
$$a = \tfrac{1}{2}\omega_3, \quad b = \omega_1 + \tfrac{1}{2}\omega_3,$$

so that
$$b - a = \omega_1, \quad b + a = \omega_1 + \omega_3.$$

Referring to the previous treatment of $\mu = 4$ in § 14, we take

$$S = s - s_1 \,.\, s - s_2 \,.\, s - s_3,$$

where
$$s_1 = (1+c)^2, \quad s_2 = c^2, \quad s_3 = 0;$$

and
$$s_a = -c - c^2, \quad \sqrt{(-S_a)} = (1+2c)(c+c^2), \quad P_a = 1 + 2c,$$

$$s_b = c + c^2, \quad \sqrt{(-S_b)} = c + c^2, \quad P_b = 1.$$

Then $\lambda = c+c^2, \quad s = (c+c^2)\cos\theta,$

and $\cos\alpha = 0, \quad \cos\beta = \dfrac{c}{1+c}, \quad \cosh\gamma = \dfrac{1+c}{c};$

$$\frac{G+Cr}{\sqrt{(2AWgh)}} = \sqrt{-\frac{S_a}{\lambda^2}} = \frac{1+2c}{\sqrt{(c+c^2)}}, \quad \frac{G-Cr}{\sqrt{(2AWgh)}} = \frac{1}{\sqrt{(c+c^2)}};$$

$$\frac{G}{\sqrt{(2AWgh)}} = \sqrt{\frac{1+c}{c}}, \quad \frac{Cr}{\sqrt{(2AWgh)}} = \sqrt{\frac{c}{1+c}},$$

$$p_a = \frac{1+2c}{4\sqrt{(2c+2c^2)}}\,n, \quad p_b = \frac{1}{4\sqrt{(2c+2c^2)}}\,n,$$

$$2\,(p_a t - \psi_a) = I_a = \tfrac{1}{2}\int_s^{s_1} \frac{(1+2c)(s+c+c^2)-2\,(1+2c)(c+c^2)}{(s+c+c^2)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(1+2c)\sqrt{s}}{s+c+c^2} = \sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{s+c+c^2},$$

$$2\,(p_b t - \psi_b) = I_b = \tfrac{1}{2}\int_s^{s_1} \frac{(c+c^2-s)-2\,(c+c^2)}{(c+c^2-s)\sqrt{S}}$$

$$= -\cos^{-1}\frac{\sqrt{s}}{c+c^2-s} = -\sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s}.$$

Putting $p = p_a + p_b = \dfrac{n}{2}\sqrt{\dfrac{1+c}{2c}}, \quad \psi = \psi_a + \psi_b;$

$$e^{2(pt-\psi)i} = e^{iI_a}\,e^{iI_b}$$

$$= \frac{(1+2c)\sqrt{s}+i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2+s}\cdot\frac{\sqrt{s}-i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s}$$

$$= \frac{c^2(1+c)^2-2c^2s+s^2-2ic\sqrt{S}}{(c+c^2)^2-s^2}$$

$$= \frac{\left[c\sqrt{\{(1+c)^2-s\}}-i\sqrt{(c^2-s\,.\,s)}\right]^2}{(c+c^2)^2-s^2}$$

$$= \frac{\left[\sqrt{(1-\cos\beta\cos\theta)}-i\sqrt{\{\cos\theta\,(\cos\beta-\cos\theta)\}}\right]^2}{\sin^2\theta},$$

or $\sin\theta\, e^{(pt-\psi)i} = \sqrt{(1-\cos\beta\cos\theta)}-i\sqrt{\{\cos\theta\,(\cos\beta-\cos\theta)\}},$

$\sin\theta\cos(\psi-pt) = \sqrt{(1-\cos\beta\cos\theta)},$

$\sin\theta\sin(\psi-pt) = \sqrt{\{\cos\theta\,(\cos\beta-\cos\theta)\}}.$

78. But the values of G and Cr are interchanged if we take

$$b = \omega_1 - \tfrac{1}{2}\omega_3,$$

so that $$b-a = \omega_1 - \omega_3, \quad b+a = \omega_1;$$

and now we put

$$q = p_a - p_b = \frac{n}{2}\sqrt{\left(\frac{c}{2+2c}\right)}, \quad \psi = \psi_a - \psi_b;$$

so that $$e^{2(qt-\psi)i} = e^{iI_a}\, e^{-iI_b}$$

$$= \frac{(1+2c)\sqrt{s}+i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2+s} \cdot \frac{\sqrt{s}+i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s}$$

$$= \frac{-c^2(1+c)^2+2(1+c)^2 s - s^2 + 2i(1+c)\sqrt{S}}{(c+c^2)^2-s^2}$$

$$= \frac{\left[\sqrt{\{(1+c)^2-s\,.\,s\}}+i(1+c)\sqrt{(c^2-s)}\right]^2}{(c+c^2)^2-s^2}$$

$$= \frac{\left[\sqrt{\{(\sec\beta-\cos\theta)\cos\theta\}}+i\sqrt{(1-\sec\beta\cos\theta)}\right]^2}{\sin^2\theta},$$

or $$\sin\theta\, e^{(qt-\psi)i} = \sqrt{\{(\sec\beta-\cos\theta)\cos\theta\}}+i\sqrt{(1-\sec\beta\cos\theta)},$$

$$\sin\theta\cos(qt-\psi) = \sqrt{\{(\sec\beta-\cos\theta)\cos\theta\}},$$

$$\sin\theta\sin(qt-\psi) = \sqrt{(1-\sec\beta\cos\theta)}.$$

The point P on the axis of the top now describes a spherical curve, which touches the horizontal plane through O, the fixed point of the axis, and which has a series of cusps on the circle defined by $\theta = \beta$, where

$$\cos\beta = \frac{c}{1+c}, \quad \text{and} \quad \frac{Cr}{\sqrt{(2AWgh)}} = \sqrt{\frac{1+c}{c}} = \sqrt{(\sec\beta)}.$$

79. Next apply the pseudo-elliptic integrals derived from (§§ 13, 18)

$$\mu = 3, \quad \text{or} \quad 6;$$

and then we take $$S = s-s_1\,.\,s-s_2\,.\,s-s_3,$$

where $$s_1 = (1-c)^2, \quad s_2 = c^2, \quad s_3 = (c-c^2)^2,$$

$$0 < c < \tfrac{1}{2}.$$

The corresponding parameters are

$$a = \tfrac{1}{3}\omega_3 \text{ or } \tfrac{2}{3}\omega_3, \quad b = \omega_1 \pm \tfrac{1}{3}\omega_3 \text{ or } \omega_1 \pm \tfrac{2}{3}\omega_3\,;$$

and $a = \tfrac{2}{3}\omega_3, \quad s_a = 0,$

$$\sqrt{(-S_a)} = (c-c^2)^2, \qquad P_a = 1-c+c^2\,;$$

$$a = \tfrac{1}{3}\omega_3, \quad s_a = -2\,(c-c^2),$$

$$\sqrt{(-S_a)} = (1+c)(2-c)(c-c^2), \qquad P_a = (1+c)(2-c)\,;$$

$$b = \omega_1 \pm \tfrac{2}{3}\omega_3, \quad s_b = 2c^2\,(1-c),$$

$$\sqrt{(-S_b)} = (1+c)(1-2c)\,c^2\,(1-c), \quad P_b = (1+c)(1-2c)\,;$$

$$b = \omega_1 \pm \tfrac{1}{3}\omega_3, \quad s_b = 2c\,(1-c)^2,$$

$$\sqrt{(-S_b)} = (2-c)(1-2c)\,c\,(1-c)^2, \quad P_b = (2-c)(1-2c)\,;$$

$$p_a = \tfrac{1}{6}nP_a\sqrt{\frac{2}{\lambda}}, \quad p_b = \tfrac{1}{6}nP_b\sqrt{\frac{2}{\lambda}}\,.$$

80. Let us begin by taking $a = \tfrac{2}{3}\omega_3$, $b = \omega_1 - \tfrac{1}{3}\omega_3$; so as to make

$$b-a = \omega_1 - \omega_3,$$

and $G - Cr$ negative; also

$$\cos\beta = \frac{G}{Cr}, \quad \text{and} \quad \tan\tfrac{1}{2}\beta = \tan\tfrac{1}{2}a\,\tanh\tfrac{1}{2}\gamma.$$

For $$\frac{Cr-G}{Cr+G} = \sqrt{\frac{S_b}{S_a}} = \frac{(2-c)(1-2c)}{c},$$

$$\frac{1-\cos\beta}{1+\cos\beta} = \frac{s_b-s_2}{s_2-s_a} = \frac{2c\,(1-c)^2-c^2}{c^2} = \frac{2-5c+2c^2}{c}\,;$$

so that $$\cos\beta = \frac{G}{Cr} = \frac{-1+3c-c^2}{(1-c)^2}\,;$$

$$\frac{1-\cos a}{1+\cos a} = \frac{s_b-s_2}{s_2-s_a} = \frac{2-c}{c},$$

$$\frac{\cosh\gamma-1}{\cosh\gamma+1} = \frac{s_1-s_b}{s_1-s_a} = 1-2c.$$

Also
$$\lambda = c(1-c)^2,$$
so that
$$\frac{Cr+G}{\sqrt{(2AWgh)}} = \frac{c}{(1-c)\sqrt{c}},$$
$$\frac{Cr-G}{\sqrt{(2AWgh)}} = \frac{2-5c+2c^2}{(1-c)\sqrt{c}},$$
$$\frac{Cr}{\sqrt{(2AWgh)}} = \frac{1-c}{\sqrt{c}} = \frac{1}{\sqrt{(1+\cos\beta)}}; \quad Cr = \sec\tfrac{1}{2}\beta\sqrt{(AWgh)}.$$

Now we take
$$I_b - I_a = 3\{(p_b-p_a)t+\psi_a-\psi_b\} = 3(pt+\psi),$$
where
$$p = p_b - p_a = \frac{1-4c+c^2}{3(1-c)\sqrt{(2c)}}n,$$
or
$$e^{3(pt+\psi)i} = e^{iI_b}e^{-iI_a}.$$

But
$$s^{\frac{3}{2}}e^{iI_a} = (1-c+c^2)s-(c-c^2)^2+i\sqrt{S},$$
$$\{2c(1-c)^2-s\}^{\frac{3}{2}}e^{-iI_b}$$
$$= (1-2c)(2-c)\sqrt{\{(1-c)^2-s\,.\,s-(c-c)^2\}}$$
$$+i\{s-(1-c)^2(2-3c+2c^2)\}\sqrt{(c^2-s)},$$
so that, multiplying these equations,
$$\lambda^3\sin^3\theta\,e^{-3(pt+\psi)i}$$
$$= \{2c(1-c)^2s-s^2\}^{\frac{3}{2}}e^{iI_a}e^{-iI_b}$$
$$= \{(1-c+c^2)s-(c-c^2)^2+i\sqrt{S}\}\big[(1-2c)(2-c)\sqrt{(s_1-s\,.\,s-s_3)}$$
$$+i\{s-(1-c)^2(2-3c+2i)\}(s_2-s)\big]$$
$$= \big[(s-c^2)\{s-(1-c)^2(2-3c+2c^2)\}$$
$$+(1-2c)(2-c)\{(1-c+c^2)s-(c-c^2)^2\}\big]\sqrt{(s_1-s.s-s_3)}$$
$$-i\big[\{(1-c+c^2)s-(c-c^2)^2\}\{s-(1-c)^2(2-3c+2c^2)\}$$
$$+(1-2c)(2-c)\{s-(1-c)^2\}\{s-(c-c^2)\}\big]\sqrt{(s_2-s)}$$
$$= \{s^2-c^2s+c^3(1-c)^2\}\sqrt{(s_1-s\,.\,s-s_3)}$$
$$+i\{-(1-4c+c^2)s-4c^2(1-c)^2s+2c^3(1-c)^4\}\sqrt{(s_2-s)},$$
while, as a verification, we find that
$$\{2c(1-c)^2s-s^2\}^{\frac{3}{2}}e^{iI_a}e^{iI_b} = \{\sqrt{(s_1-s\,.\,s-s_3)}+i(1-c)^2\sqrt{(s_2-s)}\}^3.$$

81. Again, take $G-Cr$ negative, and

$$a=\tfrac{1}{3}\omega_3,\quad b=\omega_1-\tfrac{2}{3}\omega_3;$$

then
$$s_a=-2(c-c^2),\quad s_b=2c^2(1-c),$$

$$\lambda=\tfrac{1}{2}(s_b-s_a)=c-c^3,$$

$$\frac{Cr+G}{Cr-G}=\sqrt{\frac{S_a}{S_b}}=\frac{(1+c)(2-c)(c-c^3)}{(1+c)(1-2c)(c^2-c^3)}=\frac{2-c}{c-2c^2},$$

$$\frac{1+\cos\beta}{1-\cos\beta}=\frac{s_2-s_a}{s_b-s_2}=\frac{c^2+2c-2c^3}{2c^2-2c^3-c^2}=\frac{2-c}{c-2c^2},$$

$$\frac{Cr+G}{\surd(2AWgh)}=\frac{2-c}{\surd(c-c^3)},\quad \frac{Cr-G}{\surd(2AWgh)}=\frac{c-2c^2}{\surd(c-c^3)},$$

$$P_a=(1+c)(2-c),\quad P_b=(1+c)(1-2c),$$

$$p_a=\tfrac{1}{6}n(1+c)(2-c)\sqrt{\frac{2}{c-c^3}}=\tfrac{1}{3}n(2-c)\sqrt{\left(\frac{1}{2c}\cdot\frac{1+c}{1-c}\right)},$$

$$p_b=\tfrac{1}{6}n(1+c)(1-2c)\sqrt{\frac{2}{c-c^3}}=\tfrac{1}{3}n(1-2c)\sqrt{\left(\frac{1}{2c}\cdot\frac{1+c}{1-c}\right)},$$

$$p=p_a-p_b=\tfrac{1}{3}n(1+c)\sqrt{\left(\frac{1}{2c}\cdot\frac{1+c}{1-c}\right)},$$

$$I_a-I_b=3(pt-\psi),$$

where
$$p=p_a-p_b=\tfrac{1}{3}n\,\frac{(1+c)^{\frac{3}{2}}}{\surd(2c-2c^2)},$$

$$e^{3(pt-\psi)i}=e^{iI_a}.e^{-iI_b},$$

and
$$(s-s_a)^{\frac{3}{2}}e^{iI_a}$$
$$=(s-2+c-c^2)\surd\{s-(c-c^2)^2\}+i(1+c)(2-c)\surd\{(1-c)^2-s.c^2-s\},$$

$$(s_b-s)^{\frac{3}{2}}e^{-iI_b}$$
$$=(s-c^2-c^3+2c^4)\surd\{(1-c)^2-s\}+i(1+c)(1-2c)\surd\{c^2-s.s-(c-c^2)^2\}.$$

Multiplying these equations, we obtain a result of the form

$$\lambda^3\sin^3\theta\,e^{3(pt-\psi)}=A\surd(s_1-s.s-s_3)+iB\surd(s_2-s).$$

So also combinations can be made of $a=\frac{1}{3}\omega_3$, $b=\omega_1+\frac{1}{3}\omega_3$, and of $a=\frac{2}{3}\omega_3$, $b=\omega_1+\frac{2}{3}\omega_3$.

$$\mu = 8.$$

82. The corresponding parameters are

$$a = \tfrac{1}{4}\omega_3 \text{ or } \tfrac{3}{4}\omega_3, \quad \text{and} \quad b = \omega_1 \pm \tfrac{1}{4}\omega_3 \text{ or } \omega_1 \pm \tfrac{3}{4}\omega_3;$$

and now (§ 30) we can put, with $c(1-c)(1-2c)$ positive,

$$s_1 = \tfrac{1}{4}(1-2c)^2(1-2c+2c^2)^2,$$
$$s_2 = c^2(1-c)^2(1-2c+2c^2)^2,$$
$$s_3 = c^2(1-c)^2(1-2c)^2,$$

$$a = \tfrac{1}{4}\omega_3, \qquad s_a = -c(1-c)^2(1-2c)(1-2c+2c^2),$$
$$\sqrt{(-S_a)} = \tfrac{1}{2}c(1-c)^3(1-2c)(1-2c+2c^2)(1-2c^2),$$
$$P_a = (1-2c^2)(3-4c+2c^2);$$

$$a = \tfrac{1}{2}\omega_3, \qquad s_a = 0,$$
$$\sqrt{(-S_a)} = \tfrac{1}{2}c^2(1-c)^2(1-2c)^2(1-2c+2c^2)^2,$$
$$P_a = (1-2c+2c^2)^2;$$

$$a = \tfrac{3}{4}\omega_3, \qquad s_a = c^3(1-c)(1-2c)^2(1-2c+2c^2),$$
$$\sqrt{(-S_a)} = \tfrac{1}{2}c^3(1-c)(1-2c)^3(1-2c+2c^2)(1-2c^2),$$
$$P_a = (1-2c^2)(1-4c+6c^2);$$

$$b = \omega_1 + \tfrac{1}{4}\omega_3, \; s_b = c(1-c)^3(1-2c)^2(1-2c+2c^2),$$
$$\sqrt{(-S_b)} = \tfrac{1}{2}c(1-c)^3(1-2c)^3(1-2c+2c^2)(1-4c+2c^2),$$
$$P_b = (3-8c+6c^2)(1-4c+2c^2);$$

$$b = \omega_1 + \tfrac{3}{4}\omega_3, \; s = c^3(1-c)(1-2c)(1-2c+2c^2),$$
$$\sqrt{(-S_b)} = \tfrac{1}{2}c^3(1-c)(1-2c)(1-2c+2c^2)(1-4c+2c^2),$$
$$P_b = (1+2c^2)(1-4c+2c^2).$$

83. Then, if $\quad a = \tfrac{3}{4}\omega_3, \quad b = \omega_1 + \tfrac{1}{4}\omega_3,$

$$P_b + P_a = 4(1-2c)^2,$$
$$P_b - P_a = 2(1-2c+2c^2)(1-6c+6c^2).$$

We can make $\quad P_b - P_a = 0, \quad \text{or} \quad q = 0,$

if $$1-6c+6c^2 = 0, \quad c = \frac{3 \pm \sqrt{3}}{6};$$

and then $$c(1-c)(1-2c) = \frac{1-2c}{6} = \pm \frac{\sqrt{3}}{18},$$

so that we take $c = \tfrac{1}{6}(3-\sqrt{3})$ to make $c(1-c)(1-2c)$ positive.

Then
$$\frac{1-\cos\beta}{1+\cos\beta} = \frac{s_\delta - s_2}{s_2 - s_a} = \frac{(1-c)^3(1-4c+2c^2)}{c^3(1-2c^2)},$$

$$\cos\beta = \frac{4c^4-8c^3+10c^2-6c+1}{(2c-1)^3} = \frac{G}{Cr},$$

so that
$$\cos\beta = -\frac{1}{\sqrt{3}},$$

if
$$c = \tfrac{1}{6}(3-\sqrt{3}).$$

Also
$$\frac{1-\cos a}{1+\cos a} = \frac{s_b - s_2}{s_2 - s_a} = \frac{(1-c)^2(1-2c)}{c^2(1-2c)} = \left(\frac{1-c}{c}\right)^2,$$

so that, if
$$c = \tfrac{1}{6}(3+\sqrt{3}),$$

$$\frac{1-\cos a}{1+\cos a} = \frac{2+\sqrt{3}}{2-\sqrt{3}};$$

$$\cos a = -\tfrac{1}{2}\sqrt{3},$$

$$a = 150^\circ.$$

Then
$$\frac{Cr}{\sqrt{(AWgh)}} = \sqrt[4]{3}, \quad \frac{G}{\sqrt{(AWgh)}} = -\frac{1}{\sqrt[4]{3}}.$$

$$\mu = 5 \text{ or } 10.$$

84. The corresponding parameters are

$$a = \tfrac{1}{5}\omega_3,\ \tfrac{2}{5}\omega_3,\ \tfrac{3}{5}\omega_3,\ \tfrac{4}{5}\omega_3;$$

$$b = \omega_1 + \tfrac{1}{5}\omega_3,\ \omega_1 + \tfrac{2}{5}\omega_3,\ \omega_1 + \tfrac{3}{5}\omega_3,\ \omega_1 + \tfrac{4}{5}\omega_3,$$

and, according to the preceding investigations of §§ 36–39, with suffixes 1, 2, 3, instead of a, β, γ, we put

$$s_1 = 4(c^2+\sqrt{C})^2, \quad s_2 = (c+1)^2(c-1)^4, \quad s_3 = 4(c^2-\sqrt{C})^2,$$

where
$$C = c^3+c^2-c, \text{ and } 2+\sqrt{5} > c > 1;$$

so that
$$s_1 - s_2 \,.\, s_2 - s_3 = (c^2-1)^5(-c^2+4c+1) \text{ is positive},$$

and
$$s_1 > s_2 > s_3.$$

We find that we must take

$$v = \omega_1 + \tfrac{3}{5}\omega_3,$$

and now [illegible] $= \omega_1 - \frac{1}{3}\omega_3$, [illegible]

[illegible] $= 4c^2$ [illegible] $- 1)$,

[illegible]

[illegible] $= \omega_1 - \frac{1}{3}\omega_3$, [illegible] $- 1)$,

$= 4c$ [illegible] $- 1 - \sqrt{C})$,

$\sqrt{(-S}$ [illegible] $= 4c$ [illegible] $- 1 - 4c\sqrt{C})\sqrt{C}$,

[illegible] $- 1 - 3c\sqrt{C})$;

[illegible] $= \omega_1 - \frac{1}{3}\omega_3$, [illegible]

$\sqrt{(-S}$ [illegible] $= 4c$ [illegible] $+ 1)$,

[illegible] $- 1)$;

[illegible] $= \omega_1 - \frac{1}{3}\omega_3$, [illegible] $= 4$ [illegible] $(\sqrt{C} - c)$,

$\sqrt{(-S}$ [illegible] $= 4$ [illegible] $- 1 - 4\sqrt{C})\sqrt{C}$,

[illegible] $- 2 - 3\sqrt{C})$.

Now $c = \frac{1}{3}\omega_3$, [illegible] $= -4$ [illegible] $(\sqrt{C} - c)$,

[illegible] $= 4$ [illegible] $- 1 - 4\sqrt{C})\sqrt{C}$,

[illegible] $- 2 - 3\sqrt{C})$;

[illegible] $= \frac{1}{3}\omega_3$, [illegible]

[illegible] $= 4c$ [illegible]

[illegible]

[illegible] $= \frac{1}{3}\omega_3$, [illegible] $= -4$ [illegible] $(\sqrt{C} - 1)$,

$= -4c$ [illegible] $- 1 + \sqrt{C})$,

[illegible] $= 4c$ [illegible] $- 1 + 4c\sqrt{C})\sqrt{C}$,

[illegible] $- 1 + 3c\sqrt{C})$;

[illegible] $= \frac{1}{3}\omega_3$, [illegible] $= -4c$ [illegible] $- 1)^2$,

[illegible] $(-S_2) = 4c^2$ [illegible]

[illegible] $- 1$.

85. Take [illegible] § 72 [illegible]

[illegible] $= \frac{1}{3}\omega_3$, [illegible] $= \omega_1 - \frac{1}{3}\omega_3$, [illegible] $= \omega_1 - \omega_3$,

$\lambda = \frac{1}{2}$ [illegible] $= 4c^2$ [illegible] $+ 1)(c - 1)^2$,

$G = C$ [illegible]

$$\frac{Cr-G}{Cr+G}=\sqrt{\frac{-S_b}{-S_a}}=\frac{4c^3\,(c+1)(c-1)^3\,(-c^2+4c+1)}{4c^3\,(c+1)^4\,(c-1)^2}$$

$$=\frac{(c-1)(-c^2+4c+1)}{(c+1)^3}=\frac{c^3-5c^2+3c+1}{c^3+3c^2+3c+1},$$

$$\frac{Cr}{G}=\frac{c^3-c^2+3c+1}{4c},$$

$$\frac{1-\cos\beta}{1+\cos\beta}=\frac{s_b-s_2}{s_2-s_a}=\frac{4c\,(c+1)(c-1)^3-(c+1)^2\,(c-1)^4}{(c+1)^2\,(c-1)^4+4c\,(c+1)^2\,(c-1)^2}$$

$$=\frac{(c+1)(c-1)^3\,(4c-c^2+1)}{(c+1)^2(c-1)^2(c+1)^2}=\frac{(c-1)(c^2-4c-1)}{(c+1)^3},$$

$$\frac{1-\cos a}{1+\cos a}=\frac{s_b-s_3}{s_3-s_2}=\frac{c^3-c^2-5c+1+4\sqrt{C}}{(c+1)^3},$$

$$\cos a=2\,\frac{c^2-\sqrt{C}}{(c+1)(c-1)^2}=\frac{2c}{c^2+\sqrt{C}}.$$

[86. In the above applications ψ_a and ψ_b are both pseudo-elliptic, so that the curve described by a point P on the axis of the top can be written down when G and Cr are interchanged.

But, by the rule for the addition of elliptic integrals of the third kind,

$$\psi_a+\psi_b-\psi_e,$$

where

$$e=a+b,$$

can be expressed by means of an inverse circular function of $\cos\theta$, and by a secular term pt, so that ψ_e alone is required.

In fact, putting, in § 75,

$$\frac{d}{h}-\frac{G^2-C^2r^2}{2AWgh}=E,$$

so that

$$\Theta=(1-\cos^2\theta)(E-\cos\theta)-\frac{(Cr-G\cos\theta)^2}{2AWgh},$$

we find that $\psi=\psi_a+\psi_b$

$$=\frac{Gt}{2A}+\psi_e+\tan^{-1}\frac{Cr-G\cos\theta}{\sqrt{(2AWgh)}\sqrt{\Theta}},$$

where

$$\psi_e=\frac{Cr-GE}{2A}\int\frac{dt}{E-\cos\theta}=\frac{Cr-GE}{\sqrt{(2AWgh)}}\int\frac{\sin\theta\,d\theta}{(E-\cos\theta)\sqrt{\Theta}};$$

this can be verified by a differentiation, remembering that

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} = -n\sqrt{2}\sqrt{\Theta};$$

and thence we find that

$$\frac{d\psi}{dt} = \frac{G - Cr\cos\theta}{A(1-\cos^2\theta)},$$

as in equation (2) of § 70.

87. Now, when $$e = \omega_1 + \frac{f\omega_3}{\mu},$$

and I_e is the pseudo-elliptic integral of order μ, corresponding to ψ_e,

$$I_e = \tfrac{1}{2}\int \frac{P_e(s_e - s) - \mu\sqrt{(-S_e)}}{(s_e - s)\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}P_e nt\sqrt{\frac{2}{\lambda}} - \mu\psi_e,$$

if $$s_e - s = \lambda(E - \cos\theta).$$

Then

$$\psi = \left\{\frac{G}{2\sqrt{(AWgh)}} + \frac{P_e}{\mu\sqrt{(2\lambda)}}\right\}nt - \frac{I_e}{\mu} + \tan^{-1}\frac{Cr - G\cos\theta}{\sqrt{(2AWgh)}\sqrt{\Theta}};$$

so that, putting $$\frac{G}{2\sqrt{(AWgh)}} + \frac{P_e}{\mu\sqrt{(2\lambda)}} = \frac{p}{n},$$

then $\mu(\psi - pt)$ can be expressed by an inverse circular function of $\cos\theta$.

If μ is an odd number, the relation can be written in the form

$$(\sin\theta)^\mu e^{\mu(\psi - pt)i}$$

$$= \left\{(\cos\theta)^{\mu-1} + C(\cos\theta)^{\mu-2} + D(\cos\theta)^{\mu-3} + \dots\right\}$$
$$\sqrt{(\cosh\gamma - \cos\theta\,.\,\cos\theta - \cos\alpha)}$$
$$+ i\left\{P(\cos\theta)^{\mu-1} + Q(\cos\theta)^{\mu-2} + R(\cos\theta)^{\mu-3} + \dots\right\}\sqrt{(\cos\beta - \cos\theta)},$$

when f is an odd number; but $\cosh\gamma - \cos\theta$ and $\cos\beta - \cos\theta$ must change places if f is an even number; and if μ is even and f therefore odd, then $\cos\theta - \cos\alpha$ and $\cos\beta - \cos\theta$ must be interchanged; but in every case

$$P = \mu\frac{p}{n}\sqrt{2} = \frac{\mu m + P_e}{\sqrt{\lambda}},$$

where m is defined in the next article; and thence the values of $Q, R, \dots, C, D, \dots$ can be determined by a verification.

88. It is convenient to put

$$\frac{G^2}{2AWgh} = \frac{m^2}{\lambda},$$

so that

$$\frac{p}{n} = \frac{\mu m + P_e}{\mu\sqrt{(2\lambda)}};$$

also to write σ_1, σ_2, σ_3 for $s_e - s_1$, $s_e - s_2$, $s_e - s_3$, respectively; and now, since

$$\cos\alpha + \cos\beta + \cosh\gamma = E + \frac{G^2}{2AWgh} = E + \frac{m^2}{\lambda};$$

$$\sigma_1 = \lambda(E - \cosh\gamma), \quad \sigma_2 = \lambda(E - \cos\beta), \quad \sigma_3 = \lambda(E - \cos\alpha);$$

therefore

$$m^2 + \sigma_1 = \lambda(\cos\alpha + \cos\beta),$$

$$m^2 + \sigma_2 = \lambda(\cos\alpha + \cosh\gamma),$$

$$m^2 + \sigma_3 = \lambda(\cos\beta + \cosh\gamma);$$

$$2\lambda\cos\alpha = m^2 + \sigma_1 + \sigma_2 - \sigma_3,$$

$$2\lambda\cos\beta = m^2 + \sigma_1 - \sigma_2 + \sigma_3,$$

$$2\lambda\cosh\gamma = m^2 - \sigma_1 + \sigma_2 + \sigma_3;$$

$$3m^2 + \sigma_1 + \sigma_2 + \sigma_3 = 2\lambda(\cos\alpha + \cos\beta + \cosh\gamma) = 2m^2 + 2\lambda E,$$

or

$$2\lambda E = m^2 + \sigma_1 + \sigma_2 + \sigma_3.$$

Again,

$$\sigma_1\sigma_2\sigma_3 = S_e = \lambda^3\Theta_e = -\lambda^3\frac{(Cr - GE)^2}{2AWgh},$$

$$\sqrt{(-S_e)} = \lambda^{\frac{3}{2}}\frac{Cr - GE}{\sqrt{(2AWgh)}} = \frac{Cr\lambda^{\frac{3}{2}}}{\sqrt{(2AWgh)}} - m\lambda E;$$

or

$$\frac{Cr\lambda^{\frac{3}{2}}}{\sqrt{(2AWgh)}} = \tfrac{1}{2}m^3 + \tfrac{1}{2}m(\sigma_1 + \sigma_2 + \sigma_3) + \sqrt{(-S_e)}.$$

Also

$$(m^2 + \sigma_1)^2 - (\sigma_2 - \sigma_3)^2 + (m^2 + \sigma_2)^2 - (\sigma_3 - \sigma_1)^2 + (m^2 + \sigma_3)^2 - (\sigma_1 - \sigma_2)^2$$

$$= 4\lambda^2(\cos\alpha\cos\beta + \cos\alpha\cosh\gamma + \cos\beta\cosh\gamma)$$

$$= 4\lambda^2\left(-1 + \frac{GCr}{AWgh}\right) = -4\lambda^2 + \frac{8mCr\lambda^{\frac{3}{2}}}{\sqrt{(2AWgh)}};$$

and therefore

$$4\lambda^2 = (m^2 + \sigma_1 + \sigma_2 + \sigma_3)^2 + 8m\sqrt{(-S_e)} - 4(\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2);$$

thence λ and G, Cr, E are given in terms of m, σ_1, σ_2, σ_3.]

Application of Pseudo-Elliptic Integrals to the Motion of a Rigid Body about a Fixed Point under no Forces.

89. In the herpolhode of a body moving *à la Poinsot* about a fixed point under no forces, the parameter of the corresponding elliptic integral of the third kind is always of the form b, as previously employed for the top; so that the integral for ψ_b, when pseudo-elliptic, can be utilized for constructing solvable degenerate cases of algebraical herpolhodes; also of tortuous elastic wires.

The integral for ψ_a, when pseudo-elliptic, will serve in a similar manner for a tortuous revolving chain.

