PROCEEDINGS

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY

VOLUME XIX

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

A self-recording electrometer for Atmospheric Electricity. By W. A. DOUGLAS RUDGE, M.A., St John's College.

[*Received* 18 October 1915.]

In the course of the writer's work on the local variations of the atmospheric potential gradient, the need was felt for a simple self-recording electrometer. Most of those in use are costly and at the same time rather elaborate in construction. A new arrangement has therefore been devised which has answered the purpose in view, and as the apparatus may be useful in other directions a description is now given.

It has been shown* that very considerable variations of the normal potential gradient are produced by clouds of dust raised by the wind, etc.; and also by clouds of steam escaping under pressure from steam boilers†. These variations are very sudden and do not last for a long time, so that an instrument used for recording them must be fairly quick acting. After a considerable amount of preliminary work, the type of instrument adopted was a modified form of the quadrant electrometer, the record being photographed upon a piece of bromide paper attached to a revolving cylinder. One special use to which the electrometer was to have been applied was to find the relation between the potential gradient and the altitude of the place of observation, and for this purpose it was proposed to construct ten or more instruments, so that a number of observations could have been carried out simultaneously. Some work of this kind has already been done in South Africa from which it appears that the potential gradient near to the ground diminishes with the height of the place of observation above sea level‡. In order to get satisfactory results it is necessary for the

* *Proc. Roy. Soc.* A, Vol. 90. † *Proc. Roy. Soc.* A, Vol. 90.
‡ *Trans. Roy. Soc. South Africa*, Vol. VI, part 5.

stations to be chosen as far removed as possible from the disturbing influence of manufacturing operations, and of railways, and it was intended to have taken a set of observations in the neighbourhood of the Dead Sea, as in that district, stations for the instruments might have been chosen with altitudes varying from 1400 ft below sea level to 3000 ft above, and in an open country. As a number of instruments were required it was necessary to keep the cost of construction low, and this has been achieved in the instrument to be described, so that the cost of material is less than ten shillings and a moderate amount only of mechanical skill is required in the construction.

The complete apparatus consists of:

(1) The Electrometer.
(2) The recording cylinder.
(3) The illuminating arrangement.
(4) The charging battery.
(5) The collecting system.

(1) *The Electrometer.* This consists of four curved pieces of brass cut from a tube of 3 cm. diameter, and attached to a block of ebonite. The alternate pieces were connected together in the usual way. Each conductor subtended an angle at the centre of the mirror of about 60°, and the adjacent conductors were about 1 mm. apart. The "needle" was formed from a piece of silvered paper, $2{\cdot}5 \times 1{\cdot}5$ cm. carrying a small mirror, or a piece of silvered thin "cover" glass could be used for both needle and mirror. A fine wire was attached to the needle to support a piece of wire gauze which was immersed in a small bottle containing paraffin oil, for damping the motion of the needle. The system was suspended by a fine phosphor bronze wire by means of which the needle could be charged, Fig. 1. The whole was enclosed in a thin wooden case having a small window in front, and ebonite plugs to allow of connection being made to the quadrants.

(2) *The recording cylinder.* This is the most novel feature of the instrument, and is constructed from one of those small round clocks which may be bought from a shilling upwards. Two sizes of clock-case are common, of diameters 10 cm. and 6 cm., and both of these sizes have been used. A brass tube is substituted for the hour hand at the front of the clock, and a similar piece of brass tubing is attached to the arbor at the back of the clock, which is attached to the minute hand and used for setting the clock. These two tubes are in the same straight line and furnish a convenient axis about which the clock as a whole can rotate. If the tube attached to the hour hand is fixed, the clock-case will turn round once in *twelve* hours, whilst if the minute hand is fixed, the clock

rotates once in *one* hour. Two scales of measurement are thus possible and both have been employed. No difficulty was found in taking twenty-four, or two hour records, for although the records overlapped it was quite easy to distinguish one part from the other. A light zinc tube was slipped over the clock-case to give a good support for the bromide paper which was wrapped round outside. The whole clock was made to balance by fastening small pieces of lead to the inside of the case, but during the working a little

Fig. 2.
A, hour hand arbor fixed by the pin *P*.

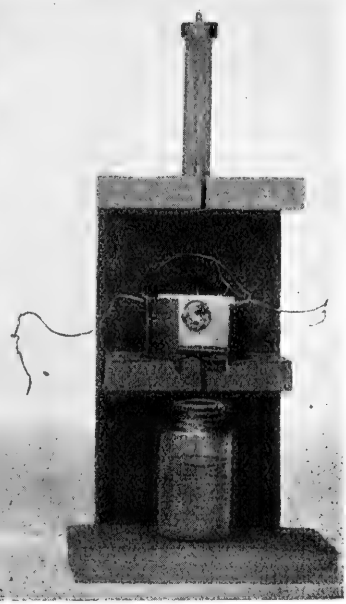


Fig. 1.
The electrometer.

irregularity occurs as a consequence of the unwinding of the spring; this however is not very great and a number of clocks could be made to keep time together. The recording cylinder was enclosed in a light tight case with a long narrow slit in front, Fig. 2.

(3) *Illuminating system.* As the apparatus was used out of doors, a lamp was unsuitable as a means of illumination, so that daylight was used and found to be very suitable. The electrometer and recording cylinder were placed at the opposite ends of a light tight box measuring $20 \times 17 \times 14$ cm. A hole was made in the

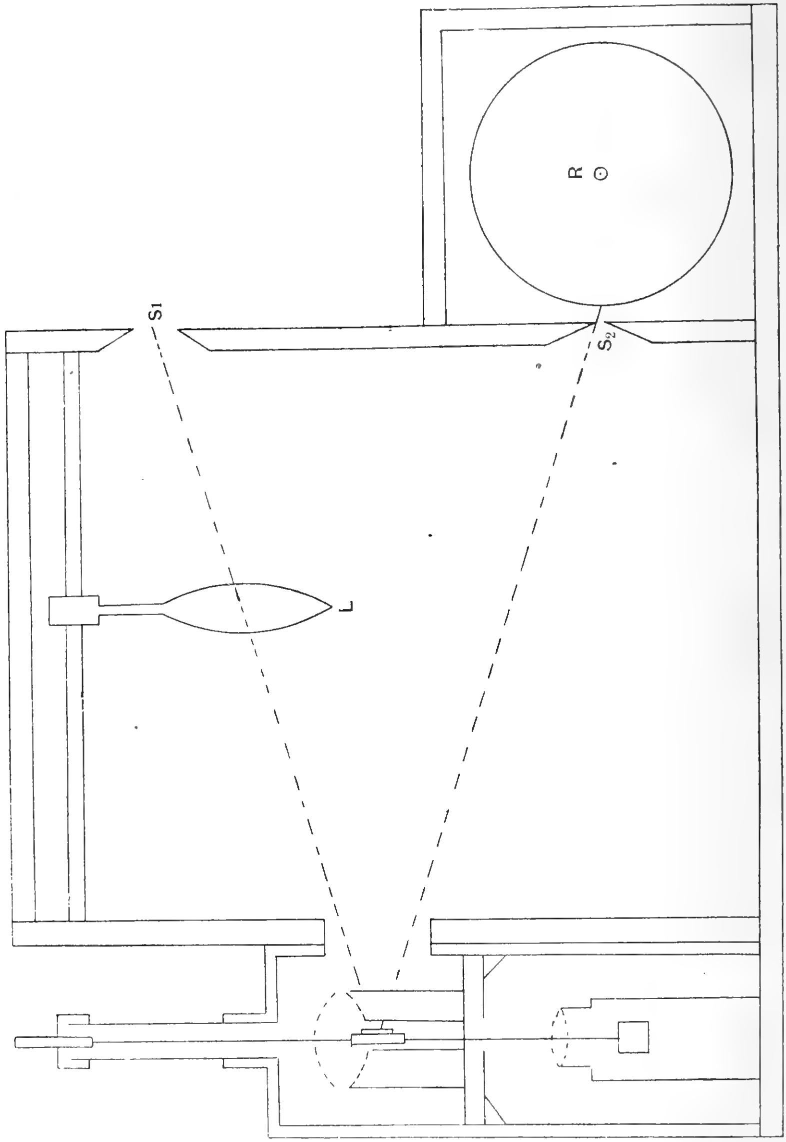


Fig. 3. *L* lens. *R* recording drum. S_1 slit for admitting light. S_2 slit in clock case.

top of one of the ends of the box, and covered over with a piece of silvered glass, upon which a fine vertical scratch—to serve as a slit—had been made. A lens, which could slide upon a rod inside the box, was employed to project the light upon the electrometer mirror, whence after reflection it was returned to the same end of the box as the slit, but at a lower level, and fell upon the horizontal slit in the case of the recording cylinder. By this means a point of light impinged upon the bromide paper, and as the latter rotated, traced out the curve which appears after developing the paper in the usual manner. Fig. 3.

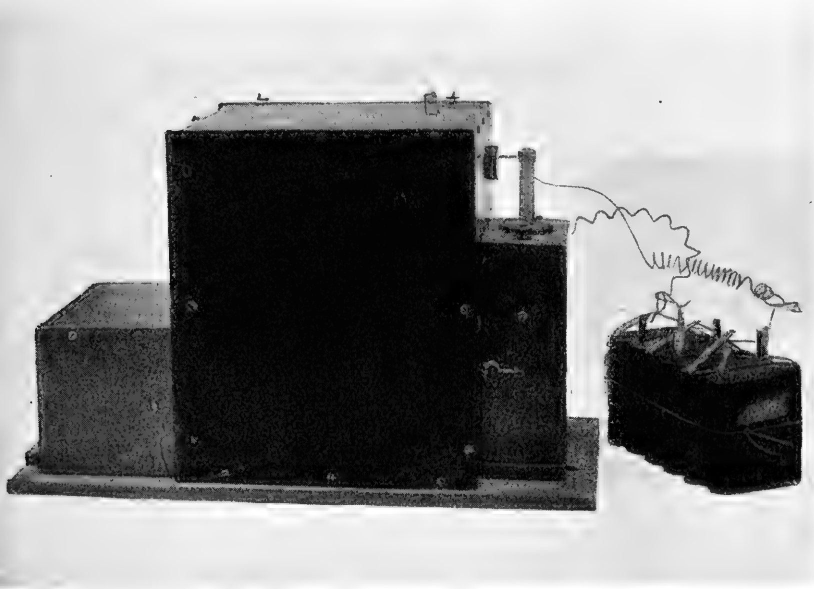


Fig. 4. *E* electrometer. *R* recording apparatus. *S* slit.

(4) *Charging battery.* In using the electrometer the opposite pairs of quadrants were kept charged to a fixed potential by means of a battery of the small Leclanché cell used for "flash" lamps. These cells are sold in sets of three and a batch of eight, giving about 35 volts, is quite sufficient for atmospheric observations. The centre of the battery was earthed. The complete apparatus is shown in Fig. 4.

(5) *The Collector.* This consisted of a small plate of brass coated with a radioactive preparation. The plate was fixed in the centre of a very short piece of brass tubing and the open ends of the

tubing covered over with wire gauze, so as to prevent loss of radium by rubbing, etc.; whilst allowing it to take up the potential of the air. The collecting plate was supported at the end of an insulated wire, and at such a height above the ground as would give a deflection suitable to the sensibility of the electrometer.

Up to the present time the apparatus has been used for the purpose of taking records of the variations in the potential gradient, due to the presence of clouds of dust raised by traffic on the roads, or to the variation caused by the steam escaping from passing trains. A number of representative curves are given.

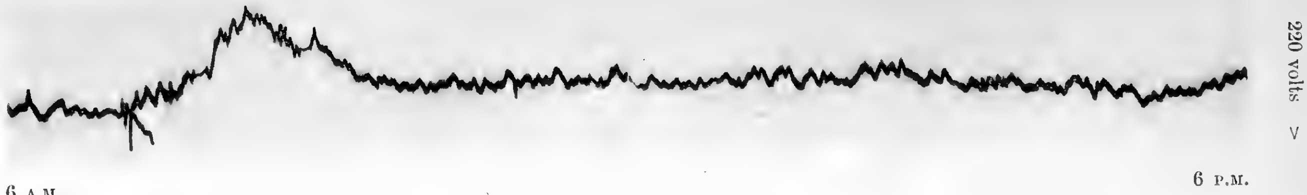
No. 1. This is a twelve hour record, taken at a station on the Gog Magog Hill about four miles from Cambridge, and so far from the railway and roadway that traffic had no disturbing influence.

No. 2 is a *simultaneous* record taken in Hills Road at a distance of less than a quarter of a mile from the railway, so that every passing train shows its influence in *increasing* the positive potential.

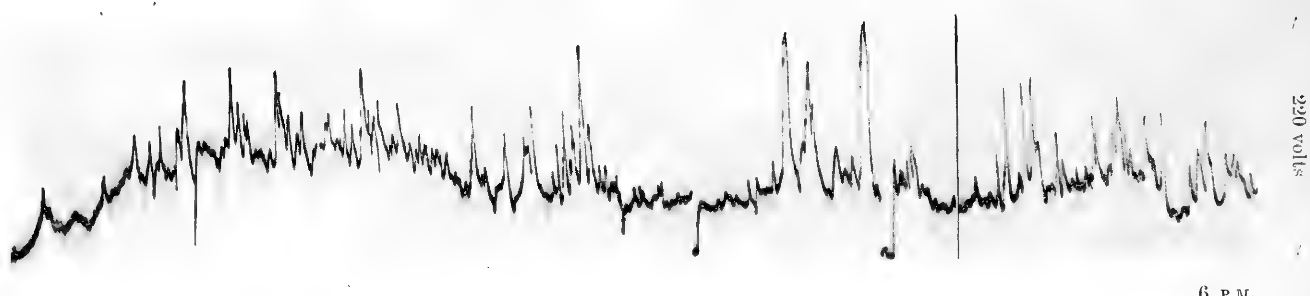
Nos. 3 and 4 are a pair of simultaneous *hour* records, three being taken at Cherryhinton reservoir, and four at about 300 yards from the railway. The "peaks" in the latter indicate the passing of a train.

No. 5 is a one hour record taken on Hills Road and shows the remarkable influence of the dust raised by passing vehicles. Every vehicle, even an ordinary bicycle, if it raises dust, disturbs the normal electrification. Nos. 6 and 7 are simultaneous records taken at some little distance from the road. Nos. 8 and 9 were taken near the "Long" road railway crossing and show the influence of passing trains. Nos. 10 and 11 are a pair of simultaneous records, 10 being taken in the Railway yard, and showing the effect of passing train and "shunting" operations; 11 was taken about a mile away from the line.

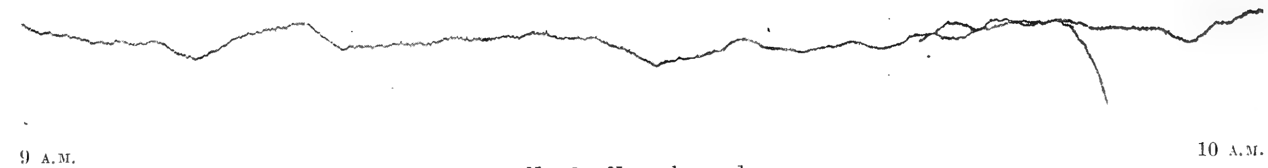
All the potentials indicated are positive and the records are reduced in reproduction, but an equal range of negative potentials could be recorded, as only one half of the width of the photographic paper was used in the records given. The sensibility of the instrument may be changed by varying the number of cells of the charging battery.



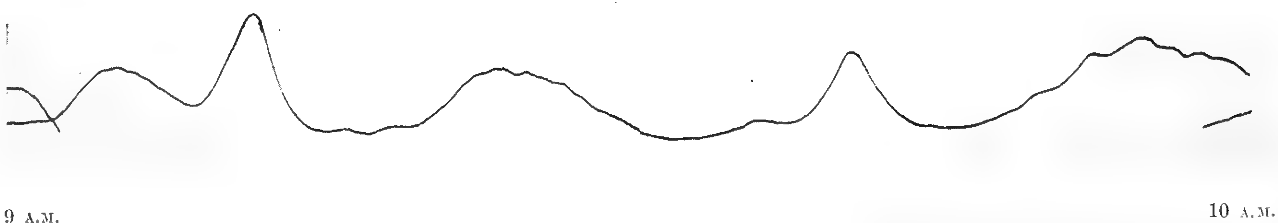
No. 1. Normal 12 hour record.



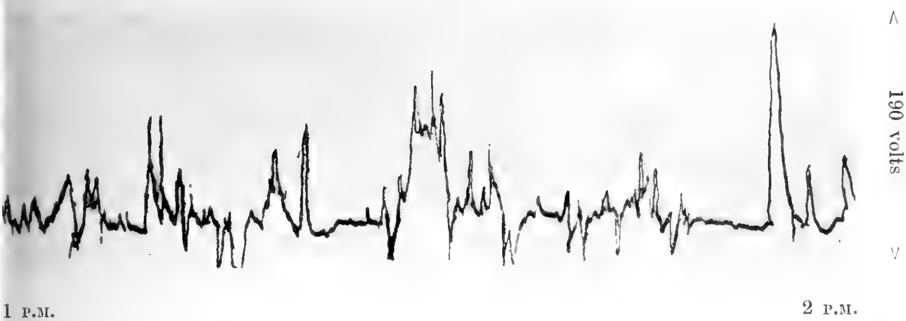
No. 2. Simultaneous 12 hour record taken near to the railway.



No. 3. Normal record.

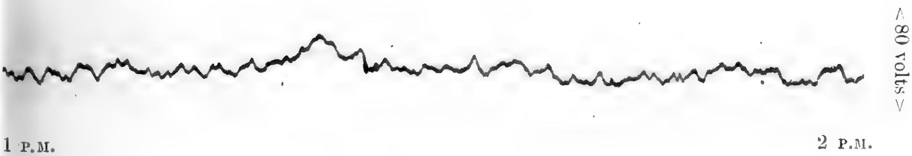


No. 4. Simultaneous record showing rise of positive potential caused by trains passing with steam escaping.

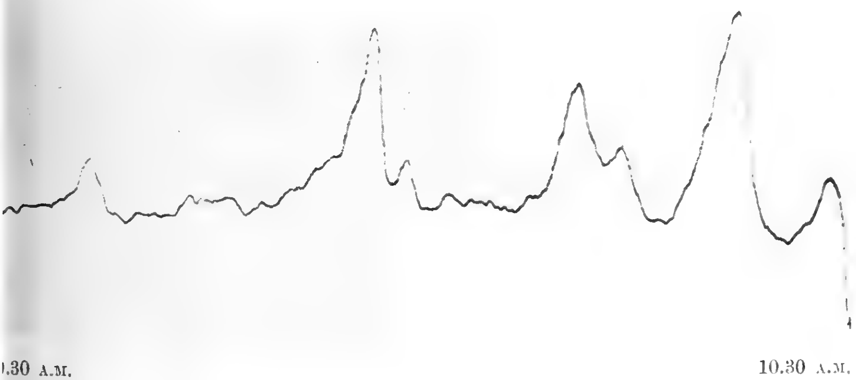


No. 5. Variation in positive potential due to the clouds of dust raised by traffic on the roads.

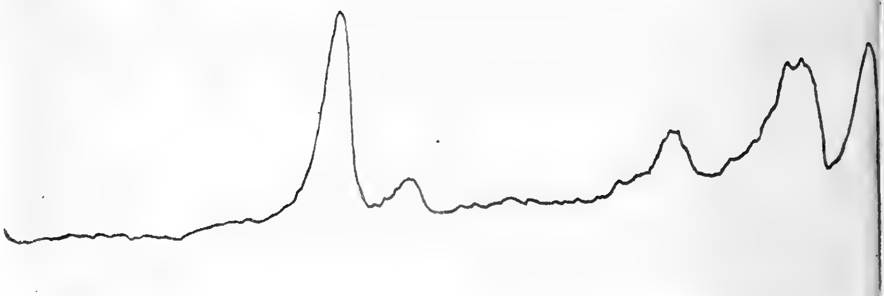
No. 6. Taken simultaneously with 5.



No. 7. Taken simultaneously with 5.

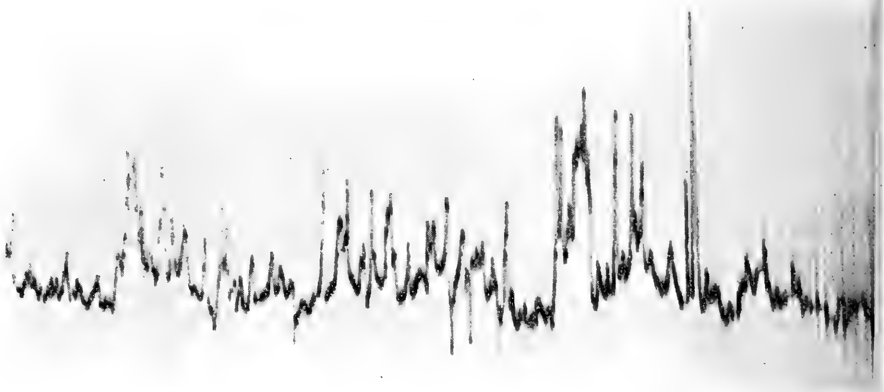


No. 8. Variation in potential due to steam from passing trains.

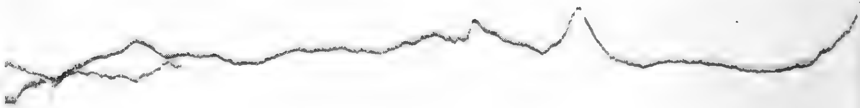


2.15 P.M. 3.15 P.M.

No. 9. Variation in potential due to steam from passing trains.



No. 10. Variation in positive potential due to "shunting" of trains.



No. 11. Taken simultaneously with No. 10, but at a distance of more than a mile from the railway.

On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$.
By S. RAMANUJAN, B.A., Trinity College. (Communicated by Mr G. H. HARDY.)

[*Received* 19 September 1916; *read* October 30, 1916.]

1. It is well known that all positive integers can be expressed as the sum of four squares. This naturally suggests the question: *For what positive integral values of a, b, c, d can all positive integers be expressed in the form*

$$ax^2 + by^2 + cz^2 + du^2\ ? \qquad (1{\cdot}1)$$

I prove in this paper that there are only 55 sets of values of a, b, c, d for which this is true.

The more general problem of finding all sets of values of a, b, c, d, for which all integers *with a finite number of exceptions* can be expressed in the form (1·1), is much more difficult and interesting. I have considered only very special cases of this problem, with two variables instead of four; namely, the cases in which (1·1) has one of the special forms

$$a(x^2 + y^2 + z^2) + bu^2 \qquad (1{\cdot}2),$$

and

$$a(x^2 + y^2) + b(z^2 + u^2) \qquad (1{\cdot}3).$$

These two cases are comparatively easy to discuss. In this paper I give the discussion of (1·2) only, reserving that of (1·3) for another paper.

2. Let us begin with the first problem. We can suppose, without loss of generality, that

$$a \leqslant b \leqslant c \leqslant d \qquad (2{\cdot}1).$$

If $a > 1$, then 1 cannot be expressed in the form (1·1); and so

$$a = 1 \qquad (2{\cdot}2).$$

If $b > 2$, then 2 is an exception; and so

$$1 \leqslant b \leqslant 2 \qquad (2{\cdot}3).$$

We have therefore only to consider the two cases in which (1·1) has one or other of the forms

$$x^2 + y^2 + cz^2 + du^2, \qquad x^2 + 2y^2 + cz^2 + du^2.$$

In the first case, if $c > 3$, then 3 is an exception; and so

$$1 \leqslant c \leqslant 3 \qquad (2{\cdot}31).$$

In the second case, if $c > 5$, then 5 is an exception; and so

$$2 \leqslant c \leqslant 5 \qquad (2{\cdot}32).$$

We can now distinguish 7 possible cases.

(2·41) $x^2 + y^2 + z^2 + du^2$.

If $d > 7$, 7 is an exception; and so

$$1 \leqslant d \leqslant 7 \qquad (2{\cdot}411).$$

(2·42) $x^2 + y^2 + 2z^2 + du^2$.

If $d > 14$, 14 is an exception; and so

$$2 \leqslant d \leqslant 14 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}421).$$

(2·43) $\quad x^2 + y^2 + 3z^2 + du^2.$

If $d > 6$, 6 is an exception; and so

$$3 \leqslant d \leqslant 6 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}431).$$

(2·44) $\quad x^2 + 2y^2 + 2z^2 + du^2.$

If $d > 7$, 7 is an exception; and so

$$2 \leqslant d \leqslant 7 \ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}441).$$

(2·45) $\quad x^2 + 2y^2 + 3z^2 + du^2.$

If $d > 10$, 10 is an exception; and so

$$3 \leqslant d \leqslant 10 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}451).$$

(2·46) $\quad x^2 + 2y^2 + 4z^2 + du^2.$

If $d > 14$, 14 is an exception; and so

$$4 \leqslant d \leqslant 14 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}461).$$

(2·47) $\quad x^2 + 2y^2 + 5z^2 + du^2.$

If $d > 10$, 10 is an exception; and so

$$5 \leqslant d \leqslant 10 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}471).$$

We have thus eliminated all possible sets of values of a, b, c, d, except the following 55:

1, 1, 1, 1	1, 2, 3, 5	1, 2, 4, 8
1, 1, 1, 2	1, 2, 4, 5	1, 2, 5, 8
1, 1, 2, 2	1, 2, 5, 5	1, 1, 2, 9
1, 2, 2, 2	1, 1, 1, 6	1, 2, 3, 9
1, 1, 1, 3	1, 1, 2, 6	1, 2, 4, 9
1, 1, 2, 3	1, 2, 2, 6	1, 2, 5, 9
1, 2, 2, 3	1, 1, 3, 6	1, 1, 2, 10
1, 1, 3, 3	1, 2, 3, 6	1, 2, 3, 10
1, 2, 3, 3	1, 2, 4, 6	1, 2, 4, 10
1, 1, 1, 4	1, 2, 5, 6	1, 2, 5, 10
1, 1, 2, 4	1, 1, 1, 7	1, 1, 2, 11
1, 2, 2, 4	1, 1, 2, 7	1, 2, 4, 11
1, 1, 3, 4	1, 2, 2, 7	1, 1, 2, 12
1, 2, 3, 4	1, 2, 3, 7	1, 2, 4, 12
1, 2, 4, 4	1, 2, 4, 7	1, 1, 2, 13
1, 1, 1, 5	1, 2, 5, 7	1, 2, 4, 13
1, 1, 2, 5	1, 1, 2, 8	1, 1, 2, 14
1, 2, 2, 5	1, 2, 3, 8	1, 2, 4, 14
1, 1, 3, 5		

Of these 55 forms, the 12 forms

1, 1, 1, 2	1, 1, 2, 4	1, 2, 4, 8
1, 1, 2, 2	1, 2, 2, 4	1, 1, 3, 3
1, 2, 2, 2	1, 2, 4, 4	1, 2, 3, 6
1, 1, 1, 4	1, 1, 2, 8	1, 2, 5, 10

have been already considered by Liouville and Pepin*.

3. I shall now prove that all integers can be expressed in each of the 55 forms. In order to prove this we shall consider the seven cases (2·41)—(2·47) of the previous section separately. We shall require the following results concerning ternary quadratic arithmetical forms.

The necessary and sufficient condition that a number *cannot* be expressed in the form

$$x^2 + y^2 + z^2 \quad \text{.......................(3·1)}$$

is that it should be of the form

$$4^\lambda(8\mu + 7), \qquad (\lambda = 0, 1, 2 \ldots, \ \mu = 0, 1, 2, \ldots) \text{......(3·11)}.$$

Similarly the necessary and sufficient conditions that a number *cannot* be expressed in the forms

$$x^2 + y^2 + 2z^2 \quad \text{....................(3·2)},$$
$$x^2 + y^2 + 3z^2 \quad \text{....................(3·3)},$$
$$x^2 + 2y^2 + 2z^2 \quad \text{....................(3·4)},$$
$$x^2 + 2y^2 + 3z^2 \quad \text{....................(3·5)},$$
$$x^2 + 2y^2 + 4z^2 \quad \text{....................(3·6)},$$
$$x^2 + 2y^2 + 5z^2 \quad \text{....................(3·7)},$$

are that it should be of the forms

$$4^\lambda(16\mu + 14) \text{ ....(3·21)},$$
$$9^\lambda(9\mu + 6) \text{ ....(3·31)},$$
$$4^\lambda(8\mu + 7) \text{ ....(3·41)},$$
$$4^\lambda(16\mu + 10) \text{ ....(3·51)},$$
$$4^\lambda(16\mu + 14) \text{ ....(3·61)},$$
$$25^\lambda(25\mu + 10) \ \textit{or} \ 25^\lambda(25\mu + 15)\dagger \text{ ....(3·71)}.$$

* There are a large number of short notes by Liouville in vols. V–VIII of the second series of his journal. See also Pepin, *ibid.*, ser. 4, vol. VI, pp. 1–67. The object of the work of Liouville and Pepin is rather different from mine, viz. to determine, in a number of special cases, explicit formulae for the number of representations, in terms of other arithmetical functions.

† Results (3·11)—(3·71) may tempt us to suppose that there are similar simple results for the form $ax^2 + by^2 + cz^2$, whatever are the values of a, b, c. It appears, however, that in most cases there are no such simple results. For instance,

The result concerning $x^2+y^2+z^2$ is due to Cauchy: for a proof see Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 550. The other results can be proved in an analogous manner. The form $x^2+y^2+2z^2$ has been considered by Lebesgue, and the form $x^2+y^2+3z^2$ by Dirichlet. For references see Bachmann, *Zahlentheorie*, vol. IV, p. 149.

4. We proceed to consider the seven cases (2·41)—(2·47). In the first case we have to show that any number N can be expressed in the form

$$N = x^2+y^2+z^2+du^2 \quad \text{.....................(4·1)},$$

d being any integer between 1 and 7 inclusive.

If N is not of the form $4^\lambda(8\mu+7)$, we can satisfy (4·1) with $u=0$. We may therefore suppose that $N=4^\lambda(8\mu+7)$.

First, suppose that d has one of the values 1, 2, 4, 5, 6. Take $u=2^\lambda$. Then the number

$$N-du^2 = 4^\lambda(8\mu+7-d)$$

is plainly not of the form $4^\lambda(8\mu+7)$, and is therefore expressible in the form $x^2+y^2+z^2$.

Next, let $d=3$. If $\mu=0$, take $u=2^\lambda$. Then

$$N-du^2 = 4^{\lambda+1}.$$

the numbers which are not of the form $x^2+2y^2+10z^2$ are those belonging to one or other of the *four* classes

$$25^\lambda(8\mu+7), \quad 25^\lambda(25\mu+5), \quad 25^\lambda(25\mu+15), \quad 25^\lambda(25\mu+20).$$

Here some of the numbers of the first class belong also to one of the next three classes.

Again, the even numbers which are not of the form $x^2+y^2+10z^2$ are the numbers

$$4^\lambda(16\mu+6),$$

while the odd numbers that are not of that form, viz.

3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, ...

do not seem to obey any simple law.

I have succeeded in finding a law in the following six simple cases:

$$x^2+y^2+4z^2,$$
$$x^2+y^2+5z^2,$$
$$x^2+y^2+6z^2,$$
$$x^2+y^2+8z^2,$$
$$x^2+2y^2+6z^2,$$
$$x^2+2y^2+8z^2.$$

The numbers which are not of these forms are the numbers

$$4^\lambda(8\mu+7) \quad \textit{or} \quad (8\mu+3),$$
$$4^\lambda(8\mu+3),$$
$$9^\lambda(9\mu+3),$$
$$4^\lambda(16\mu+14), \quad (16\mu+6), \quad \textit{or} \quad (4\mu+3),$$
$$4^\lambda(8\mu+5),$$
$$4^\lambda(8\mu+7) \quad \textit{or} \quad (8\mu+5).$$

If $\mu \geqslant 1$, take $u=2^{\lambda+1}$. Then

$$N-du^2=4^\lambda(8\mu-5).$$

In neither of these cases is $N-du^2$ of the form $4^\lambda(8\mu+7)$, and therefore in either case it can be expressed in the form $x^2+y^2+z^2$.

Finally, let $d=7$. If μ is equal to 0, 1, or 2, take $u=2^\lambda$. Then $N-du^2$ is equal to 0, $2.4^{\lambda+1}$, or $4^{\lambda+2}$. If $\mu \geqslant 3$, take $u=2^{\lambda+1}$. Then

$$N-du^2=4^\lambda(8\mu-21).$$

Therefore in either case $N-du^2$ can be expressed in the form $x^2+y^2+z^2$.

Thus in all cases N is expressible in the form (4·1). Similarly we can dispose of the remaining cases, with the help of the results stated in § 3. Thus in discussing (2·42) we use the theorem that every number not of the form (3·21) can be expressed in the form (3·2). The proofs differ only in detail, and it is not worth while to state them at length.

5. We have seen that all integers without any exception can be expressed in the form

$$m(x^2+y^2+z^2)+nu^2 \quad \text{..................(5·1)},$$

when $\quad m=1, \quad 1 \leqslant n \leqslant 7,$

and $\quad m=2, \quad n=1.$

We shall now consider the values of m and n for which all integers *with a finite number of exceptions* can be expressed in the form (5·1).

In the first place m must be 1 or 2. For, if $m>2$, we can choose an integer ν so that

$$nu^2 \not\equiv \nu \pmod{m}$$

for all values of u. Then

$$\frac{(m\mu+\nu)-nu^2}{m},$$

where μ is any positive integer, is not an integer; and so $m\mu+\nu$ can certainly not be expressed in the form (5·1).

We have therefore only to consider the two cases in which m is 1 or 2. First let us consider the form

$$x^2+y^2+z^2+nu^2 \quad \text{....................(5·2)}.$$

I shall show that, when n has any of the values

$$1, 4, 9, 17, 25, 36, 68, 100 \quad \text{...........(5·21)},$$

or is of any of the forms

$$4k+2, \quad 4k+3, \quad 8k+5, \quad 16k+12, \quad 32k+20 \ldots (5{\cdot}22),$$

then all integers save a finite number, and in fact all integers from $4n$ onwards at any rate, can be expressed in the form (5·2); but that for the remaining values of n there is an infinity of integers which cannot be expressed in the form required.

In proving the first result we need obviously only consider numbers of the form $4^\lambda(8\mu+7)$ greater than n, since otherwise we may take $u=0$. The numbers of this form less than n are plainly among the exceptions.

6. I shall consider the various cases which may arise in order of simplicity.

(6·1) $n \equiv 0 \pmod 8$.

There are an infinity of exceptions. For suppose that

$$N = 8\mu + 7.$$

Then the number

$$N - nu^2 \equiv 7 \pmod 8$$

cannot be expressed in the form $x^2+y^2+z^2$.

(6·2) $n \equiv 2 \pmod 4$.

There is only a finite number of exceptions. In proving this we may suppose that $N = 4^\lambda(8\mu+7)$. Take $u=1$. Then the number

$$N - nu^2 = 4^\lambda(8\mu+7) - n$$

is congruent to 1, 2, 5, or 6 to modulus 8, and so can be expressed in the form $x^2+y^2+z^2$.

Hence the only numbers which cannot be expressed in the form (5·2) in this case are the numbers of the form $4^\lambda(8\mu+7)$ not exceeding n.

(6·3) $n \equiv 5 \pmod 8$.

There is only a finite number of exceptions. We may suppose again that $N = 4^\lambda(8\mu+7)$. First, let $\lambda \neq 1$. Take $u=1$. Then

$$N - nu^2 = 4^\lambda(8\mu+7) - n \equiv 2 \text{ or } 3 \pmod 8.$$

If $\lambda = 1$ we cannot take $u=1$, since

$$N - n \equiv 7 \pmod 8;$$

so we take $u=2$. Then

$$N - nu^2 = 4^\lambda(8\mu+7) - 4n \equiv 8 \pmod{32}.$$

In either of these cases $N - nu^2$ is of the form $x^2+y^2+z^2$.

Hence the only numbers which cannot be expressed in the form (5·2) are those of the form $4^\lambda(8\mu+7)$ not exceeding n, and those of the form $4(8\mu+7)$ lying between n and $4n$.

(6·4) $n \equiv 3 \pmod 4$.

There is only a finite number of exceptions. Take

$$N = 4^\lambda (8\mu + 7).$$

If $\lambda \geqslant 1$, take $u = 1$. Then

$$N - nu^2 \equiv 1 \text{ or } 5 \pmod 8.$$

If $\lambda = 0$, take $u = 2$. Then

$$N - nu^2 \equiv 3 \pmod 8.$$

In either case the proof is completed as before.

In order to determine precisely which are the exceptional numbers, we must consider more particularly the numbers between n and $4n$ for which $\lambda = 0$. For these u must be 1, and

$$N - nu^2 \equiv 0 \pmod 4.$$

But the numbers which are multiples of 4 and which cannot be expressed in the form $x^2 + y^2 + z^2$ are the numbers

$$4^\kappa (8\nu + 7), \qquad (\kappa = 1, 2, 3, \ldots, \ \nu = 0, 1, 2, 3, \ldots).$$

The exceptions required are therefore those of the numbers

$$n + 4^\kappa (8\nu + 7) \quad \text{......(6·41)}$$

which lie between n and $4n$ and are of the form

$$8\mu + 7 \quad \text{......(6·42)}.$$

Now in order that (6·41) may be of the form (6·42), κ must be 1 if n is of the form $8k + 3$ and κ may have any of the values 2, 3, 4, ... if n is of the form $8k + 7$. Thus the only numbers which cannot be expressed in the form (5·2), in this case, are those of the form $4^\lambda (8\mu + 7)$ less than n and those of the form

$$n + 4^\kappa (8\nu + 7), \qquad (\nu = 0, 1, 2, 3, \ldots),$$

lying between n and $4n$, where $\kappa = 1$ if n is of the form $8k + 3$, and $\kappa > 1$ if n is of the form $8k + 7$.

(6·5) $n \equiv 1 \pmod 8$.

In this case we have to prove that

(i) if $n \geqslant 33$, there is an infinity of integers which cannot be expressed in the form (5·2);

(ii) if n is 1, 9, 17, or 25, there is only a finite number of exceptions.

In order to prove (i) suppose that $N = 7 \cdot 4^\lambda$. Then obviously u cannot be zero. But if u is not zero u^2 is always of the form $4^\kappa (8\nu + 1)$. Hence

$$N - nu^2 = 7 \cdot 4^\lambda - n \cdot 4^\kappa (8\nu + 1).$$

Since $n \geqslant 33$, λ must be greater than or equal to $\kappa + 2$, to ensure that the right-hand side shall not be negative. Hence

$$N - nu^2 = 4^\kappa (8k + 7),$$

where $$k = 14 \,.\, 4^{\lambda-\kappa-2} - n\nu - \tfrac{1}{8}(n+7)$$
is an integer; and so $N - nu^2$ is not of the form $x^2 + y^2 + z^2$.

In order to prove (ii) we may suppose, as usual, that
$$N = 4^{\lambda}(8\mu + 7).$$
If $\lambda = 0$, take $u = 1$. Then
$$N - nu^2 = 8\mu + 7 - n \equiv 6 \pmod 8.$$
If $\lambda \geqslant 1$, take $u = 2^{\lambda-1}$. Then
$$N - nu^2 = 4^{\lambda-1}(8k+3),$$
where $$k = 4(\mu+1) - \tfrac{1}{8}(n+7).$$
In either case the proof may be completed as before. Thus the only numbers which cannot be expressed in the form (5·2), in this case, are those of the form $8\mu + 7$ not exceeding n. In other words, there is no exception when $n = 1$; 7 is the only exception when $n = 9$; 7 and 15 are the only exceptions when $n = 17$; 7, 15 and 23 are the only exceptions when $n = 25$.

(6·6) $n \equiv 4 \pmod{32}$.

By arguments similar to those used in (6·5), we can show that

(i) if $n \geqslant 132$, there is an infinity of integers which cannot be expressed in the form (5·2);

(ii) if n is equal to 4, 36, 68, or 100, there is only a finite number of exceptions, namely the numbers of the form $4^{\lambda}(8\mu + 7)$ not exceeding n.

(6·7) $n \equiv 20 \pmod{32}$.

By arguments similar to those used in (6·3), we can show that the only numbers which cannot be expressed in the form (5·2) are those of the form $4^{\lambda}(8\mu + 7)$ not exceeding n, and those of the form $4^2(8\mu + 7)$ lying between n and $4n$.

(6·8) $n \equiv 12 \pmod{16}$.

By arguments similar to those used in (6·4), we can show that the only numbers which cannot be expressed in the form (5·2) are those of the form $4^{\lambda}(8\mu + 7)$ less than n, and those of the form
$$n + 4^{\kappa}(8\nu + 7), \qquad (\nu = 0, 1, 2, 3, \ldots),$$
lying between n and $4n$, where $\kappa = 2$ if n is of the form $4(8k+3)$ and $\kappa > 2$ if n is of the form $4(8k+7)$.

We have thus completed the discussion of the form (5·2), and determined the exceptional values of N precisely whenever they are finite in number.

7. We shall proceed to consider the form
$$2(x^2 + y^2 + z^2) + nu^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7\cdot1).$$

In the first place n must be odd; otherwise the odd numbers cannot be expressed in this form. Suppose then that n is odd. I shall show that all integers save a finite number can be expressed in the form (7·1): and that the numbers which cannot be so expressed are

(i) the odd numbers less than n,
(ii) the numbers of the form $4^\lambda(16\mu+14)$ less than $4n$,
(iii) the numbers of the form $n+4^\lambda(16\mu+14)$ greater than n and less than $9n$,
(iv) the numbers of the form

$$cn+4^\kappa(16\nu+14), \quad (\nu=0,1,2,3,\ldots),$$

greater than $9n$ and less than $25n$, where $c=1$ if $n\equiv 1 \pmod 4$, $c=9$ if $n\equiv 3 \pmod 4$, $\kappa=2$ if $n^2\equiv 1 \pmod{16}$, and $\kappa>2$ if $n^2\equiv 9 \pmod{16}$.

First, let us suppose N even. Then, since n is odd and N is even, it is clear that u must be even. Suppose then that

$$u=2v, \quad N=2M.$$

We have to show that M can be expressed in the form

$$x^2+y^2+z^2+2nv^2 \quad \text{.....................(7·2)}.$$

Since $2n\equiv 2 \pmod 4$, it follows from (6·2) that all integers except those which are less than $2n$ and of the form $4^\lambda(8\mu+7)$ can be expressed in the form (7·2). Hence the only *even* integers which cannot be expressed in the form (7·1) are those of the form $4^\lambda(16\mu+14)$ less than $4n$.

This completes the discussion of the case in which N is even. If N is odd the discussion is more difficult. In the first place, all odd numbers less than n are plainly among the exceptions. Secondly, since n and N are both odd, u must also be odd. We can therefore suppose that

$$N=n+2M, \quad u^2=1+8\Delta,$$

where Δ is an integer of the form $\frac{1}{2}k(k+1)$, so that Δ may assume the values 0, 1, 3, 6, ... And we have to consider whether $n+2M$ can be expressed in the form

$$2(x^2+y^2+z^2)+n(1+8\Delta),$$

or M in the form

$$x^2+y^2+z^2+4n\Delta \quad \text{.....................(7·3)}.$$

If M is not of the form $4^\lambda(8\mu+7)$, we can take $\Delta=0$. If it is of this form, and less than $4n$, it is plainly an exception. These numbers give rise to the exceptions specified in (iii) of section 7. We may therefore suppose that M is of the form $4^\lambda(8\mu+7)$ and greater than $4n$.

8. In order to complete the discussion, we must consider the three cases in which $n \equiv 1 \pmod 8$, $n \equiv 5 \pmod 8$, and $n \equiv 3 \pmod 4$ separately.

(8·1) $n \equiv 1 \pmod 8$.

If λ is equal to 0, 1, or 2, take $\Delta = 1$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 4n$$

is of one of the forms

$$8\nu + 3, \quad 4(8\nu + 3), \quad 4(8\nu + 6).$$

If $\lambda \geqslant 3$ we cannot take $\Delta = 1$, since $M - 4n\Delta$ assumes the form $4(8\nu + 7)$; so we take $\Delta = 3$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 12n$$

is of the form $4(8\nu + 5)$. In either of these cases $M - 4n\Delta$ is of the form $x^2 + y^2 + z^2$. Hence the only values of M, other than those already specified, which cannot be expressed in the form (7·3), are those of the form

$$4^{\kappa}(8\nu + 7), \qquad (\nu = 0, 1, 2, \ldots, \kappa > 2),$$

lying between $4n$ and $12n$. In other words, the only numbers greater than $9n$ which cannot be expressed in the form (7·1), in this case, are the numbers of the form

$$n + 4^{\kappa}(8\nu + 7), \quad (\nu = 0, 1, 2, \ldots, \kappa > 2),$$

lying between $9n$ and $25n$.

(8·2) $n \equiv 5 \pmod 8$.

If $\lambda \neq 2$, take $\Delta = 1$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 4n$$

is of one of the forms

$$8\nu + 3, \quad 4(8\nu + 2), \quad 4(8\nu + 3).$$

If $\lambda = 2$, we cannot take $\Delta = 1$, since $M - 4n\Delta$ assumes the form $4(8\nu + 7)$; so we take $\Delta = 3$. Then

$$M - 4n\Delta = 4^{\lambda}(8\mu + 7) - 12n$$

is of the form $4(8\nu + 5)$. In either of these cases $M - 4n\Delta$ is of the form $x^2 + y^2 + z^2$. Hence the only values of M, other than those already specified, which cannot be expressed in the form (7·3), are those of the form $16(8\mu + 7)$ lying between $4n$ and $12n$. In other words, the only numbers greater than $9n$ which cannot be expressed in the form (7·1), in this case, are the numbers of the form $n + 4^2(16\mu + 14)$ lying between $9n$ and $25n$.

$$(8\cdot3)\quad n\equiv 3 \pmod 4.$$

If $\lambda \neq 1$, take $\Delta=1$. Then

$$M-4n\Delta=4^{\lambda}(8\mu+7)-4n$$

is of one of the forms

$$8\nu+3,\quad 4(4\nu+1).$$

If $\lambda=1$, take $\Delta=3$. Then

$$M-4n\Delta=4(8\mu+7)-12n$$

is of the form $4(4\nu+2)$. In either of these cases $M-4n\Delta$ is of the form $x^2+y^2+z^2$.

This completes the proof that there is only a finite number of exceptions. In order to determine what they are in this case, we have to consider the values of M, between $4n$ and $12n$, for which $\Delta=1$ and

$$M-4n\Delta=4(8\mu+7-n)\equiv 0 \pmod{16}.$$

But the numbers which are multiples of 16 and which cannot be expressed in the form $x^2+y^2+z^2$ are the numbers

$$4^{\kappa}(8\nu+7),\qquad (\kappa=2,3,4,\ldots,\ \nu=0,1,2,\ldots).$$

The exceptional values of M required are therefore those of the numbers

$$4n+4^{\kappa}(8\nu+7)\ldots\ldots\ldots\ldots(8\cdot31)$$

which lie between $4n$ and $12n$ and are of the form

$$4(8\mu+7)\ldots\ldots\ldots\ldots(8\cdot32).$$

But in order that (8·31) may be of the form (8·32), κ must be 2 if n is of the form $8k+3$, and κ may have any of the values 3, 4, 5, ... if n is of the form $8k+7$. It follows that the only numbers greater than $9n$ which cannot be expressed in the form (7·1), in this case, are the numbers of the form

$$9n+4^{\kappa}(16\nu+14),\qquad (\nu=0,1,2,\ldots),$$

lying between $9n$ and $25n$, where $\kappa=2$ if n is of the form $8k+3$, and $\kappa>2$ if n is of the form $8k+7$.

This completes the proof of the results stated in section 7.

An Axiom in Symbolic Logic. By C. E. VAN HORN, M.A. (Communicated by Mr G. H. HARDY.)

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Philosophy's task is a search for the primal and fundamental elements of the world. Its face is turned in the opposite direction to that of science and mathematics. Philosophy hands back to them its results, and they as best they can construct systematic bodies of doctrine that purport to show us what the world may be on the one hand (science) and what the world might be on the other (mathematics). As philosophy advances in the pursuit of its task it is continually vacating old ground to science and mathematics. The history of this change of boundary can be traced in the changes in the nomenclature of human knowledge: Natural Philosophy has become Physics; Mental Philosophy has become Psychology; Moral Philosophy is becoming the inductive science of Ethics. Thus (paradoxically speaking) philosophy's advance is to be marked by the retreat of her boundaries.

It is interesting to watch this retreat in a field occupied by philosophy from its very beginning, and until recently supposed to be its permanent possession. I refer to the field of the foundations of mathematics. Here large areas once occupied by philosophy by sovereign right of long control are slowly passing into the possession of pure mathematics; and by the way both are gainers by the transfer*.

To facilitate the mathematical treatment of these new areas a new instrument of investigation had to be invented, namely, Mathematical, or Symbolic, Logic. This new logic, which is infinitely more powerful than the traditional logic, and which embraces all that is really self-consistent in the old logic, makes possible a precise and easy handling of all the highly abstract and complex ideas occurring in the new fields. For example, both philosophy and the old logic found themselves involved in many a tangle on questions concerning classes and relations because neither possessed the requisite instruments of analysis. Again, philosophy had wandered into a veritable labyrinth of difficulties concerning infinity, quantity, continuity, and so on. Here too the secret of the trouble lay in the inadequacy of the instruments of analysis afforded by the traditional logic.

* Much valuable light is thrown upon the details of this process in the writings of Bertrand Russell, especially in the preface and introductory chapters of the *Principia Mathematica*, Vol. I. 1910; and more recently in his *Scientific Method in Philosophy*, 1914.

Now however the matter is all changed. Philosophy, equipped with the latest instruments of mathematical logic, is able to deal successfully with the problems of these fields. In fact so fully have these ideas been analysed that at last philosophy as such has relinquished these fields to pure mathematics. Even more, the whole field of deduction has now become the foundation-branch of mathematics and has developed into a precise Calculus of Propositions. Out of it grow by easy stages the Calculus of Classes and the Calculus of Relations, and these in turn grow by equally easy stages into all the manifold branches of pure mathematics as more commonly known. It is in these and similar ways that philosophy and pure mathematics are both gainers by the transfer of the fields recently acquired by mathematics from philosophy.

It is now easy to understand why the axioms of mathematical logic (and so of all pure mathematics) lie in the borderland between philosophy and mathematics, and are thus the concern of the philosopher equally with the mathematician. To depart entirely from our figures and adopt others, the rootage of mathematics is in philosophy. It is here too that we meet the innovations of mathematical logic that appear so fantastic to the philosopher trained only in the old logic. Its definitions and treatment of some of the common terms of language seem so at variance with what the traditional logician is familiar with that he often views the new logic as the victim of some delusion. It appears however from the nature of the case itself that many of those peculiarities, which from the view-point of traditional logic would be described as abnormal, do not deserve to be so described; that in fact it is in the theories of the traditional logician and philosopher that the abnormalities really occur*.

In order to indicate what seems to me a possible simplification of the axiomatic basis of mathematical logic I wish to introduce in a new form an idea advocated by Sheffer. Its importance lies in the fact that in terms of it Sheffer was able to define the four fundamental operations of logic, namely, Negation, Disjunction, Implication, and Conjunction or Joint Assertion. It is a familiar fact that Kronecker found the use of certain auxiliary quantities (let us call them 'parameters') of great value in his algebraic investigations, the chief value lying in the fact that their disappearance led to desired relations among numbers essential to his investigations. It is a precisely similar use of Sheffer's idea that I desire to make in the field of the philosophy of logic. In terms of it I define, after him, the four fundamental operations of logic. Then, unlike him, I work by means of an *axiom* to *eliminate* that idea from the formulae, and in so doing to arrive at the desired

* Cf. Russell, *Scientific Method in Philosophy*, chap. I.

properties and relations of the four fundamental operations. The chief excellence of my method seems to reside in the fact that proceeding as indicated above I have been able to prove *as propositions of mathematical logic* some of the axioms hitherto laid down at the basis of this logic.

In its most satisfactory form the axiomatic basis of mathematical logic has been stated by Bertrand Russell in the first volume of the *Principia Mathematica*†. In *1 of Vol. I., pp. 98–101, of the *Principia* will be found the primitive propositions required for the theory of deduction as applied to elementary propositions. I confine myself to these purposely, for it is here that I have succeeded, I believe, in simplifying the axiomatic basis of mathematical logic.

Let p and q be any two elementary propositions. The four fundamental operations give us (1) $\sim p$ (not-p), (2) $p \vee q$ (either p or q), (3) $p \supset q$ (p implies q), and (4) $p \cdot q$ (both p and q). After Sheffer, I define these four results in terms of a single undefinable operation. I will call this undefinable operation *Deltation*. The result of performing this operation upon two elementary propositions p and q is symbolized, after Sheffer, '$p \triangle q$' (read 'p *deltas* q'). The four fundamental operations of logic can be expressed as logical functions of this parameter thus:

Negation:	$\sim p . = . p \triangle p$	Df.
Disjunction:	$p \vee q . = . \sim p \triangle \sim q$	Df.
Implication:	$p \supset q . = . p \triangle \sim q$	Df.
Conjunction:	$p \cdot q . = . \sim (p \triangle q)$	Df.

These definitions of the four fundamental operations of logic as functions of the one undefined parameter, Deltation, are made relevant to our discussion by means of the following axiom.

Axiom. *If p and q are of the same truth-value, then '$p \triangle q$' is of the opposite truth-value; but if p and q are of opposite truth-values, then '$p \triangle q$' is true.*

For convenience of reference it might be well for me to state at this point Russell's primitive propositions concerning elementary propositions as he enunciates them in *1 of the first volume of the *Principia*.

*1.1 Anything implied by a true elementary proposition is true. Pp‡.

† Whitehead and Russell, *Principia Mathematica*, Vol. I. 1910, Vol. II. 1912, Vol. III. 1913 (Cambridge University Press).

‡ Russell uses the letters "Pp" to stand for "primitive proposition," as does Peano.

*1.11 When ϕx can be asserted, where x is a real variable, and '$\phi x \supset \psi x$' can be asserted, where x is a real variable, then ψx can be asserted, where x is a real variable. Pp.

*1.2 $\vdash : p \vee p . \supset . p$ Pp.

*1.3 $\vdash : q . \supset . p \vee q$ Pp.

*1.4 $\vdash : p \vee q . \supset . q \vee p$ Pp.

*1.5 $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$ Pp.

*1.6 $\vdash : . q \supset r . \supset : p \vee q . \supset . p \vee r$ Pp.

*1.7 If p is an elementary proposition, $\sim p$ is an elementary proposition. Pp.

*1.71 If p and q are elementary propositions '$p \vee q$' is an elementary proposition. Pp.

*1.72 If ϕp and ψp are elementary propositional functions which take elementary propositions as arguments, '$\phi p \vee \psi p$' is an elementary propositional function. Pp.

These are all the primitive propositions that are needed for the development of the theory of deduction, as applied to elementary propositions, according to Russell's method of treatment.

It is my purpose to show that by means of my axiom Russell's primitive propositions *1.2 to *1.71 can be demonstrated. I do this by starting at the very beginning and developing the immediate consequences of three of the axioms which I lay down as the basis of the theory of deduction as applied to elementary propositions. The resulting deductive development at length reaches a point where it includes among its theorems Mr Russell's seven primitive propositions and two others that can take the place of his definitions of Implication and Conjunction. Altogether I prove seventeen theorems. Some of these theorems occur as propositions in the first volume of the *Principia*. Although many more theorems can be proved as simply as the ones given, to economize space I shall stop at the point where my development of Mathematical Logic includes the nine theorems mentioned above.

I will now state the three axioms used in this paper. The first is *1.1 given above, the last is my axiom as already enunciated.

Axiom 1. *Anything implied by a true elementary proposition is true.*

Axiom 2. *If p and q are elementary propositions, then '$p \triangle q$' is an elementary proposition.*

Axiom 3. *If p and q are of the same truth-value, then '$p \triangle q$' is of the opposite truth-value; but if p and q are of opposite truth-values, then '$p \triangle q$' is true.*

Theorem 1

If p is an elementary proposition, $\sim p$ is an elementary proposition.

Dem.

Axiom 2 gives us '$p \triangle p$' elementary when p is elementary; '$p \triangle p$' is $\sim p$, by Definition of Negation. Hence the theorem.

This is a proof of Mr Russell's primitive proposition *1.7 given above.

Theorem 2

If p and q are elementary propositions, '$p \vee q$' is an elementary proposition.

Dem.

By Theorem 1, if p and q are elementary so also are $\sim p$ and $\sim q$. Therefore, by Axiom 2, '$\sim p \triangle \sim q$' is elementary; but this, by Definition of Disjunction, is '$p \vee q$'. Hence the theorem.

This is Mr Russell's primitive proposition *1.71 quoted above.

Theorem 3

The propositions p and $\sim p$ are of opposite truth-values.

Dem.

Two possibilities can occur:

1°: p true. By Axiom 3, '$p \triangle p$' is false; but this by Definition of Negation is $\sim p$; hence in this case p and $\sim p$ are opposite in truth-value.

2°: p false. By Axiom 3, '$p \triangle p$' is true; but this by Definition of Negation is $\sim p$; hence in this case also p and $\sim p$ are opposite in truth-value. Hence the theorem.

This theorem states in precise form the information usually given in text-books on logic in more or less vague statements that are called 'definitions' of negation.

Theorem 4

$\vdash . p \supset p.$

Dem.

[Th. 3]	$\vdash .$ p and $\sim p$ of opposite truth-values	(1)
[(1). Ax. 3]	$\vdash . p \triangle \sim p$	(2)
[(2). Def. of Implication]	$\vdash .$ theorem.	

This is proposition *2.08† of the *Principia*.

† *Op. cit.* Vol. I. p. 105.

Theorem 5

If p is false, '$p \vartriangle q$' is always true.

Dem.

Two possibilities can occur: either q true, or q false. In either case '$p \vartriangle q$' is true by Ax. 3.

Theorem 6

If q is false, '$p \vartriangle q$' is always true.

Proof similar to that of preceding theorem.

Theorem 7

The propositions '$p \vartriangle q$' and '$q \vartriangle p$' have the same truth-value.

Dem.

If p and q are of the same truth-value then, by Ax. 3, '$p \vartriangle q$' and '$q \vartriangle p$' are both of the opposite truth-value. If p and q are of opposite truth-values then, by Ax. 3, '$p \vartriangle q$' and '$q \vartriangle p$' are both true. Hence the theorem.

Theorem 8

The proposition

$$\sim p \vartriangle \sim (\sim q \vartriangle \sim r)$$

is true if any one or more of the propositions p, q, r are true; but if all of these propositions are false then the proposition

$$\sim p \vartriangle \sim (\sim q \vartriangle \sim r)$$

is false.

Dem.

Eight possibilities can occur:

1°: p, q, r all true. Then (Th. 3) $\sim p$, $\sim q$, $\sim r$ are all false. Hence (Ax. 3) '$\sim q \vartriangle \sim r$' is true. Hence (Th. 3) $\sim (\sim q \vartriangle \sim r)$ is false. Hence (Ax. 3) the proposition '$\sim p \vartriangle \sim (\sim q \vartriangle \sim r)$' is true in this case.

2°: p and q true, but r false. By Th. 3, $\sim p$ and $\sim q$ are false, while $\sim r$ is true. Hence (Ax. 3) '$\sim q \vartriangle \sim r$' is true. Hence (Th. 3) $\sim (\sim q \vartriangle \sim r)$ is false. Hence (Ax. 3) the proposition is true in this case. In a similar manner in the following cases:

3°: p true, q false, r true;
4°: p false, q, r true;
5°: p true, q, r false;
6°: p false, q true, r false;
7°: p, q false, r true;

we have '$\sim p \vartriangle \sim (\sim q \vartriangle \sim r)$' true.

But in 8°: p, q, r false, we have $\sim p$, $\sim q$, $\sim r$ all true, by

Th. 3. Hence (Ax. 3) '$\sim q \triangle \sim r$' is false, making $\sim(\sim q \triangle \sim r)$ true (Th. 3). Hence (Ax. 3) in this case the proposition is false.

Hence the theorem.

Theorem 9

The propositions

$$`\sim p \triangle \sim(\sim q \triangle \sim r)\text{'}, \quad `\sim q \triangle \sim(\sim p \triangle \sim r)\text{'},$$

always have the same truth-value.

This follows at once from Th. 8.

At this point I introduce Mr Russell's definition of Equivalence† as it occurs in the *Principia*.

Equivalence: $p \equiv q . = . p \supset q . q \supset p$ Df.

Theorem 10

$\vdash . p \equiv \sim(\sim p)$.

Dem.

We first prove $\vdash . p \supset \sim(\sim p)$. Two cases arise:

1°: p true. By Theorem 3, $\sim p$ is false, $\sim(\sim p)$ is true, and $\sim[\sim(\sim p)]$ is false. Hence

[Ax. 3]	$\vdash . p \triangle \sim[\sim(\sim p)]$	(1)
[(1). Def. Implica.]	$\vdash . p \supset \sim(\sim p)$	(2)

2°: p false. By Th. 3, $\sim p$ is true, $\sim(\sim p)$ is false, and $\sim[\sim(\sim p)]$ is true.

[Ax. 3]	$\vdash . p \triangle \sim[\sim(\sim p)]$	(3)
[(1). Implica.]	$\vdash . p \supset \sim(\sim p)$	(4)

Hence in all cases we have

	$\vdash . p \supset \sim(\sim p)$	(5)
We now prove	$\vdash . \sim(\sim p) \supset p$.	
[Th. 3]	$\vdash . q$ and $\sim q$ of opposite truth-values	(6)
[(6). Ax. 3]	$\vdash . \sim q \triangle q$	(7)
[(7). $\dfrac{\sim p}{q}$]	$\vdash . \sim(\sim p) \triangle \sim p$	(8)
[(8). Def. Implica.]	$\vdash . \sim(\sim p) \supset p$	(9)
[(5). (9). Def. Equiv.]	$\vdash .$ theorem.	

This is proposition *4.13 of the *Principia*‡. It is the Principle of Double Negation, and asserts that any proposition is logically equivalent to the denial of its negation.

† *Op. cit.* Vol. I. p. 120, *4.01.
‡ *Op. cit.* Vol. I. p. 122.

Theorem 11

$\vdash : p \vee p . \supset . p.$

Dem.

[Ax. 3] $\vdash . \sim p$ and '$\sim p \triangle \sim p$' of opposite truth-values (1)

[(1). Ax. 3] $\vdash : \sim p \triangle \sim p . \triangle . \sim p$ (2)

[(2). Def. Disjunc. Implica.] $\vdash$. theorem.

This is Mr Russell's primitive proposition *1.2 given above.

Theorem 12

$\vdash : q . \supset . p \vee q.$

Dem.

Two cases need only be treated:

1°: q true. Then (Th. 3) $\sim q$ is false. Hence (Th. 6) '$\sim p \triangle \sim q$' is true. Hence $\sim (\sim p \triangle \sim q)$ is false, by Th. 3. Hence

[Ax. 3] $\vdash : q . \triangle . \sim (\sim p \triangle \sim q)$ (1)

2°: q false.

[Th. 5. $\dfrac{q, \sim (\sim p \triangle \sim q)}{p, \quad q}$] $\vdash . q . \triangle . \sim (\sim p \triangle \sim q)$ (2)

[(1). (2). Def. Disjunc. Implica.] $\vdash$. theorem.

This is Mr Russell's primitive proposition *1.3 given above.

Theorem 13

$\vdash : p \vee q . \supset . q \vee p.$

Dem.

[Th. 7] $\vdash :$ '$\sim p \triangle \sim q$' and '$\sim q \triangle \sim p$' of the same truth-value (1)

[(1). Th. 3. Ax. 3] $\vdash : \sim p \triangle \sim q . \triangle . \sim (\sim q \triangle \sim p)$ (2)

[(2). Def. Disjunc. Implica.] $\vdash :$ theorem.

This is Mr Russell's primitive proposition *1.4 given above.

Theorem 14

$\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r).$

Dem.

[Th. 9] $\vdash :$ '$\sim p \triangle \sim (\sim q \triangle \sim r)$' and '$\sim q \triangle \sim (\sim p \triangle \sim r)$' of the same truth-value (1)

[(1). Th. 3. Ax. 3] $\vdash : \sim p \triangle \sim (\sim q \triangle \sim r)$
$. \triangle . \sim [\sim q \triangle \sim (\sim p \triangle \sim r)]$ (2)

[(2). Def. Disjunc. Implica.] $\vdash :$ theorem.

This is Mr Russell's primitive proposition *1.5 given above.

Theorem 15

$\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r.$

Dem.

There are three cases to be discussed:

1°: If p is true, or if r is true, or if both p and r are true, q being *any* elementary proposition.

[Th. 8] $\quad \vdash : \sim l . \triangle . \sim (\sim p \triangle \sim r)$ (1)

[(1). $\frac{\sim l}{l}$. Th. 10] $\quad \vdash : l . \triangle . \sim (\sim p \triangle \sim r)$ (2)

[(2). $\frac{\sim p \triangle \sim q}{l}$] $\quad \vdash : \sim p \triangle \sim q . \triangle . \sim (\sim p \triangle \sim r)$ (3)

[(3). Th. 3. Th. 6]

$\vdash : q \triangle \sim r . \triangle . \sim [\sim p \triangle \sim q . \triangle . \sim (\sim p \triangle \sim r)]$ (4)

Taken together with the Definitions of Implication and Disjunction, (4) gives the theorem in this case.

2°: If both p and r are false, but q true. In this case $\sim p$ and $\sim r$ are true by Th. 3. Hence (Ax. 3) '$\sim p \triangle \sim r$' is false. The proof in this case proceeds as follows:

[Th. 3] $\quad \vdash : \sim (\sim p \triangle \sim r)$ (5)

Since q is true, $\sim q$ is false (Th. 3).

[Th. 6] $\quad \vdash . \sim p \triangle \sim q$ (6)

[(5). (6). Th. 3. Ax. 3] $\vdash : \sim [\sim p \triangle \sim q . \triangle . \sim (\sim p \triangle \sim r)]$ (7)

By Ax. 3, '$q \triangle \sim r$' is in this case false.

[(7). Ax. 3]

$\vdash : q \triangle \sim r . \triangle . \sim [\sim p \triangle \sim q . \triangle . \sim (\sim p \triangle \sim r)]$ (8)

As in the previous case this result gives the theorem.

3°: All three false. Hence $\sim p$ and $\sim r$ true as before. In this case '$\sim p \triangle \sim q$' is false by Ax. 3. The proof in this last case proceeds thus:

[Th. 3, as in 2°] $\quad \vdash . \sim (\sim p \triangle \sim r)$ (9)

[(9). Ax. 3] $\quad \vdash : \sim p \triangle \sim q . \triangle . \sim (\sim p \triangle \sim r)$ (10)

In this case q and $\sim r$ are of opposite truth-values.

[Ax. 3] $\quad \vdash : q \triangle \sim r$ (11)

[(10). Th. 3. (11). Ax. 3]

$\vdash : q \triangle \sim r . \triangle . \sim [\sim p \triangle \sim q . \triangle . \sim (\sim p \triangle \sim r)]$ (12)

As in the two preceding cases, this result, together with the Definitions of Implication and Disjunction, gives the theorem.

No other cases can arise. Hence the theorem.

This is Mr Russell's primitive proposition *1.6 given above. It asserts that an alternative may be added to both premise and conclusion in any implication without impairing the truth of the implication.

This completes the list of Mr Russell's primitive propositions that I proposed for proof by means of my axiom, on the basis of the definitions given in this paper of the four fundamental operations of logic.

I now propose to prove two propositions which can take the place of his definitions of Implication† and Conjunction‡, or Joint Assertion.

THEOREM 16

$\vdash : p \supset q \, . \equiv . \sim p \vee q.$

Dem.

[Th. 4 $\frac{p \triangle \sim q}{p}$]	$\vdash : p \triangle \sim q \, . \supset . \, p \triangle \sim q$	(1)
[Th. 10]	$\supset . \sim (\sim p) \triangle \sim q$	(2)
[(2). Def. Implica. Disjunc.]	$\vdash : p \supset q \, . \supset . \sim p \vee q$	(3)
[(1). Th. 10]	$\vdash : \sim (\sim p) \triangle \sim q \, . \supset . \, p \triangle \sim q$	(4)
[(4). Def. Implica. Disjunc.]	$\vdash : \sim p \vee q \, . \supset . \, p \supset q$	(5)
[(3). (5). Def. Equiv.]	$\vdash :$ theorem.	

THEOREM 17

$\vdash : p \, . \, q \, . \equiv . \sim (\sim p \vee \sim q).$

Dem.

[Th. 4 $\frac{\sim (p \triangle q)}{p}$]	$\vdash : \sim (p \triangle q) \, . \supset . \sim (p \triangle q)$	(1)
[Th. 10]	$\supset . \sim [\sim (\sim p) \triangle \sim (\sim q)]$	(2)
[(2). Def. Conjunc. Disjunc.]	$\vdash : p \, . \, q \, . \supset . \sim (\sim p \vee \sim q)$	(3)
[(1). Th. 10]	$\vdash : \sim [\sim (\sim p) \triangle \sim (\sim q)] \, . \supset . \sim (p \triangle q)$	(4)
[(4). Def. Conjunc. Disjunc.]	$\vdash : \sim (\sim p \vee \sim q) \, . \supset . \, p \, . \, q$	(5)
[(3). (5). Def. Equiv.]	$\vdash :$ theorem.	

With these theorems established the development of the *Principia Mathematica* can proceed as given by its authors. All that I have done is to reduce the number of axioms needed for that development.

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† *Op. cit.* Vol. I. p. 98, *1.01. ‡ *Ibid.* p. 116, *3.01.

A Reduction in the number of the Primitive Propositions of Logic. By J. G. P. NICOD, Trinity College. (Communicated by Mr G. H. HARDY.)

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Of the four elementary truth-functions needed in logic, only two are taken as indefinables in *Principia Mathematica*. These two have now been defined by Mr Sheffer[†] in terms of a single new function $p \mid q$, "p stroke q." I propose to make use of Mr Sheffer's discovery in order to reduce the number of the primitive propositions needed for the logical calculus.

There are two slightly different forms of the new indefinable, for we may treat $p \mid q$ as meaning the same thing as either $\sim p \,.\, \sim q$, or $\sim p \vee \sim q$[‡]. The definition of $\sim p$ is the same in both cases, namely $p \mid p$, while that of $p \vee q$ simply changes from $p/q \mid p/q$ with the *AND*-form into $p/p \mid q/q$ with the *OR*-form.

However, the best course is for us to define all the four truth-functions directly in terms of the new one. In so doing, we find that, while the definition of $\sim p$ remains the same, and those of $p \vee q$, $p \,.\, q$ simply permute, as we pass from the *AND*-form to the *OR*-form, the definition of $p \supset q$ is simpler in the latter form. It is $p \,\|\, q/q$, as against $p/p \mid q \,\|\, p/p \mid q$.

The *OR*-form is therefore to be preferred[§].

DEFINITIONS.

$$\sim p \,.=.\, p \mid p \quad \text{Df.} \qquad p \vee q \,.=.\, p/p \mid q/q \quad \text{Df.}$$
$$p \supset q \,.=.\, p \mid q/q \quad \text{Df.} \qquad p \,.\, q \,.=.\, p/q \mid p/q \quad \text{Df.}$$

REMARKS ON THESE DEFINITIONS.

One ought not to aim at retaining before one's mind the complex translation into the usual system, "$\sim p \vee \sim q$," as the "real meaning" of the stroke. For the stroke, in the stroke-system, is simpler than either $\sim$ or $\vee$, and from it both of them arise. We may not be able to think otherwise than in terms of the four usual functions; it will then be more in accordance with the nature of the new system to think of the $\mid$, not as some fixed compound of $\sim$ and $\vee$, but as a bare structure, out of which, in various ways, $\sim$ and $\vee$ will grow.

† Sheffer, *Trans. Amer. Math. Soc.* Vol. XIV. pp. 481—488.

‡ Sheffer, *loc. cit.*, footnote †, p. 488.

§ $p \mid q$ thus corresponds to what is termed the Disjunctive relation in Mr W. E. Johnson's writings.

The above definitions give clear expression to the symmetry between *OR* and *AND*; and this, notwithstanding the choice that we had to make between an *OR*-form, and an *AND*-form. This is of some interest, because, in general, the very symmetry forces upon us an arbitrary choice, which, in turn, quite obscures the symmetry.

I shall use $\bar{q}$ for $q \mid q$ whenever convenient. Observe that $p \mid \bar{q}$, i.e. $p \supset q$, forms a *natural symbol* $|\overline{}$ for implication, allowing of permutation $\overline{q}| \, p$. We may notice in general that the new system brings the four functions into relations far closer than those in Mr Russell's system. For instance, in

$$p/p \mid p/p \,.\!\mid\!.\, p/p$$

the two propositions $p \vee p \,.\supset.\, p$ and $\sim p \vee p$ coincide.

Every stroke-formula falls into two parts on the right and left of a central stem. It will, therefore, add to clearness to use black type instead of dots to indicate the central symbol. Further, slanting strokes are covered by straight ones: thus $p/q \mid p/q$ stands for $(p \mid q) \mid (p \mid q)$.

The definition of the two primitive notions of the *Principia* in terms of a single new one *tends* to reduce the number of the primitive propositions needed. But how far does this reduction actually occur? Does it extend beyond the obvious substitution of "If p and q are elementary propositions, $p \mid q$ is an elementary prop." (Sheffer, p. 488) for $*1{\cdot}7$ and $*1{\cdot}71$, stating the same for $\sim p$ and $p \vee q$ respectively? The reduction goes, as we shall presently find, very much farther.

It has first to be said, in order that we may be as precise as possible, that the *whole* amount gained in applying the stroke-definitions cannot with complete certainty be attributed to them. For Mr Russell's system, as it now stands, has not said its last word in that matter.

Incidentally, I found that $*1{\cdot}4$, $p \vee q \,.\supset.\, q \vee p$, can be proved by means of the other four, with the unimportant change of $*1{\cdot}3$, $q \,.\supset.\, p \vee q$ into $q \,.\supset.\, q \vee p$. In "Association," $*1{\cdot}5$, write p for r:

$$p \vee (q \vee p) \,.\supset.\, q \vee (p \vee p).$$

The left-hand side, by the help of $q \,.\supset.\, q \vee p$ and "Summation," will be found to be implied in $p \vee q$. The right-hand side, likewise, by $p \vee p \,.\supset.\, p$, and "Summation," will be found to imply $q \vee p$. The result then follows by using "Syllogism" (obtained from "Summation" with the transformation $\dfrac{\sim p}{p}$ †) twice.

† By $\frac{p}{q}$ or $\frac{p,\, p'}{q,\, q'}$ I mean (following Mr Russell) the substitution of p for q or p, p' for q, q'. By (*e.g.*) $P\frac{p}{q}$ I mean the result of effecting the substitution in P.

Let us, however, take Mr Russell's eight propositions in the form given in *Principia*. It is my object to reduce them to three —two non-formal and one formal—by means of the stroke-definitions given above.

It can be shown, as a first stage, that *two formal* propositions are enough, namely:

$$(1)\quad p \mid p/p.$$
$$(2)\quad p \mid q/q \,\pmb{|}\, \overline{s/q \mid \overline{p/s}}.$$

The first proposition is the form of "Identity" ($p \supset p$) in the stroke-system. It would, at first sight, appear more natural to adopt the order $q/s \mid \overline{p/s}$ in the left-hand side of (2), since

$$p \mid q/q \,.\supset.\, q/s \mid \overline{p/s}$$

is the syllogistic principle of the stroke-system, giving "Syllogism," $p \supset q \,.\supset:\, q \supset s \,.\supset.\, p \supset s$ when $s \mid s$ is written for s.

It will however be found that the inverted order, $s/q \mid \overline{p/s}$, is much more advantageous than the normal syllogistic order, $q/s \mid \overline{p/s}$. For, owing to this "twist," Identity and (2) yield "Permutation," $s/p \mid \overline{p/s}$, which now enables us to eliminate the twist in (2), and revert to the normal order. From the three propositions thus obtained, the rest follow.

This, by the way, illustrates the following fundamental fact. Which form of a given principle is the most general, and contains the maximum assertion, is a function of the symbolic system used. Thus, for instance, in Mr Russell's system,

$$p \,.\supset.\, q \vee p \qquad (a)$$

is more general than $\quad p \,.\supset.\, q \supset p \qquad (b)$

since (b) is (a) with $\sim q$ for q. In the stroke-system, on the contrary, $p \,\pmb{|}\, \overline{q/q \mid p/p}$, meaning the same thing as (a), is less general than $p \,\pmb{|}\, \overline{q \mid p/p}$, whose meaning is that of (b), since it is obtained from it by writing $q \mid q$ for q.

A further step has to be made in order to be left with *only one formal* primitive proposition. It consists in adapting to better advantage the form of the primitive propositions to the properties of the stroke-symbolism where implication is concerned. We had above

$$p \supset q \,.=.\, p \mid q/q \quad \text{Df.}$$

If we look for the meaning of the general form $p \mid r/q$, we find this to be $\sim p \vee \sim(\sim r \vee \sim q)$, i.e. $p \,.\supset.\, r \,.\, q$. We thus come to the fundamental property that, in the new system, $p \supset q$ is a case of $p \,.\supset.\, s \,.\, q$, whereas in *Principia* the contrary relation of course holds.

This leads us to substitute $p \mid r/q$ for $p \mid q/q$ in the "left-hand sides" of both the non-formal rule of implication and the syllogistic proposition (2) above. The reform may be further extended to the proposition (2) as a whole, which might be given the form $P \mid S/Q$ instead of $P \mid Q/Q$, with the proviso, if the proposition is to remain true, that S must be implied in P. Now, for S, write the proposition (1) above, $p \mid p/p$; for (as we at this early stage know "unofficially") a true proposition will be implied by everything.

We then have the three primitive propositions of the stroke-system:

Non-formal { I. If p is an elementary proposition, and q is an elementary proposition, then $p \mid q$ is an elementary proposition†.

II. If $p \mid r/q$ is true, and p is true, then q is true.

This is the non-formal rule of implication, *1·1, with the modification just explained.

Formal III. $p \mid q/r \,\boldsymbol{|}\, t \mid t/t \,.\!\mid\!.\, s/q \mid \overline{p/s}.$

I shall call II "the Rule," and III "the Prop."

Remarks on these Primitive Propositions.

Observe $p \mid r/q$ in II, while $p \mid q/r$ in III. This alternance will prove essential for the working of the calculus.

In III, I shall use π for $t \mid t/t$, P for $p \mid q/r$, Q for $s/q \mid \overline{p/s}$, and shall speak of III as $P \mid \pi/Q$.

$P \mid \pi/Q$, by the Rule, yields the same result as the syllogistic proposition (2) above, when the left-hand side P is a truth of logic. This restriction of the syllogistic form to its categorical use with an asserted premiss is a peculiar character of the first proofs to follow, and is of some philosophical interest.

One feels inclined to think that III *merely* asserts together (1) and (2) above. This view, whatever may be the amount of truth it contains, takes *AND* too much as a matter of course, and tends to lose sight of (α) the fact that III, as a structure, is simpler than (2) alone: for III is (2) with $t \mid t/t$ instead of $s/q \mid \overline{p/s}$; and (β) the very real step from $p \,.\, q$ to q, together with the philosophical difference between two assertions and only one.

The main steps in the formal deduction are:

1. Proof of "Identity," $t \mid t/t$.
2. Passage from $P \mid \pi/Q$ to the usual implicative form $P \mid Q/Q$.
3. Elimination of the twist $s/q \mid \overline{p/s}$ in Q, and return to the normal order $q/s \mid \overline{p/s}$.

† This is the proposition shown by Sheffer to imply the analogous propositions *1·7 and *1·71 in *Principia*.

4. Proof of "Association," $p \mid \overline{q/r} \,.\supset.\, q \mid \overline{p/s}$.

5. Theorems equivalent to the definitions of $p \,.\, q$, $p \supset q$ in *Principia*.

Proof of Identity, $t \boldsymbol{|} t \mid t$.

As this first proof from a single formal premiss stands in a unique position, I shall, without in any way obscuring the precise play of the symbols, expound it after a more heuristic order than is usually followed.

We start with the Prop. $P \boldsymbol{|} \pi \mid Q$, and the Rule enabling us to pass from the truth of P to that of Q; and we have to prove π. This can only be reached through some proposition of the form $A \boldsymbol{|} B \mid \pi$, where A is a truth of logic†. The proof will thus consist in passing from $P \boldsymbol{|} \pi \mid Q$ to $A \boldsymbol{|} B \mid \pi$ by some permutative process.

A simple two-terms permutative law $s \mid q \boldsymbol{|} \overline{q \mid s}$, we do not yet possess. Our Prop. yields only a roundabout three-terms permutation, $s \mid q \boldsymbol{|} \overline{p \mid s}$, subject to the condition of $p \boldsymbol{|} q \mid r$ being a truth of logic†. This, however, is enough for our purpose.

In the Prop., write t for p, q, r:

(*a*) $$\pi \boldsymbol{|} \pi \mid Q_1,$$

Q_1 being $s \mid t \boldsymbol{|} \overline{t \mid s}$. Write now π for p, q; Q_1 for r: then by (*a*) and the Rule,

(*b*) $$s \mid \pi \boldsymbol{|} \overline{\pi \mid s}.$$

From (*b*), in the same manner,

(*c*) $$u \mid \pi/s \boldsymbol{|} \overline{s/\pi \mid u}.$$

This enables us to pass, by the Rule, from $P \boldsymbol{|} \pi \mid Q$ to

(*d*) $$Q \mid \pi \boldsymbol{|} P.$$

In order to complete the proof of π, we need only find some expression which: (α) can be a value for P, i.e. is a case of $p \boldsymbol{|} q \mid r$, and ($\beta$) is implied in some truth of logic, say T. For, by $T \boldsymbol{|} P \mid P$, the Prop., and the Rule, as above,

(*e*) $$s \mid P \boldsymbol{|} \overline{T \mid s}.$$

In (*e*), write $Q \mid \pi$ for s: first by (*d*) and the Rule, then by T and the Rule, we obtain $T \boldsymbol{|} Q \mid \pi$, and so

(*f*) $$\pi.$$

† This use of the Rule by anticipation, with still undetermined P's and Q's, is in truth contrary to the nature of a non-formal rule, which must never be used to build up the structure of an argument. It must always be possible to dispense with all such 'anticipated' assertions in the final form of a proof. This will be seen to be very easy in the present case.

Now, $Q_1 \mid \pi \,\boldsymbol{|}\, \pi$ fulfils (α) and (β). For (α) π being the complex expression $t \,\boldsymbol{|}\, t \mid t$, is a case of the form $q \mid r$, and (β) we have, by (c) above, $\pi \mid \pi/Q_1 \,\boldsymbol{|}\, \overline{Q_1/\pi \mid \pi}$, and by ($a$) $\pi \,\boldsymbol{|}\, \pi \mid Q_1$.

To obtain the strictest development of the proof we have only to write $Q_1/\pi \mid \pi$ for P and $\pi \mid \pi/Q_1$ for T all through the preceding argument.

PERMUTATION, $s \mid p \,\boldsymbol{|}\, \overline{p \mid s}$

$\left[\text{Gives } s \vee p \,.\supset.\, p \vee s \text{ by } \frac{p \mid p \quad s \mid s}{p \qquad s}.\right]$

Dem.: Prop. $\frac{p \quad p \quad p}{p \quad q \quad r}$, Id., and Rule.

TAUTOLOGY, $p/p \mid p/p \,\boldsymbol{|}\, p/p$

i.e. $p \vee p \,.\supset.\, p$

Dem.: Id. $\frac{p/p}{p}$, Perm., and Rule.

ADDITION, $s \,\boldsymbol{|}\, \overline{p \mid s/s}$

$\left[\text{Gives } s \,.\supset.\, p \vee s \text{ by } \frac{p/p}{p}.\right]$

Dem.: By Perm. (twice), $\overline{p \mid s/s} \,\boldsymbol{|}\, s/s \mid p$ (a)

By Prop. $\frac{\overline{p \mid s/s} \quad s/s \quad p \quad s}{p \qquad q \qquad r \quad s}$, $\vdash (a)$, $\vdash$ Id.†, $\overline{p \mid s/s} \,\boldsymbol{|}\, s$

By Perm., result.

RETURN FROM GENERALISED IMPLICATION $P \;\; \pi/Q$ TO $P \;\; Q/Q$.

Lemma, $p/p \mid \overline{s/p}$

Dem.: By Perm. (twice), $\overline{s/p} \mid p/s$ (a)

By Prop. $\frac{\overline{s/p} \quad p \quad s \quad u}{p \quad q \quad r \quad s}$, $\vdash (a)$,

$u \mid p \,\boldsymbol{|}\, \overline{\overline{s/p} \mid u}$

Write p/p for u: by Id. and Perm. (twice), result.

† $\vdash (a)$ means the use of the Rule to pass from a to b in $a \mid s/b$.

Theorem, $$P \mid \pi/Q \,\pmb{|}\, \overline{Q/Q \mid P}$$

Dem.: Prop. $\dfrac{Q \mid Q \quad \pi/Q \quad P}{p \quad q, r \quad s}$, $\vdash$ Lemma, result.

Hence, by Perm., $P \mid Q/Q$, i.e.

$$p \mid q/r \,\pmb{|}\, \overline{s/q \mid \overline{p/s}} \qquad (S')$$

SYLLOGISM, $$p \mid q/r \,\pmb{|}\, \overline{q/s \mid \overline{p/s}}$$

[Gives $p \supset q \,.\supset:\, q \supset s \,.\supset.\, p \supset s$ for $\dfrac{q \quad s/s}{r \quad s}$.]

Dem.: In this Dem., Permutation is used to correct the twisting action of S', much as handwriting has first to be inverted, if it is to be seen right in a mirror.

By $S' \dfrac{q/s \quad s/q \quad u}{p \quad q, r \quad s}$, $\vdash$ Perm., and Perm.,

$$\overline{q/s \mid u} \,\pmb{|}\, u \mid s/q \qquad (a)$$

By $S' \dfrac{s/q \mid u \quad u \mid s/q \quad \overline{q/s \mid u}}{p \quad q, r \quad s}$, $\vdash$ Perm., $\vdash a$, and Perm.,

$$\overline{q/s \mid u} \,\pmb{|}\, s/q \mid u \qquad (b)$$

By $S' \dfrac{p \mid q/r \quad s/q \mid \overline{p/s} \quad \overline{q/s \mid \overline{p/s}}}{p \quad q, r \quad s}$, $\vdash S'$, $\vdash b$, result.

ASSOCIATION, $$p \mid \overline{q/r} \,\pmb{|}\, \overline{q \mid \overline{p/r}}$$

The structure of the proof is this:

$$\text{Syll.} \dfrac{p \quad q/r \quad r}{p \quad q, r \quad s}$$

gives $$p \mid \overline{q/r} \,.\supset:\, q/r \mid r \,.\, \mid \overline{p/r}.$$

We now need only the Lemma $q \mid \overline{q/r \mid r}$ for our result to follow by Syll. twice.

Lemma, $$q \mid \overline{q/p \mid p}$$

The proof of this lemma—call it L—is as follows: We prove (*a*) $q \mid L/L$, (*b*) $L/L \mid q/q$. From this, by Syll. and Tautol., the result follows.

Dem.: (*a*) By Syll. $\dfrac{q, p}{r, s}$,

$$p \mid q/q \,.\supset.\, q/p \mid \overline{p/p} \qquad (1)$$

By Add., Syll., $\vdash$(1),

$$q\,.\supset:q/p\,|\,\overline{p/p} \qquad (2)$$

The right side of (2) implies, by Syll.,

$$p/p\,|\,p\,.\supset.\,q/p\,|\,p \qquad (3)$$

By Id., Perm., Add. $\dfrac{p/p\,|\,p, \quad q}{p, \quad q}$,

$$q\,.\supset:p/p\,|\,p \qquad (4)$$

By Syll. twice, $\vdash$(2), $\vdash$(3), $\vdash$(4),

$$q\supset:q\,.\supset.\,q/p\,|\,p, \quad \text{i.e. } q\,|\,L/L.$$

(*b*) By lemma to Syll., $q/q\,\big|\,\overline{s/q}$; by Perm. and Syll., $q/q\,|\,\overline{q/s}$. Hence, $q/q\,|\,L/L$; by Perm., $L/L\,|\,q/q$.

Now, by Syll.:

$$L/L\,|\,q/q\,.\supset:q\,|\,L/L\,.\supset.\,L/L\,|\,L/L.$$

By $\vdash b$, $\vdash a$, and Taut. $\dfrac{L}{p}$, result. We can now complete the proof of 'Association.'

Association, $\qquad p\,|\,\overline{q/r}\,\big|\,\overline{q\,|\,\overline{p/r}}$

Dem.: By Syll., $\quad p\,|\,\overline{q/r}\,.\supset:q/r\,|\,r\,.\,|\,\overline{.\,p/r}$

By Syll. twice, $\vdash$ Lemma, result.

SUMMATION, $\qquad q\supset r\,.\supset:p\vee q\,.\supset.\,p\vee r$

Dem.: By Syll., Assoc.,

$$q\,|\,s\,.\supset:p\,|\,q/r\,.\supset.\,p\,|\,s \qquad (1)$$

By (1) $\dfrac{s/s, \quad q, \quad p/p}{s, \quad r, \quad p}$, result.

THEOREMS EQUIVALENT TO THE DEFINITIONS OF $p\supset q$, $p\,.\,q$, IN *Principia*.

$p\supset q\,.\supset.\sim p\vee q$, and reciprocal theorem.

That is, $\qquad p\,|\,q/q\,.\supset.\,\overline{p/p}\,|\,q/q.$

Dem.: Taut., and Syll.

Reciprocal theorem by Add. $\dfrac{s/s, \quad p}{s, \quad p}$, and Syll.

$p\,|\,q\,.\supset.\sim p\vee\sim q$, and reciprocal theorem.

That is, $\qquad p\,|\,q\,.\supset.\,\overline{p/p}\,|\,\overline{q/q}.$

Dem.: Taut. Syll.; then, Perm., Taut., and Syll., or S'.

Reciprocal theorem by Add. $\dfrac{p/p}{p,\ s}$ instead of Taut.

$p \,.\, q \,.\, \supset \,.\, \sim(\sim p \vee \sim q)$ and reciprocal theorem.

That is, $$p \,.\, q \,.\, \supset \,.\, p/q \,|\, p/q.$$

Dem.: Id., Def. of $\sim$, preceding theorem, and Syll.

Reciprocal theorem in the same manner.

Appendix.

After the substance of this paper had been written, I was given the opportunity of seeing Mr Van Horn's very interesting and original paper dealing with what is practically the same subject. Mr Van Horn recognises clearly the superiority of what has been called above the *OR*-form over the *AND*-form chosen in Sheffer's text. This deserves the more notice, as Mr Van Horn, I understand, had not Sheffer's article at hand in the time he was writing his own paper. His Δ, as will be seen from the definitions he gives, is indistinguishable from $|$. I was much attracted by the harmonious character of Mr Van Horn's third Axiom. It seems to me therefore all the more desirable that certain objections, which Mr Van Horn's proofs in their present form naturally suggest to the reader, should be dealt with.

(α) It is not quite plain to me whether "of the same truth-value" (say S for short), "of opposite truth-values" (say O), are used as indefinables, or as abbreviations. If the former, we have no right to go, e.g., from $p\,O\,q$, and $\sim p$, to q, etc., without some axiom to that effect, connecting O and S with Δ. If, on the other hand, S and O are abbreviations—as it seems to me they are—the two parts of Axiom 3 stand for not less than four propositions:

1. If p and q, $\sim(p \Delta q)$.
2. If $\sim p$ and $\sim q$, $p \Delta q$.
3. If p and $\sim q$, $p \Delta q$.
4. If $\sim p$ and q, $p \Delta q$.

We cannot assert the first two, or the last two, or all four, propositions together, because we should then need $p \,.\, q \,.\, \supset \,.\, p$, $p \,.\, q \,.\, \supset \,.\, q$, before we could make any use of such a synthetic Axiom.

This uncertainty as to the status of S and O is not without its effect upon the proofs. Consider, for instance, Th. 3. In the proof, "1°: p true. By Axiom 3, $p \Delta p$ false" will be seen to require $p S p$, concerning the origin of which, and the relation it has to $p \supset p$ (Th. 4), which it indirectly serves to prove, Mr Van Horn says nothing.

(β) In his extensive use of the Principle of Excluded Middle, Mr Van Horn makes no explicit mention of the last steps, that lead from $p \supset q$, $\sim p \supset q$, to q. These steps would seem to require several propositions: (1) those carrying us from $\sim p \vee p$ to $q \vee q$ —"Summation," plus "Permutation," presumably—and (2) "Tautology" $q \vee q . \supset . q$. As Mr Van Horn uses the principle of Excluded Middle in this particular way in the first formal proof given—that of Th. 3—both the principle itself and the propositions required for its use ought, I think, to be deduced immediately from Axiom 3; and I do not see how this is possible.

Bessel functions of equal order and argument. By G. N. WATSON, M.A., Trinity College.

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1. A proof of the approximate formula

$$J_n(n) \sim \frac{\Gamma(\frac{1}{3})}{\pi\, 2^{\frac{2}{3}}\, 3^{\frac{1}{6}}\, n^{\frac{1}{3}}},$$

(the order and argument of the Bessel function being equal and large) was apparently first published by Graf and Gubler*, although the formula had been stated by Cauchy† many years before. The formula has been discussed more recently by Nicholson‡ and by Lord Rayleigh§, while Debye|| has given a complete asymptotic expansion of $J_n(n)$ in descending powers of n; this expansion is obtained by the aid of the elaborate and powerful machinery which is provided by the mode of contour integration known as the "Methode der Sattelpunkte¶" (Méthode du Col, method of steepest descents).

The earlier writers, just mentioned, employed Bessel's formula

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta,$$

valid when n is an integer, and it is by no means obvious to what extent their methods of approximating are valid**.

As the correctness of the approximation can be established without the use of contour integration on the one hand and without appealing to physical arguments†† on the other hand, it seems to be worth while to write out a formal and rigorous proof (based on comparatively elementary reasoning) that, when n is large and real, then

* *Einleitung in die Theorie der Besselschen Funktionen*, I. (1898), pp. 96—107.
† *Comptes Rendus*, XXXVIII. (1854), p. 993; *Oeuvres* (1), XII. p. 163.
‡ *Phil. Mag.*, August 1908, pp. 273—279.
§ *Phil. Mag.*, December 1910, pp. 1001—1004.
|| *Mathematische Annalen*, LXVII. (1909), pp. 535—538.
¶ This method of discussing $\int e^{nf(s)}\phi(s)\,ds$ consists in choosing a contour on which $If(s)$ is constant, and so $Rf(s)$ falls away from its maximum as rapidly as possible ($f(s)$ being monogenic); it is to be traced to a posthumous paper by Riemann, *Werke*, 1876, p. 405.
** See § 4 below.
†† For example Kelvin's "Principle of stationary phase" (*Phil. Mag.*, March 1887, pp. 252—255; *Math. Papers*, IV. pp. 303—306) is really based on the theory of interference. See also Stokes, *Camb. Phil. Trans.* IX. (1850), p. 175, foot-note (*Math. Papers*, II. p. 341).

$$J_n(n) = \frac{\Gamma(\frac{1}{3})}{\pi 2^{\frac{2}{3}} 3^{\frac{1}{6}} n^{\frac{1}{3}}} - \frac{2^{\frac{2}{3}} 3^{\frac{1}{6}} \Gamma(\frac{2}{3})}{140\pi n^{\frac{5}{3}}} + o(n^{-\frac{5}{3}}),$$

$$J_n'(n) = \frac{\Gamma(\frac{2}{3})}{\pi 2^{\frac{1}{3}} 3^{-\frac{1}{6}} n^{\frac{2}{3}}} + o(n^{-\frac{2}{3}}).$$

2. In order not to restrict ourselves to the case in which n is a positive integer, we take the Bessel-Schläfli integral*, namely

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\theta - x\sinh\theta}\,d\theta,$$

(which is valid whether n be an integer or not), and, after writing n for x, we integrate by parts. This process gives

$$J_n(n) = \frac{1}{n\pi}\int_0^\pi \frac{1}{1-\cos\theta}\frac{d}{d\theta}\{\sin n(\theta - \sin\theta)\}\,d\theta$$

$$+ \frac{\sin n\pi}{\pi}\int_0^\infty \frac{1}{1+\cosh\theta}\frac{d}{d\theta}\{e^{-n(\theta+\sinh\theta)}\}\,d\theta$$

$$= \frac{1}{n\pi}\left[\frac{\sin n(\theta - \sin\theta)}{1-\cos\theta}\right]_0^\pi + \frac{\sin n\pi}{\pi}\left[\frac{e^{-n(\theta+\sinh\theta)}}{1+\cosh\theta}\right]_0^\infty$$

$$+ \frac{1}{n\pi}\int_0^\pi \frac{\sin n(\theta-\sin\theta)}{(1-\cos\theta)^2}\sin\theta\,d\theta$$

$$+ \frac{\sin n\pi}{\pi}\int_0^\infty \frac{\sinh\theta}{(1+\cosh\theta)^2}e^{-n(\theta+\sinh\theta)}\,d\theta.$$

The integrated parts cancel; and

$$\int_0^\infty \frac{\sinh\theta}{(1+\cosh\theta)^2}e^{-n(\theta+\sinh\theta)}\,d\theta < \int_0^\infty (1+\cosh\theta)\,e^{-n(\theta+\sinh\theta)}\,d\theta$$

$$= 1/n;$$

and so, when n is large and real,

$$J_n(n) = \frac{1}{n\pi}\int_0^\pi \frac{\sin\theta\sin n\phi}{(1-\cos\theta)^3}\,d\phi + O(n^{-2}),$$

where ϕ has been written in place of $\theta - \sin\theta$. It is obvious that ϕ increases steadily from 0 to π as θ increases from 0 to π.

When θ is small, $\phi \sim \frac{1}{6}\theta^3$ and $\sin\theta\,.\,(1-\cos\theta)^{-3} \sim 8\theta^{-5}$. Hence, as $\theta \to 0$,

$$\frac{\phi^{\frac{5}{3}}\sin\theta}{(1-\cos\theta)^3} \to \frac{8}{6^{\frac{5}{3}}}.$$

Now write $\frac{1}{8}(6\phi)^{\frac{5}{3}}\sin\theta\,.\,(1-\cos\theta)^{-3} \equiv f_1(\phi)$;

* Schläfli, *Math. Ann.* III. (1871), p. 148.

then it is fairly evident* that when $0 \leqslant \theta \leqslant \pi$ (i.e. when $0 \leqslant \phi \leqslant \pi$), $f_1(\phi)$ is bounded and has only a finite number of maxima and minima (and therefore it has limited total fluctuation). Consequently, since† $\int_0^\infty \psi^{-\frac{5}{3}} \sin\psi\, d\psi$ *is convergent*, we have‡

$$\lim_{n\to\infty} n^{-\frac{2}{3}} \int_0^\pi \phi^{-\frac{5}{3}} \sin(n\phi) \,.\, f_1(\phi)\, d\phi = f_1(0) \int_0^\infty \psi^{-\frac{5}{3}} \sin\psi\, d\psi.$$

Therefore, since $f_1(0) = 1$, we have

$$n^{-\frac{2}{3}} \int_0^\pi \phi^{-\frac{5}{3}} \sin(n\phi) \,.\, f_1(\phi) d\phi = \frac{3\sqrt{3}}{4} \Gamma(\tfrac{1}{3}) + o(1),$$

and so $$J_n(n) = 2^{-\frac{2}{3}} 3^{-\frac{1}{6}} \pi^{-1} \Gamma(\tfrac{1}{3}) n^{-\frac{1}{3}} + o(n^{-\frac{1}{3}}).$$

To obtain the second approximation to $J_n(n)$, we observe that, when θ is small,

$$\frac{(6\phi)^{\frac{5}{3}} \sin\theta}{8(1-\cos\theta)^3} - 1 = \frac{(1 - \frac{1}{6}\theta^2 + \frac{1}{120}\theta^4 - \ldots)(1 - \frac{1}{20}\theta^2 + \frac{1}{840}\theta^4 - \ldots)^{\frac{5}{3}}}{(1 - \frac{1}{12}\theta^2 + \frac{1}{360}\theta^4 - \ldots)^3} - 1$$

$$\sim -\tfrac{1}{280}\theta^4.$$

Consequently, if $\phi^{-\frac{4}{3}} \left\{ \frac{\phi^{\frac{5}{3}} \sin\theta}{(1-\cos\theta)^3} - \frac{8}{6^{\frac{5}{3}}} \right\} \equiv -f_2(\phi),$

we have $f_2(0) = 6^{-\frac{1}{3}} \div 35$. Also, as in the case of $f_1(\phi)$, we assume§ for the moment that $f_2(\phi)$ has limited total fluctuation in the range $(0, \pi)$. The application of Bromwich's theorem is therefore permissible, and we deduce that

$$\lim_{n\to\infty} n^{\frac{2}{3}} \int_0^\pi \phi^{-\frac{1}{3}} \sin(n\phi) \,.\, f_2(\phi) d\phi = 3^{\frac{1}{6}}\, 2^{-\frac{4}{3}}\, \Gamma(\tfrac{2}{3})/35,$$

* A formal proof will be given in § 5A that $f_1(\phi)$ is, in fact, monotonic and decreasing (we use the term decreasing to mean non-increasing).

† Euler's result that $\int_0^\infty \psi^{m-1} \sin\psi\, d\psi = \Gamma(m) \sin(\frac{1}{2}m\pi)$, when $-1 < m < 1$, is well known.

‡ Bromwich, *Infinite Series*, p. 444, proves that, *if $f(\phi)$ has limited total fluctuation in the range* $(0, b)$, where $b > 0$, and if $U_n = \int_0^b \frac{\sin n\phi}{\phi} f(\phi)\, d\phi$, then

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \int_0^{nb} \frac{\sin\psi}{\psi} f(\psi/n)\, d\psi = f(0) \int_0^\infty \frac{\sin\psi}{\psi} d\phi\,;$$

but his analysis is equally applicable to the more general integral

$$V_n = n^m \int_0^b \phi^{m-1} \sin(n\phi) \,.\, f(\phi)\, d\phi \qquad (-1 < m < 1),$$

and hence

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \int_0^{nb} \psi^{m-1} \sin\psi \,.\, f(\psi/n)\, d\psi = f(0) \int_0^\infty \psi^{m-1} \sin\psi\, d\psi.$$

§ A formal proof will be given in § 5B that $f_2(\phi)$ is monotonic and increasing.

that is

$$\int_0^{\pi} \phi^{-\frac{1}{3}} \sin(n\phi) . f_2(\phi)\, d\phi = 3^{\frac{1}{6}}\, 2^{-\frac{4}{3}}\, n^{-\frac{2}{3}}\, \Gamma(\tfrac{2}{3})/35 + o(n^{-\frac{2}{3}}),$$

and so

$$J_n(n) = \frac{8}{n^{\frac{1}{3}}\pi}\left\{\int_0^{\infty} \frac{\sin\psi}{(6\psi)^{\frac{5}{3}}}\, d\psi - \int_{n\pi}^{\infty} \frac{\sin\psi}{(6\psi)^{\frac{5}{3}}}\, d\psi\right\}$$

$$- \frac{1}{n\pi}\int_0^{\pi} \phi^{-\frac{1}{3}} f_2(\phi) \sin(n\phi) . d\phi + O(n^{-2})$$

$$= \frac{\Gamma(\frac{1}{3})}{\pi\, 2^{\frac{2}{3}}\, 3^{\frac{1}{6}}\, n^{\frac{1}{3}}} - \frac{2^{\frac{2}{3}}\, 3^{\frac{1}{6}}\, \Gamma(\frac{2}{3})}{140\pi n^{\frac{5}{3}}} - \frac{8}{n^{\frac{1}{3}}\pi}\int_{n\pi}^{\infty} \frac{\sin\psi}{(6\psi)^{\frac{5}{3}}}\, d\psi + o(n^{-\frac{5}{3}}).$$

Now, by the second mean-value theorem, there exists a number α exceeding $n\pi$ such that

$$\left|\int_{n\pi}^{\infty} \frac{\sin\psi}{\psi^{\frac{5}{3}}}\, d\psi\right| = \frac{1}{(n\pi)^{\frac{5}{3}}}\left|\int_{n\pi}^{\alpha} \sin\psi\, d\psi\right| < 2(n\pi)^{-\frac{5}{3}},$$

and so we have at once that, when n is large and real,

$$J_n(n) = \frac{\Gamma(\frac{1}{3})}{\pi\, 2^{\frac{2}{3}}\, 3^{\frac{1}{6}}\, n^{\frac{1}{3}}} - \frac{2^{\frac{2}{3}}\, 3^{\frac{1}{6}}\, \Gamma(\frac{2}{3})}{140\pi n^{\frac{5}{3}}} + o(n^{-\frac{5}{3}}),$$

which is the result to be established. To obtain a closer approximation by these methods would necessitate some very tedious integrations by parts.

3. We next consider the approximate formula for $J_n'(n)$. It is immediately deduced from the Bessel-Schläfli integral that

$$J_n'(n) = \frac{1}{\pi}\int_0^{\pi} \sin\theta . \sin n(\theta - \sin\theta) . d\theta$$

$$+ \frac{\sin n\pi}{\pi}\int_0^{\infty} \sinh\theta\, e^{-n(\theta + \sinh\theta)}\, d\theta.$$

Now we get, on integrating by parts,

$$\int_0^{\infty} \sinh\theta\, e^{-n(\theta+\sinh\theta)}\, d\theta$$

$$= -\frac{1}{n}\int_0^{\infty} \frac{\sinh\theta}{1+\cosh\theta} \cdot \frac{d}{d\theta}\{e^{-n(\theta+\sinh\theta)}\}\, d\theta$$

$$= \frac{1}{n}\int_0^{\infty} e^{-n(\theta+\sinh\theta)} . \tfrac{1}{2}\operatorname{sech}^2 \tfrac{1}{2}\theta\, d\theta$$

$$< \frac{1}{2n}\int_0^{\infty} e^{-n\theta}\, d\theta = O(n^{-2}).$$

Hence $$J_n'(n) = \frac{1}{\pi}\int_0^\pi \frac{\sin\theta}{1-\cos\theta}\sin n\phi\, d\phi + O(n^{-2}),$$

where ϕ, as previously, stands for $\theta - \sin\theta$.

Now, if $f_3(\phi) \equiv \phi^{\frac{1}{3}}\sin\theta\,.\,(1-\cos\theta)^{-1}$, then $f_3(0) = 2^{\frac{2}{3}}3^{-\frac{1}{3}}$ and $f_3(\phi)$ has limited total fluctuation* in the range $(0, \pi)$.

Hence, applying Bromwich's theorem we have

$$n^{\frac{2}{3}}\int_0^\pi \frac{\sin n\phi}{\phi^{\frac{1}{3}}} f_3(\phi)\, d\phi = f_3(0)\int_0^\infty \frac{\sin\psi}{\psi^{\frac{1}{3}}}\, d\psi + o(1),$$

and so $$J_n'(n) = \frac{2^{-\frac{1}{3}}3^{\frac{1}{6}}\Gamma(\frac{2}{3})}{\pi n^{\frac{2}{3}}} + o(n^{-\frac{2}{3}}) + O(n^{-2}),$$

when n is large and real; and this is equivalent to the result stated in § 1. The approximation could be carried one stage further (as in § 2), but it seems hardly necessary to give the analysis.

4. As an example of the necessity for the caution which has to be taken in approximating to integrals with rapidly oscillating integrands, it may be remarked that some of the earlier writers mentioned in § 1 assumed that when x and n are large and nearly equal [in fact, when $|x - n| = o(n^{\frac{1}{3}})$], then Airy's integral

$$A_n(x) \equiv \frac{1}{\pi}\int_0^\infty \cos\{n\theta - x(\theta - \tfrac{1}{6}\theta^3)\}\, d\theta$$

is an approximation to Bessel's integral for $J_n(x)$. This assumption is correct, and it happens that the first *two* terms in the asymptotic expansions of $A_n(x)$ and $J_n(x)$ are the same.

