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On Rational Quadratic Transformations.

BY

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ON RATIONAL QUADRATIC TRANSFORMATIONS.

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In my inaugural dissertation[1] certain quadratic transformations played an important part. Their generation by the combination of conjugate imaginary collineations (in point co-ordinates) was a novelty that seemed capable of a generalization which I sought in vain at that time. It was also desirable to invert the transformations. This inversion was accomplished for some of the transformations, but not for all, although it was evident that they were really birational. I had always intended to investigate the inversion of quadratic transformations in general, but a press of other work had caused the subject to be abandoned until very recently, when Dr. Dickson's interesting construction[2] of a special transformation recalled my interest to this field.

The main object of this paper was to have been the inversion of the general quadratic transformation, which is accordingly presented at the close of the paper. But the method of treatment led me also to the generalization mentioned above, furnishing a simple geometrical construction for the general transformation, and this is to be found in § 2.

§ 1. *Transformations in the reduced form.*

That a quadratic transformation

(1) $\rho y_1 = \phi\ (x_1, x_2, x_3),\ \rho y_2 = \psi\ (x_1, x_2, x_3),\ \rho y_3 = \chi(x_1, x_2, x_3)$

shall admit a rational inversion, it is necessary and suffi-

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[1] "Ueber die zu der Curve: $\lambda^3\mu + \mu^3\nu + \nu^3\lambda = o$ im projectiven Sinne gehörende mehrfache Ueberdeckung der Ebene."—*American Journal of Mathematics*, Vol. XIII, 1890.

[2] L. E. Dickson, "A quadratic Cremona transformation defined by a conic."—*Rendiconti del Circolo Matematico di Palermo*, T. IX, 1895. See also his paper in the current volume of the *Proceedings of the California Academy of Sciences*.

 January 31, 1898.

cient that the three conics $\phi = o$, $\psi = o$ and $\chi = o$ have three points, and only three, in common. If these points be all distinct (and of course not collinear) and the equations of the lines joining them in pairs be $\lambda = o$, $\mu = o$, $\nu = o$, then the equation of any conic through the three points must be of the form

$$p\,\mu\,\nu + q\,\nu\,\lambda + r\,\lambda\,\mu = o\,,$$

and the formulæ of transformation may be written in what we shall call the *reduced form:*

(2) $\rho\,y_i = p_i\mu\,\nu + q_i\nu\,\lambda + r_i\lambda\,\mu\,,\ (i = 1, 2, 3).$

When the formulæ are given in this form, it is clear that the general quadratic transformation is equivalent to the product $S_1\,Q\,S_2$ of a linear transformation S_1, of what we may call a *normal* quadratic transformation Q, and of a second linear transformation S_2. I append the explicit formulæ of these transformations:

$$(3)\quad S_1:\quad \begin{aligned} \rho\,\lambda &= l_1\,x_1 + l_2\,x_2 + l_3\,x_3 \\ \rho\,\mu &= m_1\,x_1 + m_2\,x_2 + m_3\,x_3 \\ \rho\,\nu &= n_1\,x_1 + n_2\,x_2 + n_3\,x_3 \end{aligned}$$

$$(4)\quad Q:\quad \sigma\,Y_1 = \mu\,\nu\,,\quad \sigma\,Y_2 = \nu\,\lambda\,,\quad \sigma\,Y_3 = \lambda\,\mu$$

$$(5)\quad S_2:\quad \begin{aligned} \tau\,y_1 &= p_1\,Y_1 + q_1\,Y_2 + r_1\,Y_3 \\ \tau\,y_2 &= p_2\,Y_1 + q_2\,Y_2 + r_2\,Y_3 \\ \tau\,y_3 &= p_3\,Y_1 + q_3\,Y_2 + r_3\,Y_2 \end{aligned}$$

The transformation can now be readily inverted, for the transformation inverse to $S_1\,Q\,S_2$ is $S_2^{-1}\,Q^{-1}\,S_1^{-1}$, where S_2^{-1}, Q^{-1}, S_1^{-1} are given by the formulæ:

$$(6)\quad S_2^{-1}:\quad \begin{aligned} \rho\,Y_1 &= P_1\,y_1 + P_2\,y_2 + P_3\,y_3 \\ \rho\,Y_2 &= Q_1\,y_1 + Q_2\,y_2 + Q_3\,y_3 \\ \rho\,Y_3 &= R_1\,y_1 + R_2\,y_2 + R_3\,y_3 \end{aligned}$$

P_1, Q_1, etc., denoting as usual the co-factors of p_1, q_1, etc., in the determinant of S_2, viz.:

$$(p\,q\,r) = \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}$$

$$(7)\quad Q^{-1}:\qquad \sigma\,\lambda = Y_2\,Y_3\,,\quad \sigma\,\mu = Y_3\,Y_1\,,\quad \sigma\,\nu = Y_1\,Y_2$$

$$(8)\quad S_1^{-1}:\qquad \begin{aligned} \tau\,x_1 &= L_1\,\lambda + M_1\,\mu + N_1\,\nu \\ \tau\,x_2 &= L_2\,\lambda + M_2\,\mu + N_2\,\nu \\ \tau\,x_3 &= L_3\,\lambda + M_3\,\mu + N_3\,\nu \end{aligned}$$

Compounding these formulæ, the inverse transformation may be conveniently written:

$$(9)\quad \begin{aligned} \rho\,x_1 &= L_1\,Y_2\,Y_3 + M_1\,Y_3\,Y_1 + N_1\,Y_1\,Y_2 \equiv \Phi\,(y_1\,, y_2\,, y_3) \\ \rho\,x_2 &= L_2\,Y_2\,Y_3 + M_2\,Y_3\,Y_1 + N_2\,Y_1\,Y_2 \equiv \Psi\,(y_1\,, y_2\,, y_3) \\ \rho\,x_3 &= L_3\,Y_2\,Y_3 + M_3\,Y_3\,Y_1 + N_3\,Y_1\,Y_2 \equiv \mathrm{X}\,(y_1\,, y_2\,, y_3) \end{aligned}$$

The above inversion is of course sufficiently simple, but I have preferred to write out the formulæ in detail, as some of the results will be convenient for comparison in the investigation of transformations in the general form which is to follow.

§ 2. *Geometrical theorems and constructions.*

Before proceeding to the problem of the inversion of transformations given in the general form, I will point out that the formulæ of § 1 lead to a theorem which seems to me to be a fundamental theorem in the geometrical theory of quadratic transformations.

From (7) we have

$$(10)\qquad \lambda : \nu = Y_3 : Y_1 \quad\text{and}\quad \mu : \nu = Y_3 : Y_2.$$

Hence, to the intersection of

$$(11)\qquad \lambda - k\,\nu = o \quad\text{and}\quad \mu - k'\,\nu = o$$

corresponds the intersection of

$$(12)\qquad Y_3 - k\,Y_1 = o \quad\text{and}\quad Y_3 - k'\,Y_2 = o,$$

where k and k' may have any values whatever. But $\lambda - k\,\nu = o$ and $Y_3 - k\,Y_1 = o$ represent for variable k two homographic pencils, while $\mu - k'\,\nu = o$ and $Y_3 - k'\,Y_2 = o$ represent a second pair of homographic pencils. A slight generalization of these formulæ leads us to the following:

THEOREM I: *Let A and A' be homographic pencils of rays, and let B and B' be two other homographic pencils of rays. If to each intersection of a ray of A with a ray of B correspond the intersection of the corresponding rays of A' and B', this correspondence can be represented by a quadratic transformation.*

The transformation is evidently birational, and the most general birational quadratic transformation can be represented in this way. It might indeed seem at first that the position of the pencils is slightly too general, but this is not the case. For, denoting the rays of the various pencils by

$$(13) \qquad \begin{array}{llll} (A) & p - \lambda q = o; & (A') & k - \lambda l = o; \\ (B) & r - \mu t = o; & (B') & m - \mu n = o, \end{array}$$

it is easily shown that

$$\begin{aligned} &(\lambda_1 - \lambda_2)(p - \lambda q) \\ &\equiv (\lambda - \lambda_2)(p - \lambda_1 q) - (\lambda - \lambda_1)(p - \lambda_2 q), \end{aligned}$$

with like formulæ for the other pencils. If now we choose $\lambda_1, \lambda_2, \mu_1, \mu_2$ so that

$$\begin{aligned} p - \lambda_2 q \equiv r - \mu_2 t \equiv g \\ k - \lambda_1 l \equiv m - \mu_1 n \equiv h \end{aligned}$$

and furthermore write

$$\frac{\lambda - \lambda_1}{\lambda - \lambda_2} = \lambda' \quad \frac{\mu - \mu_1}{\mu - \mu_2} = \mu',$$

then formulæ (13) become

$$(14) \qquad \begin{array}{ll} (A) & (p - \lambda_1 q) - \lambda' g = o; \\ (A') & h - \lambda'(k - \lambda_2 l) = o; \\ (B) & (r - \mu_1 t) - \mu' g = o; \\ (B') & h - \mu'(m - \mu_2 n) = o, \end{array}$$

and these are of the same form as (11) and (12).

Theorem I can also be proved directly without difficulty by the method of synthetic geometry. To a ray of A or of B corresponds a ray of A' or of B' respectively. But a straight line in general will be generated by pencils A and B in perspective. The corresponding pencils A' and B'

are then homographic, and generate a conic. But the birational one-to-one correspondence in which a straight line in general is converted into a conic is a quadratic Cremona transformation.

Since a pair of homographic pencils determine a conic through their vertices as the locus of the intersections of corresponding rays, and, *vice versa*, a conic determines a pair of homographic pencils whose vertices are any two points of the conic, we can restate Theorem I in a form which may be more convenient for practical work:

Theorem II: *Let* α *and* β *be any two conics. Let* A *and* A' *be any two points on* α, B *and* B' *any two points on* β. P *and* P' *are corresponding points in a birational quadratic transformation if* PA *and* $P'A'$ *meet on* α, *while* PB *and* $P'B'$ *meet on* β.

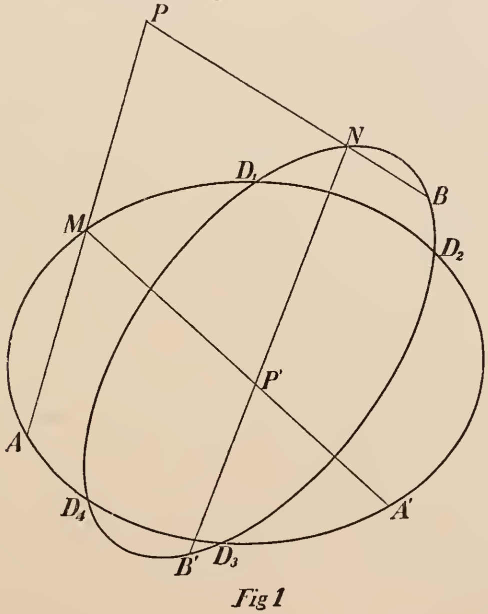


Fig 1

The construction is illustrated by fig. 1. The four points, D_1, D_2, D_3, D_4, in which α and β meet are evidently self-corresponding points of the transformation. They are in general the only such points, but if α and β coincide, all the points of these coincident conics are self-corresponding. This special case is Dr. Dickson's transformation.

A and B are vertices of one of the fundamental triangles, A' and B' the corresponding vertices of the other. The remaining vertices C and C' are readily found by the following construction (see fig. 2):

Let $A\,B$ meet α again in Q_1, and β again in P_1; let $A'\,B$

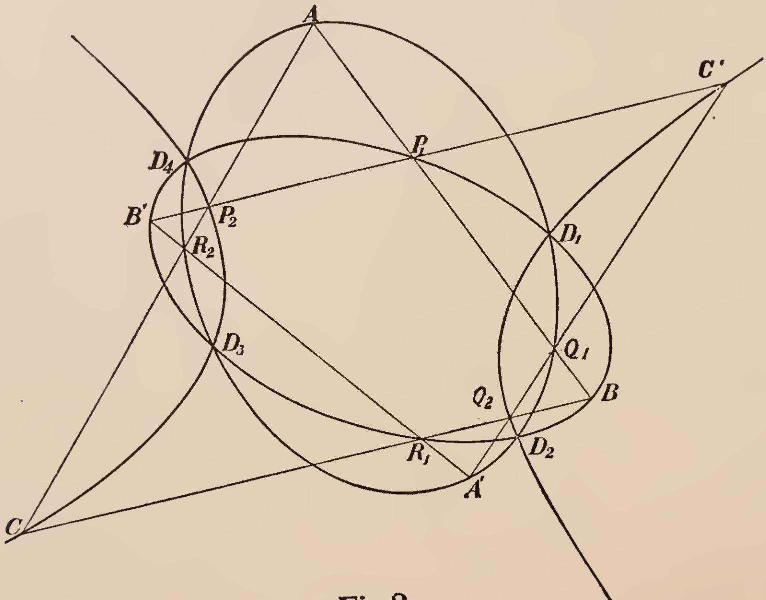


Fig. 2

meet α again in R_2 and β again in R_1. Then $A\,R_2$ and $B\,R_1$ meet in C, and $A'\,Q_1$, and $B'\,P_1$ meet in C'.

It is obvious that C and C' will lie on a third conic γ through D_1, D_2, D_3, D_4. In the case of Dr. Dickson's transformation, however, where α and β coincide, the above construction shows that C and C' coincide, being the intersection of $A\,B'$ and $A'\,B$.

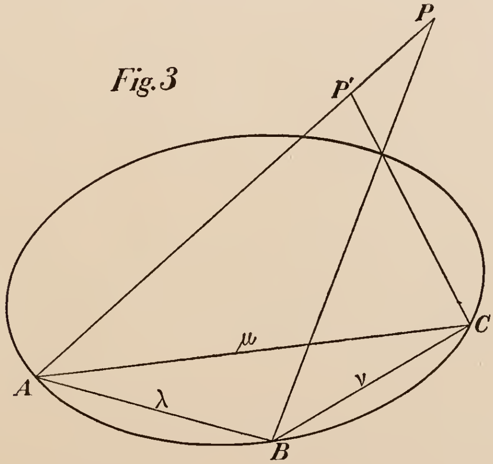
Finally, the transformation is fully determined if the double points D_1, D_2, D_3, D_4 and the vertices of one of the fundamental triangles, say A, B, C, be given. These points suffice to determine the conics α, β and γ. The vertices of the other triangle can be found by the following theorem, which is easily deduced from the relations given above:

Theorem III: *Let α, β, γ be three conics of a pencil through four points, and let A, B, C be three points chosen arbitrarily on α, β, γ respectively. Let AB meet α again in Q_1 and β again in P_1; let BC meet β again in R_1 and γ again in Q_2; let CA meet γ again in P_2 and α again in R_2. Then P_1P_2, Q_1Q_2, R_1R_2 are the sides of a triangle whose vertices A' B' C' lie on α, β, γ respectively.*

This construction is illustrated in Figure 2.

A simple construction for a special case in which, as in Dr. Dickson's transformation, there is a self-corresponding conic, is shown in Fig. 3. A, B and C are any three points on a conic. All pairs of corresponding points P

Fig. 3



and P^1 are collinear with A. Furthermore, PB and $P'C$ meet on the conic. All points on the conic evidently correspond to themselves. A counts twice as a vertex of each fundamental triangle, B and C are the other vertices. If $\lambda:\mu:\nu$ are the co-ordinates of P with reference to the triangle ABC, $\lambda':\mu':\nu'$ those of P' and $\mu\nu+\nu\lambda+\lambda\mu=o$ the equation of the conic, the formulæ of the transformation will be

$$\lambda' : \mu' : \nu' = \mu(\nu+\lambda) : -\mu\nu : -\nu^2$$

and those of the inverse transformation will be

$$\lambda : \mu : \nu = \nu' (\lambda' + \mu') : -\mu'^2 : -\mu' \nu'.$$

§ 3. *Transformations in the general form.*

Suppose the formulæ of transformation to be given in the general form

$$(15) \quad \begin{aligned} \rho\, y_1 = \phi\, (x_1, x_2, x_3) &\equiv a_1 x_1^2 + b_1 x_2^2 + c_1 x_3^2 + \\ &\quad 2 f_1 x_2 x_3 + 2 g_1 x_3 x_1 + 2 h_1 x_1 x_2 \\ \rho\, y_2 = \psi\, (x_1, x_2, x_3) &\equiv a_2 x_1^2 + b_2 x_2^2 + c_2 x_3^2 + \\ &\quad 2 f_2 x_2 x_3 + 2 g_2 x_3 x_1 + 2 h_2 x_1 x_2 \\ \rho\, y_3 = \chi\, (x_1, x_2, x_3) &\equiv a_3 x_1^2 + b_3 x_2^2 + c_3 x_3^2 + \\ &\quad 2 f_3 x_2 x_3 + 2 g_3 x_3 x_1 + 2 h_3 x_1 x_2 \end{aligned}$$

It is then in given cases frequently very difficult to transmute these formulæ into the reduced form (3), even when we know that they are really birational. The Jacobian of ϕ, ψ, χ must have a double point at each of their intersections, and must therefore degenerate into the triangle of which these intersections are the vertices; that is to say, the equation of the Jacobian will be

$$k \lambda \mu \nu = o.$$

The test that the Jacobian shall so degenerate, is that its Hessian shall do likewise, so that the coefficients of the Jacobian and of its Hessian must be proportional. But, supposing this test to be satisfied, it is usually by no means easy to factor the Jacobian. The formulæ of the inverse transformation can however be established in another way, without actually finding the coefficients of the reduced form (3), and to accomplish this is the object of the present paper.

If we write $\lambda = o$ in equations (3), we find

$$y_1 : y_2 : y_3 = p_1 : p_2 : p_3.$$

Similarly, $\mu = o$ gives

$$y_1 : y_2 : y_3 = q_1 : q_2 : q_3$$

and $\nu = o$ gives

$$y_1 : y_2 : y_3 = r_1 : r_2 : r_3.$$

In other words, the sides of the triangle λ, μ, ν are converted into the above three points by the transformation. These points are then the vertices of the triangle $\Upsilon_1 \Upsilon_2 \Upsilon_3$; that is to say, they are the three points common to the conics $\Phi = o$, $\Psi = o$, $X = o$ of the inverse transformation. Indeed, this can readily be seen by substituting the co-ordinates of these points in (6). Our first problem will be to establish the equations which determine these co-ordinates.

Again, writing $x_2 = o$, $x_3 = o$ in (15), we find

$$y_1 : y_2 : y_3 = a_1 : a_2 : a_3 .$$

In other words, $\Psi = o$ and $X = o$ meet also in the point $(a_1 : a_2 : a_3)$. Similarly, $X = o$ and $\Phi = o$ meet also in $(b_1 : b_2 : b_3)$, and $\Phi = o$ and $\Psi = o$ meet in $(c_1 : c_2 : c_3)$. We have therefore found five points on each of these conics.

In what follows, we shall frequently denote these points by the single letters p, q, r, a, b, c.

§ 4. *Determination of the equations for p, q, r.*

We have just seen that the points p, q, r are the fundamental points of the inverse transformation. Our first problem is therefore to establish the equations for these points. We shall find that it will not be necessary to solve the equations, but that they appear directly in the formulæ of the inverse transformation.

A comparison of the general form (15) with the reduced form (3) gives us the following identities:

$$(16)\quad \begin{aligned}
a_1 &= p_1\, m_1\, n_1 + q_1\, n_1\, l_1 + r_1\, l_1\, m_1 \\
b_1 &= p_1\, m_2\, n_2 + q_1\, n_2\, l_2 + r_1\, l_2\, m_2 \\
c_1 &= p_1\, m_3\, n_3 + q_1\, n_3\, l_3 + r_1\, l_3\, m_3 \\
2 f_1 &= p_1\,(m_2\, n_3 + m_3\, n_2) + q_1\,(n_2\, l_3 + n_3\, l_2) + \\
&\qquad r_1\,(l_2\, m_3 + l_3\, m_2) \\
2 g_1 &= p_1\,(m_3\, n_1 + m_1\, n_3) + q_1\,(n_3\, l_1 + n_1\, l_3) + \\
&\qquad r_1\,(l_3\, m_1 + l_1\, m_3) \\
2 h_1 &= p_1\,(m_1\, n_2 + m_2\, n_1) + q_1\,(n_1\, l_2 + n_2\, l_1) + \\
&\qquad r_1\,(l_1\, m_2 + l_2\, m_1)
\end{aligned}$$

with similar identities for the other coefficients.

Solving these equations for the quantities $m_1 n_1$, $m_2 n_2$, etc., we have, denoting again by $(p\,q\,r)$ the determinant of the transformation S_2, and by P_1, P_2, etc., the co-factors of p_1, p_2, etc., in that determinant:

$$
\begin{aligned}
&(p\,q\,r)\, m_1 n_1 = P_1 a_1 + P_2 a_2 + P_3 a_3 \equiv P_a\,, \\
&(p\,q\,r)\, m_2 n_2 = P_1 b_1 + P_2 b_2 + P_3 b_3 \equiv P_b\,, \\
(17)\quad &(p\,q\,r)\, m_3 n_3 = P_1 c_1 + P_2 c_2 + P_3 c_3 \equiv P_c\,, \\
&(p\,q\,r)\,(m_2 n_3 + m_3 n_2) = 2\,P_f \\
&(p\,q\,r)\,(m_3 n_1 + m_1 n_3) = 2\,P_g \\
&(p\,q\,r)\,(m_1 n_2 + m_2 n_1) = 2\,P_h \qquad \text{etc., etc.}
\end{aligned}
$$

From these identities we readily deduce the following:

$$
\begin{aligned}
&(p\,q\,r)^2\, n_2 n_3\,(l_3 m_2 - l_2 m_3) = P_b Q_c - P_c Q_b \equiv \\
&\qquad (p\,q\,r)\,(r\,b\,c) \\
(18)\quad &(p\,q\,r)^2 n_2^2\,(l_3 m_2 - l_2 m_3) = 2\,(P_b Q_f - P_f Q_b) \equiv \\
&\qquad 2\,(p\,q\,r)\,(r\,b\,f) \\
&(p\,q\,r)^2\, n_3^2\,(l_3 m_2 - l_2 m_3) = 2\,(P_f Q_c - P_c Q_f) \equiv \\
&\qquad 2\,(p\,q\,r)\,(r\,f\,c)
\end{aligned}
$$

and a comparison of the left-hand members leads immediately to the result:

$$(19)\qquad (r\,b\,c)^2 = 4\,(r\,b\,f)\,(r\,f\,c).$$

In the same way we can show that

$$(20)\qquad (r\,c\,a)^2 = 4\,(r\,c\,g)\,(r\,g\,a),$$

and finally that

$$(21)\qquad (r\,a\,b)^2 = 4\,(r\,a\,h)\,(r\,h\,b).$$

But a slight consideration of the symmetry of the identities (16) and (17), shows us that equations (19), (20) and (21) are equally true if either p or q be substituted for r, and hence that the three conics

$$
\begin{aligned}
&(y\,b\,c)^2 - 4\,(y\,b\,f)\,(y\,f\,c) = o \\
(22)\quad &(y\,c\,a)^2 - 4\,(y\,c\,g)\,(y\,g\,a) = o \\
&(y\,a\,b)^2 - 4\,(y\,a\,h)\,(y\,h\,b) = o
\end{aligned}
$$

all pass through the three points p, q and r.

We can then infer that the conics $\Phi = o$, $\Psi = o$ and

$X = o$ of the inverse transformation are at most linear functions of the conics of (22). We shall see that they are identical with the latter.

§ 5. *Identification of the equations just obtained with those of the inverse transformation.*

From the form of equations (22), we see that the first of these equations is also satisfied by the points b and c, the second by c and a, the third by a and b. But these are exactly the conditions which were to be satisfied (§ 3) by the curves $\Phi = o$, $\Psi = o$, $X = o$.

The left-hand members of equations (22) *are therefore, except for eventual numerical factors, identical with the forms* Φ, Ψ, X *of the inverse transformation.*

These factors shall now be determined. Let

$$(23)\qquad \begin{aligned} \rho\, x_1 &= k_1\,[(y\,b\,c)^2 - 4\,(y\,b\,f)\,(y\,f\,c)] \\ \rho\, x_2 &= k_2\,[(y\,c\,a)^2 - 4\,(y\,c\,g)\,(y\,g\,a)] \\ \rho\, x_3 &= k_3\,[(y\,a\,b)^2 - 4\,(y\,a\,h)\,(y\,h\,b)] \end{aligned}$$

The consideration of a single pair of corresponding points will be sufficient to determine the ratios of $k_1 : k_2 : k_3$. Let us write $y_1 : y_2 : y_3 = 1 : o : o$, when

$$(24)\qquad \begin{aligned} \rho\, x_1 &= k_1\,[(b_2 c_3 - b_3 c_2)^2 - 4\,(b_2 f_3 - b_3 f_2)\,(f_2 c_3 - f_3 c_2)] \\ \rho\, x_2 &= k_2\,[(c_2 a_3 - c_3 a_2)^2 - 4\,(c_2 g_3 - c_3 g_2)\,(g_2 a_3 - g_3 a_2)] \\ \rho\, x_3 &= k_3\,[(a_2 b_3 - a_3 b_2)^2 - 4\,(a_2 h_3 - a_3 h_2)\,(h_2 b_3 - h_3 b_2)] \end{aligned}$$

which can be reduced by substituting from (16) into

$$(25)\qquad \begin{aligned} \rho\, x_1 &= -\,k_1 L_1 M_1 N_1\,(L_1 Q_1 R_1 + M_1 R_1 P_1 + N_1 P_1 Q_1) \\ \rho\, x_2 &= -\,k_2 L_2 M_2 N_2\,(L_2 Q_1 R_1 + M_2 R_1 P_1 + N_2 P_1 Q_1) \\ \rho\, x_3 &= -\,k_3 L_3 M_3 N_3\,(L_3 Q_1 R_1 + M_3 R_1 P_1 + N_3 P_1 Q_1) \end{aligned}$$

But, substituting $y_1 : y_2 : y_3 = 1 : o : o$ in formulæ (9), the equivalent of (23), and comparing the result with (25), we find that

$$(26)\qquad k_1 L_1 M_1 N_1 = k_2 L_2 M_2 N_2 = k_3 L_3 M_3 N_3 .$$

It is now easy to show by means of (16) that

$$
(27) \qquad
\begin{aligned}
2\,(b\,f\,c) &= (p\,q\,r)\; L_1\, M_1\, N_1 \\
2\,(c\,g\,a) &= (p\,q\,r)\; L_2\, M_2\, N_2 \\
2\,(a\,h\,b) &= (p\,q\,r)\; L_3\, M_3\, N_3
\end{aligned}
$$

and hence by (26) that

$$k_1 : k_2 : k_3 = (b\,f\,c)^{-1} : (c\,g\,a)^{-1} : (a\,h\,b)^{-1}$$

Substituting these values in (23) *we have the formulæ of the inverse transformation in the following final form:*

$$
(28) \qquad
\begin{aligned}
\rho\, x_1 &= (b\,f\,c)^{-1}\,[(y\,b\,c)^2 - 4\,(y\,b\,f)\,(y\,f\,c)] \\
\rho\, x_2 &= (c\,g\,a)^{-1}\,[(y\,c\,a)^2 - 4\,(y\,c\,g)\,(y\,g\,a)] \\
\rho\, x_3 &= (a\,h\,b)^{-1}\,[(y\,a\,b)^2 - 4\,(y\,a\,h)\,(y\,h\,b)]
\end{aligned}
$$

With these formulæ the object of the present paper is accomplished. The method will fail in special cases, but in such cases the numerical relations between the coefficients are always such as to make the inversion by other methods comparatively easy. For instance, we see by (28) that the point f is the polar of the line $b\,c$ with respect to $\Phi = o$. Now if it happen that f lies on this line our formula for Φ may vanish identically; but in this case evidently Φ degenerates into two lines of which $b\,c$ is one. Again, the method will sometimes fail when two of the points p, q, r coincide; but in this case the Jacobian has a square factor, which can therefore readily be found, and the formulæ reduced at once.

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1. THE importance of the quadratic Cremona (i. e., birational) transformation is enhanced by the fact that any Cremona transformation of arbitrary order can be obtained by applying a succession of quadratic ones. The theorem that the deficiency of an algebraic curve is unchanged under a Cremona transformation thus requires proof only for quadratic transformations.

A quadratic transformation T

$$x':y':z'=U\,(x,y,z):V\,(x,\,y,\,z):\;W\,(x,\,y,\,z)$$

will be birational (i. e., to any point $x':y':z'$ will correspond but one point $x:y:z$) if and only if the three conics $U=o$, $V=o$, $W=o$ have in common three points of intersection, say I_1, I_2, I_3, called principal points. The most general correspondence thus required between the straight lines of one plane and the conics through three fixed points in a second plane may be obtained geometrically by Steiner's Construction. An arbitrary line of the first plane, together with two fixed lines a and b not lying in either plane, determine an hyperboloid of one sheet which is cut by the second plane in a conic passing through three fixed points, one on a, one on b, and a third on that line of the first plane meeting both a and b.

In applications to the resolution of higher singularities of plane curves, we desire geometric constructions in a single plane.

2. *Analytic reduction of T to a normal type Q.*

By a proper choice of $p:q:r$, the line $px+qy+rz=o$ is transformed by T^{-1} (the inverse of T) into a conic $pU+qV+rW=o$ which breaks up into a pair of lines through

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I_1, I_2 I_3. Thus if the latter are distinct points, we can find three lines

$$(1)\qquad \begin{aligned} \xi &\equiv p_1 x + q_1 y + r_1 z = o \\ \eta &\equiv p_2 x + q_2 y + r_2 z = o \\ \zeta &\equiv p_3 x + q_3 y + r_3 z = o \end{aligned}$$

which T^{-1} transforms into the degenerate conics $L_2 \cdot L_3 = o$, $L_1 \cdot L_3 = o$, $L_1 \cdot L_2 = o$ respectively, where $L_1 \ (x, y, z) = o$ is the line joining I_2 with I_3, etc. We may write, after solving (1),

$$L_i \ (x, y, z) \equiv \lambda_i \ (\xi, \eta, \zeta) \quad (i = 1, 2, 3).$$

Hence

$$(2)\ \rho\xi' \equiv \rho(p_1 x' + q_1 y' + r_1 z') = p_1 U + q_1 V + r_1 W = L_2 \cdot L_3 \equiv \lambda_2 \cdot \lambda_3,$$

and similarly for η' and ζ'. Thus T written in the coördinates ξ, η, ζ becomes

$$(3)\qquad \xi' : \eta' : \zeta' = \lambda_2 \lambda_3 : \lambda_1 \lambda_3 : \lambda_1 \lambda_2,$$

which we denote as the transformation $\overline{T}$, geometrically identical with T. If S denotes the linear transformation [see (1)]:

$$\rho \xi = p_1 x + q_1 y + r_1 z \text{ , etc.,}$$

we have $$\overline{T} = S^{-1} T S \ ;$$

for S^{-1} transforms the point $(\xi : \eta : \zeta)$ into $(x : y . z)$, which T transforms into $(x' : y' : z')$, which finally S transforms to $(\xi' : \eta' : \zeta')$, the intermediate eliminations being expressed by (2).

To obtain a further reduction, denote by S_1 the linear transformation

$$\xi' = \lambda_1 \ (\xi, \eta, \zeta), \ \eta' = \lambda_2 \ (\xi, \eta, \zeta), \ \zeta' = \lambda_3 \ (\xi, \eta, \zeta).$$

Then $S_1^{-1} \ \overline{T}$ becomes the normal type of transformation Q:

$$\xi' : \eta' : \zeta' = \eta \ \zeta : \xi \ \zeta : \xi \ \eta = \frac{1}{\xi} : \frac{1}{\eta} : \frac{1}{\zeta} \ ,$$

as we may readily verify that $\overline{T} = S_1 \ Q$.

The most general quadratic Cremona transformation with three distinct principal points may thus be written

$$T = S \ S_1 \ Q \ S^{-1}.$$

3. An elegant geometric construction for T follows readily from (3) as remarked recently by Prof. Haskell.[1] Thus $\lambda_2(\lambda_3 - \rho\lambda_1) = o, \;\; \lambda_1(\lambda_3 - \sigma\lambda_2) = o$

are transformed by (3) into respectively

$$\xi - \rho\zeta = o, \;\; \eta - \sigma\zeta = o.$$

Hence if we set up two projective pencils

$$\lambda_3 - \rho\lambda_1 = o, \;\; \xi - \rho\zeta = o$$

by means of a conic, and similarly for the second pair of pencils, we find for an arbitrary point, determined as the intersection of the rays $\lambda_3 - \rho'\lambda_1 = o$, $\lambda_3 - \sigma'\lambda_2 = o$, the transformed point by taking the intersection of the transformed rays $\xi - \rho'\zeta = o$, $\eta - \sigma'\zeta = o$.

4. The normal type of transformation Q transforms

$$a\xi + b\eta + c\zeta = o \text{ into } a\eta\zeta + b\xi\zeta + c\xi\eta = o.$$

In particular, the line $b\eta + c\zeta = o$ through a vertex is transformed into the line-pair $\xi = o$, $b\zeta + c\eta = o$, the latter evidently making with the sides $\eta = o$, $\zeta = o$ angles equal to those made by the given line. The vertex $\zeta = \eta = o$ is transformed into $\xi = o$, the side opposite. Neglecting the vertices, every bisector of an interior or exterior angle of the triangle is a self-corresponding line.

If we make the construction on a sphere, we have a simple division into forty-eight compartments by the sides and bisectors. Each of the four centers of the inscribed and escribed circles of the triangle of reference and each of the four analogous points for the symmetrical triangle is a vertex for six triangular compartments. These eight centers are the fixed points under Q, which produces a "projective rotation" through 180° of the six compartments about each center.

5. An immediate generalization of the transformation Q is obtained by making a correspondence between any point

[1] Proc. Cal. Acad. Sci., 3d Ser., Math.-Physics, Vol. I.

P and the intersection P' of the straight lines harmonically separated from P by each pair of opposite sides of a complete quadrilateral $A\ C\ B\ D$. Taking its diagonal triangle $G\ E\ F$ as our triangle of reference, viz.:

$$GF \equiv x = o\,, \quad FE \equiv y = o\,, \quad EG \equiv z = o\,,$$

the vertices of the quadrilateral may be defined by

$$A\ (a:b:c),\ B\ (-a:b:c),\ D\ (a:b:-c),\ C\ (a:-b:c).$$

The line GP' separated from $P\ (x_1 : y_1 : z_1)$ and hence from $G\ P$, viz.,

$x - \frac{x_1}{z_1} z = o$, harmonically by

$$G\ D : x + \frac{a}{c} z = o\,, \quad G\ A : x - \frac{a}{c} z = o$$

has the equation $\qquad x - \frac{a^2}{c^2} \frac{z_1}{x_1} \cdot z = o\,.$

Similarly for $F\ P' : x - \frac{a^2}{b^2} \frac{y_1}{x_1} \cdot y = o\,.$

Hence P' has the coördinates

$$\frac{a^2}{x_1} : \frac{b^2}{y_1} : \frac{c^2}{z_1}.$$

The only fixed points are the four vertices. If one of these should be the unit point $1:1:1$, the transformation is Q.

6. Another normal type of quadratic Cremona transformation[1] is defined geometrically by keeping fixed one pair of opposite sides of a hexagon inscribed in a given "base conic" and making correspond the two variable points of intersection of the other two pairs of opposite sides. Let the fixed pair, $A\ C$ and $D\ B$, intersect in the point O, and take as triangle of reference:

$$D\ O \equiv \xi = o,\ A\ O \equiv \eta = o,\ A\ D \equiv \zeta = o.$$

and as base conic $\qquad a\,\xi\,\eta + b\,\xi\,\zeta + c\,\eta\,\zeta + d\,\zeta^2 = o.$

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[1] L. E. Dickson, Rendiconti del Circolo Matematico di Palermo, Vol. IX, 1895; also, American Mathematical Monthly, 1895.

The point P' $(\xi' : \eta : \zeta')$ corresponding by the construction to P $(\xi : \eta : \zeta)$ is readily found to have the coördinates[1]

$$T_1: \qquad \xi' : \eta' : \zeta' = \xi\,\zeta : \eta\,\zeta : (-1/d)\,(a\,\xi\,\eta + b\,\xi\,\zeta + c\,\eta\,\zeta).$$

The vertices and sides of triangle $A\,O\,D$ are transformed into the sides and vertices of triangle $C\,O\,B$. Repeating the transformation, we find for T_1^2:

$$\xi' = \xi,\ \eta' = \eta,\ \zeta' = \frac{a}{\dfrac{a}{\zeta} + \dfrac{b}{\eta} + \dfrac{c}{\xi}} - \frac{(b\,\xi + c\eta)}{d},$$

the proportionality factor in T_1 dropping out of T_1^2.

The point which T_1^2 leaves fixed are given by

$$(b\,\xi + c\,\eta)\ (a\,\xi\,\eta + b\,\xi\,\zeta + c\,\eta\,\zeta + d\,\zeta^2) = o,$$

viz., the base conic and the line $O\,K$ joining O to the intersection K of $A\,D$ and $B\,C$, upon which therefore an involution is marked out by T_1. For the special case $b = c = o$, when C coincides with A, B with D, we have Hirst's Construction[2], by which P and P' are conjugate points with respect to the base conic; thus $T_1^2 = 1$ in this case.

In general, if S' is the linear transformation

$$\xi' : \eta' : \zeta' = \eta : \xi : \frac{-1}{a}\,(b\,\xi + c\,\eta + d\,\zeta)$$

we find that $T_1\,S'$ reduces to the normal type Q.

7. *Investigation of the most general quadratic Cremona transformation,*

$$T: \qquad x' = \frac{U\,(x, y)}{W\,(x, y)}\,,\quad y' = \frac{V\,(x, y)}{W\,(x, y)}$$

such that corresponding points P: (x, y) and P': (x', y') are

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[1] I exclude the trival case $a = o$, since then the whole plane is transformed into $B\,C$, a part of the degenerate base conic. Also I exclude $d = o$, when O is on the base conic. If A and D have coincided (viz.: at O), every point of the plane is transformed into the base conic; if C and B have coincided (viz.: with O), every point is transformed with the single point O. Not trivial are the cases when D and C, or when A and B, coincide at O. The two cases are quite analogous. For either, the transformation and its inverse have but two distinct principal points. The formulæ are readily obtained, taking for example $A\,O\,B$ as triangle of reference.

[2] Cf. Scott, "Modern Analytical Geometry," pp. 219-224.

collinear with a given point O: (a, b). We must then have identically in x and y:

$$(1) \qquad \frac{x-a}{y-b} \equiv \frac{x'-a}{y'-b} \equiv \frac{U-aW}{V-bW}$$

Hence there must exist a linear function $L\ (x, y)$ such that

$$(2) \qquad U-aW \equiv (x-a)\ L\ ; \quad V-bW \equiv (y-b)\ L$$

from which, $\qquad bU-aV \equiv (bx-ay)\ L.$

For the case in which $U=o$, $V=o$, $W=o$ have three distinct points of intersection, these can not all lie on $L=o$, since T would then reduce to a linear transformation. Hence one point of intersection is O: (a, b) and the other two lie on $L=o$, say A and D. Hence by virtue of (2) our transformation T may be written

$$(3) \qquad x'-a = \frac{(x-a)\ L}{W}, \quad y'-b = \frac{(y-b)\ L}{W}.$$

The line $L=o$ is transformed into the point (a, b), so that the the latter is not fixed. Hence the fixed points of (3) all lie on the "base conic,"

$$(4) \qquad W=L.$$

T contains nine arbitrary constants, depending on the arbitrary point (a, b), the arbitrary conic through it

$$(5) \qquad W \equiv \alpha\,(x-a)^2 + \beta\,(x-a)\,(y-b) + \gamma\,(y-b)^2 + \delta\,(x-a) + \epsilon\,(y-b) = o,$$

and the arbitrary points A and D on $W=o$ (distinct and different from O) to determine the line $L=o$. We will fix A and D as the intersections of $W=o$ with

$$(6) \qquad \left.\begin{array}{ll} OA: & X \equiv p_1\,(x-a) + q_1\,(y-b) = o \\ OD: & Y \equiv p_2\,(x-a) + q_2\,(y-b) = o \end{array}\right\}$$

Solving, $\quad \begin{cases} \rho\,(x-a) = q_2 X - q_1 Y \\ \rho\,(y-b) = -p_2 X + p_1 Y \end{cases} \qquad \rho = p_1 q_2 - p_2 q_1$

Hence (7) $\quad W \equiv a_2 X^2 + \lambda XY + a_1 Y^2 + \delta_2 X - \delta_1 Y = o$

where $\quad \begin{cases} a_1 \equiv \alpha q_1^2 - \beta p_1 q_1 + \gamma p_1^2, & \delta_1 \equiv \rho\,(\delta q_1 - \epsilon p_1) \\ a_2 \equiv \alpha q_2^2 - \beta p_2 q_2 + \gamma p_2^2, & \delta_2 \equiv \rho\,(\delta q_2 - \epsilon p_2) \\ \lambda \equiv \beta\,(p_1 q_2 + p_2 q_1) - 2\,\alpha\, q_1 q_2 - 2\,\gamma p_1 p_2 & \end{cases}$

For the X, Y coördinates of A and D we have

$$A: \quad \left(\frac{-\delta_2}{a_2}, o\right); \quad D: \quad \left(o, \frac{\delta_1}{a_1}\right)$$

so that the equation to $A\,D$ is

$$L \equiv \frac{a_1}{\delta_1} Y - \frac{a_2}{\delta_2} X - 1 = o,$$

the multiplicative constant in L/W of (3) being taken into W. The base conic $W = L$ is met by $X = o$ and $Y = o$ in respectively

$$B: \left(o, \frac{1}{\delta_1}\right); \; C: \left(\frac{-1}{\delta_2}, o\right).$$

Our transformation (3) becomes for the X, Y coördinates:

$$(3') \qquad X' = \frac{L\,(X, Y)}{W(X, Y)} \cdot X\,, \quad Y' = \frac{L\,(X, Y)}{W(X, Y)} \cdot Y.$$

We may verify that $A\,P$ and $B\,P'$ meet in a point on the base conic (4), From Pascal's Theorem on the inscribed hexagon, it follows that $D\,P$ and $C\,P'$ meet on this base conic. We may thus invert the relations (3′) by interchanging A with C, B with D, and P with P', the base conic remaining fixed, viz.:

$$a_2 X^2 + \lambda X Y + a_1 Y^2 - L - M - 1 = o$$

where the equation of $B\,C$ has been written in the form

$$M \equiv \delta_1 Y - \delta_2 X - 1 = o.$$

We have therefore to replace in (3′) δ_1 by $\frac{a_1}{\delta_1}$, δ_2 by $\frac{a_2}{\delta_2}$ (leaving a_2, λ and a_1 unchanged) and interchange X with X', Y with Y'.

Hence

$$(3'_1) \qquad X = \frac{M\,(X', Y')}{W_1\,(X', Y')} \cdot X'\,, \quad Y = \frac{M\,(X'\; Y')}{W_1\,(X', Y')} \cdot Y'$$

where

$$W_1(X', Y') \equiv a_2 X'^2 + \lambda X' Y' + a_1 Y' + \frac{a_2}{\delta_2} X' - \frac{a_1}{\delta_1} Y'.$$

To pass now to the inverse of our transformation (3), we note that

$$\delta_1 Y' - \delta_2 X' - 1 = -\rho^2 \delta (x'-a) - \rho^2 \epsilon (y'-b) - 1$$

$$a_2 X'^2 + \lambda X' Y' + a_1 Y'^2 = \rho^2 [\alpha (x'-a)^2 + \beta (x'-a)(y'-b) + \gamma (y'-b)^2]$$

$$\frac{a_2}{\delta_2} X' - \frac{a_1}{\delta_1} Y' =$$

$$(x'-a) \left\{ \frac{\alpha \delta q_1 q_2 - \alpha \epsilon (p_1 q_2 + q_1 p_2) + (\beta \epsilon - \gamma \delta) p_1 p_2}{(\delta q_1 - \epsilon p_1)(\delta q_2 - \epsilon p_2)} \right\}$$
$$+ (y'-b) \left\{ \frac{(\alpha \epsilon - \beta \delta) q_1 q_2 + \gamma \delta (p_1 q_2 + q_1 p_2) - \gamma \epsilon p_1 p_2)}{(\delta q_1 - \epsilon p_1)(\delta q_2 - \epsilon p_2)} \right\}.$$

Subtracting 1 from the last expression on the right, we have L written in x', y'. Taking for convenience $\rho = 1$, we have for M in terms of x', y': $M' \equiv -\delta (x'-a) - \epsilon (y'-b) - 1$. If now we denote

$$C \equiv \alpha (x-a)^2 + \beta (x-a)(y-b) + \gamma (y-b)^2 - 1,$$

our transformation T and its inverse may be written[1]

$$(3) \qquad T: \quad x'-a = \frac{(x-a) L}{C-M}, \quad y'-b = \frac{(y-b) L}{C-M}$$

$$(3_1) \qquad T^{-1}: x'-a = \frac{(x-a) M}{C-L}, \quad y'-b = \frac{(y-b) M}{C-L}.$$

The base conic (every point of which T and T^{-1} leave fixed) has the equation

$$C - M - L = o.$$

The line joining O to the intersection K of AD with BC is

$$M = L.$$

An arbitrary straight line

$$(y-b) = m (x-a) + l$$

is transformed by T into the conic

$$M \{ (y-b) - m (x-a) \} = l (C-L).$$

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[1] The symmetry of the formulæ suggests that we give initially two pairs of points to fix $L = o$, $M = o$ (introducing eight parameters) and a base conic through these four points (introducing a ninth parameter).