Writing the equation of the momental ellipsoid in the form

$$Ax^2+By^2+Cz^2 = Dh^2, \quad \text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,$$

where

$$Aa^2 = Bb^2 = Cc^2 = Dh^2,$$

and supposing the motion *à la Poinsot* to be produced by rolling this ellipsoid on a fixed plane at a distance h from its centre, which is fixed, then, as shown in my "Applications of Elliptic Functions," § 104, where the notation employed here is defined, we put

$$G = D\mu, \quad T = D\mu^2,$$

so that

$$G^2/T = D;$$

and then ρ, ϕ, the polar coordinates of a point on the herpolhode in this fixed plane, are given by

$$\frac{d\rho^2}{dt} = \frac{\mu}{h}\sqrt{(4\,.\,\rho_\alpha^2-\rho^2\,.\,\rho_\beta^2-\rho^2\,.\,\rho_\gamma^2-\rho^2)},$$

$$\frac{d\phi}{dt} = \mu+\tfrac{1}{2}\frac{\sqrt{(-\rho_\alpha^2\rho_\beta^2\rho_\gamma^2)}}{\rho^2\sqrt{(\rho_\alpha^2-\rho^2\,.\,\rho_\beta^2-\rho^2\,.\,\rho_\gamma^2-\rho^2)}}\cdot\frac{d\rho^2}{dt},$$

where

$$\frac{\rho_\alpha^2}{h^2} = \frac{B-D\,.\,D-C}{BC},$$

$$\frac{\rho_\beta^2}{h^2} = \frac{C-D\,.\,D-A}{CA},$$

$$\frac{\rho_\gamma^2}{h^2} = \frac{A-D\,.\,D-B}{AB}.$$

90. We take $A>B>C$, or $a^2<b^2<c^2$; and now, with

$$\frac{\rho^2}{h^2}=\frac{n^2}{\mu^2}(\wp b-\wp u),$$

(i.) $A>B>D>C$, or $a^2<b^2<h^2<c^2$,

$$\wp b-e_\alpha=\frac{\mu^2}{n^2}\,\frac{\rho_\alpha^2}{h^2}=\frac{\mu^2}{n^2}\,\frac{B-D\,.\,D-C}{BC}\ \text{(positive)},$$

$$\wp b-e_\beta=\frac{\mu^2}{n^2}\,\frac{\rho_\beta^2}{h^2}=\frac{\mu^2}{n^2}\,\frac{A-D\,.\,D-C}{AC}\ \text{(positive)},$$

$$\wp b-e_\gamma=\frac{\mu^2}{n^2}\,\frac{\rho_\gamma^2}{h^2}=-\frac{\mu^2}{n^2}\,\frac{A-D\,.\,B-D}{AB}\ \text{(negative)},$$

so that $$e_\alpha=e_2,\quad e_\beta=e_3,\quad e_\gamma=e_1,$$

and $$b=\omega_1+f\omega_3.$$

In Jacobi's notation, with

$$b=\mathrm{K}+f\mathrm{K}'i,$$

$$\mathrm{sn}^2 b=\frac{\wp b-e_\gamma}{\wp b-e_\beta}=-\frac{C}{B}\,\frac{B-D}{D-C},$$

$$\mathrm{dn}^2 b=\frac{\wp b-e_\alpha}{\wp b-e_\beta}=\frac{A}{B}\,\frac{B-D}{A-D},$$

$$\mathrm{sn}^2 b=\frac{e_\gamma-e_\beta}{\wp b-e_\beta}=\frac{D}{B}\,\frac{B-C}{D-C},$$

and then, to the complementary modulus κ',

$$\mathrm{sn}^2 f\mathrm{K}'=\frac{\dfrac{1}{D}-\dfrac{1}{A}}{\dfrac{1}{C}-\dfrac{1}{A}}=\frac{h^2-a^2}{c^2-a^2},$$

$$\mathrm{cn}^2 f\mathrm{K}'=\frac{\dfrac{1}{C}-\dfrac{1}{D}}{\dfrac{1}{C}-\dfrac{1}{A}}=\frac{c^2-h^2}{c^2-a^2},$$

$$\mathrm{dn}^2 f\mathrm{K}'=\frac{\dfrac{1}{C}-\dfrac{1}{D}}{\dfrac{1}{C}-\dfrac{1}{B}}=\frac{c^2-h^2}{c^2-b^2}.$$

Therefore
$$\operatorname{sn}^2(1-f)\,K' = \frac{c^2-b^2}{c^2-a^2},$$

$$\operatorname{cn}^2(1-f)\,K' = \frac{b^2-a^2}{c^2-a^2},$$

$$\operatorname{dn}^2(1-f)\,K' = \frac{b^2-a^2}{h^2-a^2}.$$

Denoting by β, γ the semi-axes of the focal ellipse of the momental ellipsoid, and by δ the distance from its centre of the revolving plane on which it rolls; then, according to Sylvester's theorem of correlated bodies, $(\delta>\beta)$,

$$\operatorname{sn} fK' = \frac{\delta}{\gamma},$$

$$\operatorname{cn} fK' = \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)},$$

$$\operatorname{dn} fK' = \frac{\gamma^2-\delta^2}{\gamma^2-\beta^2};$$

$$\operatorname{sn}(1-f)\,K' = \sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)},$$

$$\operatorname{cn}(1-f)\,K' = \frac{\beta}{\gamma},$$

$$\operatorname{dn}(1-f)\,K' = \frac{\beta}{\delta}.$$

(ii.) $\qquad A>D>B>C$, or $a^2<h^2<b^2<c^2$, or $\delta<\beta$;

a similar procedure now shows that $\wp b-e_\alpha$ is negative, $\wp b-e_\beta$ is positive, $\wp b-e_\gamma$ is positive; so that

$$e_\alpha = e_1, \quad e_\beta = e_3, \quad e_\gamma = e_2;$$

and with
$$b = \omega_1+f\omega_3 \quad \text{or} \quad K+fK'i,$$

$$\operatorname{cn}^2 b = \frac{\wp b-e_\alpha}{\wp b-e_\beta} = -\frac{A}{B}\,\frac{D-B}{A-D},$$

$$\operatorname{dn}^2 b = \frac{\wp b-e_\gamma}{\wp b-e_\beta} = \frac{C}{B}\,\frac{D-B}{A-D},$$

$$\operatorname{sn}^2 b = \frac{e_\alpha-e_\beta}{\wp b-e_\beta} = \frac{D}{B}\,\frac{A-B}{A-D};$$

and to the complementary modulus κ',

$$\operatorname{sn} f\mathrm{K}' = \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)},$$

$$\operatorname{cn} f\mathrm{K}' = \frac{\delta}{\gamma};$$

$$\operatorname{dn} f\mathrm{K}' = \frac{\delta}{\beta};$$

$$\operatorname{sn}(1-f)\,\mathrm{K}' = \frac{\beta}{\gamma},$$

$$\operatorname{cn}(1-f)\,\mathrm{K}' = \sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)},$$

$$\operatorname{dn}(1-f)\,\mathrm{K}' = \sqrt{\frac{\gamma^2-\beta^2}{\gamma^2-\delta^2}}.$$

91. Confining our attention to the herpolhode of the focal ellipse of the momental ellipsoid, and employing ρ, ϕ as polar coordinates, then

$$\phi - pt = \tfrac{1}{2}\int \frac{\sqrt{(-\rho_a^2\rho_\beta^2\rho_\gamma^2)}\,d\rho^2}{\rho^2\sqrt{(\rho_a^2-\rho^2\,.\,\rho_\beta^2-\rho^2\,.\,\rho_\gamma^2-\rho^2)}}.$$

To employ the previous notation, we put

$$s_b - s = k\rho^2,$$

$$s - s_a = k(\rho_a^2-\rho^2),\ \&c.,$$

so that $$s_b - s_a = k\rho_a^2,\ \&c.$$

When $\delta<\beta$, the value of ρ^2 in the herpolhode oscillates between its maximum value $\gamma^2-\delta^2$ and its minimum value $\beta^2-\delta^2$; but, when $\delta>\beta$, the minimum value of ρ^2 is $\dfrac{\gamma^2-\delta^2\,.\,\delta^2-\beta^2}{\delta^2}$.

Therefore, with $s_1>s_2>s_3$, using suffixes 1, 2, 3, instead of α, β, γ,

(i.) $\delta>\beta$, $$s_b - s_1 = -k(\delta^2-\beta^2),$$

$$s_b - s_2 = k\frac{\gamma^2-\delta^2\,.\,\delta^2-\beta^2}{\delta^2},$$

$$s_b - s_3 = k(\gamma^2-\delta^2),$$

so that

$$k\delta^2 = \frac{s_1-s_b \,.\, s_b-s_3}{s_b-s_2},$$

$$s_1-s = k\,(\rho^2+\delta^2-\beta^2),$$

$$s_2-s = k\left(\mu^2 - \frac{\gamma^2-\delta^2 \,.\, \delta^2-\beta^2}{\delta^2}\right),$$

$$s-s_3 = k\,(\gamma^2-\delta^2-\rho^2).$$

Then

$$\frac{\gamma^2-\delta^2}{\delta^2} = \frac{s_b-s_2}{s_1-s_b}, \quad \frac{\gamma^2}{\delta^2} = \frac{s_1-s_2}{s_1-s_b},$$

$$\frac{\delta^2-\beta^2}{\delta^2} = \frac{s_b-s_2}{s_b-s_3}, \quad \frac{\beta^2}{\delta^2} = \frac{s_2-s_3}{s_b-s_3};$$

and therefore

$$\mathrm{sn}\, f\mathrm{K}' = \sqrt{\frac{s_1-s_b}{s_1-s_2}},$$

$$\mathrm{cn}\, f\mathrm{K}' = \sqrt{\frac{s_b-s_2}{s_1-s_2}},$$

$$\mathrm{dn}\, f\mathrm{K}' = \sqrt{\frac{s_b-s_3}{s_1-s_3}},$$

$$\mathrm{sn}\,(1-f)\,\mathrm{K}' = \sqrt{\frac{s_1-s_3}{s_1-s_2} \,.\, \frac{s_b-s_2}{s_b-s_3}},$$

$$\mathrm{cn}\,(1-f)\,\mathrm{K}' = \sqrt{\frac{s_2-s_3}{s_1-s_2} \,.\, \frac{s_1-s_h}{s_b-s_3}},$$

$$\mathrm{dn}\,(1-f)\,\mathrm{K} = \sqrt{\frac{s_2-s_3}{s_b-s_3}}.$$

(ii.) $\delta < \beta$,

$$s_b-s_1 = -k\,\frac{\gamma^2-\delta^2 \,.\, \beta^2-\delta^2}{\delta^2},$$

$$s_b-s_2 = k\,(\beta^2-\delta^2),$$

$$s_b-s_3 = k\,(\gamma^2-\delta^2),$$

so that

$$k\delta^2 = \frac{s_b-s_2 \,.\, s_b-s_3}{s_1-s_b},$$

$$s_1-s = k\left(\rho^2 + \frac{\gamma^2-\delta^2 \,.\, \beta^2-\delta^2}{\delta^2}\right),$$

$$s_2-s = k\,(\rho^2-\beta^2+\delta^2),$$

$$s-s_3 = k\,(\gamma^2-\delta^2-\rho^2).$$

Then
$$\frac{\gamma^2-\delta^2}{\delta^2}=\frac{s_1-s_b}{s_b-s_2},\quad \frac{\gamma^2}{\delta^2}=\frac{s_1-s_2}{s_b-s_2};$$
$$\frac{\beta^2-\delta^2}{\delta^2}=\frac{s_1-s_b}{s_b-s_3},\quad \frac{\beta^2}{\delta^2}=\frac{s_1-s_3}{s_b-s_3};$$

and the values of the elliptic functions of fK′ and $(1-f)$ K′ with respect to the modulus k' are the same functions of s as before.

$$\mu = 4.$$

92. Here (§ 14)
$$s_1=(1+c)^2,\quad s_2=c^2,\quad s_3=0,\quad s_b=c+c^2;$$

and therefore
$$\kappa^2=\frac{c^2}{(1+c)^2},\quad \kappa'^2=\frac{1+2c}{(1+c)^2},$$
$$\text{sn}^2\tfrac{1}{2}\text{K}'=\frac{1+c}{1+2c},\quad \text{cn}^2\tfrac{1}{2}\text{K}'=\frac{c}{1+2c},\quad \text{dn}^2\tfrac{1}{2}\text{K}'=\frac{c}{1+c}.$$

Then
$$b=\omega_1+\tfrac{1}{2}\omega_3,$$

and (i.), $\delta>\beta$,
$$k\delta^2=(1+c)^2,$$
$$\frac{\gamma^2}{\delta^2}=\frac{1+2c}{1+c},\quad \frac{\beta^2}{\delta^2}=\frac{c}{1+c}.$$

(ii.) $\delta<\beta$,
$$k\delta^2=c^2,$$
$$\frac{\gamma^2}{\delta^2}=\frac{1+2c}{c},\quad \frac{\beta^2}{\delta^2}=\frac{1+c}{c},$$

and
$$\delta^2=\gamma^2-\beta^2,$$

in each case, so that the focal ellipse of the momental ellipsoid rolls upon a plane at a distance from its centre equal to the distance of a focus from the centre.

Since
$$(c+c^2-s)\,e^{iI_b}=i\surd s+\surd\{(1+c)^2-s\,.\,c^2-s\}$$
$$=i\surd(s-s_3)+\surd(s_1-s\,.\,s_2-s),$$

the herpolhode is given by the equation
$$\rho^2e^{2(\phi-pt)i}=\surd(\rho^2-\rho_1^2\,.\,\rho^2-\rho_2^2)+i\surd(-\rho_1^2-\rho_2^2\,.\,\rho_3^2-\rho^2).$$

With respect to axes revolving with angular velocity p, we can put

$$x^2-y^2 = \rho^2 \cos 2(\phi - pt),$$

so that $$(x^2-y^2)^2 = (x^2+y^2)^2-(\rho_1^2+\rho_2^2)(x^2+y^2)+\rho_1^2\rho_2^2,$$

or $$(4x^2-\rho_1^2-\rho_2^2)(4y^2-\rho_1^2-\rho_2^2) = (\rho_1^2-\rho_2^2)^2.$$

This algebraical herpolhode is due originally to Halphen (*F. E.*, II., Chap. VI.).

$$\mu = 6.$$

93. Here (§ 18)

$$s_1 = (1-c)^2, \quad s_2 = c^2, \quad s_3 = (c-c^2)^2,$$

if $$c < \tfrac{1}{2};$$

and if $$f = \tfrac{1}{3}, \quad b = \omega_1 + \tfrac{1}{3}\omega_3, \quad s_b = 2c(1-c)^2;$$

$$f = \tfrac{2}{3}, \quad b = \omega_1 + \tfrac{2}{3}\omega_3, \quad s_b = 2c^2(1-c).$$

Taking $f = \frac{1}{3}$, then

$$\text{cn}\,\tfrac{2}{3}\text{K}' = c, \quad \text{sn}\,\tfrac{1}{3}\text{K}' = 1-c,$$

so that $$\text{cn}\,\tfrac{2}{3}\text{K}' + \text{sn}\,\tfrac{1}{3}\text{K}' = 1,$$

a well known relation (§ 20).

(i.) $\delta > \beta$, $$\frac{\delta}{\gamma} = \text{sn}\,\tfrac{1}{3}\text{K}' = 1-c,$$

$$\frac{\beta}{\gamma} = \text{cn}\,\tfrac{2}{3}\text{K}' = c,$$

so that $$\delta = \gamma - \beta,$$

or "the focal ellipse of the momental ellipsoid rolls upon a plane at a distance from its centre equal to the difference of the semi-axes."

Now $$k\delta^2 = (1-c)^4,$$

$$s_b - s = (1-c)^4 \frac{\rho^2}{\delta^2},$$

so that, writing $3(\phi - pt)$ for $I(\omega_1 + \frac{1}{3}\omega_3)$,

$$\{2c(1-c)^2 - s\}^{\frac{3}{2}} e^{3i(\phi - pt)} = -\{s-(1-c)^2(2-3c+2c^2)\}\sqrt{(s_1-s)}$$
$$+ i(1-2c)(2-c)\sqrt{(s_1-s\,.\,s-s_3)},$$

we obtain

$$\rho^3 e^{3i(\phi-pt)} = \left\{\rho^2 + \frac{(2-c)(1-2c)}{(1-c)^2}\,\delta^2\right\}\sqrt{\left(\rho^2 - \frac{\gamma^2-\delta^2\,.\,\delta^2-\beta^2}{\delta^2}\right)}$$

$$+i\,\frac{(2-c)(1-2c)}{(1-c)^2}\,\delta\sqrt{(\rho^2+\delta^2-\beta^2\,.\,\gamma^2-\delta^2-\rho^2)}$$

as the equation of the herpolhode.

(ii.) $\delta<\beta$,

$$\sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)} = \text{cn}\,\tfrac{2}{3}\text{K}' = c, \quad \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)} = \text{sn}\,\tfrac{1}{3}\text{K}' = 1-c,$$

so that
$$\sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)}+\sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)} = 1.$$

Now
$$k\delta^2 = (2c-c^2)^2,$$

and the herpolhode is given by

$$\rho^3 e^{3i(\phi-pt)} = \left(\rho^2+\frac{1-2c}{c}\,\delta^2\right)\sqrt{(\rho^2-\beta^2+\delta^2)}$$

$$+i\,\frac{1-2c}{c}\,\delta\sqrt{\left(\rho^2+\frac{\gamma^2-\delta^2\,.\,\beta^2-\delta^2}{\delta^2}\,\gamma^2-\delta^2-\rho^2\right)}.$$

If we take $f=\tfrac{2}{3}$, and (i.) $\delta>\beta$, (ii.) $\delta<\beta$, we shall obtain similar results.

$$\mu = 8.$$

94. Here (§§ 30, 82)
$$b = \omega_1+f\omega_3,$$
where
$$f=\tfrac{1}{4} \text{ or } \tfrac{3}{4};$$
$$s_1 = \tfrac{1}{4}(1-2c)^2(1-2c+2c^2)^2,$$
$$s_2 = c^2(1-c)^2(1-2c+2c^2)^2,$$
$$s_3 = c^2(1-c)^2(1-2c)^2;$$
and
$$f=\tfrac{1}{4}, \quad s_b = c\,(1-c)^2(1-2c)^2(1-2c+2c^2),$$
$$f=\tfrac{3}{4}, \quad s_b = c^2(1-c)(1-2c)(1-2c+2c^2).$$

(i.) $f=\tfrac{1}{4}$, $\delta>\beta$,

$$\frac{\delta^2}{\gamma^2} = \text{sn}^2\,\tfrac{1}{4}\text{K}' = \frac{(1-2c)^2}{(1-2c+2c^2)(1-2c^2)},$$

$$\frac{\beta^2}{\gamma^2} = \text{cn}^2\,\tfrac{3}{4}\text{K}' = \frac{4c^3(1-c)}{(1-2c+2c^2)(1-2c^2)},$$

(ii.) $f=\frac{1}{4},\ \delta<\beta,$

$$\frac{\beta^2}{\gamma^2}=\text{sn}^2\,\tfrac{3}{4}\text{K}'=\frac{1-2c}{(1-2c+2c^2)(1-2c^2)},$$

$$\frac{\delta^2}{\gamma^2}=\text{cn}^2\,\tfrac{1}{4}\text{K}'=\frac{2c\,(1-c)(1-c+c^2)}{(1-2c+2c^2)(1-2c^2)},\ \text{\&c.}$$

$$\mu=5 \text{ or } 10.$$

95. Then (§§ 38, 84)

$$s_1=4\,(c^2+\sqrt{C})^2,$$

$$s_2=(c+1)^2(c-1)^4,$$

$$s_3=4\,(c^2-\sqrt{C})^2;$$

and

$$f=\tfrac{1}{5},\quad s_b=4c\,(c+1)(c-1)^3,$$

$$f=\tfrac{2}{5},\quad s_b=8c\,(c+1)^2(c+1).$$

(i.) $f=\frac{1}{5},\ \delta>\beta,$

$$\frac{\delta}{\gamma}=\text{sn}\,\tfrac{1}{5}\text{K}'=\sqrt{\frac{s_1-s_b}{s_1-s_3}}=\frac{2c\,(\sqrt{C}-1)}{(c+1)^2\,(c-1)},$$

$$\frac{\beta}{\gamma}=\text{cn}\,\tfrac{4}{5}\text{K}'=\sqrt{\frac{s_2-s_3}{s_1-s_2}\,\frac{s_1-s_b}{s_b-s_3}}=\frac{\sqrt{C}-1}{\sqrt{C}+1},$$

$$\frac{\beta}{\delta}=\text{dn}\,\tfrac{4}{5}\text{K}'=\sqrt{\frac{s_2-s_3}{s_b-s_3}}=\frac{\sqrt{C}-1}{2c}.$$

Therefore

$$\frac{\gamma}{\delta}=\frac{\sqrt{C}+1}{2c},\quad \frac{\beta}{\delta}=\frac{\sqrt{C}-1}{2c},$$

$$\frac{\gamma+\beta}{\delta}=\frac{\sqrt{C}}{c},\quad \frac{\gamma-\beta}{\delta}=\frac{1}{c}.$$

and

$$\left(\frac{\gamma+\beta}{\delta}\right)^2=\frac{c^3+c^2-c}{c^2}=c+1-\frac{1}{c}$$

$$=\frac{\gamma-\beta}{\delta}+1-\frac{\delta}{\gamma-\beta}.$$

Also

$$\sqrt{\frac{\beta}{\gamma}}=\frac{\sqrt{C}-1}{(c+1)\sqrt{(c-1)}},\quad \sqrt{\frac{\gamma}{\beta}}=\frac{\sqrt{C}+1}{(c+1)\sqrt{(c-1)}};$$

and therefore

$$\sqrt{\frac{\gamma}{\beta}}+\sqrt{\frac{\beta}{\gamma}}=2\frac{\sqrt{C}}{(c+1)\sqrt{(c-1)}},\quad \sqrt{\frac{\gamma}{\beta}}-\sqrt{\frac{\beta}{\gamma}}=\frac{2}{(c+1)\sqrt{(c-1)}}.$$

(ii.) $f = \frac{1}{5}$, $\delta < \beta$; then

$$\frac{\delta}{\gamma} = \text{cn}\,\tfrac{1}{5}\text{K}', \quad \frac{\delta}{\beta} = \text{dn}\,\tfrac{1}{5}\text{K}', \quad \frac{\beta}{\gamma} = \text{sn}\,\tfrac{4}{5}\text{K}'.$$

(iii.) $f = \frac{3}{5}$, $\delta > \beta$,

$$\frac{\delta}{\gamma} = \text{sn}\,\tfrac{3}{5}\text{K}' = \frac{2c}{c^2+\sqrt{C}} = 2\,\frac{c^2-\sqrt{C}}{(c+1)(c-1)^2},$$

$$\frac{\beta}{\gamma} = \text{cn}\,\tfrac{2}{5}\text{K}' = \frac{c^2-\sqrt{C}}{c^2+\sqrt{C}},$$

$$\frac{\beta}{\delta} = \text{dn}\,\tfrac{2}{5}\text{K}' = \frac{c^2-\sqrt{C}}{2c},$$

$$\frac{\gamma}{\delta} = \text{ns}\,\tfrac{3}{5}\text{K}' = \frac{c^2+\sqrt{C}}{2c}.$$

Therefore $$\frac{\gamma-\beta}{\delta} = \frac{\sqrt{C}}{c}, \quad \frac{\gamma+\beta}{\delta} = c;$$

and therefore $$\left(\frac{\gamma-\beta}{\delta}\right)^2 = \frac{c^3+c^2-c}{c^2} = c+1-\frac{1}{c}$$

$$= \frac{\gamma+\beta}{\delta} + 1 - \frac{\delta}{\gamma+\beta},$$

the well known relation for the existence of *poristic pentagons.*

Also $$\sqrt{\frac{\gamma}{\beta}} = \frac{c^{\frac{3}{2}}+\sqrt{(c^2+c-1)}}{(c-1)\sqrt{(c+1)}}, \quad \sqrt{\frac{\beta}{\gamma}} = \frac{c^{\frac{3}{2}}-\sqrt{(c^2+c-1)}}{(c-1)\sqrt{(c+1)}};$$

and therefore

$$\sqrt{\frac{\gamma}{\beta}} + \sqrt{\frac{\beta}{\gamma}} = \frac{2c^{\frac{3}{2}}}{(c-1)\sqrt{(c+1)}}, \quad \sqrt{\frac{\gamma}{\beta}} - \sqrt{\frac{\beta}{\gamma}} = \frac{2\sqrt{(c^2+c-1)}}{(c-1)\sqrt{(c+1)}}.$$

Since $$\text{sn}\,\tfrac{1}{5}\text{K}' + \text{cn}\,\tfrac{4}{5}\text{K}' = \frac{c^2+2c-1+2\sqrt{C}}{(c+1)^2},$$

$$\text{sn}\,\tfrac{3}{5}\text{K}' + \text{cn}\,\tfrac{2}{5}\text{K}' = \frac{c^2+2c-1-2\sqrt{C}}{(c-1)^2},$$

therefore $$(\text{sn}\,\tfrac{1}{5}\text{K}' + \text{cn}\,\tfrac{4}{5}\text{K}')(\text{sn}\,\tfrac{3}{5}\text{K}' + \text{cn}\,\tfrac{2}{5}\text{K}') = 1.$$

Also $$\text{sn}\,\tfrac{1}{5}\text{K}' - \text{cn}\,\tfrac{4}{5}\text{K}' = \frac{-c^2-1+2\sqrt{C}}{c^2-1} = \text{sn}\,\tfrac{3}{5}\text{K}' - \text{cn}\,\tfrac{2}{5}\text{K}'.$$

(iv.) $f = \frac{2}{5}$, $\delta < \beta$,

$$\frac{\delta}{\gamma} = \text{cn}\,\tfrac{2}{5}\text{K}', \quad \frac{\delta}{\beta} = \text{dn}\,\tfrac{2}{5}\text{K}', \quad \frac{\beta}{\gamma} = \text{sn}\,\tfrac{3}{5}\text{K}'.$$

96. For $\mu = 7$ and higher values of μ, the resolution of S into its factors introduces analytical difficulties depending on the solution corresponding to 2μ, and the complexity of the formulas in the dynamical applications is considerably increased.

But the case of $\mu = 12$ (§ 46) will serve for the parameters $\omega_1 + \frac{1}{6}\omega_3$ and $\omega_1 + \frac{5}{6}\omega_3$; and $\mu = 16$ (§ 58) for parameters

$$\omega_1 + \frac{1,\ 3,\ 5,\ 7}{8}\omega_3.$$

As stated in § 7, an essential part of the method of this paper consists in assigning a first place to the elliptic functions of aliquot parts of the periods; and thence the value of the modulus can be deduced, if required.

Memoirs on the subject of Pseudo-Elliptic Integrals will be found from the following list of references:—

Legendre.—*Fonctions elliptiques*, I., Chap. XXVI.
Abel.—*Œuvres complètes*, t. I., p. 164, t. II., p. 139.
Jacobi.—*Werke*, t. I., p. 329.
Tchebicheff.—*St. Petersburg Acad. Sci. Bulletin*, III., 1861.
Raffy.—*Bulletin de la Société Mathématique de France*, t. XII.
Goursat.—*Bulletin de la Société Mathématique de France*, t. XV.
Halphen.—*Fonctions elliptiques*, t. II., Chap. XIV.
Dolbnia.—*Liouville*, 1890.
Burnside.—*Messenger of Mathematics.*

ERRATA.

p. 201, line 4, *read* $\gamma_7 = (y-x)\,x - y^3$.
p. 213, line 11, *read* $S = 4s\,(s+x)^2 - \{(1+x)\,s + x^2\}^2$.
p. 218, line 15, *read* $s - c^2 + c^3 - 2c^4$, in the numerator.
p. 222, last line, *read* $s_3 = y - x = z^2\,(1-z)$.
p. 238, line 6, *read* $= -4a^2\left(\frac{1}{a} - 1 - a\right)$, &c.
p. 239, line 7 from bottom, *read* $a = \frac{1+c}{1-c}$.
p. 240, lines 9, 10, *read* 80 *instead of* 20;
line 11, *read* $c^3 - c^2 - 13c - 3$;
lines 13, 16, *read* 20 *instead of* 5.

Thursday, April 12*th,* 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

The following communications were made:—

On Regular Difference-Terms: the President (Professor Greenhill, *pro tem.*, in the Chair).

Theorems concerning Spheres: Mr. S. Roberts.

Second Memoir on the Expansion of certain Infinite Products: Professor L. J. Rogers.

A Property of the Circum-circle (ii.): Mr. R. Tucker.

A Proof of Wilson's Theorem: Mr. J. Perott (communicated by Dr. H. Taber, Clark University, U.S.A.).

On the Sextic Resolvent of a Sextic Equation: Professor W. Burnside.

Mr. Perigal exhibited some diagrams illustrating circle-squaring by dissection.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xviii., St. 3; Leipzig, 1894.

Zeuthen, H. G.—"Note sur la résolution géometrique d'une équation du 3ᵉ degré par Archimède," pamphlet (No. 4, *Bibliothecæ Mathematicæ*, Stockholm).

Zeuthen, H. G.—"Notes sur l'histoire des mathématiques," 2 and 3, two pamphlets, 8vo; Kjöbenhavn, 1894.

"Bulletin of the New York Mathematical Society," Vol. iii., No. 6.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," No. 1; 1894.

"Transactions of the Texas Academy of Science," Vol. i., No. 2; Austin, 1893.

"Bulletin de la Société Mathématique de France," Tome xxii., Nos. 1, 2.

"Sitzungsberichte der Königl. Preussischen Akademie der Wissenschaften zu Berlin," xxxix.-liii., and Jahrgang 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. iii., Fasc. 4, 5, 6, Sem. 1.

"Acta Mathematica," xviii., No. 1; Stockholm, 1893.

"Transactions of the Cambridge Philosophical Society," Vol. xv., Pt. 4.

"Educational Times," April, 1894.

"Annals of Mathematics," Vol. viii., No. 3; Virginia.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. viii., Fasc. 1, 2.

"Indian Engineering," Vol. xv., Nos. 7-11.

"Mathematical Questions and Solutions," edited by W. J. C. Miller, Vol. lx.

Williamson, B.—"Introduction to the Mathematical Theory of the Stress and Strain of Elastic Solids," 8vo; London, 1894.

"Smithsonian Report of 1891"; Washington.

"Premiers fondements pour une théorie des transformations périodiques univoques," par M. S. Kantor. (Mémoire couronné par l'Académie des Sciences physiques et Mathématiques de Naples dans le concours pour 1883.) Naples, 1891.

Theorems concerning Spheres. By SAMUEL ROBERTS.

Read April 12th, 1894. Received April 30th, 1894.

1. I must first of all mention some results relative to plane space, which are suggestively analogous to those referred to in the heading of this paper.

The following theorem was the subject of a question by Professor Mannheim (*Educational Times*, Quest. 10145, *Reprint*, Vol. LII., p. 48), and has been discussed at considerable length by the late M. Eugène Catalan (*Memorie della Pontificia Accademia dei Nuevo Lincei*, Vol. VI., pp. 223–233, 1890).

Let A, B, C be the vertices of a given triangle (Fig. 1). Through A let a circle be drawn meeting the side AB a second time in a

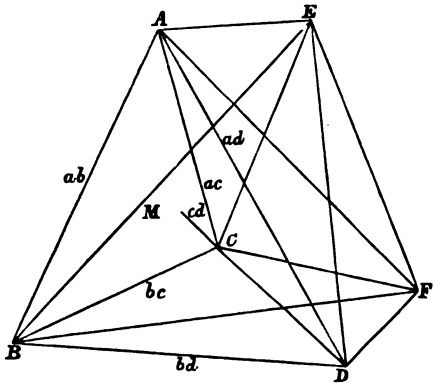

FIG. 1.

point taken at will thereon, which may be conveniently denoted by ab, and meeting the side AC a second time at a point taken at will thereon, and similarly denoted by ac.

Through B let a second circle be drawn meeting the side AB a second time in the point ab, the side BC again in the point bc, and the circle first drawn in a point M; then the points C, ac, bc, M are concyclic. Take, further, an arbitrary point D in the plane of the triangle ABC, and draw the straight lines, AD meeting the circle through A again in a point ad, BD meeting the circle through B again in a point bd, and CD meeting the circle through C again in cd; then the points D, ad, bd, cd, M are concyclic.

M. Catalan and others have proved this theorem simply enough by means of the condition that a quadrilateral may be inscribable in a circle.

The method is not available for the establishment of analogues in three-dimensioned space. However, we readily arrive at the same results by the repeated application of the first part of the theorem which may be stated in the familiar form—" If an arbitrary point be taken on each side of a given triangle, and through each vertex and the points on the adjacent sides a circle be drawn, these three circles intersect in a point."

Assuming the truth of this theorem as to the triangle ABC, and the points ab, ac, bc, we can next apply it to the triangle ABD, so that D, ad, bd, M lie on one circle, and then to the triangle ACD, so that D, ad, cd, M lie on one circle, and consequently the five points D, ad, bd, cd, M lie on one circle.

2. In like manner we may take another point E at will in the plane of the triangle, and, forming the linear connexions EA, EB, EC, ED, and denoting the intersections of these with the four previously constructed circles in their order by ae, be, ce, de, we conclude that the six points E, ae, be, ce, de, M also lie on one circle. Continuing the process, we arrive at a system of n circles, and $\frac{n\,\overline{n-1}}{1.2}$ lines connecting two-and-two together n points, so that there are n intersections of $n-1$ straight lines and one circle, $\frac{n.\overline{n-1}}{1.2}$ intersections of one straight line and two circles, and one common intersection of the n circles. On each line will lie two multiple points of the first class, and one of the second, while on each circle will lie the common intersection of the circle, one point of the first class and $n-1$ points of the second class. Otherwise regarded, the conclusion is that, if the system of straight lines is given, and also $n-1$ of the circles, $n+1$ points are determined of the n^{th} circle.

The foregoing result is one with the theorem that, if $n-1$ circles intersect in one point, and a polygon be constructed so that each side not being an extreme one passes through a single intersection of two circles, and the two vertices terminating the side lie one on each of the two circles, and if the two extreme sides pass through fixed points on the two final circles, then, the polygon being varied subject to the conditions stated, the locus of the last vertex is a circle through the common point and the fixed points. If we connect any point on

[redacted]

this locus with the vertices of the polygon, there will be $n-1$ points determined on the locus by the intersections of the connexion with the corresponding circles. Thus, including the common point and the point selected on the locus, $n+1$ points are determined. We may suppress in a variety of ways all the $n-1$ circles but two, and all the lines but three, and obtain the same locus by the variation of the triangle formed by the three lines under the conditions (*Quart. Journal of Math.*, Vol. IV., p. 361, 1861).