But Airy's integral for $A_n(n\alpha)$ is *not* an approximation† to $J_n(n\alpha)$ when α is fixed and $0 < \alpha < 1$, while $n \to \infty$.

To establish this statement we use Carlini's formula‡

$$J_n(n\alpha) \sim \frac{\alpha^n e^{n\sqrt{(1-\alpha^2)}}}{\{1+\sqrt{(1-\alpha^2)}\}^n\,.\,(1-\alpha^2)^{\frac{1}{4}}\sqrt{(2\pi n)}},$$

(valid when $0 < \alpha < 1$), and after observing that we may write

$$A_n(n\alpha) = \frac{1}{\pi}\left(\frac{3\pi}{n\alpha}\right)^{\frac{1}{3}}\int_0^\infty \cos\{\tfrac{1}{2}\pi(mw + w^3)\}\, dw,$$

* A formal proof will be given in § 5 c that $f_3(\phi)$ is monotonic and decreasing.

† For example, the arguments given in the *Phil. Mag.*, August 1908, p. 274, to justify the approximation seem to me to be as applicable to the second case as to the first.

‡ A translation of Carlini's memoir (published at Milan, 1817) was given by Jacobi, *Astr. Nach.* xxx. (1850); *Ges. Werke*, VII. pp. 189—245. See p. 240 for the formula quoted.

where $$m = \frac{2n(1-\alpha)}{\pi}\left(\frac{3\pi}{n\alpha}\right)^{\frac{1}{3}}, \qquad w = \left(\frac{n\alpha}{3\pi}\right)^{\frac{1}{3}}\theta,$$

we use Stokes' asymptotic formula*

$$\int_0^\infty \cos\{\tfrac{1}{2}\pi(mw + w^3)\}\,dw \sim 2^{-\frac{1}{2}}(3m)^{-\frac{1}{4}}\exp\{-\pi(\tfrac{1}{3}m)^{\frac{3}{2}}\},$$

valid for large values of m.

This process gives

$$A_n(n\alpha) \sim \frac{\exp\{-\frac{1}{3}n\,2^{\frac{3}{2}}\,\alpha^{-\frac{1}{2}}(1-\alpha)^{\frac{3}{2}}\}}{\{2\alpha(1-\alpha)\}^{\frac{1}{4}}\sqrt{(2\pi n)}}.$$

Hence $$\frac{J_n(n\alpha)}{A_n(n\alpha)} \sim \left(\frac{2\alpha}{1+\alpha}\right)^{\frac{1}{4}} e^{n\chi(\alpha)},$$

where

$$\chi(\alpha) = \sqrt{(1-\alpha^2)} + \log\alpha - \log\{1+\sqrt{(1-\alpha^2)}\} + \tfrac{1}{3}\alpha^{-\frac{1}{2}}(2-2\alpha)^{\frac{3}{2}}.$$

Since $$\chi(\tfrac{1}{2}) = \cdot 02047,$$

a rough approximation to $J_{1000}(500)/A_{1000}(500)$ is $(\frac{2}{3})^{\frac{1}{4}}e^{20\cdot 47}$.

5. We now prove the monotonic properties (valid for $0 \leqslant \theta \leqslant \pi$) stated in §§ 2, 3:

(A) To prove that $f_1(\phi) \equiv \frac{1}{8}(6\phi)^{\frac{5}{3}}\sin\theta\,.\,(1-\cos\theta)^{-3}$ is a decreasing function, we have

$$\frac{d}{d\theta}\{8f_1(\phi)/6^{\frac{5}{3}}\} = \frac{(3+2\cos\theta)\,\phi^{\frac{2}{3}}\,g_1(\theta)}{(1-\cos\theta)^3},$$

where $g_1(\theta) \equiv [5\sin\theta(1-\cos\theta)/(9+6\cos\theta)] - \theta + \sin\theta$,

so that

$$g_1'(\theta) = -6(1-\cos\theta)^3/(9+6\cos\theta)^2 \leqslant 0, \text{ and } g(0) = 0.$$

We now see that $g_1(\theta) \leqslant 0$, and so $f_1'(\phi) \leqslant 0$, which is the result stated.

(B) To prove that

$$f_2(\phi) \equiv \phi^{-\frac{4}{3}}[(8/6^{\frac{5}{3}}) - \{\phi^{\frac{5}{3}}\sin\theta/(1-\cos\theta)^3\}]$$

is an increasing function, we first prove two subsidiary theorems, namely:

B (i). *If* $c \equiv \cos\theta$, $s \equiv \sin\theta$, *then the function*

$$g_4(\theta) \equiv (85 + 163c + 84c^2 + 18c^3)\,\phi - \tfrac{1}{3}s(1-c)(149 + 157c + 44c^2)$$

is not positive.

* *Math. Papers*, II. p. 343. The result may also easily be derived from Nicholson's expression of Airy's integral in terms of Bessel functions of order $\pm\frac{1}{3}$, *Phil. Mag.*, July 1909, pp. 6—17.

B (ii). *The function*

$$g_5(\theta) \equiv 2s(7+3c)\phi^2 - 3(3+2c)(1-c)^2\phi + \tfrac{5}{9}s(1-c)^3$$

is not positive.

To prove B (i) we observe that

$$\frac{d}{d\theta}\{g_4(\theta)/(85+163c+84c^2+18c^3)\}$$

$$= - s^2(1-c)^3(644+416c+60c^2)/(85+163c+84c^2+18c^3)^2$$

$$\leqslant 0.$$

The denominator may be written in the form

$$18\gamma^3 + 30\gamma^2 + 49\gamma - 12$$

where $\gamma \equiv 1+c$, and so the denominator changes sign once only when $0 < \theta < \pi$, say at $\theta = \beta$. Hence

$$g_4(\theta)/(85+163c+84c^2+18c^3)$$

decreases from 0 to $-\infty$ and then from $+\infty$ to π as θ increases from 0 to β and then from β to π. Hence $g_4(\theta)$ cannot be positive.

To prove B (ii) we observe that

$$\frac{d}{d\theta}[\tfrac{1}{2}g_5(\theta)/\{s(7+3c)\}] = (1-c)\,g_4(\theta)/\{2(1+c)(7+3c)^2\} \leqslant 0,$$

by B (i); and so $g_5(\theta) \leqslant g_5(0) = 0$, as was to be proved.

To prove the main theorem, we have

$$\frac{df_2(\phi)}{d\theta} = \{-(64/6^{\frac{8}{3}}) + g(\theta)\}\,\phi^{-\frac{7}{3}}(1-\cos\theta),$$

where $\qquad g(\theta) \equiv \{(3+2c)\phi^{\frac{8}{3}} - s(1-c)\phi^{\frac{5}{3}}\}(1-c)^{-4}.$

Now $\qquad g'(\theta) = -g_5(\theta)\,\phi^{\frac{2}{3}}(1-c)^{-5} \geqslant 0,$

so that $\qquad g(\theta) \geqslant g(0) = 64/6^{\frac{8}{3}},$

and so $f_2'(\phi) \geqslant 0$, as was to be proved.

(C) To prove that $f_3(\phi) \equiv \phi^{\frac{1}{3}}\sin\theta\,.\,(1-\cos\theta)^{-1}$ is a decreasing function, we have

$$\frac{df_3(\phi)}{d\theta} = \tfrac{1}{3}\phi^{-\frac{2}{3}}(1-\cos\theta)^{-1}g_3(\theta),$$

where $\qquad g_3(\theta) \equiv \sin\theta(1-\cos\theta) - 3(\theta - \sin\theta).$

Since $g_3'(\theta) = -2(1-\cos\theta)^2$ we may use the arguments of (A) to prove the truth of theorem (C).

The limits of applicability of the Principle of Stationary Phase. By G. N. WATSON, M.A., Trinity College.

[*Received* 22 November 1916.]

1. The method of approximating to the value of the integral

$$u = \frac{1}{2\pi} \int_0^\infty \cos [m \{x - tf(m)\}] \, dm,$$

where x and t are large, by considering the contribution to the integral of the range of values of m in the immediate vicinity of the stationary values of $m \{x - tf(m)\}$, is due to Kelvin*, though the germ of the idea may be traced in a paper published nearly forty years earlier by Stokes†.

Kelvin's result is that, if $m \{x - tf(m)\}$ has a minimum when $m = \mu > 0$, then, as $t \to \infty$,

$$u \sim (2\pi t)^{-\frac{1}{2}} \{-\mu f''(\mu) - 2f'(\mu)\}^{-\frac{1}{2}} \cos \{t\mu^2 f'(\mu) + \tfrac{1}{4}\pi\};$$

and this result has important applications in connexion with various problems of mathematical physics‡.

Kelvin, in his analysis of this interesting asymptotic formula, takes for granted, on physical grounds, the validity of a certain passage to the limit. This process requires justification from the purely mathematical point of view; and the necessary justification is afforded by a convergence theorem due to Bromwich§. This theorem plays the same part in dealing with integrals as an analogous theorem, due to Tannery||, plays in connexion with series.

The special form of Bromwich's theorem, which is required in the rigorous investigation of Kelvin's theorem, may be enunciated as follows:

If $f(x)$ be a function of x with limited total fluctuation in the range $x \geqslant 0$, and if γ be a function of n such that $n\gamma \to \infty$ as $n \to \infty$, then, if $-1 < m < 1$,

* *Phil. Mag.*, March 1887, pp. 252—255 (*Math. and Physical Papers*, IV. pp. 303—306).

† *Camb. Phil. Trans.* IX. (1851), p. 175 (*Math. and Physical Papers*, II. p. 341).

‡ See Macdonald, *Phil. Trans.* 210 A. (1910), pp. 134—145.

§ Bromwich, *Theory of Infinite Series*, p. 444. In the special case $m = 0$, which is explicitly considered by Bromwich, the result is important in the investigation of Fourier series by the method of Dirichlet. The theorem given by Bromwich on p. 443 is equally applicable to the more general case.

|| *Fonctions d'une variable*, p. 183.

$$n^m \int_0^{\gamma} x^{m-1} f(x) \sin nx\, dx \to f(+0) \int_0^{\infty} t^{m-1} \sin t\, dt$$
$$= f(+0)\,\Gamma(m) \sin \tfrac{1}{2} m\pi.$$

[*If* $0 < m < 1$, *the sines may be replaced throughout by cosines; and, if* $n\gamma \to a$ *as* $n \to \infty$, *where* a *is finite, the infinity in the upper limit of the integral must be replaced by* a.]

As the formal analytical proof of a theorem* slightly more general than Kelvin's theorem is quite simple, and as sufficient general restrictions to be satisfied by the function $f(m)$ are apparent in the course of the investigation, it seems to be worth while to place the theorem on record. It is applicable to all kinds of stationary points, whereas Kelvin considered only cases of true maxima or minima of the simplest type.

2. The main theorem which will be proved in this paper is as follows†:

Let α, β *be any numbers (infinity not excluded), possibly depending on the variable* n, *such that the real function* $bt - tf(t)$ *has only one stationary value in the range* $\alpha \leqslant t \leqslant \beta$, *at* $t = \mu$, b *being independent of* n. *Let the first* r *differential coefficients with regard to* t *of* $bt - tf(t)$, *be continuous‡ in a range of values of* t *of which* $t = \mu$ *is an* interior *point, it being supposed that the last of them is the lowest which does not vanish at* $t = \mu$, *so that* $r \geqslant 2$.

Let $F(t)$ *be a real function, continuous when* $\alpha < t < \beta$, *except possibly at* $t = \mu$, *and let*

$$\lim_{t \to \mu+0} F(t) \,.\, (t-\mu)^{\lambda} = A, \quad \lim_{t \to \mu-0} F(t) \,.\, (\mu - t)^{\lambda} = A_1,$$

where A, A_1 *are not zero; for brevity, let* $(1-\lambda)/r = m$.

Then, if the function

$$F(t) \,.\, |bt - tf(t) - \mu^2 f'(\mu)|^{1-m} \,.\, |b - tf'(t) - f(t)|^{-1}$$

has limited total fluctuation§ in the range $\alpha \leqslant t \leqslant \beta$, *and if*

$$|nb\beta - n\beta f(\beta) - n\mu^2 f'(\mu)|, \quad |nb\alpha - n\alpha f(\alpha) - n\mu^2 f'(\mu)|$$

both tend to infinity with n, *the approximate value of the integral*

$$I \equiv \frac{1}{2\pi} \int_{\alpha}^{\beta} F(t) \cos\{bnt - ntf(t)\}\, dt,$$

* For the connexion between this theorem and a problem, due to Riemann (*Werke*, p. 260), which has been discussed by Fejér (*Comptes Rendus*, November 30, 1908, and a memoir published by the Academy of Budapest in 1909) and by Hardy (*Quarterly Journal*, XLIV. 1913, pp. 1—40 and 242—263), see § 4 below.

† It is convenient to modify Kelvin's notation.

‡ It is necessary for $f(t)$ to have a continuous first differential coefficient when $\alpha < t < \beta$.

§ If the fluctuation depends on n, it must be a bounded function of n as $n \to \infty$.

when n is large, is

$$\frac{(r-1)!(r!)^{m-1}\Gamma(m)[A\cos\{n\mu^2f'(\mu)+\frac{1}{2}\epsilon m\pi\}+A_1\cos\{n\mu^2f'(\mu)+\frac{1}{2}\eta m\pi\}]}{2\pi n^m\{|\mu f^{(r)}(\mu)+rf^{(r-1)}(\mu)|\}^m},$$

provided that $0<1-\lambda<r$; where $\epsilon=\pm 1$ according as $bt-tf(t)$ is an $\begin{matrix}\textit{increasing}\\ \textit{decreasing}\end{matrix}$ function when $t>\mu$, and $\eta=\pm 1$ according as the same function is $\begin{matrix}\textit{decreasing}\\ \textit{increasing}\end{matrix}$ when $t<\mu$. When $n\to\infty$ by only such values that $\cos\{n\mu^2f'(\mu)\}$ is always zero, λ may lie in the extended range $-r<1-\lambda<r$. And, finally, $F(t)$ and $bt-tf(t)$ may be infinite at $t=\alpha$, β, provided only that the integral converges for all sufficiently large values of n.

3. For brevity, write $tf(t)\equiv\phi(t)$. Then μ is given by the equation

$$b-\phi'(\mu)=0,$$

so that, when $|t-\mu|$ is sufficiently small,

$$\begin{aligned} bt-tf(t)&=t\phi'(\mu)-\phi(t)\\ &=\{\mu\phi'(\mu)-\phi(\mu)\}-(t-\mu)^r\phi^{(r)}(t')/r!, \end{aligned}$$

where, by Taylor's theorem, t' lies between μ and t.

Now define a new variable ψ by the equation

$$bt-tf(t)=\mu\phi'(\mu)-\phi(\mu)+\psi,$$

and let γ, Γ be the values of ψ corresponding to $t=\alpha$, $t=\beta$.

Noticing that $\mu\phi'(\mu)-\phi(\mu)=\mu^2f'(\mu)$, we have

$$I=\frac{\cos\{n\mu^2f'(\mu)\}}{2\pi}\int_\alpha^\beta F(t)\cos n\psi\,dt \\ -\frac{\sin\{n\mu^2f'(\mu)\}}{2\pi}\int_\alpha^\beta F(t)\sin n\psi\,dt;$$

also $$\int_\alpha^\beta F(t)\begin{matrix}\cos\\ \sin\end{matrix}(n\psi)dt=\left\{\int_\gamma^0+\int_0^\Gamma\right\}F(t)\begin{matrix}\cos\\ \sin\end{matrix}(n\psi)\frac{dt}{d\psi}d\psi,$$

ψ being a monotonic function of t when $\alpha\leqslant t\leqslant\mu$ and also when $\mu\leqslant t\leqslant\beta$.

Now ϵ, η have been so chosen that $\epsilon\psi$ and $\eta\psi$ are positive when $t>\mu$ and $t<\mu$ respectively; hence, when $t\to\mu+0$, we have

$$F(t)\frac{dt}{d\psi}\sim -A(t-\mu)^{-\lambda-r+1}(r-1)!\div\phi^{(r)}(\mu),$$

$$\epsilon\psi\sim(t-\mu)^r|\phi^{(r)}(\mu)|\div r!.$$

It follows that

$$F(t)\frac{dt}{d\psi}\Big/(\epsilon\psi)^{(-\lambda-r+1)/r}\to AK,$$

as $t\to\mu+0$, where

$$K\equiv\frac{-(r-1)!\,(r!)^{m-1}}{\phi^{(r)}(\mu)\,\{|\,\phi^{(r)}(\mu)\,|\}^{m-1}}.$$

Since $|\,n\Gamma\,|\to\infty$ with n, by hypothesis, we deduce from Bromwich's theorem that

$$\int_0^\Gamma F(t)\,{\cos\atop\sin}\,n\psi\,\frac{dt}{d\psi}\,d\psi\sim n^{-m}AK\int_0^{\epsilon\infty}(\chi\epsilon)^{m-1}\,{\cos\atop\sin}\,\chi d\chi.$$

Writing $\chi\epsilon\equiv\omega$, we get

$$\int_0^{\epsilon\infty}(\chi\epsilon)^{m-1}\cos\chi d\chi=\epsilon\int_0^\infty\omega^{m-1}\cos\omega d\omega=\epsilon\Gamma(m)\cos\tfrac{1}{2}m\pi,$$

and similarly

$$\int_0^{\epsilon\infty}(\chi\epsilon)^{m-1}\sin\chi d\chi=\Gamma(m)\sin\tfrac{1}{2}m\pi.$$

In like manner, when $t\to\mu-0$,

$$F(t)\frac{dt}{d\psi}\Big/(\eta\psi)^{(-\lambda-r+1)/r}\to(-)^{r-1}A_1K,$$

and so, since $|\,n\gamma\,|\to\infty$ with n, we have

$$\int_\gamma^0 F(t)\,{\cos\atop\sin}\,n\psi\,\frac{dt}{d\psi}\,d\psi\sim(-)^r n^{-m}A_1K\int_0^{\eta\infty}(\chi\eta)^{m-1}\,{\cos\atop\sin}\,\chi d\chi.$$

Collecting our results, we see that the first approximation to I is

$$\begin{aligned}I\sim{}&[\{AK\epsilon+(-)^rA_1K\eta\}\cos\tfrac{1}{2}m\pi\cos\{n\mu^2f'(\mu)\}\\&-\{AK+(-)^rA_1K\}\sin\tfrac{1}{2}m\pi\sin\{n\mu^2f'(\mu)\}]\\={}&A\,[\cos\{n\mu^2f'(\mu)+\tfrac{1}{2}\epsilon m\pi\}+A_1\cos\{n\mu^2f'(\mu)+\tfrac{1}{2}\eta m\pi\}]\\&\times\frac{(r-1)!\,(r!)^{m-1}\,\Gamma(m)}{2\pi n^m\{|\,\phi^{(r)}(\mu)\,|\}^m},\end{aligned}$$

and this is the result stated.

The formula fails to be effective in the neighbourhood of those values of n for which the expression in [ ] vanishes, as the error in the approximation then becomes comparable with the approximation obtained.

[It is evident that if the cosine in the integral defining I may be replaced by a sine, then the cosines in the approximation are replaced by sines.]

Cases of practical importance are those in which $A = A_1$ and $t=\mu$ is a true minimum or maximum of $bt - tf(t)$, so that ϵ and η are both $+1$ or both -1. The formula then is

$$I \sim \frac{A \cos\{n\mu^2 f'(\mu) \pm \tfrac{1}{2}m\pi\}\,.\,(r-1)\,!\,(r\,!)^{m-1}\,\Gamma(m)}{\pi n^m \{|\phi^{(r)}(\mu)|\}^m}.$$

If $n\Gamma$ or $n\gamma$ tend to finite limits, the gamma functions have to be replaced by incomplete gamma functions; and if one or other tends to zero, we modify the approximation by writing zero for A or A_1 respectively in the general formula.

The general result reduces to Kelvin's formula when $r=2$, $\lambda = 0$, $m=\frac{1}{2}$, and $\epsilon=\eta=1$, provided that (with Kelvin's notation) x/t is constant. In that case, a sufficient condition for the validity of the formula is that

$$\frac{d}{dm}\,[\{(mx/t) - mf(m) - \mu^2 f'(\mu)\}^{\frac{1}{2}}]^{-1}$$

should have limited total fluctuation when $m \geqslant 0$.

If x were a function of t, Bromwich's general theorem (*loc. cit.*, p. 443) would have to be used, and the enunciation of sufficient conditions (even in their simplest form) for the validity of the formula, would be exceedingly laborious. The reason for this is that (with the notation employed in this paper) ψ and $F(t)\,dt/d\psi$ would both be functions of n.

4. The problem of Riemann (see § 1 above) essentially consists in obtaining an approximation for integrals of the type

$$\int_0^\infty \rho(t) \begin{matrix}\cos\\ \sin\end{matrix}\, \sigma(t)\, \frac{\sin nt}{t}\, dt,$$

when n is large and $\sigma'(t) \to \infty$ as $t \to 0$.

These integrals are expressible by integrals of the type

$$\int_0^\infty t^{-1}\rho(t) \begin{matrix}\cos\\ \sin\end{matrix}\, \{nt \pm \sigma(t)\}\, dt,$$

so that the problem is, at first sight, very similar to that discussed in §§ 2—3.

There is however an essential difference, namely that, in the problem we have discussed, $ntf(t)$ owes its large rate of increase (which balances the rate of increase of nbt at the stationary point) to the large factor n, whereas, in the problem attacked by Fejér and Hardy, the function $\sigma(t)$ owes its large rate of increase to the infinity of $\sigma'(t)$ at $t=0$. In our problem μ is *fixed*, whereas in the other problem the stationary point of $nt - \sigma(t)$ tends to zero as $n \to \infty$. It seems to be this difference which accounts for the somewhat elaborate investigation given by Hardy and which makes the theorems of Fejér and Hardy rather deeper than the theorem of §§ 2—3.

It should be pointed out that there is one integral which can be regarded as coming under either head, namely*,

$$\int_0^\infty x^{-\lambda} \frac{\sin}{\cos} (nx + ax^{-r})\, dx,$$

where n is large, a, λ, and r are positive and λ and r are chosen so that the integral converges. [For the sine-integral, the conditions for convergence are $0 < \lambda < r + 1$.] As the integral stands it is of the type discussed by Fejér and Hardy, with a variable stationary point where $x^{r+1} = ar/n$. But if we make the substitution

$$nx^{r+1} = t^{r+1},$$

and then write ν for $n^{r/(r+1)}$, it becomes

$$\nu^{(\lambda-1)/r} \int_0^\infty t^{-\lambda} \frac{\sin}{\cos} \{\nu (t + at^{-r})\}\, dt,$$

which is of the type discussed in this paper, having a *fixed* stationary point where $t = (ra)^{1/(r+1)}$. The reader will have no difficulty in deducing the approximate formula by either method.

5. As an example of the apparent inapplicability of the methods of this paper consider the integral of Bessel for $J_n(x)$ when n and x are both large and $x - n$ is $O(n^{\frac{1}{3}})$.

The integral is

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin\theta)\, d\theta,$$

and the stationary point is given by $\cos\theta = n/x$; let the root of this equation be $\theta = \mu$, and let $x = n + an^{\frac{1}{3}}$ where $a > 0$; when n is large we have

$$\mu \sim (2a)^{\frac{1}{2}} n^{-\frac{1}{3}}.$$

In considering $\int_0^\mu \cos(n\theta - x\sin\theta)\, d\theta$, we write

$$\chi = n\theta - x \sin\theta - (n\mu - x\sin\mu),$$

and the last integral is expressible by integrals of the type

$$\int_0^{n(\tan\mu - \mu)} \frac{\cos}{\sin} \chi \frac{d\theta}{d\chi}\, d\chi.$$

* I am indebted to Mr Hardy for suggesting that the integral in which $\sigma(t) = 1/t$ can be reduced to an integral of Kelvin's type.

Now $$\frac{d\chi}{d\theta} = n - x\cos\theta \sim x(\theta-\mu)\sin\mu,$$

when $\theta \sim \mu$ and $\chi \sim \frac{1}{2}x\sin\mu\,.\,(\theta-\mu)^2$.

Hence, as $\chi \to 0$,

$$\chi^{\frac{1}{2}}\frac{d\theta}{d\chi} \to \frac{-1}{\sqrt{(2x\sin\mu)}},$$

and

$$\int_0^{n(\tan\mu-\mu)} \frac{\cos}{\sin}\chi \frac{d\theta}{d\chi}$$

$$= \frac{1}{\sqrt{(2x\sin\mu)}}\int_0^{n(\tan\mu-\mu)} \left\{\sqrt{(2x\sin\mu)}\,.\,\chi^{\frac{1}{2}}\frac{d\theta}{d\chi}\right\}\chi^{-\frac{1}{2}}\frac{\cos}{\sin}\chi\,d\chi.$$

Now, as $n\to\infty$, $n(\tan\mu-\mu)\to\frac{1}{3}(2a)^{\frac{3}{2}}$ and so the limiting range of integration is of *finite* length.

Moreover, $\sqrt{(2x\sin\mu)}\,.\,\chi^{\frac{1}{2}}\dfrac{d\theta}{d\chi}\to -1$ as $n\to\infty$ *when* χ *is zero, that is, when* $\theta=\mu$. *But, when* $\theta\to 0$, *the limit of* $\sqrt{(2x\sin\mu)}\,.\,\chi^{\frac{1}{2}}\dfrac{d\theta}{d\chi}$ *is*

$$-\{2\sin\mu(\sin\mu-\mu\cos\mu)\}^{\frac{1}{2}}/(1-\cos\mu),$$

and, as $n\to\infty$, *the limit of this is not* -1 *but* $-2\sqrt{(\frac{2}{3})}$; and so we *cannot* infer that

$$-\int_0^{n(\tan\mu-\mu)}\left\{\sqrt{(2x\sin\mu)}\,.\,\chi^{\frac{1}{2}}\frac{d\theta}{d\chi}\right\}\chi^{-\frac{1}{2}}\frac{\cos}{\sin}\chi\,d\chi \sim \int_0^b \chi^{-\frac{1}{2}}\frac{\cos}{\sin}\chi\,d\chi,$$

where b is $\operatorname{Lim} n(\tan\mu-\mu)$.

The evaluation of the approximate formula for $J_n(x)$ in the circumstances under consideration consequently seems to require more elaborate analysis than is afforded by the methods contained in this paper.

On the Functions of the Mouth-Parts of the Common Prawn. By L. A. Borradaile, M.A., Selwyn College.

[*Read* 30 October 1916.]

The food is seized by either pair of chelipeds, or by the third maxillipeds, and is usually placed by them within the grasp of the second maxillipeds, though sometimes it is passed directly to deeper-lying structures. The second maxillipeds are the most important of the food-grasping organs. They have three principal movements; in one, the broad flaps in which they end open downwards like a pair of doors, and with their stout fringes gather up the food; in another, they rotate in the horizontal plane to and from the middle line of the body, and thus narrow or widen the gap through which the food passes; in the third, the bent distal part of the limb tends to straighten, so as to brush forward any object which lies between them. Frequently these movements are combined. Owing to the facts that the second maxillipeds cover the mouth-parts anterior to them, and that if they be removed feeding is not properly performed and usually not attempted, it is difficult to trace the food beyond them, but the following seems to be its fate. If it be small in bulk, or finely divided, or very soft, it is passed to the maxillules, by whose strong, fringed laciniae it is swept forwards, and probably caused to enter through the slit between the paragnatha, into the chamber which is guarded by the upper and lower lips. If it be tough or in large masses, the second maxillipeds and maxillules brush it forwards towards the incisor processes of the mandibles. The action of the latter is, by rotating in a vertical plane, to tuck the food into the gap between the paragnatha and the labrum. If the mass be large, pieces are torn off it by this action. Finally, to enter the gullet, the food must pass between the molar processes and be pounded by them.

The mandibular palps, maxillae, and first maxillipeds appear to play parts of little importance in regard to the food. The palps are present and absent in closely related genera, and appear to be disappearing in the higher Carides. The same is true of the lobes of the maxillae, which are in constant regular motion to and from the middle line, and probably serve to restrain the action of the scaphognathite. The large laciniae of the first maxilliped may have as their function the covering of the maxillae and protecting them from the food. The labrum undergoes active movements, whose function is probably to aid in keeping the food under the action of the mandibles. The exopodites of the maxillipeds set up a strong current forwards from the mouth. No doubt this aids in carrying away the exhausted water from the gill chamber and the excreta from the tubercles of the green glands. Into the same current particles which have been taken as food are from time to time rejected by the forward kicking of the second maxillipeds.

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PROCEEDINGS

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY

VOL. XIX. PARTS II., III.

[LENT AND EASTER TERMS 1917.]

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

The Direct Solution of the Quadratic and Cubic Binomial Congruences with Prime Moduli. By H. C. POCKLINGTON, M.A., St John's College.

[*Received* 22 January 1917: *read* 5 February 1917.]

1. The solution of congruences by exclusion methods, although easy enough when the modulus is moderately large, becomes impracticable for large moduli because the labour varies as the modulus or its square root. In a direct method the labour varies roughly as the cube of the number of digits in the modulus, and so remains moderate for large moduli. The object of this paper is to develop the direct method. We take $x^2 \equiv a$, mod. p, first, discussing the cases where $p = 4m + 3$ and $p = 8m + 5$ in § 2 and that where $p = 8m + 1$ in § 3. We next take $x^3 \equiv a$ and discuss the cases where $p = 3m + 2$, $p = 9m + 4$ and $p = 9m + 7$ in § 4 and that where $p = 9m + 1$ in § 5.

2. Throughout the paper we suppose the modulus to be p where p is prime*. If p is of the form $4m + 3$ the solution of $x^2 \equiv a$ is $x \equiv \pm a^{m+1}$. If p is of the form $8m + 5$ the solution is $x \equiv \pm a^{m+1}$ provided that $a^{2m+1} \equiv 1$. But if not, $a^{2m+1} \equiv -1$, and as 2 is a non-residue $4^{2m+1} \equiv -1$; so that $(4a)^{2m+1} \equiv 1$ and we have $y \equiv \pm (4a)^{m+1}$ as the solution of $y^2 \equiv 4a$. Hence

$$x \equiv \pm y/2 \text{ or } x \equiv \pm (p + y)/2$$

is the solution of $x^2 \equiv a$. These values of x can be calculated without serious difficulty by repeated squaring (followed by division by the modulus to find the remainder) and multiplication of the numbers so found (again followed by division).

* Hence if it is composite we must factorize it and solve the congruence for each of the different prime factors.

3. Put $D \equiv -a$, so that we have to solve $x^2 + D \equiv 0$ where D is positive or negative but not divisible by p. Let t_1 and u_1 be so chosen* that $t_1^2 - Du_1^2 = N$ is a quadratic non-residue of p, and let

$$t_n = \{(t_1 + u_1\sqrt{D})^n + (t_1 - u_1\sqrt{D})^n\}/2,$$
$$u_n = \{(t_1 + u_1\sqrt{D})^n - (t_1 - u_1\sqrt{D})^n\}/2\sqrt{D}.$$

These numbers are clearly integral. Also

$$t_{m+n} = t_m t_n + D u_m u_n, \quad u_{m+n} = t_m u_n + t_n u_m,$$

by use of which (at first with $m = n$) we can find the remainders of t_n and u_n to our modulus without serious difficulty even when n is large. We also have $t_n^2 - Du_n^2 = N^n$.

Supposing that p is of the form $4m+1$, we have D a quadratic residue of p, and $t_p \equiv t_1^p \equiv t_1$, $u_p \equiv u_1^p D^{(p-1)/2} \equiv u_1$; and now

$$t_1 \equiv t_{p-1} t_1 + D u_{p-1} u_1, \quad u_1 \equiv t_{p-1} u_1 + t_1 u_{p-1}$$

give on solution $t_{p-1} \equiv 1$, $u_{p-1} \equiv 0$. Let $p - 1 = 2r$. Then

$$0 \equiv u_{p-1} \equiv 2t_r u_r$$

shows that either t_r or u_r is divisible by p. If it is u_r we put $r = 2s$ and proceed similarly. We cannot have every u divisible by p, for u_1 is not. We cannot be stopped by having $u_m \equiv 0$ with m odd, for we always have $t_m^2 - Du_m^2 \equiv N^m$, and this would then give t_m^2 congruent to a non-residue. But if m is even we can proceed further. Hence when we are stopped we must have $t_m \equiv 0$. This gives $-Du_m^2 \equiv N^m$, and as $-D$ is a residue m must be even. Putting $m = 2n$ we have $0 \equiv t_m \equiv t_n^2 + Du_n^2$, so that the solution of $x^2 + D \equiv 0$ is got by solving the linear congruence $u_n x \equiv \pm t_n$.

In applying the method, if n is the largest odd number contained in $p - 1$ we first work to get the suffixes n, and then the suffixes $2n$, $4n$, $8n$, etc. Thus in the case of $x^2 + 2 \equiv 0$, mod. 41, we see that $t_1 \equiv 3$, $u_1 \equiv 1$ is suitable, and we find $t_2 \equiv 11$, $u_2 \equiv 6$; $t_4 \equiv 29$, $u_4 \equiv 9$; $t_5 \equiv 23$, $u_5 \equiv 15$; $t_{10} \equiv 36$, $u_{10} \equiv 34$; $t_{20} \equiv 0$. The solution of $34x \equiv 36$ is $x \equiv 30$; and so the two solutions of $x^2 + 2 \equiv 0$ are $x \equiv \pm 30$, mod. 41.

4. If p is of the form $3m + 2$ the only solution of $x^3 \equiv a$ is $x \equiv 1/a^m$. If p is of the form $9m + 4$ one solution is $x \equiv 1/a^m$, and if of the form $9m + 7$ one is $x \equiv a^{m+1}$. The other solutions are got from this by multiplying by $(-1 + \theta)/2$ and $(-1 - \theta)/2$, where $\theta^2 + 3 \equiv 0$, a congruence which we have shown how to solve.

* We have to do this by trial, using the Law of Quadratic Reciprocity, which is a defect in the method. But as for each value of u half the values of t are suitable, there should be no difficulty in finding one.

5. Let δ be the arithmetical cube root of a, which we assume* not to be a cube. Find† t_1, u_1, v_1 such that the norm $N = t_1^3 + au_1^3 + a^2v_1^3 - 3at_1u_1v_1$ of the algebraic number

$$U = t_1 + u_1\delta + v_1\delta^2$$

is a cubic non-residue of p. We see that as a is a cubic residue of p we have $U^p \equiv t_1 + u_1\delta + v_1\delta^2$, so that if

$$U^{p-1} \equiv t_{p-1} + u_{p-1}\delta + v_{p-1}\delta^2$$

we have $u_{p-1} \equiv v_{p-1} \equiv 0$. Now taking U^m where m is in turn $(p-1)/3$, $(p-1)/9$, etc. we see that we cannot always have $u_m \equiv v_m \equiv 0$. Let U^m be the last of this series for which this happens. Then m is divisible by 3, for otherwise the norm of U^m, which reduces to t_m^3, would be congruent to the non-residue N^m. Putting $m = 3n$ we have

$$t_{3n} \equiv t_n^3 + au_n^3 + a^2v_n^3 + 6at_nu_nv_n,$$

$$0 \equiv u_{3n} \equiv 3(t_n^2u_n + at_nv_n^2 + au_n^2v_n),$$

$$0 \equiv v_{3n} \equiv 3(t_nu_n^2 + t_n^2v_n + au_nv_n^2).$$

The last two give $t_n(av_n^3 - u_n^3) \equiv 0$; so that if t_n is not divisible by p we have $x \equiv u_n/v_n$ as one solution of $x^3 \equiv a$, for as u_n and v_n are not both divisible by p this shows that neither is. They also give $v_n(au_n^3 - t_n^3) \equiv 0$, and so $x \equiv t_n/u_n$ is a solution. Eliminating a from the same two congruences we see that the ratio λ of the two x's satisfies $\lambda^2 + \lambda + 1 \equiv 0$, so that they are distinct. The third solution follows immediately.

If however t_n is divisible by p the two congruences show that either u_n or v_n is divisible by p. We now have $au_n^3 \equiv N^n$ or $a^2v_n^3 \equiv N^n$. In either case n must be divisible by 3 as before, and we have as one solution $x \equiv N^r/u_n$ or $x \equiv av_n/N^r$ respectively, where $r = n/3$.

* Simply because of the way in which for the sake of shortness we are stating the method.

† This again must be done by trial. In order to use the Law of Cubic Reciprocity we must express p in the form $u^2 + uv + v^2$, which requires the solution of $\theta^2 + 3 \equiv 0$.

On a theorem of Mr G. Pólya. By G. H. Hardy, M.A., Trinity College.

[*Received and read* 5 February 1917.]

1. Mr G. Pólya has recently discovered a number of very beautiful theorems concerning Taylor's series with integral coefficients and 'ganzwertige ganze Funktionen'. The latter functions are integral functions which assume integral values for all integral (or for all positive integral) values of the independent variable. One of the most remarkable of these theorems is the following*:

Suppose that $g(x)$ is an integral function, and $M(r)$ the maximum of $|g(x)|$ for $|x| \leqslant r$. Suppose further that

$$g(0),\ g(1),\ g(2), \ldots$$

are integers, and that

$$\lim_{r \to \infty} 2^{-r} \sqrt{r}\, M(r) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then $g(x)$ is a polynomial.

Mr Pólya observes that, if it were possible to get rid of the factor $\sqrt{r}$ from the equation (1), the theorem could be enunciated in a notably more pregnant form, viz.:

Among all transcendental integral functions, which assume integral values for all positive integral values of the variable, that of least increase† is the function 2^x.

Mr Pólya states, however, that he has not been able to effect this generalisation. And my object in writing this note is to show that the generalisation desired may be obtained by a slight modification of Mr Pólya's own argument, and without the addition of any essentially new idea to those which he employs.

2. Mr Pólya‡ reduces the proof of the theorem to a proof that the integral

$$J(n) = \frac{n!}{2\pi i} \int \frac{g(x)\, dx}{x(x-1)(x-2)\ldots(x-n)},$$

extended over the circle $|x| = r = 2n$, tends to zero when $n \to \infty$.

* G. Pólya, 'Über ganzwertige ganze Funktionen', *Rendiconti del Circolo Matematico di Palermo*, vol. 40, 1915, pp. 1—16. See also 'Über Potenzreihen mit ganzzahligen Koeffizienten', *Mathematische Annalen*, vol. 77, 1916, pp. 497—513, where Mr Pólya refers to a third memoir ('Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen', *Journal für Mathematik*) which I have not been able to consult.

† *Croissance, Wachstum.*

‡ *Loc. cit.*, p. 7.

This he proves by observing that the modulus of J_n does not exceed

$$\frac{n!\,M(r)}{(r-1)(r-2)\dots(r-n)} = \frac{\Gamma(n+1)\,\Gamma(n)}{\Gamma(2n)} M(r),$$

and by an application of Stirling's Theorem. In order to complete the proof in this manner it is necessary to assume the condition (1).

If however we suppose only that

$$\lim_{r\to\infty} 2^{-r} M(r) = 0 \quad\dots\dots\dots\dots(2),$$

or

$$M(r) = o(2^r) \quad\dots\dots\dots\dots(2'),$$

the proof may be completed as follows. We have

$$J_n = o\left\{n!\,2^{2n}\int_{-\pi}^{\pi}\frac{d\theta}{|(x-1)(x-2)\dots(x-n)|}\right\},$$

where $x = 2ne^{i\theta}$. Now

$$|x-s| = \sqrt{(4n^2 - 4ns\cos\theta + s^2)} \geqslant 2n - s\cos\theta$$

for $1 \leqslant s \leqslant n$, so that

$$\left|\prod_1^n (x-s)\right| \geqslant \prod_1^n (2n - s\cos\theta) = (\cos\theta)^n \prod_1^n (2n\sec\theta - s)$$

if $\cos\theta > 0$, and

$$\left|\prod_1^n (x-s)\right| \geqslant \prod_1^n (2n - s\cos\theta) = |\cos\theta|^n \prod_1^n (2n\,|\sec\theta| + s)$$

if $\cos\theta < 0$. Hence

$$J_n = o(K_n) + o(L_n),$$

where

$$K_n = n!\,2^{2n}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\Gamma(2n\sigma - n)}{\Gamma(2n\sigma)}\sigma^n\,d\theta,$$

$$L_n = n!\,2^{2n}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\Gamma(2n\sigma + 1)}{\Gamma(2n\sigma + n + 1)}\sigma^n\,d\theta,$$

and $\sigma = \sec\theta$.

A straightforward application of Stirling's Theorem shows that

$$n!\,2^{2n}\sigma^n\frac{\Gamma(2n\sigma - n)}{\Gamma(2n\sigma)} = O\left\{e^{\Phi n}\sqrt{\left(\frac{2n\sigma}{2\sigma - 1}\right)}\right\},$$

and

$$n!\,2^{2n}\sigma^n\frac{\Gamma(2n\sigma + 1)}{\Gamma(2n\sigma + n + 1)} = O\left\{e^{\Psi n}\sqrt{\left(\frac{2n\sigma}{2\sigma + 1}\right)}\right\},$$

uniformly in θ, where

$$\Phi = \Phi(\theta) = (2\sigma - 1)\log(2\sigma - 1) - 2\sigma\log 2\sigma + \log\sigma + 2\log 2,$$

$$\Psi = \Psi(\theta) = 2\sigma\log 2\sigma - (2\sigma + 1)\log(2\sigma + 1) + \log\sigma + 2\log 2.$$

3. When θ increases from 0 towards $\frac{1}{2}\pi$, or decreases towards $-\frac{1}{2}\pi$, σ increases from 1 towards ∞. Also

$$\frac{d\Phi}{d\sigma} = 2\log(2\sigma - 1) - 2\log 2\sigma + \frac{1}{\sigma} = 2\log\left(1 - \frac{1}{2\sigma}\right) + \frac{1}{\sigma} < 0,$$

$$\frac{d\Psi}{d\sigma} = 2\log 2\sigma - 2\log(2\sigma + 1) + \frac{1}{\sigma} = \frac{1}{\sigma} - 2\log\left(1 + \frac{1}{2\sigma}\right) > 0.$$

Thus Φ steadily decreases and Ψ steadily increases. Moreover

$$\Phi(0) = 0, \quad \Psi(0) = 4\log 2 - 3\log 3;$$

and it is easily verified that both Φ and Ψ tend to the limit

$$\log 2 - 1$$

when θ tends to $\frac{1}{2}\pi$.

We thus obtain, in the first place,

$$L_n = O\left\{e^{-n(1-\log 2)}\sqrt{n}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{\left(\frac{2\sigma}{2\sigma+1}\right)}\,d\theta\right\} = o(1).$$

Secondly, we observe that, if δ is any positive number, we have

$$\Phi(\theta) < \Phi(\delta) = -\eta < 0$$

for

$$\delta \leqslant \theta \leqslant \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi \leqslant \theta < -\delta.$$

Hence we may replace the limits in K_n by $-\delta$ and δ, the remainder of the integral being of the form

$$O\left\{e^{-\eta n}\sqrt{n}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{\left(\frac{2\sigma}{2\sigma-1}\right)}\,d\theta\right\} = o(1).$$

4. All that remains, then, is to prove that

$$I_n = n!\,2^{2n}\int_{-\delta}^{\delta}\frac{\Gamma(2n\sigma - n)}{\Gamma(2n\sigma)}\sigma^n\,d\theta = O(1);$$

and we have

$$I_n = O\left\{\sqrt{n}\int_{-\delta}^{\delta}e^{\Phi n}\sqrt{\left(\frac{2\sigma}{2\sigma-1}\right)}\,d\theta\right\} = O\left(\sqrt{n}\int_{-\delta}^{\delta}e^{\Phi n}\,d\theta\right).$$

The function $\Phi(\theta)$ may now be expanded in powers of θ. We find without difficulty that

$$\Phi = -A\theta^2 + O(\theta^4),$$

where

$$A = \log 2 - \tfrac{1}{2} > 0.$$

It follows that

$$I_n = O\left\{\sqrt{n}\int_{-\delta}^{\delta}e^{-An\theta^2 + O(n\theta^4)}\,d\theta\right\}$$

$$= O\left(\sqrt{n}\int_{-\infty}^{\infty}e^{-\frac{1}{2}An\theta^2}\,d\theta\right) = O(1).$$

The proof of the theorem conjectured by Mr Pólya is thus completed.

5. Mr Pólya has also proved an analogous theorem concerning integral functions which assume integral values for *all* integral values of x, viz.:

If $\ldots, g(-2), g(-1), g(0), g(1), g(2) \ldots$

are integers, and

$$\lim_{r \to \infty} \left(\frac{3+\sqrt{5}}{2}\right)^{-r} \sqrt{r}\, M(r) = 0 \quad \ldots\ldots\ldots\ldots\ldots (3),$$

then $g(x)$ *is a polynomial.*

His proof applies, as it stands, to *odd* functions only, its application to a completely general function demanding the more stringent condition

$$\lim_{r \to \infty} \left(\frac{3+\sqrt{5}}{2}\right)^{-r} r^{\frac{3}{2}} M(r) = 0 \quad \ldots\ldots\ldots\ldots\ldots (3').$$

He states that it is possible to replace the index $\frac{3}{2}$ by $\frac{1}{2}$ in all cases, but that, as he has not been able to reduce the condition to

$$\lim_{r \to \infty} \left(\frac{3+\sqrt{5}}{2}\right)^{-r} M(r) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3''),$$

he has not thought it worth while to publish the details of his work.

A modification of Mr Pólya's argument, in every way similar to that which I have made in the proof of his first theorem, enables us to replace (3) by (3″) when $g(x)$ is odd. The same modification in his unpublished argument would, I presume, be equally effective in general.

That the number

$$\frac{3+\sqrt{5}}{2}$$

cannot be replaced by any larger number, and so really is the number which ought to occur in any theorem of this character, is shown by Mr Pólya by the example of the function

$$\frac{1}{\sqrt{5}} \left\{ \left(\frac{3+\sqrt{5}}{2}\right)^{x} - \left(\frac{3+\sqrt{5}}{2}\right)^{-x} \right\},$$

which assumes integral values for all integral values of x.

Submergence and glacial climates during the accumulation of the Cambridgeshire Pleistocene Deposits. By J. E. MARR, Sc.D., F.R.S., St John's College.

[*Read* 5 February 1917.]

A. *Introductory.*

The sequence of events during palaeolithic times is still a subject surrounded by much uncertainty. The area of the Great Ouse Basin is one in which considerable light has already been thrown on vexed questions, and as the examination of the area is carried out in greater detail, important results will be obtained, for in this area we get evidence of the relationship of the palaeolithic deposits to those which were formed during a period of submergence and re-emergence, and also to accumulations which give evidence of the occurrence of more than one cold period.

The general distribution of the palaeolithic deposits of the district around Cambridge, and their main characters, have long been known, and an account of the deposits, with references to the previous literature, is given in the Geological Survey Memoir *The Geology of the Neighbourhood of Cambridge*, published in 1881.

Since that memoir appeared, further light has been thrown on the deposits, especially by Professor Hughes, who has given his latest views in a paper entitled *The Gravels of East Anglia* (Cambridge University Press, 1916).

I have devoted much attention to this subject during the last six years and hope to describe my detailed results elsewhere. The present paper is concerned with a discussion of the main problems involved, in hopes that it may direct the attention of workers to the importance of further observations, for the deposits with which we are concerned are only exposed temporarily during the working of gravel-pits and the digging of foundations and drains, and it is desirable that all temporary excavations should be carefully studied, and the objects obtained rendered available for study by deposit in Museums, for isolated specimens in private collections are usually mere objects of curiosity devoid of scientific value.

B. *Submergence and its effects. The actual sequence of deposits.*

In the fenland and on its borders we meet with marine deposits above sea-level, which have long been known around March and Narborough. They occur above and below fen-level at March and undoubted marine deposits containing sea-shells are found to a height of at least 50 feet above sea-level in the Nar Valley, and deposits up to 80 feet above sea-level have been claimed as marine. Unfortunately no exposure of these Nar Valley beds has been seen for a very long time, and their exact upward limit is a matter which must remain unsettled until new excavations are made. It is held, with good reason, that the beds of March and the Nar Valley are geologically contemporaneous in the sense that they belong to the same period of sea-invasion, which was subsequent to the accumulation of the chalky Boulder Clay; and as there is good evidence that much of the fenland was low-lying

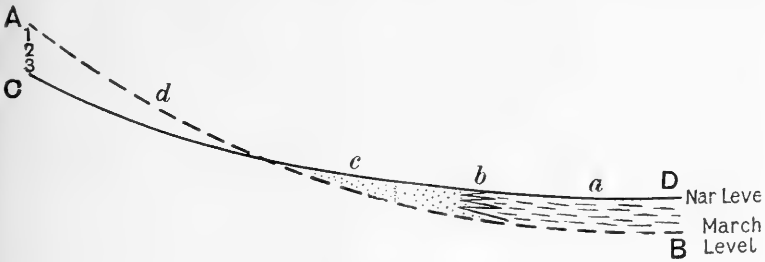

Fig. 1.

AB.	Slope of ground *before* marine gravels were deposited.
CD.	,, ,, *after* ,, ,, ,,
a.	Tract of marine gravels.
b.	,, interdigitating marine and fluviatile gravels.
c.	,, fluviatile deltaic deposits.
d.	,, erosion in valley towards its head, during period of deposit of *a*, *b*, *c*.
1, 2, 3.	Order of formation of deposits in tracts *c* and *d* respectively. (1 is oldest.)

Vertical scale greatly exaggerated.

ground after this boulder-clay was formed, it would appear probable that the March gravels are earlier than those of the Nar Valley, and therefore that a gradual silting up of a bay of the sea took place, until the sediments reached a height of at least 50 feet above present sea-level.

During this period of silting the rivers Ouse, Cam and others would build delta-deposits along the lower parts of their courses, with interdigitation of marine and fluviatile deposits in an intermediate belt of ground as shewn in figure 1. In this delta-material, the chronological sequence of deposit would be from below upward, as shewn by 1, 2 and 3 in the belt *c*. The upper waters of the rivers would still be eroding, and the sequence would be from above downwards (see figs. in tract *d*).

After submergence had ceased, it would be replaced by re-emergence, as shewn by the erosion of the rivers to their present levels, and new deposits 4, 5, ... (not shewn in the diagram) would be banked against or laid down upon those formed during the period of subsidence and general accumulation in tracts *c* and *a*. It will be seen therefore that relative height of deposits above the present river-level is not in itself a necessary indication of age.

The geological surveyors gave the following classification of the Cam gravels:

Gravels of the Present River System: Lowest Terrace; Intermediate Terrace; Highest Terrace

Gravels of the Ancient River System.

I shall treat of three of these, leaving out of account the gravels of the Intermediate Terrace, which I have not studied extensively owing to poor and infrequent exposure of recent years. I shall speak of the gravels of the 'Ancient River System' as the Observatory gravels, those of the highest terrace of the 'present river system' as the Barnwell village gravels, and those of the lowest terrace as the Barnwell Station gravels. The ages of these deposits will ultimately be accurately determined by an examination of the fossil evidence, including implements of human manufacture. So far, the evidence of this kind points to the Barnwell village deposits being of two ages, the older formed during the period of delta-growth, the newer during the period of re-emergence and erosion. At the end of the period of delta-growth, and therefore of an age intermediate between those of the supposed two Barnwell village deposits, I would place the Observatory gravel, and certain loams, to be referred to later, and after all of these, the Barnwell Station gravel marking the culmination of the period of re-erosion, for there is evidence of a later period of sinking and deposit after this was formed. This succession is represented in Fig. 2, which shews a section across the Cam valley at Cambridge, before the edges of the valley sides had been destroyed leaving the Observatory gravels as a ridge with lower ground on either side.

In the figure the terms Upper, Middle and Lower Palaeolithic indicate the ages of the various gravels as inferred by me from the palaeontological evidence. I am using the term Middle Palaeolithic in the sense in which it was used by Prof. Sollas in the first edition of *Ancient Hunters* as equivalent to Mousterian. I believe therefore that the older Barnwell village gravel is pre-Mousterian, that of the Observatory (in part at any rate) Mousterian, and the newer Barnwell village gravel and that of Barnwell Station post-Mousterian, the former being of earlier date than the latter.

Mr Jukes-Browne, in an essay on the Post Tertiary Deposits of

Cambridgeshire, advocated a change in the direction of the rivers near Cambridge between the formation of the Observatory gravels, and those which he regarded as belonging to the 'present river system.' That such a change occurred is admitted, but the evidence points to all the deposits save those of the Barnwell Station terrace having been formed before the river diversion occurred.

I may now pass on to consider briefly the palaeontological evidence in favour of the order of age indicated above, leaving details for a future paper.

In the pits of Barnwell village and of the Milton Road near Chesterton, loams are sometimes exposed at the base of the overlying gravels. These loams contain *Corbicula fluminalis*, and with it are associated *Unio littoralis*, *Belgrandia marginata*, and *Hippopotamus*. On the continent this is recognised as an early

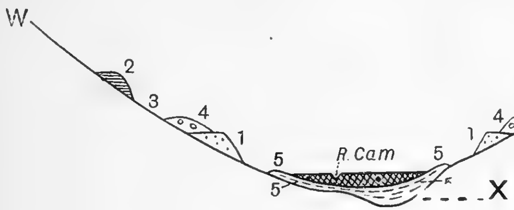

Fig. 2.

Section across Cam N. of Cambridge, with higher valley-slopes restored.

The figures shew the suggested order of formation of the deposits. Cross-hatching represents modern alluvium of Cam.

5. Barnwell Station gravels (Upper Palaeolithic 2).
4. Newer Barnwell village gravels (Upper Palaeolithic 1).
3. Loams of Huntingdon Road area.
2. Observatory gravels (Middle Palaeolithic).
1. Older Barnwell village gravel and loam (Lower Palaeolithic).

X = Buried channel.

Vertical scale greatly exaggerated.

palaeolithic fauna of Chellean or pre-Chellean date, and there seems to be no evidence of the reappearance of this fauna at a later date.

In the *Geological Magazine* for 1878 (p. 400) Mr A. F. Griffith described the occurrence of a palaeolithic implement from one of the Barnwell pits. A cast of this is in the Sedgwick Museum, and it appears to be of Chellean type.

Further afield, the occurrence of similar implements at or near fen-level in Swaffham and Soham fens, and at West Row near Mildenhall, and at Shrub Hill near Feltwell, indicates that rivers had excavated their channels to fen-level in those times.

There are patches of gravel between the higher Chesterton terrace which corresponds to the Barnwell village terrace and the

Observatory level, but no sections are now seen in them, so we may pass on to the Observatory deposits. In these shells and mammalian bones are very rare, though the former have been found in concretions, indicating that they once lay in the gravels, but have since been dissolved. Implements are relatively abundant, and I have found a large number during recent years. Many of them are of Chellean type, others probably Acheulean, but there are a large number of Mousterian type, some having the facetted platform which, as shewn by M. Commont, came into use in Northern France in Mousterian times. It may be noted that the implements of Mousterian type are patinated differently to and in a less degree than those of Chellean type, and I regard the two series as of distinct ages. Either the deposits, which are thick and varied in character, are of two dates, or implements of different ages lying upon the surface were washed into the deposits contemporaneously. This can only be settled by finding a number *in situ*, a work of great difficulty, but the evidence is in favour of the latter view.

I may note that when a valley is being deepened implements of one age only are likely to lie in abundance near the spot where the gravels were accumulating, but when there is general aggradation, the highest deposits of the delta-growth are likely to receive washings of implements of various ages which have been lying together, at or near the surface. In any case the age of the newest gravel of a terrace will be determined by the implements of latest date.

Lying on this gravel in channels are reddish sandy loams, which must have spread over the gravel, but have since been destroyed by erosion except where so preserved. There is also a deposit of somewhat similar loam but of a lighter colour flanking the gravel at a lower level on either side. It is rarely exposed, and only in shallow sections, but I believe it may be of the same general date as that lying on the gravel.

No relics have been found in it, though two implements of possible Upper Palaeolithic date were found on the loam when draining the Christ's Cricket Ground, but they may well have been surface finds. Many other surface finds, some of apparent palaeolithic type, are found on this loam belt, and will be referred to later.

Those gravels of the terraces of Barnwell village age, which I would refer to a date later than that of the *Corbicula* gravels, are now exposed in a pit near the Milton Road and in another on the Newmarket Road near Elfleda House, $2\frac{1}{4}$ miles from Cambridge. These contain a fauna differing from the *Corbicula* fauna, and including the mammoth, woolly rhinoceros, horse and red deer, the horse being abundant.

Implements are scarce, but in both pits I have found some suggestion of upper palaeolithic forms, and in each pit a water-worn pot-boiler has been discovered.

In the Barnwell Station pit the common mammal is the reindeer, associated with the mammoth, tichorhine rhinoceros and horse. In the *Geological Magazine* for 1916 (p. 339), Miss E. W. Gardner and I recorded the occurrence of an arctic flora in this deposit, with abundance of leaves of *Betula nana*. A long preliminary list of the other plants which indicate arctic conditions was made by the late Mr Clement Reid, F.R.S., but has not yet been published. A few worked flints of undeterminable date have been found, but the fauna indicates the late palaeolithic period, and the late date of these deposits seems to be shewn by the fact that whereas all the others are apparently connected with the old drainage line extending from Cambridge to Somersham, these are almost certainly parallel to the present course of the Cam: they appear indeed to be the upper portion of the deposits filling an old buried channel of the Cam, evidence for the occurrence of which is borne out by certain observations made by Prof. Hughes in the paper to which reference has been given.

C. *Climatic Changes.*

There is much difference of opinion as regards the occurrence of alternating glacial and interglacial periods in Pleistocene times, and it would seem that some light is thrown upon this question by the Cambridgeshire deposits and those of adjoining counties.

I take the prevalent view that the implement-bearing deposits from the beginning of Chellean times post-date the period of the Chalky Boulder Clay, though others hold a different view, but as the local evidence bearing upon this question has already been recorded I need not enlarge upon this point.

If the succession as outlined above be correct the following climatic changes seem to have occurred after the cold period marked by the accumulation of the Boulder Clay:

(*a*) A warm period during the formation of the *Corbicula*-bearing strata. Arguments in favour of this are well known.

(*b*) A cold period during the accumulation of the Observatory gravels(?) and the newer loams. No evidence of this has been advanced in this area, and a few remarks are necessary.

The fauna of the Observatory gravels tells us nothing, and the loams have hitherto furnished no organic remains, but a widespread development of loam marks the Mousterian period, and N.W. Europe is believed to have been subjected to a cold climate during part of the period.

The sections recently seen near Cambridge tell us little, but a brickpit in *stratified* loam with 'race' nodules similar to those found in the Cambridge sections has long been worked near the railway between Longstanton and Swavesey. It contains boulders, and is actually mapped as boulder-clay. A somewhat similar loam with boulders at High Lodge near Mildenhall has long been known for its implements of Mousterian type. These deposits are at an elevation just below that of the highest palaeolithic gravels, as are those of Cambridge.

Further afield there is the very significant section at Hoxne, described in detail in a paper drawn up by the late Clement Reid, F.R.S., and published in the *Report of the British Association* for 1896.

At that locality we have a stratigraphical sequence. Above the boulder-clay lies an aquatic deposit marked by a temperate fauna. It is succeeded by loams with an arctic flora, and above that are loams with palaeolithic implements. They have been usually regarded as Acheulean, but there is one specimen in the Sedgwick Museum which is of a distinct Mousterian type. Taking these facts into consideration, a period of cold climate in this country in Mousterian times seems probable. In any case, the evidence points to a difference of date of the arctic plant-beds of Hoxne and Barnwell Station.

(*c*) The fauna of the beds of the Barnwell village terrace claimed here as of newer date than those containing *Corbicula* suggests an amelioration of the climate, but in the absence of a well preserved flora, this is doubtful.

(*d*) The Barnwell Station flora, as before observed, is distinctly arctic, and when this flora lived here, we can hardly suppose that our higher hills escaped glaciation. The same remark may be made of the Hoxne flora.

This series of changes would accord with the classification of the beds on the continent thus:

European Continent	Cambridgeshire
Pleistocene	
Würm glaciation	Barnwell Station beds.
Warm period	Newer Barnwell village deposits.
Riss glaciation	Observatory gravels and loams.
Warm period	*Corbicula* gravels.
Mindel glaciation	Chalky Boulder Clay.
Pliocene	
Warm period	Cromer 'Forest' series.
Günz glaciation	Chillesford beds.

I merely put this forward tentatively, claiming however that we have in Cambridge proofs of two if not three Pleistocene cold periods.

D. *Surface Implements.*

Implements of all ages from earlier palaeolithic to recent times are found lying together on the surface. Some no doubt have got there from the erosion of deposits which contained them, others belong to the surface. My object is to insist on their careful collection, with exact records of their localities, even to the particular position in a field where they lay.

If they can be shewn to be limited to heights above those of a particular deposit, they may yield valuable information as to geological changes.

Two areas in which surface implements are abundant are found very near Cambridge, one on the tract between Castle End and Girton on either side of the Huntingdon Road, on the ground occupied by the Observatory gravels and loams, the other a little south of Fen Ditton, between the railway and the river, and at no great height above the latter. They have not been yet sufficiently studied to enable one to draw definite conclusions, but the former group does not seem to occur below the level of the Barnwell village terrace, which suggests that the river may have eroded its valley below that level to its present position since those implements were made. The other set marks the position of a site on a terrace, which is I believe the terrace of the Barnwell Station deposits, and would indicate the formation of that terrace before this set of implements was manufactured.

As the above is merely a preliminary account of these deposits, I have not burdened it with references, nor have I acknowledged the many friends who have helped in the collection of implements and other objects.

The bulk of the implements on which my conclusions are based were collected by myself, and the rest by friends chiefly under my supervision, and in no case has any implement been purchased from workmen, so that the collection, which will be deposited in the Sedgwick Museum, is of value, inasmuch as each implement is known to have been obtained from the locality assigned to it.

On the Hydrodynamics of Relativity. By C. E. WEATHERBURN, M.A. (Camb.), D.Sc. (Sydney), Ormond College, Parkville, Melbourne.

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I. THE EQUATIONS OF MOTION.

§ 1. Relativistic equations for the adiabatic motion of a frictionless fluid have been found by Lamla* and Laue† in the form

$$\left.\begin{aligned}\frac{\partial}{\partial t}(\kappa u)+u\frac{\partial}{\partial x}(\kappa u)+v\frac{\partial}{\partial y}(\kappa u)+w\frac{\partial}{\partial z}(\kappa u)+\frac{1}{\gamma}\frac{\partial P}{\partial x}&=X\\ \frac{\partial}{\partial t}(\kappa v)+u\frac{\partial}{\partial x}(\kappa v)+v\frac{\partial}{\partial y}(\kappa v)+w\frac{\partial}{\partial z}(\kappa v)+\frac{1}{\gamma}\frac{\partial P}{\partial y}&=Y\\ \frac{\partial}{\partial t}(\kappa w)+u\frac{\partial}{\partial x}(\kappa w)+v\frac{\partial}{\partial y}(\kappa w)+w\frac{\partial}{\partial z}(\kappa w)+\frac{1}{\gamma}\frac{\partial P}{\partial z}&=Z\end{aligned}\right\}\ldots(1);$$

where u, v, w are the components of velocity at the point (x, y, z) relative to a definite system of reference S; X, Y, Z those of the impressed force per unit of normal rest-mass; and

$$\gamma=\frac{c}{\sqrt{c^2-(u^2+v^2+w^2)}}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

c being the constant velocity of light. The significance of the symbols P and κ is as follows.

Since the motion is adiabatic the rest-mass of an element of fluid is determined by one variable only, say the pressure p.

If we choose some definite pressure p_0 as the normal or standard pressure, the element has a definite constant normal rest-mass δm_0. If the element occupies a volume δV relative to the system of reference S, the *density* k relative to that system is defined by

$$k=\frac{\delta m_0}{\delta V}.$$

* *Ann. der Physik*, Vol. 37, p. 772 (1912).

† *Das Relativitätsprinzip*, § 36 (2nd ed. 1913). For a more general discussion of the mechanics of deformable bodies from the standpoint of Relativity, cf. Herglotz, *Ann. der Physik*, Vol. 36, p. 493 (1911); also a paper by Ignatowsky, *Phys. Zeit.*, Vol. 12, p. 441 (1911).

Using a dash to refer in every case to the rest-system S', we have for the rest-density

$$k' = \frac{\delta m_0}{\delta V'} = \frac{\delta m_0}{\gamma \delta V} = \frac{k}{\gamma} \quad \text{.....................(3).}$$

The function P is defined by the integral

$$P = \int_{p_0}^{p} \frac{dp}{k'} \quad \text{..........................(4),}$$

and in terms of this function κ is given by

$$\kappa = \gamma \left(1 + \frac{P}{c^2}\right) \quad \text{.......................(5).}$$

For the rest-system S' the quantity γ has the value unity, while κ becomes

$$\kappa' = 1 + \frac{P}{c^2} \quad \text{..........................(5').}$$

The constancy of normal rest-mass leads, as in the classical theory, to an *equation of continuity*

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(ku) + \frac{\partial}{\partial y}(kv) + \frac{\partial}{\partial z}(kw) = 0 \quad \text{...........(6).}$$

§ 2. Using $\mathbf{F}$ and $\mathbf{v}$ for the force and velocity vectors, we may write the equations of motion more conveniently

$$\frac{\partial}{\partial t}(\kappa \mathbf{v}) + \mathbf{v} \cdot \nabla (\kappa \mathbf{v}) + \frac{1}{\gamma} \nabla P = \mathbf{F} \quad \text{..............(7).}$$

Then because the gradient of the scalar product of two vectors is given by

$$\nabla (\mathbf{a} \cdot \mathbf{b}) = \mathbf{b} \cdot \nabla \mathbf{a} + \mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{b} \times \operatorname{curl} \mathbf{a} + \mathbf{a} \times \operatorname{curl} \mathbf{b},$$

the second term of (7) is equivalent to

$$\frac{1}{2\kappa} \nabla (\kappa^2 \mathbf{v}^2) - \mathbf{v} \times \operatorname{curl} (\kappa \mathbf{v}),$$

while, in virtue of (5'), $\nabla P = c^2 \nabla \kappa'$. Hence the equation may be expressed in the form

$$\frac{\partial}{\partial t}(\kappa \mathbf{v}) + \frac{1}{2\kappa} \nabla (\kappa^2 \mathbf{v}^2 + c^2 \kappa'^2) - \mathbf{v} \times \operatorname{curl} (\kappa \mathbf{v}) = \mathbf{F}.$$

But again the second term is equal to

$$\frac{1}{2\kappa} \nabla \left[\kappa^2 \left(\mathbf{v}^2 + \frac{c^2}{\gamma^2}\right)\right] = \frac{c^2}{2\kappa} \nabla \kappa^2 = c^2 \nabla \kappa,$$

and the equation of motion takes the very convenient form

$$\frac{\partial}{\partial t}(\kappa \mathbf{v}) + c^2 \nabla \kappa + 2\mathbf{w} \times \mathbf{v} = \mathbf{F} \quad \text{..............(8)},$$

where we have written

$$2\mathbf{w} = \operatorname{curl}(\kappa \mathbf{v}).$$

In cases where the impressed force $\mathbf{F}$ admits a potential, so that $\mathbf{F} = -\nabla V$, our equation reduces to

$$\frac{\partial}{\partial t}(\kappa \mathbf{v}) + \nabla (c^2 \kappa + V) + 2\mathbf{w} \times \mathbf{v} = 0 \quad \text{...........(8')}.$$

§ 3. *Clebsch's transformation**. The equation of motion may be expressed in terms of functions analogous to those of Clebsch if we write

$$\kappa \mathbf{v} = \nabla \phi + \lambda \nabla \mu \quad \text{.......................(9)},$$

ϕ, λ, μ being three independent functions of x, y, z and t. Taking the curl of both members we find immediately that

$$2\mathbf{w} = \nabla \lambda \times \nabla \mu \quad \text{....................(10)}.$$

The function $\mathbf{w} \equiv \frac{1}{2}\operatorname{curl}(\kappa \mathbf{v})$ plays the same part in the present analysis as $\frac{1}{2}$ curl $\mathbf{v}$ in classical hydrodynamics. It will therefore, by analogy, be called the *vorticity*; and a line whose direction at any point is the direction of $\mathbf{w}$ at that point, a *vortex line*. Since by (10) $\mathbf{w}$ is perpendicular to both $\nabla\lambda$ and $\nabla\mu$ it is clear that the vortex lines are the intersections of the surfaces

$$\lambda = \text{const.}, \quad \mu = \text{const.}$$

Using then dots to denote partial differentiation with respect to t, and assuming the existence of a force potential, we may write (8′) as

$$\begin{aligned} -\nabla (V + c^2\kappa) &= \nabla (\dot{\phi} + \lambda\dot{\mu}) + \dot{\lambda}\nabla\mu - \dot{\mu}\nabla\lambda \\ &\qquad + (\mathbf{v} \cdot \nabla\lambda)\nabla\mu - (\mathbf{v} \cdot \nabla\mu)\nabla\lambda \\ &= \nabla (\dot{\phi} + \lambda\dot{\mu}) + \frac{d\lambda}{dt}\nabla\mu - \frac{d\mu}{dt}\nabla\lambda, \end{aligned}$$

which may be neatly expressed in the form

$$\frac{d\lambda}{dt}\nabla\mu - \frac{d\mu}{dt}\nabla\lambda + \nabla H = 0 \quad \text{..............(11)},$$

where the function H is given by the equation

$$H = \dot{\phi} + \lambda\dot{\mu} + V + c^2\kappa \quad \text{.................(12)}.$$

* Cf. Basset, *Treatise on Hydrodynamics*, Vol. 1, p. 28; also Silberstein, *Vectorial Mechanics*, p. 146.

On scalar multiplication of (11) by $\mathbf{w}$, it follows in virtue of (10) that

$$\mathbf{w} \cdot \nabla H = 0,$$

showing that H *is constant along a vortex line.* It can also be shown that H is independent of x, y, z and is therefore a function of t only. For taking the curl of (11) we deduce

$$\nabla \left(\frac{d\lambda}{dt}\right) \times \nabla\mu - \nabla \left(\frac{d\mu}{dt}\right) \times \nabla\lambda = 0.$$

On scalar multiplication by $\nabla\lambda$ it follows, by (10), that

$$\mathbf{w} \cdot \nabla \left(\frac{d\lambda}{dt}\right) = 0,$$

and similarly that

$$\mathbf{w} \cdot \nabla \left(\frac{d\mu}{dt}\right) = 0.$$

From these we deduce as in the old theory* that

$$\frac{d\lambda}{dt} = \frac{d\mu}{dt} = 0 \quad \text{.......................(13)}.$$

Thus the first two terms disappear from (11), which becomes simply $\nabla H = 0$, showing that H is constant in space and is therefore a function of t only; or

$$\dot{\phi} + \lambda\dot{\mu} + V + c^2\kappa = H(t) \quad \text{..............(14)}.$$

From (13) it is clear that the surfaces $\lambda = \text{const.}$ and $\mu = \text{const.}$, and therefore also the *vortex lines* which are their lines of intersection, *are always composed of the same particles of fluid.*

§ 4. *Steady motion.* When the motion is steady partial derivatives with respect to t are zero. If then the impressed force is derivable from a potential V, (8′) becomes

$$2\mathbf{v} \times \mathbf{w} = \nabla (V + c^2\kappa) \quad \text{.................(15)},$$

and the equation of continuity

$$\text{div}\,(k\mathbf{v}) = 0 \quad \text{.......................(16)}.$$

If we multiply (15) scalarly by $\mathbf{v}$ the first member vanishes, showing that

$$\mathbf{v} \cdot \nabla (V + c^2\kappa) = 0.$$

Thus the function $V + c^2\kappa$ *is constant along a line of flow.* Similarly scalar multiplication of (15) by $\mathbf{w}$ gives

$$\mathbf{w} \cdot \nabla (V + c^2\kappa) = 0,$$

* Cf. Basset, *loc. cit.* p. 29.

and therefore $V + c^2\kappa$ *is constant also along a vortex line.* This is a particular case of the more general theorem, proved in the preceding section, that H is constant along a vortex line. Thus the surface

$$V + c^2\kappa = \text{const.}$$

is composed of a double system of vortex lines and lines of flow.

II. Irrotational Motion.

§ 5. When the vorticity $\frac{1}{2}$ curl $(\kappa\mathbf{v})$ is zero the motion will be termed *irrotational* or *non-vortical*, being analogous to the motion of that name in the older theory. In this case $\kappa\mathbf{v}$ can be expressed as the gradient of a scalar function ϕ, which may be called the *velocity potential*: *i.e.*

$$\kappa\mathbf{v} = \nabla\phi \quad\text{.........................(17).}$$

The lines of flow are orthogonal to the surfaces of equal velocity potential.

The equation of motion can always be integrated when a force and a velocity potential exist. For (8′) then becomes

$$\nabla(\dot{\phi} + c^2\kappa + V) = 0.$$

The function in brackets is therefore constant throughout the liquid, and will be a function of t only; *i.e.*

$$\dot{\phi} + c^2\kappa + V = f(t) \quad\text{....................(18).}$$

This is the required integral of the equation of motion. An arbitrary function of t may, however, be incorporated in the velocity potential ϕ, and this equation then written without loss of generality

$$\dot{\phi} + c^2\kappa + V = 0 \quad\text{....................(18′).}$$

When the irrotational motion is *steady* $(c^2\kappa + V)$ is constant throughout the liquid, and is also invariable in time. In the preceding section, where $\mathbf{w}$ was not assumed to be zero, this function was only proved constant along vortex lines and lines of flow.

The *equation of continuity* (6), or as it may be written

$$\frac{dk}{dt} + k \operatorname{div} \mathbf{v} = 0,$$

may be expressed in terms of ϕ, if we write $\kappa\mathbf{v}/\kappa$ for $\mathbf{v}$, and expand the divergence of the quotient. The equation then becomes

$$\frac{d}{dt}\log k + \nabla\left(\frac{1}{\kappa}\right)\cdot\nabla\phi + \frac{1}{\kappa}\nabla^2\phi = 0 \quad\text{..........(19).}$$

This form is not so short as in the ordinary theory, nor can we obtain Laplace's equation, as there, by assuming the fluid incompressible, for such an assumption is inconsistent with the theory of relativity*.

§ 6. *Steadily rotating fluid.* Suppose that the fluid is in a state of steady rotation about the z-axis, and that the angular velocity of rotation Ω is a function of the distance r from that axis. We shall now determine what must be the form of this function in order that a velocity potential may exist†. If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the directions of the coordinate axes

$$\mathbf{v} = -\mathbf{i}\Omega y + \mathbf{j}\Omega x,$$
$$v = r\Omega.$$

For irrotational motion this velocity must satisfy the equation

$$\operatorname{curl}(\kappa\mathbf{v}) = 0,$$

that is
$$2\kappa\Omega + r\frac{d}{dr}(\kappa\Omega) = 0,$$

the integral of which is

$$\kappa\Omega r^2 = \text{const.} = \mu,$$

say, so that
$$\kappa\Omega = \frac{\mu}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

The velocity potential ϕ is then given by

$$\frac{d\phi}{dr} = 0, \qquad \frac{1}{r}\frac{d\phi}{d\theta} = \kappa v = \frac{\mu}{r},$$

showing that
$$\phi = \mu\theta + \text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(B),$$

which is an example of a *cyclic* velocity potential. The integral of the equation of motion is by (18′)

$$c^2\kappa + V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

But κ involves v^2 and therefore Ω, which is itself expressed in terms of κ by (A). This equation however gives

$$\Omega^2 r^4 = \frac{\mu^2}{\kappa^2} = \frac{\mu^2(c^2 - \Omega^2 r^2)}{\kappa'^2 c^2},$$

whence
$$\Omega^2 = \frac{\mu^2 c^2}{r^2(\mu^2 + r^2 c^2 \kappa'^2)},$$

* Cf. § 10 below. It will be shown, however, in § 11 that $\nabla^2\phi = 0$ is the equation of continuity for the steady irrotational motion of a fluid of minimum compressibility.

† Cf. Lamb, *Hydrodynamics*, § 28 (1st ed.).

κ' being given by (5'). On substitution of this value in (C) the integral of the equation of motion, viz.

$$V + \frac{c^3 \kappa'}{\sqrt{c^2 - v^2}} = 0,$$

becomes

$$V + \frac{c}{r}\sqrt{\mu^2 + r^2 c^2 \kappa'^2} = 0 \quad \text{.................(D)}.$$

§ 7. *Flow and circulation.* We define the *flow* from a point P to another Q, along a path of which $d\mathbf{s}$ denotes an element, as the quantity

$$\int_P^Q \kappa \mathbf{v} \cdot d\mathbf{s}.$$

Whenever a velocity potential exists this is equal to $\phi_Q - \phi_P$. The *circulation* round a closed curve is the line integral

$$I = \int_0 \kappa \mathbf{v} \cdot d\mathbf{s} \quad \text{.......................(20)}$$

taken round that closed curve. This, by Stokes' theorem, is equal to the surface integral

$$I = \int \operatorname{curl}(\kappa \mathbf{v}) \cdot \mathbf{n}\, dS \quad \text{................(20')}$$

taken over any surface drawn in the region and bounded by the closed curve. When the motion is irrotational the integrand is zero, and the circulation round the closed curve vanishes. It follows that, for a simply-connected region, the velocity potential is single-valued.

III. Vortex Motion.

§ 8. When the vorticity $\mathbf{w}$ is not zero the motion will be called *vortical* or *vortex motion.* A vortex tube is one bounded by vortex lines. Considering the portion of a vortex tube between any two cross sections, we find as usual on equating the volume and surface integrals

$$0 = \int \operatorname{div} \operatorname{curl}(\kappa \mathbf{v})\, d\tau = \int_S 2\mathbf{w} \cdot \mathbf{n}\, dS,$$

that the *moment* of the vortex tube $\int \mathbf{w} \cdot \mathbf{n}\, dS$, where the integration is extended over the cross section, is the same for all sections. And hence, as in the classical theory, the vortex lines either form closed lines, or else end in the surface of the fluid.

I shall now show that, on the assumption of a force potential, *Kelvin's theorem** of the constancy of the circulation in a closed filament moving with the fluid is true in the present case also. Consider a closed filament consisting always of the same particles, and let $d\mathbf{s}$ be a vector element of its length and ds the corresponding scalar. Then the circulation round it is

$$I = \int_0 \kappa \mathbf{v} \cdot d\mathbf{s}.$$

The time rate of change of this is

$$\frac{dI}{dt} = \int_0 \left[\frac{d}{dt}(\kappa \mathbf{v}) \cdot d\mathbf{s} + \kappa \mathbf{v} \cdot \left(\frac{d\mathbf{s}}{dt}\right)\right]$$

$$= \int_0 \left[d\mathbf{s} \cdot \left(-\nabla V - \frac{1}{\gamma} \nabla P + \kappa \mathbf{v} \cdot (d\mathbf{s} \cdot \nabla) \mathbf{v}\right)\right]$$

$$= \int_0 \left[-\frac{\partial}{\partial s} V + \kappa \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial s}\right] ds - \int_0 d\mathbf{s} \cdot \left(\frac{c^2}{\gamma} \nabla \frac{\kappa}{\gamma}\right) \quad \ldots (21).$$

Now the last integral is

$$\int_0 d\mathbf{s} \cdot \left[\frac{c^2}{\gamma^2} \nabla \kappa - \frac{c}{2\gamma} \frac{\kappa}{\sqrt{c^2 - v^2}} \nabla v^2\right] = \int_0 \left[(c^2 - v^2) \frac{\partial \kappa}{\partial s} - \frac{\kappa}{2} \frac{\partial v^2}{\partial s}\right] ds.$$

On substitution of this value in (21) that equation reduces to

$$\frac{dI}{dt} = \int_0 \left[c^2 \frac{\partial \kappa}{\partial s} - \frac{\partial V}{\partial s} + \frac{\partial}{\partial s}(\kappa v^2)\right] ds.$$

Hence, since the path of integration is closed and κ, V, and κv^2 are single-valued functions, the integral vanishes, showing that

$$\frac{dI}{dt} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Thus the circulation does not alter with the time.

Corollary. If I is zero at any instant it will remain zero. In particular, if the motion is irrotational at any instant it will remain so, provided that the impressed forces have a potential.

§ 9. *Helmholtz's theorems*†. That these theorems are true in the present theory also follows without difficulty from the form (8′) of the equation of motion. For taking the curl of both members we have

$$\frac{\partial \mathbf{w}}{\partial t} + \operatorname{curl}(\mathbf{w} \times \mathbf{v}) = 0.$$

* Cf. Silberstein, *loc. cit.* p. 161, for the proof of the ordinary theorem.
† *Ibid.* pp. 163—65.

Expanding the second term and using the equation of continuity, we find

$$\frac{d\mathbf{w}}{dt} - \frac{\mathbf{w}}{k}\frac{dk}{dt} - \mathbf{w}\cdot\nabla\mathbf{v} = 0,$$

which, after division by k, may be written

$$\frac{d}{dt}\left(\frac{\mathbf{w}}{k}\right) = \frac{\mathbf{w}}{k}\cdot\nabla\mathbf{v} \quad \text{.....................(23).}$$

Differentiation with respect to t gives

$$\frac{d^2}{dt^2}\left(\frac{\mathbf{w}}{k}\right) = \left(\frac{d}{dt}\frac{\mathbf{w}}{k}\right)\cdot\nabla\mathbf{v} + \frac{\mathbf{w}}{k}\cdot\left(\frac{d}{dt}\nabla\mathbf{v}\right).$$

If then $\mathbf{w}$ vanishes at any instant it follows from (23) that the first derivative of $\mathbf{w}/k$ also vanishes, and from the next equation likewise the second derivative at that instant. Similarly all the derivatives with respect to t vanish at that instant, and the quantity $\mathbf{w}/k$ remains permanently zero, so that *the motion continues irrotational.*

Further, *the moment of a vortex filament does not vary with the time.* For if $d\mathbf{s}$ is an element of such a filament moving with the fluid

$$d\mathbf{s} = \mathbf{w}\,ds/w,$$

and

$$\frac{d}{dt}(d\mathbf{s}) = d\mathbf{s}\cdot\nabla\mathbf{v} = \frac{ds}{w}\mathbf{w}\cdot\nabla\mathbf{v},$$

so that (23) is equivalent to

$$\frac{d}{dt}\left(\frac{\mathbf{w}}{k}\right) = \frac{w}{k\,ds}\frac{d}{dt}(d\mathbf{s}) \quad \text{..................(24).}$$

Now if μ is the moment of the filament, dm_0 the constant normal rest-mass of the element considered, and α the cross-sectional area

$$\mu = \alpha w, \quad dm_0 = k\alpha ds,$$

so that

$$\frac{w}{k} = \mu\frac{ds}{dm_0} \quad \text{.......................(25).}$$

Substituting this value in (24), and remembering that dm_0 is constant, we have

$$\frac{d}{dt}(\mu d\mathbf{s}) = \mu\frac{d}{dt}(d\mathbf{s}),$$

and therefore

$$\frac{d\mu}{dt} = 0,$$

showing that the moment of the filament remains constant.

It has been proved already that *a vortex filament consists always of the same particles of fluid*, though this can also be now deduced from (24) and (25), using the invariability of μ.

IV. Fluid of Minimum Compressibility*.

§ 10. According to the theory of Relativity no velocity can exceed that of light. Hence there is no such thing as an incompressible fluid; for such a fluid would admit a wave propagation with infinite velocity. A fluid of minimum compressibility is one in which a wave can attain a velocity equal to that of light; and for such a fluid the quantity κ is directly proportional to the density†

$$\kappa = k/k_0', \quad \kappa' = k'/k_0' \quad \ldots\ldots\ldots\ldots\ldots\ldots(27),$$

where k_0' is a constant representing the normal rest-density, *i.e.* the rest-density corresponding to the normal pressure p_0.

For a fluid of minimum compressibility the equations of motion, energy and continuity may by (27) be expressed in terms of the velocity $\mathbf{v}$ and the rest-density k'. The equation of motion, viz.

$$\frac{d}{dt}(\kappa \mathbf{v}) + \frac{1}{\gamma}\nabla P = \mathbf{F},$$

becomes on substitution

$$k\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dk}{dt} + \frac{c^2}{\gamma}\nabla k' = k_0'\mathbf{F}.$$

Dividing by γ and using the equation of continuity to transform the second term, we have at once

$$k'\left(\frac{d\mathbf{v}}{dt} - \mathbf{v}\operatorname{div}\mathbf{v}\right) + (c^2 - v^2)\nabla k' = k_0'\mathbf{F}/\gamma \quad \ldots\ldots(28),$$

which is the *equation of motion* in the required form.

Multiplying this equation scalarly by $\mathbf{v}$, and transforming $\mathbf{v}\cdot\nabla k'$, we obtain

$$k'\left(\frac{1}{2}\frac{dv^2}{dt} - v^2\operatorname{div}\mathbf{v}\right) + (c^2 - v^2)\left(\frac{dk'}{dt} - \frac{\partial k'}{\partial t}\right) = \frac{k_0'\mathbf{F}\cdot\mathbf{v}}{\gamma}.$$

Now
$$\frac{dk'}{dt} = \frac{d}{dt}\left(\frac{k\sqrt{c^2 - v^2}}{c}\right)$$

$$= \frac{dk}{dt}\frac{\sqrt{c^2 - v^2}}{c} - \frac{k}{2c\sqrt{c^2 - v^2}}\frac{dv^2}{dt}$$

$$= -k'\operatorname{div}\mathbf{v} - \frac{1}{2}\frac{\gamma^2 k'}{c^2}\frac{dv^2}{dt},$$

* Lamla, *loc. cit.* p. 788; Laue, *loc. cit.* § 37.

† Laue, *loc. cit.* p. 241.

in virtue of the equation of continuity. On substitution of this value in the last equation it becomes simply

$$k' \operatorname{div} \mathbf{v} + \frac{c^2 - v^2}{c^2} \frac{\partial k'}{\partial t} = -\frac{k_0'}{c^2 \gamma} \mathbf{F} \cdot \dot{\mathbf{v}} \quad \text{.............(29)},$$

which is the *energy equation* in terms of k' and $\mathbf{v}$. These equations (28) and (29) are identical with those found otherwise by Lamla* and Laue†. The equation of continuity is as before

$$\frac{dk}{dt} + k \operatorname{div} \mathbf{v} = 0 \quad \text{....................(30)},$$

which takes the required form if k is replaced by $\gamma k'$.

§ 11. *Steady irrotational motion.* In virtue of (27) the equation of continuity may also be written

$$\frac{\partial \kappa}{\partial t} + \operatorname{div} (\kappa \mathbf{v}) = 0 \quad \text{....................(31)},$$

and therefore when the motion is irrotational

$$\frac{\partial \kappa}{\partial t} + \nabla^2 \phi = 0 \quad \text{....................(31')}.$$

If it is also steady the first term is zero, and we have (as in the older theory for the case of an incompressible fluid)

$$\nabla^2 \phi = 0 \quad \text{....................(31'')}.$$

Thus *for steady irrotational motion of a fluid of minimum compressibility the velocity potential satisfies Laplace's equation.*

It follows immediately that for such a fluid, filling a simply-connected region within a hollow shell, which is fixed relative to some system of reference S, steady irrotational motion relative to that system is impossible. For by Green's theorem

$$\int \kappa^2 v^2 d\tau = \int (\nabla \phi)^2 \, d\tau = -\int \phi \kappa \mathbf{v} \cdot \mathbf{n} \, dS - \int \phi \nabla^2 \phi \, d\tau.$$

Now the last integral vanishes by the equation of continuity. So also does the last but one: for $\mathbf{v} \cdot \mathbf{n}$ is zero, being the normal velocity at the surface of the fluid. Hence

$$\int \kappa^2 v^2 d\tau = 0,$$

showing that $\mathbf{v}$ must vanish identically throughout the fluid.

* *Loc. cit.* p. 792.
† *Loc. cit.* p. 244.

In the present case* the *integral of the equation of motion* found in § 5, viz.

$$c^2\kappa + V = 0,$$

takes the form

$$c^2 k + k_0' V = 0,$$

or, in terms of the rest-density k',

$$c^3 k' + V k_0' \sqrt{c^2 - v^2} = 0.$$

§ 12. *Steady motion in two dimensions.* Supposing the fluid of minimum compressibility, let its steady motion be parallel to one plane—the plane xy. Introduce a function ψ satisfying the relations

$$\left.\begin{aligned} \kappa u &= -\frac{\partial\psi}{\partial y} \\ \kappa v &= \frac{\partial\psi}{\partial x} \end{aligned}\right\} \quad \text{.......................(32)},$$

u, v being, as in § 1, the components of velocity parallel to the axes of x and y respectively. Such a function ψ exists, the equation of continuity

$$\operatorname{div}(\kappa\mathbf{v}) = 0$$

being satisfied identically. The function ψ is proportional to the flux of matter across a line AP drawn from a fixed point A to the variable point $P(x, y)$. For owing to an infinitesimal displacement δx of P the increment in the flux of matter is

$$kv\,\delta x = k_0'\kappa v\,\delta x = k_0'\frac{\partial\psi}{\partial x}\delta x.$$

Thus if Ψ denote the flux

$$\frac{\partial\Psi}{\partial x}\delta x = k_0'\frac{\partial\psi}{\partial x}\delta x.$$

Similarly $$\frac{\partial\Psi}{\partial y}\delta y = k_0'\frac{\partial\psi}{\partial y}\delta y,$$

showing that $$\Psi = k_0'\psi,$$

as stated. The part played by this function ψ is exactly similar to that of the stream function in the two-dimensional motion of a liquid in the classical theory. The present function also is a true *stream function.* Its value is independent of the path chosen from A to P provided the region is simply-connected. For, if ABP and

* Lamla considers only the case of free motion (V=const.); *loc. cit.* p. 795.

ACP are two different paths, the flux across the complete boundary $ACPBA$ is

$$\int k\mathbf{v}\cdot\mathbf{n}ds = \int \operatorname{div}(k\mathbf{v})\,d\tau = 0,$$

as is also obvious because the motion is steady. The lines $\psi =$ const. are the actual *stream lines*: for if P moves subject to this condition there is no flux across the path traced out by that point.

The above is true whether the motion is irrotational or vortical. The vorticity w is equal to

$$\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=\tfrac{1}{2}\nabla^2\psi\,;$$

and therefore for *irrotational motion* ψ must satisfy Laplace's equation

$$\nabla^2\psi = 0 \quad\text{..........................(33).}$$

If this relation is satisfied there is a velocity potential ϕ, and (32) may then be expressed in the form

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=-\frac{\partial\psi}{\partial y}\\ \frac{\partial\phi}{\partial y}&=-\frac{\partial\psi}{\partial x}\end{aligned}\right\}\quad\text{........................(34).}$$

These are identical with the relations subsisting between the stream function and the velocity potential in the classical theory of the two-dimensional irrotational motion of a liquid. They are the conditions that $\phi+i\psi$ should be a function of the complex variable $x+iy$. The theory of such functions may then be used as in the theory referred to*, to give various possible forms of stream lines and lines of equal velocity potential.

§ 13. *Source, sink and doublet.* Similarly the irrotational motion of a fluid of minimum compressibility defined by the velocity potential

$$\phi = -\frac{m}{k_0'}\cdot\frac{1}{r}\quad\text{........................(35),}$$

where r is the distance from a fixed point O, corresponds to the assumption of a continual creation of matter at the point O, of amount $4\pi m$ per unit time. For

$$\kappa\mathbf{v} = \nabla\phi = -\frac{m}{k_0'}\nabla\left(\frac{1}{r}\right)=\frac{m}{k_0'}\cdot\frac{\mathbf{r}}{r^3},$$

so that

$$k\mathbf{v} = \frac{m\mathbf{r}}{r^3}.$$

* Cf. Lamb, *loc. cit.* chap. IV.

The velocity is therefore radial from O, and kv is inversely proportional to r^2. The flow of matter per unit time across the surface of a sphere of radius r is $4\pi m$, equal to the rate of creation of matter at O. Such a motion is then that due to a *source* of strength m at the point O. If the negative sign in (35) were replaced by a positive one, we should have the motion due to a *sink* at O of strength m. And finally the velocity potential representing a doublet at O of moment M and with its axis along the unit vector $\mathbf{n}$ is

$$\phi = \frac{M}{k_0'} \mathbf{n} \cdot \nabla \left(\frac{1}{r}\right).$$

On the convergence of certain multiple series. By G. H. HARDY, M.A., Trinity College.

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1. In a paper published in 1903 in the *Proceedings of the London Mathematical Society**, and bearing the same title as this one, I proved a theorem concerning the convergence of multiple series, of the type

$$\Sigma a_{i_1, i_2, \ldots i_k} u_{i_1, i_2, \ldots i_k},$$

which is given (with an improvement in the conditions) on p. 89 of Dr Bromwich's *Theory of infinite series.* This theorem is one of a class of some importance; and I propose now to state and prove the leading theorems of this class in a form more systematic and general than has been given to them before. I shall begin by recapitulating, with certain changes of form, some known theorems concerning simply infinite series; and I shall then obtain the corresponding theorems for double series in a form as closely analogous as possible. The generalisation from double series to multiple series of any order may well be left to the reader.

Simply infinite series.

2. I shall say that a function a_m, real or complex, of a positive integral variable m is *of bounded variation* if

$$\sum_1^\infty |a_m - a_{m+1}|$$

is convergent. It is plain that this condition involves the existence of $a = \lim a_m$.

THEOREM 1. *The necessary and sufficient condition that a_m should be of bounded variation is that its real and imaginary parts should be of bounded variation.*

This follows at once from the inequalities

$$|\alpha_m - \alpha_{m+1}| \leqslant |a_m - a_{m+1}|, \quad |\beta_m - \beta_{m+1}| \leqslant |a_m - a_{m+1}|,$$

$$|a_m - a_{m+1}| \leqslant |\alpha_m - \alpha_{m+1}| + |\beta_m - \beta_{m+1}|,$$

where
$$a_m = \alpha_m + i\beta_m.$$

* Ser. 2, vol. 1, pp. 124—128. See also 'Note in addition to a former paper on conditionally convergent multiple series', *ibid.*, vol. 2, 1904, pp. 190—191.

THEOREM 2. *The necessary and sufficient condition that a real function a_m should be of bounded variation is that it should be of the form $A_m - A_m'$, where A_m and A_m' are positive and decrease steadily as m increases.*

The sufficiency of the condition follows at once from the inequality

$$|a_m - a_{m+1}| \leqslant (A_m - A_{m+1}) + (A_m' - A'_{m+1}).$$

In order to prove that it is necessary, let us suppose that a_m is of bounded variation, and let us write

$$p_m = |a_m - a_{m+1}| \ (a_m - a_{m+1} \geqslant 0), \quad p_m = 0 \ (a_m - a_{m+1} < 0),$$
$$p_m' = |a_m - a_{m+1}| \ (a_m - a_{m+1} \leqslant 0), \quad p_m' = 0 \ (a_m - a_{m+1} > 0),$$
$$B_m = \sum_m^\infty p_n, \quad B_m' = \sum_m^\infty p_n'.$$

Then B_m and B_m' are positive and decrease steadily as m increases; and

$$B_m - B_m' = \sum_m^\infty (a_n - a_{n+1}) = a_m - a.$$

We may therefore take $A_m = B_m + C$ and $A_m' = B_m' + C'$, where C and C' are suitably chosen constants.

THEOREM 3. *If a_m is of bounded variation, and Σu_m is convergent, then $\Sigma a_m u_m$ is convergent.*

Theorem 1 shews that it is enough to prove this theorem when a_m is real. Theorem 2 shews that it is enough to prove it when a_m is positive and steadily decreasing. In this form the theorem is classical*.

LEMMA α. *If Σc_m is a divergent series of positive terms, we can find a sequence of positive numbers ϵ_m, tending steadily to the limit zero, such that $\Sigma \epsilon_m c_m$ is divergent.*

LEMMA β. *If Σc_m is a divergent series of positive terms, we can find a sequence of integers m_i such that the series $\Sigma c_m'$, where $c_m' = 0$ if $m = m_i$ and $c_m' = c_m$ otherwise, is divergent.*

Lemma α is due to Abel†. Lemma β is quite trivial, and the proof may be left to the reader.

* See Bromwich, *Infinite Series*, p. 48. Theorem 3 is given by Dedekind in his editions of Dirichlet's *Vorlesungen über Zahlentheorie*: see *e.g.* p. 255 of the third edition. The central idea of all such theorems is of course Abel's. The line of argument followed here is due substantially to Hadamard, 'Deux théorèmes d'Abel sur la convergence des séries', *Acta Mathematica*, vol. 27, 1903, pp. 177—184.

† 'Sur les séries', *Œuvres*, vol. 2, pp. 197—205.

THEOREM 4. *If $\Sigma a_m u_m$ is convergent whenever Σu_m is convergent, then a_m is of bounded variation.*

This theorem is due to Hadamard*. We have to shew that, if $\Sigma |a_m - a_{m+1}|$ is divergent, u_m can be so chosen that Σu_m is convergent and $\Sigma a_m u_m$ is not. By Lemma α, we can choose a sequence of positive and steadily decreasing numbers ϵ_m so that $\epsilon_m \to 0$ and Σc_m, where

$$c_m = \epsilon_m |a_m - a_{m+1}|,$$

is divergent. By Lemma β, we can then choose the sequence m_i so that $\Sigma c_m'$ is divergent. We take

$$u_1 = U_1, \quad u_m = U_m - U_{m-1} \qquad (m > 1),$$

where

$$U_{m_i} = 0,$$

and

$$U_m = \epsilon_m \frac{|a_m - a_{m+1}|}{a_m - a_{m+1}}$$

if $m \neq m_i$, the last expression being interpreted as meaning ϵ_m if $a_m = a_{m+1}$. We have then

$$\sum_1^{m_i} a_m u_m = \sum_1^{m_i - 1} (a_m - a_{m+1}) U_m + a_{m_i} U_{m_i} = \sum_1^{m_i - 1} c_m',$$

which tends to infinity with i. Thus $\Sigma a_m u_m$ is not convergent, while Σu_m converges to zero.

We may call a_m *a convergence factor* if $\Sigma a_m u_m$ is convergent whenever Σu_m is so. Theorems 3 and 4 may then be combined concisely in

THEOREM 5. *The necessary and sufficient condition that a_m should be a convergence factor is that it should be of bounded variation.*

Double series.

3. The convergence of a double series, in Pringsheim's sense†, does not necessarily involve the convergence of any of its rows or columns ‡. In this paper I shall confine my attention to convergent series whose rows and columns are convergent separately: in this case I shall say that the series is *regularly convergent.* A regularly convergent double series is also convergent when summed by rows or by columns, and its sum by rows or by columns is equal to its sum as a double series §.

Similarly I shall say that $a_{m,n}$ tends *regularly* to a limit if

$$\lim_{m \to \infty} a_{m,n} = a_n, \quad \lim_{n \to \infty} a_{m,n} = a_m,$$

* *l.c. supra.*

† Bromwich, *Infinite Series*, p. 72.

‡ Bromwich, *ibid.*, p. 74.

§ Bromwich, *ibid.*, p. 75.

and the double limit

$$\lim_{m,n\to\infty} a_{m,n} = a,$$

all exist. In this case a_m and a_n tend to a when m and n tend to infinity.

LEMMA γ. *If $\Sigma\Sigma u_{m,n}$ is regularly convergent, to the sum s, and*

$$s_{m,n} = \sum_1^m \sum_1^n u_{\mu,\nu},$$

then, given any positive number ϵ, we can find ω so that

$$|s_{m,n} - s| < \epsilon$$

if either m or n is greater than ω.

We may suppose $s = 0$ without loss of generality. Since the double limit exists, we can choose ω_1 so that $|s_{m,n}| < \epsilon$ if m and n are *both* greater than ω_1. When ω_1 is fixed we can choose ω_2 and ω_3 so that the inequality is satisfied for $1 \leqslant m \leqslant \omega_1$, $n > \omega_2$ and for $m > \omega_3$, $1 \leqslant n \leqslant \omega_1$. We can then take ω to be the greatest of ω_1, ω_2, and ω_3.

LEMMA δ. *In the same circumstances, we can choose ω so that*

$$\left| \sum_m^p \sum_n^q u_{\mu,\nu} \right| < \epsilon$$

if $p \geqslant m$, $q \geqslant n$, and either m or n is greater than ω.

This follows at once from Lemma γ and the identity

$$\sum_m^p \sum_n^q u_{\mu,\nu} = s_{p,q} - s_{p,n-1} - s_{m-1,q} + s_{m-1,n-1}.$$

4. I shall say that $a_{m,n}$ is *of bounded variation in* (m, n) if

(1) $a_{m,n}$ is, for every fixed value of m or n, of bounded variation in n or m,

(2) the series

$$\Sigma\Sigma\, | a_{m,n} - a_{m,n+1} - a_{m+1,n} + a_{m+1,n+1} |$$

is convergent. And I shall say that $a_{m,n}$ is *a convergence factor* if $\Sigma\Sigma a_{m,n} u_{m,n}$ is regularly convergent whenever $\Sigma\Sigma u_{m,n}$ is regularly convergent. My main object is to prove the analogue of Theorem 5 for double series, *i.e.* to establish the equivalence of these two notions.

It will be convenient to write

$$\Delta_m a_{m,n} = a_{m,n} - a_{m+1,n}, \quad \Delta_n a_{m,n} = a_{m,n} - a_{m,n+1},$$
$$\Delta_{m,n} a_{m,n} = a_{m,n} - a_{m,n+1} - a_{m+1,n} + a_{m+1,n+1}.$$

The condition that $a_{m,n}$ should be of bounded variation is then that the series $\Sigma \mid \Delta_m a_{m,n} \mid$, $\Sigma \mid \Delta_n a_{m,n} \mid$, and $\Sigma\Sigma \mid \Delta_{m,n} a_{m,n} \mid$ should all be convergent. It is clear that these conditions involve the regular convergence of $a_{m,n}$ to a limit a.

THEOREM 6. *If the condition (2) is satisfied, and $a_{m,1}$ and $a_{1,n}$ are of bounded variation in m and n respectively, then $a_{m,n}$ is of bounded variation in (m, n).*

For
$$\Delta_\mu a_{\mu,n} = \Delta_\mu a_{\mu,1} - \sum_{\nu=1}^{n-1} \Delta_{\mu,\nu} a_{\mu,\nu},$$
$$\sum_{\mu=1}^{m-1} \mid \Delta_\mu a_{\mu,n} \mid \leqslant \sum_{\mu=1}^{m-1} \mid \Delta_\mu a_{\mu,1} \mid + \sum_{\mu=1}^{m-1} \sum_{\nu=1}^{n-1} \mid \Delta_{\mu,\nu} a_{\mu,\nu} \mid,$$
so that
$$\Sigma \mid \Delta_{\mu,} a_{\mu,n} \mid$$
is convergent.

THEOREM 7. *If $a_{m,n}$ is of bounded variation in (m, n), then*
$$a_m = \lim_{n\to\infty} a_{m,n}, \quad a_n = \lim_{m\to\infty} a_{m,n}$$
are of bounded variation in m and n respectively.

For
$$a_\nu = a_{1,\nu} - \sum_{\mu=1}^{\infty} \Delta_\mu a_{\mu,\nu},$$
$$a_\nu - a_{\nu+1} = \Delta_\nu a_{1,\nu} - \sum_{\mu=1}^{\infty} \Delta_{\mu,\nu} a_{\mu,\nu},$$
$$\sum_{\nu=1}^{n-1} \mid a_\nu - a_{\nu+1} \mid \leqslant \sum_{\nu=1}^{n-1} \mid \Delta_\nu a_{1,\nu} \mid + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{n-1} \mid \Delta_{\mu,\nu} a_{\mu,\nu} \mid,$$
and so
$$\Sigma \mid a_\nu - a_{\nu+1} \mid$$
is convergent.

THEOREM 8. *The necessary and sufficient condition that $a_{m,n}$ should be of bounded variation is that its real and imaginary parts should be of bounded variation.*

This follows from Theorem 1 and the inequalities
$$\mid \Delta_{m,n} \alpha_{m,n} \mid \leqslant \mid \Delta_{m,n} a_{m,n} \mid, \quad \mid \Delta_{m,n} \beta_{m,n} \mid \leqslant \mid \Delta_{m,n} a_{m,n} \mid,$$
$$\mid \Delta_{m,n} a_{m,n} \mid \leqslant \mid \Delta_{m,n} \alpha_{m,n} \mid + \mid \Delta_{m,n} \beta_{m,n} \mid,$$
where
$$a_{m,n} = \alpha_{m,n} + i\beta_{m,n}.$$

THEOREM 9. *The necessary and sufficient condition that a real function $a_{m,n}$ should be of bounded variation is that it should be of the form $A_{m,n} - A'_{m,n}$, where*

$$A_{m,n} \geqslant 0, \quad \Delta_m A_{m,n} \geqslant 0, \quad \Delta_n A_{m,n} \geqslant 0, \quad \Delta_{m,n} A_{m,n} \geqslant 0,$$

and $A'_{m,n}$ satisfies similar conditions.

Suppose first that $a_{m,n}$ is of the form indicated. It is plain that the series

$$\sum_m \Delta_m A_{m,n}, \quad \sum_n \Delta_n A_{m,n}, \quad \sum\sum \Delta_{m,n} A_{m,n},$$

and the corresponding series formed from $A'_{m,n}$, are all convergent. Further we have

$$|\Delta_m a_{m,n}| \leqslant \Delta_m A_{m,n} + \Delta_m A'_{m,n},$$

and similar inequalities for $\Delta_n a_{m,n}$ and $\Delta_{m,n} a_{m,n}$. Hence $a_{m,n}$ is of bounded variation.

Next suppose that $A_{m,n}$ is of bounded variation, and let

$$p_{m,n} = |\Delta_{m,n} a_{m,n}| \ (\Delta_{m,n} a_{m,n} \geqslant 0), \quad p_{m,n} = 0 \ (\Delta_{m,n} a_{m,n} < 0),$$

$$p'_{m,n} = |\Delta_{m,n} a_{m,n}| \ (\Delta_{m,n} a_{m,n} \leqslant 0), \quad p'_{m,n} = 0 \ (\Delta_{m,n} a_{m,n} > 0).$$

Suppose also that

$$B_{m,n} = \sum_m^\infty \sum_n^\infty p_{\mu,\nu}, \quad B'_{m,n} = \sum_m^\infty \sum_n^\infty p'_{\mu,\nu}.$$

Then it is plain that

$$\Delta_m B_{m,n} \geqslant 0, \quad \Delta_n B_{m,n} \geqslant 0, \quad \Delta_{m,n} B_{m,n} \geqslant 0,$$

and that $B'_{m,n}$ satisfies similar conditions.

Also

$$B_{m,n} - B'_{m,n} = \sum_m^\infty \sum_n^\infty \Delta_{\mu,\nu} a_{\mu,\nu} = a_{m,n} - a_m - a_n + a,$$

$$a_{m,n} = B_{m,n} - B'_{m,n} + a_m + a_n - a.$$

But, by Theorems 7 and 2, we have

$$a_m = C_m - C_m', \quad a_n = D_n - D_n',$$

where C_m, C_m', D_n, and D_n' are positive and steadily decreasing functions. Thus

$$a_{m,n} = A_{m,n} - A'_{m,n},$$

where

$$A_{m,n} = B_{m,n} + C_m + D_n + E, \quad A'_{m,n} = B'_{m,n} + C_m' + D_n' + E',$$

E and E' being suitably chosen constants; and it is clear that $A_{m,n}$ and $A'_{m,n}$ will satisfy the conditions of the theorem if E and E are sufficiently large.