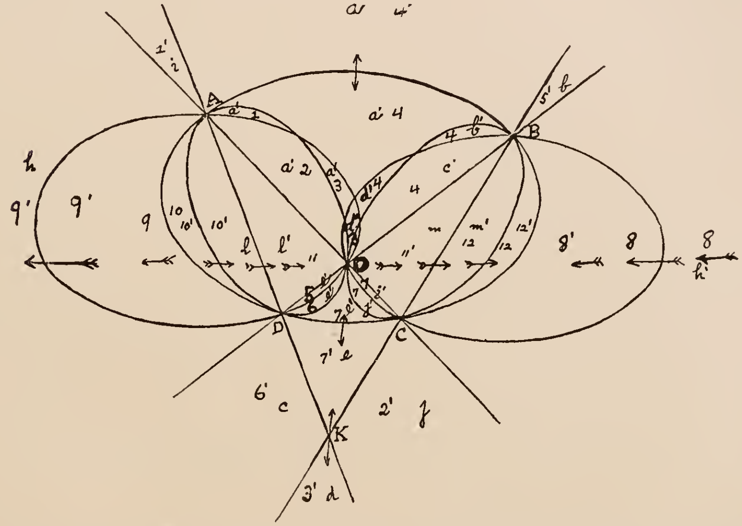
The conic which T transforms into this straight line is

$$L\{(y-b)-m(x-a)\}=l(C-M).$$

These two conics meet in four points given by

$$(L-M)\{(y-b)-m(x-a)-l\}\doteq o,$$

two being the intersections of the straight line with the base conic and two lying on OK, viz.: O itself and, say, H. The point H' in which the given line intersects OK and H mutually correspond, so that an involution is marked out on OK.



The compartments of the plane under the transformation T given by (3) and (3_1) are given in the accompanying figure.

The line BC transforms into a conic tangent to the base conic $ABCD$ at B and C and tangent to OK at O. The conic transforming into AD is tangent to the base conic at A and D and to OK at O.

The line at infinity transforms into the conic $C=L$ (lying at the right in the figure) whose tangent at O is parallel to AD ($L=o$). The conic $C=M$ (lying at the left) with its tangent at O parallel to BC ($M=o$) transforms into the line at infinity.

One region, as a, may transform into several, all designated a'; calling them individually 1, 2, 3, 4, they transform into respectively 1′, 2′, 3′, 4′. The arrows on a line through O indicate the directions in which its points are transformed.

8. *Transformations of period two.*

The division into compartments is simplest when the correspondence between every pair of corresponding points is mutual, *i. e.*, when the period of the transformation is 2. Let us find the most general quadratic Cremona transformation T of period 2 and having three distinct principal points. The condition that $T = S S_1 Q S^{-1}$ be its own reciprocal is $S_1 Q = Q S_1^{-1}$. Let

$$S_1 = \begin{cases} \rho x' = a x + b y + c z \\ \rho y' = a' x + b' y + c' z \\ \rho z' = a'' x + b'' y + c'' z \end{cases} \quad S_1^{-1} = \begin{cases} \sigma x = A x' + A' y' + A'' z' \\ \sigma y = B x' + B' y' + B'' z' \\ \sigma z = C x' + C' y' + C'' z' \end{cases}$$

For $Q S_1^{-1}$ and $S_1 Q$ we have respectively

$$\lambda x' = \frac{A}{x} + \frac{A'}{y} + \frac{A''}{z}, \text{ etc.}; \quad \mu x' = \frac{1}{a x + b y + c z}, \text{ etc.}$$

The condition that $S_1 Q = Q S_1^{-1}$ is thus

$(a x + b y + c z)\ (A y z + A' x z + A'' x y) \equiv (a' x + b' y + c' z) \times (B y z + B' x z + B'' x y) \equiv (a'' x + b'' y + c'' z)\ (C y z + C' x z + C'' x y)$. Hence

(1) $a A + b A' + c A'' = a' B + b' B' + c' B'' = a'' C + b'' C' + c'' C''$

(2) $a A' = a' B' = a'' C'$; (3) $c A' = c' B' = c'' C'$;

(4) $a A'' = a' B'' = a'' C''$; (5) $b A'' = b' B'' = b'' C''$;

(6) $b A = b' B = b'' C$; (7) $c A = c' B = c'' C$.

Case I: $a \gtrless o$. From (2) and (3), $B' B'' = o$; from (4) and (5), $C'' B'' = o$.

(I_a) Suppose first $B'' \gtrless o$. Thus $B' = o$, $C'' = o$. From (2) and (4), $A' = o$, $a'' C' = o$ and $A'' = o$, $a' = o$. Since $A' = B' = o$, $C' \gtrless o$. Thus $a'' = o$. Likewise $A \gtrless o$. By (5), (6), (3), (7) in turn, $b' = o$, $b = o$, $c'' = o$, $c = o$. Hence S_1 is

$$x' : y' : z' = a x : c' z : b'' y.$$

Thus $S_1 Q$ (which is geometrically equivalent to $T = S(S_1 Q) S^{-1}$) becomes

$$x' : y' : z' = \frac{yz}{a} : \frac{xy}{c'} : \frac{xz}{b''}.$$

For $b'' = c'$, $z'/y' = z/y$, and the transformation is virtually that of Dr. Hirst (§ 6), the fixed conic here being $x^2 - \frac{b''}{a} yz = 0$.

The general case $b'' \gtrless c'$ is reduced to the latter by first applying the transformation $x' : y' : z' = x : y : \alpha z, \alpha = b''/c'$, which transforms any point P into a point P' such that the cross-ratio

$$\frac{OP}{RP} : \frac{OP'}{RP'} = \alpha,$$

O being the vertex ($x = o$, $y = o$) and R the intersection of OP with the side $z = o$.

(I_b) Suppose however $B'' = o$. Then by (4), $A'' = o$, $a'' C'' = o$. Thus $C'' \gtrless o$, $a'' = o$. By (5) $b'' = o$. Thus $B' = a c'' \gtrless o$, so that by (2), $a' = o$, $A' = o$. Thus $A \gtrless o$, whence by (6), $b = o$. From $B'' = o, A'' = o$ follows $c' = o$, $c = o$. Hence

$$S_1 : \qquad x' : y' : z' = ax : b'y : c''z.$$

Thus $S_1 Q$ becomes $x' : y' : z' = \frac{yz}{a} : \frac{xz}{b'} : \frac{xy}{c''}$.

If a, b', c'' have the same sign, this is the transformation of § 5 by means of harmonic conjugates. If, for example, a alone be negative we apply first the transformation $x' : y' : z' = -x : y : z$, which transforms any point P into a point P' separating harmonically the vertex ($y = o$, $z = o$) and the side opposite, $x = o$, from the point P.

Case II: $a = o$. By a similar investigation we find that S_1 is either $x' : y' : z' = by : a'x : c''z$ or $x' : y' : z' = cz : b'y : a''x$, as the symmetry of the problem for x, y, z would lead us to expect from the result of the case (I_a).

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On Curvilinear Asymptotes.

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ON CURVILINEAR ASYMPTOTES.[1]

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THE treatment of curvilinear asymptotes in all our text-books is extremely unsatisfactory. It is sixty years since Plücker[2] called attention to the defects in Euler's discussion of this subject, and stated clearly the true point of view. With the exception of a single memoir by Stolz[3], I have not found any evidence that Plücker's theory has attracted the attention of mathematicians, but this is doubtless due to the fact that the development of projective geometry has diverted the attention of mathematicians from metrical geometry to the undue neglect of the latter.

That Plücker's views have not found a place in our text-books is presumably owing to the lack of a simple method of deriving the results in given numerical cases. For, important and exhaustive as is the memoir of Stolz referred to above, it can not be denied that his method is inconvenient in actual practice, since it depends entirely on expansion in series of decreasing powers. The object of this brief article is to propose a simple method of treatment, which seems to me a natural sequence of Plücker's theory. This method is merely an extension of the partial fractions method for rectilinear asymptotes—a method that is unrivaled for simplicity, but has never received half the attention it deserves.

I. *Ordinary rectilinear asymptotes.* For this case the method may be summarized as follows. A little fuller treatment can be found in Edwards' Calculus.

Let the equation of a given curve of the n^{th} order be

(1) $$o = U_n \equiv u_n + u_{n-1} + u_{n-2} + \quad . \quad . \quad . + u_1 + u_0 ,$$

where U_n, or V_n, W_n, etc., will be used to denote rational

[1] Read before the California Academy of Sciences, Sept. 20, 1897.

[2] *Liouville's Journal*, Vol. I, p. 229. See also the first part of Plücker's "Theorie der algebraischen Curven."

[3] *Mathematische Annalen*, Vol. XI, p. 41.

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integral functions of the n^{th} degree in the co-ordinates, while u_n, u_{n-1}, etc., or v_n, w_n, etc., shall denote homogeneous functions of the degree indicated.

Now, suppose p to be a (unrepeated) linear factor of u_n, say

(2) $$u_n \equiv p \cdot v_{n-1}.$$

Then, by the method of partial fractions, we can find

(3) $$\frac{u_{n-1}}{u_n} = \frac{a}{p} + \frac{v_{n-2}}{v_{n-1}},$$

and $p + a = o$ is the equation of an asymptote. For, since

$$u_n = p \cdot v_{n-1}$$
$$u_{n-1} = a \cdot v_{n-1} + p \cdot v_{n-2}$$

it follows that

(4) $$U_n \equiv (p + a)(v_{n-1} + v_{n-2}) + V_{n-2}.$$

Therefore the line: $p + a = o$ meets the curve: $U_n = o$ only when also $V_{n-2} = o$, that is, in but $n - 2$ finite points. The remaining two points of intersection are accordingly at infinity, and $p + a = o$ is an asymptote.

II. *Parabolic Asymptotes.* Suppose u_n contain a square factor, say

(5) $$u_n = p^2 \cdot v_{n-2}.$$

Then, applying the method of partial fractions,

(6) $$\frac{u_{n-1}}{u_n} = \frac{q}{p^2} + \frac{v_{n-3}}{v_{n-2}},$$

where q is a linear homogeneous expression, and the parabola: $p^2 + q = o$ is asymptotic to the curve. For, since

$$u_n = p^2 \cdot v_{n-2}$$
$$u_{n-1} = q \cdot v_{n-2} + p^2 \cdot v_{n-3},$$

it follows that

(7) $$U_n \equiv (p^2 + q)(v_{n-2} + v_{n-3}) + V_{n-2}.$$

The parabola: $p^2 + q = o$ meets $U_n = o$ only when $V_{n-2} = o$, that is, in $2\,(n-2)$ points. The remaining four points of intersection are at infinity. But any parabola of the family

$$p^2 + q + k = o,$$

where k is any constant, has the same property. For we can easily transmute (7) into the form

$$U_n \equiv (p^2 + q + k)\,(v_{n-2} + v_{n-3}) + W_{n-2},$$

and the same analysis holds.

There is therefore a family of asymptotic parabolas, meeting $U_n = o$ in four coincident points at infinity. If, however, k be so chosen that one of the intersections of $p^2 + q + k = o$ with $W_{n-2} = o$ be also at infinity, the parabola in question will have 5-point contact, or will *osculate* $U_n = o$. The necessary and sufficient condition for this osculation is that the axis of the parabola ($p = o$) should meet V_{n-2} at infinity, that is, that the terms of highest degree in V_{n-2} should contain p as a factor. The value of k in question can be determined by resolving

(8)
$$\frac{u_{n-2} - q \,.\, v_{n-3}}{p \,.\, v_{n-2}} = \frac{k}{p} + \frac{w_{n-3}}{v_{n-2}}$$

For in this case

$$u_{n-2} = k \,.\, v_{n-2} + q \,.\, v_{n-3} + p \,.\, w_{n-3},$$

and hence

(9)
$$U_n \equiv (p^2 + q + k)\,(v_{n-2} + v_{n-3}) + p \,.\, w_{n-3} + V_{n-3}.$$

The parabola thus determined will then be regarded as *the asymptote.* This distinction is not made, so far as I have been able to discover, except by Plücker and Stolz in the memoirs referred to.

In particular, if q is a multiple of p, say $q = \mu p$, we have the case of parallel asymptotes. The given curve has a double point at infinity on the line: $p = o$, and the tangents at this double point, or the asymptotes, are given by the equation:

$$p^2 + \mu p + k = o.$$

III. *Cubic Asymptotes.* Suppose u_n contain a cubic factor, say

(10)
$$u_n = p^3 \,.\, v_{n-3}\,.$$

Then, proceeding as before,

$$(11) \qquad \frac{u_{n-1}}{u_n} = \frac{v_2}{p^3} + \frac{v_{n-4}}{v_{n-3}}.$$

Every cubic curve: $p^3 + v_2 + q + k = o$, where q is an arbitrary linear function and k an arbitrary constant, will meet the given curve in *six* points at infinity. But if q be determined by the resolution of

$$(12) \qquad \frac{u_{n-2} - v_2 \cdot v_{n-4}}{p^2 \cdot v_{n-3}} = \frac{q}{p^2} + \frac{w_{n-4}}{v_{n-3}},$$

then every such cubic curve, where k alone is arbitrary, will meet the given curve in *eight* coincident points at infinity. For we shall have

$$\begin{aligned} u_n &= p^3 . v_{n-3} \\ u_{n-1} &= v_2 \cdot v_{n-3} + p^3 . v_{n-4} \\ u_{n-2} &= q \cdot v_{n-3} + v_2 \cdot v_{n-4} + p^2 . w_{n-4} \end{aligned}$$

and consequently

$$(13) \quad U_n \equiv (p^3 + v_2 + q + k)(v_{n-3} + v_{n-4}) + p^2 . w_{n-4} + V_{n-3}$$

There will in general be no particular one of this family of cubics having closer than 8-point contact with the given curve. But if v_2 contain p, say $v_2 = p . r$, there is 9-point contact with all cubics of the family, and there will be one member of the family having 10-point contact, namely that one for which k is determined by the resolution of

$$(14) \qquad \frac{u_{n-3} - q \cdot v_{n-4}}{p \cdot v_{n-3}} = \frac{k}{p} + \frac{w'_{n-4}}{v_{n-3}}.$$

For then we shall find that

$$(15) \qquad U_n \equiv (p^3 + p\, r + q + k)\ (v_{n-3} + v_{n-4}) + p^2 . w_{n-4} \\ + p \cdot w'_{n-4} + V_{n-4} .$$

The extension of the method to higher cases is so obvious that I will not carry it further, especially since the number of special cases to be considered soon becomes inconveniently large, and it does not seem worth while to discuss them.

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Systems of Simple Groups derived from the Orthogonal Group.

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SYSTEMS OF SIMPLE GROUPS DERIVED FROM THE ORTHOGONAL GROUP.[1]

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1. IN THE application of group theory to problems of geometry and analysis, *simple groups* play the fundamental rôle. The question of the resolvability of an algebraic equation by radicals or of the integration of a differential equation by quadratures is answered by the structure of the group (discontinuous or continuous) of the equation, and thus depends on the character of the simple groups obtained in its decomposition. It has been shown by Killing and Cartan that all finite, continuous, simple groups (five isolated ones excepted) are given by four singly-infinite systems, set up by Sophus Lie. To each of these systems corresponds a triply-infinite system of discontinuous groups. Two of the latter were obtained[2] by the writer as generalizations of Jordan's linear groups. The remaining two systems are set up in this paper for the cases $p^n = 8l + 3$ and $8l + 5$ with $m > 4$. It is hoped that the remaining cases $p^n = 8l \pm 1$ can be reported on in the near future.

2. A linear substitution on m indices in a Galois Field[3]

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[1] Read before the Chicago Section of the Amer. Math. Society, December 30, 1897, under the title, "On the decomposition of the Orthogonal Group."

[2] For abstract with references see "Systems of continuous and discontinuous simple groups," Bulletin of the American Mathematical Society, May, 1897.

[3] A set of quantities form a Galois Field when the sum, difference, product or quotient (except by zero) of any two of the quantities gives a quantity belonging to the set. Galois introduced the set of p^n distinct quantities (p being prime),

$$a_0 + a_1 X + a_2 X^2 + \cdots + a_{n-1} X^{n-1} \quad (a_i = 0, 1, \ldots p-1),$$

where X is a root of an irreducible congruence modulo p of degree n. They form a Galois Field of order p^n.

 March 8, 1898.

of order p^n will be called *orthogonal* if it leaves invariant the function

$$\xi_1^2 + \xi_2^2 + \cdots + \xi_m^2.$$

Thus for an orthogonal substitution,

$$(1) \qquad S: \quad \xi'_i = \sum_{j=1}^{m} a_{ij}\, \xi_j \qquad (i = 1, 2, \ldots m),$$

we have the conditions for the coefficients (when $p > 2$):

$$(2) \qquad \sum_{i=1}^{m} a^2_{ij} = 1 \quad , \quad \sum_{i=1}^{m} a_{ij}\, a_{ik} = o$$

$$(j, k = 1, \ldots . m, j \gtreqless k).$$

Generalizing the work of Jordan, Traité des Substitutions, pp. 155–170, I have announced[1] the theorem:

The group G of orthogonal substitutions of determinant unity on m indices in the Galois Field of order $p^n \gtreqless 5, p > 2$, is generated by the substitutions (each affecting only two indices):

$$O^{\alpha,\beta}_{i,j}: \qquad \begin{cases} \xi'_i = \alpha\, \xi_i + \beta\, \xi_j \\ \xi'_j = -\beta\, \xi_i + \alpha\, \xi_j \end{cases} \qquad (\alpha^2 + \beta^2 = 1).$$

For m odd, the order of G was found to equal the order ω of the linear Abelian group on m-1 indices, with factors of composition 2 and $\omega/2$. But it is here proved that, when $p^n = 8\,l \pm 3$, G has the same factors (m being odd and >3). Thus are reached two triply-infinite systems of simple groups of the same order, which are probably not isomorphic when $m > 3$, judging by the corresponding simple continuous groups of Lie.

3. The orthogonal substitutions $O^{\alpha,\beta}_{1,2}$, affecting only the indices ξ_1 and ξ_2, form a commutative group $O_{1,2}$ of order $p^n - \epsilon$ (the number of solutions of $\alpha^2 + \beta^2 = 1$), where $\epsilon = \pm 1$ according as -1 is a square or a not-square. A sub-group $Q_{1,2}$ of index 2 is formed by the substitutions

[1] "Orthogonal group in a Galois Field," Bulletin of the American Mathematical Society, Feb., 1898. The order of the group is there given, also its structure for the case $p = 2$.

$$Q^{\alpha,\beta}_{1,2}: \qquad \begin{cases} \xi'_1 = (\alpha^2-\beta^2)\,\xi_1 - 2\,\alpha\,\beta\,\xi_2 \\ \xi'_2 = \qquad 2\,\alpha\,\beta\,\xi_1 + (\alpha^2-\beta^2)\,\xi_2. \end{cases}$$

Thus, $$Q^{\gamma,\delta}_{1,2}\,Q^{\alpha,\beta}_{1,2} = Q^{\alpha\gamma-\beta\delta,\ \alpha\delta+\beta\gamma}_{1,2}.$$

Since $Q^{\alpha,\beta}_{1,2} = Q^{\alpha',\beta'}_{1,2}$ if and only if $\alpha = \pm\alpha'$, $\beta = \pm\beta'$, the order of the group $Q_{1,2}$ is $\frac{1}{2}\,(p^n-\epsilon)$.

Let C_i denote the orthogonal substitution affecting only ξ_i, whose sign it changes. Then $C_1 C_2$ belongs to the group $Q_{1,2}$. Let T_{12} denote the transposition $(\xi_1\ \xi_2)$. Then $T_{12}\,C_1$ belongs to $Q_{1,2}$ if and only if 2 is a square, so that we may have $\alpha^2 = \beta^2 = \frac{1}{2}$.

For $p^n = 8\,l \pm 3$, $2 =$ not-square, $T_{12}\,C_1$ serves to extend the group $Q_{1,2}$ to the group $O_{1,2}$. Since $T_{12}\,C_1$ transforms $Q^{\alpha,\beta}_{1,j}$ into $Q^{\alpha,\beta}_{2,j}$, $Q^{\alpha,\beta}_{2,j}$ into $Q^{\alpha,-\beta}_{1,j}$, for $j>2$, it is commutative with the group Q of all the $Q_{i,j}$. Hence the group H resulting from extending Q by the alternating group on the m indices is a sub-group of index 2 under the group G. For $p^n = 11$, 13, 19, 29, I find (see appendix) that every $T_{ij}\,T_{kl}$ is already contained in Q; while for $p^n = 3$ or 5 this is not the case, the $Q_{i,j}$ being merely products of an even number of the C_k.

4. Theorem. *For $m>4$, $p^n>5$, $p>2$, the group H is simple if m be odd and has the factors of composition* 2 *and one-half its order if m be even.*

For m even, H contains the invariant sub-group of order 2 generated by N, viz.:

$$N: \qquad \xi'_i = -\xi_i \qquad (i = 1, 2, \ldots m).$$

Suppose H has an invariant sub-group I containing a substitution S, not the identity and (if m be even) not N. It will be proved that I coincides with H.

5. Suppose first that S, given by formulæ (1) and (2), is commutative with every product $C_i\,C_j$. Then, for $p>2$,

$$\alpha_{ij} = 0, \quad \alpha^2_{ii} = 1 \quad (i, j = 1, \ldots m, i \gtrless j).$$

Hence S is simply a product of an even number of the C_i, in which certain ones, as C_k, are lacking, since S differs from N. Let $S = C_i C_j C_r C_s C_t \cdots$. Its transform by $T_{ij} T_{ik}$ gives $S' = C_j C_k C_r C_s C_t \cdots$. Hence I contains $S' S^{-1} = C_i C_k$, and, by transforming by suitable even substitutions, every product of two C's.

Lemma. *Having every $C_i C_j$, the group I contains every $Q^{\alpha,\beta}_{i,j}$.* Thus, 2 being a not-square, H contains one of the two:

$$O^{\alpha,\beta}_{1,2}\,,\quad O^{\beta,-\alpha}_{1,2} \equiv O^{\alpha,\beta}_{1,2}\, T_{12}\, C_1,$$

which transform $C_1 C_3$ into $Q^{\alpha,\beta}_{1,2} C_1 C_3$ and $Q^{\alpha,\beta}_{1,2} C_2 C_3$ respectively. In either case I contains $Q^{\alpha,\beta}_{1,2}$.

Thus if $p^n > 5$, I contains a $Q^{\alpha,\beta}_{1,2}$ different from the identity and from $C_1 C_2$. Taking it in place of S, we are led to the case following.

6. Suppose on the contrary that S is not commutative with every $C_i C_j$, for example[1] not with $C_1 C_2$. Then I contains the substitution, not the identity,

$$S_1 \equiv S^{-1} C_1 C_2 S C_2 C_1 = S_\alpha C_1 C_2,$$

where $S_\alpha \equiv S^{-1} C_1 C_2 S$, of period 2, is found to be:

$$\xi'_i = \xi_i - 2\,\alpha_{i1} \sum_{j=1}^{m} \alpha_{j1}\, \xi_j - 2\,\alpha_{i2} \sum_{j=1}^{m} \alpha_{j2}\, \xi_j\,. \tag{3}$$
$$(i = 1, \ldots m).$$

The next problem is to simplify $S_\alpha C_1 C_2$ by transforming it by substitutions belonging to H.

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[1] We may assume for later use that $\alpha^2_{11} + \alpha^2_{21} \lesseqgtr 1$. Indeed if S be commutative with $C_1 C_2$ then S is merely a product of a substitution affecting ξ_1 and ξ_2 by a substitution affecting $\xi_3, \ldots \xi_m$, both of the same determinant ± 1. For $m > 3$ we may make, by a suitable transformation of S, $\alpha'^2_{11} + \alpha'^2_{21} \gtrless 1$, whence it can not be commutative with $C_1 C_2$.

7. It may be readily verified that $O^{\lambda,\ \mu}_{i,\ j}$ transforms S_a, given by (3), into $S_{a'}$ whose coefficients are

$$a'_{i1} = \lambda\, a_{i1} + \mu\, a_{j1}\,,\quad a'_{j1} = -\mu\, a_{i1} + \lambda\, a_{j1}\,,$$
$$a'_{i2} = \lambda\, a_{i2} + \mu\, a_{j2}\,,\quad a'_{j2} = -\mu\, a_{i2} + \lambda\, a_{j2}\,,$$
$$a'_{k1} = a_{k1}\,,\ a'_{k2} = a_{k2}\ (k = 1\,,\ .\ .\ m,\ k \gtreqless i, j).$$

Since $\lambda^2 + \mu^2 = 1$, we have

$$(4) \qquad \sum_{s=1}^{m} a'^{2}_{s1} = 1\,,\quad \sum_{s=1}^{m} a'^{2}_{s2} = 1\,,\quad \sum_{s=1}^{m} a'_{s1} a'_{s2} = o\,,$$

or precisely the same conditions as those for a_{s1}, a_{s2}.

If $a^2_{i1} + a^2_{j1} = \delta^2$, a square not zero, we may make $a'_{i1} = o$, $a'_{j1} = \delta$ by taking $\lambda = a_{j1}/\delta$, $\mu = -a_{i1}/\delta$.

The condition $a'_{i1} = a'_{j1}$ requires

$$\lambda\ (a_{i1} - a_{j1}) = -\mu\ (a_{i1} + a_{j1}),\ \lambda^2 + \mu^2 = 1,$$
$$\text{whence}\quad \lambda^2 = \frac{(a_{i1} + a_{j1})^2}{2\ (a^2_{i1} + a^2_{j1})}.$$

It can be satisfied if $2\ (a^2_{i1} + a^2_{j1})$ is a square $\gtreqless o$.

If $a^2_{i1} + a^2_{j1} = o$, with a_{i1} and a_{j1} not zero, an arbitrary value $\tau \gtreqless o$ may be assigned to a'_{i1}. Thus, $\lambda\, a_{i1} + \mu\, a_{j1} = \tau$ requires $\mu^2\ (a^2_{i1} + a^2_{j1}) - 2\, a_{j1}\, \mu\, \tau = a^2_{i1} - \tau^2$.

8. Similarly the orthogonal substitution:

$$O_{ijk}\ :\qquad \begin{aligned} \xi'_i &= \lambda\, \xi_i + \mu\, \xi_j + v\, \xi_k \\ \xi'_j &= \lambda' \xi_i + \mu' \xi_j + v' \xi_k \\ \xi'_k &= \lambda'' \xi_i + \mu'' \xi_j + v'' \xi_k \end{aligned}$$

with $\lambda^2 + \mu^2 + v^2 = 1$, etc., transforms S_a into $S_{a'}$ with the coefficients

$$\begin{aligned} a'_{i1} &= \lambda\, a_{i1} + \mu\, a_{j1} + v\, a_{k1} \\ a'_{j1} &= \lambda' a_{i1} + \mu' a_{j1} + v' a_{k1} \\ a'_{k1} &= \lambda'' a_{i1} + \mu'' a_{j1} + v'' a_{k1} \\ a'_{s1} &= a_{s1}\ \ (s = 1,\ .\ .\ m,\ s \gtreqless i, j, k), \end{aligned}$$

with analogous formulæ for a'_{i2}, a'_{j2}, etc. We may verify that the relations (4) hold here.

Suppose that a_{i1} is not zero. Can we choose λ, μ, v, satisfying $\lambda^2 + \mu^2 + v^2 = 1$, so that $a'_{i1} = o$?

If $a^2_{i1} + a^2_{j1} = o$, and therefore $a_{j1} \gtreqless o$, this is accomplished by the values

$$\lambda = \frac{-a_{k1}}{2\, a_{i1}}\ , \ \mu = \frac{-a_{k1}}{2\, a_{j1}}, \ v = 1.$$

If $a^2_{i1} + a^2_{j1} \gtreqless o$, we derive the equivalent condition

$$a^2_{i1}\,(a^2_{i1} + a^2_{j1}) = \{\, \mu\,(a^2_{i1} + a^2_{j1}) + v\, a_{j1}\, a_{k1} \,\}^2 + v^2\, a^2_{i1}\,(a^2_{i1} + a^2_{j1} + a^2_{k1}),$$

which, by the generalization of the theorem of Jordan, p. 159, has solutions μ, v except when $a^2_{i1} + a^2_{j1} + a^2_{k1} = o$, for which case the condition may be written

$$a^2_{i1} = -(\mu\, a_{k1} - v\, a_{j1})^2.$$

If -1 is a square, this has solutions; or we may apply § 7 since $a^2_{i1} + a^2_{k1} = -a^2_{j1}$, a square.

9. The transformed of S_a by O_{ijkl} gives $S_{a'}$, where $a'_{i1} = \lambda\, a_{i1} + \mu\, a_{j1} + v\, a_{k1} + \sigma\, a_{l1}$, etc.
If $a^2_{i1} + a^2_{j1} + a^2_{k1} = o$, the values

$$\lambda = \frac{-a_{l1}}{a_{i1}}\ , \ \mu = \frac{a_{k1}\, a_{l1}}{a^2_{i1}}\ , \ v = \frac{-a_{j1}\, a_{l1}}{a^2_{i1}}\ , \ \sigma = 1$$

make $a'_{i1} = o\,,\ \lambda^2 + \mu^2 + v^2 + \sigma^2 = 1.$

10. Our invariant subgroup I of H was shown to contain the substitution $S \equiv S_a\, C_1\, C_2$ not the identity. Applying §§ 7–9, we transform S successively by O_{i345} ($i = m$, $m-1$, . . 6) or else by $O_{i345}\, T_{12}\, C_1$ when the former does not belong to the group H. We thus obtain in I a substitution S_1 in which $a'_{m1} = a'_{m-1,1} = \text{- - - -} = a'_{61} = o$ and also $a'_{51} = o$. The latter follows by § 8 since $a'^2_{11} + a'^2_{21} \equiv a^2_{11} + a^2_{21}$ differs from 1, whence $a'^2_{51} + a'^2_{41} + a'^2_{31} \gtreqless o$.

11. Next transform S_1 successively by O_{j567} ($j = m, \ldots 8$), giving a substitution $S' \equiv S_\beta\, C_1\, C_2$ leaving ξ_8, . . ξ_m invariant. If β_{52}, β_{62}, β_{72} all differ from zero, we may by § 8 suppose $\beta_{72} = o$ unless $\beta^2_{52} + \beta^2_{62} + \beta^2_{72} = o$, with -1 a not-square. In the latter case, we transform S' by O_{34567} and require that (since $\beta_{51} = \beta_{61} = \beta_{71} = o$):

$$\beta'_{72} \equiv \lambda\, \beta_{32} + \mu\, \beta_{42} + v\, \beta_{52} + \sigma\, \beta_{62} + \rho\, \beta_{72} = o,$$
$$\beta'_{71} \equiv \lambda\, \beta_{31} + \mu\, \beta_{41} = o.$$

Taking[1] $\lambda = -\mu \frac{\beta_{41}}{\beta_{31}}$, $v = -\sigma \frac{\beta_{62}}{\beta_{52}}$, the first condition gives

$$\rho\,\beta_{72} = \mu\left(\frac{\beta_{41}}{\beta_{31}}\cdot\beta_{32}-\beta_{42}\right).$$

The final condition $\lambda^2+\mu^2+v^2+\sigma^2+\rho^2=1$ becomes

$$\sigma^2\left(1+\frac{\beta^2_{62}}{\beta^2_{52}}\right)+\mu^2\left\{1+\frac{\beta^2_{41}}{\beta^2_{31}}+\frac{1}{\beta^2_{72}}\left(\frac{\beta_{41}}{\beta_{31}}\beta_{32}-\beta_{42}\right)^2\right\}=1,$$

which can always be satisfied. Indeed by an earlier transformation of S' by O_{567} we can avoid the two values of β_{72} which make the coefficient of μ^2 zero.

12. We have thus reached, in the group I, a substitution $S_\gamma C_1 C_2$ affecting only $\xi_1, \ldots \xi_6$, with $\gamma^2_{11}+\gamma^2_{21} \gtreqless 1$. As in § 10, we may suppose $\gamma_{51}=\gamma_{61}=o$. Our substitution will thus be commutative with T_{56} if $\gamma_{52}=\gamma_{62}$, which we may suppose the case from § 7 when 2 and $\gamma^2_{52}+\gamma^2_{62}$ are both not-squares. If $\gamma^2_{52}+\gamma^2_{62}$ be a square not zero, we may by § 7 suppose that $\gamma_{62}=o$, when we are led to the case treated in the next paragraph.

There remains the case $\gamma^2_{52}+\gamma^2_{62}=o$, occurring when -1 is a square. If $\gamma_{31}=o$, we transform by O_{356} and have, by § 8, $\gamma'_{62}=\gamma'_{61}=o$, a case treated in the next paragraph. The case $\gamma^2_{31}+\gamma^2_{41}=$ a square is therefore solved; while $\gamma^2_{31}+\gamma^2_{41}=o$ is excluded since then $\gamma^2_{11}+\gamma^2_{21}=1$. With $\gamma^2_{52}+\gamma^2_{62}=o$, $\gamma^2_{31}+\gamma^2_{41}=$ a not-square, we may make $S_\gamma\ C_1 C_2$ commutative with T_{36}. Transforming by $O^{\lambda\,\mu\,v}_{345}$, the conditions are

$$\begin{aligned}\gamma'_{31}&=\lambda\,\gamma_{31}+\mu\,\gamma_{41} &&=\gamma_{61}=o\\ \gamma'_{32}&=\lambda\,\gamma_{32}+\mu\,\gamma_{42}+v\,\gamma_{52}&&=\gamma_{62}.\end{aligned}$$

Writing $\delta\equiv\gamma_{42}-\frac{\gamma_{41}}{\gamma_{31}}\gamma_{32}$, these conditions give

$$\lambda=-\mu\frac{\gamma_{41}}{\gamma_{31}},\ v=\frac{\gamma_{62}}{\gamma_{52}}-\mu\frac{\delta}{\gamma_{52}}.$$

---

[1] If either $\beta_{31}=o$ or $\beta_{41}=o$ we could at once have made $\beta_{72}=o$.

The final condition $\lambda^2 + \mu + \nu^2 = 1$, becomes, on applying $\gamma^2_{52} = -\gamma^2_{62}$ and setting $\omega \equiv 1 + \frac{\gamma^2_{41}}{\gamma^2_{31}} - \frac{\delta^2}{\gamma^2_{62}}$,

$$\mu^2\omega + 2\mu\frac{\delta}{\gamma_{62}} = 2,$$

or $$\left(\mu\omega + \frac{\delta}{\gamma_{62}}\right)^2 = \frac{2\gamma^2_{62}(\gamma^2_{31} + \gamma^2_{41}) - \delta^2\gamma^2_{31}}{\gamma^2_{31}\gamma^2_{62}}.$$

Here $\omega \gtreqless o$ and $2(\gamma^2_{31} + \gamma^2_{41})$ is a square not zero. Thus $\delta = o$ gives an immediate solution for μ. To prove that there is always a solution, we note that (§ 7) an arbitrary value except zero may be assigned to γ_{62} by a previous transformation by O_{56}. Further,[1] in a Galois Field of order p^n for which -1 is a square, $(p^n - 5)/4$ of the squares are followed by squares (not zero). Hence, if $p^n > 5$, there exists values of ρ^2 making $\rho^2 - 1$ a square. Thus the right member of our condition may be made a square.

Consider the case that our substitution is commutative[2] with T_{56}. Now the conditions that S, given by formula (1), shall be commutative with every T_{ij} are seen to be

$$a_{ii} = a_{11},\ a_{ij} = a_{12}\quad (i, j = 1 \text{ - - } m).$$

The conditions (2) that S be orthogonal then give

$$a^2_{11} + (m-1)a^2_{12} = 1,\ 2a_{11}a_{12} + (m-2)a^2_{12} = o.$$

Thus $$a_{11} = a_{12} \pm 1,\ m a^2_{12} = \mp 2a_{12}.$$

Hence S is the identity, or $N \equiv C_1 C_2 \cdots C_m$, or, if m is not a multiple of p, the substitution Σ of determinant $-(\pm 1)^m$:

$$\xi'_i = \pm\xi_i \mp \frac{2}{m}(\xi_1 + \xi_2 + \ldots + \xi_m)\quad (i = 1 \ldots m).$$

Thus our substitution of determinant $+1$ affecting only six indices can not be Σ, and, by our first hypothesis, is neither

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[1] By a generalization of Jordan, Traité des Substitutions, § 198, or Gauss, Commentatio prima, 16.

[2] Of course the same treatment applies if it be commutative with T_{36}.

the identity nor N. We may thus suppose it to be not commutative with T_{12}, for example. Thus I contains

$$S^{-1}\,T_{12}\,T_{56}\,S\,T_{56}\,T_{12} = S^{-1}\,T_{12}\,S\,T_{12} = S_\delta\,T_{12},$$

where S_δ denotes[1] the substitution

$$\xi'_i = \xi_i - \delta_i \sum_{j=1}^{6} \delta_j\,\xi_j \quad (i = 1 \,\text{--}\, 6),$$

where $\delta_i \equiv a_{i2} - a_{i1}$. By virtue of (2) we have

$$\sum_{i=1}^{6} \delta^2_i = 2.$$

By transforming $S_\delta\,T_{12}$ by O_{3456} we may take $\delta_6 = o$. We are thus led to the case treated in the next paragraph.

13. We have shown that the invariant subgroup I must contain a substitution

$$S: \quad \xi'_i = \sum_{j=1}^{5} a_{ij}\,\xi_j \quad (i = 1 \ldots 5),$$

leaving $\xi_6, \ldots \xi_m$ fixed.

By § 5 we may suppose that S is not commutative with every C_i $(i = 1 \ldots 5)$. Then if S be not commutative with C_1, for example, and if we suppose $m \overline{\gtrless} 6$, I will contain

$$S^{-1}\,C_1\,C_6\,S\,C_1\,C_6 = S^{-1}\,C_1\,S\,C_1 = T_a\,C_1,$$

not the identity, where T_a denotes the substitution

$$T_a: \quad \xi'_i = \xi_i - 2\,a_{i1} \sum_{j=1}^{5} a_{j1}\,\xi_j \qquad (i = 1 \ldots 5).$$

Transforming $T_a\,C_1$ by O_{2345}, we may take $a_{51} = o$. If $T_a\,C_1$ be commutative with every $T_{ij}\,T_{56}$ $(i, j = 1 \ldots 4)$, then $a_{11} = a_{21} = a_{31} = a_{41}$, and $4\,a^2_{11} = 1$. Thus T_a becomes

$$\xi'_i = \xi_i - \tfrac{1}{2}\,(\xi_1 + \xi_2 + \xi_3 + \xi_4).$$

---

[1] Thus S replaces ξ_i by $\sum_{j=1}^{6} a_{ij}\,\xi_j$, which T_{12} replaces by

$$\sum_{j=1}^{6} a_{ij}\,\xi_j + (a_{i2} - a_{i1})\,(\xi_1 - \xi_2).$$

This S^{-1} replaces by

$$\xi_i + (a_{i2} - a_{i1}) \sum_{j=1}^{6} (a_{j1} - a_{j2})\,\xi_j.$$

In this case, the transformed of $T_\alpha C_1$ by $C_2 C_3$ gives $T_\beta C_1$, where T_β replaces ξ_1 by

$$\xi_1 - 1/2\ (\xi_1 - \xi_2 - \xi_3 + \xi_4).$$

Thus $T_\beta C_1$ is not commutative with every $T_{ij} T_{56}$. Hence I contains a substitution S, given by (1), not commutative with $T_{12} T_{56}$, for example, but leaving $\xi_5, \ldots \xi_m$ fixed; hence I contains the substitution, not the identity,

$$S^{-1} T_{12} T_{56} S T_{12} T_{56} = S^{-1} T_{12} S \cdot T_{12} = S_\delta T_{12},$$

where S_δ denotes the substitution (see end of § 12):

$$S_\delta: \quad \xi'_i = \xi_i - \delta_i \sum_{j=1}^{4} \delta_j \xi_j \quad (i = 1, \ldots 4).$$

14. If $\delta_3 = \delta_4 = o$, $S_\delta T_{12}$ takes the form

$$\begin{cases} \xi'_1 = -\delta_1 \delta_2 \xi_1 + (1 - \delta^2_2) \xi_2 \equiv (a^2 - \beta^2) \xi_1 + 2 a \beta \xi_2 \\ \xi'_2 = (1 - \delta^2_1) \xi_1 - \delta_1 \delta_2 \xi_2 \equiv -2 a \beta \xi_1 + (a^2 - \beta^2) \xi_2, \end{cases}$$

where $\quad a = \tfrac{1}{2} (\delta_1 - \delta_2),\ \beta = \tfrac{1}{2} (\delta_1 + \delta_2)$

so that $\quad a^2 + \beta^2 = \tfrac{1}{2} (\delta^2_1 + \delta^2_2) = 1.$

Hence $S_\delta T_{12}$ becomes $Q^{a,\beta}_{2,1}$. Therefore I contains $Q^{a,\beta}_{1,3}$, the transformed of its reciprocal $Q^{a,\beta}_{1,2}$ by $T_{23} T_{45}$ and thus I contains

$$Q^{a,\beta}_{2,1} Q^{a,\beta}_{1,3} = S_\sigma T_{23},$$

with $\quad \sigma_1 = 2 a \beta,\ \sigma_2 = a^2 - \beta^2,\ \sigma_3 = -1.$

If $a\beta \gtrless o$, its transformed by $T_{13} T_{45}$ gives $S_{\sigma'} T_{12}$, in which $\sigma'_3 \gtrless o$. But if $a\beta = o$, $S_\delta T_{12}$, not being the identity, reduces to $C_1 C_2$, when by § 5 the group I contains every Q_{ij} and therefore (for $p^n > 5$) a substitution $Q^{a,\beta}_{2,1} Q^{a,\beta}_{1,3}$ in which $a\beta \gtrless o$.

15. We have thus reached in the group I a substitution $S_\delta T_{12}$, not the identity, having δ_3 and δ_4 not both zero, say $\delta_4 \gtrless o$, and leaving $\xi_5, \ldots \xi_m$ fixed. Transforming it by $O^{a,\beta,\gamma}_{3,4,6}$ we can make $\delta'_6 = \delta'_1$, the conditions being (since $\delta_6 = o$)

$$a \delta_3 + \beta \delta_4 = \delta_1,\ a^2 + \beta^2 + \gamma^2 = 1.$$

These combine into the single condition

$$a^2\,(\delta^2_3+\delta^2_4)-2\,a\,\delta_1\,\delta_3+\delta^2_4\,\gamma^2=\delta^2_4-\delta^2_1\,.$$

For $\delta^2_3+\delta^2_4=o$, and therefore $\delta_3 \gtrless o$, a solution is given by $\gamma=o$ when $\delta_1 \gtrless o$, and by $\gamma=1$ when $\delta_1=o$. For $\delta^2_3+\delta^2_4 \gtrless o$, there exist solutions of the equivalent equation of condition,

$$\{\,a\,(\delta^2_3+\delta^2_4)-\delta_1\,\delta_3\,\}^2+\delta^2_4\,(\delta^2_3+\delta^2_4)\,\gamma^2=\delta^2_4\,(\delta^2_3+\delta^2_4-\delta^2_1).$$

Similarly by transforming the result by O_{345} we can (since $\delta'_5=\delta_5=o$) make $\delta''_5=\delta''_6$.

Hence I contains a substitution (not the identity) $S_\omega\,T_{12}$ commutative with both T_{16} and T_{56}. Thus I contains the substitution

$$(S_\omega\,T_{12})\,T_{26}T_{56}\,(S_\omega T_{12})^{-1}\,T_{56}T_{26}=S_\omega\,T_{16}\,S_\omega^{-1}\,T_{26}=T_{16}\,T_{26}.$$

Since the alternating group on $m>4$ letters is simple, the group I containing $T_{16}\,T_{26}$ contains the whole alternating group.

Now $C_1\,C_2$ transforms $T_{12}\,T_{13}$ into $T_{12}\,T_{13}\,C_1\,C_3$, so that I contains $C_1\,C_3$ and thus every $C_i\,C_j$. Hence, by the lemma of § 5, the invariant subgroup I, under H, coincides with H, whose maximal invariant subgroup is thus of order 1 or 2 according as m is odd or even.