3. The diagram of Fig. 1 may be regarded as representing straight lines and planes in general space. Viewing it so, let $ABCD$ be a tetrahedron. On each of the edges AB, AC, AD, BC, BD, CD, in their order, let there be taken a point at will represented according to the previous notation by ab, ac, ad, bc, bd, cd. It is known that, if a sphere be constructed through the vertex A and the points ab, ac, ad, a second through the vertex B and the points ab, bc, bd, and a third through the vertex C and the points ac, bc, cd, then the points D, ad, bd, cd and a triple intersection of the three so constructed spheres lie on one sphere, *i.e.*, the four spheres meet in a point M (*Proc.*, Vol. XII.).

Take another arbitrary point in space E, and connect linearly with A, B, C, D by AE, BE, CE, DE, meeting the four spheres through A, B, C, D, respectively, in the points ae, be, ce, de; the six points E, ae, be, ce, de, M lie on one sphere. For we can apply the previous result to the tetrahedron $ABCE$, showing that E, ae, be, ce, M are on one sphere, and next to the tetrahedron $BCDE$, showing that the points E, be, ce, de, M are on one sphere. Again, we may take any other point F, and connecting as before with A, B, C, D, E, determine seven points F, af, &c., M on a sphere. In this way, we arrive at a system of n spheres and n points connected two-and-two by $\frac{n \,.\, n-1}{1.2}$ edges, formed by $\frac{n \,.\, n-1 \,.\, n-2}{1.2.3}$ planes. There are n intersections of $n-1$ planes and one sphere, $\frac{n(n-1)}{1.2}$ intersections of two planes and two spheres, n intersections of one plane and three spheres, and one common intersection of n spheres.

4. According to the foregoing reckoning we shall determine $n+1$ points on the n^{th} sphere. But, in reality, there are determined $\frac{n^2-n+4}{2}$ multiple intersections on each sphere, viz., one intersection of $n-1$ planes and one sphere, $n-1$ intersections of two planes and

two spheres, $\frac{n-1\,.\,n-2}{2}$ intersections of one plane and three spheres, and the common point of n spheres.

A certain number of triple intersections and simple intersections are not here taken into account.

For example, in the case of a tetrahedron, the intersection of the sphere through a vertex with the opposite face and all the triple intersections depending on it are left out. We omit, in fact, twenty-four intersections of two planes and one sphere, and twenty-four intersections of one plane and two spheres.

It is not necessary to work out the numbers generally. Moreover, the multiplicities of the omitted intersections may be increased in special cases, and their numbers will be consequently modified.

5. We will examine a little more in detail the case of the tetrahedron ($n = 4$). The arbitrary points taken on the several edges determine more than at first appears.

There are eight quadruple intersections on each of the four spheres, determining in each case a hexahedron with plane quadrilateral faces. The tetrahedron is thus formed exteriorly or interiorly into four hexahedra each inscribable in a sphere. The diagram (Fig. 2) will give a fairly good idea of the arrangement when the figure is divided

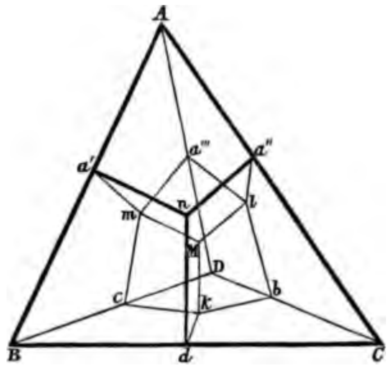

FIG. 2.

interiorly. The hexahedra are, of course, $Aa'a''a'''lmnM$, $Ba'cdkmnM$, $Ca''bdklnM$, $Da'''bcklmM$.

We may consider the spheres as given, while the tetrahedron is altered by displacement in accordance with the conditions imposed. The figure $klmnM$, forming three trihedral angles whose sum is measured by 4π, will remain fixed.

That such variation of the tetrahedron may be effected appears as follows.

Let $ABCD$ be the original tetrahedron (Fig. 3).

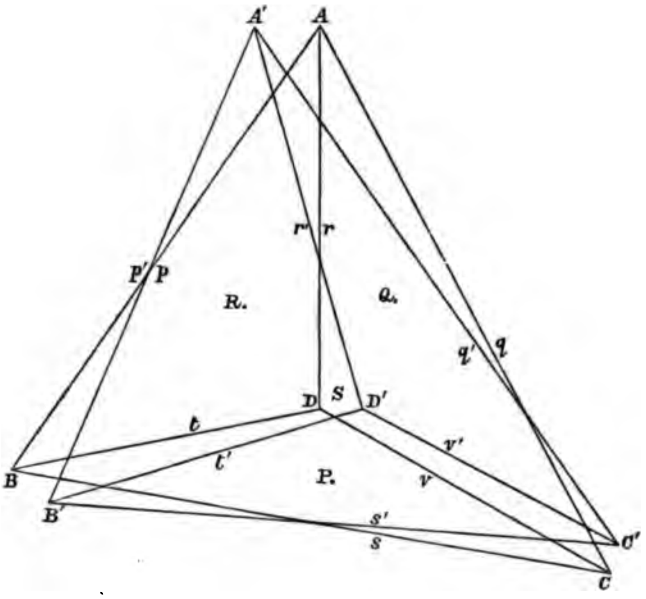

FIG. 3.

Let $A'p'B'$ be drawn so that A' is on the sphere A, and B' on the sphere B, and p' is on the circle of intersection of A and B. Here I use the letters of the vertices to denote the respective spheres passing through them. Then, draw $A'D'$ in the plane $A'B'R$ and meeting in r' and D' the sphere D, R being a triple intersection of A, B, D and r' on the intersection of A, D. Join $B'D'$. The sphere A determines a circle through A', p', R, meeting $A'D'$ in r'. The sphere B determines a circle through B', p', R, meeting $B'D'$ in t'. The points D', r', t', R lie on one circle. But the points D', r', R determine a circle on the sphere D. so that t' lies on D.

Let M be the intersection of A, B, C, D, and let P, Q, S be the intersections of the triads of spheres (B, C, D), (A, C, D), (A, B, C). Planes through the edge $A'B'$ and the point S, through the edge $B'D'$ and the point P, and through the edge $A'D'$ and the point Q determine the edges $A'C'$, $D'C'$, $B'C'$. The sphere A will pass through Q and also meet $A'C'$ in q', the sphere B will pass through P and also meet $B'C'$ in s', the sphere D will pass through P and meet $D'C'$

in v'. Hence a sphere will pass through C, q', s', v', M, P, Q, S; but this is the sphere D, which passes through C, q, s, v, M, P, Q, S; so that, as we make further displacements, the locus of the last vertex is the sphere C.

6. There are a few particular inferences which may be noticed: (*a*) When a vertex of the figure (Fig. 1) is considered as generating a sphere, and the number of director spheres is greater than three, it becomes unnecessary to retain all the plane faces of the figure, just as in the plane analogue we may suppress certain of the double chords; in fact, we obtain in a variety of ways the same locus when all but three of the spheres are suppressed.

(*b*) We may regard the diagram of Fig. 1 as a flat evanescent figure in solid space. The triple intersections of three circles in each face coalesce, so that we fall back upon M. Mannheim's theorem, when we regard only the sections of the spheres.

(*c*) In Fig. 1 suppose that the vertex F is removed to an infinite distance. It follows that, if we draw from the vertices A, B, C, D, E parallel straight lines, they will again meet the respective spheres in points which lie on a plane passing through the common point of intersection M.

7. A more important particular case is the figure of five planes, of which a form is given in Fig. 4. Here the six arbitrary points on

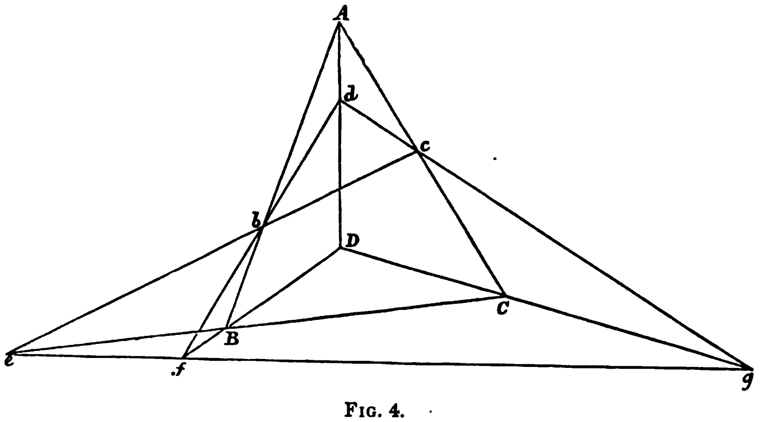

FIG. 4.

the edges of the tetrahedron $ABCD$ lie in one plane. In the figure, b, c, d, e, f, g lie in the fifth plane, and there are formed five tetrahedra. Of the five spheres circumscribing these, four meet each plane in circles circumscribing the triangles formed by the intersections of four straight lines. These circles meet in a quadruple intersection of the four spheres, and the centres of the circles are concyclic with that point. Thus the spheres about ABC and D, Abc and d, Bbe and f, Ddf and g meet in the plane ABD; the spheres about ABC and D, Abc and d, Bbe and f, Cce and g meet in the plane ABC. The two multiple intersections are therefore the triple intersections of the three spheres common to both sets. If we select a tetrahedron, and omit the sphere circumscribing it, the four spheres through the vertices intersect in the fifth plane.

If we add another plane, the sections of the spheres circumscribing the tetrahedron by a plane will consist of circles circumscribing the triangle formed by five straight lines, and, if there are n planes, the sections of the spheres by a plane will be circles circumscribing the triangles formed by the $n-1$ intersections of the plane with the remainder of the planes. To these systems of lines Miquel's theorem and the extensions by Clifford, Longchamps, &c., apply, and we need not occupy ourselves further with them in the present connexion.

We may consider a figure representing the intersections of five planes as reduced in the limit to one plane. Thus, by inspection of Fig. 5, we see that, if bcd, BCD are homologous triangles, A their

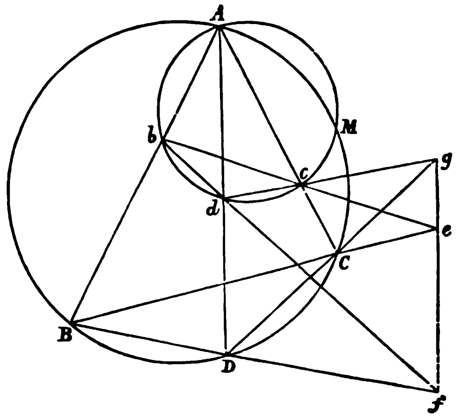

Fig. 5.

centre of homology, and efg the axis, and if $Abcd$, $ABCD$ are inscribable in circles, then the points $cCeg$, $bBef$, $dDfg$ are sets of concyclic points and the five circles meet in a point M. The symmetry of the figure shows that each intersection of three lines is the centre of homology with respect to two of the triangles, and each circle circumscribing a triangle passes through its centre of homology with regard to another triangle.

8. The inversion by reciprocal radii vectores of Fig. 1 in the simplest case, that of the tetrahedron, introduces more symmetry.

Taking the centre of inversion at an arbitrary point in space, we get, for the four faces, four spheres passing through the centre and intersecting in six circles which have four triple intersections. These form a tetrahedron with circular edges and spherical sides. We have also four spheres, meeting in a point. Each of these passes through the inverses of the arbitrary points on the edges of the original tetrahedron adjacent to the vertex through which the sphere passes. These inverses may themselves be regarded as arbitrary points, one on each circular edge. This is the direct interpretation of the original theorem, but does not fully express the symmetry.

There are eight spheres intersecting in sixteen quadruple points, the radical centres of sets of four spheres. Let us say the spheres are A, B, C, D, a, b, c, d. We have to take no account of the intersections of A and a, B and b, C and c, D and d.

The quadruple intersections may be denoted as in the following scheme:—

Abcd,	*ABCD*,	*abcd*,	*aBCD*,
ABcd,	*ABCd*,	*aBcd*,	*aBCd*,
AbCd,	*AbCD*,	*abCd*,	*abCD*,
AbcD,	*ABcD*,	*abcD*,	*aBcD*,

showing that there are eight such points on each sphere. It follows that, if we take six spheres B, C, D, b, c, d, and from the triple intersections as indicated; bearing in mind that *Abcd*, *abcd* must mean that A passes through one of the triple intersections of *bcd* and a passes through the other, then, if the eight points of one set form the apices of a hexahedron inscribable in the sphere A, the other eight form a hexahedron inscribable in the sphere a. The analogue in plane space is—" The circles which have for chords the four sides of a quadrilateral inscribable in a circle form by the other intersections of the same pairs of circles a quadrilateral inscribable in a circle " (Catalan, *Théorèmes et Problèmes*, sixième édition, p. 39).

The limiting case may be noted in which the six spheres meet in one point, the tangents at which are parallel to the faces of a hexahedron with quadrilateral faces and inscribable in a sphere, and to this also there is a plane analogue.*

9. Invert now the figure of five planes and its five associated spheres. This gives us ten spheres and sixteen points of quintuple intersections. Let the spheres derived from the five planes be denoted by a, b, c, d, e, and the other five spheres by A, B, C, D, E. The quintuple intersections will be duly represented by

$$AbcdE,\quad ABCdE,\quad ABcDE,\quad AbCDE,$$
$$AbcDE,\quad ABcde,\quad AbCde,\quad ABCDe,$$
$$abCdE,\quad aBcdE,\quad abcDE,\quad aBCDE,$$
$$aBCde,\quad aBcDe,\quad abCde,\quad abcde.$$

The total number of quintuple arrangements containing the first five letters of the alphabet would be $2^5 = 32$. But, if we take any one of the set, say $AbcdE$, its complementary form $aBCDe$ does not appear. There are left sixteen sets.

From the scheme it appears that eight of the quintuple intersections lie on the sphere A, and eight others on the sphere a. Also the hexahedra have not only six plane quadrilateral faces but also two diagonal planes. Thus as to the hexahedron circumscribed by the sphere A, the circular sections AE, AB, &c., Ae, Ab, &c., pass severally through four intersections, and the same is the case with the circular sections aB, aC, &c., ab, ac, &c.

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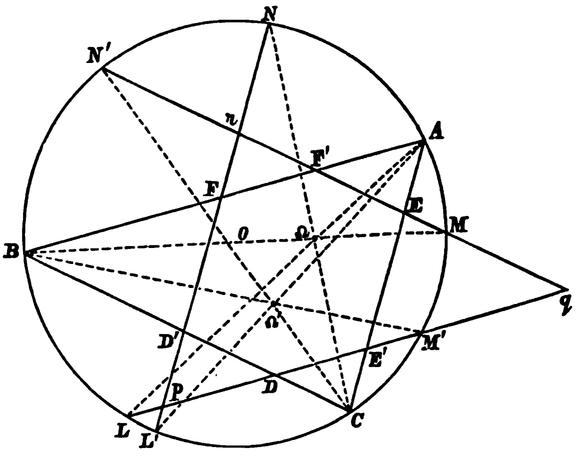
* Having proceeded so far, I happened to refer to a paper by M. Auguste Miquel, and in the second part (Liouville, *Journal*, t. x., 1e série, 1845) I unexpectedly found the theorem of this article in the second form. Accepting Miquel's proof, we may evidently by inversion with respect to one of the quadruple points pass back to the original theorem relative to the tetrahedron, from which we set out. For we shall have four planes whose six intersections correspond to the six circles of intersection through the point. We shall have also the six arbitrary points, one on each linear edge, and finally the four spheres each passing through a vertex (intersection of three planes) and the arbitrary points on the three adjacent edges. The proposition is the last in the second part of Miquel's paper, and differs rather in character from the other contents, which relate to circles in the plane and on a spherical surface. Inversion by reciprocal "radii vectores" was at that date of recent introduction. In fact, the theory of "images" is given by Professor W. Thomson (Lord Kelvin) in the same volume. Accordingly, M Miquel makes no use of the method which is directly applicable to some of his propositions. It appears that Mr. Stubbs employed the substitution ρ^{-1} for ρ in space equations towards the end of 1843.

A Property of the Circum-circle (ii.). *By* R. Tucker, M.A.

Received April 9th, 1894. Read April 12th, 1894.

1. If through the Brocard points lines AL, AL'; BM, BM'; CN, CN' are drawn to meet the circum-circle in L, L', M, M'; N, N'; and then through these points straight lines are drawn parallel to AB, AC; BC, BA; CA, CB; these straight lines by their intersection determine a triangle pqr congruent to ABC.



2. Using the notation of my former note (*cf.* Vol. xxii., pp. 470–472), the coordinates of the points are, viz.,

$$L, \quad -c\,(a^2+b^2)\,\alpha = abc\beta = a^2\gamma\,;$$

$$L', \quad -b\,(c^2+a^2)\,\alpha = a^2\beta = abc\gamma\,;^*$$

and the equations to the lines are, viz.,

$$\left.\begin{aligned} Lpq, \quad & a\alpha+b\beta-a^2b^2\gamma/c\,(a^2+b^2) = 0 \\ L'pr, \quad & a\alpha+c\gamma-c^2a^2\beta/b\,(c^2+a^2) = 0 \end{aligned}\right\} \quad \text{..............(i.).}$$

* [The construction in Art. 1 is seen to be legitimate because the α-coordinates of M and N' are $2\Delta a\,(b^2+c^2)/\lambda^2$; and so for the other points. Hence MN' parallel to BC. From the present article it is also evident that p, q, r are the images of A, B, C with respect to the mid-point of $\Omega\Omega'$.]

The points p, q, r are

$$\left.\begin{aligned}\frac{a\alpha}{-b^2c^2} &= \frac{\beta}{b(c^2+a^2)} = \frac{\gamma}{c(a^2+b^2)} = \frac{2\Delta}{\lambda^2} \\ \frac{\alpha}{a(b^2+c^2)} &= \frac{b\beta}{-c^2a^2} = \frac{\gamma}{c(a^2+b^2)} = \frac{2\Delta}{\lambda^2} \\ \frac{\alpha}{a(b^2+c^2)} &= \frac{\beta}{b(c^2+a^2)} = \frac{c\gamma}{-a^2b^2} = \frac{2\Delta}{\lambda^2}\end{aligned}\right\} \text{ .... (ii.).}$$

From these equations we get altitude of Δpqr equal that of ABC, and the triangles, by construction, are similar; hence they are congruent and co-Brocardal.

Also, from (ii.), we see that the centre of perspective is the mid-point of the join of $\Omega\Omega'$, *i.e.*, the centre of the Brocard ellipse.

3. The equation to the circle pqr is

$$\alpha\beta\gamma + \ldots + \ldots = (a\alpha + \ldots + \ldots)\, K\, (bc\omega_a \alpha + \ldots + \ldots)/\lambda^4 \text{......(iii.)};$$

i.e., it is the circle, equal, of course, to the circum-circle, obtained in the former note; hence this is a nine-point circle, passing through p, q, r; α, β, γ; α', β', γ' (*cf.* p. 470, *l.c.*).

4. It is readily verified that the Brocard ellipse, whose equation is

$$\sqrt{\frac{\alpha}{a}} + \sqrt{\frac{\beta}{b}} + \sqrt{\frac{\gamma}{c}} = 0,$$

touches the sides of pqr, as well as of ABC.

5. The conic through $ABCpqr$ has for its equation

$$bc\,(b^2+c^2)\,\beta\gamma + ca\,(c^2+a^2)\,\gamma\alpha + ab\,(a^2+b^2)\,\alpha\beta = 0 \text{.........(iv.),}$$

whose centre evidently coincides with the Brocard ellipse centre.

This conic cuts the circum-circle in the Steiner point, viz.,

$$a\alpha\,(b^2-c^2) = b\beta\,(c^2-a^2) = c\gamma\,(a^2-b^2);$$

hence its axes are parallel to the axes of the minimum circum-ellipse of ABC.

The Steiner point for the triangle pqr is given by the equations

$$a\omega_a\alpha = b\omega_b\beta = c\omega_b\gamma \text{ ....(v.).}$$

6. The tangents to the conic (iv.) at A, B are

$$b\,(a^2+b^2)\,\beta + c\,(c^2+a^2)\,\gamma = 0,$$

$$c\,(b^2+c^2)\,\gamma + a\,(a^2+b^2)\,\alpha = 0;$$

if they intersect in r', *i.e.*,

$$b\beta/(c^2+a^2) = a\alpha/(b^2+c^2) = -c\gamma/(a^2+b^2),$$

then Ap', Bq', Cr' intersect in

$$\frac{b^2+c^2}{a},\ \frac{c^2+a^2}{b},\ \frac{a^2+b^2}{c} \quad \text{....................(vi.)},$$

the Lemoine point of the medial triangle of ABC (*cf. Proc.*, Vol. XVIII., p. 395).

If, in like manner, the tangents at p, q, r intersect in p'', q'', r'', then Ap'', Bq'', Cr'' meet in

$$\frac{a\alpha\,(a^2b^2-b^2c^2+c^2a^2)}{b^2+c^2} = \ldots = \ldots \quad \text{...............(vii.)}.$$

7. The joins of L, L'; M, M'; N, N' are evidently parallel to BC, CA, AB, respectively, and the triangles LMN, $L'M'N'$, are congruent to ABC.

Ω is the negative Brocard point of LMN, and Ω' is the positive Brocard point of $L'M'N'$.*

We proceed to find the other Brocard points. Let Ω_1 be the positive point for LMN; then, since

$$\Omega_1 LL' = 3\omega,$$

the coordinates are (*cf.* § 2)

$$(ac^2 \sin 3\omega - a^3 \sin \omega)/\lambda,\ \ldots,\ \ldots,$$

i.e., $$a \sin \omega\,(b^2\lambda^2 - \omega_c k)/\lambda^3,\ \ldots,\ \ldots,$$

i.e., $$a\,(b^2\lambda^2-\omega_c k),\quad b\,(c^2\lambda^2-\omega_a k),\quad c\,(a^2\lambda^2-\omega_b k).$$

In like manner, the coordinates of Ω_2, the negative point for $L'M'N'$, are

$$a\,(c^2\lambda^2-\omega_b k),\quad b\,(a^2\lambda^2-\omega_c k),\quad c\,(b^2\lambda^2-\omega_a k).$$

8. The equations (for reference) to LM, $L'M'$, are

$$b^2(a^2+b^2)\,\alpha + ab\,(b^2+c^2)\,\beta - a^3c\gamma = 0,$$

$$ab\,(c^2+a^2)\,\alpha + a^2(a^2+b^2)\,\beta - b^3c\gamma = 0.$$

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* These triangles have been discussed by me in "Uni-Brocardal Triangles," *Proc.*, Vol. XVIII., pp. 393-398.

9. In the former figure (Vol. XXII., *l.c.*) we may mention that $\alpha\alpha'$, $\beta\beta'$ intersect on the symmedian line through C.

10. Let pq cut BC, CA, in D, E'; qr cut CA, AB in E, F'; and rp cut AB, BC in F, D'.

The points are given thus

$$D,\ 0,\ a^2b,\ c(a^2+b^2); \qquad D',\ 0,\ b(c^2+a^2),\ ca^2;$$

$$E,\ a(b^2+c^2),\ 0,\ b^2c; \qquad E',\ ab^2,\ 0,\ c(a^2+b^2);$$

$$F,\ c^2a,\ b(c^2+a^2),\ 0; \qquad F',\ a(b^2+c^2),\ bc^2,\ 0.$$

Hence we obtain $DE'^2 = a^4b^2c^2(b^2+c^2+2bc\cos A)/\lambda^4$;

$$EE' = a^2bc^2/\lambda^4 = D'F,$$

i.e.,
$$EE'.AC = a^2b^2c^2/\lambda^2 = FF'.AB = DD'.BC.$$

11. The conic round $DEFD'E'F'$ is

$$b^2c^2(c^2+a^2)(a^2+b^2)\alpha^2+\ldots+\ldots = (2a^4+\lambda^2)\,bc(b^2+c^2)\,\beta\gamma+\ldots+\ldots.$$

12. We see that $AE = b^3c^2/\lambda^2$, $AF' = b^2c^3/\lambda^2$;

hence the hexagon

$$D'DE'EF'F = \Delta(1-\Sigma a^4b^4/\lambda^4) = 2\Delta a^2b^2c^2/\lambda^4.$$

Also the diagonals pass through the mid-points of $\Omega\Omega'$, which is therefore the centre of the conic in § 11.

Second Memoir on the Expansion of certain Infinite Products. By L. J. ROGERS.

Received April 2nd, 1894. Read April 12th, 1894.

1. If $A_r(\theta)$ denote the coefficient of $x^r/(1-q)(1-q^2)\ldots(1-q^r)$ in the expansion of

$$1 \div (1-2x\cos\theta+x^2)(1-2xq\cos\theta+x^2q^2)\ldots,$$

we have seen that the value of $A_r(\theta)$ is

$$2\cos r\theta+\frac{1-q^r}{1-q}\,2\cos(r-2)\,\theta+\frac{1-q^r}{1-q}\,\frac{1-q^{r-1}}{1-q^2}\,2\cos(r-4)\,\theta+\ldots \qquad \ldots\ldots\ldots(1),$$

and that certain series and infinite products have been expanded according to ascending orders of A's. Now suppose that any such series

$$a_0 + a_1 A_1(\theta) + a_2 A_2(\theta) + \dots$$

be equivalent to the Fourier series

$$b_0 + 2b_1 \cos\theta + 2b_2 \cos 2\theta + \dots.$$

We obtain, by equating coefficients of cosines of even multiples of θ,

$$\left.\begin{array}{ll}
a_0 + a_2 \dfrac{1-q^2}{1-q} + a_4 \dfrac{1-q^4}{1-q}\,\dfrac{1-q^3}{1-q^2} + a_6 \dfrac{1-q^6}{1-q}\,\dfrac{1-q^5}{1-q^2}\,\dfrac{1-q^4}{1-q^3} + \dots & = b_0 \\
a_2 + a_4 \dfrac{1-q^4}{1-q} + a_6 \dfrac{1-q^6}{1-q}\,\dfrac{1-q^5}{1-q^2} + a_8 \dfrac{1-q^8}{1-q}\,\dfrac{1-q^7}{1-q^2}\,\dfrac{1-q^6}{1-q^3} + \dots & = b_2 \\
a_4 + a_6 \dfrac{1-q^6}{1-q} + a_8 \dfrac{1-q^8}{1-q}\,\dfrac{1-q^7}{1-q^2} + \dots & = b_4 \\
a_6 + a_8 \dfrac{1-q^8}{1-q} + \dots & = b_6 \\
a_8 + \dots & = b_8 \\
\&c. &
\end{array}\right\} \quad \dots\dots\dots\dots(2).$$

We may evidently, by multiplying these equations by suitable quantities, obtain a relation connecting the a's and b's in the form

$$a_0 + a_1\lambda + a_2\lambda^2 + \dots = b_0 + b_1\mu_1 + b_2\mu_2 + \dots \quad \dots\dots\dots(3).$$

Now, since $\quad a_0 + a_1 A_1(\theta) + \dots = b_0 + 2b_1 \cos\theta + \dots,$

and $\quad 2\cos\theta \, . \, A_{r-1}(\theta) = A_r(\theta) + (1-q^{r-1}) A_{r-2}(\theta)$

(see footnotes on p. 344, Vol. XXIV.), we get, after multiplying by $2\cos\theta$, which is $A_1(\theta)$,

$$a_0 A_1(\theta) + a_1\{A_2(\theta) + (1-q)\} + a_2\{A_3(\theta) + (1-q^2) A_1(\theta)\} + \dots$$
$$= 2b_1 + (b_0 + b_2)\cos\theta + (b_1 + b_3)\cos 2\theta + \dots \quad \dots\dots\dots(4),$$

and hence, by (3),

$$a_1(1-q) + \{a_0 + a_2(1-q^2)\}\lambda + \{a_1 + a_3(1-q^3)\}\lambda^2 + \dots$$
$$= 2b_1 + (b_0 + b_2)\mu_1 + (b_1 + b_3)\mu_2 + \dots;$$

i.e.,
$$\lambda\{a_0+a_1\lambda+a_2\lambda^2+\dots\}$$
$$+\frac{1}{\lambda}\{a_0+a_1\lambda+a_2\lambda^2+\dots\}-\frac{1}{\lambda}\{a_0+a_1\lambda q+a_2\lambda^2q^2+\dots\}$$
$$=b_0\mu_1+b_1(2+\mu_2)+b_2(\mu_1+\mu_3)+\dots.$$

If, however, as in Vol. XXIV., p. 337, δ_λ denote the operation which turns $f(\lambda)$ into $\frac{f(\lambda)-f(\lambda q)}{\lambda}$, we see that
$$(\lambda+\delta_\lambda)(b_0+b_1\mu_1+b_2\mu_2+\dots)$$
is identically the same series in the b's as
$$b_0\mu_1+b_1(2+\mu_2)+\dots.$$

Hence
$$\left.\begin{aligned}\mu_1&=\lambda\\ 2+\mu_2&=(\lambda+\delta_\lambda)\,\mu_1=\lambda^2+1-q\\ \mu_1+\mu_3&=(\lambda+\delta_\lambda)\,\mu_2\\ \dots\quad&\dots\quad\dots\quad\dots\end{aligned}\right\}\quad\dots\dots\dots\dots(5).$$

From these equations we may successively obtain the values of μ_1, μ_2, ..., and by substituting in (3), and equating coefficients of powers of λ, we may obtain the values of a_0, a_1, ... It is, moreover, obvious that the terms containing a's with even suffixes may be equated to those containing b's with even suffixes.

The actual values of μ_1, μ_2, ..., however, may be best determined by means of the identity
$$1+\frac{x\mu_1}{1-q}+\frac{x^2\mu_2}{1-q^2}+\frac{x^3\mu_3}{1-q^3}+\dots$$
$$=1+\frac{x}{1-q}(\lambda-x)+\frac{x^2}{1-q^2}(\lambda-x)(\lambda-xq)$$
$$+\frac{x^3}{1-q^3}(\lambda-x)(\lambda-xq)(\lambda-xq^2)+\dots\quad\dots\dots(6),$$
which we will now proceed to establish.

Calling the latter series F, we see that
$$\delta_\lambda F=x+x^2(\lambda-x)+x^3(\lambda-x)(\lambda-xq)+\dots\quad\dots\dots\dots(7),$$
since
$$\delta_\lambda(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-1})$$
$$=\frac{1}{\lambda}\{(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-1})$$
$$-q^{n-1}(\lambda q-x)(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-2})\}$$
$$=(1-q^n)(\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-2}).$$

Again,

$$\delta_x F = \{\lambda - x(1+q)\} + x(\lambda - xq)\{\lambda - x(1+q^2)\}$$
$$+ x^2(\lambda - xq)(\lambda - xq^2)\{\lambda - x(1+q^3)\} + \dots,$$

as is easily seen, and this

$$= (\lambda - xq) + x(\lambda - xq)(\lambda - xq^2) + x^2(\lambda - xq)(\lambda - xq^2)(\lambda - xq^3) + \dots$$
$$- x - x^2(\lambda - xq) - x^3(\lambda - xq)(\lambda - xq^2) - \dots \quad \dots\dots\dots\dots (8),$$

while, by (2),

$$\delta_x\delta_\lambda F = 1 + x(\lambda - x) + x^2(\lambda - x)(\lambda - xq) + \dots$$
$$- q - xq(\lambda - xq) - x^2q^2(\lambda - xq)(\lambda - xq^2) - \dots \quad \dots (9).$$

Now $x(\lambda - x)\,\delta_x F = x(\lambda - x)(\lambda - xq) + x^2(\lambda - x)(\lambda - xq)(\lambda - xq^2) + \dots$

$$- x^2(\lambda - x) - x^3(\lambda - x)(\lambda - xq) - \dots,$$

and, since $\delta_x F$ may also be written in the form

$$(\lambda - x) + x(\lambda - x)(\lambda - xq) + x^2(\lambda - x)(\lambda - xq)(\lambda - xq^2) + \dots,$$
$$- xq - x^2q^2(\lambda - xq) - x^3q^3(\lambda - xq)(\lambda - xq^2) - \dots,$$

we see that

$$(1 - \lambda x + x^2)\,\delta_x F = (\lambda - x)$$
$$+ x^2(\lambda - x) + x^3(\lambda - x)(\lambda - xq) + \dots$$
$$- xq - x^2q^2(\lambda - xq) - x^3q^3(\lambda - xq)(\lambda - xq^2) - \dots,$$

which, by (9), $= \lambda - 2x + x\delta_x\delta_\lambda F$ $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10).$

If, then, F be arranged in powers of x in the form

$$1 + \frac{m_1}{1-q} + \frac{x^2 m_2}{1-q^2} + \dots,$$

we see that $\qquad \delta_x F = m_1 + m_2 x + m_3 x^2 + \dots,$

and (10) becomes

$$(1 + x^2)(m_1 + m_2 x + m_3 x^2 + \dots)$$
$$= \lambda - 2x + x(\lambda + \delta_\lambda)(m_1 + m_2 x + m_3 x^2 + \dots).$$

Equating coefficients of powers of x, we get

$$m_1 = \lambda,$$
$$2 + m_2 = (\lambda + \delta_\lambda)\, m_1,$$
$$m_1 + m_3 = (\lambda + \delta_\lambda)\, m_2,$$
$$\dots \quad \dots \quad \dots$$

so that the m's are derived from one another in precisely the same way as the μ's in (5), and are therefore identical.