THEOREM 10. *If $a_{m,n}$ is of bounded variation, and $\Sigma\Sigma u_{m,n}$ is regularly convergent, then $\Sigma\Sigma a_{m,n} u_{m,n}$ is regularly convergent.*

In virtue of Theorem 8, it is enough to prove this when $a_{m,n}$ is real. In virtue of Theorem 9, it is enough to prove it when

$$a_{m,n} \geqslant 0, \quad \Delta_m a_{m,n} \geqslant 0, \quad \Delta_n a_{m,n} \geqslant 0, \quad \Delta_{m,n} a_{m,n} \geqslant 0.$$

In the first place, by Theorem 3, every row and column of the series $\Sigma\Sigma a_{m,n} u_{m,n}$ is convergent.

In the second place, we have

$$\sum_m^p \sum_n^q a_{\mu,\nu} u_{\mu,\nu} = \sum_m^{p-1} \sum_n^{q-1} \Delta_{\mu,\nu} a_{\mu,\nu} \sum_m^\mu \sum_n^\nu u_{i,j}$$
$$+ \sum_m^{p-1} \Delta_\mu a_{\mu,q} \sum_m^\mu \sum_n^q u_{i,j} + \sum_n^{q-1} \Delta_\nu a_{p,\nu} \sum_m^p \sum_n^\nu u_{i,j} + a_{p,q} \sum_m^p \sum_n^q u_{i,j}\text{*}.$$

It follows that, if $p \geqslant m$, $q \geqslant n$, we have

$$\left| \sum_m^p \sum_n^q a_{\mu,\nu} u_{\mu,\nu} \right| \leqslant a_{m,n} H_{m,n},$$

where $H_{m,n}$ is the upper bound of

$$\left| \sum_m^\mu \sum_n^\nu u_{i,j} \right| \quad (\mu \geqslant m, \ n \geqslant \nu).$$

Now

$$\sum_1^p \sum_1^q a_{\mu,\nu} u_{\mu,\nu} - \sum_1^m \sum_1^n a_{\mu,\nu} u_{\mu,\nu} = \left(\sum_{m+1}^p \sum_{n+1}^q + \sum_{m+1}^p \sum_1^n + \sum_1^m \sum_{n+1}^q \right) a_{\mu,\nu} u_{\mu,\nu},$$

and so

$$\left| \sum_1^p \sum_1^q a_{\mu,\nu} u_{\mu,\nu} - \sum_1^m \sum_1^n a_{\mu,\nu} u_{\mu,\nu} \right| \leqslant (a_{m+1,n+1} + a_{m+1,1} + a_{1,n+1})\, h_{m,n},$$

where $h_{m,n}$ is the upper bound of

$$\left| \sum_k^\mu \sum_l^\nu u_{i,j} \right|,$$

for all values of k, l, μ, and ν such that $\mu \geqslant k$, $\nu \geqslant l$, and $k > m$ or $l > n$.

* See pp. 124—125 of my paper quoted in § 1, where the general form of this identity, for multiple series of any order, is given. Similar transformations of double series were given independently by M. Krause, 'Über Mittelwertsätze im Gebiete der Doppelsummen und Doppelintegrale', *Leipziger Berichte*, vol. 55, 1903, pp. 240—263. See also Bromwich, 'Various extensions of Abel's Lemma', *Proc. London Math. Soc.*, ser. 2, vol. 6, 1907, pp. 58—76, where further interesting applications of the identity are made.

Hence, by Lemma δ, we can choose ω so that

$$\left|\sum_1^p\sum_1^q a_{\mu,\nu}u_{\mu,\nu} - \sum_1^m\sum_1^n a_{\mu,\nu}u_{\mu,\nu}\right| \leqslant (a_{m+1,n+1}+a_{m+1,n}+a_{1,n+1})\,\epsilon < 3a_{1,1}\,\epsilon,$$

if m and n are greater than ω. Thus the double series is convergent, and, since its rows and columns are convergent, it is regularly convergent.

When $a_{m,n}$ and its various differences are positive, this theorem is nearly the same as that referred to in § 1. It is related to the latter theorem, in fact, as what Dr Bromwich calls 'Abel's test' for ordinary convergence is related to 'Dirichlet's test'.* The more direct generalisation is as follows.

THEOREM 11. *If $a_{m,n}$ is of bounded variation and tends regularly to zero, and*

$$\sum_1^m\sum_1^n u_{\mu,\nu}$$

is bounded, then $\Sigma\Sigma a_{m,n}u_{m,n}$ is regularly convergent.

The proof is similar to that of Theorem 10, and I need hardly write it out at length. The theorem shews, for example, that the series

$$\Sigma\Sigma\frac{\cos(m\theta+n\phi)}{(a+m\omega+n\omega')^s},$$

where θ and ϕ are real, ω'/ω is positive or complex, and the real part of s is positive, is regularly convergent except for certain special values of θ, ϕ, and a; or again that the series

$$\Sigma\Sigma\frac{\cos(m\theta+n\phi)}{(am^2+2bmn+cn^2)^s},$$

* Theorem 10 itself does not seem to have been enunciated before, even in the specialised form. The nearest theorem which I have been able to find is one given by C. N. Moore, 'On convergence factors in double series and the double Fourier's series', *Trans. Amer. Math. Soc.*, Vol. 14, 1913, pp. 73—104. Moore's theorem (a particular case of a theorem concerning Cesàro summability) is as follows: *if*

(1) $\Sigma\Sigma u_{m,n}$ is convergent as a double series in Pringsheim's sense,

(2) $$\left|\sum_1^m\sum_1^n u_{\mu,\nu}\right| < K,$$

(3) $$a_{m,n}\to 0,$$

(4) $$\lim_{m\to\infty}\sum_{n=1}^{\infty}|a_{m,n}|=0,\quad \lim_{n\to\infty}\sum_{m=1}^{\infty}|a_{m,n}|=0,$$

(5) $$\Sigma\Sigma\,|u_{m,n}\,a_{m,n}|$$

is convergent, then $\Sigma\Sigma a_{m,n}u_{m,n}$ is convergent.

where θ, ϕ, a, b, and c are real, and a, $ac-b^2$, and the real part of s are positive, is regularly convergent except for certain special values of θ and ϕ. In either of these series, of course, the cosine may be replaced by a sine.

In order to prove the converse of Theorem 10 we require two lemmas analogous to Lemmas α and β.

LEMMA ϵ. *If $\Sigma\Sigma c_{m,n}$ is a divergent series of positive terms, we can find $\epsilon_{m,n}$ so that* (1) *$\epsilon_{m,n}$ decreases when m or n increases,* (2) *$\epsilon_{m,n}$ tends regularly to zero, and* (3) *the series $\Sigma\Sigma\epsilon_{m,n}c_{m,n}$ is divergent.*

(1) Suppose first that at least one row or column of the original series, say the νth row $\Sigma c_{m,\nu}$, is divergent. By Lemma α, we can choose a steadily decreasing sequence η_m, with limit zero, so that $\Sigma\eta_m c_{m,\nu}$ is divergent. We take

$$\epsilon_{m,n}=\eta_m\,(n\leqslant\nu),\quad \epsilon_{m,n}=0\,(n>\nu),$$

and it is plain that the conditions of the lemma are satisfied.

(2) Suppose that every row and column is convergent; and let

$$\sum_{(m)} c_{m,n}=\gamma_n,\quad \sum_{(n)} c_{m,n}=\gamma_m.$$

Then $\Sigma\gamma_m$ is divergent. We choose a steadily decreasing sequence η_m so that $\Sigma\eta_m\gamma_m$ is divergent. Then $\Sigma\Sigma c'_{m,n}$, where

$$c'_{m,n}=\eta_m\,c_{m,n},$$

is divergent; and so $\Sigma\gamma_n'$, where

$$\gamma_n'=\sum_{(m)}\eta_m c_{m,n},$$

is divergent. We now choose a steadily decreasing sequence ζ_n, with limit zero, so that $\Sigma\zeta_n\gamma_n'$ is divergent. It is clear that, if we write

$$c''_{m,n}=\eta_m\zeta_n c_{m,n}=\epsilon_{m,n}c_{m,n},$$

all the conditions of the lemma will be satisfied.

LEMMA ζ. *If $\Sigma\Sigma c_{m,n}$ is a divergent series of positive terms, we can choose a sequence of pairs of integers (m_i, n_i), tending to infinity with i, so that the series $\Sigma\Sigma c'_{m,n}$, where $c'_{m,n}=0$ if $m=m_i$, $n\leqslant n_i$ or $m\leqslant m_i$, $n=n_i$, and $c'_{m,n}=c_{m,n}$ otherwise, is divergent.*

The modification to be made in the series is effected by drawing perpendiculars on to the axes from the points (m_i, n_i), and annulling all terms which correspond to points on these perpendiculars. Let σ_m denote the sum of the terms whose representative points lie on the perpendiculars from (m, m) on to the axes. Then $\Sigma\sigma_m$ is divergent. Applying Lemma β to this series we obtain the construction required, m_i being in fact always equal to n_i.

THEOREM 12. *If $\Sigma\Sigma a_{m,n}u_{m,n}$ is regularly convergent whenever $\Sigma\Sigma u_{m,n}$ is regularly convergent, then $a_{m,n}$ is of bounded variation.*

In the first place it follows from Theorem 4 that $u_{m,n}$ is, for every value of n (or m), of bounded variation in m (or n). It remains only to shew that $\Sigma\Sigma\,|\,\Delta_{m,n}a_{m,n}\,|$ is convergent.

Suppose, on the contrary, that it is divergent. By Lemma ϵ, we can choose a sequence of positive numbers $\epsilon_{m,n}$, tending regularly to zero, so that $\Sigma\Sigma c_{m,n}$, where

$$c_{m,n} = \epsilon_{m,n}\,|\,\Delta_{m,n}a_{m,n}\,|,$$

is divergent. We can then modify this series as in Lemma ζ without destroying its divergence.

Now let

$$U_{m,n} = \sum_1^m\sum_1^n u_{\mu,\nu}$$

and suppose that

$$U_{m,n} = 0$$

if $m = m_i$, $n \leqslant n_i$ or $m \leqslant m_i$, $n = n_i$, and that otherwise

$$U_{m,n} = \epsilon_{m,n}\frac{|\,\Delta_{m,n}a_{m,n}\,|}{\Delta_{m,n}a_{m,n}}:$$

this last formula being interpreted as meaning $\epsilon_{m,n}$ if

$$\Delta_{m,n}a_{m,n} = 0.$$

These equations define $u_{m,n}$ uniquely for all values of m and n, and it is plain that $U_{m,n}$ tends regularly to zero, so that $\Sigma\Sigma\,u_{m,n}$ is regularly convergent. On the other hand

$$\sum_1^{m_i}\sum_1^{n_i} a_{m,n}u_{m,n} = \sum_1^{m_i-1}\sum_1^{n_i-1}\Delta_{m,n}a_{m,n}U_{m,n} = \sum_1^{m_i-1}\sum_1^{n_i-1}c'_{m,n},$$

which tends to infinity with i. Thus $\Sigma\Sigma a_{m,n}u_{m,n}$ is not convergent.

This proves Theorem 12. Combining it with Theorem 10 we obtain the analogue of Theorem 5, viz.

THEOREM 13. *The necessary and sufficient condition that $a_{m,n}$ should be a convergence factor is that it should be of bounded variation.*

Bessel functions of large order. By G. N. WATSON, M.A., Trinity College.

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1. When the order of a Bessel function is large, the asymptotic expansion of the function assumes various forms depending on the values of the ratio of the argument to the order of the function. The dominant terms of the asymptotic expansions are given by the formulae:

(i) When n is large, x is fixed and $0 \leqslant x < 1$, then

$$J_n(nx) \sim (2\pi n)^{-\frac{1}{2}} (1-x^2)^{-\frac{1}{4}} x^n \{1+\sqrt{(1-x^2)}\}^{-n} \exp \{n\sqrt{(1-x^2)}\}.$$

(ii) When n is large, x is fixed and $x > 1$, then

$$J_n(nx) \sim (\tfrac{1}{2}\pi n)^{-\frac{1}{2}} (x^2-1)^{-\frac{1}{4}} \cos \{n\sqrt{(x^2-1)} - n \sec^{-1} x - \tfrac{1}{4}\pi\}.$$

(iii) When n is large and $\epsilon = O(n^{-\frac{2}{3}})$, then

$$J_n(n+n\epsilon) \sim \Gamma(\tfrac{1}{3})/\{\pi 2^{\frac{2}{3}} 3^{\frac{1}{6}} n^{\frac{1}{3}}\}.$$

The corresponding complete asymptotic expansions, valid for general complex values of n and x, have been given by Debye*. Accounts of the history of the approximate formulae are to be found in Debye's memoirs and also in two papers† which I have published recently.

It is evident that there are transition stages between the domains of validity of the three formulae quoted; and not much is known about the behaviour of $J_n(nx)$ in these transition stages. Consequently I propose to establish approximate formulae (involving Bessel functions of orders‡ $\pm\frac{1}{3}$) which exhibit the behaviour of the Bessel function right through the transition stages. These formulae are more exact forms of some approximations which Nicholson§ obtained some years ago without estimating the margin of error or the precise ranges in which the results were valid.

* *Math. Ann.*, LXVII. (1909), pp. 535—558. *München. Sitzungsberichte* [5], 1910.

† *Proceedings*, XIX. (1916), pp. 42—48. *Proc. London Math. Soc.* (2), XVI. (1917), pp. 150—174.

‡ These functions have been tabulated by Dinnik, *Archiv der Math. und Phys.*, XVIII. (1911), p. 337.

§ *Phil. Mag.*, Feb. 1910, pp. 228—249.

The approximations which I shall obtain are derived by shewing that certain integrals of Airy's type* are effective approximations to the integrals which occur in Debye's analysis. It will be assumed that the reader is familiar with Debye's memoirs, although it seems desirable to modify the notation to a considerable extent. The formula for $J_n(nx)$, when $x \geqslant 1$, is of importance in connexion with the maxima of the Bessel function†.

The two formulae which will be obtained in this paper are as follows:

(I) When $\alpha \geqslant 0$,

$$J_n(n \operatorname{sech} \alpha) \sim 2\pi^{-1} 3^{-\frac{1}{2}} \tanh \alpha \\ \times \exp\{n(\tanh \alpha + \tfrac{1}{3} \tanh^3 \alpha - \alpha)\} \,.\, K_{\frac{1}{3}}(\tfrac{1}{3} n \tanh^3 \alpha),$$

where the error is less than $3n^{-1} \exp\{n(\tanh \alpha - \alpha)\}$, and $K_m(z)$ denotes the Bessel function of Basset's type (see § 6).

(II) When $0 \leqslant \beta \leqslant \frac{1}{4}\pi$,

$$\begin{aligned} J_n(n \sec \beta) \sim \tfrac{1}{3} \tan \beta \cos\{n(\tan \beta - \tfrac{1}{3} \tan^3 \beta - \beta)\} \\ \times [J_{-\frac{1}{3}}(\tfrac{1}{3} n \tan^3 \beta) + J_{\frac{1}{3}}(\tfrac{1}{3} n \tan^3 \beta)] \\ + 3^{-\frac{1}{2}} \tan \beta \sin\{n(\tan \beta - \tfrac{1}{3} \tan^3 \beta - \beta)\} \\ \times [J_{-\frac{1}{3}}(\tfrac{1}{3} n \tan^3 \beta) - J_{\frac{1}{3}}(\tfrac{1}{3} n \tan^3 \beta)], \end{aligned}$$

where the error is less than $24/n$.

PART I. *The value of $J_n(nx)$ when $0 \leqslant x \leqslant 1$.*

2. We take Sommerfeld's integral

$$J_n(nx) = \frac{1}{2\pi i} \int_{\infty - \pi i}^{\infty + \pi i} e^{n(x \sinh w - w)} \, dw.$$

The stationary points of $x \sinh w - w$, *qua* function of w, are given by $\cosh w = 1/x$; accordingly we replace x by $\operatorname{sech} \alpha$, where $\alpha \geqslant 0$; and then, putting $w = \alpha + t$, we have

$$J_n(nx) = \frac{1}{2\pi i} e^{n(\tanh \alpha - \alpha)} \int_{\infty - \pi i}^{\infty + \pi i} \exp\{n \tanh \alpha (\cosh t - 1) \\ + n(\sinh t - t)\} \, dt.$$

The exponent has a stationary point at $t = 0$, and the method of steepest descents provides us with the contour whose equation is

$$I\{\tanh \alpha (\cosh t - 1) + (\sinh t - t)\} = 0.$$

* These integrals have been expressed in terms of Bessel functions by Nicholson, *Phil. Mag.*, July 1909, pp. 6—17, and by Hardy, *Quarterly Journal*, XLI. (1910), pp. 226—240.

† *Proc. London Math. Soc.* (2), XVI. (1917), p. 169.

The portion of this curve which is suitable for our purposes consists of an arc* on the right of the imaginary axis in the t-plane with its vertex at the origin and with the lines $I(t) = \pm \pi$ as asymptotes.

If we write $t \equiv u + iv$, where u, v are real, the equation of the curve becomes

$$\cosh(\alpha + u) = v \operatorname{cosec} v \cosh \alpha.$$

We shall put

$$\tanh \alpha (\cosh t - 1) + (\sinh t - t) = -\tau,$$

so that as t traverses the contour τ diminishes from $+\infty$ to 0, and then increases to $+\infty$; and therefore

$$J_n(nx) = \frac{1}{2\pi i} e^{n(\tanh \alpha - \alpha)} \left\{ \int_\infty^0 + \int_0^\infty \right\} e^{-n\tau} (dt/d\tau)\, d\tau;$$

in the first integral $v \leqslant 0$ and in the second integral $v \geqslant 0$.

Now define T by the equation†

$$\tfrac{1}{2} T^2 \tanh \alpha + \tfrac{1}{6} T^3 = -\tau.$$

A contour in the T-plane on which τ is real is a semi-hyperbola touching the imaginary axis at the origin and going off to infinity in directions inclined $\pm \frac{1}{3}\pi$ to the real axis. If we write

$$T \equiv U + iV,$$

where U, V are real, the equation of the hyperbola becomes

$$U \tanh \alpha + \tfrac{1}{2} U^2 = \tfrac{1}{6} V^2.$$

Taking the semi-hyperbola as the T-contour, we shall shew that an approximation to

$$\int_{\infty - \pi i}^{\infty + \pi i} e^{-n\tau}\, dt \text{ is } \int_{\infty \exp(-\frac{1}{3}\pi i)}^{\infty \exp(\frac{1}{3}\pi i)} e^{-n\tau}\, dT.$$

It is easy to see that the difference of these integrals is

$$\left\{ \int_\infty^0 + \int_0^\infty \right\} e^{-n\tau} \left\{ \frac{dt}{d\tau} - \frac{dT}{d\tau} \right\} d\tau,$$

and so the problem before us is reduced to the determination of an upper bound for $|d(t - T)/d\tau|$.

* This curve is derived from the curve shewn in fig. 4 (p. 541) of Debye's first paper by turning it through a right angle and taking the origin at the vertex. The degenerate case when α is zero is shewn in fig. 5.

† Since $\tau = \frac{1}{2} t^2 \tanh \alpha + \frac{1}{6} t^3 + O(t^4)$ when $|t|$ is small, the curve in the T-plane closely resembles the curve in the t-plane near the origin; and, the parts of the curves near the origin being the most important when n is large, we are obviously able to anticipate that the integrals under consideration are approximately equal.

3. We shall now shew that, whenever $\tau \geqslant 0$ and when, corresponding to any given value of τ, we choose V to have the same sign as v, we have the inequality

$$|d(t-T)/d\tau| \leqslant 3\pi.$$

Since, corresponding to any value of τ, the two values of t are conjugate complex numbers (and similarly for T), it is evidently sufficient to prove this inequality when v and V are both positive.

On comparing the values for τ in terms of t and T, we perceive that

$$\begin{aligned}(T-t)\{\tfrac{1}{2}(T+t)\tanh\alpha + \tfrac{1}{6}(T^2+Tt+t^2)\} \\ = \tanh\alpha(\cosh t - 1 - \tfrac{1}{2}t^2) + (\sinh t - t - \tfrac{1}{6}t^3).\end{aligned}$$

Also

$$\begin{aligned}d(t-T)/d\tau &= \{T\tanh\alpha + \tfrac{1}{2}T^2\}^{-1} - \{\sinh t\tanh\alpha + (\cosh t - 1)\}^{-1} \\ &= \frac{t-T}{T\{\sinh t\tanh\alpha + (\cosh t-1)\}} \\ &\quad + \frac{\tfrac{1}{2}t(t-T) + (\sinh t - t)\tanh\alpha + (\cosh t - 1 - \tfrac{1}{2}t^2)}{T(\tanh\alpha + \tfrac{1}{2}T)\{\sinh t\tanh\alpha + (\cosh t - 1)\}}.\end{aligned}$$

Now

$$\begin{aligned}|\sinh t\tanh\alpha + (\cosh t - 1)| \\ = \operatorname{sech}\alpha\sqrt{[(\cosh u - \cos v)\{\cosh(2\alpha+u) - \cos v\}]};\end{aligned}$$

and since

$$\begin{aligned}\{\cosh(2\alpha+u) - \cos v\} - \cosh^2\alpha(\cosh u - \cos v) \\ = \sinh^2\alpha(\cosh u + \cos v) + \sinh 2\alpha\sinh u \\ \geqslant 0,\end{aligned}$$

and

$$\begin{aligned}\{\cosh(2\alpha+u) - \cos v\} - \sinh^2\alpha(\cosh u + \cos v) \\ = \cosh^2\alpha(\cosh u - \cos v) + \sinh 2\alpha\sinh u \\ \geqslant 0,\end{aligned}$$

we see that $|\sinh t\tanh\alpha + (\cosh t - 1)|$ *exceeds both*

$$\cosh u - \cos v \equiv |\cosh t - 1|$$

and also $\quad \tanh\alpha\sqrt{(\cosh^2 u - \cos^2 v)} \equiv \tanh\alpha|\sinh t|$.

We now divide the range of integration into two parts, namely $\tau \geqslant 1$ and $0 \leqslant \tau \leqslant 1$.

4. Consider first what happens when $\tau \geqslant 1$.

If $|T| \leqslant 1$, we have (on the T-contour)

$$\tau = |\tfrac{1}{2}T^2\tanh\alpha + \tfrac{1}{6}T^3| < \tfrac{1}{2} + \tfrac{1}{6} < 1.$$

Also, if $|t| \leqslant 1$, we have (on the t-contour)

$$\tau = |(\cosh t - 1)\tanh\alpha + (\sinh t - t)| \leqslant \sum_{m=2}^{\infty} |t|^m/m! \leqslant e - 2 < 1.$$

Hence, when $\tau \geqslant 1$, *we must have both* $|T| \geqslant 1$ *and also* $|t| \geqslant 1$.

But, when $|T| \geqslant 1$, since $U \geqslant 0$, we have

$$|(d\tau/dT)| = |T\tanh\alpha + \tfrac{1}{2}T^2| \geqslant |\tanh\alpha + \tfrac{1}{2}T| \geqslant \tfrac{1}{2}.$$

Also, when $|t| \geqslant 1$, we have u or v (or both) greater than $1/\sqrt{2}$, and we always have v less than π.

Hence, by the result of § 3,

$$|(d\tau/dt)| = |\sinh t\tanh\alpha + (\cosh t - 1)| \geqslant 2(\sinh^2\tfrac{1}{2}u + \sin^2\tfrac{1}{2}v),$$

and this exceeds the smaller of

$$2\sinh^2(1/\sqrt{8}), \quad 2\sin^2(1/\sqrt{8}).$$

Consequently

$$|(dt/d\tau)| \leqslant \tfrac{1}{2}\operatorname{cosec}^2(1/\sqrt{8}) = 4{\cdot}14 < 2\pi - 2.$$

Therefore, when $\tau \geqslant 1$, *we have* $|d(t-T)/d\tau| < 2\pi$.

We shall make use of this inequality in § 6.

5. Consider next what happens when $0 \leqslant \tau \leqslant 1$.

If $|T| \geqslant 2$, we have (on the T-contour)

$$\tau = |\tfrac{1}{2}T^2\tanh\alpha + \tfrac{1}{6}T^3| \geqslant 4|\tfrac{1}{2}\tanh\alpha + \tfrac{1}{6}T| \geqslant \tfrac{2}{3}|T| \geqslant \tfrac{4}{3}.$$

Also noting that u and v increase together, when $v \geqslant \tfrac{1}{2}\pi$, we have (on the t-contour)

$$\begin{aligned}\tau &= u + \tanh\alpha - \cos v \operatorname{sech}\alpha \sinh(\alpha + u)\\ &\geqslant u + \tanh\alpha\\ &\geqslant \tanh\alpha - \alpha + \log\{\tfrac{1}{2}\pi\cosh\alpha + \sqrt{(\tfrac{1}{4}\pi^2\cosh^2\alpha - 1)}\},\end{aligned}$$

on expressing u in terms of v and noting that $v \operatorname{cosec} v$ exceeds $\tfrac{1}{2}\pi$. This function of α increases with α and so it exceeds

$$\log\{\tfrac{1}{2}\pi + \sqrt{(\tfrac{1}{4}\pi^2 - 1)}\} > 1.$$

Hence, when $\tau \leqslant 1$, *we must have both* $|T| \leqslant 2$ *and also* $v \leqslant \tfrac{1}{2}\pi$.

Further, when $v \leqslant \tfrac{1}{2}\pi$, we have

$$\cosh u \leqslant \operatorname{sech}\alpha\cosh(\alpha + u) = v\operatorname{cosec} v \leqslant \tfrac{1}{2}\pi < \cosh 1{\cdot}1,$$

so that $|t|^2 \leqslant \tfrac{1}{4}\pi^2 + (1{\cdot}1)^2 < 4$, *and therefore* $|t| < 2$.

That is to say, *when* $\tau \leqslant 1$, *neither* $|T|$ *nor* $|t|$ *exceeds* 2.

Also, for all values of t,

$$\begin{aligned}du/dv &= (1 - v\cot v)/\sqrt{(v^2 - \sin^2 v\operatorname{sech}^2\alpha)}\\ &\geqslant (1 - v\cot v)/v \geqslant \tfrac{1}{3}v,\end{aligned}$$

and so $u \geqslant \tfrac{1}{6}v^2$.

Further, when $v \leqslant \frac{1}{2}\pi$, we have* $1 - v\cot v \leqslant \sqrt{(v^2 - \sin^2 v)}$, and so $du/dv \leqslant 1$, i.e. $v \geqslant u$, whence at once we have $v\sqrt{2} \geqslant |t|$.

Next, when $|t| \leqslant 2$, we have

$$\begin{aligned} |\sinh t| &\geqslant |t| \{1 - \tfrac{1}{6}|t|^2 - \tfrac{1}{120}|t|^4 - \ldots\} \\ &\geqslant |t| \{1 - \tfrac{2}{3}(1 + \tfrac{1}{5} + \tfrac{1}{25} + \ldots)\} \\ &= \tfrac{1}{6}|t|. \end{aligned}$$

In like manner we may prove that, when $|t| \leqslant 2$,

$$|\cosh t - 1| \geqslant 4|t|^2/13 \geqslant \tfrac{1}{4}|t|^2, \quad |\cosh t - 1 - \tfrac{1}{2}t^2| \leqslant |t|^4/20,$$
$$|\sinh t - t| \leqslant 5|t|^3/24, \quad |\sinh t - t - \tfrac{1}{6}t^3| \leqslant |t|^5/108.$$

We are now in a position to obtain an upper bound for $|T - t|$. It is first evident that

$$\begin{aligned} &|\tfrac{1}{2}(T+t)\tanh\alpha + \tfrac{1}{6}(T^2 + Tt + t^2)| \\ &\qquad \geqslant I\{\tfrac{1}{2}(T+t)\tanh\alpha + \tfrac{1}{6}(T^2 + Tt + t^2)\} \\ &\qquad \geqslant \tfrac{1}{2}v\tanh\alpha + \tfrac{1}{3}uv \\ &\qquad \geqslant \tfrac{1}{2}v(\tanh\alpha + \tfrac{1}{9}v^2) \\ &\qquad \geqslant |t|(\tanh\alpha + \tfrac{1}{18}|t|^2)/\sqrt{8} \\ &\qquad \geqslant |t|^2(\tfrac{1}{2}\tanh\alpha + \tfrac{1}{18}|t|)/\sqrt{8}. \end{aligned}$$

Hence, by the result stated in § 3,

$$|T - t| \leqslant 8^{\frac{1}{2}}|t|^2 \frac{(\tanh\alpha)/20 + |t|/108}{\frac{1}{2}\tanh\alpha + |t|/18} \leqslant \tfrac{1}{3}\sqrt{2}|t|^2 \leqslant \tfrac{1}{2}|t|^2.$$

To obtain a stronger inequality, we write the equation of § 3 in the modified form

$$\begin{aligned} &(T-t)(T+t)\{\tfrac{1}{2}\tanh\alpha + \tfrac{1}{8}(T+t)\} \\ &\quad = -(T-t)^3/24 + \tanh\alpha(\cosh t - 1 - \tfrac{1}{2}t^2) + (\sinh t - t - \tfrac{1}{6}t^3). \end{aligned}$$

The expression on the right does not numerically exceed

$$|t|^6/192 + \tanh\alpha \,.\, |t|^4/20 + |t|^5/108 \leqslant \tanh\alpha \,.\, |t|^4/20 + |t|^5/48;$$

and since $|(T+t)\{\frac{1}{2}\tanh\alpha + \frac{1}{8}(T+t)\}|$ exceeds both $\frac{1}{2}|t|\tanh\alpha$ and also $\frac{1}{8}|t|^2$, we see that

$$|T - t| \leqslant (\tfrac{1}{10} + \tfrac{1}{6})|t|^3 = 4|t|^3/15.$$

If we now further restrict t *so that* $|t| \leqslant 1$, the last inequality gives

$$\begin{aligned} &|\{-(T-t)^3/24 + \tanh\alpha(\cosh t - 1 - \tfrac{1}{2}t^2) + (\sinh t - t - \tfrac{1}{6}t^3)\}| \\ &\qquad \leqslant 4^3|t|^9/(24 \,.\, 15^3) + \tanh\alpha \,.\, |t|^4/20 + |t|^5/108 \\ &\qquad \leqslant \tanh\alpha \,.\, |t|^4/20 + |t|^5/104, \end{aligned}$$

* Since $\frac{d}{dv}\{\sin^2 v(1 - v\cot v)^2 - \sin^2 v(v^2 - \sin^2 v)\} = -2\sin 2v \,.\, (v^2 - \sin^2 v) \leqslant 0$ when $0 \leqslant v \leqslant \frac{1}{2}\pi$.

and this inequality, combined with the modified form of the equation of § 3, gives

$$|T-t| \leqslant (1/10+1/13)|t|^3 \leqslant |t|^3/5.$$

Hence, when $|t| \leqslant 1$, $|T-t| \leqslant \frac{1}{5}|t|$, and so $|T| \geqslant \frac{4}{5}|t|$; and therefore, when $|t| \geqslant 1$, $|T|$ exceeds the value which it has when $|t|=1$, and so, *a fortiori*, $|T| \geqslant \frac{4}{5}$.

It now follows that, when both τ and $|t|$ do not exceed 1, we have

$$\begin{aligned}\left|\frac{dt}{d\tau}-\frac{dT}{d\tau}\right| &\leqslant \frac{\frac{1}{5}|t|^3}{\frac{4}{5}|t|\,.\,|\{\sinh t \tanh\alpha+(\cosh t-1)\}|} \\ &\quad+\frac{|t|^4/10+5\tanh\alpha\,|t|^3/24+|t|^4/20}{(8|t|^2/25)\,|\{\sinh t\tanh\alpha+(\cosh t-1)\}|} \\ &=\{23|t|^2/32+125\tanh\alpha\,|t|/192\} \\ &\quad\div|\{\sinh t\tanh\alpha+(\cosh t-1)\}|.\end{aligned}$$

Since the denominator exceeds both $\frac{2}{3}|t|\tanh\alpha$ and $\frac{1}{4}|t|^2$, we see that

$$|d(t-T)/d\tau| \leqslant (23/8)+(125/128) < 2\pi.$$

If $\tau \leqslant 1$ and $1 \leqslant |t| \leqslant 2$ we use the second expression of § 3 for $d(t-T)/d\tau$. Replacing $|T-t|$ in the numerator by $4|t|^3/15$ and $|T|$ in the denominator by $\frac{4}{5}$, we get in a similar manner

$$\begin{aligned}|d(t-T)/d\tau| &\leqslant \{|t|^3/3+125\tanh\alpha\,.\,|t|^3/192+11|t|^4/60\} \\ &\quad\div|\{\sinh t\tanh\alpha+(\cosh t-1)\}| \\ &\leqslant 4\{|t|/3+11|t|^2/60\}/13+125|t|^2/128 \\ &< 3\pi.\end{aligned}$$

6. It is obvious, from the results of §§ 4, 5, that, *whenever* $\tau \geqslant 0$, we have

$$|d(t-T)/d\tau| < 3\pi;$$

and from this result we have

$$\frac{1}{2\pi}\left|\left\{\int_\infty^0+\int_0^\infty\right\}e^{-n\tau}\left\{\frac{dt}{d\tau}-\frac{dT}{d\tau}\right\}d\tau\right| < 3\int_0^\infty e^{-n\tau}\,d\tau=3/n.$$

The evaluation of $\int_{\infty\exp(-\frac{1}{3}\pi i)}^{\infty\exp(\frac{1}{3}\pi i)} e^{-n\tau}\,dT$ presents no special points of interest; the simplest procedure is to modify the contour into two rays, starting from the point at which $T=-\tanh\alpha$ and making angles $\pm\frac{1}{3}\pi$ with the real axis.

If we write $T = -\tanh\alpha + \xi e^{\pm\frac{1}{3}\pi i}$ on the respective rays, the integral becomes

$$e^{\frac{1}{3}\pi i}\exp\left(\tfrac{1}{3}n\tanh^3\alpha\right).\int_0^\infty \exp\left\{-\tfrac{1}{6}n\xi^3 - \tfrac{1}{2}n\xi e^{\frac{1}{3}\pi i}\tanh^2\alpha\right\}d\xi$$

$$-e^{-\frac{1}{3}\pi i}\exp\left(\tfrac{1}{3}n\tanh^3\alpha\right).\int_0^\infty \exp\left\{-\tfrac{1}{6}n\xi^3 - \tfrac{1}{2}n\xi e^{-\frac{1}{3}\pi i}\tanh^2\alpha\right\}d\xi.$$

These are integrals of Airy's type; on expanding

$$\exp\left(-\tfrac{1}{2}n\xi e^{\pm\frac{1}{3}\pi i}\tanh^2\alpha\right)$$

in powers of $\tanh\alpha$ and integrating term-by-term—a procedure which is easily justified—we get on reduction

$$\tfrac{2}{3}\pi i\tanh\alpha.\exp\left(\tfrac{1}{3}n\tanh^3\alpha\right).\left[I_{-\frac{1}{3}}\left(\tfrac{1}{3}n\tanh^3\alpha\right) - I_{\frac{1}{3}}\left(\tfrac{1}{3}n\tanh^3\alpha\right)\right],$$

where, in accordance with the ordinary notation,

$$I_m(z) = i^{-m}J_m(iz).$$

On introducing Basset's function $K_m(z)$, defined as

$$\tfrac{1}{2}\pi\cot m\pi\left[I_{-m}(z) - I_m(z)\right],$$

we obtain the final formula

$$J_n(n\operatorname{sech}\alpha) = \frac{2}{\pi\sqrt{3}}\left[\tanh\alpha\exp\left\{n\left(\tanh\alpha + \tfrac{1}{3}\tanh^3\alpha - \alpha\right)\right\}\right.$$
$$\left.\times K_{\frac{1}{3}}\left(\tfrac{1}{3}n\tanh^3\alpha\right)\right] + 3\theta_1 n^{-1}\exp\left\{n(\tanh\alpha - \alpha)\right\},$$

where $|\theta_1| < 1$.

When n is large the ratio of the error term to the dominant term is of order $n^{-\frac{1}{2}}\sqrt{\tanh\alpha}$, $n^{-\frac{2}{3}}$, $n^{-\frac{2}{3}}$, according as $n\tanh^3\alpha$ is large, finite or small.

The formulae (i) and (iii) of § 1 agree with this result when α is finite and when $n\tanh^3\alpha$ is small, respectively.

Part II. *The value of $J_n(nx)$ when $x \geqslant 1$.*

7. It is convenient to regard Hankel's solutions of Bessel's equation, $H_n^{(1)}$ and $H_n^{(2)}$, as fundamental. The ordinary solutions are expressed in terms of these functions by the equations

$$J_n(nx) = \tfrac{1}{2}\left\{H_n^{(1)}(nx) + H_n^{(2)}(nx)\right\},$$
$$J_{-n}(nx) = \tfrac{1}{2}\left\{e^{n\pi i}H_n^{(1)}(nx) + e^{-n\pi i}H_n^{(2)}(nx)\right\}.$$

The integral formulae of Sommerfeld's type are

$$H_n^{(1)}(nx) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{n(x\sinh w - w)}\,dw,$$

$$H_n^{(2)}(nx) = -\frac{1}{\pi i}\int_{-\infty}^{\infty-\pi i} e^{n(x\sinh w - w)}\,dw.$$

The stationary points of $x \sinh w - w$, *qua* function of w, are given by $\cosh w = 1/x$. As $0 < 1/x \leqslant 1$, we put $x = \sec \beta$ where $0 \leqslant \beta < \frac{1}{2}\pi$; and two stationary points are given by $w = \pm \beta i$.

Now it has been shewn by Debye that a branch of the curve*

$$I(x \sinh w - w) = I(x \sinh i\beta - i\beta)$$

is a suitable contour for $H_n^{(1)}$, and the reflexion of this contour in the real axis† is a suitable contour for $H_n^{(2)}$.

On making a change of variable by writing $w = t + i\beta$, we have

$$H_n^{(1)}(nx) = \frac{1}{\pi i} e^{ni(\tan\beta - \beta)} \int_{-\infty - i\beta}^{\infty + \pi i - i\beta} e^{-n\tau}\, dt,$$

where $$i \tan\beta (\cosh t - 1) + \sinh t - t \equiv -\tau.$$

If we put $t \equiv u + iv$, where u, v are real, the equation of the contour is

$$\cosh u = (\sin\beta + v \cos\beta) \operatorname{cosec}(v + \beta),$$

and, on the contour,

$$\tau = u - \sec\beta \sinh u \cos(v + \beta).$$

When v is given, $\cosh u$ is given and the sign of u is ambiguous; *we take u to have the same sign as v*, in order that the contour may be of the requisite type.

Next define T by the equation

$$\tfrac{1}{2}T^2 i \tan\beta + \tfrac{1}{6}T^3 = -\tau.$$

We write $T \equiv U + iV$, where U, V are real; a contour in the T-plane on which τ is positive is that branch of the cubic‡, whose equation is

$$(U^2 - V^2)\tan\beta + \tfrac{1}{3}V(3U^2 - V^2) = 0,$$

which passes from $-\infty - i\tan\beta$ through the origin to $\infty \exp(\frac{1}{3}\pi i)$.

Taking this curve as the contour, we shall shew that an approximation to

$$\int_{-\infty - i\beta}^{\infty + \pi i - i\beta} e^{-n\tau}\, dt \text{ is } \int_{-\infty - i\tan\beta}^{\infty \exp(\frac{1}{3}\pi i)} e^{-n\tau}\, dT.$$

* This curve is derived from the curve shewn in fig. 2 (p. 540) of Debye's first paper by turning it through a right angle and taking the origin at the node. The reader will observe that the character of the contour has changed with the passage of x through the value unity.

† Since $H_n^{(1)}$, $H_n^{(2)}$ are conjugate complex numbers when n and x are real, it will be sufficient to confine our attention to $H_n^{(1)}$.

‡ Of course τ is *real* on the whole cubic; as T traverses the specified portion of it, τ decreases from $+\infty$ to 0 and then increases to $+\infty$.

8. Before proceeding further, we shall shew that *the slopes of the contours in the t-plane and in the T-plane never* exceed* $\sqrt{3}$.

If we write

$$(\sin\beta + v\cos\beta)\operatorname{cosec}(v+\beta) \equiv \psi(v),$$

we have
$$\frac{dv}{du} = \frac{\sinh u}{\psi'(v)} = \pm\frac{[\{\psi(v)\}^2-1]^{\frac{1}{2}}}{\psi'(v)}.$$

Now

$$\psi'(v) = \operatorname{cosec}(\beta+v)\{\cos\beta - \cot(\beta+v)(\sin\beta + v\cos\beta)\},$$

and so $\psi'(v)$ is positive when $\beta+v$ is an obtuse angle. When $0 \leqslant \beta + v \leqslant \frac{1}{2}\pi$, however, we find that

$$\cos\beta\tan(\beta+v) - (\sin\beta + v\cos\beta)$$

is an increasing function which vanishes with v. Hence $\psi'(v)$ has the same sign as v (and therefore the same sign as u), and consequently

$$\frac{dv}{du} = \frac{[\{\psi(v)\}^2-1]^{\frac{1}{2}}}{|\psi'(v)|}.$$

It is therefore necessary to prove that

$$\{\psi(v)\}^2 - 1 \leqslant 3\{\psi'(v)\}^2,$$

i.e. that
$$\chi(v) \equiv 3\{\psi'(v)\}^2 - \{\psi(v)\}^2 + 1 \geqslant 0.$$

Now $\chi(0)=0$, and it is consequently sufficient to shew that $\chi'(v)$ has the same sign as v. Since

$$\chi'(v) = 2\psi'(v)\{3\psi''(v) - \psi(v)\}$$

and $\psi'(v)$ has the same sign as v, it is sufficient to prove that

$$3\psi''(v) - \psi(v) \geqslant 0.$$

Since $\psi(v)\sin(v+\beta)$ reduces to a linear function of v, its second derivate vanishes, and so the inequality to be proved reduces to

$$\psi(v) - 3\psi'(v)\cot(v+\beta) \geqslant 0,$$

i.e. to

$$(\sin\beta + v\cos\beta)\{1 + 3\cot^2(v+\beta)\} - 3\cos\beta\cot(v+\beta) \geqslant 0.$$

But

$$\sin\beta + v\cos\beta - 3\cos\beta\cot(v+\beta)/\{1+3\cot^2(v+\beta)\}$$

has the positive derivate $4\cos\beta\{1+3\cot^2(v+\beta)\}^{-2}$, and is positive when $v=-\beta$; hence it is positive throughout the range $-\beta \leqslant v \leqslant \pi - \beta$. And this is the result which had to be proved.

* In the limiting case when $\beta=0$, the t-contour has slope $\sqrt{3}$ immediately on the right of the origin, and the T-contour consists of the rays $\arg T = 0$, $\arg T = \frac{1}{3}\pi$; so there is no better inequality of the form stated.

In like manner, we find that

$$0 \leqslant \frac{dV}{dU} = \frac{(\tan\beta + V)^{\frac{3}{2}}(\tan\beta + \frac{1}{3}V)^{\frac{1}{2}}}{\tan^2\beta + V\tan\beta + \frac{1}{3}V^2},$$

and it may be proved by quite simple algebra that the square of this last fraction does not exceed 3.

From the results just proved it follows on integration that

$$|v| \leqslant |u|\sqrt{3}, \quad |V| \leqslant |U|\sqrt{3},$$

and hence

$$|u| \geqslant \tfrac{1}{2}|t|, \quad |v| \leqslant \tfrac{1}{2}|t|\sqrt{3}, \quad |U| \geqslant \tfrac{1}{2}|T|, \quad |V| \leqslant \tfrac{1}{2}|T|\sqrt{3}.$$

9. We now return to the integrals of § 7. As in the corresponding work of §§ 2—3, we have to obtain an upper bound for $|d(T-t)/d\tau|$; we shall in fact shew that this function does not exceed 12π.

We notice that formulae corresponding to those given in § 3 are

$$(T-t)\{\tfrac{1}{2}(T+t)\,i\tan\beta + \tfrac{1}{6}(T^2+Tt+t^2)\} = i\tan\beta(\cosh t - 1 - \tfrac{1}{2}t^2) + \sinh t - t - \tfrac{1}{6}t^3,$$

$$\begin{aligned} d(t-T)/d\tau &= \{iT\tan\beta + \tfrac{1}{2}T^2\}^{-1} - \{i\sinh t\tan\beta + (\cosh t - 1)\}^{-1} \\ &= \frac{t-T}{T\{i\sinh t\tan\beta + (\cosh t - 1)\}} \\ &\quad + \frac{\frac{1}{2}t(t-T) + i(\sinh t - t)\tan\beta + (\cosh t - 1 - \frac{1}{2}t^2)}{T(i\tan\beta + \frac{1}{2}T)\{i\sinh t\tan\beta + (\cosh t - 1)\}}. \end{aligned}$$

Now

$$|i\sinh t\tan\beta + \cosh t - 1| = \sec\beta\sqrt{[(\cosh u - \cos v)\{\cosh u - \cos(2\beta + v)\}]},$$

and since

$$\begin{aligned} \{\cosh u - \cos(2\beta+v)\} - \cos^2\beta(\cosh u - \cos v) &= \sin^2\beta(\cosh u + \cos v) + \sin 2\beta \sin v \\ &\geqslant (1+\cos v)\{\sin^2\beta + \sin 2\beta\tan\tfrac{1}{2}v\} \\ &\geqslant (1+\cos v)\{\sin^2\beta - \sin 2\beta\tan\tfrac{1}{2}\beta\} \\ &\geqslant 0, \end{aligned}$$

we have

$$|i\sinh t\tan\beta + (\cosh t - 1)| \geqslant \cosh u - \cos v = |\cosh t - 1|.$$

Also

$$\cosh u - \cos(2\beta + v) \geqslant 2\sin^2(\beta + \tfrac{1}{2}v) \geqslant 2\sin^2\tfrac{1}{2}\beta,$$

and so

$$|i \sinh t \tan \beta + (\cosh t - 1)| \geqslant \sin \tfrac{1}{2}\beta \sec \beta \sqrt{\{2(\cosh u - \cos v)\}}$$
$$\geqslant \tan \beta \,|\sinh \tfrac{1}{2}t|.$$

That is to say $|i \sinh t \tan \beta + (\cosh t - 1)|$ *exceeds both* $|\cosh t - 1|$ *and also* $\tan \beta\,|\sinh \tfrac{1}{2}t|$.

In order to simplify the subsequent analysis, it is convenient to place a restriction on β. *We shall consequently assume in future that* $0 \leqslant \beta \leqslant \tfrac{1}{4}\pi$, *so that* $\tan \beta \leqslant 1$. This restriction is not of importance so far as the final result is concerned, because Debye's formula, quoted in § 1 (ii), is effective whenever $\sec \beta \geqslant 1 + \delta$, where δ is any positive constant; and so it is certainly effective when $\sec \beta \geqslant \sqrt{2}$. The importance of the analysis in the present investigation is due to the fact that it is valid for *small* values of β.

10. Consider what happens when $\tau \geqslant \tfrac{1}{2}$, whether v, V are both positive or both negative.

When $|T| \leqslant \tfrac{3}{4}$, we have (on the T-contour)

$$\tau = |\tfrac{1}{2}iT^2 \tan \beta + \tfrac{1}{6}T^3| \leqslant |T^2|(\tfrac{1}{2} + \tfrac{1}{6}|T|) < \tfrac{1}{2},$$

and if $|t| \leqslant \tfrac{3}{4}$, we have (on the t-contour)

$$\tau = |\{i \tan \beta (\cosh t - 1) + (\sinh t - t)\}|$$
$$\leqslant \sum_{m=2}^{\infty} |t|^m/m! \leqslant e^{\frac{3}{4}} - 1 - \tfrac{3}{4} = 2{\cdot}12 - 1{\cdot}75 < \tfrac{1}{2}.$$

Hence, when $\tau \geqslant \tfrac{1}{2}$, *we must have both* $|T| \geqslant \tfrac{3}{4}$ *and* $|t| \geqslant \tfrac{3}{4}$.

But, when $|T| \geqslant \tfrac{3}{4}$, we have

$$|(d\tau/dT)| = |T|\,.\,|i \tan \beta + \tfrac{1}{2}T| \geqslant |T|\,.\,|\tfrac{1}{2}R(T)| \geqslant \tfrac{1}{4}|T|^2 \geqslant \tfrac{9}{64}.$$

Also (as in § 4) when $|t| \geqslant \tfrac{3}{4}$, we have

$$|(d\tau/dt)| = |i \sinh t \tan \beta + (\cosh t - 1)|$$
$$\geqslant |\cosh t - 1| = \cosh u - \cos v \geqslant 2 \sin^2(\tfrac{3}{16}\sqrt{2}) = 0{\cdot}137,$$

and so $|(dt/d\tau)| \leqslant 7{\cdot}3$.

From these results we see that, when $\tau \geqslant \tfrac{1}{2}$,

$$|d(t - T)/d\tau| < 15 < 5\pi.$$

We shall make use of this inequality in § 12.

11. Consider next what happens when $0 \leqslant \tau \leqslant \tfrac{1}{2}$, whether v, V are both positive or both negative.

When $|T| \geqslant 2$, we have (on the T-contour)

$$\tau = |T|^2\,.\,|\tfrac{1}{2}i \tan \beta + \tfrac{1}{6}T| \geqslant |T|^2\,.\,|\tfrac{1}{6}R(T)| \geqslant \tfrac{1}{12}|T|^3 > \tfrac{1}{2}.$$

Also, when $|t| \geqslant 2$ and $v + \beta \geqslant \tfrac{1}{2}\pi$, we have $u \geqslant \tfrac{1}{4}\pi\sqrt{3}$, and then

$$\tau = u - \sec \beta \sinh u \cos (v + \beta) \geqslant u \geqslant 1.$$

Next, when $|t| \geqslant 2$ and $\beta \leqslant v + \beta \leqslant \frac{1}{2}\pi$, we have

$$\cosh u = (\sin\beta + v\cos\beta)\operatorname{cosec}(v+\beta)$$
$$\leqslant \sin\beta + (\tfrac{1}{2}\pi - \beta)\cos\beta < \tfrac{1}{2}\pi < \cosh 1{\cdot}1,$$

since $\sin\beta + (\frac{1}{2}\pi - \beta)\cos\beta$ is a decreasing function of β.

This gives $2 \leqslant |t| < \sqrt{\{(\frac{1}{2}\pi)^2 + (1{\cdot}1)^2\}} < \sqrt{3{\cdot}7}$, which is impossible; so that, when $|t| \geqslant 2$, we cannot have $\beta \leqslant v + \beta \leqslant \frac{1}{2}\pi$.

Lastly, when $|t| \geqslant 2$ and $0 \geqslant v \geqslant -\beta$, we have $u \leqslant 0$, and so

$$-u \geqslant \sqrt{(4-\beta^2)} \geqslant \sqrt{\{4 - (\tfrac{1}{4}\pi)\}^2} > 1{\cdot}8,$$

and

$$\tau = \sec\beta\sinh(-u)\cos(v+\beta) - (-u)$$
$$\geqslant \sinh(-u) - (-u) > \tfrac{1}{6}(1{\cdot}8)^3 > \tfrac{1}{2}.$$

Therefore, whenever $|t| \geqslant 2$, we have $\tau \geqslant \frac{1}{2}$.

Hence, when $0 \leqslant \tau \leqslant \frac{1}{2}$, *we must have both* $|t| \leqslant 2$ *and* $|T| \leqslant 2$.

Next we shall shew that $R\{\frac{1}{2}t + T^2/(T+t)\}$ has the same sign as u and U.

The function under consideration is equal to

$$[\tfrac{1}{2}u\{(U+u)^2 + (V+v)^2\} + (U^2 - V^2)(U+u)$$
$$+ 2UV(V+v)] \div [(U+u)^2 + (V+v)^2].$$

Taking U, V, u, v positive for the sake of definiteness, we see that the numerator of this fraction exceeds

$$\tfrac{1}{2}u(U^2+V^2) + u(U^2 - V^2) = \tfrac{1}{2}u(3U^2 - V^2) \geqslant 0.$$

Similarly we can prove that the numerator is negative when U, V, u, v are all negative. It follows from this result that

$$|R\{t + T^2/(T+t)\}| \geqslant \tfrac{1}{2}|R(t)| \geqslant \tfrac{1}{4}|t|.$$

We are now in a position to obtain an upper bound for $|T - t|$ *when* $|t|$ *and* $|T|$ *are both less than* 2.

First suppose that $|t| \leqslant \frac{1}{2}$.

Then, from the formula quoted at the beginning of § 9,

$$|(T-t)|\,.\,|(T+t)|\,.\,|\{\tfrac{1}{2}i\tan\beta + \tfrac{1}{6}t + \tfrac{1}{6}T^2/(T+t)\}|$$
$$= |i\tan\beta(\cosh t - 1 - \tfrac{1}{2}t^2) + (\sinh t - t - \tfrac{1}{6}t^3)|$$
$$\leqslant \sum_{m=4}^{\infty} |t|^m/m! \leqslant 5|t|^4/119.$$

But $|T+t| \geqslant |t|$ and

$$|\{\tfrac{1}{2}i\tan\beta + \tfrac{1}{6}t + \tfrac{1}{6}T^2/(T+t)\}| \geqslant \tfrac{1}{6}|R\{t + T^2/(T+t)\}| \geqslant \tfrac{1}{24}|t|.$$

Hence, when $|t| \leqslant \frac{1}{2}$, we have $|(T-t)| \leqslant 120|t|^2/119$.

Next, keeping $|t| \leqslant \frac{1}{2}$, we take the formula

$$\tfrac{1}{2}(T-t)(T+t)\{i\tan\beta + \tfrac{1}{4}(T+t)\}$$
$$= -\tfrac{1}{24}(T-t)^3 + i\tan\beta(\cosh t - 1 - \tfrac{1}{2}t^2) + (\sinh t - t - \tfrac{1}{6}t^3)$$

and observe that

$$|i\tan\beta + \tfrac{1}{4}(T+t)| \geqslant \tfrac{1}{4}\ R(T+t)| \geqslant \tfrac{1}{8}|t|;$$

and also, in view of the fact that, as τ varies through positive values, $t+T$ traces out in the Argand diagram a curve, through the origin, whose slope obviously never exceeds $\sqrt{3}$, the distance of all points of this curve from $-4i\tan\beta$ must exceed $2\tan\beta$. Hence $|i\tan\beta+\frac{1}{4}(T+t)| \geqslant \frac{1}{2}\tan\beta$.

Using these two inequalities, combined with the fact that $|(T-t)| \leqslant 120|t|^2/119$, and the obvious inequalities

$$|T+t| \geqslant |t|, \quad |\cosh t - 1 - \tfrac{1}{2}t^2| \leqslant |t|^4/23,$$
$$|\sinh t - t - \tfrac{1}{6}t^3| \leqslant |t|^5/119,$$

we deduce from the last equation for $T-t$ that

$$|T-t| \leqslant \tfrac{2}{3}|t|\{120|t|/119\}^3 + 4|t|^3/23 + 16|t|^3/119 \leqslant |t|^3.$$

Using now the inequality $|T-t| \leqslant |t|^3$ in place of

$$|T-t| \leqslant 120|t|^2/119,$$

we get

$$|T-t| \leqslant \tfrac{2}{3}|t|^7 + 4|t|^3/23 + 16|t|^3/119$$
$$\leqslant (1/24 + 4/23 + 16/119)|t|^3 \leqslant \tfrac{1}{2}|t|^3.$$

Using now the inequality $|T-t| \leqslant \frac{1}{2}|t|^3$, we get, in place of the last result,

$$|T-t| \leqslant (1/192 + 4/23 + 16/119)|t|^3 \leqslant \tfrac{1}{3}|t|^3.$$

From this result it follows that, when $|t| \leqslant \frac{1}{2}$, $|T-t| \leqslant \frac{1}{12}|t|$, and so $|T| \geqslant \frac{11}{12}|t|$.

Consequently, from the formula for $d(t-T)/d\tau$ given at the beginning of § 9, we see that, when $|t| \leqslant \frac{1}{2}$,

$$\left|\frac{dt}{d\tau} - \frac{dT}{d\tau}\right| \leqslant \frac{\frac{1}{3}|t|^3}{\frac{11}{12}|t|\,.\,|(\cosh t - 1)|} + \frac{\frac{1}{6}|t|^4 + \frac{1}{5}|t|^3\tan\beta + \frac{1}{23}|t|^4}{\frac{1}{4}(\frac{11}{12}|t|)^2\,|\{i\sinh t\tan\beta + (\cosh t - 1)\}|}.$$

Now, when $|t| \leqslant 2$,

$$|\cosh t - 1| \geqslant \tfrac{1}{2}|t|^2[1 - \tfrac{4}{12} - \tfrac{16}{360} - \ldots] \geqslant \tfrac{1}{4}|t|^2,$$

and $$|\sinh t| \geqslant |t|[1 - \tfrac{4}{6} - \tfrac{16}{120} - \ldots] \geqslant \tfrac{1}{6}|t|;$$

and so, using the results of § 9, we get

$$|d(t-T)/d\tau| \leqslant 16/11 + (576/121)[4(1/6 + 1/23) + 6/5]$$
$$< 12,$$

when $|t| \leqslant \frac{1}{2}$.

Lastly, when $\frac{1}{2} \leqslant |t| \leqslant 2$, we have $|T| \geqslant 11/24$, and so, by the method of § 10, we get

$$|d(t-T)/d\tau| \leqslant 4(24/11)^2 + \tfrac{1}{2}\operatorname{cosec}^2(\tfrac{1}{8}\sqrt{2})$$
$$< 35{\cdot}3 < 12\pi.$$

12. It follows from the results of §§ 10, 11 that, for *all* positive values of τ,

$$|d(t-T)/d\tau| < 12\pi,$$

and consequently

$$\left|\left\{\int_\infty^0 + \int_0^\infty\right\} e^{-n\tau}\left\{\frac{dt}{d\tau} - \frac{dT}{d\tau}\right\} d\tau\right| < 24\pi/n,$$

so that

$$H_n^{(1)}(n\sec\beta) = \frac{1}{\pi i} e^{ni(\tan\beta-\beta)} \int_{-\infty - i\tan\beta}^{\infty\exp(\frac{1}{3}\pi i)} e^{-n\tau}\, dT + 24\theta_2/n,$$

where $|\theta_2| < 1$.

To evaluate this integral, where $-\tau = \frac{1}{2}T^2 i\tan\beta + \frac{1}{6}T^3$, we take the contour to consist of the two rays $\arg(T + i\tan\beta) = \pi$, $\frac{1}{3}\pi$; on writing $T = -i\tan\beta - \xi$, $-i\tan\beta + \xi e^{\frac{1}{3}\pi i}$ on the respective rays, expanding the integrand in powers of ξ and integrating term by term we find that

$$\begin{aligned}\int_{-\infty - i\tan\beta}^{\infty\exp(\frac{1}{3}\pi i)} & e^{-n\tau}\, dT \\ &= \tfrac{2}{3}\pi i\tan\beta\exp(-\tfrac{1}{3}ni\tan^3\beta) \\ &\quad\times[e^{-\frac{1}{3}\pi i} J_{-\frac{1}{3}}(\tfrac{1}{3}n\tan^3\beta) + e^{\frac{1}{3}\pi i} J_{\frac{1}{3}}(\tfrac{1}{3}n\tan^3\beta)] \\ &= 3^{-\frac{1}{2}}\pi i\tan\beta\exp(\tfrac{1}{6}\pi i - \tfrac{1}{3}ni\tan^3\beta)\, H_{\frac{1}{3}}^{(1)}(\tfrac{1}{3}n\tan^3\beta).\end{aligned}$$

Since
$$J_n(n\sec\beta) = R[H_n^{(1)}(n\sec\beta)],$$
$$J_{-n}(n\sec\beta) = R[e^{n\pi i} H_n^{(1)}(n\sec\beta)],$$

it follows at once that, when $0 \leqslant \beta \leqslant \frac{1}{4}\pi$,

$$\begin{aligned}J_n(n\sec\beta) &= 3^{-1}\tan\beta\cos\{n(\tan\beta - \tfrac{1}{3}\tan^3\beta - \beta)\}\,.\,[J_{-\frac{1}{3}} + J_{\frac{1}{3}}] \\ &\quad + 3^{-\frac{1}{2}}\tan\beta\sin\{n(\tan\beta - \tfrac{1}{3}\tan^3\beta - \beta)\}\,.\,[J_{-\frac{1}{3}} - J_{\frac{1}{3}}] + 24\theta/n,\\ J_{-n}(n\sec\beta) &= 3^{-1}\tan\beta\cos\{n(\pi + \tan\beta - \tfrac{1}{3}\tan^3\beta - \beta)\}\,.\,[J_{-\frac{1}{3}} + J_{\frac{1}{3}}] \\ &\quad + 3^{-\frac{1}{2}}\tan\beta\sin\{n(\pi + \tan\beta - \tfrac{1}{3}\tan^3\beta - \beta)\}\,.\,[J_{-\frac{1}{3}} - J_{\frac{1}{3}}] + 24\theta'/n,\end{aligned}$$

where the arguments of the Bessel functions $J_{\pm\frac{1}{3}}$ on the right are all equal to $\frac{1}{3}n\tan^3\beta$, and $|\theta|$, $|\theta'|$ are both less than 1. It is easy to see that, except near the zeros of the dominant terms on the right, the ratios of the error terms to the dominant terms are of orders $\sqrt{(n^{-1}\tan\beta)}$, $n^{-\frac{2}{3}}$, $n^{-\frac{2}{3}}$, according as $n\tan^3\beta$ is large, finite or small.

A particular case of a theorem of Dirichlet. By H. TODD, B.A., Pembroke College. (Communicated, with a prefatory note, by Mr H. T. J. NORTON.)

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[The following note is an extract from an essay submitted to the Smith's Prize Examiners.

It will, perhaps, be convenient if I preface Mr Todd's argument by explaining its relation to the theory of algebraical numbers. The principal theorem is a famous one of Dirichlet's on the unities of an algebraic corpus or order. It will be remembered that if ϑ is a root of an irreducible equation of the nth degree, the coefficients of which are integers, then, if the coefficient of the nth power of the unknown is 1, ϑ is an algebraic integer, and if in addition the absolute term is ± 1, ϑ is a unity; and further, that if ϑ is an integer of the nth degree, then the order of ϑ is the aggregate of numbers w of the form

$$x_0 + x_1 \vartheta + \dots x_{n-1} \vartheta^{n-1},$$

where $x_0 \dots x_{n-1}$ are rational whole numbers, every member of the order of ϑ being an integer of the nth or some lower degree. Dirichlet's theorem*, as modified by Dedekind and others, asserts that if the irreducible equation satisfied by ϑ has r real and $2s$ imaginary roots, then the order of ϑ contains $r+s-1$ fundamental unities, $\epsilon_1, \dots, \epsilon_{r+s-1}$, which are such that every unity contained in the order is expressible in one and only one way as a product

$$\eta \,.\, \epsilon_1^{m_1} \dots \epsilon_{r+s-1}^{m_{r+s-1}},$$

where η is a root of unity contained in the order and $m_1, \dots, m_{r+s-1}$ are rational integers; and that, conversely, every such product is a unity and a member of the order. The simplest cases of this theorem are those in which the equation satisfied by ϑ is (i) a quadratic with two imaginary roots, (ii) a quadratic with two real roots, (iii) a cubic with one real and two imaginary roots and (iv) a quartic of which all the roots are imaginary. In the first case, and in this alone, there are only a finite number of unities in the order, and they are all roots of unity; in the other cases

* The theorem, when stated completely, has a wider scope, corresponding to a wider definition of an 'order' than is given above: what is there defined is more properly called a 'regular order'. A general statement and proofs are given in Bachmann, *Zahlentheorie*, vol. v., ch. 8.

mentioned there is one and only one fundamental unity and in cases (ii) and (iii) ± 1 are the only roots of unity which the order contains. In case (i) the theorem is easy to prove. In case (ii), if $t^2 + 2bt + c = 0$ is the equation satisfied by ϑ, the unities of the order are essentially the same as the solutions of the Pellian Equation

$$x^2 - (b^2 - c)\,y^2 = \pm 1,$$

and Dirichlet's results can be deduced from the theory of this equation. In other cases the proof of the theorem is much more difficult. Mr Todd is concerned with the case in which ϑ is the cube root of an integer—which comes under the heading (iii) above. If $\vartheta^3 = n$, the general theorem asserts (a) that the order of ϑ contains an infinity of unities, (b) that they are all expressible in the form

$$\pm \gamma^m,$$

where γ is a particular one among them and m is a positive or negative whole number, and (c) that every number of this form is a unity of the order. Mr Todd's essay contained an elementary proof of (b) and (c); the proof of (c) does not essentially differ from that given in text-books, though this was not known to him at the time, but the proof of (b) appears to be new and forms the subject of the following note.—H. T. J. N.]

If $\vartheta^3 = n$, and $\Gamma = x + y\vartheta + z\vartheta^2$ is a member of the order of ϑ, then

$$\Gamma\vartheta = nz + x\vartheta + y\vartheta^2,$$

$$\Gamma\vartheta^2 = ny + nz\vartheta + x\vartheta^2,$$

so that Γ satisfies the cubic equation

$$\begin{vmatrix} x - t, & y, & z \\ nz, & x - t, & y \\ ny, & nz, & x - t \end{vmatrix} = 0;$$

hence it follows that Γ is a unity of the order if and only if x, y, z satisfy the Diophantine equation

$$\begin{vmatrix} x, & y, & z \\ nz, & x, & y \\ ny, & nz, & x \end{vmatrix} \equiv x^3 + ny^3 + n^2z^3 - 3nxyz = \pm 1 \text{.........(i)},$$

It will be the object of this short note to give a simple elementary proof of the fact that, if the existence of unities is assumed, then every unity of the order of ϑ can be expressed in the form

$$\pm \Gamma^m,$$

where Γ is one particular unity of the order, and m is a positive or negative integer or zero.

In what follows we shall restrict ourselves to the positive sign on the right-hand side of equation (i), since the negative sign merely replaces (x, y, z) by $(-x, -y, -z)$. Also when x, y, z are all positive, we shall refer to $(x + y\vartheta + z\vartheta^2)$ as a "unity of positive integers".

Suppose that

$$\Gamma = x + y\vartheta + z\vartheta^2$$

is any unity of the order of ϑ: we shall first prove the following inequalities, viz.:

$$|x - y\vartheta|, \ |y\vartheta - z\vartheta^2|, \ |z\vartheta^2 - x| \leqslant 2/\sqrt{(3\Gamma)} \text{ ....(ii).}$$

For, if we write

$$\alpha = x - y\vartheta,$$

$$\beta = y\vartheta - z\vartheta^2,$$

and

$$\gamma = z\vartheta^2 - x,$$

we see that the equation satisfied by x, y, z can be thrown into the form

$$\Gamma(\alpha^2 + \beta^2 + \gamma^2) = 2:$$

so that we have

and

$$\left.\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 2/\Gamma \\ \alpha + \beta + \gamma &= 0 \end{aligned}\right\}.$$

From these two equations, assuming Γ to be constant, we find that the maxima and minima for each of α, β, γ are

$$\pm 2/\sqrt{(3\Gamma)};$$

from which the truth of the statement (ii) follows immediately.

Further, we have the fact that if $\Gamma = x + y\vartheta + z\vartheta^2$ is any unity of the order and $\Gamma > 1$, then x, y, z will be positive.

For, since $\Gamma > 1$, we have the inequalities

$$|x - y\vartheta|, \ |y\vartheta - z\vartheta^2|, \ |z\vartheta^2 - x| < 2/\sqrt{3} < 1{\cdot}15.$$

But, Γ being positive, the only possibilities of negative signs occurring amongst x, y, z are either (a) one negative and two positive or (b) two negative and one positive; and in each case two of the inequalities given would take the form

$$|\lambda + \mu\phi| < 1{\cdot}15,$$

where λ and μ are positive integers and $\phi \geqslant \sqrt[3]{2}$, which is obviously impossible, except in the trivial case of one or more of the quantities x, y, z vanishing: it will be seen, on examining the inequalities, that the only possibility is $x = 1$, $y = 0$, $z = 0$, which gives $\Gamma = 1$ and so is excluded. Hence x, y, z must be positive. From this we can easily shew that if there exists any unity in the order other than ± 1, then there exists a unity of positive integers other than $+1$ of which any other unity of positive integers is a positive integral power. For suppose that Γ is any unity of the order other than ± 1: then by definition of a unity it follows that the three numbers

$$-\Gamma, \quad 1/\Gamma, \quad -1/\Gamma,$$

will be unities of the order also: and of these four it is plain that one will be positive and greater than 1, *i.e.* it will be a unity of positive integers.

Now take any number $\kappa > 1$; then there will be only a finite number of Γ's for which $\kappa > \Gamma > 1$, since for any such Γ we must have $\kappa > x > 0$, $\kappa > y > 0$, $\kappa > z > 0$. Hence there must be a unity of positive integers which is greater than $+1$ and less than any other; let this one be γ.

Suppose that Γ is any unity of positive integers which is, if possible, not a positive integral power of γ. Then we shall have $\Gamma > \gamma$, so that we can assume that Γ is intermediate in magnitude between γ^p and γ^{p+1}, where p is some positive integer. But by the last part of Dirichlet's Theorem we know that

$$\Gamma/\gamma^p$$

is also a unity of the order, *i.e.* we have found a unity of the order which is less than γ and greater than $+1$, which contradicts the assumption that γ was the least unity greater than $+1$. Hence Γ must be a positive integral power of γ. Finally we have the result that, if Γ is any unity of the order, it can be expressed in the form

$$\pm \gamma^p,$$

where γ has its previous significance and p is any positive or negative integer or zero. For if Γ is any unity of the order, other than ± 1, the numbers

$$-\Gamma, \quad 1/\Gamma, \quad -1/\Gamma$$

also will be unities, and one of these will be positive and greater than 1, and so will be expressible in the form

$$\gamma^q,$$

where q is a positive integer. Hence Γ can be expressed in the form

$$\pm \gamma^p,$$

where p is some positive or negative integer or zero.

The result obtained can be put into an interesting geometrical form as we shall proceed to shew.

It is evident that any rational point (x, y, z) in space of three dimensions can be regarded as being determined by its affix $\Gamma = x + y\vartheta + z\vartheta^2$, where ϑ is the real root of the equation $\vartheta^3 = n$: also the affix of any point determines a plane through that point and parallel to the asymptotic plane of the surface whose equation is $\Delta \equiv x^3 + ny^3 + n^2z^3 - 3nxyz = 1$; such a plane we shall call a "Γ-plane".

We shall now prove the following proposition:

The Γ-planes of any two consecutive integral points on the surface $\Delta = 1$, together with the surface itself, enclose a space of constant volume.

The equation $\Delta = 1$ can be written in the form

$$\{x + y\vartheta + z\vartheta^2\}\ \{(x - y\vartheta)^2 + (y\vartheta - z\vartheta^2)^2 + (z\vartheta^2 - x)^2\} = 2;$$

so that the section by the Γ-plane of the point (ξ, η, ζ) will be given by the equations

$$x^2 + y^2\vartheta^2 + n\vartheta z^2 - nyz - \vartheta^2 zx - \vartheta xy = 1/\Gamma \quad \text{.........(i)}$$

and
$$x + y\vartheta + z\vartheta^2 = \Gamma.$$

Evidently the quadric (i) and the surface $\Delta = 1$ are cut in a common section by the Γ-plane of the point (ξ, η, ζ). It is this quadric that we shall now examine.

If by any rotation of axes it becomes $ax^2 + by^2 + cz^2 = 1$, we shall have (from the usual properties of invariants)

$$\left.\begin{aligned} a + b + c &= \Gamma(1 + n\vartheta + \vartheta^2), \\ ab + bc + ca &= \tfrac{3}{4}\Gamma^2\vartheta^2(1 + n\vartheta + \vartheta^2), \\ abc &= 0; \end{aligned}\right\}$$

so that the quadric is evidently a cylinder, and the direction of its axis is the line $x = y\vartheta = z\vartheta^2$.

Suppose that $c = 0$; then the area of a right section of the cylinder will be

$$\pi/\sqrt{(ab)} = \frac{2\pi}{\vartheta\Gamma}\Big/\sqrt{3(1 + n\vartheta + \vartheta^2)}.$$

But the angle between the normals to the right section and the Γ-plane is the same as the angle between the two lines

$$x/\vartheta^2 = y/\vartheta = z,$$

and
$$x = y/\vartheta = z/\vartheta^2;$$

i.e., is
$$\cos^{-1}\{3\vartheta^2/(1 + n\vartheta + \vartheta^2)\}:$$

hence the area of the section made by the Γ-plane will be

$$2\pi \sqrt{(1 + n\vartheta + \vartheta^2)}/3n\sqrt{3}\,\Gamma.$$

Now the perpendicular distance between two near Γ-planes, Γ and $\Gamma + \delta\Gamma$, is $\delta\Gamma/\sqrt{(1 + n\vartheta + \vartheta^2)}$, and so the element of volume enclosed by these two planes and the surface $\Delta = 1$ will be, to the first order,

$$\frac{2\pi}{3n\sqrt{3}} \cdot \frac{\delta\Gamma}{\Gamma}.$$

Integrating this between the limits $\Gamma = \gamma^{p+1}$ and $\Gamma = \gamma^p$ (*i.e.* the Γ-planes of any two consecutive integral points), we find that the volume of the space enclosed is $2\pi \log \gamma/3n\sqrt{3}$; and since this is independent of the integer p, our proposition is proved.

On Mr Ramanujan's Empirical Expansions of Modular Functions. By L. J. MORDELL, Birkbeck College, London. (Communicated by Mr G. H. HARDY.)

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In his paper* "On Certain Arithmetical Functions" Mr Ramanujan has found empirically some very interesting results as to the expansions of functions which are practically modular functions. Thus putting

$$\left(\frac{\omega_2}{2\pi}\right)^{12} \Delta(\omega_1, \omega_2) = r[(1-r)(1-r^2)(1-r^3)\ldots]^{24} = \sum_{n=1}^{\infty} T(n)\, r^n,$$

he finds that

$$T(mn) = T(m)\, T(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

if m and n are prime to each other; and also that

$$\sum_{n=1}^{\infty} \frac{T(n)}{n^s} = \Pi\, 1/(1 - T(p)\, p^{-s} + p^{11-2s}) \ldots\ldots\ldots\ldots(2),$$

where the product refers to the primes 2, 3, 5, 7 .... He also gives many other results similar to (2).

My attention was directed to these results by Mr Hardy, and I have found that results of this kind are a simple consequence of the properties of modular functions. In the case above

$$\Delta(\omega_1, \omega_2) \quad (r = e^{2\pi i\omega},\ \omega = \omega_1/\omega_2)$$

is the well-known modular invariant of dimensions -12 in ω_1, ω_2, which is unaltered by the substitutions of the homogeneous modular group defined by

$$\omega_1' = a\omega_1 + b\omega_2,\ \ \omega_2' = c\omega_1 + d\omega_2,$$

where a, b, c, d are integers satisfying the condition $ad - bc = 1$.

Theorems such as $T(mn) = T(m)\, T(n)$ had already been investigated by Dr Glaisher† for other functions; but the theorems typified by equation (2) seem to be of a new type, and it is very remarkable that they should have been discovered empirically. The proof of Mr Ramanujan's formulae is as follows.

Let $f(\omega_1, \omega_2)$ be a modular‡ form of dimensions $-\kappa$ in ω_1, ω_2, which is a relative invariant of the homogeneous modular group, so that $f(\omega_1', \omega_2')/f(\omega_1, \omega_2)$ is a constant independent of ω_1, ω_2.

* *Transactions of the Cambridge Philosophical Society*, vol. XXII., no. IX., 1916.

† See, for example, his paper "The Arithmetical Functions $P(m)$, $Q(m)$, $\Omega(m)$", *Quarterly Journal of Mathematics*, vol. XXXVII., p. 36.

‡ For an elementary introduction to the modular functions, see Hurwitz, *Mathematische Annalen*, vol. 18, p. 520.

Let also p be any prime number; then we may take

$$(\omega_1, p\omega_2),\ (\omega_1 + \omega_2, p\omega_2) \ldots (\omega_1 + (p-1)\,\omega_2, p\omega_2),\ (p\omega_1, \omega_2)$$

as the reduced substitutions of order p. Then for many modular forms* it is well known that unities $\xi, \xi_0, \xi_1, \ldots, \xi_{p-1}$ can be found so that

$$\phi = \xi f(p\omega_1, \omega_2) + \xi_0 f(\omega_1, p\omega_2) \ldots \\ + \xi_{p-1} f(\omega_1 + (p-1)\,\omega_2, p\omega_2)$$

is also a relative invariant of the modular group.

This is also true of the quotient $Q = \phi/f(\omega_1, \omega_2)$, which is a modular function of ω. Q is really an automorphic function whose fundamental polygon (putting $\omega = x + \iota y$) is that part of the upper ω plane bounded by the lines $x = \pm\frac{1}{2}$ and external to the circle $x^2 + y^2 = 1$, but we reckon only half the boundary as belonging to the fundamental polygon. The only infinities of Q are given by the zeros of $f(\omega_1, \omega_2) = 0$, and if these zeros are also zeros of the numerator of at least the same order as of the denominator, it follows that Q has no infinities in the fundamental polygon. Hence Q is a constant, so that $\phi = Qf(\omega_1, \omega_2)$.

Suppose now that

$$f(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{\kappa} \sum_{s=1}^{\infty} A_s r^s,$$

where $A_1 = 1$. Then

$$\xi_0 f(\omega_1, p\omega_2) + \xi_1 f(\omega_1 + \omega_2, p\omega_2) + \ldots$$

becomes

$$\left(\frac{2\pi}{p\omega_2}\right)^{\kappa} \sum_{s=1}^{\infty} \sum_{\lambda=0}^{p-1} \xi_\lambda A_s r^{s/p} e^{2\pi\iota\lambda/p},$$

and in the examples with which we are concerned all the terms will vanish, because of the summation in λ, except those for which $s \equiv 0 \pmod p$, and then the sum will become

$$\left(\frac{2\pi}{p\omega_2}\right)^{\kappa} \sum_{s=1}^{\infty} pA_{sp} r^s.$$

Hence we have

$$\left(\frac{2\pi}{\omega_2}\right)^{\kappa} \xi \sum_{s=1}^{\infty} A_s r^{sp} + \left(\frac{2\pi}{p\omega_2}\right)^{\kappa} \sum_{s=1}^{\infty} pA_{sp} r^s = Q\left(\frac{2\pi}{\omega_2}\right)^{\kappa} \sum_{s=1}^{\infty} A_s r^s.$$

Equating coefficients, we find, if s is prime to p,

$$pA_{sp} = Qp^{\kappa} A_s.$$

* This fact is intimately connected with the transformation equations in the theory of the modular functions. We may note that it is often more convenient to select the reduced substitutions in different ways.

Taking $s=1$, $$pA_p = Qp^\kappa,$$

so that $$A_{sp} = A_s A_p \quad\text{.........................(3).}$$

If no restrictions are placed on s we find, by equating coefficients of r^{ps},

$$\xi A_s + \frac{1}{p^{\kappa-1}} A_{sp^2} = QA_{sp}.$$

From this

$$A_{sp^2} - A_p A_{sp} + \xi p^{\kappa-1} A_s = 0 \quad\text{.................(4).}$$

From equations (3) and (4), we can prove that $A_{mn} = A_m A_n$ if m and n are prime to each other. For all we really have to shew is that, if p is a prime and s is prime to p, then $A_{sp^\lambda} = A_s A_{p^\lambda}$. But from equation (4), we have

$$A_{sp^{\lambda+2}} - A_p A_{sp^{\lambda+1}} + \xi p^{\kappa-1} A_{sp^\lambda} = 0,$$

and $$A_{p^{\lambda+2}} - A_p A_{p^{\lambda+1}} + \xi p^{\kappa-1} A_{p^\lambda} = 0 \quad\text{...........(4a).}$$

Hence the theorem follows by induction, for if it is true for λ and $\lambda+1$ it is true for $\lambda+2$. But it is true for $\lambda=0$ and for $\lambda=1$ (equation 3): hence it holds universally.

We notice also that equation (4a) is a linear difference equation of the second order with constant coefficients*. Hence, since $A_1 = 1$,

$$1 + A_p x + A_{p^2} x^2 + A_{p^3} x^3 + \ldots = 1/(1 - A_p x + \xi p^{\kappa-1} x^2),$$

from which, by putting $x = 1/p^s$,

$$1 + \frac{A_p}{p^s} + \frac{A_{p^2}}{p^{2s}} + \frac{A_{p^3}}{p^{3s}} + \ldots = 1\bigg/\left(1 - \frac{A_p}{p^s} + \frac{\xi p^{\kappa-1}}{p^{2s}}\right).$$

Putting for p in succession the primes 2, 3, 5 ..., multiplying together the corresponding equations, and remembering that $A_{mn} = A_m A_n$ if m and n are prime to each other, we have

$$\frac{A_1}{1^s} + \frac{A_2}{2^s} + \frac{A_3}{3^s} + \frac{A_4}{4^s} + \ldots = \Pi\, 1\bigg/\left(1 - \frac{A_p}{p^s} + \frac{\xi p^{\kappa-1}}{p^{2s}}\right) \quad\text{......(5),}$$

where the product refers to the primes 2, 3, 5 ....

The simplest application of these results is given by the function

$$f_a(\omega_1, \omega_2) = \left[\Delta\left(\frac{12}{a}\omega_1, \omega_2\right)\right]^{a/12},$$

* This is obvious if we put $\mu_\lambda = A_{p^\lambda}$.

where a is a divisor of 12. Its expansion in powers of r involves only positive integral powers of r and starts with $\left(\frac{2\pi}{\omega_2}\right)^a r$. $f_a(\omega_1, \omega_2)$ is not however an invariant of the modular group. We can avoid this difficulty by taking $f(\omega_1, \omega_2) = [\Delta(\omega_1, \omega_2)]^{a/12}$. In this case*

$$\xi_\kappa f(\omega_1 + \kappa\omega_2, p\omega_2) = \left[e^{-\frac{\kappa p \pi i}{6}} \sqrt[12]{\Delta(\omega_1 + \kappa\omega_2, p\omega_2)}\right]^a,$$

$$\xi f(p\omega_1, \omega_2) \qquad = \left[(-1)^{\frac{p-1}{2}} \sqrt[12]{\Delta(p\omega_1, \omega_2)}\right]^a,$$

provided we exclude $p = 2$ and $p = 3$. Putting for the moment

$$\left(\frac{\omega_2}{2\pi}\right)^a [\sqrt[12]{\Delta(\omega_1, \omega_2)}]^a = \sum_{s=0}^{\infty} B_s r^{\frac{a}{12}+s},$$

we find

$$\left(\frac{\omega_2}{2\pi}\right)^a \sum_{\kappa=0}^{p-1} \xi_\kappa f(\omega_1 + \kappa\omega_2, p\omega_2)$$

$$= \sum_{s=0}^{\infty} \sum_{\kappa=0}^{p-1} e^{-\frac{a\kappa p\pi\iota}{6} + \left(\frac{a}{12}+s\right)\frac{2\kappa\pi\iota}{p}} B_s r^{\left(\frac{a}{12}+s\right)\frac{1}{p}}.$$

But since $p \neq 2$ or 3, $p^2 - 1 \equiv 0 \pmod{12}$. Hence

$$\sum_{\kappa=0}^{p-1} e^{\frac{2\kappa\pi\iota}{p}\left(\frac{a-ap^2}{12}+s\right)} = 0,$$

unless $a(1-p^2)/12 + s \equiv 0 \pmod p$, that is $a + 12s \equiv 0 \pmod p$, and is then equal to p. Hence ϕ is a power series in $r^{1/12}$ (really of the form $r^{d/12}(A + Br + Cr^2 \ldots)$), starting with $r^{(a+12s)/12p}$, where s is the smallest positive integer for which $a + 12s \equiv 0 \pmod p$. Now the only zeros of $f(\omega_1, \omega_2) = 0$ in the fundamental polygon are at $\omega = \iota\infty$ or $r = 0$, and

$$f(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^a r^{\frac{a}{12}}(1 + Dr + Er^2 \ldots).$$

But putting $a = 12/b$, so that b is an integer,

$$\frac{a + 12s}{12p} = \frac{1 + bs}{bp} \geqslant \frac{1}{b} \geqslant \frac{a}{12},$$

since $$1 + bs \equiv 0 \pmod p.$$

Hence $\phi/f(\omega_1, \omega_2)$ is a constant, and equations (3), (4), (5) apply to the function

$$\left[\Delta'\left(\frac{12}{a}\omega_1, \omega_2\right)\right]^{a/12}.$$

We note also that $\qquad \xi = (-1)^{a(p-1)/2}$.

* Hurwitz, *l.c.*, p. 572, or Weber, *Lehrbuch der Algebra*, vol. 3, p. 252.

When $p=2$, these theorems hold if $a=4$ or 12. For the functions $\xi_\lambda f_\lambda$* are selected as before, and it is clear that the argument above applies, as $a(1-p^2)/12$ is an integer.

Lastly, when $p=3$, these theorems hold if $a=3, 6, 12$, and the functions $\xi_\lambda f_\lambda$* are selected as before.

Hence, altering our notation, we have the following theorems. If a is a divisor of 12 and

$$r\left[\left(1-r^{\frac{12}{a}}\right)\left(1-r^{\frac{24}{a}}\right)\left(1-r^{\frac{36}{a}}\right)\ldots\right]^{2a}=\sum_{n=1}^{\infty} f_a(n)\,r^n,$$

then
$$f_a(m)\,f_a(n)=f_a(mn)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

if m and n are prime to each other; and

$$\sum_{n=1}^{\infty}\frac{f_a(n)}{n^s}=\Pi\, 1\Bigg/\left(1-\frac{f_a(p)}{p^s}+\frac{(-1)^{\frac{p-1}{2}a}\,p^{a-1}}{p^{2s}}\right)\ldots\ldots\ldots(7).$$

The product refers to the primes 2, 3, 5, etc., except that

$p=2$ is excluded except when $a=4, 12$,

and $p=3$ is excluded except when $a=3, 6, 12$.

We notice that when $a=1, 2, 3$, or 6, $p=2$ is not excluded as a factor of say m in (6), as in this case $f_a(m)$ and $f_a(mn)$ are both zero. Similarly for $p=3$ when $a=1, 2, 4$.

The result (6) is given by Mr Ramanujan when $a=12$, as are most of the cases of (7). We shall now shew how in many cases we can find simple expressions for $f_a(p)$.

If $a=1$, it is known that, by a result due to Euler†,

$$r\left[(1-r^{12})(1-r^{24})\ldots\right]^2=\left[\sum_{-\infty}^{\infty}(-1)^n r^{\frac{(6n+1)^2}{2}}\right]^2$$
$$=\Sigma\Sigma(-1)^{m+n}\,r^{\frac{(6m+1)^2+(6n+1)^2}{2}}$$
$$=\Sigma\Sigma(-1)^{\eta}\,r^{\xi^2+9\eta^2},$$

where $\xi=3(m+n)+1$, $\eta=n-m$, so that ξ, η take all integer values satisfying $\xi\equiv 1 \pmod 3$, $\xi+\eta\equiv 1 \pmod 2$.

Hence $f_1(p)=2(-1)^\eta$ if $p=\xi^2+9\eta^2$ and we take both ξ and η to be positive. If $p\equiv -1$ or $\pm 5 \pmod{12}$, $f_1(p)$ is obviously zero. This is Mr Ramanujan's result (118).

If $a=2$, it is known (Klein-Fricke, vol. 2, page 374) that

$$r\left[(1-r^6)(1-r^{12})\ldots\right]^4=\tfrac{1}{3}\Sigma(-1)^{\xi}\,\xi r^{\xi^2+3\xi\eta+3\eta^2},$$

where ξ, η take all integer values satisfying

$$\xi\equiv 2 \pmod 3,\quad \eta\equiv 1 \pmod 2.$$

* Hurwitz, *l.c.*, vol. 18.

† See also Klein-Fricke, *Modulfunktionen*, vol. 2, p. 374.

Hence $$f_2(p) = \Sigma \tfrac{1}{3}(-1)^{\xi}\xi$$
extended to the solutions of $p = \xi^2 + 3\xi\eta + 3\eta^2$ for which
$$\xi \equiv 2 \pmod 3, \quad \eta \equiv 1 \pmod 2.$$
This* can be written as $f_2(p) = 2v$, where $p = 3u^2 + v^2$, u is positive and $v \equiv 1 \pmod 3$. Also $f_2(p) = 0$ if $p \equiv -1 \pmod 3$. This is Mr Ramanujan's result (127).

If $a = 3$ we have, from Klein-Fricke, vol. 2, page 377,
$$r[(1-r^4)(1-r^8)\ldots]^6 = -\tfrac{1}{2}\Sigma(\xi^2 - \eta^2)r^{\xi^2+\eta^2},$$
where ξ takes all even values and η all odd values. Hence
$$f_3(p) = -2(\xi^2 - \eta^2)$$
if $p = \xi^2 + \eta^2$, ξ is even, η is odd, and both ξ and η are positive. Also $f_3(p) = 0$ if $p \equiv 3 \pmod 4$. This is Mr Ramanujan's result (123).

If $a = 4$, then by Klein-Fricke, vol. 2, page 373,
$$r[(1-r^3)(1-r^6)\ldots]^8 = \tfrac{1}{6}\Sigma\xi^3 r^{\xi^2+3\xi\eta+3\eta^2},$$
where ξ, η take all values for which $\xi \equiv 2 \pmod 3$.

Hence $f_4(p) = \frac{1}{6}\Sigma\xi^3$ extended to all the solutions of
$$p = \xi^2 + 3\xi\eta + 3\eta^2,$$
where $\xi \equiv 2 \pmod 3$. This† can be written as $f_4(p) = 2(v^3 - 9vu^2)$, where $p = 3u^2 + v^2$, u is positive, and $v \equiv 1 \pmod 3$. This is Mr Ramanujan's result (128).

When $a = 6$, $f_6(n)$ is known by means of the representations of n as a sum of four squares. Mr Ramanujan has overlooked the fact that in his result (159) $2c_p$ is $-f_6(p)$. The theorem
$$f_6(m)f_6(n) = f_6(mn),$$
is due to Dr Glaisher.

When $a = 12$, we have Mr Ramanujan's results given as equations (1) and (2) in this paper.

He also gives results when $a = \frac{1}{2}, \frac{3}{2}$.

Thus $$\sum_{n=1}^{\infty} \frac{f_{\frac{1}{2}}(n)}{n^s} = \Pi\, 1\Big/\left(1 - \left(\frac{3}{p}\right)\frac{1}{p^{2s}}\right), \qquad (p = 3, 5, 7\ldots),$$
$$\sum_{n=1}^{\infty} \frac{f_{\frac{3}{2}}(n)}{n^s} = \Pi\, 1\Big/\left(1 - \left(\frac{-1}{p}\right)\frac{1}{p^{2s-1}}\right), \quad (p = 3, 5, 7\ldots),$$

* When ξ is even put $\xi = 2v$, $\eta = u - v$, and when ξ is odd put $\xi = 3u - v$, $\eta = v - u$. Both these cases are admissible, and we find that $p = v^2 + 3u^2$ and $v \equiv 1 \pmod 3$. Also $\Sigma(-1)^{\xi}\xi = 2v + 2v - (3u - v) - (-3u - v) = 6v$, where now u is taken as positive.

† See the last footnote. In addition to the two cases there considered, η even is admissible. Put then $\eta = 2u$, $\xi = -v - 3u$, from which $p = v^2 + 3u^2$ and $v \equiv 1 \pmod 3$.

where $\left(\frac{3}{p}\right)$ and $\left(\frac{-1}{p}\right)$ are symbols of quadratic reciprocity, so that $\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}$, $\left(\frac{3}{p}\right)=1$ if $p\equiv\pm1 \pmod{12}$, and $\left(\frac{3}{p}\right)=-1$ if $p\equiv\pm5 \pmod{12}$. If $p=3$, $\left(\frac{3}{p}\right)=0$.

These are particular cases of Euler's theorem that

$$\Sigma\frac{f(n)}{n^s}=\Pi\, 1\Big/\left(1-\frac{f(p)}{p^s}\right)$$

if the function f satisfies the condition

$$f(mn)=f(m)\,f(n),$$

the product refers to any group of primes, and the summation to all numbers whose prime factors are included in the group. Thus

$$r(1-r^{24})(1-r^{48})\ldots=\sum_{-\infty}^{\infty}(-1)^n r^{(6n+1)^2}=\sum_{1,3,5\ldots}^{\infty}\left(\frac{3}{n}\right)r^{n^2}$$

and $\quad r[(1-r^8)(1-r^{16})\ldots]^3=\sum\limits_{1,3,5\ldots}^{\infty}(-1)^{\frac{n-1}{2}}nr^{n^2}=\sum\limits_{1,3,5\ldots}^{\infty}\left(\frac{-1}{n}\right)nr^{n^2}.$

Finally, Mr Ramanujan gives two results, equations (155) and (162), of which the first is

$$\sum_1^{\infty}\frac{f_{10}(n)}{n^s}=\frac{1}{1+2^{2-s}}\Pi\, 1/(1-2c_p p^{-s}+(-1)^{\frac{p-1}{2}}p^{4-2s}),$$
$$(p=3,5\ldots),$$

where $c_p=u^2-(4v)^2$ and u and v are the positive integers satisfying $u^2+(4v)^2=p^2$. But if $p\equiv3 \pmod 4$, c_p is taken to be zero. $f_{10}(n)$ is defined* by

$$\sum_1^{\infty}f_{10}(n)r^n$$
$$=r[(1-r^2)(1-r^4)(1-r^6)\ldots]^{14}/[(1+r)(1-r^2)(1+r^3)(1-r^4)\ldots]^4,$$

and this is equal to†

$$\tfrac{1}{4}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(x+\iota y)^4 r^{x^2+y^2}.$$

The second result is

$$\sum_1^{\infty}\frac{f_{16}(n)}{n^s}=\frac{1}{1+2^{3-s}}\Pi\, 1/(1-2c_p p^{-s}+p^{7-2s}),\quad (p=3,5\ldots),$$

* The functions $f_{10}(n)$, $f_{16}(n)$ arise in finding the number of representations of n as a sum of 10 and 16 squares respectively and the series $\Sigma\,\Sigma\,(x+\iota y)^4 r^{x^2+y^2}$ is well known in this connection.

† From this, it follows that the result can be also proved as a particular case of Euler's product.

where $f_{16}(n)$ is defined* by

$$\sum_1^\infty f_{16}(n)\, r^n$$

$$= r\,[(1+r)(1-r^2)(1+r^3)(1-r^4)\ldots]^8/[(1-r^2)(1-r^4)(1-r^6)\ldots]^8.$$

Mr Ramanujan overlooks the fact that $c_p = \frac{1}{2} f_{16}(p)$.

These results can be proved by aid of the principles used in finding equations (3) and (4). We should however have to consider now invariants of a sub-group of the modular group, and it seems hardly worth while to go into details.

* The functions $f_{10}(n)$, $f_{16}(n)$ arise in finding the number of representations of n as a sum of 10 and 16 squares respectively and the series $\Sigma\Sigma\,(x+\iota y)^4\, r^{x^2+y^2}$ is well known in this connection.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1916—1917.

ANNUAL GENERAL MEETING.

October 30, 1916.

In the Comparative Anatomy Lecture Room.

PROFESSOR NEWALL, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Dr Marr.

Vice-Presidents:

Dr Fenton.
Prof. Eddington.
Prof. Newall.

Treasurer:

Prof. Hobson.

Secretaries:

Mr A. Wood.
Mr G. H. Hardy.
Mr H. H. Brindley.

Other Members of the Council:

Dr Duckworth.
Dr Crowther.
Dr Bromwich.
Dr Doncaster.
Mr C. G. Lamb.
Mr J. E. Purvis.
Dr Shipley.
Dr Arber.
Prof. Biffen.
Mr L. A. Borradaile.
Mr W. H. Mills.
Mr F. F. Blackman.

The following was elected an Associate of the Society:

W. Morris Jones, Emmanuel College.

The following Communications were made:

1. Methods of investigation in atmospheric electricity. By C. T. R. Wilson, M.A., Sidney Sussex College.

2. On the functions of the mouth parts of the Common Prawn. By L. A. Borradaile, M.A., Selwyn College.

3. On the growth of Daphne. By J. T. Saunders, M.A., Christ's College.

4. A self-recording electrometer for Atmospheric Electricity. By W. A. D. Rudge, M.A., St John's College.

5. An axiom in Symbolic Logic. By C. E. Van Horn. (Communicated by Mr G. H. Hardy.)

6. On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$. By S. Ramanujan, Trinity College. (Communicated by Mr G. H. Hardy.)

7. A reduction in the number of primitive propositions of Logic. By J. G. P. Nicod, Trinity College. (Communicated by Mr G. H. Hardy.)

November 13, 1916.

In the School of Agriculture.

Dr Marr, President, in the Chair.

The following were elected Fellows of the Society:

F. W. Green, M.A., Jesus College.
R. I. Lynch, M.A.

The following was elected an Associate of the Society:

N. Yamaga, Fitzwilliam Hall.

The following Communications were made:

1. The surface law of heat loss in animals. By Professor Wood.

2. Inheritance of henny plumage in cocks. By Professor Punnett and Capt. P. G. Bailey.

3. On extra mammary glands and the reabsorption of milk sugar. By Dr MARSHALL and K. J. J. MACKENZIE, M.A., Christ's College.

4. Experimental work on clover sickness. By A. AMOS, M.A., Downing College. (Communicated by Professor Biffen.)

5. Bessel's functions of equal order and argument. By G. N. WATSON, M.A., Trinity College.

February 5, 1917.

In the Sedgwick Museum.

DR MARR, PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

F. W. H. Oldham, B.A., Trinity College.

The following Communications were made:

1. Submergence and glacial climates during the accumulation of the Cambridgeshire Pleistocene Deposits. By Dr MARR.

2. Glacial Phenomena near Bangor, North Wales. By P. LAKE, M.A., St John's College.

3. The Cretaceous Faunas of New Zealand. By H. WOODS, M.A., St John's College.

4. Exhibition of the Fruit of Chocho *Sechium edule*: remarkable in the Nat. Order Cucurbitaceae, native of the West Indies and cultivated also in Madeira as a vegetable. By R. I. LYNCH, M.A.

5. The limits of applicability of the Principle of Stationary Phase. By G. N. WATSON, M.A., Trinity College.

6. The Direct Solution of the Quadratic and Cubic Binomial Congruences with Prime Moduli. By H. C. POCKLINGTON, M.A., St John's College.

7. On the Hydrodynamics of Relativity. By C. E. WEATHERBURN, M.A., Trinity College.

8. The Character of the Kinetic Potential in Electromagnetics. By R. HARGREAVES, M.A., St John's College.

9. On the Fifth Book of Euclid's Elements. (Fourth Paper.) By Dr M. J. M. HILL.

10. On a theorem of Mr G. Pólya. By G. H. HARDY, M.A., Trinity College.

February 19, 1917.

In the Botany School.

DR MARR, PRESIDENT, IN THE CHAIR.

The following Communications were made:

1. (1) On an Australian specimen of *Clepsydropsis*.

(2) Observations on the Evolution of Branching in the Ferns. By B. SAHNI, B.A., Emmanuel College. (Communicated by Professor Seward.)

2. On some anatomical characters of coniferous wood and their value in classification. By C. P. DUTT, B.A., Queens' College. (Communicated by Professor Seward.)

CONTENTS.

PROCEEDINGS

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY

VOL. XIX. PART IV.

[MICHAELMAS TERM 1917—EASTER TERM 1918.]

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July 1918.

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2. Separate copies of certain parts of Volumes I.—XI. of the *Transactions* may be had on application to Messrs BOWES & BOWES or Messrs DEIGHTON, BELL & Co., Limited, Cambridge.

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5. Letters and Communications for the Society should be addressed to one of the Secretaries,

Mr G. H. HARDY, Trinity College. [Mathematical.]
Mr A. WOOD, Emmanuel College. [Physical.]
Mr H. H. BRINDLEY, St John's College. [Biological.]

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New Museums,
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8. Members of the Society are requested to inform the Secretaries of any change of address.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

Extensions of Abel's Theorem and its converses. By Dr A. KIENAST, Küsnacht, Zürich, Switzerland. (Communicated by Mr G. H. Hardy.)

[*Received* 26 September 1917.]

INTRODUCTION.

Abel proved in 1826 the theorem:

"*If* $\lim_{n\to\infty} \sum_1^n a_\kappa$ *exists and is finite, then*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = \lim_{n\to\infty} \sum_1^n a_\kappa."$$

Let us write

$$\left.\begin{aligned} s_n &= \sum_1^n a_\kappa \\ t_n^{(1)} &= \frac{1}{n}\sum_1^n s_\kappa \\ &\cdots\cdots\cdots\cdots \\ t_n^{(\lambda+1)} &= \frac{1}{n}\sum_1^n t_\kappa^{(\lambda)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then Hölder* proved in 1882

Theorem 1. *If* $\lim_{n\to\infty} t_n^{(\lambda)}$ *exists and is finite, then*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = \lim_{n\to\infty} t_n^{(\lambda)}.$$

* Bromwich, *Infinite series*, p. 313.

In 1897 Mr Tauber, and in 1900 Mr Pringsheim, published the following converse of Abel's theorem:

Theorem 2. *The two conditions*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l \quad (\textit{finite}),$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n \kappa a_\kappa = 0$$

are each necessary for the convergence of Σa_κ, *i.e. for the existence of*

$$\lim_{n\to\infty} \sum_1^n a_\kappa = l;$$

and, taken together, they are sufficient.*

In the present paper I replace the means (1) by

$$\left.\begin{array}{ll} s_n = \sum\limits_1^n a_\kappa & (n = 1, 2, \ldots) \\ s_n^{(1)} = \dfrac{1}{n} \sum\limits_1^{n-1} s_\kappa & (n = 2, 3, \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ s_n^{(\lambda+1)} = \dfrac{1}{n} \sum\limits_{\lambda+1}^{n-1} s_\kappa^{(\lambda)} & (n = \lambda + 2, \lambda + 3, \ldots) \end{array}\right\} \ldots\ldots(2).$$

Defining $r_n^{(\lambda)}$ by

$$\left.\begin{array}{ll} r_n^{(1)} = \sum\limits_1^n \kappa a_\kappa & (n = 1, 2, \ldots) \\ r_n^{(2)} = \sum\limits_1^{n-1} \dfrac{1}{\kappa} r_\kappa^{(1)} & (n = 2, 3, \ldots) \\ \ldots\ldots\ldots\ldots\ldots & \\ r_n^{(\lambda+1)} = \sum\limits_\lambda^{n-1} \dfrac{1}{\kappa} r_\kappa^{(\lambda)} & (n = \lambda + 1, \lambda + 2, \ldots) \end{array}\right\} \ldots\ldots(3),$$

I prove, in Part I,

Theorem 3. *The two conditions*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l \quad (\textit{finite}),$$

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0$$

* Bromwich, *Infinite series*, p. 251.

are each necessary for the existence of

$$\lim_{n\to\infty} s_n^{(\lambda)} = l\,;$$

and, taken together, they are sufficient.

This theorem includes the analogue of 1:

Theorem 4. *If* $\lim_{n\to\infty} s_n^{(\lambda)} = l$ *exists and is finite, then*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l.$$

It is easy to verify that $\lim_{n\to\infty} t_n^{(\lambda)} = \lim_{n\to\infty} s_n^{(\lambda)}$ if $\lambda = 1$ or 2; for higher values of λ this relation certainly holds if both limits exist, as follows from Theorems 1 and 4.

In Part II, I propose to extend Theorem 3 to certain other mean values; and Part III contains some general remarks about the converse of Abel's theorem.

Part I.

1. In the researches which follow I have to make use of the following theorems.

Theorem 5. *If* $\lim_{n\to\infty} \sum_1^n a_\kappa = \lim_{n\to\infty} s_n = l$ *(finite),*

and b_κ *is positive and*

$$\lim_{n\to\infty} \sum_1^n b_\kappa = \lim_{n\to\infty} t_n = \infty,$$

then

$$\lim_{n\to\infty} \frac{1}{t_n} \sum_1^n b_\kappa s_\kappa = \lim_{n\to\infty} s_n = l \quad \text{..............(4).}$$

This theorem is due to Stolz*.

Theorem 6. *Suppose that* b_κ *is positive and* Σb_κ *divergent; and let* D *be the region defined by*

$$\rho < 2\cos\psi \quad (|\psi| \leqslant \psi_0 < \tfrac{1}{2}\pi),$$

where $1 - x = \rho e^{-i\psi}$. *Further suppose that*

$$\Sigma b_\kappa |x^\kappa| / |\Sigma b_\kappa x^\kappa| < G,$$

where G *is a finite constant, for all values of* x *inside the region* D.

* Bromwich, *Infinite series*, p. 378.

Finally suppose that a_n/b_n tends to the limit l when n tends to infinity. Then

$$\lim_{x \to 1} (\Sigma\, a_\kappa x^\kappa / \Sigma\, b_\kappa x^\kappa) = l \qquad\qquad (5),$$

when x approaches 1 along any path inside D.

This theorem is due to Pringsheim*. It is to be supposed throughout this paper that, when x tends to 1, its approach to 1 is along some path inside D.

Theorem 7. *If the radius of convergence of $P(x) = \sum_1^\infty a_\kappa x^\kappa$ is r, then*

$$\lim_{n \to \infty} a_n x^n = 0 \quad (|x| < r).$$

If the radius of convergence of $Q(x) = \sum_1^\infty a_\kappa x^\kappa$ is unity, it will remain unchanged if $Q(x)$ be transformed in any of the following ways:

(i) by suppressing a limited number of terms,

(ii) by multiplying by x^σ, σ being an integer,

(iii) by multiplying by $\dfrac{1}{1-x} = \Sigma\, x^n$,

(iv) by integrating term by term,

(v) by differentiating a limited number of times.

Using in succession one or other of these operations, there result the following power-series, all with radius of convergence unity:

$$xP'(x) = \sum_1^\infty \kappa a_\kappa x^\kappa,$$

$$P_1(x) = \frac{1}{1-x}[xP'(x)] = \sum_1^\infty r_\kappa^{(1)} x^\kappa,$$

$$P_2(x) = \frac{1}{1-x}\left[\int_0^x \frac{1}{x} P_1(x)\,dx\right] = \sum_1^\infty r_{\kappa+1}^{(2)} x^\kappa,$$

$$P_3(x) = \frac{1}{1-x}\left[\int_0^x P_2(x)\,dx\right] = \sum_1^\infty r_{\kappa+2}^{(3)} x^{\kappa+1},$$

$$P_4(x) = \frac{1}{1-x}\left[\int_0^x P_3(x)\,dx\right] = \sum_1^\infty r_{\kappa+3}^{(4)} x^{\kappa+2},$$

....

* *Acta Mathematica*, vol. 28, p. 7.

Thus the series $\sum_{\mu=1}^{\infty} r_{\mu+\lambda-1}^{(\lambda)} x^{\mu+\lambda-2}$ converges if $|x| < 1$; the same is the case with

$$\sum_{p=\kappa+1}^{\infty} r_{p+\lambda-\kappa-1}^{(\lambda)} x^{p+\lambda-\kappa-2}$$

or

$$\sum_{p=\kappa+1}^{\infty} r_{p+\lambda-1}^{(\lambda+\kappa)} x^{p+\lambda-2}.$$

Differentiating the last series $(\lambda-2)$ times, we obtain

$$\Sigma (p+1)(p+2)\dots(p+\lambda-2) r_{p+\lambda-1}^{(\lambda+\kappa)} x^p;$$

which gives

Theorem 8. *If the radius of convergence of $P(x) = \sum_1^{\infty} a_\kappa x^\kappa$ is unity, then for every $|x| < 1$*

$$\lim_{n\to\infty} r_{n+\lambda-1}^{(\lambda+\kappa)} x^{n+\lambda-2} = 0,$$

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(\lambda+\kappa)} x^n = 0,$$

$$\lim_{n\to\infty} (n+1)(n+2)\dots(n+\lambda-2) r_{n+\lambda-1}^{(\lambda+\kappa)} x^n = 0.$$

2. The demonstration of Theorem 3 depends on certain identities. The formula

$$s_n^{(1)} = s_n - \frac{1}{n} r_n^{(1)}$$

leads, by successive summation, to the series of equations

$$\left.\begin{array}{l} s_n^{(2)} = s_n^{(1)} - \dfrac{1}{n} r_n^{(2)} \\ \dots\dots\dots\dots\dots\dots\dots \\ s_n^{(\lambda)} = s_n^{(\lambda-1)} - \dfrac{1}{n} r_n^{(\lambda)} \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots (6).$$

If $\lim_{n\to\infty} s_n^{(\lambda)} = l$ exists, then, by Theorem 5, $\lim_{n\to\infty} s_n^{(\lambda+1)}$ also exists and is equal to l, and therefore one of these identities gives

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0.$$

Theorem 9. *If $\lim_{n\to\infty} s_n^{(\lambda)} = l$ exists and is finite, then*

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0.$$

Thus the second condition of Theorem 3 is necessary.