16. Consider the orthogonal group H on 5 indices. Suppose an invariant subgroup I contains a substitution

$$S:\qquad \xi'_i=\sum_{j=1}^{5} a_{ij}\,\xi_j\qquad (i=1\,.\,.\,5)$$

not the identity and not a product of the C_i (see § 5). By a previous transformation of S we can suppose that $a^2_{11}\gtrless 1$. Transforming S by

$$O_{5432}:\qquad \begin{cases}\xi'_5=\lambda\,\xi_5+\mu\,\xi_4+v\,\xi_3+\rho\,\xi_2\\ \xi'_4=\lambda'\xi_5+\mu'\xi_4+v'\xi_3+\rho'\xi_2\\ \dots\dots\dots\dots\dots\dots\end{cases}$$

we have $a'_{11}=a_{11}$, $a'_{15}=\lambda\,a_{15}+\mu\,a_{14}+v\,a_{13}+\rho\,a_{12}$. By §§ 8–9 we can make $a'_{15}=o$. Transform the resulting substitution S' by O_{432}. We reach the substitution

$$S_1:\qquad \xi'_i=\sum_{j=1}^{5}\beta_{ij}\,\xi_j$$

where $\beta_{11}=a'_{11}$, $\beta_{15}=a_{15}=o$, $\beta_{14}=\lambda a_{14}+\mu a'_{12}+\nu a_{12}$. Since $a'^2_{14}+a'^2_{13}+a'^2_{12}=1-a^2_{11}\gtrless o$, we may by § 8 make $\beta_{14}=o$.

If S_1 be commutative with both $C_1 C_2$ and $C_1 C_3$, S_1 becomes simply,

$$\xi'_1=\pm\xi_1,\ \xi'_2=\pm\xi_2,\ \xi'_3=\pm\xi_3,$$
$$\xi'_4=\rho\,\xi_4+\sigma\,\xi_5,\ \xi'_5=\mp(\sigma\,\xi_4-\rho\,\xi_5),$$

where $\sigma\gtrless o$ since S and therefore S_1 is not a mere product of the C_i. But the transformed of S_1 by $T_{14}\,T_{35}$ gives a substitution

$$\xi'_1=\rho\,\xi_1+\sigma\,\xi_3,\ \xi'_2=\pm\xi_2,\ \ldots$$

which is not commutative with $C_1 C_2$.

In any case we have in the group I a substitution

$$S:\quad \xi'_i=\sum_{j=1}^{5}\delta_{ij}\,\xi_j\quad (i=1\text{--}5)$$

with $\delta_{14}=\delta_{15}=o$ and not commutative with $C_1 C_2$, for example. Hence I contains the substitution

$$S\,C_1 C_2\,S^{-1}\,C_1 C_2=R_\delta\,C_1 C_2,$$

$$R_\delta:\ \xi'_i=\xi_i-2\,\delta_{1i}\sum_{j=1}^{5}\delta_{1j}\,\xi_j-2\,\delta_{2i}\sum_{j=1}^{5}\delta_{2j}\,\xi_j.$$

17. Suppose first that $\delta^2_{24}+\delta^2_{25}$ is a square not zero. Transforming $R_\delta\,C_1 C_2$ by O_{45} we may make $\delta_{25}=o$. We have therefore in I a substitution affecting only ξ_1, ξ_2, ξ_3, ξ_4, from which as in § 16 we obtain a substitution

$$S:\quad \xi'_i=\sum_{j=1}^{4}\gamma_{ij}\,\xi_j\quad (i=1\text{--}4)$$

with $\gamma^2_{11}\gtrless 1$, $\gamma_{14}=o$. It is thus not commutative with C_1. Hence I contains the substitution, not the identity,

$$S\,C_1 C_5\,S^{-1}\,C_1 C_5=S\,C_1\,S^{-1}\,C_1=T_\gamma\,C_1,$$

$$T_\gamma:\ \xi'_i=\xi_i-2\,\gamma_{1i}\sum_{j=1}^{3}\gamma_{1j}\,\xi_j\quad (i=1,2,3).$$

As in § 13, we find that I contains the substitution $S_\delta\,T_{12}$, where

$$S_\delta:\ \xi'_i=\xi_i-\delta_i\sum_{j=1}^{3}\delta_j\,\xi_j\quad (i=1,2,3),$$

with the single condition $\delta^2_1+\delta^2_2+\delta^2_3=2$.

By § 14 we may suppose that δ_3 is not zero. Since 2 is a not-square, we can not have $\delta^2_1 + \delta^2_2 = o$.

If $\delta^2_1 + \delta^2_2$ is a not-square, we may make $\delta_1 = \delta_2$ by § 7. Transforming by O_{345}, we can make $\delta'_3 = \lambda\,\delta_3 = \delta_1$; for the condition $\mu^2 + \nu^2 = 1 - \delta^2_1/\delta^2_3$ can be satisfied. Since $\delta_1 = \delta_2 = \delta_3$, S_δ is commutative with T_{12} and T_{23}. Hence I contains the substitution

$$(S_\delta\, T_{12})\, T_{13}\, T_{12}\, (T_{12}\, S_\delta)\, T_{12}\, T_{13} = S_\delta\, T_{23}\, S_\delta\, T_{13} = T_{23} T_{13}.$$

If, however, $\delta^2_1 + \delta^2_2$ is a square, we can make $\delta_2 = o$. Then S_δ will be commutative with T_{25} and T_{45}. Thus

$$(S_\delta\, T_{12})\, T_{15}\, T_{45}\, (T_{12}\, S_\delta)\, T_{45}\, T_{15} = T_{25}\, T_{15}$$

belongs to group I.

In either case it follows (see end of § 15) that H is simple if m be odd, but has a maximal invariant sub-group of index two if m be even.

18. Suppose next that $\delta^2_{24} + \delta^2_{25}$ is a not-square. By § 7 we may thus suppose that $R_\delta\, C_1\, C_2$ is commutative with T_{45}. If it be commutative with every T_{ij} $(i, j = 1, \ldots 5)$ it has the form (see end of § 12),

$$\xi_i = -\xi_i + \frac{2}{5}(\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5) \quad (i = 1, \ldots 5).$$

But the transformed of this substitution by $C_1 C_2$ is not of the same form. We may thus assume that $R \equiv R_\delta\, C_1\, C_2$ is commutative with T_{45} but not with T_{12}, for example. Thus I contains

$$R^{-1}\, T_{12}\, T_{45}\, R\, T_{45}\, T_{12} = S_\rho\, T_{12},$$

not the identity. As at the end of § 12, we have

$$S_\rho: \qquad \xi'_i = \xi_i - \rho_i \sum_{j=1}^{5} \rho_j\, \xi_j \qquad (i = 1 - - 5).$$

If $\rho_5 = o$ we are led to the case solved in § 17. By § 8 we may make $\rho_5 = o$ unless $\rho^2_3 + \rho^2_4 + \rho^2_5 = o$ with -1 a not-square. In the latter case, $\rho^2_1 + \rho^2_2 = 2$, when we may make $\rho_1 = \rho_2 = +1$. Further, we can not have $\rho_3 = \rho_4 = \rho_5$;

for in that case $p=3$ so that -1 is a square. Let $\rho_4 \gtrless \rho_5$. The group I contains the substitution

$$(S_\rho\, T_{12})^{-1}\, T_{45}\, T_{12}\, (S_\rho\, T_{12})\, T_{12}\, T_{45} = S_\rho\, T_{45}\, S_\rho\, T_{45}.$$

We may verify that $S_\sigma \equiv S_\rho\, T_{45}\, S_\rho$ has the form

$$S_\sigma \;:\qquad \xi'_i = \xi_i - \sigma_i \sum_{j=1}^{5} \sigma_j\, \xi_j \qquad (i = 1 \text{ - - } 5),$$

where $\sigma_4 = 1 + \rho_4\,(\rho_5 - \rho_4)$, $\sigma_5 = -1 + \rho_5\,(\rho_5 - \rho_4)$,

$$\sigma_i = \rho_i\,(\rho_5 - \rho_4) \qquad (i = 1, 2, 3).$$

Thus $$\sigma_1^2 + \sigma_2^2 + \text{ - - } + \sigma_5^2 = 2.$$

If $S_\sigma\, T_{45}$ be the identity, $S_\rho\, T_{12} = T_{45}\, T_{12}$; for we must have $$\rho_1 = \rho_2 = \rho_3 = o\,,\quad \rho_4 = -\rho_5 = \pm 1\,.$$

If $S_\sigma\, T_{45}$ be not the identity, we obtain from it a substitution affecting four indices only unless

$$\sigma^2_1 + \sigma^2_2 + \sigma^2_3 = o\,.$$

In the latter case, $\rho^2_1 + \rho^2_2 + \rho^2_3 = o$, $\rho^2_4 + \rho^2_5 = 2$. We may thus by § 7 make $\rho_4 = \rho_5 = +1$. Hence $\rho_1 = \rho_2 = \rho_4 = \rho_5 = +1$. Thus S_σ is commutative with both T_{25} and T_{45}. The theorem then follows by the proof at the end of § 17.

19. Suppose finally that $\delta^2_{24} + \delta^2_{25} = o$, $\delta_{25} \gtrless o$. If $\delta_{11} = \delta_{12} = o$, $\delta^2_{13} = 1$. Then $\delta_{23} = o$ follows from the relation

$$(5) \qquad \sum_{j=1}^{5} \delta_{1j}\, \delta_{2j} = o\,.$$

Thus R_δ (given at end of § 16) becomes

$$\xi'_3 = -\xi_3\,,\qquad \xi'_i = \xi_i - 2\,\delta_{2i} \sum_j \delta_{2j}\, \xi_j \qquad (i, j = 1, 2, 4, 5).$$

Hence I contains a substitution $R_\delta\, C_1\, C_2$ commutative with C_3. Our theorem follows as in § 17.

With either δ_{11} or δ_{12} not zero, we may make each one $\gtrless o$ by transforming by O_{12}.

Similarly the theorem follows if $\delta_{13} = o$. Thus $\delta^2_{11} + \delta^2_{12} = 1$, so that we can make $\delta_{12} = o$. Then $\delta_{21} = o$ by (5). Thus $R_\delta\, C_1\, C_2$ leaves ξ_1 fixed.

We may thus suppose that δ_{11}, δ_{12}, δ_{13} all differ from zero. Transforming $R_\delta\, C_1\, C_2$ by O_{345} we can obtain a substitution commutative with T_{13} or else with T_{23}. Our theorem will then follow by the proof in § 18. The conditions that it be commutative with T_{13} are

$$\begin{aligned} \delta'_{23} &\equiv \lambda\, \delta_{23} + \mu\, \delta_{24} + v\, \delta_{25} = \delta_{21} \\ \delta'_{13} &\equiv \lambda\, \delta_{13} \qquad\qquad\qquad\quad\; = \delta_{11} \\ &\lambda^2 + \mu^2 + v^2 = 1\,. \end{aligned}$$

These three conditions combine into a single one:

$$\mu^2\left(1 + \frac{\delta^2_{24}}{\delta^2_{25}}\right) - 2\,\mu\,\omega\,\frac{\delta_{24}}{\delta^2_{25}} + \frac{\omega^2}{\delta^2_{25}} + \frac{\delta^2_{11}}{\delta^2_{13}} = 1$$

where
$$\omega \equiv \delta_{21} - \frac{\delta_{23}}{\delta_{13}}\,\delta_{11}\,.$$

Since the coefficient of μ^2 is zero, μ is determined unless $\omega = o$. Similarly the conditions that our substitution be commutative with T_{23} may be satisfied unless

$$\omega' \equiv \delta_{22} - \frac{\delta_{23}}{\delta_{13}}\,\delta_{12} = o.$$

It can not happen that both ω and ω' vanish; for

$$\frac{\delta_{21}}{\delta_{11}} = \frac{\delta_{22}}{\delta_{12}} = \frac{\delta_{23}}{\delta_{13}} = \tau.$$

would require $\tau = \pm 1$ since we have

$$\delta^2_{\kappa 1} + \delta^2_{\kappa 2} + \delta^2_{\kappa 3} = 1 \qquad (\kappa = 1, 2).$$

Then would follow

$$o = \sum_{j=1}^{5} \delta_{1j}\,\delta_{2j} = \pm\,(\delta^2_{11} + \delta^2_{12} + \delta^2_{13}) = \pm 1.$$

APPENDIX.

1. For $p^n = 3$, the Q_{ij} are simply $C_i\, C_j$. If $m = 3$, the group of the Q_{ij} is as follows:

$$G_4: \qquad \{\, 1,\ C_1 C_2,\ C_1 C_3,\ C_2 C_3 \,\}.$$

This is extended by $T_{12}\,C_1$ to a G_8, which $T_{13}\,C_1$ extends to the total group:

$$G_{24}:\begin{cases} 1, \quad C_1\,C_2\,(\text{three}), \quad T_{12}\,T_{13}\,(\text{two}), \quad T_{12}\,T_{13}\,C_i\,C_j\,(\text{six}), \\ \qquad T_{12}\,C_i\,(\text{nine}), \quad T_{12}\,C_1\,C_2\,C_3\,(\text{three}), \end{cases}$$

of which the first line constitutes our group H.

2. For $p^n=5$, $m=3$, the groups Q and O are the above G_4 and G_{24} respectively. The latter is extended to the total group G_{120} by the substitution

$$R=\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad R^2=1.$$

Indeed the left-hand multipliers in a rectangular table for G_{120} with G_{24} as first row are

$$1, \quad R, \quad C_1\,C_2\,R, \quad C_1\,C_3\,R, \quad C_2\,C_3\,R.$$

This follows from the relations,

$$C_1\,C_2\,C_3\,R=R\,C_1\,C_2\,C_3, \quad C_1\,R=C_2\,C_3\,R\,C_1\,C_2\,C_3,$$
$$T_{12}\,C_1\,R=C_2\,R\,T_{23}, \quad T_{12}\,C_3\,R=C_3\,R\,T_{23},$$
$$T_{12}\,T_{13}\,R=R\,T_{13}\,T_{12}, \text{ etc.}$$

We might express the G_{120} by the formula

$$(R\,C_3)^i\,\{G_{24}\}\,C_1 \qquad (i=0, \cdots 4).$$

Indeed $\quad (R\,C_3)^5=C_1\,C_2\,C_3,$

while no lower power of $R\,C_3$ is contained in $\{G_{24}\}\,C_1$. Thus

$$(R\,C_3)^2=\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}.$$

3. For $p^n=11$, $m=3$, the Q_{ij} are $C_i\,C_j$ or $O^{\pm 5,\,\pm 3}_{i,\,j}$. The group $Q_{1,2}$ is of order six, since $O^{5,\,3}_{1,\,2}$ is of period three. The left hand multipliers extending Q to the total group are of the types (where $O_{ij}\equiv O^{5,\,3}_{i\,j}$):

$$O_{12}\,O_{13}=\begin{pmatrix} 3 & 4 & 3\\ -3 & 5 & 0\\ -4 & 2 & 5\end{pmatrix},\ (O_{12}\,O_{13})^2=\begin{pmatrix} -4 & 5 & 2\\ -2 & 2 & 2\\ -5 & 4 & 2\end{pmatrix},$$

$$O_{23}O_{13}O_{12}=\begin{pmatrix} 3 & 3 & -4\\ -4 & -3 & 3\\ -3 & -4 & 3\end{pmatrix},\ (O_{12}O_{13})^3O_{31}=\begin{pmatrix} -4 & -2 & 5\\ 2 & -5 & 4\\ -5 & 4 & 2\end{pmatrix}.$$

Since $$(O_{12}\,O_{13})^3=\begin{pmatrix} -2 & 2 & -2\\ 2 & -5 & 4\\ -2 & 4 & -5\end{pmatrix},$$

$O_{12}\,O_{13}$ is of period six; also it follows that

$$(O_{12}\,O_{13})^3\,C_2\,C_3\,T_{13}\,T_{12}=(O_{23}\,O_{13}\,O_{12})^2,$$

so that $T_{13}\,T_{12}$ belongs to the group Q.

4. For $p^n=13$, the Q_{ij} are $C_i\ C_j$ or $O^{\pm 6,\,\pm 2}_{i,\ j}$.

$$O^{6,\,2}_{1,\,2}\,O^{6,\,2}_{1,\,3}=\begin{pmatrix} -3 & -1 & 2\\ -2 & 6 & 0\\ 1 & -4 & 6\end{pmatrix},\ (O_{12}\,O_{13})^2=\begin{pmatrix} 0 & 2 & 6\\ -6 & -1 & -4\\ -2 & 3 & -1\end{pmatrix},$$

$$(O_{12}\,O_{13})^4=\begin{pmatrix} 2 & 3 & -1\\ 1 & 3 & -2\\ -3 & 3 & 3\end{pmatrix},\ O_{23}O_{12}O_{13}=\begin{pmatrix} -3 & 3 & -3\\ -2 & -3 & -1\\ 1 & 3 & 2\end{pmatrix}.$$

Thus $$(O_{12}\,O_{13})^4\ T_{12}\,T_{13}=C_1\,C_2\,(O_{23}\,O_{12}\,O_{13})\,C_1\,C_3,$$

so that $T_{12}\,T_{13}$ belongs to the group Q. Note that $O_{12}\,O_{13}$ is of period 7 since $(O_{12}\,O_{13})^8=O_{12}\,O_{13}$.

5. For $p^n=19$, the Q_{ij} are $C_i\,C_j$, $O^{\pm 2,\,\pm 4}_{i,\ j}$, $O^{\pm 7,\,\pm 3}_{i,\ j}$.

$$S\equiv O^{2,\,4}_{1\,2}\,O^{2,\,4}_{1\,3}=\begin{pmatrix} 4 & 8 & 4\\ -4 & 2 & 0\\ -8 & 3 & 2\end{pmatrix},\ S^2=\begin{pmatrix} 9 & 3 & 5\\ -5 & -9 & 3\\ -3 & 5 & -9\end{pmatrix},$$

$$S^4=\begin{pmatrix} -6 & 6 & 9\\ -9 & 5 & -3\\ -6 & -4 & 5\end{pmatrix},\ S^8=S^{-1},\ S^9=1.$$

$$T\equiv O^{2,\,4}_{2\,3}\,O^{2,\,4}_{1\,3}\,O^{2,\,4}_{1\,2}=\begin{pmatrix} 4 & -5 & -6\\ -8 & -8 & -5\\ -4 & -8 & 4\end{pmatrix},\ T^2=T^{-1}.$$

$$O^{7,\,3}_{2\,3}\,O^{7,\,3}_{1\,3}\,O^{7,\,3}_{1\,2}=\begin{pmatrix} -8 & -4 & 4\\ -2 & 0 & -4\\ -3 & -2 & -8\end{pmatrix}=T_{23}\,T_{12}\,C_2\,C_3\,O^{2,\,4}_{1\,2}\,O^{2,\,4}_{1\,3},$$

so that $T_{23}\,T_{12}$ belongs to the group Q.

6. For $p^n = 29$, the Q_{ij} are $C_i\,C_j$, $O^{\pm 9,\,\pm 6}_{i,\,j}$, $O^{\pm 11,\,\pm 5}_{i,\,j}$, $O^{\pm 13,\,\pm 8}_{i,\,j}$.

$$O^{9\,6}_{1\,2}\,O^{9\,6}_{1\,3} = \begin{pmatrix} -6 & -4 & 6 \\ -6 & 9 & 0 \\ 4 & -7 & 9 \end{pmatrix},\quad O^{9\,6}_{2\,3}\,O^{9\,6}_{1\,3}\,O^{9\,6}_{1\,2} = \begin{pmatrix} -6 & -9 & 0 \\ 4 & 7 & -9 \\ -6 & 4 & -6 \end{pmatrix},$$

whence the latter is $C_2\,C_3\,O^{9,\,6}_{1,\,2}\,O^{9,\,6}_{1,\,3}\,T_{13}\,T_{12}$. Thus $T_{13}\,T_{12}$ belongs to Q.

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Systems of Simple Groups derived from the Orthogonal Group.

(Second Paper.)

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SYSTEMS OF SIMPLE GROUPS DERIVED FROM THE ORTHOGONAL GROUP.

(Second Paper.)

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1. This paper completes the investigations begun in the Proceedings of the California Academy of Sciences, 3d Ser., Math.-Physics, Vol. I, No. 4, pp. 29–46. The final result may be stated as follows:—

The squares of the linear substitutions on m indices in the Galois Field of order p^n, $p>2$, which leave the sum of the squares of the m indices invariant, generate a group which is simple if m be odd (with the exception $p^n=3$, $m=3$), but has the factors of composition 2 and one-half its order if m be even and >4.

2. The case $p^n = 8\,l \pm 1$ remains to be treated. Let $O_{1,2}^{x,\beta}$ denote a particular substitution not in the group $Q_{1,2}$. We study the group H_1 given by extending[1] Q by all the products $O_{i,j}^{x,\beta}\,Q_{k,l}^{x,\beta}$.

Theorem. *H_1 contains half*[2] *of the substitutions of G, the group of all orthogonal substitutions of determinant unity.*

In fact, every substitution of G is of the form

$$h_1\; O_{i,j}^{x,\beta}\; h_2\; O_{k,l}^{x,\beta}\; h_3 \ldots,$$

$h_1, h_2, \ldots$ being substitutions of H_1. This product can be put into the form $h\;O_{1,2}^{x,\beta}$. Indeed, $O_{i,j}^{x,\beta}$ can be carried to the right of every $Q_{i,j}^{\lambda,\mu}$ and every $Q_{k,l}^{\lambda,\mu}$ (k and $l \lesseqgtr i,j$).

[1] For $p^n = 17$ this extension is unnecessary [see appendix].

[2] Compare § 8.

Nov. 16, 1899.

Since $\left(O^{x,\beta}_{i,k}\right)^2 = Q^{x,-\beta}_{i,k}$, we have

$$O^{x,\beta}_{i,j}\, Q^{\lambda,\mu}_{i,k} = O^{x,\beta}_{i,j} \left(O^{x,\beta}_{i,k}\right)^2 Q^{x,\beta}_{i,k} \bullet Q^{\lambda,\mu}_{i,k}$$

$$= O^{x,\beta}_{i,j}\, O^{x,\beta}_{i,k} \bullet Q^{x,\beta}_{i,k}\, Q^{\lambda,\mu}_{i,k} \bullet O^{x,\beta}_{i,k} = h'\, O^{x,\beta}_{i,k}$$

Our product will finally take the form

$$h''\, O^{x,\beta}_{r,s} = h''\, O^{x,\beta}_{r,s}\, O^{x,\beta}_{2,1} \bullet O^{x,\beta}_{1,2} = h\, O^{x,\beta}_{1,2}.$$

3. Theorem. *For $m>4$ the maximal invariant subgroup I of H_1 is of order* 1 *or* 2 *according as m is odd or even.*

The proof is quite similar to that given for the case $p^n = 8l \pm 3$ in the paper cited. The lemma of § 6 is now replaced by the following

Lemma. *If I contains $C_i\, C_j$ it coincides with H_1.*

Thus $O^{x,\beta}_{2,4}\, O^{x,\beta}_{1,2}$ transforms $C_1\, C_3$ into $Q^{x,\beta}_{1,2}\, C_1\, C_3$.

Hence I contains every $Q^{x,\beta}_{i,j}$ and in particular $T_{23}\, C_3$ (2 being a square). Thus I contains

$$(T_{23}\, C_3)\,(O_{12}\, O_{14})\,(T_{23}\, C_3)^{-1}\,(O_{12}\, O_{14})^{-1} = O_{13}\, O_{21}.$$

$$(T_{23}\, C_3\;\; T_{14}\, C_4)\,(O_{12}\, O_{14})\,(T_{23}\, C_3\; T_{14}\, C_4)^{-1}\,(O_{12}\, O_{14})^{-1} = O_{43}\, O_{21}.$$

In §10 replace $O_{1345}\, T_{12}\, C_1$ by $O_{1345}\, O^{x,\beta}_{1,2}$. The development at the end of § 18 is unnecessary when 2 is a square. Thus from

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = 0, \quad \rho_1^2 + \rho_2^2 = 2,$$

it follows that $\rho_3^2 = -2$; while -1 was supposed to be a not-square.

4. Under the subdivision $p^n = 8\, l \pm 3$, the cases $p^n = 3$ and 5 were excluded in the earlier paper.

Consider the case $p^n = 5$, $m>4$. [See p. 41 of paper cited.] $T_{12}\, C_1$ extends H to the total group G; indeed, it transforms R into $C_2\, C_3\, R\;\; T_{12}\, T_{13}\, C_2\, C_3$, viz.:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix} = R^{-1}\, C_1\, C_2\, R.$$

The variations from the proof of the theorem of § 4 of my paper, necessary for the case $p^n = 5$, are the following. Instead of applying the lemma in § 6 (which becomes trivial for $p^n = 5$), we note that the transformed of $C_1\, C_2$ by R is not a product of the C_i (as shown by the above formula). We may omit § 11. For p. 36 we note that the value $\rho^2 = 1$ may be chosen, the right member of the equation of condition being then zero. At the end of § 14, when $x.\beta = 0$, I contains $C_1\, C_2$ and thus, by the above formula, also the substitution

$$R\, T_{12}\, T_{13} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = S_\sigma\, T_{23}\,,$$

if we take $\sigma_1 = \sigma_2 = \sigma_3 = \pm 2$, so that

$$1 - \sigma_i^2 = 2\,, \quad -\sigma_i\,\sigma_j = 1\,.$$

5. We pass next to the case $p^n = 3$.

For $m = 3$, the group H is the composite group

$$G_{12} = \left\{ 1,\ C_i\, C_j\ (\text{three}),\ T_{ij}\, T_{ik}\ (\text{two}),\ T_{ij}\, T_{ik}\, C_t\, C_l\ (\text{six}) \right\}.$$

For $m > 3$, the total group G may be generated by the $C_i\ C_j$ and $T_{ij}\ C_k$ together with the substitution* of determinant $+1$:

$$W = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \qquad W^2 = W^{-1} = C_1\ W C_1\,.$$

The $C_i\ C_j$ generate a group of order

$$2^{m-1} = 1 + \frac{m\,(m-1)}{1\,.\,2} + \frac{m\,(m-1)\,(m-2)\,(m-3)}{1\,.\,2\,.\,3\,.\,4} + \cdots,$$

---

*In the theoram given on p. 30 of my earlier paper, $p^n \geqq 5$ should read $p^n > 5$. The case $p^n = 3$ was not observed to be special until it was noticed that in Jordan, § 211, case 2^0 is proved only when -1 is a not-square. The case 3^n, n > 1, can be readily disposed of.

which the alternating group on m letters extends to a group H' of order $\frac{1}{2}\, m\,!\; 2^{m-1}$. This group is extended by W to our group H of index 2 under G. Indeed $T_{12}\, C_1$ transforms W into $W^2\, C_1\, C_2$, so that $T_{12}\, C_1$ will extend H to G.

For $m = 4$, the left-hand multipliers in a rectangular table for H (of order $2^5 .\, 3^2$) with H' (of order $2^5 .\, 3$) as first row may be taken to be $1,\ W,\ W^2$. Indeed, we have

$$T_{13}\, T_{12}\, W = W\, T_{13}\, T_{12}\, C_1\, C_3\,,\quad C_1\, C_2\, W = W\, T_{12}\, T_{34}\, C_1\, C_2\, C_3\, C_4 .$$

The factors of composition of H are thus 3, 3, 2, 2, 2, 2, 2.

6. Theorem. *For* $m > 4$, $p^n = 3$, *the maximal invariant sub-group* I *of* H *is of order* 1 *or* 2 *according as* m *is odd or even.*

If I contains a substitution different from $C_1\, C_2 \ldots C_m$ but commutative with every $C_i\, C_j$, it will contain $C_1\, C_5$ (by § 5 of the earlier paper) and hence also

$$W^{-1}\, C_1\, C_5\, W = W^{-2}\, C_1\, C_5 = W\, C_1\, C_5\,,$$

which is not commutative with every $C_i\, C_j$.

Applying § § 6, 9, 10 and the first of § 11, we find that I contains S' which leaves $\xi_8, \ldots \xi_m$ fixed and has $\beta_{51} = \beta_{61} = \beta_{71} = o$. Of the three quantities $\beta_{52}, \beta_{62}, \beta_{72}$, two may be chosen, say β_{52} and β_{62}, such that both are zero or both not zero (viz., ± 1). Transforming by $C_1\, C_5$, if necessary, we may assume that $\beta_{52} = \beta_{62}$. Thus our substitution is commutative with T_{56}. If commutative with every T_{ij} it becomes [see p. 36]:

$$\xi'_i = -\xi_i + 2\, (\xi_1 + \ldots + \xi_7) \quad (i = 1 \ldots 7)\,.$$

Its transformed by $C_1\, C_2$ is not of this form. Hence I contains a substitution S commutative with T_{56} but not with T_{12} for example. Thus [see top of p. 37] I contains a substitution leaving ξ_6 and ξ_7 fixed. We are thus led to the case of a substitution affecting only 5 indices:

$$S \;:\; \xi'_i = \sum_{j=1}^{5} x_{ij}\, \xi_j \quad (i = 1 \ldots 5),$$

which is not a mere product of the $C_i\, C_j$. Thus, if every $x_{ii} \gtreqless o$, must $x_{12} \gtreqless o$, for example. Hence S is not commutative with $C_1\ C_2$; for we can not have $x_{11}^2 + x_{12}^2 = 1$ as required by the note to p. 32. Thus I contains $S^{-1}\ C_1\ C_2\ S\ C_2\ C_1$, not the identity and having the coefficient

$$x_{11}' = -(1 - 2\ x_{11}^2 - 2\ x_{12}^2) \equiv o \pmod{3},$$

We may therefore suppose that $x_{11} = o$ in the substitution S. Transforming by O_{5432} we may make a second coefficient $x_{1j} = o$. Then from

$$\sum_{j=1}^{5} x_{1j}^2 \equiv 1 \pmod{3}$$

it follows that four of the x_{1j}'s are zero. Let $x_{15} \gtreqless o$. Thus S is not commutative with C_5 so that I contains $R_x\ C_1\ C_5$ not the identity (p. 40). Four of the x_{2j}'s are all zero or all not zero. Transforming by certain $C_5\ C_j$, when necessary, we may suppose that, for example,

$$x_{21} = x_{22} = x_{23}\ ,\ x_{11} = x_{12} = x_{13} = o\ .$$

Hence $R_x\ C_1\ C_5$ is commutative with $T_{12}\ T_{13}$.

Hence I contains the substitutions

$$(C_1\, C_5\, R_x)\ T_{13}\, T_{12}\, (R_x\, C_1\, C_5)\ T_{12}\, T_{13} = C_1\, C_5 \bullet C_2\, C_5 = C_1\, C_2\ ,$$
$$W^{-1}\, C_1\, C_5\, W\, C_1\, C_5 = W\ ,\ T_{12}\, T_{34} = W^{-1}\, C_1\, C_2\, W\, C_1\, C_2\, C_3\, C_4.$$

Hence I coincides with H.

7. Theorem. *The Orthogonal Group on 3 indices in the* $G\ F\ [p^n]$, $p > 2$, *has a sub-group* H *of order* $\frac{1}{2}\, p^n\, (p^{2n} - 1)$ *which is simply isomorphic to the group of linear fractional substitutions of determinant unity on one index.*

Let i be a root, real or imaginary, of $i^2 = -1$; so that i belongs to our $G\ F\ [p^n]$ only when $-1 =$ square.

Introduce in place of ξ_1, ξ_2, ξ_3 the new indices

$$\eta_1 = -i\ \xi_1\ ,\ \eta_2 = \xi_2 - i\ \xi_3\ ,\ \eta_3 = \xi_2 + i\ \xi_3\ ,$$

whence $$-\eta_1^2 + \eta_2\ \eta_3 = \xi_1^2 + \xi_2^2 + \xi_3^2\ .$$

The orthogonal substitution

$$S\ :\ \ \xi'_i = \sum_{j=1}^{3} x_{ij}\ \xi_j \quad (i = 1,\ 2,\ 3)$$

takes the form S_1:

$$\eta_1' = x_{11}\,\eta_1 + \tfrac{1}{2}\,(x_{13} - i\,x_{12})\,\eta_2 - \tfrac{1}{2}\,(x_{13} + i\,x_{12})\,\eta_3,$$
$$\eta_2' = (x_{31} + i\,x_{21})\,\eta_1 + \tfrac{1}{2}\,(x_{22} - i\,x_{32} + i\,x_{23} + x_{33})\,\eta_2 + \tfrac{1}{2}\,(x_{22} - i\,x_{32} - i\,x_{23} - x_{33})\,\eta_3,$$
$$\eta_3' = (-x_{31} + i\,x_{21})\,\eta_1 + \tfrac{1}{2}\,(x_{22} + i\,x_{32} + i\,x_{23} - x_{33})\,\eta_2 + \tfrac{1}{2}\,(x_{22} + ix_{32} - ix_{23} + x_{33})\,\eta_3.$$

It is proven below that the coefficient

$$\tfrac{1}{2}\,(x_{22} + x_{33}) + i/2\,(x_{23} - x_{32}) = \alpha^2,$$

α being a complex of the form $\rho + \sigma\,i$, where ρ and σ are marks of the $G\,F\,[p^n]$. It follows that the coefficient of η_3 in η_2' is a square, viz.:

$$\tfrac{1}{2}\,(x_{22} - x_{33}) - i/2\,(x_{23} + x_{32}) = \beta^2.$$

In fact, by virtue of the orthogonal relations (2), p. 30,

$$\alpha^2\,\beta^2 = \tfrac{1}{4}\,(x_{22}^2 + x_{23}^2 - x_{32}^2 - x_{33}^2) - i/4\,(2\,x_{22}\,x_{32} + 2\,x_{23}\,x_{33}) = \tfrac{1}{4}\,(x_{31} + i\,x_{21})^2.$$

The coefficients of η_2 and η_3 in η_3' are squares, say γ^2 and δ^2, since they are the conjugates of β^2 and α^2 respectively. We may verify that

$$\alpha^2\,\gamma^2 = \tfrac{1}{4}\,(x_{13} - i\,x_{12})^2,$$

whose conjugate is therefore $\beta^2\,\delta^2$. Further, we find[1]

$$\begin{vmatrix} \alpha^2 & \beta^2 \\ \gamma^2 & \delta^2 \end{vmatrix} = x_{22}\,x_{33} - x_{32}\,x_{23} = x_{11}.$$

Also $(\alpha\,\delta + \beta\,\gamma)^2$ when expanded reduced to x_{11}^2.

Hence $\qquad \alpha\,\delta - \beta\,\gamma = 1.$

Our substitution S_1 thus becomes[2]

$$\begin{pmatrix} \alpha\,\delta + \beta\,\gamma & \alpha\,\gamma & \beta\,\delta \\ 2\,\alpha\,\beta & \alpha^2 & \beta^2 \\ 2\,\gamma\,\delta & \gamma^2 & \delta^2 \end{pmatrix} \qquad [\alpha\,\delta - \beta\,\gamma = 1].$$

Giving S_1 the notation $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ we verify the composition formula for $S_1'\,S_1$:

---

[1] Baltzer, Determinanten, § 14, 5, page 189.

[2] Compare Fricke-Klein, Automorphe Functionen, I, p. 14; also Weber, Algebra, II, p. 190.

$$\begin{bmatrix} x' & \beta' \\ \gamma' & \delta' \end{bmatrix} \begin{bmatrix} x & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} x\,x' + \beta\,\gamma' & x\,\beta' + \beta\,\delta' \\ \gamma\,x' + \delta\,\gamma' & \gamma\,\beta' + \delta\,\delta' \end{bmatrix}.$$

Thus if $\frac{1}{2}\,(x_{22} + x_{33}) + i/2\,(x_{23} - x_{32})$, which is one of the coefficients in S_1, be a square x^2, and if the corresponding coefficient in S'_1 be a square x'^2, then will that of $S'_1\,S_1$ be the square $(x\,x' + \beta\,\gamma')^2$.

For the substitution $O^{\lambda,\mu}_{2,3}$ the above expression is

$$\lambda + i\,\mu = (\rho + i\,\sigma)^2$$

if $\lambda = \rho^2 - \sigma^2,\ \mu = 2\,\rho\,\sigma$, *i. e.*, if $O^{\lambda,\mu}_{2,3} = Q^{\rho,\sigma}_{2,3}$.

For $O^{\lambda,\mu}_{1,2}$, the expression is $\frac{1}{2}\,(\lambda + 1) = \rho^2$ if $O^{\lambda,\mu}_{1,2} = Q^{\rho,\sigma}_{1,2}$.

For $O^{\lambda,\mu}_{1,2}\,O^{\lambda,\mu}_{1,3}$ the expression is

$$\tfrac{1}{2}\,(2\,\lambda + i\,\mu^2) = \left\{\ \tfrac{1}{2}\,(1 + i)\,(1 - i\,\lambda)\right\}^2.$$

For $O^{\lambda,\mu}_{1,2}\,O^{\lambda,\mu}_{3,1}$, it is the conjugate of the latter. For the special case $O^{0,-1}_{1,2}\,O^{0,-1}_{3,1} = T_{12}\,T_{13}$ it is $\frac{-i}{2} = \left(\frac{1-i}{2}\right)^2$.

For the generator R (necessary when $p^n = 5$) it is 3^2. Since the group H can be generated from the above substitutions, it follows[1] that our expression is always a square.

We have therefore proven that H is simply isomorphic to the group Γ of linear fractional substitutions of determinant unity on one index. When -1 is a square, the coefficients x, β, γ, δ belong to the $G\,F\,[p^n]$ and, if $p^n > 3$, the group Γ is simple.[2] When -1 is a not-square, the coefficients

---

[1] A direct proof by induction could doubtless be made. Thus the expression, denoted by x^2 for the substitution S_1, when built for the product $O^{\lambda,\mu}_{2,3}\,S_1$ is seen to be $(\lambda + i\,\mu)\,x^2$.

[2] Moore, *A doubly-infinite system of simple groups*, Mathematical Papers of the Chicago Congress of 1893. Other proofs have since been given by Burnside and Dickson.

α, δ and also β, γ are conjugate imaginaries in i, so that[1] Γ is the "imaginary form" of the former simple group. Hence the ternary orthogonal group H is simple if $p^n > 3$.

8. It follows from what precedes that for $m = 3$ the sub-group H does not coincide with the total group G, but is of index 2 under it. A similar result doubtless holds for any m. To prove that our group H is generated by the squares of the substitutions of G (as stated in § 1), we note that

$$\left(O^{\alpha,-\beta}_{1,2}\right)^2 = Q^{\alpha,\beta}_{1,2}, \quad \left(O^{\alpha,\beta}_{1,2}\, T_{13}\, T_{24}\right)^2 = O^{\alpha,\beta}_{1,2}\, O^{\alpha,\beta}_{3,4}$$

$$\left(O^{\alpha,\beta}_{1,2}\, T_{13}\, C_1\, C_2\, C_3\right)^2 = O^{\alpha,\beta}_{1,2}\, O^{\alpha,\beta}_{3,2}.$$

The orders of the simple groups reached are as follows:

For $m = 2k + 1 \geqq 3$, $p > 2$, and $p^n > 3$ when $m = 3$,

$$\tfrac{1}{2}\,(p^{2nk}-1)\,p^{2nk-n}\,(p^{2nk-2n}-1)\,p^{2nk-3n}\ldots\,(p^{2n}-1)\,p^n;$$

for $m = 2k > 4$,

$$\tfrac{1}{4}\,(p^{nk}-\omega)\,p^{nk-n}\cdot(p^{2nk-2n}-1)\,p^{2nk-3n}\ldots\,(p^{2n}-1)\,p^n,$$

where $\omega = (\pm 1)^k$ according as $p^n = 4l \pm 1$.

APPENDIX.

9. For $p^n = 7$, $m = 3$, the Q_{ij} constitute the group G_{24} of page 44. This is extended by $O^{2,2}_{1,2}$ to the group of all 14.24 orthogonal substitutions of determinant $+1$. For a rectangular table of the latter with G_{24} as first line we may choose as left hand multipliers

$$1,\ O_{13}\,O_{23},\ O_{23}\,O_{12},\ O_{23}\,O_{13},$$

$$O_{12}\ O_{13} = \begin{pmatrix} 4 & 4 & 2 \\ -2 & 2 & 0 \\ -4 & -4 & 2 \end{pmatrix}, \quad O_{13}\ O_{12} = \begin{pmatrix} 4 & 2 & 4 \\ -4 & 2 & -4 \\ -2 & 0 & 2 \end{pmatrix},$$

$$O_{12}\ O_{23} = \begin{pmatrix} 2 & 2 & 0 \\ -4 & 4 & 2 \\ 4 & -4 & 2 \end{pmatrix},$$

---

[1] Moore, l. c. § 6.

$$O_{12}\ ;\ O_{13}\ ,\ O_{23}\ ,\quad O_{13}\,O_{12}\,O_{23} = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 4 & -4 \\ 4 & -4 & -2 \end{pmatrix},$$

$$O_{23}\,O_{12}\,O_{13} = \begin{pmatrix} 4 & 4 & -2 \\ -2 & 4 & 4 \\ -4 & 2 & -4 \end{pmatrix},\quad O_{23}\,O_{12}\,O_{23} = \begin{pmatrix} 2 & 4 & 4 \\ -4 & 4 & -2 \\ 4 & 2 & -4 \end{pmatrix}.$$

$$O_{13}\,O_{23}\,O_{13} = \begin{pmatrix} -4 & -4 & -2 \\ -4 & 2 & 4 \\ 2 & -4 & 4 \end{pmatrix}.$$

It may be verified that the whole group has been reached. Thus, for example,

$$O_{13}\,O_{12}\,O_{13} = O_{23}\,O_{12}\,O_{13}\,T_{23}\,C_2,$$
$$O_{23}\,O_{13}\,O_{12} = O_{13}\,O_{12}\,O_{23}\,T_{12}\,T_{13}\,C_2\,C_3,$$
$$\left(O_{13}\,O_{12}\right)^2 = O_{23}\,O_{12}\,T_{12}\,T_{13}\,C_2\,C_3,$$
$$O_{13}\,O_{12}\bullet O_{23}\,O_{13} = O_{23}\,O_{13}\,T_{23}\,C_3\ ,$$
$$O_{13}\,O_{12}\bullet O_{23}\,O_{12} = O_{12}\,O_{23}\,T_{23}\,C_1\,C_2\,C_3.$$

Our subgroup H of index 2 is obtained by taking only the seven multipliers in the first three lines.

10. For $p^n = 9$, $m = 3$, the Q_{ij} constitute the same G_{24}. Define the $G F [3^2]$ by the congruence $i^2 \equiv -1 \pmod{3}$. Then $O^{i,i}_{1,2}$ extends G_{24} to the total group $G_{30\cdot 24}$, for a rectangular table of which with G_{24} as first line we may choose the left hand multipliers

$$1\ ,\ O^{i,i}_{1,2}\ ,\ O^{i,i}_{1,3}\ ,\ O^{i,i}_{2,3}\ ,$$

$$O_{12}\,O_{13} = \begin{pmatrix} -1 & -1 & i \\ -i & i & 0 \\ 1 & 1 & i \end{pmatrix},\ O_{23}\,O_{12}\,O_{13} = \begin{pmatrix} -1 & 1-i & -1-i \\ -i & -1 & -1 \\ 1 & 1+i & -1+i \end{pmatrix},$$

$$(O_{12}\,O_{13})^2 = \begin{pmatrix} 1-i & 1 & -1-i \\ 1+i & -1+i & 1 \\ -1 & -1-i & -1+i \end{pmatrix},\quad (O_{23}\,O_{12}\,O_{13})^2 =$$

$$\begin{pmatrix} -1+i & 1 & -1-i \\ -1-i & -1+i & 1 \\ -1 & 1+i & 1-i \end{pmatrix},$$

there being six with a single element o, twelve with a single element $\pm i$, and eight with neither o nor $\pm i$. The 15 multipliers giving our simple group H are those with an even number of the O_{ij}.

11. For $p^n = 17$, the Q_{ij} are $C_i\ C_j$, $T_{ij}\ C_i$ or $O_{ij}^{\pm 3, \pm 3}$.

$$S \equiv O_{1,2}^{3,3}\ O_{1,3}^{3,3} = \begin{Bmatrix} 9 & 9 & 3 \\ -3 & 3 & 0 \\ -9 & -9 & 3 \end{Bmatrix}, \quad S^2 = \begin{Bmatrix} -7 & -4 & 2 \\ -2 & -1 & -9 \\ 4 & 1 & -1 \end{Bmatrix}$$

$$S^4 = O_{3,1}^{3,3}\ T_{23}\ C_1\ C_2\ C_3\ , \quad S^3 = S^{-1}\ , \quad S^9 = 1.$$

$$O_{1,2}^{6,4}\ O_{1,3}^{6,4} = \begin{Bmatrix} 2 & 7 & 4 \\ -4 & 6 & 0 \\ -7 & 1 & 6 \end{Bmatrix} = \begin{Bmatrix} 2 & 7 & 4 \\ 1 & 4 & 1 \\ -9 & 2 & 1 \end{Bmatrix} \bullet O_{3,2}^{3,3}$$

$$= T_{13}\ T_{12}\ C_1\ C_2\ (O_{1,2}^{3,3}\ O_{1,3}^{3,3})^2\ T_{23}\ C_2 \bullet O_{3,2}^{3,3}\ .$$

Also, the square of $O_{1,2}^{6,4}$ gives $O_{1,2}^{3,-3}$. Hence the product of any two of the $O_{i,\ j}^{\pm 6,\ \pm 4}$ belongs to Q.