Hence the truth of (6) is established.

If we write q_n as an abbreviation for $1-q^n$, we may easily observe the formation of the coefficients.

In fact, since $\qquad (\lambda-x)(\lambda-xq)\dots(\lambda-xq^{n-1})$

$$=\lambda^n-\frac{q_n}{q_1}\lambda^{n-1}x+q\frac{q_nq_{n-1}}{q_1q_2}\lambda^{n-2}x^2-q^3\frac{q_nq_{n-1}q_{n-2}}{q_1q_2q_3}\lambda^{n-3}x^3+\dots,$$

we get, from (6), $\qquad a_0+a_1\lambda+a_2\lambda^2+\dots$

$$=b_0+q_1b_1\left\{\frac{\lambda}{q_1}\right\}+q_2b_2\left\{\frac{\lambda^2}{q_2}-\frac{1}{q_1}\right\}$$

$$+q_3b_3\left\{\frac{\lambda^3}{q_3}-\frac{\lambda}{q_2}\,\frac{q_2}{q_1}\right\}$$

$$+q_4b_4\left\{\frac{\lambda^4}{q_4}-\frac{\lambda^2}{q_3}\,\frac{q_3}{q_1}+q\frac{1}{q_2}\right\}$$

$$+q_5b_5\left\{\frac{\lambda^5}{q_5}-\frac{\lambda^3}{q_4}\,\frac{q_4}{q_1}+q\frac{\lambda}{q_3}\,\frac{q_3}{q_1}\right\}$$

$$+q_6b_6\left\{\frac{\lambda^6}{q_6}-\frac{\lambda^4}{q_5}\,\frac{q_5}{q_1}+q\frac{\lambda^2}{q_4}\,\frac{q_4q_3}{q_1q_2}-q^3\frac{1}{q_3}\right\} \quad \dots\dots\dots(11),$$

$$+\dots,$$

where the coefficients in the bracketed series are formed on the analogy of those appearing in the development of $2\cos n\theta$ in powers of $2\cos\theta$, and q occurs in the r^{th} term of any series raised to the $\frac{1}{2}(r-1)(r-2)^{\text{th}}$ power.

Equating the several powers of λ, we see that

$$a_0=b_0-(1+q)\,b_2+q\,(1+q^2)\,b_4-q^3\,(1+q^3)\,b_6+\dots \quad \dots\dots(12),$$

the general term being

$$(-1)^r q^{\frac{1}{2}r(r-1)}(1+q^r)\,b_{2r},$$

while $(1-q)\,a_1=(1-q)\,b_1-(1-q^3)\,b_3+q\,(1-q^5)\,b_5-q^3\,(1-q^7)\,b_7-\dots,$

the general term being

$$(-1)^r q^{\frac{1}{2}r(r-1)}(1-q^{2r+1})\,b_{2r+1}\dots\dots\dots\dots\dots\dots\dots\dots(13).$$

Similarly, series may be obtained for a_2, a_3, ... in terms of the b's, but for our present investigations it will not be necessary to quote them here.

Either of these relations gives us a method of expanding the square of the infinite product $(1-q)(1-q^2)(1-q^3)\ldots$ in powers of q.

For from the identity which gives $1 \div \Theta\left(\frac{2K\theta}{\pi}\right)$ as a series of partial fractions, which, on changing q^2, 2θ into q, θ, becomes

$$\frac{\Pi\,[1-q^n]^2}{\Pi\,[1-2q^{n+\frac{1}{2}}\cos\theta+q^{2n-1}]}$$

$$=\frac{1-q}{1-2q^{\frac{1}{2}}\cos\theta+q}-\frac{q\,(1-q^3)}{1-2q^{\frac{3}{2}}\cos\theta+q^3}+\frac{q^3\,(1-q^5)}{1-2q^{\frac{5}{2}}\cos\theta+q^5}-\ldots$$

$$\begin{aligned}
= \quad & (1-q+q^3-q^6+\ldots) \\
& +2q^{\frac{1}{2}}\cos\theta\,(1-q\,.\,q+q^3\,.\,q^3-q^6\,.\,q^6+\ldots) \\
& +2q\cos 2\theta\,(1-q\,.\,q^2+q^3\,.\,q^4-q^6\,.\,q^6+\ldots) \\
& +\ldots,
\end{aligned}$$

we get, from (13),

$$\begin{aligned}
\Pi\left[1-q^n\right]^2 = \quad & (1-q)(1-q^2+q^5-q^9+\ldots) \\
& -q\,(1-q^3)(1-q^4+q^9-q^{15}+\ldots) \\
& +q^3\,(1-q^5)(1-q^6+q^{13}-q^{21}+\ldots) \\
& -q^6\,(1-q^7)(1-q^8+q^{17}-q^{27}+\ldots) \\
& +\ldots.
\end{aligned}$$

Multiplying out the binomial factors on the right-hand side, and arranging the series in two blocks, it will be found that horizontal and vertical series are equal in pairs, starting from a series of terms running parallel to the diagonals of the blocks, so that

$$\begin{aligned}
\Pi\left[1-q^n\right]^2 = \quad & 1-2q+2q^3-2q^6+\ldots \\
& -q^2\,(1-2q^3+2q^7-2q^{12}+\ldots) \\
& +q^4\,(1-2q^4+2q^9-2q^{15}+\ldots) \\
& -q^{10}\,(1-2q^5+2q^{11}-2q^{18}+\ldots) \\
& +q^{14}\,(1-2q^6+2q^{13}+2q^{21}+\ldots) \\
& -\ldots,
\end{aligned}$$

where the indices in the terms outside the brackets are of the form $n\,(3n\pm 1)$, while those in the bracketed series form series whose differences are in arithmetic progression.

2. The series on the right-hand side of § 1, (6) can, in the cases where $\lambda = 1$ or where $\lambda = q^2$, be very easily arranged according to powers of x, by means of a functional equation which it satisfies.

Let $F(\mu, x) = 1 + \frac{\mu}{1-q}(\lambda - x) + \frac{\mu^2}{1-q^2}(\lambda - x)(\lambda - xq) + \dots$.

Then
$$F(\mu, x) - F(\mu q, x)$$
$$= \mu(\lambda - x) + \mu^2(\lambda - x)(\lambda - xq) + \dots$$
$$= \mu(\lambda - x) + \mu(\lambda - x)\{F(\mu, xq) - F(\mu q, xq)\} \quad \dots\dots\dots (1).$$

Moreover
$$\mu x\{F(\mu q, x) - F(\mu q^2, x)\}$$
$$= \mu^2 qx(\lambda - x) + \mu^3 q^2 x(\lambda - x)(\lambda - xq) + \dots$$
$$= -\mu^2(\lambda - qx)(\lambda - x) - \mu^3(\lambda - q^2 x)(\lambda - x)(\lambda - xq) - \dots$$
$$+ \mu^2\lambda(\lambda - x) + \mu^3\lambda(\lambda - x)(\lambda - xq) + \dots$$
$$= \mu(\lambda - x) - (1 - \mu\lambda)\{F(\mu, x) - F(\mu q, x)\} \quad \dots\dots\dots\dots (2).$$

Again,
$$F(\mu, xq) - F(\mu, x) = \mu x + \mu x\{F(\mu, xq) - F(\mu q, xq)\} \quad \dots\dots (3),$$
i.e.,
$$F(\mu, xq)(1 - \mu x) = F(\mu, x) - \mu x F(\mu q, xq) + \mu x.$$

By the help of (1), (2), and (3), we may eliminate all the functions except $F(\mu, x)$, $F(\mu q, xq)$, and $F(\mu q^2, xq^2)$, and obtain the equation
$$\frac{1-\mu\lambda}{1-\mu x}\{F(\mu, x) - F(\mu q, xq) + \mu x\}$$
$$+ \mu^2 xq^2 \frac{\lambda - xq}{1 - \mu xq^2}\{F(\mu q, xq) - F(\mu q^2, xq^2) + \mu xq^2\}$$
$$= \mu(1 - \mu xq^2)(\lambda - xq) \quad \dots\dots\dots\dots\dots\dots\dots\dots (4).$$

Let $\mu = x$, and write $F(x)$ for $F(x, x)$. Then
$$\frac{1-\lambda x}{1-x^2}\{F(x) - F(xq) + x^2\} + x^3 q^2 \frac{\lambda - xq}{1 - x^2 q^2}\{F(xq) - F(xq^2) + x^2 q^2\}$$
$$= x(1 - x^2 q^2)(\lambda - xq).$$

Let $\lambda = 1$, and write $\psi(x)$ for $\frac{F(x) - F(xq) + x^2}{1+x}$, so that
$$\psi(x) + x^3 q^2 \psi(xq) = x - x^2 q - x^3 q^2 + x^4 q^3.$$

From this relation we easily get
$$\psi(x) = x - x^2 q - x^3 q^3 + x^5 q^5 + x^6 q^7 - x^8 q^{12} - x^9 q^{15} + \dots \quad \dots\dots (5),$$

a series which is worthy of notice for its resemblance to the expansion of $\Pi_1^\infty\left[1-q^n\right]$ in powers of q.

Substituting for $\psi(x)$ and multiplying up, we get

$$F(x)-F(xq) = x-x^2q-x^3(q+q^2)-x^4q^2+x^5q^5+x^6(q^5+q^7)+\ldots,$$

and, finally,

$$F(x)=1+\frac{x}{1-q}-\frac{qx^2}{1-q^2}-\frac{(q+q^2)\,x^3}{1-q^3}-\frac{q^2x^4}{1-q^4}+\frac{q^5x^5}{1-q^5}+\frac{(q^5+q^7)\,x^6}{-q^6}+\ldots \qquad \ldots\ldots\ldots\ldots(6).$$

Hence, by § 1, (6), $\qquad a_0+a_1+a_2+\ldots$

$$=b_0+b_1-qb_2-(q+q^2)\,b_3-q^2b_4+q^5b_5+(q^5+q^7)\,b_6+q^7b_7-q^{10}b_8-\ldots \qquad \ldots\ldots\ldots\ldots(7).$$

The formation of these coefficients is sufficiently obvious. The coefficient of b_{3n} is $(-1)^n\left\{q^{\frac{1}{2}n(3n-1)}+q^{\frac{1}{2}n(3n+1)}\right\}$, while that of b_{3n-1} is $(-1)^n q^{\frac{1}{2}n(3n-1)}$, and that of b_{3n+1} is $(-1)^n q^{\frac{1}{2}n(3n+1)}$. Moreover we may, of course, equate the series of terms with even suffixes to each other, and those with odd.

Again, in (4), let $\lambda = q^{\frac{1}{2}}$, so that

$$\lambda-xq = q^{\frac{1}{2}}(1-\lambda x).$$

Then, in a manner similar to the above, we may show

$$F(x) = 1+\frac{xq^{\frac{1}{2}}}{1-q}-\frac{x^2}{1-q^2}-\frac{q^{\frac{1}{2}}(1+q^2)\,x^3}{1-q^3}-\frac{q^4x^4}{1-q^4}+\frac{q^{\frac{1}{2}}x^5}{1-q^5}+\frac{q^4(1+q^4)\,x^6}{1-q^6}+\ldots \qquad \ldots\ldots\ldots\ldots(8),$$

so that, by (1), $\qquad a_0+q^{\frac{1}{2}}a_1+qa_2+\ldots$

$$= b_0+q^{\frac{1}{2}}b_1-b_2-q^{\frac{1}{2}}(1+q^2)\,b_3-q^4b_4+q^{\frac{1}{2}}b_5+q^4(1+q^4)\,b_6+q^{\frac{11}{2}}b_7-q^8b_8-\ldots$$

The series of alternate terms giving

$$a_0+qa_2+q^2a_4+\ldots$$

$$= b_0-b_2-q^4b_4+q^4(1+q^4)\,b_6-q^8b_8-q^{20}b_{10}+q^{20}(1+q^8)\,b_{12}-\ldots \qquad \ldots(9)$$

is of special interest, as will be seen later.

The coefficient of b_{6n} is $q^{2n(3n-1)}+q^{2n(3n+1)}$, while that of b_{6n-2} and b_{6n+2} are respectively $-q^{2n(3n-1)}$ and $-q^{2n(3n+1)}$.

3. Closely analogous to the results obtained in the preceding sections are those derived from the coefficients of $x^r/(1-q^r)!$ in the expansion of

$$(1+2xq\cos\theta+x^2q^2)(1+2xq^2\cos\theta+x^2q^4)\ldots,$$

which in Vol. xxiv., p. 352, was denoted by $B_r(\theta)$.

If the q series $\quad a_0+a_1B_1(\theta)+a_2B_2(\theta)+\dots$
be equivalent to the same Fourier series

$$b_0+2b_1\cos\theta+2b_2\cos 2\theta+\dots,$$

we have, as in § 1, (2),

$$\left.\begin{array}{ll}
a_0+a_2q^{1+1}\dfrac{q_2}{q_1}+a_4q^{3+3}\dfrac{q_4q_3}{q_1q_2}+a_6q^{6+6}\dfrac{q_6q_5q_4}{q_1q_2q_3}+\dots & =b_0 \\
a_2q^3+a_4q^{6+1}\dfrac{q_4}{q_1}+a_6q^{10+3}\dfrac{q_6q_5}{q_1q_2}+\dots & =b_2 \\
a_4q^{10}+a_6q^{15+1}\dfrac{q_6}{q_1}+a_8q^{21+3}\dfrac{q_8q_7}{q_1q_2}+\dots & =b_4 \\
a_6q^{21}+a_8q^{28+1}\dfrac{q_8}{q_1}+\dots & =b_6 \\
a_8q^{36}+\dots & =
\end{array}\right\}\quad\dots\dots(1),$$

whence a_0, a_2, ... may be expressed in terms of the b's. However, it is easy to see, from the formation of the indices of the powers of q which occur in the coefficient, that if by § 1, (2), we obtain a relation

$$m_0a_0+m_2a_2+m_4a_4+\dots=n_0b_0+n_2b_2+n_4b_4+\dots,$$

then, from (1) above, we get

$$m_0a_0+m_2q^2a_2+m_4q^6a_4+m_6q^{12}a_6+\dots$$
$$=n_0b_0+n_2q^{-1}b_2+n_4q^{-4}b_4+n_6q^{-9}b_6+\dots\quad\dots\dots(2),$$

while, in a similar manner, if

$$m_1a_1+m_3a_3+\dots=n_1b_1+n_3b_3+\dots,$$

then

$$m_1qa_1+m_3q^4a_3+m_5q^9a_5+\dots=n_1b_1+n_3q^{-2}b_3+n_5q^{-6}b_5+n_7q^{-12}b_7+\dots.$$

4. By the results obtained in § 1 it is obvious that, if conditions for convergency are satisfied, any cosine-series in θ may be expanded uniquely in the form

$$a_0+a_1A_1(\theta)+a_2A_2(\theta)+\dots.$$

Let us therefore expand the product

$$(1+2\lambda q\cos\theta+\lambda^2q^2)(1+2\lambda q^2\cos\theta+\lambda^2q^4)\dots$$

in this form.

Now it has been seen in Vol. XXIV., p. 345, that, if

$$f(\theta)=C_0+C_1A_1(\theta)+C_2A_2(\theta)+\dots,$$

and $$\frac{f(\theta)}{P(\lambda)} = K_0 + K_1 A_1(\theta) + K_2 A_2(\theta) + \ldots,$$

where $$P(\lambda) = (1-2\lambda\cos\theta+\lambda^2)(1-2\lambda q\cos\theta+\lambda^2 q^2)\ldots,$$

then $$K_0 + K_1 x + K_2 x^2 + \ldots = \frac{1}{(x\lambda)}\,\frac{1}{(x\delta)}\,(C_0 + C_1\lambda + \ldots),$$

where δ operates only on the λ's not contained in the C's; or, since

$$\frac{1}{(x\delta_\lambda)}\,\phi(\lambda) = \frac{1}{(\lambda\delta_x)}\,\phi(x),$$

we may write this relation in the form

$$K_0 + K_1 x + K_2 x^2 + \ldots = \frac{1}{(\lambda x)}\,\frac{1}{(\lambda\delta_x)}\,(C_0 + C_1 x + C_2 x^2 + \ldots),$$

where the C's are independent of x.

Hence $$C_0 + C_1 x + C_2 x^2 + \ldots = (\lambda\delta_x)(\lambda x)(K_0 + K_1 x + \ldots).$$

If, then, $$P(\lambda) = C_0 + C_1 A_1(\theta) + \ldots,$$

we have $$C_0 + C_1 x + C_2 x^2 + \ldots = (\lambda\delta_x)(\lambda x),$$

because $$K_0 + K_1 A_1(\theta) + \ldots = 1.$$

Now $$(\lambda\delta_x)(\lambda x)$$

$$= \left\{1 - \frac{\lambda\delta_x}{1-q} + \frac{q\lambda^2\delta_x^2}{(1-q)(1-q^2)} - \ldots\right\}\left\{1 - \frac{\lambda x}{1-q} + \frac{q\lambda^2 x^2}{(1-q)(1-q^2)} - \ldots\right\},$$

and since $$\delta_x x^r = (1-q^r)\,x^{r-1}$$

by definition, we see that the coefficient of x^r is the series

$$(-1)^r \frac{q^{\frac{1}{2}r(r-1)}\lambda^r}{(1-q^r)!}\,\Sigma\,\frac{\lambda^{2s} q^{rs+s(s-1)}}{(1-q^s)!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $$s = 0,\ 1,\ 2,\ \ldots.$$

Changing λ in $-\lambda q$, we have

$$(1+2\lambda q\cos\theta+\lambda^2 q^2)(1+2\lambda q^2\cos\theta+\lambda^2 q^4)\ldots$$

$$= 1 + \frac{\lambda^2 q^2}{1-q} + \frac{\lambda^4 q^6}{(1-q)(1-q^2)} + \frac{\lambda^6 q^{12}}{(1-q)(1-q^2)(1-q^3)} + \ldots$$

$$+ \frac{q\lambda}{1-q}\,A_1(\theta)\left\{1 + \frac{\lambda^2 q^4}{1-q} + \frac{\lambda^4 q^{8}}{(1-q)(1-q^2)} + \ldots\right\}$$

$$+ \frac{q^3\lambda^2}{(1-q)(1-q^2)}\,A_2(\theta)\left\{1 + \frac{\lambda^2 q^4}{1-q} + \ldots\right\}$$

$$+ \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Moreover, if we write $\chi(\lambda^2)$ for the series

$$1+\frac{\lambda^2q^2}{1-q}+\frac{\lambda^4q^6}{(1-q)(1-q^2)}+\dots,$$

which is the coefficient of $A_0(\theta)$, we can write this expansion in the form

$$\chi(\lambda^2)+\frac{q\lambda A_1(\theta)}{1-q}\chi(\lambda^2q)+\frac{q^3\lambda^2A_2(\theta)}{(1-q)(1-q^2)}\chi(\lambda^2q^2)$$
$$+\frac{q^6\lambda^3A_3(\theta)}{(1-q)(1-q^2)(1-q^3)}\chi(\lambda^2q^3)+\dots \quad \dots\dots(3).$$

The function $\chi(\lambda^2)$ also satisfies the relation

$$\chi(\lambda^2)-\chi(\lambda^2q)=\lambda^2q^2\chi(\lambda^2q^2),$$

so that

$$\frac{\chi(\lambda^2)}{\chi(\lambda^2q)}=\frac{1}{1+}\,\frac{\lambda^2q^2}{1+}\,\frac{\lambda^2q^3}{1+}\,\frac{\lambda^2q^4}{1+}\dots \quad \dots\dots\dots\dots\dots(4).$$

5. When $\lambda=q^{-\frac{1}{2}}$, we get some very interesting results. For then the equation § 4, (2), gives

$$(1+2q^{\frac{1}{2}}\cos\theta+q)(1+2q^{\frac{3}{2}}\cos\theta+q^3)\dots$$
$$=1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\dots$$
$$+\frac{q^{\frac{1}{2}}A_1(\theta)}{1-q}\left\{1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\dots\right\}$$
$$+\dots$$
$$=\phi(q)+\frac{q^{\frac{1}{2}}A_1(\theta)}{1-q}\psi(q)+\dots,\text{ say.}$$

But, by the theory of theta functions, the above product

$$=\frac{1}{\Pi[1-q^n]}(1+2q^{\frac{1}{2}}\cos\theta+2q^2\cos 2\theta+2q^{\frac{9}{2}}\cos 3\theta+2q^8\cos 4\theta+\dots).$$

Hence, by § 1, (12),

$$\Pi[1-q^n]\,\phi(q)=1-q^2(1+q)+q^9(1+q^3)-q^{21}(1+q^5)-\dots$$
$$=1-q^{\frac{5}{2}}(q^{-\frac{1}{2}}+q^{\frac{1}{2}})+q^{10}(q^{-1}+q^1)-q^{\frac{45}{2}}(q^{-\frac{3}{2}}+q^{\frac{3}{2}})+\dots$$
$$=\Pi[1-q^{5n}]\,\Pi[1-q^{5n\pm2}];$$

therefore

$$\phi(q)=1\div(1-q)(1-q^4)(1-q^6)(1-q^9)\dots$$
$$=1\div\Pi[1-q^{5n\pm1}]\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

Similarly, by § 1, (13),

$$\Pi\,[1-q^n]\,\psi(q) = 1-q-q^4(1-q^3)+q^{12}(q-q^6)-\dots$$
$$= 1-q^{\frac{5}{2}}(q^{-\frac{3}{2}}+q^{\frac{3}{2}})+\dots$$
$$= \Pi\,[1-q^{5n}]\,\Pi\,[1-q^{5n\pm 1}];$$

therefore $$\psi(q) = 1 \div \Pi\,[1-q^{5n\pm 2}] \quad \dots\dots\dots\dots\dots\dots(2).$$

Combining these results, we see that

$$\phi(q)\,\psi(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\dots}{(1-q)(1-q^2)(1-q^3)\dots},$$

and that

$$\frac{\phi(q)}{\psi(q)} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\dots = \frac{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots}$$
$$= (1-q)(1+q^2)(1+q^3)(1+q^7)(1+q^8)(1-q^9)(1-q^{11})(1+q^{12})\dots,$$

where the indices in the binomial factors include all numbers whose final digits are 1, 2, 3, 7, 8, or 9, the first and last being combined with minus signs, and the rest with plus signs.

Similarly,

$$\frac{\psi(q)}{\phi(q)} = (1+q)(1-q^3)(1+q^4)(1+q^6)(1-q^7)(1+q^9)(1+q^{11})(1-q^{13})\dots.$$

6. The series

$$\chi(\lambda) = 1+\frac{\lambda q^2}{1-q}+\frac{\lambda^2 q^6}{(1-q)(1-q^2)}+\dots$$

may be expressed in another form by means of Lemma IV. on p. 340, Vol. XXIV. For, since this lemma gives

$$(\lambda\mu\eta_1)(\lambda_1\mu)\,\frac{1}{(\lambda\delta_1)}\,\frac{\psi(\lambda_1)}{(\lambda_1\mu)} = \frac{1}{(\lambda\delta_1)}\,\psi(\lambda_1),$$

we get, by putting $\psi(\lambda_1) \equiv (-\lambda_1 q)$, and $\mu = -q$,

$$\frac{1}{(\lambda\delta_1)}(-\lambda_1 q) = (-\lambda q\eta_1)(-\lambda_1 q).$$

Expanding each of these products and performing the operations involved, we get

$$1+qH_1(\lambda,\lambda_1)+q^3H_2(\lambda,\lambda_1)+q^6H_3(\lambda,\lambda_1)+\dots$$
$$=(-\lambda_1 q)\left\{1+\frac{\lambda q}{(1-q)(1+\lambda_1 q)}+\frac{\lambda^2 q^3}{(1-q)(1-q^2)(1+\lambda_1 q)(1+\lambda_1 q^2)}+\dots\right\},$$

which, by symmetry,

$$=(-\lambda q)\left\{1+\frac{\lambda_1 q}{(1-q)(1+\lambda q)}+\frac{\lambda_1^2 q^3}{(1-q)(1-q^2)(1+\lambda q)(1+\lambda q^2)}+\dots\right\}$$

............(1).

Let $\lambda_1 = \lambda q^{\frac{1}{2}}$, so that

$$1+\Sigma x^r H_r(\lambda, \lambda_1) = 1 \div (1-\lambda x)(1-\lambda x q^{\frac{1}{2}})(1-\lambda x q)\dots,$$

and $$H_r(\lambda, \lambda_1) = \lambda^r \div (1-q^{\frac{1}{2}})(1-q)\dots(1-q^{\frac{1}{2}r});$$

then (1) becomes, after changing q into q^2,

$$1+\frac{\lambda q^2}{1-q}+\frac{\lambda^2 q^6}{(1-q)(1-q^2)}+\dots,$$

i.e., $$\chi(\lambda) = \Pi[1+q^{2n+1}\lambda]\times$$

$$\left\{1+\frac{\lambda q^2}{(1-q^2)(1+\lambda q^3)}+\frac{\lambda^2 q^6}{(1-q^2)(1-q^4)(1+\lambda q^3)(1+\lambda q^5)}+\dots\right\}\ \dots(2),$$

$$=\Pi[1+q^{2n}\lambda]\times$$

$$\left\{1+\frac{\lambda q^3}{(1-q^2)(1+\lambda q^2)}+\frac{\lambda^2 q^8}{(1-q^2)(1-q^4)(1+\lambda q^2)(1+\lambda q^4)}+\dots\right\}\ \dots(3).$$

Again, in (1), let $\lambda_1 = -\lambda$, so that

$$1+\Sigma x^r H_r(\lambda, \lambda_1) = 1 \div (1-\lambda^2 x^2)(1-\lambda^2 q^2 x^2)\dots,$$

and $$H_{2r}(\lambda, \lambda_1) = \lambda^{2r} \div (1-q^2)(1-q^4)\dots(1-q^{2r}),$$

and $$H_{2r+1}(\lambda, \lambda_1) = 0;$$

then (1) becomes

$$1+\frac{q^3\lambda^2}{1-q^2}+\frac{q^{10}\lambda^4}{(1-q^2)(1-q^4)}+\dots = (-\lambda q)\left\{1-\frac{\lambda q}{(1-q)(1+\lambda q)}+\dots\right\}.$$

Changing q into q^2, and λ into λq, we get

$$1+\frac{\lambda^2 q^8}{1-q^4}+\frac{\lambda^4 q^{24}}{(1-q^4)(1-q^8)}+\dots = \Pi[1+\lambda q^{2n+1}]\times$$

$$\left\{1-\frac{\lambda q^3}{(1-q^2)(1+\lambda q^3)}+\frac{\lambda^2 q^8}{(1-q^2)(1-q^4)(1+\lambda q^3)(1+\lambda q^5)}-\dots\right\}\dots(4),$$

the left side of which is $\chi(\lambda^2)$ in which q has been changed into q^4.

If, now, in (2), we put $\lambda = q^{-1}$, we get

$$\phi(q) = \Pi[1+q^{2n}]\left\{1+\frac{q}{1-q^4}+\frac{q^4}{(1-q^4)(1-q^8)}+\dots\right\};$$

therefore $$\phi(q)+\phi(-q) = 2\Pi[1+q^{2n}]\times$$

$$\left\{1+\frac{q^4}{(1-q^4)(1-q^8)}+\frac{q^{16}}{(1-q^4)(1-q^8)(1-q^{12})(1-q^{16})}+\dots\right\}.$$

But in (4), if $\lambda = -q^{-2}$, we get

$$\phi(q^4) = \Pi[1-q^{2n-1}]\times$$

$$\left\{1+\frac{q}{(1-q)(1-q^3)}+\frac{q^4}{(1-q)(1-q^3)(1-q^5)(1-q^4)}+\dots\right\}.$$

Comparing these results, we see that

$$\phi(q)+\phi(-q) = 2\frac{\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}\phi(q^{16}) \quad\dots\dots\dots\dots\dots(5).$$

Similarly,

$$\phi(q)-\phi(-q) = 2\Pi[1+q^{2n}]\left\{\frac{q}{1-q^4}+\frac{q^9}{(1-q^4)(1-q^8)(1-q^{12})}+\dots\right\},$$

which, by putting $l=+1$, and $-q^4$ for q in (4),

$$= \frac{2\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}q\psi(-q^4) \quad\dots\dots\dots\dots\dots\dots\dots\dots(6).$$

Again, by (3), we shall get

$$\psi(q) = \Pi[1+q^{2n}]\left\{1+\frac{q^3}{1-q^4}+\frac{q^8}{(1-q^4)(1-q^8)}+\dots\right\},$$

whence, by (4),

$$\psi(q)-\psi(-q) = 2\frac{\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}q^3\psi(q^{16}) \quad\dots\dots\dots\dots(7),$$

and similarly, by (3), used twice,

$$\psi(q)+\psi(1-q) = 2\frac{\Pi[1+q^{2n}]}{\Pi[1-q^{8n-4}]}\phi(-q^4) \quad\dots\dots\dots\dots(8).$$

These four identities (5), (6), (7), (8) are sufficiently remarkable in themselves to call for mention at this point, although they may all be derived from the Θ-function values of the series $\phi(q)$, $\psi(q)$ obtained in the last section.

For instance, $$\phi(q)+\phi(-q)$$

$$= \frac{(1-q)(1-q^9)(1-q^{11})(1-q^{19})\dots+(1+q)(1+q^9)(1+q^{11})(1+q^{19})\dots}{(1-q^2)(1-q^{18})(1-q^{22})\dots\times(1-q^4)(1-q^6)(1-q^{14})(1-q^{16})},$$

by § 5, (1).

This numerator multiplied by

$$(1-q^{10})(1-q^{20})(1-q^{30})\ldots$$

$$= 2\left\{1+q^{20}(q^{-8}+q^{8})+q^{80}(q^{-16}+q^{16})+\ldots\right\}$$

$$= 2\Pi(1-q^{40n})(1+q^{12})(1+q^{28})(1+q^{52})(1+q^{68})\ldots.$$

In this manner we reduce the left hand of (5) to an infinite product, which is easily seen to include identically all the factors in the right-hand side, after substituting for $\phi(q^{16})$ by § 5, (1).

It is not, however, in these identities that the special interest in the series $\phi(q)$ and $\psi(q)$ lies. These relations may be considerably simplified by substituting the functions

$$\phi(q)\times(1-q^2)(1-q^4)(1-q^6)\ldots \text{ and } \psi(q)\times(1-q^2)(1-q^4)(1-q^6)\ldots.$$

Let $u_{\pm r}$ denote $\phi(\pm q^r)\,\Pi\left\{1-q^{2rn}\right\}$,

and $v_{\pm r}$ „ $\psi(\pm q^r)\,\Pi\left\{1-q^{2rn}\right\}$.

Then (5), (6), (7), and (8) become

$$u_1+u_{-1} = 2\Pi\left[1-q^{2n}\right]\phi(q^{16}) = 2\frac{\Pi\left[1-q^{8n}\right]}{\Pi\left[1-q^{32n}\right]}u_{16} \quad\ldots\ldots\ldots(9),$$

$$u_1-u_{-1} = 2qv_{-4} \quad\ldots\ldots\ldots(10),$$

$$v_1-v_{-1} = 2q^3\frac{\Pi\left[1-q^{8n}\right]}{\Pi\left[1-q^{32n}\right]}v_{16} \quad\ldots\ldots\ldots(11),$$

$$v_1+v_{-1} = 2u_{-4} \quad\ldots\ldots\ldots(12).$$

Now, if we put $\lambda = q^{-1}$ in (3), we get

$$\phi(q) = (1+q)(1+q^3)(1+q^5)\ldots$$

$$\times\left\{1+\frac{q^2}{(1+q)(1-q^2)}+\frac{q^6}{(1+q)(1-q^2)(1+q^3)(1-q^4)}+\ldots\right\},$$

so that

$$u_{-1}$$

$$=\Pi[1-\lambda(-q)^n]\left\{1+\frac{q^2}{(1-q)(1-q^2)}+\frac{q^6}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\ldots\right\} \quad\ldots\ldots\ldots(13).$$

By § 2, (9), and § 3, (2), we see that, if

$$a_0+a_1B_1(\theta)+a_2B_2(\theta)+\ldots \equiv b_0+2b_1\cos\theta+2b_2\cos 2\theta+\ldots,$$

then

$$a_0+a_2q^3+a_4q^8+a_6q^{15}+\ldots$$

$$= b_0-\frac{b_2}{q}-q^4\frac{b_4}{q^4}+q^4(1+q^4)\frac{b_6}{q^9}-q^8\frac{b_8}{q^{16}}-\ldots \quad\ldots\ldots(14).$$

Now, by the definition of $B_r(\theta)$,

$$(1+2q^{\frac{1}{2}}\cos\theta+q)(1+2q^{\frac{3}{2}}\cos\theta+q^3)\dots$$

$$=1+\frac{B_1(\theta)\,q^{-\frac{1}{2}}}{1-q}+\frac{B_2(\theta)\,q^{-1}}{(1-q)(1-q^2)}+\frac{B_3(\theta)\,q^{-\frac{3}{2}}}{(1-q)(1-q^2)(1-q^3)}+\dots$$

$$=\frac{1}{\Pi\,[1-q^n]}\,(1+2q^{\frac{1}{2}}\cos\theta+2q^2\cos 2\theta+2q^{\frac{9}{2}}\cos 3\theta+2q^8\cos 4\theta+\dots).$$

Hence, by (13), u_{-1} is the right-hand side of (14), where $b_{2r}=q^{3r^2}$.