3. I proceed to prove some other identities. We have

$$\left.\begin{array}{ll} r_n^{(1)} = \sum_1^n \kappa a_\kappa, & a_n = \frac{1}{n}\{r_n^{(1)} - r_{n-1}^{(1)}\} \\ r_n^{(2)} = \sum_1^{n-1} \frac{1}{\kappa} r_\kappa^{(1)}, & \frac{r_n^{(1)}}{n} = r_{n+1}^{(2)} - r_n^{(2)} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ r_n^{(\lambda+1)} = \sum_1^{n-1} \frac{1}{\kappa} r_\kappa^{(\lambda)}, & \frac{r_n^{(\lambda)}}{n} = r_{n+1}^{(\lambda+1)} - r_n^{(\lambda+1)} \end{array}\right\} \quad \ldots\ldots(7);$$

and by successive substitution we find

$$a_n = \frac{1}{n}\{r_n^{(1)} - r_{n-1}^{(1)}\} = r_{n+1}^{(2)} - 2r_n^{(2)} + r_{n-1}^{(2)} + \frac{1}{n}\{r_n^{(2)} - r_{n-1}^{(2)}\}$$
$$= (n+1)\,r_{n+2}^{(3)} - 3nr_{n+1}^{(3)} + 3(n-1)\,r_n^{(3)} - (n-2)\,r_{n-1}^{(3)}$$
$$+ \frac{1}{n}\{r_n^{(3)} - r_{n-1}^{(3)}\} = \ldots.$$

Writing

$$\sum_{\mu=0}^{\lambda} (-1)^\mu \binom{\lambda}{\mu} (n+1-\mu)(n+2-\mu)\ldots$$
$$\ldots(n+\lambda-\mu-2)\,r_{n+\lambda-\mu-1}^{(\lambda+\kappa)} = G_{\lambda,\kappa,n},$$
$$G_{1,\kappa,n} = 0, \quad G_{2,\kappa,n} = r_{n+1}^{(2+\kappa)} - 2r_n^{(2+\kappa)} + r_{n-1}^{(2+\kappa)},$$

we can easily verify that

$$G_{\lambda,\kappa,n} = G_{\lambda+1,\kappa,n} - (\lambda-1)\,G_{\lambda,\kappa+1,n} \quad \ldots\ldots\ldots\ldots(8).$$

Moreover

$$\frac{1}{n}\{r_n^{(\kappa)} - r_{n-1}^{(\kappa)}\} = G_{2,\kappa-1,n} + \frac{1}{n}\{r_n^{(\kappa+1)} - r_{n-1}^{(\kappa+1)}\} \quad \ldots\ldots(9).$$

Developing a_n in this way we obtain, after a finite number ρ of steps, the formula

$$a_n = \sum_{\lambda+\kappa=\rho} c_{\lambda,\kappa}\,G_{\lambda,\kappa,n} + \frac{1}{n}\{r_n^{(\rho)} - r_{n-1}^{(\rho)}\} \quad \ldots\ldots\ldots(10).$$

The upper index of all the r's is the same throughout this expression. For the present purpose it is not necessary to determine the coefficients $c_{\lambda,\kappa}$, which are integers.

In consequence of the definitions of $s_m^{(n)}$ and $r_m^{(n)}$ we have

$$r_m^{(n)} = 0 \quad (n > m).$$

But it is not difficult to see that the recurrence formulae (8) and (9) still hold, if the number ρ of steps exceeds the index n. It is only necessary to put $r_m^{(n)} = 0$ whenever $n > m$. The form of the relations (7), viz.

$$\frac{r_n^{(\lambda)}}{n} = r_{n+1}^{(\lambda+1)} - r_n^{(\lambda+1)},$$

is the cause why the coefficients of the remaining terms are not influenced by the fact that some terms disappear. Thus

$$\sum_1^m a_n x^n = \sum_{\lambda+\kappa=\rho} c_{\lambda,\kappa} \left\{ \sum_{n=1}^m G_{\lambda,\kappa,n} x^n \right\} + \sum_{n=\rho}^m \frac{1}{n} x^n \{ r_n^{(\rho)} - r_{n-1}^{(\rho)} \} \quad \text{............(11).}$$

To evaluate the first of these sums we have

$$(1-x)^\lambda \sum_{n=1}^m (n+1)\dots(n+\lambda-2)\, r_{n+\lambda-1}^{(\lambda+\kappa)} x^n = \sum_{n=1}^m G_{\lambda,\kappa,n} x^n + \sum_{\nu=1}^{\lambda} D_{\lambda,\kappa,m+\nu} x^{m+\nu},$$

say. Each of the $\frac{1}{2}\lambda(\lambda+1)$ terms contained in the second sum has the form

$$K(p+1)(p+2)\dots(p+\lambda-2)\, r_{p+\lambda-1}^{(\lambda+\kappa)} x^{p+\mu},$$

$$(\mu = \nu, \nu+1, \dots \lambda; \quad \nu = 1, 2, \dots \lambda; \quad p = m - \nu + 1).$$

Therefore, by Theorem 8, we have, for every $|x| < 1$ and any finite λ,

$$\lim_{m\to\infty} \sum_1^m G_{\lambda,\kappa,n} x^n = (1-x)^\lambda \sum_1^\infty (n+1)(n+2)\dots(n+\lambda-2)\, r_{n+\lambda-1}^{(\lambda+\kappa)} x^n.$$

The second sum in (11) gives

$$\sum_\rho^m \frac{1}{n} \{ r_n^{(\rho)} - r_{n-1}^{(\rho)} \} x^n = \sum_\rho^{n-1} \frac{1}{n+1} r_n^{(\rho)} \left[\frac{n+1}{n} - x \right] x^n + \frac{1}{m} r_m^{(\rho)} x^m$$

$$= (1-x) \sum_\rho^{m-1} \frac{1}{n+1} r_n^{(\rho)} x^n + \sum_\rho^{m-1} \frac{1}{n(n+1)} r_n^{(\rho)} x^n + \frac{1}{m} r_m^{(\rho)} x^m \quad \text{.........(12);}$$

and again, by Theorem 8, we have, for every $|x| < 1$ and any finite ρ,

$$\lim_{m\to\infty} \sum_\rho^m \frac{1}{n} x^n \{ r_n^{(\rho)} - r_{n-1}^{(\rho)} \} = (1-x) \sum_\rho^\infty \frac{1}{n+1} r_n^{(\rho)} x^n + \sum_\rho^\infty \frac{1}{n(n+1)} r_n^{(\rho)} x^n.$$

Thus we have established

Theorem 10. *If $\Sigma a_\kappa x^\kappa$ has unity as radius of convergence, then*

$$\sum_1^\infty a_n x^n = \sum_{\lambda+\kappa=\rho} c_{\lambda,\kappa}\left[(1-x)^\lambda \sum_{n=1}^\infty (n+1)\ldots(n+\lambda-2)\, r_{n+\lambda-1}^{(\rho)} x^n\right]$$

$$+(1-x)\sum_\rho^\infty \frac{1}{n+1} r_n^{(\rho)} x^n + \sum_\rho^\infty \frac{1}{n(n+1)} r_n^{(\rho)} x^n \quad (|x|<1) \qquad \text{.........(13)}$$

4. Equation (13) has now to be considered when $x \to 1$. To the first terms on the right-hand side we apply Theorem 6, which gives

$$\lim_{x\to 1} \frac{\sum_1^\infty (n+1)\ldots(n+\lambda-2)\, r_{n+\lambda-1}^{(\rho)} x^n}{\frac{1}{(\lambda-1)!}\sum_1^\infty (n+1)\ldots(n+\lambda-1)x^n} = \lim_{n\to\infty} \frac{r_{n+\lambda-1}^{(\rho)}}{n+\lambda-1}.$$

Again, by Theorem 6,

$$\lim_{x\to 1}(1-x)\sum_\rho^\infty \frac{1}{n+1} r_n^{(\rho)} x^n = \lim_{n\to\infty}\frac{1}{n} r_n^{(\rho)};$$

and finally Theorem 8 gives

Theorem 11. *If $\Sigma a_\kappa x^\kappa$ has unity as radius of convergence, and if*

$$\lim_{n\to\infty}\frac{1}{n} r_n^{(\rho)} = 0,$$

then

$$\lim_{x\to 1}\sum_1^\infty a_\kappa x^\kappa = \lim_{x\to 1}\sum_\rho^\infty \frac{1}{n(n+1)} r_n^{(\rho)} x^n.$$

5. Furthermore, equations (7) and (12) lead to

$$\sum_\rho^m \frac{1}{n}\left\{r_n^{(\rho)} - r_{n-1}^{(\rho)}\right\} x^n = \sum_{\rho-1}^{m-1} \frac{1}{n(n+1)} r_n^{(\rho-1)} x^{n+1}$$

$$= (1-x)\sum_\rho^{m-1}\frac{1}{n+1} r_n^{(\rho)} x^n + \sum_\rho^{m-1}\frac{1}{n(n+1)} r_n^{(\rho)} x^n + \frac{1}{m} r_m^{(\rho)} x^m.$$

Putting $x=1$, it follows that

$$\sum_\rho^{m-1}\frac{1}{n(n+1)} r_n^{(\rho)} = \sum_{\rho+1}^{m-1}\frac{1}{n(n+1)} r_n^{(\rho+1)} + \frac{1}{m} r_m^{(\rho+1)}.$$

Hence

Theorem 12. *If $\Sigma a_\kappa x^\kappa$ has unity as radius of convergence, and if*

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(\rho+1)} = 0,$$

then
$$\lim_{m\to\infty} \sum_{\rho}^{m-1} \frac{1}{n(n+1)} r_n^{(\rho)} = \lim_{m\to\infty} \sum_{\rho+1}^{m-1} \frac{1}{n(n+1)} r_n^{(\rho+1)}.$$

Another identity is acquired by developing $s_n^{(\rho)}$ $(n = \rho + 1, \rho + 2, \ldots)$ in the form

$$s_n^{(\rho)} = \frac{s_\rho^{(\rho-1)}}{\rho+1} + \sum_{\rho+2}^{n} \{s_m^{(\rho)} - s_{m-1}^{(\rho)}\}$$

$$= \frac{s_\rho^{(\rho-1)}}{\rho+1} + \sum_{\rho+2}^{n} \left[\frac{1}{m} s_{m-1}^{(\rho-1)} - \frac{1}{m(m-1)} \sum_{\rho}^{m-2} s_\kappa^{(\rho-1)}\right]$$

$$= \sum_{\rho}^{n-1} \frac{1}{m(m+1)} r_m^{(\rho)}.$$

Theorem 13. *If $s_n^{(\rho)}$ and $r_m^{(\rho)}$ are defined as in (2) and (3), then*

$$s_n^{(\rho)} = \sum_{\rho}^{n-1} \frac{1}{m(m+1)} r_m^{(\rho)}.$$

If $\lim_{n\to\infty} s_n^{(\lambda)} = l$ exists and is finite, then, by Theorem 5,

$$\lim_{n\to\infty} s_n^{(\lambda+1)} = l;$$

and by Theorem 13

$$\sum_{\lambda+1}^{\infty} \frac{1}{m(m+1)} r_m^{(\lambda+1)} = l.$$

Therefore by Abel's theorem

$$\lim_{x\to 1} \sum_{\lambda+1}^{\infty} \frac{1}{m(m+1)} r_m^{(\lambda+1)} x^m = \sum_{\lambda+1}^{\infty} \frac{1}{m(m+1)} r_m^{(\lambda+1)} = l.$$

On the same assumption, Theorem 9 gives $\lim_{n\to\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0$; and therefore Theorem 11 gives

$$\lim_{x\to 1} \sum_{1}^{\infty} a_\kappa x^\kappa = \lim_{x\to 1} \sum_{\lambda+1}^{\infty} \frac{1}{n(n+1)} r_n^{(\lambda+1)} x^n = l.$$

Thus we obtain

Theorem 4. *If* $\lim\limits_{n\to\infty} s_n^{(\lambda)} = l$ *exists and is finite, then*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l.$$

The first condition of Theorem 3 is therefore necessary.

6. To demonstrate the rest of the assertion in Theorem 3, it follows from the hypothesis $\lim\limits_{n\to\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0$ that Theorem 11 is applicable. Thus the assumptions are transformed into

$$\lim_{x\to 1} \sum_{\lambda+1}^\infty \frac{1}{n(n+1)} r_n^{(\lambda+1)} x^n = l,$$

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(\lambda+1)} = 0.$$

From this last equation follows

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda+1}^{n} \frac{1}{\kappa+1} r_\kappa^{(\lambda+1)} = \lim_{n\to\infty} \frac{r_n^{(\lambda+1)}}{n+1} = 0.$$

Hence Theorem 2 can be applied to the series $\Sigma \frac{r_n^{(\lambda+1)}}{n(n+1)} x^n$; and the conclusion is that

$$\lim_{n\to\infty} \sum_{\lambda+1}^{n} \frac{r_\kappa^{(\lambda+1)}}{\kappa(\kappa+1)} = l.$$

Theorems 12 and 13 now yield

$$\lim_{n\to\infty} s_n^{(\lambda)} = l,$$

with which the proof of Theorem 3 is completed.

7. The foregoing deductions are valid for $\lambda = 1, 2, \ldots$. For $\lambda = 0$ they still hold, except those in § 6. This case requires the proof of the following special case of Theorem 2:

Theorem 14. *If*

$$\lim_{x\to 1} \sum_1^\infty \frac{1}{n(n+1)} r_n^{(1)} x^n = l,$$

$$\lim_{n\to\infty} \frac{1}{n} r_n^{(1)} = 0,$$

then
$$\sum_1^\infty \frac{1}{n(n+1)} r_n^{(1)} = l.$$

This proof is actually given by Mr Tauber, and is therefore the basis of the theorems of this paper.

Part II.

8. Let b_κ denote the terms of an infinite sequence of positive real numbers, which have the properties

$$\left.\begin{array}{ll} (1) & \lim\limits_{n\to\infty} \sum\limits_1^n b_\kappa = \lim\limits_{n\to\infty} t_n = \infty \\ (2) & \dfrac{1}{n}\sum\limits_1^n \kappa \dfrac{t_{\kappa+1}-t_\kappa}{t_\kappa} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(14),$$

tends to a limit or oscillates between finite limits. Then

Theorem 15. *The two conditions*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l \qquad (\textit{finite}),$$

$$\lim_{n\to\infty} \frac{1}{t_n}\sum_1^n t_\lambda a_\lambda = 0,$$

are each necessary for the convergence of Σa_κ, i.e. for the existence of

$$\lim_{n\to\infty} \sum_1^n a_\kappa = l:$$

and, taken together, they are sufficient.

Abel's theorem states that the first of these conditions is necessary.

If $\lim\limits_{n\to\infty} s_n = l$, then $\lim\limits_{n\to\infty} a_n = 0$, and by Theorem 5

$$\lim_{n\to\infty} \frac{1}{t_n}\sum_2^n b_\lambda s_{\lambda-1} = \lim_{n\to\infty}\left[\frac{1}{t_n}\sum_1^n b_\lambda s_\lambda - \frac{1}{t_n}\sum_1^n b_\lambda a_\lambda\right] = l.$$

The identity

$$\frac{1}{t_n}\sum_2^n b_\lambda s_{\lambda-1} = s_n - \frac{1}{t_n}\sum_1^n t_\lambda a_\lambda$$

now gives, as a consequence of $\lim\limits_{n\to\infty} s_n = l$,

$$\lim_{n\to\infty} \frac{1}{t_n}\sum_1^n t_\lambda a_\lambda = 0.$$

Therefore the second condition is necessary too.

9. To prove the converse, we require two identities. If

$$\sum_1^n t_\lambda a_\lambda = p_n, \quad a_n = \frac{1}{t_n}\{p_n - p_{n-1}\},$$

we have

$$\begin{aligned}\sum_1^n a_\kappa x^\kappa &= \frac{p_1}{t_1}x + \sum_2^n \frac{1}{t_\kappa}\{p_\kappa - p_{\kappa-1}\}x^\kappa \\ &= \sum_1^{n-1}\frac{p_\kappa}{t_{\kappa+1}}\left[\frac{t_{\kappa+1}}{t_\kappa} - x\right]x^\kappa + \frac{p_n}{t_n}x^n \\ &= (1-x)\sum_1^{n-1}\frac{p_\kappa}{t_{\kappa+1}}x^\kappa + \sum_1^{n-1}\frac{t_{\kappa+1}-t_\kappa}{t_\kappa}\frac{p_\kappa}{t_{\kappa+1}}x^\kappa + \frac{p_n}{t_n}x^n \quad \text{.........(15).}\end{aligned}$$

Putting $x = 1$, this gives the identity

$$s_n = \sum_1^{n-1}\frac{t_{\kappa+1}-t_\kappa}{t_\kappa}\frac{p_\kappa}{t_{\kappa+1}} + \frac{p_n}{t_n} \quad \text{...............(16).}$$

If we suppose $\lim\limits_{n\to\infty}\dfrac{p_n}{t_n} = 0$, it follows that

$$\lim_{n\to\infty}\frac{p_n}{t_n}x^n = 0$$

for every $|x| < 1$; and, by Theorem 6,

$$\lim_{x\to1}(1-x)\sum_1^\infty \frac{p_\kappa}{t_{\kappa+1}}x^\kappa = 0.$$

Now passing in (15) to the limit (first $n\to\infty$ and then $x\to1$ we find that *if* $\lim\limits_{n\to\infty}\dfrac{p_n}{t_n} = 0$, *then*

$$\lim_{x\to1}\sum_1^\infty a_\kappa x^\kappa = \lim_{x\to1}\sum_1^\infty \frac{t_{\kappa+1}-t_\kappa}{t_\kappa}\frac{p_\kappa}{t_{\kappa+1}}x^\kappa \quad \text{............(17).}$$

Theorem 15 starts from the assumptions

$$\lim_{n\to\infty}\frac{p_n}{t_n} = 0,$$

$$\lim_{x\to1}\sum_1^\infty a_\kappa x^\kappa = l.$$

The first assumption shows that (17) is available; and this equation gives, with the second assumption,

$$\lim_{x\to1}\sum_1^\infty \frac{t_{\kappa+1}-t_\kappa}{t_\kappa}\frac{p_\kappa}{t_{\kappa+1}}x^\kappa = l.$$

Now Theorem 2 can be applied to the series $\Sigma \frac{t_{\kappa+1}-t_\kappa}{t_\kappa} \frac{p_\kappa}{t_{\kappa+1}} x^\kappa$, provided that

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n \kappa \frac{t_{\kappa+1}-t_\kappa}{t_\kappa} \frac{p_\kappa}{t_{\kappa+1}} = 0 \quad \dots\dots\dots\dots\dots(18).$$

Assuming for a moment that this condition is satisfied, Theorem 2 leads to

$$\lim_{n\to\infty} \sum_1^n \frac{t_{\kappa+1}-t_\kappa}{t_\kappa} \frac{p_\kappa}{t_{\kappa+1}} = l;$$

and (16) gives finally

$$\lim_{n\to\infty} s_n = l,$$

proving the theorem, which is the analogue to Theorem 2.

Condition (18) depends on the b's as well as on the a's; but since $\lim \frac{p_\kappa}{t_{\kappa+1}} = 0$, it will certainly be fulfilled when

$$\frac{1}{n} \sum_1^n \kappa \frac{t_{\kappa+1}-t_\kappa}{t_\kappa}$$

tends to a limit or oscillates finitely. For, ϵ being given, we can choose κ so that

$$\left| \frac{1}{n} \sum_1^n \lambda \frac{t_{\lambda+1}-t_\lambda}{t_\lambda} \frac{p_\lambda}{t_{\lambda+1}} \right| < \frac{1}{n} \sum_1^{\kappa-1} \lambda \frac{t_{\lambda+1}-t_\lambda}{t_\lambda} \frac{p_\lambda}{t_{\lambda+1}} + \frac{\epsilon}{n} \sum_\kappa^n \lambda \frac{t_{\lambda+1}-t_\lambda}{t_\lambda}.$$

We may suppose, for example, that

$$t_n = n^\sigma; \quad \log n; \quad \log\log n; \dots$$

10. Adding to the notations used hitherto

$$\frac{1}{t_n} \sum_2^n b_\lambda s_{\lambda-1} = s_n^{(1)},$$

$$\frac{1}{t_n} \sum_3^n b_\lambda s_{\lambda-1}^{(1)} = s_n^{(2)},$$

$$\sum_2^n b_\lambda \frac{p_{\lambda-1}}{t_{\lambda-1}} = q_n,$$

and restricting the choice of the numbers b_κ not only as done in 8, but further by supposing that the two limits

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{t_{n+1}}{b_{n+2}}, \quad \lim_{n\to\infty} \frac{b_{n+2}-b_{n+1}}{b_{n+1}} \frac{t_{n+1}}{b_{n+2}} \quad \dots\dots\dots(19)$$

shall exist, or at any rate that the functions under the limit sign shall oscillate finitely, I proceed to prove

Theorem 16. *The two conditions*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l \quad (\textit{finite}),$$

$$\lim_{n\to\infty} \frac{1}{t_n} \sum_2^n b_\lambda \frac{p_{\lambda-1}}{t_{\lambda-1}} = 0,$$

are each necessary for the existence of the limit

$$\lim_{n\to\infty} s_n^{(1)} = l:$$

and, taken together, they are sufficient.

It is not possible to demonstrate this theorem for every set of numbers b_κ. The following example shows this.

Mr Riesz has pointed out* that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_1^n \frac{1}{\kappa} s_\kappa$$

exists and is finite in the case of

$$s_n = \sum_1^n \kappa^{-1-ai}.$$

However, Abel's limit

$$\lim_{x\to 1} \sum_1^\infty \kappa^{-1-ai} x^\kappa$$

does not exist, as the function behaves like

$$\Gamma(ai)\left(\log\frac{1}{x}\right)^{ai}$$

when $x\to 1$.

11. The demonstration depends on some identities analogous to those employed in the case of the arithmetic means, viz.

$$s_n^{(1)} = s_n - \frac{1}{t_n} p_n,$$

$$s_n^{(2)} = s_n^{(1)} - \frac{1}{t_n} \sum_2^n b_\lambda \frac{p_{\lambda-1}}{t_{\lambda-1}},$$

* See G. H. Hardy, 'Slowly oscillating series', *Proc. London Math. Soc.*, ser. 2, vol. 8 (1910), p. 310.

which series of relations might be continued. They show (in conjunction with Theorem 5) that $\lim s_n^{(2)} = l$ whenever $\lim s_n^{(1)} = l$, from which we deduce

Theorem 17. *If* $\lim\limits_{n\to\infty} s_n^{(1)} = l$ *exists and is finite, then*

$$\lim_{n\to\infty} \frac{1}{t_n} \sum_2^n b_\lambda \frac{p_{\lambda-1}}{t_{\lambda-1}} = 0.$$

Thus the second condition of Theorem 16 is necessary.

12. We have also

$$b_n \frac{p_{n-1}}{t_{n-1}} = q_n - q_{n-1},$$

and thus

$$\begin{aligned}
\sum_1^n a_\kappa x^\kappa &= \sum_1^n \frac{q_{\kappa+1} - q_\kappa}{b_{\kappa+1}} x^\kappa - \sum_2^n \frac{t_{\kappa+1}}{t_\kappa} \frac{q_\kappa - q_{\kappa-1}}{b_\kappa} x^\kappa \\
&= \sum_1^n \frac{q_{\kappa+1} - q_\kappa}{b_{\kappa+1}} x^\kappa - x \sum_1^{n-1} \frac{q_{\kappa+1} - q_\kappa}{b_{\kappa+1}} x^\kappa + \sum_2^n \frac{t_\kappa - t_{\kappa-1}}{t_\kappa} \frac{q_\kappa - q_{\kappa-1}}{b_\kappa} x^\kappa \\
&= (1-x)\left[\sum_1^{n-1} \frac{q_{\kappa+1} - q_\kappa}{b_{\kappa+1}} x^\kappa - \frac{q_n x^n}{b_{n+1}}\right] + \frac{q_{n+1} x^n}{b_{n+1}} - \frac{q_n x^{n+1}}{b_{n+1}} \\
&\quad + \sum_2^n \frac{t_\kappa - t_{\kappa-1}}{t_\kappa} \frac{q_\kappa - q_{\kappa-1}}{b_\kappa} x^\kappa \\
&= (1-x)\left\{(1-x)\sum_1^{n-1} \frac{q_{\kappa+1}}{b_{\kappa+2}} x^\kappa + \sum_1^{n-1} \frac{b_{\kappa+2} - b_{\kappa+1}}{b_{\kappa+1}} \frac{q_{\kappa+1}}{b_{\kappa+2}} x^\kappa - \frac{q_1 x}{b_2}\right\} \\
&\quad + \frac{q_{n+1} x^n}{b_{n+1}} - \frac{q_n x^{n+1}}{b_{n+1}} + \sum_2^n \frac{t_\kappa - t_{\kappa-1}}{t_\kappa} \frac{q_\kappa - q_{\kappa-1}}{b_\kappa} x^\kappa.
\end{aligned}$$

Now the series

$$\sum \frac{q_n}{b_n} x^n = \sum \frac{q_n}{t_{n-1}} \left(\frac{1}{n-1} \frac{t_{n-1}}{b_n}\right)(n-1)x^n$$

has a radius of convergence at least as great as 1, since $\lim \dfrac{q_n}{t_{n-1}} = 0$ and $\dfrac{1}{n-1}\dfrac{t_{n-1}}{b_n}$ tends to a limit or oscillates finitely. Thus

$$\lim_{n\to\infty} \frac{q_n}{b_n} x^n = 0$$

for every $|x| < 1$, and therefore

$$\sum_1^\infty a_\kappa x^\kappa = (1-x)^2 \sum_1^\infty \frac{q_{\kappa+1}}{b_{\kappa+2}} x^\kappa + (1-x) \sum_1^\infty \frac{b_{\kappa+2}-b_{\kappa+1}}{b_{\kappa+1}} \frac{q_{\kappa+1}}{b_{\kappa+2}} x^\kappa$$
$$- (1-x)\frac{q_1 x}{b_2} + \sum_2^\infty \frac{q_\kappa - q_{\kappa-1}}{t_\kappa} x^\kappa.$$

Taking account of the conditions (19), it follows from Theorem 6 that

$$\lim_{x\to 1} (1-x)^2 \sum_1^\infty \frac{q_{\kappa+1}}{b_{\kappa+2}} x^\kappa = 0,$$

and
$$\lim_{x\to 1} (1-x) \sum_1^\infty \frac{b_{\kappa+2}-b_{\kappa+1}}{b_{\kappa+1}} \frac{q_{\kappa+1}}{b_{\kappa+2}} x^\kappa = 0,$$

so that
$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = \lim_{x\to 1} \sum_1^\infty \frac{q_{\kappa+1}-q_\kappa}{t_{\kappa+1}} x^{\kappa+1} \qquad \text{.............(20).}$$

13. Lastly we have the identity

$$s_n^{(1)} = \frac{b_2 s_1}{t_2} + \sum_3^n [s_\kappa^{(1)} - s_{\kappa-1}^{(1)}] = \frac{b_2 p_1}{t_1 t_2} + \sum_3^n \left\{\frac{b_\kappa s_{\kappa-1}}{t_\kappa} - s_{\kappa-1}^{(1)} t_{\kappa-1}\left(\frac{1}{t_{\kappa-1}} - \frac{1}{t_\kappa}\right)\right\}$$
$$= \sum_2^n \frac{b_{\kappa+1} p_\kappa}{t_{\kappa+1} t_\kappa} = \sum_1^{n-1} \frac{q_{\kappa+1} - q_\kappa}{t_{\kappa+1}} \qquad \text{..................................(21).}$$

14. If $\lim_{n\to\infty} s_n^{(1)} = l$ exists and is finite, then, by (21), $\sum_1^\infty \frac{q_{\kappa+1}-q_\kappa}{t_{\kappa+1}}$ converges to the sum l. Therefore by Abel's theorem

$$\lim_{x\to 1} \sum_1^\infty \frac{q_{\kappa+1}-q_\kappa}{t_{\kappa+1}} x^{\kappa+1} = l,$$

and since (Theorem 17) $\lim_{n\to\infty} \frac{q_n}{t_n} = 0$, equation (20) is valid, and thus

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l.$$

We have therefore

Theorem 18. *Let the coefficients b_κ be chosen so as to satisfy the conditions* (19). *Then, if $\lim_{n\to\infty} s_n^{(1)} = l$ exists and is finite,*

$$\lim_{x\to 1} \sum_1^\infty a_\kappa x^\kappa = l.$$

The first condition of Theorem 16 is consequently necessary.

15. The proof of the converse begins with equation (20), which is valid since $\lim\limits_{n\to\infty}\frac{q_n}{t_n}=0$. Therefore

$$\lim_{x\to 1}\sum_1^\infty \frac{q_{\kappa+1}-q_\kappa}{t_{\kappa+1}}x^{\kappa+1}=l.$$

This is equivalent to the first condition of Theorem 15. But the second is satisfied too, viz.

$$\lim_{n\to\infty}\frac{1}{t_n}\sum_2^n t_\kappa\frac{q_\kappa-q_{\kappa-1}}{t_\kappa}=\lim_{n\to\infty}\frac{1}{t_n}[(q_2-0)+\dots+(q_n-q_{n-1})]$$

$$=\lim_{n\to\infty}\frac{q_n}{t_n}=0.$$

Thus

$$\lim_{n\to\infty}\sum_1^n\frac{q_{\kappa+1}-q_\kappa}{t_{\kappa+1}}=l,$$

and, by equation (21),

$$\lim_{n\to\infty}s_n^{(1)}=l,$$

which completes the demonstration.

The conditions (14) and (19) imposed on the numbers b_κ are not necessary but only sufficient. The conditions necessary and sufficient would depend also on the coefficients a_κ of the power series considered, so that for a given series $\Sigma\, a_\kappa x^\kappa$ a given set b_κ may be admitted which must be excluded for other series $\Sigma\, c_\kappa x^\kappa$.

PART III.

16. Theorem 2 is in a sense a perfect converse of Abel's theorem, from which all these researches originated.

Series for which Abel's limit exists may be divided into two classes, those which are convergent and those which are divergent, series for which the limit does not exist being excluded. Theorem 2 shows that the first class consists of those, and those only, which satisfy the condition

$$\lim_{n\to\infty}\frac{1}{n}\sum_1^n \kappa a_\kappa=0 \quad \dots\dots\dots\dots\dots\dots\dots(22).$$

The second class consists of those, and those only, which do not satisfy the condition.

The condition (22) is satisfied, in particular, if

$$\lim_{n\to\infty} na_n=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(23)$$

But this condition, unlike (22), is not a *necessary* condition for convergence.

Recent investigators have generalised the condition (23) in a different manner. Thus Mr J. E. Littlewood proved* the theorem:

$$\text{“}\sum_1^\infty a_\kappa \textit{ is convergent, provided } \lim_{x\to 1}\sum_1^\infty a_\kappa x^\kappa = A \textit{ and } |na_n| < K.\text{”}$$

And still more recently Mr G. H. Hardy and Mr J. E. Littlewood† proved

Theorem 19. *If* $\lim_{x\to 1}\sum a_n x^n = A$, *and* $a_n > -\frac{1}{n}K$, *then* $\sum a_n$ *converges to the sum* A.

But however interesting in themselves these two theorems and their proofs may be they are less perfect than Theorem 2. For the conditions $|na_n| < K$ and $na_n > -K$ are neither necessary for convergence nor is either, together with $\lim_{x\to 1}\sum a_\kappa x^\kappa = A$, necessary, nor do they characterise the non-converging series for which Abel's limit exists. Their interest is in fact of a quite different character from that of Theorem 2.

It is not difficult to state similar theorems which are open to the same objection but which give information in cases where the last two theorems fail.

17. The terms a_κ of any sequence can be written in the form $a_\kappa = \frac{c_\kappa}{t_\kappa}$, where t_κ is subject to the same conditions as in Theorem 15. This theorem then shows that

$$\text{“}\sum_1^\infty \frac{c_\kappa}{t_\kappa} \textit{ is convergent, provided } \lim_{x\to 1}\sum_1^\infty \frac{c_\kappa}{t_\kappa} x^\kappa = l \textit{ and } \lim_{n\to\infty}\frac{1}{t_n}\sum_1^n c_\kappa = 0.\text{”}$$

Now the second condition is certainly satisfied if $\lim_{n\to\infty}\sum_1^n c_\kappa$ tends to a limit or oscillates finitely. The only limitation thus imposed upon the order of magnitude of a_κ is that $|c_\kappa| < K$, *i.e.* that the order of $\kappa|a_\kappa|$ does not exceed that of $\frac{\kappa}{t_\kappa}$. Instead of the condition

* J. E. Littlewood, 'The converse of Abel's Theorem on power-series', *Proc. London Math. Soc.*, ser. 2, vol. 9 (1911), p. 438.

† G. H. Hardy and J. E. Littlewood, 'Tauberian theorems concerning power-series and Dirichlet's series whose coefficients are positive', *Proc. London Math. Soc.*, ser. 2, vol. 13 (1914), p. 188. See also E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (Berlin, 1916), pp. 45 *et seq.*: the actual theorem is stated in § 9 and finally proved in § 10 (Die Hardy-Littlewoodsche Umkehrung des Abelschen Stetigkeitssatzes).

$\kappa a_\kappa > -K$ of Theorem 19 we have $\left|\sum_1^n t_\kappa a_\kappa\right| < K$, a condition which allows κa_κ to tend to infinity in either direction.

That such cases exist, in which Σa_κ is convergent, is shown by the fact that

$$\Sigma \frac{e^{\kappa i\phi}}{t_\kappa} \quad (0 < \epsilon < \phi < 2\pi - \epsilon)$$

is convergent if t_κ is any function of κ which tends steadily to infinity with κ.

18. A similar result can be obtained from another theorem of Messrs Hardy and Littlewood, viz.:

Theorem 20. *If $f(x) = \Sigma a_\kappa x^\kappa$ is a power series with positive coefficients, and $f(x) \sim \dfrac{1}{1-x}$ as $x \to 1$, then*

$$\sum_1^n a_\kappa \sim n.*$$

From this theorem it is possible to deduce Theorem 19 (see above) of the same authors.

Now the hypothesis is equivalent to

$$\lim_{x\to 1} (1-x) f(x) = \lim_{x \to 1} \sum_1^\infty (a_\kappa - a_{\kappa-1}) x^\kappa = 1,$$

and the conclusion is

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n a_\kappa = \lim_{n\to\infty} \frac{1}{n} \sum_1^n \left\{ \sum_1^\kappa (a_\lambda - a_{\lambda-1}) \right\} = 1.$$

Thus Theorem 20 is equivalent to

Theorem 21. *If $\lim_{x\to 1} \sum_1^\infty b_\kappa x^\kappa = 1$, and if the sums $s_n = \sum_1^n b_\kappa$ are all positive, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n s_\kappa = 1.$$

Here again is a condition which, in case the series converges, does not prevent the real numbers κb_κ from tending to infinity in both directions.

* G. H. Hardy and J. E. Littlewood, *l.c.* See also E. Landau, *l.c.*, § 9.

Sir George Stokes and the concept of uniform convergence. By
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1. The discovery of the notion of uniform convergence is generally and rightly attributed to Weierstrass, Stokes, and Seidel. The idea is present implicitly in Abel's proof of his celebrated theorem on the continuity of power series; but the three mathematicians mentioned were the first to recognise it explicitly and formulate it in general terms*. Their work was quite independent, and it would be generally agreed that the debt which mathematics owes to each of them is in no way diminished by any anticipation on the part of the others. Each, as it happens, has some special claim to recognition. Weierstrass's discovery was the earliest, and he alone fully realised its far-reaching importance as one of the fundamental ideas of analysis. Stokes has the actual priority of publication; and Seidel's work is but a year later and, while narrower in its scope than that of Stokes, is even sharper and clearer.

My object in writing this note is to call attention to and, so far as I can, explain two puzzling features in the justly famous memoir† in which Stokes announces his discovery. The memoir is remarkable in many respects, containing a general discussion of the possible modes of convergence, both of series and of integrals, far in advance of the current ideas of the time. It contains also two serious mistakes, mistakes which seem at first sight almost inexplicable on the part of a mathematician of so much originality and penetration.

The first mistake is one of omission. It does not seem to have occurred to Stokes that his discovery had any bearing whatever on the question of term by term integration of an infinite series. The same criticism, it is true, may be made of Seidel's paper. But Seidel is merely silent on the subject. Stokes, on the other hand, quotes the false theorem that a convergent series may always be integrated term by term, and refers, apparently with approval, to the erroneous proof offered by Cauchy and Moigno‡.

Of this there is, I think, a fairly simple and indeed a double

* The idea was rediscovered by Cauchy, five or six years after the publication of the work of Stokes and Seidel. See Pringsheim, 'Grundlagen der allgemeinen Funktionenlehre', *Encykl. der Math. Wiss.*, II A 1, § 17, p. 35.

† 'On the critical values of the sums of periodic series', *Trans. Camb. Phil. Soc.*, vol. 8, 1847, pp. 533–583 (*Mathematical and physical papers*, vol. 1, pp. 236–313).

‡ See p. 242 of Stokes's memoir (as printed in the collected papers).

explanation. In the first place it must be remembered that Stokes was primarily a mathematical physicist. He was also a most acute pure mathematician; but he approached pure mathematics in the spirit in which a physicist approaches natural phenomena, not looking for difficulties, but trying to explain those which forced themselves upon his attention. The difficulties connected with continuity and discontinuity are of this character. The theorem that a convergent series of continuous functions has necessarily a continuous sum is one whose falsity is open and aggressive: examples to the contrary obtrude themselves on analyst and physicist alike. The falsity of this theorem Stokes therefore observed and corrected. The falsity of the corresponding theorem concerning integration lies somewhat deeper. It is easy enough, when one's attention has been called to it, to see that the proof of Cauchy and Moigno is invalid. But there are no particularly obvious examples to the contrary: simple and natural examples are indeed somewhat difficult to construct*. And Stokes, his suspicions never having been excited, seems to have accepted the false theorem without examination or reflection.

This is half the explanation. The second half, I think, lies in the distinctions between different modes of uniform convergence which I shall consider in a moment.

Stokes's second mistake is more obvious and striking. He proves, quite accurately, that uniform convergence implies continuity†. He then enunciates and offers a proof‡ of the converse theorem, which is false. The error is not one merely of haste or inattention. The argument is as explicit and as clearly stated in one case as in the other; and, up to the last sentence, it is perfectly correct. He proves that continuity involves *something*, and then states, without further argument, that this something is what he has just defined as uniform convergence. It is merely this last statement that is false.

Stokes's mistake seems at first sight so palpable that I was for some time quite at a loss to imagine how he could have made it. A closer examination of his memoir, and a comparison of his work with other work of a very much later date, has made the lapse a good deal more intelligible to me; and my attempts to understand it have led me to a number of remarks which, although they contain very little that is really novel, are, I think, of some historical and intrinsic interest.

2. There are no less than *seven* different senses, all important, in which a series may be said to be uniformly convergent.

* See Bromwich, *Infinite series*, pp. 116–118; Hardy, 'Notes on some points in the integral calculus', XL, *Messenger of Mathematics*, vol. 44, 1915, pp. 145–149.

† p. 282. I use 'uniform' instead of Stokes's 'not infinitely slow'.

‡ p. 283.

I shall write the series in the form

$$\sum_1^\infty u_n(x);$$

and I shall suppose, for simplicity, that every term of the series is continuous, and the series convergent, for every x of the interval $a \leqslant x \leqslant b$. I shall denote the sum of the series by $s(x)$; and I shall write

$$s_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x), \quad s(x) = s_n(x) + r_n(x).$$

The fundamental inequality in all my definitions will be of the type

$$|r_n(x)| \leqslant \epsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{A}).$$

I shall refer to this inequality simply as (A).

When we define uniform convergence, in one sense or another, we have to choose various numbers in a definite logical order, those which are chosen later being, in general, functions of those which are chosen before. I shall write each number in a form in which all the arguments of which it is a function appear explicitly: thus $n_0(\xi, \epsilon)$ is a function of ξ and ϵ, $n_0(\epsilon)$ one of ϵ alone.

It will sometimes happen that one of the later numbers depends upon several earlier numbers *already connected by functional relations*, so that it is really a function of a selection of these numbers only. Thus δ may have been determined as a function of ϵ; and n_0 may have to be determined as a function of ξ, ϵ, and δ, so that it is in reality a function of ξ and ϵ only. I shall express this by writing

$$n_0 = n_0(\xi, \epsilon, \delta) = n_0(\xi, \epsilon);$$

and I shall use a similar notation in other cases of the same kind.

3. The first three senses of uniform convergence are as follows.

A 1: Uniform convergence throughout an interval. *The series is said to be uniformly convergent throughout the interval* (a, b) *if to every positive* ϵ *corresponds an* $n_0(\epsilon)$ *such that* (A) *is true for* $n \geqslant n_0(\epsilon)$ *and* $a \leqslant x \leqslant b$.

This is the ordinary or 'classical', and most important, sense, the sense in which uniform convergence is defined in every treatise on the theory of series.

A 2: Uniform convergence in the neighbourhood of a point. *The series is said to be uniformly convergent in the neighbourhood of the point* ξ *of the interval* (a, b) *if an interval* $(\xi - \delta(\xi), \xi + \delta(\xi))$* *can be found throughout which it is uniformly convergent; that is to say if a positive* $\delta(\xi)$ *exists such that* (A) *is true for every positive* ϵ, *for* $n \geqslant n_0(\xi, \delta, \epsilon) = n_0(\xi, \epsilon)$, *and for* $\xi - \delta(\xi) \leqslant x \leqslant \xi + \delta(\xi)$.

* A trivial change is of course required in the definition if $\xi = a$ or $\xi = b$. The same point naturally arises in the later definitions.

A 3: Uniform convergence at a point. *The series is said to be uniformly convergent at the point $x=\xi$ (or for $x=\xi$) if to every positive ϵ correspond a positive $\delta(\xi, \epsilon)$ and an $n_0(\xi, \epsilon, \delta)=n_0(\xi, \epsilon)$ such that* (A) *is true for $n \geqslant n_0(\xi, \epsilon)$ and for $\xi-\delta(\xi, \epsilon) \leqslant x \leqslant \xi+\delta(\xi, \epsilon)$.*

4. Before proceeding further it will be well to make a few remarks concerning these definitions and their relations to one another.

The idea of uniform convergence *in the neighbourhood of a particular point* (Definition **A 2**) is substantially that defined by Seidel in 1848*. It is clear, however, that definitions **A 1** and **A 2** were both familiar to Weierstrass as early as 1841 or 1842†. It is obvious that a series uniformly convergent throughout an interval is uniformly convergent in the neighbourhood of every point of the interval. The converse theorem is important and by no means obvious, and was first proved by Weierstrass‡ in a memoir published in 1880. This theorem would now be proved by a simple application of the 'Heine-Borel Theorem', and is a particular case of a theorem which will be referred to in a moment.

Definition **A 3** appears first, in the form in which I state it, in a paper of W. H. Young published in 1903§; but the idea is present in an earlier paper of Osgood‖. The essential difference between definitions **A 2** and **A 3** is that in the latter δ is chosen *after* ϵ and is a function of ξ and ϵ, while in the former it is chosen before ϵ and is a function of ξ alone. In each case n_0 is a function of two independent variables, ξ and ϵ. It is plain that uniform convergence in the neighbourhood of ξ involves uniform convergence at ξ, and at (and indeed in the neighbourhood of) all points sufficiently near to ξ. But uniform convergence at ξ does not involve uniform convergence in the neighbourhood of ξ.

It is important, however, to observe that *uniform convergence at every point of an interval involves uniform convergence throughout the interval.* This important theorem is proved very simply by

* 'Note über eine Eigenschaft der Reihen, welche discontinuirliche Functionen darstellen', *Münchener Abhandlungen*, vol. 7, 1848, pp. 381–394. This memoir has been reprinted in Ostwald's *Klassiker der exakten Wissenschaften*, no. 116. The reference there given to vol. 5, 1847, is incorrect.

† For detailed references bearing on this and similar historical points, see Pringsheim's article already quoted.

‡ See the memoir 'Zur Functionenlehre' (*Abhandlungen aus der Funktionenlehre*, pp. 69–104 (pp. 71–72)).

§ 'On non-uniform convergence and term-by-term integration of series', *Proc. London Math. Soc.*, ser. 2, vol. 1, pp. 89–102.

‖ 'Non-uniform convergence and the integration of series', *American Journal of Math.*, vol. 19, 1897, pp. 155–190. See Prof. Young's remarks on this point at the beginning of his later paper 'On uniform and non-uniform convergence of a series of continuous functions and the distinction of right and left', *Proc. London Math. Soc.*, ser. 2, vol. 6, 1907, pp. 29–51.

Young, in his paper already quoted, by means of the Heine-Borel Theorem*; and it plainly includes, as a particular case, Weierstrass's theorem referred to above.

5. It seems to me that the definition given by Stokes is not any one of **A 1**, **A 2**, **A 3**; and that, if we are to understand him rightly, we must consider another parallel group of definitions. These definitions differ from those given above in that (A) is supposed to be satisfied, not for *all* sufficiently large values of n, but only for *an infinity of* values.

B 1: Quasi-uniform convergence throughout an interval. *The series is said to be quasi-uniformly convergent throughout* (a, b) *if to every positive* ϵ *and every* N *corresponds an* $n_0(\epsilon, N)$ *greater than* N *and such that* (A) *is true for* $n = n_0(\epsilon, N)$ *and* $a \leqslant x \leqslant b$.

B 2: Quasi-uniform convergence in the neighbourhood of a point. *The series is said to be quasi-uniformly convergent in the neighbourhood of* ξ *if an interval* $(\xi - \delta(\xi), \xi + \delta(\xi))$ *can be found throughout which it is quasi-uniformly convergent; i.e., if a positive* $\delta(\xi)$ *exists such that* (A) *is true for every positive* ϵ, *every* N, *an* $n_0(\xi, \delta, \epsilon, N) = n_0(\xi, \epsilon, N)$ *greater than* N, *and* $\xi - \delta(\xi) \leqslant x \leqslant \xi + \delta(\xi)$.

B 3: Quasi-uniform convergence at a point. *The series is said to be quasi-uniformly convergent for* $x = \xi$ *if to every positive* ϵ *and every* N *correspond a positive* $\delta(\xi, \epsilon, N)$ *and an*

$$n_0(\xi, \epsilon, \delta, N) = n_0(\xi, \epsilon, N),$$

greater than N, *such that* (A) *is true for* $n = n_0(\xi, \epsilon, N)$ *and for* $\xi - \delta(\xi, \epsilon, N) \leqslant x \leqslant \xi + \delta(\xi, \epsilon, N)$.

Definition **B 1** is to be attributed to Dini or to Darboux†. Another form of it has been given by Hobson‡. As Arzelà and Hobson§ have pointed out, a series is quasi-uniformly convergent throughout an interval if, and only if, it can be made uniformly convergent by an appropriate bracketing of its terms.

Definition **B 2** is for us at the moment of peculiar interest, for (as I shall show in a moment) it is really *this* definition that is given by Stokes.

Definition **B 3** is also of great interest, both in itself and in

* Choose ϵ and determine $\delta(\xi, \epsilon)$ and $n_0(\xi, \epsilon)$, as in definition **A 3**, for every ξ of the interval. Every point of (a, b) is included in an interval $(\xi - \delta, \xi + \delta)$. By the Heine-Borel Theorem, every point of (a, b) is included in one or other of a finite sub-set of these intervals. If $N(\epsilon)$ is the largest of the n_0's corresponding to each of the intervals of this finite sub-set, then (A) is true for $n \geqslant N$ and $a \leqslant x \leqslant b$.

This is the essence of the proof, though, like all proofs of the same character, it requires a somewhat more careful statement if all appearance of dependence upon Zermelo's *Auswahlsprinzip* is to be avoided.

† See Pringsheim, *l. c.*

‡ 'On modes of convergence of an infinite series of functions of a real variable', *Proc. London Math. Soc.*, ser. 2, vol. 1, 1903, pp. 373–387. Hobson (following Dini) uses the expression 'simply uniformly'.

§ *L. c.*, p. 375.

relation to Stokes's memoir. For *the necessary and sufficient condition that $s(x)$ should be continuous for $x = \xi$ is that the series should be quasi-uniformly convergent for $x = \xi$.* This theorem is in substance due to Dini*. I give the proof, as it is essential for the criticism of Stokes's memoir.

(1) *The condition is sufficient.* For

$$|s(x) - s(\xi)| \leqslant |s_n(x) - s_n(\xi)| + |r_n(x)| + |r_n(\xi)|.$$

Choose ϵ, N, $\delta(\xi, \epsilon, N)$, and $n = n_0(\xi, \epsilon, N)$ as in definition **B 3**. Then $|r_n(x)| < \epsilon$ for $\xi - \delta \leqslant x \leqslant \xi + \delta$. Now that n is fixed we can choose δ_1 less than δ and such that $|s_n(x) - s_n(\xi)| < \epsilon$ for $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$. And thus

$$|s(x) - s(\xi)| < 3\epsilon$$

for $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$, so that $s(x)$ is continuous for $x = \xi$.

It is plain that this argument proves, *a fortiori*, that **A 2**, **A 3**, and **B 2** all furnish sufficient conditions for continuity at a point, and **A 1** and **B 1** sufficient conditions for continuity throughout an interval.

(2) *The condition is necessary.* For

$$|r_n(x)| \leqslant |s(x) - s(\xi)| + |r_n(\xi)| + |s_n(x) - s_n(\xi)|.$$

Suppose that ϵ and N are given. Then we can choose $\delta(\xi, \epsilon)$ so that $|s(x) - s(\xi)| < \epsilon$ for $\xi - \delta \leqslant x \leqslant \xi + \delta$, and $n_0(\xi, \epsilon, N)$ so that $n_0 > N$ and $|r_{n_0}(\xi)| < \epsilon$. And, when n_0 has thus been fixed, we can choose $\delta_1(\xi, \epsilon, n_0) = \delta_1(\xi, \epsilon, N)$ so that $\delta_1 < \delta$ and $|s_{n_0}(x) - s_{n_0}(\xi)| < \epsilon$ for $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$. Thus $|r_n(x)| < 3\epsilon$ for $n = n_0 > N$ and $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$, so that the series is quasi-uniformly convergent for $x = \xi$.

6. If a series is uniformly convergent at every point ξ of an interval, it is (as we saw in § 4) uniformly convergent throughout the interval: definition **A 3** (and *a fortiori* definition **A 2**) passes over, in virtue of the Heine-Borel Theorem, into definition **A 1**. It is important to observe that this relation does not hold between **B 3** (or **B 2**) and **B 1**: a series quasi-uniformly convergent at every point of an interval (or in the neighbourhood of every such point) is not necessarily quasi-uniformly convergent throughout the interval. We can apply the Heine-Borel Theorem in the manner indicated in the first sentences of the footnote * to p. 152; but the last stage of the argument, in which every one of a finite number of different integers is replaced by the largest of them, fails. What we obtain is *the necessary and sufficient condition that $s(x)$ should be continuous throughout the interval*; and this is not

* *Fondamenti...*, p. 107 (German translation, *Grundlagen...*, pp. 143–145).

the condition **B 1** but a condition first formulated by Arzelà*, viz.:

C: Quasi-uniform convergence by intervals (*convergenza uniforme a tratti*). *The series is said to be quasi-uniformly convergent by intervals if to every positive* ϵ *and every* N *correspond a division of* (a, b) *into a finite number* $\nu(\epsilon, N)$ *of intervals* $\delta_r(\epsilon, N)$, *and a corresponding number of numbers* $n_r(\epsilon, N)$, *all greater than* N, *and such that* (A) *is true for* $n = n_r$ $(r = 1, 2, \ldots, \nu)$ *and all values of* x *which belong to* δ_r.

The deduction of Arzelà's criterion from **B 3**, in the manner sketched above, was first made by Hobson†.

There is one further point which seems worth noticing here, although it is not directly connected with Stokes's memoir. Dini‡ proved that *if* $u_n(x) \geqslant 0$ *for all values of* n *and* x, *and* $s(x)$ *is continuous throughout* (a, b), *then the series is uniformly convergent throughout* (a, b). This theorem is now almost intuitive. For it is obvious that, for series of positive terms, quasi-uniform convergence in any one of the senses **B 1**, **B 2**, or **B 3** involves uniform convergence in the corresponding sense **A 1**, **A 2**, or **A 3**. If then $s(x)$ is continuous throughout (a, b) it is continuous for every ξ of (a, b); and therefore the series is quasi-uniformly convergent for every ξ; and therefore uniformly convergent for every ξ; and therefore uniformly convergent throughout (a, b).

7. Let us now consider Stokes's definitions and proofs in the light of the preceding discussion.

It is clear, in the first place, that Stokes has in his mind some phenomenon characteristic of *a small, but fixed, neighbourhood of a point.*

'Let $u_1 + u_2 + \ldots$ (66)', he says§, 'be a convergent infinite series having U for its sum. Let $v_1 + v_2 + \ldots$ (67) be another infinite series of which the general term v_n is a function of the positive variable h and becomes equal to u_n when h vanishes. Suppose that *for a sufficiently small value of* h *and all inferior values* the series (67) is convergent, and has V for its sum. It might at first sight be supposed that the limit of V for $h = 0$ was necessarily equal to U. This however is not true....

'THEOREM. The limit of V can never differ from U unless the convergency of the series (67) becomes infinitely slow when h vanishes.

* 'Sulle serie di funzioni', *Memorie di Bologna*, ser. 5, vol. 8, 1900, pp. 131–186, 701–744.

† *L. c.*, pp. 380–382.

‡ *L. c.* (German edition), pp. 148–149. See also Bromwich, *Infinite series*, p. 125 (Ex. 3).

§ p. 279.

'The convergency of the series is here said to become infinitely slow when, if n be the number of terms which must be taken in order to render the sum of the neglected series numerically less than a given quantity e, which may be as small as we please, n increases beyond all limit as h decreases beyond all limit.

'DEMONSTRATION. If the convergency do not become infinitely slow it will be possible to find *a number n, so great that for the value of h we begin with and for all inferior values greater than zero the sum of the neglected terms shall be numerically less than e....*'

Stokes's words, and in particular those which I have italicised, seem to me to make two things perfectly clear.

(1) Stokes is considering neither a property of an interval (a, b) *im Grossen* (such as is contemplated in **A 1** or **B 1**), nor a property of a single point which (as in **A 3** or **B 3**) need not be shared by any neighbouring point, but a property of an interval *im Kleinen*, that is to say *a small but fixed interval chosen to include a particular point.* His definition is therefore one of the type of **A 2** or **B 2**.

Stokes's failure to perceive the bearing of his discovery on problems of integration is made much more natural when we realise that he is considering throughout a neighbourhood of a point and not an interval *im Grossen.* And this remark applies to Seidel as well.

(2) Stokes is considering an inequality satisfied for a special value of n, or at most an infinite sequence of values of n, and *not* necessarily for all values of n from a certain point onwards. In this respect there is a quite sharp distinction between Stokes's work and Seidel's. What Stokes defines is (to use the language of this note) a mode of *quasi-uniform* convergence and not one of strictly uniform convergence.

It seems to me, then, that what Stokes defines is what I have called *quasi-uniform convergence in the neighbourhood of a point* (**B 2**).

8. If we adopt this view, Stokes's mistake becomes very much more intelligible. He proves, quite correctly, that uniform convergence in his sense implies continuity: his proof, stated quite formally and by means of inequalities, is substantially that given in § 5, under (1). He then continues* as follows.

'Conversely, if (66) is convergent, and if $U = V_0$†, the convergency of the series (67) cannot become infinitely slow when h

* p. 282. The italics are mine.

† V_0 is what Stokes calls 'the value of V for $h=0$', by which he means, of course, its limit when h tends to 0.

vanishes. For if U_n', V_n' represent the sums of the terms after the nth in the series (66), (67) respectively, we have

$$V = V_n + V_n',\ U = U_n + U_n';$$

whence

$$V_n' = V - U - (V_n - U_n) + U_n'.$$

Now $V - U$, $V_n - U_n$ vanish with h, and U_n' vanishes when n becomes infinite. Hence *for a sufficiently small value of h and all inferior values, together with a value of n sufficiently large and independent of h,* the value of V_n' may be made numerically less than any given quantity e however small; *and therefore, by definition, the convergency of the series* (67) *does not become infinitely slow when h vanishes.*'

Now this argument is, until we reach the last sentence, perfectly accurate, and indeed, if we translate it into inequalities, substantially identical with that given in § 5, under (2). Stokes proves, in fact, that continuity at ξ involves quasi-uniform convergence at ξ. Where he falls into error is simply in his final assertion that this property is that which he has previously defined, the mistake being due to a failure to observe that his intervals of values of h depend upon a prior choice of ϵ. In a word, he confuses, momentarily, **B 2** and **B 3**. The ordinary view that Stokes defined uniform convergence in the same sense as Weierstrass compels us to suppose that he confused **B 3** with **A 1**, or at any rate with **A 2**: and this is hardly credible.

I add one final remark. If we could identify Stokes's idea with **B 3**, instead of with **B 2**, we could acquit him of having made any mistake at all, since **B 3** really is a necessary and sufficient condition for continuity. We could then regard Stokes as having anticipated Dini's theorem. This view, however, does not seem to me to be tenable.

Shell-deposits formed by the flood of January, 1918. By PHILIP LAKE, M.A., St John's College.

[*Read* 18 February 1918.]

The heavy snow of the third week in January 1918 was followed by a very rapid thaw and a considerable fall of rain, and the Cam, in consequence, rose to an exceptional height. In the neighbourhood of Cambridge the floods were the most extensive of recent years, the water reaching its highest level on Sunday, Jan. 20.

The traces of the flood remained visible for several weeks, its limits being marked in most places by straws, twigs, silt, etc., with a sprinkling of land and fresh-water shells. But below the town, near the railway-bridge, the shells were so abundant as to form a remarkable deposit, which seems to deserve a special record. It was not till the 25th Jan. that I saw it, and the following notes are drawn up from the observations made on that day and on two or three subsequent visits.

The deposit lay partly upon the tow-path and partly in the shallow ditch on the inner side of the path, and it extended with little interruption from the immediate neighbourhood of the 'Pike and Eel' to a point about 350 yards below the railway-bridge, a total distance of approximately 850 yards. Occasional patches occurred still farther down, and scattered shells even as far as Ditton Corner. Beyond Ditton the tow-path was in several places covered with a thick layer of silt, but I saw no more shells until within sight of the lock at Baitsbite.

The deposit was somewhat irregular and it was difficult to form an estimate of its average width, but this can hardly have been less than a foot, and was probably much more.

Above the railway-bridge the shells were mixed with silt, especially in the ditch on the inner side of the path; but even here the proportion of shells was large, and in places they formed the bulk of the deposit. Below the railway-bridge the deposit was free from silt and consisted entirely of shells. In the shallow hollows formed by the irregularities of the surface, it was often an inch or two deep, so that it was possible to scoop up the shells by the handful. Owing to its colour it showed conspicuously as light streaks upon the slightly darker path.

By far the greater part of the deposit consisted of *Limnaea*, *L. stagnalis* and *L. peregra* being the most abundant species; but other fresh-water shells also occurred and land-snails were by no means rare. Mr C. E. Gray, of the Sedgwick Museum, went down shortly after my first visit, and in a very short time obtained most of the following species, but a few names have been added to the list from specimens collected subsequently:

Sphaerium corneum (L.),
Bithynia tentaculata (L.),
Vivipara contecta (Millet),
Valvata piscinalis (Müller),
Limnaea stagnalis (L.),
 ,, *peregra* (Müller),
 ,, *auricularia* (L.),
Planorbis corneus (L.),
 ,, *umbilicatus* Müller,
 ,, *carinatus* Müller,
 ,, *vortex* (L.),
 ,, *contortus* (L.),
Physa fontinalis (L.),
Helix nemoralis L.,
Theba cantiana (Mont.),
Hygromia striolata (Pfr.),
Vitrea draparnaldi (Beck),
 ,, *cellaria* (Müller).

Even now the list is probably far from complete, and a closer examination would no doubt reveal the presence of many other forms.

The last five species are land-shells, and, with the exception of *Vitrea cellaria*, they occurred in Mr Gray's first collection and were identified by Mr Hugh Watson. *Vitrea draparnaldi* does not appear to be a native of the county, but is found in and near greenhouses; for instance, in the Botanical Gardens. In Mr Gray's first collection, which was made below the railway-bridge, it was represented only by a single specimen, which we supposed to have come from the florist's greenhouses close by. But at a later date he found it to occur abundantly at the beginning of the tow-path, some five or six hundred yards above the greenhouses. In order to make sure that the specimens really belong to this species they were sent to Mr Watson, who agreed with the identification.

Since there were so many specimens of *Vitrea draparnaldi* at the beginning of the tow-path, and so few (at least comparatively) below the railway-bridge, it seems clear that they cannot have been carried far, for otherwise they would have been more evenly distributed. It is most probable indeed that there was a colony of this species in the immediate neighbourhood. The nearest greenhouse that I have been able to find above the locality where the species was so abundant is five or six hundred yards off, and stands well away from the river. The specimens can hardly have come from there, and it is more likely that the colony lived out of doors and nearer to the river. Nevertheless its progenitors may have been 'escapes'. The greenhouses below the railway-bridge have now been out of use for some time, and the snails that were in them must have been forced to seek new quarters.

Most of the shells, both land and fresh-water, were perfect or nearly so, and all of them were empty. Neither Mr Gray nor myself found a single specimen with any remains of its former inhabitant. The greater number were very fresh in appearance, but some of the land-shells had evidently been exposed to the weather for some time, and some of the fresh-water shells had lain in the mud long enough to become discoloured or incrusted as if the process of fossilization had begun. The specimens of *Vitrea draparnaldi*, it may be noted, were all fresh-looking.

Apart from the extent of the shelly deposit, its freedom from silt below the railway-bridge was perhaps its most important feature, for it shows that even a muddy river like the Cam may produce a purely calcareous deposit.

The fact that the shells were all empty indicates that those belonging to the river must have lain in its bed for some time; and in this connection an observation made by Mr Gray is of interest. Some years ago at Bottisham, when dredging operations were going on, he noticed that the mud brought up by the dredger was full of fresh-water shells.

During floods the river digs up its bed and, as on the occasion here described, it may deposit the shells in one place and the silt in another. In the case of an artificially controlled stream like the Cam, floods are comparatively rare; but in an unrestrained river we may reasonably expect them to be both more numerous and more extensive. It seems quite possible therefore that neither the clayey fresh-water limestones of the Wealden nor the purer fresh-water limestones of the Purbeck series required lagunary conditions for their formation.

Is the Madreporarian Skeleton an Extraprotoplasmic Secretion of the Polyps? By G. MATTHAI, M.A., Emmanuel College, Cambridge. (Communicated by Professor Stanley Gardiner.)

[*Read* 18 February 1918.]

In 1881 von Heider (**5**) suggested that the calcareous skeleton of the Madreporaria is formed by the deposition of carbonate of lime within certain specialised ectodermal cells (calicoblasts*) constituting an outer layer, and repeated this conclusion in a subsequent paper (**6**). In 1882 von Koch (**8**) inferred from embryological observations that the skeleton is deposited outside the living tissues, i.e. is extraprotoplasmic in origin. In 1896 Ogilvie (**9**) supported von Heider's view and argued that, by repeated calcification of "cells" of the calicoblastic layer of ectoderm, successive strata of calcareous "scales" are formed, and slightly modified her opinion in 1906 (**10**). Fowler (**4**) had previously accepted von Koch's view. In 1899 Bourne (**2**), from his studies on the Anthozoan skeleton, supported von Koch's conclusions and entirely disagreed with von Heider and Ogilvie. He further held that, whilst in *Heliopora* and the Madreporaria the corallum is formed outside the living calicoblastic layer, the spicules of the Alcyonaria are formed within certain ectodermal cells or scleroblasts which either remain in the ectoderm or wander into the mesoglæa (**2**, p. 506). Following von Koch and Bourne, it is now generally believed that the Madreporarian skeleton is an extraprotoplasmic formation and that Alcyonarian spicules are entoplastic products.

After a ground-down section of an Astræid corallite has been slowly decalcified on a slide, somewhat homogeneous organic remains (distinguishable from algal filaments penetrating the skeleton) are left which react to any of the common stains. This is clear indication that the calcareous matter has been deposited in an organic matrix. Bourne regards this matrix as due to the "disintegration of calicoblasts" (**2**, pp. 520 and 521, fig. 21), assuming that the organic basis was not part of the living calicoblastic ectoderm. His view is that carbonate of lime is secreted by the calicoblastic layer and is passed through its outer border (the "limiting membrane") into the decaying part outside, exactly as the Alcyonarian spicule is "from its early origin, separated from the protoplasm which elaborated the material necessary for its further growth by a layer of some cuticular material" (**2**, p. 537),

* Von Heider's original rendering of this word is *chalicoblast*, of which the first half, I am informed, is derived from the Greek χάλιξ, which in Roman characters should be spelt *chalix*. Subsequently, Fowler changed the spelling to *calycoblast*, and in 1888 both this author and Bourne adopted the present form *calicoblast*.

viz., the spicule-sheath. At the same time, Bourne contends that the spicule is entoplastic in formation whilst the Madreporarian corallum is exoplastic. To be consistent, both the spicule and the corallum would have to be regarded as formed either within living protoplasm or outside it, but spicules could not be viewed as intraprotoplasmic products whilst assuming the extraprotoplasmic origin of the corallum.

Duerden (**3**) held that the organic basis of the corallum of *Siderastrea radians* is a "secretion" of the calicoblastic layer of ectoderm to which it is closely adherent (pl. 8, fig. 45) and is "a homogeneous, mesoglæa-like matrix within which the minute calcareous crystals forming the skeleton are laid down" (p. 34). Since he refers to the skeleton as "ectoplastic" in origin (p. 113), it is evident that he agrees with Bourne in the view that the organic matrix was not part of the living tissues when calcareous matter began to be deposited in it. But in the account of these authors there is no more evidence to show that, in the Madreporaria, the organic ground substance or "colloid matrix" (**2**, p. 539) was non-living at every phase of skeleton formation than that the areas of the scleroblasts of the Alcyonaria in which the deposition of spicular matter took place had not, at least at the initial stages of this process, formed part of the living protoplasm.

Further if, in the Madreporaria, the calcareous matter were deposited outside the living calicoblastic ectoderm, it is difficult to understand how the manifold patterns of coralla so characteristic of this group of organisms can have been built up*. But if the matrix in which carbonate of lime is laid down is part of the living calicoblastic sheet, it follows that the protoplasm must regulate the arrangement of the calcareous matter into the various skeletal types which, in large measure, maintain their respective form independent of changes in environmental conditions. Similarly, the formation of the various kinds of spicules of the Alcyonaria can be adequately explained only if calcareous deposition takes place within living protoplasm, and indeed, Bourne has drawn attention to the phenomenon that "the spicules of the Alcyonaria show a definite and complex crystalline structure, the details of which are, indeed, moulded upon and dominated by an equally complex organic matrix..." (**2**, p. 517).

The intraprotoplasmic origin of spicules in the Alcyonaria might, without difficulty, be ascertained since sections can be made without decalcification, whereas in *Heliopora* and the Madreporaria possessing massive coralla, satisfactory sections are possible only after decalcification, and in this condition the skeleton may appear

* In explanation of this phenomenon, Bourne suggests that "the general arrangement of the fasciculi of crystals is dominated, in some manner of which we are ignorant, by the living tissues which clothe the corallum" (**2**, p. 539).

as though formed outside the living tissues. A further difficulty with regard to the Madreporaria is that, except perhaps at the growing points, the skeleton would secondarily lose its intraprotoplasmic character and appear to be external to the living tissues by having displaced most of the protoplasm in which it was deposited, just as the discrete condition of fully developed Alcyonarian spicules is due to the increase of calcareous matter at the expense of the protoplasm in which it was formed.

From the above considerations it would appear to be highly probable that von Heider was right in regarding the Madreporarian skeleton as formed within the calicoblastic protoplasm. Bourne directs much of his criticism to von Heider's suggestion that the striæ in the calicoblastic layer (i.e., in the processes of attachment) are calcareous fibres, but it is not improbable that, in the undecalcified condition, some of these processes of attachment might be partially calcified.

When thin sections of Astræid coralla are examined under a microscope, they frequently appear to consist of calcareous pieces united by sutures resembling the "laminæ" or "trabeculæ" of the skeleton of *Heliopora* (**1**, p. 463, pl. 11, figs. 7 and 8) and the "trabecular parts" of the Madreporarian skeleton as figured by Ogilvie (**9**, p. 124, figs. 13, 19, etc.). Each piece is composed of calcareous strands radiating from a dark centre or line which, as Ogilvie suggested, appears to be the organic remains of the protoplasm in which the calcareous needles were laid down. There is some similarity between these elements and the spicules of *Tubipora* (**7**, figs. 9 and 10) which, according to Hickson, are not fused together but dovetailed into one another as in the membrane bones of Mammals (p. 562). The resemblance is also marked in the case of the scale-like spicules of *Plumarella* (**2**, figs. 6 and 7) containing dark centres from which calcareous fibres or rods radiate.

It is difficult to gather from Bourne's account what he considers to be the unit of skeletal structure in the Alcyonaria. Are spicules such units*? But spicules are not all homologous elements since they are formed in protoplasmic areas containing one or more nuclei and no limit can be set to their size in the various genera (**2**, pp. 508–517), an extreme case being the scale-like spicules of *Primnoa* and *Plumarella*, each of which is "formed by several cells, or at least by a comparatively large cœnocytial investment containing many nuclei" (p. 510). Or, is a spicule a calcareous piece which behaves like a single crystal when examined under crossed Nicols? The same confusion prevails with regard to skeletal units in the Madreporaria—whether they are represented by "fibro-crystals" (Bourne), "crystalline sphæroids" (von Koch) or

* Bourne applies the term spicule to "an entoplastic product *of a single cell or of a cœnocyte*" (**2**, p. 504). The italics are mine.

"calcareous scales" (Ogilvie). The latter are not calcified calicoblastic "cells" as Ogilvie contended since the calicoblastic ectoderm is now found to be a multinucleated sheet of protoplasm devoid of cell-limits, i.e., a syncytium.

In fact, there is hardly any evidence to show that the skeleton of the Anthozoa is made up of homologous units just as it is highly doubtful if their soft parts are composed of uninucleated units or cells. The significance of the Anthozoan skeleton would consist in its probable formation within syncytial protoplasm according to physical laws under the presiding activity of the living protoplasm which would direct the complex skeletal architecture. The calcareous deposit further appears to be differentiated into elements which remain separate as spicules in most Alcyonarians but are united to form a compact skeleton in certain Alcyonarians, e.g., *Tubipora, Corallium, Heliopora*, and in all the Madreporaria (in which the calcareous matter may undergo subsequent rearrangement). From this point of view, a separate calcareous piece of an Alcyonarian might be regarded as a diminutive corallum, and the corallum of a Madreporarian as a massive spicule, and finally, the formation of the Anthozoan skeleton would be essentially similar to the formation of membrane bone in Vertebrates*.

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* It is interesting to note that structures analogous to fibrous connective tissue, tendon and bone of Vertebrates, occur in the Madreporaria, viz., the middle lamina (=mesoglæa), processes of attachment and the calcareous corallum, a matter which will be discussed in a future communication.

On Reactions to Stimuli in Corals. By G. MATTHAI, M.A., Emmanuel College, Cambridge. (Communicated by Professor Stanley Gardiner.)

[*Read* 18 February 1918.]

The following is a brief record of feeding-experiments made on living Astræid colonies during a short stay at the Carnegie Biological Station at Tortugas (July 16—Aug. 2) and at the Bermuda Biological Station on Agar's Island (Aug. 20—Sep. 14) in the summer of 1915, which, though necessarily incomplete as they had to be undertaken in the midst of other work, gave some indication of the nature of reactions to stimuli in the Madreporaria. In order to watch the behaviour of living Corals, colonies of most of the recent species recorded from those localities were kept in aquaria of running sea-water, viz.:

Mœandra labyrinthiformis (Linn.), *Mœandra strigosa* (Dana), *Mœandra clivosa* (Ell. and Sol.), *Manicina areolata* (Linn.), *Colpophyllia gyrosa* (Ell. and Sol.), *Isophyllia dipsacea* (Dana), *Isophyllia fragilis* (Dana), *Dichocœnia Stokesi*, Ed. and H., *Eusmilia aspera* (Dana), *Favia fragum* (Esp.), *Orbicella cavernosa* (Linn.), *Orbicella annularis* (Ell. and Sol.), *Stephanocœnia intersepta* (Esp.), *Oculina diffusa*, Lam., *Mycetophyllia lamarckana*, Ed. and H., *Siderastræa radians* (Pallas), *Siderastræa siderea* (Ell. and Sol.), *Agaricia purpurea*, Les., *Porites astreoides*, Lam., *Porites furcata*, Lam., *Porites clavaria*, Lam., *Madracis decactis* (Ly.), and *Acropora muricata* (Linn.).

In *Isophyllia dipsacea* (Dana), when a particle of meat was placed on the oral disc with contracted mouths, the oral lip was slowly directed towards the particle and the mouth became dilated, to an extent depending on the size of the food-particle. The latter was, in the meantime, slowly moved into the oral opening by ciliary action. To facilitate this event, the periphery of the oral disc was drawn over towards the dilated mouth and the disc itself was somewhat depressed, thus deepening the peristomial cavity. During distention of the mouth, the stomodæum was everted and, consequently, the cœlenteric cavity with its convolutions of mesenteries became exposed. After the food-particle had passed into the cœlenteric cavity, it was caught in the mesenterial coils. If the fragment of meat was large, the mouth remained widely open till the former had been reduced in size by the digestive action of the mesenterial filaments. The stomodæum was subsequently withdrawn and the mouth opening gradually narrowed. But if, before this, the oral lip was touched with a glass needle, it did not contract as it would do instantaneously if no food-particle had previously been swallowed. Every mouth that was tested could thus take in particles of meat. The touch of the food-particle on the oral disc was also a stimulus for the expansion of the tentacles around the mouth and of those around the neighbouring oral openings.

When a particle of meat was placed on the tentacles of a colony of *Mœandra labyrinthiformis* (Linn.), it was slowly passed on to the oral disc, but the tentacles did not show any sign of contraction. At the same time, the oral disc was depressed and arched over the mouth opening till finally its margin closed over the peristome. In the meantime, the tentacles were fully distended, the entocœlic ones were directed obliquely towards the oral opening, those of one side passing between those of the opposite side. The food-particle was now hidden from view. After it had passed into the cœlenteric cavity and had presumably undergone partial digestion, the periphery of the oral disc gradually moved outwards carrying the tentacles with it, thus again exposing the peristomial cavity.

The principal movements in these two cases are:

(1) Ciliary movement passing the food-particle into the nearest oral aperture.

(2) The direction of the oral lip towards the food-particle *pari passu* with the dilatation of the mouth.

(3) The narrowing and deepening of the peristomial cavity, which help to roll the food-particle into the oral opening.

(4) The expansion of the tentacles of the affected oral disc and of those of adjacent oral discs.

(5) The eversion of the stomodæum and consequent exposure of the cœlenteric cavity and mesenterial coils.

(6) The return of the soft parts to their original condition by the retraction of the stomodæum into the cœlenteric cavity, recoil of the oral lip to its normal extent, shortening of the tentacles, flattening of the oral disc and withdrawal of its periphery carrying the tentacles outwards.

When a drop of meat-juice was gently placed on a colony of *Favia fragum* (Esp.), the oral apertures in the neighbourhood were slowly distended after a short pause. The inner or entocœlic row of tentacles was then extended and directed over the oral disc, meeting or intercrossing over the mouth as had been noticed in the case of *Mœandra labyrinthiformis* (Linn.), thus hiding the oral region, whilst the exocœlic tentacles were arched outwards. Similar movements were observed in *Mœandra strigosa* (Dana).

When meat-juice was spurted by a pipette on sea-water containing a colony of *Orbicella cavernosa* (Linn.), strong contraction of the soft parts was set up in the neighbourhood, the polyps entirely closing up. This was followed by the protrusion of convolutions of mesenteries through mouth openings, oral discs and especially through edge-zones, combined with secretion of mucus over the polyps, the former obviously to paralyse prey and the latter to entangle food-particles. Shortly afterwards, the oral apertures were widely distended to let in the meat-juice but the process was unaccompanied by eversion of stomodæa. Similar events were observed in *Manicina areolata* (Linn.).

When finely powdered carmine was scattered in sea-water containing a colony of *Manicina areolata* (Linn.), it was partly taken into the stomodæa, the oral lips becoming conspicuously stained. The carmine was, however, subsequently passed out of the stomodæa, showing thereby, that the mouth openings could function as inhalent and exhalent apertures.

When a tentacle of any of the Astræid colonies was touched with a fine glass needle, it was suddenly withdrawn in a manner resembling pseudopodial movement and the neighbouring tentacles were also retracted. In *Porites* and *Madracis*, whose soft parts are composed of small polyps, the instantaneous contraction of a polyp due to mechanical stimulation caused the contraction of its neighbours as well. In all these cases, the wave of contraction started from a centre, viz., the point of stimulation, but remained local and did not spread over the entire colony.

Series of movements such as the above, made in response to chemical and tactile stimuli, are reminiscent of amœboid or streaming movement of protoplasm, the soft parts of the colonies appearing to serve as the medium for the transmission of stimuli*. If the initial stimulus be too strong, the sudden contraction of the soft parts, due to the mechanical impact, is followed by slow purposive movements.

The amœboid character of the movements of the soft parts of Astræid Corals is in conformity with their histological structure which, on examination, revealed neither a muscular nor a nervous system, although a neuro-muscular apparatus has been supposed by most authors to exist in Madreporaria. The so-called muscular fibres at the base of the ectoderm and endoderm seem to be of the nature of specialised connective tissue fibres, for in both teased preparations and in sections of 4μ—10μ thicknesses these are found to be without nuclei and to form part of the middle lamina (= mesoglæa) which is itself composed of fine fibres cemented together by a homogeneous matrix containing a few scattered nucleated cells. Fibrils pass into the middle lamina through the granular stratum present at the base of the ectoderm (and less frequently at the base of the endoderm), but these fibrils do not show any histological differentiation which would justify us in regarding them as belonging to nerve elements†.

* Carpenter regarded the feeding reactions of *Isophyllia* as muscular in nature and as brought about by the transmission of impulses of a "nervoid character," but he had not investigated the histological structure of its soft parts (*vide Contributions Bermuda Biol. Station*, No. 20, Cambridge, Mass., U.S.A., p. 149, 1910).

† For a detailed account of the minute structure of coral polyps *vide* "The Histology of the Soft Parts of Astræid Corals" to be published shortly.

Notes on certain parasites, food, and capture by birds of the Common Earwig (Forficula auricularia). By H. H. BRINDLEY, M.A., St John's College.

[*Read* 18 February 1918.]

(*a*) *Effects of parasitism.*

In a paper entitled "The effects of Parasitic and other kinds of castration in Insects" (*Jour. Exper. Zool.* VIII. Philadelphia, 1910) Wheeler expresses the opinion (p. 419) that Giard has given good reasons for supposing that the dimorphism exhibited by the forcipes of male earwigs from the Farn Islands, Northumberland (Bateson and Brindley, "On some cases of variation in secondary sexual characters statistically examined," *Proc. Zool. Soc. Lond.* 1892, p. 585), is due to "differences in the number of gregarines they harbour in their alimentary tract." The reference to Giard is *C.R. Acad. Sci.* CXVIII. 1894, p. 872, where he writes "J'ai tout lieu de croire qu'une interprétation du même genre (referring to the changes evoked in *Carcinus* by the action of parasites) peut s'appliquer pour la distribution des longueurs des pinces des Forficules mâles. Il est possible, en effet, d'après la longueur de la pince, de prévoir qu'une Forficule mâle possède des Grégarines et qu'elle en possède une plus ou moins grande quantité."

In criticism of the above statements Capt. F. A. Potts and myself published a letter in *Science*, Philadelphia, Dec. 9, 1910, p. 836, in which we gave reasons for disagreeing with Wheeler's conclusion: viz., (i) that in the absence of any further account by Giard the above passage could not be taken as direct evidence that he had examined the intestine of *Forficula* for gregarines and found a correspondence between their presence and the condition of the male forcipes; (ii) that out of several thousand earwigs collected by us on the Farn Islands in 1907 over 50 males of different forceps lengths were carefully dissected with the results that the gregarine *Clepsydrina ovata* was found to occur commonly in the alimentary canal, that it occurred indifferently and was absent indifferently in "low" and "high" males, and that no correlation could be traced between the number of parasites and the length of its forcipes. Moreover, no difference in the development of the testes or other internal sexual organs could be detected in low and high males respectively.

Since the above was written I have (August 1917) examined the alimentary canal of 51 earwigs out of a large batch obtained at Porthcressa, St Mary's, Isles of Scilly, where the males exhibit well-marked dimorphism (*Camb. Phil. Soc. Proc.* XVII. part 4, 1914, p. 331).

The results summarised are as follows:

Infection by *Clepsydrina ovata.*

	Number examined	Not infected	Infected	Number of gregarines found	Average number of gregarines in the infected individuals
Low males	23	12	11	323	29
High males	23	11	12	238	20
Females	5	1	4	53	13

Thus the evidence so far obtained is that the dimorphism of the forcipes in *F. auricularia* ♂ is not a result of or influenced by gregarine infection—though in view of the well-established effects of such parasitism on the secondary sexual characters of another arthropod in Geoffrey Smith's case of *Inachus dorsettensis* modified by the gregarine *Aggregata* (*Mitt. Zool. Stat. Neap.* XVII. 1905, p. 406), the absence of positive evidence to the contrary at the time Wheeler wrote, but now obtained, certainly afforded ground for his support of Giard.

In this connection I may quote a letter from Geoffrey Smith, whose recent death at the battle front brings us into common mourning with Oxford zoologists for a friend and colleague. Writing to me about 1907 he said, "Have you noticed that Giard attributes all cases of High and Low Dimorphism to parasitic castration? I am sure this is not right, but there is no doubt that parasitic castration is a much more frequent occurrence than is commonly supposed." These words, and a footnote to the same effect in his paper "High and Low Dimorphism" (*Mitt. Zool. Stat. Neap.* XVII. 1905, p. 321), are typical of the writer's insight and balanced judgment.

It may be stated that the gregarines in the Porthcressa earwigs fell roughly into categories of small, medium, and large, but they all seemed to be *C. ovata.* Rather more than half were small individuals, and those of medium size were slightly in excess of the large, but the sizes were not recorded in the case of the first few earwigs examined. Very large numbers were found in syzygy, and such associated individuals were of all three sizes. One instance of syzygy of a large with quite a small individual was observed. There was no noteworthy difference between the numbers of gregarines of different sizes or between the proportion of free gregarines to those in syzygy in their low and high male hosts respectively.

During our stay on the Scilly Islands in 1912 Capt. Potts and myself, in company with Capt. J. T. Saunders, found in St Martin's several earwigs parasitised by a gordiid larva (*sp. incert.*), the coils of which, though projecting between the terga of the abdomen, seemed to have no effect on the health and activity of their hosts. The same apparent absence of deleterious effects was noticed in three of the Porthcressa batch of 1917 which were found to be similarly infected. In one, a low male, a large gordiid occupied most of the body, and no portion of the alimentary canal posterior to the crop could be found; in a high male similarly infested by a large gordiid there was very little of the hind gut left; and an adult female contained three or four gordiids of various sizes, the gut in this case being intact and apparently healthy. A fourth individual, a low male, was not parasitised when examined, but as the gut was partially atrophied, it had probably been recently deserted by a gordiid. All these infected individuals seemed as active and healthy and to possess fat bodies as large as those not infected; the earwig's resistance to such extensive destruction of internal organs is very noteworthy. As *Clepsydrina ovata* inhabits the chylific ventricle and hind gut and as the presence of gordiids evidently often results in destruction of these portions of the alimentary tract, the latter parasite is likely to be exclusive of gregarines, and these were absent in all three of the males mentioned above (including that with the hind gut intact), while only two were found in the female.

That the presence of parasitic worms has sometimes serious effects on the insect's health is suggested by the recent observations of Jones recorded in "The European Earwig and its control," a report on the invasion of Newport, R.I., in 1911 by *Forficula auricularia* and its subsequent spread (*U.S. Dept. Agric. Bull.* 566, Washington, June, 1917), from which it appears that 10 per cent. of earwigs kept in the laboratory were killed by the infection of a worm identified as *Filaria locustae*, whose average length is given as 83 mm. This however is a size exceeding considerably that of the gordiids in the Scilly earwigs, which I have called "large" when attaining a length of 50 mm.

In southern Russia *Forficula tomis*, Kolenati, is parasitised by the tachinid fly, *Rhacodineura antiqua* (Pantel, *Bull. Soc. Entom. France*, No. 8, Paris, 1916, p. 150), but I do not know if it attacks the common earwig. The paper quoted mentions the capture of the adult fly in Holland and Portugal.

Lucas (*Entom.* XXXVII. 1904, p. 213) reports *F. auricularia* (or ? *lesnei*) attacked by scarlet acarine mites.

Among fungoid parasites, *Entomophthora forficulae* diminishes the number of earwigs (Picard, *Bull. Soc. Étude Vulg. Zool. Agric. Bordeaux*, Jan.—April, 1914, pp. 1, 25, 37, 62). It is possibly this species which has caused heavy mortality among the earwigs which I have kept in captivity in the Zoological Laboratory during recent years. Infection by the above or other fungus is a very frequent result of damp in the soil or in the plaster of Paris cells bedded with coco fibre which I have employed. The most effective preventive of fungus has so far been keeping the earwigs in roomy glass dishes lined with virtually dry sand and supplying water only by wetting the vegetable food given.

(*b*) *Food.*

In "The Wild Fauna and Flora of the Royal Botanic Gardens, Kew," 1906 (*Kew Bull. Add. Series V*), Lucas writes (p. 23) of the Common Earwig, "It is an animal feeder. Does it do as much damage as is supposed?" And Ealand in "Insects and Man," 1915, p. 266, states "most gardeners would assert that the insect is destructive to cultivated plants. Careful observation and experiment, however, show that it is carnivorous and that it devours caterpillars, snails, slugs, etc.... its habit of hiding in such flowers as the sunflower and dahlia have earned it an undeserved reputation for evil."

I find that seven out of nine recent and more or less comprehensive manuals of Economic Entomology do not mention earwigs at all, which is fair evidence for considerable doubt as to their being harmful insects. Of the two works in which earwigs are mentioned one speaks of them as destructive to mangolds, turnips, cabbage crops, and plant blossoms, while the other states dahlias as attacked, "but nearly all plants suffer." Virtually every fruit grower and horticulturist of whom we make enquiry assures us that earwigs are most destructive pests, but is the general belief thus expressed really well founded?

Recent literature leaves the impression that in certain localities earwigs may be specially harmful to plants of economic value, though an explanation of this capriciousness is wanting. Theobald (*Rep. on Econ. Zool.*, South-Eastern Agric. Coll., Wye, April 1914) gives hops as attacked by *F. auricularia*. Lind and others in a summary of the diseases of agricultural plants in 1913 (79 *Beretning fra Statens Forsögsvirksamded i Plantekultur*, no. 30, Copenhagen, 1914) state that in one locality in Denmark cauliflowers were completely destroyed by the Common Earwig, which seems a very exceptional event. Schøyen in *Beretning om skadeinsekter og plantesygdommer i land og havebruket* 1915 (Report on the injurious insects and fungi of the field and the orchard in 1916), Kristiania, 1916, mentions that in many parts of Norway different vegetables, cabbage in particular, were extensively damaged by *F. auricularia*. Tullgren, in a report on injurious animals in Sweden during 1912—1916 (*Meddelande från Centralanstalten för Jorsbruksförsök*, no. 152; *Entomologiska Avdelningen*, no. 27, p. 104), records damage by *F. auricularia* to ornamental plants, barley, wheat, and cabbage. In the case of the invasion of Newport, R.I., by the Common Earwig, Jones (*op. cit.*) reports that the quite young individuals eat tender shoots of clover and grass, and possibly grass roots; while later on shoots of Lima Bean and dahlia and blossoms of Sweet William and early roses are attacked, with a general preference for the bases of petals and stamens rather than for green shoots. Adults are recorded as feeding almost wholly on petals and stamens, though clover, grass and terminal buds of chrysanthemums and other "fall flowers" are also devoured. Sopp, "The Callipers of Earwigs" (*Lancs. and Ches. Entom. Soc. Proc.* 1904, p. 42), records having seen a female earwig using her forcipes to repeatedly pierce damp decaying seaweed on which she was apparently feeding. Lüstner (*Centralbl. Bakt. Parasit. u. Infektionskrankheiten*, XL. nos. 19–21, Jena, April 1914, p. 482) has summarised the work of over thirty observers of the contents of the crop of the Common Earwig. Altogether 162 individuals were thus examined, and the conclusion was arrived at that earwigs normally feed on dead portions of plants and on fungi such as *Capnodium*, living leaves and flowers being attacked when circumstances favoured the change. Dahlia leaves and petals were very readily devoured. How far earwigs are a pest to ripe fruit seems not to have been investigated, but it was concluded that as a rule they may be regarded as harmless save in special cases. It was admitted however that the further the enquiry went the less definite were the results.

In view of the diversity of reports as to the favourite food plants of earwigs and the general want of exact information as to the damage likely to be done by earwigs in a flower or kitchen garden I carried out a small series of observations on the earwigs obtained last August from St Mary's, Isles of Scilly, which were kept in captivity in the Zoological Laboratory for some weeks, primarily for the purpose of examining their alimentary canal for parasites. These earwigs, several dozen in number, were kept in a large glass dish bedded with sand slightly damped occasionally. They had no animal food save that afforded by those which died. In order to obtain information as to preference for one kind of plant above another they were given three different species, taken haphazard, at a time for a period of two days or more.

A summary of the results is as follows:—

Aug. 20 and 21. Vegetable marrow leaves were very much eaten; horse-radish leaves very little touched; Michaelmas Daisy leaves and flowers hardly, if at all, touched.

Aug. 22 and 23. Beetroot leaves were much eaten, the leaf stalks in particular, these being opened out and the pith taken: white phlox leaves and flowers, the petals much gnawed and pollen grains were found in the gut: dwarf bean leaves, little touched.

Aug. 24 to 26. Blue Anchusa leaves and flowers, the petals were much eaten but the leaves neglected: white rose leaves and flowers, petals devoured but leaves untouched: golden rod (*Solidago*) leaves and flowers, leaves nibbled at sides here and there but flowers apparently neglected.

Aug. 27 to 29. Yellow Oenothera flowers and pods, the petals were much eaten but the pods remained untouched: white Japanese anemone leaves and flowers, petals eaten to some extent, leaves neglected: raspberry foliage, the leaves were not nibbled, but the earwigs congregated in numbers on their hairy undersides, an action much more pronounced than in the case of any of the other plants given throughout the observations.

Aug. 30 and 31. Cabbage leaves were destroyed by the blade being gnawed down between the veins to the midrib, while the ends of the veins were shorn off: rhubarb leaves, eaten a good deal: scarlet runner leaves, flowers, and pods, apparently quite neglected.

Sept. 1 to 3. Plum fruit unskinned was much attacked: potato tuber and rather unripe apple, both unskinned, were not touched at all.

Sept. 4 to 10. On the 4th the plum was removed, but the apple and potato were not attacked during the seven days.

Sept. 11 to 15. On the 11th the apple was cut across, with the result that it was slightly gnawed during the five days: the potato remained untouched.

Sept. 16 to 23. On the 16th the potato was cut across, which was followed by its being very thoroughly attacked, though the apple was not entirely deserted.

Of the 51 earwigs whose alimentary canals were examined for gregarine 7 contained spores of *Puccinea graminis* (one had as many as 180 and another 100), while the food of another individual included numerous unidentified entomophilous pollen grains. Both spores and pollen grains appeared to be very slightly if at all digested. It is hoped to extend the observations in the coming summer, as those recorded above were limited to only a few of the possible food plants and only adult earwigs were kept. It may well be that there are differences in the preferences of nymphs and adults, and as the former are in the majority till about the end of July, it is possible that they may be harmful to certain plants in particular, as Jones's observations (*op. cit.*) suggest.

It seems established that a large number of ordinary garden species are liable to serious attack by earwigs, and that the latter can continue healthy on a purely vegetable diet. But much further information of a detailed kind is required before we can explain why in a given locality a particular kind of plant is attacked while in another it is neglected. Does it mean that the presence or absence of suitable animal food is a factor?

As regards animal food, there is a considerable amount of evidence that earwigs are often carnivorous by choice, very possibly they are so usually (cf. Rühl, *M.T. Schweiz. Ges.* VII. 1887, p. 310). In respect of eating dead animal matter I have found that when kept in captivity they devour the soft parts of their fellows who have died even when fresh vegetable food is available. In this necrophagous habit they resemble cockroaches. Jones (*op. cit.*) states that dead flies and dead or dying comrades are devoured. Lüstner (*op. cit.*) finds that only dead animal matter is taken. This conclusion points to too limited an inquiry and want of taking into account the possible presence of food plants which were more attractive than available living prey. In any case his opinion that earwigs should not be regarded as beneficial is traversed by the records of their killing certain insect pests of plants.

Round Island, the northernmost islet of the Scilly group, is swarming with earwigs, and they congregate in vast numbers in the light-keepers' midden inside the discarded pressed beef tins. If, as seems probable, they reached the islet before the lighthouse was built a change of diet seems to have occurred, as the indigenous vegetation is chiefly *Armeria maritima*, *Cochlearia officinalis* and *Mesembryanthemum edule.* There is no turf. It is of course possible that they seek the potato peelings also thrown into the midden and that their numbers inside the discarded tins mean that the latter are frequented partly for shelter. If the Round Island earwigs have really turned during comparatively recent years from a herbivorous to an extensively carnivorous diet, Rosevear, another islet of the Scilly group may, in a sense, be a converse case. It is the other locality in the Scilly group in which (as far as I know) the earwig population is densest. Like Round Island, it is very small, but differs from it in being uninhabited. But from 1850 to 1858 it was occupied by the builders of the Bishop Rock Lighthouse, so is it possible that the abundance of earwigs is due to the animal food available in the past? However this may be the present diet of the Rosevear earwigs appears likely to be vegetarian in the main, unless the islet harbours some insect or other small arthropod suitable for food. The commonest plants are *Armeria maritima* and *Lavatera arborea*, the latter growing luxuriously. But before the abundance of earwigs on Rosevear can be discussed adequately something must be known of the conditions obtaining on Rosevean and Gorregan, its small and only immediate neighbours. Of these islets I possess no information at present. Also, there are other peculiarities as regards the earwigs of Rosevear and Round Island which are beyond the scope of the present paper.

There is no doubt that earwigs sometimes kill and devour other insects larger than themselves, though the event is probably somewhat exceptional. Chapman ("Notes on Early Stages and Life History of the Earwig," *Entom. Record*, XXIX. no. 2, Jan. 1917) states that "animal food, such as dead insects, seemed always acceptable" to earwigs in captivity. Sopp (*op. cit.* p. 42) regards earwigs as probably "omnivorous feeders, largely carnivorous by choice, but often phytophagous, frugivorous, or even necrophagous of necessity." Whether attack on living animals as prey is common I cannot say, I have no observations of my own to record; but it appears that occasionally the forcipes, organs of much disputed function, are used for this purpose. Sopp (*op. cit.*) has seen them employed to seize and crush large flies which were subsequently devoured and quotes an instance of a larva similarly attacked from the records of another observer. Burr (*Entom. Record*, Sept. 1903) saw a blue-bottle seized by the forcipes of a male *Labidura riparia* kept in captivity. Lucas (*Entom.* XXXVIII. 1905, p. 267) records a female of this species as using the forcipes to capture a cinnabar moth larva, which was afterwards devoured. Jones (*op. cit.*) records that the Newport, R.I., earwigs attack and devour "certain sluggish unprotected larvae."

There are many observations which show that earwigs in some localities prey upon small insect larvae, and in certain instances they have been recommended as a means of diminishing plant pests. Thus the following references, as also others quoted in this paper, have appeared in issues of *The Review of Applied Entomology*, 1913—1918. Bernard (*Technique des traitements contre les Insectes de la Vigne*, Paris, 1914) states that they devour the pupae of one or more of *Clysia ambiguella*, *Polychrosis botrana*, and *Sparanothis pilleriana* (*v.* also Kirkaldy, *Entom.* XXXIII. 1900, p. 87). Dobrodeev (*Mem. Bur. Entom. of Cent. Board of Land Administration and Agric.*, Petrograd, XI. no. 5, 1915) makes a similar report as regards the destruction of the first two Tortricidae named above by earwigs. Molz (*Zeits. Angewandte Chemie*, Leipzig, XXVI. nos. 77, 79, 1913, pp. 533, 587) speaks of earwigs as natural enemies of the vine moth. Feytaud (*Bull. Soc. Étude Vulg. Zool. Agric. Bordeaux*, XV. nos. 1—8, Jan.—Aug. 1916, pp. 1, 21, 43, 52, 65, 88) states that earwigs destroy the eggs and larvae of the coccid vine pests *Eulecanium persica* and (probably) *Pulvinaria vitis*. Harrison in "An unusual parsnip pest" (*Entomologist*, XLVI. Feb. 1913, p. 59) reports them as most effective in killing and eating *Depressaria heracliana*, the "parsnip web-worm." Brittain and Gooderham (*Canad. Entom.*, London, Ont., XLVII. no. 2, Feb. 1916, p. 37) make a similar statement.

There is no doubt that our knowledge of the bionomics of the earwig is at present very imperfect. As in the case of other very common animals far too much has been taken for granted. The earwig's nocturnal habit, its tendency to assemble in great numbers between two closely apposed surfaces, and its "frightening attitude" of flexing its abdomen dorsalwards with opened forcipes all tend to give it a reputation for evil which very probably is but partially deserved. We all know how the habit of entering crevices is responsible for the belief that it gnaws through the tympanic membrane with the result of mania or even death. *Perce-oreille* speaks for itself. It seems fairly established that its universally bad reputation among gardeners is founded on tradition and want of judgment combined with neglect of the increasing evidence that its presence is sometimes beneficial by its destructiveness to more harmful insects than itself. That it eats the petals of dahlias and chrysanthemums to some extent is true, but as far as my own observations go the outlay of time and material devoted to the traditional protection of the flowers by inverted flower pots stuffed with straw seems hardly worth while. The great attraction which the flowers have for earwigs seems to be the closeness and number of their petals, which provide a daytime shelter whence nightly excursions for feeding are made. Anyone possessing a garden may greatly add to our knowledge of favourite foods; observation at night is particularly needed. As regards garden varieties of roses the case against earwigs is probably more severe.

(*c*) *Capture by birds.*

During the last decade systematic investigation of the contents of the alimentary canal of British wild birds by several observers has resulted in most useful information as to which should be regarded as harmful and which as neutral or beneficial to agriculture. It is manifest from the laborious and painstaking work now at our disposal that many of the reputations, good or evil, which certain common birds have in the eyes of farmers and gardeners need considerable revision, in some cases even reversal.

As regards the capture of earwigs by birds, it appears that they are not a favourite food when we bear in mind how numerous they are sometimes and that they are large enough to be easily seized. No doubt their nocturnal habit affords much protection from capture.

Collinge in "The Food of some British Wild Birds" (London, 1913) reports on the contents of the crop, etc., of 29 of the commonest species, among which only four contained earwigs, and these were very few in number. Thus in 404 House Sparrows 2 earwigs were found, 1 in each of 2 birds; in 721 Rooks 2 earwigs were found, 1 in each of 2 birds; in 40 Skylarks 3 earwigs were found among 2 birds; in 64 Song Thrushes 7 earwigs were found among 2 birds.

Newstead in "The Food of some British Birds" (*Supp. to Journ. of Board of Agric.* no. 9, Dec. 1908) records observations on the swallowed food of 128 species, the outcome of 871 post-mortem and pellet examinations carried out in various years from 1894 to 1908. He finds that 10 species had eaten earwigs, the numbers of birds examined and the numbers of earwigs found being: 1 Whimbrel, 40 earwigs; 2 Green Woodpeckers, 24 earwigs; 2 Starlings, 3 earwigs; 1 Nuthatch, 3 earwigs; 1 Chaffinch, 1 Great Titmouse, 1 Redbreast, 1 Song Thrush, 1 Whinchat, 1 Woodcock, 1 earwig each.

Theobald and McGowan in "The Food of the Rook, Chaffinch and Starling" (*Supp. to Journ. of Board of Agric.* no. 15, May 1916) put on record a particularly valuable and interesting series of observations, as they examined the food month by month during nearly $2\frac{1}{2}$ years, viz., from Jan. 1912 to May 1914, the inquiry covering 277 Rooks, 748 Starlings, and 527 Chaffinches. An analysis of their results as regards earwigs for the $2\frac{1}{2}$ years is as follows:

		Birds examined	Earwigs found	Average number of earwigs taken by each bird
April to September	Starling	372	154	·41
	Chaffinch	277	7	·025
	Rook	121	3	·025
October to March	Starling	376	199	·53
	Chaffinch	248	5	·020
	Rook	156	3	·019

I have divided the year into two periods of six months conformably with the seasonal presence or absence of earwigs on the surface of the ground. From October to March most male earwigs die and the females are hibernating. In view of this it is curious that earwigs should be taken as numerously during this period as during the six months when both nymphs and adults can be found easily. The numbers recorded for Rook and Chaffinch are small, though a large number of birds were examined. The Starling is a great insect eater; is it possible that it habitually searches for buried insects during the colder months and devours earwigs found with the rest? This action may be true for the other two birds also. The figures for all three are certainly curious.

So we find only 13 species of birds reported as having captured earwigs, and most of them as very sparingly. The Starling is not recorded by Collinge as an earwig eater.

The above quoted reports certainly suggest that wild birds cannot be relied upon to diminish earwigs in a garden. Many of the most insectivorous are not reported as feeding upon earwigs at all. They may be distasteful, and a large number together emit a well-defined odour, and the same is true of a number preserved in alcohol. Be this as it may, domestic fowls always eat them readily, a fact which is noted by Jones (*op. cit.*) in the case of the invasion of Newport, R.I. He also mentions that toads will eat them.

Miss Maud D. Haviland, Hon. Mem. B.O.U., to whom I am indebted for assistance with regard to the literature of the subject and for kind advice in the preparation of these notes, informs me that she has noticed a Redbreast take earwigs in preference to earthworms.

ADDENDA.

Under (*b*).

Mr H. Ling Roth informs me that he has found earwigs very destructive to iris pods, with resulting premature fall of seeds, in a garden at Halifax, Yorks.

Under (*c*).

Gurney, in "Ornithological Notes from Norfolk for 1916" (*British Birds*, x. 1917, p. 242), records that his father in October, 1843, found several earwigs in a Stone Curlew.

Reciprocal Relations in the Theory of Integral Equations. By Major P. A. MacMahon and H. B. C. Darling.

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1. Let
$$\int_{a_1}^{b_1} f_1(x)\,\kappa(xt)\,dx = \psi_1(t)$$
and
$$\int_{a_2}^{b_2} f_2(x)\,\kappa(xt)\,dx = \psi_2(t);$$
then, if we suppose the functions f_1, f_2 and κ to be such that the order of integration is indifferent, we have
$$\int_{a_1}^{b_1} f_1(x)\,\psi_2(xt)\,dx = \int_{a_2}^{b_2} dy \int_{a_1}^{b_1} f_1(x) f_2(y)\,\kappa(xyt)\,dx$$
$$= \int_{a_2}^{b_2} f_2(y)\,\psi_1(yt)\,dy,$$
or, as it may be written,
$$\int_{a_1}^{b_1} f_1(x)\,\psi_2(xt)\,dx = \int_{a_2}^{b_2} f_2(x)\,\psi_1(xt)\,dx \quad \text{.........(1).}$$

In the *Messenger of Mathematics*, May 1914, p. 13, Mr Ramanujan has employed this result to deduce a number of interesting relations between definite integrals. The method is very suggestive and appears capable of considerable extension. For example, if
$$\left.\begin{aligned} &\int_{a_1}^{b_1} f_1(x)\,\kappa\{\theta(x,t)\}\,dx = \psi_1(t) \\ \text{and}\quad &\int_{a_2}^{b_2} f_2(x)\,\kappa\{\theta(x,t)\}\,dx = \psi_2(t) \end{aligned}\right\} \quad \text{...............(2),}$$
then
$$\int_{a_1}^{b_1} f_1(x)\,\psi_2\{\theta(x,t)\}\,dx = \int_{a_2}^{b_2} f_2(x)\,\psi_1\{\theta(x,t)\}\,dx \quad \text{...(3),}$$
provided that
$$\theta\{x, \theta(y,t)\} = \theta\{y, \theta(x,t)\} \quad \text{..................(4).}$$

The functional equation (4) is satisfied by
$$\theta(x,t) = \phi^{-1}\{f(x) + \phi(t)\} \quad \text{..................(5),}$$
where f and ϕ are arbitrary functions; which is a general form of solution and includes among others such solutions as
$$\theta(x,t) = \phi^{-1}\{f(x)\,.\,\phi(t)\} \quad \text{..................(6),}$$
$$\theta(x,t) = \phi^{-1}\left\{\frac{f(x)\,\phi(t) + 1}{f(x) + \phi(t)}\right\} \quad \text{...............(7),}$$
$$\theta(x,t) = \phi^{-1}\{f(x) + \phi(t) + f(x)\,\phi(t)\} \quad \text{.........(8).}$$

Thus, to derive (7) from (5) let

$$f(x) = \coth^{-1}\{F(x)\}, \quad \phi(t) = \coth^{-1}\{\phi_1(t)\}:$$

then (5) becomes

$$\phi^{-1}[\coth^{-1}\{F(x)\} + \coth^{-1}\{\phi_1(t)\}].$$

Now let $\phi^{-1}(z) = u,$

then $z = \phi(u) = \coth^{-1}\{\phi_1(u)\},$

whence $\phi_1(u) = \coth z,$

and $u = \phi_1^{-1}(\coth z);$

that is $\phi^{-1}(z) = \phi_1^{-1}(\coth z),$

and therefore (5) reduces to

$$\phi_1^{-1}\left\{\frac{F(x)\,\phi_1(t)+1}{F(x)+\phi_1(t)}\right\},$$

which is of the form (7).