Note that $T \equiv O_{2,3}^{3,3}\ O_{1,2}^{3,3}\ O_{1,3}^{3,3}$ is of period six; thus,

$$T = \begin{Bmatrix} 9 & 1 & 2 \\ -3 & 9 & 9 \\ -9 & -2 & -1 \end{Bmatrix}, \quad T^2 = \begin{Bmatrix} -8 & -3 & 8 \\ 1 & -8 & -2 \\ 2 & -8 & -1 \end{Bmatrix}, \quad T^4 = T^{-2}.$$

12. Theorem. *There exists in the group* Q_{12} *a substitution of the form* $O_{1,2}^{\lambda,\lambda}$ *if and only if* $p^n = 16\,l \pm 1$.

Since 2 must be a square $(2\,\lambda^2 = 1)$ the group Q_{12} has the subgroup $\left\{ 1,\ C_1\ C_2\ ,\ T_{12}\ C_1,\ T_{12}\ C_2 \right\}$. Multiplying on the left by $O_{1,2}^{x,\beta}$ we reach

$$O_{1,2}^{x,\beta},\quad O_{1,2}^{-x,-\beta},\quad O_{1,2}^{\beta,-x},\quad O_{1,2}^{-\beta,x}\ .$$

Similarly the multiplier $O_{1,2}^{x,-\beta}$ gives a set of four, different from the former only when β differs from $\pm x$. Hence there are $8\,k$ or $8\,k + 4$ distinct substitutions in Q_{12}, according as $O_{1,2}^{\lambda,\lambda}$ occurs or not. The order of the group being $\frac{1}{2}(p^n - \epsilon)$, where $\epsilon = \pm 1$ according as $p^n = 4l \pm 1$, the condition for $O_{1,2}^{\lambda,\lambda}$ is that $p^n - \epsilon$ be divisible by 16.

Since the condition that $O_{1,2}^{x,\beta}$ be a $Q_{1,2}$ is that $2(1 \pm x)$ be squares, the condition that $O_{1,2}^{\lambda,\lambda}$ occur among the $Q_{1,2}$ is that $2 \pm \sqrt{2}$ be squares.

Corollary. $2 + \sqrt{2}$ is a square in the $GF[p^n]$ only when p^n is of the form $16l \pm 1$.

Thus for $p^n = 17$, $\lambda = 3$, $2 + \sqrt{2} \equiv 2 + 6 \equiv 5^2 \pmod{17}$;

for $p^n = 31$, $\lambda = 4$, $2 + \sqrt{2} \equiv 2 + 8 \equiv 14^2 \pmod{31}$.

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On an $m\,n^2$ Parameter Group of Linear Substitutions in $m\,n$ Variables.

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ON AN $m\ n^2$ PARAMETER GROUP OF LINEAR SUBSTITUTIONS IN $m\ n$ VARIABLES.

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§ 1. *All substitutions of the form*

$$(1)\left\{\begin{aligned}
\eta_i &= \sum_{k=1}^{n} a_{ik}\ y_k \\
\eta_i^{(1)} &= \sum_{k=1}^{n} (a_{ik}'\ y_k + a_{ik}\ y_k') \\
\eta_i^{(2)} &= \sum_{k=1}^{n} (a_{ik}''\ y_k + a_{ik}'\ y_k' + a_{ik}\ y_k'') \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\eta_i^{(m-1)} &= \sum_{k=1}^{n} (a_{ik}^{(m-1)}\ y_k + a_{ik}^{(m-2)}\ y_k' + \ldots + a_{ik}\ y_k^{(m-1)})
\end{aligned}\right.$$

$$(i = 1, 2, \ldots n)$$

belong to a group G. There are $m\ n$ variables and $m\ n^2$ parameters. If D denotes the determinant

$$D = \left| a_{ik} \right|, \ (i, k = 1, 2, \ldots n)$$

then the determinant of (1) is D^m.

The $m\ n^2$ independent infinitesimal transformations of (1) are

$$\left\{\begin{aligned}
U_{ik}^{(0)} f &= \frac{df}{dy_i}\ y_k + \frac{df}{dy_i'}\ y_k' + \ldots + \frac{df}{dy_i^{(m-1)}}\ y_k^{(m-1)} = \sum_{\lambda=0}^{m-1} \frac{df}{dy_i^{(\lambda)}}\ y_k^{(\lambda)} \\
U_{ik}^{(1)} f &= \frac{df}{dy_i'}\ y_k + \ldots + \frac{df}{dy_i^{(m-1)}}\ y_k^{(m-2)} = \sum_{\lambda=1}^{m-1} \frac{df}{dy_i^{(\lambda)}}\ y_k^{(\lambda-1)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
U_{ik}^{(m-2)} f &= \frac{df}{dy_i^{(m-2)}}\ y_k + \frac{df}{dy_i^{(m-1)}}\ y_k' = \sum_{\lambda=m-2}^{m-1} \frac{df}{dy_i^{(\lambda)}}\ y_k^{[\lambda-(m-2)]} \\
U_{ik}^{(m-1)} f &= \frac{df}{dy_i^{(m-1)}}\ y_k\ ,
\end{aligned}\right.$$

or in general

$$(2)\quad U_{ik}^{(\rho)} f = \sum_{\lambda=\rho}^{m-1} \frac{df}{dy_i^{(\lambda)}} y_k^{(\lambda-\rho)} \quad \begin{pmatrix} i, k = 1, 2, \ldots n \\ \rho = 1, 2, \ldots m \end{pmatrix}.$$

§ 2. *Relations between the Symbols $U_{ik}^{(\rho)} f$.*

If we wish to investigate questions as to the invariants of the group, we must know how many of the equations $U_{ik}^{(\rho)} f = o$ are independent. It is therefore necessary to find the relations between the different symbols. We find at once

$$(3)\quad y_j\, U_{ik}^{(m-1)} f - y_k\, U_{ij}^{(m-1)} f = o$$
$$\begin{pmatrix} i = 1, 2, \ldots n \\ k = 1, 2, \ldots j-1, j+1, \ldots n \end{pmatrix},$$

i. e., $n\,(n-1) = n^2 - n$ relations. Therefore only n of the equations $U_{ik}^{(m-1)} f = o$ are independent, and there are obviously not less than n.

Further we find

$$(4)\quad y_j\, U_{ik}^{(m-2)} f - y_k\, U_{ij}^{(m-2)} f = y_k'\, U_{ij}^{(m-1)} f - y_j'\, U_{ik}^{(m-1)} f,$$

again $n^2 - n$ relations, which show that just n of the equations $U_{ik}^{(m-2)} f = o$ are independent.

Generally we have

$$(5)\quad y_j\, U_{ik}^{(\rho)} f - y_k\, U_{ij}^{(\rho)} f = \sum_{\lambda=\rho+1}^{m-1} \frac{df}{dy_i^{(\lambda)}} \left(y_k^{(\lambda-\rho)}\, y_j - y_j^{(\lambda-\rho)}\, y_k \right),$$

where the right member can evidently be expressed in terms of $U_{ik}^{(\rho+1)} f, \ldots U_{ik}^{(m-1)} f$. Therefore, by induction, for every value of ρ there are precisely n equations of the form $U_{ik}^{(\rho)} f = o$ which are independent. Altogether, therefore, there are $m\,n$ independent equations, which form a complete system in $m\,n$ variables. This system therefore has no solution except $f =$ const. *The general group G therefore has no invariants, i. e., it is transitive.*

§ 3. *Composition of the Group. Commutator-relations.*

We have

$$(6)\quad U_{ik}^{(\rho)} f = \sum_{\lambda=\rho}^{m-1} \frac{df}{dy_i^{(\lambda)}} y_k^{(\lambda-\rho)}$$

(7) $U_{jl}^{(\sigma)} f = \sum_{\lambda=\sigma}^{m-1} \frac{df}{dy_j^{(\lambda)}} y_l^{(\lambda-\sigma)}$.

Now $U_{ik}^{(\rho)} \left(y_l^{(\lambda-\sigma)} \right) = o$ for $\lambda = \sigma, .. m-1$, if $\rho > m-1-\sigma$,

and $U_{jl}^{(\sigma)} \left(y_k^{(\lambda-\rho)} \right) = o$ for $\lambda = \rho, .. m-1$, if $\sigma > m-1-\rho$.

Therefore

(8) $\left(U_{ik}^{(\rho)}, U_{jl}^{(\sigma)} \right) f = o$

for $\rho + \sigma > m-1$.

So we have

$$(U_{ik}^{(m-1)}, U_{jl}^{(1)}) = (U_{ik}^{(m-1)}, U_{jl}^{(2)}) = \ldots = (U_{ik}^{(m-1)}, U_{jl}^{(m-1)}) = o$$

$$(U_{ik}^{(m-2)}, U_{jl}^{(2)}) = \ldots = (U_{ik}^{(m-2)}, U_{jl}^{(m-1)}) = o$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(U_{ik}^{(1)}, U_{jl}^{(m-1)}) = o$$

It remains to consider those commutators, the sum of whose indices is not greater than $m-1$.

We find

(9) $\left(U_{ik}^{(\rho)}, U_{jl}^{(\sigma)} \right) f = o$

if $i \gtrless l$ and $j \gtrless k$.

Now let $i = l$, but $j \gtrless k$. Then

(10) $\left(U_{ik}^{(\rho)}, U_{ji}^{(\sigma)} \right) f = U_{jk}^{(\rho+\sigma)} f$, $(\rho + \sigma < m-1, j \gtrless k.)$

If $i = l$ and $j = k$,

(11) $\left(U_{ik}^{(\rho)}, U_{ki}^{(\sigma)} \right) f = U_{kk}^{(\rho+\sigma)} f - U_{ii}^{(\rho+\sigma)} f$

$$(\rho + \sigma < m-1).$$

In particular, if $i = k$,

(12) $\left(U_{ii}^{(\rho)}, U_{ii}^{(\sigma)} \right) f = o$.

Now let $i \gtrless l$ but $j = k$. Then we find in the same manner

(13) $\left(U_{ik}^{(\rho)}, U_{kl}^{(\sigma)} \right) f = - U_{il}^{(\rho+\sigma)} f$.

It is now easy to describe the structure of the group. *The infinitesimal transformations* $U_{ik}^{(\rho)} f$ *for* $\rho = 1, 2, \ldots m-1$ *generate a self-conjugate sub-group* G_1 *of* G. For if we form $(U_{ik}^{(\rho)}, U_{jl}^{(\sigma)}) f$, whenever it is not zero, it is found to be expressed in terms of $U_{\lambda\mu}^{(\rho+\sigma)} f$. Compounding therefore all of the infinitesimal transformations of G with those of G_1, only operations of G_1 result. Therefore G_1 is a self-conjugate sub-group of G. It has $(m-1)n^2$ parameters. Similarly taking $U_{ik}^{(\rho)} f$ for $\rho = 2, 3 \ldots m-1$, we find that these transformations generate a self-conjugate sub-group G_2, with $(m-2)n^2$ parameters, of G_1, and so on. We are thus finally led to an n^2 parameter group G_{m-1} which is self-conjugate sub-group of a $2\,n^2$ parameter group G_{m-2}, and which is generated by the infinitesimal transformations $U_{ik}^{(m-1)} f$. The finite equations of the group are

$$\eta_i = y_i\ ,\ \eta_i' = y_i'\ ,\ \ldots\ \eta_i^{(m-2)} = y_i^{(m-2)}\ ,$$
$$\eta_i^{(m-1)} = a_{i1}\, y_1 + a_{i2}\, y_2 + \ldots + a_{in}\, y_n + y_i^{(m-1)}.$$
$$(i = 1, 2, \ldots n).$$

§ 4. *Linear Differential Equations.*

If the transformation group of a linear differential equation of the $m\,n$ th order is the group G, it is easily seen that the integration of that equation can be reduced to the successive integration of the m equations of the n th order, out of which the equation of the $m\,n$ th order is compounded, and quadratures. The same is true for the more general group which is found if in equations (1) the coefficient of $y_k^{(\lambda)}$ in the expression for $\eta_i^{(\lambda)}$ is arbitrary instead of being equal to a_{ik}. The group G however has a special significance owing to the fact that if $\eta_1, \ldots \eta_n$ and a_{ik} are considered as functions of a variable x, the group of linear substitutions

$$\eta_i = \sum_{k=1}^{n} a_{ik}\ y_k$$

extended $m-1$ times has the form of the group G.

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The Apparent Projection of Stars upon the bright Limb of the Moon at Occultation, and Similar Phenomena at Total Solar Eclipses, Transits of Venus and Mercury, Etc., Etc.

BY

GEORGE DAVIDSON, PH.D., SC.D.

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Read before the California Academy of Sciences, August 9, 1897.

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 November 15, 1900.

INTRODUCTION.

THIS paper is prompted mainly by the many references made by astronomers and astronomical publications to "Baily's Beads" in cases of total solar eclipses; to the "black drop" and "ligament" in transits of Mercury and Venus over the Sun's disc; to the projection of a star upon the bright body of the Moon at immersion or emersion of occultation; to stars and the satellites of Jupiter seen through the limb or body of that planet at occultation; to planets surrounded by an atmosphere of great depth; and to empirical corrections to the semidiameters of the Sun and Moon at meridian transits.

We have lately examined the reports of the observers of the transits of Venus in 1874 and 1882; of the transits of Mercury since 1878, and the reports of the total solar eclipses since 1860; and we are amazed at the crude and apparently indigested explanations given for certain appearances attending the phenomena. There would seem to have been a desire to seek explanations in some obscure cause or causes—such as the poor character of the objective used, the lack of sharpness in the limbs of the planets (owing to their possessing an atmosphere), diffraction, telescopic and occular irradiation, astigmatism, "sympathetic attraction," etc.

These explanations are very unsatisfactory, and the experienced observer, who has observed occultations of stars by the Moon or by Jupiter, meridian transits of the Moon, transits of Mercury and Venus across the Sun's disc, and total solar eclipses in a serene atmosphere, knows that the outlines of the planets, the Moon, and the Sun are remarkably sharp and steady, even under high magnifying power; that the observations made under such conditions are satisfactory; and that there are no extraordinary phenomena present.

In such an atmosphere the spurious disc of a star, large or small, moves along the horizontal thread of the Meridian Circle, or the Zenith Telescope, so steadily that the error of observation should not exceed one-tenth of a second of arc. And the double-star observer certainly expects his measures of distance to differ little from that amount. In the telescope the diffraction rings of the stars are beautifully delicate. On a quiet night the stars appear, to the normal eye, fixed on the sky like minute dots of light, without a sign of twinkling or irradiation, and the intensity of the light of each star, large or small, is continuous.

The observer knows that in an atmosphere which is remarkably unsteady, when every star to the naked eye is dancing madly, the smallest stars have disappeared, the smaller ones appear and disappear spasmodically, and the larger ones near the horizon rapidly change color, and even the planets have much irradiation and are blinking; the star in the instrument has lost its diffraction rings, the spurious disc is broken up, and the confused mass jumps wildly across the micrometer thread and frequently expands into a nebulous film many seconds in diameter. We have measured such an apparition of Polaris forty-seven seconds in diameter.

"Burnham has remarked that an object-glass of six inches will one night show the companion of Sirius perfectly; on the next night, just as good in every respect, so far as one can tell with the unaided eye, the largest telescope in the world will show no more trace of the small star than if it had been blotted out of existence." (Webb's "Celestial Objects.")

With a serene atmosphere the transit observer is certain of his times. With a disturbed atmosphere he can make only an unsatisfactory estimate by eye and ear, and a still less reliable one by the chronograph.

In the course of fifty-four years' experience as an observer, largely in the field, we have encountered a range of physical conditions that falls to the lot of very few. We have made astronomical and geodetic observations in all climates, at all seasons, from the low ocean coast to elevations reaching 12,566 feet, and with instruments of precision of various character. Our latest astronomical observations were made in perhaps the worst locality possible. The peninsula of San Francisco near the Golden Gate is unique for general unsteadiness of the atmosphere, yet marked on rare occasions by some supreme exhibitions of quietness.

GEODETIC OBSERVATIONS IN THE SIERRA NEVADA.

In the high Sierras of eastern California, from 7,250 to 10,430 feet elevation, after a great southeast storm with clearing weather in the night, a strong wind from the northwest, and a wonderfully pellucid sky and steady atmosphere to windward that gave astonishing clearness, sharpness, steadiness and apparent nearness to all objects, we have, at the greater elevation, seen star-like heliotrope images with the naked eye, at stations 120, 130, 140 and 160 miles distant to the west and northwest, whence the cold, keen wind was blowing. At this station, with the whole range covered with snow, the mountain face was precipitous for 1,356 feet to the north. To the southeastward and eastward, over the very deep valley of the Mokelumne River, directly under us, subject to the Sun's early warmth, the atmosphere became excessively disturbed and the heliotrope images in that direction were not visible to the eye at distances of forty miles. In the telescope the former images were steady, round, and so minute that the micrometer thread nearly covered them; the latter images were very diffuse, irregular and unsteady, and ranged from twenty seconds to thirty seconds in diameter. In the first case the pointing was quickly and satisfactorily made; in the second the pointings were unsatisfactory, even with half a dozen occular pointings as checks.

VISIBILITY OF JUPITER'S SATELLITES.

In astronomical work we have observed the Sun, Moon, planets and stars when the Sun was serenely steady; and at night not a star twinkled to the naked eye, and no irradiation was visible to Jupiter. Half a dozen times in our experience we have been able to distinguish with the naked eye[1] two of Jupiter's satellites, when close together, as one; and at one mountain station (3,200 feet elevation) all the members of our party witnessed a similar phenomenon. Very curiously, that evening was slightly hazy, and very faint stars were not visible; but the atmosphere was remarkably steady. Under atmospheric condition of supreme quietness we have placed eleven stars in the Pleiades.

The visibility of Jupiter's satellites by the naked eye has generally been received with much doubt. Humboldt, in his Cosmos, has related a well authenticated case, but we can come to our own observers for confirmation. At the time of the total solar eclipse of August, 1878, one of the United States Naval Observatory observers in Colorado saw with the naked eye three satellites of Jupiter separately—two very distinctly and one less so. The atmosphere was supremely serene, and the station (Idaho Springs) 7,548 feet above the sea. Two other persons, not observers, also witnessed the phenomenon. In 1874 the Professor and his colleagues had doubted my seeing even two satellites close together as one.

Conditions of Steadiness and Unsteadiness of the Atmosphere.

Under similar atmospheric conditions the limbs of the Sun, Moon and planets are sharply defined and remarkably steady; the Sun-spots are beautifully distinct and their changes of form readily followed; the irregularities of the outline of the Moon are unmistakable; the spurious discs of the stars are encircled by delicate diffraction rings, and the overlapping spurious discs of very close double stars present new forms. Under conditions of supreme steadiness of the atmosphere, we believe that an observer with keen eyesight should see, with unassisted vision, the larger satellites of Jupiter, when favorably located, and from ten to twelve stars in the Pleiades. Either is a severe and unique test of steadiness and eyesight. In these periods of great steadiness the observer aches for finer micrometers, higher power, and more stable instruments.

[1] Except with spectacles for short-sightedness.

When the steadiness of the atmosphere begins to break, the images of the Sun, Moon and planets exhibit occasional tremors or shiverings; and even when this vibratory movement has become nearly continuous, but not rapid, the actual and apparently more condensed outline of the image can be selected for measures of precision in moments of quietude. As the unsteadiness of the atmosphere increases, the vibration of the limbs increases in rapidity and amplitude until there is finally created a blurred outline of Sun, Moon or planet, which necessarily presents an increased diameter of the object. The stars are similarly affected; with the first signs of unsteadiness in the atmosphere the diffraction rings are broken, and disappear as the unsteadiness increases; the spurious disc is broken into pieces; its march is peculiar, in presenting one fragment after another with more or less frequency, but may continue with a waltzing movement until the unsteadiness has increased and the spurious disc is without form, dilated, jumping wildly in every direction, and a wretched object for observation.

Similar conditions are exhibited by geodetic signals of all classes. In a serene atmosphere poles are seen very steady as fine, sharp images, sometimes finer than the micrometer threads. When this atmosphere is disturbed, tremors or shiverings carry the images to right or left one or more diameters, and the observer selects moments of quietude for first-class observations; finally, as the unsteadiness of the atmosphere increases, the images become large, confused, blurred and faint, and observations are difficult to make and unsatisfactory. In the quiet atmosphere, white poles five feet long and four and six-tenths inches in diameter, projected upon a dark background, have been seen with the naked eye at distances of four and one-half miles across water. In times of great unsteadiness and equally clear atmosphere they are difficult to see in the telescope.

Heliotrope images behave precisely as those of the stars. On a certain line of the main triangulation of the Pacific Coast, where plains, several mountain ridges and intervening valleys were crossed by the line of sight, and where the cold ocean air sweeping through the Karquinez Strait added ten-fold confusion, the heliotrope image from Mt. Diablo, distant forty-five miles, was seen in the telescope like the waving flame from the stack of a smelting furnace. Sometimes the image exceeded sixty seconds in irregular diameter, and its horizontal direction was very abnormal. This condition lasted for many days. When the air became of more uniform temperature the image was small, star-like, and steady, and its direction normal.

In the remarkably quiet atmosphere of the mountains, before sunset, with a pure sky for a background, we once saw, with the two and one-eighth inches objective of the theodolite, the observer leave his station, which was thirty-three miles distant from Mt. Tamalpais. With a salmon-colored sky in the west, and the atmosphere remarkably clear and steady, Superintendent Bache observed upon the signal-pole of Mt. Wachusett, distant sixty-one miles. The image was nearly as dark and sharp as the cross-threads.

In winter, at great elevation, we once saw, with a three-inch objective, just after sunrise, with a serene atmosphere, the heliotroper projected against a snow background at a distance of fifty-one miles. From Mt. Diablo, 3,849 feet elevation, with the same size objective, just before sunrise, in a perfectly quiet atmosphere, when the crest-line of the Sierra Nevada was projected like a dark-blue saw against the warm eastern sky, and apparently only thirty or forty miles distant, we saw upon several mornings the observing hut on the pinnacle of Mt. Conness. The hut was six feet wide by seven and a half feet high, and the distance one hundred and forty-three miles; the width therefore subtended $1''.64$. After the hut had been cut down to about four feet we still made it out upon an extraordinarily clear and steady morning. On account of the distance and great height of Mt. Conness (12,566 feet) the Sun rose to the heliotroper about eight minutes earlier than to Mt. Diablo. The sharpness and steadiness of Mt. Conness were supreme, and the image of the heliotrope was like a minute star; but so soon as the Sun's rays struck the eastern flank of Mt.

Diablo the atmosphere around the observing station became so unsteady that observations were made with difficulty and little satisfaction.

We had a similar experience at the transit of Venus, Station Cerro Roblero, in New Mexico, in 1882. Before sunrise the atmosphere was so serene that we could see, with the five-inch equatorial, the small branches of the scant scrub growth on the dark, sharp, blue crest-line of the Organ Mountains, twenty miles distant. Immediately after sunrise the suddenly heated air, rising up the steep slope of the Cerro Roblero, just in front of the observatory, caused great waves of disturbed air to obliterate the branches of the brush and to exhibit the border of the Sun as a remarkably confused outline. The direction of these waves was very marked. At the time of the first contact of the images of Venus and the Sun the observations were made with difficulty and doubt, and micrometer measures were almost impracticable. The station was about 5,676 feet above the sea, and 1,655 feet above the Rio Grande and the Jornado del Muerte.

We have been thus prolix in order to indicate that the principal disturbance exhibited in the telescope, as well as to the unassisted eye, is in the immediate vicinity of the observing station. At trigonometrical stations, in disturbed conditions of the air, the confused and unsteady image of a heliotrope will exhibit a movement of $5''$ to $15''$ of arc. The observer could not see such movement if it took place rapidly at the distant station. If the heliotrope image at Mt. Conness were moved bodily sideways forty-four inches, the observer at Mt. Diablo would see it change only one second in amplitude, supposing the movement were slow and therefore capable of cognizance, instead of being very rapid and not then capable of detection. This exhibition of the locally disturbed atmosphere at the observing station can be readily produced artificially.

At Round Top station, in the Sierra Nevada, 10,435 feet elevation, we counted forty-seven forest fires that arose on the flanks of the forested mountains over which we were observing. The smoke gradually filled the Sacramento Valley to a height of 10,000 feet, and obscured the outline of the Coast Range Mountains; but we observed the heliotropes through this irregularly heated atmosphere with little difficulty, and at the close of the operations the probable error of a direction was only seven hundredths of a second of arc.

Besides the disturbance of the outer atmosphere near the observer, there is another source of disturbance in the air within the tubes of the great telescopes, especially in sunlight observations or at the sudden change of temperature at sunset. This is visibly exhibited in the blurred, unsteady and confused outline of the Sun and the Sun-spots projected upon the ground-glass plate of the horizontal heliograph tubeless telescope of forty-feet focus. In the transit of Venus of 1882 the image of the Sun at Cerro Roblero was projected through a forty-feet tin tube of large diameter, and was much blurred; but this was largely corrected when the tube was covered with a wooden roof not in contact.

Professor A. E. Douglass, of the Lowell Observatory, has shown that the waves of the disturbed atmosphere outside and inside of the large telescope tubes are distinctly visible to the naked eye when it is placed in the focus of the objective, as in the Foucault test. And even more than one series of such waves of disturbance at different distances and moving in different directions is not infrequent.

It is well known that to obtain the best results with the great telescopes it is imperative that the air inside the dome and inside the telescope tube shall have the temperature of the outside atmosphere.

At Arequipa, Peru, the Harvard Observatory was situated close to the valley of a mountain stream or arroyo, down which, on clear nights, a swift stream of cold air descends. Whenever this cold stream overflowed the banks and enveloped the observatory, and rose to the height of the objective, the seeing was immediately ruined for the rest of the night. In such circumstances, if the eye-piece of the telescope was removed and the eye placed in the focus, fine parallel, dark lines could be seen moving swiftly lengthwise across the illuminated lens in the direction of the wind.[1]

At San Francisco, in the strong "wind gap" of the Golden Gate, it sometimes happens at sunset, after a calm, moderate day, that star images are disc-like and steady; in a few minutes the westerly wind rises and brings in the fog-chilled air from the ocean, with flying patches of fog overhead, and the star images at once become diffused and nebulous, with an unsteadiness reaching an amplitude of 5″ to 10″ of arc.

Large volumes of air, each of different but uniform temperature, projected across a line of sight at a great distance from the observer, do not necessarily produce abnormal images of a star or heliotrope; they seem to act as prisms, and deflect the line of light so as to exhibit abnormal horizontal or vertical refraction without any marked unsteadiness. Along the face of a great mountain wall we have measured a slow change of azimuth of a signal amounting to 65″ of arc, due to the gradual rising of the morning temperature of the air immediately adjacent to the eastern rocky face of Mt. Constitution.

There are so many conditions conspiring to a disturbed atmosphere that it necessarily acts in various ways difficult to predict. Where there are regular, irregular or confused atmospheric waves moving at different distances from the objective, and in different directions, they necessarily give different images of the star at different foci, and at each change they bodily shift the image near the focus, and also change its form. Professor Douglass' observations are very instructive in this matter.

When volumes of unusually heated air pass over the objective they may act as a species of air lens which suddenly and irregularly spreads out the image of the star at the focus; but no change of focus will bring the image to condensation. This we first experienced at Point Concepcion in 1850, where the perceptibly hot volumes of air from

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[1] Douglass, "The American Meteorological Journal," March, 1895, p. 395.

the higher gulches were driven over the observatory by an incoming cold north wind from the Sierra Santa Inez.

It is sometimes reported that stars become very unsteady in a strong wind; yet we have determined micrometer values by observations upon a close circumpolar star during a fierce "norther." In geodetic observations between Mt. Diablo and Mt. Conness, on a line of 143 miles across the Sacramento Valley, we witnessed a strong norther blowing down the valley and carrying vast quantities of dust across the line of sight; but the wind did not reach the observing station on the first day, when the heliotrope image was fairly good, but was moved bodily by horizontal refraction two or more seconds of arc from the normal direction.

Observed Occultations of Red Stars at the Bright Limb of the Moon.

Referring now more particularly to the details of some astronomical observations, we give a few instances from our experience and that of others.

On September 18, 1848, about seven and one-half hours in the morning, we observed the occultation of α Tauri by the bright limb of the Moon, using a telescope of probably two and one-half inches aperture and moderate power. The red star touched the bright limb, and we noted the time mentally, because it did not instantly disappear. We continued the watch until the star was unmistakably *within the apparent limb*, when it disappeared instantly, about two and a half seconds after the first apparent contact. We at once submitted the case to our chief, R. H. Fauntleroy, who said the disappearance was the true time of the occultation, but he gave no explanation of the phenomenon. We then submitted the observation to the Superintendent of the United States Coast Survey, Professor A. D. Bache, who wrote to the Royal Astronomer (Airy) on the subject. The reply was that he personally had never observed the phenomenon, but there were similar cases on record. He gave no explanation.

In the trigonometrical survey of Admiralty Inlet and Puget Sound, during the approach of a southeast storm which finally brought up dense clouds, a heavy squall and rain, we observed α Scorpii (April 22, 1856,) with a three-inch Fraunhofer telescope and astronomical eye-piece, power about 70. The disappearance of the star was behind the bright limb, and the record states that "the time denotes the instant of the disappearance of the star *after it had been projected* upon the body of the Moon for about two and one-half seconds, certainly not less. The ruddy color of the star showed distinctly upon the body of the Moon, and the instant of disappearance was as accurately noted as if it had disappeared behind the dark limb. Having once before observed the same phenomenon, we were fully prepared for it."

In November, 1886, we presented a paper to the California Academy of Sciences upon the occultation of α Tauri on the twelfth of November. Three instruments were used at the disappearance: two three-inch Fraunhofers, power 100, and a two-inch, power 55. The atmosphere was clear and moderately steady, the stars showed signs of unsteadiness, and the Moon's border appeared reasonably sharp. The telescopes were not large enough for such observations, but an intervening house prevented the use of the six and four-tenths inches Equatorial.

The star was visible to the unassisted eye to within four minutes of contact. When it was a few seconds from the Moon's bright limb the star became invisible in the two-inch telescope. In the three-inch Fraunhofers the star did not disappear when it touched the apparent limb of the Moon, but continued to move upon the disc until it was fully one and one-half times the diameter of its spurious disc on the Moon, when it disappeared with the instantaneity of such phenomena. The star grew somewhat difficult to see after it entered upon the factitious limb, but its reddish hue left no doubt in the minds of the two observers, who differed 0.07 second in the time of its disappearance. They estimated that it was three seconds of time on the limb. At emersion from behind the narrow, dark edge of the Moon (which was one day past the full) the 6.4-inch Equatorial noted the reappearance 0.17 second earlier than the two three-inch telescopes.

At the Chabot Observatory, thirty-six seconds of time east of the Davidson Observatory, the observer used the Clark eight-inch Equatorial for the same occultation. The atmospheric conditions were nearly the same as mentioned. The star advanced upon the apparent limb of the Moon, but not so far as above mentioned, and some of the apparent rays of the star projected outside the Moon's border.

On March 29, 1887, we observed the immersion of *α* Tauri at the dark limb, and the emersion at the bright limb; the former by daylight, and the ash-gray limb not visible. The disappearance was instantaneous. At reappearance the Moon's limb was blurred and very unsteady, and the star reappeared upon this blurred bright image inside of the outermost limit, which was very fuzzy and decreasing in brightness outward. The star was not so bright as anticipated, but its almost sparkling red color left no doubt whatever in our mind as to the nearest tenth of a second of the epoch. The star left the limb in two and one-half or three seconds, when it appeared much brighter. At emersion a power of 250 was employed.

We do not mention the hundred and more occultations which we have observed when the above phenomenon was not present, or when the smallness and whiteness of the star may have prevented its detection.

Other Observed Occultations of the Stars by the Moon, and of Stars and Satellites by Jupiter.

In 1890, the Astronomer Royal called the attention of the Royal Astronomical Society to the occultations of *ζ* Tauri (3.0 magn.) that presented matter of "interest" to observers and astronomers. On September 16, 1889, Mr. Turner, of the Greenwich Observatory, observed the occultation of the same star by the bright limb of the Moon with the Lassell Reflector of thirty-four inches, and Mr. Lewis with a refractor of three and three-fourths inches. Mr. Turner noted: "No projection; disappeared instantaneously at bright limb." Mr. Lewis noted that "the star touched the limb of the Moon five seconds before the observation, and was slightly inside the limb." It appeared "as a brilliant spot in the Moon, and disappeared suddenly at the time given above." In the same number of the "Observatory" where this report is made Mr. Ranyard says that "in the case of Jupiter there are instances * * * where one or two stars have apparently been seen through the limb of the planet. There are such a number of these observations that we cannot doubt the planets have not a sharp limb; they seem to be surrounded by an atmosphere of great depth, or rather a gaseous envelope, in which clouds or dusty matter float in irregular masses." This is an extraordinary statement, and, in our judgment, based upon a misconception of the cause of these phenomena.

The outline of any cloud system at such enormous distances from the Earth as Mars, Jupiter and Saturn would subtend so minute an angle that the outline of the planet would necessarily appear sharp, even if the assumed cloud were irregularly distributed. The Sun's outline appears remarkably sharp when our atmosphere is steady, and yet we know that extraordinary disturbances of the solar surface, far beyond our experience, are constantly taking place.

On May 10, 1895, Mr. John Tebbutt observed the occultation of Antares at Windsor, New South Wales. The steadiness and definition of objects were reported very satisfactory. The instrument was an eight-inch objective, with a magn. power of 74, and "Antares, quite brilliant to the last moment, was closely watched till it came into contact with the bright limb; but then, instead of instantly disappearing, it seemed to cut its way into the disc during two or three seconds, and vanished instantaneously. The reappearance of Antares (62 1-3 minutes) later at the dark limb was remarkably instantaneous." It should be noted however, that the observer was not looking at the spot where the star reappeared, and thereby lost the earlier reappearance of the well-known companion; and at any rate he could not have seen the star reappear with sudden brightness unless the dark limb was undisturbed. The same occultation was observed at Waverly, Sydney, by Walter F. Gale, under fairly good conditions and thick haze. With a six and a quarter-inch objective and magn. power 140 "the disappearance was instantaneous at a notch formed in the Moon's limb by two peaks." This description of the point of ingress indicates that the limb of the Moon was steady and sharp.

The same occultation was also observed at Warrickville, near Sydney, by C. J. Merfield, with a seven and a half-inch reflector, with power 170. "The definition was fair, although hazy at the time of disappearance, which was not instantaneous, as is usual."

Professor Young, in his "General Astronomy" (p. 246), refers to the projection of a star on the Moon's dark limb in the following paragraph: "In some cases observers have reported that a star, instead of disappearing instantaneously when struck by the Moon's limb (faintly visible by earth-shine), has appeared to cling to the limb for a second or two before vanishing, and in a few instances they have even reported it as having reappeared and disappeared a second time, as if it had been for a moment visible through a rift in the Moon's crust. Some of these anomalous phenomena have been explained by the subsequent discovery that the star was double, or had a faint companion." No further explanation is attempted. We have purposely watched stars occulted at the ash-gray limb of the Moon, but have never been able to see a sufficiently bright and defined limb to note whether it was invaded by the star before its sudden disappearance.

In the publications of the Astronomical Society of the Pacific for November 30, 1889, Professor Barnard, of the Lick Observatory, says that "at a number of occultations of the satellites [of Jupiter] I watched carefully for any evidences of their being seen through the edges of the planet, but saw nothing of the kind. Professor Holden informs me, however, that with the thirty-six-inch Equatorial the whole disc of a satellite has been visible within the planet's atmosphere at every occultation he has observed." In the "Astronomical Journal," volume VIII, page 64, there is a record by these observers of the occultation of 47 Libræ (6.4 magn.) by Jupiter, on June 9, 1888. With the thirty-six-inch objective and power 672, the images were below the average of good seeing. Director Holden reports that "the star entered the limb and was seen bisected" at 1^m 30^s after it had touched the outer edge. "The entire image of the star was seen, inside as well as outside the limb, being easily distinguishable from the planet's surface by its brilliancy and peculiar color." In 2^m 35^s "the star was entirely inside the limb, but still visible. For the next ten seconds the star was alternately visible and invisible, and the planet's limb was quite unsteady." In 2^m 51^s from the star's first touching the limb it was certainly gone. Professor Barnard observed the same occultation with the twelve-inch objective, reduced to eight inches because "the images were too unsteady with the full aperture" (magn. power 240). He noted the time of "contact of the following edge of the star with the preceding limb" of Jupiter. At 1^m 12^s after, he star was partly on the limb; at 1^m 51^s the star was seen by glimpses to that epoch, and three seconds later it had certainly disappeared when it had advanced about three-quarters of its disc within the limb. On the usual scale of seeing, the conditions were below the average; but even when the star had encroached on the limb the disc was small, round and bright, and as clearly defined as that of a satellite entering transit. No certain diminution of its light was observed.

Professor Barnard makes no allusion to the color of the star, and as its spectrum is recorded as that of the first type, there seems no reason why the first mentioned observer should attribute a "peculiar color" to it.

Under date of April, 1894, Professor Barnard thinks "astronomers should reject the idea that the satellites of Jupiter can be seen through his limb"; since, under good conditions, with the Lick thirty-six-inch objective, "the limb of Jupiter has appeared perfectly opaque, as in all my previous observations with smaller telescopes." But he makes no reference to the character of the factitious limb of the planet in unfavorable conditions of the atmosphere.

At Melbourne, September 14, 1879, the occultation of 64 Aquarii (6.9 magn.) by Jupiter was watched in the great reflector. The star was thirty-five seconds in disappearing, and remained visible ten seconds longer as a speck of light seen through ground glass. This speck "also disappeared gradually." Proctor says the observers personally assured him the phenomenon was not due to irradiation.

In 1878, at the Adelaide Observatory, two observers (Messrs. Todd and Ringwood) saw the occultations of satellites I, II and III upon certain occasions "within the disc of the planet" at disappearance, and once at reappearance. "In every other case the occultation was perfect at the limb." This latter sentence was not quoted by Proctor.

Dr. E. C. Pickering, Director of Harvard Observatory, has described an occultation of D. M., 23° 1087 (7.3 magn.), by Jupiter, April 14, 1883. He used the fifteen-inch objective, with achromatic eye-piece, power 400. "For a period of about ten minutes after disappearance was observed, the star is recorded as 'suspected.' The record states that for some time previous to final disappearance 'the star alternately disappeared and reappeared without cause; seeing pretty good and uniform throughout.' At the reappearance, the limb of the planet was watched continuously for forty-nine seconds, at the beginning of which the star was not visible, and at the end of which it was distinctly seen, remaining visible thereafter."

In our experience in observing Jupiter, we have never seen the border sharply defined when using powers of 500, but atmospheric conditions have never been supremely good at the times of observation. In observing the occultation of Jupiter's satellites through an unsteady atmosphere, our fellow-observer, with a smaller telescope and objective of two and one-half or three inches, lost the satellite much sooner than we observed its disappearance in the 6.4-inch Clark Equatorial, when we followed it into the factitious limb; and with a disturbed atmosphere this advantage should always pertain to the larger objective and higher power.

Numerous observers might be quoted who have seen stars and satellites projected within the apparent limb of Jupiter at occultation. Some of the details are rather startling, and perhaps the most notable is that by Admiral Smyth, in 1828, supported in part by two other observers at stations a few miles distant. The satellite II remained for some minutes projected upon the apparent edge of the planet, and thirteen minutes afterwards it was outside of the disc; then it remained visible four minutes and "suddenly vanished." If the observers had been experts, observing with instrumental means adequate to give sufficient light, definition, and size to the images — noting the times with accuracy, and fully recognizing the effect of the locally disturbed atmosphere — we are satisfied that some satisfactory explanation would have been reached.

Such details as these, and others upon different occasions, are rather astonishing. Such remarkable conditions must be due to imperfect instrumentation, to inexperience, to weariness or lack of quick and positive accommodation of the eye, to erroneous estimates of small intervals of time, to extraordinarily abnormal conditions of the atmosphere, to intervals of unconscious cerebration, or to gross mistakes. It was from the abnormal observations at Melbourne that Proctor undertook to deduce the depth of Jupiter's atmosphere, and concluded that the star 64 Aquarii was seen through a range of 10,000 miles of atmosphere below a depth of 300 miles.

We may here mention incidentally that for many years we have been on the outlook for Saturn and his ring system to occult some star of moderate magnitude; but the phenomenon has not occurred in our experience. We were anxious to learn how the ring system would treat the star.

Similar Phenomena at Total Solar Eclipses.

Many reports of observations of total solar eclipses and transits of Venus and of Mercury over the Sun's disc are painfully discordant, misleading and uninstructive. They betray inexperience and lack of knowledge of the local physical disabilities that may present themselves. It would almost appear as if those who had the least experience were the most authoritative in their opinions. And certainly, when the results are published, there is little or no discrimination made in assigning proper weights to them. Some observers make no mention of their defective eyesight, and we have personally known observers, quoted as authority, that should never have been trusted to attempt any observation of precision.

The elaborate reports of the total solar eclipse of 1869, in the United States, by many observers in different localities, are quite instructive in the unconscious exhibition of contradictory statements and inappropriate adjectives. Two observers at the same station reported directly opposite conditions of the steadiness of the limbs of the Sun and Moon. One said they were defined with the utmost clearness; another sent us drawings to explain his statement that they were "boiling excessively." And yet the observer who reported steadiness was in doubt twelve seconds of the time of contact, and took the mean of his limits of doubt. And it is evident that the many observers had different conceptions of what constituted the phenomenon of "Baily's beads." Some looked for, and evidently expected, the exhibition as a necessary and normal physical condition. The tabulated descriptions of the phenomena at totality range wildly; we find recorded: "no trace of Baily's beads;" "the phenomenon of Baily's beads lasted but a few moments;" "Baily's beads beautifully distinct and spreading over an arc of 30°;" "Baily's beads were seen fifteen seconds or more; they were long and thin, moved with the moon, and were unquestionably the effect of the irregularities of the Moon's contour exaggerated by irradiation." One observer has given detailed drawings of the form of the "beads," and exhibits them with mechanical precision and hardness; as a matter of fact he was not a draughtsman nor a skilled observer, and yet his name carried great weight.

SIMILAR PHENOMENA AT TRANSITS OF VENUS AND MERCURY ACROSS THE SUN'S DISC.

In our observations of the transit of Venus, December 1882, at Cerro Roblero, New Mexico, 5,676 feet above the sea, and 1,655 feet above the Rio Grande del Norte and the great arid plain of the Jornado del Muerte, there were, towards the close, occasional tremors or shiverings of the borders of the discs of the Sun and planet from slight tremulousness of our atmosphere; but the eye was not confused, as would have been the case if the discs had continued ill-defined and unsteady by excessive vibration. In the first case the eye could and did select its opportunity for micrometric measurements. But at this very period the assistant astronomer in charge of the photographic exposures reported the results unfavorably affected by this slight unsteadiness. It was a case where the eye selected a moment of steadiness, but the mechanical movement of the photographic shutter permitted no selection. The unsteadiness of the atmosphere at the time between the third and fourth contacts permitted a beautiful and unique exhibition of the fine, white, faint, crescent of coronal light apparently surrounding part of the disc of Venus as an illuminated atmosphere. The crescent was long, extremely thin, white, and as fine, sharp and regular as if cut by the finest graver; and we watched it die away in excessive minuteness. We made drawings of the exhibition. Another observer of this beautiful phase of the transit at a different station saw a similar phenomenon "shortly after the first external contact, when the limb of the planet was 'boiling,'" and "it was difficult to be certain whether it lay within or without the planet's contour * * * but it certainly appeared to extend at any rate within the circumference; and indeed it presented the appearance of 'Baily's beads' at the time of a total solar eclipse. Its whole depth was one-fourth of the planetary radius"; and the published drawing showed the broader part of the coronal light partly inside the planet's circumference when the planet was half on the Sun's disc. Astr. Nach. 2,481. Here we have exhibitions of the same or a similar phenomenon at the same transit under widely different atmospheric conditions; and the first is surely to be taken as better representing the true or normal phase of the phenomenon, and the latter is to be reckoned as abnormal and misunderstood.