Thus $\qquad u_{-1}=1-q-q^8+q^{13}(1+q^4)-q^{34}-q^{45}+q^{56}(1+q^8)- \quad \dots(15).$

This series may be systematically arranged, if we notice that by taking every fourth term we get powers of q whose indices are in hyper-arithmetic progression.

We then see that we may write the series in the form

$$1+q^{15.1^2}(q^{-2}+q^2)+q^{15.2^2}(q^{-4}+q^4)+\dots$$
$$-q\,\{1+q^{15.1^2}(q^{-8}+q^8)+q^{15.2^2}(q^{-16}+q^{16})+\dots\},$$

consisting of two Θ-series of the 15th order.

Changing q into $-q$, we arrive finally at the remarkable identity

$$\Pi\,[1-q^{2n}]\left\{1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\dots\right\}$$
$$=1-q^{15.1^2}(q^{-2}+q^2)+q^{15.2^2}(q^{-4}+q^4)-\dots$$
$$+q\,\{1-q^{15.1^2}(q^{-8}+q^8)+q^{15.2^2}(q^{-16}+q^{16})-\dots\}$$
$$=\Pi\,[1-q^{30n}]\left\{\begin{array}{l}(1-q^{13})(1-q^{17})(1-q^{43})(1-q^{47})\dots\\ \qquad +q\,(1-q^7)(1-q^{23})(1-q^{37})(1-q^{53})\dots\end{array}\right\},$$

and, remembering the product value of $\phi(q)$ obtained in § 7, we see that

$$\Pi\,[1-q^{2n}]\div\Pi\,[1-q^{30n}]\,\Pi\,[1-q^{5n\pm 1}]$$
$$=\quad(1-q^{13})(1-q^{17})(1-q^{43})(1-q^{47})(1-q^{73})(1-q^{77})\dots$$
$$+q\,(1-q^7)(1-q^{23})(1-q^{37})(1-q^{53})(1-q^{67})(1-q^{83})\dots \quad \dots(16).$$

We may, moreover, obtain from (10) a similar expression for v_1, after changing $-q^4$ into q,

$$v_1=1-q^{15.1^2}(q^{-4}+q^4)+q^{15.2^2}(q^{-8}+q^8)-\dots$$
$$+q^3\,\{1-q^{15.1^2}(q^{-14}+q^{14})+q^{15.2^2}(q^{-28}+q^{28})-\dots\},$$

whence $\Pi\,[1-q^{2n}] \div \Pi\,[1-q^{30n}]\,\Pi\,[1-q^{6n\pm 2}]$

$$= (1-q^{11})(1-q^{19})(1-q^{41})(1-q^{49})(1-q^{71})(1-q^{79})\dots$$
$$+q^{3}(1-q)(1-q^{29})(1-q^{31})(1-q^{59})(1-q^{61})(1-q^{89})\dots .$$

Again, since, by (9),

$$u+u_{-1} = \Pi\,[1-q^{8n}]\,\phi\,(q^{16}),$$

we have $\Pi\,[1-q^{8n}]\,\phi\,(q^{16})$

$$= 1+q^{60\cdot 1^2}(q^{-4}+q^{4})+q^{60\cdot 2^2}(q^{-8}+q^{8})$$
$$-q\,\{q^{15\cdot 1^2}(q^{-8}+q^{8})+q^{15\cdot 3^2}(q^{-24}+q^{24})$$

$$= \Pi\,[1-q^{120n}]\left\{\begin{array}{l}(1+q^{56})(1+q^{64})(1+q^{176})(1+q^{184})\dots \\ \quad -q^{16}(q^{-8}+q^{8})(1+q^{104})(1+q^{136})(1+q^{224})(1+q^{256})\dots\end{array}\right\};$$

therefore, changing q^8 into q,

$$\Pi\,[1-q^{n}]\,\phi(q^{2}) \div \Pi\,[1-q^{15n}] = (1-q)(1-q^{3})(1-q^{4})(1-q^{5})\dots(1-q^{N})\dots,$$

where N is any integer which is not a multiple of 15, or whose last digit is not 2 or 8,

$$= (1+q^{7})(1+q^{8})(1+q^{22})(1+q^{23})\dots$$
$$-q\,(1+q^{2})(1+q^{13})(1+q^{17})(1+q^{28})(1+q^{32})\dots .$$

Similarly, from (11),

$$\Pi\,[1-q^{n}]\,\psi\,(q^{2}) \div \Pi\,(1-q^{15n}] = \Pi\,[1-q^{N}],$$

where N is not a multiple of 15, and does not end with a 4 or 6,

$$= (1+q^{4})(1+q^{11})(1+q^{19})(1+q^{26})\dots$$
$$-q\,(1+q)(1+q^{14})(1+q^{16})(1+q^{29})\dots .$$

7. We have seen that, if

$$a_0 \pm a_1 A_1(\theta) + a_2 A_2(\theta) \pm \dots = b_0 \pm 2b_1 \cos\theta + 2b_2 \cos 2\theta \pm \dots \quad \dots(1),$$

then a_0, a_1, ... can separately be expanded in § 1 in series containing b's with simple coefficients. Moreover the series

$$a_0+a_1+a_2+\dots \quad \text{and} \quad a_0+a_1 q^{\frac{1}{4}}+a_2 q+\dots$$

have been similarly expanded in § 2.

These identities are only a few of a very large number of relations connecting simple series in the a's with simple series in the b's, which will be established in the subsequent sections of this memoir.

These can all be treated in the manner of § 6, (14), where by § 3 (2), we see that, if

$$a_0+m_1a_1+m_2a_2+\ldots = b_0+n_1b_1+n_2b_2+\ldots \qquad \ldots\ldots\ldots(2),$$

then $$a_0+m_2q^2a_2+m_4q^6a_4+\ldots = b_0+n_2q^{-1}b_2+n_4q^{-4}b_4+\ldots$$

and $$m_1q^{\frac34}a_1+m_3q^{\frac{15}{4}}a_3+\ldots = n_1q^{-\frac14}b_1+n_3q^{-\frac94}b_3+\ldots,$$

which, applied to the Θ-function identity

$$\Pi[1-q^n]\left\{1+\frac{B_1(\theta)\,q^{-\frac14}}{1-q}+\frac{B_2(\theta)\,q^{-1}}{(1-q)(1-q^2)}+\ldots\right\}$$
$$= 1+2q^{\frac14}\cos\theta+2q\cos 2\theta+\ldots,$$

gives

$$\Pi[1-q^n]\left\{1\pm\frac{q^{\frac14}m_1}{1-q}+\frac{qm_2}{(1-q)(1-q^2)}\pm\frac{q^{\frac94}m_3}{(1-q)(1-q^2)(1-q^3)}+\ldots\right\}$$
$$= 1\pm q^{\frac14}n_1+qn_2\pm q^{\frac94}n_3+q^4n_4\pm\ldots \qquad \ldots\ldots\ldots\ldots\ldots(3).$$

Many of the relations obtained will only lead to well-known identities, and in such cases the application of this section will not be quoted.

8. We have seen in § 4, (2), that, if

$$1+\frac{B_1(\theta)\,\lambda}{1-q}+\frac{B_2(\theta)\,\lambda^2}{(1-q)(1-q^2)}+\ldots,$$

be expanded according to $A(\theta)$'s, the coefficient of $A_r(\theta)$ is

$$\Sigma\,\frac{\lambda^{r+2s}\,q^{\frac12r(r+1)+rs+s(s+1)}}{(1-q^r)!\,(1-q^s)!},$$

when s has all integral values from 0 to ∞.

Now $\frac12r(r+1)+rs+s(s+1) = \frac14r^2+\frac14(r+2s)(r+2s+2)$,

so that, if any power λ^m of λ be changed into $\lambda q^{-\frac14m(m+2)}$, the corresponding coefficient of $A_r(\theta)$ would be

$$\Sigma\,\frac{\lambda^{r+2s}\,q^{\frac14r^2}}{(1-q^r)!\,(1-q^s)!}.$$

This is precisely the same thing as saying that

$$1+\frac{B_1(\theta)\,q^{-\frac34}\lambda}{1-q}+\frac{B_2(\theta)\,q^{-2}\lambda^2}{(1-q)(1-q^2)}+\ldots$$
$$=\left\{1+\frac{A_1(\theta)\,q^{\frac14}\lambda}{1-q}+\frac{A_2(\theta)\,q\lambda^2}{(1-q)(1-q^2)}+\ldots\right\}$$
$$\times\left\{1+\frac{\lambda^2}{1-q}+\frac{\lambda^4}{(1-q)(1-q^2)}+\ldots\right\} \quad \ldots(1).$$

Now suppose that, in consequence of a relation

$$a_0+a_1A_1(\theta)+\ldots = \alpha_0+\alpha_1B_1(\theta)+\ldots \quad \ldots\ldots\ldots\ldots(2),$$

we can establish some relation of the form

$$a_0+a_1\lambda_1+a_2\lambda_2+\ldots = \alpha_0+\alpha_1n_1q^{\frac{1}{2}}+\alpha_2n_2q^2+\ldots \quad \ldots\ldots\ldots(3);$$

then (1) gives relations connecting the A's and B's by equating coefficients of powers of λ; (2) establishes connexions between a_0, a_1, ... and α_0, α_1, ... by equating coefficients of the A's; on substituting for the α's in (3), we get relations connecting λ_1, λ_2, ... with m_1, m_2.

It will be easy to see, however, that these are simply expressed by substituting λ_r for $A_r(\theta)$, and m_r for $B_r(\theta)\,q^{-\frac{1}{4}r(r+2)}$, so that

$$1+\frac{m_1\lambda}{1-q}+\frac{m_2\lambda^2}{(1-q)(1-q^2)}+\ldots$$

$$=\left\{1+\frac{\lambda_1q^{\frac{1}{2}}\lambda}{1-q}+\frac{\lambda_2q\lambda^2}{(1-q)(1-q^2)}+\ldots\right\}\left\{1+\frac{\lambda^2}{1-q}+\ldots\right\} \quad \ldots(4),$$

Now, in § 3, we have seen that, if

$$a_0+m_1a_1+m_2a_2+\ldots = b_0+n_1b_1+n_2b_2+\ldots \quad \ldots\ldots\ldots\ldots(5),$$

then $$a_0+m_1q^{\frac{1}{2}}a_1+m_2q^2a_2+\ldots = b_0+n_1q^{-\frac{1}{2}}b_1+n_2q^{-1}b_2+\ldots,$$

i.e., $$a_0+a_1\lambda_1+a_2\lambda_2+\ldots = b_0+n_1q^{-\frac{1}{2}}b_1+n_2q^{-1}b_2+\ldots \quad \ldots(6).$$

If, then, we know a relation of the form (5), we can by (4) obtain the coefficients λ_1, λ_2, ... which establish the equation (6), and *vice versâ*.

Example 1.—Let $\lambda_1=\lambda_2=\ldots=0$,

so that, by § 1, (12), we know the values of $n_1q^{-\frac{1}{2}}$, n_2q^{-1}, &c. Then substituting in (5) the values of the m's given by (4), we get

$$a_0+a_2(1-q^2)+a_4(1-q^3)(1-q^4)+a_6(1-q^4)(1-q^5)(1-q^6)+\ldots$$
$$= b_0-q(1+q)\,b_2+q^5(1+q^2)\,b_4-q^{12}(1+q^3)\,b_6+\ldots.$$

Example 2.—In § 7, (1), by putting $\theta=\frac{\pi}{2}$, we get

$$a_0-a_2(1-q)+a_4(1-q)(1-q^3)-\ldots = b_0-2b_2+2b_4-\ldots.$$

The present section gives

$$a_0+a_2(1-q)+a_4(1-q)(1-q^3)+\ldots = b_0-2qb_2+2q^4b_4-\ldots.$$

Example 3.—By putting $\pm 2\cos\theta = q^{-\frac12}+q^{\frac12}$ in § 7, (1), we get

$$a_0 \pm a_1 q^{-\frac12}(1+q^{\frac12}) + a_2 q^{-1}(1+q^{\frac12})(1+q) \pm \dots$$
$$= b_0 \pm b_1 q^{-\frac12}(1+q^{\frac12}) + b_2 q^{-1}(1+q) \pm \dots,$$

and the derived form is

$$a_0 \pm a_1 (1+q^{\frac12}) + a_2 (1+q^{\frac12})(1+q) \pm a_3 (1+q^{\frac12})(1+q)(1+q^{\frac32}) + \dots$$
$$= b_0 \pm b_1 (1+q^{\frac12}) + b_2 q^{\frac12}(1+q) \pm b_3 q^{\frac32}(1+q^{\frac32}) + \dots .$$

Example 4.—If $\qquad -2\cos 2\theta = q^{-1}+q,$

$$a_0 - a_2 q^{-1}(1-q) + a_4 q^{-2}(1-q)(1-q^3) - \dots$$
$$= b_0 - q^{-1}(1+q^2)\, b_2 + q^{-2}(1+q^4)\, b_4 - \dots,$$

and the derived form is

$$a_0 + a_2 q(1-q) + a_4 q^2 (1-q)(1-q^3) + \dots$$
$$= b_0 - (1+q^2)\, b_2 + q^2(1+q^4)\, b_4 - q^6(1+q^6)\, b_6 + \dots .$$

9. Quadratic transformation of q.

We have already seen, on pp. 175 and 343 of Vol. xxiv., that

$$\frac{(\mu\nu)}{P(\mu)\,P(\nu)} = 1 + H_1(\mu, \nu)\, A_1(\theta) + \dots,$$

from which relation, by putting $\nu = \mu q^{\frac12}$, and afterwards changing q into q^2, we have

$$\left\{ 1 - \frac{\mu^2 q}{1-q^2} + \frac{\mu^4 q^4}{(1-q^2)(1-q^4)} - \dots \right\} \left\{ 1 + \frac{A_1(\theta)\,\mu}{1-q} + \frac{A_2(\theta)\,\mu^2}{(1-q)(1-q^2)} + \dots \right\}$$
$$= 1 + \frac{A_1(\theta, q^2)}{1-q}\,\mu + \frac{A_2(\theta, q^2)}{(1-q)(1-q^2)}\,\mu^2 + \dots \qquad \dots\dots\dots\dots(1),$$

where $A_r(\theta, q^2)$ is what $A_r(\theta)$ becomes when q is changed into q^2.

By equating coefficients of powers of μ, we get $A_r(\theta, q^2)$ in terms of $A_r(\theta)$, $A_{r-2}(\theta)$ ... .

Now suppose that we have some function expanded according to both kinds of A, *i.e.*,

$$a_0 + a_1 A_1(\theta) + \dots = \gamma_0 + \gamma_1 A_1(\theta, q^2) + \dots \qquad \dots\dots\dots\dots(2).$$

Substituting from (1) in the right-hand side of (2), and equating coefficients of like A's, we get relations connecting the a's and γ's.

If this is written in the form

$$a_0+m_1a_1+m_2a_2+\ldots = \gamma_0+c_1\gamma_1+c_2\gamma_2+\ldots \qquad \ldots\ldots\ldots(3),$$

it will not be difficult to see, just as in the last section, that the m's and c's will be connected by the relation

$$1+\frac{c_1\mu}{1-q}+\frac{c_2\mu^2}{(1-q)(1-q^2)}+\ldots$$

$$=\left\{1+\frac{m_1\mu}{1-q}+\frac{m_2\mu^2}{(1-q)(1-q^2)}+\ldots\right\}\left\{1-\frac{\mu^2q}{1-q^2}+\frac{\mu^4q^4}{(1-q^2)(1-q^4)}-\ldots\right\}$$

$$\ldots\ldots\ldots\ldots(4).$$

It is evident, moreover, that if we know a relation connecting a series of γ's with a series of b's, we may change q to $q^{\frac{1}{2}}$ throughout, provided we change γ_r into a_r.

In this manner we may extend very considerably the number of relations connecting series of a's with series of b's. It will not be necessary to work them all out in detail, since the method of deduction is the same for all.

Example 1.—Let $\qquad m_r = q^{\frac{1}{2}r}$,

so that the right-hand side of (4) becomes

$$\frac{(1-\mu^2q)(1-\mu^2q^3)(1-\mu^2q^5)\ldots}{(1-\mu q^{\frac{1}{2}})(1-\mu q^{\frac{3}{2}})(1-\mu q^{\frac{5}{2}})},$$

which
$$=(1+\mu q^{\frac{1}{2}})(1+\mu q^{\frac{3}{2}})\ldots$$

$$=1+\frac{\mu q^{\frac{1}{2}}}{1-q}+\frac{\mu^2q^2}{(1-q)(1-q^2)}+\ldots.$$

Hence $\quad a_0+a_1q^{\frac{1}{2}}+a_2q+a_3q^{\frac{3}{2}}+a_4q^2+\ldots$

$$=\gamma_0+\gamma_1q^{\frac{1}{2}}+\gamma_2q^2+\gamma_3q^{\frac{9}{2}}+\gamma_4q^8+\ldots.$$

But since the left-hand side has been given in § 2, (8), and changing q into $q^{\frac{1}{2}}$, and γ into a, we have

$$a_0+a_1q^{\frac{1}{2}}+a_2q+a_3q^{\frac{3}{2}}+a_4q^4+\ldots$$

$$=b_0+b_1q^{\frac{1}{2}}-b_2-q^{\frac{1}{2}}(1+q)\,b_3-q^2b_4+q^{\frac{5}{2}}b_5+\ldots,$$

where, of course, the series of terms with even suffixes are equal, and those with odd.

Example 2.—Let $$m_r = 0;$$

then $a_0 = \gamma_0 - q(1-q)\gamma_2 + q^4(1-q)(1-q^3)\gamma_4 - \dots$

$$= b_0 - (1+q)b_2 + q(1+q^2)b_4 - q^3(1+q^3)b_6 - \dots, \text{ by § 1, (12)};$$

therefore $a_0 - q^{\frac12}(1-q^{\frac12})a_2 + q^2(1-q^{\frac12})(1-q^{\frac32})a_4 - \dots$

$$= b_0 - (1+q^{\frac12})b_2 + q^{\frac12}(1+q)b_4 - q^{\frac32}(1+q^{\frac32})b_6 + \dots .$$

By § 7, (3), we get the identity

$$\Pi[1-q^n]\left\{1 - \frac{q^{\frac32}(1-q^{\frac12})}{(1-q)(1-q^2)} + \frac{q^6(1-q^{\frac12})(1-q^{\frac32})}{(1-q)(1-q^2)(1-q^3)(1-q^4)} - \dots\right\}$$

$$= \text{the } \Theta\text{-function } 1 - q(1+q^{\frac12}) + q^{\frac72}(1+q) - \dots .$$

Example 3. $a_0 + a_2(1-q) + a_4(1-q)(1-q^3) + \dots$

$$= \gamma_0 + \gamma_2(1-q)^2 + \gamma_4(1-q)^2(1-q^3)^2 + \dots$$

$$= b_0 - 2qb_2 + 2q^4b_4 - \dots, \text{ by the preceding section};$$

therefore $a_0 + a_2(1-q^{\frac12})^2 + a_4(1-q^{\frac12})^2(1-q^{\frac32})^2 + \dots$

$$= b_0 - 2q^{\frac12}b_2 + 2q^2b_4 - \dots,$$

whence, by § 7,

$$\Pi[1-q^n]\left\{1 + \frac{(1-q^{\frac12})^2q}{(1-q)(1-q^2)} + \frac{(1-q^{\frac12})^2(1-q^{\frac32})^2q^4}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \dots\right\}$$

$$= 1 - 2q^{\frac32} + 2q^6 - \dots .$$

Example 4. $a_0 + a_2(1-q^2) + a_4(1-q^3)(1-q^4) + \dots$

$$= \gamma_0 + \gamma_2(1-q) + \gamma_4(1-q)(1-q^3) + \dots$$

$$= b_0 - q(1+q)b_2 + q^5(1+q^2)b_4 - \dots, \text{ by § 9, Ex. 1};$$

therefore $a_0 + a_2(1-q^{\frac12}) + a_4(1-q^{\frac12})(1-q^{\frac32}) + \dots$

$$= b_0 - q^{\frac12}(1+q^{\frac12})b_2 + q^{\frac52}(1+q)b_4 - q^3(1+q^{\frac32})b_6 + \dots .$$

By § 7, this gives

$$\Pi[1-q^n]\left\{1 + \frac{q(1-q^{\frac12})}{(1-q)(1-q^2)} + \frac{q^4(1-q^{\frac12})(1-q^{\frac32})}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \dots\right\}$$

$$= 1 - q^{\frac52}(1+q^{\frac12}) + q^{\frac{11}{2}}(1+q) - q^{15}(1+q^{\frac32}) + \dots$$

$$= \Pi[1-q^{\frac72 n}]\,\Pi[1-q^{\frac12(7n\pm3)}].$$

Hitherto we have deduced results by changing q^2 into q. We may derive other identities by assuming values for the c's in § 4.

Example 5.—Let $$c_r = 0;$$

then $$1+\frac{m_1\mu}{1-q}+\ldots = 1 \div (1-\mu^2 q)(1-\mu^2 q^3)\ldots$$
$$= 1+\frac{q\mu^2}{1-q^2}+\frac{q^2\mu^4}{(1-q^2)(1-q^4)}+\ldots;$$

therefore $$\gamma_0 = a_0 + a_2 q\,(1-q) + a_4 q^2\,(1-q)(1-q^3)+\ldots$$
$$= b_0 - b_2\,(1+q^2) + b_4 q^2\,(1+q^4) - b_6 q^6\,(1+q^6)+\ldots,$$

by changing q into q^2, and a_0 into γ_0 in § 1, (12).

This result agrees with § 8, Ex. 4.

10. Second quadratic transformation of q.

By putting $r = -\mu$ in the identity quoted at the beginning of the last section, we get a relation

$$\left\{1+\frac{A_2(\theta)}{1-q^2}\mu^2+\frac{A_4(\theta)}{(1-q^2)(1-q^4)}\mu^4+\ldots\right\}$$
$$\times\left\{1-\frac{\mu^2}{1-q}+\frac{\mu^4}{(1-q)(1-q^3)}-\ldots\right\}$$
$$= 1+\frac{A_2(2\theta, q^2)}{1-q^2}\mu^2+\frac{A_4(2\theta, q^2)}{(1-q^2)(1-q^4)}\mu^4+\ldots.$$

As in the preceding sections, we see that, if in consequence of a known equation

$$a_0 + a_2 A_2(\theta) + a_4 A_4(\theta)+\ldots = e_0 + e_1 A_1(2\theta, q^2) + e_2 A_2(2\theta, q^2)+\ldots,$$

we can derive a relation

$$a_0 + a_2 m_2 + a_4 m_4 + \ldots = e_0 + e_1 k_1 + e_2 k_2 + \ldots,$$

then $$1+\frac{k_1\mu^2}{1-q^2}+\frac{k_2\mu^4}{(1-q^2)(1-q^4)}+\ldots$$
$$=\left\{1+\frac{m_2\mu^2}{1-q^2}+\frac{m_4\mu^4}{(1-q^2)(1-q^4)}+\ldots\right\}$$
$$\times\left\{1-\frac{\mu^2}{1-q}+\frac{\mu^4}{(1-q)(1-q^3)}-\ldots\right\} \quad \ldots(1).$$

Example 1.—Let $$k_r = 0;$$

$$e_0 = a_0 + a_2\,(1+q) + a_4 q\,(1+q)(1+q^3)+\ldots.$$

But the expansion of e_0 in terms of the b's is evidently obtained by changing q into q^2, and b_r into b_{2r} in the expansion of a_0; therefore

$$a_0+a_2(1+q)+a_4q(1+q)(1+q^2)+a_6q^3(1+q)(1+q^2)(1+q^3)+\dots$$
$$=b_0-b_4(1+q^2)+b_8q^2(1+q^4)-b_{12}q^6(1+q^6)-\dots.$$

By § 7, we get

$$\Pi[1-q^n]\left\{1+\frac{q(1+q)}{(1-q)(1-q^2)}+\frac{q^5(1+q)(1+q^2)}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}$$
$$=\text{the }\Theta\text{-function }1-q^4(1+q^2)+q^{16}(1+q^4)-\dots.$$

Example 2.—From Ex. 1, § 8, we get

$$a_0+a_2(1+q)+a_4(1+q)(1+q^2)+\dots$$
$$=b_0-b_4q^2(1+q^2)+b_8q^{10}(1+q^4)-\dots;$$

whence

$$\Pi[1-q^n]\left\{1+\frac{q(1+q)}{(1-q)(1-q^2)}+\frac{q^4(1+q)(1+q^2)}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}$$
$$=\text{the }\Theta\text{-function }1-q^7(q^{-1}+q)+q^{28}(q^{-2}+q^2)-\dots.$$

Example 3.—Let $k_1=-q,\quad k_2=q^3,\quad k_3=-q^6\dots;$

then $$a_0+a_2+a_4q^2+a_6q^6+a_8q^{12}+\dots$$
$$=e_0-e_1q+e_2q^3-e_3q^6+\dots$$
$$=b_0+qb_2-b_4-q(1+q^4)b_6-q^8b_8+\dots,\text{ by § 2, (8).}$$

Similarly, $$a_0+a_2q+a_4q^4+\dots$$
$$=e_0-e_1+e_2-e_3+\dots$$
$$=b_0-b_2-q^2b_4+q^2(1+q^2)b_6-\dots,\text{ as in § 9, Ex. 1.}$$

Example 4.—Let $m_2=1+q,\quad m_4=(1+q)(1+q^3),\ \dots;$

then, since

$$1+\frac{1+q}{1+q^2}\mu^2+\frac{(1+q)(1+q^3)}{(1-q^2)(1-q^4)}\mu^4+\dots=\frac{(1+\mu^2q)(1+\mu^2q^3)\dots}{(1-\mu^2)(1-\mu^2q^2)\dots},$$

we have $$1+\frac{k_1\mu^2}{1-q^2}+\dots=1\div(1-\mu^4)(1-\mu^4q^4)\dots;$$

therefore $$a_0+a_2(1+q)+a_4(1+q)(1+q^3)+\dots$$
$$=e_0+e_2(1-q^2)+e_4(1-q^2)(1-q^6)+\dots$$
$$=b_0-2q^2b_4+2q^8b_8-\dots,\text{ by § 8, Ex. 2.}$$

Example 5.—Let $m_r = 0,$

$$e_0-(1+q)\,e_1+(1+q)(1+q^2)\,e_2-\dots$$
$$= a_0 = b_0-(1+q)\,b_2+q\,(1+q^2)\,b_4-\dots, \text{ by § 1, (12)};$$

therefore
$$a_0-(1+q^{\frac{1}{2}})\,a_1+(1+q^{\frac{1}{2}})(1+q)\,a_2-\dots$$
$$= b_0-(1+q^{\frac{1}{2}})\,b_1+q^{\frac{1}{2}}\,(1+q)\,b_2-q^{\frac{3}{2}}\,(1+q^{\frac{3}{2}})\,b_3-\dots.$$

11. Cubic transformation of q.

The identity

$$\frac{(\lambda\mu)(\mu\nu)(\nu\lambda)}{P(\lambda)\,P(\mu)\,P(\nu)} = 1+\Sigma H_r\,(\lambda\mu\nu/\lambda,\,\mu,\,\nu)\,A_r(\theta),$$

given in Vol. XXIV., p. 346, when

$$\mu = \lambda q^{\frac{1}{3}},\quad \nu = \lambda q^{\frac{2}{3}},$$

and q is changed into q^3, becomes

$$1+\frac{A_1(\theta)}{1-q}\lambda+\frac{A_2(\theta)}{(1-q)(1-q^2)}\lambda^2+\dots$$
$$=\left\{1+\frac{\lambda^3 q}{1-q}+\frac{\lambda^6 q^2}{(1-q)(1-q^2)}+\dots\right\}\{1+\Sigma H_r\,.\,A_r(\theta,\,q^3)\},$$

where
$$1+\Sigma H_r x^r \equiv \frac{(1-\lambda^3 q^3 x)(1-\lambda^3 q^6 x)\dots}{(1-\lambda x)(1-\lambda q x)(1-\lambda q^2 x)\dots}.$$

If, then, $a_0+a_1A_1(\theta)+\dots = f_0+f_1A_1(\theta,\,q^3)+\dots,$

we see that
$$a_0+a_2 q\,(1-q^2)+a_4 q^2\,(1-q^3)(1-q^4)+\dots$$
$$= b_0-(1+q^3)\,b_2+q^3\,(1+q^6)\,b_4-q^9\,(1+q^9)\,b_6+\dots.$$

Again, by § 9, we get

$$a_0+a_2 q\,(1-q^2)+a_4 q^2\,(1-q^3)(1-q^4)+\dots$$
$$= \gamma_0+\gamma_2 q^2\,(1-q)+\gamma_4 q^4\,(1-q)(1-q^3)+\dots,$$

so that
$$a_0+a_2 q\,(1-q^{\frac{1}{3}})+a_4 q^2\,(1-q^{\frac{1}{3}})(1-q^{\frac{2}{3}})+\dots$$
$$= b_0-(1+q^{\frac{2}{3}})\,b_2+q^{\frac{2}{3}}\,(1+q^2)\,b_4-q^{\frac{4}{3}}\,(1+q^{\frac{4}{3}})\,b_6+\dots,$$

and, by § 7,

$$\Pi[1-q^n]\left\{1+\frac{q^2(1-q^{\frac{1}{3}})}{(1-q)(1-q^2)}+\frac{q^6(1-q^{\frac{1}{3}})(1-q^{\frac{2}{3}})}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\dots\right\}$$
$$= 1-q-q^{\frac{4}{3}}+q^{\frac{11}{3}}+q^{\frac{17}{3}}-\dots$$
$$= \Pi\,[1-q^{\frac{1}{3}n}]\,\Pi\,[1-q^{\frac{1}{3}(7n\pm 2)}] \quad (\textit{cf.}\ \text{Ex. 4, § 9}).$$

The foregoing examples will illustrate the great fertility of the method employed for deducing identities which are difficult to prove by other means. It may be noticed that, when all the b's are equated to unity, the expression for a_0 vanishes identically. The equation § 1, (4) would lead us to infer that a_1 would also vanish identically on the same supposition, as indeed is obvious from § 1, (13). Similarly, it may be shown that all the a's vanish identically when the b's are equated to unity. Consistently with this fact, it will be then seen that, if in any relation connecting an a-series with a b-series the coefficients of the a's form a convergent series, then the b-series vanishes identically, as in § 2, (9), § 8, Ex. 4, &c.; but, if the b-series does not vanish identically, then the coefficients of the a's form a divergent series, as in § 2, (7), § 8, Ex. 1, 2, 3, &c.

On Regular Difference Terms. By A. B. Kempe, M.A., F.R.S.

Read and Received April 12th, 1894.

1. Let $\alpha, \beta, \gamma, \ldots$ be a system S_n of n quantities, which may be termed *roots;* and let w differences $\alpha-\beta$, $\alpha-\beta$; $\beta-\gamma$, $\alpha-\gamma$; &c., be formed with these, each root entering into v of the differences. Then the product of these w differences will be called a *regular difference term* of the system S_n, and will be said to be of *degree n*, *order v*, and *weight w.*

2. The expression

$$(\alpha-\beta)^2(\beta-\gamma)(\gamma-\delta)^2(\delta-\alpha)$$

affords an example of a regular difference term of degree 4, order 3, and weight 6.

3. We may have difference terms into which the different roots do not all enter the same number of times; such difference terms are, however, *irregular.* A difference term will be irregular although each of the roots which enters into it enters the same number of times as the others, provided that there are other roots of the system under consideration which do not enter at all. Such a difference term will,

however, be a regular difference term of the reduced system which consists only of the roots which do enter into the term.

4. Where the degree of a regular difference term is even, the order may be as low as unity; but, where the degree is odd, the order cannot be less than 2, for we have

$$vn = 2w,$$

and thus both degree and order cannot be odd.

Regular difference terms of even degree and order 1, or of odd degree and order 2, will be called *elemental terms* of the system of roots considered. Elemental terms of order 1 may be called *linear elements*, and those of order 2 *quadratic elements*.

5. The product of two or more regular difference terms of S_n will, of course, be also a regular difference term of S_n, and its order will be the sum of the orders of the factors. A given regular difference term of S_n may therefore be such as to admit of being expressed as the product of two or more regular difference terms of S_n of lower order; but, on the other hand, it may not be so expressible. Thus

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)(\beta-\zeta)(\gamma-\eta)(\gamma-\theta)(\delta-\iota)(\delta-\kappa) \times(\epsilon-\zeta)^2(\eta-\theta)^2(\iota-\kappa)^2,$$

a regular difference term of degree 10, order 3, and weight 15, of the system of roots

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \theta,\ \iota,\ \kappa,$$

does not admit of being expressed as the product of regular difference terms of the same system of lower order.

6. Regular difference terms which admit of being expressed as the product of others of the same system, of lower orders, may be said to be *decomposable*.

7. Regular difference terms which are not decomposable may be said to be *primitive*. Elemental terms are, of course, primitive.

8. Regular difference terms which are so completely decomposable that they can be expressed as the product of elemental terms may be designated *pure composite terms*.