As an example of the use of (2) and (3) in the determination of relations between integrals, let

$$f_1(x) = \sin x, \quad f_2(x) = \cos x,$$

and, using the form (6) for θ, let

$$\theta(x, t) = e^{x \log t},$$

and $\kappa(x) = x.$

Then, putting $b_1 = b_2 = a, \quad a_1 = a_2 = 0,$

we have from (2)

$$\psi_1(t) = \int_0^a \sin x \,.\, e^{x \log t}\,dx$$

$$= \frac{(\log t \,.\, \sin a - \cos a)\, e^{a \log t} + 1}{1 + (\log t)^2},$$

and

$$\psi_2(t) = \int_0^a \cos x \,.\, e^{x \log t}\,dx$$

$$= \frac{(\log t \,.\, \cos a + \sin a)\, e^{a \log t} - \log t}{1 + (\log t)^2}.$$

Substituting these values in (3), and then putting $\log t = 1/r$ for brevity, we obtain

$$r\left[\int_0^a \frac{\{x \sin(x-a) + r\cos(x-a)\}\, e^{ax/r}}{r^2 + x^2}\,dx - \int_0^a \frac{x \sin x + r\cos x}{r^2 + x^2}\,dx\right] = 0;$$

so that, provided r is not zero, we have

$$\int_0^a \frac{\{x \sin(x-a) + r\cos(x-a)\}\, e^{ax/r}}{r^2+x^2}\, dx$$

$$= \int_0^a \frac{x\sin x + r\cos x}{r^2+x^2}\, dx \quad\ldots\ldots\ldots\ldots\ldots (9),$$

an identity which may be verified by differentiation with respect to a. Putting $x = r\tan\xi$ and then replacing ξ by x, (9) becomes

$$\int_0^{\tan^{-1} a/r} \frac{\cos(x+a-r\tan x)}{\cos x}\, e^{a\tan x}\, dx$$

$$= \int_0^{\tan^{-1} a/r} \frac{\cos(x - r\tan x)}{\cos x}\, dx \quad\ldots\ldots (10),$$

which admits of ready verification by differentiation with respect to a. The identities (9) and (10) hold generally, provided that the constants are finite; we have seen that r must not be zero. It will be noticed that both (9) and (10) are of the form

$$\int_0 \chi(x, a)\, dx = \int_0 \chi(x, 0)\, dx,$$

where the upper limits of integration involve a.

2. As another illustration of how the method admits of generalisation, let

$$\int_{a_1}^{b_1} f_1(x)\, \kappa\{\theta(x, t)\}\, dx = \psi_1(t).$$

and

$$\int_{a_2}^{b_2} f_2(x)\, \kappa\{\theta(x, t)\}\, dx = \psi_2(t);$$

then

$$\int_{a_1}^{b_1} f_1(x)\, \psi_2\{\lambda(x, t)\}\, dx = \int_{a_2}^{b_2} f_2(x)\, \psi_1\{\lambda(x, t)\}\, dx$$

when

$$\lambda(x, t) = \phi_2^{-1}\{f(x) + \phi_1(t)\}$$

and

$$\theta(x, t) = g\{f(x) + \phi_2(t)\},$$

f, g, ϕ_1 and ϕ_2 being any functions. It should be observed that λ becomes θ when $\phi_1 = \phi_2$ and $g = \phi_2^{-1}$. Other corresponding pairs of functions are

$$\lambda(x, t) = \phi_2^{-1}\{f(x) \,.\, \phi_1(t)\},$$

$$\theta(x, t) = g\{f(x) \,.\, \phi_2(t)\},$$

and

$$\lambda(x, t) = \phi_2^{-1}\left\{\frac{f(x)\,\phi_1(t) + 1}{f(x) + \phi_1(t)}\right\},$$

$$\theta(x, t) = g\left\{\frac{f(x)\,\phi_2(t) + 1}{f(x) + \phi_2(t)}\right\}.$$

3. A further extension is obtained when the kernel κ includes more than one parameter t; thus let

$$\int_{a_1}^{b_1} f_1(x)\,\kappa\,\{\theta(x, t_1, t_2)\}\,dx = \psi_1(t_1, t_2),$$

$$\int_{a_2}^{b_2} f_2(x)\,\kappa\,\{\theta(x, t_1, t_2)\}\,dx = \psi_2(t_1, t_2),$$

so that

$$\int_{a_1}^{b_1} f_1(y)\,\kappa\,[\theta\,\{y, \mu(x, t_1, t_2), \nu(x, t_1, t_2)\}]\,dy$$
$$= \psi_1\,\{\mu(x, t_1, t_2), \nu(x, t_1, t_2)\}$$

and

$$\int_{a_2}^{b_2} f_2(y)\,\kappa\,[\theta\,\{y, \mu(x, t_1, t_2), \nu(x, t_1, t_2)\}]\,dy$$
$$= \psi_2\,\{\mu(x, t_1, t_2), \nu(x, t_1, t_2)\}.$$

Now consider

$$\int_{a_1}^{b_1} f_1(x)\,\psi_2\,\{\mu(x, t_1, t_2), \nu(x, t_1, t_2)\}\,dx$$
$$= \int_{a_1}^{b_1} f_1(x)\left(\int_{a_2}^{b_2} f_2(y)\,\kappa\,[\theta\,\{y, \mu(x, t_1, t_2), \nu(x, t_1, t_2)\}]\,dy\right)dx.$$

This double integral is equal to

$$\int_{a_2}^{b_2} f_2(x)\left(\int_{a_1}^{b_1} f_1(y)\,\kappa\,[\theta\,\{y, \mu(x, t_1, t_2), \nu(x, t_1, t_2)\}]\,dy\right)dx,$$

if $\quad \theta\,\{y, \mu(x, t_1, t_2), \nu(x, t_1, t_2)\} = \theta\,\{x, \mu(y, t_1, t_2), \nu(y, t_1, t_2)\}.$

Now suppose

$$\mu(x, t_1, t_2) = \phi_1^{-1}\,\{f(x) + \phi_1(t_1) + \phi_1(t_2)\},$$
$$\nu(x, t_1, t_2) = \phi_2^{-1}\,\{f(x) + \phi_2(t_1) + \phi_2(t_2)\},$$
$$\theta(x, t_1, t_2) = \phi^{-1}\,\{2f(x) + \phi_1(t_1) + \phi_2(t_2)\};$$

then $\quad \theta\,\{y, \mu(x, t_1, t_2), \nu(x, t_1, t_2)\}$

$$= \phi^{-1}\,\{2f(y) + 2f(x) + \phi_1(t_1) + \phi_1(t_2) + \phi_2(t_1) + \phi_2(t_2)\}.$$

This is symmetrical in x and y, so that we may write

$$\mu(x, t_1, t_2) = \phi_1^{-1}\,\{f_3(x) + \phi_3(t_1, t_2)\},$$
$$\nu(x, t_1, t_2) = \phi_2^{-1}\,\{f_4(x) + \phi_4(t_1, t_2)\},$$
$$\theta(x, t_1, t_2) = g\,\{f_3(x) + f_4(x) + \phi_1(t_1) + \phi_2(t_2)\},$$

leading to

$$g\,\{f_3(y) + f_4(y) + f_3(x) + f_4(x) + \phi_3(t_1, t_2) + \phi_4(t_1, t_2)\},$$

which is symmetrical in x and y; and hence it follows that

$$\int_{a_1}^{b_1} f_1(x)\,\psi_2\{\mu(x, t_1, t_2),\ \nu(x, t_1, t_2)\}\,dx$$

$$= \int_{a_2}^{b_2} f_2(x)\,\psi_1\{\mu(x, t_1, t_2),\ \nu(x, t_1, t_2)\}\,dx.$$

As a particular case we may write

$$\mu(x, t_1, t_2) = \phi_1^{-1}\{f(x) + \phi_1(t_1) + \phi_1(t_2)\},$$

$$\nu(x, t_1, t_2) = \phi_2^{-1}\{f(x) + \phi_2(t_1) + \phi_2(t_2)\},$$

$$\theta(x, t_1, t_2) = g\{2f(x) + \phi_1(t_1) + \phi_2(t_2)\},$$

and again

$$\mu(x, t_1, t_2) = \phi^{-1}\{\tfrac{1}{2}f(x) + \phi(t_1) + \phi(t_2)\},$$

$$= \nu(x, t_1, t_2) = \phi^{-1}\{\tfrac{1}{2}f(x) + \phi(t_1) + \phi(t_2)\},$$

$$\theta(x, t_1, t_2) = \phi^{-1}\{f(x) + \phi(t_1) + \phi(t_2)\},$$

the case where $\mu \equiv \nu$ and each resembles θ as much as possible. It is evident that the case in which the kernel includes any number of parameters may be treated in the same manner and presents little difficulty.

4. The method may also be extended to double integrals. Thus let

$$\int_{a_1}^{b_1}\int_{a_1'}^{b_1'} f_1(x, y)\,\kappa\{\theta(x, y, t_1, t_2)\}\,dx\,dy = \psi_1(t_1, t_2),$$

$$\int_{a_2}^{b_2}\int_{a_2'}^{b_2'} f_2(x, y)\,\kappa\{\theta(x, y, t_1, t_2)\}\,dx\,dy = \psi_2(t_1, t_2);$$

then
$$\int_{a_1}^{b_1}\int_{a_1'}^{b_1'} f_1(x, y)\,\psi_2\{\mu(x, y, t_1, t_2), \nu(x, y, t_1, t_2)\}\,dx\,dy$$

$$= \int_{a_2}^{b_2}\int_{a_2'}^{b_2'} f_2(x, y)\,\psi_1\{\mu(x, y, t_1, t_2), \nu(x, y, t_1, t_2)\}\,dx\,dy$$

if
$$\theta\{z, w, \mu(x, y, t_1, t_2), \nu(x, y, t_1, t_2)\}$$

$$= \theta\{x, y, \mu(z, w, t_1, t_2), \nu(z, w, t_1, t_2)\}.$$

If A, B, C, D, E be functional symbols, one solution is

$$\mu(x, y, t_1, t_2) = A^{-1}\{B(x, y) + C(t_1, t_2)\}$$

$$\nu(x, y, t_1, t_2) = D^{-1}\{B(x, y) + E(t_1, t_2)\}$$

$$\theta(x, y, t_1, t_2) = B(x, y) + \tfrac{1}{2}A(t_1) + \tfrac{1}{2}D(t_2).$$

5. Let us next consider the case of three integral equations

$$\int_{a_1}^{b_1} f_1(x)\,\kappa\{\theta(x, t)\}\,dx = \psi_1(t),$$

$$\int_{a_2}^{b_2} f_2(x)\,\kappa\{\theta(x, t)\}\,dx = \psi_2(t),$$

$$\int_{a_3}^{b_3} f_3(x)\,\kappa\{\theta(x, t)\}\,dx = \psi_3(t).$$

We have

$$\left.\begin{aligned}&\int_{a_1}^{b_1} f_1(x)\,\psi_2\{\theta(x, t)\}\,\psi_3\{\theta(x, t)\}\,dx\\ &=\int_{a_2}^{b_2} f_2(x)\,\psi_3\{\theta(x, t)\}\,\psi_1\{\theta(x, t)\}\,dx\\ &=\int_{a_3}^{b_3} f_3(x)\,\psi_1\{\theta(x, t)\}\,\psi_2\{\theta(x, t)\}\,dx\end{aligned}\right\}\quad\ldots\ldots\ldots(11),$$

if certain conditions are satisfied. For

$$\int_{a_1}^{b_1} f_1(x)\,\psi_2\{\theta(x, t)\}\,\psi_3\{\theta(x, t)\}\,dx$$

$$=\int_{a_1}^{b_1} f_1(x)\int_{a_2}^{b_2} f_2(y)\,\kappa\{\theta(y, t)\}\,dy\int_{a_3}^{b_3} f_3(z)\,\kappa\{\theta(z, t)\}\,dz\,dx,$$

and the equalities (11) will hold good if, for example, $\kappa(x) = x^s$ and

$$\theta\{y, \theta(x, t)\}\,.\,\theta\{z, \theta(x, t)\}$$

is unaltered by the circular substitution (xyz).

Now suppose that

$$\theta(x, t) = f(x)\,t^{\frac{1}{2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12);$$

then

$$\begin{aligned}\theta\{y, \theta(x, t)\}\,.\,\theta\{z, \theta(x, t)\} &= f(y)\,f(z)\,\theta(x, t)\\ &= f(x)\,f(y)\,f(z)\,t^{\frac{1}{2}}.\end{aligned}$$

Hence if $\kappa(x) = x^s$ the relation (12) satisfies the conditions. The generalisation to the equality of n integrals is apparent, and in that case

$$\theta(x, t) = f(x)\,t^{\frac{1}{n-1}}$$

is a solution.

We have also

$$\int_{a_1}^{b_1} f_1(x)\,\psi_2\{\lambda(x, t)\}\,\psi_3\{\lambda(x, t)\}\,dx$$

$$=\int_{a_2}^{b_2} f_2(x)\,\psi_3\{\lambda(x, t)\}\,\psi_1\{\lambda(x, t)\}\,dx$$

$$=\int_{a_3}^{b_3} f_3(x)\,\psi_1\{\lambda(x, t)\}\,\psi_2\{\lambda(x, t)\}\,dx$$

if $$\lambda(x, t) = f(x) t^r, \quad \theta(x, t) = \{f(x)\}^{2r'} t^{r'},$$
and in particular if
$$\lambda(x, t) = f(x) t^{\frac{1}{2}}, \quad \theta(x, t) = \{\lambda(x, t)\}^{2r'}.$$

A solution may also be obtained when $\kappa(x) = p^x$, in which case
$$\kappa[\theta\{y, \theta(x, t)\}] \,.\, \kappa[\theta\{z, \theta(x, t)\}] = e^{\theta\{y, \theta(x, t)\} + \theta\{z, \theta(x, t)\}}.$$

Putting $$\theta(x, t) = f(x) + \tfrac{1}{2} t,$$
we have
$$\theta\{y, \theta(x, t)\} + \theta\{z, \theta(x, t)\} = f(y) + f(z) + f(x) + \tfrac{1}{2} t,$$
which is of the symmetrical form required.

6. In the cases investigated above the kernels of the several integral equations have been functions of the same form. It is, however, easy to extend the method to the case where the kernels are functions of different form. Thus if
$$\int_{a_1}^{b_1} f_1(x)\, \kappa_1\{\theta(x, t)\}\, dx = \psi_1(t),$$
$$\int_{a_2}^{b_2} f_2(x)\, \kappa_2\{\theta(x, t)\}\, dx = \psi_2(t),$$
we are led to the condition
$$\kappa_1[\theta\{y, \lambda(x, t)\}] = \kappa_2[\theta\{x, \lambda(y, t)\}].$$

Case 1. Let $\kappa_1(z) = z$, $\kappa_2(1/z) = z$; then the condition becomes
$$\theta\{y, \lambda(x, t)\} \,.\, \theta\{x, \lambda(y, t)\} = 1;$$
a solution of which is
$$\theta(x, t) = \chi\{F(x), \phi(t)\} \div \chi\{\phi(t), F(x)\},$$
where $$\lambda(x, t) = \phi^{-1} F(x),$$
and χ is any function.

Case 2. Let $\kappa_1(z) = z$, $\kappa_2(-z) = z$; then the condition is
$$\theta\{y, \lambda(x, t)\} + \theta\{x, \lambda(y, t)\} = 0;$$
a solution of which is
$$\theta(x, t) = \chi\{F(x), \phi(t)\} - \chi\{\phi(t), F(x)\}.$$

Case 3. Let $\kappa_1(z) = z^s$, $\kappa_2(z) = (1 - z^s)^{1/s}$; then the condition is
$$[\theta\{y, \lambda(x, t)\}]^s + [\theta\{x, \lambda(y, t)\}]^s = 1;$$
a solution of which is
$$\theta(x, t) = \chi\{F(x), \phi(t)\} \div [(\chi\{F(x), \phi(t)\})^s + (\chi\{\phi(t), F(x)\})^s]^{1/s}.$$

Fish-freezing. By Professor STANLEY GARDINER and Professor NUTTALL.

[*Read* 18 February 1918.]

Fish-freezing commenced in 1888, in connection with Western American salmon. It was started to preserve the excess of fish caught during the runs for canning in the slack season. The business proved so profitable that fish began to be distributed all over North America and exported to Europe, the chief market in the latter being Germany. The fish are, as soon as possible after catching, brought to the refrigerator, frozen dry on trays at about 10° F., this process taking about 36 hours. The fish then are drawn into a room at 20° F., where they are dipped into fresh water, their surfaces being thus covered with a glaze of ice. They are then packed in parchment paper in strong wooden cases and exported to Europe by refrigerator cars and cold storage steamers. The process is also applied to halibut, haddock, cod, pollack and various flat fish in America. It succeeds in preserving the fish for an indefinite period of time, but the product breaks up in cooking, tending to become rather woolly and loses flavour and aroma.

To meet this a fresh process has now been developed, freezing the fish in brine consisting of about 18 per cent. of salt at a temperature of 5° to 20° F. The brine is an excellent conductor of heat and cold. A large fish freezes thoroughly in three hours, a herring in twenty minutes. After freezing, the fish returns to the same condition as it was when placed into the brine; there is no woolliness, no loss of flavour or aroma. The difference is due to the fact that, whereas in dry freezing there is a breaking up of the actual muscular fibres, due to the formation of ice crystals, in brine freezing the ice crystals are so small that the muscular fibres are entirely unaffected and on thawing return to the normal. In neither form of freezing is there danger from moulds or putrefaction if the fish is stored below 20° F.

The authors advocate the creation of a vast store of frozen herrings against time of scarcity, instead of the herrings being pickled and exported. The value of fish as food is weight for weight about the same as meat, containing the same constituents. If the excess of the herring catch were stored in this way, there would be, on pre-war figures, a store of herrings in this country to meet the necessity for albuminous food in the British Isles for at least eight weeks.

On the branching of the Zygopterideun Leaf, and its relation to the probable Pinna-nature of Gyropteris sinuosa, *Goeppert.* By B. Sahni, M.A., Emmanuel College. (Communicated by Professor Seward.)

[*Read* 20 May 1918.]

(1) The supposed quadriseriate "pinnae" of forms like *Stauropteris* and *Metaclepsydropsis* are *tertiary* raches, the vascular strands of the secondary raches (pinna-trace-bar, Gordon) being completely embedded in the cortex of the primary rachis. All Zygopterideae therefore have a single row of pinnae on each side of the leaf. (2) This revives the suggestion that *Gyropteris sinuosa* Goepp. is a *free* secondary rachis of a form like *Metaclepsydropsis.* (3) The genus *Clepsydropsis* should include *Ankyropteris* because: *a.* A fossil described in 1915 (Mrs Osborn, *Brit. Ass. Rep.*, p. 727) combines the leaf-trace of *Clepsydropsis* with the stem of *Ankyropteris*, the leaf-trace in both arising as a closed ring. *b.* In *C. antiqua* Ung. also the leaf-trace arose similarly, as shown by a section figured by Bertrand (*Progressus* 1912, fig. 21, p. 228) in which a row of small tracheides connecting the inner ends of the peripheral loops represents those lining the ring before it became clepsydroid by median constriction.

The Structure of Tmesipteris Vieillardi *Dang.* By B. Sahni, M.A., Emmanuel College. (Communicated by Professor Seward.)

[*Read* 20 May 1918.]

The most primitive (least reduced) of the Psilotales. Specifically distinct from *T. tannensis* in (1) erect terrestrial habit, (2) distinct vascular supply to scale-leaves, (3) medullary xylem in lower part of aerial stem.

On Acmopyle, *a Monotypic New Caledonian Podocarp.* By B. Sahni, M.A., Emmanuel College. (Communicated by Professor Seward.)

[*Read* 20 May 1918.]

Indistinguishable from *Podocarpus* in habit, vegetative anatomy, drupaceous seed, megaspore-membrane, young embryo, male cone, stamen, two-winged pollen and probably male gametophyte. Chief differences: (1) seed nearly erect; (2) epimatium nowhere free from integument, even partaking in formation of micropyle; (3) outer flesh with a continuous tracheal mantle covering the basal two-thirds of the stone.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1917—1918.

ANNUAL GENERAL MEETING.

October 29, 1917.

In the Comparative Anatomy Lecture Room.

DR MARR, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Dr Marr.

Vice-Presidents:

Prof. Newall.
Dr Doncaster.
Mr W. H. Mills.

Treasurer:

Prof. Hobson.

Secretaries:

Mr A. Wood.
Mr G. H. Hardy.
Mr H. H. Brindley.

Other Members of Council:

Dr Bromwich.
Mr C. G. Lamb.
Mr J. E. Purvis.
Dr Shipley.
Dr Arber.
Prof. Biffen.
Mr L. A. Borradaile.
Mr F. F. Blackman.
Prof. Sir J. Larmor.
Prof. Eddington.
Dr Marshall.

The following Communications were made to the Society:

1. On the convergence of certain multiple series. By G. H. HARDY, M.A., Trinity College.

2. Bessel functions of large order. By G. N. WATSON, M.A., Trinity College.

3. A particular case of a theorem of Dirichlet. By H. Todd, B.A., Pembroke College. (Communicated by Mr H. T. J. Norton.)

4. On Mr Ramanujan's Empirical Expansions of Modular Functions. By L. J. Mordell. (Communicated by Mr G. H. Hardy.)

5. Extensions of Abel's Theorem and its converses. By Dr A. Kienast. (Communicated by Mr G. H. Hardy.)

November 12, 1917.

In the Comparative Anatomy Lecture Room.

Professor Marr, President, in the Chair.

The following Communications were made to the Society:

1. Some experiments on the inheritance of weight in rabbits. By Professor Punnett and the late Major P. G. Bailey.

2. The Inheritance of Tight and Loose Paleae in *Avena nuda* crosses. By A. St Clair Caporn. (Communicated by Professor Biffen.)

February 4, 1918.

In the Comparative Anatomy Lecture Room.

Professor Marr, President, in the Chair.

The following Communications were made to the Society:

1. On certain integral equations. By Major P. A. MacMahon.

2. (1) Sir George Stokes and the concept of uniform convergence.
   (2) Note on Mr Ramanujan's Paper entitled: On some definite integrals.

By G. H. Hardy, M.A., Trinity College.

3. Asymptotic expansions of hypergeometric functions. By G. N. Watson, M.A., Trinity College.

4. (1) On certain trigonometrical sums and their applications in the theory of numbers.
   (2) On some definite integrals.

By S. Ramanujan, B.A., Trinity College. (Communicated by Mr G. H. Hardy.)

February 18, 1918.

In the Comparative Anatomy Lecture Room.

Professor Marr, President, in the Chair.

The following were elected Fellows of the Society:

E. Lindsay Ince, B.A., Trinity College.
S. Ramanujan, B.A., Trinity College.

The following Communications were made to the Society:

1. Fish-freezing. By Professor Stanley Gardiner and Professor Nuttall.

2. Shell deposits formed by the flood of January 1918. By P. Lake, M.A., St John's College.

3. (1) Reactions to Stimuli in Corals.
(2) Is the Madreporarian Skeleton an Extraprotoplasmic Secretion of the Polyps?

By G. Matthai, M.A., Emmanuel College. (Communicated by Professor Stanley Gardiner.)

4. Notes on certain parasites, food, and capture by birds of *Forficula auricularia*. By H. H. Brindley, M.A., St John's College.

May 20, 1918.

In the Botany School.

Professor Marr, President, in the Chair.

The following was elected a Fellow of the Society:

C. Stanley Gibson, Sidney Sussex College.

The following Communications were made to the Society:

1. (1) On the branching of the Zygopteridean Leaf, and its relation to the probable Pinna-nature of *Gyropteris sinuosa*, Goeppert.
(2) The Structure of *Tmesipteris Vieillardi* Dang.
(3) On *Acmopyle*, a Monotypic New Caledonian Podocarp.

By B. Sahni, M.A., Emmanuel College. (Communicated by Professor Seward.)

2. Asymptotic Satellites in the problem of three bodies. By D. Buchanan. (Communicated by Professor Baker.)

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PROCEEDINGS

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY

VOL. XIX. PART V.

[MICHAELMAS TERM 1918 AND LENT TERM 1919.]

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Mr A. WOOD, Emmanuel College. [Physical.]
Mr H. H. BRINDLEY, St John's College. [Biological.]

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8. Members of the Society are requested to inform the Secretaries of any change of address.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

On Certain Trigonometrical Series which have a Necessary and Sufficient Condition for Uniform Convergence. By A. E. JOLLIFFE.

(Communicated by Mr G. H. HARDY.)

[*Received* 1 June 1918; *read* 28 October 1918.]

1. The series $\Sigma a_n \sin n\theta$, where (a_n) is a sequence decreasing steadily to zero, is convergent for all real values of θ, and it has been proved by Mr T. W. Chaundy and myself* that the series is uniformly convergent throughout any interval if $na_n \to 0$, this condition being necessary as well as sufficient.

A generalization of this theorem is as follows:

If (λ_n) *is a sequence increasing steadily to infinity and* (a_n) *is a sequence decreasing steadily to zero, then the necessary and sufficient condition that the series* $\Sigma\, a_{n+1} (\cos \lambda_n \theta - \cos \lambda_{n+1} \theta)/\theta$, *which is convergent for all real values of* θ, *should be uniformly convergent, throughout any interval of values of* θ, *is* $\lambda_n a_n \to 0$.

I shall prove rather more than this, viz. that the condition is *sufficient* for uniform convergence and necessary for continuity.

When $\theta = 0$, it is understood that the value assigned to any term of the series is its limit as θ tends to zero, so that for $\theta = 0$ the sum of the series, which I shall denote by Σu_n, is zero. Since, by Abel's lemma,

$$|u_{n+1} + \ldots + u_p| < 2a_{n+1}/\theta,$$

it is evident that there is continuity and uniform convergence throughout any interval which does not include $\theta = 0$, so that it is only intervals which include $\theta = 0$ that we have to consider.

* *Proc. London Math. Soc.* (2), Vol. 15, p. 214.

A very trifling modification of the analysis which follows will show that, so far as an interval which includes $\theta = 0$ is concerned, the same results hold for the series

$$\Sigma a_{n+1} (\cos \lambda_n \theta - \cos \lambda_{n+1} \theta) \operatorname{cosec} b\theta,$$

where b is any fixed number. If either $\frac{1}{2}(\lambda_{n+1} - \lambda_n)$ or $\frac{1}{2}(\lambda_{n+1} + \lambda_n)$ is always an integral multiple of some fixed number b, then λ_n differs by a constant from an integral multiple of $2b$, and the series is periodic with a period π/b. In this case the results which are true for an interval which includes $\theta = 0$ are true for any interval. The particular series $\Sigma a_n \sin n\theta$ corresponds to $b = \frac{1}{2}$, $\lambda_n = n + \frac{1}{2}$.

2. Since the sum of the series when $\theta = 0$ is zero, it follows that, for continuity at $\theta = 0$, the sum of the series, when θ is different from zero, must tend to zero as θ tends to zero in any manner. In particular, the sum when $\theta = \pi/2\lambda_n$ must tend to zero, as n tends to infinity.

When $\theta = \pi/2\lambda_n$, let m be the integer such that

$$\lambda_{m-1}\theta \leqslant \pi < \lambda_m \theta.$$

It should be noticed that we may have $m - 1 = n$, and that

$$\lambda_m > 2\lambda_n.$$

When $m - 1 > n$, $\cos \lambda_{p-1}\theta - \cos \lambda_p \theta$ is positive, so long as p is not greater than $m - 1$, and consequently

$$\begin{aligned}\theta (u_1 + u_2 + \dots + u_{m-1})\\ > a_n (\cos \lambda_1 \theta - \cos \lambda_n \theta) + a_{m-1} (\cos \lambda_n \theta - \cos \lambda_{m-1} \theta).\end{aligned}$$

Also, by Abel's lemma,

$$\theta (u_m + u_{m+1} + \dots + u_{m+q}) > a_m (\cos \lambda_m \theta - 1)$$

for all values of q.

Hence the sum of the series is greater than

$$\{a_n \cos \lambda_1 \theta - (a_{m-1} - a_m) \cos \lambda_{m-1} \theta - a_m\}/\theta,$$

which, since $a_{m-1} \geqslant a_m$ and $\cos \lambda_{m-1} \theta$ is negative, is greater than

$$(a_n \cos \lambda_1 \theta - a_m)/\theta = 2\lambda_n (a_n - a_m)/\pi + b_n,$$

where b_n denotes $a_n (1 - \cos \lambda_1 \theta)/\theta$ and consequently tends to zero as n tends to infinity.

When $m - 1 = n$, we can divide the series up into

$$(u_1 + u_2 + \dots + u_n) + (u_m + u_{m+1} + \dots),$$

and, noticing that $\cos \lambda_{m-1} \theta = 0$, we see that the sum is greater than $(a_n \cos \lambda_1 \theta - a_m)/\theta$, as before.

Hence the sum of the series, when $\theta = \pi/2\lambda_n$, can in no case tend to zero, as n tends to infinity, unless $\lambda_n(a_n - a_m) \to 0$.

If $\lambda_n(a_n - a_m) \to 0$, then, given any positive number ϵ, we can find ν such that $\lambda_n(a_n - a_m) < \epsilon$ for $n \geqslant \nu$. Denote m by $(n, 1)$ and let $(n, 2)$ be the integer formed from $(n, 1)$ in the same way that $(n, 1)$ is formed from n, and so on. Then

$$a_n - a_{n,1} < \epsilon/\lambda_n,\ a_{n,1} - a_{n,2} < \epsilon/\lambda_{n,1}, \ldots\ldots,$$

for $n \geqslant \nu$, and by addition

$$a_n < \epsilon\,(1/\lambda_n + 1/\lambda_{n,1} + \ldots + 1/\lambda_{n,p}) + a_{n,p}.$$

Now $\lambda_{n,1} > 2\lambda_n$, $\lambda_{n,2} > 2\lambda_{n,1}$, and so on, so that $a_n < 2\epsilon/\lambda_n + a_{n,p}$. Also when n is fixed we can choose p so that $a_{n,p} < \epsilon/\lambda_n$, and we shall have therefore

$$\lambda_n a_n < 3\epsilon\ (n \geqslant \nu).$$

Hence $\lambda_n a_n \to 0$ is a necessary condition that the sum of the series should be continuous at $\theta = 0$, and *à fortiori* that it should be continuous throughout any interval which includes $\theta = 0$.

3. To show that this condition is sufficient for uniform convergence in any interval, and *à fortiori* for continuity at any point, it is sufficient to show that

$$|u_{n+1} + \ldots + u_p| < AM,$$

for all values of θ, where A is some fixed number and M is the greatest value of $\lambda_r a_r$ for $r \geqslant n+1$.

Since the value of the series is changed in sign only by changing the sign of θ, it is sufficient to consider positive values of θ only. By Abel's lemma

$$|u_{n+1} + \ldots + u_p| < 2a_{n+1}/\theta < 2\lambda_{n+1}a_{n+1}/\pi,$$

if $\theta \geqslant \pi/\lambda_{n+1}$. If $\theta \leqslant \pi/\lambda_p$, every term of $u_{n+1} + \ldots + u_p$ is positive; and, if u_r is one of these terms,

$$\begin{aligned} u_r &\leqslant M(\cos\lambda_{r-1}\theta - \cos\lambda_r\theta)/\lambda_r\theta \\ &\leqslant 2M\sin\tfrac{1}{2}(\lambda_r - \lambda_{r-1})\,\theta\sin\tfrac{1}{2}(\lambda_r + \lambda_{r-1})\,\theta/\lambda_r\theta < M\theta\,(\lambda_r - \lambda_{r-1}),\end{aligned}$$

so that $$u_{n+1} + \ldots + u_p < M\theta\lambda_p < \pi M.$$

If $\pi/\lambda_p < \theta < \pi/\lambda_{m+1}$, let $\pi/\lambda_{q+1} < \theta \leqslant \pi/\lambda_q$, and divide $u_{n+1} + \ldots + u_p$ up into $u_{n+1} + \ldots u_q$ and $u_{q+1} + \ldots + u_p$.

Then $|u_{n+1} + \ldots + u_q| < \pi M$, and

$$|u_{q+1} + \ldots + u_p| < 2a_{q+1}/\theta < 2a_{q+1}\lambda_{q+1}/\pi < 2M/\pi.$$

Therefore $$|u_{n+1} + \ldots + u_p| < (\pi + 2/\pi)\,M.$$

Hence for all values of θ

$$|u_{n+1} + \dots + u_p| < (\pi + 2/\pi) M,$$

and therefore the condition $\lambda_n a_n \to 0$ is sufficient for uniform convergence and *à fortiori* for continuity in any interval.

4. If λ_n tends to infinity more rapidly than n, the series does not seem to be capable of any modification. If $\lambda_n = An + B$, where A and B are fixed, we obtain practically the series $\Sigma a_n \sin n\theta$ and nothing more. But when λ_n tends to infinity more slowly than n, and with a certain measure of regularity, the theorem can be transformed in an interesting manner. We have, in fact, the following theorem:

If λ_n tends steadily to infinity and $\lambda_{n+1} - \lambda_n$ tends steadily to zero, then the necessary and sufficient condition for the uniform convergence of

$$\Sigma a_n (\lambda_{n+1} - \lambda_n) \sin \lambda_n \theta$$

is $\lambda_n a_n \to 0$.

As before, I prove rather more, viz. that the condition is sufficient for uniform convergence and necessary for continuity.

This theorem will follow at once from the theorem just proved, if we can show that the series

$$\Sigma a_n \{(\cos \lambda_n \theta - \cos \lambda_{n+1} \theta)/\theta - (\lambda_{n+1} - \lambda_n) \sin \lambda_n \theta\}$$

is uniformly convergent throughout any interval. Here the condition $\lambda_{n+1} a_n \to 0$ is equivalent to $\lambda_n a_n \to 0$, since $\lambda_{n+1} - \lambda_n \to 0$.

We can verify immediately that

$$\cos y - \cos x - \sin y \sin (x - y)$$
$$= \sin^2 \tfrac{1}{2}(x - y)(\cos y - \cos x) + \tfrac{1}{2} \sin (x - y)(\sin x - \sin y).$$

It follows by Abel's lemma that, if $\lambda_{n+1} - \lambda_n$ decreases steadily, so that $\sin (\lambda_{n+1} - \lambda_n)\theta$ and $\sin \frac{1}{2}(\lambda_{n+1} - \lambda_n)\theta$ decrease steadily, then

$$\left| \sum_{n+1}^{p} \{\cos \lambda_n \theta - \cos \lambda_{n+1} \theta - \sin (\lambda_{n+1} - \lambda_n) \theta \sin \lambda_n \theta\} \right|$$
$$< 2 \sin^2 \tfrac{1}{2} (\lambda_{n+1} - \lambda_n)\theta + \sin (\lambda_{n+1} - \lambda_n)\theta.$$

Also, given any ϵ, we can choose ν so that $\lambda_{n+1} - \lambda_n < \epsilon$ for $n \geqslant \nu$. Hence, for $n \geqslant \nu$, we have

$$\left| \sum_{n+1}^{p} \{\cos \lambda_n \theta - \cos \lambda_{n+1} \theta - \sin (\lambda_{n+1} - \lambda_n) \theta \sin \lambda_n \theta\} \right|$$
$$< 2\epsilon^2 \theta^2 + \epsilon\theta < 3\epsilon\theta,$$

for any interval of values of θ, if ϵ is sufficiently small.

It follows also that

$$\left| \sum_{n+1}^{p} \sin(\lambda_{n+1} - \lambda_n)\theta \sin \lambda_n \theta \right| < 2 + 3\epsilon\theta < 3,$$

for $n \geqslant \nu$. Now

$$(\lambda_{n+1} - \lambda_n)\,\theta \operatorname{cosec}(\lambda_{n+1} - \lambda_n)\,\theta - 1$$

decreases steadily to zero, and is less than

$$\tfrac{1}{3}(\lambda_{n+1} - \lambda_n)^2\theta^2 < \tfrac{1}{3}\epsilon^2\theta^2 \quad (n \geqslant \nu).$$

Therefore

$$\left| \sum_{n+1}^{p} \theta(\lambda_{n+1} - \lambda_n) \sin \lambda_n\theta - \sum_{n+1}^{p} \sin(\lambda_{n+1} - \lambda_n)\theta \sin \lambda_n \theta \right| < \epsilon^2\theta^2.$$

Hence

$$\left| \sum_{n+1}^{p} \{(\cos \lambda_n \theta - \cos \lambda_{n+1}\theta)/\theta - (\lambda_{n+1} - \lambda_n) \sin \lambda_n \theta\} \right|$$
$$< 3\epsilon + \epsilon^2\theta < 4\epsilon \quad (n \geqslant \nu).$$

Hence the series

$$\Sigma a_n \{(\cos \lambda_n \theta - \cos \lambda_{n+1}\theta)/\theta - (\lambda_{n+1} - \lambda_n) \sin \lambda_n \theta\}$$

is uniformly convergent throughout any interval, and hence the result enunciated follows.

5. If instead of a sequence (λ_n) we have a function $\lambda(x)$ such that, as $x \to \infty$, $\lambda(x)$ increases steadily to infinity and $\lambda'(x)$ decreases steadily to zero, then $\lambda_{n+1} - \lambda_n$ decreases steadily to zero. The series $\Sigma(\lambda'_n - \lambda_{n+1} + \lambda_n)$, where λ'_n denotes the value of $\lambda'(x)$ when $x = n$, is convergent and is moreover *absolutely* convergent, since $\lambda'_n - \lambda_{n+1} + \lambda_n$ is positive. Hence, by Weierstrass' M test*, the series $\Sigma a_n(\lambda'_n - \lambda_{n+1} + \lambda_n) \sin \lambda_n \theta$ is uniformly convergent throughout every interval. It follows then that $a_n\lambda_n \to 0$ is the necessary and sufficient condition that the series $\Sigma a_n \lambda'_n \sin \lambda_n \theta$ should be continuous at every point and uniformly convergent throughout every interval.

In particular the series $\Sigma a_n n^{t-1} \sin(n^t \theta)$, where t is any real number not exceeding 1, is continuous at every point and uniformly convergent throughout every interval if $n^t a_n \to 0$, this condition being necessary as well as sufficient.

* Bromwich, *Infinite series*, p. 113.

Some Geometrical Interpretations of the Concomitants of Two Quadrics. By H. W. Turnbull, M.A.

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§ 1. In the *Mathematische Annalen*, Vol. LVI, Gordan has given a system of 580 invariants for two quaternary quadratics. It appears that by carrying out the processes of reduction a little further, the irreducible forms can be shewn to number 123 at most. That is to say, the system is about as complicated as the ternary system for three conics which Ciamberlini* first established. It is therefore worth while to give geometrical interpretations to members of the system for two quadratics. In the following pages about a hundred of them are shewn. The geometrical significance of the residue appears to be remote.

Using the classification introduced by Gordan, the numbers of forms of each type J which have not been reduced are shewn in the subjoined Table. The rows of the Table give the numbers of forms of each particular order in the three sets of coordinates x, p, u, which define points, straight lines, and planes respectively. Detailed lists of these forms will be found at the heads of the paragraphs which deal with separate types.

References	Order in x, p, u	Type J^1	J^2	J^3	J^4	J^5	Total
§ 5	Invariants	5					5
§ 6	Covariants	4		1			5
,,	Contravariants	4		1			5
§§ 7–14	Complexes	6	1	1	4	4	16
§ 15	Mixed (1, 0, 1)	1		2			3
§ 21	(1, 0, 3)			4			4
,,	(3, 0, 1)			4			4
§ 22	(2, 0, 2)	1		6			7
§§ 17–20	(0, 1, 2)		1	6	1	1	9
,,	(2, 1, 0)		1	6	1	1	9
§ 18	(0, 3, 2)			1			1
,,	(2, 3, 0)			1			1
§ 23	(0, 2, 2)			4			4
,,	(2, 2, 0)			4			4
§ 16	(1, 1, 1)		4	12			16
§ 23	(1, 2, 1)			12	6	6	24
,,	(2, 1, 2)			4			4
,,	(3, 0, 3)			2			2
	Totals	21	7	71	12	12	123

* Ciamberlini, *Giornale di Matematiche*, Vol. XXIX.

Notation.

§ 2. Let u_1, u_2, u_3, u_4 be plane coordinates; and let v, w be cogredient with u. We may then typify line coordinates by

$$p_{ij} = (uv)_{ij} = u_i v_j - v_i u_j, \quad (i, j = 1, 2, 3, 4);$$

and x or point coordinates by $x_1 = (uvw)_{234}$ and three similar expressions for x_2, x_3, x_4. Then the symbolic system of Gordan can be exhibited as follows.

Let the point equations of the quadrics be

$$f = a_x^2 = a_x'^2 = \ldots,$$

and

$$f' = b_x^2 = b_x'^2 = \ldots.$$

Let the line equations be

$$\Pi = (Ap)^2 = (A'p)^2 = \ldots,$$

$$\Pi' = (Bp)^2 = (B'p)^2 = \ldots.$$

Let the tangential equations be

$$\Sigma = u_\alpha^2 = u_{\alpha'}^2 = \ldots,$$

$$\Sigma' = u_\beta^2 = u_{\beta'}^2 = \ldots.$$

Then the connections between the symbols are

$$A = aa', \quad B = bb', \quad \alpha = aa'a'', \quad \beta = bb'b''.$$

And all concomitants of the system can be expressed in terms of factors

$$d_x, \quad (dd'p), \quad (dd'd''u), \quad (dd'd''d'''),$$

where d signifies a or b. But the irreducibles can be shewn to be composed of the following types,

$$a_\alpha^2,\ b_\beta^2,\ a_x,\ b_x,\ (Ap),\ (Bp),\ u_\alpha,\ u_\beta,\ (abp),\ (Abu),\ (Bau),\ (AB),$$

$$a_\beta,\ b_\alpha,\ (A\beta x),\ (B\alpha x),\ (\alpha\beta p),\ (AB)',\ F_1,\ F_2;$$

where

$$(A\beta x) = a_\beta a_x' - a_\beta' a_x = \dot{a}_\beta \dot{a}_x',$$

say, and

$$(\alpha\beta p) = u_\alpha v_\beta - u_\beta v_\alpha = \dot{u}_\alpha \dot{v}_\beta,$$

$$(AB)' = (A\dot{b}u)\,\dot{b}_x',$$

$$F_1 = (abp)\,a_\beta' - (a'bp)\,a_\beta = (Abp\beta).$$

Reciprocation.

§ 3. By interchanging the symbols (a, α), (b, β), (u, x) without altering A, B or p, we obtain from any given concomitant the reciprocal form. Thus the bracket factors $(A\beta x)$ and (Abu) are reciprocals. So also would be $(abcu)$ and $(\alpha\beta\gamma x)$, the latter being of a type not arising for less than three quadrics. Though the process by which Gordan arrived at such symbols as $(A\beta x)$ and $(\alpha\beta p)$ was purely analytic, it is interesting to observe that from the geometrical point of view such analytical results were almost inevitable. Below will be found several examples of the use of this principle of duality.

The fundamental forms.

§ 4. A brief investigation would reveal the importance of the following forms, to which special symbols are therefore attached.

Let	f denote	a_x^2,	f'	denote	b_x^2,
	Σ	,,	u_α^2,	Σ'	,,	u_β^2,
	Π	,,	$(Ap)^2$,	Π'	,,	$(Bp)^2$,
	k	,,	$(A\beta x)^2$,	k'	,,	$(B\alpha x)^2$,
	χ	,,	$(Abu)^2$,	χ'	,,	$(Bau)^2$,
	π_{12}	,,	$(abp)^2$,	Π_{12}	,,	$(\alpha\beta p)^2$,
and	C	,,	$(AB)(Ap)(Bp)$.			

Some account of these forms may be found in Salmon, *Analytical Geometry of Three Dimensions* (revised by Rogers), Vol. I, Ch. IX. There Σ, χ, χ', Σ' are denoted by σ, τ, τ', σ' (§ 214): k, k' are the T, T' of § 215; Π, π_{12}, Π' are the Ψ, Ψ_1, Ψ' of § 217.

Invariants.

§ 5. The irreducible invariants are a_α^2, b_α^2, $(AB)^2$, a_β^2, b_α^2 or the Δ, Θ, Φ, Θ', Δ' of Salmon, § 200. In fact, there are no other types, for two quadrics of any dimension n, than the $n+1$ coefficients of λ in the discriminant of

$$\lambda f + f'^*.$$

The five covariants and contravariants.

§ 6. The covariants ($n+1$ in the case of n-ary forms*) are the four quadrics f, k, k', f' and the quartic J defined by

$$a_\beta b_\alpha a_x b_x (AB)(A\beta x)(B\alpha x).$$

This is indeed the jacobian of the four quadrics, and represents the four planes of the self-conjugate tetrahedron (cf. Salmon, § 233).

* Cf. Turnbull, 'Quadratics in n variables' (pp. 235-238), *Camb. Phil. Trans.*, Vol. XXI, No. viii.

Correlatively, Σ, χ, χ', Σ' are the four quadrics in u which make the system of contravariants together with

$$j = a_\beta b_\alpha u_\alpha u_\beta (AB)(Abu)(Bau).$$

This latter represents the four vertices of the same tetrahedron. In fact, the jacobian of u_α^2, u_β^2, $(Abu)^2$, $(Bau)^2$ is

$$(\alpha\beta\, \overline{Ab}\, \overline{Ba})\, u_\alpha u_\beta (Abu)(Bau),$$

where $A = a'a''$, say; expanding the first bracket this becomes

$$\dot{a}_\alpha' \dot{a}_\beta'' (bBa) M - \dot{a}_\alpha' b_\beta (\dot{a}''Ba) M + b_\alpha \dot{a}_\beta' (\dot{a}''Ba) M,$$

where each term represents two, with a', a'' permuted, and M is short for $u_\alpha u_\beta (Abu)(Bau)$. But the factor a_α' is reducible to a_α^2 (Gordan, II, § 6); which means in this case that the symbols u of the factors u_α, (Abu) would be bracketed. Hence the product involving a_α' is zero. Thus the jacobian is equal to

$$\begin{aligned} &b_\alpha \dot{a}_\beta' (\dot{a}''Ba) M,\\ &= b_\alpha a_\beta (a''Ba') M + b_\alpha \dot{b}_\beta' (a''a'\dot{b}''a) M \quad (\text{if } B = b'b'')\\ &= - b_\alpha a_\beta (a'a''B) M \text{ (as before)}\\ &= - j. \end{aligned}$$

A correlative reduction applies to the case of J.

The complexes.

§ 7. A complex is a function of p, or line coordinates, but not explicitly of u or x. There are eight quadratic and eight cubic complexes in the system. The quadratics are

$$(Ap)^2 \text{ or } \Pi,\ (Bp)^2 \text{ or } \Pi',\ (abp)^2 \text{ or } \pi_{12},\ (\alpha\beta p)^2 \text{ or } \Pi_{12},$$
$$(AB)(Ap)(Bp) \text{ or } C,\ (abp)(\alpha\beta p)\, a_\beta b_\alpha,\ F_1^2 \text{ and } F_2^2.$$

Differentiation.

§ 8. Let p be any symbolic product belonging to the whole system; then $\dfrac{\partial P}{\partial x_i}$ $(i = 1, 2, 3, 4)$ would be composed of terms each with one odd symbol a_i or b_i left over. Thus the four symbols $\dfrac{\partial P}{\partial x_i}$ may be considered as the coordinates of a certain *plane*. For example the coordinates of the polar plane of a point (x) with regard to a_x^2 are $(a_x a_1, a_x a_2, a_x a_3, a_x a_4)$. Likewise $\dfrac{\partial P}{\partial u_i}$ would give a set of point coordinates.

Again, $\frac{\partial P}{\partial p_{ij}}$ would give six quantities which would symbolise the coordinates of a certain linear complex: and, in some special cases, the coordinates of a straight line. For example,

$$\frac{1}{2}\frac{\partial (Ap)^2}{\partial p} = (Ap)(A)$$

is a useful way of denoting the six quantities $(Ap)A_{ij}$ $(i, j = 1, 2, 3, 4)$, which represent a straight line, since they satisfy the identical relation existing between the six p-coordinates of a straight line.

Line coordinates.

§ 9. This identity satisfied by line coordinates (p) is

$$\Sigma p_{ij}p_{kl} = 0 \quad \text{..........................(1)},$$

which we denote by $\omega(p) = 0$. Symbolically, the condition that two lines p and q should intersect is $(pq) = 0$. If p is the line common to two planes u, v, and q is that common to u', v', then this condition is $(uvu'v') = 0$.

If two lines p, q intersect, then $\kappa p_{ij} + \lambda q_{ij}$ represents the coordinates of any line of the plane p, q passing through the common point of p, q. Since the line p touches the quadric f if $(Ap)^2 = 0$, it follows that the line (κ, λ) touches this quadric if

$$\kappa^2 (Ap)^2 + 2\kappa\lambda (Ap)(Aq) + \lambda^2 (Aq)^2 = 0.$$

Hence $(Ap)(Aq)$ vanishes if p intersects the conjugate of q in f; for then p and q are harmonic conjugates of the two tangents to f in this pencil of lines (κ, λ). This shews that the coordinates $\frac{\partial \Pi}{\partial p}$, *i.e.* $(Ap)A_{ij}$, are those of the line conjugate to p in the quadric f. Analytically it is evident that these coordinates represent a line and not a linear complex, since they satisfy the required condition (1). In fact

$$(Ap)(A'p)(AA') = \tfrac{1}{3}(AA')^2\,\omega(p)^*.$$

But the left member of this equation is the symbolic equivalent of substituting $(Ap)A_{ij}$ for p in (1): which proves the statement.

Complexes and their polars.

§ 10. Let $(Dp)^2 = 0$ represent one of the quadratic complexes of § 7. Then $(Dp)D_{ij}$ gives the coordinates of a linear complex

* Cf. Gordan, II, § 6.

polar to (p) in $(Dp)^2$. If (p) is a member of the complex $(Dp)^2$, the polar is called the tangential linear complex.

The complex $(Dp)\,D_{ij}$ is not usually a special linear complex. The preceding case was exceptional. For in that case the quadratic complex was $(Ap)^2=0$, and all the rays touched the quadric f.

The complexes π_{12}, Π_{12}, C, $(abp)(\alpha\beta p)\,a_\beta b_\alpha$.

§ 11. The principal quadratic complexes which occur are

$$\pi_{12}\equiv(abp)^2,\quad \Pi_{12}\equiv(\alpha\beta p)^2,\quad C\equiv(AB)(Ap)(Bp).$$

The two former are well known, π_{12} being the aggregate of lines cutting the quadrics harmonically, and Π_{12} being the correlative complex. The third, C, is the complex of lines whose conjugates, in f and f' respectively, intersect. For the conjugate of p in f is $(Ap)(A)$ and in f' is $(Bp)(B)$. Again, C is satisfied too by the singular lines of the complex π_{12}. For if p is a line of $(abp)^2=0$, its tangent linear complex (§ 10) is $(abp)(abq)=0$, q representing current coordinates: further, p is a singular line if this tangent linear complex is special, *i.e.* if

$$(abp)(aba'b')(a'b'p)=0,$$

which reduces to $(AB)(Ap)(Bp)=0$. Correlatively C also contains the singular lines of the complex Π_{12}.

Again, the singular lines of the complex C belong to the complex $(abp)(\alpha\beta p)\,a_\beta b_\alpha$. This follows in the same way as in the above case. But a more direct interpretation of this last form arises from the apolar* condition for two linear complexes; if the polar linear complexes of a line (p) with regard to π_{12} and Π_{12} are apolar, then $(abp)(\alpha\beta p)\,a_\beta b_\alpha$ vanishes.

The complexes F_1^2, F_2^2.

§ 12. Besides the original complexes $(Ap)^2$ and $(Bp)^2$, and the four complexes of § 11, there remain two more quadratics, F_1^2 and F_2^2. Just as $(abp)^2$ is the harmonic complex between f and f', so F_1^2 is the harmonic complex between f' and k, while F_2^2 is that between f and k'. To prove this we build up a form $(f',k)^2$ from f' and k, in the same way as $(f,f')^2$, *i.e.* $(abp)^2$, is built from f and f'. Then

$$\begin{aligned}(f',k)^2&=(b_x^2,\,(ABx)^2)^2\\&=(b_x^2,\,2a_\beta^2a_x'^2-2a_\beta a_\beta' a_x a_x')^2\\&=2a_\beta^2(a'bp)^2-2a_\beta a_\beta'(abp)(a'bp)\\&=[(abp)\,a_\beta'-(a'bp)\,a_\beta]^2=F_1^2\quad(\S\ 2).\end{aligned}$$

* The linear complexes $(Dp)=0$, $(Ep)=0$ are apolar if $(DE)=0$.

The eight cubic complexes:

$$F_2(abp)\,b_\alpha(Bp), \qquad F_2(\alpha\beta p)\,a_\beta(Bp),$$
$$F_2(abp)\,b_\alpha(AB)(Ap),\ F_2(\alpha\beta p)\,a_\beta(AB)(Ap);$$

and four involving F_1.

§ 13. If a_x^2, b_x^2, c_x^2 are three quadrics, the lines p cutting them in involution are given by the cubic complex

$$(bcp)(cap)(abp)=0.$$

Let us denote this complex by the symbol (a_x^2, b_x^2, c_x^2). Then (f, f', k') may be formulated, and we shall have

$$\begin{aligned}(a_x^2, b_x^2, k') &= ((abp)\,a_x b_x,\ (B\alpha x)^2)^2\\ &= ((abp)\,a_x b_x,\ -2b_\alpha' b_x'' b_\alpha'' b_x' + 2b_x'^2 b_\alpha''^2)^2\\ &= -2\,(abp)(ab''p)(bb'p)\,b_\alpha' b_\alpha'' + 2\,(abp)(ab'p)(bb'p)\,b_\alpha''^2.\end{aligned}$$

The second term is zero, since b, b' are interchangeable. The first term is $F_2(abp)\,b_\alpha(Bp)$ to a constant coefficient.

Reciprocally (Σ, Σ', χ') represents $F_2(\alpha\beta p)\,a_\beta(Bp)$; and there are two like forms involving F_1.

§ 14. This leaves four complexes such as $F_2(abp)\,b_\alpha(AB)(Ap)$ to be interpreted, but the geometrical significance is not at all immediate. If however we write (f, f', k') as $(Dp)^3$, then the line (p) has a polar linear complex

$$(Dp)^2(Dq)=0.$$

And if $q=(Ap)(A)$, *i.e.* if q is the conjugate line of p in the quadric f, then

$$(Dp)^2(DA)(Ap)=0.$$

This latter form is equivalent to $F_2(abp)\,b_\alpha(AB)(Ap)$: and similar results follow for the other three forms, as in § 13.

The mixed concomitants.

§ 15. To denote the order of a form, let (i, j, k) mean that the order is i in x, j in p, and k in u. Then there are three linear forms $(1, 0, 1)$ and sixteen linear forms $(1, 1, 1)$.

The three linear forms $(1, 0, 1)$:

$$u_\beta a_\beta a_x, \quad u_\alpha b_\alpha b_x, \quad (AB)(Abu)\,b_x'.$$

If (v) is the polar plane of a point (x) in f, then $(v)=a_x(a)$. Hence $u_\beta a_\beta a_x=0$ is the condition that a conjugate plane of u in f'

should be the polar of x in f. Similarly for $u_\alpha b_\alpha b_x$. Again, $(AB)(Abu)b_x'$ vanishes if the polar of x in f' is conjugate to u in χ, *i.e.* in $(Abu)^2=0$.

The sixteen forms (1, 1, 1):

two like $a_x(Bau)(Bp)$,	*two like* $u_\alpha(B\alpha x)(Bp)$,
,, ,, $a_x a_\beta(\alpha\beta p)u_\alpha$,	,, ,, $a_x(abp)b_\alpha u_\alpha$,
,, ,, $a_x(Bau)(AB)(Ap)$,	,, ,, $u_\alpha(B\alpha x)(AB)(Ap)$,
,, ,, $(abp)(Abu)(A\beta x)a_\beta$,	,, ,, $(\alpha\beta p)(Abu)(A\beta x)b_\alpha$.

§ 16. The polar plane of a point (x), with regard to f, meets a plane (u) in a straight line whose coordinates are $(au)a_x$. If $a_x(aBu)(Bp)=0$, this line cuts the conjugate of p in f'. Let us denote this relation by (f_x, Π'). The significance of the reciprocal of this, viz. (Σ_u, Π'), is obvious. This accounts for four forms since either f or f' can be employed.

Suppose we word this relation differently and say that the plane (u) cuts the polar of (x) in f in a line which lies in the linear complex polar of (p) in Π': then a like meaning attached to (f_x, Π_{12}) interprets $a_x a_\beta(\alpha\beta p)u_\alpha$. So also

$$(\Sigma_u, \pi_{12})=a_x(abp)b_\alpha u_\alpha,$$
$$(f_x, (AB)(Ap)(Bp))=a_x(aBu)(AB)(Ap),$$

with reducible terms, and

$$(\Sigma_u, (AB)(Ap)(Bp))=u_\alpha(B\alpha x)(AB)(Ap),$$

while (k_x, π_{12}), (χ_u, Π_{12}) denote the remaining two forms of the above list. To complete the set of sixteen forms we merely write Σ' for Σ, k' for k, and so on.

The polar quadrics (0, 1, 2) *and* (2, 1, 0).

§ 17. There are nine forms of order (2, 1, 0), any one of which represents a quadric associated with a given line (p); or, from another point of view, represents a linear complex associated with a given point (x). The simplest of these is $(abp)a_x b_x$. Let this denote the *polar quadric* of the line (p) with regard to the system $f+\lambda f'$. It is convenient to use the symbol $p(f, f')$ for this relation.

The equation $(abp)a_x b_x=0$ is the analytical condition required when the polar planes of a point (x) with regard to f and f' meet in a line which intersects (p). For the coordinates of these polar planes of x are denoted by $a_i a_x$, $b_i b_x$ $(i=1, 2, 3, 4)$. Hence the coordinates of their line of intersection are $a_x b_x(ab)_{ij}$; and this line cuts (p) if $(abp)a_x b_x=0$.

Forming the invariant of the polar quadric, we obtain an expression which reduces directly to $\{(AB)(Ap)(Bp)\}^2$. Hence if p belongs to the complex C, its polar quadric is a cone.

§ 18. Again, the tangential equation of the polar quadric $(abp)\,a_x b_x = 0$ is formed in the same way as $u_a{}^2$ is formed from $a_x{}^2$. A simple reduction leads to

$$(Ap)(Bp)(abp)(aBu)(bAu).$$

Likewise the point equation of $(\alpha\beta p)\,u_\alpha u_\beta$ involves the form

$$(Ap)(Bp)(\alpha\beta p)(A\beta x)(B\alpha x).$$

This interprets the two forms of orders (0, 3, 2) and (2, 3, 0).

§ 19. Again, if we form the polar quadric of (p) with regard to each pair of quadrics f, f', k, k', we obtain the following results:

$p(f, k)$ equivalent to $(Ap)(A\beta x)\,a_\beta a_x$, with a like form for $p(f', k')$,

$p(f, k')$ „ $F_2 a_x (B\alpha x)$, „ „ $p(f', k)$,

$p(k, k')$ „ $(A\beta x)(\alpha\beta p)(B\alpha x)(AB)$.

If, further, (q') is the conjugate line of (p) in $(Bp)^2$, *i.e.* in f', then

$q'(f, k)$ is equivalent to $a_x a_\beta (A\beta x)(AB)(Bp)$,

and $q'(f', k')$ „ $b_x b_\alpha (B\alpha x)(AB)(Ap)$.

All these equivalences are readily verified, but we give a special proof for the case of $p(k, k')$. In fact, the polar plane of x in $k=0$, *i.e.* in $(A\beta x)^2=0$, has coordinates $\dfrac{\partial k}{\partial x_i}$, which may be symbolised as $(A\beta x)(A\beta)$*. So also the coordinates of the polar of x in k' are denoted by $(B\alpha x)(B\alpha)$. Hence the line of intersection of these polars is denoted by $(A\beta x)(B\alpha x)\,[AB\alpha\beta]$, which is equal to $(A\beta x)(B\alpha x)(AB)(\alpha\beta)$*; and the line cuts p if

$$(A\beta x)(B\alpha x)(AB)(\alpha\beta p)=0.$$

§ 20. These eight polar quadrics now enumerated, viz. $p(f, f')$, $p(f, k)$, ..., $q'(f', k')$, must be supplemented with one more form, $(\alpha\beta p)\,a_\beta b_\alpha a_x b_x$, to complete the set of nine forms (2, 1, 0) belonging to the irreducible system of two quadrics f and f'. The geometrical significance of this last form is as follows: the line joining the two points, x_1 and x_2, cuts p; x_1 being the pole in f of the plane whose pole in f' is x, and x_2 being the pole in f' of the plane whose pole in f is x.

* $(A\beta)=a_\beta a'-a'_\beta a$, and the combination of $(A\beta)$ with $(B\alpha)$, as a transvectant, into $[AB\alpha\beta]$ is essentially the reduction of Ch. II, § 15 in the paper of Gordan.

Correlatively there are nine forms (0, 1, 2), quadratic in u, exactly parallel with the above, of which $(\alpha\beta p)\,u_\alpha u_\beta$ is the simplest.

The four forms (3, 0, 1) *and their correlatives*:

$$(Abu)(A\beta x)\,a_\beta a_x b_x, \qquad (A\beta x)(Abu)\,b_\alpha u_\alpha u_\beta,$$
$$(AB)(A\beta x)(B\alpha x)\,a_\beta a_x u_\alpha, \; (AB)(Abu)(Bau)\,b_\alpha u_\alpha a_x,$$

and four similar forms interchanging f and f'.

§ 21. If a_x^2, b_x^2, c_x^2 signify any three quadrics, then $(abcu)\,a_x b_x c_x$ vanishes when the common point of the polars of (x) in the three quadrics lies on the plane (u). Applied to the quadrics f, f', k, k' taken three at a time, this condition involves the four forms (3, 0, 1) indicated above. The correlative condition, applied to each set of three from among Σ, Σ', χ, χ', gives rise to the four forms (1, 0, 3). For example, if we select f, f', k as the three quadrics, then the condition is $(Abu)(A\beta x)\,a_\beta a_x b_x = 0$.

The polars of (x) in all four quadrics f, f', k, k' meet in a point if (x) lies on any face of the self-conjugate tetrahedron

$$(A\beta x)(B\alpha x)\,a_\beta b_\alpha a_x b_x = 0.$$

The remaining forms of the system.

§ 22. None of the remaining forms appear to have any special geometrical importance: but we give a few examples. First, as to the forms of order (2, 0, 2), we may exhibit them as follows:

$a_x(aBu)(B\alpha x)\,u_\alpha$ and a similar form,

$(Abu)(A\beta x)\,b_\alpha a_\beta u_\alpha a_x$ " " "

$(AB)(Abu)(Bau)\,a_x b_x$ and a correlative form,

and $[(AB)']^2$.

Suppose (q) to denote the common line of the plane (u) and the polar of (x) in f, and (q') to denote the line joining (x) to the pole of (u) in f. Then the condition that q, q' should satisfy the harmonic relation $(Bq)(Bq')=0$ becomes on substitution

$$a_x(aBu)(B\alpha x)\,u_\alpha = 0.$$

Thus the first in the above group of forms is interpreted. The second form vanishes if two lines (q), (q') satisfy the harmonic relation $(abq)(abq')=0$, where (q) now denotes the intersection of the plane (u) with the polar of (x) in k, while (q') is the same as before.

Again, the third form of the set vanishes if the lines in which the polars of (x) in f and f' cut the plane (u) satisfy the harmonic relation for the complex $C=(AB)(Ap)(Bp)$.

Finally the last form $[(AB)']^2$, which is equivalent, except for reducible terms, to $(Abu)\,b_x'\,(Ab'u)\,b_x$ (§ 2), is involved in the condition that the line common to (u) and the polar of (x) in f' should touch f.

§ 23. Next there are four forms of order (0, 2, 2), such as $(Ap)(Abu)(abp)\,a_\beta u_\beta$, $(Ap)(AB)(Bau)(\alpha\beta p)\,a_\beta u_\alpha$, and four correlatives of order (2, 2, 0). All of these have obscure geometrical properties, though they present no difficulty to identify.

After this there are twenty-four forms of order (1, 2, 1). The simplest of these is $(Abu)\,b_x'\,(Ap)(Bp)$, which vanishes when u, x, p satisfy the following conditions: if the polar of (x) in f' meets (p) at a point (y), and if the polar of (y) in f' cuts the plane (u) in a line (q), then p, q satisfy the harmonic relation $(Ap)(Aq)=0$. The remainder of these (1, 2, 1) forms are of like nature.

Beyond this there are four forms (2, 1, 2), and two forms (3, 0, 3), none of which present concise geometrical interpretations.

Some properties of $p(n)$, the number of partitions of n. By S. RAMANUJAN, B.A., Trinity College.

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§ 1. A recent paper by Mr Hardy and myself* contains a table, calculated by Major MacMahon, of the values of $p(n)$, the number of unrestricted partitions of n, for all values of n from 1 to 200. On studying the numbers in this table I observed a number of curious congruence properties, apparently satisfied by $p(n)$. Thus

(1) $p(4),\ p(9),\ p(14),\ p(19),\ \ldots \equiv 0 \pmod{5}$,

(2) $p(5),\ p(12),\ p(19),\ p(26),\ \ldots \equiv 0 \pmod{7}$,

(3) $p(6),\ p(17),\ p(28),\ p(39),\ \ldots \equiv 0 \pmod{11}$,

(4) $p(24),\ p(49),\ p(74),\ p(99),\ \ldots \equiv 0 \pmod{25}$,

(5) $p(19),\ p(54),\ p(89),\ p(124),\ \ldots \equiv 0 \pmod{35}$,

(6) $p(47),\ p(96),\ p(145),\ p(194),\ \ldots \equiv 0 \pmod{49}$,

(7) $p(39),\ p(94),\ p(149),\ \ldots \equiv 0 \pmod{55}$,

(8) $p(61),\ p(138),\ \ldots \equiv 0 \pmod{77}$,

(9) $p(116),\ \ldots \equiv 0 \pmod{121}$,

(10) $p(99),\ \ldots \equiv 0 \pmod{125}$.

From these data I conjectured the truth of the following theorem:

If $\delta = 5^a 7^b 11^c$ *and* $24\lambda \equiv 1 \pmod{\delta}$, *then*

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta),\ \ldots \equiv 0 \pmod{\delta}.$$

This theorem is supported by all the available evidence; but I have not yet been able to find a general proof.

I have, however, found quite simple proofs of the theorems expressed by (1) and (2), viz.

(1) $$p(5m+4) \equiv 0 \pmod{5}$$

and (2) $$p(7m+5) \equiv 0 \pmod{7}.$$

* G. H. Hardy and S. Ramanujan, 'Asymptotic formulae in Combinatory Analysis', *Proc. London Math. Soc.*, ser. 2, vol. 17, 1918, pp. 75—115 (Table IV, pp. 114—115).

From these

(5) $$p(35m+19) \equiv 0 \pmod{35}$$

follows at once as a corollary. These proofs I give in § 2 and § 3. I can also prove

(4) $$p(25n+24) \equiv 0 \pmod{25}$$

and (6) $$p(49n+47) \equiv 0 \pmod{49},$$

but only in a more recondite way, which I sketch in § 3.

§ 2. *Proof of* (1). We have

(11) $$\begin{aligned} x\{(1-x)(1-x^2)(1-x^3)\ldots\}^4 &= x(1-3x+5x^3-7x^6+\ldots)(1-x-x^2+x^5+\ldots) \\ &= \Sigma(-1)^{\mu+\nu}(2\mu+1)x^{1+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(3\nu+1)}, \end{aligned}$$

the summation extending from $\mu=0$ to $\mu=\infty$ and from $\nu=-\infty$ to $\nu=\infty$. Now if

$$1+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(3\nu+1) \equiv 0 \pmod{5}$$

then $$8+4\mu(\mu+1)+4\nu(3\nu+1) \equiv 0 \pmod{5},$$

and therefore

(12) $$(2\mu+1)^2+2(\nu+1)^2 \equiv 0 \pmod{5}.$$

But $(2\mu+1)^2$ is congruent to 0, 1, or 4, and $2(\nu+1)^2$ to 0, 2, or 3. Hence it follows from (12) that $2\mu+1$ and $\nu+1$ are both multiples of 5. That is to say, the coefficient of x^{5n} in (11) is a multiple of 5.

Again, all the coefficients in $(1-x)^{-5}$ are multiples of 5, except those of 1, x^5, x^{10}, ..., which are congruent to 1: that is to say

$$\frac{1}{(1-x)^5} \equiv \frac{1}{1-x^5} \pmod{5},$$

or $$\frac{1-x^5}{(1-x)^5} \equiv 1 \qquad \pmod{5}.$$

Thus all the coefficients in

$$\frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^5}$$

(except the first) are multiples of 5. Hence the coefficient of x^{5n} in

$$\frac{x(1-x^5)(1-x^{10})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} = x\{(1-x)(1-x^2)\ldots\}^4 \frac{(1-x^5)(1-x^{10})\ldots}{\{(1-x)(1-x^2)\ldots\}^5}$$

is a multiple of 5. And hence, finally, the coefficient of x^{5n} in

$$\frac{x}{(1-x)(1-x^2)(1-x^3)\ldots}$$

is a multiple of 5; which proves (1).

§ 3. *Proof of* (2). The proof of (2) is very similar. We have

$$(13)\quad x^2\{(1-x)(1-x^2)(1-x^3)\dots\}^6$$
$$= x^2(1-3x+5x^3-7x^6+\dots)^2$$
$$= \Sigma(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)\,x^{2+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)},$$

the summation now extending from 0 to ∞ for both μ and ν. If

$$2+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(\nu+1)\equiv 0 \pmod{7},$$

then
$$16+4\mu(\mu+1)+4\nu(\nu+1)\equiv 0 \pmod{7},$$
$$(2\mu+1)^2+(2\nu+1)^2\equiv 0 \pmod{7},$$

and $2\mu+1$ and $2\nu+1$ are both divisible by 7. Thus the coefficient of x^{7n} in (13) is divisible by 49.

Again, all the coefficients in

$$\frac{(1-x^7)(1-x^{14})(1-x^{21})\dots}{\{(1-x)(1-x^2)(1-x^3)\dots\}^7}$$

(except the first) are multiples of 7. Hence (arguing as in § 2) we see that the coefficient of x^{7n} in

$$\frac{x^2}{(1-x)(1-x^2)(1-x^3)\dots}$$

is a multiple of 7; which proves (2). As I have already pointed out, (5) is a corollary.

§ 4. The proofs of (4) and (6) are more intricate, and in order to give them I have to consider a much more difficult problem, viz. that of expressing

$$p(\lambda)+p(\lambda+\delta)x+p(\lambda+2\delta)x+\dots$$

in terms of Theta-functions, in such a manner as to exhibit explicitly the common factors of the coefficients, if such common factors exist. I shall content myself with sketching the method of proof, reserving any detailed discussion of it for another paper.

It can be shown that

$$(14)\quad \frac{(1-x^5)(1-x^{10})(1-x^{15})\dots}{(1-x^{\frac{1}{5}})(1-x^{\frac{2}{5}})(1-x^{\frac{3}{5}})\dots}=\frac{1}{\xi^{-1}-x^{\frac{1}{5}}-\xi x^{\frac{2}{5}}}$$
$$=\frac{\xi^{-4}-3x\xi+x^{\frac{1}{5}}(\xi^{-3}+2x\xi^2)+x^{\frac{2}{5}}(2\xi^{-2}-x\xi^3)+x^{\frac{3}{5}}(3\xi^{-1}+x\xi^4)+5x^{\frac{4}{5}}}{\xi^{-5}-11x-x^2\xi^5},$$

where
$$\xi=\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)\dots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\dots},$$

the indices of the powers of x, in both numerator and denominator of ξ, forming two arithmetical progressions with common difference 5. It follows that

$$(15)\quad (1-x^5)(1-x^{10})(1-x^{15})\ldots\{p(4)+p(9)x+p(14)x^2+\ldots\}$$
$$=\frac{5}{\xi^{-5}-11x-x^2\xi^5}.$$

Again, if in (14) we substitute $\omega x^{\frac{1}{5}}$, $\omega^2 x^{\frac{1}{5}}$, $\omega^3 x^{\frac{1}{5}}$, and $\omega^4 x^{\frac{1}{5}}$, where $\omega^5=1$, for $x^{\frac{1}{5}}$, and multiply the resulting five equations, we obtain

$$(16)\quad \left\{\frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}\right\}^6=\frac{1}{\xi^{-5}-11x-x^2\xi^5}.$$

From (15) and (16) we deduce

$$(17)\quad p(4)+p(9)x+p(14)x^2+\ldots$$
$$=5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^6};$$

from which it appears directly that $p(5m+4)$ is divisible by 5.

The corresponding formula involving 7 is

$$(18)\quad p(5)+p(12)x+p(19)x^2+\ldots$$
$$=7\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^3}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^4}$$
$$+49x\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^7}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^8},$$

which shows that $p(7m+5)$ is divisible by 7.

From (16) it follows that

$$\frac{p(4)x+p(9)x^2+p(14)x^3+\ldots}{5\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^4}$$
$$=\frac{x}{(1-x)(1-x^2)(1-x^3)\ldots}\,\frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^5}.$$

As the coefficient of x^{5n} on the right-hand side is a multiple of 5, it follows that $p(25m+24)$ is divisible by 25.

Similarly

$$\frac{p(5)x+p(12)x^2+p(19)x^3+\ldots}{7\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^2}$$
$$=x(1-3x+5x^3-7x^6+\ldots)\frac{(1-x^7)(1-x^{14})\ldots}{\{(1-x)(1-x^2)\ldots\}^7}$$
$$+7x^2\frac{\{(1-x^7)(1-x^{14})\ldots\}^5}{\{(1-x)(1-x^2)\ldots\}^8};$$

from which it follows that $p(49m+47)$ is divisible by 49.

[Another proof of (1) and (2) has been found by Mr H. B. C. Darling, to whom my conjecture had been communicated by Major MacMahon. This proof will also be published in these *Proceedings*. I have since found proofs of (3), (7), and (8).]

Proof of certain identities in combinatory analysis: (1) by Prof. L. J. Rogers; (2) by S. Ramanujan, B.A., Trinity College. (Communicated, with a prefatory note, by Mr G. H. Hardy.)

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[The identities in question are those numbered (10) and (11) in each of the two following notes, viz.

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\dots$$
$$=\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})}\dots\quad(1),$$

and

$$1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\dots$$
$$=\frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})}\dots\quad(2).$$

On the left-hand side the indices of the powers of q in the numerators are n^2 and $n(n+1)$, while in each of the products on the right hand side the indices of the powers of q form two arithmetical progressions with difference 5.

The formulae were first discovered by Prof. Rogers, and are contained in a paper published by him in 1894*. In this paper they appear as corollaries of a series of general theorems, and, possibly for this reason, they seem to have escaped notice, in spite of their obvious interest and beauty. They were rediscovered nearly 20 years later by Mr Ramanujan, who communicated them to me in a letter from India in February 1913. Mr Ramanujan had then no proof of the formulae, which he had found by a process of induction. I communicated them in turn to Major MacMahon and to Prof. O. Perron of Tübingen; but none of us were able to suggest a proof; and they appear, unproved, in Ch. 3, Vol. 2, 1916, of Major MacMahon's *Combinatory Analysis*†.

Since 1916 three further proofs have been published, one by

* L. J. Rogers, 'Second memoir on the expansion of certain infinite products', *Proc. London Math. Soc.*, ser. 1, vol. 25, 1894, pp. 318—343 (§ 5, pp. 328—329, formulae (1) and (2)).

† Pp. 33, 35.

Prof. Rogers* and two by Prof. I. Schur of Strassburg†, who appears to have rediscovered the formulae once more.

The proofs which follow are very much simpler than any published hitherto. The first is extracted from a letter written by Prof. Rogers to Major MacMahon in October 1917; the second from a letter written by Mr Ramanujan to me in April of this year. They are in principle the same, though the details differ‡. It seemed to me most desirable that the simplest and most elegant proofs of such very beautiful formulae should be made public without delay, and I have therefore obtained the consent of the authors to their insertion here.

It should be observed that the transformation of the infinite products on the right-hand sides of (1) and (2) into quotients of Theta-series, and the expression of the quotient of the series on the left-hand sides as a continued fraction, exhibited explicitly in Prof. Rogers' original paper and in Mr Ramanujan's present note, offer no serious difficulty. All the difficulty lies in the expression of these series as products, or as quotients of Theta-series.—G. H. H.]

1. (*By L. J. Rogers.*)

Suppose that $|q|<1$, and let V_m denote the convergent series

$$(1-x^m)-x^n q^{n+1-m}(1-x^m q^{2m})\,C_1 + x^{2n} q^{4n+3-2m}(1-x^m q^{4m})\,C_2 - \ldots\ldots\ldots\ldots(1),$$

where

$$C_r=\frac{(1-x)(1-xq)(1-xq^2)\ldots(1-xq^{r-1})}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^r)}:$$

the general term being

$$(-1)^r x^{nr} q^{nr^2+\frac{1}{2}r(r+1)-mr}(1-x^m q^{2mr})\,C_r.$$

Then

$$V_m-V_{m-1}=x^{m-1}(1-x)-x^n q^{n+1-m}\{(1-q)+x^{m-1}q^{2m-1}(1-xq)\}\,C_1 + x^{2n}q^{4n+3-2m}\{(1-q^2)+x^{m-1}q^{4m-2}(1-xq^2)\}\,C_2+\ldots\ \ldots\ldots(2).$$

Suppose now that the symbol η is defined by the equation

$$\eta f(x)=f(xq).$$

Then $(1-q^r)\,C_r=(1-x)\,\eta C_{r-1}$, $(1-xq^r)\,C_r=(1-x)\,\eta C_r$.

* L. J. Rogers, 'On two theorems of Combinatory Analysis and some allied identities', *Proc. London Math. Soc.*, ser. 2, vol. 16, 1917, pp. 315—336 (pp. 315—317).

† I. Schur, 'Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche', *Berliner Sitzungsberichte*, 1917, No. 23, pp. 301—321.

‡ I have altered the notation of Mr Ramanujan's letter so as to agree with that of Prof. Rogers.

Hence, arranging (2) in terms of ηC_1, ηC_2, ..., we obtain

$$\frac{V_m - V_{m-1}}{1-x}$$
$$= (x^{m-1} - x^n q^{n-m+1}) - x^{n+m-1} q^{n+m} (1 - x^{n-m+1} q^{3n-3m+3}) \eta C_1 + \dots$$
$$= x^{m-1} \{(1 - x^{n-m+1} q^{n-m+1}) - x^n q^{n+m} (1 - x^{n-m+1} q^{3n-3m+3}) \eta C_1 + \dots\}$$
$$= x^{m-1} \eta V_{n-m+1} \quad \dots\dots\dots (3).$$

If we write $$v_m \prod_{r=0}^{\infty} (1 - xq^r) = V_m \quad \dots\dots\dots (4),$$

then (3) becomes $$v_m - v_{m-1} = x^{m-1} \eta v_{n-m+1} \quad \dots\dots\dots (5).$$

It should be observed that V_0 and v_0 vanish identically.

In particular take $n = 2, m = 1$, and $n = 2, m = 2$. We then obtain $$v_1 = \eta v_2, \quad v_2 - v_1 = x\eta v_1;$$

and so $$v_1 - \eta v_1 = xq\eta^2 v_1 \quad \dots\dots\dots (6).$$

Now let $$v_1 = 1 + a_1 x + a_2 x^2 + \dots \quad \dots\dots\dots (7).$$

Then from (5)

$$1 + a_1 x + a_2 x^2 + \dots - (1 + a_1 xq + a_2 xq^2 + \dots)$$
$$= xq (1 + a_1 xq^2 + a_2 x^2 q^4 + \dots);$$

and so $$a_1 = \frac{q}{1-q}, \quad a_2 = \frac{q^4}{(1-q)(1-q^2)} \quad \dots\dots\dots (8).$$

But when $x = q$, $C_r = 1$; and so

$$V_1 = (1-q) - q^4 (1-q^3) + q^{13} (1-q^5) - \dots \quad \dots\dots (9).$$

From (4), (6), (7), and (8) it follows that

$$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \dots$$
$$= \frac{(1-q) - q^4(1-q^3) + q^{13}(1-q^5) - \dots}{(1-q)(1-q^2)(1-q^3)\dots} \quad \dots\dots\dots (10).$$

Similarly we have

$$v_2 = \frac{1}{\eta} v_1 = 1 + \frac{x}{1-q} + \frac{x^2 q^2}{(1-q)(1-q^2)} + \dots;$$

and, when $x = q$,

$$v_2 = 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \dots$$

and $$V_2 = (1-q^2) - q^3(1-q^6) + q^{11}(1-q^{10}) - \dots.$$

Thus

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\dots$$
$$=\frac{(1-q^2)-q^3(1-q^6)+q^{11}(1-q^{10})-\dots}{(1-q)(1-q^2)(1-q^3)} \qquad \dots\dots(11).$$

2. (*By S. Ramanujan.*)

Let

$$G(x)=1+\sum_1^\infty(-1)^\nu x^{2\nu}q^{\frac{1}{2}\nu(5\nu-1)}(1-xq^{2\nu})\frac{(1-xq)(1-xq^2)\dots(1-xq^{\nu-1})}{(1-q)(1-q^2)(1-q^3)\dots(1-q^\nu)}$$
$$=1-x^2q^2(1-xq^2)\frac{1}{1-q}$$
$$+x^4q^9(1-xq^4)\frac{1-xq}{(1-q)(1-q^2)}-\dots \qquad \dots\dots(1).$$

If we write $\qquad 1-xq^{2\nu}=1-q^\nu+q^\nu(1-xq^\nu)$,

every term in (1) is split up into two parts. Associating the second part of each term with the first part of the succeeding term, we obtain

$$G(x)=(1-x^2q^2)-x^2q^3(1-x^2q^6)\frac{1-xq}{1-q}$$
$$+x^4q^{11}(1-x^2q^{10})\frac{(1-xq)(1-xq^2)}{(1-q)(1-q^2)}-\dots \qquad \dots\dots(2).$$

Now consider $\qquad H(x)=\dfrac{G(x)}{1-xq}-G(xq) \qquad \dots\dots\dots\dots\dots\dots(3).$

Substituting for the first term from (2) and for the second term from (1), we obtain

$$H(x)=xq-\frac{x^2q^3}{1-q}\{(1-q)+xq^4(1-xq^2)\}$$
$$+\frac{x^4q^{11}(1-xq^2)}{(1-q)(1-q^2)}\{(1-q^2)+xq^7(1-xq^3)\}$$
$$-\frac{x^6q^{24}(1-xq^2)(1-xq^3)}{(1-q)(1-q^2)(1-q^3)}\{(1-q^3)+xq^{10}(1-xq^4)\}+\dots.$$

Associating, as before, the second part of each term with the first part of the succeeding term, we obtain

$$H(x) = xq(1-xq^2)\left\{1 - x^2q^6(1-xq^4)\frac{1}{1-q}\right.$$

$$+ x^4q^{17}(1-xq^6)\frac{1-xq^3}{(1-q)(1-q^2)}$$

$$\left. - x^6q^{33}(1-xq^8)\frac{(1-xq^3)(1-xq^4)}{(1-q)(1-q^2)(1-q^3)} + \dots\right\}$$

$$= xq(1-xq^2)\,G(xq^2) \quad \dots\dots(4).$$

If now we write $\quad K(x) = \dfrac{G(x)}{(1-xq)\,G(xq)},$

we obtain, from (3) and (4),

$$K(x) = 1 + \frac{xq}{K(xq)},$$

and so

$$K(x) = 1 + \frac{xq}{1+}\,\frac{xq^2}{1+}\,\frac{xq^3}{1+\dots} \quad \dots\dots(5).$$

In particular we have

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\dots} = \frac{1}{K(1)} = \frac{(1-q)\,G(q)}{G(1)} \quad \dots\dots(6);$$

or

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\dots} = \frac{1-q-q^4+q^7+q^{13}-\dots}{1-q^2-q^3+q^9+q^{11}-\dots} \quad \dots\dots(7).$$

This equation may also be written in the form

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\dots} = \frac{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\dots}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\dots} \quad \dots\dots(8).$$

If we write

$$F(x) = \frac{G(x)}{(1-xq)(1-xq^2)(1-xq^3)\dots},$$

then (4) becomes $\quad F(x) = F(xq) + xq\,F(xq^2),$

from which it readily follows that

$$F(x) = 1 + \frac{xq}{1-q} + \frac{x^2q^4}{(1-q)(1-q^2)} + \frac{x^3q^9}{(1-q)(1-q^2)(1-q^3)} + \dots \quad (9).$$

In particular we have

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\ldots=\frac{G(1)}{(1-q)(1-q^2)(1-q^3)\ldots}$$

$$=\frac{1-q^2-q^3+q^9+q^{11}-\ldots}{(1-q)(1-q^2)(1-q^3)\ldots}$$

$$=\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\ldots} \quad\ldots\ldots(10),$$

and

$$1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\ldots=\frac{(1-q)\,G(q)}{(1-q)(1-q^2)(1-q^3)\ldots}$$

$$=\frac{1-q-q^4+q^7+q^{13}-\ldots}{(1-q)(1-q^2)(1-q^3)\ldots}$$

$$=\frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\ldots} \quad\ldots\ldots(11).$$

On Mr Ramanujan's congruence properties of $p(n)$. By H. B. C. DARLING. (Communicated by Mr G. H. Hardy.)

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1. *Proof that* $p(5m+4) \equiv 0 \pmod 5$.

Let $$u = (1-x)(1-x^2)(1-x^3)\ldots;$$
then by Jacobi's expansion
$$u^3 = \sum_{n=0}^{n=\infty} (-1)^n (2n+1)\, x^{\frac{1}{2}n(n+1)};$$
so that in $\partial^2 u^3$, where ∂ denotes differentiation with respect to x, the coefficients are of the form
$$\tfrac{1}{4}(n-1)\,n\,(n+1)(n+2)\{2(n+3)-5\},$$
and therefore
$$\partial^2 u^3 \equiv 0 \pmod 5 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, in $\partial^4 u^3$ the coefficients are of the form
$$\tfrac{1}{16}(n^2+n-4)(n-2)(n-1)\,n\,(n+1)(n+2)\{2(n+4)-7\},$$
and therefore
$$\partial^4 u^3 \equiv 0 \pmod 7 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now $$\partial^2\left(\frac{1}{u}\right) = -\frac{1}{u^2}\,\partial^2 u + \frac{2}{u}(\partial u)^2;$$
also $\partial u^3 = 3u^2\partial u$, and $\partial^2 u^3 = 3u^2\partial^2 u + 6u(\partial u)^2$. Hence
$$\partial^2\left(\frac{1}{u}\right) = -\frac{1}{3u^4}\,\partial^2 u^3 + \frac{4}{9u^7}(\partial u^3)^2 \quad\ldots\ldots\ldots\ldots(3);$$
and thus, by (1), we have
$$\partial^3\left(\frac{1}{u}\right) \equiv -\frac{28}{9u^8}(\partial u^3)^2\,\partial u \equiv -\frac{28}{27u^{10}}(\partial u^3)^3 \pmod 5;$$
so that $$\partial^4\left(\frac{1}{u}\right) \equiv 0 \pmod 5 \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again if $1/u$ be expanded in powers of x, and the operator ∂^4 be applied to the resulting series, it is evident that the coefficients of all powers of x of the forms $5m$, $5m+1$, $5m+2$ and $5m+3$ will be multiplied by a factor divisible by 5; but that the coefficients of the powers of x of the form $5m+4$ will be multiplied by a factor which is not divisible by 5. Hence it follows at once from (4) that
$$p(5m+4) \equiv 0 \pmod 5.$$

2. *Proof that* $p(7m+5) \equiv 0 \pmod 7$.

Differentiating (3), we have

$$\partial^3\left(\frac{1}{u}\right) \equiv -\frac{1}{3u^4}\partial^3 u^3 + \frac{4}{3u^5}\partial u \partial^2 u^3 + \frac{4}{9u^7}\partial(\partial u^3)^2 \pmod 7$$

$$\equiv -\frac{1}{3u^4}\partial^3 u^3 + \frac{6}{9u^7}\partial(\partial u^3)^2 \pmod 7.$$

Similarly, having regard to (2),

$$\partial^4\left(\frac{1}{u}\right) \equiv \frac{4}{9u^7}\partial u^3 \partial^3 u^3 + \frac{6}{9u^7}\partial^2(\partial u^3)^2 \pmod 7,$$

$$\partial^5\left(\frac{1}{u}\right) \equiv \frac{4}{9u^7}\partial^2 u^3 \partial^3 u^3 + \frac{6}{9u^7}\partial^3(\partial u^3)^2 \pmod 7 \quad \text{.........(5)},$$

$$\partial^6\left(\frac{1}{u}\right) \equiv \frac{4}{9u^7}(\partial^3 u^3)^2 + \frac{6}{9u^7}\partial^4(\partial u^3)^2 \pmod 7 \quad \text{...........(6)}.$$

Again $\partial^6\left(\frac{x}{u}\right) = x\partial^6\left(\frac{1}{u}\right) + 6\partial^5\left(\frac{1}{u}\right) = x^{-5}\partial\left\{x^6\partial^5\left(\frac{1}{u}\right)\right\};$

so that, by (5) and (6),

$$\partial^6\left(\frac{x}{u}\right) = \frac{x^{-5}}{9u^7}\left[4\partial\left(x^6\partial^2 u^3 \partial^3 u^3\right) + 6\partial\left\{x^6\partial^3(\partial u^3)^2\right\}\right] \quad \text{......(7)}.$$

Now
$$\partial(\partial u^3)^2 = 2\partial^2 u^3 \partial u^3,$$
$$\partial^2(\partial u^3)^2 = 2\partial^3 u^3 \partial u^3 + 2(\partial^2 u^3)^2.$$

Thus, by (2),
$$\partial^3(\partial u^3)^2 \equiv 6\partial^3 u^3 \partial^2 u^3 \pmod 7;$$

and therefore, by (7), we see that

$$\partial^6\left(\frac{x}{u}\right) \equiv \partial\left\{x^6\partial^3 u^3 \partial^2 u^3\right\} \pmod 7;$$

that is, by (2),

$$\partial^6\left(\frac{x}{u}\right) \equiv x^6\partial^3 u^3 \partial^3 u^3 + 6x^5\partial^3 u^3 \partial^2 u^3 \pmod 7$$

$$\equiv \partial^3 u^3 \partial\left(x^6\partial^2 u^3\right) \pmod 7 \quad \text{....................(8)}.$$

But the coefficients in $\partial(x^6\partial^2 u^3)$ are of the form

$$\tfrac{1}{8}(n-1)n(n+1)(n+2)\{2(n-3)+7\}\{(n-2)(n+3)+14\},$$

and are therefore divisible by 7; and therefore, by (8),

$$\partial^6\left(\frac{x}{u}\right) \equiv 0 \pmod 7.$$

Hence, by considerations similar to those in the latter part of § 1, we see that

$$p(7m+5) \equiv 0 \pmod 7.$$

On the exponentiation of well-ordered series. By Miss DOROTHY WRINCH. (Communicated by Mr G. H. HARDY.)

[*Read* 29 October 1918.]

The problem before us in this paper is the investigation of the necessary and sufficient conditions that P^Q should be Dedekindian or semi-Dedekindian when P and Q are well ordered series.

The field of P^Q is the class of Cantor's *Belegungen* and consists of those relations which cover all the members of the field of Q with members of the field of P: several members of the field of Q may be covered with the same member of the field of P, but every member of the field of Q is covered with one member of the field of P and one only. In order to prove that P^Q is Dedekindian it is necessary to prove that every sub-class of the field of P^Q has a lower limit or minimum with respect to P^Q. If there is a last term of the series P^Q it is the lower limit of the null class. Unit sub-classes have their unique members as minima. It remains, then, to consider sub-classes with two or more members.

Now the relation P^Q orders two relations R and S by putting R before S, if R covers the first Q-term, which is not covered with the same P-term by both R and S, with a P-term occurring earlier in the P-series than the term with which S covers it. Suppose λ is a sub-class of the field of P^Q with at least two members. We will call $Q_{\mathrm{m}}\text{‘}\lambda$ the first Q-term which is not covered with the same P-term by all λ's; and $\breve{T}_P\text{‘}\lambda$ that subset of λ which consists of those members of λ which cover $Q_{\mathrm{m}}\text{‘}\lambda$ with that term, in the class of P-terms with which various λ's cover $Q_{\mathrm{m}}\text{‘}\lambda$, which occurs earliest in the P-order. $\breve{T}_P\text{‘}\lambda$ will therefore be contained in λ and not identical with it. It will be seen that P^Q-terms belonging to $\breve{T}_P\text{‘}\lambda$ come earlier in the P^Q-order than terms of λ not belonging to it. Constructing

$$\breve{T}_P\text{‘}\breve{T}\text{‘}\lambda$$

we get a smaller subset of λ: members of this subset occur earlier in P^Q than other members of λ. Continuing this process with

$$\lambda,\ \breve{T}_P\text{‘}\lambda,\ \breve{T}_P\text{‘}\breve{T}_P\text{‘}\lambda,\ \breve{T}_P\text{‘}\breve{T}_P\text{‘}\breve{T}_P\text{‘}\lambda, \ldots\ \mu, \ldots\ \nu, \ldots,$$

we obtain smaller and smaller sub-classes of λ: if μ precedes ν in this order, members of ν occur earlier in the P^Q-order than members of μ which are not members of ν. We take the common part of all these subsets of λ, i.e. the class of relations which belong to all the sets

$$\lambda,\ \breve{T}_P`\lambda,\ \breve{T}_P`\breve{T}_P`\lambda \ldots;$$

and get a subset of λ

$$p`(\overleftarrow{T_P})_*`\lambda$$

which, again, consists of members of λ which come earlier in the P^Q-order than members of λ not belonging to it. Repeating the original procedure we get

$$\breve{T}_P`p`(\overleftarrow{T_P})_*`\lambda,\ \breve{T}_P`\breve{T}_P`p`(\overleftarrow{T_P})_*`\lambda, \ldots,$$

and so obtain a series of sub-classes of λ ordered by the serial relation

$$A\,(T_P, \lambda),$$

where A is the relation between μ and ν when ν is contained in μ but not identical with it. And this is a well-ordered relation: consequently it will have an end, viz.

$$p`(T_P{*}A)`\lambda.$$

If this is not null, it consists of a single member, which will be the minimum of λ in P^Q. But if it is null we will put

$$PQ`\lambda = \dot{s}`\hat{N}\,\{\exists\mu\,.\,\mu\epsilon\,(T_P{*}A)`\lambda\,.\,N = (\dot{p}`\mu)\upharpoonright\overrightarrow{Q}`Q_{\mathrm{m}}`\mu\}.$$

Then $PQ`\lambda$ is a relation covering a certain section of the $Q$-terms with $P$-terms: $PQ`\lambda$ agrees in the way it covers the Q-spaces with each member μ of $A\,(T_P, \lambda)$ as far as $Q_{\mathrm{m}}`\mu$. $PQ`\lambda$ will therefore cover Q-spaces up to z, if there is a μ which is a member of the field of

$$A\,(T_P, \lambda)$$

such that z precedes $Q_{\mathrm{m}}`\mu$ in the $Q$-order. If no member of the field of $A\,(T_P, \lambda)$ agrees in the covering of $Q$-spaces beyond a certain member $z$ of the field of $Q$, $PQ`\lambda$ covers no spaces beyond z with P-terms and for this reason is not a member of the field of P^Q.

If R is a P^Q-term which agrees with $PQ`\lambda$ in the covering of $Q$-spaces as far as it goes, $R$ precedes all the members of $\lambda$ in the $P^Q$-order; further, any member of the field of $P^Q$, following $R$ and all relations agreeing with $PQ`\lambda$ as far as it goes, follows at least one member of λ. Hence, if there were a maximum in the P^Q-order in the class ρ of members of the field of P^Q which agree with $PQ`\lambda$ as far as it goes, this relation would precede all $\lambda$'s and any relation following it would follow at least one member of $\lambda$. If the class consists of one term $R$, it will have a maximum, namely $R$ itself: $R$ will then be equal to $PQ`\lambda$ and $PQ`\lambda$ will, therefore, be the lower limit of $\lambda$. But $\rho$ is a unit class only when $PQ`\lambda$ covers the *whole* of the Q-terms with P-terms. When $PQ`\lambda$ does not cover the whole of the Q-terms, but covers Q-terms only up to z (say), all ρ's will agree in their covering of Q-spaces up to z, and the remaining Q-spaces will be covered differently by different members of ρ. To get a maximum of the ρ's with respect to P^Q, we want a relation S which is a ρ such that no member of ρ comes later in the P^Q-order. Now, if P has no last term, every P-term is followed by other P-terms. However S covers z and the Q-spaces after z, by replacing the term covering any member of the field of Q after z by a member of the field of P following it in the P-order, we obtain a relation T which is a ρ and follows S in the P^Q-order. S is, consequently, not the maximum of ρ in the P^Q-order. Now if z in the field of Q is covered by $PQ`\lambda$, the term immediately following $z$ will also be covered by $PQ`\lambda$. Therefore, if Q is a finite series or an ω, $PQ`\lambda$ will always cover the whole of the Q-terms; since, as λ has at least two members, it will always cover one Q-term. Any λ will then have a lower limit or minimum with respect to P^Q. In such cases, P^Q will certainly be Dedekindian with the addition of a last term, whether P has a last term itself or not.

But if $\mathrm{Nr}`Q$ is greater than $\omega$, it is possible to find a subclass $\lambda$ of the field of $P^Q$ which is such that $PQ`\lambda$ does not cover the whole of the field of Q.

For, let 1 and 2 represent the first and second terms in the P-series and let (e.g.)

$$1 \ldots \vdash 1\,(\zeta)\,2 \ldots \vdash 2\,(\xi),\ 111 \ldots\ldots$$

represent a relation which covers the first ζ Q-terms with 1, subsequent terms up to (but not including) the ξth term with 2, and all remaining terms with 1. Such a relation is clearly a member of the field of P^Q. Consider the class of relations λ which cover all Q-spaces up to z with 1, and all the Q-spaces following z with 2, as z is varied from the second Q-term to the ζth, where ζ is an ordinal number with no immediate predecessor. We will arrange this class of relations in the P^Q-order.

$$\begin{array}{ll}
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
1 \ldots \vdash 1\,(\varpi)\ 2 \ldots \vdash 2\,(\zeta),\ 22 \ldots\ldots & (\varpi < \zeta) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
11112 \ldots\ldots\ldots\ldots \vdash 2\,(\zeta),\ 22 \ldots\ldots\ldots\ldots & \\
11122 \ldots\ldots\ldots\ldots \vdash 2\,(\zeta),\ 22 \ldots\ldots\ldots\ldots & \\
11222 \ldots\ldots\ldots\ldots \vdash 2\,(\zeta),\ 22 \ldots\ldots\ldots\ldots & \\
12222 \ldots\ldots\ldots\ldots \vdash 2\,(\zeta),\ 22 \ldots\ldots\ldots\ldots &
\end{array}$$

This class has no minimum in the P^Q-order, and $PQ`\lambda$ covers all the Q-places up to the ζth with 1 and does not cover the subsequent Q-places at all. It is therefore not a member of the field of P^Q. But, as we have seen, every relation which agrees with PQ‘λ as far as it goes, and covers the other Q-places with any P-terms whatever, precedes all λ's: and any member of the field of P^Q following this relation, and all relations agreeing with PQ‘λ as far as it goes, follows at least one member of λ. Thus, e.g., the relation

$$11 \ldots \vdash 1\,(\zeta),\ 2111 \ldots\ldots\ldots$$

precedes all λ's, and any relation following it and all relations agreeing with PQ‘λ as far as it goes (as e.g. the relation

$$11211 \ldots \vdash 1\,(\zeta),\ 21211 \ldots)$$

follows at least one relation belonging to λ, e.g. the relation

$$11122 \ldots \vdash 2\,(\zeta),\ 222 \ldots$$

Thus λ will have a lower limit if and only if there is a maximum among the relations covering all places up to the ζth with 1. And this is the case when and only when P has a last term u (say). For then the relation

$$111 \ldots\ldots \vdash 1\,(\zeta)\, uuu \ldots$$

will be the lower limit of λ. Thus if Nr‘Q is greater than ω, it will be the case that all existent sub-classes of the field of P^Q will have a lower limit or minimum when and only when P has a last term. A non-existent subclass (i.e. a subclass with no members) will have a lower limit or minimum when and only when P has a last term. If Nr‘Q is greater than ω, P^Q is Dedekindian when P has a last term, and if P has no last term P^Q even with the addition of a last term is not Dedekindian. We thus arrive at the following conclusions. When P and Q are well-ordered series, (1) P^Q is Dedekindian when and only when P has a last term; (2) if Nr‘Q is greater than ω, P^Q with the addition of a last term is Dedekindian if and only if P has a last term; (3) if P^Q is made Dedekindian by the addition of a last term when and only when P has a last term, Nr‘Q is greater than ω.

These propositions will now be established.

[*The symbols used are those of* Principia Mathematica. *Among the propositions referred to, those whose numbers are greater than 1 are proved in* P.M., *while the others are established in the course of this paper.*]

$*01.\quad Q_{\mathrm{m}}‘\lambda = \min_Q‘\hat{y}\,(\overrightarrow{\dot{s}‘\lambda‘y} \sim \epsilon\, 0 \cup 1)$ Df

$*02.\quad \breve{T}_P‘\lambda = \lambda \cap \hat{M}\,(M‘Q_{\mathrm{m}}‘\lambda = \min_P‘\overrightarrow{\dot{s}‘\lambda}‘Q_{\mathrm{m}}‘\lambda)$ Df

$*{\cdot}03.\quad A = \hat{\lambda}\,\hat{\mu}\,(\mu \subset \lambda \,.\, \mu \neq \lambda)$ Df

$*{\cdot}04.\quad PQ`\lambda = \dot{s}`\hat{N}\,(\exists\mu \,.\, \mu \in (T_P * A)`\lambda \,.\, N = (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\mu)$ Df

$*{\cdot}1.\quad \vdash : P, Q \in \Omega \,.\, \lambda \in 0 \,.\, \supset .\, \overrightarrow{B}`\mathrm{Cnv}`P^Q = \overrightarrow{\min}`(P^Q)`\lambda$ [$*207{\cdot}17$]

$*{\cdot}11.\quad \vdash : P, Q \in \Omega \,.\, \lambda \in 1 \,.\, \lambda \subset C`P^Q \,.\, \supset \breve{\iota}`\lambda = \min\,(P^Q)`\lambda$

Dem.

$\vdash .\, *176{\cdot}19 \,.\quad \supset \vdash : R \in C`P^Q \,.\, \supset_R .\, \sim (RP^QR)$ (1)

$\vdash .\, (1) \,.\, *205{\cdot}18 \,.\quad \supset \vdash .\,$ Prop

$*{\cdot}201.\quad \vdash : P, Q \in \Omega \,.\, \lambda \subset C`P^Q \,.\, \mathrm{E}\,!\,\breve{T}_P`\lambda \,.\, \supset .\, p`(T_P * A)`\lambda$
$= B`\mathrm{Cnv}`A\,(T_P, \lambda) \,.\, A\,(T_P, \lambda) \in \Omega$

Dem.

[$\cdot 02$] $\vdash .\, T_P \in Rl`A \cap \mathrm{Cls} \to 1$ (1)

$\vdash .\, (1) \,.\, *258{\cdot}231 \,.\quad \supset \vdash .\,$ Prop

$*{\cdot}202.\quad \vdash : \mathrm{E}\,!\,\breve{T}_P`\lambda \,.\, \supset .\, p`(T_P * A)`\lambda \sim \in \mathrm{D}`T_P$ [$*{\cdot}201$]

$*{\cdot}203.\quad \vdash : P, Q \in \Omega \,.\, \lambda \subset C`P^Q \,.\, \lambda \sim \in 0 \cup 1 \,.\, \supset .\, \mathrm{E}\,!\,\breve{T}_P`\lambda$

Dem.

$\vdash :.\, \lambda \subset C`P^Q \,.\, R, S \in \lambda \,.\, x \in \mathrm{G}`R \,.\, \supset_{R,S} .\, R`x = S`x : \supset .\, \lambda \in 0 \cup 1 :.$

[Transp] $\supset \vdash :.\, \mathrm{Hp} \,.\, \supset : R, S \in \lambda \,.\, \supset .\, \exists x \,.\, R`x \neq S`x \,.\, x \in \mathrm{G}`R :.$

[$*250{\cdot}121$] $\supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \mathrm{E}\,!\,\min_Q`\hat{y}\,(\dot{s}`\overrightarrow{\lambda}`y \sim \in 0 \cup 1) :.$

[$*{\cdot}01{\cdot}02$] $\supset \vdash :.\, \mathrm{Hp} \,.\, \supset : \mathrm{E}\,!\,Q_m`\lambda \,.\, \mathrm{E}\,!\,\breve{T}_P`\lambda$

$*{\cdot}2031.\quad \vdash : \mathrm{E}\,!\,T_P`\lambda \,.\, \supset .\, \mathrm{E}\,!\,Q_m`\lambda$ [$*{\cdot}02$]

$*{\cdot}204.\quad \vdash : P, Q \in \Omega \,.\, \lambda \subset C`P^Q \,.\, \lambda \sim \in \mathrm{D}`\breve{T}_P \,.\, \supset .\, \lambda \in 0 \cup 1$
[$*{\cdot}203$. Transp]

$*{\cdot}205.\quad \vdash .\, \mathrm{Hp}\, *{\cdot}203 \,.\, \supset .\, p`(T_P * A)`\lambda \in 0 \cup 1$ [$**{\cdot}202{\cdot}203{\cdot}204$]

$*{\cdot}211.\quad \vdash .\, \mathrm{E}\,!\,Q_m`\lambda \,.\, \supset .\, \dot{s}`\overrightarrow{\lambda}`Q_m`\lambda \sim \in 0 \cup 1$ [$*{\cdot}01$]

$*{\cdot}212.\quad \vdash :.\, \mathrm{E}\,!\,Q_m`\lambda : \supset : (\dot{s}`\lambda) \upharpoonright \overrightarrow{Q}`Q_m`\lambda \in 1 \to \mathrm{Cls} : R \in \lambda \,.$
$\supset .\, R \in 1 \to \mathrm{Cls}$

Dem.