This. phenomenon of the so-called "atmosphere of Venus," though not connected with the disturbance of our atmosphere, is thus seen to be affected by it just as much as the outline of Sun or planet. Mouchez has called it the "pâle auréole * * * to be attributed in part to the atmosphere of the Sun rendered visible by contrast, and also in part by the atmosphere of Venus." Heraud and Bonify designated it as the "filet lumineuse;" and Airy says this "penumbra must, of course, be considered a part of Venus;" and elsewhere he says "the partial illumination of the atmosphere of Venus introduces difficulties of observation" etc. One observer says that this ring of light around Venus enabled him to see the planet twenty-four minutes before contact. One of our astronomers who had witnessed the transits of Venus of 1874 and 1882 asserts that "there seems to be absolutely no way of escaping from a new difficulty—the planet's atmosphere causes it to be surrounded by a luminous ring—as it enters upon the Sun's disc, and thus renders the time of contact uncertain by at least five or six seconds." Another admits "the disturbance of the image by irregular refraction in the Earth's atmosphere," but declares that "we must look to the chief cause of the great discordances" of the times of contact of Venus and the Sun, "in the atmosphere which surrounds Venus." And he further says, that "the cruel manner in which all the phenomenon are modified by the atmosphere of Venus is not easy to explain." Another declares "the planet was surrounded by a ring of light, as bright as the Sun, that pushed the edge away from it; and no actual contact could be observed.'' And yet there was no black drop, no ligament and no distortion reported by the observer.

It is admitted that there is no atmosphere enveloping the Moon, and yet Royal Astronomer Stone at the Cape of Good Hope observed the outline of the Moon projected against the Sun's corona five minutes and eight seconds before the total phase of a solar eclipse. We reported a very faint exhibition of this phenomenon January 11th, 1880. If the corona of the Sun is the visible arc of illumination in this case through contrast, we are entirely justified in assuming it to be the agent of illumination at the circumference of Venus; and on account of the irregularity and inequality of the brightest parts of the corona, it may be seen at one part of the planet's circumference and not at another, or it may be seen all around the planet. A critical examination of the reports clearly demonstrates that this faint illumination on the border of the dark body of the planet is seen as (1) part of a circumference; (2) at different parts of the circumference; (3) of varying breadth at different points of its length by the same observer; (4) as wholly encircling the planet. If it were the atmosphere of Venus it would be uniform and continuous, unless a Venus cloud system interfered irregularly. Professor Young says (p. 159) that at a total solar eclipse there is ''not any ring of light running out on the edge of the Moon like that which encircles the disc of Venus at the time of the transit.''

With a slight digression, we mention one or two vagaries of observers of the transits of Mercury.

At one of the recent transits where the observers were side by side, one reported the ''definition of Mercury sharp and steady,'' and the ''limb of the Sun undulating,'' as if the two objects were unequally affected. The other observer described the ''limbs of the Sun and Mercury loosely connected by a patch of haziness oscillating between them;'' and at another phase the '' undulations of the Sun's limb occasionally reached Mercury;'' leaving the inference that the planet was steady. At yet another station the ''planet appeared with rippling fiery and black balls chasing each other between the planet and the Sun.'' In the transit of 1878 one observer ''was surprised at the absence of glare and tremor; and the edge of the Sun's disc was clearly defined and remarkably steady with the exception of a few notches and scratches.''

We had been fortunate in our observations of the two transits of Venus, and of several transits of Mercury, never to have seen an exhibition of ''black drop'' or ''ligament,'' except at the transit of Mercury in 1891, when the atmospheric conditions of San Francisco were very unfavorable although the sky was clear; the limbs of the Sun and planet were both spurious or factitious on account of excessive vibration; and the observation was necessarily difficult and doubtful. Under such adverse conditions the observation may be very wild, or it may turn out very good; but certainly the observer cannot assign much weight to the epoch noted.

Instructions for Observing the Transit of Venus.

In re-examining the large number of reports of the transits of Venus of 1874 and 1882, we have gathered forty or fifty expressions of the observers in their endeavors to describe the phenomenon usually known as ''black drop,'' ''ligament,'' or ''filament.''

This need hardly be wondered at. The British Instructions of 1882 assert that the phenomena seen by most observers near the time of contact are of a complex character, and extend over considerable intervals of time. These instructions surely conveyed the impression that these complex phenomena were the normal physical conditions. They added confusion to previous descriptions by mention of the ''light of the cusps;'' and evidently regarded the exhibition of haze, or ligament, as ''the glimmering of the light of the 'auréole,' 'penumbra', or 'sunlight' refracted through the atmosphere of Venus across the dark space between the cusps.'' And further, if a ''pure geometrical contact'' (which is the only line that can be given for contact) is frustrated by haze, shadow, ligament, or black drop, then the last marked discontinuity of the illumination of the Sun near the point of contact is the epoch to be recorded if it ''is distinctly recognized as independent of mere atmospheric tremor.''

The American Instructions were mainly based upon the appearances of 1769; and declared the ''atmosphere of Venus clearly illuminated.'' They refer to the difficulty of observing the moment of tangency on account of imperfection of vision, irradiation produced by the Earth's atmosphere, and imperfections in the telescope; and then mention the difficulties to be expected by atmospheric undulation.

One of our eminent astronomers accepts the explanation of Lalande, that the phenomenon of ''black drop,'' etc., is caused by irradiation at the bright object; and that this irradiation arises from a number of causes, imperfections of the eye, imperfections of the telescope, and the softening effect of the atmosphere upon a celestial object when seen near the horizon. He likens it to a narrow and less bright false edge around the bright object; and says that this band will appear narrower the better the telescope and the steadier the atmosphere. He furthermore says that in the observations of the transit of Venus of 1874, with the improved instruments, very few of the experienced observers noticed any distortion at all. Later on he adds that ''in the varied forms presented 'when the air is not still,' we recognize all the peculiar appearances described by the observers of 1769.'' In 1874 the Mexican observers at Yokohama reported to us a decided exhibition of ''black drop.''

Colonel Tennant has remarked of the black drop phenomenon, '' there is no doubt in my mind that the outer part of the Sun is never free from the result of outstanding astigmatism.'' We have astigmatism of the human eye, and astigmatism of the telescopic objective and ocular, but we do not understand to which he applies the term, if to either of them. And one of our best astronomers, in reviewing the physical and other conditions of the 1761, 1769 and 1874 transits, summed up his deductions by saying that "the black drop, and the atmospheres of Venus and the Earth, had again produced a series of complicated phenomena extending over many seconds of time."

Curious Descriptions of the Phenomena.

With much more speculation of similar import we should be prepared for the crude descriptions and attempted explanations given by the observers. They range from a 'Chinaman's hat' to a 'pear-shaped planet,' and are even stretched out to a 'gourd shape.' The descriptions and explanations by the same observer are sometimes curiously contradictory or vague, as: 'the limbs were boiling violently,' 'yet quite sharp and doubt only three or four seconds;' there was 'black drop' but 'no distortion;' 'no distortion' yet the 'limbs of the Sun and Moon were spinning;' 'interference lines;' 'Venus serrated;' 'the Sun's limb had lost its sharpness from the overlapping of Venus' atmosphere;' the 'shadow of contact;' from the overlapping of the two 'penumbras of imperfect definition;' and most remarkable and incomprehensible, 'the sympathetic attraction or assimilation' of the limbs of the Sun and the planet at the second contact but no 'mutual attraction' at the third.

In Egypt, in 1882, one observer paid particular attention to detect the black drop and could not see it; another observer at the same locality observed the black drop. The imagination would seem to have played a part in the observations. One observer saw Venus twenty-four minutes before contact; and near the first contact he saw "a distinct cone of shadow thrown away into space;" and his drawing is stronger and stranger than his description. In 1874 the chief astronomer of one of the foreign expeditions declared to us that the phenomenon was simply and solely a case of diffraction.

ERRORS OF OBSERVATION.

It is therefore astonishing to note that under such adverse conditions for accurate observation the times of contacts are noted to tenths of seconds. As two seconds of time mark the apparent relative movement of the bodies about one-tenth of a second of arc, we need not wonder that some of the observers were not sure of the minute; and in one case an error of three minutes could not be fixed. This occasional and accidental distortion of the planet's disc in such transits had called for the determination of the epoch of the geometrical contact of the limbs of the images of the two bodies and for other appearances; but under such unfavorable conditions the judgment of the best observer may reasonably have been at fault, and his recorded epoch of contact very wild. This is shown by many illustrations of the distortion where the outlines of the "black drop" are drawn with a hardness and positiveness that are certainly uninstructive, and unsuggestive of the cause.

The difficulty of observation is made painfully manifest in the experiments at Washington early in 1874, when most of the observers practised upon an artificial transit of Venus to determine the times of the four contacts. The observations range from four seconds before the first contact to twenty-eight seconds after; and even at the second and third contacts, from twenty-one seconds before to seventeen seconds after the epochs. These extraordinary personal equations must surely have been due in part to inexperience, and mainly to the disturbed conditions of the atmosphere. And yet the mean value of a series of such observations was applied to the actual observations in the field upon the Sun and planet, to reduce them to a systematic series. Had these preliminary observations been made in a serene atmosphere, the apparent personal equations would have been brought to normal and reasonable limits, notwithstanding one observer declares that "the optical edge of a bright body is not, and in the nature of things, cannot be, absolutely sharp in the eye or in the telescope." The great discrepancies arose in large measure from the unsteadiness of the short line of intervening atmosphere near the ground; and we then related our experience at great elevations and over heated plains, and urged the occupation of elevated and isolated stations as the most effective means of avoiding an unsteady atmosphere in such costly and important observations.

Observations of Solar Eclipses under Different Atmospheric Conditions.

Professor Dr. Schaeberle of the Lick Observatory observed the total solar eclipse of October 9, 1893, at 6,600 feet elevation, on the western flank of the Andes in Chili. He was eminently successful in all his observations on account of the extreme steadiness of the atmosphere. The Sun's image, projected by the six-inch equatorial upon a white cardboard, looked "more like an engraving than an optical image." Aubertin, at the same station, said that during the eclipse "the atmosphere was absolutely pellucid"; he had had experience at Gibraltar in 1870.

Our experience in observing the above eclipse at San Francisco as a partial phase was the reverse of Professor Schaeberle's, but very instructive. We condense the report. The images of the Sun, and the micrometer thread were projected by the 6.4-inch equatorial upon a white sheet of paper. The apparent diameter of the Sun's image was about twenty-two inches. The atmosphere was unsteady, and the border of the Sun was confused and blurred, and lacked the solid or consistent brightness of the disc. This factitious border was about half a millimeter broad at the least. The solar spots were confused and their details were not distinguishable. As the time of first contact approached we watched the predicted point of contact, not for the first indentation of the Sun's disc, but for the first commingling or overlapping, so to speak, of the factitious image of the Moon's border with the factitious image of the Sun's border. There first appeared a very faint darkening of the confused and apparently expanded border of the Sun's limb. We noticed it when it was about three millimeters long; the disturbance of the limb was very great. The dark commingling or overlapping increased in length, breadth, and darkness; and at last changed to a line of blackness when the actual limb of the Moon's image touched upon the Sun's condensed, brighter and actual border. There was no hesitation in noting the time of this contact by several observers. At the time of the last contact, four of the observers watched the phenomenon. The borders of the two discs were more unsteady than at the first contact, and the reversed order of the first contact-appearances took place. A great mountain range of the Moon was the last to disappear. The two exhibitions were very instructive.

At the total solar eclipse of January 11, 1880, on the Sierra Santa Lucia, 6,100 feet above the sea, we observed the phenomenon under peculiar circumstances fully detailed in the official report. The atmosphere was beautifully serene after a prolonged and terrific storm of wind and snow. The limb of the Sun was not absolutely steady but exhibited occasional tremors or shiverings, and there was no disturbance of the limb at first contact. "The cusps were very sharp and clear, and whenever a tremor occurred on account of any slight atmospheric disturbance these cusps were apparently doubled." (At Oakland where the atmosphere was much disturbed the cusps appeared confused and blunted.) As totality approached, the crescent of the Sun was remarkably long and narrow on account of the slight difference of the apparent semi-diameters of the Sun (16′ 18″.1) and the Moon (16′ 23″.5). The last line of the crescent was 30° to 40° long before it broke. It exhibited no distortion from atmospheric disturbances except an occasional tremor or shivering. The cusps, before the crescent was reduced to a line, were remarkably sharp, curved points, as if cut by the finest graver. The breaking of this last thin line of sunlight was occasioned by the intrusion of the lunar mountains and the inequalities of outline. It presented the appearance of a line of bright dots and dashes, and black spaces. There was no wavy motion to interfere with this exhibition; whenever a bright spot or a line disappeared it was instantaneous, and gone forever. The moving dots of the "Baily's beads" were absolutely wanting. At the last contact of the eclipse the atmospheric conditions were wholly changed. The atmosphere was in a remarkable state of undulation and the limbs of the Moon and Sun were moving in great rapid waves, and the observation was unsatisfactory. The Sun set below the ocean horizon about ten minutes later and we should have expected a quiet atmosphere, but the warm Sun rays heating the air upon the steep, snow-covered ocean-side of the mountain mass caused rapid evaporation of the snow, and irregularly heated currents of air to flow over the station. At this eclipse we called attention to the less darkness of the sky adjacent to the advancing body of the Moon, due to contrast with coronal effects.

At the solar eclipse of October 30, 1883, we projected the images of the Sun and Moon upon a sheet of white paper, and studied the disturbed outline of both objects. The greatest obscuration occurred just before sunset, when the Moon was half way over the Sun's disc. Owing to the extreme refraction at this zenith distance, and the "exceedingly great boiling" of the borders of the two images, the "distortion of the cusps was striking and peculiar."

In contrast with these two exhibitions of great atmospheric unsteadiness, we introduce an experience approaching that of Professor Schaeberle's above mentioned.

We observed the total solar eclipse of August 7, 1879, on the Chilkaht River in Alaska, near the end of heavy, cloudy weather, when the atmosphere had become clear and steady. At the first contact "the limb of the Moon was very sharply defined, and the outline of the Sun was very steady and sharp." Just before totality "the crescent was very beautiful to the unassisted eye, and in the telescope." In the three-inch Fraunhofer, with moderate power, the "borders of the Sun and Moon were remarkably steady and very sharply defined. In twenty-four years of practice in observing we have rarely, if ever, seen them under such favorable circumstances. We observed them without any shade, and followed the Sun's border to the instant of disappearance. The bright, long, narrow crescent was sharply and regularly defined throughout; the extremities were clear-cut and pointed. As the width and length of the crescent decreased, this sharpness of outline and regularity of form were maintained until it became a fine line of *living* white; and, shortening rapidly, it disappeared as a very short, fine, distinct line, and not as a star at its disappearance. There was no breaking of this line into points or heads; no wave motion along it; no disturbance whatever of continuity or regularity of form."

"A feature of the phenomenon of totality was the vivid impression that the dark body of the Moon stood out clearly and unmistakably in relief in the space between the observer and the coronal brightness around the obscured body of the Sun, and did not lie flat and upon it, as in a picture." This perspective effect resulted from the serenity of the atmosphere, the sharpness of the Moon's outline, and an impression that the dark Moon was very close to us.

Graphic Exhibitions of Unsteadiness.

If it were necessary to add graphic demonstration to the effect of the disturbed atmosphere already described, we have it in the photographs referred to in the following note by Warren de la Rue,[1] where he describes the method of photographing the Sun's disc, in a very small fraction of a second. He says, "So rapid is the delineation of the Sun's image that fragments of the limb, optically detached by the 'boil' of our atmosphere, are frequently depicted on the collodion, completely separated from the remainder of the Sun's disc; more frequently still from the same cause the contour of the Sun presents an undulating line." To this case can be added a photograph of the Sun taken at the Lick Observatory January 14, 1897; wherein the factitious limb is like a flocculent border of cotton or wool with ragged holes through the relatively faint, irregular outline

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[1] Proc. Brit. Assn. Adv't. Sci., Aberdeen, 1859, page 151.

of the disc; and several cloud-like areas actually detached. The breadth of the thin border is about 6″, and the whole factitious breadth must be more.

It seems a physical impossibility that we should be able to detect any irregularities of even a dense atmosphere at the border of the Sun. If that body were to decrease 45 miles in diameter, our instrumentation would not be able to measure it under present conditions. Much less could we detect it if there were actually rolling billows of the exterior matter of the Sun 45 miles high around its border. They would subtend but 0.1″ in height; a quantity covered by the finest spider thread in the telescope. Much less could we detect any changes of atmospheric conditions at the distances of Jupiter, Mars, Venus, or Mercury. Some great deep Sun spot just on the edge of the Sun would doubtless show a depression, but not any imaginable storm disturbance of the densest atmosphere, as we understand storms and waves.

Official Recognition of the Factitious Borders of the Moon and Sun.

In still further confirmation of the existence of the spurious limbs of the Moon and Sun under unfavorable atmospheric conditions, it is only necessary to appeal to the representative of astronomical authority in the American Ephemeris or Nautical Almanac. In that government publication an empirical correction of 2″.5 is applied to the Moon's semi-diameter in order to represent the observed value in meridian instruments. But this "constant is omitted in the computation of solar eclipses and occultations as due to telescopic and ocular irradiation" (p. 528, 1899). This 2″.5 of increased semi-diameter may represent the average measure of the factitious border, but certainly it is frequently much greater.

In many years of experience upon the Pacific Coast of the United States in observing lunar transits, we have learned to expect wild results when a markedly factitious bright limb of the Moon was observed, the resulting longitude being always affected as if the diameter of the Moon were too great. At times, when the vibration or undulation of the disc of the Moon was not rapid, the sharp, solid limb could be observed even with a faint, spurious, vibrating outline beyond it. In 1889-'90, during lunar transit observations for longitude at San Francisco, our party had a case of spurious disc from excessive diffusiveness, with a predicted error in the resulting longitude that was verified upon reduction. This clearly indicated that even the official empirical correction may frequently be too small; while in times of supreme steadiness of the atmosphere and resulting sharpness of the limb of the Moon, this correction is not required.

If this factitious diameter belongs to the Moon, the Sun is much more likely to carry a more pronounced factitious diameter, and every observer must have frequently watched its wildly boiling limb flaring and leaping across the threads or rolling forward like waves of flame along the horizontal thread. This factitious border is in effect tacitly acknowledged in the American Ephemeris above referred to, where "the adopted semi-diameter of the Sun at the Earth's mean distance is 16′ 02″. In the computations pertaining to eclipses, Bessel's semi-diameter 15′ 59″.788 has been adopted." It appears to us that the explanation in the American Ephemeris of the cause of the acknowledged spurious diameter of the Moon is erroneous.

GREAT ELEVATIONS FOR OBSERVATION.

Although our experience has led us to advocate the selection of great elevations for astronomical observations of precision and research, yet elevation alone is not a panacea. The locality must present other favorable orographical conditions. Unfavorable conditions are great gulches, cañons, or narrow valleys lying directly under a summit and leading to it. These gulches become filled with highly heated air during two or three days of clear, calm weather, and the first wind from below drives this heated air over the summit and spreads stars out into unsteady nebulous films; the the rings of Saturn become woolly girdles, the crape ring is invisible and the details of the belts of Jupiter are indistinguishable. And in such a location during daytime, observations of precision on the Sun are absolutely useless. The seeing upon a mountain peak may be generally good when there is no snow on the surface; and even remarkably good with snow, when the temperature is low and the wind is strong; but when the Sun shines and the temperature rises, and rapid evaporation takes place with light airs, the seeing becomes very bad. (Experience at 10,450 to 7,250 feet elevation.)

On Mt. Conness, in the Sierra Nevada (12,566 feet elevation), the geodetic heliotrope images were perfect until the afternoon Sun shone upon the 2,300 feet nearly vertical western face of that curious buttress; and then the uprising, irregularly warmed atmosphere caused the images to become unsteady. During the quieter moods of the atmosphere, we observed Polaris for azimuth in the middle of the day very satisfactorily with a telescope that would not show it in daytime at lower elevations or under less favorable conditions.

In 1872 we experimented in the Sierra Nevada at elevations from 7,200 to 9,500 feet with remarkable success; while another party of the United States Coast and Geodetic Survey sent for the same purpose to Sherman, 8,300 feet elevation, in the Rocky Mountains, had at times a very unsteady atmosphere. The Lick Observatory at Mount Hamilton, in the Diablo Range, at 4,209 feet elevation, is surrounded by unfavorable orographical conditions, especially for observations of precision in daytime.

It would appear that Newton attributed the apparent unsteadiness of celestial objects to the disturbance of our own atmosphere. "Telescopes," he writes, "cannot be formed so as to take away that confusion of rays which arises from the tremor of the atmosphere. The only remedy is a most serene and quiet air. Such as may perhaps be found on the tops of the highest mountains above the grosser clouds."

PRECISION OBSERVATIONS AT LOW ELEVATIONS.

And yet there are climatic conditions at comparatively low elevations, which are supremely favorable for observations of precision at night. The experience of the observers of the United States Coast and Geodetic Survey on the immense, elevated, arid plains from El Paso to San Diego is very instructive. During the hot cloudless days the atmosphere over the immediate surface of the parched earth was in violent unsteadiness, and the heliotrope images at even short distances flared out wildly like burning houses. They were the worst possible objects for observations of precision. After sunset the conditions were suddenly changed. With a cloudless sky and a minimum of aqueous vapor in the attenuated air (Barometer about 26 inches) radiation was quickly effective. The temperature fell many degrees in a short time, and the air became supremely quiet. The latitude telescope showed stars as minute discs with diffraction rings running along the micrometer thread with such extreme steadiness that it was impossible to be in doubt more than one-tenth of a second of arc in measurement; and in the transit instrument the star marched with absolute regularity across the reticule. It was an experience that an observer of over thirty years in the field declared he had never before enjoyed.

Per contra, on the low Yolo plains of the Sacramento Valley in California, we have had the azimuth signal-light (distance eleven miles), in a calm night, running wildly up and down the vertical thread of the theodolite through five minutes of arc, in an ever changing line of broken stars of the prismatic colors, and yet showing little or no horizontal motion. The range of height of this column of images was eighty-seven feet.

In the great Gangetic plains of India the night signals of the triangulation parties, when shown from towers of fifty feet elevation, frequently appeared as continuous columns of light sixty feet high.

In the triangulation of the western coast of Lower California by the United States Steamer "Thetis," the observer informs us that the unsteadiness of the atmosphere and the irregularity of the refraction apparently lengthened the signal poles to a hundred feet and more; and the tripod supports were like great writhing snakes. Even upon some small low island in the ocean, where the climatic conditions are favorable, the atmosphere at night becomes very quiet and serene. From a vessel's deck we have watched the larger stars nearly reach the horizon with very little twinkling.

Recapitulation of Phenomena at Observation.

These examples are a few of the many experiences in our geodetic and astronomical observations, and in the descriptive experiences of a few other observers. And yet from recent publications there appear to be many observers who do not understand the cause of the phenomena, and who seem to think there must be something occult and unexplainable. We have therefore felt constrained to repeat this explanation which we have held and announced for many years. It seems to cover every phase of the phenomena.

In the occultation of stars by the Moon, a spurious and factitious limb of the Moon can be formed only and solely by the unsteadiness of our atmosphere more immediately surrounding the station of the observer.

In a serene atmosphere, the outline of the Moon is so sharp and clear-cut that the mountains and valleys thereon are very distinctively exhibited, and will bear the largest magnifying power. When the atmosphere begins to change to unsteadiness, the sharp outline of the Moon (or other object) is first affected by tremors or shiverings, which are so infrequent that the observer is able to select the actual border and its features. As the unsteadiness of the atmosphere increases, the tremors increase in frequency and in amplitude, until the border of the Moon becomes a confused, blurred outline, in which no serrations can be detected. The early shiverings usually denote the direction of movement of the disturbed air. Under the atmospheric conditions of clearness and serenity, the apparently approaching image of the star is a beautiful, steady, minute disc with one or more delicate diffraction rings. The Moon's white and sharply defined border exhibits all its serrations and irregularities. As it approaches the star, the diffraction rings disappear in the increasing light in the field of view, but the disc remains; and at their visible contact the disappearance of even the largest and brightest star is absolutely instantaneous; and no observer can be in doubt of the epoch beyond the tenth of a second. Under such favorable atmospheric conditions, the image of the star never enters upon the visible limb of the Moon, no matter what the size of the objective or the magnifying power employed. And with such conditions, the reappearance of a star from the bright limb of the Moon will be instantaneous. If the star were apparently to make a near approach to the Moon along the dark north or south limb, where there are serrations, it might pass so close as to disappear behind the first mountain and reappear in the next valley (each phenomenon being absolutely instantaneous) to be swallowed up by the next mountain, or in its absence, to continue its visible course. We have heard (1846) the elder Bond describe such a phenomenon in his experience and it has happened once to ourselves. But when the atmosphere begins to change to unsteadiness, the star loses its diffraction rings, the nucleus is broken up and gradually diffused into a nebulous image, fuzzy and unsteady, with a brighter, irregular, dancing nucleus, or is spread out as a nebulous film as much as forty or fifty seconds of arc in diameter. This unsteadiness of the atmosphere throws the disc of the Moon into irregular vibrations or displacements of equal amplitude and duration, giving it a factitious border. With this disturbed and spurious limb the Moon approaches the diffused image of the star, both being in a state of great unsteadiness; but if the nucleus of the star be sufficiently large, bright, and colored, like Antares and Aldebaran, the impression of its image upon the retina of the eye is naturally more intense as an isolated spot than the image of the whitish, confused, spurious border of the Moon, even when the former is within the latter. The action of the eye is unconsciously selective, but a certain effort of selection is made under the less favorable condition when the star is small or white. There is no effort at selection needed to receive an impression of the Moon's limb, although the actual limb may doubtless be detected, under high powers, inside this factitious border. The impression of the image of the star is, therefore, continuous within the range of amplitude of the excursions of the disc of the Moon, but is instantly lost when the limit of this range of vibration is actually reached by the apparent nucleus of the equally unsteady star, and the true limb of the Moon occults it. With a large colored star all the phases of this phenomenon are unmistakable; with a large white star they may be somewhat in doubt, as in the case of 47 Libræ and other stars elsewhere quoted; with a small white star they will probably not be noted, especially in small telescopes with low power.

The exhibition of the phenomenon at the reappearance of a star on the blurred and factitious bright limb of the Moon should be in exactly the reverse order of the foregoing phenomenon of disappearance; and we have been fortunate in looking over our old record to find the original notes of the disappearance of Aldebaran behind the dark limb of the Moon, and its reappearance from the bright limb, on March 29, 1887, as elsewhere mentioned.

In this résumé we have confined ourselves to the exhibition of the normal and abnormal conditions in our atmosphere attending the phenomenon of occultations of stars by the bright limb of the Moon, or by Jupiter or any other planet, because the explanation virtually covers all associated phenomena in solar eclipses, lunar transits, transits of Venus and Mercury, and the vagaries of star images.

When our atmosphere is absolutely serene there is not and cannot be the slightest abnormal exhibition of form, size, march, or steadiness of images; or of doubt of instantaneity in the epochs observed. Under such a condition there would be no empirical corrections in the Nautical Almanac to the semidiameters of the Sun and Moon; and the diameters of the planets measured by different observers, and the right ascensions and delineations of the stars would be consistent within the instrumental and personal equations.

OTHER IMPORTANT RESULTS AFFECTED.

There is another and very important class of astronomical results affected by these factitious borders of the Sun and planets and the unsteadiness of the stars.

Dr. Newcomb says that the observed right ascensions of the mean of the Sun's two limbs, relative to the fixed stars, are affected by personal errors for which no means of elimination have been tried. He suggests personal error and possibly the effect of the Sun on the telescope. To these causes should be added the larger effects of the disturbed and factitious limbs; and in cases of steadiness, to diffraction of the limb at the spider line. He elsewhere notes "very large errors, both accidental and systematic, to which observations of the Sun are liable;" and again he regards "the constant error in declination of the Sun as something peculiar to the observer and the instrument." In another place he says: "There is a remarkable systematic difference in the observed A. R. of Mercury, according as the planet is east or west of the Sun, and therefore according to the illuminated side. The sign of the result shows that the reduction to the center of the planet was apparently too small, and then it is of interest to learn according to what law this error changed as the planet moved around its relative orbit."

Furthermore, the hurtful effect of observing upon the outer limb of the factitious limb of the disturbed Moon can be shown in the determination of the parallactic inequality of the Moon from meridian observations. Dr. Newcomb says the method is peculiarly liable to systematic error, owing to the fact that observations have to be made on one limb of the Moon when the inequality is positive, and on the other limb when it is negative. Hence if we determine the inequality (125″.5) by the comparison of its extreme observed effects on the Moon's right ascension, any error in the adopted semidiameter will affect the result by its full amount. This suggests that this gravitational method of determining the Solar Parallax would be more accurately employed by observing occultations of stars large enough to be seen through the spurious bright limb of the Moon. Moreover, the observations themselves would afford some data for the elimination of the error depending upon a factitious limb.

Nearly all Measured Diameters Too Large.

In all instrumental observations for the right ascension of the Sun or Moon, and for their declination and diameter, it must be evident that the observer obtains accurate measures only when the disc of either body is sharply defined, devoid of tremor or unsteadiness, and unaffected by irregular and extraordinary refraction, without reckoning diffraction at the spider line, unknown instrumental errors, and peculiarities of observation. As these supreme conditions of steadiness are seldom obtained, it necessarily follows that the mean of any number of observations taken under different atmospheric conditions (say for example the diameter of the Sun, Moon, or planets) must be too large. The diameter can only be too small through error of observation, instrumental errors, diffraction of the spider line, or abnormal refraction.

With a disturbed atmosphere in observations for the determination of the right ascension of the Moon during the first half of a lunation, the observed A. R. of the limb will be too small and for the second half too large. Similar results will follow from observations of the I and II limbs of the Sun and planets. Therefore all published diameters of the Sun, Moon and planets derived directly from actual observations must be too large. Moreover, this presents a case where the mean of measured quantities is not the most probable value. The mean of all the minimum measures would be nearer the truth.

The amplitude of the excursions of the images of the Sun, Moon, and planets in a disturbed atmosphere is a variable quantity that can not be determined but should be avoided if practicable. To obtain the most trustworthy results in reasonable time, we need the fixed observatories located at points which conspire to give the best atmospheric conditions determined upon after systematic and exhaustive trial for special lines of research; the highest class of instrumentation; observers with ideal eyes (of which unfortunately there are very few); discrimination in the selection of proper times of observation; and a wise rejection of observations made under abnormal conditions.

Note. Since this paper was presented to the California Academy of Sciences, Professor A. E. Douglass of the Lowell Observatory at Flagstaff, Arizona, has published the second of his interesting papers upon "The Atmosphere, Telescope and Observer," and "Scales of Seeing." Pop. Astr. 1898.

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Determination of The Constant of Refraction from Observations made with The Repsold Meridian Circle of The Lick Observatory

BY

RUSSELL TRACY CRAWFORD

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PREFACE.

In March 1899, Dr. Keeler, late Director of the Lick Observatory, suggested that I should undertake, for a thesis, to reduce Professor Schaeberle's Meridian Circle observations for refraction, to discuss them completely, and to end with the construction of tables for the Lick Observatory. According to Astronomer Tucker, who is in charge of the Meridian Circle work at the Lick Observatory, the Pulkowa tables are not fully satisfactory for the reduction of observations made there, and those of Bessel are still less so.

After consultation with the late Professor Keeler, and with his consent, it was decided to forego the reduction of Professor Schaeberle's observations for the present, and to determine the Constant of Refraction from observations to be made by myself and to be reduced according to the method set forth in the following pages.

I wish to express my thanks to our late Director, Professor Keeler, for kind suggestions, for permission to use the Meridian Circle, and for absolute freedom in the method of procedure and in the reduction of the observations.

Russell Tracy Crawford.

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DETERMINATION OF THE CONSTANT OF REFRACTION FROM OBSERVATIONS MADE WITH THE REPSOLD MERIDIAN CIRCLE OF THE LICK OBSERVATORY.

BY RUSSELL TRACY CRAWFORD.

CONTENTS.

INTRODUCTION.

1. The Meridian Circle.—The instrument with which these observations for refraction were made has been fully described by Astronomer Tucker in Volume IV of the "Publications of the Lick Observatory, 1900." For the sake of completeness, however, it will be described again in this paper.

The instrument was made by Messrs. Repsold and Sons, and was described by Professors Auwers and Krueger to be "in its construction in every way suited to be the chief instrument in an observatory of the first class." (cf. Vol. I, "Publications of the Lick Observatory.")

The aperture of the object glass, which was made by Clark and Sons, is 6.4 inches. Its focal length is 6 feet

4 inches. The tube of the telescope is in two parts, each of which is attached to a central cube. Their diameters decrease from 8.1 inches at the cube to 6.5 inches near their outer ends. An eyepiece giving a power of 90 and a field of 12′ was used for these observations. The star images formed are not exactly round, but are slightly elongated in a direction parallel to the horizontal (declination) thread. There being no component of this elongation parallel to the vertical threads, it can have no effect upon observations for zenith distance.

The axis is 3 feet 2½ inches long, the distance between the counterpoises being 2 feet 2 inches. The pivots are 3.6 inches in diameter and are protected by brass covers. The telescope is furnished with clamps which, however, were never used during these observations. After the telescope was once set for a star it was not moved again to make the bisection, this being done by means of the declination micrometer. The value of one revolution of the screw of this micrometer is 48″.10. This value has been adopted as the result of many determinations made in past years. The micrometer thread is single.

The instrument has two circles, one of which can be rotated about the axis of the instrument while the other is rigidly fixed to it. They are both graduated to 2′. The degrees, as numbered, increase counter-clockwise. The diameter of the silver circle, upon which the graduations are marked, is 26 inches. There are 130 graduations to the inch. The fixed circle was used throughout these observations.

The four reading microscopes on each side are alike in all respects. They are 26.5 inches long and have clear apertures of 0.55 of an inch. Their powers are 40 and their fields are nearly one degree. The objectives are 5 inches from the circle and their eye ends project 8 inches from the frame holding them. The micrometer heads are divided into 60 parts. One revolution of a micrometer head carries the threads over one minute of arc of the circle. There are two pairs of threads in every micrometer, but one of which is generally used.

There is a separate broken telescope for setting. This is supported on wyes attached to either pier and is at the level of the lower rim of the circle. By means of this the circle can be seen either from the north or from the south, so that the settings can be made very conveniently.

The illumination for both the field of view and for the circles under the microscopes is furnished by a Rochester lamp placed in a cylindrical case 9 feet from the axis of the instrument. This light also illuminates the heads of the microscope micrometers. Most of the heat from this lamp is carried out of the room by a pipe which extends from directly over the lamp through the roof to the outside air.

A simple mechanism enables the observer to change the system of illumination from a bright field with dark wires to a dark field with bright wires and *vice versa;* he can also reduce the amount of illumination at will.

The brick piers supporting the instrument are 34 inches by 44 inches at the floor of the room and 22 inches square at the top. The sides next to the telescope are vertical. They are cased in wood with a layer of felt between the surfaces. The platforms for the microscope reader are entirely disconnected from the casing of the piers.

The microscope bearers are 23 inches in diameter and 17 inches long. The wyes for the pivots of the instrument are attached to the inner faces of these frames.

The weights of the counterpoises hang from levers 26 inches long. The fulcra are in the centers of the levers and are 6 inches from the inner faces of the microscope bearers.

Two collimators, of same aperture and focal length as the Meridian Circle, are suitably mounted. The collimator micrometers are 35 feet 6 inches apart.

2. *The Room.*—The Meridian Circle house on Mount Hamilton has been most admirably designed. Its efficiency will become apparent from the meteorological data to be given later.

The observing room is 43 feet long (north and south) and 38 feet wide. All of the walls are double. The outer of the two is a louvre-work of galvanized iron which prevents the sunlight from touching any part of the building proper. The inner wall is of California redwood, and is separated from the outer by a two foot air space. The ceiling is also of redwood. It is about 16 feet above the floor. Above the ceiling is an air space 8 feet high at the observing slit and sloping to meet the east and the west walls.

The observing slit is slightly over three feet in width. The covering for the slit is in four parts which open outward. The ends are closed by shutters, each of which is in two parts opening inwards. Each end is also provided with a single shutter which slides up and down. For stars at zenith distances greater than 72 degrees these shutters have to be lifted. When down they are very efficient wind breaks.

There is a large canopy which can be rolled over the instrument to serve as an additional protection in stormy weather or when the instrument is not in use.

For a more detailed account of the instrument and room see Astronomer Tucker's account of them in Volume IV of the "Publications of the Lick Observatory, 1900."

3. Meteorology.—To make quite sure of the condition of the atmosphere at any time during the observations, the thermometers were read, on the average, three times an hour (at nearly equal intervals); and the barometer was observed every hour. The reading of the wet bulb thermometer was also taken when the dry was read. The relative humidity has not been introduced into the reductions, but it was thought desirable to have it for possible future reductions.

The barometer, Green 2839, hangs on the north wall of the observing room. It reads to one two-hundredth of an inch. The dry and the wet bulb thermometers (F) hang in the air space between the north walls. The dry bulb thermometer, used to indicate the external temperatures, is Green 494. This thermometer has been calibrated at the

Yale Observatory. The corrections which have been applied to all the readings have been taken from the following table sent from Yale Observatory:—

t (F)	*Cor.*
0°	+0°.1
32	—0 .2
52	—0 .1
72	—0 .2
112	—0 .1

The table which follows contains the *uncorrected* temperatures (t), the readings of the attached thermometer (T), of the barometer (B), and the times at which they were taken. The readings of the wet bulb thermometer are not given here.

TABLE OF BAROMETER AND THERMOMETER READINGS.

Sid.	June 7				June 8				June 9				June 12				June 13			
T		*t*	*T*	*B*		*t*	*T*	*B*		*t*	*T*	*B*		*t*	*T*	*B*		*t*	*T*	*B*
h		°	°			°	°			°	°			°	°			°	°	
14.7		60.2	60	5169		66.0	65½	5175		69.0	68	5178		55.1	57	5147		61.8	62	5165
15.0		62.9				66.0				69.8				56.0				62.5		
.3		62.8				66.6				69.2				57.0				63.0		
.6		63 8				66.7				69.4				58.8				62.9		
.9	.0	63.8	62	5170	.0	66.5	66	5174		69.7	69	5178		58.2	58	5148		63.2	62½	5164
16.3					.4	66.2			.4	68.8			.4	58.3			.4	63.0		
.6	.7	63.9			.7	66.0			.7	68.7			.7	58.2				62.9		
17.0		63.8	62½	5169		66.0	66	5172		57.2	68½	5177		58.6	58½	5148		62.5	62½	5165
.4		63.8			.5	66.1				68.0				57.8				62.8		
.7	.8	64.0			.8	66.0				67.5			.8	57.5	58	5145		62.8		
18.1		63.3	63	5167		65.9	66	5170		67.5	68	5176						62.8	62½	5163
.4		61.0			.5	65.8				67.5								62.8		
.7										68.0								62.6		
19.0		61.0	62	5166		65.5	65½	5169		67.9	68	5174						62.0	62½	5163

Sid.	June 14				June 19				June 21				June 22				June 27			
T		*t*	*T*	*B*		*t*	*T*	*B*		*t*	*T*	*B*		*t*	*T*	*B*		*t*	*T*	*B*
h		°	°			°	°			°	°			°	°			°	°	
14.7		70.1	69½	5194	.8	57.7	59	5185		64.9	65½	5178	.8	68.2	68½	5170		66.0	67½	5187
15.0		70.6				57.8				65.8				68.4				66.7		
.3		70.8				56.4				65.3				68.0				66 7		
.6		70.4			.7	56.3				66.4				67.5				66.9		
.9		70.9	70	5194		56.3	57½	5182		65.8	65½	5178		67.0	68	5167		66.8	67½	5187
16.3	.4	70.9				55.9				66.2				67.2				66.8		
.6		70.9				55.5				66.0				67.2				66.8		
17.0		70.9	70	5194		55.4	56½	5179		66.2	65½	5178		66.0	68	5166		66.1	67	5187
.4		70.6				55.8				66.0				65.9				65.0		
.7		69.5				56.2				66.1				66.1				65 0		
18.1		69.2	70	5193		56.0	56½	5177		66.2	66	5176		66.1	67	5163		65.6	66	5185
.4		68.9				56.2				66.2				66.4				65.8		
.7		69.5				57.0				66.1				66.6				65.8		
19.0		68.8	69	5193		56.4	56½	5175		66.2	66½	5173		66.3	66½	5161		65.8	66½	5184

Table of Barometer and Thermometer Readings.—(*Con.*)

Sid.	June 28			June 29			June 30			July 3			July 4		
T	*t*	*T*	*B*	*t*	*T*	*B*	*t*	*T*	*B*	*t*	*T*	*B*	*t*	*T*	*B*
h	°	°		°	°		°	°		°	°		°	°	
14.7	68.2	70	5188	69.0	69	5191	66.5	70½	5178	74.0	76½	5164	71.2	71½	5167
15.0	68.0			69.0			67.2			73.7			70.0		
.3	67.0			68.2			66.8			73.7			67.9		
.6	67.2			68.8			66.8			72.8			69.0		
.9	67.2	67½	5189	69.3	68½	5193	65.8	68	5176	72.0	74	5165	69.0	70	5167
16.3	66.2			69.6			66.3			71.2			68.9		
.6	66.8			69.3			66.3			70.6			68.8		
17.0	67.6	67	5186	70.5	68½	5191	66.2	67½	5174	72.3	72½	5161	67.7	69	5167
.4	68.4			70.8			66.3			72.2			67.8		
.7	68.4			70.0			66.5			71.8			67 3		
18.1	68.2	67½	5187	69.4	69	5190	66.4	67	5172	71 6	72½	5160	67.0	67½	5165
.4	67.3			69.2			66.4			72.0			67.8		
.7	67.2			69.1			66.8			72.0			68.1		
19.0	67.6	67	5187	68.7	69	5188	66.6	67	5169	71.3	72½	5158	68.4	68	5163

Sid.	July 5			July 6		
T	*t*	*T*	*B*	*t*	*T*	*B*
h	°	°		°	°	
14.7	62.8	66	5166	58.0	62	5170
15.0	61.8			58.4		
.3	62.2			57.6		
.6	61.8			57.1		
.9	61.0	68	5167	57.2	59⅓	5170
16.3	61.0			57.8		
.6	61.0			57.8		
17.0	61.0	62	5166	58.2	59	5168
.4	61.0			58.2		
.7	60.9			58.2		
18.1	60.5	61½	5165	58.3	59	5168
.4	60.5			58.3		
.7	60.7			58.2		
19.0	60.4	61	5163	58.0	58½	5167

Note.—On the nights of June 7, 8, 9, 12, 13, 14, 19 and 22 some of the observations were made at times a little different from those given in the column *Sid. T.* The actual times of such observations are indicated just before the column *t*. Thus, on June 7 before the column *t* occur the numbers .0, .7, and .8, which indicate that the times of the corresponding observations were 16.0 instead of 15.9, as given in the column *Sid. T.*, 16.7 instead of 16.6, 17.8 instead of 17.7, etc.

In this table the unit of B is one two-hundredth of an inch.

From this table the following data have been taken:

Maximum temperature	= 74°.0,	July 3
Minimum temperature	= 55°.1,	June 12
Maximum range	= 18°.9	
Maximum barometer	= 5194,	June 14
Minimum barometer	= 5145,	June 12
Maximum range	= 49.	

During this period of observing, the maximum difference between the dry and the wet bulb thermometers was $75°.5 - 48°.0 = 22°.5$. This was on June 29. The minimum was $65°.0 - 56°.0 = 9°.0$, which occurred June 27.

Concerning the maximum temperature noted above, 74°.0, it should be remarked that this was the first reading of the period, and was taken several minutes before the sun had set.

Besides the regular thermometers in the air space between the north walls, three other thermometers were suspended from the ceiling of the observing room. All three were swung under the observing slit, near the plane of the meridian. One was directly over the instrument, and three or four feet from the ceiling. The other two were hung, one north and one south, about half way between the instrument and the north and south walls respectively, and at such a distance above the floor that the plane of the axis of the instrument and the line of sight of the telescope, pointed at about 83° zenith distance (north and south respectively), would intersect the thermometers near their bulbs.

Before being thus placed, these thermometers were compared with Green 494, so that their readings could be reduced for comparison with those of the external thermometer (Green 494).

During the course of an evening's observations these three thermometers were read just after reading the regular thermometer. The average difference between the inside and the outside thermometers was found to be the same for all three, and is 0°.3 (F). It is nearly always the case (in this hemisphere) that the southern part of a room is a trifle warmer than the northern. But this is not the case on Mount Hamilton. The temperature of the air inside is, on the average, very uniform and but very little (0°.3) warmer than the air outside. In his "Untersuchung über die Astronomische Refraction u. s. w.," Dr. Bauschinger notes that the southern part of his observing room in Munich was warmer than the northern, and that at night the average difference between the inside and the outside temperatures is 1°.3 (C). From his investigation, he concludes that the temperature of the air *within* the observing room should be taken into account.

Because of these difficulties, many observers have seriously considered the idea of mounting their instruments under a movable house, so that when at work the instrument will be entirely out of doors, and thus completely obviate this difficulty. But this would needlessly endanger the instrument. To accomplish the same purpose, the Meridian Circle house being built at Kiel is to be constructed in the shape of a cylinder whose axis coïncides with the axis of the instrument. This is undoubtedly the best form of construction.

For the efficiency of the Meridian Circle house on Mount Hamilton, the difference between the inside and the outside thermometers can speak. As has been said, the average difference (in the sense Inside–Outside) is + 0°.3 (F). The *maximum* difference noted was one evening, a few minutes before the sun had set, when the difference was + 1°.1 (F). *The maximum difference noted here is less than half the average at Munich.* After this Meridian Circle house has been completely opened for an hour and a half, the temperature inside is practically the same as it is outside.