9. It is known that, for a given system of n roots, the number of

primitive regular difference terms is limited;* so that every regular difference term is either one of this limited number of primitive terms, or is the product of two or more of these, or of their powers. Some progress has also been made towards the specification of the orders and forms of primitive terms.†

10. It does not, however, appear to have been hitherto observed that *every regular difference term*, whether decomposable or primitive, *of a system of roots S_n, can be expressed* as the sum of pure composite terms of S_n, and therefore *as a rational integral function of elemental terms of S_n.*

11. For example, the primitive regular difference term referred to in § 5, viz.:—

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)(\beta-\zeta)(\gamma-\eta)(\gamma-\theta)(\delta-\iota)(\delta-\kappa) \times (\epsilon-\zeta)^2(\eta-\theta)^2(\iota-\kappa)^2,$$

can be expressed in the form

$$\begin{aligned}
&\left[(\alpha-\beta)(\gamma-\theta)(\delta-\iota)(\epsilon-\kappa)(\zeta-\eta)\right] \\
&\qquad \times\left[(\alpha-\gamma)(\beta-\delta)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right] \\
&\qquad \times\left[(\alpha-\delta)(\beta-\gamma)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right] \\
&+\left[(\alpha-\beta)(\gamma-\zeta)(\delta-\iota)(\epsilon-\kappa)(\eta-\theta)\right] \\
&\qquad \times\left[(\alpha-\gamma)(\beta-\delta)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right] \\
&\qquad \times\left[(\alpha-\delta)(\beta-\eta)(\gamma-\theta)(\epsilon-\zeta)(\iota-\kappa)\right] \\
&+\left[(\alpha-\beta)(\gamma-\theta)(\delta-\epsilon)(\zeta-\eta)(\iota-\kappa)\right] \\
&\qquad \times\left[(\alpha-\gamma)(\beta-\kappa)(\delta-\iota)(\epsilon-\zeta)(\eta-\theta)\right] \\
&\qquad \times\left[(\alpha-\delta)(\beta-\gamma)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right] \\
&+\left[(\alpha-\beta)(\gamma-\zeta)(\delta-\epsilon)(\eta-\theta)(\iota-\kappa)\right] \\
&\qquad \times\left[(\alpha-\gamma)(\beta-\kappa)(\delta-\iota)(\epsilon-\zeta)(\eta-\theta)\right] \\
&\qquad \times\left[(\alpha-\delta)(\beta-\eta)(\gamma-\theta)(\epsilon-\zeta)(\iota-\kappa)\right],
\end{aligned}$$

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* See "Ueber die Endlichkeit des Invarientensystems fur binären Grundformen," by D. Hilbert, in the *Mathematische Annalen*, Vol. xxxiii., where the result is obtained by the aid of a theorem of Professor Gordan with regard to a class of diophantine equations.

† See "Die Theorie der Regularen Graphs," by Julius Petersen, in the *Acta Mathematica*, Vol. xv.

that is, as the sum of four pure composite terms, each of which is the product of three linear elemental terms.

12. The object of the present paper is to demonstrate the theorem of § 10. This theorem is one of some importance. Thus, to confine ourselves to one example, let Q_n be a quantic the roots of which are those of the system S_n; then, if T be any regular difference term of S_n, the expression

$$\Sigma T,$$

where the summation extends to all terms derivable from T by transpositions *inter se* of the n roots, is an invariant of Q_n, and every rational integral invariant of Q_n is a rational integral function of invariants of that form. Now, if T be expressible as the sum of pure composite terms of S_n, every rational integral invariant of Q_n is expressible as a rational integral function of invariants, such as

$$\Sigma E_1^{e_1} \,.\, E_2^{e_2} \,.\, E_3^{e_3}, \ldots,$$

where E_1, E_2, E_3, ... are all elemental terms of S_n, being linear or quadratic according as n is even or odd, and the summation, as before stated, is of all terms obtainable by transposition (not of the elemental terms E_1, E_2, &c., but) of the roots $\alpha, \beta, \gamma, \ldots$. From this result the proof by Hilbert's method (see foot-note § 9) of the finiteness of the number of the invariants of a quantic Q_n in terms of which the whole system of its invariants may be expressed follows immediately.

13. A regular difference term of degree 2 is of the form

$$(\alpha-\beta)^v,$$

and is clearly a pure composite term, for each factor $(\alpha-\beta)$ is a linear elemental term.

14. A regular difference term of degree 3 is of the form

$$(\alpha-\beta)^m (\beta-\gamma)^m (\gamma-\alpha)^m \equiv \left[(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\right]^m,$$

where $2m = v$, and is clearly also a pure composite term, for each factor $\left[(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\right]$ is a quadratic elemental term.

15. We have

$$(\alpha-\beta)(\gamma-\delta) \equiv (\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\gamma-\beta),$$

and similar identities in the case of any other four roots. By the use of these identities any difference term A' of a degree >3 can be expressed in a variety of ways as the sum

$$A'_1+A'_2+A'_3+\ldots$$

of a number of other difference terms. Each of the latter is derived from A' by transpositions of roots, and therefore the number of α's, of β's, &c., in each is the same as the number in A'. If, then, A' be a regular difference term, each of the terms A'_1, A'_2, ... must also be regular.

16. The transpositions of the roots referred to in the last section are not transpositions such as those referred to in § 12, viz.:—of the whole set of α's with the whole set of β's, and so on; but are transpositions of individual α's with individual β's, and so on. Thus to each α in A' there will correspond a definite α in A'_1, in A'_2, and so on. Consequently, if for any root α in A' we substitute a new root ξ, not one of those in A', and if we also substitute ξ for the corresponding α in each of the terms A'_1, A'_2, A'_3, ..., the identity of § 15 will be converted into another identity; and, if any term A'_s of the former identity breaks up into factors in any particular way, there will be a corresponding term in the new identity which will break up into factors in the same way, one of those factors containing a root ξ in place of a root α.

17. In precisely the same way, we may substitute ξ's for any number of α's in A', and thus obtain a new term A, and if we also substitute a root ξ for each of the corresponding α's in each of the terms A'_1, A'_2, A'_3, ..., and thus obtain corresponding terms A_1, A_2, A_3, ..., we shall have

$$A \equiv A_1+A_2+A_3+\ldots,$$

and, if A'_s breaks up into factors in any particular way, A_s will break up into factors in the same way, these factors, however, containing in some cases ξ's in lieu of certain of the α's. This result will be found of importance in the sequel (§ 41).

18. In the demonstration which follows it will be shown that every regular difference term of order v of a system S_n may, by the use of the identities of § 15, be expressed as the sum of certain regular difference terms of S_n of order v, designated *uncrossed terms;* that each of these uncrossed terms may, by the same means, be expressed as the sum of certain other regular difference terms of S_n of order v, called *reducible terms;* and that each of these reducible terms may be expressed as the sum of *pure composite terms* of order v, provided

that every regular difference term of the same order v of any system S_{n-2} of $(n-2)$ roots can be expressed as the sum of pure composite terms of S_{n-2}. Since, then, we know that regular difference terms of order v and of degree 2 or 3 can be expressed as the sum of pure composite terms, being, in fact, themselves pure composite terms (§§ 13, 14), it will follow that every regular difference term of order v and degree n of a system S_n can be expressed as the sum of pure composite terms, and therefore as a rational integral function of elemental terms of S_n.

19. For the purpose of the demonstration let the roots of S_n, taken in any order, be represented respectively by the symbols

$$[1],\ [2],\ [3],\ \ldots\ [n-1],\ [n];$$

and let it be supposed that

$$[n+r] = [r],$$

the roots being thus regarded as composing a cycle of period n. The numbers contained in these symbols may be termed the *places* of the roots they respectively represent.

20. In the case of any difference

$$\pm\{[s]-[r]\},$$

the number $(s-r)$ (which may, of course, be negative) may be termed the *distance* between the roots composing the difference.

21. A difference $\{[q]-[p]\}$

may be thrown into the equivalent forms

$$-\{[p]-[q]\},$$
$$\{[n+q]-[p]\},$$
$$-\{[m+p]-[q]\}.$$

Of these four forms, that will always be supposed to be employed in which the distance between the roots is positive and a minimum. Thus, where a difference

$$\{[s]-[r]\}$$

is considered, it is to be understood that

$$s > r,$$

and $$s-r \leqq (n+r)-s.$$

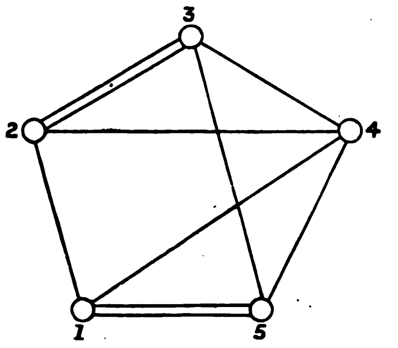
22. We may graphically represent the n roots of a regular difference term by small circular nuclei arranged at the angular points of a regular polygon of n sides, and numbered successively 1, 2, 3, ... n; and any difference

$$\{[s]-[r]\}$$

which is a factor of the term may then be represented by a line lying along a side or diagonal of the polygon connecting the angular points numbered s and r. Thus the regular difference term

$$\{[5]-[4]\}\{[4]-[3]\}\{[3]-[2]\}^2\{[2]-[1]\}\{[5]-[1]\}^2$$
$$\times\{[5]-[3]\}\{[4]-[2]\}\{[4]-[1]\}$$

may be represented by the regular graph



23. If p, q, r, s be four numbers such that

$$s > r > q > p,$$

the two differences

$$\{[s]-[q]\} \text{ and } \{[r]-[p]\}$$

may be said to *cross*, or to be *a crossed pair*; other pairs of differences,

e.g., $\{[s]-[r]\}$ and $\{[q]-[p]\}$,

or $\{[s]-[p]\}$ and $\{[r]-[q]\}$,

or $\{[s]-[r]\}$ and $\{[s]-[p]\}$,

being said to be *uncrossed*.

24. In the graphical representation two differences which cross will be represented by two lines lying along diagonals which intersect.

25. A difference term may be said to be *crossed* or *uncrossed* according as it does or does not contain any crossed pairs of factor differences.

26. By the aid of the identity of § 15, we may express the product

$$\{[s]-[q]\}\,.\,\{[r]-[p]\}$$

in the form

$$\{[s]-[r]\}\cdot\{[q]-[p]\}+\{[s]-[p]\}\cdot\{[r]-[q]\};$$

i.e., we may express the product of a crossed pair as the sum of two products of uncrossed pairs.

27. Consider now the identity

$$\begin{aligned} &\{[s]-[q]\}\cdot\{[r]-[p]\}\cdot\{[y]-[x]\} \\ \equiv\ &\{[s]-[r]\}\cdot\{[q]-[p]\}\cdot\{[y]-[x]\} \\ +&\{[s]-[p]\}\cdot\{[r]-[q]\}\cdot\{[y]-[x]\}. \end{aligned}$$

Here, if $\{[s]-[r]\}$ and $\{[y]-[x]\}$

are a crossed pair, we have either

$$y > s > x > r > q > p,$$

or $$s > y > r > x > q > p,$$

or $$s > y > r > q \geqq x > p,$$

or $$s > y > r > q > p \geqq x;$$

and therefore either

$$\{[s]-[q]\} \quad\text{and}\quad \{[y]-[x]\}$$

are a crossed pair, or

$$\{[r]-[p]\} \quad\text{and}\quad \{[y]-[x]\}$$

are so.

So, if $\{[q]-[p]\}$ and $\{[y]-[x]\}$

are a crossed pair, we have either

$$y \geqq s > r > q > x > p,$$

or $$s > y \geqq r > q > x > p,$$

or $$s > r > y > q > x > p,$$

or $$s > r > q > y > p > x;$$

and therefore, again, either

$$\{[s]-[q]\} \quad \text{and} \quad \{[y]-[x]\}$$

are a crossed pair, or

$$\{[r]-[p]\} \quad \text{and} \quad \{[y]-[x]\}$$

are so.

Furthermore, if both

$$\{[s]-[r]\} \quad \text{and} \quad \{[y]-[x]\},$$

and

$$\{[q]-[p]\} \quad \text{and} \quad \{[y]-[x]\},$$

are crossed pairs, we have

$$s > y > r > q > x > p,$$

and therefore both $\{[s]-[q]\}$ and $\{[y]-[x]\}$,

and $\{[r]-[p]\}$ and $\{[y]-[x]\}$,

are crossed pairs.

Since, then, the pair

$$\{[s]-[q]\} \quad \text{and} \quad \{[r]-[p]\}$$

are a crossed pair, and the pair

$$\{[s]-[r]\} \quad \text{and} \quad \{[q]-[p]\}$$

are not, we see that the number of crossed pairs formed by the three factor differences of

$$\{[s]-[q]\}\cdot\{[r]-[p]\}\cdot\{[y]-[x]\}$$

must be greater than the number of crossed pairs formed by the three factor differences of

$$\{[s]-[r]\}\cdot\{[q]-[p]\}\cdot\{[y]-[x]\}.$$

In precisely the same way we may prove that the number of crossed pairs formed by the three factor differences of

$$\{[s]-[q]\}\cdot\{[r]-[p]\}\cdot\{[y]-[x]\}$$

must be greater than the number of crossed pairs formed by the three factor differences of

$$\{[s]-[p]\}\cdot\{[r]-[q]\}\cdot\{[y]-[x]\}.$$

28. Suppose now that the regular difference term T of the system S_n contains as a factor a crossed pair

$$\{[s]-[q]\}\cdot\{[r]-[p]\},$$

so that we may put

$$\begin{aligned} T = & \; L\{[s]-[q]\}\cdot\{[r]-[p]\} \\ = & \; L\{[s]-[r]\}\cdot\{[q]-[p]\} \quad (=T_\lambda) \\ & +L\{[s]-[p]\}\cdot\{[r]-[q]\} \quad (=T_\mu) \\ = & \; T_\lambda+T_\mu; \end{aligned}$$

then, since L consists of factor differences, such as $\{[y]-[x]\}$, it follows immediately from § 27 that both T_λ and T_μ must contain a smaller number of crossed pairs than T.

Taking any crossed pair in T_λ, we may by the same process put

$$T_\lambda = T_\pi + T_\rho,$$

and, similarly, we may put $T_\mu = T_\sigma + T_\tau$,

and therefore $$T = T_\pi + T_\rho + T_\sigma + T_\tau,$$

where T_π and T_ρ contain a smaller number of crossed pairs than T_λ, and T_σ and T_τ a smaller number than T_μ.

Proceeding to deal with T_π, T_ρ, T_σ, and T_τ in the same way, and continuing the process on the derived terms, we shall at each stage obtain terms containing a smaller number of crossed pairs than are contained in the terms from which they are derived; and we can continue the process on the derived terms so long as we obtain terms which contain any crossed pairs. In this way we shall ultimately be able to put

$$T \equiv U_1 + U_2 + U_3 + \ldots,$$

where the terms on the right-hand side of the identity are all uncrossed regular difference terms of S_n, none of them containing any crossed pairs.

29. We proceed next to consider a special property possessed by any uncrossed regular difference term U of S_n; and to show that, by means of the identities of § 15, U can be expressed as the sum of certain other regular difference terms of S_n, designated *reducible terms*.

30. The differences under consideration may be divided nto two classes, viz.:—we have differences of the form

$$\{[r+1]-[r]\},$$

in which the distance between the roots is unity, and differences in which the distance is greater than unity.

In the graphical representation of a regular difference term, made in accordance with § 22, differences of the former description will be represented by lines lying along the sides of the polygon. Such differences may accordingly be called *side differences.*

Differences of the other sort will similarly be represented by lines lying along the diagonals of the polygon, and may therefore be called *diagonal differences.*

31. The special property of U which we have to consider is this—there must be one root of the system S_n which enters only into factor differences of U which are side differences, and does not enter into any which are diagonal. There must, in fact, be two such roots; but the existence of one is sufficient for our purposes.

In other words, there must be a root $[r]$ which enters only into differences

$$\{[r+1]-[r]\} \quad \text{and} \quad \{[r]-[r-1]\},$$

and consequently U contains a factor

$$\{[r+1]-[r]\}^k \cdot \{[r]-[r-1]\}^{v-k},$$

where the sum of the indices is v, the order of U.

32. The proof of this presents no difficulty. If there are any diagonal differences which are factors of U, there must be one or more in which the roots are at a minimum distance apart d, where

$$d \geqq 2.$$

Let $$\{[s+d]-[s]\}$$

be one of these. Then each of the roots

$$[s+1],\ [s+2],\ \ldots\ [s+d-1],$$

enters only into side differences. For, since U is uncrossed, any root $[s+c]$, where $c<d$ and >0, cannot enter into differences which cross

$$\{[s+d]-[s]\};$$

i.e., $[s+c]$ can only enter into differences such as

$$\{[s+c]-[e]\},$$

where $$e \overset{=}{>} s,$$

or such as $$\{[f]-[s+c]\},$$

where $$f \overset{=}{<} s+d.$$

In the former case the distance between the roots is

$$s+c-e,$$

which, since $c > d$ and $e \overset{=}{>} s$, must be $< d$.

In the latter case, the distance between the roots is

$$f-s-c,$$

which, since $$f \overset{=}{<} s+d,$$

must also be $< d$.

Thus $[s+c]$ can only enter into differences in which the distance between the roots is $< d$.

But the minimum distance between the roots in the case of the diagonal differences of U is d. Thus $[s+c]$ cannot enter into any diagonal differences, but only into side differences.

33. The result arrived at might also have been obtained from a consideration of the graphical representation of U, in which there will be no intersecting diagonals (§ 24); and from a recognition of the fact that, in a regular polygon in which there are no intersecting diagonals, there must be at least two summits from which no diagonals proceed.

34. Since, then, U contains a factor

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k},$$

we may put

$$U = C\{[r+1]-[r-1]\}^h \{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k},$$

where C has no factors containing $[r]$, and no factor

$$\{[r+1]-[r-1]\}.$$

35. Now, each root of S_n enters v times into U; thus there will be $(v-h-k)$ factor differences in C containing $[r+1]$, and $(k-h)$ containing $[r-1]$, and these differences will be distinct from each other.

Let the product of the former be denoted by C_{v-h-k}, and of the latter by C_{k-h}. Then we may put

$$U = D.C_{v-h-k}\,C_{k-h}\{[r+1]-[r-1]\}^h\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k},$$

where D is the product of factor differences which do not contain either $[r+1]$, $[r]$, or $[r-1]$. The number of these will be

$$\begin{aligned} & w-(v-h-k)-(k-h)-h-k-(v-k) \\ &= w-2v+h \\ &= \frac{vn}{2}-2v+h \quad (\S\,4) \\ &= \frac{v}{2}(n-4)+h. \end{aligned}$$

Now, in cases where $n>3$, and it is with such that we are now dealing (§ 15), this last number must be at least h, *i.e.*, there are as many factors in D as in

$$\{[r+1]-[r-1]\}^h.$$

36. Let $$\{[b]-[a]\}$$

be any factor of D; then, by the identity of § 15, we have

$$\begin{aligned} &\{[r+1]-[r-1]\}.\{[b]-[a]\} \\ &\equiv \{[r+1]-[b]\}.\{[r-1]-[a]\} \\ &\quad + \{[r+1]-[a]\}.\{[b]-[r-1]\}; \end{aligned}$$

i.e., since $[a]$ and $[b]$ are both different from $[r+1]$ and $[r-1]$, D being the product of factor differences which contain neither of the latter roots (§ 35), we can express

$$\{[r+1]-[r-1]\}\{[b]-[a]\}$$

as the sum of two terms which do not contain

$$\{[r+1]-[r-1]\}$$

as a factor.

In the same way, by taking h factor differences of D, we can express each of the h factors of

$$\{[r+1]-[r-1]\}^h,$$

when multiplied by one of those factors of D, as the sum of two terms which do not contain

$$\{[r+1]-[r-1]\}.$$

as a factor, and thus we can express

$$D\{[r+1]-[r-1]\}^k$$

as the sum of terms which do not contain

$$\{[r+1]-[r-1]\}$$

as a factor.

37. We may therefore put

$$U\equiv\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k}\{R_1+R_2+R_3+\ldots\},$$

where none of the terms R_1, R_2, R_3, ... contain $[r]$ or the factor

$$\{[r+1]-[r-1]\}.$$

38. Any regular difference term

$$\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k}R,$$

where R does not contain

$$\{[r+1]-[r-1]\}$$

as a factor, and does not contain $[r]$, may be said to be a ***reducible*** regular difference term of the system of roots S_n.

39. Each of the roots of S_n other than $[r-1]$, $[r]$, and $[r+1]$ enters v times into R; the root $[r]$ does not enter at all; the root $[r-1]$ enters k times, and the root $[r+1]$ enters $(v-k)$ times. If, then, we were in R to put the root $[r+1]$ in place of the root $[r-1]$, $[r+1]$ would enter v times, and R would become a regular difference term R', of order v and degree $(n-2)$, of the system of roots S_{n-2} obtained by withdrawing the roots $[r]$ and $[r-1]$ from the system S_n.

40. Consequently the difference term R may be obtained by substituting for k properly selected roots $[r+1]$ in R', k roots $[r-1]$, where $[r-1]$ is not a root which enters into R'.

41. Suppose, then, that any regular difference term R' of order v of a system S_{n-2} of $(n-2)$ roots can be expressed as the sum

$$R'_1+R'_2+R'_3+\ldots$$

of a number of pure composite terms of that system. It follows immediately from § 17 that we can put

$$R\equiv R_1+R_2+R_3+\ldots,$$

where R is obtained by substituting k roots $[r-1]$ for k properly selected roots $[r+1]$ in R', and R_1, R_2, R_3, ... are obtained by making corresponding substitutions in R'_1, R'_2, R'_3, ... It also follows that R_1, R_2, R_3, ... break up into factors corresponding to the elemental factors of the pure composite terms R'_1, R'_2, R'_3, ...

42. Taking first the case where n is even, and therefore $n-2$ is even, and consequently the elemental factors of any term R'_s are linear (§ 4) and v in number, we see that R_s, the corresponding term, breaks up into v factors, into each of which each of the roots of S_{n-2} other than $[r+1]$ enters once and once only. Into k of these factors $[r+1]$ will not enter, but $[r-1]$ will do so in the case of each once and once only; while $[r-1]$ will not enter into the remaining $v-k$, but $[r+1]$ will do so in the case of each once and once only.

Now, considering the term

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s,$$

we see that, corresponding to each of the k factors of R_s into which $[r-1]$ enters, there is a factor

$$\{[r+1]-[r]\},$$

and, corresponding to each of the $v-k$ factors of R_s into which $[r+1]$ enters, there is a factor

$$\{[r]-[r-1]\},$$

and in each case the product of the two corresponding factors gives a term into which each of the n roots of S_n enters once and once only, *i.e.*, gives a linear elemental term of S_n.

Thus $$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s$$

is the product of v linear elemental terms of S_n.

Hence, provided that R'_s is a pure composite term of the system S_{n-2},

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s$$

will be a pure composite term of the system S_n; and, consequently, provided that R' can be expressed as the sum of terms such as R'_s, which are pure composite terms of S_{n-2}, R can be expressed as the sum of terms such as

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s,$$

which are pure composite terms of S_n.

43. We can arrive at the same result in the case where n, and therefore also $n-2$, is odd. Here the elemental factors of R'_r are quadratic, and are $\frac{v}{2}$ in number, v being necessarily even, (§ 4). Consequently R_r breaks up into $\frac{v}{2}$ factors, into each of which each of the roots of S_{n-2} other than $[r+1]$ enters twice and twice only. Into f of these factors (where f is zero or some integer not $> \frac{v}{2}$, and such that $k-f$, and therefore also $v-k-f$, is even) the roots $[r-1]$ and $[r+1]$ both enter once and once only; into $\frac{k-f}{2}$ of the remaining factors $[r-1]$ enters twice and twice only, and $[r+1]$ does not enter at all; and into the rest of the factors, $\frac{(v-k-f)}{2}$ in number, $[r+1]$ enters twice, and twice only, and $[r-1]$ does not enter at all.

Hence, considering the term

$$\{[r+1]-[r]\}^k\,\{[r]-[r-1]\}^{v-k}R_r,$$

we see that, corresponding to each of the f factors of the first description, we may take a factor

$$\{[r+1]-[r]\}\cdot\{[r]-[r-1]\};$$

corresponding to each of the $\frac{(k-f)}{2}$ factors of the second description, we may take a factor

$$\{[r+1]-[r]\}^2;$$

and, corresponding to each of the $\frac{(v-k-f)}{2}$ factors of the third description, we may take a factor

$$\{[r]-[r-1]\}^2;$$

and in so doing we shall exactly take all the factors of

$$\{[r+1]-[r]^k\,\{[r]-[r-1]\}^{v-k}.$$

In each case the product of the two corresponding factors gives a term into which each of the n roots of S_n enters twice and twice only, *i.e.*, gives a quadratic elemental term of S_n.

Thus $$\{[r+1]-[r]\}^k\,\{[r]-[r-1]\}^{v-k}R_r$$

is the product of $\frac{v}{2}$ quadratic elemental terms of S_n, and is therefore a pure composite term of S_n.

Consequently, as in the preceding section, we see that, provided that R'_s is a pure composite term of the system S_{n-2},

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s$$

will be a pure composite term of S_n, and therefore, provided that R' can be expressed as the sum of terms such as R'_s, which are pure composite terms of S_{n-2}, R can be expressed as the sum of terms such as

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s,$$

which are pure composite terms of S_n.

44. Whether, then, n be even or odd,

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R,$$

i.e., any reducible term of S_n, can be expressed as the sum of terms which are pure composite terms of S_n, provided that R' can be expressed as the sum of terms which are pure composite terms of S_{n-2}.

45. Hence the general regular difference term T of S_n of order v, being expressible as the sum of uncrossed terms of S_n (§ 28), which uncrossed terms are expressible as the sum of reducible terms of S_n (§ 37), can be expressed as the sum of pure composite terms of S_n, provided that any regular difference term of S_{n-2} of order v can be expressed as the sum of pure composite terms of S_{n-2}.

46. In other words, any regular difference term of order v and degree n can be expressed as a rational integral function of elemental terms of the system of roots to which it belongs, provided that any regular difference term of order v and degree $n-2$ can be so expressed. But we have seen (§§ 13, 14) that regular difference terms of order v and of degrees 2 and 3 can be so expressed. Therefore every regular difference term can be expressed as a rational integral function of the elemental terms of the system of roots to which it belongs.

47. We may carry the matter a step further. If we represent the roots, as hitherto, by the symbols [1], [2], [3], ... [n], the elemental terms may be divided into crossed and uncrossed terms. Now, every crossed elemental term may be expressed as the sum of uncrossed elemental terms (§ 28). Hence *every regular difference term can be expressed as a rational integral function of the uncrossed elemental terms of the system of roots*

$$[1],\ [2],\ \dots\ [n]$$

to which it belongs.

Thursday, May 10th, 1894.

Prof. GREENHILL, F.R.S., Vice-President, in the Chair.

The following communications were made:—

On the Kinematical Discrimination of the Euclidean and non-Euclidean Geometries: Mr. A. E. H. Love.

Permutations on a Regular Polygon: Major MacMahon.

The Stability of a Tube: Professor Greenhill (Dr. J. Larmor in the Chair).

Researches in the Calculus of Variations—Part V., The Discrimination of Maxima and Minima Values of Integrals with Arbitrary Values of the Limiting Variations; Part VI., The Theory of Discontinuous or Compounded Solutions: Mr. E. P. Culverwell.

The following present was made for the Album:—cabinet likeness of Mr. E. P. Culverwell.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LV., No. 332.

"Philosophical Transactions of the Royal Society," Vols. 180–184, 1889–1893, and a list of Fellows of the Society, dated November 30th, 1893.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 4; Leipzig, 1894.

"Seventh Annual Report of the Canadian Institute"; Toronto, 1894.

"Bulletin of the New York Mathematical Society," Vol. III., No. 7.

"Bulletin des Sciences Mathématiques," Tome XVIII., Fév. and Mars, 1894; Paris, 1894.

Macfarlane, Alex.—"Principles of Elliptic and Hyperbolic Analysis," 8vo; Boston.

"Transactions of the Russian Mathematical Society," Tome XV.; Odessa, 1893.

"Transactions of the Canadian Institute," No. 7, Vol. IV., Pt. I., March, 1894; Toronto.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 7, 1 Sem.; Roma, 1894.

"Annali di Matematico," Serie 2, Tomo XXII., Fasc. 1 and 2, April, 1894; Milano.

"Educational Times," May, 1894.

"Annales de la Faculté des Sciences de Toulouse," Tome VIII., Fasc. 1; Paris, 1894.

"Journal für die reine und angewandte Mathematik," Bd. CXIII., Heft 2; Berlin, 1894.

"Annals of Mathematics," Vol. VIII., No. 4; Virginia University, May, 1894.

"Indian Engineering," Vol. xv., Nos. 12–15.
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Wright, J. M. F.—"Commentary on Newton's 'Principia,'" 2 vols., 8vo; London, 1828.
Brougham, Henry Lord, and E. J. Routh.—"Analytical View of Newton's 'Principia,'" 8vo; London, 1856.
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Byerly, W. E.—"Fourier's Series and Spherical Harmonics," 8vo; Boston, 1893.

Researches in the Calculus of Variations—Part V., The Discrimination of Maxima and Minima Values of Integrals with Arbitrary Values of the Limiting Variations.

By E. P. Culverwell, M.A., F.T.C.D. Received May 8th, 1894. Read May 10th, 1894.

1. Discussions of true maxima and minima of integrals with variable limits, as distinguished from merely stationary solutions, are rare in the standard text-books. Moigno has none; Jellett, Todhunter, and Carll have each obtained different and erroneous results in the one example they all give, that of the maximum solid of revolution for given superficial area (see Jellett, *Cal. of Var.*, pp. 161–165; Todhunter, *History of Cal. of Var.*, p. 408; Carll, *Cal. of Var.*, pp. 122 and 129). The only other problem with variable limits I can find attempted in those text-books is one selected by Mr. Todhunter in his *History*, p. 328, in order to show that the ordinary method is insufficient when the limits themselves enter into the quantity to be integrated. Mr. Carll adopts Mr. Todhunter's view, insisting even more strongly on the inadequacy of the ordinary method. But the ordinary method, though clumsy, is in every case adequate.

The absence of examples is doubtless due to the fact that writers on the calculus of variations have considered the variability of the constants as introducing only a problem of the differential calculus, and have contented themselves by saying that, if the stationary value of the integral be expressed in terms of the arbitrary constants, the rule for ascertaining whether the solution is a maximum or a minimum is well known. But the direct solution springs so naturally from the equations of the calculus of variations that reference to the differential calculus is superfluous.

2. Following the notation on pp. 242–244 of *Proc. of the Lond. Math. Soc.*, Vol. XXIII., I use Δ to denote variations of the *limiting* values of x and y. Then, if U be the integral, we may write the reduced form of the first variation as

$$\delta U = \Big|_{x'}^{x''} (u\Delta x + A_0 \Delta y + A_1 \Delta \dot{y} + \&c. + A_{n-1} \Delta y^{(n-1)}) + \int_{x'}^{x''} M \delta y \, dx \quad ...(1).$$

Now, if we suppose S to be the stationary value of the integral expressed as a function of the limiting values of x, y, $\dot{y}$, ... $y^{(n-1)}$, then the change in the stationary value as we pass from one set of limiting values to another is clearly

$$\Delta S = \frac{dS}{dx'}\Delta x' + \frac{dS}{dx''}\Delta x'' + \frac{dS}{dy'}\Delta y' + \&c. + \frac{dS}{dy''^{(n-1)}}\Delta y''^{(n-1)} \quad ......(2).$$

Since therefore each stationary value makes $M = 0$, we get from comparing (1) and (2) the following values of the partial differential coefficients of S with regard to limits :—

$$\frac{dS}{dx'} = -u', \quad \frac{dS}{dy'} = -A_0', \quad \frac{dS}{d\dot{y}'} = -A_1', \ \&c.,$$

$$\frac{dS}{dx''} = +u'', \quad \frac{dS}{dy''} = +A_0'', \quad \frac{dS}{d\dot{y}''} = +A_1'', \ \&c.,$$

Hence $\frac{d^2S}{dx'^2}$, $\frac{d^2S}{dx'dy'}$, &c., are obtained by differentiating these values, and the ordinary method of calculating the stationary values by direct integration, and then finding by differentiation the first and second partial differential coefficients, requires us to take a lot of trouble to obtain what the equations of the calculus give us at once.

3. If the stationary value S is a maximum for fixed limits, and if S itself is a maximum when the limits are variable, then evidently S is a true maximum among all neighbouring values of the integral. But if while S is a maximum among integrals with the *same* limits, it is a minimum among *stationary* integrals with *consecutive* limits, then it is neither a maximum nor a minimum among *all* consecutive integrals.

4. *Example I.*—To find the curve which generates the solid of revolution of maximum volume for given superficial area, one extremity $x'y'$ to lie on the axis of revolution, and the other $x''y''$ to lie on the curve $y = \theta(x)$. (The references to Jellett, Todhunter, and Carll on this problem have already been given.)

Here, taking $y = 0$ as the axis of revolution, we get for U

$$U = \int_{x'}^{x''} \left(y^2 + ay\sqrt{1+\dot{y}^2}\right) dx,$$

a being Euler's multiplier, and from $\delta U = 0$ we get

$$y^2 + \frac{ay}{\sqrt{1+\dot{y}^2}} = C \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

but, since $y' = 0$, we get $C = 0$, which gives for the solution

$$y^2 + (x-b)^2 = a^2,$$

and the Jacobian condition, as extended to this case (see *Proc. Lond. Math. Soc.*, Vol. XXIII., p. 249) shows that for fixed limits U is a true maximum.