[$*176{\cdot}19$] $\vdash :.\, R \in C`P^Q \,.\, \supset : z \in C`Q \,.\, \supset .\, \overrightarrow{R}`z \sim \in 0$ (1)

$\vdash .\, (1) \,.\, *{\cdot}02 \,.\quad \supset \vdash .\,$ Prop

$*\cdot 213$. $\vdash : S \in 1 \rightarrow \mathrm{Cls} \,.\, R \mathrel{\mathsf{G}} S \,.\, \supset \,.\, S \upharpoonright \mathrm{G}{}^{\backprime}R = R$

Dem.

$$\vdash \,.\, \exists u, v \,.\, uSv \,.\, v \in \mathrm{G}{}^{\backprime}R \,.\, \sim(uRv) \,.\, \supset \,.\, \exists u, v, u' \,.\, u' \neq u \,.\, uSv \,.\, u'Rv \,.\, \sim(uRv) \,.\, v \in \mathrm{G}{}^{\backprime}R :$$

$$\supset \vdash :.\, R \mathrel{\mathsf{G}} S \,.\, \supset : \exists u, v \,.\, uSv \,.\, v \in \mathrm{G}{}^{\backprime}R \,.\, \sim(uRv) \,.\, \supset \,.\, \exists u, v, u' \,.\, u' \neq u \,.\, uSv \,.\, u'Sv :$$

$$\supset \vdash :.\, R \mathrel{\mathsf{G}} S \,.\, S \in 1 \rightarrow \mathrm{Cls} \,.\, \supset : \sim \{\exists u, v \,.\, uSv \,.\, v \in \mathrm{G}{}^{\backprime}R \,.\, \sim(uRv)\}$$

$$\supset \vdash :.\, R \mathrel{\mathsf{G}} S \,.\, S \in 1 \rightarrow \mathrm{Cls} \,.\, \supset : uSv \,.\, v \in \mathrm{G}{}^{\backprime}R \,.\, \supset_{u,v} \,.\, uRv :$$

$$\supset \vdash :.\, R \mathrel{\mathsf{G}} S \,.\, S \in 1 \rightarrow \mathrm{Cls} \,.\, \supset : S \upharpoonright \mathrm{G}{}^{\backprime}R = R$$

$*\cdot 2131$. $\vdash :.\, \exists\,!\, \varpi \,.\, \alpha \subset \mathrm{G}{}^{\backprime}\dot{s}{}^{\backprime}\varpi \,.\, (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha \in 1 \rightarrow \mathrm{Cls} : \supset : R \in \varpi \,.\, \supset \,.\, R \upharpoonright \alpha = (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha = (\dot{p}{}^{\backprime}\varpi) \upharpoonright \alpha$

Dem.

$$\vdash \,.\, *40\cdot 13 \,.\, *41\cdot 44 \,.\, \supset \vdash :.\, R \in \varpi \,.\, \supset : xRy \,.\, \supset \,.\, x(\dot{s}{}^{\backprime}\varpi)y : \mathrm{G}{}^{\backprime}R \subset \mathrm{G}{}^{\backprime}(\dot{s}{}^{\backprime}\varpi) :.$$

$$\supset \vdash : \alpha \subset \mathrm{G}{}^{\backprime}R \,.\, R \in \varpi \,.\, \supset \,.\, R \upharpoonright \alpha \mathrel{\mathsf{G}} (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha \quad (1)$$

$$\vdash *\cdot 213 \,.\, (1) \,.\, \supset \vdash : \alpha \subset \mathrm{G}{}^{\backprime}R \,.\, R \in \varpi \,.\, (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha \in 1 \rightarrow \mathrm{Cls} \,.\, \supset \,.\, R \upharpoonright \alpha = (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha \quad (2)$$

$$\vdash :.\, \exists\,!\, \varpi \,.\, R \in \varpi \,.\, \supset_R \,.\, xRy \,.\, y \in \alpha : \supset : \exists S \,.\, S \in \varpi \,.\, xSy \,.\, y \in \alpha \quad (3)$$

$$\vdash \,.\, (3) \,.\, \supset \vdash : \exists\,!\, \varpi \,.\, \supset \,.\, (\dot{p}{}^{\backprime}\varpi) \upharpoonright \alpha \mathrel{\mathsf{G}} (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha \quad (4)$$

$$\vdash \,.\, (4) \,.\, *\cdot 213 \,.\, \supset \vdash :.\, \exists\,!\, \varpi \,.\, \supset : (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha \in 1 \rightarrow \mathrm{Cls} \,.\, \supset \,.\, (\dot{s}{}^{\backprime}\varpi) \upharpoonright \alpha = (\dot{p}{}^{\backprime}\varpi) \upharpoonright \alpha \quad (5)$$

$$\vdash \,.\, (2) \,.\, (5) \,.\, \supset \vdash \,.\, \mathrm{Prop}$$

$*\cdot 214$. $\vdash :.\, \mathrm{Hp}\, *\cdot 203 : \supset : R \in \lambda \,.\, \supset \,.\, R \upharpoonright \overrightarrow{Q}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda = (\dot{s}{}^{\backprime}\lambda) \upharpoonright \overrightarrow{Q}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda = (\dot{p}{}^{\backprime}\lambda) \upharpoonright \overrightarrow{Q}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda$ $\quad [*40\cdot 13 \,.\, *41\cdot 44 \,.\, *\cdot 2131]$

$*\cdot 215$. $\vdash :.\, \mathrm{Hp}\, *\cdot 203 : \supset : R \in \lambda - \breve{T}_P{}^{\backprime}\lambda \,.\, S \in \breve{T}_P{}^{\backprime}\lambda \,.\, \supset \,.$

$$R \upharpoonright \overrightarrow{Q}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda = S \upharpoonright \overrightarrow{Q}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda \,.\, (S{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda)\, P\, (R{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda)$$

Dem.

$$[*\cdot 02] \quad \vdash : S \in \breve{T}_P{}^{\backprime}\lambda \,.\, \supset \,.\, S \in \lambda \quad (1)$$

$$[**\cdot 01\cdot 02] \quad \vdash : E\,!\, Q_{\mathrm{m}}{}^{\backprime}\lambda \,.\, S \in \breve{T}_P{}^{\backprime}\lambda \,.\, R \in \lambda - \breve{T}_P{}^{\backprime}\lambda \,.$$

$$\supset \,.\, S{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda = \min_P{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\lambda}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda \,.\, R{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda \neq \min{}^{\backprime}\overrightarrow{\dot{s}{}^{\backprime}\lambda}{}^{\backprime}Q_{\mathrm{m}}{}^{\backprime}\lambda \quad (2)$$

$$\vdash \,.\, (1) \,.\, (2) \,.\, *\cdot 214 \,.\, \supset \vdash \,.\, \mathrm{Prop}$$

$*{\cdot}\mathbf{216}$. $\vdash :. \mathrm{Hp}\, *{\cdot}203 : \supset : R \in \lambda - \breve{T}_P{'}\lambda \,.\, S \in \breve{T}_P{'}\lambda \,.\supset .\, SP^Q R$ [$*{\cdot}215$]

$*{\cdot}\mathbf{217}$. $\vdash :. \mathrm{Hp}\, *{\cdot}203 : \supset : \mu \in (T_P{*}A){'}\lambda \,.\, R \in \lambda - \mu \,.\, S \in \mu \,.\supset .\, SP^Q R$

Dem.

$\vdash . *40{\cdot}23 . \quad \supset \vdash :. \rho \subset (T_P{*}A){'}\lambda \,.\, \exists !\, \rho : \mu \in \rho \,.\, S \in \mu \,.\, R \in \lambda - \mu \,.$
$\supset .\, SP^Q R : \supset : S \in p{'}\rho \,.\, R \in \lambda - p{'}\rho \,.\supset_{R,S} .\, SP^Q R \quad (1)$

$\vdash . (1) . *{\cdot}216 . *258{\cdot}241 . \supset \vdash .$ Prop

$*{\cdot}\mathbf{218}$. $\vdash : \mathrm{Hp}\, *{\cdot}203 \,.\, \exists !\, p{'}(T_P{*}A){'}\lambda \,.\supset .\, \breve{\iota}{'}p{'}(T_P{*}A){'}\lambda = \min (P^Q){'}\lambda$

Dem.

$\vdash . **{\cdot}217{\cdot}201 . \quad \supset \vdash : \mathrm{Hp} \,.\supset : S \in \lambda - p{'}(T_P{*}A){'}\lambda \,.$
$R \in p{'}(T_P{*}A){'}\lambda \,.\supset .\, RP^Q S \quad (1)$

$\vdash . (1) . \quad \supset \vdash .$ Prop

$*{\cdot}\mathbf{31}$. $\vdash :. S \in \lambda : \supset : k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$k \in (T_P{*}A){'}\lambda \,.\, S \sim\in \breve{T}_P{'}k$

Dem.

[$*22{\cdot}43$] $\quad \vdash :. S \in \lambda \,.\supset : k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$\exists \rho \,.\, \rho \subset (T_P{*}A){'}\lambda \,.\, \exists !\, \rho \,.\, \rho = \{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\, k = p{'}\rho$

[$*257{\cdot}125$] $\supset \vdash :. S \in \lambda \,.\supset : k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$\exists \rho \,.\, \mathrm{lt}_A{'}\rho \in (T_P{*}A){'}\lambda \,.\, \rho = \{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\, k = p{'}\rho$

[$*258{\cdot}211$] $\supset \vdash :. S \in \lambda \,.\supset .\, k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$k \in (T_P{*}A){'}\lambda \quad (1)$

$\vdash . (1) . *257{\cdot}125 .$
$\supset \vdash :. S \in \lambda \,.\supset .\, k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$\breve{T}_P{'}k \in (T_P{*}A){'}\lambda \quad (2)$

[$*40{\cdot}12$] $\quad \supset \vdash :. S \in \lambda \,.\supset .\, k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$\varpi \in (T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S \,.\supset_{\varpi} .\, k \subset \varpi :$

[Transp] $\quad \supset \vdash :. S \in \lambda \,.\supset .\, k = p{'}\{(T_P{*}A){'}\lambda \cap \overleftarrow{\epsilon}{'}S\} \,.\supset .$
$\varpi \in (T_P{*}A){'}\lambda \,.\, \sim(k \subset \varpi) \,.\supset .\, S \sim\in \varpi \quad (3)$

$\vdash . *22\cdot43 . *\cdot04 . (2) .$

$$\supset \vdash :. S \in \lambda . \supset . k = p`\{(T_P * A)`\lambda \cap \overleftarrow{\epsilon}`S\} : \supset .$$
$$\sim (k \subset \breve{T}_P`k) . \breve{T}_P`k \in (T_P * A)`\lambda \quad (4)$$

$\vdash . (3) . (4) . \quad \supset \vdash :. S \in \lambda . \supset . k = p`\{(T_P * A)`\lambda \cap \overleftarrow{\epsilon}`S\} . \supset .$

$$\sim (S \in \breve{T}_P`k) \quad (5)$$

$\vdash . (1) . (5) . \qquad \supset \vdash . \text{Prop}$

$*\mathbf{32}.$ $\vdash : \text{Hp} *\cdot203 . \mu \breve{T}_P \lambda . \text{E} ! \breve{T}_P`\mu . \supset . (Q_m`\lambda) \, Q \, (Q_m`\mu)$

Dem.

$[*\cdot212]$ $\quad \vdash : \text{Hp} *\cdot203 . R \in \lambda . \supset . z \, Q \, (Q_m`\lambda) \supset . R`z \in 1 \quad (1)$

$[**\cdot02\cdot203]$ $\quad \vdash . \mu \breve{T}_P \lambda . \supset . \mu \subset \lambda \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash :. \text{Hp} *\cdot203 . \mu \breve{T}_P \lambda . S \in \mu .$

$$\supset : z \, Q \, (Q_m`\lambda) . \supset . S`z \in 1 \quad (3)$$

$[*\cdot02]$ $\quad \vdash : \text{Hp} *\cdot203 . \mu \breve{T}_P \lambda . S \in \mu .$

$$\supset . S`Q_m`\lambda = \min_P`\hat{y} \, \{\overrightarrow{\dot{s}`\lambda}`y \sim \in 0 \cup 1\} \quad (4)$$

$\vdash . (3) . (4) . \quad \vdash :: \text{Hp} *\cdot203 . \mu \breve{T}_P \lambda . S \in \mu .$

$$\supset :. \exists z' : (Q_m`\lambda) \, Qz' : uQz' . \supset_u . S`u \in 1 \quad (5)$$

$\vdash . (5) . \quad \vdash : \text{Hp} *\cdot203 . \mu \breve{T}_P \lambda . \text{E} ! \breve{T}_P`\mu . \supset . (Q_m`\lambda) \, Q \, (Q_m`\mu)$

$*\mathbf{321}.$ $\vdash : \text{Hp} *\cdot203 . \mu \, (A \, (T_P, \lambda)) \, \nu . \text{E} ! \breve{T}_P`\nu . \supset . (Q_m`\mu) \, Q \, (`Q_m`\nu)$

Dem.

$[**\cdot02\cdot203]$ $\quad \vdash : \rho \subset (T_P * A)`\lambda . \exists ! \rho . \exists ! p`\rho .$

$$\supset . \overrightarrow{\dot{s}`p`\rho}`Q_m`p`\rho \sim \in 1 \cup 0 \quad (1)$$

$[*40\cdot12]$ $\quad \vdash : \lambda \in \rho . \supset . p`\rho \subset \lambda \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash :. \rho \subset (T_P * A)`\lambda . \exists ! \rho . \exists ! p`\rho :$

$$\supset : \lambda \in \rho . \supset . \overrightarrow{\dot{s}`\lambda}`Q_m`p`\rho \sim \in 1 \cup 0 \quad (3)$$

$\vdash . (3) . *\cdot02 . \quad \supset \vdash :. \rho \subset (T_P * A)`\lambda . \exists ! \rho . \exists ! p`\rho .$

$$\supset : \breve{T}_P`\lambda, \lambda \in \rho . \supset . \overrightarrow{\dot{s}`\breve{T}_P`\lambda}`Q_m`\lambda \in 1 . \overrightarrow{\dot{s}`\breve{T}_P`\lambda}`Q_m`p`\rho \sim \in 1 \cup 0 :$$
$$\supset \vdash :. \rho \subset (T_P * A)`\lambda . \exists ! \rho . \exists ! p`\rho :$$
$$\supset : \breve{T}_P`\lambda, \lambda \in \rho . \supset . Q_m`\lambda \neq Q_m`p`\rho \quad (4)$$

$\vdash . (1) . *01 . (4) . \supset \vdash : \rho \subset (T_P * A)`\lambda . \exists ! \rho . \exists ! p`\rho .$

$\breve{T}_P`\lambda, \lambda \in \rho . \supset . Q_m`p`\rho \neq Q_m`\lambda .$

$Q_m`p`\rho \in \hat{y} \{\overrightarrow{\dot{s}`\lambda}`y \sim \in 0 \cup 1\} . Q_m`\lambda = \min_Q`\hat{y} \sim \in 0 \cup 1 \quad (5)$

$\vdash . (5) . \qquad \supset \vdash : \rho \subset (T_P * A)`\lambda . \exists ! \rho . \exists ! p`\rho .$

$\breve{T}_P`\lambda, \lambda \in \rho . \supset . (Q_m`\lambda) \, Q \, (Q_m`p`\rho) \quad (6)$

$\vdash . (6) . *32 . *258\cdot241 . \supset \vdash . \text{Prop}$

***33.** $\vdash : \text{Hp} * 203 . \supset . PQ`\lambda \in 1 \to \text{Cls}$

Dem.

$[*04] \qquad \vdash : \text{Hp} . \varpi = \hat{\rho} \{\exists \mu . \mu \in (T_P * A)`\lambda .$

$N = \dot{p}`\mu \upharpoonright \overrightarrow{Q}`Q_m`\mu\} . \supset . PQ`\lambda = \dot{s}`\varpi \quad (1)$

$\vdash :. \text{Hp}(1) . \supset : M, N \in \varpi . \supset_{M, N} .$

$\exists \mu, \nu . \mu, \nu \in (T_P * A)`\lambda . N = \dot{p}`\nu \upharpoonright \overrightarrow{Q}`Q_m`\nu . M = \dot{p}`\mu \upharpoonright \overrightarrow{Q}`Q_m`\mu :$

$[*201 . *250\cdot113] \quad \supset \vdash :. \text{Hp}(1) . \supset : M, N \in \varpi . \supset_{M, N} . \exists \mu, \nu .$

$N = \dot{p}`\nu \upharpoonright \overrightarrow{Q}`Q_m`\nu . M = \dot{p}`\mu \upharpoonright \overrightarrow{Q}`Q_m`\mu .$

$\mu \{A \, (T_P, \lambda)\} \, \nu . \vee . \nu \{A \, (T_P, \lambda)\} \, \mu :$

$[**321\cdot214] \qquad \supset \vdash :: \text{Hp}(1) :. \supset :. M, N \in \varpi . \supset_{M, N} .$

$\exists \mu, \nu : \mu, \nu \in (T_P * A)`\lambda . N = (\dot{p}`\nu) \upharpoonright \overrightarrow{Q}`Q_m`\nu . M = (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\mu :$

$(Q_m`\mu) \, Q \, (Q_m`\nu) . (\dot{p}`\nu) \upharpoonright \overrightarrow{Q}`Q_m`\mu = (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\mu : \vee :$

$(Q_m`\nu) \, Q \, (Q_m`\mu) . (\dot{p}`\nu) \upharpoonright \overrightarrow{Q}`Q_m`\nu = (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\nu ::$

$\supset \vdash :. \text{Hp}(1) : \supset . M, N \in \varpi . \supset . \text{Cl}`M \subset \text{Cl}`N .$

$M = N \upharpoonright \text{Cl}`M . \vee . \text{Cl}`N \subset \text{Cl}`M . N = M \upharpoonright \text{Cl}`N :.$

$\supset :. \text{Hp}(1) : \supset . M, N \in \varpi .$

$\supset . y \in \text{Cl}`M \cap \text{Cl}`N . \supset . M`y = N`y \quad (2)$

$\vdash . (2) . \qquad \supset . \text{Prop}$

***34.** $\vdash : \mu \in (T_P * A)`\lambda . E ! \breve{T}_P`\lambda . \text{Hp} * 203 .$

$\supset . (PQ`\lambda) \upharpoonright \overrightarrow{Q}`Q_m`\mu = (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\mu$

Dem.

$[*04] \quad \vdash . \text{Hp} * 203 . E ! \breve{T}_P`\mu . \mu \in (T_P * A)`\lambda .$

$\supset . (\dot{p}`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\mu \subset PQ`\lambda \quad (1)$

[*·33] $\vdash . \mathrm{Hp} . \supset . PQ`\lambda \in 1 \rightarrow \mathrm{Cls}$

$\vdash . (1) . (2) . *213 . \supset \vdash . \mathrm{Prop}$

***·341.** $\vdash : \mu \in (T_P * A)`\lambda . \mathrm{Hp} *·203 . p`(T_P * A)`\lambda .$

$$\supset . PQ`\lambda \upharpoonright \overrightarrow{Q}`Q_m`\mu = \dot{p}`\mu \upharpoonright \overrightarrow{Q}`Q_m`\mu$$

Dem.

[*·201] $\vdash : \mathrm{Hp} . \supset . \overrightarrow{B}`\mathrm{Cnv}`A\,(T_P, \lambda) = \Lambda :$

$$\supset \vdash : \mathrm{Hp} . \supset . \mathrm{E} ! \,\breve{T}_P`\mu \quad (1)$$

$\vdash . (1) . *34 . \supset \vdash . \mathrm{Prop}$

***·35.** $\vdash :. R \upharpoonright \mathrm{C\!l}`PQ`\lambda = PQ`\lambda . \mathrm{Hp} *·203 . p`(T_P * A)`\lambda = \Lambda .$

$$R \in C`P^Q : \supset_R : S \in \lambda . \supset_S . RP^Q S$$

Dem.

[**·34·31] $\vdash . \mathrm{Hp} . S \in \lambda . k = p`\{(T_P * A)`\lambda \cap \overleftarrow{\epsilon}`S\} .$

$$\supset_R . R \upharpoonright \overrightarrow{Q}`Q_m`\lambda = S \upharpoonright \overrightarrow{Q}`Q_m`\lambda \quad (1)$$

[**·34·02] $\vdash . \mathrm{Hp} . S \in \lambda . k = p`\{(T_P * A)`\lambda \cap \overleftarrow{\epsilon}`S\} .$

$$\supset_R . \exists T . T \in \breve{T}_P`k . R`Q_m`\lambda = T`Q_m`\lambda .$$
$$(T`Q_m`\lambda)\, P\, (S, Q_m`\lambda) \quad (2)$$

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset_R : S \in \lambda . \supset_S . RP^Q S$

***·4.** $\vdash :: \mathrm{Hp} *·203 . p`(T_P * A)`\lambda = \Lambda . z \in \mathrm{C\!l}`PQ`\lambda :. \supset :. S \upharpoonright \overrightarrow{Q}`z$

$$= (PQ`\lambda) \upharpoonright \overrightarrow{Q}`z : (PQ`\lambda`z)\, P\, (S`z) . \supset . \exists U . U \in \lambda . UP^Q S$$

Dem.

[*·04] $\vdash : z \in \mathrm{C\!l}`PQ`\lambda . \supset . \exists \mu . \mu \in (T_P * A)`\lambda . z \in \overrightarrow{Q}`Q_m`\mu :$

[*·341] $\supset \vdash :. \mathrm{Hp} . \supset : \exists \mu . \nu = \breve{T}_P`\mu . \supset . (PQ`\lambda) \upharpoonright \overrightarrow{Q}`Q_m`\nu$

$$= (\dot{p}`\breve{T}_P`\mu) \upharpoonright \overrightarrow{Q}`Q_m`\nu . zQ_*(Q_m`\nu) :.$$

[*·341] $\supset \vdash : \mathrm{Hp} . \supset . \exists \mu, U . U \in \breve{T}_P`\mu . (PQ`\lambda) \upharpoonright \overrightarrow{Q}`Q_m`\breve{T}_P`\mu$

$$= U \upharpoonright \overrightarrow{Q}`Q_m`\breve{T}_P`\nu . zQ_*(Q_m`\nu) :$$

[*176·19] $\supset \vdash : \mathrm{Hp} . \supset : S \upharpoonright \overrightarrow{Q}`z = PQ`\lambda \upharpoonright \overrightarrow{Q}`z .$

$$(PQ`\lambda`z)\, P\, (S`z) . \supset . \exists U . U \in \lambda . UP^Q S$$

***·41.** $\vdash :. \text{Hp} *·203 . p\text{‘}(T_P * A)\text{‘}\lambda = \Lambda . R \in C\text{‘}P^Q .$
$R \upharpoonright \text{Ɑ}\text{‘}PQ\text{‘}\lambda = PQ\text{‘}\lambda . \supset . RP^Q V : \supset : \exists S . S \in \lambda . SP^Q V$

Dem.

$[*176·19] \quad \vdash : \text{Hp} . \supset . \sim (V \upharpoonright \text{Ɑ}\text{‘}PQ\text{‘}\lambda = PQ\text{‘}\lambda) :$

$[*176·19] \quad \supset \vdash : \text{Hp} . \supset . \exists z . z \in \text{Ɑ}\text{‘}PQ\text{‘}\lambda . V \upharpoonright \overrightarrow{Q}\text{‘}z$
$= (PQ\text{‘}\lambda) \upharpoonright \overrightarrow{Q}\text{‘}z . (PQ\text{‘}\lambda\text{‘}z) \, P \, (V\text{‘}z)$

$[*4] \quad \vdash . \text{Hp} . \supset . \exists S . S \in \lambda . SP^Q V$

***·42.** $\vdash :. P, Q \in \Omega . \lambda \sim \in 0 \cup 1 . \lambda \subset C\text{‘}P^Q . p\text{‘}(T_P * A)\text{‘}\lambda :$
$\supset : \rho = C\text{‘}P^Q \cap \hat{R}[R \upharpoonright \text{Ɑ}\text{‘}PQ\text{‘}\lambda = PQ\text{‘}\lambda] . \supset . \rho \subset p\text{‘}\overrightarrow{P^Q}\text{‘‘}\lambda .$
$p\text{‘}\overleftarrow{P^Q}\text{‘‘}\rho \subset s\text{‘}\overleftarrow{P^Q}\text{‘‘}\lambda \quad [**·35·41]$

***·43.** $\vdash : E! \, Q_m\text{‘}\lambda . \supset . \dot{s}\text{‘}\overrightarrow{\lambda}\text{‘}Q_m\text{‘}\lambda \sim \in 1 . \dot{s}\text{‘}\overrightarrow{\breve{T}_P\text{‘}\lambda}\text{‘}Q_m\text{‘}\lambda \in 1 \quad [**·01·02]$

***·431.** $\vdash : T_P \Subset A . (T_P)_{\text{po}} \Subset A$

Dem.

$\vdash . *·43 . \quad \supset \vdash : \mu \, \breve{T}_P \, \lambda . \supset . \mu \subset \lambda . \mu \neq \lambda \quad (1)$

$\vdash . (1) . *201·18 . \supset \vdash . \text{Prop}$

***·432.** $\vdash : \text{Hp} *·203 . \supset . \overrightarrow{\min} (P^Q)\text{‘}\lambda \subset p\text{‘}(T_P * A)\text{‘}\lambda$

Dem.

$[*·217] \quad \vdash :. \text{Hp} . R \in \mu . \mu \in (T_P * A)\text{‘}\lambda . \mu \, \{A \, (T_P, \lambda)\} \, \nu .$
$R \sim \in \nu : \supset : \exists ! \, \nu . \supset . \exists S . SP^Q R :.$
$\supset \vdash :. \text{Hp} . R \in \mu . \mu \in (T_P * A)\text{‘}\lambda . R \min (P^Q) \lambda :$
$\supset : \mu \, \{A \, (T_P, \lambda)\} \, \nu . \exists ! \, \nu . \supset . R \in \nu \quad (1)$

$[*·431] \quad \vdash : \text{Hp} . R \in \mu . \nu \, \{A \, (T_P, \lambda)\} \, \mu . \supset . R \in \nu \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \text{Hp} . \supset . R \in \overrightarrow{\min} \, P^Q\text{‘}\lambda : \supset : \mu \in (T_P * A)\text{‘}\lambda .$
$\supset_\mu . R \in \mu \quad (3)$

$\vdash . (3) . \quad \supset \vdash . \text{Prop}$

***·433.** $\vdash : \text{Hp} *·42 . \rho = C\text{‘}P^Q \cap \hat{R}(R \upharpoonright \text{Ɑ}\text{‘}PQ\text{‘}\lambda = PQ\text{‘}\lambda) .$
$\supset . \overrightarrow{\max} (P^Q)\text{‘}\rho = \overrightarrow{\text{tl}} (P^Q)\text{‘}\lambda$

Dem.

$\vdash :: \text{Hp} *·42 . \supset :. S \in C\text{‘}P^Q : \supset : S \upharpoonright \text{Ɑ}\text{‘}PQ\text{‘}\lambda = PQ\text{‘}\lambda .$
$\vee . \exists z . z \in \text{Ɑ}\text{‘}PQ\text{‘}\lambda . S \upharpoonright \overrightarrow{Q}\text{‘}z = PQ\text{‘}\lambda \upharpoonright \overrightarrow{Q}\text{‘}z .$
$(S\text{‘}z) \, P \, (PQ\text{‘}\lambda\text{‘}z) . \vee . (PQ\text{‘}\lambda\text{‘}z) \, P \, (S\text{‘}z) :.$

[*·4] $\supset \vdash :: \text{Hp} . \supset :. S \in C`P^Q . \supset . S \in \rho : \text{v} : T \in \rho . \supset_T .$
$SP^Q T : \text{v} : \exists U . U \in \lambda . UP^Q S ::$

$\supset \vdash :: \text{Hp} . \supset :. V \in \lambda . \supset_V . SP^Q V : S \in C`P^Q :$
$\supset : S \in \rho . \text{v} . T \in \rho . \supset_T . SP^Q T :.$

$\supset \vdash :: \text{Hp} . \supset :. \sigma = \hat{U}(V \in \lambda . \supset_V . UP^Q V) - \overrightarrow{\max}(P^Q)`\rho .$
$S \in \sigma : \supset : \text{E} ! \max(P^Q)`\rho . \supset . SP^Q (\max(P^Q)`\rho) .$

$\max(P^Q)`\rho \in \rho : \text{v} : \overrightarrow{\max}`P^Q``\rho = \Lambda . \supset . \exists T . SP^Q T . T \in \rho ::$

$\supset \vdash :: \text{Hp} . \supset :. \sigma = \hat{U}(V \in \lambda . \supset_V . UP^Q V) - \overrightarrow{\max}(P^Q)`\rho .$
$\supset . \sigma \subset P^Q``\rho$ (1)

$\vdash . (1) . *205 \cdot 193 .$ $\supset \vdash :. \text{Hp} . \supset : \sigma = \hat{U}(V \in \lambda . \supset_V . UP^Q V)$
$- \overrightarrow{\max}(P^Q)`\rho . \supset . \overrightarrow{\max}(P^Q)`\rho \cup \sigma = \overrightarrow{\max}(P^Q)`\rho$ (2)

$\vdash . *206 \cdot 02 .$ $\supset \vdash :. \text{Hp} . \supset : \overrightarrow{\text{prec}}(P^Q)`\lambda$
$= \overrightarrow{\max}` \hat{U}(V \in k . \supset . VP^Q U)$ (3)

$\vdash . (3) .$ $\supset \vdash :. \text{Hp} . \supset : \sigma = \hat{U}(V \in \lambda . \supset_V . UP^Q V)$
$- \overrightarrow{\max}(P^Q)`\rho . \supset . \overrightarrow{\text{prec}}(P^Q)`\lambda = \overrightarrow{\max}(P^Q)(`\rho \cup \sigma) :$

$\vdash . (2) .$ $\supset \vdash :. \text{Hp} . \supset : \sigma = \hat{U}(V \in \lambda . \supset_V . UP^Q V)$
$- \overrightarrow{\max}(P^Q)`\rho . \supset . \overrightarrow{\text{prec}}(P^Q)`\lambda = \overrightarrow{\max}(P^Q)`\rho$ (4)

$\vdash . *\cdot 432 .$ $\supset \vdash : \text{Hp} . \supset . \overrightarrow{\min}(P^Q)`\lambda = \Lambda$ (5)

$\vdash . (4) . (5) . *207 \cdot 02 .$ $\supset \vdash : \text{Hp} . \supset . \overrightarrow{\max}(P^Q)`\rho = \overrightarrow{\text{tl}}(P^Q)`\lambda$

***·44.** $\vdash : \text{Hp} *\cdot 42 . \rho = C`P^Q \cap \hat{R}(R \upharpoonright \text{Cl}`PQ`\lambda = PQ`\lambda) . \rho \in 1 .$
$\supset . \breve{\iota}`\rho = \text{tl}(P^Q)`\lambda$ [*205·18 . *·433]

***·45.** $\vdash :. \text{Hp} *\cdot 42 . \rho = C`P^Q \cap R(\hat{R} \upharpoonright \text{Cl}`PQ`\lambda = PQ`\lambda) :$
$\supset : \text{Cl}`PQ`\lambda = C`Q . \equiv . \rho = \iota`PQ`\lambda$

Dem.

$\vdash :: \text{Hp} . \supset :. \text{Cl}`PQ`\lambda = C`Q . \supset . PQ`\lambda \in C`P^Q :$
$R \in \rho . \supset . \text{Cl}`R = \text{Cl}`PQ`\lambda ::$

$\supset \vdash :. \text{Hp} . \supset : \text{Cl}`PQ`\lambda = C`Q . \supset . PQ`\lambda = \breve{\iota}`\rho$ (1)

$\vdash :. \text{Hp} . \supset : \rho = \iota`PQ`\lambda . \supset . PQ`\lambda \in C`P^Q :.$

[$*176{\cdot}19$] $\supset \vdash :. \text{Hp} . \supset : \rho = \iota{}^{\backprime}PQ{}^{\backprime}\lambda . \supset . \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda = C{}^{\backprime}Q$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*{\cdot}451$. $\vdash :. P, Q \in \Omega . \lambda \subset C{}^{\backprime}P^Q . \lambda \sim\in 0 \cup 1 . p{}^{\backprime}(T_P * A){}^{\backprime}\lambda = \Lambda .$
$\supset . \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda = C{}^{\backprime}Q : \supset . \text{E}! \, \text{tl}\,(P^Q){}^{\backprime}\lambda$ [$**45{\cdot}44$]

$*{\cdot}46$. $\vdash :. P, Q \in \Omega . \lambda \subset C{}^{\backprime}P^Q . \lambda \sim\in 0 \cup 1 . p{}^{\backprime}(T_P * A){}^{\backprime}\lambda = \Lambda .$
$\supset_\lambda . \text{Cl}{}^{\backprime}TQ{}^{\backprime}\lambda = C{}^{\backprime}Q : \supset : \mu \sim\in 0 . \mu \subset C{}^{\backprime}P^Q .$
$\supset_\mu . \text{E}! \, \text{limin}\,(P^Q){}^{\backprime}\mu$ [$**{\cdot}11{\cdot}218{\cdot}451$]

$*{\cdot}5$. $\vdash . \text{Hp}\, *{\cdot}42 . \supset . \overrightarrow{\text{max}_Q}{}^{\backprime}\text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda = \Lambda$

Dem.

[$*{\cdot}04$] $\vdash . \text{Hp} . z \in \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda .$
$\supset . \exists \mu . \mu \in (T_P * A){}^{\backprime}\lambda . z \, Q \, (Q_{\text{m}}{}^{\backprime}\mu)$ (1)

[$*{\cdot}04$] $\vdash . \text{Hp} . \mu \in (T_P * A){}^{\backprime}\lambda .$
$\supset . \exists \nu . \nu = T_P{}^{\backprime}\mu . \overrightarrow{Q}{}^{\backprime}Q_{\text{m}}{}^{\backprime}\nu . \subset \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda$ (2)

$\vdash . *{\cdot}216 . (1) . (2) . \supset \vdash . \text{Hp} . z \in \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda .$
$\supset . \exists z' . zQz' . z' \in \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda$ (3)

$\vdash . (3) . \quad \supset \vdash . \text{Prop}$

$*{\cdot}51$. $\vdash :. \text{Hp}\, *{\cdot}42 : \supset : \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda = C{}^{\backprime}Q .$
$\vee . \exists z . z \in C{}^{\backprime}Q - \text{Cl}{}^{\backprime}Q_1 . \text{Cl}{}^{\backprime}PQ{}^{\backprime}\lambda = \overrightarrow{Q}{}^{\backprime}z$ [$*{\cdot}5$]

$*{\cdot}52$. $\vdash :. P, Q \in \Omega . C{}^{\backprime}Q \subset \text{Cl}{}^{\backprime}Q_1 . \supset : \mu \sim\in 0 . \mu \subset C{}^{\backprime}P^Q .$
$\supset_\mu . \text{E}! \, \text{limin}\,(P^Q){}^{\backprime}\mu$ [$**{\cdot}46{\cdot}51$]

$*{\cdot}53$. $\vdash : P, Q \in \Omega . C{}^{\backprime}Q \subset \text{Cl}{}^{\backprime}Q_1 . \text{E}! \, B{}^{\backprime}\text{Cnv}{}^{\backprime}P^Q . \supset . P^Q \in \text{Ded}$
[$**{\cdot}52{\cdot}1$]

$*{\cdot}531$. $\vdash : P, Q \in \Omega . C{}^{\backprime}Q \subset \text{Cl}{}^{\backprime}Q_1 . \overrightarrow{B}{}^{\backprime}\text{Cnv}{}^{\backprime}P^Q = \Lambda .$
$\supset . P^Q \in \text{semi-Ded}$ [$**{\cdot}52{\cdot}1$]

$*{\cdot}5401$. $\vdash :. P, Q \in \Omega : \supset : \rho = \overrightarrow{B}{}^{\backprime}\text{Cnv}{}^{\backprime}P^Q . \equiv . R \in \rho .$
$\supset_\rho . \text{D}{}^{\backprime}R = \iota{}^{\backprime}B{}^{\backprime}\breve{P}$ [$*176{\cdot}19$]

$*{\cdot}541$. $\vdash :. P, Q \in \Omega . \supset : \text{E}! \, B{}^{\backprime}\text{Cnv}{}^{\backprime}P^Q . \equiv . \text{E}! \, B{}^{\backprime}\breve{P}$ [$*{\cdot}5401$]

$*{\cdot}55$. $\vdash :. P, Q \in \Omega . C{}^{\backprime}Q \subset \text{Cl}{}^{\backprime}Q_1 : \supset : P^Q \in \text{semi-Ded} :$
$P^Q \in \text{Ded} . \equiv . \text{E}! \, B{}^{\backprime}\breve{P}$ [$**{\cdot}541{\cdot}53{\cdot}531$]

$*{\cdot}56$. $\vdash :. P, Q \in \Omega . \text{Nr}{}^{\backprime}Q \leqslant \omega . \supset : P^Q \in \text{semi-Ded} :$
$P^Q \in \text{Ded} . \equiv . \text{E}! \, B{}^{\backprime}\breve{P}$ [$*{\cdot}55$]

$$*6. \quad \vdash :. \mathrm{Hp} \, *{\cdot}42 \,.\, \rho = C`P^Q \cap \hat{R}\,[R \upharpoonright \text{Ɑ}`PQ`\lambda = PQ`\lambda] \,.\, \rho \sim\in 1 :$$
$$\supset : \mathrm{E}\,!\, B`\breve{P} \,.\equiv.\, \mathrm{E}\,!\, \max(P^Q)`\rho$$

Dem.

$$[*176{\cdot}19] \quad \vdash :. \mathrm{Hp} \,.\supset:\, R \upharpoonright \text{Ɑ}`PQ`\lambda = PQ`\lambda \,.\, \text{Ɑ}`R = C`Q \,.$$
$$\mathrm{D}`(R \upharpoonright - \text{Ɑ}`PQ`\lambda) = \iota`B`\breve{P} \,.\equiv_R.\, R = \max(P^Q)`\rho \quad (1)$$

$$\vdash .(1). \quad \supset \vdash :. \mathrm{Hp} \,.\supset:\, \mathrm{E}\,!\, B`\breve{P} \,.\equiv.\, \mathrm{E}\,!\, \max(P^Q)`\rho$$

$$*61. \quad \vdash :. P, Q \in \Omega \,.\, \exists a \,.$$
$$\lambda = \hat{R}\,(\exists z \,.\, zQ_* a \,.\, R = \iota`1_P \uparrow \overrightarrow{Q}`z \cup \iota`2_P \uparrow \overleftarrow{Q_*}`z) \,.$$
$$a \in C`Q - \text{Ɑ}`Q_1 :\supset:\, \mathrm{E}\,!\, \mathrm{limin}\,(P^Q)`\lambda \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P}$$

Dem.

$$\vdash :. \mathrm{Hp} :\supset: \sim(\exists R, S \,.\, R, S \in \lambda \,.\, R \neq S \,.\, R \upharpoonright \overrightarrow{Q_*}`a$$
$$= S \upharpoonright \overrightarrow{Q_*}`a) :$$
$$\vdash :. \mathrm{Hp} \,.\supset:\, \mu \in (T_P * A)`\lambda \,.\supset.\, Q_{\mathrm{m}}`\mu \, Q \, a \quad (1)$$

$$\vdash .(1).*{\cdot}04. \quad \supset \vdash :. \mathrm{Hp} \,.\supset.\, \text{Ɑ}`PQ`\lambda \subset \overrightarrow{Q}`a \quad (2)$$
$$\vdash .(2). \quad \supset \vdash : \mathrm{Hp} \,.\supset.\, \rho \sim\in 1 \quad (3)$$

$$\vdash .**{\cdot}6{\cdot}433. \quad \supset \vdash :. \mathrm{Hp} \,.\supset:\, \mathrm{E}\,!\, \mathrm{limin}\,(P^Q)`\lambda \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P}$$

$$*62. \quad \vdash :. P, Q \in \Omega \,.\, \exists!\, C`Q - \text{Ɑ}`Q_1 \,.\supset.\, \lambda \subset C`P^Q \,.\, \lambda \sim\in 0 :$$
$$\supset : \mathrm{E}\,!\, \mathrm{limin}\,(P^Q)`\lambda \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P} \quad [*{\cdot}61]$$

$$*63. \quad \vdash :. P, Q \in \Omega \,.\, \mathrm{Nr}`Q > \omega :\supset:\, P^Q \in \mathrm{Ded} \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P} \,.$$
$$\overrightarrow{B}`\breve{P} = \Lambda \,.\equiv.\, P^Q \sim\in \text{semi-Ded} \quad [**{\cdot}61{\cdot}1]$$

$$*7. \quad \vdash :. P, Q \in \Omega \,.\supset:.\, P^Q \in \mathrm{Ded} \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P} \quad [**{\cdot}56{\cdot}63]$$

$$*8. \quad \vdash :: P, Q \in \Omega \,.\supset:.\, P^Q \in \text{semi-Ded} \,.\equiv.\, \mathrm{E}!\, B`\breve{P} :\equiv: \mathrm{Nr}`Q > \omega$$

Dem.

$$\vdash .**{\cdot}63{\cdot}7. \quad \supset \vdash :: P, Q \in \Omega \,.\supset:.\, \mathrm{Nr}`Q > \omega :$$
$$\supset : P^Q \in \text{semi-Ded} \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P} \quad (1)$$

$$\vdash .*56. \quad \supset \vdash :: P, Q \in \Omega \,.\supset:.\, \mathrm{Nr}`Q \leqslant \omega \,.$$
$$\supset : P^Q \in \text{semi-Ded} : \mathrm{E}\,!\, B`\breve{P} \,.\vee.\, \overrightarrow{B}`\breve{P} = \Lambda \quad (2)$$

$$\vdash .(1).(2). \quad \supset \vdash :: P, Q \in \Omega :.\, \mathrm{Nr}`Q > \omega :$$
$$\equiv : P^Q \in \text{semi-Ded} \,.\equiv.\, \mathrm{E}\,!\, B`\breve{P}$$

The definitions and method used in the earlier part of this paper (**·01—·341) are suggested in *Principia Mathematica* *276. There it is stated tentatively that

$$\exists ! \, p`(T_P * A)`\lambda \,.\, \supset \,.\, \breve{\iota}`p`(T_P * A)`\lambda = \min(P^Q)`\lambda$$
$$\sim \exists ! \, p`(T_P * A)`\lambda \,.\, \supset \,.\, PQ`\lambda = \mathrm{prec}\,(P^Q)`\lambda$$

The first of these propositions is established in **·1—·218: the second seems to be untrue. If in the field of Q there is a term a with no immediate predecessor (as for example the term ω if Q were the series of ordinals less than $\omega + 4$), there is a λ, a subclass of the field of P^Q, for which $PQ`\lambda$ is a relation covering with $P$-terms only the $Q$-terms which precede $a$ (*cp.* *·61). In such a case $PQ`\lambda$ is not a P^Q term and so is not prec $(P^Q)`\lambda$. If $P$ has a last term $z$, the relation agreeing with $PQ`\lambda$ as far as a and covering a and all subsequent Q places with z will be prec $(P^Q)`\lambda$, and therefore the lower limit of λ with respect to P^Q.

Thus, while agreeing with the proposition *if P and Q are well-ordered series and P has a last term, P^Q is Dedekindian*, and extending it to the proposition *if P and Q are well-ordered series, P^Q is Dedekindian when and only when P has a last term*, we disagree with the conclusion that *if P and Q are well-ordered series, P^Q with the addition of a term at the end is Dedekindian even if P has no last term.* Instead we would substitute the propositions *when P and Q are well-ordered series, and* $\mathrm{Nr}`Q \leqslant \omega$, *$P^Q$ with the addition of a term at the end is Dedekindian whether or not P has a last term*, and *if* $\mathrm{Nr}`Q > \omega$, *$P^Q$ with the addition of a term at the end is Dedekindian when and only when P has a last term.*

The Gauss-Bonnet Theorem for Multiply-Connected Regions of a Surface. By ERIC H. NEVILLE, M.A., Trinity College.

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Among the most delightful passages of differential geometry is the use of Green's theorem to prove the relation discovered by Bonnet between the integral curvature of a bounded region on any bifacial surface and the integrated geodesic curvature of the boundary. The fundamental equation is

$$\int \kappa_g ds + \iint K d^2S = \int \frac{d\xi}{ds} ds,$$

where the line integrals are taken round the whole boundary and the surface integral over the region contained, κ_g is the geodesic curvature of the boundary, K the Gaussian curvature of the surface, and ξ an angle to the direction of the boundary from the direction of one of the curves of reference. Though there is no allusion to curves of reference on the left of this equation, not only do these curves appear explicitly on the right, but the use of Green's theorem implies that there does exist some system of curvilinear coordinates valid throughout the region and upon the boundary, an assumption of which it is difficult to gauge the exact force. The primary object of this note is to express Bonnet's theorem in a form purely intrinsic.

In the case of a simply-connected region not extending to infinity, whose boundary has continuous curvature at every point, the value of $\int (d\xi/ds)\, ds$ is 2π*. If the region is simply-connected and does not extend to infinity, but the boundary is a curvilinear polygon, formed of a finite number of arcs of continuous curvature, the sum of the external angles must be added to the integral to make the total of 2π; in other words, $\int (d\xi/ds)\, ds$ is then the amount by which the sum of the external angles falls short of 2π. In the particular case of a curvilinear triangle, the amount by which the sum of the three external angles falls short of 2π is the amount by which the sum of the three internal angles exceeds π, and is called the *angular excess* of the triangle. The name is adopted to serve a wider purpose: whether a connected region of a surface is bounded by a single closed curve or by a number of

* See a paper by G. N. Watson, "A Problem of Analysis Situs", *Proc. Lond. Math. Soc.*, ser. 2, vol. 15, p. 227 (1916).

curves, the amount by which the sum of all the external angles of the boundary falls short of 2π is called the angular excess of the boundary.

Whatever the number of curves forming the boundary of a region, the addition to the boundary of a simple cut, joining a point of the boundary either to a point of the cut or to a point of the boundary and described once in each direction, increases the sum of the external angles by 2π. If the cut divides the region into two parts, the angular excess of each part is the amount by which the sum of the external angles of that part falls short of 2π, and therefore the sum of the two angular excesses is the amount by which the sum of the external angles of the composite boundary falls short of 4π; this, being as we have just seen the amount by which the sum of the external angles of the original boundary falls short of 2π, is the angular excess of the original boundary. If on the other hand the cut leaves the region undivided, there is an actual decrease of 2π in the excess. It follows that if by a succession of n simple cuts the region is divided into m distinct parts, the sum of the angular excesses of the boundaries of the parts is less than the angular excess of the original boundary by $2(n-m+1)\pi$. Suppose now that each of these parts is simply-connected and that there are no singular points of the surface in the original region or upon its boundary. Then since Bonnet's theorem in its simplest form is applicable to each of the parts, addition of the sum of the integral curvatures of the parts to the sum of the integral geodesic curvatures of the boundaries of these parts gives the sum of the angular excesses of the individual boundaries. But the sum of the integral curvatures of the parts is the integral curvature of the original region, and the sum of the integral geodesic curvatures of the boundaries of the parts is the integral geodesic curvature of the original boundary, since an arc described once in each direction adds nothing to $\int \kappa_g ds$. Hence the sum of the integral geodesic curvature of the original boundary and the integral curvature of the bounded region is less than the angular excess of the original boundary by $2(n-m+1)\pi$. This result affords a *proof* that if only the dissection has reached a stage at which every part is simply-connected, the difference $n-m$ is independent alike of the form of the cuts and of their number. Since a simply-connected region is divided by one cut into two pieces, the integer used to measure connectivity is not $n-m$ but $n-m+2$, and Bonnet's theorem in its most general form asserts that

If a bounded bifacial region of any surface has finite connectivity k and neither extends to infinity nor includes within it or upon its boundary any singularities of the surface, the sum of the integral geodesic curvature of the boundary and the integral curva-

ture of the region bounded is less than the angular excess of the boundary by $2(k-1)\pi$.

In other words, the sum of the two integrals and the external angles of the boundary is $2(2-k)\pi$.

Gauss' famous theorem on the integral curvature of a geodesic triangle, which may be regarded either as the simplest case or as the ultimate basis of Bonnet's theorem, is in no less need of modification if the region contemplated is multiply-connected.

If a geodesic triangle on any surface has internal angles A, B, C *and connectivity* k, *and if the surface is regular throughout the triangle and on its perimeter, the integral curvature of the triangle is* $A+B+C-(2k-1)\pi$.

The application to the whole of a surface which, like a sphere and an anchor-ring, does not extend to infinity, but has no boundary, is interesting. A simple closed curve can always be drawn to divide such a surface into two distinct parts, and since its direction as the boundary of one part is opposite to its direction as the boundary of the other part, the sum of the external angles of the two boundaries is zero, and so also is the sum of their integral geodesic curvatures. It follows from Bonnet's theorem that, if there are no singular points on the surface and the connectivities of the two parts are i, j, the integral curvature of the complete surface is $2(4-i-j)\pi$. Hence $i+j$ is constant; in order that a surface which, like a sphere, is cut by any simple closed curve into two simply-connected parts may be described as of unit connectivity, the connectivity is measured by the integer $i+j-1$, and

If the connectivity of a bifacial surface which has no boundary and no singular points and does not extend to infinity is k, *the integral curvature of the surface is* $2(3-k)\pi$.

A striking deduction made by Darboux from Bonnet's theorem may be mentioned here. If on a complete surface there is any family of curves such that the surface can be divided into a finite number of parts throughout each of which this family provides one set of curves of reference, the angle ξ of our first paragraph can be measured from the curve belonging to this family, and $\int(d\xi/ds)\,ds$ taken once in each direction over every part of an imposed boundary is necessarily zero. Hence

For there to exist on an unbounded bifacial surface, which does not extend to infinity and is everywhere regular, a family of curves which covers the surface and is wholly without singularities, the surface must have integral curvature zero and must therefore be triply-connected.

In conclusion the subject may be presented in another form. Let the angular excess of the boundary of a region of connectivity k reduced by $2(k-1)\pi$ be called the *effective* angular excess. If a simple cut which is added to the boundary does not divide the region, the angular excess is reduced by 2π, and, since the connectivity is reduced by unity, the effective angular excess is unaltered. If, on the other hand, the cut divides the region into parts of connectivities i, j, not only is the sum of the actual angular excesses of the boundaries of the parts the actual angular excess of the original boundary, but, since k is $i+j-1$, the sum of $i-1$ and $j-1$ is $k-1$: the effective angular excess of the boundary of the whole is the sum of the effective angular excesses of the boundaries of the parts. Effective angular excess is therefore additive in precisely the same way as the surface integral of a single-valued function. If then Bonnet's theorem for a simply-connected region is expressed in the form that the sum of the integral curvature and the integral geodesic curvature is the *effective* angular excess, the restriction on the connectivity is seen at once to be superfluous. But to take this course implies a previous acquaintance with the theory of connectivity, whereas it is arguable that if Bonnet's theorem is used to *establish* the theory of connectivity the extent to which there is an appeal to intuition is materially reduced.

On an empirical formula connected with Goldbach's Theorem.
By N. M. Shah, Trinity College, and B. M. Wilson, Trinity College. (Communicated by Mr G. H. Hardy.)

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§ 1. The following calculations originated in a request recently made to us by Messrs G. H. Hardy and J. E. Littlewood, that we should check a suggested asymptotic formula for the number of ways $\nu(n)$ of expressing a given even number n as the sum of two primes. The formula in question is

$$\nu(n) \sim \lambda(n) = 2A \frac{n}{(\log n)^2} \frac{p-1}{p-2} \frac{q-1}{q-2} \cdots \quad \ldots\ldots(1),$$

where
$$n = 2^\alpha p^a q^b \ldots \quad (\alpha \geqslant 1)$$

and A denotes the constant

$$\prod_{p=3}^{\infty} \left\{1 - \frac{1}{(p-1)^2}\right\},$$

p assuming, in this product, the odd prime values 3, 5, 7, 11, 13, ...

The formula (1) was deduced from another conjectured asymptotic formula, namely

$$\sum_{m+m'=n} \Lambda(m) \Lambda(m') \sim 2An \frac{p-1}{p-2} \frac{q-1}{q-2} \cdots \quad \ldots\ldots(2),$$

where $\Lambda(m)$ is the arithmetical function equal to $\log p$ when m is a prime p, or a power of p, and to zero otherwise, and the summation on the left is extended to all pairs of positive integers m, m' such that

$$m + m' = n.$$

Formula (1) arises from (2) by replacing in the latter $\Lambda(m)$ and $\Lambda(m')$ each by $\log n$. It is natural, however, to expect a more accurate result if we replace $\Lambda(m)$ and $\Lambda(m')$ not by $\log n$ but by $\log \frac{1}{2} n$, or, better still, if we replace the left-hand member of (2) by

$$\frac{\nu(n)}{n} \int_0^n \log x \log(n-x)\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The exact value of the expression (3) is found to be

$$\nu(n) \{(\log n)^2 - 2\log n + 2 - \tfrac{1}{6}\pi^2\} \quad \ldots\ldots\ldots\ldots(4).$$

The various formulae thus obtained from (2) are, of course, all asymptotically equivalent; but the modified formulae are likely to give more accurate results than (1) for comparatively small values of n. We used the formula

$$\nu(n) \sim \rho(n) = 2A \frac{n}{(\log n)^2 - 2\log n} \frac{p-1}{p-2} \frac{q-1}{q-2} \cdots \quad \ldots(5),$$

obtained by ignoring the constant $2 - \frac{1}{6}\pi^2$ in (4).

§2. For the numerical data used we are indebted to two different sources. The most complete numerical results are contained in the tables compiled and published* by R. Haussner, which give the values of $\nu(n)$ for all values of n not exceeding 5000. Tables extending up to 1000 and 2000 had been calculated earlier by G. Cantor and V. Aubry. Further data, less systematic, indeed, than those of Haussner, but extending to considerably larger values of n, were given by L. Ripert† in a number of short papers in *l'Intermédiaire des mathématiciens*.

The values given for $\nu(n)$ in the accompanying table differ, in several respects, from those given by Haussner or Ripert. In the first place, $m+m'$ and $m'+m$ are here counted as different decompositions, whereas the above two writers regard them as identical; secondly we do not (as do Haussner and Ripert) regard 1 as a prime; and thirdly we increase the values of $\nu(n)$ obtained from their tables by addition of the number of ways in which n may be expressed as the sum of two powers of primes, *i.e.* the number of ways in which

$$n=p^a+q^b,$$

where p and q are primes, and either a or b is greater than unity. The last two modifications make, of course, no difference to the asymptotic formula, but it seems natural to make them when the genesis of the formula (1) or (5) is considered.

As regards the choice and arrangement of the numbers n in the table, the smaller numbers—*i.e.* the numbers not exceeding 5000—are intended to be "typical"; that is, they are specially selected numbers, taken in groups so as best to test or illustrate the accuracy of formula (1). Thus, for example, if the formula in question is true, a multiple of 6 may be expected, in general, to allow of an unusually large number of decompositions‡. On the other hand a power of 2 may be expected to allow of an unusually small number. The numbers below 5000 have therefore been selected in groups of four or five, all the numbers of each group being as nearly equal as possible; and each group of numbers contains, in general, one highly composite number (*i.e.* 2 . 3 . 5 . 7 . 11 ....), one power of 2, and one number which is the product of 2 and a prime.

For values of n exceeding 5000, such choice of "typical" numbers was, unfortunately, impossible without a large amount of fresh calculation. Ripert, indeed, selected his numbers according to a system, and they, too, occur, in general, in groups of approximately equal magnitude; but he selected them with different objects, so that his numbers are, from our point of view, neither "typical" nor arbitrary.

* *Nova Acta der Akad. der Naturforscher* (Halle), vol. 72 (1897), pp. 5–214.
† See, for example, vol. 10 (1903), pp. 76–77, 166–167.
‡ It was first pointed out by Cantor, on the evidence of his numerical results previously mentioned, that this is actually so.

The accompanying table gives the number of decompositions—actual and theoretical—for thirty-five numbers; the value found for the constant A was 0·66016. In the second column the first number is the number of decompositions, using prime numbers only, and the second the number of decompositions involving powers of primes higher than the first.

§ 3. *Table of decompositions.*

n	$\nu(n)$	$\rho(n)$	$\nu(n):\rho(n)$
$30=2.3.5$	$6+4=10$	22	·45...
$32=2^5$	$4+7=11$	8	1·38...
$34=2.17$	$7+6=13$	9	1·44...
$36=2^2.3^2$	$8+8=16$	17	·94
$210=2.3.5.7$	$42+0=42$	49	·85
$214=2.107$	$17+0=17$	16	1·07
$216=2^3.3^3$	$28+0=28$	32	·88
$256=2^8$	$16+3=19$	17	1·10
$2{,}048=2^{11}$	$50+17=67$	63	1·06
$2{,}250=2.3^2.5^3$	$174+26=200$	179	1·11
$2{,}304=2^8.3^2$	$134+8=142$	136	1·04
$2{,}306=2.1153$	$67+20=87$	69	1·26
$2{,}310=2.3.5.7.11$	$228+16=244$	244	1·00
$3{,}888=2^4.3^5$	$186+24=210$	197	1·06
$3{,}898=2.1949$	$99+6=105$	99	1·06
$3{,}990=2.3.5.7.19$	$328+20=348$	342	1·02
$4{,}096=2^{12}$	$104+5=109$	102	1·06
$4{,}996=2^2.1249$	$124+16=140$	119	1·18
$4{,}998=2.3.7^2.17$	$288+20=308$	305	1·01
$5{,}000=2^3.5^4$	$150+26=176$	157	1·12
$8{,}190=2.3^2.5.7.13$	$578+26=604$	597	1·01
$8{,}192=2^{13}$	$150+32=182$	171	1·06
$8{,}194=2.17.241$	$192+10=202$	219	·92
$10{,}008=2^3.3^2.139$	$388+30=418$	396	1·06
$10{,}010=2.5.7.11.13$	$384+36=420$	384	1·09
$10{,}014=2.3.1669$	$408+8=416$	396	1·05
$30{,}030=2.3.5.7.11.13$	$1{,}800+54=1{,}854$	1,795	1·03
$36{,}960=2^5.3.5.7.11$	$1{,}956+38=1{,}994$	1,937	1·03
$39{,}270=2.3.5.7.11.17$	$2{,}152+36=2{,}188$	2,213	·99
$41{,}580=2^2.3^3.5.7.11$	$2{,}140+44=2{,}184$	2,125	1·03
$50{,}026=2.25013$	$702+8=710$	692	1·03
$50{,}144=2^5.1567$	$674+32=706$	694	1·02
$170{,}166=2.3.79.359$	$3{,}734+46=3{,}780$	3,762	1·00
$170{,}170=2.5.7.11.13.17$	$3{,}784+8=3{,}792$	3,841	·99
$170{,}172=2^2.3^2.29.163$	$3{,}732+48=3{,}780$	3,866	·98

§ 4. Goldbach asserted that every even number is the sum of two primes, and this unproved proposition is usually called 'Goldbach's Theorem'. It is evident that the truth of Hardy and Littlewood's formula would imply that of Goldbach's theorem, at any rate for all numbers from a certain point onwards.

Previous writers, from Cantor onwards, had noted that the irregularity in the variation of $\nu(n)$ depends on the structure of n as a product of primes. In a short abstract in the *Proceedings* of the London Mathematical Society, Sylvester* suggested the formula

$$\nu(n) \sim \frac{2n}{\log n} \Pi \frac{p-2}{p-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where, in the product on the right p assumes all prime values from 3 to $\sqrt{n}$, except those which are factors of n. Sylvester gives but little indication as to how he arrived at the formula, and indeed there is much in his paper which is not very clear. It is at once obvious that if n, n' are two large, but approximately equal, even numbers, the values furnished for the ratio $\nu(n):\nu(n')$ by formulae (1) and (6) will be the same. For if

$$n = 2^a p^a q^b \ldots$$

and

$$n' = 2^{a'} p'^{a'} q'^{b'} \ldots$$

both formulae will give, as an approximate expression for this ratio, the quotient

$$\frac{p-1}{p-2}\frac{q-1}{q-2}\cdots\bigg/\frac{p'-1}{p'-2}\frac{q'-1}{q'-2}\cdots.$$

The actual values of $\nu(n)$ would however be different. For from formula (6) we should deduce

$$\nu(n) \sim \frac{2n}{\log n}\frac{p-1}{p-2}\frac{q-1}{q-2}\cdots \prod_{h \leqslant \sqrt{n}} \frac{h-2}{h-1}.$$

Now

$$\prod_{p \leqslant \sqrt{n}} \frac{p-2}{p-1} = \prod_{p \leqslant \sqrt{n}} \frac{p(p-2)}{p(p-1)}$$

$$= \prod_{p \leqslant \sqrt{n}} \frac{(p-1)^2-1}{(p-1)^2} \prod_{p \leqslant \sqrt{n}} \left(1-\frac{1}{p}\right)$$

$$\sim A \prod_{p \leqslant \sqrt{n}} \left(1-\frac{1}{p}\right),$$

where A is the same constant as in formula (1). Also it is known† that

$$\prod_{p \leqslant \sqrt{n}} \left(1-\frac{1}{p}\right) \sim \frac{2e^{-\gamma}}{\log n},$$

* *Proc. London Math. Soc.*, vol. 4 (1871), pp. 4–6 (*Math. Papers*, vol. 2, pp. 709–711). See also *Math. Papers*, vol. 4, pp. 734–737.

† Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 140.

so that (6) is equivalent to

$$\nu(n) \sim 4Ae^{-\gamma} \frac{n}{(\log n)^2} \frac{p-1}{p-2} \frac{q-1}{q-2} \cdots \qquad \text{.........(7).}$$

Hence the asymptotic values furnished for $\nu(n)$ by (6) and by (1) are in the ratio $2e^{-\gamma} : 1$, *i.e.* in the ratio 1·123 : 1.

A quite different formula was suggested by Stäckel*, viz.

$$\nu(n) \sim \frac{n^2}{(\log n)^2 \phi(n)} \qquad \text{....................(8),}$$

where $\phi(n)$ denotes, as usual, the number of numbers less than n and prime to n. This is equivalent to

$$\nu(n) \sim \frac{n}{(\log n)^2} \frac{p}{p-1} \frac{q}{q-1} \cdots \qquad \text{...........(9).}$$

Since $p/(p-1)$ is nearer to unity than $(p-1)/(p-2)$, the oscillations of $\nu(n)$ would, if Stäckel's formula were correct, be decidedly less pronounced than they would be if (1) were correct. As between the two formulae, the numerical evidence seems to be decisive. Thus the ratio $\nu(8190) : \nu(8192)$ is 3·32, whereas according to (1) it should be 3·48, and according to Stäckel's formula it should be 2·37. Stäckel's result is obtained by considerations of probability which ignore entirely the irregularity of the distribution of the primes in a given interval $n \leqslant N$, and it is not surprising, therefore, that it should be seriously in error.

On the other hand it should be observed that Sylvester's formula (7) gives, within the range of the table on p. 240, very good results, not much worse than those given by (5), and decidedly better than those given by (1). This is shown by the table which follows, in which decompositions into powers of primes higher than the first are neglected.

n	*Formula* (7) $\nu(n) : 2e^{-\gamma}\lambda(n)$	*Formula* (1) $\nu(n) : \lambda(n)$
$2{,}048 = 2^{11}$	·95	1·06
$2{,}250 = 2 . 3^2 . 5^3$	1·17	1·31
$2{,}304 = 2^8 . 3^2$	1·18	1·33
$2{,}306 = 2 . 1153$	1·17	1·31
$2{,}310 = 2 . 3 . 5 . 7 . 11$	1·12	1·26
$10{,}008 = 2^3 . 3^2 . 139$	1·11	1·25
$10{,}010 = 2 . 5 . 7 . 11 . 13$	1·12	1·27
$10{,}014 = 2 . 3 . 1669$	1·17	1·32
$170{,}166 = 2 . 3 . 79 . 359$	1·06	1·19
$170{,}170 = 2 . 5 . 7 . 11 . 13 . 17$	1·05	1·18
$170{,}172 = 2^2 . 3^2 . 29 . 163$	1·04	1·16

* *Göttinger Nachrichten* (1896), pp. 292–299.

§ 5. It has been shown by Landau* that

$$\sum_{1}^{n} \nu(h) \sim \frac{n^2}{2(\log n)^2} \quad \text{.....................(10)};$$

and that Stäckel's formula (8) is inconsistent with (10), and accordingly incorrect.

The same test can be applied to the formula (1) and Sylvester's formula (7). In fact Messrs Hardy and Littlewood have shown† that (10) is a consequence of (1): from which it follows, of course, that the asymptotic formula of the type of (10), furnished by Sylvester's formula, would be in error to the extent of a factor $2e^{-\gamma} = 1{\cdot}123$; that Sylvester's formula is therefore also incorrect; and that if *any* formula of this type is correct, it must be (1).

It may seem at first surprising that, in these circumstances, Sylvester's formula should give, for fairly large values of n, results actually better (as is shown by the results in the table on p. 242) than those given by (1). The explanation is to be found in the nature of the error term in (1). The modified formula (5), which we have already shown to be likely to give better results than (1), for moderately large values of n, differs from (1) by a factor of the type

$$1 + \frac{2}{\log n} + \ldots.$$

This factor does not affect the *asymptotic* value of $\nu(n)$, but it makes a great deal of difference within the limits throughout which verification is possible: thus when $n = 170{,}170$ it is equal to $1{\cdot}166$. When $n = 10^{10}$, it is equal to $1{\cdot}087$, and its difference from unity is negligible only when n is quite outside the range of computation. It is only such values of n that would reveal the superiority of the unmodified formula (1) over Sylvester's formula.

§ 6. Shortly after the writing of the preceding sections had been completed, Mr Hardy informed us of the existence of a third proposed asymptotic formula for $\nu(n)$, given more recently by V. Brun‡. The formula to which Brun's argument leads is

$$\nu(n) \sim 2Bn \frac{p-1}{p-2} \frac{q-1}{q-2} \cdots \quad \text{...............(11)},$$

where

$$B = \left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right) \cdots \left(1 - \frac{2}{h}\right)$$

$$= \prod_{h=3}^{h<\sqrt{n}} \left(1 - \frac{2}{h}\right).$$

* *Göttinger Nachrichten* (1900), pp. 177–186.

† See their note which follows this paper.

‡ *Archiv for Mathematik* (Christiania), vol. 34, 1917, no. 8. See also § 4 of Hardy and Littlewood's note.

By an argument similar to that used in § 4, in the reduction of Sylvester's formula, it may be shown that this is equivalent to the formula

$$\nu(n) \sim 8Ae^{-2\gamma} \frac{n}{(\log n)^2} \frac{p-1}{p-2} \frac{q-1}{q-2} \ldots = 4e^{-2\gamma}\lambda(n) \quad (12).$$

Thus this asymptotic value for $\nu(n)$, and the Hardy-Littlewood value, are in the ratio $4e^{-2\gamma} : 1 = 1{\cdot}263\ldots : 1$. Sylvester's is their geometric mean.

The formulae (11) and (12) would furnish a quite close approximation for $\nu(n)$ for those values of n on which it could be, in practice, tested. Thus, for $n = 170{,}170$, we find that

$$\nu(n)/4e^{-2\gamma}\lambda(n) = {\cdot}93 \ldots.$$

But the ultimate incorrectness of the formula may be proved in the same way as that of Sylvester's formula, namely by use of Landau's asymptotic formula (10).

Brun knew of the memoirs of Stäckel and Landau, but appears to have been unacquainted with Sylvester's work.

Note on Messrs Shah and Wilson's paper entitled: 'On an empirical formula connected with Goldbach's Theorem'. By G. H. HARDY, M.A., Trinity College, and J. E. LITTLEWOOD, M.A., Trinity College.

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1. The formulae discussed by Messrs Shah and Wilson were obtained in the course of a series of researches which have occupied us at various times during the last two years. A full account of our method will appear in due course elsewhere*; but it seems worth while to give here some indication of the genesis of these particular formulae, and others of the same character. We have added a few words about various questions which are suggested by Shah and Wilson's discussion.

The genesis of the formulae.

2. Let

$$f(x) = \Sigma \Lambda(n) x^n = \Sigma \Lambda(n) e^{-ny} = F(y)$$

and

$$f_\kappa(x) = F_\kappa(y) = \Sigma \chi_\kappa(n) \Lambda(n) e^{-ny},$$

where $\Lambda(n)$ is equal to $\log p$ when n is a prime p, or a power of p, and to zero otherwise, and $\chi_\kappa(n)$ is one of Dirichlet's 'characters to modulus q'†. Also let

$$x = \mathbf{x} e^{2p\pi i/q},$$

where p is positive, less than q, and prime to q; and suppose that $\mathbf{x}$ tends to unity by positive values.

It is known that

$$\sum_1^n \chi_\kappa(\nu) \Lambda(\nu) = o(n),$$

unless χ_κ is the 'principal' character χ_1, in which case

$$\sum_1^n \chi_\kappa(\nu) \Lambda(\nu) \sim \sum_1^n \Lambda(\nu) \sim n.$$

It follows that

$$f_1(\mathbf{x}) \sim \frac{1}{1-\mathbf{x}} \tag{2·1}$$

and

$$f_\kappa(\mathbf{x}) = o\left(\frac{1}{1-\mathbf{x}}\right) \quad (\kappa > 1). \tag{2·2}$$

* An outline of one of its most important applications is contained in a paper entitled 'A new solution of Waring's Problem', which will be published shortly in the *Quarterly Journal of Mathematics*.

† See Landau, *Handbuch*, pp. 391 *et seq.*

Now

$$(2\cdot3)\qquad f(x) = \Sigma\Lambda(n)\,\mathbf{x}^n e^{2np\pi i/q} = \sum_{j=1}^{q} e^{2jp\pi i/q} \sum_{n\equiv j} \Lambda(n)\,\mathbf{x}^n.$$

If j is prime to q, we have*

$$(2\cdot4)\qquad \sum_{n\equiv j} \Lambda(n)\,\mathbf{x}^n = \frac{1}{\phi(q)} \sum_{\kappa=1}^{\phi(q)} \bar{\chi}_\kappa(j)\, f_\kappa(\mathbf{x}),$$

where $\bar{\chi}_\kappa$ is the character conjugate to χ_κ, and $\phi(q)$ is the number of numbers less than and prime to q. It follows from (2·1) and (2·2) that

$$(2\cdot5)\qquad \sum_{n\equiv j} \Lambda(n)\,\mathbf{x}^n \sim \frac{\bar{\chi}_1(j)}{\phi(q)}\frac{1}{1-\mathbf{x}} = \frac{1}{\phi(q)}\frac{1}{1-\mathbf{x}}.$$

If on the other hand j is not prime to q, the formula (2·4) is untrue, as its right-hand side is zero. But in this case $\Lambda(n)=0$ unless n is a power of q, so that

$$(2\cdot6)\qquad \sum_{n\equiv j} \Lambda(n)\,\mathbf{x}^n = o\left(\frac{1}{1-\mathbf{x}}\right).$$

From (2·3), (2·5), and (2·6) it follows that

$$(2\cdot7)\qquad f(x) \sim \frac{A_q}{1-\mathbf{x}},$$

where

$$(2\cdot71)\qquad A_q = \frac{1}{\phi(q)} \sum_j e^{2jp\pi i/q} = \frac{1}{\phi(q)} \sum_j e^{2j\pi i/q},$$

the summation extending over all values of j less than and prime to q. The sum which appears in (2·71) has been evaluated by Jensen and Ramanujan†, and its value is $\mu(q)$, the well-known arithmetical function of q which is equal to zero unless q is a product $p_1 p_2 \ldots p_\rho$ of different primes, and then equal to $(-1)^\rho$. Thus

$$(2\cdot8)\qquad f(x) \sim \frac{\mu(q)}{\phi(q)}\frac{1}{1-\mathbf{x}}\ddagger.$$

3. The sum

$$(3\cdot1)\qquad \omega(n) = \sum_{m+m'=n} \Lambda(m)\,\Lambda(m'),$$

* Landau, *l.c.*, p. 421.

† J. L. W. V. Jensen, 'Et nyt Udtryk for den talteoretiske Funktion $\sum_1^m \mu(n) = M(m)$', *Saertryk af Beretning om den 3 Skandinaviske Matematiker-Kongres*, Kristiania, 1915; S. Ramanujan, 'On certain trigonometrical sums and their applications in the theory of numbers', *Trans. Camb. Phil. Soc.*, vol. 22, 1918, pp. 259–276.

‡ If $\mu(q)$ is zero, this formula is to be interpreted as meaning

$$f(x) = o\left(\frac{1}{1-\mathbf{x}}\right).$$

which appears on the left-hand side of Shah and Wilson's equation (2), is the coefficient of x^n in the expansion of $\{f(x)\}^2$. And

$$\{f(x)\}^2 \sim \left\{\frac{\mu(q)}{\phi(q)}\right\}^2 \frac{1}{(1-\mathbf{x})^2} = \left\{\frac{\mu(q)}{\phi(q)}\right\}^2 \Sigma n x^n e^{-2np\pi i/q},$$

when $x \to e^{2p\pi i/q}$ along a radius vector. Our general method accordingly suggests to us to take

$$\Omega(n) = n\,\Sigma \left\{\frac{\mu(q)}{\phi(q)}\right\}^2 e^{-2np\pi i/q},$$

where the summation extends over $q = 1, 2, 3, \ldots$ and all values of p less than and prime to q, as an approximation to $\omega(n)$. Using Ramanujan's notation, this sum may be written

(3·2) $$\Omega(n) = n\,\Sigma \left\{\frac{\mu(q)}{\phi(q)}\right\}^2 c_q(n).$$

The series (3·2) can be summed in finite terms. We have

(3·3) $$c_q(n) = \Sigma \delta \mu\left(\frac{q}{\delta}\right),$$

the summation extending over all common divisors δ of q and n*; and it is easily verified, either by means of this formula or by means of the definition of $c_q(n)$ as a trigonometrical sum, that

$$c_{qq'}(n) = c_q(n)\, c_{q'}(n)$$

whenever q and q' are prime to one another. We may therefore write

$$\Omega(n) = n\Sigma A_q = n\Pi \chi_{\varpi},$$

where the product extends over all primes ϖ, and

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + A_{\varpi^3} + \ldots = 1 + A_{\varpi},$$

since A_q contains the factor $\mu(q)$ and $A_{\varpi^2}, A_{\varpi^3}, \ldots$ are accordingly zero.

If n is not divisible by ϖ, we have $c_{\varpi}(n) = \mu(\varpi) = -1$ and

$$A_{\varpi} = -\frac{1}{\{\phi(\varpi)\}^2} = -\frac{1}{(\varpi-1)^2};$$

while if n is divisible by ϖ we have

$$c_{\varpi}(n) = \mu(\varpi) + \varpi\mu(1) = \varpi - 1,$$

$$A_{\varpi} = \frac{1}{\varpi - 1}.$$

Hence

$$\Omega(n) = n\Pi'\left(1 + \frac{1}{\varpi - 1}\right)\Pi''\left\{1 - \frac{1}{(\varpi-1)^2}\right\},$$

* Ramanujan, *l.c.*, p. 260.

where Π' applies to primes which divide n and Π'' to primes which do not.

It is evident that $\Omega(n)$ is zero if n is odd. On the other hand, if n is even, we have

$$\Omega(n) = 2n\Pi\left\{1 - \frac{1}{(\varpi - 1)^2}\right\} \Pi\left[\left(1 + \frac{1}{\mathbf{p} - 1}\right)\Big/\left\{1 - \frac{1}{(\mathbf{p} - 1)^2}\right\}\right]$$
$$= 2n\Pi\left\{1 - \frac{1}{(\varpi - 1)^2}\right\} \Pi\left(\frac{\mathbf{p} - 1}{\mathbf{p} - 2}\right),$$

where ϖ now runs through all odd primes and $\mathbf{p}$ through odd prime divisors of n.

The formula
$$\omega(n) \sim \Omega(n)$$
is formula (2) of Shah and Wilson's paper*.

The incorrectness of Sylvester's formula.

4. It is easy to prove that *if any formula of the type*

(4·1) $$\omega(n) \sim C\Omega(n)$$

be true, then C must be unity. In other words, our formula is the only formula of this type which can possibly be correct. This may be shown as follows.

Let

(4·2) $$f(s) = \Sigma \frac{\Omega(n)}{n^s},$$

where n runs through all even values; and let $s - 1 = t$. The series is absolutely convergent if $s > 2$, $t > 1$. Replacing $\Omega(n)$ by its expression in terms of the prime divisors of n, and splitting up $f(s)$ into factors in the ordinary manner, we obtain

$$f(s) = \frac{2^{1-t}A}{1 - 2^{-t}} \Pi\left(1 + \frac{\varpi - 1}{\varpi - 2}\frac{\varpi^{-t}}{1 - \varpi^{-t}}\right) = \frac{2^{1-t}A\chi(t)}{1 - 2^{-t}},$$

say, where A is the same constant as in Shah and Wilson's paper, and ϖ runs through all odd primes.

Let

$$\psi(t) = \Pi\left(1 + \frac{\varpi^{-t}}{1 - \varpi^{-t}}\right) = \Pi\left(\frac{1}{1 - \varpi^{-t}}\right) = (1 - 2^{-t})\,\zeta(t),$$

and suppose that $t \to 1$. Then

$$\frac{\chi(t)}{\psi(t)} = \Pi\left\{\left(1 + \frac{\varpi - 1}{\varpi - 2}\frac{\varpi^{-t}}{1 - \varpi^{-t}}\right)\Big/\left(1 + \frac{\varpi^{-t}}{1 - \varpi^{-t}}\right)\right\}$$
$$\to \Pi\left\{\left(1 + \frac{1}{\varpi - 2}\right)\Big/\left(1 + \frac{1}{\varpi - 1}\right)\right\}$$
$$= \Pi\left\{\frac{(\varpi - 1)^2}{\varpi(\varpi - 2)}\right\} = \Pi\left\{\frac{(\varpi - 1)^2}{(\varpi - 1)^2 - 1}\right\} = \frac{1}{A};$$

* When $\Omega(n) = 0$, the formula is to be interpreted as meaning $\omega(n) = o(n)$.

and so

$$(4\cdot3) \qquad f(s) \sim 2A\chi(t) \sim 2(1-2^{-t})\zeta(t) \sim \frac{1}{t-1} = \frac{1}{s-2}.$$

This is a consequence of our hypothesis: the corresponding consequence of the hypothesis (4·1) would be

$$(4\cdot31) \qquad f(s) \sim \frac{C}{s-2}.$$

On the other hand, it is easy to prove* that

$$(4\cdot4) \qquad \omega(1) + \omega(2) + \ldots + \omega(n) \sim \tfrac{1}{2}n^2;$$

and from this to deduce that

$$\phi(s) = \Sigma \frac{\omega(n)}{n^s} \sim \frac{1}{s-2}$$

when $s \to 2$. This equation is inconsistent with (4·1) and (4·31), unless $C = 1$.

It follows that Sylvester's suggested formula is definitely erroneous.

It is more difficult to make a definite statement about the formula given by Brun. The formula to which his argument naturally leads is Shah and Wilson's formula (12); and this formula, like Sylvester's, is erroneous. But in fact Brun never enunciates this formula explicitly. What he does is rather to advance reasons for supposing that *some* formula of the type (4·1) is true, and to determine C on the ground of empirical evidence†. The result to which he is led is equivalent to that obtained by taking $C = 1\cdot5985/1\cdot3203 = 1\cdot2107$ ‡. The reason for so substantial a discrepancy is in effect that explained in the last section of Shah and Wilson's paper.

Further results.

5. The method of § 2 leads to a whole series of results concerning the number of decompositions of n into 3, 4, or any number of primes. The results suggested by it are as follows. Suppose

* Since
$$\Sigma\Lambda(n)x^n \sim \frac{1}{1-x}$$
as $x \to 1$, we have
$$\Sigma\omega(n)x^n = \{\Sigma\Lambda(n)x^n\}^2 \sim \frac{1}{(1-x)^2};$$
and the desired result follows from Theorem 8 of a paper published by us in 1912 ('Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive', *Proc. London Math. Soc.*, ser. 2, vol. 13, pp. 174–192). This, though the shortest, is by no means the simplest proof.

The formula (4·4) is substantially equivalent to Landau's formula (10) in Shah and Wilson's paper.

† Evidence connected not with Goldbach's theorem itself but with a closely related problem concerning pairs of primes differing by 2. See § 7.

‡ 1·5985 is Brun's constant, while 1·3203 is $2A$.

that $\nu_r(n)$ is the number of expressions of n as the sum of r primes Then if r is *odd* we have

$$(5\cdot11) \qquad \nu_r(n) = o(n^{r-1})$$

if n is even, and

$$(5\cdot12) \qquad \nu_r(n) \sim \frac{2B}{(r-1)!} n^{r-1} \Pi \left\{\frac{(\mathbf{p}-1)^r - (\mathbf{p}-1)}{(\mathbf{p}-1)^r + 1}\right\}$$

if n is odd, $\mathbf{p}$ being an odd prime divisor of n, and

$$(5\cdot13) \qquad B = \Pi \left\{1 + \frac{1}{(\varpi - 1)^r}\right\},$$

where ϖ runs through all odd primes. On the other hand, if r is *even*, we have

$$(5\cdot21) \qquad \nu_r(n) = o(n^{r-1})$$

if n is odd, and

$$(5\cdot22) \qquad \nu_r(n) \sim \frac{2C}{(r-1)!} n^{r-1} \Pi \left\{\frac{(\mathbf{p}-1)^r + (\mathbf{p}-1)}{(\mathbf{p}-1)^r - 1}\right\},$$

where

$$(5\cdot23) \qquad C = \Pi \left\{1 - \frac{1}{(\varpi - 1)^r}\right\},$$

if n is even. The last formula reduces to (1) of Shah and Wilson's paper when $r = 2$.

We have not been able to find a rigorous proof, independent of all unproved hypotheses, of any of these formulae. But we are able to connect them in a most interesting manner with the famous 'Riemann hypothesis' concerning the zeros of Riemann's function $\zeta(s)$. The Riemann hypothesis may be stated as follows: *$\zeta(s)$ has no zeros whose real part is greater than $\frac{1}{2}$.* If this be so, it follows easily that all the zeros of $\zeta(s)$, other than the trivial zeros $s = -2$, $s = -4, \ldots$, lie on the line $\sigma = \mathbf{R}(s) = \frac{1}{2}$. It is natural to extend this hypothesis as follows: *no one of the functions defined, when $\sigma > 1$, by the series*

$$L(s) = \Sigma \frac{\chi_\kappa(n)}{n^s},$$

possesses zeros whose real part is greater than $\frac{1}{2}$. We may call this the *extended Riemann hypothesis.* This being so, what we can prove is this, that *if the extended Riemann hypothesis is true, then the formulae* (5·11)—(5·23) *are true for all values of r greater than* 4.

The reasons for supposing the extended hypothesis true are of the same nature as those for supposing the hypothesis itself true. It should be observed, however, that it is necessary, before we generalise the hypothesis, to modify the form in which it is usually stated; for it is not proved (as it is for $\zeta(s)$ itself) that $L(s)$ can have no real zero between $\frac{1}{2}$ and 1.

6. A modification of our method enables us to attack a closely related problem, that of the existence of pairs of primes differing by a constant even number k.

We have

$$\Sigma \Lambda(n) \Lambda(n+k) r^{2n+k} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 e^{-ki\theta} \, d\theta,$$

where $f(x)$ is the same function as in § 1, and r is positive and less than unity. We divide the range of integration into a number of small arcs, correlated in an appropriate manner with a certain number of the points $e^{2p\pi i/q}$, and approximate to $|f(re^{i\theta})|^2$ on each arc by means of the formula (2·8). The result thus suggested is that

$$\Sigma \Lambda(n) \Lambda(n+k) r^{2n} \sim \frac{2A}{1-r^2} \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),$$

where A has the same meaning as in § 2 and $\mathbf{p}$ is an odd prime divisor of k. From this it would follow that

$$(6\cdot1) \qquad \sum_{\nu<n} \Lambda(\nu) \Lambda(\nu+k) \sim 2An \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right);$$

and that, if $N_k(n)$ is the number of prime pairs less than n, whose difference is k, then

$$(6\cdot2) \qquad N_k(n) \sim \frac{2An}{(\log n)^2} \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right).$$

This formula is of exactly the same form as (1), except that $\mathbf{p}$ is now a factor of k and not of n. In particular we should have

$$(6\cdot3) \qquad N_2(n) \sim \frac{2An}{(\log n)^2},$$

and

$$(6\cdot4) \qquad N_6(n) \sim \frac{4An}{(\log n)^2}.$$

We should therefore conclude that there are about two pairs of primes differing by 6 to every pair differing by 2. This conclusion is easily verified. In fact the numbers of pairs differing by 2, below the limits*

100, 500, 1000, 2000, 3000, 4000, 5000,

are

9, 24, 35, 61, 81, 103, 125;

while the numbers of pairs differing by 6 are

16, 47, 73, 125, 168, 201, 241.

* To be precise, the numbers of pairs (p, p') such that $p' = p + 2$ and p' does not exceed the limit in question.

The numbers of pairs differing by 4, which should be roughly the same as those of pairs differing by 2, are

$$9,\ 26,\ 41,\ 63,\ 86,\ 107,\ 121.$$

7. Brun, in his note already referred to, recognises the correspondence between the problem of §§ 2—4 and that of the prime-pairs differing by 2, and realises the identity of the constants involved in the formulae; but does not allude to the more general problem of prime-pairs differing by k. He does not determine the fundamental constant A, attempting only to approximate to it empirically by means of a count of prime-pairs differing by 2 and less than 100000, made by Glaisher in 1878*. The value of the constant thus obtained is, as was pointed out in § 4, seriously in error. The truth is that when we pass from (6·1), which, when $k=2$, takes the form

$$\sum_{\nu<n} \Lambda(\nu)\Lambda(\nu+2) \sim 2An,$$

to (6·3), the formula which presents itself most naturally is not (6·3) but

$$N_2(n) \sim 2A\int^n \frac{dx}{(\log x)^2}. \tag{7·1}$$

This formula is of course, in the long run, equivalent to (6·3). But

$$\int^n \frac{dx}{(\log x)^2} = \frac{n}{(\log n)^2}\left(1+\frac{2\,!}{\log n}+\frac{3\,!}{(\log n)^2}+\dots\right)^{\dagger};$$

and the second factor on the right-hand side is, for $n=100000$, far from negligible. Thus (6·3) may be expected, for such values of n, to give results considerably too small.

If we take the lower limit of integration in (7·1) to be 2, we find that the value of the right-hand side for $n=100000$ is, to the nearest integer, 1249, whereas the actual value of $N_2(n)$ is, according to Glaisher, 1224‡. The ratio is 1·02, and the agreement seems to be as good as can reasonably be expected.

The calculation of prime-pairs has been carried further by Mrs Streatfeild, whose results are exhibited in the following table:

* J. W. L. Glaisher, 'An enumeration of prime-pairs', *Messenger of Mathematics*, vol. 8, 1878, pp. 28–33. The number of pairs below 100000 is 1225.

† The series is naturally divergent, and must be closed, after a finite number of terms, with an error term of lower order than the last term retained.

‡ Glaisher reckons 1 as a prime and (1, 3) as a prime-pair, making 1225 in all.

n	$N_2(n)$	$2A\int_2^n \frac{dx}{(\log x)^2}$	Ratio
100,000	1224	1249	1·020
200,000	2159	2180	1·010
300,000	2992	3035	1·014
400,000	3801	3846	1·012
500,000	4562	4625	1·014
600,000	5328	5381	1·010

8. In a later paper* Brun gives a more general formula relating to prime-pairs (p, p') such that $p = ap' + 2$. This formula also involves an undetermined constant k. It is worth pointing out that our method is equally applicable to this and to still more general problems. Suppose, in the first place, that $\nu(n)$ is the number of expressions of n in the form

$$n = ap + bp',$$

where p and p' are primes†. We may suppose without loss of generality that a and b have no common factor.

The results suggested by our method are as follows. If n has any factor in common with a and b, then

$$\nu(n) = o\left\{\frac{n}{(\log n)^2}\right\};$$

and this is true even when n is prime to both a and b, unless *one* of n, a, b is even‡. But if n, a and b are coprime, and one of them even, then

$$\nu(n) \sim \frac{2A}{ab}\frac{n}{(\log n)^2}\Pi\left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),$$

where A is the constant of § 2, and the product is now extended over all odd primes which divide n or a or b.

* 'Sur les nombres premiers de la forme $ap+b$', *Archiv for Mathematik*, vol. 24, 1917, no. 14.

† We might naturally include powers of primes.

‡ These results are trivial. If n and a have a common factor, it divides bp', and is therefore necessarily p', which can thus assume but a finite number of values. If n, a, b are all odd, either p or p' must necessarily be 2.

Similarly, suppose $N(n)$ to be the number of pairs of solutions of the equation

$$ap' - bp = k$$

such that $p' < n$. It is supposed that a and b have no common factor. Then

$$N(n) = o\left\{\frac{n}{(\log n)^2}\right\}$$

unless k is prime to both a and b, and one of the three is even. If these conditions are satisfied

$$N(n) \sim \frac{2A}{a} \frac{n}{(\log n)^2} \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),$$

where $\mathbf{p}$ is now an odd prime factor of k, a, or b.

The distribution of Electric Force between two Electrodes, one of which is covered with Radioactive Matter. By W. J. HARRISON, M.A., Fellow of Clare College.

[*Read* 17 February 1919.]

It has been shown by Rutherford* that it is probable that the ionisation due to an α particle per unit length of its path is inversely proportional to its velocity, provided the velocity exceeds a certain minimum necessary to effect ionisation. It follows that the ionisation per unit time is constant at all points of the path.

Suppose radioactive matter distributed uniformly over the surface of a large plane electrode assumed to be infinite in order to obtain simplicity in calculation. Consider the α particles projected from a point P of the electrode. These particles are projected equally in all directions, hence the rate of ionisation per unit volume at a point Q will be proportional to $1/PQ^2$, provided $PQ < R$, where R is the range of the particles. The total rate of ionisation at a point Q distance $x\,(x < R)$ from the electrode will be proportional to

$$\int_0^{\sqrt{R^2-x^2}} \frac{2\pi r\,dr}{x^2+r^2},$$

where r is the distance of a point P on the electrode from the foot of the perpendicular from Q. Now

$$\int_0^{\sqrt{R^2-x^2}} \frac{2r\,dr}{x^2+r^2} = \Big[\log(x^2+r^2)\Big]_0^{\sqrt{R^2-x^2}}$$

$$= \log\frac{R}{x}.$$

Hence rate of ionisation

$$q = q_0 \log\frac{R}{x}.$$

The equations determining the distribution of electric force are given by Thomson, *Conduction of Electricity through Gases*, 1906, chap. III. The notation of this book is adopted as being sufficiently well known. The differential equation for the electric force X is of the form

$$\frac{d^2X^2}{dx^2} + \frac{a}{X^2}\left(\frac{dX^2}{dx}\right)^2 + \frac{b}{X^2} = cq_0 \log\frac{R}{x},\quad x < R,$$

$$\frac{d^2X^2}{dx^2} = 0, \qquad x > R.$$

* *Radioactive Substances and their Radiations*, 1913, p. 158.

The numerical solution may be obtained for any particular values of the constants a, b, c, q_0, R by approximate methods. In the absence of any definite experimental results with which to compare the calculations, the labour involved in integration is not worth undertaking.

The case, however, of the saturation current is the most important, and the integration is simple. It is assumed that recombination of ions does not take place in this case, and therefore the equations reduce to

$$\frac{d^2X^2}{dx^2} = 8\pi e\left(\frac{1}{k_1}+\frac{1}{k_2}\right) q_0 \log\frac{R}{x}, \quad x < R,$$

$$= 0, \qquad x > R.$$

Write $$8\pi e q_0\left(\frac{1}{k_1}+\frac{1}{k_2}\right) = K.$$

Then, for $x < R$,

$$X^2 = K\left[\frac{x^2}{2}\log\frac{R}{x} + \tfrac{3}{4}x^2 + Bx + C\right],$$

for $x > R$,

$$X^2 = \frac{8\pi i x}{k_1} + A$$

(*vide* Rutherford, *Radioactive Substances, etc.*, p. 67), A, B, C are constants of integration.

Now the conditions are

(1) at $x=0$, $n_1=0$, if $x=0$ be the positive plate,

(2) at $x=R$, $n_2=0$,

(3) at $x=R$, n_1 is continuous,

(4) at $x=R$, X is continuous.

From (1) $$\frac{i}{X} + \frac{k_2}{4\pi}\frac{dX}{dx} = 0$$

(*vide Conduction of Electricity through Gases*, chap. III.).

$$\therefore \quad \frac{dX^2}{dx} = -\frac{8\pi i}{k_2},$$

$$\therefore \quad K.B = -\frac{8\pi i}{k_2}.$$

(2) and (3) lead to the same condition, which is the same as (1), if

$$i = eRq_0.$$

Now since there is no recombination

$$i = \int_0^R e q_0 \log\frac{R}{x}\,dx = eRq_0.$$

Hence conditions (1), (2), (3) are identical and determine B. Condition (4) supplies a relation between C and A,

$$A = K(C - \tfrac{1}{4}R^2).$$

Hence

$$X^2 = K\left[\tfrac{1}{2}x^2 \log \frac{R}{x} + \tfrac{3}{4}x^2 - \frac{k_1}{k_1 + k_2} Rx + DR^2\right],$$

$$0 < x < R, \text{ where } DR^2 = C,$$

$$X^2 = K\left[\frac{k_2}{k_1 + k_2} Rx + R^2(D - \tfrac{1}{4})\right], \quad x > R.$$

The constant D can be determined when the potential difference between the electrodes is given*.

The general character of these results can be shown by numerical calculation for the cases $k_1 = k_2$, $1{\cdot}25\,k_1 = k_2$, $k_1 = 1{\cdot}25\,k_2$ (corresponding to the case in which the positive ion moves more slowly, as usual, than the negative ion, and the radioactive matter is spread on the negative plate), and for distances R, $2R$, $3R$ between the electrodes, and for $D = 0{\cdot}1$, $0{\cdot}5$, $1{\cdot}0$. In order that the current may be the saturation current it is necessary in practice that D should exceed a certain limit. This limit is dependent on the particular conditions of any given experiment.

The distribution of the electric force X is shown on the graph below. The curves marked (1), (2), (3) are for the cases

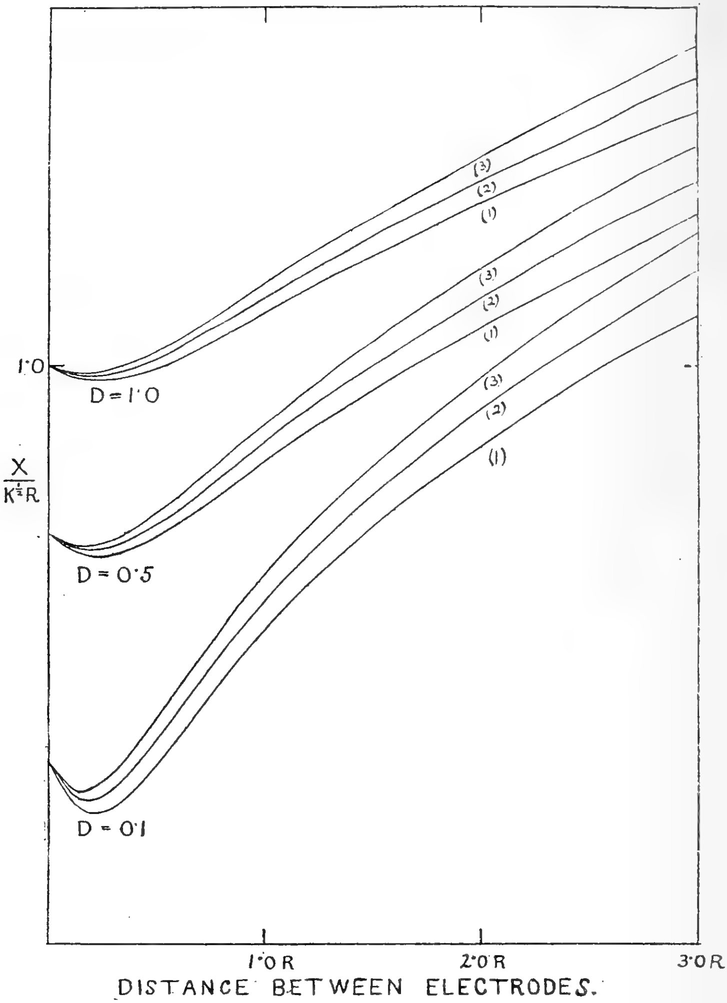
$$k_1 = 1{\cdot}25\,k_2, \quad k_1 = k_2, \quad k_2 = 1{\cdot}25\,k_1, \text{ respectively.}$$

The potential difference V between the electrodes is given in the following table, d being the distance between the plates.

$$\frac{V}{R^2 . K^{\frac{1}{2}}}.$$

		$k_1 = 1{\cdot}25\,k_2$	$k_1 = k_2$	$k_2 = 1{\cdot}25\,k_1$
$D = 0{\cdot}1$	$d = R$	·343	·379	·412
	$d = 2R$	1·056	1·147	1·232
	$d = 3R$	2·034	2·193	2·343
$D = 0{\cdot}5$	$d = R$	·725	·743	·762
	$d = 2R$	1·676	1·740	1·800
	$d = 3R$	2·841	2·963	3·078
$D = 1{\cdot}0$	$d = R$	1·014	1·027	1·041
	$d = 2R$	2·203	2·250	2·298
	$d = 3R$	3·566	3·663	3·759

* These forms of X are not strictly valid in the immediate neighbourhood of the electrodes, as the natural agitation of the ions has been neglected in this theory. *Vide* Pidduck, *Treatise on Electricity*, 1916, p. 505.



(3)
(2)
(1)
(3)
(2)
(1)
(3)
(2)
(1)
1·0
D = 1·0
X / K^½ R
D = 0·5
D = 0·1
1·0 R
2·0 R
3·0 R
DISTANCE BETWEEN ELECTRODES.

The conversion of saw-dust into sugar. By J. E. PURVIS, M.A.

[*Read* 17 February 1919.]

The production of sugar from wood is well known. In the Classen process, saw-dust is digested in closed retorts with a weak solution of sulphurous acid under a pressure of between six and seven atmospheres. The products contain about 25 °/₀ of dextrose, and other substances are pentose, acetic acid, furfurol and formaldehyde. Cellulose material can also be converted into sugar by other acids.

The following results were obtained by digesting saw-dust from ordinary deal with different acids of varying concentrations; estimating the amount of sugar in the liquid in the usual way from the amount of cuprous oxide precipitated from Fehling's solution, and converting this oxide of copper to cupric oxide. The numbers were then calculated in terms of dextrose.

(1) 25 grams of saw-dust were digested with 300 c.c. distilled water and 50 c.c. strong H_2SO_4 (1 c.c. $H_2SO_4 = 1{\cdot}78$ grms. H_2SO_4) for $5\frac{1}{4}$ hours in a sand bath at a temperature just below the boiling point and the mixture was constantly stirred. This was then filtered; the residue well washed and the filtrate made up to a litre; 10 c.c. of the filtrate were neutralised with sodium carbonate and the cuprous oxide from Fehling's solution was precipitated, filtered, dried and ignited to cupric oxide. This gave 0·215 grm. CuO which is equivalent to 39 °/$_0$ of dextrose on the original amount of saw-dust.

(2) 25 grams of saw-dust to which were added 500 c.c. of distilled water and 25 c.c. of strong H_2SO_4 of the same strength as in experiment (1) and digested for 5 hours under the same conditions. This gave 13 °/$_0$ of dextrose.

(3) 50 grams of saw-dust were digested with 500 c.c. of distilled water and 50 c.c. of the strong H_2SO_4 for $5\frac{3}{4}$ hours. The yield was 11·5 °/$_0$ dextrose.

(4) 25 grams of saw-dust were digested with 250 c.c. of tap water and 10 c.c. of strong H_2SO_4 for 2 hours. This yielded 10·5 °/$_0$ dextrose.

(5) 25 grams of saw-dust were digested with 720 c.c. of tap water and 10 c.c. strong H_2SO_4 for 2 hours. This produced 3·35 °/$_0$ dextrose.

(6) 50 grams of saw-dust were digested with 500 c.c. water and 50 c.c. N/1 HCl (= 1·825 grms. HCl) for 3 hours. This gave 3·35 °/$_0$ dextrose.

(7) 50 grams of saw-dust were digested with 500 c.c. water and 100 c.c. N/1 H_2SO_4 (= 2·45 grms. H_2SO_4) for 2 hours. This produced 1·82 °/₀ dextrose.

(8) 25 grams of saw-dust were digested with 700 c.c. water and 5 grams P_2O_5 for 12 hours at the temperature of the room (about 15° C.), and then for 3 hours just below the boiling point. This gave 12·66 °/₀ dextrose.

The results show that the amount of sugar which can be obtained depends on the nature of the acid and its strength relative to the amount of saw-dust, and on the time of digestion. The greatest amount was obtained when the strongest sulphuric acid acted for a considerable time. In the other experiments not so much was obtained as by the Classen process. For the commercial production of sugar from such a cheap material as saw-dust the question to be decided would be the relative cost of the Classen process compared with the cost under the conditions of these experiments. That would include a comparison of the cost of the various acids and the recovery of these acids for further use. The conversion of sugar into alcohol and acetone presents no difficulty; and it would be important to consider whether such useful chemical substances could not be produced from a waste product like saw-dust at a cheaper rate than by the present costly methods.

Bracken as a source of potash. By J. E. Purvis, M.A.

[*Read* 17 February 1919.]

The Master of Christ's College, Cambridge, in the autumn of 1917, had some correspondence with Mr J. A. A. Williams of Aberglaslyn Hall, Beddgelert, in regard to the use of bracken as a fertiliser. Mr Williams had burnt the bracken growing on a peaty soil on his estate at Beddgelert, ploughed in the ashes and obtained highly satisfactory crops of potatoes. It seemed to be of some importance to find out what amount of potash could be obtained from the ash; and in October 1917 a sample of bracken from the Botanic Gardens, Cambridge, was analysed. This grows on a poor sandy soil.

It is known that bracken contains larger quantities of potash in the summer months than in the autumn and more complete investigations were deferred till the summer of 1918. Meanwhile in the April (1918) number of the *Journal of Agriculture* (vol. 25, no. 1, p. 1) Messrs Berry, Robinson and Russell published an article on "Bracken as a source of potash" which contained the results of the analyses of material collected from various districts in England, Scotland and Wales from May to October 1916, and from June to October 1917. The numbers show that the amount of potash is much higher in the summer months than in the autumn. For example, bracken gathered June 1st, 1917, from Harpenden Common, Rothamsted, which is mainly gravel and clay, produced 4·1 °/₀ of potash (K_2O) on the dried material and only 1·8 °/₀ when gathered September 1st, 1917. The authors also considered that their evidence indicates a more rapid falling off of the potash from bracken growing on sandy and peaty soils than on heavier soils rich in potash: and that, therefore, its chances of success as a fertiliser would be greater in these heavier soils.

In view of these results the investigations were continued with the bracken growing in the Botanic Gardens, Cambridge, and also with that on Mr Williams's Welsh estate. The following tables summarise the results.

Generally, the numbers are of the same order as those obtained by Messrs Berry, Robinson and Russell, and confirm the opinion that in the summer months there is more potash than in the later months. Also there is a clear indication that, on an average, the Welsh peaty soil yields more potash than the Cambridge poor sandy soil.

CAMBRIDGE BRACKEN.

Date when sample was gathered	Percentage of dry matter in fresh bracken	Percentage of ash in dry matter	Percentage of potash (K_2O) in	
			fresh bracken	dry bracken
16 October, 1917	27·60	7·51	0·29	0·82
1 June, 1918	15·34	6·81	0·46	3·00
2 July, 1918	21·58	5·02	0·52	2·45
1 August, 1918	30·26	5·96	0·50	1·70
31 August, 1918	26·50	7·86	0·30	1·07
1 October, 1918	29·06	7·93	0·33	1·13
WELSH BRACKEN.				
3 June, 1918	24·4	6·55	0·77	3·19
4 July, 1918	25·8	5·78	0·83	3·22
31 July, 1918	40·7	3·84	0·42	1·45
1 September, 1918	30·97	7·02	0·53	1·71
3 October, 1918	34·54	4·82	0·45	1·32

To estimate the cost of collection is difficult as the conditions of transit and labour are variable and estimates for one locality would be useless for another. It is evident, however, that bracken is a valuable source of potash: but its economic application as a fertiliser will be controlled by the requirements and conditions of the neighbourhood where it grows.

I have to thank Mr Williams for supplying the Welsh bracken, and Mr Lynch, of the Cambridge Botanic Gardens, for samples from the gardens.

The action of electrolytes on the electrical conductivity of the bacterial cell and their effect on the rate of migration of these cells in an electric field. By C. SHEARER, Sc.D., F.R.S., Clare College. (From the Pathological Laboratory, Cambridge.)

[*Read* 17 February 1919.]

If a thick creamy emulsion of the meningococcus or *B. coli* is made up in neutral Ringer's solution (that is, one in which the sodium bicarbonate is left out), and the conductivity measured by means of a Kohlrausch bridge and cell; it is found that its resistance is more than treble that of the same solution without the bacteria: that is the greater part of the resistance is due to the presence of the bacteria.

This determination was made as follows: a 24 hour culture of the meningococcus or *B. coli* on trypagar (24 plates) was washed off in a considerable quantity of Ringer's solution, centrifuged down and rewashed several times in a similar manner to remove all traces of serum or any salts derived from the culture medium. The centrifuged deposit was then made up to standard strength in neutral Ringer's solution, so that it was not too thick to be sucked up in a medium sized pipette and transferred to a Hamburger cell and its conductivity determined. It was found that the conductivity of such standard emulsions when measured under similar conditions of temperature was fairly uniform*. When sufficient care was taken to get the emulsions of the right thickness, resistances of 110 ohms could be pretty constantly obtained. The same quantity of Ringer's solution alone had about 26·7 ohms resistance under the same conditions.

If, however, in place of the Ringer's solution we make up the bacterial emulsions in pure sodium chloride of the same conductivity as that of the Ringer's solution, *i.e.* one in which the resistance is 26·7 ohms (which corresponds to a NaCl solution of about 0·85 °/₀), we obtain as in the case of the emulsion in Ringer's solution an initial resistance of 110 ohms. Within a few minutes, however, this gradually drops and at the end of 30 or 40 minutes the emulsion now has the same conductivity as that of the bare sodium chloride solution without the bacteria, *i.e.* 26·7 ohms resistance. Thus pure sodium chloride of about the concentration as that present in the blood gradually destroys the resistance of the bacterial cell. If the bacteria are allowed to lie in this solution for several hours it will be found that at the end of this time, on subculture, they are

* All measurements were made at constant temperature 25° C. Resistance constant of conductivity cell $= 29 \cdot 8 \times 10^{-1}$.

dead. If they are only allowed to remain in the NaCl for a short time and then transferred to neutral Ringer again they immediately return to their normal resistance and grow freely on subculture.

If when the resistance of the bacterial emulsion has fallen in NaCl solution a little $CaCl_2$ is added it again regains its normal conductivity and is uninjured. Thus we get the usual antagonistic action of $CaCl_2$ to NaCl. It was found that KCl, LiCl, $MgCl_2$ acted like NaCl in reducing the resistance offered by the bacteria, while $BaCl_2$, $SrCl_2$ have no action on the resistance but act like $CaCl_2$. Thus it is clear that in the bacteria as with so many other plant and animal cells the entrance of the ions of NaCl, KCl, LiCl, $MgCl_2$ is prevented by the presence of very small quantities of $CaCl_2$, $BaCl_2$ or $SrCl_2$. Bacterial emulsions made up in $BaCl_2$, $SrCl_2$ and $CaCl_2$, having the same conductivity as Ringer's solution, showed no change in resistance on being kept in these solutions for some time, invariably remaining normal.

The interest of these experiments consists in that they agree completely with the results obtained by Loeb, Osterhout and a large number of other workers on animal and plant cells.

In Laminaria, Osterhout finds with $CaCl_2$ and presumably also with $BaCl_2$ and $SrCl_2$ there is invariably a brief temporary rise in resistance when placed in these solutions of the same conductivity as sea-water which is followed by a gradual fall. With the bacterial cell no such preliminary rise can be distinguished, while the fall due to the toxic action of the solution is much delayed and slower.

In view of the remarkable action of tri-valent ions on artificial membranes as shown by the work of Perrin, Girard and Mines, and the action on the permeability of cell wall as shown by the work of Mines, Osterhout and Gray, it is of great interest to consider their action on the bacterial cell.

While the tri-valent positive ion of lanthanium nitrate brings about a rapid rise of resistance in Laminaria according to Osterhout and in the Echinoderm egg according to Gray, when this salt is used in such dilution as not to affect the conductivity of the solution itself, no such action can be distinguished in the case of bacteria by means of the Kohlrausch method. The resistance remains unchanged until it begins to fall on account of the increasing strength of the salt added. In the same way the positive tri-valent ions of $CeCl_3$, neo-ytterbium chloride and the tri-valent negative ions of sodium citrate appear to have no action in increasing or decreasing the resistance of the bacterial cell as determined by the conductivity method. It should be pointed out that these salts can only be used in very dilute solutions. In the case of lanthanium nitrate this salt readily flocculates living bacteria when used in stronger solutions than $\frac{1}{5000}$ M.

It would seem remarkable in view of the sharp action of La on the Echinoderm egg when used in a strength of $\frac{1}{5000}$ M. that some similar action should not be found with bacteria, but repeated experiments with centrifuged solid bacterial deposits of both the meningococcus and *B. coli* using the same type of electrodes used by Gray for the Echinoderm egg and obtaining resistances as high as 150 ohms failed to show any initial rise of resistance. It was possible that in the case of bacteria, their enormous surface would render the preliminary rise of resistance so temporary that, before the electrodes could be placed in position and the bridge readings adjusted, it would be over and passed. To test this point a small quantity of La was added while the bridge telephone was kept to the ear, but in every instance no change could be detected. It would seem that the bacterial cell is normally in a state of maximum impermeability and that this can not be further increased by the presence of $CaCl_2$ and the tri-valent salts.