During the months October to December, inclusive, a similar set of observations was secured. For these months the average difference between the inside and the outside temperatures is even less than for the summer months. But the range of the difference is much greater for the fall and the winter months. The maximum differences observed were — 2°.0 (F) and + 2°.1 (F). There was one still larger difference, viz. — 3°.7 (F), which can hardly be counted in the series, for it occurred on a poor night, immediately after observing had been suspended because of clouds and poor "seeing." The hot wave, which caused the outside temperature to rise suddenly, undoubtedly destroyed the "seeing." Although the winter months present conditions not so favorable as those of the summer months, nevertheless they also speak well for the efficiency of the Lick Observatory Meridian Circle house.

4. *Plan for Observing.*—The method of determining the refractions here may be stated as being a quasi converse to Talcott's method of determining the latitude. Instead of eliminating the refractions to get the latitude, the method is to determine the refractions by eliminating the latitude, as follows:

Let

z_s = the zenith distance of a southern star,
z_n = the zenith distance of a northern star,
z'_s = the apparent zenith distance of the southern star,
z'_n = the apparent zenith distance of the northern star,
δ_s = the declination of the southern star,
δ_n = the declination of the northern star,
r_s = the refraction of the southern star,
r_n = the refraction of the northern star,
φ = the latitude of the Meridian Circle.

Then

$$\delta_n = \varphi + z_n = \varphi + (z'_n + r_n) \qquad (1)$$
$$\delta_s = \varphi - z_s = \varphi - (z'_s + r_s) \qquad (2)$$
$$\delta_n - \delta_s = z'_s + z'_n + r_s + r_n \qquad (3)$$

Let

$$A = \delta_n - \delta_s \qquad (4)$$
$$B = z'_s + z'_n \qquad (5)$$

Then

$$A = B + r_s + r_n \qquad (6)$$

or

$$r_s + r_n = A - B \qquad (7)$$

If now, the southern and northern zenith distances were the same, and if, at the times of observing them, the conditions of the atmosphere were the same, the two refractions would be the same, *i. e.*,

$$r_s = r_n.$$

In this case we have

$$2r = A - B \qquad \text{(I)}$$

In practice these ideal conditions are only approximately satisfied. We therefore proceed as follows:

From (7) we have

$$2r_s - r_s + r_n = A - B \qquad (8)$$

whence

$$2r_s = (A - B) + (r_s - r_n)$$

and

also

$$\left.\begin{aligned} r_s &= \tfrac{1}{2}(A-B) + \tfrac{1}{2}(r_s - r_n) \\ r_n &= \tfrac{1}{2}(A-B) + \tfrac{1}{2}(r_n - r_s) \end{aligned}\right\} \qquad \text{(II)}$$

In case the northern star is at lower culmination we shall have:

$$\delta_n = 180^\circ - z_n - \varphi \qquad (9)$$

$$\delta_s = \varphi - z_s \qquad (10)$$

$$\delta_n + \delta_s = 180^\circ - z_n - z_s \qquad (11)$$

$$= 180^\circ - [z'_n + r_n + z'_s + r_s]. \qquad (12)$$

Hence

$$r_n + r_s = 180^\circ - [z'_n + z'_s] - [\delta_n + \delta_s] \qquad (13)$$

and

$$2r_s = 180^\circ - [z'_n + z'_s] - [\delta_n + \delta_s] + [r_s - r_n]. \qquad (14)$$

Calling

$$A' = \delta_n + \delta_s \qquad (15)$$

and since

$$B = z'_s + z'_n \qquad (5)$$

we have

and

$$\left.\begin{aligned} r_s &= 90^\circ - \tfrac{1}{2}[A' + B] + \tfrac{1}{2}[r_s - r_n] \\ r_n &= 90^\circ - \tfrac{1}{2}[A' + B] + \tfrac{1}{2}[r_n - r_s] \end{aligned}\right\} \qquad \text{(III)}$$

In order to obtain the refractions from (II) and (III) it is necessary to know the declinations of the stars, their apparent zenith distances (or rather the sums of the zenith distances of the pairs of north and south stars), and the differences between the refractions of the pairs. The stars chosen for this work are all fundamental, and in a first approximation their declinations are to be considered absolute. The list of stars, given later, has been taken from Professor Newcomb's "Catalogue of Fundamental Stars for 1875 and 1900, reduced to an absolute System." The apparent zenith distances, or the sums of the zenith distances of the several pairs, are obtained from the Meridian Circle observations; and the differences in the refractions are found by computing the refractions from some standard table. In this work the Pulkowa tables have been used. The term $\frac{1}{2}(r_s - r_n)$ being of the nature of a differential refraction, any error in the constant of refraction of the table used will have practically no effect upon this difference. The more nearly ideal conditions (*i. e.*, when $r_s = r_n$) are approached, of course, the better the determination of the refractions will be.

This method has both its advantages and its disadvantages. Among the former, the most important are: first, the total elimination of the latitude and hence also of its variation; second, the elimination of the nadir, since $(z'_s + z'_n)$ is nothing more nor less than the difference between the circle readings, and is therefore independent of the zenith point; third, there is no wait of twelve hours or of six months in order to observe a star at both culminations, as is usually done; and fourth, the simplicity of the reductions.

The greatest disadvantage in this method lies in the fact that the declinations of the stars have to be considered known. But by taking fundamental stars, such as those whose places are given by Professor Newcomb's new Fundamental Catalogue, and by taking a large number of these stars, this difficulty will be nearly completely eliminated.

Having now the new refractions, the correction to the constant of the table used (Pulkowa) is found from the following equation [eq. (701) pg. 672, Vol. I, Chauvenet, "Spherical and Practical Astronomy"]:

$$dr = A d\alpha + B d\beta,$$

where

$$A = \frac{r}{\alpha}$$

and

$$B = \sin^2 z \sqrt{\frac{2}{\beta}} \left(\frac{dQ}{d\beta} - \frac{Q}{2\beta}\right).$$

For this observatory, whose altitude is 4,209 feet and where the mean annual pressure is less than 26 inches, an investigation into the effect of the higher powers of $\triangle\beta$ involved in the factor $\beta = \frac{b}{B} = 1 + \frac{b - B}{B} = 1 + \frac{\triangle b}{B}$ (in Bessel's notation for r) was necessary. In his memoir, "Untersuchungen über die Constitution der Atmosphäre und die Strahlenbrechung in Derselben," St. Petersburg, 1866, Gyldén has neglected the squares and higher powers of $\frac{\triangle b}{B}$, since for places at low altitudes $\frac{\triangle b}{B}$ is a very small quantity. This investigation was made by Professor Comstock (Vol. I, "Publications of the Lick Observatory"). From his investigation the conclusion is drawn that "the Pulkowa Refraction Tables may be used for atmospheric pressures as low as 25 inches without taking into account the squares and higher powers of $\triangle b$, and the quantities so neglected will not be sensible at zenith distances less than 80°." The minimum reading of the barometer during these observations was 25.72 inches, so that in these reductions no modification of the factor of the refraction depending upon the barometer need be made.

This question having been disposed of, the assumption is here made that all of the error in the refractions is due to an error in the constant of refraction. This amounts to assuming the constant β to be correct or that $d\beta = 0$. The equation above then reduces to the very simple expression

$$dr = A d\alpha = \frac{r}{a} d\alpha ;$$

hence

$$\frac{d\alpha}{\alpha} = \frac{dr}{r},$$

or

$$d\log\alpha = d\log r.$$

Having dlogr from the reductions, we thus have $d\log\alpha$, and hence $d\alpha$.

This assumption would perhaps seem somewhat risky for stars whose zenith distances are greater than 80°. But at the conclusion of the reductions, the value of $d\log\alpha$ deduced from such stars was found to fit in very well with those deduced from the other stars. Furthermore, down to 85° zenith distance the observing was very good. In consequence of these facts it was decided to take into account all the stars observed. The zenith distances of the stars in this list range from 21° 21′ to 89° 12′ (apparent).

From 85° zenith distance down, the quality of the "seeing" decreases quite rapidly. This can be seen from the following table of average weights. These weights were derived from the probable errors of the individual determinations of dlog$a$.

Z. D.	*Av. Wt.*
20° to 30°	2.0
50 to 60	7.5
60 to 70	7.5
70 to 80	11.8
80 to 85	14.8
85 to 90	3.6

The small weight for the small zenith distances is due to the fact that in the expression for da the refraction occurs in the denominator. The small weight for the stars at zenith distances greater than 85° is, of course, due to uncertainties in observing at such low altitudes.

Observations.

1. List.—The following list of 31 stars was observed on seventeen nights, from 1899 June 7 to 1899 July 6, inclusive, and have been reduced according to the plan outlined in the preceding section. Eleven other stars were on the same observing list, but they have not been used here. They were put on to obtain data for determining bisection error, and for other purposes.

The numbers of the stars are those of Newcomb's "Catalogue of Fundamental Stars for 1875 and 1900, reduced to an Absolute System."

No.	α (1900)			δ (1900)		
948	14h	51m	59s	—42°	43′	52″.30
190	2	57	33	+53	6	53 .92
959	15	5	6	—51	43	6 .62
968	15	13	29	+67	43	35 .08
977	15	21	9	+15	46	46 .45
984	15	28	28	—40	49	50 .61
225	3	33	28	+62	53	33 .74
997	15	39	21	+ 6	44	24 .53
1005	15	47	32	—19	52	5 .65
1009	15	51	50	+15	59	16 .46
1019	16	0	1	+58	49	56 .19
264	4	5	6	+85	17	29 .06
1032	16	12	21	—49	54	36 .79
282	4	24	6	+53	41	37 .37
1084	16	52	56	+ 9	31	49 .32
1094	17	8	30	+65	50	15 .88
1105	17	15	52	—24	53	59 .07
1110	17	20	58	—29	46	35 .61
349	5	26	21	+74	58	39 .95
356	5	29	54	+85	8	49 .60
1135	17	40	35	—40	5	17 .65
377	5	46	28	+55	41	1 .68
1156	17	58	51	—50	5	53 .20
1162	18	3	48	—45	58	18 .07
406	6	10	48	+59	2	50 .18
1179	18	19	34	—46	1	24 .50
1182	18	21	48	—25	28	37 .40
424	6	29	10	+79	40	22 .10
438	6	45	29	+77	6	17 .47
444	6	48	37	+58	33	14 .18
1225	19	0	42	—27	48	53 .80

2. *Details of Observations.*—A night's program consisted in observing the above list, together with three nadirs, one before, one during, and one after the observing of the stars. As has been pointed out, the nadirs are not necessary for the refraction determinations, but were taken for the reduction of the latitude, which is a problem practically inseparable from the main one undertaken here.

No transits were observed during these observations, the whole attention being devoted to the observations for zenith distance. The telescope was set to the nearest 2′ and not disturbed until the observation had been completed. The bisection was made (with but a very few exceptions) at the central transit wire, by means of the declination micrometer. For the sake of uniformity every star was bisected but once during its transit. Because of unavoidable circumstances a few of the stars had passed the meridian before the bisection could have been made. In these cases the readings have been reduced to the meridian.

For the position of the circle four microscopes were read. Settings were made upon two scratches under every microscope. The circle microscopes were usually read after the star had been bisected. In a few cases, because of a following star culminating very soon, the microscopes were read before the bisection. In such cases the position of the circle was quickly checked after the bisection.

The correction for runs for a night was obtained from all of the microscope readings of the night. This correction has been applied to all of the observations. Its values for the several nights of observing are given in the following table:—

Date	*R*	*Date*	*R*	*Date*	*R*
June 7	+0″.06	June 19	+0″.02	June 30	+0″.06
8	+0 .08	21	+0 .03	July 3	+0 .07
9	+0 .08	22	+0 .03	4	+0 .08
12	+0 .05	27	+0 .04	5	+0 .05
13	+0 .03	28	+0 .07	6	+0 .08
14	+0 .07	29	+0 .06		

These corrections were applied to the circle readings to reduce them to the mean position of the two scratches; so that for a reading of 0″ the correction is +R, for 60″ it is 0, and for 120″ it is —R.

In the few cases where the bisections were made a little late the reductions to the meridian were computed from the formula,

$$\delta = \delta' - \frac{\sin^2 \frac{1}{2}(\tau - m)}{\sin 1''} \sin 2\delta'$$

The horizontal flexure in this instrument is very small. In his work published in Vol. IV, "Publications of the Lick Observatory," Astronomer Tucker adopts the correction 0″.1sin Z. D., which was determined from a series of observations extending over two and a half years. In this work but two observations for flexure were made, one on 1899 June 3, and the other, 1899 July 8. The mean of the two gives the correction —0″.01sin Z. D.; so that for these observations the flexure correction has been considered zero. The mean of the values of one revolution of the declination micrometer, determined at the same time, is 48″.05. The value adopted, as noted before, is 48″.10.

For the computation of the preliminary refractions (called r′ in the reductions) the Pulkowa tables have been used. The reductions for the barometer, for the attached, and for the external thermometers were taken from Vol. I, "Publications of the Lick Observatory."

The graduation errors of the 1° divisions of the fixed circle have been determined by Astronomer Tucker. His results are given in Vol. IV, "Publications of the Lick Observatory." He says there, in part: "The probable error of a reading upon four divisions of the fixed circle due to graduation may be adopted as ±0″.15. * * * There is some evidence of periodic character in the errors, and it may be assumed, in absence of further data, that the probable error due to errors of graduation is not diminished by reading upon two adjoining divisions under each microscope. * * * The largest error measured is 0″.7 for the mean of four divisions."

The errors are not sufficiently systematic to warrant interpolating for undetermined divisions, so that no correction for division error has been applied.

Three nadirs were observed every night. The changes during a night were usually very small. The following table gives the means of the three determinations on the several nights:

Date	*Nadir* 134° 57′	*t*	*Date*	*Nadir* 134° 57′	*t*
June 7	22″.87	62°	June 27	20″.95	66°
8	22 .18	66	28	21 .32	67
9	22 .14	69	29	21 .40	69
12	24 .41	57	30	21 .70	66
13	22 .70	62	July 3	21 .43	72
14	21 .61	70	4	21 .46	69
19	23 .81	57	5	22 .91	61
21	22 .36	66	6	22 .10	58
22	21 .59	67			

All of the observations were taken with the fixed circle west. Had more time been available the instrument would have been reversed.

Weights, ranging from 5, the highest, to 1 (occasionally ½), the lowest, were arbitrarily assigned to all the observations. Judgment on a weight was formed from the steadiness of the image during the observation. These weights have been applied all through the reductions.

3. Reduction of Observations.— The first thing done on the reductions was to take the means of the microscope readings and to apply the micrometer corrections, giving the circle readings (called C′ in the tables following). The means of the microscopes were checked by taking the difference of every microscope reading from the mean of the four. If the sums of these differences for the two opposite pairs of microscopes was the same, the mean was correct. The corrections for the micrometers were checked by duplicating this part of the work.

From the readings C′ the quantity B [equations (II) and (III)] is obtained. The terms A and A′ of these equations are obtained from the declinations.

The declinations have been reduced to 1899.0 by means of the data furnished in Newcomb's Catalogue. The reductions to apparent places were computed by using the Besselian Star Numbers from the American Ephemeris. The factors a′, b′, c′ and d′ were computed from the American Ephemeris data. The reductions to apparent places for the first night (June 7) were computed by means of the Independent Star Numbers also. The places for the remaining nights were checked by differences. The apparent declinations are placed in the columns δ of the tables given later.

The following table exhibits the stars' approximate zenith distances and the stars with which they are grouped in the reductions for the refractions:

Star No.	Z. D. South		Z. D. North		Grouped with Star No.
	°	′	°	′	
948	79	59.9			225 *l. c.*
190 *l. c.*			89	12.0	959
959	88	45.5			190 *l. c.*, 282 *l. c.*
968			30	22.9	997
977	21	33.2			1019
984	78	6.6			225 *l. c.*
225 *l. c.*			79	41.9	948, 984, 1135
997	30	35.5			968
1005	57	11.3			264 *l. c.*, 356 *l. c.*
1009	21	20.7			1019
1019			21	29.3	977, 1009
264 *l. c.*			57	21.0	1005
1032	87	3.1			377 *l. c.*
282 *l. c.*			88	40.0	959
1084	27	48.1			1094
1094			28	29.4	1084
1105	62	12.9			424 *l. c.*
1110	67	5.1			349 *l. c.*
349 *l. c.*			67	39.0	1110
356 *l. c.*			57	29.5	1005
1135	77	22.1			225 *l. c.*
377 *l. c.*			86	47.2	1032, 1156
1156	87	13.9			377 *l. c.*
1162	83	12.4			406 *l. c.*, 444 *l. c.*
406 *l. c.*			83	30.2	1162, 1179
1179	83	15.5			406 *l. c.*, 444 *l. c.*
1182	62	47.5			424 *l. c.*
424 *l. c.*			62	57.5	1105, 1182
438 *l. c.*			65	31.4	1225
444 *l. c.*			83	59.3	1162, 1179
1225	65	7.7			438 *l. c.*

It will be noticed from this table that some of the stars are grouped with two others and that one is grouped with three others.

The following tables show the reductions for the new refractions. The column p contains the means of the weights of the pairs of stars. The other columns have already been explained. In the grouping of the pairs on the several dates the northern star is written first and the southern star below it. The numbers of the stars given at the tops are arranged in this same order. The pairs which have their northern stars at upper culmination are placed first. It will be noticed that the headings of the columns for these pairs are slightly different from the later ones containing the lower culmination stars.

Because of very bad "seeing" or of occasional accidents, some of the stars were not observed on some nights. In such cases blanks appear after the dates. No observations have been rejected.

Stars No. { 968, 997

Date	δ	A	C′	B	r′	½ $(r'_s - r'_n)$	½(A—B)	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 67 43 47.61		284 34 29.57		0 28.61			0 28.96	4
	+ 6 44 26.73	60 59 20.88	345 32 52.32	60 58 22.75	0 28.80	+ 0.10	0 29.06	0 29.16	4
8	47.88		28.48		28.43			29.21	4
	26.87	21.01	50.84	22.36	28.65	+ 0.11	29.32	29.43	4
9	48.12		27.26		28.28			28.60	3
	26.98	21.14	50.97	23.71	28.51	+ 0.11	28.71	28.82	3
12	48.80		30.50		28.83			29 37	1
	27.29	21.51	53.14	22.64	28.96	+ 0.06	29.43	29.49	1
13	49.04		28.03		28.58			28.99	1½
	27.41	21.63	51.46	23.43	28.81	+ 0.11	29.10	29.21	1½
14	49.31		26.24		28.30			28.68	3
	27.55	21.76	50.39	24.15	28.54	+ 0.12	28.80	28.92	3
19	50.67		28.00		29.02			29.43	3
	28.36	22.31	51.19	23.19	29.29	+ 0.13	29.56	29.69	3
21	51.13		25.83		28.50			29.09	3½
	28.64	22.49	49.95	24.12	28.69	+ 0.09	29.18	29.27	3½
22	51.35		24.44		28.30			28.92	5
	28.76	22.59	48.93	24.49	28.57	+ 0.13	29.05	29.18	5
27	52.22		22.44		28.48			28.61	2½
	29.23	22.99	48.00	25.56	28.70	+ 0.11	28 72	28.83	2½
28	52.41		23.76		28.45			28.88	3½
	29.34	23.07	48.82	25.06	28.70	+ 0.12	29.00	29.12	3½

STARS NO. {968, 997} (*Con.*)

Date	δ	A	C′	B	r′	½ ($r'_s - r'_n$)	½ (A−B)	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 67 43 52.62 + 6 44 29.48	60 59 23.14	284 34 22.98 345 32 48.24	60 58 25.26	0 28.40 0 28.62	+ 0.11	0 28.94	0 28.83 0 29.05	3½ 3½
30									
July 3	53.48 30.08	23.40	21.77 48.07	26.30	27.96 28.25	+ 0.14	28.55	28.41 28.69	3 3
4	53.66 30.22	23.44	22.04 48.38	26.34	28.26 28.47	+ 0.10	28.55	28.45 28.65	5 5
5	53.83 30.34	23.49	24.63 49.05	24.42	28.61 28.87	+ 0.13	29.53	29.40 29.66	3 3
6	53.96 30.44	23.52	23.27 47.69	24.42	28.88 29.17	+ 0.14	29.55	29.41 29.69	1½ 1½

Stars No. $\begin{cases} 1019 \\ 977 \end{cases}$

Date	δ	A	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A-B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 58 50 3.28		293 28 3.46		0 19.18			0 19.19	4
	+ 15 46 50.66	43 3 12.62	336 30 37.61	43 2 34.15	0 19.27	+ 0.04	0 19.23	0 19.27	4
8	3.57		3.07		19.09			19.39	4
	50.83	12.74	36.96	33.89	19.15	+ 0.03	19.42	19.45	4
9	3.84		2.73		18.99			19.05	3
	50.98	12.86	37.43	34.70	19.06	+ 0.03	19.08	19.11	3
12	4.64		4.79		19.31			19.52	1
	51.39	13.25	38.89	34.10	19.41	+ 0.05	19.57	19.62	1
13	4.91		3.14		19.18			19.66	1½
	51.54	13.37	37.13	33.99	19.25	+ 0.03	19.69	19.72	1½
14	5.20		1.52		19.00			19.48	3
	51.70	13.50	36.00	34.48	19.06	+ 0.03	19.51	19.54	3
19	6.79		28 1.98		19.51			19.59	3
	52.67	14.12	36.86	34.88	19.57	+ 0.03	19.62	19.65	3
21	7.35		27 59.71		19.13			19.33	3½
	53.00	14.35	35.33	35.62	19.20	+ 0.03	19.36	19 39	3½
22	7.62		58.28		19.04			19.10	3½
	53.14	14.48	34.54	36.26	19.07	+ 0.01	19.11	19.12	3½
27	8.75		56.75		19.12			19.33	5
	53.72	15.03	33.05	36.30	19.19	+ 0.03	19.36	19.39	5
28	9.00		57.16		19.12			19.23	2½
	53.87	15.13	33.76	36.60	19.18	+ 0.03	19.26	19.29	2½

STARS No. {1019 / 977} (*Con.*)

Date	δ	A	C′	B	r′	½ $(r'_s - r'_n)$	½ (A—B)	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 58 50 9.25		293 27 56.55		0 19.05			0 19.26	3½
	54.02	43 3 15.23	336 30 33.15	43 2 36.60	0 19.15	+ 0.05	0 19.31	0 19.36	3½
30	9.53		56.74		19.12			19.32	4
	54.20	15.33	33.41	36.67	19.14	+ 0.01	19.33	19.34	4
July 3	10.33		55.76		18.85			19.13	3
	54.71	15.62	33.11	37.35	18.84	± 0.00	19.13	19.13	3
4	10.59		55.62		18.97			19.31	5
	54.85	15.74	32.66	37.04	19.06	+ 0.04	19.35	19.39	5
5	10.80		58 01		19.26			19.76	3
	54.99	15.81	34.28	36.27	19.28	+ 0.01	19.77	19.78	3
6	11.01		55.72		19.43			19.40	2
	55.09	15.92	32.80	37.08	19.47	+ 0.02	19.42	19.44	2

Stars No. { 1019
1009

Date	δ	A	C′	B	r′	½ $(r'_s - r'_n)$	½(A—B)	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 58 50 3.28 + 15 59 20.28	42 50 43.00	293 28 3.46 336 18 8.55	42 50 5.09	0 19.18 0 19.04	— 0.07	0 18.95	0 19.02 0 18.88	4 4
8	3.57 20.45	43.12	3.07 7.81	4.74	19 09 18.95	— 0.07	19.19	19.26 19.12	4 4
9	3.84 20.61	43.23	2.73 7.58	4.85	18.99 18.84	— 0.07	19.19	19.26 19.12	3 3
12	4.64 21.06	43.58	4.79 9.90	5.11	19.31 19.16	— 0.07	19.23	19.30 19.16	1 1
13	4.91 21.22	43.69	3.13 8.30	5.16	19.18 19.04	— 0.07	19.26	19.33 19.19	2 2
14	5.20 21.38	43.82	1.52 6.86	5.34	19.00 18.86	— 0.07	19.24	19.31 19.17	3 3
19	6.79 22.41	44.38	28 1.98 7.38	5.40	19.51 19.37	— 0.07	19.49	19.56 19.42	3 3
21	7.35 22.72	44.63	27 59.71 6.09	6.38	19.13 18.99	— 0.07	19.12	19.19 19.05	4 4
22	7.62 22.93	44.69	58.28 5.29	7.01	19.04 18.90	— 0.07	18.84	18.91 18.77	4 4
27	8.75 23.56	45.19	56 75 3.24	6.49	19.12 18.98	— 0.07	19 35	19.42 19.28	5 5
28	9.00 23.73	45.27	57.16 4.07	6.91	19.12 18.98	— 0.07	19.18	19.25 19.11	2 2

Stars No. $\left\{ \begin{matrix} 1019 \\ 1009 \end{matrix} \right\}$ (*Con.*)

Date	δ	A	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A-B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 58 50 9.25		293 27 56.55		0 19.05			0 19.26	4
	+ 15 59 23.88	42 50 45.37	336 18 3.53	42 50 6.98	0 18.91	— 0.07	0 19.19	0 19.12	4
30	9.53		56.74		19.12			19.36	4
	24.07	45.46	3.61	6.87	18.97	— 0.07	19.29	19.22	4
July 3	10.33		55.76		18.85			18.89	3
	24.61	45.72	3.83	8.07	18.70	— 0.07	18.82	18.75	3
4	10.59		55.62		18.97			19.11	5
	24.78	45.81	3.34	7.72	18.82	— 0.07	19.04	18.97	5
5	10.80		58.01		19.26			19.40	3
	24.94	45.86	5.21	7.20	19.11	— 0.07	19.33	19.26	3
6	11.01		55.72		19.43			19.64	2
	25.07	45.94	2.51	6.79	19.28	— 0.07	19.57	19.50	2

Stars No. $\begin{cases}1094\\1084\end{cases}$

Date	δ	A	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A-B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 65 50 17.02 + 9 31 49.64	56 18 27.38	286 27 57.43 342 45 32.35	56 17 34.92	0 26.43 0 25.68	— 0.37	0 26.23	0 26.60 0 25.86	5 5
8	17.36 49.83	27.53	56.49 32.86	36.37	26.32 25.58	— 0.37	25.58	25.95 25.21	4 4
9	17.69 49.99	27.70	56.30 32.08	35.78	26.27 25.51	— 0.38	25.96	26.34 25.58	3 3
12	18.60 50.41	28.19	57.58 33.41	35.83	26.60 25.84	— 0.38	26.18	26.56 25.80	1 1
13	18.91 50.56	28.35	56.18 32.20	36.02	26.47 25.72	— 0.37	26.16	26.53 25.79	2 2
14	19.23 50.73	28.50	53.84 30.92	37.08	26.19 25.44	— 0.37	25.71	26.08 25.34	3 3
19	21.00 51.71	29.29	54.58 30 91	36.33	26.92 26.17	— 0.37	26.48	26.85 26.11	3 3
21	21.68 52.10	29.58	52.55 30.00	37.45	26.35 25.60	— 0.37	26.06	26.43 25.69	4 4
22	22.03 52.27	29.76	50.95 29.20	38.25	26.29 25.53	— 0.38	25.75	26.13 25.37	4 4
27	23.44 52.93	30.51	49.29 27.70	38.41	26.41 25.63	— 0.39	26.05	26.44 25.66	5 5
28	23.74 53.08	30.66	49.58 28.23	38.65	26 30 25.58	— 0.36	26.00	26.36 25.64	2 2

STARS No. $\left\{ \begin{matrix} 1094 \\ 1084 \end{matrix} \right\}$ (*Con.*)

Date	δ	*A*	*C′*	*B*	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A-B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 65 50 24.05		286 27 49.17		0 26.19			0 26.13	3
	+ 9 31 53.25	56 18 30.80	342 45 28.42	56 17 39.25	0 25.46	— 0.36	0 25.77	0 25.41	3
30	24.38		49.41		26.32			26.45	4
	53.44	30.94	28.18	38.77	25.57	— 0.37	26.08	25.71	4
July 3	25.38		47.32		25.94			25.76	3
	54.01	31.37	27.88	40.56	5.22	— 0.36	25.40	25.04	3
4	25.70		47.75		26.21			26.20	5
	54.19	31.51	27.61	39.86	25.45	— 0.38	25.82	25.44	5
5	26.01		47.90		26.56			26.14	4
	54.36	31.65	28.03	40.13	25.80	— 0.38	25.76	25.38	4
6	26.29		47.79		26.72			26.46	3
	54.50	31.79	27.42	39.63	25.96	— 0.38	26.08	25.70	3

Stars No. $\begin{cases} 225 \ l.\ c. \\ 948 \end{cases}$

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 62 53 22.12 − 42 43 57.82	20 9 24.30	235 15 29.18 394 57 19.73	159 41 50.55	4 19.13 4 27.56	+ 4.21	4 22.58	4 18.37 4 26.79	4 4
8	21.96 57.96	24.00	25.86 22.21	56.35	17.63 25.46	+ 3.91	19.84	15.93 23.75	4 4
9	21.81 58.10	23.71	23.88 24.35	60.47	16.42 23.79	+ 3.68	17.91	14.23 21.59	3 3
12	21.44 58.54	22.90	30.10 21.28	51.18	20.72 29.74	+ 4.51	22.96	18.45 27.47	1 1
13	21.30 58.65	22.65	27.85 21.04	53.19	19.17 27.04	+ 3.93	22.08	18.15 26.01	2 2
14	21.15 58.75	22.40	22.87 23.80	60.93	16.62 24.05	+ 3.71	18.34	14.63 22.05	3 3
19	20.31 59.10	21.21	31.55 20.24	48.69	23.64 30.48	+ 3.42	25.05	21.63 28.47	3 3
21	20.04 59.28	20.76	24.63 23.39	58.76	18.11 25.94	+ 3.91	20.24	16.33 24.15	4 4
22	19.93 59.39	20.54	22.86 22.84	59.98	16.84 23.93	+ 3.54	19 74	16.20 23.28	3½ 3½
27	19.54 43 59.95	19.59	22.57 21.64	59.07	18.15 25.82	+ 3.83	20.67	16.84 24.50	5 5
28	19.45 44 0.03	19.42	24.60 24.18	59.58	18.08 24.93	+ 3.42	20.50	17.08 23.92	2½ 2½

STARS No. $\left\{ \begin{matrix} 225 \; l.c. \\ 948 \end{matrix} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 62 53 19.34 − 42 44 0.07	20 9 19.27	235 15 21.40 394 57 23.07	159 41 61.67	4 17.46 4 24.62	+ 3.58	4 19.53	4 15.95 4 23.11	3½ 3½
30									
July 3	18 86 0.20	18.66	20.47 26.39	65.92	13.88 20.62	+ 3.37	17.71	14.34 21.08	2½ 2½
4	18.76 0.25	18.51	21.15 25.22	64.07	16.12 22.52	+ 3.20	18.71	15.51 21.91	5 5
5	18.69 0.32	18.37	26.87 22.36	55.49	19.66 26.96	+ 3.65	23.07	19.42 26.72	3 3
6	18.63 0.38	18.25	27.04 22.34	55.30	22.46 29.41	+ 3.47	23.23	19.76 26.70	1½ 1½

Stars No. $\left\{ \begin{array}{l} 190\ l.\ c. \\ 959 \end{array} \right.$

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
June 7									
8									
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″	′ ″	′ ″	
9	+ 53 6 39.95 — 51 43 12.97	1 23 26.98	225 44 42.86 403 43 8.96	177 58 26.10	20 47.65 18 11.02	—1 18.31	19 3.46	20 21.77 17 45.15	3 3
12	39.80 13.54	26.26	45 53.72 43 34.42	57 40.70	21 23.85 18 42.23	—1 20.81	26.52	20 47.33 18 5.71	1 1
13	39.74 13.72	26.02	45 33.61 42 31.90	56 58.29	21 5.65 18 25.95	—1 19.85	47.85	21 7.70 18 28.00	1½ 1½
14	39.66 13.86	25.80	44 45.18 42 49.63	58 4.45	20 47.94 18 9.87	—1 19.03	14.88	20 33.91 17 55.85	3 3
19									
21	39.08 14.70	24.38	45 4.24 42 56.99	57 52.75	20 59.18 18 21.67	—1 18.75	21.44	20 40.19 18 2.69	4 4
22	39.05 14.86	24.19	45 8.65 42 56.61	57 47.96	20 45.74 18 11.00	—1 17.37	23.93	20 41.30 18 6.56	3½ 3½
27	39.05 15.65	23.40	45 25.81 43 0.57	57 34.76	20 54.62 18 19.97	—1 17.32	30.92	20 48.24 18 13.60	5 5
28	39.02 15.76	23.26	44 55.03 43 4.29	58 9.26	20 52.94 18 17.28	—1 17.83	13.74	20 31.57 17 55.91	3 3

Stars No. $\left\{ \begin{array}{l} 190\ l.\ c. \\ 959 \end{array} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″	′ ″	′ ″	
June 19	+ 53 6 38.98 — 51 43 15.85	1 23 23.13	225 44 40.97 403 43 14.79	177 58 33.82	20 51.65 18 16.12	—1 17.76	19 1.53	20 19.29 17 43.77	3 3
30	38.94 15.93	23.03	45 18.75 42 58.07	57 39.32	20 51.06 18 16.05	—1 17.50	28.83	20 46.33 18 11.33	4 4
July 3									
4	38.82 16.23	22.59	45 1.07 43 12.58	58 11.51	20 38.49 18 8.10	—1 15.19	12.95	20 28.14 17 57.76	5 5
5	38.82 16.33	22.49	45 13.44 42 48.50	57 35.06	21 6.51 18 27.69	—1 19.41	31.23	20 50.64 18 11.82	3 3
6									

Stars No. $\begin{cases} 282 \ l.\ c. \\ 959 \end{cases}$

Date	δ	A′	C′	B	r′	½ $(r'_s - r'_n)$	90°—½ (A′+B)	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 53 41 30.49		226 17 17.18		17 57 70			17 47.23	4½
	— 51 43 12.60	1 58 17.89	403 42 53.57	177 25 36.39	18 28.97	+ 15.63	18 2.86	18 18.49	4½
8	30.34		17 20.57		17 50.61			17 43.70	3½
	12 78	17.56	43 5.72	25 45.15	18 20.52	+ 14.95	17 58.65	18 13.60	3½
9	30.20		16 47.39		17 46.70			17 28.44	3
	12.97	17.23	43 8.96	26 21.57	18 11.02	+ 12.16	17 40.60	17 52.76	3
12	29.85		17 25.93		18 8.67			18 0.82	1
	13.54	16.31	42 34.42	25 8.49	18 42.23	+ 16.78	18 17.60	18 34.38	1
13									
14	29.60		16 53.23		17 43.20			17 40.60	3
	13.86	15.74	42 49.63	25 56.40	18 9.87	+ 13.33	17 53.93	18 7.26	3
19	28.76		17 31.26		18 22.57			18 12.83	3
	14.44	14.32	42 27.57	24 56.31	18 46.29	+ 11.86	18 24.69	18 36.55	3
21	28.48		17 2.06		17 53.07			17 41.35	4
	14.70	13.78	42 56.99	25 54.93	18 21.67	+ 14.30	17 55.65	18 9.95	4
22	28.36		17 1.17		17 46.95			17 43.51	3½
	14.86	13.50	42 56.61	25 55.44	18 11.00	+ 12.02	17 55.53	18 7.55	3½
27	27.94		17 3.58		17 51.97			17 41.36	5
	15.65	12.29	43 0.57	25 56.99	18 19.97	+ 14.00	17 55.36	18 9.36	5
28	27.84		16 48.37		17 54.98			17 34.85	2½
	15.76	12.08	43 4.29	26 15.92	18 17.28	+ 11.15	17 46.00	17 57.15	2½

STARS No. $\left\{ \begin{matrix} 282\ l.\ c. \\ 959 \end{matrix} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 53 41 27.72		226 16 42.70		17 46.97			17 23.45	3
	− 51 43 15.85	1 58 11.87	403 43 14.79	177 26 32.09	18 16.12	+ 14.57	17 38.02	17 52.59	3
30	27.59		16 58.31		17 51.25			17 41.89	4½
	15.93	11.66	42 58.07	25 59.76	18 16.05	+ 12.40	17 54.29	18 6.69	4½
July 3									
4	27.06		16 46.51		17 42.59			17 28.79	5
	16.23	10.83	43 12.60	26 26.09	18 8.10	+ 12.75	17 41.54	17 54.29	5
5	26.97		17 4.06		18 4.20			17 50.72	3½
	16.33	10.64	42 48.50	25 44.44	18 27.69	+ 11.74	18 2.46	18 14.20	3½
6	26.89		17 10.44		18 14.13			17 59.45	2½
	16.45	10.44	42 35.29	25 24.85	18 39.95	+ 12.91	18 12.36	18 25.27	2½

Stars No. { 225 *l. c.* / 984

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
June 7									
8									
9									
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
12	+ 62 53 21.44		235 15 30.10		4 20.72			4 18.50	1
	− 40 49 55.27	22 3 26.17	390 4 0.54	157 48 30.44	3 47.12	− 16.80	4 1.70	3 44.90	1
13	21.30		15 27.85		4 19.17			4 18.02	2½
	55.40	25.90	3 59.47	31.62	3 45.61	− 16.78	4 1.24	3 44.46	2½
14	21.15		15 22.87		4 16.62			4 15.14	3
	55.52	25.63	4 0.20	37.33	3 43.37	− 16.62	3 58.52	3 41.90	3
19	20.31		15 31.55		4 23.64			4 23.11	3
	55.86	24.45	3 55.07	23.52	3 49.46	− 17.09	4 6.02	3 48.93	3
21	20.04		15 26.63		4 18.11			4 17.43	3
	56.03	24.01	3 59.11	34.48	3 44.77	− 16.67	4 0.76	3 44.09	3
22	19.93		15 22.86		4 16.84			4 16.15	3½
	56.15	23.78	4 0.07	37.21	3 43.55	− 16.64	3 59.51	3 42.87	3½
27	19.54		15 22.57		4 18.15			4 17.00	5
	56.75	22.79	3 59.16	36.59	3 44.77	− 16.69	4 0.31	3 43.62	5
28	19.45		15 24.60		4 18.08			4 18.14	2
	56.83	22.62	3 59.07	34.47	3 44.71	− 16.68	4 1.46	3 44.78	2

STARS NO. $\left\{ \begin{matrix} 225 \; l.c. \\ 984 \end{matrix} \right\}$ (*Con.*)

Date	δ	A′	C′	B	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 62 53 19.34		235 15 21.40		4 17.46			4 16.53	4
	− 40 49 56.91	22 3 22.43	393 3 59.20	157 48 37.80	3 44.18	− 16.64	3 59.89	3 43.25	4
30									
July 3	18.86		15 20.47		4 13.88			4 14.90	3
	57.07	21.79	4 1.88	41.41	3 40.88	− 16.50	3 58.40	3 41.90	3
4	18.76		15 21.15		4 16.12			4 15.44	5
	57.12	21.64	4 1.68	40.53	3 43.07	− 16.52	3 58.92	3 42.40	5
5	18.69		15 26.87		4 19.66			4 19.16	3
	57.18	21.51	4 0.69	33.82	3 46.02	− 16.82	4 2.34	3 45.52	3
6	18.63		15 27.04		4 22.46			4 20.93	1½
	57.29	21.34	3 57.89	30.85	3 48.41	− 17.02	4 3.91	3 46.89	1½

Stars No. { 225 *l. c.* / 1135

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{3}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 62 53 22.12 — 40 5 16.59	22 48 5.53	235 15 29.18 392 19 32.82	157 4 3.64	4 19.13 3 32.32	— 23.40	3 55.42	4 18.82 3 32.02	4½ 4½
8	21.96 16.68	5.28	25.86 33.46	7.60	4 17.63 3 31.52	— 23.05	53.56	4 16.61 3 30.51	4 4
9	21.81 16.78	5.03	23.88 34.82	10.94	4 16.42 3 31.07	— 22.67	52.02	4 14.69 3 29.35	3 3
12	21.44 17.15	4.29	30.10 33.42	3.32	4 20.72 3 34.09	— 23.31	56.20	4 19.51 3 32.89	1 1
13	21.30 17.27	4.03	27.85 35.03	7.18	4 19.17 3 32.65	— 23.26	54.40	4 17.66 3 31.14	2 2
14	21.15 17.39	3.76	22.87 34.06	11.19	4 16.62 3 30.86	— 22.88	52.53	4 15.41 3 29.65	3 3
19	20.31 17.74	2.57	31.55 32.18	0.63	4 23.64 3 36.10	— 23.77	58.40	4 22.17 3 34.63	3 3
21	20.04 17.87	2.17	24.63 33.59	8.96	4 18.11 3 31.73	— 23.19	54.44	4 17.63 3 31.25	4 4
22	19.93 17.96	1.97	22.86 33.87	11.01	4 16.84 3 31.17	— 22.83	53.51	4 16.34 3 30.68	4 4
27	19.54 18.59	0.95	22.57 33.37	10.80	4 18.15 3 32.55	— 22.80	54.13	4 16.93 3 31.33	5 5
28	19.45 18.71	0.74	24.60 35.52	10.92	4 18.08 3 31.16	— 23.46	54.17	4 17.63 3 30.71	2½ 2½

STARS NO. $\left\{\begin{matrix} 225 \textit{ l. c.} \\ 1135 \end{matrix}\right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 62 53 19.34		235 15 21.40		4 17.46			4 15.53	4
	— 40 5 18.81	22 48 0.53	392 19 36.68	157 4 15.28	3 30.60	— 23.43	3 52.10	3 28.67	4
30									
July 3	18.86		20.47		4 13.88			4 14 03	3
	19.06	47 59.80	37.90	17.43	3 28.60	— 22.64	51.39	3 28.75	3
4	18.76		21.15		4 16.12			4 15.22	5
	19.13	59.63	36.45	15.30	3 30.76	— 22.68	52.54	3 29.86	5
5	18.69		26.87		4 19.66			4 19.64	3½
	19.20	59.49	34.31	7.44	3 33.46	— 23.10	56.54	3 33.44	3½
6	18.63		27.04		4 22.46			4 20.30	2
	19.30	59.33	34.80	7.76	3 34.78	— 23.84	56.46	3 32.62	2

Stars No. { 264 *l. c.* / 1005 }

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 85 17 20.29		257 31 33.27		1 15.82			1 15.48	4
	— 19 52 6.90	65 25 13.39	372 8 39.39	114 32 16.12	1 15.35	— 0.23	1 15.25	1 15 02	4
8	20.01		21.80		15.47			14.93	4
	6.90	13.11	39.32	17.52	14.98	— 0.24	14.69	14.45	4
9	19.77		20.20		15.10			14.12	3
	6.93	12.84	39.62	19.42	14.59	— 0.25	13.87	13.62	3
12	19.08		24.35		16.36			15.85	1
	7.02	12.06	41.12	16.77	15.84	— 0.26	15.59	15.33	1
13	18.84		21.88		15.84			15.21	2
	7.04	11.80	40.15	18.24	15.37	— 0.23	14.98	14.75	2
14	18.60		20.42		15.09			15.33	3
	7.05	11.55	38.63	18.21	14.66	— 0.21	15.12	14.91	3
19	17.22		22.04		17.17			16.60	3
	6.87	10.35	38.99	16.95	16.66	— 0.25	16.35	16.10	3
21	16.72		18.55		15.60			14.99	4
	6.84	9.88	39.15	20.60	15.13	— 0.23	14.76	14.53	4
22	16.49		16.97		15.26			14.55	4
	6.85	9.64	38 72	21.75	14.78	— 0.24	14.31	14.07	4
27	15.56		16.45		15.60			15.43	5
	6.97	8 59	37.47	21.02	15.13	— 0.23	15.20	14.97	5
28	15.36		16.32		15.64			14.83	2
	6.96	8.40	38.81	22.49	15.10	— 0.27	14.56	14.29	2

STARS No. $\left\{ \begin{array}{l} 264 \ l.\ c. \\ 1005 \end{array} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 85 17 15.14		257 36 16.07		1 15.29			1 14.97	4
	— 19 52 6.93	65 25 8.21	372 8 38.32	114 32 22.25	1 14.88	— 0.20	1 14.77	1 14.57	4
30	14.92		16.33		15.55			15.45	4
	6.89	8.03	37.88	21.55	15.07	— 0.24	15.21	14.97	4
July 3	14.23		14.94		14.54			14.41	3
	6.76	7.47	39.19	24.25	13.99	— 0.27	14.14	13.87	3
4	14.02		15.35		14.98			14.86	5
	6.72	7.30	38.80	23.45	14.51	— 0.23	14.63	14.40	5
5	13.84		18.17		16.14			16.07	3
	6.71	7.13	39.43	21.26	15.62	— 0.26	15.81	15.55	3
6	13.66		16.36		16.77			16.09	2
	6.71	6.95	37.68	21.32	16.33	— 0.22	15.87	15.65	2