The limiting terms are

$$\Big|_{x'}^{x''} \left\{ \left(y^2 + ay\sqrt{1+\dot{y}^2}\right)\Delta x + \frac{ay\dot{y}}{\sqrt{1+\dot{y}^2}}\left(\Delta y - \dot{y}\Delta x\right) \right\} \;\ldots\ldots\ldots (4),$$

and when we substitute for $\dot{y}$ its value $\sqrt{a^2-y^2}/y$, from (3), the coefficients of $\Delta x'$ and $\Delta x''$ disappear, and, since $\Delta y'$ is always zero by the conditions of the problem, the limiting terms reduce to

$$ay''\dot{y}'' / \sqrt{1+\dot{y}''^2}\,\Delta y'';$$

or

$$y''\sqrt{a^2-y''^2}\,\Delta y''.$$

But, if $2\pi k^2$ be the given value of the superficial area, we have

$$a^2 = \frac{k^4}{2k^2-y^2};$$

and therefore, finally,

$$\frac{dS}{dy''} = \frac{y''(k^2-y''^2)}{\sqrt{(2k^2-y''^2)}}.$$

Hence $y'' = 0$ and $y'' = \pm k$ give us stationary solutions. To ascertain if the stationary solutions give maxima values to the integral, take $\frac{d^2S}{dy''^2}$, and substitute these values. Now the sign of

d^2S/dy''^2 is easily seen to be the same as that of

$$\frac{k^2-3y''^2}{\sqrt{(2k^2-y''^2)}},$$

in which the positive sign is to be given to the square root. Hence $y''=0$ gives a minimum value to S, and therefore neither a maximum nor a minimum value to U, but $y''=k$ gives a maximum value to S, and therefore also to U.

So far, we have not considered the form of the limiting curve $y=\theta(x)$. To find for what values of x'' we get maxima values of U we must express S as a function of x''. Of course, we get

$$\frac{dS}{dx''}=\frac{dS}{dy''}\frac{dy''}{dx''}=\theta'(x'')\frac{dS}{dy''}.$$

Hence we get stationary values for the same values of y'' as before, and also when $\theta'(x'')=0$. Again,

$$\frac{d^2S}{dx''^2}=\theta''(x'')\frac{dS}{dy''}+\left[\theta'(x'')\right]^2\frac{d^2S}{dy''^2}.$$

Therefore, if the curve $y=\theta(x)$ has at any point P a *minimum* ordinate *greater* than k, or a *maximum* ordinate less than k, then these ordinates give maxima values to U, for dS/dy'' is negative if $y>k$, and positive if $y<k$.*

Example II.—To find the brachistochrone for a particle descending from a curve $y'=\theta(x')$ to another curve $y''=\phi(x'')$, the initial velocity being that due to a height h.

Mr. Todhunter gives this as a problem in which the fact that the limit appears in the quantity to be integrated introduces a difficulty not provided for in the ordinary method, and after three pages of work he leaves the second variation in a form which cannot be calculated because it still contains arbitrary variations under the integral

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* Prof. Jellett's result is that, if the ordinate of $y=\theta(x)$ be a minimum, the volume will be a true maximum, while, if the ordinate be a maximum, the volume will only be a maximum compared with others obtained by the revolution of a *circular arc*. He does not give his work, nor does he give the value $y=k$ at all.

Mr. Todhunter, in amending Prof. Jellett's conclusions, seems to have forgotten that, if S be a minimum, U is not also a minimum, a point which Prof. Jellett evidently had in view, and he must also have made some error in his clerical work, since his result is that, if $y=k$ be a maximum or a minimum ordinate, U is also a maximum or a minimum respectively, whereas $y=k$ always makes U a maximum, independently of the curve $y=\theta(x)$.

Mr. Carll, in reprodu·ing Mr. Todhunter's result, says that he has carefully checked the work, but he also has fallen into some clerical error in calculating his d^4v/dx^4 (p. 130), which does not change sign, as he states, but is always negative.

sign. He then abandons the problem as insoluble by known methods (see his *History*, p. 328).

Taking the axis of x vertically downwards, the velocity at any point x is proportional to $\sqrt{h+x-x'}$, which I shall write as $\sqrt{H}$. Hence the problem is to make

$$U = \int_{x'}^{x''} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{H}}\, dx$$

a minimum.

The stationary curve gives

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = \frac{\sqrt{H}}{\sqrt{a}} \quad \text{..........................} (4),$$

where a is a constant of integration; and the limiting conditions give

$$\Delta y' = \theta'(x')\,\Delta x' \quad \text{and} \quad \Delta y'' = \phi'(x'')\,\Delta x''.$$

Hence, eliminating $\dot{y}'$ and $\dot{y}''$ from the limiting terms by means of (4), we easily reduce δU to the form

$$\delta U = \left(-\frac{\theta'(x')}{\sqrt{a}} + \frac{\sqrt{a-H'}}{\sqrt{aH'}}\right)\Delta x' + \left(\frac{\phi'(x'')}{\sqrt{a}} + \frac{\sqrt{a-H''}}{\sqrt{aH''}}\right)\Delta x'' + \int_{x'}^{x''} M\delta y\, dx.$$

Hence
$$\frac{dS}{dx'} = -\left(\frac{\theta'(x')}{\sqrt{a}} - \frac{\sqrt{a-H'}}{\sqrt{aH'}}\right),\quad \frac{dS}{dx''} = \frac{\phi'(x'')}{\sqrt{a}} + \frac{\sqrt{a-H''}}{\sqrt{aH''}} \ldots (5),$$

and it is only necessary to find $\frac{d^2S}{dx'^2}$, $\frac{d^2S}{dx'\,dx''}$, and $\frac{d^2S}{dx''^2}$, and put their values into

$$\frac{d^2S}{dx'^2}\frac{d^2S}{dx''^2} - \left(\frac{d^2S}{dx'\,dx''}\right)^2,$$

in order to complete the solution. Thus the work of the calculus of variations is complete.

But the further differentiation is very complicated, because the constant a is a function of x' and x'' determined by the condition that the solution passes through the points $x'y'$ and $x''y''$. The equation which determines a is

$$\phi(x'') - \theta(x') + \sqrt{aH'' - H''^2} - \sqrt{ah - h^2}$$
$$- \tfrac{1}{2}a\left(\text{versin}^{-1}\frac{2H''}{a} - \text{versin}^{-1}\frac{2h}{a}\right) = 0 \ldots (6),$$

and the utmost simplification we can make is to get rid of transcendental functions of a from the second differential coefficients; a itself cannot be eliminated completely. I have calculated them out in this form, but, as the result is still extremely complicated, it is not given here.

The ordinary method appears to promise well. The time along the stationary curve is

$$t = \sqrt{a}\left(\text{versin}^{-1}\frac{2H''}{a} - \text{versin}^{-1}\frac{2h}{a}\right),$$

and the problem is to make this a minimum where a is a function of x' and x'' determined by (6). The problem is more difficult than it looks, however, the work required to find even the first differential coefficients in (5) being very long.

5. The criteria for distinguishing maxima and minima values for fixed limits of x and y when s is the independent variable, and the length of the curve is not given, were not included in the previous paper in Vol. XXIII., and the discussion there promised must now be given.

Let
$$U = \int_Q^P u\,ds;$$

then the extended Jacobian criterion only applies when we suppose s, as well as the xy-limits, to be unchanged by the variation given, and therefore even for fixed limits of P and Q it is necessary to investigate the effect of a variation of s. The paper referred to enables us to ascertain whether the stationary curve is a maximum when compared with any other curve of the same length, and therefore, if we express the stationary value, S, of the integral for a curve of given length with the given xy-limits, we have only to ascertain whether that stationary value is a maximum when the length be varied, that is, we have to find the signs of

$$\frac{d^2S}{ds'^2} \text{ and } \left(\frac{d^2S}{ds'\,ds''}\right)^2 - \frac{d^2S}{ds'^2}\,\frac{d^2S}{ds''^2}.$$

We can see, too, that this must include that part of the criterion for fixed xy-limits which relates to the conjugate point (see Vol. XXIII., p. 247). For the value of the integral, taken along a stationary curve from any point to the conjugate point, differs only from its value taken along a consecutive stationary curve by terms of the *third* order. Hence the portion quadratic in Δs must vanish when the integration extends from any point to its conjugate, *i.e.*, the second differential coefficients of S with regard to the s-limits change sign, and, if the maximum property held within those limits, it holds no longer when the integral is extended beyond them.

The following problem illustrates this.

Example III.—To find the minimum surface of revolution round an axis $y = 0$ which passes through two points whose common distance from the axis is y'. Here

$$U \equiv \int_{s'}^{s''} \left\{ y + \tfrac{1}{2}\lambda\,(\dot{x}^2+\dot{y}^2-1) \right\} ds,$$

and $\Delta x'$, $\Delta y'$, &c., are zero, since the limits of x and y are fixed,

$$\delta U = \Big|_{s'}^{s''} (y-\lambda)\,\Delta s + \int_{s'}^{s''} \left\{ \left(1-\frac{d}{ds}(\lambda\dot{y})\right)\delta y - \frac{d}{ds}(\lambda\dot{x})\,\delta x \right\} ds.$$

Hence $$1-\frac{d}{ds}(\lambda\dot{y}) = 0, \quad \frac{d}{ds}(\lambda\dot{x}) = 0,$$

from which we obtain

$$y-\lambda = a, \quad (y-a)^2 = (s+b)^2+c^2,$$

$$x+d = c\log\left\{(s+b)+\sqrt{(s+b)^2+c^2}\right\},$$

a, b, c, and d being constants. Taking the x-axis symmetrically, $d = 0$; also $2b = -s''-s'$, so that we obtain

$$(y'-a)^2 = \tfrac{1}{4}(s''-s')^2+c^2 \quad\text{.......................(7)},$$

$$s''-s' = c\,(e^{x'/c}-e^{-x'/c}) \quad\text{.......................(8)},$$

$$y'-a = \lambda = \frac{c}{2}\,(e^{x'/c}+e^{-x'/c}) \quad\text{.......................(9)}.$$

From (9) we see that λ is positive throughout, so that, when all limits are fixed, the integral is a maximum. Again, from the expression for δU, we have

$$dS/ds'' = +(y'-\lambda) = a.$$

Hence $$\frac{d^2S}{ds''^2} = \frac{da}{ds''}, \quad \frac{d^2S}{ds'^2} = -\frac{da}{ds'}, \quad \frac{d^2S}{ds'ds''} = \frac{da}{ds'} = -\frac{da}{ds''},$$

and the sign of the first of these coefficients gives us all we require.

Differentiating (7) and (8), we obtain

$$-2\,(y'-a)\,\frac{da}{ds''} = \tfrac{1}{2}\,(s''-s')+2c\,\frac{dc}{ds''},$$

$$1 = \frac{dc}{ds''}\left\{ e^{x'/c}-e^{-x'/c}-\frac{x'}{c}\,(e^{x'/c}+e^{-x'/c}) \right\}.$$

Hence, putting $a = 0$, and eliminating y' by (9), we obtain, after one or two reductions,

$$\frac{d^2S}{ds''^2} = \frac{da}{ds''} = -\frac{c}{2}\,\frac{c\,(e^{x'/c}+e^{-x'/c})-x'\,(e^{x'/c}-e^{-x'/c})}{c\,(e^{x'/c}-e^{-x'/c})-x'\,(e^{x'/c}+e^{-x'/c})}.$$

The denominator of this fraction is always negative (this is evident when x' is greater than c, and when x' is less than c it may be seen by expanding). The numerator is the well-known quantity whose sign distinguishes whether the "conjugate point" has been included in the integration, in which case the minimum property no longer holds.

In the following problem all the limits are variable. It is a modification of the well known problem of describing the curve which, with given perimeter, shall contain the greatest area, and is remarkable in that it has an indefinite number of maxima and minima solutions.

To make $U = \int_{s'}^{s''}\left(\frac{s^2}{l} - 2s + \frac{2y-y'-y''}{2}\,\dot{x}\right) ds$ a maximum or a minimum.

Adding $\frac{1}{2}\lambda\,(\dot{x}^2+\dot{y}^2-1)$ to the bracket, we get for δU,

$$\delta U = \Big|_{s'}^{s''}\left(\frac{s^2}{l} - 2s - \lambda\right)\Delta s + \Big|_{s'}^{s''}\left(\frac{2y-y'-y''}{2} + \lambda\dot{x}\right)\Delta x + \Big|_{s'}^{s''}\lambda\dot{y}\,\Delta y$$

$$-\frac{x''-x'}{2}\,(\Delta y' + \Delta y'') + \int_{s'}^{s''}\left[\delta x\,\frac{d}{ds}\,(-y-\lambda\dot{x}) + \delta y\,\frac{d}{ds}\,(x-\lambda\dot{y})\right] ds \ldots (10).$$

Hence, from equating to zero the quantity under the integral sign, we get

$$y + \lambda\dot{x} = b, \quad x - \lambda\dot{y} = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

a and b being constants of integration. Combined with

$$\dot{x}^2 + \dot{y}^2 = 1,$$

these give readily enough

$$\lambda = r, \quad x = r\cos\left(\frac{s}{r} + c\right) + a, \quad y = r\sin\left(\frac{s}{r} + c\right) + (b) \quad \ldots (12),$$

r, a, b, and c being the four constants of integration, which with s' and s'' enable us to satisfy the six conditions at the limits. Using (11) and (12) to simplify the limiting terms in (10), and writing, as before, S for the stationary value of U, we get for the six conditions at the limits the equations

$$\frac{dS}{ds'}=0,\quad \frac{dS}{ds''}=0,\quad \&c. =0,$$

where
$$\frac{dS}{ds'}=-\left(\frac{s'^2}{l}-2s'-r\right),\quad \frac{dS}{ds''}=\frac{s''^2}{l}-2s''-r \quad \text{.........(13)},$$

$$\frac{dS}{dx'}=+\frac{y'+y''}{2}b=-\frac{dS}{dx''}\text{....................(14)},$$

$$\frac{dS}{dy'}=-\lambda y'-\frac{x''}{2}+\frac{x'}{2}=a-\frac{x'+x''}{2}=-\frac{dS}{dy''}\quad\text{.........(15)}.$$

Hence, for the stationary value, we get, taking $s''>s'$,

$$s''=l+\sqrt{l^2+lr},\quad s'=l-\sqrt{l^2+lr}\text{ ....(16)}.$$

(The fact that s' and s'' are given by the same quadratic equation arises from the constancy of λ, and is, of course, a mere accidental peculiarity in the problem.)

Again, (14) and (15) taken with (12) give us

$$\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)=0\text{..................(17)},$$

$$\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)=0\text{..................(18)}.$$

These equations combined give us

$$\frac{s''-s'}{r}=(2n+1)\pi\text{...........................(19)}.$$

From (19) and (16), we get

$$2\sqrt{l^2+lr}=(2n+1)r\pi\text{........................(20)},$$

from which we obtain approximately, if we take $n=0$,

$$r=\cdot 86\times l \quad\text{or}\quad r=-\cdot 46\times l\text{ ....(21)},$$

and, whatever value we take for n, we always get one positive and one negative root.

When the limits are *fixed*, we always get either a maximum or a minimum value of U, which is evident geometrically, since the problem is then to draw between two points a curve of given length which shall contain with the chord the greatest area, or the least area (*i.e.*, the maximum *negative* area). The solution, as is well known, is a circular arc, and when rds or $r(s''-s')$ is positive it gives the minimum, and when negative the maximum, value. Hence, by (19), when n is a positive integer the solution is a minimum, and when n is a negative integer it is a maximum.

There is, however, nothing in the problem to fix the positions or directions of the axes of x and y coördinates, and it is evidently permissible to fix the origin at the centre of the circle which gives the solution of the problem, and to take y' and y'' as zero, *i.e.*, to choose the y-axis so as to pass through the two extremities of the semicircle. If also we choose the positive direction of the x-axis from x' to x'', we get for the constants a, b, and c the following equations :—

$$a = 0, \quad b = 0 \quad \text{.........................(22)},$$

$$\sin\left(\frac{s'}{r}+c\right) = 0, \quad \sin\left(\frac{s''}{r}+c\right) = 0 \quad \text{...............(23)},$$

$$\cos\left(\frac{s'}{r}+c\right) = -1, \quad \cos\left(\frac{s''}{r}+c\right) = +1 \quad \text{.........(24)};$$

but when r is negative the signs on the right-hand of the equation (24) must be changed if x'' is to be positive, and x' negative. That the four equations (22) and (23) are consistent is evident either by the geometry of the solution or from (17), (18), and (19). These equations, with

$$x''-x' = +2\sqrt{r^2}, \quad y''-y' = 0 \quad \text{..................(26)},$$

enable us greatly to simplify the work of obtaining the second differential coefficients of S. We do not require to determine the constant c.

In testing whether the solution is a maximum or a minimum, we must, of course, use the expressions for $\frac{dS}{ds'}$, &c., in (13), (14), and (15) in their *unreduced* forms, reducing by the values corresponding to the stationary solution only when all the differentiations have been performed. Before differentiating (13), (14), and (15), it is convenient to obtain the differential coefficients of r, a, and b, with respect to the limits.

From (12), we get

$$\sqrt{(x''-x')^2+(y''-y')^2} = 2r\sin\left(\frac{s''-s'}{2r}\right) \quad \text{............(27)};$$

therefore $0 = \frac{dr}{ds'}\left(2\sin\frac{s''-s'}{r} - \frac{s''-s'}{r}\cos\frac{s''-s'}{2r}\right) - \cos\frac{s''-s'}{2r}$;

but, by (19), $\cos\frac{s''-s'}{2r} = 0,$

wherefore, for the stationary value

$$\frac{dr}{ds'} = \frac{dr}{ds''} = 0 \quad\text{.........................(28),}$$

Again, from (27),

$$\frac{x''-x'}{\sqrt{(x''-x')^2+(y''-y')^2}} = -\frac{dr}{dx'}\left(2\sin\frac{s''-s'}{2r} - \frac{s''-s'}{r}\cos\frac{s''-s'}{2r}\right),$$

which becomes for the stationary value, by (19) and (26),

$$\frac{dr}{dx'} = -\tfrac{1}{2} = -\frac{dr}{dx''} \quad\text{.....................(28);}$$

again, for the stationary value

$$\frac{dr}{dy'} = -\frac{dr}{dy''} = -\frac{y''-y'}{4r} = 0 \quad\text{.................(29).}$$

Again, from (12), we get

$$x''+x'-r\left[\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)\right] = 2a.$$

Hence

$$1-\frac{dr}{dx'}\left[\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)+\frac{s''}{r}\sin\left(\frac{s''}{r}+c\right)+\frac{s'}{r}\sin\left(\frac{s'}{r}+c\right)\right]$$
$$+r\frac{dc}{dx'}\left[\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)\right] = 2\frac{da}{dx'},$$

in which the quantities inside the brackets vanish by (23) and (24).

Hence $$\frac{da}{dx'} = \frac{da}{dx''} = \tfrac{1}{2} \quad\text{.........................(30),}$$

and, similarly, we show that

$$\frac{da}{dy'} = \frac{da}{dy''} = 0 \quad\text{.........................(31).}$$

Again, $y''+y'-2r\left[\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)\right] = 2b;$

therefore

$$-\frac{dr}{dx'}\left[\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)-\frac{s''}{r}\cos\left(\frac{s''}{r}+c\right)-\frac{s'}{r}\cos\left(\frac{s'}{r}+c\right)\right]$$

$$-2r\frac{dc}{dx'}\left[\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)\right]=2\frac{db}{dx'},$$

Reducing this by (19), (23), (24), and (28), we get

$$\left.\begin{aligned}\frac{db}{dx'}&=-\frac{s''-s'}{4r}=-\frac{\pi}{4}\\ \frac{db}{dx''}&=\frac{\pi}{4}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32),$$

and, similarly, we may easily show that

$$\frac{db}{dy'}=\frac{db}{dy''}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

It is now easy to find the second differential coefficients. From (13), we get

$$\frac{d^2S}{ds'^2}=-2\frac{S'}{l}+2+\frac{dr}{ds'}=-2\frac{s'}{l}+2=2\frac{\sqrt{l^2+lr}}{l}=\frac{r\pi}{l}=\cdot 86\pi$$

$$\ldots\ldots\ldots\ldots\ldots(34),$$

taking the positive root. This is also the value of d^2S/ds''^2, and

$$\frac{d^2S}{ds'\,ds''}=-\frac{dr}{ds''}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

Again, from (13) and (28), we get

$$\frac{d^2S}{ds'\,dx'}=+\frac{dr}{dx'}=-\tfrac{1}{2}=\frac{d^2S}{ds''\,dx''}\quad\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Similarly, we get

$$\frac{d^2S}{ds'\,dx''}=\frac{d^2S}{ds''\,dx'}=+\tfrac{1}{2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

and, from (14) and (32),

$$\frac{d^2S}{dx'^2}=-\frac{d^2S}{dx'\,dx''}=\frac{d^2S}{dx''^2}=-\frac{db}{dx'}=\frac{\pi}{4}\ldots\ldots\ldots\ldots\ldots(39),$$

while it is easy to see that the remaining coefficients, which all have dy' or dy'' in the denominator, vanish.

Hence, from (34) to (39), the part depending on the limits may be written as

$$\left.\begin{array}{l} \cdot 86\pi\Delta s'^2+\Delta s'\,(\Delta x'-\Delta x'')+\dfrac{\pi}{8}\,(\Delta x'-\Delta x'')^2 \\ +\cdot 86\pi\Delta s''^2+\Delta s''\,(\Delta x''-\Delta x')+\dfrac{\pi}{8}\,(\Delta x''-\Delta x')^2 \end{array}\right\} \quad \text{.........(40)},$$

each line of which is evidently positive.

Hence the integral is a true minimum for these values of the limits.

There are, however, other solutions with $n>0$ or $n<0$, and these must be examined. Since we take $s''-s'$ positive, it is evident from (20) that $2n+1$ and r must have the same sign, and hence, if n be taken positively, only the positive root of r can be used. If we work out the condition analogous to (40), we shall find that every positive root for r got from putting $n = 1, 2, 3$, &c., in (20) gives a new minimum solution in which the integration extends over an arc 3π, 5π, &c., round the circle, so that the area may be counted many times over in the integral, while, similarly, each negative value of r got from (20) by putting $n = -1$, $n = -2$, &c., gives us a maximum value of the integral. Thus there are to this integral an indefinite number of maxima and minima solutions.

As this result is in direct opposition to the principles laid down in Moigno, Jellett, and other text-books, which state that, if the limits are all variable, it is impossible that there should be a maximum or a minimum value, it is well to point out the error in the arguments used by these authors, and thus remove a *primâ facie* doubt as to the correctness of the above solution. Jellett gives two reasons: first, he counts the disposable constants and finds them insufficient in number, but the insufficiency is due to his leaving out the two disposable limits x' and x'' (or in the above problem s' and s''); secondly, he says that we may see *a priori* that there can be no maximum or minimum because the integral must be susceptible of all ranges of values if everything but its form be arbitrary, an argument which is obviously invalid, as it would equally show that the expression $(y-a)(y-b)(y-c)$ had no maximum or minimum. Moigno's reason is that when all the limits are variable the solution requires $x' = x''$, and the whole integral disappears, the error here being that he has omitted all the other solutions which give x' different from x''; in the present example we have taken a root s' different from the s'' root. But, besides this fatal objection, there are two [illegible] that even if we admit his statement, the resulting value zero for the integral would be a true stationary value, and might be a true maximum or minimum; and, second, that there are many integrals in which the initial and final coordinates do not appear symmetrically.

. . . . . .

6. When we examine the maxima and minima of double integrals with variable limits, we find an entirely new problem before us, because the ordinary rules of the calculus of variations for ascertaining the maxima and minima values of single integrals are not applicable to the single integrals which we obtain as limiting terms in dealing with double integrals. In fact, none of the rules for the discrimination of maxima and minima values already given, either in the present paper or the preceding one, are applicable where y is a single-valued function of x.

Thursday, June 14th, 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

There not being the number of members present required by Rule XLIV. to constitute a "Special" Meeting, Mr. Tucker was called upon to communicate abstracts of the following papers which had been received:—

The Solutions of

$$\sinh\left(\lambda \frac{d}{dx}\right) y = f(x), \quad \cosh\left(\lambda \frac{d}{dx}\right) y = f(x),$$

λ a Constant: Mr. F. H. Jackson.

A Theorem in Inequalities: Mr. A. R. Johnson.

Some Properties of Two Circles: Mr. Tucker.

Note on Four Special Circles of Inversion of a System of "Generalized Brocard" Circles of a Plane Triangle: Mr. J. Griffiths.

On the Order of the Eliminant of Two or More Equations: Dr. R. Lachlan.

Impromptu communications were then made by Prof. Greenhill (on a Gyrostatic Top), Dr. Larmor, and Prof. M. J. M. Hill.

There being at this time a quorum, the meeting was made

Special.

The President read out five resolutions which had been approved by the Council, and invited discussion upon them, and upon the "Memorandum and Articles of Association of the London Mathematical Society," printed copies of which were put into the hands of the members present.

After some discussion, the resolutions, having been slightly amended, were submitted to the meeting and carried unanimously in the following forms:—

1. That the London Mathematical Society be incorporated under § 23 of the Companies Act, 1867, with the Memorandum and Articles of Association submitted by the Council.
2. That before such Incorporation all the liabilities of the present Society be discharged by the Treasurer, and all persons dealing with the Society as creditors be informed that the Society will be incorporated under § 23 of the Companies Act, 1867, and the present constitution be terminated.
3. That the Incorporation be effected by the Council.
4. That immediately after the Incorporation of the London Mathematical Society all the property now held by trustees for the benefit of the Society be transferred to the Society itself, and that the Council take all necessary steps to effect this transfer.
5. That upon such incorporation and transfer being completed, the present constitution of the Society shall be terminated.

The following presents were received:—

"Proceedings of the Royal Society," Vol. lv., No. 333.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xviii., St. 5; Leipzig, 1894.

"Proceedings of the Edinburgh Mathematical Society," Vol. i., Session 1883.

"Berichte über die Verhandlungen der Koniglich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1894, i.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. viii., No. 2; Manchester, 1893-4.

"Nyt Tidsskrift for Mathematik," A. Femte Aargang, Nos. 2, 3; and B. Femte Aargang, No. 1.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxvii., Livr. 4 and 5, and Tome xxviii., Livr. 1; Harlem, 1894.

"Jahrbuch über die Fortschritte der Mathematik," Bd. xxiii., Heft 2; Berlin, 1894.

"Bulletin de la Société Mathématique de France," Tome xxii., Nos. 3 and 4; Paris, 1894.

"Bulletin of the New York Mathematical Society," Vol. III., No. 8; May, 1894.

"Rendiconti del Circolo Matematico di Palermo," Tomo VIII., Fasc. 1, 2, and 3; 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," Serie 5, Vol. III., Fasc. 8, Sem. 1; Roma, 1894.

D'Ocagne, M.—"Sur la Composition des Lois d'Erreurs de Situation d'un Point" (from *Comptes Rendus*).

"Educational Times," June, 1894.

"Indian Engineering," Vol. XV., Nos. 16–20.

"Royal Society Catalogue of Scientific Papers," Gir.-Pet, Vol. X., 4to; London, 1894.

Mannheim, Le Col. A.—"Principes et Développements de Géométrie Cinématique," 4to; Paris, 1894.

"Memorie della Regia Accademia di Scienze, Lettere ed Arti in Modena," Serie 11, Vol. IX.; Modena, 1893.

Note on Four Special Circles of Inversion of a System of Generalized Brocard Circles of a Plane Triangle. By John Griffiths, M.A. Received May 26th, 1894. Read June 14th, 1894.

Connected with a system of generalized Brocard circles of a triangle there are four circles—say, J, J_1, J_2, J_3—with respect to each of which the inverse of every circle of the system is a circle of the same system. Or we may briefly say that a system of generalized Brocard circles is self-inverse with regard to four different centres.

In a note recently communicated by me to the Society (see *Proceedings*, Vol. XXV., Nos. 479, 480), it was shown that a triangle ABC has three systems of what may be called generalized Brocard circles, or, shortly, G.B. circles. Every circle in each of the three systems in question possesses properties analogous to the Brocard circle of ABC, and can be constructed by means of a certain number of points dependent on a variable primary point U, taken on one of three given circles connected with the triangle ABC.

If U be a point on the circular arc BUC which touches AC in C,* and the angle UBC be denoted by ω, the equation in isogonal coordinates of the G.B. circle of the first system corresponding to U is

$$GBC\,(x,\ y,\ z,\ \cot\omega) = \lambda x + \mu y + \nu z - \delta = 0,$$

---

* See Fig., page 381.

where $\lambda = \sin B \sin C \left\{\operatorname{cosec}^2 A - (\cot \omega - \cot \Omega)^2\right\}$;

$\mu = \sin B (\cot \omega - \cot C)$;

$\nu = \sin C (\cot \omega - \cot B)$;

$\delta = \dfrac{2 \sin B \sin C}{\sin A} \cot \omega,$

and $$\cot \Omega = \cot A + \cot B + \cot C.$$

For example, let $\omega = \Omega$, *i.e.*, let U be the positive Brocard point of ABC, then the corresponding circle is expressed by the equation

$$\Sigma x \operatorname{cosec} A = 2 \cot \Omega.$$

This is satisfied by

$$x = \frac{\sin C}{\sin B}, \quad y = \frac{\sin A}{\sin C}, \quad z = \frac{\sin B}{\sin A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x = \frac{\sin B}{\sin C}, \quad y = \frac{\sin C}{\sin A}, \quad z = \frac{\sin A}{\sin B} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$x = 2 \cos A, \quad y = 2 \cos B, \quad z = 2 \cos C \ldots\ldots\ldots\ldots\ldots (3).$$

In other words, the corresponding circle is the Brocard circle, since it passes through the Brocard points and the centre of the circum-circle of the triangle ABC.

[The expression $GBC\,(U)$ is used in this note as an abbreviation denoting the generalized Brocard circle of the first system whose primary point is U, taken on the fixed circular arc BUC touching AC in C.]

Section I.

The Envelope of $GBC\,(U)$ is a Bicircular Quartic.

Since the equation $GBC\,(x, y, z, \cot \omega)$ given above can be written in the form

$$X \cot^2 \omega - Y \cot \omega + Z = 0,$$

it is seen at once that its envelope, as U moves along the fixed circular arc BUC, is expressed by

$$Y^2 = 4XZ,$$

where $$X = x \sin B \sin C;$$

$$Y = 2x \sin B \sin C \cot \Omega + y \sin B + z \sin C - 2 \frac{\sin B \sin C}{\sin A};$$

$$Z = x \sin B \sin C (\cot^2 \Omega - \operatorname{cosec}^2 A) + y \sin B \cot C + z \sin C$$

Hence, since a linear relation

$$\Sigma lx + k = 0$$

in isogonal coordinates denotes a circle, and

$$\Sigma lx = 0$$

a right line, it follows that the envelope of GBC (U) is a bicircular quartic, two of whose bitangents are the lines

$$x = 0,$$

$$x \sin B \sin C (\cot^2 \Omega - \operatorname{cosec}^2 A) + y \sin B \cot C + z \sin C \cot B = 0.$$

Now a bicircular quartic may be regarded as the envelope of a circle whose centre moves on a fixed conic, and which cuts a fixed circle orthogonally. Supposing then the primary point U to move along the given arc BUC, it is seen that GBC (U) cuts a fixed circle J orthogonally, and its centre describes a conic H (a hyperbola). The asymptotes of H are perpendicular respectively to the two bitangents given above whose intersection is the centre of J. It is thus shown that the centre of J is the point

$$(x = 0, \quad y \sin B \cot C = -z \sin C \cot B).$$

In other words, the centre of the orthogonal circle of the first system of generalized Brocard circles lies on the side BC of the triangle ABC.

Section II.

A System of Generalized Brocard Circles is Self-Inverse with regard to four Different Centres.

It is known that a bicircular quartic—considered as the envelope of a variable circle which is orthogonal to a fixed circle J, and whose centre moves on a fixed conic H—is self-inverse with regard to four different circles J, J_1, J_2, J_3, each of which cuts the remaining three orthogonally. The centres of J_1, J_2, J_3 are the vertices of the common self-polar triangle of the curves J and H. Hence we may assume that every G.B. circle of a given system can be inverted into a G.B. circle of the same system with respect to each of four different centres.*

* GBC (U) cuts J orthogonally, and its envelope is a certain bicircular quartic, Q say. Conversely, a circle which satisfies these two conditions is a G.B. circle. Now J and Q are both self-inverse with respect to each of the circles J_1, J_2, J_3. Hence the inverse of GBC (U) with regard to each of the three is a G.B. circle.

Section III.

Geometrical Constructions for J, H, J_1, J_2, J_3 in the case of a System of G.B. Circles; and Miscellaneous Theorems.

The constructions given below are for the first system only, but the results will, of course, apply, *mutatis mutandis*, to the second and third systems.

(1) Construction for J.

Let M be the mid-point of the side BC of a triangle ABC, and D the foot of the perpendicular from A upon BC; draw MR parallel to the perpendicular DA, and take

$$MR = MB = MC;$$

join D and R, and through R draw RJ perpendicular to RD, to inter-

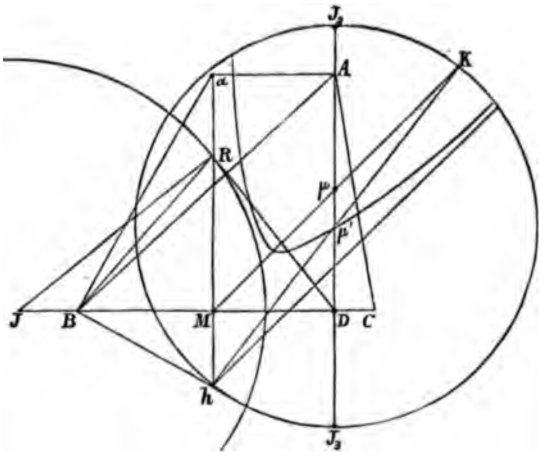


Fig. 1.