In distinction to the absence of effect of the tri-valent salts on bacteria as demonstrated by the conductivity method, is the marked action of these salts and especially lanthanium nitrate in changing the rate of migration of these cells in an electric field. This can be determined by the ultramicroscopic or still better the U tube method.

If 10 c.c. of a thick growth of *B. coli* in spleen broth be run into a U tube under neutral Ringer's solution of the same conductivity as the broth, then on passing an electric current through the tube, the temperature being constant, an even rapid migration of the bacteria takes place towards the anode.

That practically all bacteria carry a negative charge and migrate to the anode has been repeatedly confirmed by numerous workers, but what is of interest here is that this charge can be materially modified by various tri-valent salts, especially La. If to the 10 c.c. of *B. coli* emulsion in spleen broth run into the U tube in the above experiment 1 c.c. of a $\frac{1}{5000}$ M. lanthanium nitrate solution is added, it will be found that the rate of migration of the bacilli under the same conditions of electric field and temperature is now halved. If 2 c.c. of the solution is added, little or no migration takes place and the emulsion soon flocculates and is precipitated to the bottom of the tube.

In terms of the Helmholtz-Lamb theory of the double electric layer the addition of the La has considerably altered the nature of the charge on the bacterial cell wall. The conductivity method however fails to show any change under this condition. This result is possibly of some interest in view of Mines' theory of the polarising action of certain ions on the cell membrane. It is of course possible that the resistances obtained in the conductivity experiments were too low to bring out the real changes taking place.

The bionomics of Aphis grossulariae *Kalt., and* Aphis viburni *Schr.* By MAUD D. HAVILAND, Bathurst Student of Newnham College. (Communicated by Mr H. H. Brindley.)

[*Read* 17 February 1919.]

Aphis grossulariae Kalt. is a serious pest of currant and gooseberry bushes in this country. It attacks the young shoots in May, and when present in numbers, it distorts them to such an extent that growth ceases and a dense cluster of leaves is formed, under which the aphides swarm.

The bionomics of this aphis are incompletely known. It appears on red currants in May, and remains there until the middle or end of July. The sexuales have never been found. In 1912 Theobald (*Journ. Econ. Biol.*, vol. VII. p. 100) first pointed out its resemblance to *Aphis viburni* Schr., a common species, which is found on the guelder rose (*Viburnum opulus*) in spring and summer, while the sexual forms have been recorded from the same plant in the autumn. *Aphis viburni* has a very characteristic appearance, owing to the row of lateral tubercles on the abdomen. Such tubercles are not very common among the Aphidinae, but they are prominent likewise in *Aphis grossulariae.* In fact there seems to be no structural difference between the two species; though in spirit specimens, the guelder rose aphis frequently stains the alcohol dark brown, while the currant form has no such property.

In May 1918, I had under observation some red and black currant bushes, and two guelder rose shrubs, which all grew close together. Early in the month all were free from aphid attack, but on May 31st three colonies, each consisting of a single winged female with a few new-born young, appeared on the guelder roses, and the same evening four sprigs of currant were likewise each infected. During the following week, numerous other winged forms appeared both on the guelder roses and on the currants. The method of attack was the same in both cases. The migrant crept into the axil of a leaf, and from thence her progeny gradually spread up the stem and along the midrib. About the same time, I found a *Viburnum* tree swarming with winged females of *Aphis viburni* in a shrubbery a hundred yards away; and as these were indistinguishable from the migrants on the *Viburnum* and currants, I have little doubt that this was the source of infection.

Assuming that *A. viburni* and *A. grossulariae* are identical, I began experiments to test how far the host plants were interchangeable. Unfortunately, owing to heavy rains, the experiments with the original winged migrants were all inconclusive, and during

June and July I worked with alate and apterous individuals of later generations. The results are set out in the accompanying tables from which it will be seen that out of thirteen attempts to transfer *A. viburni* to *Ribes rubrum*, in only two cases did the resulting colonies survive more than ten days, while reproduction was very feeble and never occurred beyond the third generation. In one case (Table A, Number IX) an attempt was made to re-transfer the third generation back from the currant to the guelder rose, but the result was that the aphides all died within twenty-four hours.

Similar attempts were made to transfer *A. grossulariae* from currant to guelder rose, but the colonies never survived more than six days, and reproduction was very feeble. Meanwhile the natural colonies on guelder rose and currant flourished from the end of May to the middle of August and end of July respectively.

Aphis grossulariae has not been recorded from other food plants, but during June I observed three instances where winged migrants had established themselves on the flower heads of the Canterbury Bell (*Campanula*) and the resulting colonies persisted for two or three weeks.

The conclusions suggested by the foregoing observations are that, as Theobald points out, *A. grossulariae* is probably identical with *A. viburni*. The first migrant from the birth plant (*Viburnum*) can form colonies either on *Viburnum*, which is the natural host, or else on *Ribes*. The descendants of the migrants to *Viburnum* may with some difficulty be established on currant although the resulting colonies are not so strong as those derived from an early migrant. On the other hand the descendants of the migrants to currant cannot be re-established on *Viburnum*. It seems as if in two or three generations some change takes place in the currant form which prevents it from flourishing on the guelder rose. One explanation is that there is some change in the constitution of the guelder rose plant—an increase of tannins for instance—and that the strain on guelder rose can gradually adapt itself to altered conditions which the newly transferred currant reared stock cannot tolerate. But this explanation is not wholly satisfactory because the dates show that unsuccessful transferences took place in the second and third generations while the plants were still young, while the most successful attempt was made in July when the shoots were mature. It is also worth noticing that while the more successful attempts were made with winged parents, yet in several of the *Viburnum*-to-currant experiments, wingless females were found to feed and reproduce on the new host.

Theobald (*op. cit.* p. 100) suggests that *A. grossulariae* may be the alternating form of *A. viburni*, but says that he has twice failed to transfer the former to *Viburnum*—a result confirming my own experiments in Table B. On the other hand, it is possible that

TABLE A.

Results of transference of *Aphis viburni* from *Viburnum opulus* to *Ribes rubrum.*

Number	Date of transference	Forms transferred	Death of last survivor	Number of Generations born on new host
I	12 . VI . 18	alate and apterous	21 . VI . 18	?2
II	13 . VI . 18	alate	17 . VI . 18	1
III	17 . VI . 18	alate and apterous	22 . VI . 18	?2
IV	24 . VI . 18	apterous	29 . VI . 18	1
V	29 . VI . 18	apterous	2 . VII . 18	0
VI	13 . VI . 18	alate and apterous	26 . VI . 18	?3
VII	5 . VII . 18	apterous	6 . VII . 18	1
VIII	30 . VI . 18	—	9 . VII . 18	2
IX	9 . VII . 18	alate and apterous	25 . VII . 18	3
X	6 . VII . 18	apterous	12 . VII . 18	2
XI	5 . VII . 18	apterous	6 . VII . 18	0
XII	6 . VII . 18	—	7 . VII . 18	0
XIII	9 . VII . 18	—	13 . VII . 18	?1

TABLE B.

Table of transference of *Aphis viburni,* self-established on *Ribes rubrum,* to *Viburnum.*

Number	Date of transference	Forms transferred	Death of last survivor	Number of Generations born on new host
I	5 . VI . 18	apterous	12 . VI . 18	1
II	2 . VI . 18	alate and apterous	8 . VI . 18	1
III	8 . VI . 18	—	10 . VI . 18	—
IV	10 . VI . 18	—	14 . VI . 18	1
V	22 . VI . 18	apterous	24 . VI . 18	—
VI	30 . VI . 18	alate and apterous	1 . VII . 18	—
VII	1 . VII . 18	apterous	2 . VII . 18	—
VIII	24 . VII . 18	alate and apterous	25 . VII . 18	—

A. grossulariae is not the natural summer form of *A. viburni*, but is merely a casual parasite of the currant. In those of the Aphidinae which have a regular migration between two plants, the change is usually from a woody stemmed primary, to a herbaceous secondary, host; and if in the case of *A. viburni*, the currant should be found to be the normal second host, it would be a remarkable exception to this rule. Perhaps we have here a form that has not yet adapted itself to the conditions of modern fruit growing. In a natural state, the aphides are probably able to follow the whole life cycle on *Viburnum*, but the spread of the cultivated currant has presented them with an increasing supply of alternative food which induces a change that makes a return to *Viburnum* impossible. Whether sex-producing forms can arise from the currant stock, and thence return to the guelder rose, is not known. If not, and the early date of the disappearance from the currant is against this view, we must consider that the infestation of the currant is an unfortunate accident in the history of the species, which entails a waste of migrating individuals upon a cultivated plant that might otherwise have perpetuated themselves on the natural host. However this does not mitigate the danger of the pest from a fruit grower's point of view, and infected *Viburnum* ought not to be allowed in the neighbourhood of currant bushes.

Note on an experiment dealing with mutation in bacteria. By L. DONCASTER, Sc.D., King's College.

[*Read* 17 February 1919.]

(Abstract.)

It was noticed that the recorded ratio of occurrence in cases of meningitis of the four agglutination-types of Meningococcus corresponded very closely with the ratio of occurrence of the four iso-agglutinin groups of blood in a normal human population. It seemed possible, therefore, that by growing Meningococcus of one type in media containing human blood of different groups, mutation to other types might be induced. Experiment showed that considerable differences in type of agglutination resulted, but it was concluded that this was caused by the sorting out of races of different agglutinability from a mass culture, rather than by true mutation.

CONTENTS.

PROCEEDINGS

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY

VOL. XIX. PART VI.

[EASTER AND MICHAELMAS TERMS 1919.]

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

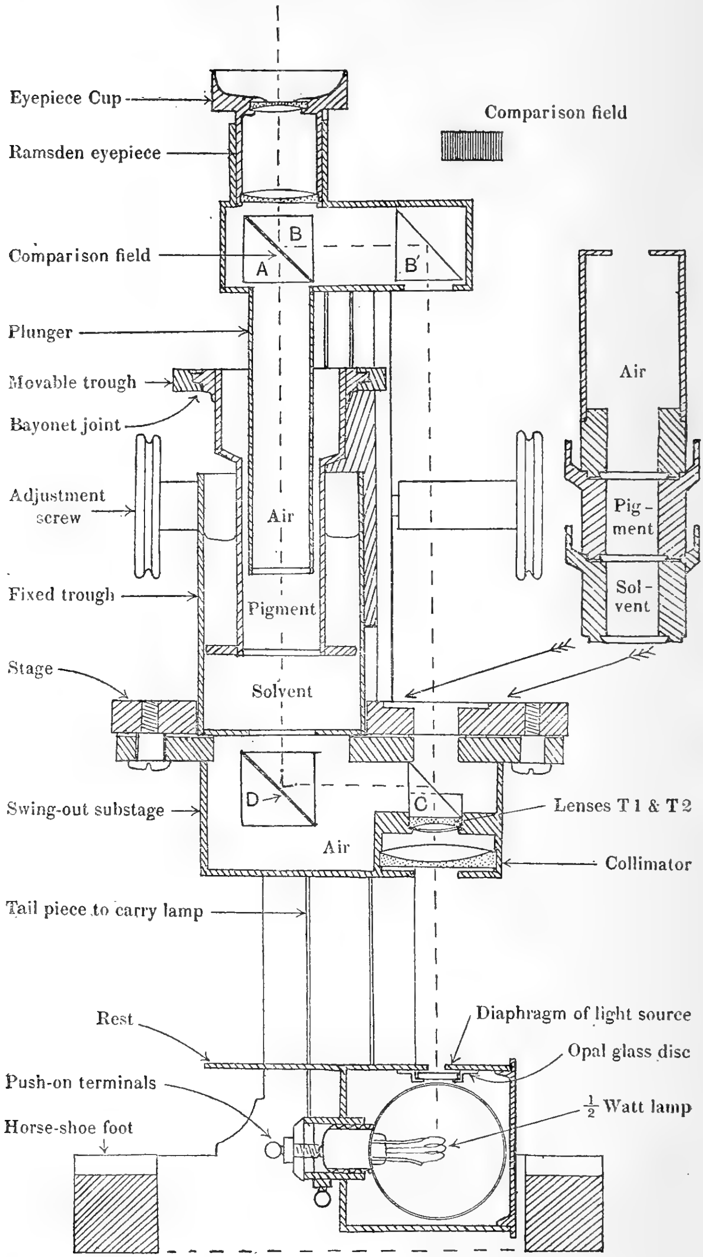
Colourimeter Design. By H. Hartridge, M.D., Fellow of King's College, Cambridge.

[*Received* 7 October 1919; *read* 10 November 1919.]

In a previous paper (1) I have described certain factors which affect the efficiency of the spectrophotometer. The colourimeter has been found to be similarly affected, so that various modifications in the usual designs are indicated.

The comparison field is in most instruments divided at a diameter, so that one half receives light which has passed through one limb, and the other half light that has passed through the other limb of the instrument. In a few designs the bull's-eye and the central strip fields have been employed. All these fields have the disadvantage that local stimulation of the retina may occur that sets up after image phenomena greater in degree in one part than in another, thus preventing accurate determinations. And, further, they do not make the best use of the effects of simultaneous contrast. A better type of field is the one which I have previously described in connection with the spectrophotometer, namely, one which is subdivided into a number of strips, of which alternate numbers receive light from the two limbs of the instrument. With this field the eye does not select any one part for examination, but tends rather to judge of the field as a whole. When the adjustment of intensity has been correctly made the whole field should become uniform. The effects of retinal fatigue therefore tend to become uniformly distributed. The contour of this type of field is of considerable length compared with its total area; the conditions are therefore beneficial for the development of contrast. The absence of visible lines of junction still further increases this effect.

The prisms A and B by which the beams of light through the two limbs of the instrument are combined at the compound field


Eyepiece Cup
Ramsden eyepiece
Comparison field
B
A
B′
Comparison field
Plunger
Movable trough
Bayonet joint
Adjustment screw
Air
Pigment
Fixed trough
Solvent
Stage
Air
Pig-ment
Sol-vent
Swing-out substage
D
C
Air
Lenses T 1 & T 2
Collimator
Tail piece to carry lamp
Rest
Diaphragm of light source
Opal glass disc
Push-on terminals
½ Watt lamp
Horse-shoe foot

described above, are similar in shape to those used in the spectrophotometer. They are shown in the diagram of the apparatus. It will be observed that the interface of the prisms is silvered, the metallic film being removed by means of a simple ruling machine, so that narrow strips of the silver alternate with strips from which the whole of the silver has been removed. Examination of the diagram will show that by this arrangement the field seen on looking down the eyepiece is formed of alternating narrow beams which have either been transmitted from one limb of the instrument through the spaces between the silver strips, or reflected from the other limb by the silver strips themselves. The lengths of the prisms A and B should be such that the two entering beams have passed through equal lengths of glass.

The troughs are adjustable on both limbs of the instrument, in colourimeters of usual design. This arrangement has the disadvantage that if there should be any backlash in the micrometer mechanism which is used for adjusting the position of the movable troughs, or error in the setting of the scale, these will affect both the thickness of the pigment solution to be estimated, and also that of the standard. Such errors can be eliminated so far as the standard is concerned by the use of a special cell, the distances between the sides of which are determined by accurately ground distance pieces, which may be made of either glass or metal. Rustless steel would appear to be a suitable metal because it resists the corrosive action of ordinary solvents.

I have shown that in the case of the spectrophotometer there are important reasons for the use of troughs with double compartments on both limbs of the instrument. In both troughs the compartment near the light source should contain the solvent only, the other being filled with the solution of the pigment. Double troughs should be used with the colourimeter for similar reasons, namely, (*a*) in order that absorption by the solvent may be compensated, since the thickness is the same on both sides of the instrument; (*b*) that pigments accompanying the one under estimation may be compensated for; (*c*) that specific surface reflection at the sides of the troughs which contain pigment may be similar on both limbs of the instrument. With regard to the type of trough that should be employed I have previously considered the advantages of the double wedge trough in conjunction with the spectrophotometer. In the case of the colourimeter the plunger type usually employed has the advantage of not requiring calibration with a micrometer microscope as wedge troughs do. The method of employing double compartment plunger troughs and standard troughs is shown in the diagram. In some colourimeters the troughs are bell mouthed, and are manufactured from black glass. These points are to be recommended. It should be noted, however, that reflection can still take place at the sides of the troughs, so that it is necessary carefully to restrict the light illuminating the troughs to narrow vertical pencils of just sufficient diameter fully to illuminate the comparison fields. Since scattered or reflected light may increase the apparent brightness of one of the fields it is essential that this be reduced to a minimum. Special care should therefore be taken in designing the instrument to prevent the entrance of stray light, and to employ an illuminating system that will limit the entering beams to the narrow pencils above referred to.

The illumination in the majority of colourimeters is obtained from the sky by means of a plane mirror. In some instruments this may be replaced at will by a finely matted white surface. The illumination therefore in either case consists of a large number of divergent pencils, which enter the lower ends of the troughs in all possible directions. Scattered light is therefore at a maximum. In the case of the microscope a similar practice used to be in vogue, but it has given way to the use of illuminating lens systems in which the corrections and alignment are well nigh as perfect as those used in the objective and eyepiece. Now, in the case of the spectrophotometer I have shown that the beams illuminating the two limbs of the instrument should proceed from identical parts of the light source. This condition should be realised in the case of the colourimeter also. The arrangement of the illuminating apparatus is shown in the diagram.

The light source is similar to that which I have applied to the microscope (2), consisting of a slab of white opal glass finely ground on both sides. This is lit from behind by means of a small half watt electric lamp, which obtains its current from a small accumulator or dry cell, or from the town supply through a suitable resistance. The lamp is enclosed in a brass box, which is silver plated inside, and is finished dead-black outside so as to radiate heat. The life of the lamp is increased by connecting it with a press switch so that it is in circuit during observation only. The lamp box is attached to the tail-piece of the instrument so that it forms an integral part of the apparatus. The whole may thus be tilted or moved from place to place without requiring readjustment. Immediately above the opal glass is a metal diaphragm, the aperture in which limits the surface exposed to a disc 4 mm. in diameter. Attached beneath the stage of the instrument and 60 mm. above the diaphragm of the light source is a plano-convex achromatic lens of 26 mm. diameter and 60 mm. focal length. The divergent rays from each point of the source are rendered parallel by this lens, and at once pass through two achromatic plano-convex lenses of 18 cms. focal length and 14 mm. diameter. These lenses have a clear aperture of 12 mm. and form a focussed image of the diaphragm of the light source, which is magnified in the ratio of the focal lengths of the lenses; since the ratio is 3 to 1 this image has a diameter of 12 mm.

The beams that emerge through the lenses $T1$ and $T2$ do not therefore anywhere exceed 12 mm. and the light does not spread, for this reason, to the sides of the troughs during its passage and therefore stray light is reduced to a minimum. The beam from the lens $T2$ passes vertically upwards through a hole in the stage to the standard trough which rests upon it. Having passed through both the layer of solvent and also that of the solution of pigment, the beam enters prism B', and is totally internally reflected at its inclined surface on to the silvered strips of the comparison field. The beam that has passed through $T1$ is deflected by internal reflection at the right angled prism C which is cemented to it, and falls on the silvered surface between the two halves of the prism D, so that the beam is directed vertically through a second hole in the stage on to the lower fixed cup of the adjustable trough, which is filled with solvent. It then passes through the movable cup which contains the pigment, and enters the prism A to fall on the silvered strips of the comparison field. The passage of this beam through the intervals between the strips, and the reflection of the beam from the other limb of the instrument at the strips themselves, has already been described. It will be noted that the reflection of the one beam by internal reflection within the prism C, and by ordinary reflection within the prism D, causes this beam to compensate for the internal reflection and reflection at a silvered surface which occurs within prism B in the case of the other beam. As it has been found that silvered surfaces vary in the intensity of rays of different wave-length which they reflect, it is advisable that both mirror D and prism B be silvered with the same solution at the same time.

The lengths of the paths of the beams through the instrument are found to be in the case of the left-hand beam an actual distance of 19·5 cms., that is an effective distance of 18 cms. since 2·2 cms. of glass is passed through; in the case of the right-hand beam the total and the equivalent lengths are the same as those on the left.

The comparison field therefore is illuminated by two superposed images of the diaphragm of the light source, one of which has passed through the standard trough and the other through the adjustable trough. When the instrument is in correct adjustment these two images exactly coincide, so that if there should be any slight inequality between the intensity of illumination of different parts of the light source both images will be similarly affected, and therefore the match between their different parts will remain unchanged. Such a condition is not secured in the usual forms of colourimeter, since it is due to the particular method of illumination described above.

The eyepiece used in the du Bosq type of colourimeter consists of a Ramsden lens system, at the upper focal plane of which has been placed a diaphragm pierced with a small aperture. This has the effect of limiting the rays reaching the eye to those which have passed as approximately parallel bundles up the limbs of the instrument. To be effective the aperture has to be small, and this has the disadvantage of making the intensity of illumination of the fields somewhat low. When this type of eyepiece is in use it is found that the eye has to be inconveniently close to the aperture in order that the whole field shall be seen at one and the same time. This is due to the fact that the diaphragm is a considerable distance below the effective pupil of the eye, even when the eye has been placed as close as possible, and as a result some of the rays which spread out from the diaphragm may not enter the pupil. The difficulty is in fact similar to that met with in high power microscopic eyepieces of the Huygenian type. To avoid this difficulty a more elaborate type of eyepiece has been devised, in which an erecting lens system has been placed above the Ramsden ocular and its diaphragm (3). This causes a sharp image to be seen on looking down the eyepiece, and at the same time the image of the small aperture is formed at a considerable distance above the top lens, so that the eye does not have to be placed inconveniently close to the eyepiece in order to obtain a full view of the field. These improvements are obtained, however, at a certain sacrifice of definition, which is unimportant in the usual types of colourimeter in which the fields are of simple design, but is of relatively greater importance if the more detailed type of field be used which has been described above. It will have been observed that in the colourimeter which I have described above the illuminating beams are formed by the special method of illumination employed. Under which circumstances it is found that the Ramsden disc of the ocular contains the overlapping focussed images of the restricting apertures of the lenses $T1$ and $T2$, which when the instrument is in correct adjustment exactly overlay one another. It is therefore unnecessary that the eyepiece should contain any diaphragm to restrict the beams, and therefore the difficulties introduced by such a diaphragm are not met with. The eyepiece itself should be achromatic and should slide in a tight-fitting jacket so that the observer may set it at the best focus. It should magnify about 3 diameters.

The angle at which the comparison field lies will be seen to be 45 degrees. But since it is enclosed between two pieces of glass, the apparent angle to the eye is reduced in the ratio of the refractive indices of glass and air. The apparent angle would therefore be about 29 degrees. Now, the dimensions of the field seen by the eye are 8 mm. by 6 mm., the latter being in the direction of the slope. The apparent difference of focus is therefore less than 4 mm., which would be equivalent to 12 mm. at a distance of 25 cms. Such a small change of focus would be at once met by a trifling change in the degree of accommodation of the eye, which would be effected subconsciously and involuntarily. No difficulty is to be met with therefore from this cause.

The Mechanical System.

The metal work of the colourimeter follows closely that of the microscope. The horse-shoe foot, stage and coarse adjustment all resemble those used in that instrument. The adjustment has a range of 40 mm. only, because, as will be shown later, the use of standard solutions of 20 mm. thickness makes a bigger movement than this unnecessary. An accuracy of one-quarter per cent. should be sufficient, and this is readily provided by a scale graduated in half mm. and reading by a vernier to one-twentieths. The adjustment should have long, well-made **V** slides so as to eliminate lost motion. The scale should be attached to the moving member, the vernier being attached to the fixed. A simple lens and 45 degree mirror should make a magnified image of this visible to the observer. To the moving member is first screwed and afterwards sweated with soft solder a strong brass ring. To this is attached by means of a three-prong bayonet catch the ring fixed to the upper lip of the movable trough. The trough is cemented into a groove turned in this ring by means of plaster of Paris or Caementium. Where plaster has been used the joint should be covered by a thin coat of Robiallac. The prisms and eyepiece are attached to a strong projection at the top of the pillar which forms the handle of the instrument.

The removal of the troughs for filling and cleaning and their replacement is a simple process which should not take more than a few seconds. To remove the adjustable troughs, first swing the substage to one side; this allows the lower trough to drop vertically through the hole in the stage until it can be removed. The upper trough is now gripped between the finger and thumb, and the trough rotated so as to free the bayonet catches; this trough is then lowered through the hole in the stage and removed. The plunger and the troughs can now be cleaned, refilled and returned. The standard double trough simply rests on its side of the stage, so that its removal takes but a moment.

The Colourimeter in Practice.

Experiment has shown that if two solutions of the same colour contain different pigments in solution, then the thicknesses required for a match vary not only with the observer and with the quality of the light, but also with the same observer from time to time. It is for this reason that the technique has been introduced of using the same pigment for the standard as that required to be estimated. Thus creatinin is no longer estimated by comparing the colour which develops when picric acid and soda are added with the colour of a solution of potassium dichromate; but a standard solution of creatinin is used, picric acid being added to it at the same time as it is added to the solution to be standardised. If, then, the thickness of the standard is 20 mm. and that of the unknown 17 mm., it is assumed that the strengths of the solutions are in the inverse ratio of those numbers. Such is not the case however, because the sodium picrate itself absorbs rays from the same part of the spectrum as does the sodium picramate, and therefore, although the light may encounter the same number of coloured radicals in both limbs of the instrument, yet the sodium picrate absorption is greater on one side than the other, because the fluids are not of the same thickness. It is principally for this reason that I have adopted an instrument in which double troughs are used, on both sides of the instrument; the lower pair on both sides being filled with sodium picrate solution in the case taken above as example, the upper pairs containing the picric acid plus creatinin. In this way the number of picrate radicals is kept approximately constant, since the total thickness of sodium picrate solution is the same on both sides of the instrument. The balance is not perfect however, because a certain amount of picric acid is used up in forming the sodium picramate, and this amount cannot be ascertained without assuming that the estimation to be done has already been accurately performed. The problem is, in fact, represented by a simultaneous equation involving two unknowns. I find that the matter can be solved in the following manner. Having diluted both the standard and the unknown solutions with equal amounts of standard picric acid and soda solutions, and having allowed the colour to develop in the ordinary manner, an estimate is made of the relative strengths of the solutions in the colourimeter. Having found that, say, a 20 mm. thickness of the standard has the same tint as 13·4 mm. of the unknown solution, a fresh sample of the unknown is taken and 13·4 c.c. of it diluted with water to bring the total to 20 c.c. The solution of the unknown has thus been brought to approximately the same concentration as the standard. (Where the approximate strength is known a preliminary dilution before making the initial estimation is beneficial.) The correctly diluted solution of the unknown is now treated, *ab initio*, with fresh picric acid solution and soda, and is then estimated against the standard in the colourimeter. It is now found that a 20 mm. thickness of the standard has the same tint as one of, say, 19·85 of the unknown after dilution. The strength of the unknown is thus ascertained, with considerable accuracy, because the conditions of equilibrium under which the sodium picramate develops and exists, and the quantities of picric acid used up in the determination are approximately constant.

It should be pointed out that the above technique presents no difficulties, and takes little longer than the ordinary method. The principle may with advantage be applied to all estimations made with the colourimeter.

The Accuracy of the Colourimeter.

Since colour is due to absorption the colourimeter depends for its utility on the fact that a change in the number of coloured radicals encountered by light causes a change in the retinal stimulus when that light falls on the eye. We may, therefore, arbitrarily state that the accuracy of the determinations depends, firstly, on the rate of change in the quality of the light which is passed through the pigment, and, secondly, on the acuteness of the perception of the eye for the change in quality of the light. The greater the rate of change and the greater the acuteness of perception of that change, the greater will be the accuracy. Many bodies which absorb light do so selectively, that is, they have a greater effect in one part of the spectrum than in another; they therefore show colour, that is, they are pigments. Under ordinary circumstances the greater the absorption the stronger the colour and the less the intensity of the transmitted light. As the concentration of a pigment is altered, and therefore the degree of absorption, the strength of colour and the brightness of the transmitted light both vary. The colourimetric determination, therefore, depends on the simultaneous occurrence of both these changes. The important questions that arise are: (1) on what do the magnitudes of these changes depend? (2) which is the more important? and (3) how can the changes be increased for a given alteration in concentration? A study of absorption band formation gives a definite answer to each of these questions as follows: (1) The changes for a given alteration of concentration are greater the flatter and broader the absorption band. If, therefore, there were two pigments of the same concentration and the same colour, one of which had a sharp well-defined band, while that of the other was broad and flat, the latter pigment would be found to give the more accurate readings in the colourimeter. (2) Of the two changes, that of colour is usually the more important, particularly with pigments showing single absorption bands. In pigments with multiple bands the intensity change may be the more important: for example, a pigment absorbing to an equal extent in two complementary parts of the spectrum will cause the light to suffer no change in colour at all, while the intensity is altered. (3) The changes in the case of any one pigment can be increased by increasing the intensity of that part of the spectrum which is suffering change or by decreasing that of parts which do not show alteration. Of the two methods the latter is the easier to carry out and the more efficient. If colour filters are used they must be carefully adjusted according to the position in the spectrum of the absorption band of the pigment to be estimated. If a spectral illuminator is used the apparatus virtually becomes a spectrophotometer, and this elaboration is hardly necessary for ordinary work. The possibility should not be overlooked of the existence of alternative colour reactions to those at present in use in which pigments having less steep absorption bands are used and which therefore permit greater accuracy in their colourimetric estimation.

The factors which influence the acuteness of perception of the eye remain for consideration. Firstly, it is clear since the accuracy of the determination depends on the correctness of the match obtained, that the eye should not be suffering from fatigue. The reading of small print and the exposure of the eyes to excessive light should, therefore, be avoided for a reasonable time before the determinations. The absence of refractional errors, eye strain, want of eye-muscle balance and the possession of good general health are all factors of importance. In my own case the period after tea is the best, provided that the morning's work has not been arduous. The presence of after images is most harmful for accurate estimations; the best method of eliminating them is, I find, to look for a few moments at a uniformly lit grey surface. All the above points may seem obvious; it is however my experience to find that they are sometimes overlooked. The apparatus itself is best placed in a dark room, or at all events where the full light of a window cannot fall on the eye of the observer. In the latter case the eyepiece cup may be made deep with advantage, so as to protect the periphery of the retina from stimulation and thus bring about an increase in the diameter of the pupil.

With regard to the use of colour filters, experiment shows that the theoretical conclusions arrived at above are amply justified, namely, that the accuracy of the determinations is increased if either the rays absorbed by the pigment are increased in intensity, or those not absorbed are decreased or removed altogether. The removal by means of colour filters is however usually attended by so great a diminution in the intensity of the light that a powerful source such as an arc lamp becomes necessary. It is a fortunate circumstance, therefore, that the retina should be even more sensitive to change in shade than it is to change in intensity. I have found, further, that the point of greatest sensitiveness is obtained when the fields are nearly neutral in colour. Such a condition is obtained by the use of a suitable colour filter which absorbs in that part of the spectrum which is occupied by the complementary colour to that absorbed by the pigment. Suppose, for example, a yellow pigment is to be estimated, then a blue solution of a dye is placed in the path of the light from the source of such a thickness and concentration that the comparison field seen in the instrument is of a neutral grey colour. Permanent colour films between glass should be used if much work is likely to be done with any given pigment. Such a technique is very simple, and I find that in my hands it increases the accuracy of the determinations by about three times (when estimating sodium picrate), the method of mean squares being used to calculate the average error of the experimental determinations both with and without the complementary filter. The probable error of the determinations was found to be 0·8 per cent., using home-made apparatus and the complementary screen. It should be possible to halve this amount if the precautions outlined above be taken and well-designed apparatus be used.

Summary.

(1) The comparison field seen on looking down the instrument should cause the greatest contrast and at the same time should not produce after images.

(2) On both limbs of the instrument double troughs should be used, so that the thickness of pigment to be measured may be varied at will, while the absorption caused by other pigments remains constant.

(3) An artificial light source should be used, and the lighting system be so designed that narrow beams are produced of just sufficient width as to completely illuminate the comparison field. The amount of reflected and scattered light may thus be reduced to a minimum.

(4) If experiment shows that the change in colour produced by a given change in thickness or concentration of the pigment can be increased by modifying the relative intensity of different parts of the spectrum of the light source, then suitable colour filters should be prepared for use during the determinations. It was found in a test case that this modification alone increased the accuracy by three times.

(5) The general design of the instrument should conform to microscopic practice, fixed troughs being supported by the stage and the movable trough actuated by the rack and pinion course

adjustment screw. The illuminating system should be fitted beneath the stage so that the instrument may be tilted or moved from place to place without disturbing the alignment.

For certain purposes it may be found beneficial to employ smaller quantities of liquid than those required in the ordinary colourimeter. I find that a modification in the design of the troughs should make 1 to 2 c.c. of liquid sufficient; and further, by modifying the optical system as well, as little as ·001 c.c. could be worked with. It should be pointed out however that such quantities could only be employed with solutions of considerably greater concentration than those usually estimated; e.g. about ten times the usual concentration for 1 c.c., and one hundred times for ·001 c.c.

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(1) Hartridge, *Journ. Physiol.* L, p. 101 (1915).
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(3) Kober, *Journ. Biol. Chem.* XXIX, p. 155 (1917).

The Natural History of the Island of Rodrigues. By H. J. SNELL (Eastern Telegraph Company) and W. H. T. TAMS[1]. (Communicated by Professor STANLEY GARDINER.)

[*Read* 10 November 1919.]

Rodrigues lies some 350 miles east of Mauritius, and is a rugged mass of volcanic rock closely resembling Mauritius and Réunion. It is surrounded by a coral reef, the edge of which at the eastern end is within 100 yards of the beach, whilst on the north and south it extends outwards to a distance of three to four miles, and on the west to two miles. There is an irregular channel inside the reef close to the shore, extending round most of the island, sufficiently deep for boats at any state of the tide, and at the south-east end a small lagoon of three to ten fathoms, with a passage through the reef. The usual anchorage is Mathurin Bay, in the reef to the north. The reef is studied with islets, those nearer the shore being mostly of volcanic nature, and situated on the north and west, whilst the rest are of limestone, modern accumulations of débris, and situated on the south.

The island itself is eleven miles long by five miles broad, and has an area of just over forty square miles. There is a central lofty ridge extending from east to west, with a break about one-third of its length from the west. The western bastion of the range is Mount Quatre-Vents, 1120 feet high, while at the eastern end is Grande Montaigne, 1140 feet. The highest point is Mount Limon (1300 feet), which lies with two other peaks a little out of the general line of mountains. The sides of these peaks are cut into numerous ravines, these being deeper and more frequent on the south side than on the north. At their upper ends these ravines are often bordered by perpendicular columnar basaltic cliffs, sometimes exceeding 200 feet in height, extensively cut into many coulées by small streams which often descend in a series of cascades.

The volcanic ridge descends on the south-west gradually, and passes into a broad coralline limestone plain, with occasional hills up to 500 feet high, indicating a comparatively recent elevation of at least a like amount. This tract of limestone is honeycombed with caves, in which stalactites and stalagmites are abundant. There are many holes and fissures, and often deep hollows occur, at the bottom of which lie large fragments of limestone in irregular heaps; these are apparently old caves, the roofs of which have fallen in. The floors of these hollows are covered with soil, often

[1] The second author is solely responsible for the names of the insects herein recorded.

with lumps of volcanic rock on the surface. The limestone is not found along the northern or southern shores, except at their eastern extremity, where patches occur at the mouths of the valleys, occasionally at some distance from the shore. Some of the patches of limestone found in the volcanic region indicate an elevation of perhaps 500 feet, and the raised beaches on the south shore, some 20 feet in height, may point to a further subsequent change of level. The position of old volcanic craters has not been accurately determined, but the main ones appear to have been situated about the Grande Montaigne and Mount Malartic.

The island is comparatively dry, and during the warm season many of the streams are dried up, though they assume in the rainy season torrential proportions. The climate is like that of Mauritius. The rainfall is very irregular; during the north-west monsoon from November to April the weather is wet and warm, and early in this season there are frequently severe hurricanes. From May to October the south-east monsoon prevails, and the weather is then cool and dry. Fogs are rare, and climatic conditions render the island healthy to live in.

Rodrigues was discovered in 1510, by a Portuguese commander, whose name it bears. In 1691 the Dutch landed several fugitive French Huguenots there, among whom was M. François Leguat, who wrote an account of the island in 1708. The island was later cultivated by the French East Indian Company, and maize and corn were grown; these, with dried fish, turtles and land tortoises, were exported to Mauritius. It was occupied by the British in 1809, and made the base of operations against Mauritius. It is still cultivated as a garden for Mauritius, its main exports being beans, acacia seed, maize, salt fish, cattle, goats and pigs. The population is about 5000, mostly settled around Port Mathurin, the only town in the island. The people are mainly French Creoles, with a few Chinese and Indians, and are subject to the Government of Mauritius, which supplies a Resident Magistrate. The island is a station of the Eastern Telegraph Company, connecting to Cocos-Keeling.

Each family usually cultivates an acre or several acres of land, whereon they grow maize, sweet potatoes, haricot beans, pumpkins, various herbs, onions, etc. They depend, in fact, largely on their own plantations for food. At one time a species of mountain-rice, which does not require an abundance of moisture, was grown in large quantities, but its cultivation was abandoned owing to the depredations of small birds. Tobacco grows well. Haricot beans are still exported. There have lately been, however, only five ships per year, and these small sailing ships of 500 tons down to 100 tons register; this makes it very difficult to market the produce of the island. The maize grown is barely enough for local consumption.

One of the most profitable products of this island is acacia seed, which is exported to Mauritius for cattle feeding. The acacia (*Lucaena glauca*), which was introduced about seventy years ago, now grows wild and flourishes everywhere, covering the ground for acres, and forming a dense almost impenetrable scrub, beneath which nothing will grow. The cattle and goats are exceedingly fond of the leaves and pods, and this is probably the reason for its spreading so extensively, the original plantation having been in a valley near Port Mathurin. Amongst other things which have been successfully grown may be mentioned coffee, vanilla, sugar-cane, oranges and lemons. Bananas and plantains, custard apples, strawberries and raspberries are found wild. Many other commodities such as ginger, safran (turmeric) and arrowroot have also been grown.

There is very little real pasturage in Rodrigues, the largest area being in Malgache Valley. Besides this there are barren tracts round the coast covered with coarse grass, which provides insufficient subsistence for the stock. Most of the inhabitants own goats and pigs, on which they rely for their milk and meat supply, and which are also exported. They were allowed to run wild, but measures have now been introduced by the Government to control them. Poultry, ducks and geese also thrive in the island.

Rodrigues was originally covered with dense forests of lofty trees, with corresponding undergrowth. Indeed, according to early descriptions its vegetation partook of the nature of a regular tropical moist woodland. Here were to be found flightless birds, the Solitaires, and giant land tortoises. When Leguat saw this island first, the scenery was such as to call forth from him such designations as "a lovely isle," "an earthly paradise." To-day its grandeur and beauty have vanished. There remains a bare parched pile, on which it is difficult if not impossible to discover any corner in its original condition. Many agencies are responsible for this destruction and denudation. It has been swept by fire many times, accidentally and intentionally. The goats devour the young shoots and leaves of any vegetation within their reach. Pigs have done their share, especially with regard to the Latanier Palm (*Pandanus*), of the nuts of which they are very fond. Then there are the introduced plants, which have in many cases crowded out the native vegetation. A notable example is seen in the acacia, previously mentioned, which has spread into almost every valley in the island. A certain amount of destruction has been done by the inhabitants, who have cut timber over large tracts without discrimination. Though a check has been placed on this by the government, there still remains a source of destruction, in that the inhabitants are in the habit of acquiring year by year fresh tracts of woodland, the undergrowth of which they cut down and burn, and here they plant their haricot beans. They utilise a tract of land for one season, and abandon it the next. Thus the work of destruction continues. Many of the older inhabitants, at present living on the island, say that they remember large tracts, which are now almost bare except for a few Vacoas (Screw-pines), being originally covered with almost impenetrable forest, but nobody remembers the large expanse of coralline limestone at the south-western end of the island in any other than its present state, though there are unmistakeable traces, in roots and stumps embedded in the ground and charred by fire, showing that this region was also at one time completely afforested. The large rifts are often thirty feet or more deep, and fifteen to twenty yards wide, and contain many fine old indigenous trees which have escaped destruction. The Valley of St François, at the north-east end of the island, is perhaps the only other tract which has escaped destruction.

The commonest trees in the island are the Vacoas or Screw-pines (*Pandanus*), of which there are two species, both endemic. Three other species have been recorded by various authorities, one being a native of Asia, and the other two Madagascar species. None of them occurs in Mauritius or Réunion, and the evidence of their occurrence in Rodrigues is faulty. There are three species of endemic palms, belonging to three genera, which are all Mascarene. Probably half the plants have been destroyed, but from what is left—297 species of Phanerogams, and 175 species of Cryptogams (excluding Marine Algae)—it is clear that the endemic flora was large and of Mascarene affinities. There are only about twenty species of ferns, the scarcity of this group being accounted for by the present dryness of the island, in confirmation of which it may be remarked that the tree-ferns of the other Mascarene islands are not represented.

The present day fauna is not large. The extinct fauna has proved to be of very great interest, particularly in the case of the Solitaire (*Pezophaps solitaria*, Gmel.), the extinct Didine bird related to the Dodo of Mauritius. Considerable collections of the remains of this bird have been made from the limestone caves, where also the remains of other extinct birds and of the giant Land Tortoise have been found. Our main knowledge of the recent fauna is due to the labours of the naturalists attached to the Transit of Venus Expeditions carried out in 1874–5.

The marine fauna is in general of the Indo-Pacific type.

The only indigenous mammal found in the island is a fruit-bat, *Pteropus rodericensis*, Dobson, which is peculiar to Rodrigues. The introduced mammals, other than those already mentioned, are deer, rabbits, rats, mice and cats, the latter being left by the Dutch to destroy the rats.

Sir Edward Newton, K.C.M.G., published a list of Rodrigues birds in his "List of the Birds of the Mascarene Islands" (*Trans. Norfolk and Norwich Naturalists' Society*, vol. IV, President's Address).

The Fresh Water Fishes, as far as known, belong to species which inhabit the fresh waters of the Mascarene Islands generally, with the exception of two Grey Mullets, which were collected by the Transit of Venus Expedition, and were described as new.

Further collections in certain groups have recently been made by Mr H. P. Thomasset and Mr H. J. Snell, who visited the island during the period August to November, 1918, with a view to improving our knowledge of the insect fauna.

Mr Snell visited practically every part of the island, with the exception of the valley of St François, and a small district round the Rivière Coco. The best collecting ground he found to be undoubtedly the Grande Rivière Valley, which he worked right up to Mount Limon. The islands on the reef were also visited, but contained very little of interest, as they have been burnt over in recent years, and are now covered with rough coarse grass and short scrub (*Tournefortia*, *Pemphis*, etc.). These islands, particularly Gombranil and Flat, were formerly nesting places for sea-birds, which seem to have disappeared, only a few white terns and boobies being found on Sandy and Coco Islands, which were some years ago planted with firs.

In the deepest ravines were commonly seen the fruit-bats or flying-foxes, feeding on the flower of a kind of aloe, of which they seem very fond, and also on wild figs, mangoes, etc. Geckos were abundant in warm and sheltered spots, particularly in all habitations. Their eggs were frequently found in nests (usually composed of dry Sow-thistle bloom) under rocks and in crevices. Two species only have been recorded: *Gehyra mutilata*, Gray, and *Phelsuma cepedianum*, Gray; the latter is common in Madagascar, Mauritius and Réunion, but is rare in Rodrigues. Freshwater fishes were found in many of the streams, in which also eels were quite common.

There are in the island a Land Planarian, *Geoplana whartoni*, Gull., and a Land Nemertean, *Tetrastemma rodericanum*, Gull. Both are peculiar to Rodrigues, but the former has not been adequately described. (Mr Thomasset subsequently obtained a Land Planarian from Mauritius, a new locality for these.) They were found under decaying logs, sometimes on the bark, under the bark, or in the wood; the Nemertean appeared to exist in far greater quantities than the Land Planarians, but they often live together in the same situation. Earthworms were not abundant. Amongst the Crustacea collected, large numbers of an Amphipod were found under stones, dead leaves, etc., wherever the ground was moist. In all the streams were to be found freshwater shrimps and a crayfish. Woodlice were abundant in decaying vegetable matter, the largest specimens being obtained from rotting banana stems.

Myriapoda were common throughout the island. Large centipedes live on the corals on the west side of the island, attaining sometimes a length of twelve inches. Hardly a lump of débris can be turned over without disclosing one or more of these creatures. The Transit of Venus Expedition obtained twelve species of Myriapods, of which eleven were new. There is a single species of scorpion, *Tityus marmoreus*, Koch, and in addition the Transit of Venus Expedition obtained twenty-seven species of Arachnida, eleven being new; unfortunately Mr Snell could not obtain a supply of alcohol adequate to preserve these.

In the Insect collections among the Orthoptera, the Forficulidae are represented by eleven specimens, probably *Anisolabis varicornis*, Smith. Of the Blattidae, *Periplaneta americana*, Linn. and *Leucophaea surinamensis*, Fab. are among the five species previously recorded, whilst there are two other species in Mr Snell's collection at present undetermined. One species of Mantidae occurs in the island, viz. *Polyspilota aeruginosa*, Goeze, of wide distribution. Of the Gryllidae there are three species in the present collection: *Acheta bimaculata*, de Geer, found also in Africa and S. Europe; *Curtilla africana*, Beauv., found also in Africa, Asia, Australia, and New Zealand (introd.?); and a species of *Ornebius* near *syrticus*, Bolivar, but larger and more brightly coloured than the Seychelles specimens of this species. Besides the first of these, the Transit of Venus Expedition obtained three other species. Among the Phasgonuridae we have *Conocephaloides differens*, Serv. and *Anisoptera iris*, Serv., both previously recorded by the Transit of Venus Expedition. In addition the present collection contains a specimen of apparently another species of *Anisoptera*, resembling *A. conocephala*, Linn., which occurs in Spain, Africa, and the Seychelles. There are two species of Locustidae: *Locusta danica*, Linn., a cosmopolitan species, and *Chortoicetes rodericensis*, Butl., described from Rodrigues, and not found elsewhere.

The Neuroptera comprise a few specimens of a Termite, and specimens of one species of Hemerobiidae and of one species of Chrysopidae. It may here be mentioned that Dr H. Scott found a species of Termite working in the wood at the bottom of a lighter in Victoria harbour, Mahé, Seychelles. This indicates a possible explanation of the existence of Termites in such a locality as Rodrigues, where any indigenous Termites would probably be exterminated by the fires which have repeatedly devastated the island. Until the Termites in Mr Snell's collection have been identified, no statement of course can be ventured regarding the distribution of this species. Mr Gulliver, on the Transit of Venus

Expedition, secured one specimen of *Myrmeleon obscurus*, Rambur. This species was described from Mauritius, and is widely distributed in Africa.

The Odonata consist of six species, as follows:

Pantala flavescens, Fab., occurs in all the warmer parts of the world, but not in Europe.

Tramea limbata, Desj., a very variable species of wide distribution, described from Mauritius.

Orthetrum brachiale, P. de Beauv. Found elsewhere in Zanzibar, Congo, etc.

Anax imperator mauricianus, Rambur. Agrees with a specimen in the Museum of Zoology, Cambridge, named by Campion. The species was also taken by Gulliver, on the Transit of Venus Expedition.

Ischnura senegalensis, Rambur. Widely distributed in tropical Asia and Africa.

Agrion ferrugineum, Rambur. One specimen was taken by Gulliver. The present collection contains several specimens.

The collection of Hymenoptera, exclusive of Ants, contains two species of Tubulifera, eleven species of Aculeata, and approximately 170 specimens (of about twenty species) of Parasitica. The two species of Tubulifera, for the identification of which I am indebted to Mr F. D. Morice of the British Museum of Natural History, are *Chrysis* (*Pentachrysis*) *lusca*, Fab., found also in India, Ceylon and Mauritius, and *Philoctetes coriaceus*, Dahlb., known also from East and South Africa. Of the Aculeata, the Formicidae are not yet determined, and a species of *Halictus* is at present unidentified. The remainder of the Aculeates are as follows:

Megachile disjuncta, Fab. Common in India; recorded also from Mauritius. (*M. lanata*, Fab., is recorded by Smith as having been taken by Gulliver on the Transit of Venus Expedition.)

Megachile rufiventris, Guér. Found elsewhere in East and South Africa, Mauritius and Seychelles; previously taken in Rodrigues by Gulliver.

Apis unicolor, Latr. Previously taken in Rodrigues by Gulliver. Found in the Seychelles, Amirantes, Chagos (Diego Garcia, Peros Banhos). Commoner in Madagascar.

Odynerus trilobus, Fab. This species has not been previously recorded from Rodrigues. It is common and widely distributed, being known from Madagascar, Mauritius, Réunion and South Africa.

Polistes macaensis, Fab. Previously taken by Gulliver and listed as *P. hebraeus*, Linn. There seems to have been considerable confusion over these names, as Cameron (*Trans. Linn. Soc.* (2), vol. XII, p. 71) lists this species as *P. hebraeus*, Fab., stating that it is known from Rodrigues. Dr R. C. L. Perkins has, however, demonstrated the differences between the male *P. macaensis* and male *P. hebraeus*. (See *Ent. Mo. Mag.* (2), vol. XII, 1901, p. 264.) *P. macaensis* is known also from Seychelles, Amirantes, Chagos (Salomon Islands, Diego Garcia), and Mauritius.

Scolia (*Dielis*) *grandidieri*, Sauss. I am indebted to Mr Rowland E. Turner of the British Museum of Natural History for the identification of this species. He states that the specimens under review are of "a form of *D. grandidieri*, Sauss. from Madagascar, with a few more punctures on the abdomen than in that species."

Ampulex compressa, Fab., not previously recorded from Rodrigues. Common from Eastern Europe to China, and also in Africa.

Passaloecus (*Polemistus*) *macilentus*, Sauss. Mr R. E. Turner has kindly identified this species for me. He states (in litt.) that "Mr Morice considers that *Philoctetes coriaceus*, Dahlb. is probably parasitic on this, as species of *Passaloecus* are often attacked by small Chrysids." The species was described from Madagascar.

Sceliphron bengalense, Dahlb. (= *Peolpaeus convexus*, Sm.). Mr Turner has confirmed my identification of this species. He adds: "This is probably an imported species, as species of the genus build mud nests on ships and are carried in that way from place to place."

Trypoxylon errans, Sauss. Not previously recorded from Rodrigues. Found also in Mauritius and the Seychelles.

There are approximately 750 specimens of Coleoptera, of possibly 100 species; 640 specimens of Diptera, of at least seventy species; and 360 specimens of Hemiptera, of some forty-five species. These have not yet been critically examined.

In the Lepidoptera, seven species of Butterflies were collected by Mr Gulliver on the Transit of Venus Expedition. Of these one species is not represented in Mr Snell's collection, viz. *Hesperia forestan*, Cr. The list of Butterflies is as follows:

**Melanitis leda*, Linn.
**Danais chrysippus*, Linn.
Precis rhadama, Boisd.
**Hypolimnas misippus*, Linn.
†**Atella phalantha*, Drury
†**Zizera lysimon*, Hübn.
†**Polyommatus boeticus*, Linn.
**Tarucus telicanus*, Lang.
Parnara borbonica, Boisd.

Among the Moths (Heterocera), exclusive of the Pyralidae, Tortricidae, and Tineidae, though Gulliver's collection contained only twelve species, five of these were species not represented in Mr Snell's collection. Mr Snell obtained three species of Sphingidae,

* Of wide distribution.
† Not previously recorded from Rodrigues.

one species of Arctiidae, twenty-five species of Noctuidae, and two species of Geometridae, as follows:

**Acherontia atropos*, Linn.
†**Herse convolvuli*, Linn.
† *Hippotion aurora*, Roth. & Jord.
†**Utetheisa pulchelloides*, Hamps.
†**Chloridea obsoleta*, Fab.
†**Agrotis ypsilon*, Linn.
†**Cirphis loreyi*, Dup.
† *Cirphis leucosticha*, Hamps. (= *insulicola*, Saalm.)
†**Perigea capensis*, Guén.
†**Eriopus maillardi*, Guén.
**Prodenia litura*, Fab.
**Spodoptera abyssinia*, Guén.
Athetis expolita, Butl.
†*Eublemma apicimacula*, Mab.
**Amyna octo*, Guén.
†**Erias insulana*, Boisd.
**Anua tirhaca*, Cr.
Achaea trapezoides, Guén.
Achaea finita, Guén.
**Parallelia algira*, Linn.
**Chalciope hyppasia*, Cr.
†**Mocis undata*, Fab.
**Phytometra chalcytes*, Esp.
**Cosmophila erosa*, Hübn.
†**Dragana pansalis*, Walk.
†**Magulaba imparata*, Walk.
†**Hydrillodes lentalis*, Guén.
†**Hypena masurialis*, Guén.
†**Hyblaea puera*, Cr.
†**Craspedia minorata*, Boisd.
†**Thalassodes quadraria*, Guén.

The five species collected by Mr Gulliver and not represented in the present collection are as follows:

**Argina cribraria*, Clerck. (Hypsidae).
**Nodaria externalis*, Walk. (redescribed as *Diomea bryophiloides*, Butl.) (Noctuidae).
Pericyma turbida, Butl. (Noctuidae). Peculiar to Rodrigues.
**Achaea catella*, Guén. (Noctuidae).
**Mocis repanda*, Fabr. (Noctuidae).

Butler listed a species as *Laphygma cycloides*, Guén., apparently in error, as Sir George Hampson has in his Catalogue placed the record under *Spodoptera abyssinia*, Guén.

There are about 180 specimens, of some thirty species, of Micro-lepidoptera. These have not yet been worked out.

The collections made by Mr Snell are of importance as showing more definitely the relations of Rodrigues with the other islands in the vicinity. Undoubtedly the fauna has, with the flora, suffered considerably from the devastating effects of the fires which have so frequently swept the island, but investigation of the collections of the groups not yet worked out, will undoubtedly show that considerable traces of the indigenous fauna still exist, and will serve to indicate with greater accuracy the affinities of Rodrigues with the neighbouring islands.

* Of wide distribution.
† Not previously recorded from Rodrigues.

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Preliminary Note on the Life History of Lygocerus (*Proctotrypidae*), *hyperparasite of* Aphidius. By MAUD D. HAVILAND, Fellow of Newnham College. (Communicated by Mr H. H. BRINDLEY.)

[*Read* 10 November 1919.]

Plant lice are frequently parasitized by certain Braconidae of the family Aphidiidae. The parasite oviposits in the haemocoele of the aphis, and the larva, during development, consumes the viscera of the host. At metamorphosis nothing remains but the dry skin, within which the *Aphidius* spins a cocoon for pupation.

At this stage, the *Aphidius* itself is liable to be parasitized in turn by certain Cynipidae, Chalcidae, and Proctotrypidae. The two former are known to be hyperparasites, but the Proctotrypidae have hitherto been considered doubtful, although some writers have suspected that they are hyperparasites of the *Aphidius*, and not parasites of the aphis. Gatenby in his paper: "Notes on the Bionomics, Embryology, and Anatomy of certain Hymenoptera Parasitica" (*Journ. Linn. Soc.* 1919, vol. XXX, pp. 387–416) says: "...I am inclined to support the view that the Proctotrypid is a parasite, and not a hyperparasite."

The following is a summary of some observations made in the summer of 1919, on two Proctotrypids of the genus *Lygocerus*. I am much indebted to Professor Kieffer, who has kindly identified them for me as *L. testaceimanus*, Kieff., hyperparasite of *Aphidius salicis*, Hal., parasite of *Aphis saliceti*, Kalt., from the willow; and *L. cameroni*, Kieff., hyperparasite of *Aphidius ervi*, Hal., parasite of *Macrosiphum urticae* from the nettle. The following notes probably apply to both species, but the observations were made more especially upon the latter. It was found also that in captivity *L. testaceimanus* would oviposit on *Aphidius ervi*. The Proctotrypids do not confine their attacks to the Aphidiidae, but their larvae may also be found feeding on the larvae of other Chalcid or Cynipid hyperparasites of that family; and indeed once or twice were observed upon dead pupae of their own species. One remarkable instance of hyperparasitism came under notice. An aphis (*Macrosiphum urticae*) was parasitized by an *Aphidius* (*A. ervi*). The latter had been hyperparasitized by a Chalcid, of species unknown, which immediately after pupation had been attacked by another hyperparasite, either Chalcid or Cynipid, whose identity is not yet determined. This second hyperparasite in turn had been attacked by *Lygocerus cameroni*, and the larva was in the second instar when the cocoon was opened. We may ask, where are the limits to this hyperparasitism?

Lygocerus cameroni was fairly common round Cambridge in 1919, from mid-July to the end of August. The female selects an aphis-cocoon containing a full-grown larva or newly transformed pupa of *Aphidius*, and runs round it with much excitement, tapping it with her antennae. Oviposition takes from 30–60 seconds, the insect meanwhile standing either on the top of the cocoon facing the anterior end, or on the leaf behind, with her back to it. Either way, the ovipositor is brought into the angle of the host's body, as it lies curled inside. Sometimes two or three eggs, the result of successive ovipositions by different females, are found on the same host.

The egg, which is laid on the upper surface of the abdomen of the *Aphidius*, measures $\cdot 25 \times \cdot 10$ mm. It is translucent, white, and elliptical, with marked longitudinal striae of the chorion, and a minute stalk at one end. Treatment of the egg with lacto-phenol and cotton-blue showed the presence of bodies resembling the symbiotes from the pseudovitellus of Aphides. The egg hatches in about twenty hours.

The larva of the first instar is a maggot shaped form, with thirteen body segments and a head furnished with two minute papillae. The mouth, which is circular and very small, contains two simple chitinous mandibles set well behind the hood-like labrum and the labium. The mid-gut, which at this stage does not communicate with the rectum, is large and globose, and its contents tinge the transparent body pale yellow. Later on, when the host dies, they become brown. The tracheal system consists of two lateral longitudinal trunks, united by an anterior and posterior commissure. When newly hatched, there are two open spiracles between the first and second and on the fourth segments, but soon afterwards the spiracles of the third and fifth segments become functional. The larva is active and crawls over the host's body. This instar lasts from twenty to twenty-four hours, and the dimensions are about $\cdot 45 \times \cdot 22$ mm.

The larva of the second instar differs from that of the first chiefly in the size, which is $\cdot 70 \times \cdot 35$ mm., and in the tracheal system. The ramifications of the latter are more numerous, the dorso-ventral branches of the second segment become visible, and the spiracular trunks of segments six, seven, and eight appear, though their spiracles are not open. The duration of this instar is about thirty-six hours, and at this time the host usually dies, and its body becomes blackened and shrunken.

In the third instar, the papillae on the head disappear, the body becomes more globose, and the greater proportionate development of the three first segments causes the head to be bent round to the ventral side. The dimensions are about $1\cdot 00 \times \cdot 75$ mm. The spiracles of the sixth, seventh and eighth segments open, and the spiracular trunk of the second segment becomes visible. In addition, two short spiracular trunks can be made out on the ninth and tenth segments; but these never become functional, and they disappear in the later stages of development. This instar lasts from about thirty-five to forty hours.

In the fourth instar, which lasts about two days, the Proctotrypid grows rapidly, and when mature measures 1·67 × ·83 mm. The remainder of the host is quickly consumed, and, just before metamorphosis, the mid-gut opens into the rectum, and its contents are voided into the cocoon. The larva is active and wriggles about freely inside the aphis skin, aided possibly by a curious caudal appendage; and by these movements the faeces, together with the host's skin, are kneaded into a moist compact pellet on the ventral side of the body.

The full grown larva is yellowish white, and each segment has a double row of short chitinous spines. The thorax is large and broad, while the abdominal segments taper away somewhat to the eleventh, which bears a short stout appendage furnished with spines. The head is turned completely under the thorax, and the tracheal system does not differ essentially from that of the preceding instar. No larval antennae nor maxillary nor labial palpi seem to exist at this stage.

Lygocerus does not produce silk, but pupates in the cocoon made previously by the *Aphidius* inside the skin of the aphis. The period of pupation is fourteen to sixteen days. When ready to emerge, the imago gnaws a hole somewhere on the upper side of the cocoon, and creeps out. So far, no parthenogenetic ovipositions have been observed, and two broods, certainly, and possibly more, may occur in the season. The life of the imagoes is generally five or six days, but they may live as many as ten. Examples in captivity were observed to feed on the sap oozing from cut leaves, and on honeydew dropped by the aphides, but they seemed to live as long and to remain as vigorous when no food was supplied.

Note on the solitary wasp, Crabro cephalotes. By CECIL WARBURTON, M.A., Christ's College.

[*Read* 10 November 1919.]

Last summer a small colony of *C. cephalotes* took possession in my garden of a log of elmwood which was kept as an example of a woodpecker's nest. The entrance hole of the woodpecker was there, and just below it the log had been sawn through so that the internal cavity could be examined.

The first advent of the wasps was not noticed, but in the first week of August a wasp was observed entering the hole, and this led to an investigation of the log, which presented signs of boring in the half-decayed heart-wood. One of the wasps had attacked the log from the top and its operations could be noted with more or less exactness, but the others passed in and out by the woodpecker's hole, and it was impossible to recognise individuals or to follow their work without constantly disturbing it by opening up the log, with the risk of inaccurately replacing the two halves. The log was nevertheless opened several times during the first half of August, but it was then thought better to let the wasps finish their work without further disturbance.

That the wasps are not easily diverted from their labours the following facts sufficiently demonstrate. The log was moved several yards, to a spot more convenient for observation. The wasp working on the top (hereafter referred to as wasp No. 1) was captured in a glass tube and examined for identification, but on being liberated continued working as before. Close observation, with a hand lens, did not deter this wasp from entering its burrow without hesitation in the course of its operations, nor were the other wasps disconcerted by the removal of the lid on several occasions at an early stage of their work. As a rule no attention was paid to anyone sitting silently near the log, but it must be recorded that on one occasion a wasp returning with a fly apparently objected to the dress—light with dark spots—of a lady sitting near at hand, and after a close investigation from many points of view, retired instead of entering the log. To ascertain if wasp No. 1 were at home or not I was in the habit of placing a stout straw in its burrow—protruding an inch or more. One would have thought that on returning home and finding such an object impeding its entrance the insect would manifest some perturbation and either refuse to enter or take some measures to remove the obstacle. It did nothing of the kind, but absolutely disregarded the straw, pushing past it even when laden with a fly. It was several times ejected together with the frass from new tunnelling operations, but never otherwise.

Continuous observation of work that went on for many hours a day for about three weeks was, of course, impossible, but on several days, especially during the week Aug. 18—25, operations were watched for spells of an hour or two at a time, and the exact times of ingress and egress carefully noted. The notes which immediately follow especially concern wasp No. 1.

The hole was sometimes clear, sometimes choked with "sawdust." After watching for a time the "sawdust" would be seen to heave up and form a mound over the hole. Then the wasp would emerge and proceed to remove the frass, butting it away from the neighbourhood of the hole with its head. Sometimes in the course of its excavations the wasp would emerge, fly away for a time, and return empty handed to resume its digging.

On Aug. 19 it was seen to be carrying home flies, and the performance was watched for an hour, and the following times were noted:

Returned with fly,	9.37, 9.48, 10.18, 10.31.
Emerged,	9.40, 9.55, 10.25, 10.39.

Thus four flies were caught in the hour, and the times spent in capturing three of them were 8′, 23′ and 6′ respectively, while 3′, 7′, 7′ and 8′ were occupied in packing the four flies into the burrows. To find, capture, paralyse and bring home the right kind of fly in six minutes strikes one as a remarkable feat. From further observations it appeared that the operation usually occupied about a quarter of an hour. None but "hover flies" (Syrphidae) were taken by any of the wasps, and the prey was generally *Syrphus balteatus*, a species almost as large as the wasp itself. It was, nevertheless, carried with perfect ease, arranged longitudinally, head foremost beneath its captor, and, I believe, venter to venter. No preliminary examination of the hole was ever made before carrying the fly in, such as Fabre has recorded in the case of some wasps. About noon on Aug. 21 this wasp apparently ceased working. There were no signs of activity that afternoon nor the following morning.

On Aug. 22 about 3 p.m. a wasp (wasp No. 2) was seen to come out of the woodpecker's hole and alight on the top of the log, which it proceeded to explore. It found No. 1's burrow and entered it for a short distance, after which it flew away. Nothing further was noted till the evening of Aug. 23, when on returning home at 5.30 I noticed a heap of frass on the top of the hole. At 6.20 a wasp arrived and after pointing at the main entrance, seemed to change its mind and alighting on the top, entered No. 1's hole. Its behaviour convinced me that it was not No. 1, but it might very well be wasp No. 2. Anyhow it entered the burrow, and by 7.50 it had turned out more "sawdust" containing several of the flies so carefully stored up by wasp No. 1! The explanation that first occurred to one was that the wasp wanted to dig, and naturally found it easier to work where someone had been before. Such a defective instinct would, however, militate against the preservation of the race. Moreover there were no further developments, and No. 2 remained satisfied with undoing some of No. 1's work. A wild suggestion did occur to me, which I will give for what it is worth. Is it possible that one of those working from the interior became aware of operations from the outside which might imperil the results of its own labours, and proceeded to put a stop to them?

With regard to the remaining wasps, which entered by the woodpecker's hole and worked from the inside, the following notes may be given.

The earlier hasty inspections of the interior showed that the cavity of the woodpecker's nest was being gradually filled with the "sawdust" of their workings, and conspicuous on the "sawdust" were a number of Syrphid flies, apparently dead. At the final investigation at the beginning of October about a hundred and twenty of these derelict flies were found in the central cavity, and as there were certainly not more than six wasps at work at any time, and as two were early captured and retained for identification, it is probably safe to estimate the average numbers of the wasps responsible for discarding them at five. This allows twenty-four discarded flies to each wasp—about six hours strenuous labour by each insect entirely wasted! As wasp No. 1 was never seen to discard a captured fly this phenomenon was apparently attributable to the conditions prevailing inside. There all the burrows commenced with a horizontal boring at the junction of the two sections of the log, at some little distance from the main opening. After alighting at the main entrance they had, therefore, either to fly across or to crawl round the central cavity, and it seems as though a number of flies had been accidentally dropped. It would be quite in keeping with what has been observed in the case of allied insects that a wasp which had accidentally dropped a fly should make no attempt to retrieve it, but should simply go away and catch another. These discarded flies were in any case very useful as evidence of the particular prey selected by *Crabro cephalotes*.

At the beginning of October some of these flies had been reduced to fragments by other predaceous creatures, but of 113 recognisable specimens 60 were *S. balteatus*.

My friend Mr N. D. F. Pearce very kindly undertook to identify the remainder for me and he finds among them five species of *Syrphus*, three of *Platychirus*, two of *Melanostoma*, and one of

Rhingia, *Catabomba* and *Helophilus* respectively. No family of flies except the Syrphidae was represented. The complete list is as follows:

Syrphus balteatus	60
S. luniger	5
S. vitripennis	4
S. corollae	4
S. auricollis	3
S. albistrictus	1
Platychirus albimanus ♀	9
P. scutatus ?	2
P. peltatus	1
Melanostoma mellinum	7
M. scalare ?	2
Rhingia campestris	13
Catabomba pyrastri	1
Helophilus pendulus	1
	113

Early in October the log was thoroughly explored, and an attempt was made to follow out the windings of the galleries, but the extreme friability of the decaying heart-wood made this very difficult.

The first thing that struck one was the absence of any attempt to seal or mask the tunnels which were entirely open to any chance intruder. Indeed a family of wood-lice was found three inches down the tunnel of wasp No. 1. There was nothing to prevent any enemy from entering. While at work the wasps had never manifested any interest in other insects in the neighbourhood of their burrows, nor did they finally make any provision for keeping them out. While watching the operations of wasp No. 1 a few insects had been seen to enter the tunnel, including Phoridae, one of which was secured, and a Muscid fly (? *Tachina*) and an Ichneumonid which unfortunately evaded capture.

The main tunnels were clear, and penetrated the wood for several inches, with abrupt turnings on no definite plan. From these proceeded side galleries in which were found "sawdust," the débris of flies, and the brown cocoons containing the fully-fed wasp larvae. Sections of the log showed that these were dotted here and there throughout the soft heart-wood precisely like the raisins in a Christmas pudding.

Neon Lamps for Stroboscopic Work. By F. W. ASTON, M.A., Trinity College (D.Sc., Birmingham), Clerk-Maxwell Student of the University of Cambridge.

[*Read* 19 May 1919.]

For the accurate graduation and testing of revolution indicators and similar technical purposes the stroboscopic method is probably the most reliable. This depends on the fact that if a rotating disc is illuminated N times per second by very short flashes, a regular figure drawn symmetrically on the disc will appear at rest when the number of revolutions of the disc per second is some exact multiple or submultiple of N depending on the number of sides of the regular figure.

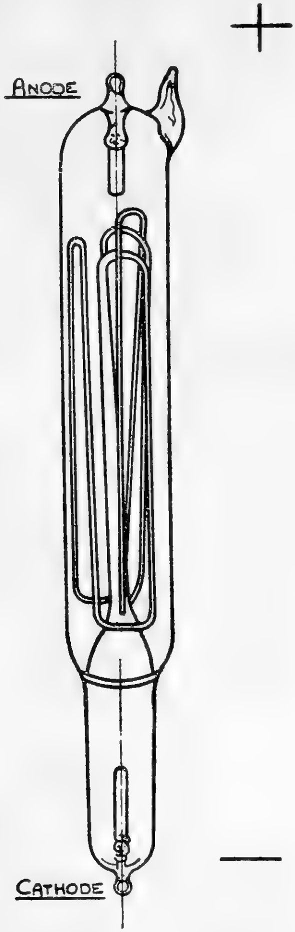
The value of N—in practice 50—can be set and easily kept extremely constant by the use of an electrically driven tuning-fork so that the success of the method rests principally upon the illuminating flashes; its accuracy will depend upon their shortness of duration and brightness; its convenience as a practical method upon their brightness and quality as affecting the eye of the observer.

The first experiments were tried with naked Leyden jar sparks obtained from the secondary of an ordinary ignition coil, the tuning-fork being introduced into the primary circuit as an interrupter. These showed the principle of the method to be excellent but spark illumination left much to be desired; it was noisy, feeble in intensity, and being mostly of short wave-length, caused rapid and excessive eye-strain even when used in a dark room.

The remarkable properties of Neon seemed to offer an almost ideal solution of the illumination problem. A form of lamp to replace the spark was therefore devised which appeared likely to give good results and several of these were filled from the author's stock of Neon at the Cavendish Laboratory. The success of these lamps was immediate, eye-strain disappearing completely. The present paper is a description of the lamps and their behaviour during continuous use.

The Form of Lamp.

The original form of the lamp, which it has not been found necessary to alter materially, is shown in the sketch. As, in the discharge in Neon, nearly all the light is in the "Positive Column" and its brightness increases with the current density, the lamp was designed to give a positive column as long and narrow as possible consistent with the potential available in the spark, and consists essentially of two relatively large spaces containing the electrodes connected by a very long capillary tube which is the counterpart of the filament in an ordinary glow lamp. In the lamps



Neon vacuum lamp for Stroboscopic work.
Two-thirds actual size.

in use the filament is about 60 cm. long by 1 mm. diameter and is coiled up inside the space containing the anode. This was done for convenience and strength, but it has another and important advantage, for this type of construction is strongly unsymmetrical to the discharge, allowing it to pass much more easily in the direction indicated in the figure than in the opposite, hence it effectually stops the "reverse" current from the secondary of the coil.

Other important results depending on the length of the filament will be discussed later, it should be roughly one hundred times the length of the spark the coil is capable of giving in air when running on the tuning-fork break.

It is hardly necessary to state that the shape into which the filament is wound is not in the least essential and could be varied to any extent in lamps for special purposes.

The electrodes are of aluminium and may be of any form so long as they are not too small.

Method of Filling Lamps.

As Neon, like the other gases of the Helium group, has the remarkable property of liberating gas from aluminium electrodes which have been completely run in for other gases, the operation of filling necessitates the contamination of a comparatively large volume of Neon, so that this can only be done economically and conveniently where liquid air is available for re-purifying.

So far all the lamps have been filled on the author's Neon fractionation apparatus at the Cavendish Laboratory[1]. The gas for filling is contained in charcoal cooled in liquid air. A quantity is admitted to the exhausted lamp which is then sparked at a pressure of 1 to 3 mm. with a small coil for a time. The dirty gas is then pumped off with a Toepler mercury pump, a fresh supply of pure gas admitted and the tube run again. These operations are repeated until spectroscopic and other observations show the desired conditions of purity have been reached and are not altered seriously by prolonged running. The full charge of 5 to 10 mm. of gas is now let in and the lamp sealed off. The whole operation takes about 3 hours, three lamps being filled at once. The pressure, purity and time of running in are all matters of some nicety as will be seen from consideration of the life of the lamp.

Life of the Lamps.

Apart from accident the lamps are serviceable until the pressure of gas within them becomes too low for the spark to light them adequately. Their life appears to consist of two distinct periods, the first during which chemically active impurities derived from the electrodes and walls of the tube are being slowly and completely eliminated (at least as far as a spectroscopic observation goes) and the second during which sputtering of the cathode takes place and the inactive Neon itself slowly disappears until the pressure gets too low for use. During the first period the luminosity steadily

[1] v. Lindemann and Aston, *Phil. Mag.* XXXVII, May 1919, p. 527.

improves, remaining almost constant afterwards till near the end of the second period when it rapidly decreases.

The first set of lamps were filled with very carefully purified Neon at 1–2 mm. pressure and run till sputtering had commenced before being used; they may therefore be considered to have had no first period at all. These lamps had a life of 500–1000 hours.

Experiments soon showed that the less preliminary running and the higher the pressure of filling the longer the life would be, but on the other hand, if the preliminary running is not sufficient the impurities derived from the electrodes turn the light of the lamp a dull grey and render it absolutely useless and pressures above 10 mm. are not advisable as these increase the spark potential of the lamp too much.

One lamp was actually so nicely balanced in these respects that though it became grey and useless after about 1 hour's use it completely recovered its original brightness after a day's rest. This is clearly a case of carbon compounds being given off by the electrodes while running, which are reabsorbed on standing and there is little doubt that were it worth while very prolonged running would render this lamp quite satisfactory. Very slow production of gases from the electrodes is advantageous, as prolonging the first period of the life, so that these should be of a fairly solid pattern.

So far, the best results have been obtained from a batch of lamps filled at about 10 mm. pressure, some with pure Neon, some with a mixture of Neon and about 10 per cent. Helium.

One of the latter had a working life of well over 3000 working hours, Helium disappearing from its spectrum after the first few hundred.

As there is every reason to assume that for any given lamp the life is determined by the total number of coulombs passed through it, the light obtained per coulomb should be arranged to be a maximum. This will be the case when the filament is made as long as possible, consistent with the potential available from the coil.

Cause of Disappearance of Gas from the Lamps.

The exhaustion of gas by continuous running has long been observed in the case of spectrum discharge tubes. It is doubtless allied to the phenomenon of "Hardening" in X-ray bulbs, but differs from the latter in that under the relatively high pressures in spectrum tubes, and the Neon lamps under consideration, the mean free-path of a charged molecule is so small that it can only fall freely through a potential of a few hundred volts and so never attain the very high velocities reached in the X-ray bulbs which are supposed to cause the gas molecules to become permanently embedded in the glass walls.

The disappearance of gases of the Helium group in spectrum tubes is invariably associated with sputtering of the electrodes which, at high pressures, only takes place when the gas is spectroscopically free from chemically active gases. It is generally supposed that the gas so disappearing remains embedded or adsorbed in the layer of sputtered aluminium on the sides of the tube near the cathode, the idea of true chemical combination not being acceptable without very rigorous proof.

In order to obtain information on this point, a completely run out specimen of the first batch of lamps, which was of course very heavily sputtered, was taken for test. First the sputtered cathode end was gradually heated to near the softening point of the glass (when it cracked) without any substantial or apparent increase in the internal pressure of Neon. The end was then cut off, broken into small pieces and heated in a quartz tube in a high vacuum apparatus provided with a spectrum tube. At a temperature about the softening point of the glass a good deal of gas was released which showed the hydrocarbon spectrum (but may nevertheless have contained some Neon as this is easily masked); this gas was pumped off and on heating further to a red heat, as the glass started to melt, Neon was given off, the spectrum showing quite clearly.

Apparatus for measurement and analysis of the gas so released was not available, but it is hoped to repeat this interesting experiment, which shows definitely that the Neon is contained either in the sputtered aluminium or very near the surface of the glass so that it is released by heat.

Use of other Gases instead of Neon.

Ordinary chemically active gases give very feeble illumination, CO being about the best. Helium gives a bright discharge but not nearly so valuable in quality for visual work as Neon; its presence as an impurity in the latter gas renders the discharge more rosy red but up to 10 per cent. does not affect its brightness seriously. Mercury vapour as used by C. T. R. Wilson in his photography of ionisation tracks would probably give very bright flashes, but the fact that the lamp has to be kept very hot is a serious objection.

Reason for Superiority of Neon.

The brilliant orange-red glow of the discharge in Neon is composed almost entirely of lines in the region 5700–6700 A.U. and is in such striking contrast to sunlight that stroboscopic observations can even be done in broad daylight if necessary, the ordinary appearance of the rotating disc having merely a grey background added, looking bluish by contrast.

The actual amount of light radiated per unit of energy, i.e.

the real efficiency of the discharge in Neon, is not markedly greater than that in e.g. mercury vapour, but the apparent efficiency is enormously enhanced by the fact that it consists so largely of red light. Victor Henri and J. L. des Bancels have shown ("Photochemie de la Rétine," *Jl. Phys. Path.* XIII, 1911) that the Fovea Centralis of the eye is immensely more sensitive to red light than the outlying portions of the retina[1], thus a Neon lamp as a source of general illumination is very disappointing, but when viewed directly appears surprisingly bright. As the spinning disc of the stroboscope subtends a comparatively small angle the Fovea is the only part of the observer's eye used in testing, which is probably the reason for the eye strain with the spark.

Nature and Duration of the "Working Flash."

If one analyses the flash of a short spectrum type Neon tube in a rotating mirror it is seen to consist of two separate parts, an extremely short flash followed by a flame or "arc." The first is probably due to the simultaneous ionisation of the gas throughout the whole length of the tube, the second to the further carriage of current by the ions formed during the first. The structure of the latter, which appears to consist of bright striations travelling from anode to cathode at velocities of the order of that of sound in the gas, is of great theoretical interest and is at present under investigation. Discussion of its nature is needless in the present paper for its duration being of the order of thousandths of a second it is useless for stroboscopic work and, by the employment of a sufficiently long filament tube, it can be eliminated altogether. In a lamp properly proportioned to the power of the coil in use the whole energy of the discharge is absorbed in the first flash. In order to get some idea of the duration of this "working flash" the following experiment was performed.

A plain mirror, silvered outside to avoid double images, was mounted vertically on the axis of a large centrifuge and the image in this of the Neon lamp at a distance of 3 metres was observed by means of a telescope with a micrometer eye-piece. Each division in the micrometer subtended $4{\cdot}2 \times 10^{-4}$ radians and when the centrifuge was running at 3500 revolutions per minute corresponded to $5{\cdot}75 \times 10^{-7}$ seconds.

The lamps were run with the tuning-fork attachment used in actual testing and were viewed directly and also through ground glass with a V-shaped slit to be certain of getting the effect of the

[1] The difference of retinal effect between red and green light can be easily observed by looking at an ordinary luminous wrist watch in the faint red light of a photographic dark room. On shaking the watch so sluggish is the green light in recording its position on the retina compared with the red that the figures seem to be shaken completely off the dial, giving a most curious and striking effect.

total duration of the flash. In neither case was the fuzziness of the image of a measurable order. After careful observation under good conditions the conclusion of three observers was, that it was probably less than one-tenth of a division and certainly less than one-fifth. This gives the maximum duration of the working flash as one-ten-millionth of a second, so that it can be taken as perfectly instantaneous for the purpose employed.

Other Technical Applications.

Of the many uses besides measuring velocity of rotation to which Neon lamps may be put with advantage in engineering and other problems it is sufficient to mention two in which they have been very successful. Any rapidly rotating mechanism such as an airscrew, if illuminated by a lamp the break of which is operated mechanically at each revolution, will appear *at rest*, flicker being small at speeds well over 1000 R.P.M., so that strains or movement of parts can be examined with great accuracy under actual working conditions.

A still more striking effect can be obtained by illuminating a high speed internal combustion engine by a lamp whose break is operated mechanically at e.g. 99 breaks per 100 revolutions of the engine shaft by the use of a creeping gear. The engine then appears to be rotating quite smoothly at one-hundredth its normal speed so that such instructive details as the movements of the valves and springs, the bouncing of the former on their seats, etc., can be studied with ease.

It is of course necessary for the speed of rotation to be fairly rapid to give appearance of continuity to the eye and in consequence one cannot apply this method to the analysis of such a thing as the movement of a chronometer escapement.

As the technical importance of Neon lamps is rapidly on the increase it is very desirable that liquid air engineers in this country should consider the erection of a fractionating plant for recovering the gas from the air (which contains ·00123 per cent. by volume) such as has been used with such success by Mons. Georges Claude of Paris, to whom the author is indebted for the Neon with which these experiments were performed.

The pressure in a viscous liquid moving through a channel with diverging boundaries. By W. J. HARRISON, M.A., Fellow of Clare College, Cambridge.

[*Read* 24 November 1919.]

If non-viscous liquid is flowing along a tube having a cross-section which is increasing in area in the direction of flow, the pressure will also increase, in general, in the same direction. On the basis of this remark an explanation has been given of the secretory action of the kidneys. The author's attention was drawn to this explanation by Dr Ffrangcon Roberts. The physiological aspect of the question and a more detailed numerical consideration will be dealt with by Dr Roberts and the author in a separate paper.

In the present paper two problems are considered, viz. the flow of liquid in two and three dimensions when the stream lines are straight lines diverging from a point.

Two-dimensional problem.

Let the boundaries of the channel be $\theta = \pm \alpha$, where (r, θ) are two-dimensional polar coordinates. The motion in which the stream lines are straight lines passing through the origin has been obtained by G. B. Jeffery[1]. With a slight change of notation the results of his solution are as follows.

Let the velocity at any point be u/r, where u is a function of θ only. Then

$$u^2 = -4\nu u - \nu \frac{d^2u}{d\theta^2} + a,$$

where ν is the kinematic coefficient of viscosity, and a is a constant of integration. Whence

$$u = -2\nu(1 - m^2 - m^2k^2) - 6\nu k^2m^2 \operatorname{sn}^2(m\theta, k),$$

where k and m are constants, which may be determined from the conditions that u must vanish at $\theta = \pm \alpha$, and that the total rate of flux may have a given value. Instead of the latter condition it is simpler to assume that the velocity is given for $\theta = 0$, i.e. $u = u_0$ for $\theta = 0$.

Thus the conditions are

$$-2\nu(1 - m^2 - m^2k^2) = u_0,$$
$$(1 - m^2 - m^2k^2) + 3k^2m^2 \operatorname{sn}^2(m\alpha, k) = 0.$$

[1] *Phil. Mag.* (6), vol. XXIX, p. 459.

These may be written

$$m^2 = (1 + u_0/2\nu)/(1 + k^2),$$

$$\mathrm{sn}^2 \left\{ \left(\frac{1 + u_0/2\nu}{1 + k^2} \right)^{\frac{1}{2}} \alpha, k \right\} = \frac{1 + k^2}{3k^2 (1 + 2\nu/u_0)}.$$

If the values of u_0 and α be given, the last equation serves for the determination of k. Writing $k_1 = 1/k$, the equation has the same form in k_1 as in k. Hence, if k is a solution, $1/k$ is also a solution. Therefore, of real values of k, it is only necessary to consider such that satisfy $0 \leqslant k \leqslant 1$.

Treat α as small, and assume that $\left(\frac{1 + u_0/2\nu}{1 + k^2} \right)^{\frac{1}{2}} \alpha$ is also small.

We have
$$\alpha^2 = \frac{(1 + k^2)^2}{3k^2 (2 + 2\nu/u_0 + u_0/2\nu)}.$$

The least value of α for a given value of $u_0/2\nu$, if k is real, is given by $k = 1$. In this case, if $u_0/2\nu = 1$, $\alpha^2 = \frac{1}{3}$, $\alpha = \cdot 58$. This value of α is not small enough for the approximation to hold good. Put $k = 1$ and $2\nu/u_0 = 1$ in the original equation, and we find $\alpha = \cdot 65$, approximately. For smaller values of α, k will be a complex imaginary quantity. As $u_0/2\nu$ is either increased or decreased, a real value for k can be obtained for smaller values of α.

It will be found sufficient for the purposes of the present paper to restrict the consideration of the solution to the ranges of values of α and $u_0/2\nu$ for which k has a real value. We proceed to discuss the pressure variation in the case for which k is real; the variation in the case for which k is complex can be inferred by considerations of continuity.

Let p be the mean pressure at the point (r, θ) in the liquid, and ρ its density. We obtain from the two-dimensional polar equations of motion

$$-\frac{u^2}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu}{r^3}\frac{\partial^2 u}{\partial \theta^2},$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\frac{\partial u}{\partial \theta}.$$

Hence
$$\frac{p}{\rho} = -\frac{1}{2}\frac{u^2}{r^2} - \frac{1}{2}\frac{\nu}{r^2}\frac{\partial^2 u}{\partial \theta^2} + F(\theta)$$

$$= \frac{2\nu u}{r^2} - \frac{a}{2r^2} + F(\theta),$$

substituting for $\frac{\partial^2 u}{\partial \theta^2}$ from the differential equation satisfied by u.

Also
$$\frac{p}{\rho} = \frac{2\nu u}{r^2} + f(r).$$

Hence $\frac{p}{\rho} = \frac{2\nu u}{r^2} - \frac{a}{2r^2} + C$, where C is a constant. Now the lateral stress in the liquid is $p_{\theta\theta}$, where

$$\frac{p_{\theta\theta}}{\rho} = -\frac{p}{\rho} + 2\nu\frac{u}{r}$$
$$= \frac{a}{2r^2} + C.$$

Hence $p_{\theta\theta}$ is independent of θ, and is the normal stress (of the nature of a tension) exerted by the liquid on the boundary. If a is negative the normal pressure on the boundary decreases as the channel widens, and if a is positive the normal pressure increases.

Now by substitution of the solution for u given above in the differential equation satisfied by u, we find

$$a = 4\nu^2 [-1 + m^4 (1 - k^2 + k^4)]$$
$$= 4\nu^2 [-1 + (1 + u_0/2\nu)^2 (1 + k^6)/(1 + k^2)^3].$$

(1) Writing $a = 0$, we can immediately discriminate between those cases for which the pressure on the boundary decreases and those for which it increases.

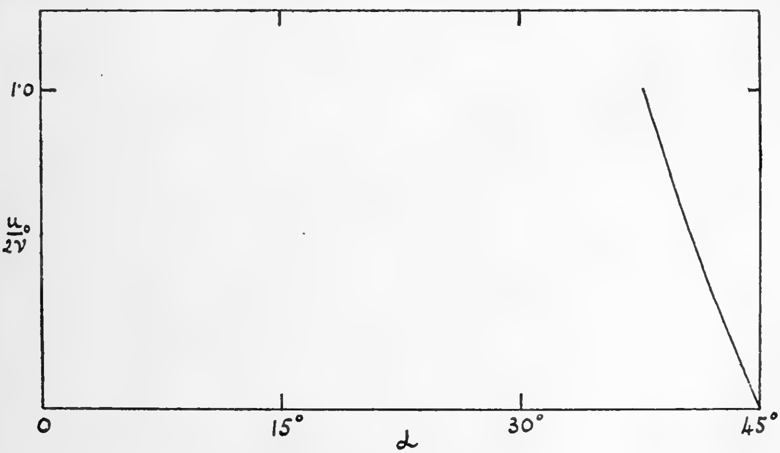
If $a = 0$, we have

$$1 + u_0/2\nu = (1 + k^2)^{\frac{3}{2}}/(1 + k^6)^{\frac{1}{2}},$$

and $$1 + 2\nu/u_0 = (1 + k^2)^{\frac{3}{2}}/\{(1 + k^2)^{\frac{3}{2}} - (1 + k^6)^{\frac{1}{2}}\}.$$

Hence $$\mathrm{sn}^2 \left\{\left(\frac{1 + k^2}{(1 + k^6)^{\frac{1}{2}}}\right)^{\frac{1}{4}} \alpha, k\right\} = \frac{(1 + k^2)^{\frac{3}{2}} - (1 + k^6)^{\frac{1}{2}}}{3k^2 (1 + k^2)^{\frac{1}{2}}}.$$

The following diagram shows how the value of $u_0/2\nu$ for which $p_{\theta\theta}$ is independent of r varies with α, for those cases in which k is



real. It clearly indicates that when α is small the critical value of $u_0/2\nu$ may be somewhat large.

If $\alpha > \pi/4$, the lateral pressure increases for all values of u_0.

(2) It is a simple matter to discuss the variation of the pressure when $u_0/2\nu$ is large. We have, approximately

$$\mathrm{sn}^2\left\{\left(\frac{u_0/2\nu}{1+k^2}\right)^{\frac{1}{2}}\alpha,\ k\right\} = \frac{1+k^2}{3k^2},$$

$$m^2 = \frac{u_0/2\nu}{1+k^2},$$

$$a = \frac{u_0^2\,(1+k^6)}{(1+k^2)^3},$$

$$p_{\theta\theta} = \frac{u_0^2\,(1+k^6)}{2r^2\,(1+k^2)^3} + C.$$

k will be real provided $k^2 > \frac{1}{2}$, and, corresponding to real values of k, α will be small.

In the absence of viscosity, so that $u = u_0$ for all values of θ,

$$p_{\theta\theta} = \frac{u_0^2}{2r^2} + C'.$$

Thus the lateral pressure increases at a rate which is $\dfrac{1+k^6}{(1+k^2)^3}$ of the rate for a non-viscous liquid.

The following table will indicate the character of the results when k is real.

$u_0/2\nu$	α	$(1+k^6)/(1+k^2)^3$
100	10° 30′ 9° 30′	·30 ·27
1000	1° 3′ 0° 57′	·30 ·27
10,000	0° 6′	·28

For larger values of α than those given above, and for the corresponding values of $u_0/2\nu$, k is unreal.

When α is small there is apparently an approximation which Jeffery gives, viz.

$$u = -\,2\nu\,(1 - m^2 - m^2k^2) - 6\nu k^2 m^4\theta^2,$$

where $$0 = -(1 - m^2 - m^2k^2) - 3k^2m^4\alpha^2,$$

leading to $$u = -\frac{6\nu m^2(1-m^2)}{1-3m^2\alpha^2}(\theta^2 - \alpha^2)$$
$$= u_0(1 - \theta^2/\alpha^2).$$

This gives $$a = -2\nu u_0/\alpha^2,$$

and $$\frac{p_{\theta\theta}}{\rho} = -\frac{\nu u_0}{\alpha^2 r^2} + C.$$

Thus the lateral pressure apparently decreases for all values of u_0. But if
$$u_0 = -2\nu(1 - m^2 - m^2k^2),$$
and $$0 = -(1 - m^2 - m^2k^2) - 3k^2m^4\alpha^2,$$
we have $$u_0/2\nu = 3k^2m^4\alpha^2,$$
and therefore $m\alpha$ is not necessarily small. Hence the approximation is only valid for values of $u_0/2\nu$ below some limiting value. If this condition be satisfied the expression for $p_{\theta\theta}$ given above is an approximation to its value for small values of α.

Three-dimensional problem.

Let the boundary of the channel be $\theta = \alpha$, where (r, θ, ϕ) are polar coordinates. This problem has been considered by Prof. A. H. Gibson[1]. In his solution Cartesian and Polar Coordinates are confused, and he assumes that the stream lines are straight lines diverging from the origin, a state of motion which is impossible if the inertia terms are retained in the equations of motion, as he retains them. One result of these errors is that in his solution p is a function of θ although the preliminary assumption is virtually made that p is independent of θ. His expression for the pressure appears to be quite wrong.

Assume, in the first place, that the stream lines are straight lines diverging from the origin, so that $u = f(\theta)/r^2$, $v = 0$, $w = 0$. The polar equations of motion reduce to

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2u}{r^2}\right],$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{r\partial\theta} + \frac{2\nu}{r^2}\frac{\partial u}{\partial\theta},$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial\phi}.$$

[1] *Phil. Mag.* (6), vol. XVIII, p. 36, 1909.

We have

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{2f^2}{r^5} + \frac{\nu}{r^4}[\cot\theta \,.\, f' + f''],$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \theta} = \frac{2\nu}{r^3}f'.$$

Hence eliminating p,

$$\frac{4ff'}{r^5} + \frac{\nu}{r^4}[f''' + f''\cot\theta - f'\operatorname{cosec}^2\theta + 6f'] = 0.$$

Therefore $ff' = 0$, and

$$f''' + f''\cot\theta - f'\operatorname{cosec}^2\theta + 6f' = 0.$$

Hence $f'(\theta) = 0$, and the boundary conditions cannot be satisfied, since u becomes independent of θ.

For slow motion, or any motion in which the inertia terms can be neglected, we have

$$f''' + f''\cot\theta - f'\operatorname{cosec}^2\theta + 6f' = 0 \quad\text{.........(1).}$$

A first integral is

$$f'' + f'\cot\theta + 6f + C = 0 \quad\text{..............(2).}$$

The solution of (2) suitable for the present purpose is

$$f(\theta) = D(2 - 3\sin^2\theta) - \tfrac{1}{6}C.$$

Let $\quad f(\theta) = u_0, \quad \theta = 0,$

$\quad\quad f(\theta) = 0, \quad \theta = \alpha.$

We have $\quad D = u_0/3\sin^2\alpha,$

$\quad\quad C = 2(2 - 3\sin^2\alpha)\,u_0/\sin^2\alpha.$

Hence $\quad u = u_0(\sin^2\alpha - \sin^2\theta)/r^2\sin^2\alpha.$

Integrating the equations of motion, we have

$$\frac{p}{\rho} = -\frac{1}{3}\frac{\nu}{r^3}(f'\cot\theta + f'') + F_1(\theta)$$

$$= -\frac{1}{3}\frac{\nu}{r^3}(-6f - C) + F_1(\theta),$$

and

$$\frac{p}{\rho} = \frac{2\nu}{r^3}f + F_2(r).$$

Hence

$$\frac{p}{\rho} = \frac{2\nu}{r^3}f(\theta) + \frac{1}{3}\frac{C\nu}{r^3} + B,$$

and

$$\frac{p_{\theta\theta}}{\rho} = -\frac{1}{3}\frac{C\nu}{r^3} - B.$$

The lateral pressure will continually increase as the channel widens if C be negative, that is, if $\sin\alpha > (\frac{2}{3})^{\frac{1}{2}}$, or $\alpha > 54°\ 45'$. If $\alpha < 54°\ 45'$, for sufficiently small values of u_0 the pressure will continually diminish.

The Effect of Ions on Ciliary Motion. By J. Gray, M.A., Fellow of King's College, Cambridge.

[*Read* 10 November 1919.]

The ciliary mechanism of the gills of *Mytilus edulis* has been described by Orton[1]. There are at least four distinct sets of cilia, whose movements form a complex but highly coordinated system by which food particles are filtered from the sea-water and passed up to the mouth. This coordinated system is entirely free from any nervous control and continues for many days in detached portions of the gill. These gill fragments therefore form an admirable material for the physiological study of ciliary motion.

The effect of the hydrogen ion on ciliary action is very easily studied. Normal sea-water has a P_H of about 7·8; when the concentration of hydrogen ions is increased to about 6·5 rapid cessation of movement occurs. In sea-water of P_H 6·7 the rate of ciliary movement is checked at first, but within $\frac{3}{4}$–$1\frac{1}{2}$ hours complete recovery takes place. If gill fragments whose cilia have been stopped by the more acid solution are returned to normal sea-water, complete recovery takes place in less than 20 minutes although the cilia may have been motionless for several hours. A large number of experiments have been performed from which it is clear that if the concentration of hydrogen ions is only slightly greater than normal, the cells can react to the environment and recovery take place in the acid solution. In stronger acid, however, recovery only takes place on removing the gills to a more alkaline solution. In still stronger acid the cells become opaque and are killed.

Gills which are exposed to an abnormally high concentration of hydroxyl ions behave in a remarkable manner. In such solutions ciliary action is either not affected at all or proceeds at an abnormally rapid rate, but the individual cells of the ciliated epithelia break away from each other and move about in the solution owing to the movement of their cilia. Since such cells are no longer in their normal environment, it is impossible to determine any upper limit of hydroxyl ions which will permit normal ciliary action to go on.

Since the hydrogen ion has a most marked effect on ciliary activity, it is necessary to adjust the hydrogen ion concentration of all artificial solutions during a study of the effects of various salts on ciliary action. In the case of the salts of the alkali metals this is satisfactorily performed by the addition of an appropriate

[1] *Journ. Marine Biol. Assoc.* vol. IX, p. 444 (1912).

buffer such as sodium bicarbonate. In the case of the salts of the alkaline earths it is impossible to obtain pure isotonic solution of the same hydrogen ion concentration as sea-water, and it is therefore necessary to compare the effects of the pure solutions with that of sea-water whose hydrogen ion concentration is abnormally high.

A number of experiments have been performed which prove that sodium, potassium, calcium and magnesium are all necessary to maintain gill fragments in a normal state of ciliary activity for a protracted period, viz. four days. If one or more metals are omitted, the individual cells of the ciliated epithelia show the same disruptive phenomenon as in sea-water of abnormally high concentration of hydroxyl ions. Solutions containing only one metal show this phenomenon to a very marked degree although they may be more acid than normal sea-water; the effect of solutions containing two metals is less marked than that of solutions containing only one metal, but more marked than that of solutions containing three metals. No evidence was obtained of specific ion action or of antagonistic action between monovalent and divalent ions.

These experiments afford another example of the intense action of the hydrogen ion upon physiological activity and of its reversible nature if the acid treatment is not too severe. The same action of acids is found in the activity of the heart and in the movement of spermatozoa.

A Note on Photosynthesis and Hydrogen Ion Concentration. By J. T. SAUNDERS, M.A., Christ's College.

[*Read* 10 November 1919.]

Last April (1919) I was testing the hydrogen ion concentration of the water of Upton Broad, a small broad in Norfolk. I had determined the hydrogen ion concentration of the water of the broad itself to be 8·3 and I found this varied very little whether the water was taken from the surface or the bottom, from near the edge or the centre of the broad. The determination of the hydrogen ion concentration was made by the use of standard solutions and indicators as recommended by Clark and Lubs.

When however the water in the shallow lodes and ditches surrounding the broad was tested, great variations in the hydrogen ion concentration occurred. The water became more acid as soon as the broad was left and the ditches entered. At one end of the broad where the water was shallow, not more than 18 inches deep, and when there was no wind to mix it with the open waters of the broad which was 6 feet deep, the hydrogen ion concentration would fall to 8·15. In the lode itself the hydrogen ion concentration was 7·65. After boiling and rapidly cooling, water from the middle of the broad and from the shallows both showed a hydrogen ion concentration of 8·4, while that from the lode after the same treatment was 8·15.

At one point in the lode, however, I found surprising variations. Dippings of water from the same place gave readings of the hydrogen ion concentration varying from 7·7 to 8·6. At this point there was a certain amount of Spirogyra growing and I found that if I took water from the centre of a mass of Spirogyra I could get a reading as high as 9·0.

I took some of the Spirogyra back with me and placed it in test-tubes in tap-water which I coloured with indicator solutions. The hydrogen ion concentration was 7·2 at the commencement of the experiment. After standing the test-tube in a window in sunlight the hydrogen ion concentration rose after an hour to 8·6 and in two hours the phenolphthalein indicator had turned bright pink, indicating a hydrogen ion concentration of more than 9·0. I had no standard solutions with me which I could use to test higher values than 9·0 so that I was unable to determine accurately the ultimate result. I left the test-tubes until the next morning, when I found the hydrogen ion concentration had fallen to 7·6. After again placing the test-tubes in sunlight the hydrogen ion concentration rose above 9·0.

On my return to Cambridge I repeated these rough experiments. It is easy to prove that the rise in alkalinity is not due to alkali dissolved out of the glass, nor is it due alone to the abstraction of the dissolved carbon dioxide out of the water. The hydrogen ion concentration of the Cambridge tap-water which I used for these experiments was 7·15 when the water was tested immediately after being drawn from the tap. On standing at a temperature of 13° C. the hydrogen ion concentration rises to 7·4. After boiling and rapidly cooling the hydrogen ion concentration was 7·9 and bubbling through air free from carbon dioxide produced the same result. By incubating tap-water for 36 hours at a temperature of 40° C. and then cooling the hydrogen ion concentration could be made to rise to 8·15, but in no case did the value of the control tap-water approach near that of the tap-water containing Spirogyra filaments.

The following is a record of a typical experiment. The Spirogyra was placed in 25 c.c. of tap-water in a boiling tube and exposed to light at a window. Control boiling tubes containing tap-water only were used. All these tubes were half immersed in a glass bowl of running water so that the temperature was maintained fairly constant.

Date	Time (G.M.T.)	Temp.	Hydrogen Ion concentration		Remarks
			Control	Spirogyra	
1. v. 19	11·10 a.m.	14·0° C.	7·15	7·15	Dull day.
,,	12·10 p.m.	13·0° C.	7·4	8·3	
,,	1.10 p.m.	12·5° C.	7·4	8·6	
,,	2.10 p.m.	12·5° C.	7·4	8·6	
,,	3.10 p.m.	12·5° C.	7·4	8·8	
,,	5.30 p.m.	13·0° C.	7·4	8·5	

I have tried using Elodea instead of Spirogyra and it gives much the same result.

Both in darkness and in daylight the contents of the living cell of Spirogyra show an acid reaction when stained with neutral red. When Spirogyra is killed by heating to 40° C. and then placed in tap-water the hydrogen ion concentration falls considerably since the cell membranes are broken or dead and the contents of the cell are now free to pass out into the water.

In a large pond the mass of the plants in proportion to the water is not sufficiently great to affect the hydrogen ion concentration very much. I have however found slight variations. On one occasion I noticed a fall in the hydrogen ion concentration of 0·1 after several dull days and a subsequent rise of 0·2 after sunny days. This variation may possibly be due in some degree to the photosynthetic activity of the plants present.

The distribution of intensity along the positive ray parabolas of atoms and molecules of hydrogen and its possible explanation. By

F. W. ASTON, M.A., Trinity College (D.Sc., Birmingham). Clerk-Maxwell Student of the University of Cambridge.

[*Read* 19 May 1919.]

No one working with positive rays analysed by Sir J. J. Thomson's method can fail to notice the very remarkable intensity variation along the molecular and atomic parabolas described by him under the term 'beading.' It will be sufficient for the reader to refer to Plate III of his monograph on the subject (*Rays of positive electric*, p. 52) to realise how striking these can be. Beadings at points corresponding to energy greater than the normal have been quite satisfactorily accounted for by multiple charges (*l.c.*, p. 46), but the ones with which this paper is concerned have a smaller energy than the normal, actually half, and fractional charges are presumably impossible. Nevertheless they seem capable of a simple explanation and an opportunity of putting this to the test occurred recently while making some experiments to determine the best form and position of the cathode preliminary to the design of an apparatus to carry the analysis to higher degrees of precision.

The observations were made with an apparatus essentially of the form now well known (*l.c.*, p. 20) the discharge tube being arranged to be removable with the minimum trouble to change or move the cathode. As no camera suitable for photographic recording was immediately available or necessary a willemite screen and visual observation was employed. This form has many obvious disadvantages and in addition, owing to the enormous difference in sensitivity between the parabolas of hydrogen and those due to heavier elements the latter can only be seen with difficulty. It has however one notable advantage, namely that sudden and even momentary changes in intensity can be observed and *correlated in time* with changes in the discharge or in the intensity of other lines. As no accurate measurements were intended a large canal ray tube was employed so that the H_1 and H_2 parabolas could be easily seen even with the less effective types of cathode.

It was soon realised that the appearance on the screen was in general the sum of two superposed effects which could be only unravelled like the writings on a palimpsest by eliminating one of them. This by good fortune it was found possible to do under certain conditions. For the sake of clearness it is proposed to consider these two extreme types and their explanation before going on to describe the conditions under which they may be attained or approached. In the diagrams the fields of electric and magnetic forces are horizontal and such that positive ions will be deflected to the right and up, negative ones to the left and down. *Brightness* is roughly indicated by the *width* of the parabolic patch drawn.

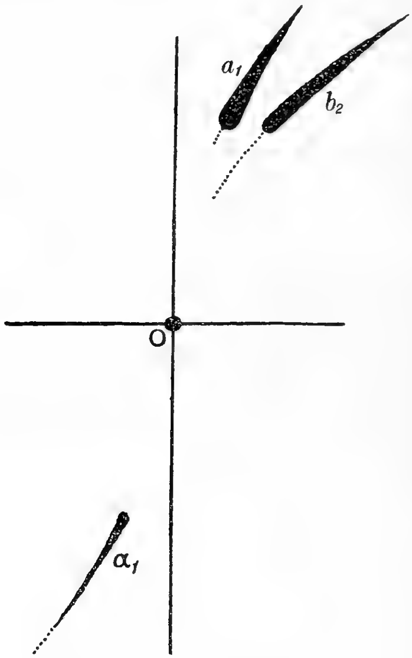


Fig. 1. Atomic Type.

Atomic type of discharge.

Fig. 1 illustrates the first or 'Atomic' type in which apparently the whole of the discharge is carried up to the face of the cathode by ions of atomic mass. Those which pass through the fields without collision produce the true primary streak on parabola $m = 1$, the head of which corresponds in energy to that obtained by the charge e falling through the full potential of the discharge. Now the pressure in the canal ray tube is never negligible being on the average at least half that in the discharge tube, and the ionisation along its length very intense so that in passing through it a large number will collide with electrons, atoms or molecules. The collision and capture of a single negative electron will result in a neutral atom striking the screen at the central undeflected spot O while the capture of two will cause the faint negative parabolic streak a_1 as has already been described (*l.c.*, p. 39).

But besides these forms of collision by which the velocity of the atom is practically unaffected there is distinct evidence that it may collide with and capture another hydrogen atom. If the atom struck is negatively charged the resulting molecule will strike the central spot but if it is neutral and the collision is inelastic the resulting positive ray will have the same momentum (the atom struck being relatively at rest) but double the mass so that it will strike the *molecular parabola* at a point the same height above the X-axis as would the atom which generated it. Molecular rays formed in this manner will therefore form the streak b_2 which, allowing for the geometrical difference in the curves will show a similar distribution of intensity to a_1. Collision with a positively charged atom will obviously be unlikely to result in capture and those with heavier atoms will be referred to later. It is to be noted in connection with the brightness of these secondary streaks a_1 and b_2, which may conveniently be called 'satellites' to distinguish them from the 'secondary lines' already fully described (*l.c.*, p. 32), that a_1 is always very much fainter than its primary but b_2 can be equally bright.

This atomic type of discharge with its pendant bright arc on the molecular parabola corresponding to similar momentum and half normal energy is most beautifully illustrated in Fig. 29 of Plate III already referred to. It was this photograph which suggested the above theory of its explanation.

Molecular type of discharge.

The extreme form in which the whole discharge is carried up to the cathode by ions of molecular mass is unattainable so far in practice and is probably impossible but its share in the illumination of the screen can be deduced by eliminating the superimposed atomic type and is indicated in Fig. 2.