Stars No. $\left\{ \begin{array}{l} 356\ l.c. \\ 1005 \end{array} \right.$

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	p
June 7									
8									
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
9	+ 85 8 49.23		257 27 52.09		1 15.72			1 15.65	3
	— 19 52 6.93	65 16 42.30	372 8 39.62	114 40 47.53	1 14.59	— 0.56	1 15.09	1 14.53	3
12	48.41		54.49		16.80			16.47	1
	7.02	41.39	41.12	46.63	15.84	— 0.48	15.99	15.51	1
13	48.16		52.30		16.29			15.99	2½
	7.04	41.12	40.12	47.82	15.37	— 0.46	15.53	15.07	2½
14	47.87		49.87		15.60			15.68	3
	7.05	40.82	38.63	48.78	14.66	— 0.47	15.21	14.74	3
19	46.25		52.74		17.56			17.64	3
	6.87	39.38	38.99	46.25	16.66	— 0.45	17.19	16.74	3
21	45.63		48.47		15.99			15.70	4
	6.84	38.79	39.15	50.68	15.13	— 0.43	15.27	14.84	4
22	45.30		47.37		15.80			15.61	4
	6.85	38.45	38.72	51.35	14.78	— 0.51	15.10	14.59	4
27	44.02		46.47		16.27			16.55	5
	6.97	37.05	37.47	51.00	15.13	— 0.57	15.98	15.41	5
28	43.77		45.99		15.77			15.52	2
	6.96	36.81	38.81	52.82	15.10	— 0.33	15.19	14.86	2

STARS No. $\left\{ \begin{matrix} 356\ l.\ c. \\ 1005 \end{matrix} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 85 8 43.48		257 27 45.52		1 15.53			1 15.65	4
	− 19 52 6.93	65 16 36.55	372 8 38 32	114 40 52.80	1 14.88	− 0.32	1 15.33	1 15.01	4
30	43.19		45.70		15.87			16.16	4
	6.89	36.30	37.88	52.18	15.07	− 0.40	15.76	15.36	4
July 3	42.25		43.05		14.84			14.61	3
	6.76	35.49	39.19	56.14	13.99	− 0.42	14.19	13.77	3
4	41.94		44.46		15.58			15.75	5
	6.72	35.22	38.80	54.34	14.51	− 0.53	15.22	14.69	5
5	41.67		45.48		16.58			16.03	3½
	6.71	34.96	39 43	53.95	15.62	− 0.48	15.55	15.07	3½
6	41.41		45.04		17.05			16.69	2½
	6.71	34.70	37.68	52.64	16.33	− 0.36	16.33	15.97	2½

Stars No. { 377 *l. c.* / 1032 }

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	p
June 7									
8									
9									
12									
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
13	+ 55 41 1.33 − 49 54 40.61	5 46 20.72	228 10 14.90 402 0 21.08	173 50 6.18	11 22.33 12 1.13	+ 19.40	11 46.55	11 27.15 12 5.95	1 1
14	1.16 40.77	20.39	10 0.34 0 39.28	38.94	11 15.85 11 52.53	+ 18.34	30.33	11 11.99 11 48.67	3½ 3½
19	41 0.12 41.39	18.73	10 14.46 0 21.24	6.78	11 35.13 12 15.48	+ 20.17	47.25	11 27.08 12 7.42	3 3
21	40 59.71 41.65	18.06	10 5.27 0 36.34	31.07	11 18.89 11 58.40	+ 19.75	35.44	11 15.69 11 55.19	4 4
22	59 50 41.82	17.68	9 58.22 0 33.65	35.43	11 17.25 11 54.68	+ 18.71	33.45	11 14.74 11 52.16	4 4
27	58.75 42.72	16.03	10 2.48 0 33.09	30.61	11 21.65 11 58.23	+ 18.29	36.68	11 18.39 11 54.97	5 5
28	58.60 42.86	15.74	9 59.26 0 41.03	41.77	11 16.63 11 59.38	+ 21.37	31.25	11 9.88 11 52.62	2½ 2½

STARS NO. $\left\{ \begin{matrix} 377 \\ 1032 \end{matrix} \right.$ *l. c.* $\left. \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 55 40 58.43		228 9 49.65		11 15.01			11 5.21	3½
	− 49 54 42.99	5 46 15.44	402 0 43.98	173 50 54.33	11 54.84	+ 19.91	11 25.12	11 45.03	3½
30	58.23		10 0.13		11 17.75			11 7.19	4
	43.10	15.13	0 39.43	39.30	11 57.80	+ 20.02	27.21	11 47.23	4
July 3	57.61		9 44.08		11 8.03			11 2.16	3
	43.38	14.43	0 46.11	62.03	11 47.25	+ 19.61	21.77	11 41.38	3
4	57.41		10 0.23		11 15.58			11 14.53	4½
	43.48	13.93	0 40.99	40.76	11 51.85	+ 18.13	32.66	11 50.79	4½
5	57.23		10 6.49		11 25.62			11 20.73	4
	43.61	13.62	0 32.18	25.69	12 4.80	+ 19.59	40.32	11 59.91	4
6	57.05		10 4.93		11 30.43			11 20.21	2½
	43.74	13.31	0 30.21	25.28	12 11 43	+ 20.50	40.71	12 1.21	2½

Stars No. { 424 *l. c.* / 1105 }

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
June 7									
8									
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
9	+ 79 40 28.16 — 24 53 59.26	54 46 28.90	251 59 50.10 377 10 16.51	125 10 26.41	1 34.35 1 31.46	— 1.44	1 32.35	1 33.79 1 30.91	3½ 3½
12									
13	27.13 59.44	27.69	50.13 16.24	26.11	35.10 32.18	— 1.46	33.10	34.56 31.64	2 2
14	26.85 59.47	27.38	50.49 15.56	25.07	34.41 31.22	— 1.59	33.78	35.37 32.19	3½ 3½
19	25.26 59.40	25.86	52.35 15.94	23.59	36.54 33.73	— 1.40	35.28	36.68 33.88	3 3
21	24.64 59.38	25.26	47.26 16.19	28.93	34.64 31.80	— 1.42	32.91	34.33 31.49	4 4
22	24.31 59.39	24.92	45.67 15.10	29.43	34.35 31.56	— 1.39	32.83	34.22 31.44	4 4
27	22.95 59.61	23.34	43.76 14.50	30.74	34.90 32.05	— 1.42	32.96	34.38 31.54	5 5
28	22.69 59.65	23.04	44.93 15.48	30.55	34.65 31.56	— 1.54	33.21	34.75 31.67	3 3

STARS NO. { 424 *l. c.* / 1105 } (*Con.*)

Date	δ	A′	C′	B	r′	½ $(r'_s - r'_n)$	90° − ½ (A′ + B)	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 79 40 22.40		251 59 43.75		1 34.34			1 34.33	3
	− 24 53 59.65	54 46 22.75	377 10 15.50	125 10 31.75	1 31.18	− 1.58	1 32.75	1 31.17	3
30	22.10		43.70		34.51			34.34	4½
	59.65	22.45	15.41	31.71	31.66	− 1.42	32.92	31.50	4½
July 3	21.10		41.68		33.26			33.37	3
	59.57	21.53	16.32	34.64	30.36	− 1.45	31.92	30.47	3
4	20.77		42.48		34.13			34.27	5
	59.56	21.21	15.59	33.11	31.26	− 1.43	32.84	31.41	5
5	20.46		44.05		35.50			35.42	4
	59.56	20.90	15.31	31.26	32.49	− 1.50	33.92	32.42	4
6	20.16		44.16		36.02			36.02	3
	59.57	20.59	14.49	30.33	33.05	− 1.48	34.54	33.06	3

Stars No. { 349 *l. c.* / 1110

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 74 58 39.46		247 18 25.85		1 57.63			1 57.97	5
	− 29 46 35.49	45 12 3.97	382 2 29.10	134 44 3.25	1 54.46	− 1.58	1 56.39	1 54.81	5
8	39.17		24.48		57.14			57.42	4
	35.53	3.64	29.16	4.68	53.98	− 1.58	55.84	54.26	4
9	38.91		23.19		56.85			56.25	3
	35.58	3.33	30.52	7.33	53.69	− 1.58	54.67	53.09	3
12	38.19		26.02		58.54			58.06	1
	35.79	2.40	30.71	4.69	55.33	− 1.60	56.46	54.86	1
13	37.96		23.58		57.77			57.70	1
	35.86	2.10	29 27	5.69	54.59	− 1.59	56.11	54.52	1
14	37.71		22.15		56.65			57.41	3
	35.92	1.79	28.72	6.57	53.46	− 1.59	55.82	54.23	3
19	36.26		24.88		59.74			59.16	3
	35.99	12 0.27	29.45	4.57	56.57	− 1.58	57.58	56.00	3
21	35.71		20.63		57.31			57.02	4
	36.02	11 59.69	30.07	9.44	54.14	− 1.58	55.44	53.86	4
22	35.43		18.12		57.04			56.19	4
	36.07	59.36	29.55	11.43	53.88	− 1.58	54.61	53.03	4
27	34.32		17.95		57.74			57.42	5
	36.43	57.89	28.42	10.47	54.53	− 1.60	55.82	54.22	5
28	34.09		18.10		56.96			56.90	2
	36.47	57.62	29.82	11.72	53.82	− 1.57	55.33	53.76	2

STARS NO. $\left\{ \begin{matrix} 349 \ l.\ c. \\ 1110 \end{matrix} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 74 58 33.85		247 18 17.41		1 56.54			1 56.36	4
	— 29 46 36.51	45 11 57.34	382 2 30.52	134 44 13.11	1 53.37	— 1.58	1 54.78	1 53.20	4
30	33.59		16.93		57.14			56.47	4½
	36.54	57.05	30.10	13.17	53.97	— 1.58	54.89	53.31	4½
July 3	32.75		13.92		55.53			55.09	2½
	36.55	56.20	30.71	16.79	52.37	— 1.58	53.51	51.93	2½
4	32.49		16.55		56.65			56.64	5
	36.55	55.94	30.51	13.96	53.47	— 1.59	55.05	53.46	5
5	32.23		17.98		58.23			57.92	4
	36.57	55.66	29.69	11.71	55.02	— 1.60	56.32	54.72	4
6	32.01		18.15		58.95			58.35	3
	36.62	55.39	29.29	11.14	55.73	— 1.61	56.74	55.13	3

Stars No. { 377 *l. c.* / 1156

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 55 41 2.34		228 10 11.24		11 21.21			11 17.87	5
	− 50 5 50.75	5 35 11.59	402 11 14.82	174 1 3.58	12 30.32	+ 34.55	11 52.42	12 26.97	5
8	2.14		4.37		11 18.45			11 13.56	4
	50.88	11.26	18.07	13.70	12 26.38	+ 33.96	47.52	12 21.48	4
9	1.96		0.28		11 16.78			11 11.74	3
	51.03	10.93	18.41	18.13	12 24.25	+ 33.73	45.47	12 19 20	3
12									
13	1.33		14.90		11 22.33			11 20.86	½
	51.71	9.62	15.03	0.13	12 30.88	+ 34.27	55.13	12 29.40	½
14	1.16		0.34		11 15.85			11 10.00	3½
	51.89	9.27	23.26	22.92	12 23.67	+ 33.91	43.91	12 17.82	3½
19	41 0.12		14.46		11 35.13			11 23.35	3
	52.49	7.63	10.18	0 55.72	12 45.10	+ 34.98	58.33	12 33.31	3
21	40 59.71		5.27		11 18.89			11 16.84	4
	52.70	7.01	17.01	1 11.74	12 26.48	+ 33.79	50.63	12 24.42	4
22	59.50		9 58.22		11 17.25			11 13.60	4
	52.86	6.64	17.03	18.81	12 24.62	+ 33.68	47.28	12 20.96	4
27	58.75		10 2.48		11 21.65			11 17.34	5
	53.73	5.02	14.45	12.97	12 29.00	+ 33.67	51.01	12 24.68	5
28	58.60		9 59.26		11 16.63			11 10.61	3
	53.90	4.70	25.42	26.16	12 24.55	+ 33.96	44.57	12 18.53	3

STARS No. { 377 *l. c.* / 1156 } (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 55 40 58.43		228 9 49.65		11 15.01			11 5.46	4
	− 50 5 54.05	5 35 4.38	402 11 26.72	174 1 37.07	12 22.65	+ 33.82	11 39.28	12 13 10	4
30	58.23		10 0.13		11 17.75			11 13.49	4
	54.19	4.04	21.35	21.22	12 25.52	+ 33.88	47.37	12 21.25	4
July 3	57.61		9 44.08		11 8.03			11 2.59	3
	54.52	3.09	29.44	45.36	12 14.42	+ 33.19	35.78	12 8.97	3
4	57.41		10 0.23		11 15.58			11 11.90	5
	54.64	2.77	25.64	25.41	12 23.60	+ 34.01	45.91	12 19.92	5
5	57.23		6.49		11 25.62			11 19.81	3½
	54.77	2.46	15.42	8.93	12 34.62	+ 34.50	54.31	12 28.81	3½
6	57.05		4.93		11 30.43			11 23.95	2½
	54.92	2.13	6.26	1.33	12 39.07	+ 34.32	58 27	12 32.59	2½

Stars No. { 406 *l. c.* / 1162

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
June 7									
8									
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
9	+ 59 2 52.90		231 27 10.27		6 32.25			6 29.51	3½
	− 45 58 15.92	13 4 36.98	398 9 49.26	166 42 38.99	6 17.27	− 7.49	6 22.02	6 14.53	3½
12									
13	52.19		17.29		35.21			34.33	½
	16.50	35.69	47.98	30.69	20.17	− 7.52	26.81	19.29	½
14	52.02		7.78		32.19			29.51	3½
	16.65	35.37	48.45	40.67	17.13	− 7.53	21.98	14.45	3½
19	50.86		18.62		42.02			37.19	3
	17.15	33.71	45.82	27.20	26.74	− 7.64	29.55	21.91	3
21	50.39		8.82		33.35			30.76	4
	17.30	33.09	49.21	40.39	18.35	− 7.50	23.26	15.76	4
22	50.16		6.33		32.34			30.53	4
	17.43	32.73	47.46	41.13	17.41	− 7.46	23.07	15.61	4
27	49.28		8.18		34.47			32.76	5
	18.20	31.08	46.52	38.34	19.52	− 7.47	25.29	17.82	5
28	49.10		5.95		32.83			29.92	3½
	18.34	30.76	50.61	44.66	17.56	− 7.63	22.29	14.66	3½

STARS No. { 406 *l. c.* / 1162 } (*Con.*)

Date	δ			A′			C′			B			r′		½ $(r'_s - r'_n)$	90°−½(A′+B)		r		p
	°	′	″	°	′	″	°	′	″	°	′	″	′	″	″	′	″	′	″	
June 29	+ 59	2	48.90				231	27	5.26				6	31.87				6	29.25	4
	− 45	58	18.47	13	4	30.43	398	9	51.46	166	42	46.20	6	16.74	− 7.56	6	21.69	6	14.13	4
30			48.67						5.28					32.94					28.50	4
			18.57			30.10			53.21			47.93		17.91	− 7.51		20.99		13.48	4
July 3			47.97					26	59.31					27.63					24.51	3½
			18.86			29.11			55.94			56.63		12.87	− 7.38		17.13		9.75	3½
4			47.74					27	4.37					31.72					29.02	5
			18.94			28.80			51.44			47.07		17.81	− 6.95		22.07		15.12	5
5			47.53						11.77					37.27					35.46	3½
			19.05			28.48			47.60			35.83		22.05	− 7.61		27.85		20.24	3½
6			47.32						9.39					39.46					34.68	3
			19.18			28.14			47.22			37.83		24.13	− 7.66		27.02		19.36	3

Stars No. { 444 *l. c.* / 1162 }

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7									
8									
9	+ 58 33 20.44		230 58 2.07		6 58.73			6 54.88	3½
	— 45 58 15.92	12 35 4.52	398 9 49.26	167 11 47.19	6 17.27	— 20.73	6 34.15	6 13.42	3½
12									
13	19.74		8.16		7 2.88			6 59.82	½
	16.50	3.24	47.98	39.82	6 20.17	— 21.35	38.47	6 17.12	½
14	19.57		5 71		6 59.02			6 58.11	2½
	16.65	2.92	48.45	42.74	6 17.13	— 20.94	37.17	6 16.23	2½
19	18.44		12.84		7 8.99			7 3.99	3
	17.15	1.29	45.82	32.98	6 26.74	— 21.12	42.87	6 21.75	3
21	17.97		3.92		7 0.41			6 58.05	4
	17.30	0.67	49.21	45.29	6 18.35	— 21.03	37.02	6 15.99	4
22	17.72		2.44		6 58.99			6 58.14	4
	17.43	35 0.29	47.46	45.02	6 17.41	— 20.79	37.35	6 16.56	4
27	16.78		2.10		7 1.48			6 59.48	5
	18.20	34 58.58	46.52	44.42	6 19.52	— 20.98	38.50	6 17.52	5
28	16.61		0.05		7 0.43			6 57.02	
	18.34	58.27	50.61	50.56	6 17.56	— 21.43	35.59	6 14.16	4

STARS No. { 444 *l. c.* / 1162 } (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 58 33 16.40		230 58 0.79		6 58.92			6 56.79	3½
	− 45 58 18.47	12 34 57.93	398 9 51.46	167 11 50.67	6 16.74	− 21.09	6 35.70	6 14.61	3½
30	16.17		0.83		6 59.55			6 55.83	4
	18.57	57.60	53.21	52.38	6 17.91	− 20.82	35.01	6 14.19	4
July 3	15.41		57 55.07		6 54.01			6 51.86	3
	18.86	56.55	55.94	60.87	6 12.87	− 20.57	31.29	6 10.72	3
4	15.17		57 58.10		6 57.66			6 55.14	5
	18.94	56.23	51.44	53.34	6 17.81	− 19.92	35.22	6 15.30	5
5	14.94		58 6.86		7 4.44			7 2.88	3½
	19.05	55.89	47.60	40.74	6 22.05	− 21.19	41.69	6 20.50	3½
6	14.71		58 6.84		7 7.10			7 3.53	2
	19.18	55.53	47.22	40.38	6 24.13	− 21.48	42.05	6 20.57	2

Stars No. { 406 *l. c.* / 1179 }

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 59 2 53.33		231 27 14.32		6 35.64			6 33.88	4
	− 46 1 21.48	13 1 31.85	398 12 46.55	166 45 32.23	6 23.79	− 5.92	6 27.96	6 22.04	4
8	53.10		12.30		33.15			31.01	4
	21.57	31.53	50.25	37.95	21.65	− 5.75	25.26	19.51	4
9	52.90		10.27		32.25			29.06	4
	21.68	31.22	53.38	43.11	19.80	− 6.22	22.84	16.62	4
12									
13	52.19		17.29		35.21			35.28	½
	22.22	29.97	49.30	32.01	22.67	− 6.27	29.01	12.74	½
14	52.02		7.78		32.19			29.39	4
	22.37	29.65	51.76	43.98	19.78	− 6.20	23.19	16.99	4
19	50.86		18.62		42.02			38.14	3
	22.83	28.03	47.21	28.59	29.11	− 6.45	31.69	25.24	3
21	50.39		8.82		33.35			29.98	4
	22.99	27.40	54.02	45.20	20.79	− 6.28	23.70	17.42	4
22	50.16		6.33		32.34			29.74	4
	23.10	27.06	52.34	46.01	19.80	− 6.27	23.47	17.20	4
27	49.28		8.18		34.47			32.76	5
	23.84	25.44	49.89	41.71	21.81	− 6.33	26.43	20.10	5
28	49.10		5.95		32.83			29.80	3½
	23.98	25.12	53.43	47.48	20.62	− 6.10	23.70	17.60	3½

STARS NO. $\left\{ \begin{array}{l} 406 \; l. c. \\ 1179 \end{array} \right\}$ (*Con.*)

Date	δ			A'			C'			B			r'		$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$		r		p
	°	′	″	°	′	″	°	′	″	°	′	″	′	″	″	′	″	′	″	
June 29	+ 59	2	48.90				231	27	5.26				6	31.87				6	29.41	3½
	— 46	1	24.10	13	1	24.80	398	12	54.05	166	45	48.79	6	19.47	— 6.20	6	23.21	6	17.01	3½
30			48.67						5.28					32.94					29.06	3½
			24.22			24.45			55.33			50.05		20.32	— 6.31		22.75		16.44	3½
July 3			47.97					26	59.31					27.63					25.25	3½
			24.48			23.49			57.83			58.52		15.12	— 6.25		19.00		12.75	3½
4			47.74					27	4.37					31.72					30.05	4½
			24.58			23.16			53.94			49.57		18.90	— 6.41		23.64		17.23	4½
5			47.53						11.77					37.27					35.24	4
			24.68			22.85			51.19			39.42		24.53	— 6.37		28.87		22.50	4
6			47.32						9.39					39.46					35.21	3
			24.81			22.51			49.24			39.85		26.68	— 6.39		28.82		22.43	3

Stars No. { 444 *l. c.* / 1179 }

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 58 33 20.87		230 58 12.93		7 4.25			7 3.73	3
	− 46 1 21.48	12 31 59.39	398 12 46.55	167 14 33.62	6 23.79	− 20.23	6 43.50	6 23.27	3
8	20.64		5.04		7 0.37			6 57.22	4
	21.57	59.07	50.25	45.21	6 21.65	− 19.36	37.86	6 18.50	4
9	20.44		2.07		6 58.73			6 54.43	4
	21.68	58.76	53.38	51.31	6 19.80	− 19.46	34.97	6 15.51	4
12									
13	19.74		8.16		7 2.88			7 0.77	½
	22.22	57.52	49.30	41.14	6 22.67	− 20.10	40.67	6 20.57	½
14	19.57		5.71		6 59.02			6 58.00	3
	22.37	57.20	51.76	46.05	6 19.78	− 19.62	38.38	6 18.76	3
19	18.44		12.84		7 8.99			7 4.95	3
	22.83	55.61	47.21	34.37	6 29.11	− 19.94	45.01	6 25.07	3
21	17.97		3.92		7 0.41			6 57.27	4
	22.99	54.98	54.02	50.10	6 20.79	− 19.81	37.46	6 17.65	4
22	17.72		2.44		6 58.99			6 57.33	4
	23.10	54.62	52.34	49.90	6 19.80	− 19.59	37.74	6 18.15	4
27	16.78		2.10		7 1.48			6 59.47	5
	23.84	52.94	49.89	47.79	6 21.81	− 19.83	39.64	6 19.81	5
28	16.61		0.05		7 0.43			6 56.90	4
	23.98	52.63	53.43	53.38	6 20.62	− 19.90	37.00	6 17.10	4

STARS NO. { 444 *l. c.* / 1179 } (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 58 33 16.40		230 58 0.79		6 58.92			6 56.94	3
	− 46 1 24.10	12 31 52.30	398 12 54.05	167 14 53.26	6 19.47	− 19.72	6 37.22	6 17.50	3
30	16.17		0.83		6 59.55			6 56.39	3½
	24.22	51.95	55.33	54.50	6 20.32	− 19.61	36.78	6 17.17	3½
July 3	15.41		57 55.07		6 54.01			6 52.60	3
	24.48	50.93	57.83	62.76	6 15.12	− 19.44	33.16	6 13.72	3
4	15.17		57 58.10		6 57.66			6 56.17	4½
	24.58	50.59	53.94	55.84	6 18.90	− 19.38	36.79	6 17.41	4½
5	14.94		58 6.86		7 4.44			7 2.66	4
	24.68	50.26	51.19	44.33	6 24.53	− 19.95	42.71	6 22.76	4
6	14.71		58 6.84		7 7.10			7 4.06	2
	24.81	49.90	49.24	42.40	6 26.68	− 20.21	43.85	6 23.64	2

Stars No. $\left\{ \begin{array}{l} 424 \ l. c. \\ 1182 \end{array} \right.$

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+ 79 40 28.77		251 59 52.60		1 35.46			1 35.24	3
	− 25 28 36.31	54 11 52.46	377 44 50.41	125 44 57.81	1 34.72	− 0.37	1 34.87	1 34.50	3
8	28.47		51.05		34.62			34.48	4
	36.27	52.20	50.55	59.50	33.95	− 0.33	34.15	33.82	4
9	28.16		50.10		34.35			33.80	4
	36.27	51.89	51.25	61.15	33.71	− 0.32	33.48	33.16	4
12									
13	27.13		50.13		35.10			35.15	2½
	36.36	50.77	49.76	59.63	34.40	− 0.35	34.80	34.45	2½
14	26.85		50.49		34.41			35.46	4
	36.38	50.47	49.74	59.25	33.77	− 0.32	35.14	34.82	4
19	25.26		52.35		36.54			36.73	3
	36.25	49.01	50.53	44 58.18	35.89	− 0.32	36.41	36.09	3
21	24.64		47.26		34.64			34.78	4
	36.19	48.45	49.93	45 2.67	33.96	− 0.34	34.44	34.10	4
22	24.31		45.67		34.35			34.52	4
	36.17	48.14	49.16	3.49	33.68	− 0.33	34.19	33.86	4
27	22.95		43.76		34.90			34.83	5
	36.32	46.63	48.15	4.39	34.22	− 0.34	34.49	34.15	5
28	22.69		44.93		34.65			34.70	4
	36.35	46.34	49.90	4.97	33.95	− 0.35	34.35	34.00	4

Stars No. $\left\{ \begin{matrix} 424 \; l. c. \\ 1182 \end{matrix} \right\}$ (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 79 40 22.40		251 59 43.75		1 34.34			1 34.39	3
	− 25 28 36.37	54 11 46.03	377 44 49.62	125 45 5.87	1 33.65	− 0.34	1 34.05	1 33.71	3
30	22.10		43.70		34.51			34.26	3½
	36.35	45.75	50.09	6.39	33.85	− 0.33	33.93	33.60	3½
July 3	21.10		41.10		33.26			33.81	3
	36.26	44.84	49.89	8.21	32.59	− 0.33	33.48	33.15	3
4	20.77		42.48		34.13			34.09	5
	36.25	44.52	50.42	7.94	33.48	− 0.32	33.77	33.45	5
5	20.46		44.05		35.50			35.11	4
	36.23	44.23	50.27	6.22	34.83	− 0.33	34.78	34.45	4
6	20.16		44.16		36.02			35.86	3
	36.24	43.92	49.21	5.05	35.33	− 0.34	35.52	35.18	3

Stars No. { 438 *l. c.* / 1225

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 7	+77 6 25.30		249 26 0.93		1 46.90			1 46.86	3
	−27 48 57.74	49 17 27.56	380 5 1.55	130 39 0.62	1 44.99	−0.95	1 45.91	1 44.96	3
8	25.00		25 58.08		46.00			45.41	4
	57.70	27.30	1.82	3.74	44.13	−0.93	44.48	43.55	4
9	24.73		58.05		45.61			45.35	4
	57.67	27.06	2.17	4.12	43.73	−0.94	44.41	43.47	4
12									
13	23.73		58.23		46.54			46.48	1½
	57.72	26.01	1.09	2.86	44.72	−0.91	45.57	44.66	1½
14	23.46		58.61		45.68			46.76	2½
	57.75	25.71	1.15	2.54	43.91	−0.88	45.88	45.00	2½
19	21.96		59.53		48.02			47.73	3
	57.63	24.33	1.59	2.06	46.17	−0.92	46.81	45.89	3
21	21.34		54.93		46.00			46.39	4
	57.55	23.79	0.30	5.37	44.06	−0.97	45.42	44.45	4
22	21.01		53.62		45.65			45.73	4
	57.54	23.47	5 0.57	6.95	43.77	−0.94	44.79	43.85	4
27	19.69		51.56		46.27			46.15	5
	57.65	22.04	4 59.12	7.56	44.36	−0.95	45.20	44.25	5
28	19.44		52.52		46.01			46.42	4
	57.68	21.76	4 59.89	7.37	44.05	−0.98	45.44	44.46	4

STARS NO. { 438 *l. c.* / 1225 } (*Con.*)

Date	δ	A'	C'	B	r'	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A'+B)$	r	p
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″	
June 29	+ 77 6 19.15		249 25 51.85		1 45.69			1 45.72	3½
	— 27 48 57.70	49 17 21.45	380 5 0.82	130 39 8.97	1 43.83	— 0.93	1 44.79	1 43.86	3½
30	18.85		52.23		45.77			46.04	4
	57.71	21.14	0.90	8.67	43.89	— 0.94	45.10	44.16	4
July 3	17.88		49.50		44.47			44.85	3
	57.63	20.25	1.31	11.81	42.70	— 0.88	43.97	43.09	3
4	17.56		49.72		45.38			45.52	5
	57.61	19.95	0.72	11.00	43.40	— 0.99	44.53	43.54	5
5	17.24		52.23		46.94			47.05	4
	57.59	19.65	5 0.39	8.16	45.04	— 0.95	46.10	45.15	4
6	16.94		52.81		47.55			47.99	2
	57.60	19.34	4 59.38	6.57	45.66	— 0.94	47.05	46.11	2

The following tables contain the reductions for dlogr or its equivalent dlog*a*. The second column contains the logarithms of the computed refractions; the next column contains the logarithms of the observed refractions; the fourth the difference between the two preceding, in the sense of Observed—Computed; the column *p* contains the weights and the last column the weighted differences. The residuals and their weighted squares are not given. Log [*pvv*] is given in every case, as is also the resulting probable error of the weighted mean of every set. All of the results in the following tables have been checked.

STAR NO. 948.

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.42742	2.42617	— 0.00125	4	— 0.00500
8	2.42399	2.42119	— 280	4	— 1120
9	2.42126	2.41762	— 364	3	— 1092
12	2.43095	2.42727	— 368	1	— 368
13	2.42658	2.42490	— 168	2	— 336
14	2.42169	2.41838	— 331	3	— 993
19	2.43214	2.42889	— 325	3	— 975
21	2.42478	2.42185	— 293	4	— 1172
22	2.42149	2.42042	— 107	3½	— 375
27	2.42459	2.42243	— 216	5	— 1080
28	2.42313	2.42147	— 166	2½	— 415
29	2.42262	2.42014	— 248	3½	— 868
30					
July 3	2.41600	2.41678	+ 78	2½	+ 195
4	2.41916	2.41816	— 100	5	— 500
5	2.42644	2.42605	— 39	3	— 117
6	2.43042	2.42602	— 440	1½	— 660
				△	— 0.00205

[*p*] = 50½ ; *log* [*pvv*] = 5.8653

p. e. = ± 0.00015

STAR NO. 190 *l. c.*

Date	*log. r'*	*log. r*	Δ	*p*	*p* Δ
June 7					
8					
9	3.09609	3.08699	— 0.00910	3	— 0.02730
12	3.10852	3.09598	— 1254	1	— 1254
13	3.10231	3.10302	+ 71	½	+ 35
14	3.09619	3.09129	— 490	3	— 1470
19					
21	3.10009	3.09349	— 660	4	— 2640
22	3.09543	3.09387	— 156	3½	— 546
27	3.09851	3.09629	— 222	5	— 1110
28	3.09793	3.09046	— 747	3	— 2241
29	3.09748	3.08610	— 1138	3	— 3414
30	3.09728	3.09563	— 165	4	— 660
July 3					
4	3.09289	3.08925	— 364	5	— 1820
5	3.10261	3.09713	— 548	3	— 1644
6					
				Δ	— 0.00513

$[p] = 38$; $\log\,[pvv] = 6.6112$

p. e. $= \pm\ 0.00047$

STAR NO. 959.—(With 190 *l. c.*)

Date	*log. r'*	*log. r*	Δ	*p*	*p* Δ
June 7					
8					
9	3.03783	3.02741	— 0.01042	3	— 0.03126
12	3.05008	3.03571	— 1437	1	— 1437
13	3.04374	3.04454	+ 80	½	+ 40
14	3.03714	3.03175	— 539	3	— 1617
19					
21	3.04205	3.03451	— 754	4	— 3016
22	3.03782	3.03605	— 177	3½	— 619
27	3.04138	3.03886	— 252	5	— 1260
28	3.04032	3.03177	— 855	3	— 2565
29	3.03986	3.02685	— 1301	3	— 3903
30	3.03983	3.03795	— 188	4	— 752
July 3					
4	3.03667	3.03252	— 415	5	— 2075
5	3.04442	3.03815	— 627	3	— 1881
6					
				Δ	— 0.00584

$[p] = 38$; $\log\,[pvv] = 6.7298$

p. e. $= \pm\ 0.00053$

Star No. 959.—(With 282 *l. c.*)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	3.04492	3.04068	− 0.00424	4½	− 0.01908
8	3.04160	3.03886	− 274	3½	− 959
9	3.03783	3.03050	− 733	3	− 2199
12	3.05008	3.04703	− 305	1	− 305
13					
14	3.03714	3.03633	− 81	3	− 243
19	3.05165	3.04787	− 378	3	− 1134
21	3.04205	3.03741	− 464	4	− 1856
22	3.03782	3.03645	− 137	3½	− 479
27	3.04138	3.03717	− 421	5	− 2105
28	3.04032	3.03228	− 804	2½	− 2010
29	3.03986	3.03043	− 943	3	− 2829
30	3.03983	3.03611	− 372	4½	− 1674
July 3					
4	3.03667	3.03112	− 555	5	− 2775
5	3.04442	3.03910	− 532	3½	− 1862
6	3.04920	3.04347	− 573	2½	− 1432
				△	− 0.00462

[p] = 51½ ; *log* [pvv] = 6.3662

p. e. = ± 0.00027

Star No. 968.

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	1.45651	1.46180	+ 0.00529	4	+ 0.02116
8	1.45383	1.46553	+ 1170	4	+ 4680
9	1.45149	1.45637	+ 488	3	+ 1464
12	1.45985	1.46790	+ 805	1	+ 805
13	1.45614	1.46225	+ 611	1½	+ 916
14	1.45172	1.45758	+ 586	3	+ 1758
19	1.46276	1.46879	+ 603	3	+ 1809
21	1.45485	1.46374	+ 889	3½	+ 3111
22	1.45181	1.46120	+ 939	3½	+ 3286
27	1.45453	1.45652	+ 199	5	+ 995
28	1.45407	1.46060	+ 653	2½	+ 1632
29	1.45340	1.45984	+ 644	3½	+ 2254
30					
July 2	1.44653	1.45347	+ 694	3	+ 2082
4	1.45114	1.45408	+ 294	5	+ 1470
5	1.45656	1.46835	+ 1179	3	+ 3537
6	1.46066	1.46850	+ 784	1½	+ 1176
				△	+ 0.00662

[p] = 50 ; *log* [pvv] = 6.6307

p. e. = ± 0.00036

STAR NO. 977.

Date	log. r′	log. r	△	p	p △
June 7	1.28488	1.28488	± 0.00000	4	± 0.00000
8	1.28218	1.28892	+ 674	4	+ 2696
9	1.28019	1.28126	+ 107	3	+ 321
12	1.28796	1.29270	+ 474	1	+ 474
13	1.28439	1.29491	+ 1052	1½	+ 1578
14	1.28022	1.29092	+ 1070	3	+ 3210
19	1.29163	1.29336	+ 173	3	+ 519
21	1.28336	1.28758	+ 422	3½	+ 1477
22	1.28043	1.28149	+ 106	3½	+ 371
27	1.28302	1.28758	+ 456	5	+ 2280
28	1.28288	1.28533	+ 245	2½	+ 612
29	1.28206	1.28691	+ 485	3½	+ 1697
30	1.28204	1.28648	+ 444	4	+ 1776
July 3	1.27518	1.28171	+ 653	3	+ 1959
4	1.28003	1.28758	+ 755	5	+ 3775
5	1.28512	1.29623	+ 1111	3	+ 3333
6	1.28948	1.28870	— 78	2	— 156
				△	+ 0.00476

[p] = 54½ ; log [pvv] = 6.7951

p. e. = ± 0.00040

STAR NO. 984.

Date	log. r′	log. r	△	p	p △
June 7					
8					
9					
12	2.35626	2.35199	— 0.00427	1	— 0.00427
13	2.35336	2.35114	— 222	2½	— 555
14	2.34902	2.34616	— 286	3	— 858
19	2.36071	2.35971	— 100	3	— 300
21	2.35175	2.35042	— 133	3	— 399
22	2.34937	2.34805	— 132	3½	— 462
27	2.35175	2.34951	— 224	5	— 1120
28	2.35162	2.35176	+ 14	2	+ 28
29	2.35060	2.34874	— 186	4	— 744
30					
July 3	2.34417	2.34616	+ 199	3	+ 597
4	2.34844	2.34713	— 131	5	— 655
5	2.35414	2.35319	— 95	3	— 285
6	2.35871	2.35581	— 290	1½	— 435
				△	— 0.00142

[p] = 39½ ; log [pvv] = 5.8091

p. e. = ± 0.00017

STAR No. 225 *l. c.*—(With 948.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.41352	2.41224	− 0.00128	4	− 0.00512
8	2.41100	2.40812	− 288	4	− 1152
9	2.40895	2.40523	− 372	3	− 1116
12	2.41618	2.41237	− 381	1	− 381
13	2.41359	2.41187	− 172	2	− 344
14	2.40930	2.40591	− 339	3	− 1017
19	2.42102	2.41769	− 333	3	− 999
21	2.41181	2.40880	− 301	4	− 1204
22	2.40967	2.40858	− 109	3½	− 381
27	2.41188	2.40966	− 222	5	− 1110
28	2.41175	2.41007	− 168	2½	− 420
29	2.41072	2.40815	− 257	3½	− 899
30					
July 3	2.40463	2.40542	+ 79	2½	+ 197
4	2.40845	2.40741	− 104	5	− 520
5	2.41441	2.41400	− 41	3	− 123
6	2.41907	2.41457	− 450	1½	− 675
				△	− 0.00211

[*p*] = 50½ ; *log* [*pvv*] = 5.8809

p. e. = ± 0.00015

STAR No. 225 *l. c.*—(With 984.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7					
8					
9					
12	2.41618	2.41246	− 0.00372	1	− 0.00372
13	2.41359	2.41165	− 194	2½	− 485
14	2.40930	2.40678	− 252	3	− 756
19	2.42102	2.42014	− 88	3	− 264
21	2.41181	2.41066	− 115	3	− 345
22	2.40967	2.40849	− 118	3½	− 413
27	2.41188	2.40983	− 205	5	− 1025
28	2.41175	2.41186	+ 11	2	+ 22
29	2.41072	2.40914	− 158	4	− 632
30					
July 3	2.40463	2.40637	+ 174	3	+ 522
4	2.40845	2.40729	− 116	5	− 580
5	2.41441	2.41357	− 84	3	− 252
6	2.41907	2.41652	− 255	1½	− 382
				△	− 0.00126

[*p*] = 39½ ; *log* [*pvv*] = 5.6934

p. e. = ± 0.00015

STAR NO. 225 *l. c.*—(With 1135.)

Date	log. r′	log. r	△	p	p △
June 7	2.41352	2.41299	— 0.00053	4½	— 0.00238
8	2.41100	2.40928	— 172	4	— 688
9	2.40895	2.40601	— 294	3	— 882
12	2.41618	2.41416	— 202	1	— 202
13	2.41359	2.41105	— 254	2	— 508
14	2.40930	2.40724	— 206	3	— 618
19	2.42102	2.41858	— 244	3	— 732
21	2.41181	2.41100	— 81	4	— 324
22	2.40967	2.40882	— 85	4	— 340
27	2.41188	2.40981	— 207	5	— 1035
28	2.41175	2.41100	— 75	2½	— 187
29	2.41072	2.40744	— 328	4	— 1312
30					
July 3	2.40463	2.40488	+ 25	3	+ 75
4	2.40845	2.40691	— 154	5	— 770
5	2.41441	2.41437	— 4	3½	— 14
6	2.41907	2.41547	— 360	2	— 720
				△	— 0.00159

[p] = 53½ ; *log* [pvv] = 5.7856

p. e. = ± 0.00013

STAR NO. 997.

Date	log. r′	log. r	△	p	p △
June 7	1.45932	1.46479	+ 0.00547	4	+ 0.02188
8	1.45705	1.46879	+ 1174	4	+ 4696
9	1.45504	1.45969	+ 465	3	+ 1395
12	1.46180	1.46967	+ 787	1	+ 787
13	1.45953	1.46553	+ 600	1½	+ 900
14	1.45544	1.46120	+ 576	3	+ 1728
19	1.46676	1.47261	+ 585	3	+ 1755
21	1.45774	1.46642	+ 868	3½	+ 3038
22	1.45588	1.46509	+ 921	3½	+ 3223
27	1.45791	1.45984	+ 193	5	+ 965
28	1.45782	1.46419	+ 637	2½	+ 1592
29	1.45671	1.46315	+ 644	3½	+ 2254
30					
July 3	1.45099	1.45773	+ 674	3	+ 2022
4	1.45436	1.45712	+ 276	5	+ 1380
5	1.46049	1.47217	+ 1168	3	+ 3504
6	1.46492	1.47261	+ 769	1½	+ 1153
				△	+ 0.00652

[p] = 50 ; *log* [pvv] = 6.6325

p. e. = ± 0.00036

Star No. 1005.—(With 264 *l. c.*)

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7	1.87711	1.87518	— 0.00193	4	— 0.00772
8	1.87496	1.87186	— 310	4	— 1240
9	1.87269	1.86700	— 569	3	— 1707
12	1.87989	1.87697	— 292	1	— 292
13	1.87719	1.87361	— 358	2	— 716
14	1.87307	1.87454	+ 147	3	+ 441
19	1.88455	1.88138	— 317	3	— 951
21	1.87581	1.87233	— 348	4	— 1392
22	1.87378	1.86964	— 414	4	— 1656
27	1.87582	1.87489	— 93	5	— 465
28	1.87565	1.87093	— 472	2	— 944
29	1.87436	1.87256	— 180	4	— 720
30	1.87545	1.87489	— 56	4	— 224
July 3	1.86915	1.86847	— 68	3	— 204
4	1.87220	1.87157	— 63	5	— 315
5	1.87863	1.87823	— 40	3	— 120
6	1.88269	1.87881	— 388	2	— 776
				△	— 0.00215

[*p*] = 56 ; *log* [*pvv*] = 6.2452

p. e. = ± 0.00021

Star No. 1005.—(With 356 *l. c.*)

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7					
8					
9	1.87269	1.87233	— 0.00036	3	— 0.00108
12	1.87989	1.87800	— 189	1	— 189
13	1.87719	1.87547	— 172	2½	— 430
14	1.87307	1.87355	+ 48	3	+ 144
19	1.88455	1.88502	+ 47	3	+ 141
21	1.87581	1.87413	— 168	4	— 672
22	1.87378	1.87268	— 110	4	— 440
27	1.87582	1.87743	+ 161	5	+ 805
28	1.87565	1.87425	— 140	2	— 280
29	1.87436	1.87512	+ 76	4	+ 304
30	1.87545	1.87714	+ 169	4	+ 676
July 3	1.86915	1.86788	— 127	3	— 381
4	1.87220	1.87326	+ 106	5	+ 530
5	1.87863	1.87547	— 316	3½	— 1106
6	1.88269	1.88064	— 205	2½	— 512
				△	— 0.00031

[*p*] = 49½ ; *log* [*pvv*] = 6.0442

p. e. = ± 0.00019

STAR NO. 1009.

Date	log. r′	log. r	△	p	p △
June 7	1.27967	1.27600	— 0.00367	4	— 0.01468
8	1.27759	1.28149	+ 390	4	+ 1560
9	1.27515	1.28149	+ 634	3	+ 1902
12	1.28250	1.28240	— 10	1	— 10
13	1.27963	1.28307	+ 344	2	+ 688
14	1.27547	1.28262	+ 715	3	+ 2145
19	1.28706	1.28825	+ 119	3	+ 357
21	1.27843	1.27989	+ 146	4	+ 584
22	1.27648	1.27346	— 302	4	— 1208
27	1.27834	1.28511	+ 677	5	+ 3385
28	1.27819	1.28126	+ 307	2	+ 614
29	1.27674	1.28149	+ 475	4	+ 1900
30	1.27815	1.28375	+ 560	4	+ 2240
July 3	1.27187	1.27300	+ 113	3	+ 339
4	1.27473	1.27807	+ 334	5	+ 1670
5	1.28132	1.28466	+ 334	3	+ 1002
6	1.28519	1.29003	+ 484	2	+ 968
				△	+ 0.00298

$[p] = 56$; $log\,[pvv] = 6.7629$

$p.\,e. = \pm\ 0.00038$

Star No. 1019—(With 977.)