[JBC and the perpendicular from J upon μM are two bitangents of the quartic envelope.]

sect CB produced in J. Then J is the centre, and JR the radius, of the orthogonal circle J of the system of G.B. circles in question.

(2) Construction for H, the locus of the centre of a G.B. circle of the first system.

Through A draw a parallel to the base BC to intersect MRa in a; join a and B, and through B draw Bh perpendicular to Ba; the point h where Bh meets aRM is the centre of the hyperbola H.

Bisect AD in μ and produce $M\mu$ to K, so that

$$\mu K = M\mu\,;$$

let the join of K and h intersect AD in μ'; then μ' is a point on H.

Lastly, the asymptotes of H are hMR, and the parallel through h to $M\mu K$. Hence the curve is completely determined.

(3) The hyperbola H bisects the linear segment $M\mu$, and the circle upon $M\mu$ as diameter is a G.B. circle.

(4) Construction for the vertices of the common self-polar triangle of J and H.

With the point μ' as centre, and $\mu'h$ or $\mu'K$ as radius, describe a circle to intersect the perpendicular AD (produced both ways, if necessary) in J_2, J_3; then the triangle MJ_2J_3 is self-polar with regard to each of the curves J and H.

It thus follows that the circles J_1, J_2, J_3 are completely known, since each is orthogonal to J, and their centres M, J_2, J_3 are known.

It may be remarked here that the circle J_1 is imaginary, the square of its radius being $MB \cdot MC$, where $MB + MC = 0$.

(5) The circle upon J_2J_3 as diameter is a G.B. circle; *i.e.*, the inverse of the bitangent BC with respect to either J_2 or J_3. This may be proved in the following manner.

Since the polar of the point J_2 with respect to J passes through J_3, the circle whose diameter is J_2J_3 is orthogonal to J.

Again, since the polar of J_2 with regard to H passes through J_3 the pair of points in which the join of J_2 and J_3 meets H cut the linear segment J_2J_3 harmonically. But, clearly, since J_2J_3 is parallel to an asymptote of H, one of the pair of points in question must be at infinity. Hence the remaining point must bisect J_2J_3. In other words, the circle described on J_2J_3 as diameter is a G.B. circle, because it cuts J orthogonally, and its centre μ' lies on H. [K and h are the K and O points of the circle in question. See *Proc.*, quoted *supra.*] Similar reasoning will apply to the circle whose diameter is the linear segment $M\mu$. [M and μ are the O and K points of this circle.]

Section IV.

Inversion with respect to the Circles J, J_1, J_2, J_3.

(1) Since J is the orthogonal circle of the system formed by $GBC\,(U)$, any circle of the system is self-inverse with regard to J. The primary point U is unaltered.

(2) With regard to J_1, whose centre is M, and the square of whose radius $= -\frac{1}{4}(BC)^2$, the inverse of $GBC\,(U)$ is another circle, $GBC\,(U_1)$, whose primary point U_1 is collinear with U and the vertex A of the triangle. This result may be illustrated by means of a figure.

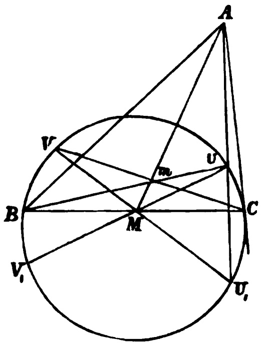


Fig. 2.

Let $BVUC$ be the circle which passes through B, C, and touches AC in C. Through the point m, where BU intersects the median AM, draw Cm to meet the circle in V; project U and V through M into V_1 and U_1; then U_1 is the primary point of the G.B. circle inverse to $GBC\,(U)$ with respect to J_1.

The points A, U, U_1 are collinear, and there is a relation

$$\cot\omega + \cot\omega_1 = \cot B + \cot C$$

between the cotangents of the angular coordinates of the primary points U, U_1. (See *Proceedings*, quoted *supra.*)

There are some other interesting relations between a pair of G.B. circles inverse to each other with respect to J_1. For example, if K denote the K point of $GBC\,(UK)$ and A_1' the A' point of $GBC\,(U_1A_1')$ the inverse of $GBC\,(UK)$ with regard to J_1, then the join A_1' and K passes through the vertex A.

(3) With reference to J_2, the inverse of $GBC\,(U)$ is another circle of the same system, viz., $GBC\,(U_2)$. The line joining the primary points U and U_2 of this pair of circles passes through a fixed point on the polar of A with respect to the circle BUC, which touches AC in C. This is expressed algebraically by the relation

$$\cot\omega\,\cot\omega_2 - \tfrac{1}{2}(\cot B + \cot C)(\cot\omega + \cot\omega_2) = \text{constant}.$$

A precisely similar result is obtained in the case of inversion with respect to J_2; viz.,

$$\cot\omega\cot\omega_2 - \tfrac{1}{2}(\cot B + \cot C)(\cot\omega + \cot\omega_2) = \text{constant}.$$

It may be remarked that the inverse of a generalized Brocard system with regard to *any* centre of inversion is a system of circles whose envelope is a bicircular quartic.

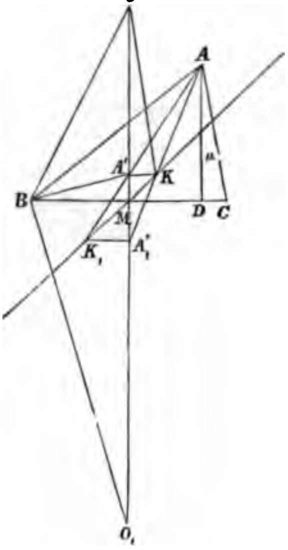


Fig. 3.

Appendix I.

Construction for a Pair of Generalized Brocard Circles Inverse to each other with respect to the circle J_1.

In the above figure M and μ are the mid-points of BC and the perpendicular AD, respectively, and MO is parallel to AD.

Take any point K on $M\mu$; through K draw KA' parallel to CB and intersecting MO in A'; let AK, AA' produced meet OM, KM, respectively, in A'_1 and K_1; then $K_1A'_1$ is parallel to BC. Again, draw BO perpendicular to BA'_1 and BO_1 to BA'; if these perpendiculars meet OMO_1 in O and O_1, the two circles $OA'K$ and $O_1A'_1K_1$ are a pair of G.B. circles inverse to each other with regard to the imaginary circle J_1, whose centre is M, and whose diameter $= \sqrt{-(BC)^2}$.

II.

Note on "Isogonal" Coordinates. [*October 24th*, 1894.]

1. The isogonal coordinates of a point P, taken, say, inside a triangle ABC, are

$$x = \frac{\sin\theta}{\sin(\theta+A)}, \quad y = \frac{\sin\phi}{\sin(\phi+B)}, \quad z = \frac{\sin\psi}{\sin(\psi+C)};$$

where $\pi-\theta$, $\pi-\phi$, $\pi-\psi$ denote the angles BPC, CPA, APB, respectively.

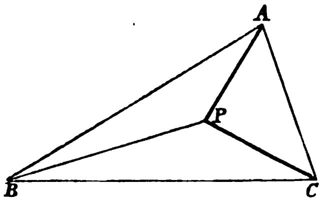


Fig. 1.

Since $\qquad \theta+\phi+\psi = \pi, \quad$ or $\quad \Sigma \cot\phi \cot\psi = 1,$

it appears that the fundamental relation between x, y, z is

$$\Sigma\,(x-yz)\sin A = 0.$$

Ex.—If $\theta = C$, $\phi = A$, and $\psi = B$, then P coincides with the positive Brocard point of the triangle ABC, and its coordinates are

$$x = \frac{c}{b}, \quad y = \frac{a}{c}, \quad z = \frac{b}{a},$$

2. There is no difficulty in proving that x, y, z are proportional to the trilinear coordinates α, β, γ of P. We have, in fact,

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{\Sigma\,a\sin A}{\Sigma\,\beta\gamma\sin A}.$$

These relations give, as before,

$$\Sigma\,(x-yz)\sin A = 0.$$

In Fig. 2, produce BP to meet the circumcircle ABC in Q, and join CQ; then

$$a\alpha = 2\Delta BPC = PB\,.\,PC\sin\theta,$$

since $\qquad \angle BPC = \pi-\theta.$

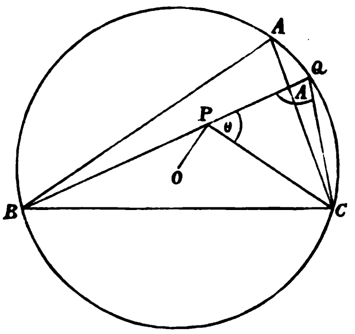


Fig. 2.

Also $$x = \frac{\sin\theta}{\sin(\theta+A)} = \frac{CQ}{PQ};$$

therefore $$\frac{x}{a\alpha} = \frac{CQ}{CP}\cdot\frac{\operatorname{cosec}\theta}{PB.PQ},$$

or $$\frac{x}{2R\sin A} = \frac{\sin\theta}{\sin A}\cdot\frac{\operatorname{cosec}\theta}{PB.PQ}\alpha.$$

It thus follows that, if R be the radius of the circle ABC (whose centre is the point O), we have

$$x = \frac{2R\alpha}{R^2-OP^2};$$

and, similarly, $$y = \frac{2R\beta}{R^2-OP^2},\quad z = \frac{2R\gamma}{R^2-OP^2}.$$

For example, the trilinear and isogonal coordinates of O are $R\cos A$, $R\cos B$, $R\cos C$, and $2\cos A$, $2\cos B$, $2\cos C$, respectively.

(3) The equation $\Sigma\,(x-yz)\sin A = 0$

remains unaltered when we write therein $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ for x, y, z. In other words, $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ denotes the isogonal of (x, y, z).

(4) A linear relation $\Sigma\lambda x+\delta = 0$

in x, y, z denotes a circle. For the trilinear equivalent is

$$\Sigma\lambda\alpha\,.\,\Sigma a\alpha+\delta\,\Sigma\, a\beta\gamma = 0.$$

Ex. $x =$ constant is the equation of a circle through the vertices B and C of ABC.

In particular, if P lies on the circle through B and C which touches AC in C, then

$$\angle PBC = \omega = \angle PCA, \quad \angle PCB = C-\omega, \quad \text{and} \quad \angle BPC = \pi - C.$$

Hence $\theta = C$, and $$x = \frac{\sin C}{\sin(C+A)} = \frac{\sin C}{\sin B},$$

also $$\frac{y}{x} = \frac{\beta}{a} = \frac{\sin\omega}{\sin(C-\omega)},$$

or $$y = \frac{\sin C}{\sin B}\,\frac{\sin\omega}{\sin(C-\omega)} = \frac{\operatorname{cosec} B}{\cot\omega - \cot C};$$

$$\frac{z}{x} = \frac{\gamma}{a} = \frac{\sin(B-\omega)}{\sin\omega},$$

or $$z = \sin C\,(\cot\omega - \cot B).$$

If $$\cot\omega = \Sigma \cot A,$$

then P coincides with the positive Brocard point of ABC, *i.e.*, with $\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$.

Equations of a Pair of Generalized Brocard Circles Inverse to each other with respect to the circle J_1. (See Sect. IV. supra.)

If (x, y, z), (x', y', z') denote a pair of points P, P' inverse to each other with respect to J, we have

$$x' = x;$$

$$y' = \frac{2\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} - z\frac{\sin C}{\sin B};$$

$$z' = \frac{2\sin B}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin C} - y\frac{\sin B}{\sin C}.$$

This being so, it is found that the linear equation

$$GBC\,(x', y', z', \cot\omega_1)$$

$$= x'\sin B\sin C\left\{\operatorname{cosec}^2 A - (\cot\omega_1 - \cot\Omega)^2\right\} + y'\sin B\,(\cot\omega_1 - \cot C)$$

$$+ z'\sin C\,(\cot\omega_1 - \cot B) - 2\frac{\sin B\sin C}{\sin A}\cot\omega_1 = 0$$

is transformed by the above substitutions into

$$GBC\,(x,\ y,\ z,\ \cot\omega)$$

$$= x\sin B\sin C\left\{\operatorname{cosec}^2 A-(\cot\omega-\cot\Omega)^2\right\}+y\sin B\,(\cot\omega-\cot C)$$

$$+z\sin C\,(\cot\omega-\cot B)-2\,\frac{\sin B\sin C}{\sin A}\cot\omega = 0,$$

where $$\cot\omega+\cot\omega_1 = \cot B+\cot C.$$

In other words, the expression $GBC\,(x,\ y,\ z,\ \cot\omega)$ is an invariant, or the inverse of a G.B. circle with respect to J_1 is another G.B. circle of the same system. The relation

$$\cot\omega+\cot\omega_1 = \cot B+\cot C$$

shows that the primary points U, U_1 of this pair of circles are collinear with A. (See Fig. 2, Sect. IV.)

The coordinates of U are

$$x = \frac{\sin C}{\sin B},\quad y\sin B = \frac{1}{\cot\omega-\cot C},$$

and $$z = \sin C\,(\cot\omega-\cot B),$$

as I have just proved above.

Similarly, for U_1, we have

$$x_1 = \frac{\sin C}{\sin B},\quad y_1\sin B = \frac{1}{\cot\omega_1-\cot C},$$

and $$z_1 = \sin C\,(\cot\omega_1-\cot B).$$

Hence it follows at once that

$$\frac{z}{y} = \sin B\sin C\,(\cot\omega-\cot B)(\cot\omega-\cot C)$$

$$= \sin B\sin C\,(\cot\omega_1-\cot B)(\cot\omega_1-\cot C) = \frac{z_1}{y_1},$$

or $$\frac{\gamma}{\beta} = \frac{\gamma_1}{\beta_1};$$

i.e., U, U_1, A are collinear.

This result may be illustrated by taking

$$\omega_1 = \pi-A \quad\text{and}\quad \omega = \Omega.$$

the Brocard angle of ABC. The coordinates of U, U_1 then become

$$x = \frac{c}{b}, \quad y = \frac{a}{c}, \quad z = \frac{b}{a};$$

$$x_1 = \frac{c}{b}, \quad y_1 = -\frac{ac}{b^2}, \quad z_1 = -\frac{c^2}{ab}.$$

Formulæ connecting the Isogonal Coordinates of a Pair of Points (x, y, z), (x', y', z') *Inverse to each other with respect to a circle represented by*

$$lx + my + nz - k = 0.$$

The formulæ in question are

$$\frac{x'}{J_0 x - x_0 J} = \frac{y'}{J_0 y - y_0 J} = \frac{z'}{J_0 z - z_0 J} = \frac{k}{(J_0 - k)\, J + J_0 k},$$

where (x_0, y_0, z_0) denotes the centre of the circle, and

$$J = lx + my + nz - k, \quad J_0 = lx_0 + my_0 + nz_0 - k.$$

Ex. 1.—Let

$$J = x\,(\sin^2 B + \sin^2 C)\sin(B-C) + y\sin B\sin^2 A - z\sin C\sin^2 A$$
$$-2\sin B\sin C\sin(B-C).$$

The centre of the circle $J = 0$ is given by

$$x_0 = 0, \quad y_0 = \frac{\sin(B-C)\tan C}{\sin A\sin B},$$

and
$$z_0 = \frac{\sin(C-B)\tan B}{\sin A\sin C}.$$

In this case the formulæ ultimately become

$$\{J\sin(B-C) + 2\rho\sin B\sin C\}\, x' = 2\rho x\sin B\sin C,$$

$$\{\dots\}\, y' = 2\rho y\sin B\sin C - 2J\frac{\sin^2 C}{\sin A}\cos B,$$

$$\{\dots\}\, z' = 2\rho z\sin B\sin C + 2J\frac{\sin^2 B}{\sin A}\cos C,$$

where $$\rho = \sin^2 A - \tfrac{1}{2} \sin 2B \sin 2C.$$

Hence, writing the equation of a G.B. circle of the first system in the form

$$\lambda x + \mu y + \nu z = \frac{2 \sin B \sin C}{\sin A} \cot \omega,$$

where
$$\lambda = \sin B \sin C \left\{ \operatorname{cosec}^2 A - (\cot \omega - \cot \Omega)^2 \right\},$$
$$\mu = \sin B (\cot \omega - \cot C),$$
$$\nu = \sin C (\cot \omega - \cot B),$$

we see that the equation of its inverse with respect to J is

$$2\rho \sin B \sin C \left\{ \lambda x + \mu y + \nu z - \frac{2 \sin B \sin C}{\sin A} \cot \omega \right\} = k_1 J,$$

where
$$k_1 = 2 \operatorname{cosec} A \left\{ \sin B \sin C \sin \overline{B-C} \cot \omega + \mu \sin^2 C \cos B - \nu \sin^2 B \cos C \right\} = 0.$$

In other words, a G.B. circle of the first system is self-inverse with respect to J.

Ex. 2.—Let

$$J = x (\sin^2 B + \sin^2 C) + y \sin B \sin A + z \sin C \sin A - 2 \sin B \sin C.$$

In this case the centre of $J = 0$ coincides with the mid-point of BC,

or $$x_0 = 0, \quad y_0 = 2 \frac{\sin C}{\sin A}, \quad z_0 = 2 \frac{\sin B}{\sin A},$$

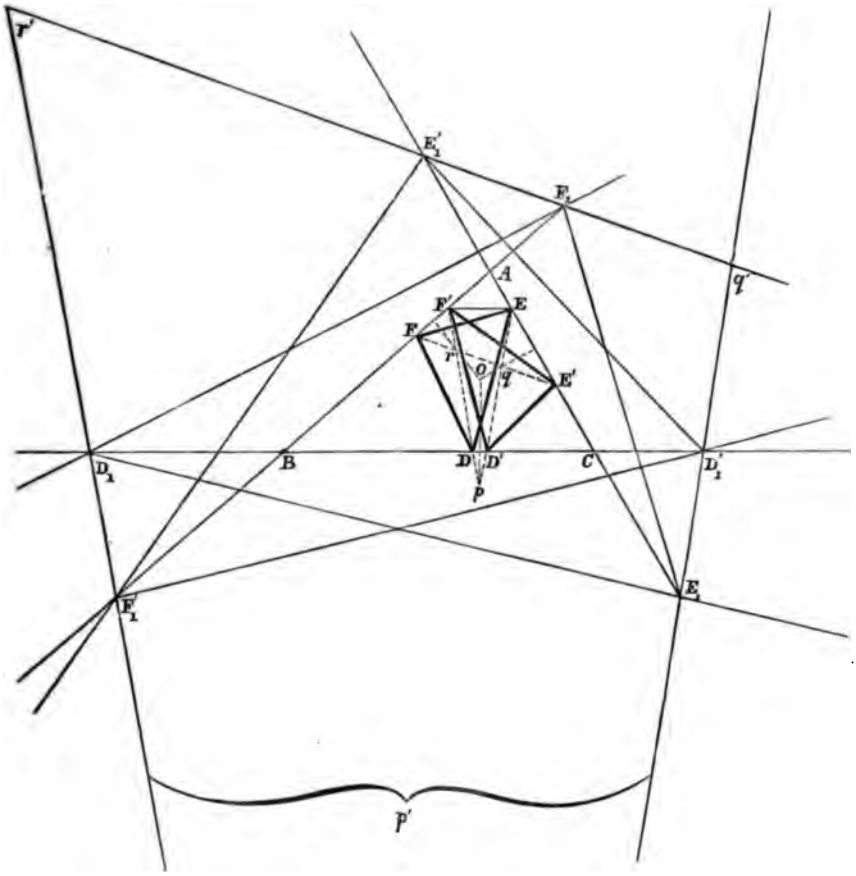
and the formulæ become

$$x' = x, \quad y' = y_0 - x \frac{\sin^2 B + \sin^2 C}{\sin A \sin B} - z \frac{\sin C}{\sin B};$$

$$z' = z_0 - x \frac{\sin^2 B + \sin^2 C}{\sin A \sin C} - y \frac{\sin B}{\sin C}.$$

[redacted]

Some Properties of Two Tucker Circles. By R. Tucker. Read June 14th, 1894. Recast October 24th, 1894.



$$[\angle F_1 = A = E'_1, \quad \angle F = A = E',$$
$$\angle D_1 = B = F'_1, \quad \angle D = B = F',$$
$$\angle E_1 = C = D'_1, \quad \angle E = C = D'.]$$

1. ABC is a triangle of which Ω, Ω' are the Brocard points. Take

$$\left.\begin{array}{c} AF = AF_1 = A\Omega = cb^2/\lambda, \quad BD = BD_1 = B\Omega = ac^2/\lambda, \\ CE = CE_1 = C\Omega = ba^2/\lambda\,; \\ AE' = AE_1' = A\Omega' = bc^2/\lambda, \quad BF' = BF_1' = B\Omega' = ca^2/\lambda, \\ CD' = CD_1' = C\Omega' = ab^2/\lambda \end{array}\right\} \ldots\ldots\text{(i.)}.$$

2. From these values, we get

$$\left.\begin{array}{lll} BF = c\,(\lambda - b^2)/\lambda, & CD = a\,(\lambda - c^2)/\lambda, & AE = b\,(\lambda - a^2)/\lambda \\ BD' = a\,(\lambda - b^2)/\lambda, & CE' = b\,(\lambda - c^2)/\lambda, & AF' = c\,(\lambda - a^2)/\lambda \end{array}\right\}\ldots\text{(ii.)},$$

$$\left.\begin{array}{lll} BF_1 = c\,(\lambda + b^2)/\lambda, & CD_1 = a\,(\lambda + c^2)/\lambda, & AE_1 = b\,(\lambda + a^2)/\lambda \\ BD_1' = a\,(\lambda + b^2)/\lambda, & CE_1' = b\,(\lambda + c^2)/\lambda, & AF_1' = c\,(\lambda + a^2)/\lambda \end{array}\right\}\ldots\text{(iii.)}.$$

Hence
$$\frac{AF'}{AE} = \frac{c}{b} = \frac{AF_1'}{AE_1}\,;$$

therefore FE, $F_1'E_1$ are parallels to BC. Also

$$\frac{AF}{AE'} = \frac{b}{c} = \frac{AF_1}{AE_1'}\,;$$

therefore FE', F_1E_1' are antiparallels to BC.

3. The circle round $FF'EE'$ passes also through D, D'. This circle we call (A). And also $F_1F_1'E_1E_1'D_1D_1'$ is circumscriptible. This circle we call (B).

4. The following results are easily got:—

$$FE' = abc/\lambda = DF' = ED' = F_1E_1' = D_1F_1' = E_1D_1'\,;$$

$$\Delta AFE' = b^2c^2\Delta/\lambda^2\,;$$

therefore
$$\Sigma AFE' = \Delta = \Sigma AF_1E_1'\,;$$

$$AF\,.\,AF' = b^2c^2\,(\lambda - a^2)/\lambda^2 = AE\,.\,AE'\,;$$

$$AF_1\,.\,AF_1' = b^2c^2\,(\lambda + a^2)/\lambda^2 = AE_1\,.\,AE_1'\,;$$

$$FF' = c\,(a^2 + b^2 - \lambda)/\lambda, \quad F_1F_1' = c\,(a^2 + b^2 + \lambda)/\lambda\,;$$

$$EF' = a\,(\lambda - a^2)/\lambda, \qquad E_1F_1' = a\,(\lambda + a^2)/\lambda.$$

5. The points are given (in trilinear coordinates) by

$$\left.\begin{array}{llll|l} D, & 0 & \lambda-c^2 & bc & 2\Delta/b\lambda \\ E, & ca & 0 & \lambda-a^2 & \times\ 2\Delta/c\lambda \\ F, & \lambda-b^2 & ab & 0 & 2\Delta/a\lambda \\ D', & 0 & bc & \lambda-b^2 & 2\Delta/c\lambda \\ E', & \lambda-c^2 & 0 & ca & \times\ 2\Delta/a\lambda \\ F', & ab & \lambda-a^2 & 0 & 2\Delta/b\lambda \end{array}\right\} \quad \ldots\ldots\ldots\ldots(\text{iv.}).$$

The subscript letters are easily seen from

$$\left.\begin{array}{llll|l} D_1, & 0 & \lambda+c^2 & -bc & \times\ 2\Delta/b\lambda \\ D_1', & 0 & -bc & \lambda+b^2 & \times\ 2\Delta/c\lambda \end{array}\right\} \quad \ldots\ldots\ldots\ldots(\text{v.}).$$

6. Assume $AF'E' = \phi$;

then $$\frac{\sin(A+\phi)}{\sin\phi} = \frac{AF'}{AE'} = \frac{\lambda-a^2}{bc},$$

whence $$\cot\phi = (2\lambda-k)/4\Delta = \tan\frac{\omega}{2},$$

i.e., $$AF'E' = 90^\circ - \frac{\omega}{2} = BD'F' = CE'D';$$

in like manner, $$= AEF = BFD = CDE.$$

7. Assume $AF_1'E_1' = \phi'$;

then $$\phi' = \frac{\omega}{2}.$$

8. Let ρ, ρ' be the radii of (A), (B) respectively; then, from § 4,

$$abc/\lambda = FE' = 2\rho\sin\phi = 2\rho\cos\frac{\omega}{2};$$

therefore $$\rho = \frac{4\Delta R}{2\lambda\cos\frac{\omega}{2}} = 2R\sin\frac{\omega}{2}$$

similarly, $$\rho' = 2R\cos\frac{\omega}{2} \qquad \ldots\ldots\ldots\ldots(\text{vi.});$$

hence $$\rho^2+\rho'^2 = 4R^2.$$

9. Since

$$EF = 2\rho \sin FE'E = 2\rho \sin B = 4R \sin B \sin \frac{\omega}{2} = 2b \sin \frac{\omega}{2} = D'E',$$

therefore the triangles DEF, $D'E'F'$ are similar to ABC, with modulus of similarity $2 \sin \frac{\omega}{2}$, and congruent to one another.

Analogous results hold for the triangles $D_1E_1F_1$, $D_1'E_1'F_1'$, with the modulus of similarity $2 \cos \frac{\omega}{2}$.

10. From § 5, we get the equations to (A) and (B) to be

$$\lambda^2 . \Sigma \alpha\beta\gamma = \Sigma a\alpha . \Sigma bc\alpha \, (\lambda \mp a^2) \quad \text{(vii.)},$$

the minus sign for (A) and the plus sign for (B).

The radical axes of these circles and the circumcircle are parallel to

$$\Sigma bc\alpha = 0,$$

i.e., their centres are on the circum-Brocardal axis. This is as it should be, for, from §§ 6 and 9, we see that (A) and (B) are Tucker circles. From a property of these circles (see Simmons in *Milne's Companion*, p. 131), the coordinates of the centres O, O_1, of (A), (B), respectively, are

$$\left.\begin{array}{l} \rho \sin\left(A + \frac{\omega}{2}\right), \quad \rho \sin\left(B + \frac{\omega}{2}\right), \quad \rho \sin\left(C + \frac{\omega}{2}\right) \\ \rho' \cos\left(A + \frac{\omega}{2}\right), \quad \rho' \cos\left(B + \frac{\omega}{2}\right), \quad \rho' \cos\left(C + \frac{\omega}{2}\right) \end{array}\right\} \ldots \text{(viii.)}.$$

11. The equations to $D'E$, $E'F$ are respectively

$$\left.\begin{array}{r} b\,(\lambda - a^2)\,\alpha + a\,(\lambda - b^2)\,\beta - abc\gamma = 0 \\ -abc\alpha + c\,(\lambda - b^2)\,\beta + b\,(\lambda - c^2)\,\gamma = 0 \end{array}\right\} \quad \text{(ix.)};$$

these intersect in

$$\frac{\alpha}{a} = \frac{\gamma}{c} = \frac{\beta\,(\lambda - b^2)}{-b\,(\lambda - c^2 - a^2)}, \quad \textit{i.e.}, \text{ in } q, \text{ say};$$

hence Ap, Bq, Cr are the symmedians of ABC, *i.e.*, the symmedian point of ABC is the centre of perspective of pqr and ABC.

From § 2, pqr is similar to the pedal triangle of ABC, and

$$qr = b \cos B \sin^2 \frac{\omega}{2} \Big/ \cos A \cos B \cos C.$$

12. If p', q', r' be the analogous points for the subscript points, analogous results hold, and the three triangles have a common centre of perspective. In this case

$$q'r' = b\cos B\cos^2\frac{\omega}{2}\Big/\cos A\cos B\cos C.$$

13. The circumcentre of ABC evidently bisects the join of O, O_1.

14. If d, e, f be the mid-points of DD', EE', FF', it is easily seen that dp, eq, fr meet in O.

The equation to eq is

$$-ab\alpha\,(\lambda+c^2-a^2)+(c^2-a^2)(\lambda-b^2)\,\beta+bc\gamma\,(\lambda-c^2+a^2)=0;$$

hence we see that it cuts the B-median in a point whose perpendicular distance from

$$CA = b\sin\omega.$$

15. From the figure we see that O, O_1 are respectively the incentres of pqr, $p'q'r'$. From § 6,

$$\angle\, OFE' = \frac{\omega}{2};$$

hence in-radius of $pqr = \rho\sin\frac{\omega}{2}$.

Similarly, in-radius of $p'q'r' = \rho'\cos\frac{\omega}{2}$.

16. Since $\qquad \angle\, AF\Omega = \frac{\pi}{2}-\frac{\omega}{2} = \angle\, BFD,$

therefore $\qquad \Omega FD = \omega,$

i.e., Ω is the positive Brocard point of DEF. In like manner, Ω' is the negative Brocard point of $D'E'F'$.

Let ω_1, ω_2 be the other Brocard points of these triangles. Then

$$\angle\, BDF = \frac{\pi}{2}-B+\frac{\omega}{2};$$

hence $\qquad \omega_1 DB = \frac{\pi}{2}-B+\frac{3\omega}{2};$

therefore ω_1 is given by the ratios

$$a^2c\cos\left(B-\frac{3\omega}{2}\right) : b^2a\cos\left(C-\frac{3\omega}{2}\right) : c^2b\cos\left(A-\frac{3\omega}{2}\right),$$

and ω_2 by the ratios

$$a^2b\cos\left(C-\frac{3\omega}{2}\right) : b^2c\cos\left(A-\frac{3\omega}{2}\right) : c^2a\cos\left(B-\frac{3\omega}{2}\right).$$

17. We can show that Ω is the negative Brocard point of $D_1E_1F_1$, and Ω' the positive point of $D_1'E_1'F_1'$.

The other points ω_1', ω_2' are given by the ratios

$$ac^2\sin\left(B-\frac{3\omega}{2}\right) : ba^2\sin\left(C-\frac{3\omega}{2}\right) : cb^2\sin\left(A-\frac{3\omega}{2}\right);$$

$$ab^2\sin\left(C-\frac{3\omega}{2}\right) : bc^2\sin\left(A-\frac{3\omega}{2}\right) : ca^2\sin\left(B-\frac{3\omega}{2}\right).$$

18. The equations to (A), (B) may be put in the form

$$(\alpha \mp a\sin\omega)(\beta \mp b\sin\omega)(\gamma \mp c\sin\omega) - \alpha\beta\gamma = 0$$

(see Simmons, *l.c.*, p. 136, or Casey, *Conics*, 2nd edition, p. 421).

The following presents were received during the Recess :—

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ERRATA.

P. 342, lines 3, 4, 5. Insert the factor $(\lambda\lambda_4)$ immediately before $(\lambda_1\lambda_2)$.

P. 352. On the last line but three, the terms in the series should contain ascending powers of x, *i.e.*, x with the B_1 term, x^2 with the B_2 term, and so on.

NOTES.

Mr. Johnson sends the following abstract of the amended form of his "Theorem in Inequalities" (p. 374). It will be seen that it is a generalization of the A.M. and G.M. theorem, and that the converse of the theorem is not true.

A Theorem in Inequalities.

The theorem is that, if p_r denote the mean value of the products of n positive quantities taken r together, then

$$p_1,\ p_2^{\frac{1}{2}},\ p_3^{\frac{1}{3}},\ \dots\ p_n^{1/n}$$

are in descending order of magnitude.

The following proof depends on the theory of equations.

The roots of the equation

$$f(x) \equiv x^n - n \,.\, p_1 x^{n-1} + \frac{n(n-1)}{2!} p_2 x^{n-2} - \dots + (-1)^n p_n = 0$$

are the n positive quantities. The arithmetic mean of the reciprocals of the n roots of $f(x) = 0$ exceeds the geometric mean. This gives

$$p_{n-1}^{1/(n-1)} > p_n^{1/n}.$$

Also, the roots of $f(x) = 0$ being real and positive, so are those of $f'(x) = 0$, $f''(x) = 0$, &c. The same consideration applied to the roots of these equations in turn gives

$$p_{n-2}^{1/(n-2)} > p_{n-1}^{1/(n-1)},\quad p_{n-3}^{1/(n-3)} > p_{n-2}^{1/(n-2)},\quad \text{\&c., \&c.}$$

Mr. Love sends the following note:—

Since my paper "On the Motion of Paired Vortices with a Common Axis" in this volume of the *Proceedings* (pp. 185–194) was published, I find that the problem there treated has been previously discussed by W. Gröbli. His dissertation, "Specielle Probleme über die Bewegung geradliniger paralleler Wirbelfäden," is published in the *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 1877. The author determines the actual path of each of the vortices in the combination, and gives figures for a number of special cases; he does not consider the relative motions of the two vortices on the same side of the axis of symmetry.

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the paper either has been printed* in extenso, *or an abstract of it given, in the "Proceedings."*]

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