Date	log. r′	log. r	△	p	p △
June 7	1.28286	1.28307	+ 0.00021	4	+ 0.00084
8	1.28075	1.28758	+ 683	4	+ 2732
9	1.27851	1.27987	+ 136	3	+ 408
12	1.28578	1.29048	+ 470	1	+ 470
13	1.28283	1.29358	+ 1075	1½	+ 1612
14	1.27867	1.28959	+ 1092	3	+ 3276
19	1.29036	1.29203	+ 167	3	+ 501
21	1.28165	1.28623	+ 458	3½	+ 1603
22	1.27970	1.28103	+ 133	3½	+ 465
27	1.28158	1.28623	+ 465	5	+ 2325
28	1.28158	1.28398	+ 240	2½	+ 600
29	1.27989	1.28466	+ 477	3½	+ 1669
30	1.28139	1.28601	+ 462	4	+ 1848
July 3	1.27536	1.28171	+ 635	3	+ 1905
4	1.27798	1.28578	+ 780	5	+ 3900
5	1.28468	1.29579	+ 1111	3	+ 3333
6	1.28837	1.28780	— 57	2	— 114
				△	+ 0.00488

$[p] = 54½$; $log\,[pvv] = 6.7903$

$p.\,e. = \pm\ 0.00040$

Star No. 1019—(With 1009.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	1.28286	1.27921	— 0.00365	4	— 0.01460
8	1.28075	1.28466	+ 391	4	+ 1564
9	1.27851	1.28466	+ 615	3	+ 1845
12	1.28578	1.28556	— 22	1	— 22
13	1.28283	1.28623	+ 340	2	+ 680
14	1.27867	1.28578	+ 711	3	+ 2133
19	1.29036	1.29137	+ 101	3	+ 303
21	1.28165	1.28307	+ 142	4	+ 568
22	1.27970	1.27669	— 301	4	— 1204
27	1.28158	1.28825	+ 667	5	+ 3335
28	1.28158	1.28443	+ 285	2	+ 570
29	1.27989	1.28466	+ 477	4	+ 1908
30	1.28139	1.28691	+ 552	4	+ 2208
July 3	1.27536	1.27623	+ 87	3	+ 261
4	1.27798	1.28126	+ 328	5	+ 1640
5	1.28468	1.28780	+ 312	3	+ 936
6	1.28837	1.29314	+ 477	2	+ 954
				△	+ 0.00290

[*p*] = 56 ; *log* [*pvv*] = 6.7654

p. e. = ± 0.00038

Star No. 264 *l. c.*

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	1.87981	1.87783	— 0.00198	4	— 0.00792
8	1.87779	1.87466	— 313	4	— 1252
9	1.87564	1.86994	— 570	3	— 1710
12	1.88285	1.87996	— 289	1	— 289
13	1.87988	1.87628	— 360	2	— 720
14	1.87560	1.87697	+ 137	3	+ 411
19	1.88743	1.88423	— 320	3	— 960
21	1.87852	1.87500	— 352	4	— 1408
22	1.87657	1.87245	— 412	4	— 1648
27	1.87853	1.87754	— 99	5	— 495
28	1.87878	1.87408	— 470	2	— 940
29	1.87673	1.87489	— 184	4	— 736
30	1.87825	1.87766	— 59	4	— 236
July 3	1.87237	1.87163	— 74	3	— 222
4	1.87492	1.87425	— 67	5	— 335
5	1.88164	1.88121	— 43	3	— 129
6	1.88517	1.88133	— 384	2	— 768
				△	— 0.00218

[*p*] = 56 ; *log* [*pvv*] = 6.2338

p. e. = ± 0.00020

STAR NO. 1032.

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7					
8					
9					
12					
13	2.85820	2.86091	+ 0.00271	1	+ 0.00271
14	2.85280	2.85044	— 236	3½	— 826
19	2.86657	2.86178	— 479	3	— 1437
21	2.85637	2.85442	— 195	4	— 780
22	2.85411	2.85258	— 153	4	— 612
27	2.85620	2.85429	— 191	5	— 955
28	2.85690	2.85286	— 404	2½	— 1010
29	2.85421	2.84821	— 600	3½	— 2100
30	2.85600	2.84956	— 644	4	— 2576
July 3	2.84957	2.84596	— 361	3	— 1083
4	2.85239	2.85174	— 65	4½	— 292
5	2.86022	2.85728	— 294	4	— 1176
6	2.86417	2.85807	— 610	2½	— 1525
				△	— 0.00317

[*p*] = 44½ ; *log* [*pvv*] = 6.2854

p. e. = ± 0.00029

STAR NO. 282 *l. c.*

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7	3.03250	3.02825	— 0.00425	4½	— 0.01912
8	3.02963	3.02682	— 281	3½	— 983
9	3.02804	3.02054	— 750	3	— 2250
12	3.03690	3.03376	— 314	1	— 314
13					
14	3.02661	3.02556	— 105	3	— 315
19	3.04241	3.03855	— 386	3	— 1158
21	3.03063	3.02586	— 477	4	— 1908
22	3.02814	3.02674	— 140	3½	— 490
27	3.03018	3.02586	— 432	5	— 2160
28	3.03140	3.02319	— 821	2½	— 2052
29	3.02815	3.01847	— 968	3	— 2904
30	3.02989	3.02608	— 381	4½	— 1714
July 3					
4	3.02636	3.02069	— 567	5	— 2835
5	3.03511	3.02968	— 543	3½	— 1900
6	3.03907	3.03320	— 587	2½	— 1467
				△	— 0.00473

[*p*] = 51½ ; *log* [*pvv*] = 6.3770

p. e. = ± 0.00027

STAR NO. 1084.

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7	1.40959	1.41263	+ 0.00304	5	+ 0.01520
8	1.40788	1.40157	— 631	4	— 2524
9	1.40678	1.40790	+ 112	3	+ 336
12	1.41235	1.41162	— 73	1	— 73
13	1.41026	1.41145	+ 119	2	+ 238
14	1.40551	1.40381	— 170	3	— 510
19	1.41774	1.41681	— 93	3	— 279
21	1.40831	1.40976	+ 145	4	+ 1256
22	1.40704	1.40432	— 272	4	— 1088
27	1.40883	1.40926	+ 43	5	+ 215
28	1.40783	1.40892	+ 109	2	+ 218
29	1.40587	1.40500	— 87	3	— 261
30	1.40780	1.41010	+ 230	4	+ 920
July 3	1.40181	1.39863	— 318	3	— 954
4	1.40565	1.40552	— 13	5	— 65
5	1.41159	1.40449	— 710	4	— 2840
6	1.41433	1.40993	— 440	3	— 1320
				△	— 0.00101

[*p*] = 58 ; *log* [*pvv*] = 6.7152

p. e. = ± 0.00035

STAR NO. 1094.

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7	1.42207	1.42488	+ 0.00281	5	+ 0.01405
8	1.42036	1.41414	— 622	4	— 2488
9	1.41943	1.42062	+ 119	3	+ 357
12	1.42492	1.42423	— 69	1	— 69
13	1.42275	1.42374	+ 99	2	+ 198
14	1.41809	1.41631	— 178	3	— 534
19	1.43008	1.42894	— 114	3	— 342
21	1.42082	1.42210	+ 128	4	+ 512
22	1.41979	1.41714	— 265	4	— 1060
27	1.42185	1.42226	+ 41	5	+ 205
28	1.41995	1.42095	+ 100	2	+ 200
29	1.41807	1.41714	— 93	3	— 279
30	1.42033	1.42243	+ 210	4	+ 840
July 3	1.41401	1.41095	— 306	3	— 918
4	1.41844	1.41830	— 14	5	— 70
5	1.42416	1.41731	— 685	4	— 2740
6	1.42679	1.42259	— 420	3	— 1260
				△	— 0.00104

[*p*] = 58 ; *log* [*pvv*] = 6.6817

p. e. = ± 0.00034

STAR NO. 1105.

Date	log. r′	log. r	△	p	p △
June 7					
8					
9	1.96122	1.95861	− 0.00261	3½	− 0.00913
12					
13	1.96464	1.96209	− 255	2	− 510
14	1.96010	1.96468	+ 458	3½	+ 1603
19	1.97190	1.97257	+ 67	3	+ 201
21	1.96285	1.96137	− 148	4	− 592
22	1.96173	1.96114	− 59	4	− 236
27	1.96404	1.96161	− 243	5	− 1215
28	1.96169	1.96223	+ 54	3	+ 162
29	1.95990	1.95985	− 5	3	− 15
30	1.96218	1.96142	− 76	4½	− 342
July 3	1.95599	1.95650	+ 51	3	+ 153
4	1.96028	1.96099	+ 71	5	+ 355
5	1.96610	1.96577	− 33	4	− 132
6	1.96874	1.96876	+ 2	3	+ 6
				△	− 0.00029

$[p] = 50½$; $log\ [pvv] = 6.1881$

$p.\ e. = \pm\ 0.00023$

STAR NO. 1110.

Date	log. r′	log. r	△	p	p △
June 7	2.05864	2.05998	+ 0.00134	5	+ 0.00670
8	2.05683	2.05790	+ 107	4	+ 428
9	2.05574	2.05342	− 232	3	− 696
12	2.06194	2.06017	− 177	1	− 177
13	2.05916	2.05888	− 28	1	− 28
14	2.05485	2.05778	+ 293	3	+ 879
19	2.06659	2.06446	− 213	3	− 639
21	2.05744	2.05637	− 107	4	− 428
22	2.05644	2.05319	− 325	4	− 1300
27	2.05892	2.05775	− 117	5	− 585
28	2.05622	2.05599	− 23	2	− 46
29	2.05450	2.05385	− 65	4	− 260
30	2.05679	2.05427	− 252	4½	− 1134
July 3	2.05066	2.04895	− 171	2½	− 427
4	2.05488	2.05484	− 4	5	− 20
5	2.06079	2.05964	− 115	4	− 460
6	2.06344	2.06119	− 225	3	− 675
				△	− 0.00084

$[p] = 58$; $log\ [pvv] = 6.1535$

$p.\ e. = \pm\ 0.00018$

STAR NO. 349 *l. c.*

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.07053	2.07177	+ 0.00124	5	+ 0.00620
8	2.06871	2.06974	+ 103	4	+ 412
9	2.06761	2.06539	— 222	3	— 666
12	2.07385	2.07210	— 175	1	— 175
13	2.07104	2.07078	— 26	1	— 26
14	2.06690	2.06971	+ 281	3	+ 843
19	2.07824	2.07613	— 211	3	— 633
21	2.06933	2.06826	— 107	4	— 428
22	2.06834	2.06517	— 317	4	— 1268
27	2.07091	2.06974	— 117	5	— 585
28	2.06804	2.06781	— 23	2	— 46
29	2.06647	2.06580	— 67	4	— 268
30	2.06870	2.06622	— 248	4½	— 1116
July 3	2.06268	2.06104	— 164	2½	— 410
4	2.06687	2.06685	— 2	5	— 10
5	2.07272	2.07159	— 113	4	— 452
6	2.07535	2.07316	— 219	3	— 657
				△	— 0.00084

$[p] = 58$; $log\ [pvv] = 6.1255$

$p.\ e. = \pm\ 0.00018$

STAR NO. 356 *l. e.*

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7					
8					
9	1.87922	1.87881	— 0.00041	3	— 0.00123
12	1.88536	1.88349	— 187	1	— 187
13	1.88248	1.88076	— 172	2½	— 430
14	1.87854	1.87898	+ 44	3	+ 132
19	1.88965	1.89009	+ 44	3	+ 132
21	1.88077	1.87910	— 167	4	— 668
22	1.87968	1.87858	— 110	4	— 440
27	1.88234	1.88395	+ 161	5	+ 805
28	1.87948	1.87806	— 142	2	— 284
29	1.87811	1.87881	+ 70	4	+ 280
30	1.88005	1.88173	+ 168	4	+ 672
July 3	1.87414	1.87280	— 134	3	— 402
4	1.87840	1.87938	+ 98	5	+ 490
5	1.88413	1.88098	— 315	3½	— 1102
6	1.88677	1.88474	— 203	2½	— 507
				△	— 0.00033

$[p] = 49\frac{1}{2}$; $log\ [pvv] = 6.0359$

$p.\ e. = \pm\ 0.00019$

STAR NO. 1135.

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.32700	2.32638	− 0.00062	4½	− 0.00279
8	2.32535	2.32327	− 208	4	− 832
9	2.32443	2.32087	− 356	3	− 1068
12	2.33060	2.32816	− 244	1	− 244
13	2.32766	2.32457	− 309	2	− 618
14	2.32400	2.32149	− 251	3	− 753
19	2.33465	2.33169	− 296	3	− 888
21	2.32578	2.32479	− 99	4	− 396
22	2.32463	2.32362	− 101	4	− 404
27	2.32747	2.32496	− 251	5	− 1255
28	2.32461	2.32368	− 93	2½	− 232
29	2.32347	2.31946	− 401	4	− 1604
30					
July 3	2.31931	2.31962	+ 31	3	+ 93
4	2.32378	2.32193	− 185	5	− 925
5	2.32932	2.32927	− 5	3½	− 17
6	2.33200	2.32760	− 440	2	− 880
				△	− 0.00193

[*p*] = 53½ ; *log* [*pvv*] = 5.9589

p. e. = ± 0.00016

STAR NO. 377 *l.* c.—(With 1032.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7					
8					
9					
12					
13	2.83400	2.83705	+ 0.00305	1	+ 0.00305
14	2.82985	2.82736	− 249	3½	− 871
19	2.84207	2.83701	− 506	3	− 1518
21	2.83180	2.82974	− 206	4	− 824
22	2.83075	2.82914	− 161	4	− 644
27	2.83356	2.83148	− 208	5	− 1040
28	2.83035	2.82600	− 435	2½	− 1087
29	2.82931	2.82296	− 635	3½	− 2222
30	2.83107	2.82425	− 682	4	− 2728
July 3	2.82480	2.82096	− 384	3	− 1152
4	2.82968	2.82900	− 68	4½	− 306
5	2.83608	2.83298	− 310	4	− 1240
6	2.83912	2.83265	− 647	2½	− 1617
				△	− 0.00336

[*p*] = 44½ ; *log* [*pvv*] = 6.2716

p. e. = ± 0.00028

Star No. 377 *l. c.*—(With 1156.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.83328	2.83115	— 0.00213	5	— 0.01065
8	2.83152	2.82838	— 314	4	— 1256
9	2.83045	2.82720	— 325	3	— 975
12					
13	2.83400	2.83306	— 94	½	— 47
14	2.82985	2.82607	— 378	3½	— 1323
19	2.84207	2.83464	— 743	3	— 2229
21	2.83180	2.83048	— 132	4	— 528
22	2.83075	2.82840	— 235	4	— 940
27	2.83356	2.83081	— 275	5	— 1375
28	2.83035	2.82647	— 388	3	— 1164
29	2.82931	2.82312	— 619	4	— 2476
30	2.83107	2.82833	— 274	4	— 1096
July 3	2.82480	2.82124	— 356	3	— 1068
4	2.82968	2.82730	— 238	5	— 1190
5	2.83608	2.83239	— 369	3½	— 1291
6	2.83912	2.83502	— 410	2½	— 1025
				△	— 0.00334

$[p] = 57$; $log\ [pvv] = 6.0815$

p. e. = ± 0.00018

Star No. 1156.

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.87524	2.87330	— 0.00194	5	— 0.00970
8	2.87296	2.87010	— 286	4	— 1144
9	2.87172	2.86876	— 296	3	— 888
12					
13	2.87557	2.87471	— 86	½	— 43
14	2.87138	2.86795	— 343	3½	— 1200
19	2.88372	2.87698	— 674	3	— 2022
21	2.87302	2.87182	— 120	4	— 480
22	2.87193	2.86980	— 213	4	— 852
27	2.87448	2.87197	— 251	5	— 1255
28	2.87189	2.86837	— 352	3	— 1056
29	2.87078	2.86516	— 562	4	— 2248
30	2.87246	2.86996	— 250	4	— 1000
July 3	2.86594	2.86271	— 323	3	— 969
4	2.87134	2.86918	— 216	5	— 1080
5	2.87773	2.87437	— 336	3½	— 1176
6	2.88028	2.87655	— 373	2½	— 933
				△	— 0.00304

$[p] = 57$; $log\ [pvv] = 6.0105$

p. e. = ± 0.00016

STAR NO. 1162.—(With 406 *l. c.*)

Date	log. r′	log. r	Δ	p	p Δ
June 7					
8					
9	2.57666	2.57349	— 0.00317	3½	— 0.01109
12					
13	2.57998	2.57897	— 101	½	— 50
14	2.57649	2.57339	— 310	3½	— 1085
19	2.58742	2.58196	— 546	3	— 1638
21	2.57789	2.57491	— 298	4	— 1192
22	2.57681	2.57474	— 207	4	— 828
27	2.57924	2.57728	— 196	5	— 980
28	2.57699	2.57364	— 335	3½	— 1173
29	2.57604	2.57302	— 302	4	— 1208
30	2.57739	2.57226	— 513	4	— 2052
July 3	2.57156	2.56791	— 365	3½	— 1277
4	2.57727	2.57417	— 310	5	— 1550
5	2.58212	2.58006	— 206	3½	— 721
6	2.58448	2.57905	— 543	3	— 1629
				Δ	— 0.00330

$[p] = 50$; $log\ [pvv] = 5.8169$

$p.\ e. = \pm\ 0.00015$

STAR NO. 1162—(With 444 *l. c.*)

Date	log. r′	log. r	Δ	p	p Δ
June 7					
8					
9	2.57666	2.57219	— 0.00447	3½	— 0.01564
12					
13	2.57998	2.57648	— 350	½	— 175
14	2.57649	2.57545	— 104	2½	— 260
19	2.58742	2.58178	— 564	3	— 1692
21	2.57789	2.57518	— 271	4	— 1084
22	2.57681	2.57583	— 98	4	— 392
27	2.57924	2.57694	— 230	5	— 1150
28	2.57699	2.57306	— 393	4	— 1572
29	2.57604	2.57358	— 246	3½	— 861
30	2.57739	2.57309	— 430	4	— 1720
July 3	2.57156	2.56904	— 252	3	— 756
4	2.57727	2.57438	— 289	5	— 1445
5	2.58212	2.58035	— 177	3½	— 619
6	2.58448	2.58043	— 405	2	— 810
				Δ	— 0.00297

$[p] = 47½$; $log\ [pvv] = 5.8851$

$p.\ e.\ \pm\ 0.00017$

Star No. 406 *l. c.*—(With 1162.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7					
8					
9	2.59357	2.59052	— 0.00305	3½	— 0.01067
12					
13	2.59683	2.59586	— 97	½	— 49
14	2.59350	2.59052	— 298	3½	— 1043
19	2.60425	2.59900	— 525	3	— 1575
21	2.59477	2.59191	— 286	4	— 1144
22	2.59366	2.59165	— 201	4	— 804
27	2.59602	2.59413	— 189	5	— 945
28	2.59420	2.59097	— 323	3½	— 1130
29	2.59315	2.59022	— 293	4	— 1172
30	2.59432	2.58939	— 493	4	— 1972
July 3	2.58842	2.58491	— 351	3½	— 1229
4	2.59297	2.58997	— 300	5	— 1500
5	2.59909	2.59711	— 198	3½	— 693
6	2.60148	2.59625	— 523	3	— 1569
				△	— 0.00318

$[p] = 50$; *log* $[pvv] = 5.7810$

$p.\ e. = \pm\ 0.00015$

Star No. 406 *l. c.*—(With 1179.)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.59730	2.59537	— 0.00193	4	— 0.00772
8	2.59456	2.59219	— 237	4	— 948
9	2.59357	2.59002	— 355	4	— 1420
12					
13	2.59683	2.59691	+ 8	½	+ 4
14	2.59350	2.59039	— 311	4	— 1244
19	2.60425	2.60003	— 422	3	— 1266
21	2.59477	2.59104	— 373	4	— 1492
22	2.59366	2.59077	— 289	4	— 1156
27	2.59602	2.59413	— 189	5	— 945
28	2.59420	2.59084	— 336	3½	— 1176
29	2.59315	2.59041	— 274	3½	— 959
30	2.59432	2.59002	— 430	3½	— 1505
July 3	2.58842	2.58574	— 268	3½	— 938
4	2.59297	2.59112	— 185	4½	— 832
5	2.59909	2.59686	— 223	4	— 892
6	2.60148	2.59683	— 465	3	— 1395
				△	— 0.00292

$[p] = 58$; *log* $[pvv] = 5.6978$

$p.\ e. = \pm\ 0.00011$

STAR NO. 1179.—(With 406 *l. c.*)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.58409	2.58211	— 0.00198	4	— 0.00792
8	2.58167	2.57922	— 245	4	— 980
9	2.57955	2.57590	— 365	4	— 1460
12					
13	2.58282	2.58290	+ 8	½	+ 4
14	2.57953	2.57633	— 320	4	— 1280
19	2.59007	2.58573	— 434	3	— 1302
21	2.58069	2.57682	— 387	4	— 1548
22	2.57955	2.57657	— 298	4	— 1192
27	2.58185	2.57990	— 195	5	— 975
28	2.58049	2.57703	— 346	3½	— 1211
29	2.57918	2.57635	— 283	3½	— 991
30	2.58015	2.57569	— 446	3½	— 1561
July 3	2.57417	2.57142	— 275	3½	— 962
4	2.57852	2.57661	— 191	4½	— 860
5	2.58493	2.58263	— 230	4	— 920
6	2.58735	2.58255	— 480	3	— 1440
				△	— 0.00301

[*p.*] = 58 ; *log* [*pvv*] = 5.7112

p. e. = ± 0.00011

STAR NO. 1179—(With 444 *l. c.*)

Date	*log. r′*	*log. r*	△	*p*	*p* △
June 7	2.58409	2.58351	— 0.00058	3	— 0.00174
8	2.58167	2.57807	— 360	4	— 1440
9	2.57955	2.57462	— 493	4	— 1972
12					
13	2.58282	2.58043	— 239	½	— 119
14	2.57953	2.57837	— 116	3	— 348
19	2.59007	2.58554	— 453	3	— 1359
21	2.58069	2.57709	— 360	4	— 1440
22	2.57955	2.57766	— 189	4	— 756
27	2.58185	2.57956	— 229	5	— 1145
28	2.58049	2.57646	— 403	4	— 1612
29	2.57918	2.57692	— 226	3	— 678
30	2.58015	2.57654	— 361	3½	— 1264
July 3	2.57417	2.57254	— 163	3	— 489
4	2.57852	2.57681	— 171	4½	— 769
5	2.58493	2.58293	— 200	4	— 800
6	2.58735	2.58392	— 343	2	— 686
				△	— 0.00276

[*p*] = 54½ ; *log* [*pvv*] = 5.9125

p. e. = ± 0.00015

Star No. 1182.

Date	log. r′	log. r	△	p	p △
June 7	1.97645	1.97543	— 0.00102	3	— 0.00306
8	1.97288	1.97230	— 58	4	— 232
9	1.97180	1.96923	— 257	4	— 1028
12					
13	1.97496	1.97520	+ 24	2½	+ 60
14	1.97208	1.97690	+ 482	4	+ 1928
19	1.98178	1.98268	+ 90	3	+ 270
21	1.97296	1.97359	+ 63	4	+ 252
22	1.97167	1.97248	+ 81	4	+ 324
27	1.97413	1.97382	— 31	5	— 155
28	1.97291	1.97313	+ 22	4	+ 88
29	1.97153	1.97179	+ 26	3	+ 78
30	1.97242	1.97128	— 114	3½	— 399
July 3	1.96655	1.96918	+ 263	3	+ 789
4	1.97073	1.97058	— 15	5	— 75
5	1.97694	1.97520	— 174	4	— 696
6	1.97923	1.97855	— 68	3	— 204
				△	+ 0.00012

[p] = 59 ; *log* [pvv] = 6.2272

p. e. = ± 0.00020

Star No. 424 *l. c.*—(With 1105.)

Date	log. r′	log. r	△	p	p △
June 7					
8					
9	1.97474	1.97216	— 0.00258	3½	— 0.00903
12					
13	1.97816	1.97571	— 245	2	— 490
14	1.97502	1.97941	+ 439	3½	+ 1536
19	1.98471	1.98534	+ 63	3	+ 180
21	1.97609	1.97465	— 144	4	— 576
22	1.97473	1.97414	— 59	4	— 236
27	1.97727	1.97488	— 239	5	— 1195
28	1.97613	1.97658	+ 45	3	+ 135
29	1.97468	1.97465	— 3	3	— 9
30	1.97549	1.97470	— 79	4½	— 356
July 3	1.96971	1.97021	+ 50	3	+ 150
4	1.97371	1.97437	+ 66	5	+ 330
5	1.98000	1.97964	— 36	4	— 144
6	1.98238	1.98236	— 2	3	— 6
				△	— 0.00031

[p] = 50½ ; *log* [pvv] = 6.1588

p. e. = ± 0.00022

STAR NO. 424 *l. c.*—(With 1182.)

Date.	*log. r'*	*log. r*	△	*p*	*p* △
June 7	1.97982	1.97882	— 0.00100	3	— 0.00300
8	1.97599	1.97534	— 65	4	— 260
9	1.97474	1.97220	— 254	4	— 1016
12					
13	1.97816	1.97841	+ 25	2½	+ 62
14	1.97502	1.97982	+ 480	4	+ 1920
19	1.98471	1.98556	+ 85	3	+ 255
21	1.97609	1.97672	+ 63	4	+ 252
22	1.97473	1.97552	+ 79	4	+ 316
27	1.97727	1.97695	— 32	5	— 160
28	1.97613	1.97635	+ 22	4	+ 88
29	1.97468	1.97493	+ 25	3	+ 75
30	1.97549	1.97433	— 116	3½	— 406
July 3	1.96971	1.97225	+ 254	3	+ 762
4	1.97371	1.97354	— 17	5	— 85
5	1.98000	1.97823	— 177	4	— 708
6	1.98238	1.98164	— 74	3	— 222
				△	+ 0.00010

[*p*] = 59 ; *log* [*pvv*] = 6.2248

p. e. = ± 0.00020

STAR NO. 438 *l. c.*

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7	2.02896	2.02882	— 0.00014	3	— 0.00042
8	2.02529	2.02288	— 241	4	— 964
9	2.02369	2.02263	— 106	4	— 424
12					
13	2.02753	2.02727	— 26	1½	— 39
14	2.02398	2.02841	+ 443	2½	+ 1107
19	2.03351	2.03234	— 117	3	— 351
21	2.02530	2.02690	+ 160	4	+ 640
22	2.02385	2.02420	+ 35	4	+ 140
27	2.02640	2.02592	— 48	5	— 240
28	2.02535	2.02702	+ 167	4	+ 668
29	2.02403	2.02415	+ 12	3½	+ 42
30	2.02436	2.02547	+ 111	4	+ 444
July 3	2.01900	2.02057	+ 157	3	+ 471
4	2.02276	2.02333	+ 57	5	+ 285
5	2.02915	2.02958	+ 43	4	+ 172
6	2.03160	2.03338	+ 178	2	+ 356
				△	+ 0.00040

[*p*] = 56½ ; *log* [*pvv*] = 6.0647

p. e. = ± 0.00017

Star No. 444 *l. c.*—(With 1162.)

Date	*log. r'*	*log. r*	Δ	p	$p\Delta$
June 7					
8					
9	2.62193	2.61791	— 0.00402	3½	— 0.01407
12					
13	2.62622	2.62306	— 316	½	— 158
14	2.62223	2.62129	— 94	2½	— 235
19	2.63245	2.62736	— 509	3	— 1527
21	2.62367	2.62123	— 244	4	— 976
22	2.62220	2.62132	— 88	4	— 372
27	2.62478	2.62271	— 207	5	— 1035
28	2.62369	2.62016	— 353	4	— 1412
29	2.62213	2.61992	— 221	3½	— 774
30	2.62278	2.61891	— 387	4	— 1548
July 3	2.61701	2.61475	— 226	3	— 678
4	2.62082	2.61819	— 263	5	— 1315
5	2.62782	2.62622	— 160	3½	— 560
6	2.63053	2.62688	— 365	2	— 730
				Δ	— 0.00268

$[p] = 47½$; *log* $[pvv] = 5.7941$

p. e. $= \pm$ 0.00015

Star No. 444 *l. c.*—(With 1179.)

Date	*log. r'*	*log. r*	Δ	p	$p\Delta$
July 7	2.62762	2.62709	— 0.00053	3	— 0.00159
8	2.62363	2.62036	— 327	4	— 1308
9	2.62193	2.61745	— 448	4	— 1792
12					
13	2.62622	2.62405	— 217	½	— 108
14	2.62223	2.62118	— 105	3	— 315
19	2.63245	2.62834	— 411	3	— 1233
21	2.62367	2.62042	— 325	4	— 1300
22	2.62220	2.62048	— 172	4	— 688
27	2.62478	2.62270	— 208	5	— 1040
28	2.62369	2.62003	— 366	4	— 1464
29	2.62213	2.62007	— 206	3	— 618
30	2.62278	2.61950	— 328	3½	— 1148
July 3	2.61701	2.61553	— 148	3	— 444
4	2.62082	2.61927	— 155	4½	— 698
5	2.62784	2.62599	— 185	4	— 740
6	2.63053	2.62743	— 310	2	— 620
				Δ	— 0.00251

$[p] = 54½$; *log* $[pvv] = 5.8266$

p. e. $= \pm$ 0.00013

STAR NO. 1225.

Date	*log. r'*	*log. r*	△	*p*	*p* △
June 7	2.02116	2.02103	— 0.00013	3	— 0.00039
8	2.01757	2.01515	— 242	4	— 968
9	2.01589	2.01481	— 108	4	— 432
12					
13	2.02004	2.01978	— 26	1½	— 39
14	2.01668	2.02119	+ 451	2½	+ 1127
19	2.02602	2.02486	— 116	3	— 348
21	2.01728	2.01891	+ 163	4	+ 952
22	2.01609	2.01641	+ 32	4	+ 128
27	2.01853	2.01804	— 46	5	— 230
28	2.01726	2.01895	+ 169	4	+ 676
29	2.01634	2.01645	+ 11	3½	+ 39
30	2.01658	2.01770	+ 112	4	+ 448
July 3	2.01155	2.01322	+ 167	3	+ 501
4	2.01453	2.01511	+ 58	5	+ 290
5	2.02135	2.02181	+ 46	4	+ 184
6	2.02390	2.02576	+ 186	2	+ 372
				△	+ 0.00042

$[p] = 56\frac{1}{2}$; $log\ [pvv] = 6.0780$

$p.\ e. = \pm\ 0.00018$

The next table contains the results collected from those preceding. The weights given in the column *p* have been derived from the probable errors as given in column *r*. The remaining columns are self-explanatory.

Star	△	*r*	*log. r*²	*log. p*	*p*	*p* △
948	— 205	±15	2.3522	1.0964	12.5	— 0.02562
190 *l. c.*	— 513	47	3.3442	0.1044	1.3	— 667
959(1)	— 584	53	3.4486	0.0000	1.0	— 584
959(2)	— 462	27	2.8627	0.5859	3.9	— 1802
968	+ 662	36	3.1126	0.3360	2.2	+ 1456
977	+ 476	40	3.2041	0.2445	1.8	+ 857
984	— 142	17	2.4609	0.9877	9.7	— 1377
225(1) *l. c.*	— 211	15	2.3522	1.0964	12.5	— 2637
225(2) *l. c.*	— 126	15	2.3522	1.0964	12.5	— 1675
225(3) *l. c.*	— 159	13	2.2279	1.2207	16.6	— 2639
997	+ 652	36	3.1126	0.3360	2.2	+ 1434
1005(1)	— 215	21	2.6444	0.8042	6.4	— 1376
1005(2)	— 31	19	2.5575	0.8911	7.8	— 242
1009	+ 298	38	3.1596	0.2890	1.9	+ 566
1019(1)	+ 488	40	3.2041	0.2445	1.8	+ 878
1019(2)	+ 290	38	3.1596	0.2890	1.9	+ 551
264 *l. c.*	— 218	20	2.6021	0.8465	7.0	— 1526
1032	— 317	29	2.9248	0.5238	3.3	— 1046
282 *l. c.*	— 473	27	2.8627	0.5859	3.9	— 1845
1084	— 101	35	3.0881	0.3605	2.3	— 232
1094	— 104	34	3.0630	0.3856	2.4	— 250
1105	— 29	23	2.7235	0.7251	5.3	— 154
1110	— 84	18	2.5105	0.9381	8.7	— 731
349 *l. c.*	— 84	18	2.5105	0.9381	8.7	— 731
356 *l. c.*	— 33	19	2.5575	0.8911	7.8	— 257
1135	— 193	16	2.4065	1.0421	11.0	— 2123
377(1) *l. c.*	— 336	28	2.8943	0.5543	3.6	— 1210
377(2) *l. c.*	— 334	18	2.5105	0.9381	8.7	— 2906
1156	— 304	16	2.4065	1.0421	11.0	— 3344
1162(1)	— 330	15	2.3522	1.0964	12.5	— 4125
1162(2)	— 297	17	2.4609	0.9877	9.7	— 2881
406(1) *l. c.*	— 318	15	2.3522	1.0964	12.5	— 3975
406(2) *l. c.*	— 292	11	2.0828	1.3658	23.2	— 6774
1179(1)	— 301	11	2.0828	1.3658	23.2	— 6983
1179(2)	— 276	15	2.3522	1.0964	12.5	— 3450
1182	+ 12	20	2.6021	0.8465	7.0	+ 84
424(1) *l. c.*	— 31	22	2.6848	0.7638	5.8	— 180
424(2) *l. c.*	+ 10	20	2.6021	0.8465	7.0	+ 70
438 *l. c.*	+ 40	17	2.4609	0.9877	9.7	+ 388
444(1) *l. c.*	— 268	15	2.3522	1.0964	12.5	— 3350
444(2) *l. c.*	— 251	13	2.2279	1.2207	16.6	— 4167
1225	+ 42	18	2.5105	0.9381	8.7	+ 365
					△	— 0.00180

$[p] = 340.6$

$[pvv] = 0.00108489$

$\triangle = -0.00180 \pm 0.00019$

4. *The Constant of Refraction.*—The value of a deduced by Gyldén for the Pulkowa Tables, as given in his "Untersuchungen über die Constitution der Atmosphäre u.s.w.," is

$$a = 0.00027985 = 57''.723.$$

This is for $B = 29.5966$ inches at 0° and $t = 7°.44$ R.

The Pulkowa Tables used here, however, are Gyldén's with μ systematically reduced by $-$ 0.00124. Combining this with the value found for $\triangle a$, the correction to Gyldén's constant becomes

$$\triangle a = - 0.00304a$$
$$= - 0''.175$$

and

$$a = 57''.548.$$

This reduced to the condition of 760 mm. pressure at 0° and 0° C temperature gives

$$a = 60''.159.$$

To this value of a correspond the following:

$$c = 0.00029182$$

and

$$\mu = 1.00029178.$$

For the sake of comparison, the most important determinations of the constant of refraction are given below. These values are for the conditions $B = 760$ mm. at 0° C and external thermometer = 0° C. (These values are taken from Professor Bauschinger's "Untersuchungen über die Astronomische Refraction u.s.w.").

		a	μ
1.	Fund. Astr.	60''.320	1.00029257
2.	Tab. Reg.	.440	29315
3.	Tab. Pulk.	.268	29232
4.	Fuss	.122	29161
5.	Greenw. 1857–1865 ..	.120	29160
6.	Pulk. 1865	.209	29203
7.	Greenw. 1877–1886 ..	.192	29195
8.	Pulk. 1885	.058	29130
9.	München	.104	29152

The first and second of these are determinations by Bessel; the third by Gyldén; the fifth by Stone; the sixth by Nyrén; the seventh by Newcomb; the eighth by Nyrén; and the last by Bauschinger.

Bauschinger gives weight zero to each of Bessel's determinations; to the first, because there was considerable uncertainty in Bradley's meteorological instruments; to the second, because of the uncertainty in reading the Meridian Circle (read by vernier to one second). He gives equal weight to the last seven, and gets for a mean

$$a = 60''.153 \quad \text{and} \quad \mu = 1.00029176.$$

5. *Latitude.*—The following table gives the value of φ deduced separately from the southern and from the northern stars. All of the stars of the list down to 84° Z. D. were used.

$\varphi = + 37° \; 20'$

Date	φ_S	p	$p\,\varphi_S$	φ_N	p	$p\,\varphi_N$
	$''$		$''$	$''$		$''$
June 7	25.38	4	101.52	24.89	4	99.56
8	25.88	4	103.52	24.71	4	98.84
9	26.49	4	105.96	24.27	4	97.08
12	26.08	1	26.08	24.96	1	24.96
13	25.99	2	51.98	25.27	2	50.54
14	25.88	4	103.52	25.26	3	75.78
19	26.55	4	106.20	24.54	3	73.62
21	25.99	5	129.95	24.66	4	98.64
22	25.65	5	128.25	24.54	4	98.16
27	25.67	7	179.69	24.59	5½	135.24
28	26.48	4	105.92	24.87	3	74.61
29	25.10	5	125.50	24.89	4	99.56
30	26.08	5	130.40	24.80	4	99.20
July 3	25.60	4	102.40	24.91	3	74.73
4	26.03	7	182.21	25.22	5½	138.71
5	25.95	5	129.75	25.07	4	100.28
6	26.80	3	80.40	24.60	3	73.80
Σ		73	1893.25		61	1513.31
Weighted mean φ			25.93			24.81

Applying the new refractions found here, the latitudes become from the

Southern Stars — $\varphi = 25''.55$

Northern Stars — $\varphi = 25\;.19$

giving for the mean φ at this epoch (1899 June 22),

$$\varphi = + 37° \; 20' \; 25''.37.$$

The remainder of the difference between the values of φ as found from the northern stars and from the southern stars (0″.36) is probably due to slight errors in the declinations of the stars used, and to bisection error.

CONCLUSION.

In conclusion it is desired to state that limitations of time have prevented the *complete* reduction of these observations and of the series taken during the fall months (1899 Oct.–Dec.). It is hoped that, in the near future, time will be available in which to carry out these reductions by correcting the declinations used and then repeating such portions of these computations as will be necessary. It is also desired to make reductions which will include the relative humidity and a term depending upon the zenith distance.

It will be noticed from the table (p. 189) that there is a large range in the values of $\triangle$, viz., from -0.00584 to $+0.00662$. This discordance is due partly to the values of the declinations adopted, but is also very clearly a function of the zenith distance. By introducing a term depending upon the zenith distance, and re-solving by Least Squares, this discordance can be greatly diminished.

From this investigation the following conclusions can be drawn:—

1. That this preliminary reduction gives for the Constant of Refraction

$$\alpha = 60''.159$$

for $B = 760$ mm. at 0° (C) and $t = 0°$ (C).

2. That for the epoch 1899 June 22, the latitude of the Lick Observatory Meridian Circle was

$$\varphi = +37° \; 20' \; 25''.37.$$

3. That the final reduction will show that the Constant of Refraction of the Pulkowa Tables is too large.

4. That the observing room of the Lick Observatory Meridian Circle is of a very good design, and that there is no need of mounting Meridian Circles in the open air.

Addendum.

The table on page 189 shows a large range in the values of $\triangle$, viz., from $+0.00662$ to -0.00584. Upon plotting these values, using the zenith distance z for abscissa, and $\triangle$ for ordinate, it is easily seen that $\triangle$ varies quite uniformally with the zenith distance. A straight line, inclined about 145° to the zenith distance axis, and cutting it at z = about 55°, appears to represent $\triangle$ very well. Therefore, assuming Z to be the zenith distance for $\triangle = 0$, we can set up an observation equation of the following type for every star:

$$\log a = \log a_0 + [Z - z]x,$$

or

$$\log a - \log a_0 = \triangle = Zx - zx = D - zx,$$

where

$$D = Zx,$$

and where a_0 is the a of the tables used (Pulkowa).

Equations of this kind were, accordingly, formed and solved for Z and x by the method of Least Squares.

Equations of Condition.

$\triangle = D - zx.$

No.	Star	D	—	zx	=	$\triangle$	*p*
1	948	D	—	80.00 x	=	—0.00205	12.5
2	190 l. c.		—	89.20	=	— 513	1.3
3	959		—	88.76	=	— 487	4.9
4	968		—	30.38	=	+ 662	2.2
5	977		—	21.55	=	+ 476	1.8
6	984		—	78.11	=	— 142	9.7
7	225 l. c.		—	79.70	=	— 167	41.6
8	997		—	30.59	=	+ 652	2.2
9	1005		—	57.19	=	— 114	14.2
10	1009		—	21.34	=	+ 298	1.9
11	1019		—	21.49	=	+ 386	3.7
12	264 l. c.		—	57.35	=	— 218	7.0
13	1032		—	87.05	=	— 317	3.3
14	282 l. c.		—	88.67	=	— 473	3.9
15	1084		—	27.80	=	— 101	2.3
16	1094		—	28.49	=	— 104	2.4
17	1105		—	62.21	=	— 29	5.3
18	1110		—	67.08	=	— 84	8.7
19	349 l. c.		—	67.65	=	— 84	8.7
20	356 l. c.		—	57.49	=	— 33	7.8
21	1135		—	77.37	=	— 193	11.0
22	377 l. c.		—	86.79	=	— 335	12.3
23	1156		—	87.23	=	— 304	11.0
24	1162		—	83.21	=	— 316	22.2
25	406 l. c.		—	83.50	=	— 301	35.7
26	1179		—	83.26	=	— 292	35.7
27	1182		—	62.79	=	+ 12	7.0
28	424 l. c.		—	62.96	=	— 9	12.8
29	438 l. c.		—	65.52	=	+ 40	9.7
30	444 l. c.		—	83.99	=	— 258	29.1
31	1225		—	65.13	=	+ 42	8.7

To reduce the number of equations, those nearly alike were combined, as follows: Equations No. 1, 6, 7 and 21; 2, 3 and 14; 4 and 8; 5, 10 and 11; 9, 12 and 20; 13, 22 and 23; 15 and 16; 17, 27 and 28; 18 and 19; 24, 25, 26 and 30; and 29 and 31, giving the 11 equations:—

No.	*a*		*b*		*n*	*p*	$\sqrt{p}$
1	D	—	79.20 x	=	—0.00174	74.8	8.6
2		—	88.78	=	— 485	10.1	3.2
3		—	30.48	=	+ 657	4.4	2.1
4		—	21.47	=	+ 385	7.4	2.7
5		—	57.31	=	— 117	29.0	5.4
6		—	87.00	=	— 320	26.6	5.2
7		—	27.15	=	— 103	4.7	2.2
8		—	62.75	=	— 7	25.1	5.0
9		—	67.36	=	— 84	17.4	4.2
10		—	83.49	=	— 291	122.7	11.1
11		—	65.34	=	+ 41	18.4	4.3

Weighted Observation Equations.

No.	*a*		*b*		*n*
1	8.6 D	—	681.1 x	=	—0.01496
2	3.2	—	284.1	=	— 1552
3	2.1	—	57.9	=	+ 1248
4	2.7	—	58.0	=	+ 1040
5	5.4	—	309.5	=	— 632
6	5.2	—	452.4	=	— 1664
7	2.2	—	59.7	=	— 227
8	5.0	—	313.7	=	— 35
9	4.2	—	282.9	=	— 353
10	11.1	—	926.7	=	— 3230
11	4.3	—	281.0	=	+ 176

To render these more nearly homogeneous, let D=D; 100x=y and multiply the absolute term by 100. Then we have the following

Weighted Homogeneous Observation Equations.

No.	*a*		*b*		*n*
1	8.6 D	—	6.811 y	=	— 1.496
2	3.2	—	2.841	=	— 1.552
3	2.1	—	0.579	=	+ 1.248
4	2.7	—	0.580	=	+ 1.040
5	5.4	—	3.095	=	— 0.632
6	5.2	—	4.524	=	— 1.664
7	2.2	—	0.597	=	— 0.227
8	5.0	—	3.137	=	— 0.035
9	4.2	—	2.829	=	— 0.353
10	11.1	—	9.267	=	— 3.230
11	4 3	—	2.810	=	+ 0.176

Combining these by the method of Least Squares we obtain the following

Normal Equations.

$$+341.28\,D - 254.512\,y = -61.7188$$
$$-254.51 \quad +197.151 \quad = +53.4383$$

Solving these, remembering that the absolute terms had been multiplied by 100, we have

$$\log D = 7.75694;\ \log y = 8.00376 \text{ or } \log x = \dot{6}.00376.$$

Now since D=Zx, we have log Z=1.75318,

Whence $x = +0.0001009$ and $Z = 56^\circ.647 = 56^\circ 38' 49''$.

Substituting the values of D and x, thus found, in the Weighted Observation Equations, we find [*pvv*] =

0.00024690, from which the following probable errors have been deduced:

$$r_x = \pm 0.0000130 \text{ and } r_Z = \pm 0^\circ.031 = \pm 0^\circ 1' 52''.$$

We, therefore, have from this solution

$$Z = 56^\circ 38'.8 \pm 1'.9 \text{ and } x = +0.000101 \pm 0.000013,$$

giving

$$\log a = \log a_0 + 0.000101 \; [56^\circ 38'.8 - z].$$

We are, therefore, led to the conclusion that the so-called Constant of Refraction needs not only a correction, but a correction for every zenith distance. In other words, the formula from which refractions are computed needs to be modified. Or, the formula may be retained unaltered, and the desired result obtained by correcting the log μ table of the refraction tables used (Pulkowa) by the amount

$$\triangle \log \mu = 0.000101 \; [56^\circ 38'.8 - z].$$

R. T. C.