The principal feature is a short and very bright spot of light b_1 on the molecular parabola at the point corresponding in energy to a fall through the full potential of the discharge. It will be shown that all the ions causing this are probably generated in the negative glow. Besides this there are two symmetrical and equally bright positive and negative satellite patches a_2 and a_2 on the atomic parabola but of half the normal energy. The proposed explanation of these is somewhat similar to that considered by Sir J. J. Thomson (*l.c.*, p. 94) and is as follows. The collision with and capture of a single negative electron by a positively charged molecule will not necessarily merely neutralise it and cause it to hit the central spot O but may result in it splitting into two atoms one with a positive one with a negative charge. The energy of impact may be itself capable of causing this, if not some other cause, *e.g.* radiation, may effect the dissociation. In any case it would give exactly the observed result, *i.e.* two bright patches lying symmetrically on the extension of the line joining the primary spot to the origin at twice its distance from the latter, corresponding to half the mass but the same velocity.

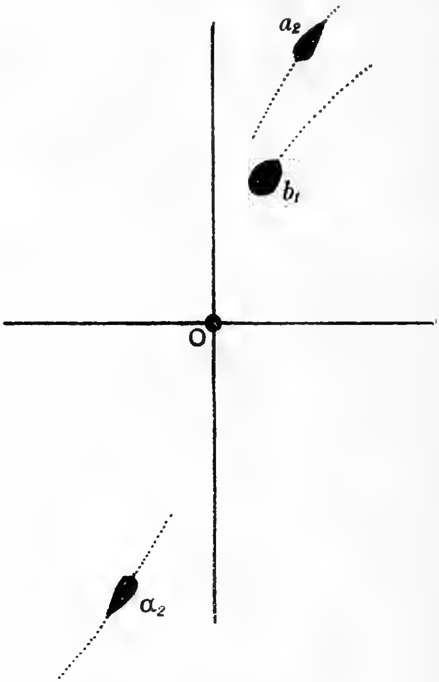


Fig. 2. Molecular Type.

The general appearance on the screen when both types of discharge are present is indicated in Fig. 3.

Effect of different forms of cathode.

Experiments were performed with plane, concave and convex cathodes. Convex cathodes are the least efficient in producing bright effects but give the molecular type with the least atomic blurring. Concave ones are most efficient and throw the maximum energy into the atomic type which can be obtained practically pure with them under a moderate range of conditions. The original shape of cathode (*l.c.*, p. 20) may be said in a sense to combine both forms and was designed to give long and bright parabolas at the same time allowing the discharge to pass easily at very low pressures. The present results however lead one to recommend a concave cathode similar to those used in X-ray focus tubes but pushed further forward into the neck of the bulb, for though this form requires a rather higher pressure this objection is more than counterbalanced by the great increase in efficiency. Plane cathodes, as was expected, give effects midway between the other forms.

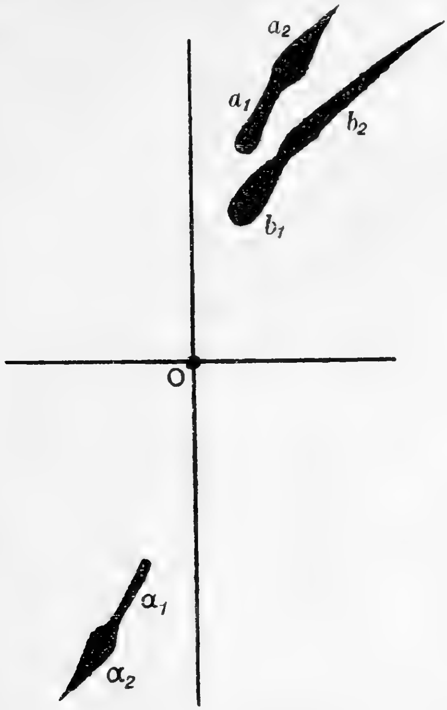


Fig. 3. General Type.

Under very exact conditions of pressure, etc. it is possible to obtain the pure atomic type with plane cathodes but no conditions have yet been found under which convex ones will give it.

These results seem to indicate that atomic ions are formed by the passage of the stream of cathode rays through the Crookes dark space molecular ones tending rather to be formed in the negative glow. The axial intensity of the cathode stream is enormously increased by the concavity of the cathode while that of the negative glow does not appear to be affected to anything like the same extent.

Behaviour during change of pressure.

The pressure in a freshly set up bulb always increases with running owing to the liberation of gas by heat etc. so that the changes due to gradual alteration of pressure can be observed most conveniently by exhausting highly, starting the coil and watching the events on the screen. Thus using a concave cathode of about 8 cms. radius of curvature set just in the neck of the discharge bulb the following sequence of events was observed. At very low pressures with a potential of about 50,000 volts the parabolas are very faint but correspond to the general type, the primary streak a_1 and spot b_1 being much brighter than their satellites (doubtless due to few collisions). As the pressure rises the discharge becomes curiously unsteady the spots on the screen become much fainter and change with flickering into the pure atomic type (Fig. 1), b_1 having practically disappeared. This form of discharge which is evidently abnormal lasts for a certain time depending on the rate of increase of pressure. Then with absolute suddenness b_1 flashes out intensely bright and with it appear *at the same instant* its satellites a_2 and α_2. At the same time the current through the bulb increases, the discharge settles down and the negative glow makes its appearance. As far as it was possible to judge the satellites a_2 and α_2 are of equal brightness and generally much brighter than the negative atomic satellite α_1.

The appearance of the discharge bulb while the pure atomic type is shown on the screen is difficult to describe but quite characteristic and different from the general. Near its critical upper limit of pressure it was found possible to effect the change to the general type by bringing a magnet near the cathode and so disturbing the discharge. On removing the magnet the discharge at once reverted to the atomic type. This form of controlled change from the one to the other gave an excellent opportunity of testing the invariable association between the primary spots and their appropriate satellites.

Possible cause of disappearance of primary molecular rays.

It is unlikely that change of pressure is itself the determining factor in the disappearance of the molecular type. This seems to be due to some disturbance in the discharge by the cathode stream (not caused by the diffuse one given by a convex cathode) which makes the formation of the negative glow impossible.

The facts so far may be brought into line fairly well by the somewhat speculative assumption that molecular rays can only originate freely in parts of the discharge where the electric force is very small, *e.g.* the negative glow, ionisation by more violent means in strong fields tending to cause simultaneous disruption of the molecule into its atomic constituents. This agrees with the observed fact that in general molecular arcs, or at least true primary molecular arcs, are shorter than atomic ones. It would also mean that a very short arc infers as origin a molecule capable of disruption. If this is so it offers interesting confirmatory evidence, if such were needed, that the substance X_3 is molecular as this body often makes its appearance on the photographic plate as a short arc.

Effects with heavier elements.

The inelastic collision of a hydrogen atomic positive ray with the atom of a heavy element would clearly result in the formation of a molecular ray of such low velocity that it might not be detected by a screen or plate and would in any case be deflected completely off the ordinary photograph.

The visual evidence on the screen although faint leaves little doubt that the formation of satellite arcs also takes place by atoms of heavier elements colliding to form molecules. There is also some evidence of this in many of the photographs, thus in Fig. 26 (*l.c.*, p. 46) taken with oxygen all four maxima are suggested. In Fig. 17 (p. 26) the satellite on the molecular parabola caused by the capture of oxygen atoms by carbon atomic rays (or vice versa, but this is less likely) is unmistakable, in fact attention is called in the text to this remarkable increase in brightness.

Should the above theory of collision with capture prove correct the formation of compound molecules by this means opens an extremely interesting field of chemical research. Another important question raised is in what form the energy of the collision is radiated off by the rapidly rotating doublet formed.

In conclusion the author wishes to express his indebtedness to the Government Grant Committee for defraying the cost of some of the apparatus used in these experiments.

Gravitation and Light. By SIR JOSEPH LARMOR, St John's College, Lucasian Professor.

[*Read* 26 January 1920.]

1. Newton's provisional thoughts on the deep questions of physical science were printed at the end of the second edition of the *Opticks* in 1717. As he explains in the Preface "...at the end of the Third Book I have added some questions. And to shew that I do not take Gravity for an Essential Property of Bodies, I have added one Question concerning its Cause, chusing rather to preface it by way of a Question, because I am not yet satisfied about it for want of Experiments." In the first and next following Queries he gives formal expression to the idea that "Bodies Act upon Light at a distance and by their action bend its Rays...."

What was thus propounded in general terms as an explanation of the diffraction of light in passing close to the edge of an obstacle, assumed a more definite but different form in the hands of the physically-minded John Michell*; in *Phil. Trans.* 1767 he insisted that the Newtonian corpuscles of light must be subject to gravitation like other bodies, therefore that the velocities of the corpuscles shot out from one of the more massive stars would be sensibly diminished by the backward pull of its gravitation, and thus that they would be deviated more than usual by a glass prism, a supposition which he proposed to test by experiment. He also speculated that the scintillation of the stars might be due to the small number of corpuscles which reach the eye from a star, amounting perhaps to only a few per second.

The forces, of molecular range, that would have to be concerned, on the lines of Newton's Query, in the diffraction of light would be of course enormously more intense than gravitation: but the other Newton-Michell theory of the gravitation of light rays is paralleled in both its aspects with curious closeness in certain modern physical speculations.

It will be observed that this notion of light being subject to gravitation makes its velocity exceed the limiting velocity c, which on electrodynamic theory could not be attained by any material body. But there need not be a discrepancy there: for the limit arises because a material body is supposed to acquire more and more inertia, belonging to energy of its motion, without limit as its velocity increases, whereas the *quantum* of energy in the hypothetical light-bundle presumably would remain sensibly the same—at any rate we would be free to make hypotheses in absence of any knowledge.

* See *Memoir of John Michell* (of Queens' College), by Sir A. Geikie, Cambridge Press, 1918.

Forty years ago there was a phase of strong remonstrance in this country against the familiar uncritical use of the phrase centrifugal force. The implication was that the term force should be restricted to intrinsic unchanging forces of nature, which are determined physically by the mutual configuration of the system of bodies between which they act: these forces are then held responsible for the accelerative effects specified by the Newtonian second law of motion. In this sense, centrifugal force so-called would not be a force of nature, but would be the reaction postulated in the scheme of the Newtonian third law to balance an imposed centripetal acceleration.

This formative principle, the Newtonian third law, of balance everywhere between applied forces and reactions against palpable changes of motion, as amplified in the Scholium annexed to it—which so widely reached forward towards modern theory as Thomson and Tait especially have remarked—would then assert that the forces of nature that act on the framework of a material body and the forces of reaction that are thereby induced in it, form together a system of forces that preserve statical equilibrium *in relation to* the constraints of that framework, as tested by the principle, also Newtonian in its origin, of virtual work. This became in time the Principle of d'Alembert (1742), who did not invent it, but exhibited its power and developed its method by applying it to a great dynamical problem of unrestricted form, that of the precession of the equinoxes. As a preliminary to its solution he had to develop in general terms the equations of static equilibrium of a system of forces considered as applied to a single rigid body such as the Earth, that is, to create a formal science of Statics: and it may be said to be the mode of development rather than the principle itself that constitutes his essential contribution to general dynamical theory. Cf. the historical introductions in Lagrange's *Mécanique Analytique*.

2. The principle of the relativity of force has recently become prominent again, and pushes along further on the same lines; it now even puts the question—Are there intrinsic forces of nature at all? May not all force, including universal gravitation, be expressible as reaction against acceleration of motion, just after the manner of the obviously unreal centrifugal? On such a view, wherever there is a force of gravitation in evidence, its presence must be replaced by an acceleration common to all of the material bodies at each place and relative to our frame of measurement, of amount equal and opposite to the intensity of the force. That would be the end of the matter, if any frame of reference could be found to satisfy this condition. There being then no forces left, the Principle of Least Action would make orbits simply the shortest paths in the frame. Newtonian uniform space and time certainly

could not permit this transformation: nor could the fourfold uniform continuum of interlaced space and time of the earlier relativity theory be adapted to it. Will such a fourfold, deformed into a non-uniform and therefore non-flat heterogeneous space, permit it? This is the problem raised by Einstein's idea of the relativity of gravitational force. Perhaps it goes even further, and asks whether if this will not do, there can be some other *corpus* of abstract differential relations invented, that will transcend the notion of spacial continuity altogether but will in compensation for that formidable complexity succeed in effecting this object.

In any case we may recognise that this merging of all the forces of nature into spacial relations satisfies one requirement which is not quite the claim that is explicitly made for it. The question is immediately insistent; why should intrinsic forces be measurable with Newton in terms of second gradients of type d^2s/dt^2 and not by a more complex formula involving others as well? The answer supplied by the theory would be that the idea of the curvature of a deranged space is expressed by a measure which does not involve higher gradients.

It is interesting to reflect nowadays that in referring to the doctrines of action at a distance in the preface to the *Electricity and Magnetism* in 1873 Maxwell classifies them as "the method which I have called the German one," and that notwithstanding Helmholtz's very powerful critical work on Maxwell's theory, beginning in 1870, that description remained substantially true until after Maxwell's death in 1879. Though he lived for nine years longer he seems to have taken no part in these discussions with exception of a reference to Helmholtz in connexion with Weber's theory (*Treatise*, § 254), but worked chiefly at the development of the theory of stresses in gases regarded as molecular media, and so in some respects parallel to his theory of an electric medium. He seems to have been content to leave his electric scheme to germinate and expand in the fulness of time. In connexion with the recent efforts to transcend *both* action at a distance and an aethereal medium, his explanations, in an Appendix to the Memoir on the determination of the ratio of the electric units, *Phil. Trans.* 1868 and the critical chapter on 'Theories of Action at a Distance' in the *Treatise*, §§ 846—866, are far from being obsolete.

This hypothesis as to gravitation, which asserts that it is essentially of the same nature as the apparent increase of weight which is experienced by an observer going up in a lift with accelerated motion, naturally involves many consequences, and raises questions regarding the relation of gravitation to physical agencies such as light, the answer to which may be ambiguous until yet further postulates intervene.

Thus in the preliminary stage it occurred to Einstein that the period of a train of light waves would be no longer uniform throughout its course. Let us consider a mass of hydrogen gas at P, say in the Sun, sending light-waves to an observer Q, both being situated in a region in which there is a field of gravitation of intensity represented by g, directed from Q to P. In terms of the postulate of the relativity of that force this statement would mean that the spacial frame to which the underlying events are referred is rushing as a whole from P toward Q with acceleration g. Let v be the velocity of the frame at the instant when a specified light-wave passes any intermediate point Q': by the time this wave has reached Q the velocity of the frame as a whole has risen to $v + g.Q'Q/c$ approximately, where g is mean intensity along the range from Q' to Q. Thus to the accelerated observers the waves emitted become longer with distance traversed, in the ratio $1 + g.Q'Q/c^2$, owing to this velocity of recession from the source: that is, the apparent wave-length undergoes change so that during the progress from Q' to Q it is altered in the ratio $1 - \delta V/c^2$, where δV is the rise of potential (or fall of gravitational potential energy) along that path.

The period of the light will thus appear to be increased to different observers on the line PQ, all of them travelling along with the same acceleration g, in different degrees according to their positions. This is what will happen if the observers and their space and optical instruments form a world of their own rushing past, or through, an underlying actual world, with this acceleration g, instead of the actual world rushing past them with the opposite acceleration produced by a force of gravitation. For these alternatives are not now the same: the *finite* velocity of propagation c is constant with respect to the actual underlying world, not the observers' moving space. If the radiating hydrogen belongs to the actual underlying world, and the spectroscopes of the observers belong to their own spacial scheme that is imposed on that world, this description is complete: the period of each wave as apparent to observers along its path will increase as the wave travels away to places of lower gravitational potential. The spectral lines of solar hydrogen as observed on the earth ought to be displaced towards the red, by the amount corresponding to the total fall of potential between Sun and Earth. But the postulate of two worlds seems to be here necessarily involved. Which of them would a mass of radiating hydrogen situated half-way to the Sun belong to?* The larger Doppler-Fizeau effect due to the motion of the source itself relative to the observers' frame has not here been

* All the bodies in the space, being subject to the same gravitation, would move along with it: the waves of light alone would seem to be regarded as independent: yet they have energy and so inertia.

mentioned: that is included satisfactorily in the earlier uniform relativity formulation.

This relation of light to gravitation is thus one of the questions raised by the postulate of the relativity of that universal force. Einstein answered in 1911* in one way, that the spectrum of solar hydrogen, when compared with terrestrial hydrogen which is connected with the observer, should be displaced slightly towards the red: but it is a question whether the consistent development of that train of ideas would not rather require that it be not displaced at all.

In connexion with his later formal theory of gravitation the same effect is described as due to varying local scales of time, which seem to be carried without change, by the pulsations of the rays, from the place of their origin to all the other parts of the universe: whereas in the above the apparent period† changes as the ray advances. The observers along the ray are supposed to be in communication with one another. In so far as their space moves forward as a whole it is not stretched or shrunk: in that case it can be only their scales of apparent duration of time that are lengthened locally by a factor, the inverse of $1 - V/c^2$. This involves that the scale of apparent velocity in the unchanged space will be altered in the direct ratio: and rays of light in a field of varying potential, if they were paths of stationary time, might be thought to be deflected. But fundamentally the path of the ray is determined by the number of *wave-lengths* in its course being made stationary, as compared with neighbouring courses: and this is, in the present case, not the same as minimum time of transit, for apparent time has lost its uniform scale while space has not.

Thus the path of a ray would be determined by the condition that $\Sigma\delta s/\lambda$ summed along it shall be stationary: but if there is correspondence between the two systems of reference which changes all lengths around each point in the same ratio then $\delta s/\lambda$ will be everywhere the same in both systems. The circumstances of the path would thus not be altered by this change of view regarding gravitation, and there ought to be no special deviation of the rays involved in it.

But if g is not uniform along the path r of the ray, is a shrinkage of the accelerated apparent space involved? The answer

* His exposition which has here been paraphrased is in *Ann. der Physik*, 35, 1911, § 3, p. 904.

The argument of this and the next two paragraphs is based on the implication that in a theory of transmission by contact, radiation like other things, the so-called clocks included, must conform to local measure: the alternative, described at the end of the paper, that radiation is extraneous in so far as it imposes an absolute scale of space-time of its own on the whole cosmos, was here taken to be excluded in advance from this type of theory.

† Measured on a fundamental scale.

is given that, passing to the general problem, the demands of the universal gravitational correspondence (to be evolved immediately, *infra*) require that the apparent space of the observers must be *constructed* so that $\delta r^2 - c'^2\delta t^2$ where c' is a function of r shall be invariant. This requires slight warping of the fourfold space, so that the section in the plane r, t is curved away from its tangent plane. But is the warped element of extension $\delta r'.c'\delta t$ thereby altered only to the second order from its corresponding previous normal value $\delta r.c\delta t$? If that be so, the scale of t must be altered in the inverse ratio to the scale of velocity c' or (what is the same in another aspect) of time t: and in fact it is partly this secondary change of scale of r that modifies the astronomical gravitation, as will presently appear.

The answer to this question might at first be imagined to be as follows: any change in the element of surface may be made in two stages, a stretching on the original plane and a displacement along the direction normal to that tangent plane: it is only the former that can produce a first-order effect: but this is only an apparent change, a mere alteration of coordinates, because in it the curvature of the plane is conserved, so it cannot affect the concatenation of relations or events which alone counts: the latter does affect them, e.g. disturb the law of gravitation, but only to the second order.

But as will appear presently this relation of conservation of extent is between coordinate systems that most closely correspond, so is a real imposed condition which cannot be adjusted by change to another set in the flat. It is the expression of, or at any rate is involved in, a restriction that in the containing fivefold the distance between corresponding points on the two systems is everywhere small, so that approximate methods can apply consistently throughout, of which otherwise, in making continuations in an uncharted extension, there would be no guarantee.

3. Now let us survey this problem of transcending gravitation from the other side, on which it originated. With Minkowski the very incomplete relativity of electrodynamics, referring only to uniform translatory convection, crystallised into the complete proposition that events occur in a uniform fourfold of mixed space and time, determined by the constitutive spacial equation

$$\delta\sigma^2 = \delta x^2 + \delta y^2 + \delta z^2 - (c\delta t)^2.$$

Here c has nothing to do with the velocity of radiation: it is simply the dimensional factor, prescribing a scale of measurement, that is needed to make time homogeneous with length and may be taken as unity. Gravitation remains outside this electrodynamic scheme, being formulated in the different Newtonian reckonings of space and time. Can it be forced in, either exactly or approximately?

The complete circumstances of the orbits in a field of force of potential energy $-V$ per unit mass (in a gravitational field V is $\Sigma\delta m/r$) are condensed into the single variational Least Action equation of Lagrange-Hamilton,

$$\delta\int\left[\tfrac{1}{2}m\left\{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right\}+mV\right]dt=0$$

with integration between limits of time fixed and unvaried. This suggests comparison with the equation for the shortest or most direct path in a modified fourfold involving Euclidean space combined with a measure of time varying from place to place: for that equation is

$$\delta\int d\sigma=0 \quad \text{where} \quad \delta\sigma^2=\delta x^2+\delta y^2+\delta z^2-c'^2\delta t^2,$$

in which c' is a function of x, y, z. Let us write

$$c'^2=c^2(1+K),$$

where K is very small on account of the greatness of c. The equation is now

$$\delta\int\left\{-c^2(1+K)+\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right\}^{\frac{1}{2}}dt=0,$$

or approximately up to the fourth order

$$c^{-1}\,\delta\int\left[-c^2-\tfrac{1}{2}Kc^2+\tfrac{1}{2}\left\{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right\}\right]dt=0.$$

The time-limits being unvaried the first term $-c^2$ can be omitted: thus this variational equation of most direct path coincides with the previous orbital equation if

$$-\tfrac{1}{2}Kc^2=V.$$

Thus the forces are absorbed into a varying scale of time; and the motion being now free under no force, the orbit is, as was anticipated, a geodesic or straightest path. The orbits have become however straightest paths, not in their original Newtonian separated space and time, but in the uniform space-time fourfold of relativity as slightly deranged by the not quite constant scale of time.

Thus the orbits in any field of attraction have actually been fitted into the mixed space-time frame of electrodynamic relativity, at the expense of doing slight violence to that frame, by making the measure of time vary from place to place while the positional specification remains uniform.

But this transformation does more than is needed. It ought somehow to be restricted to the one universal force of nature, that of gravitation with its inverse-square law. It is here that the special feature of the Einstein theory seems to come in. For velocities beyond actual astronomical experience, not small compared with that of light, mass comes to depend on speed; thus it is not any longer available as a definite dynamical constant. On the earlier uniform relativity it emerged however definitely in another way as a feature of every permanent collocation of energy and proportional to its amount E, equal in fact to E/c^2. This follows immediately if Least Action is fundamental. Thus it is grouped energy that possesses located momentum: and it is this energy that has to gravitate, mass confined to matter alone having proved inadequate to a Least Action formulation in the mixed space-time of universal limited relativity. Dynamical principles had therefore to take the form of a theory of conservation of energy and of abstract momentum as they travel through a medium, at the same time receiving additions by the operation of an internal stress to which the medium is to be subject. In other words, general dynamics cannot be more detailed than a mere description of the migration of energy and of momentum in a medium under the influence of some internal system of stress adjusted to fit the equations as simply as possible. This stress is what has to stand for or represent the agencies of nature. The theory is borrowed and generalised from the Maxwellian theory of stress in the aether, which was an isolated, apparently rather accidental, feature that did not fit well into the substance of Maxwell's scheme, because in fact it could not be connected with a strain expressive of its origin. Now however, inertia of bodies having failed as the standard measure of force, energy and momentum, and a postulated adjusting stress entirely at our choice, are promoted to occupy the vacant place. Only it is not called a stress: the idea of a physical medium is avoided, so it is named an algebraic tensor. There is no law of elasticity involved, or relation of stress to strain, such as makes elastic problems determinate. Thus the scheme may have accidental features, is perhaps far from being unique. Another parallel to it is Maxwell's theory of stresses in a gas due to varying temperature: but that continuous theory could never have been constructed in definite form without the foundation of the behaviour of the individual molecules.

When however the fourfold frame is very nearly flat, the relations of energy-momentum-stress appear to fall in with the law of gravitation, with energy as the source of its potential instead of matter.

When the deranged spacial frame nowhere differs much from the flat, it may be expected that the extent of its fourfold element will be altered from the value for coordinates of the corresponding type on the flat only to the second order, for the same kind of reason as applies in comparing a slightly deranged plane sheet with the original plane. In fact, if the displacement is everywhere small, this extent taken over a small region would have a stationary value for the flat, changing in the same direction on both sides of it. Cf. *supra*, p. 329. Thus for a spherically symmetrical field the constitution of the fourfold must be determined in polar coordinates by the equation

$$\delta\sigma^2 = (c/c')^2\,\delta r^2 + (r\delta\theta)^2 + (r\sin\theta\delta\phi)^2 - c'^2\delta t,$$

showing that the positional part of the extension is very slightly non-uniform and so not quite Euclidean. It appears to be this secondary feature, not the energy-momentum-stress tensor conditions, that modifies gravitation from the Newtonian law.

The expositions of relativity do not mention an extended fourfold, which would be foreign to the cardinal idea that space is constructed from physical origins, only in so far as it is needed—even though it has to be implied that it is reproduced unerringly each time. But the instrument of such construction or continuation of a metric space is an infinitesimal linear measuring rod supposed to have complete free mobility without change of intrinsic length: and it would seem to be a tenable view that such a mobile apparatus must determine an underlying flat space of higher dimensions* in which the physical system may be supposed imbedded.

It is to be noted here that a surface defined intrinsically in the Gaussian manner by the distance relation on it

$$\delta s^2 = f\,\delta p^2 + 2g\,\delta p\,\delta q + h\delta q^2,$$

remains the same surface when the coordinate quantities p, q are changed to others p', q' which are any assigned functions of them both, so that

$$\delta s^2 = f'\delta p'^2 + 2g'\delta p'\delta q' + h'\delta q'^2,$$

provided δs is measured by the same infinitesimal unchanging measuring rod extraneous to the surface in both cases. These two equations represent the same surface, only the generalised coordinates of the same point on it are changed from (p, q) to (p', q'). The intrinsic curvatures are the same from whichever form they be calculated: if one form represents a flat, so does the other. On this definition by an intrinsic differential relation surfaces are indistinguishable, if one can be bent to fit the other without stretching. So in the Riemann theory of spaces of more than two dimensions it is the functional forms of the *coefficients* in the quadratic function of differentials and the mobile absolute measuring rod that determine the nature of the space; any transformation of coordinates changes the coefficients (or potentials in the gravitational formulation) but so that the space remains un-

* For a radial field it need be of only one more dimension.

changed, being only referred as regards the same points to the other generalised coordinates. But the apparent extent $\delta p \delta q$ does alter when the coordinates are changed, and it would be a limitation to keep it constant. See Appendix *infra*.

The feature that remains unfathomed as yet is the fact that the velocity of transfer of energy of radiation in undisturbed regions of space is equal to the merely dimensional constant that renders time comparable with space on the fourfold frame of reference: it at any rate suggests a dynamical origin for that mixture of the effective relations of time with those of space*.

The locus in the fourfold in which σ never changes and so $\delta\sigma$ vanishes has some claim to be called the 'absolute,' in a sense parallel to the 'absolute' of Cayleyan geometry which for Euclidean space is represented by the equation $x^2 + y^2 + z^2 = 0$. Everywhere on this locus $\delta s = c'\delta t$; thus velocity of displacement is everywhere c', and the rays in it are the paths of shortest time with this velocity. It separates the disparate regions in which $\delta\sigma$ measures real distance when time is unvaried and in which $\iota\delta\sigma$ measures real time when position is unvaried.

4. It would appear (as *infra*, p. 335) that if we are prepared to replace a field of potential energy of gravitation or any other type of universal force by a field of varying time-scale without change of the uniform scale of space, on the lines sketched above, this formal change ought not sensibly to affect radiation either as regards its path or its period. To each element of extent there would be a corresponding element, and all events and measures in one pass over to the other according to rule.

But we now pass from kinematic discussion of frames of reference to physical considerations. If we are to assert, in agreement with the doctrine of relativity *plus* Least Action, that inertia is a property of organised energy and proportional to it, therefore not solely of matter, and if we are to admit with Einstein, in the same and other connexions, that light is made up of small discrete bundles or *quanta* of energy, it would appear to follow that each bundle is subject to gravitation. Therefore if a bundle comes on from infinite distance with velocity c, when it has reached a place of potential V near the Sun its velocity c' must be given by

$$\tfrac{1}{2}c'^2 - V = \tfrac{1}{2}c^2,$$

in other words, is increased in the ratio $1 + V/c^2$. It will swing round the Sun in a concave hyperbolic orbit, and as the result, the direction of its motion will suffer deflection away from the Sun by half the amount that has been astronomically observed.

This reasoning would not be estopped by the principle that c is the upper limit of possible material velocities: for that is because

* See final paragraphs.

a moving body acquires energy and therefore inertia without limit as its speed approaches c, whereas the energy of a light *quantum* is not supposed so to increase.

This is all on the older notions: the velocity c is far too great for the new approximate gravitation analysis to be applicable. But the idea of wavefronts and phases must also be introduced somehow. If we imagine a row of these corpuscles of energy coming on abreast, the more distant ones would fall behind in swinging round the Sun and their common front would become oblique to their direction of motion, the exactly transverse directions being now the loci of equal Action not of equal time. If we superposed the Huygenian principle of propagation normal to the front, the orbital deflection would thereby be just cancelled by the swinging back of the front which would retain its direction: and there would be no deflection of direction of propagation. But such ideas are plainly incoherent.

The earlier development of Einstein sketched above* was driven on other grounds to conclude that light must gain energy in a field of gravitation, but the gain was named potential energy. In the finally developed theory there seems to be no longer energy of motion or other types: energy becomes a single analytic scalar in what is left of the field of interplay of momentum, energy and stress.

These earlier considerations have doubtless crystallized into the formal theory of which also the result has been illustrated above, in a way which transforms the variational equation of free orbits in ordinary space and time into the variational equation of straightest lines in a non-uniform space-time fourfold *given differentially*. The coordinates are carried over unchanged in values, into this fourfold, but their differentials no longer express in it direct measurements of length and time; these are now imported in the Riemann manner as regards any element of arc or interval of time by the value of the absolute element $\delta\sigma$. As compared with the underlying absolute time determined by $\delta\sigma$, the element of apparent time δt of a gravitational world, which is taken over into its expression is variable, proportional to c'^{-1}, with locality.

The quantities x, y, z, t which are the measures of space and time as apparent in the world of gravitation are now mere coordinate quantities in the new differentially given world in which there are elements of absolute length and time both measured by $\delta\sigma$. The final expression for $\delta\sigma^2$ with radial symmetry

$$\delta\sigma^2 = \left(\frac{c}{c'}\right)^2 \delta r^2 + \ldots + \ldots - \left(\frac{c'}{c}\right)^2 c^2\delta t^2$$

shows that the element of apparent time in the gravitational world

* *Ann. der Physik*, 35, 1911, § 2, p. 902.

is the unchanging element of absolute time divided by c'/c, or that the scale of apparent time is variable with locality in the ratio c/c': also that the scale of apparent radial length is variable in the ratio c'/c: and therefore the scale of radial velocity is variable as their quotient c^2/c'^2. How then with respect to the velocity of rays of light whose absolute value is the same as the dimensional constant c? Referred to these variable scales its apparent value along any element of arc ought to be changed at the same rate as any other velocity along that element of arc would be changed, if rays are not to remain outside the correspondence between δx, δy, δz, δt representing time-space in the apparent gravitational world and the same quantities, now elements of mere coordinates in a differentially given world in a curved space-time which has absorbed gravitation. This maintenance of correspondence is secured if we determine the ray-velocity along any element of arc by making $\delta\sigma = 0$: and the modified theory of radiation for the apparent space of gravitation must be such as can accept this value of the velocity of propagation*. The correspondence takes over the same values of the coordinate differential elements. In the apparent gravitational world they represent its space and time, in the new world differentially specified, they belong to mere coordinates: absolute elements of space and of time are there expressed by $\delta\sigma$, but a relation of scales can be established from the formula which expresses $\delta\sigma^2$.

The transformation which changes orbits into geodesics in the differentially given space-time does not turn rays into rays: their velocity is too great and moreover their minimum property is relative to their locus $\delta\sigma = 0$. But if the ray is supposed to have a constant underlying absolute period of pulsation and a constant absolute wave-length (and therefore to be a straight line in an auxiliary uniform fivefold) its apparent period in the gravitational world must vary with locality as $(c'/c)^{-1}$, also its apparent element of length inversely as the scale of length pertaining to its direction on that locality, and its apparent velocity as before specified. Its apparent path in the gravitational world will correspond to the true absolute path $\delta\int d\sigma/\lambda_0 = 0$, therefore will be given by

$$\delta\int ds/\lambda = 0,$$

complications being avoided as fortunately t is not involved explicitly in these equations. But at the same place the scales of apparent δs and apparent λ would alter on the same ratio owing to the presence of gravitation: therefore its influence is eliminated in the quotient, and the path is not affected by the gravitation, is the same whatever be its intensity. A ray passing near the Sun ought not to be deflected on this view: an observed deflection,

* On this and the following paragraphs, cf. however the end of the paper.

which *a priori* was well worth looking for, would seem to await explanation on other lines.

Again would there be an observable change of periods of spectral lines according as the vibrating source was at the Sun or at the Earth? The underlying absolute periods of radiating hydrogen molecules would be always and everywhere the same: thus the apparent period in the gravitational world would vary inversely as the local scale of time, and be longer at the Sun. But this is a local apparent period. The waves sent out from the solar molecule are observed at the earth: we have seen that their length changes as they progress, being inversely as the local scale of length, and their speed changes also, so that their period changes inversely as the local scale of time. Thus when they have reached the Earth their period conforms to the local scale and would agree with that of the radiation of a similar terrestrial molecule. In fact if complete correspondence is established*, element for element, as above, all periods or intervals of time measured at any element are changed in the same ratio depending on the locality alone. Any other conclusion would make the pulsating rays into signals establishing absolute time throughout the apparent universe, which could hardly be a result of a theory of relativity.

The condition $\delta\sigma = 0$ prescribes a definite ray-velocity for each element of arc, the same forwards as backwards, only when $\delta\sigma^2$ involves δt^2 but no products of δt with other differentials: in other cases it gives two velocities, not equal and opposite, and this spacial scheme of rays seems to fail. If rays are to be properties of the space a very severe restriction is thus imposed on the form of $\delta\sigma^2$, but one which seems to be satisfied for the slight modifications that would be involved in the actual gravitation of experience.

In the modifications of the expression for $\delta\sigma^2$ which absorb gravitation the coefficients do not involve the time explicitly: therefore the ray-paths are fixed in the space, and it almost looks as if they were guides imposed by the nature of the space alone, as thus modified, for the alternating energies of radiation to run along.

Any inference that because a ray is fixed in space, as many waves must run in at one end as run out at another, would be at variance with the very notion of relativity, by providing a scale of absolute time throughout the universe. Such an argument seems to amount in more general form essentially to this: when the expression for $\delta\sigma^2$ does not contain t explicitly it will make no

* As has been established for the more general case in a beautiful analysis by Prof. Th. de Donder, of Brussels, *Comptes Rendus*, July 6, 1914, *Archives du Musée Teyler*, Haarlem, vol. iii, 1917, pp. 80–180. [It is merely *continuity* with non-gravitational fields, and not correspondence, that is established.]

difference to the cosmos if t is everywhere increased by the same constant: therefore the scale of time must be everywhere the same —which excludes any possibility of local scales of time. A change of origin of measurement for time is not the same as progress of events in time, unless the scale of time is everywhere the same.

The matter may be put from a different angle as follows. To obtain the time of transit of a ray from P to Q it is not possible to add elements of heterogeneous local times such as δt*. What can be done is to find the true underlying time of transit. If this homogeneous true time is delayed at the start, at one end of the path at P, it is delayed by an equal amount at arrival at the other end, as the equations of transit do not involve this time explicitly: hence apparent times at the two ends are delayed not by equal amounts, but by amounts inversely as their local scales, so that a ray cannot (as has been implied) transmit apparent time along its path.

The alternative development is, as above, that $\delta\sigma^2$ being the underlying unchanging standard there are local scales of time, and local scales of length which may involve direction, and therefore also of velocity (including that of the rays) which is their quotient. The path of a ray from point to point is determined by making the number of wave-lengths from the one to the other minimum, that is by $\delta\int ds/\lambda = 0$: but δs and λ are both altered to the same scale; thus there is no alteration due to gravitation in the variational equation determining the ray-path, so that it would suffer no deflection. The essential feature in the argument is that, whether rays may be regarded as the limiting case of free orbits or not, their specification has been postulated so that the ray-velocities correspond in the same way as all other velocities in the two frames.

Appendix.—*On Space and Time.*

Let us try for a closer realization of these abstract positions. The Gauss-Riemann theory for an ordinary curved surface will be wide enough to serve as an illustration. The theory involves coordinates p, q: they must represent something. The very least we can do for them is to regard the surface as twofold extension dotted over with points, so that the coordinates express their order of arrangement according to some plan of counting them with respect to this extension in which they lie. There is no metric idea at all in this numeration, and nothing to distinguish one surface from another. Now bring in an infinitesimal unchanging

* Yet it is just such elements of *quasi*-time dx_4 that *are* added together, *infra* p. 343. It is the so-called shifting clock-time and absolute time running parallel that are the source of all this confusion.

measuring rod, which can make play in each element of extension represented by $\delta p \delta q$ and also be transferred from place to place: and we can thereby impart or rather superpose metric quality on the twofold which hitherto was purely positional or rather tactical. The simplest plan is to follow Euclid, on the basis of the Pythagorean theorem, and expressing absolute length according to measuring rod by a symbol δs, to impose a scale-relation of form

$$\delta s^2 = \delta p^2 + \delta q^2.$$

But this metric cannot be applied consistently over a curved surface, unless it is of the very special type that can be rolled out flat: for other surfaces it is necessary to have the more general type of relation

$$\delta s^2 = f\delta p^2 + 2g\delta p\delta q + h\delta q^2,$$

in which f, g, h are functions of the coordinates p, q.

This specification of an imported metric thus determines the surface: starting from a given small region of it, the form of the surface in an outer threefold space can be gradually evolved by prolongation so as to fit in with consistent application of this metric. It is this idea of prolongation of a non-uniform manifold, equivalent to its geometrical continuation within a flat one of higher dimensions, that was Riemann's contribution to the ideas of geometry. But the manifold itself is supposed to be given only tactically or descriptively; and it is the metric that is imposed on it that, by its demand for consistency in measurements, determines for it a form, as located in a higher flat manifold. This form is expressed in detail analytically by the 'curvature' at each place, as specified by a set of functions (one in the case of a surface) of the successive gradients of the set $f, g, h, \ldots$. If we keep the system self-contained by avoiding the immersion of it in a uniform auxiliary manifold of higher dimensions, our resource is to determine the curvature as the simplest set of functions that are invariant for local changes of coordinates. But, in order of evolution at any rate, this invariance may be held to be only a derived idea.

In any case the nature of the non-uniform manifold, as thus determined by a metric imposed on formless space, has nothing to do essentially with the coordinates $p, q, \ldots$ to which it may happen to be referred: it is settled by the algebraic form of the functions $f, g, h, \ldots$ expressed in terms of $p, q, \ldots$, or in geometric terms by the 'curvature' as so expressed.

As a consequence, if we transform a surface from internal or intrinsic coordinates p, q, to others p', q', which are assigned functions of the former, so that we obtain

$$\delta s^2 = f'\delta p'^2 + 2g'\delta p'\delta q' + h'\delta q'^2,$$

and construct the surface implied in this new equation by the process of continuation, it will prove to be just the same surface as before. Whether it is expressed in terms of p', q' or of p, q is intrinsically of no consequence: the coordinates are of no account, it is only the functional forms of f, g, h that are essential.

This last statement, developed in terms of the criterion of invariance in order to avoid a representation by immersion in a uniform geometrical manifold of dimensions higher than the given four of space and time, appears to cover the general relativity of Einstein. The $f, g, h, \ldots$ can be named the potentials which determine the space. In the special relativity, before gravitation was absorbed into the metric of extension, all spaces were flat, so $f, g, h, \ldots$ were constants; which is all that is left, for that particular case, of these relations of invariance.

In this flat fourfold, relativity implied merely that a physical system is determined by its own internal relations, so that the position that may be assigned to it in the fourfold is of no account, any more than is the position of a surface or a system of bodies in space. In the later general relativity the manifold must be supposed given descriptively by coordinates, which represent numerical counts arranged to suit the number of dimensions that are involved: it only gains internal form when a metric is imposed upon it. If the Euclidean metric

$$\delta s^2 = \delta p^2 + \delta q^2 + \ldots$$

is imposed it becomes a Euclidean space everywhere uniform and also flat, in which bodies are mobile without change of form. If a metric varying with position is imposed, the expressions in this manifold of the metric relations of nature will become complicated, and the relations so changed be described as a modified set of laws.

The original non-metric continuum might be marked for instance by gradations of colour: the colour-scheme of Newton as developed by Young, Helmholtz, and Maxwell, is the standard example of a non-metric threefold extension.

May we not here have refined down to the unresolvable essence of space, as the mere possibility of descriptive continuity of threefold type which is an essential feature in our mental world? Within this *a priori datum* of threefold uncharted pure continuity we may construct types of charted spaces almost without limit, by imposing metrics of various types. Any particular space is not however determined by the system of coordinates of reference $p, q, \ldots$ but by the variable coefficients $f, g, h, \ldots$ of the imposed metric expressed as functions of them. But yet it is only under special conditions when it is uniform and flat that finite differences of these coordinates can be involved, this being part of the expression of the mobility of solid bodies in the space. It is in this narrower sense, that the system of coordinates is accidental, that relativity has now expelled general metric ideas of position. Would it be entirely wrong to assert that local or sectional relativity has been retained for nature, so far as this order of ideas extends, by transferring the laws of nature into a space-time frame which itself no longer possesses that quality?

The distinction has thus been made between an ultimate idea of space as mere threefold continuity, marked but uncharted, and the metric that may be imposed on it by which it becomes a frame fit for the purposes of description of nature. There is only one space: but its practical aspect, whether Euclidean or elliptic or merely heterogeneous, depends on the metric that we choose to assign to it. The metric would thus appear to pertain more closely to the order of nature for which it is to form the most convenient frame for description, than to space itself. For space is primarily bare threefold continuity; though a set of descriptive coordinates p, q, ... is unavoidable as a foundation of thought, any set is as valid as any other. For ultimately, the count or census of the points or marks that pervade the continuity and render it descriptively given to us, is the same count however it be made. May we say that the insistent, originally uncritical, notion of relativity reduces itself ultimately into this postulate, that as nature is presented to us, it is such that in mental operations we need attend only to one portion of the spacial continuity at a time? This makes the onefold time, or rather mere temporal succession as representable by the $\delta\sigma$ of Minkowski, the fundamental feature*, which however diverges spacially into a manifold: according to Hamilton long ago, algebra was the science of pure time.

In the above, space is given by a manifold array of points, of which the coordinates p, q, ... express one of the varieties of numerical census. Is then space-time absolute, or is it continually being constructed by physical science as it ranges over the void, for its own purposes, just to the extent that it may be required? May we say that the formless manifold is the fundamental feature, that the array of points and their census do not need to be definite in any respect *a priori*, and that the metric which is imposed on it and makes it into a definite working type of space is related to the physical world and so is to be regarded as evolved in connexion with our organic description or mapping of nature, and to be just as permanent?

What remains of the original notion of relativity after this sifting of ideas would then coincide with the principle of Newton, Faraday and Maxwell, originated by Descartes, that the operations of nature are elaborated in fourfold extension according to a scheme purely differential, that is by transmission from element to element

* The spacial sign here attached to $\delta\sigma^2$ is an accident of the order of exposition.

of the cosmos, in no case leaping across intermediate elements as action at a distance would imply. The early stage of formulation of the confused notion of relativity is the postulate that position and change of position are purely relative: the final solution is to abolish the idea of immediate *finite* change of position altogether. But that does not imply that a portion of the cosmos can evolve itself without constant interference from all the rest.

To a question as to what is gained by absorbing gravitation in space an answer would be that it need make no difference as regards gravitation; but if other relations of an assumed space-time fourfold (e.g. stress-tensor theory) have to go in also in a simple way, it may be convenient or even necessary to assist them by choosing a space which requires some alterations of the recognised laws of gravitation and, if these suggested discrepancies are verified, that may presumably have a claim to be the real type of space. The aim is not primarily to reduce gravitation to a quality of space,—perhaps is not even relativity, which has evaporated,—but is to get it out of Newtonian space and time into the mixed space-time fourfold which was strongly suggested by the form of the Maxwellian electrodynamic relations of free space, and would make that scheme valid for great velocities of convection beyond experience, even up to the speed of light.

An expansion of the Einstein ideas on general relativity has been worked out by H. Weyl (*Ann. der Physik*, 59, 1919) in which a further metric scale of vector character appears to be imposed on a non-uniform space-time, which has here been itself ascribed to the imposition of a Gauss-Riemann metric on the formless spacial threefold that is inherent in the mind. There would seem to be no *formal* obstacles to such piling up of metric upon metric, in an unlimited play of thought.

The physical analysis perhaps not very remote to this new elaboration of metric is, as I think Prof. Schouten remarks, a theory of an elastic aether in which at each point p, q, ... a vector displacement ξ, η, ... of the element of the medium is supposed, involving a strain and an elastic stress determined in terms of the strain by assigned laws. Only it is to be remembered that time is now in a fourth dimension, in which the historical world-process is all spread out once for all; so that the feature of elastic wave propagation becomes a static relation. The idea that the single fundamental electric vector is represented by a superposed metric is thus correlative with the usual dynamical hypothesis that electric force is a stress in an aether. It thus affords another illustration of this kind of speculation: the interlacing of space and time for purposes of electrodynamics having upset the historical development of dynamical principles on a Newtonian basis of separate space and time, order has to be re-constituted by piecing together a cognate analytical scheme on a symmetrical fourfold basis which *tries* to make no difference between them.

It is not improbable that these remarks merely turn over ground that has already been explored by cultivators of hyper-geometry. But it may be claimed that the interest of this range of ideas extends far beyond the analytical technique, and that their naïve expression in a form of language outside its conventions may prove to be helpful in other regions of speculation.

The argument above has been based on the supposition that the mathematical analysis must establish a complete correspondence, element for element, between the activities in the new space-time and in the Newtonian space and time. That however is not the case. There is a gravitational correspondence into which radiation and its rays do not enter. As regards the latter no conclusions could be drawn at all, except in the special circumstances in which the coordinate x_4 that stands nearest to time* does not enter explicitly into the quadratic expression determining the space. If that is postulated the equations of propagation of radiation have their solutions periodic as regards x_4 treated as a *quasi*-time, therefore every beam of radiation carries with it a scale of x_4 throughout its course†. Moreover, if the spacial quadratic contained δx_4 in a product term, the velocities of the waves of radiation in forward and backward directions would not be the same: their half difference would thus be the local velocity of the frame of reference in that direction. Where δx_4 does not occur in the first power, the frame of reference is thus fixed locally with respect to the waves of light and their assumed underlying uniform fourfold extension with regard to which they are propagated.

Thus, under these postulated circumstances of x_4 not occurring explicitly in $\delta\sigma^2$, the mere fact that isotropic vibratory radiation exists with its absolute velocity c is sufficient, not merely to determine absolute measurements both in space and time, at every locality in the extension, but also to determine the rate of motional change of the coordinates as referred to the uniform space-time of the radiation. It is gravitational correspondence, subject to this general control of the whole range of space-time by observations of light, with its isotropic and uniform qualities, that has led to verifiable conclusions. Cf. letter in *Nature*, Jan. 22, 1920: also *Monthly Notices R. Astron. Soc.*

* That is the one coordinate the square of whose differential is affected in $\delta\sigma^2$ with a negative sign, which marks it off from the others.

† It is the alleged measurement of this abstract coordinate x_4 by a travelling clock, which connotes a physical system, that is a main source of confusion.

We have absorbed gravitation into space and time by distorting the latter from its essential Newtonian uniformity: but there can be no illusion about the matter either way, for the theoretical measuring bar of the differential spacial theory is not our only instrument; in the practical world rays of light provide the essential isotropic measures, and the spectroscope is always available to reveal to us what spacial adjustments have been made, in relation to the underlying frame with regard to which the propagation of light is isotropic and has its standard absolute velocity. Light, instead of conforming to local relativity, imposes its own absolute space-time*.

The argument may be directed towards yet another type of conclusion, as follows. When change is made from Newtonian space and pure time to the uniform space-time fourfold, the equation of a straight path is altered from $\delta\int ds = 0$ to $\delta\int d\sigma = 0$. The free orbits in any field of force of potential energy function $-V$ can readily be altered so as to preserve continuity with this change, as above, that is, so that where V becomes negligible they tend to straight lines: they are then given by

$$\delta\int (d\sigma^2 + 2V dt^2)^{\frac{1}{2}} = 0.$$

The interpretation is at hand, to regard them as the analogues of straightest paths in a modified space-time, referred to a set of coordinates represented now by colourless symbols x_1, x_2, x_3, x_4 and given in terms of them by

$$\delta\sigma^2 = \delta x_1{}^2 + \delta x_2{}^2 + \delta x_3{}^2 - c^2 (1 - 2c^{-2} V)\, \delta x_4{}^2.$$

As $\delta\sigma^2$ does not here involve x_4 explicitly, the differential equations of propagation of free radiation, as expressed in this space-time in terms of these coordinates, have solutions involving the *quasi*-time x_4 only in the form $e^{\iota p x_4}$: therefore the radiation from any source, however far it has travelled, retains the same period in regard to x_4 as it had at the start. Around a radiating molecule the extension can be taken as practically uniform: therefore the interval of absolute time is equal to $(1 - c^{-2}V)\, \delta x_4$. It follows thus from the periodicity as regards x_4 that the periodic *time* of a ray alters as it travels so as to be proportional to $1 - c^{-2}V$. If the ray belongs to a definite molecular period at the Sun, it has changed when it reaches the Earth so as to agree no longer with that period as reproduced by a local vibrator.

All this is true only to the first order, but it applies to any law of potential, and is irrespective of any special energy-tensor theory. The point to be brought out is that if influence of gravitation on

* Prof. Eddington in a recent article, *Quarterly Review*, Jan. 1920, seems not to disagree with this conclusion: at any rate he contemplates the possibility of an aether.

spectral periods were definitely disproved, then it would appear that any hope of bringing orbits into direct relation with the electrodynamic space-time fourfold must be abandoned altogether*, on the threshold. This drastic conclusion is perhaps an argument in favour of the existence of the effect.

The other two verifiable effects, the influence on the planetary perihelia and the deviation of light passing near the Sun, arise in part from first order and in part from second order causes. Unlike the previous one, their exact verification is thus a test of the special theory of Einstein, or the equivalent Least Action formulation. Its original recommendation was that it restricts the universal forces of nature to the one type of gravitation: possibly it would be difficult to imagine ways in which there could be room for any different result.

* A formulation of the original Nordström type, starting from $\delta\int V d\sigma = 0$, is to some degree an exception.

On a Micro-voltameter. By C. T. R. Wilson, M.A., Sidney Sussex College.

[*Read* 19 May 1919.]

Experiments were described with a mercury voltameter, in which one elctrode consists of a sphere of mercury deposited on the end of a fine platinum wire and measured by means of a microscope. Quantities of electricity varying from a few hundred electrostatic units to about one coulomb may be measured by it. The almost instantaneous change of size of the drop when a capacity of one tenth of a microfarad, charged to 1 volt, is discharged through the instrument is easily observed. A magnet inserted in or removed from a coil connected to the terminals of the voltameter produces an easily measured effect. Experiments were also mentioned which suggest the possibility of its application in measurements of much smaller electrical quantities.

The self-oscillations of a Thermionic Valve. By R. WHIDDINGTON, M.A., St John's College.

[*Read* 19 May 1919.]

(*Abstract.*)

It has been found possible to produce oscillations of almost any frequency from a three electrode vacuum valve, without employing the usual capacity-induction circuits. Thus a valve with two suitable batteries, one in the anode circuit, another in the grid circuit, will produce quite powerful oscillations, whose frequency will be determined by the value of the grid potential.

The phenomenon can be explained by supposing that the oscillations are due to surges of mercury ions closing in on the filament from the grid with a frequency given by the approximate formula

$$n^2 = \frac{2e}{md^2} \cdot V,$$

where $\frac{e}{m}$ is the usual charge to mass ratio, d is the radial distance filament to grid and V is the positive grid voltage.

Experiments conducted so far indicate that the monatomic Hg ion with one live charge is mainly responsible.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1918—1919.

ANNUAL GENERAL MEETING.

October 28, 1918.

In the Comparative Anatomy Lecture Room.

PROF. MARR, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:
Mr C. T. R. Wilson.

Vice-Presidents:
Dr Doncaster.
Mr W. H. Mills.
Prof. Marr.

Treasurer:
Prof. Hobson.

Secretaries:
Mr Alex. Wood.
Mr G. H. Hardy.
Mr H. H. Brindley.

Other Members of Council:
Dr Shipley.
Prof. Biffen.
Mr L. A. Borradaile.
Mr F. F. Blackman.
Prof. Sir J. Larmor.
Prof. Eddington.
Dr Marshall.
Prof. Baker.
Prof. Newall.
Dr Fenton.

The following was elected an Associate of the Society:
G. A. Newgass, Trinity College.

The following Communications were made to the Society:

1. Proof of certain identities in combinatory analysis. By Prof. L. J. ROGERS and S. RAMANUJAN, B.A., Trinity College.

2. Some properties of $p(n)$, the number of partitions of n. By S. RAMANUJAN, B.A., Trinity College.

3. On the exponentiation of well-ordered series. By Miss D. WRINCH. (Communicated by Mr G. H. Hardy.)

4. On certain trigonometrical series which have a necessary and sufficient condition for uniform convergence. By A. E. JOLLIFFE. (Communicated by Mr G. H. Hardy.)

5. Some geometrical interpretations of the concomitants of two quadrics. By H. W. TURNBULL, M.A. (Communicated by Mr G. H. Hardy.)

6. On Mr Ramanujan's congruence properties of $p(n)$. By H. B. C. DARLING, B.A. (Communicated by Mr G. H. Hardy.)

7. On the correct Generic Position of *Dacrydium Bidwillii* Hook f. By B. SAHNI, M.A., Emmanuel College. (Communicated by Professor Seward.)

February 3, 1919.

In the Balfour Library.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

S. R. U. Savoor, B.A., Trinity College.
S. C. Tripathi, B.A., Emmanuel College.

The following was elected an Associate:

P. W. Burbidge, Trinity College.

The following Communications were made to the Society:

1. The Gauss-Bonnet Theorem for multiply-connected regions of a surface. By E. H. NEVILLE, M.A., Trinity College.

2. On the representations of a number as a sum of an odd number of squares. By L. J. MORDELL. (Communicated by Mr G. H. Hardy.)

3. On certain empirical formulae connected with Goldbach's Theorem. By N. M. SHAH and B. M. WILSON. (Communicated by Mr G. H. Hardy.)

4. Note on Messrs Shah and Wilson's paper entitled: On certain empirical formulae connected with Goldbach's Theorem. By G. H. HARDY, M.A., Trinity College and J. E. LITTLEWOOD, M.A., Trinity College.

February 17, 1919.

In the Comparative Anatomy Lecture Room.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

1. Note on an experiment dealing with mutation in bacteria. By Dr DONCASTER.

2. Electrical conductivity of bacterial emulsions. By Dr SHEARER.

3. The bionomics of *Aphis grossulariae,* Kalt., and *Aphis viburni,* Shrank. By Miss M. D. HAVILAND. (Communicated by Mr H. H. Brindley.)

4. (1) The conversion of saw-dust into sugar.
   (2) Bracken as a source of potash.

By J. E. PURVIS, M.A., Corpus Christi College.

5. Terrestrial magnetic variations and their connection with solar emissions which are absorbed in the earth's outer atmosphere. By S. CHAPMAN, M.A., Trinity College.

6. The distribution of Electric Force between two electrodes, one of which is covered with radioactive matter. By W. J. HARRISON, M.A., Clare College.

May 19, 1919.

In the Cavendish Laboratory.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

E. V. Appleton, M.A., St John's College.
W. G. Palmer, M.A., St John's College.
S. P. Prasad, B.A., Trinity College.

The following was elected an Associate:

Mrs Agnes Arber.

The following Communications were made to the Society:

1. (1) Use of Neon Lamps in Technical stroboscopic work.
   (2) The distribution of intensity along the positive ray parabolas of atoms and molecules of Hydrogen and its possible explanation.

By F. W. ASTON, M.A., Trinity College.

2. On a Micro-voltameter. By C. T. R. WILSON, M.A., Sidney Sussex College.

3. The self-oscillations of a Thermionic Valve. By R. WHIDDINGTON, M.A., St John's College.

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with references to the Transactions.

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PLATES.

CORRECTION.

Mordell, p. 250, line 5, after *conjugate numbers* insert *in the real conjugate fields.*

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

On the term by term integration of an infinite series over an infinite range and the inversion of the order of integration in repeated infinite integrals. By S. POLLARD, M.A., Trinity College, Cambridge. (Communicated by Prof. G. H. Hardy.)

[*Received* 1 January, 1920. *Read* 8 March, 1920.]

THE PROBLEM FOR INFINITE SERIES.

1. The problem to be solved is that of determining conditions under which the equation

$$\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\, dx = \int_a^{\infty} \sum_{n=1}^{\infty} u_n(x)\, dx, \tag{1}$$

is true. It is discussed in detail in Bromwich's *Infinite Series*, pp. 452–455, where various conditions are given. All these conditions will be found to involve uniform convergence, the fact being that the infinite integrals there considered are obtained as limits of Riemann integrals and, in the theory of the latter, considerations as to the validity of the equation

$$\lim_{m\to\infty} \int_a^b \sum_{n=1}^{m} u_n(x)\, dx = \int_a^b \sum_{n=1}^{\infty} u_n(x)\, dx, \tag{2}$$

almost always involve uniform convergence. Thus conditions for term by term integration over an infinite range, being built up from the conditions for term by term integration over a finite range, involve uniform convergence.

Now the condition of uniform convergence is by no means a necessary one: it occurs because of the lack of power in the methods of the Riemann theory. Much wider conditions can be obtained by the use of the Lebesgue theory. It is the object of this paper to give these.

CONDITIONS FOR PASSAGE TO THE LIMIT UNDER THE SIGN OF INTEGRATION, THE RANGE OF INTEGRATION BEING FINITE.

2. We give, for the sake of reference, the two principal elementary conditions.

(C 1) *If* $u_n(x)$ *is positive for* $a \leqslant x \leqslant b$; $n = 1, 2, 3 \ldots$, *then if either side of* (2) *is finite the equation holds, and if either side is infinite both are.*

(C 2) *If* $\left|\sum_{n=1}^{\nu} u_n(x)\right| < \psi(x)$ *for* $a \leqslant x \leqslant b$, $\nu = 1, 2, 3, \ldots$, *where* ψ *is summable in* (a, b), *then both sides of* (2) *exist and are finite and equal**.

RESUMÉ OF THEOREMS OF DOUBLE LIMITS.

3. As the use of double limits is fundamental in the theory about to be developed, we give a short summary of the results required.

(α) *If the double limit* $\lim\limits_{x\to\infty,\, y\to\infty} S_{x,y}$ *exists, and* $\lim\limits_{x\to\infty} S_{x,y}$ *exists for all sufficiently large* y; *then* $\lim\limits_{y\to\infty} (\lim\limits_{x\to\infty} S_{x,y})$ *exists and is equal to the double limit. Similarly for the limit* $\lim\limits_{x\to\infty} (\lim\limits_{y\to\infty} S_{x,y})$.

(β) *If* $S_{x,y}$ *is increasing in* x *and* y, *and any one of*

$$\lim_{x\to\infty,\, y\to\infty} S_{x,y}, \quad \lim_{x\to\infty} (\lim_{y\to\infty} S_{x,y}), \quad \lim_{y\to\infty} (\lim_{x\to\infty} S_{x,y})$$

exist; then all three exist and are equal.

(γ) *If* $S_{x,y}$ *can be expressed as the difference of two functions* $S'_{x,y}$, $S''_{x,y}$ *each of which is increasing in* x *and* y *and*

$$\lim_{x\to\infty,\, y\to\infty} (S'_{xy} + S''_{xy})$$

exists and is finite; then

$$\lim_{x\to\infty,\, y\to\infty} S_{x,y}, \quad \lim_{x\to\infty} (\lim_{y\to\infty} S_{x,y}), \quad \lim_{y\to\infty} (\lim_{x\to\infty} S_{x,y})$$

all exist and are finite and equal.

The condition (γ) is especially convenient when

$$S_{x,y} = \int_a^x \int_b^y f(\xi, \eta)\, d\xi d\eta.$$

* De la Vallée Poussin, *Cours d'analyse infinitésimale*, t. I., 3rd Ed., p. 264, theorems III and II.

For if
$$\lim_{x\to\infty,\,y\to\infty}\int_a^x\int_a^y |f(\xi,\eta)|\,d\xi d\eta$$
exists and is finite, then $S_{x,y}$ satisfies the condition of (γ). We have in fact
$$S_{x,y}=S'_{x,y}-S''_{x,y},$$
where
$$S'_{x,y}=\int_a^x\int_b^y |f(\xi,\eta)|\,d\xi d\eta,$$
$$S''_{x,y}=\int_a^x\int_b^y \{|f(\xi,\eta)|-f(\xi,\eta)\}\,d\xi d\eta,$$
and both $S'_{x,y}$, $S''_{x,y}$ are increasing in x and y and have a finite double limit—the former by hypothesis and the latter because
$$0\leqslant |f(\xi,\eta)|-f(\xi,\eta)\leqslant 2\,|f(\xi,\eta)|.$$

NOTE. The above results still hold when either or both of the variables x, y take only positive integral values.

DEFINITION OF INFINITE INTEGRALS.

4. Let $f(x)$ be any function which is summable in (a, X) for all X greater than a.

If
$$\lim_{X\to\infty}\int_a^X f(x)\,dx,$$
where the integral is taken in the sense of Lebesgue, exists and is finite, we say that
$$\int_a^\infty f(x)\,dx$$
converges and attribute to it the value of the limit.

This definition is evidently consistent with and more general than that usually given, where $f(x)$ is assumed to be integrable in Riemann's sense in (a, X). It has the special advantage of not being restricted to functions which are bounded in every (a, X). And we lose nothing by adopting it, as the two theorems on which the theory of infinite integrals rests, the first and second mean value theorems, are still true when we abandon the restriction that $f(x)$ is to have a Riemann integral and make only the assumption that $f(x)$ is summable*.

GENERAL THEOREMS.

5. I. *If the double limit* $\lim_{m\to\infty,\,X\to\infty}\int_a^X \sum_{n=1}^{m} u_n(x)\,dx$ *exists and*

(a) $\int_a^\infty u_n(x)\,dx$ *converges for all* n,

* *Ibid.* t. II. 2nd Ed., p. 53.

(*b*) $\sum_{n=1}^{\infty} u_n(x)$ *converges for* $X \geqslant a$,

(*c*) $\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx$, $\int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx$,

exist and are equal for all X; *then both sides of* (1) *exist and are equal.*

Proof. Write $\int_a^X \sum_{n=1}^{m} u_n(x)\,dx = S_{m,X}$,

and let $\lim_{m\to\infty,\, X\to\infty} S_{m,X} = S$.

Since $\int_a^{\infty} u_n(x)\,dx$ converges for all n

$$\lim_{X\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \text{ i.e. } \lim_{X\to\infty} S_{m,X},$$

exists for all m. Hence

$$\lim_{m\to\infty} (\lim_{X\to\infty} S_{m,X}) = S \qquad (3),$$

by (α).

In virtue of (*b*) and (*c*)

$$\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx$$

exists and is equal to $\int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx$.

Thus $\lim_{m\to\infty} S_{m,X}$ exists for $X \geqslant a$.

Hence $\lim_{X\to\infty} (\lim_{m\to\infty} S_{m,X})$ exists and is equal to S.

Taking $\lim_{m\to\infty} S_{m,X}$ in the form $\int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx$, we see that

$$\int_a^{\infty} \sum_{n=1}^{\infty} u_n(x)\,dx = S. \qquad (4)$$

And (3) and (4) give us our theorem.

II. *If* $\sum_{n=1}^{\infty} |u_n(x)|$ *converges for* $x \geqslant a$ *and the double limit*

$$\lim_{m\to\infty,\, X\to\infty} \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx$$

exists and is finite; then without further condition both sides of (1) *exist and are finite and equal.*

Proof. By (γ), if $\lim_{n\to\infty, X\to\infty} \int_a^X \sum_{n=1}^m |u_n(x)|\,dx$ exists, so does

$$\lim_{m\to\infty, X\to\infty} \int_a^X \sum_{n=1}^m u_n(x)\,dx.$$

Also
$$S_{m,X} = \int_a^X \sum_{n=1}^m |u_n(x)|\,dx \leqslant S,$$
for all X and m. But $S_{m,X}$ increases with X for each m. Hence

$$\lim_{X\to\infty} \int_a^X \sum_{n=1}^m |u_n(x)|\,dx$$

exists for each m and is less than S.

And therefore

$$\lim_{X\to\infty} \int_a^X |u_n(x)|\,dx = \lim_{X\to\infty} \left\{ \int_a^X \sum_{n=1}^m |u_n(x)|\,dx - \int_a^X \sum_{n=1}^{m-1} |u_n(x)|\,dx \right\},$$

exists for each m, *i.e.* $\int_a^\infty |u_n(x)|\,dx$ and therefore $\int_a^\infty u_n(x)\,dx$ converges for each m. This is (a) of (I).

Again, $S_{m,X}$ increases with m for each X.

Hence $\lim_{m\to\infty} S_{m,X}$ exists and is finite for each X. So from (C 1)

$$\int_a^X \sum_{n=1}^\infty |u_n(x)|\,dx$$

is finite. Thus $\sum_{n=1}^\infty |u_n(x)|$ is summable in (a, X). But

$$\left|\sum_{n=1}^m u_n(x)\right| \leqslant \sum_{n=1}^m |u_n(x)| \leqslant \sum_{n=1}^\infty |u_n(x)|,$$

and so by (C 2)

$$\lim_{m\to\infty} \int_a^X \sum_{n=1}^m u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^\infty u_n(x)\,dx,$$

exist and are finite and equal. This is (c) of (I). Now (b) of (I) is satisfied by hypothesis.

Thus all the conditions of (I) are satisfied and so both sides of (1) exist and are equal.

Deductions from the general theorems.

6. A. *If*
$$u_n(x) = \phi(x) f_n(x),$$
where
$$\sum_{n=1}^\infty f_n(x) \text{ converges for } x \geqslant a,$$
$$\left|\sum_{n=1}^\nu f_n(x)\right| < G, \text{ for } x \geqslant a \text{ and all } \nu,$$

and $\int_a^\infty |\phi(x)|\,dx$ *converges; then both sides of* (1) *exist and are finite and equal.*

B. *If either of*

$$\sum_{n=1}^{\infty} \int_a^\infty |u_n(x)|\,dx, \quad \int_a^\infty \sum_{n=1}^{\infty} |u_n(x)|\,dx,$$

exist and are finite; then both sides of (1) *exist and are finite and equal.*

C. *If*
$$\lim_{m\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx,$$

exist and are finite and equal, and

$$\Sigma \int_a^X u_n(x)\,dx$$

converges uniformly for $a \leqslant x$, *and each*

$$\int_a^\infty u_n(x)\,dx$$

converges; then both sides of (1) *exist and are finite and equal.*

D. *If* $\left|\sum_{n=1}^{\nu} u_n(x)\right| < \psi_X(x)$ *for* $a \leqslant x \leqslant X$ *and all* ν, *where* ψ_X *is summable in* (a, X), *and*

$$\Sigma \int_a^X u_n(x)\,dx$$

converges uniformly for $a \leqslant X$, X *being arbitrary, and each*

$$\int_a^\infty u_n(x)\,dx$$

converges; then both sides of (1) *exist and are finite and equal.*

D is a special case of C obtained by making use of (C 2). A, B, D may be regarded as generalisations of theorems A—C, pp. 452–455 of Bromwich's *Infinite Series.*

Proofs. A. If
$$m' > m,$$

we have
$$\sum_{n=m+1}^{m'} f_n(x) = \sum_{n=1}^{m'} f_n(x) - \sum_{n=1}^{m} f_n(x),$$

and therefore

$$\left|\sum_{n=m+1}^{m'} f_n(x)\right| \leqslant \left|\sum_{n=1}^{m'} f_n(x)\right| + \left|\sum_{n=1}^{m} f_n(x)\right| \leqslant 2G.$$

Hence

$$\left|\int_X^{X'} \sum_{n=m+1}^{m'} u_n(x)\,dx\right| \leqslant \int_X^{X'} \left|\sum_{n=m+1}^{m'} \phi(x) f_n(x)\right| dx$$
$$\leqslant \int_X^{X'} |\phi(x)| \sum_{n=m}^{m'} |f_n(x)|\,dx$$
$$\leqslant 2G \int_X^{X'} |\phi(x)|\,dx.$$

Now, given any positive number ϵ, we can, since $\int_a^\infty |\phi(x)|\,dx$ converges, find X_0 such that

$$\int_X^{X'} |\phi(x)|\,dx < \epsilon$$

for $X, X' > X_0$. Hence

$$\left|\int_X^{X'} \sum_{n=m+1}^{m'} u_n(x)\,dx\right| < \epsilon$$

for $X, X' > X_0$ and all m, m'. Thus the double limit

$$\lim_{m\to\infty,\, X\to\infty} \int_a^X \sum_{n=1}^{m} u_n(x)\,dx$$

exists. Further

$$\left|\sum_{n=1}^{\nu} u_n(x)\right| \leqslant |\phi(x)| \left|\sum_{x=1}^{\nu} u_n(x)\right| \leqslant G\,|\phi(x)|,$$

and $G\,|\phi(x)|$ is summable in (a, X) for all X greater than a. Thus by (C 2)

$$\lim \int_a^X \sum_{n=1}^{m} u_n(x)\,dx, \quad \int_a^X \sum_{n=1}^{\infty} u_n(x)\,dx$$

exist and are equal and finite.

All the conditions of (I) are now satisfied, and our theorem follows.

B. If we write
$$S_{m,X} = \int_a^X \sum_{n=1}^{m} |u_n(x)|\,dx,$$

then $\lim\limits_{m\to\infty,\, X\to\infty} S_{m,X}$ exists and is either finite or positive infinity.

In the first case our theorem follows at once by (II).

In the second case, both the repeated limits

$$\lim_{m\to\infty} (\lim_{X\to\infty} S_{m,X}), \quad \lim_{X\to\infty} (\lim_{m\to\infty} S_{m,X}),$$

are infinite. Suppose now that

$$\sum_{n=1}^{\infty} \int_a^{\infty} |u_n(x)|\,dx$$

exists and is finite. Then $\lim\limits_{m\to\infty}(\lim\limits_{x\to\infty} S_{m,X})$ exists and is finite, and we get a contradiction. And if

$$\int_a^\infty \sum_{n=1}^\infty |u_n(x)|\,dx$$

exists and is finite, then so does

$$\int_a^X \sum_{n=1}^m |u_n(x)|\,dx$$

for all X greater than a. Hence as in theorem II

$$\int_a^X \sum_{n=1}^\infty |u_n(x)|\,dx = \lim_{m\to\infty}\int_a^X \sum_{n=1}^m |u_n(x)|\,dx,$$

and it follows that

$$\int_a^\infty \sum_{n=1}^\infty |u_n(x)|\,dx = \lim_{X\to\infty}(\lim_{n\to\infty} S_{m,X}),$$

and we again get a contradiction.

Thus the first case alone is possible, and this is the case in which our theorem is true.

C. Write
$$\int_a^X u_n(x)\,dx = g_n(X),$$
$$\sum_{n=1}^m g_n(X) = S_{m,X}.$$

Since $\sum\limits_{n=1}^\infty g_n(X)$ converges uniformly for $a \leqslant X$, given $\epsilon > 0$ we can find N_0 such that

$$\Big|\sum_{n=N+1}^\infty g_n(X)\Big| < \epsilon, \qquad (X \geqslant a,\ N \geqslant N_0).$$

Thus
$$\Big|S_{m,X} - \sum_{n=1}^\infty g_n(X)\Big| < \epsilon, \quad (X \geqslant a,\ m \geqslant N_0).$$

Hence if $\lim\limits_{X\to\infty}\sum\limits_{n=1}^\infty g_n(X)$ exists and is finite, so does $\lim\limits_{m\to\infty,\,X\to\infty} S_{m,X}$, and the two are equal. Now

$$\begin{aligned}\Big|\sum_{n=1}^\infty g_n(X') - \sum_{n=1}^\infty g_n(X'')\Big| &\leqslant \Big|\sum_{n=1}^N g_n(X') - \sum_{n=1}^N g_n(X'')\Big| \\ &\quad + \Big|\sum_{n=N+1}^\infty g_n(X')\Big| + \Big|\sum_{n=N+1}^\infty g_n(X'')\Big| \\ &\leqslant \Big|\sum_{n=1}^N g_n(X') - \sum_{n=1}^N g_n(X'')\Big| + 2\epsilon.\end{aligned}$$

But since $\lim\limits_{X\to\infty} g_n(X)\ \left(=\int_a^\infty u_n(x)\,dx\right)$ exists and is finite for each n, so does $\lim_{X\to\infty} \sum_{n=1}^{N} g_n(X)$ and we can find X_0 such that

$$\left|\sum_{n=1}^{N} g_n(X') - \sum_{n=1}^{N} g_n(X'')\right| < \epsilon, \qquad (X', X'' \geqslant X_0).$$

Hence $$\left|\sum_{n=1}^{\infty} g_n(X') - \sum_{n=1}^{\infty} g_n(X'')\right| < 3\epsilon, \qquad (X', X'' \geqslant X_0)$$

and therefore, by the general principle of convergence

$$\lim_{X\to\infty} \sum_{n=1}^{\infty} g_n(X)$$

exists. Thus $\lim_{m\to\infty, X\to\infty} S_{m,X}$ exists. The other conditions of (I) are satisfied by hypothesis and our theorem follows.

The problem for infinite integrals.

7. We have to determine conditions under which the equation

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \int_b^\infty dy \int_a^\infty f(x, y)\, dx \tag{5}$$

is true. The methods adopted above apply almost without change and we get conditions almost identical with those already given. We quote them without proof, as the proofs can be made up immediately on the lines of those already given.

As regards the nature of $f(x, y)$, we assume throughout that $f(x, y)$ is summable in the region

$$(a \leqslant x \leqslant X,\ b \leqslant y \leqslant Y),$$

for all $X \geqslant a$, $Y \geqslant b$; so that, by Fubini's theorem*, the repeated integrals $\int_a^X dx \int_b^Y f(x, y)\, dy$, $\int_b^Y dy \int_a^X f(x, y)\, dx$ exist and are equal to the double integral.

General theorems.

8. I′. *If the double limit* $\lim_{X\to\infty, Y\to\infty} \int_a^X \int_b^Y f(x, y)\, dx dy$ *exists and*

(*a*) $\int_a^\infty f(x, y)\, dx$, *converges for* $y \geqslant b$,

(*b*) $\int_b^\infty f(x, y)\, dy$, *converges for* $x \geqslant a$,

(*c*) $\lim_{Y\to\infty} \int_a^X dx \int_b^Y f(x, y)\, dy$, $\int_a^X dx \int_b^\infty f(x, y)\, dy$,

* De la Vallée Poussin, *Intégrales de Lebesgue* etc., p. 53.

exist and are finite and equal for $X \geqslant a$; *then both sides of* (5) *exist and are equal.*

II′. *If the double limit* $\lim_{X\to\infty,\, Y\to\infty} \int_a^X \int_b^Y |f(x, y)|\, dx dy$ *exists and is finite; then without further condition both sides of* (5) *exist and are finite and equal.*

DEDUCTIONS FROM THE GENERAL THEOREMS.

9. A′. *If* $$f(x, y) = \phi(x)\, \theta(x, y),$$

where $$\left| \int_b^Y \theta(x, y)\, dy \right| < G \text{ for } x \geqslant a,\ y \geqslant b,$$

$$\int_b^\infty f(x, y)\, dy \text{ converges for } x \geqslant a,$$

and $\int_a^\infty |\phi(x)|\, dx$ *converges; then both sides of* (5) *exist and are finite and equal.*

B′. *If either of*

$$\int_a^\infty dx \int_b^\infty |f(x, y)|\, dy, \quad \int_b^\infty dy \int_a^\infty |f(x, y)|\, dx,$$

exist and are finite; then both sides of (5) *exist and are finite and equal*.*

C′. *If* $\lim_{Y\to\infty} \int_a^X dx \int_b^Y f(x, y)\, dy, \quad \int_a^X dx \int_b^\infty f(x, y)\, dy,$

exist and are finite and equal, and

$$\int_b^\infty dy \int_a^X f(x, y)\, dx$$

converges uniformly for $a \leqslant X$, *and*

$$\int_a^\infty f(x, y)\, dx$$

converges for $y \geqslant b$; *then both sides of* (5) *exist and are finite and equal.*

* This is de la Vallée Poussin's theorem. See Bromwich, *Infinite Series*, p. 457. The hypothesis given by Bromwich to the effect that *both* the integrals

$$\int_a^\infty f(x, y)\, dx, \quad \int_b^\infty f(x, y)\, dy,$$

are convergent is unnecessary, the existence of one (the one necessary to the existence of the repeated integral) is sufficient. That of the other is implied by the existence of the double limit, see Note 2.

D′. *If* $\left|\int_b^Y f(x, y)\, dy\right| \leqslant \psi_X(x)$ *for* $a \leqslant x \leqslant X,\ b \leqslant Y$,

where ψ_X *is summable in* (a, X), *and*

$$\int_b^\infty dy \int_a^X f(x, y)\, dx$$

converges uniformly for $a \leqslant X$, X *being arbitrary, and*

$$\int_a^\infty f(x, y)\, dx$$

converges for $y \geqslant b$; *then both sides of* (5) *exist and are finite and equal.*

10. *Note* 1. Results B are especially valuable, as they are easy to remember and convenient to apply. The power of the Lebesgue theory is shewn very clearly here in that by using it we are enabled to make the hypothesis which ensures the existence of the double limit* ensure also the passage to the limit under the sign.

Note 2. It is well to be precise as to the meaning of the word "exists" as used in connection with repeated Lebesgue integrals.

Suppose $f(x, y)$ is measurable in x, y in the rectangle

$$\begin{pmatrix} a \leqslant x \leqslant X \\ b \leqslant y \leqslant Y \end{pmatrix}.$$

We know that the function $f(x, y)$ considered as a function of x, is measurable in (a, X) for each y in (b, Y) a set of zero measure being excepted. It may not, however, be summable in (a, X), *i.e.*

$$\int_a^X f(x, y)\, dx$$

may not exist, for all y concerned. But, if $f(x, y)$ is summable over the rectangle, *i.e.*, if the double integral

$$\int_a^X \int_b^Y f(x, y)\, dx\, dy$$

exists; then it can be shewn that

$$\int_a^X f(x, y)\, dx$$

exists for all values of y in (b, Y), save possibly those of a set of measure zero.

* The existence of $\sum_{n=1}^{\infty} \int_a^\infty |u_n(x)|\, dx$ implies the existence of the double limit by (γ) of § 3; and in addition, by the use of (C 1) on $|u_n(x)|$, it will be found to imply the validity of the passage to the limit under the sign.

Now in the Lebesgue theory the integral of any summable function over a set of zero measure is zero, and consequently we may neglect a set of measure zero without affecting the value of the integral. Hence when we are faced with the problem of finding the value of a function which is indefinite or infinite at the points of a set of measure zero, we simply neglect these points and find the value of the integral over the residue. This is taken to be the value of the integral over the original set.

With the above convention it is true that, if

$$\int_a^X \int_b^Y f(x, y)\, dx dy$$

exists, so does

$$\int_b^Y dy \int_a^X f(x, y)\, dx dy,$$

although there may be points in (b, Y) at which the single integral

$$\int_a^X f(x, y)\, dx$$

does not exist.

It is always to be understood in dealing with repeated Lebesgue integrals (finite or infinite) that the inner integrals need only exist at all the points of the range of integration of the outer integral save those of a set of measure zero.

Let us apply the foregoing remarks to theorem B′. Suppose

$$\int_b^Y dy \int_a^X |f(x, y)|\, dx$$

exists. Then we know that

$$\lim_{X\to\infty,\, Y\to\infty} \int_a^X \int_b^Y |f(x, y)|\, dx dy$$

exists. It follows that

$$\int_a^X dx \int_b^Y |f(x, y)|\, dy,$$

considered as a function of Y, is bounded as Y tends to infinity. Thus

$$\lim_{Y\to\infty} \int_a^X dx \int_b^Y |f(x, y)|\, dy$$

exists and is finite. It follows that

$$\int_b^\infty |f(x, y)|\, dy$$

converges at all the points of (a, X) save possibly those of a set of measure zero, because if it did not the above limit would be infinite; and so, for our purposes

$$\int_a^X dx \int_b^\infty |f(x, y)| \, dy$$

exists.

Our convention has enabled us to infer the existence of the inner integrals from the existence of the double limit.

Note 3. A thorough treatment on different lines of the subject of this paper will be found in two papers by Prof. W. H. Young:

(1) "On the change of order of integration in an improper repeated integral," *Trans. Camb. Phil. Soc.*, XXI. p. 361.

(2) "The application of expansions to definite integrals," *Proc. Lond. Math. Soc.*, IX. (1910), p. 463.

In this paper we content ourselves with giving simple generalisations of well-known results with proofs depending on comparatively elementary theorems. There is no attempt to obtain comprehensive results.

Note on Mr Hardy's extension of a theorem of Mr Pólya. By EDMUND LANDAU. (Communicated by Prof. G. H. Hardy.)

[*Received* 10 *December* 1919. *Read* 26 *January* 1920.]

In a recent note in these *Proceedings** Mr Hardy has established an improved form of a theorem of Mr Pólya, viz.:

Suppose that $g(x)$ is an integral function, and $M(r)$ the maximum of $|g(x)|$ for $|x| \leqslant r$. Suppose further that $g(x)$ is an integer for $x = 0, 1, 2, 3, \ldots$, and that

$$M(r) = o(2^r).$$

Then $g(x)$ is a polynomial.

As Mr Hardy remarks at the beginning of his note, it is sufficient (after the analysis given already by Mr Pólya), to prove the one formula

$$n!\,2^{2n} \int_{-\pi}^{\pi} \frac{d\theta}{\prod\limits_{s=1}^{n} (2n - s\cos\theta)} = O(1).$$

Mr Hardy's proof of this formula may be replaced by the following shorter proof.

Since

$$n!\,2^{2n} \frac{1}{\prod\limits_{s=1}^{n} (2n-s)} = n!\,2^{2n} \frac{(n-1)!}{(2n-1)!} = 2 \,.\, 2^{2n} \frac{n!\,n!}{2n!} = O(\sqrt{n}),$$

it is enough to prove

$$\int_{-\pi}^{\pi} \psi(\theta, n)\, d\theta = O\left(\frac{1}{\sqrt{n}}\right),$$

where

$$\psi(\theta, n) = \prod_{s=1}^{n} \left(\frac{2n-s}{2n - s\cos\theta}\right).$$

Now

$$1 - \cos\theta = \frac{\theta^2}{2} - \frac{\theta^4}{24} + \ldots \geqslant \frac{\theta^2}{2} - \frac{\theta^4}{24} \geqslant \frac{\theta^2}{2} - \frac{10\theta^2}{24} = \frac{\theta^2}{12},$$

for $-\pi \leqslant \theta \leqslant \pi$, and

$$\frac{1}{1+y} = e^{-y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \ldots} \leqslant e^{-y + \frac{1}{2}y^2} \leqslant e^{-y + \frac{1}{2}y} = e^{-\frac{1}{2}y},$$

* Vol. XIX. (1919), pp. 60–63.

for $0 \leqslant y \leqslant 1$. Hence

$$\frac{1-\eta}{1-\eta\cos\theta} = \frac{1}{1+\dfrac{\eta}{1-\eta}(1-\cos\theta)} \leqslant \frac{1}{1+\eta(1-\cos\theta)}$$

$$\leqslant e^{-\frac{1}{2}\eta(1-\cos\theta)} \leqslant e^{-\frac{1}{24}\eta\theta^2},$$

for $0 \leqslant \eta \leqslant \frac{1}{2}$, $-\pi \leqslant \theta \leqslant \pi$; and so

$$\psi(\theta, n) = \prod_{s=1}^{n} \frac{1-\dfrac{s}{2n}}{1-\dfrac{s}{2n}\cos\theta} \leqslant e^{-\frac{\theta^2}{48n}\sum\limits_{s=1}^{n} s} = e^{-\frac{1}{96}(n+1)\theta^2} \leqslant e^{-\frac{1}{96}n\theta^2},$$

for $-\pi \leqslant \theta \leqslant \pi$ and $n = 1, 2, 3, \ldots$. Therefore

$$\int_{-\pi}^{\pi} \psi(\theta, n)\, d\theta \leqslant \int_{-\infty}^{\infty} e^{-\frac{1}{96}n\theta^2} d\theta = O\left(\frac{1}{\sqrt{n}}\right).$$

GÖTTINGEN, 4 *December* 1919.

Studies on Cellulose Acetate. By H. J. H. Fenton and A. J. Berry.

[*Read* 8 *March* 1920.]

The enormous demand for cellulose acetate and the serious shortage of acetone and certain other materials used in the manufacture of aeroplane dopes during the war originated a systematic research on cellulose acetate, especially as regards the behaviour of this material towards solvents and its chemical properties generally. The research has been pursued in a number of directions, the most important of which have been (*a*) substitutes for acetone as solvents, (*b*) the preparation of cellulose acetate and a study of the influence of the mode of preparation on the properties of the resulting product, and (*c*) the analytical chemistry of cellulose acetate. Most of our experiments, especially those relating to aeroplane dopes were necessarily of a technical character, but as a few results of general chemical interest have been obtained in the course of the work, we have thought it desirable to give a brief account of them in the present communication.

Solvents.

At the time of the difficulty caused by the serious shortage of acetone we were urged to discover efficient substitutes for this solvent for use in aeroplane dopes. It should, in passing, be observed that the properties of acetone make it an ideal solvent: its conveniently low boiling point, rapid solvent action on cellulose acetate, non-poisonous character, and, in normal times, cheap and abundant supply. All other liquids which have so far been suggested show a deficiency in some one or other of these particulars.

In August, 1917, we suggested that in case of emergency the three following solvents might be employed, viz. *acetaldehyde*, *acetonitrile*, and *nitrobenzene* with certain additions. Quite early in the investigation (October, 1916) we suggested *acetic acid* and *ethyl formate* as solvents. We also suggested the use of *cyclohexanone* and of beechwood *creosote* as substitutes for tetrachloroethane or benzyl alcohol as high boiling solvents. We were never informed whether these solvents were actually employed. It is remarkable that at considerably later dates, patents have been taken out for the use of both acetaldehyde and cyclohexanone as dope constituents. (British Patent 131647, July 4th, 1918 (acetaldehyde) and *Ibid.* 130402, February 15th, 1918 (Cyclohexanone).)

Our experiments have demonstrated that the destructive effect of acids upon fabrics is dependent on the strength of the acid in the physico-chemical sense. Hitherto it had been supposed that esters were objectionable as dope constituents on account of the possibilities of free acids resulting from hydrolysis. This, however, we found not to be the case. As far as weak acids only are concerned, tensile strength determinations gave excellent results; and fabrics doped with acetic acid as the principal solvent compared most favourably with others.

In our experiments a large number of liquids have been examined, not only from the purely practical point of view, but also from a desire to obtain if possible some information with regard to possible relationships between the nature of the liquid and its solvent action. It is of course impossible to define strictly the solubility of cellulose acetate in any given solvent owing to the colloidal nature of the products. The term "positive" is used in the following lists to imply that the liquid named has the property of gelatinizing cellulose acetate and subsequently converting it into a clear homogeneous "sol" without the aid of heat. All the results were obtained with a sample of the material which yields 54 per cent. of acetic acid on cold alkaline saponification.

Positive.

Liquid ammonia, liquid sulphur dioxide, liquid hydrogen cyanide, acetaldehyde, benzaldehyde, salicylaldehyde, acetone, methyl ethyl ketone, suberone, acetonitrile, propionitrile, formic acid, acetic acid, butyric acid, formamide, ethyl formate, ethyl oxalate, ethyl malonate, ethyl acetoacetate, aniline, phenylhydrazine, ortho-toluidine, piperidine, pyridine, tetrachloroethane, nitrobenzene*, nitromethane, cyclohexanone, guaiacol, chloroform*.

Although cellulose acetate is insoluble in water and in absolute ethyl alcohol, a mixture of these two liquids dissolves it freely on boiling. On cooling, however, precipitation takes place almost completely.

Negative.

Liquid air, liquid ethylene, liquid nitrous oxide, liquid hydrogen sulphide, benzene, toluene, turpentine, carbon disulphide, carbon tetrachloride, alcohol, ether, ethyl chloride, acetal, dimethyl acetal, nickel carbonyl, and many other liquids.

No general conclusion can be drawn as regards the chemical nature of a liquid and its solvent action on cellulose acetate. It is,

* Nitrobenzene requires certain additions. Chloroform had only a partial solvent action on this specimen of the material.

however, worthy of note that there appears to be some relation (with undoubted exceptions) between the dielectric constant and solvent action.

Influence of methods of preparation upon the properties of cellulose acetate.

The materials obtained by acetylating cellulose with acetic anhydride diluted with acetic acid in presence of various catalysts such as concentrated sulphuric acid, ferric sulphate, ortho toluidine bisulphate, may show considerable variations in properties depending upon the temperature, length of time of acetylation, and numerous other factors. When cellulose is acetylated and the product at once precipitated by water, it is nearly insoluble in acetone. Various methods have been adopted in order to convert the product so obtained into an acetone-soluble modification. The most widely used of these methods is that of Miles. This consists in heating the acetic acid solution of the cellulose acetate with water in rather greater quantity than that required to combine with the residual acetic anhydride. Sodium acetate may also be added to react with the catalyst if still present. The results are usually supposed to be due to chemical hydration.

In our experiments, cellulose was acetylated under the influence of various catalysts, and the effect of treatment by the Miles process was subjected to a critical examination. The most marked effects of this process are the changes in solubility in acetone and chloroform, most cellulose acetates being soluble in chloroform and insoluble in acetone before the treatment. This change in physico-chemical properties was found to be accompanied by a fall in the acetyl number. In one case the untreated cellulose acetate with an acetyl number of 60·9, yielded a product after the Miles process carried out at 100° for 48 hours with an acetyl number of 46·7. In another case when the treatment was carried out at the same temperature for 23 hours, the acetyl number fell from 60·5 to 50·4. The specific gravity of the cellulose acetate is also greatly reduced after the treatment. The influence on the heat test is not well marked but the decomposition point appears to be lowered somewhat.

In our view these results are to be ascribed to partial hydrolysis of the cellulose esters, not to hydration as is commonly supposed*. Apart from the diminution of the acetyl number already mentioned, we have carried out a series of experiments which have demonstrated that cellulose acetate does not form a hydrate. These

* Our view that the effect of the Miles process is essentially hydrolytic and not due to chemical hydration has been expressed subsequently by Ost (*Zeitsch. angew. Chem.* 1919, XXXII, 66, 76, and 82).

experiments originated in connexion with our determinations of the water contained in commercial samples of cellulose acetate. As is well known, the water is readily expelled by exposure of the material over concentrated sulphuric acid in a desiccator or by heating to 100°. It has frequently been supposed that the approximately constant proportion of 5 or 6 per cent. of water usually met with indicates a definite hydrate. In order to obtain positive information on this point, we determined the pressure-concentration relationship in the manner originally adopted by van Bemmelen in his well known researches on silicic acid (*Zeitsch. anorg. Chem.* 1896, XIII. 233). Weighed quantities of the material were exposed in a series of exhausted desiccators over sulphuric acid of various determined concentrations, and the corresponding vapour pressures were found by reference to Landolt and Börnstein's tables. The weights were found to be constant after 24–48 hours, and the pressure concentration relationship showed that no chemical hydration occurs. The phenomenon is to be regarded as one of adsorption, probably with subsequent diffusion, and is precisely similar to the absorption of water by cellulose itself. (Compare Masson and Richards (*Proc. Roy. Soc.* 1906, LXXVIII. 421), Trouton and Pool (*Ibid.* 1906, LXXVII. 292) and Travers (*Ibid.* 1906, LXXVIII. 21, and 1907, LXXIX. 204).)

Characterization and Analysis of cellulose acetate.

In the technical analysis of cellulose acetate, it is usual to examine the product by the heat test, solubility, acidity, and viscosity of the solutions, in addition to the determinations of acetyl (as acetic acid), copper reducing power, water, ash, and impurities. We have made an exhaustive investigation of various methods of carrying out these determinations, especially of the acetyl number, and have also carried out many ultimate analyses for carbon and hydrogen in some commercial specimens of the material.

The methods of determining the *acetyl* group may be classified under the two heads of *alkaline saponification* and *acid hydrolysis*. In the former the substance is saponified by excess of standard alkali, either at the ordinary temperature or at some higher temperature, and the excess of alkali determined by titration. In the latter, the substance is hydrolysed by strong acid, usually sulphuric or phosphoric, and the resulting acetic acid separated by steam distillation (Ost), or alcohol is added and the resulting ethyl acetate distilled off and collected in excess of standard alkali (Green and Perkin). The following is a summary of the principal results obtained in our experiments.

(1) *Cold alkaline saponification* (Ost and Katayama, *Zeitsch. angew. Chem.* 1912 (25), 1467). A known weight of the substance

is soaked with alcohol, then a measured volume of normal alkali is added and allowed to stand for 24 hours. The excess of alkali is then determined by standard acid. The mean result was 54 per cent. of acetic acid calculated for the dry substance.

(2) *Cold alkaline saponification* (Böeseken, van der Berg and Kerstjens, *Rec. Trav. Chim.* 1916, xxxv. 320). The substance is treated with strong aqueous potash for one or two days. A measured excess of normal hydrochloric acid is then added, the liquid then boiled for a moment to expel carbon dioxide and the resulting solution titrated with baryta water. The mean result calculated as above was 53·5 per cent. of acetic acid.

(3) *Hot alkaline saponification* (Barthelemy, *Moniteur Scientifique*, 1913 (3), II. 549). In this method the saponification is effected by heating the substance with normal soda for about 16 hours at 85°. The excess of alkali is then determined by titration with standard acid. Several experiments were made in which the conditions were subjected to considerable variations as regards length of heating and amount of excess of alkali. The extreme variations in the acetyl number calculated as above were 60·0 and 62·1 per cent.

(4) *Hot alkaline saponification* (Green and Perkin, *Trans. Chem. Soc.* 1906, 812). The saponification is carried out at the boiling point with semi-normal alcoholic soda and the excess of alkali titrated by standard acid. Our experiments yielded results of 60 per cent. of acetic acid, the extreme variations being 58·2 and 61·9 per cent. These numbers are in agreement with those of Green and Perkin (*loc. cit.*).

It is evident that the methods of hot alkaline saponification invariably yield results which are considerably higher than those obtained by cold saponification. There can be little doubt that the higher results are due to the action of alkali on the regenerated cellulose. Support to this contention was obtained by digesting two equal weights of filter paper with 50 c.c. of normal soda for two days, one at the ordinary temperature, the other at 85°. In the former case no alkali was consumed, while the heated product showed a loss of nearly 2 c.c. of normal alkali on titration.

(5) *Acid hydrolysis* (Ost, *loc. cit.*). The substance is first digested with 50 per cent. (by volume) sulphuric acid. After 24 hours the liquid is diluted considerably and the acetic acid separated by steam distillation, and titrated with baryta water. In our experiments phosphoric acid was substituted for sulphuric acid in order to avoid error due to possible formation of sulphur dioxide. The results varied from 51·5 to 55·0 per cent. of acetic acid.

(6) *Acid hydrolysis* (A. G. Perkin, *Trans. Chem. Soc.* 1905, 107). In this method the cellulose acetate is treated with ethyl alcohol and sulphuric acid, and the resulting ethyl acetate distilled into

excess of standard alkali. The ester is then saponified and the excess of alkali determined by titration. In our experiments phosphoric acid was used instead of sulphuric acid for the reason already mentioned. The results varied from 52·2 to 54·4 per cent. of acetic acid.

In our opinion, preference should be given to the method of cold alkaline saponification of Ost. Not only are the results more uniform, but they agree well with those obtained by acid hydrolysis. The latter methods are exceedingly tedious to carry out. We have also carried out some experiments with the use of hot baryta water as a saponifying agent and subsequent gravimetric determination of the barium, the results averaging 57–58 per cent. of acetic acid.

The materials met with in commerce known as cellulose acetate are most probably mixtures or solid solutions of various acetates, not definite chemical individuals. If, however, it were desired to represent cellulose acetate as a chemical individual, the results of our analyses of a number of specimens do not correspond with the formula of the triacetate $C_6H_7O_2\ (OCOCH_3)_3$ which is commonly supposed. They agree better with the formula of a pentacetyl derivative of $C_{12}H_{20}O_{10}$ and still better with that of a heptacetyl compound of $C_{18}H_{30}O_{15}$.

Thus

	Carbon	Hydrogen	Acetic acid
$C_6H_7O_2\ (OCOCH_3)_3$ requires	50·0	5·5	62·1 per cent.
$C_{12}H_{15}O_5\ (OCOCH_3)_5$,,	49·4	5·6	56·0 ,,
$C_{18}H_{23}O_8\ (OCOCH_3)_7$,,	49·2	5·64	53·8 ,,

Our most reliable results average carbon 49·2, hydrogen 5·5, and acetic acid 54 per cent.

Certain authors have stated that sodium ethylate may be used for the determination of acetyl in cellulose acetates. In investigating this reaction, we were surprised to find that ethyl acetate was always produced along with a yellow sodium derivative of cellulose. Quantitative experiments were performed in which the ethyl acetate was distilled into an excess of standard sodium hydroxide, and after saponification determined with standard acid. The residue was washed with alcohol to remove the unaltered sodium ethylate and this solution was titrated with standard acid. The residue was then treated with water to decompose the sodium compound and titrated also. It was found that the quantity of acetic acid converted into ethyl acetate to that becoming sodium acetate appears to depend to some extent on the proportion of sodium ethylate employed. The results can be explained, if the average commercial cellulose acetates are represented by the formula $C_{12}H_{15}O_5\ (OCOCH_3)_5$, by the equation:

$$C_{12}H_{15}O_5(OCOCH_3)_5 + C_2H_5ONa + 4C_2H_5OH$$
$$= C_{12}H_{19}O_9ONa + 5CH_3COOC_2H_5$$

which may be taken to represent the main reaction.

In support of this, the yellow sodium compound from a similar experiment, after thorough washing with alcohol, was digested for several hours in a reflux apparatus with excess of methyl iodide, and the methoxy group in the resulting product determined by Zeisel's method. The result obtained was 9·2 per cent. of methoxyl in agreement with that calculated for the formula $C_{12}H_{19}O_9OCH_3$.

The adsorption of basic dyestuffs by cellulose acetate.

Certain dyestuffs, such as gentian violet are adsorbed in considerable quantities from aqueous solution by cellulose acetate, the solid being coloured blue. Cellulose, it is true, also adsorbs the dye, but to a much smaller extent, and the solid becomes violet. This property may be utilized to identify unaltered cellulose in commercial preparations of cellulose acetate. Methyl orange gave negative results, but methyl red was adsorbed in considerable quantity, the solid becoming red. Free dimethylaminoazobenzene gave negative results, but the hydrochloride of this base was strongly adsorbed, the solid cellulose acetate assuming a pinkish yellow colour and the colour of the aqueous solution being almost completely discharged.

The authors desire to express their grateful thanks to Mr J. W. H. Oldham, M.A., of Trinity College, for much valuable assistance in connexion with this investigation. Mr Oldham has also carried out a large number of experiments on the influence of the mode of preparation upon the resulting properties of cellulose acetate, and it is hoped that his results when completed may form the subject of a future communication.

An examination of Searle's method for determining the viscosity of very viscous liquids. By Kurt Molin, Filosofie Licentiat, Physical Institute, Technical College, Trondhjem. (Communicated by Dr G. F. C. Searle.)

[*Read* 9 February 1920.]

§ 1. The determination of the coefficient of internal friction in very viscous liquids has been the object of measurements by many different methods. A review of these will be found in Reiger*. A number of more recent methods are given by Kohlrausch†, and among them is a method of Searle's‡. An examination of this method is the object of the present paper.

In his paper, "A simple viscometer for very viscous liquids," Dr Searle‡ gives an account of a viscometer he has constructed. The method consists in causing a vertical cylinder to rotate within a coaxal cylinder containing liquid, and in determining the angular velocity of the inner cylinder for a known value of the driving couple. The couple is produced by the weights of two loads acting on a drum by two threads. The time, T seconds, of one revolution of the cylinder is found, and the length, l cm., of the inner cylinder immersed in the liquid is observed.

Newton's statement is that

$$f = -\eta \frac{dV}{dn}, \quad \text{..........................(1)}$$

where f is the force per unit area which acts against the direction of motion and at right angles to the normal, n, to the surface, dV/dn is the velocity gradient, and η is the coefficient of viscosity. In this statement the motion of the liquid is supposed to take place parallel to a fixed plane. Treating the liquid as incompressible, and modifying (1), by substituting the rate of shearing for dV/dn, so as to suit the case of rotation, we obtain the following formula:

$$\eta = \frac{gD(a^2 - b^2)}{8\pi^2 a^2 b^2}\left(\frac{MT}{l}\right) = C\left(\frac{MT}{l}\right).$$

Here D is the effective diameter of the drum, a and b are the radii of the cylinders, and M is the mass of each of the two loads, which are required to move the inner cylinder with the constant angular velocity Ω, such that $2\pi/\Omega = T$.

* R. Reiger, *Ann. d. Phys.*, 19, p. 985, 1906.
† F. Kohlrausch, *Lehrbuch d. praktischen Physik*, xii. Aufl., p. 268.
‡ G. F. C. Searle, *Proc. Cambridge Phil. Soc.*, 16, p. 600, 1912.

The angular velocity of the liquid about the axis of the cylinders, at a distance r from the axis, is given by

$$\omega = \frac{2\pi}{T} \cdot \frac{b^2}{a^2 - b^2}\left(\frac{a^2}{r^2} - 1\right).$$

When $r = b$, the radius of the inner rotating cylinder,

$$\omega = \Omega = 2\pi/T,$$

and when $r = a$, the internal radius of the outer fixed cylinder, $\omega = 0$. This problem was first treated, not quite accurately, by Newton. The above results were given substantially by Stokes*, and are also given by Lamb† and by Searle‡.

The rate of shearing, $r\,d\omega/dr$, varies somewhat as r increases from b to a, as is shown by the formula

$$r\frac{d\omega}{dr} = -\frac{2\pi}{T} \cdot \frac{2a^2b^2}{(a^2 - b^2)\,r^2}.$$

We have only taken into account the friction between the coaxal cylindrical layers of the liquid and not the friction between the horizontal layers in proximity to the bottom surface of the movable cylinder, and have not considered the conditions that arise near that surface. In practice, only the lower end of the rotating cylinder is exposed to viscous action; Dr Searle makes an allowance for this end by writing

$$\eta = C \cdot \frac{MT}{l + k}, \qquad \text{.........................(2)}$$

where k is the length by which the height, l, of the liquid, in the simple theory, must be increased, in order that the increase of couple shall correspond to the viscous action in proximity to the end surface and the edge of the rotating cylinder.

Dr Searle gives a graphical method of determining k. The values of MT are plotted against l, and he says, "It will be found that the points lie on a straight line, which cuts the axis of l at a distance k from the origin." Dr Searle adds "If the corresponding total load hung from each thread be M grammes, it will be found, on repeating the observation with various loads, that MT is constant for a given level of liquid. This result confirms the fundamental assumption that the viscous stress at each point is proportional to the rate of shearing of the liquid."

* G. G. Stokes, *Brit. Ass. Report*, p. 539, 1898.

† H. Lamb, *Hydrodynamics*, Third Ed., p. 546, 1906.

‡ G. F. C. Searle, *loc. cit.*, p. 602. Compare C. Brodman, *Wied. Ann.*, 45, p. 163, 1892.

§ 2. In my experiments I used Dr Searle's viscometer, as supplied by Messrs W. G. Pye and Co., Cambridge*. I determined the viscosity of treacle, as Dr Searle refers to a determination of η for that liquid. I found $2b = 3{\cdot}74$ cm., $2a = 5{\cdot}01$ cm., and $D = 1{\cdot}95$ cm. Since $g = 982$ cm. sec.$^{-2}$ at Trondhjem, the constant C has the value

$$C = 3{\cdot}070 \pm 0{\cdot}035.$$

From the data given by Dr Searle, I find for the constant of the instrument used by him, $C_s = 3{\cdot}153$.

In my instrument the rate of shearing for radius r is given by

$$r\frac{d\omega}{dr} = -\frac{2\pi}{T}\cdot\frac{15{\cdot}80}{r^2}.$$

§ 3. To examine how MT depends upon M, when l is kept constant, six series of observations were taken with six values of l varying from 10·0 to 2·15 cm., and in each series M was made to vary from 5 to 205 grammes.

Since the viscosity of highly viscous substances diminishes very rapidly as the temperature increases, as was shown by Reiger† and by Glaser‡ for values of η of the magnitudes $4{\cdot}8 \times 10^5$ to $67{\cdot}2 \times 10^6$, and by Ladenburg§ for $\eta = 1{\cdot}3 \times 10^3$, great care must be taken to keep the temperature constant. The apparatus was, therefore, placed in a thermostat with electric temperature regulation, and a very constant temperature of 19·8° C. was maintained. The apparatus was left in the thermostat for 24 hours before the measurements were begun, and, during the short time a rotation trial was in progress, only the outer wooden door of the thermostat was opened, since one could see into the thermostat through the inner glass door. The final measurements were all carried out in the course of a day; the observations were made at intervals of about 10 minutes, so that the unavoidable disturbances of temperature, due to the manipulations, might have time to disappear.

In other respects the measurements were carried out in accordance with Dr Searle's|| instructions. The revolutions were timed by aid of a stop-watch and the times were taken for different numbers of revolutions with odd numbers up to 9, as well as the average time for one revolution. As no decrease in the time of a single revolution could be noticed as the rotation continued, the divergences from the mean lying within the limits of the errors

* Catalogue of Scientific Apparatus manufactured by W. G. Pye and Co., List No. 120, p. 39, 1914.

† R. Reiger, *loc. cit.*, p. 998.

‡ H. Glaser, *Ann. d. Phys.*, 22, p. 719, 1907.

§ R. Ladenburg, *Ann. d. Phys.*, 22, p. 309, 1907.

|| G. F. C. Searle, *loc cit.*, p. 603.

of observation, there was no observable acceleration. We may conclude that, even for the greatest values of M, the viscosity of the liquid remained sensibly constant, in spite of the fact that some potential energy was converted into heat.

The values of T found in these experiments are given in Table 1.

TABLE 1.

Time, in seconds, of one revolution of cylinder.

M grm.	$l=10{\cdot}0$ cm.	$l=8{\cdot}45$ cm.	$l=7{\cdot}65$ cm.	$l=5{\cdot}50$ cm.	$l=3{\cdot}30$ cm.	$l=2{\cdot}15$ cm.
5						129·0
7				114·6		50·7
10	120·4	108·7	100·2	71·3	44·7	
12	93·3	85·0	77·0	54·3		
15	71·5	65·3	59·0	42·3	26·6	19·5
20	52·8	47·0	41·7	29·7		
25	41·1	35·9	32·2	23·6	14·5	11·1
30	33·3	29·6	26·4	19·0		
35	28·8	24·8	22·2	16·3	10·0	
40	25·1	21·7	19·2	14·0		
45	22·0	19·0	17·1	12·3		
55	17·9	15·2	13·6	10·1	6·3	4·6
65	14·9	12·6	11·4	8·3		
75	12·8	10·9	9·8	7·2	4·5	3·4
105	9·1	7·7	7·0	5·1		
155	6·1	5·2	4·8	3·5		
205	4·6	3·9	3·6			

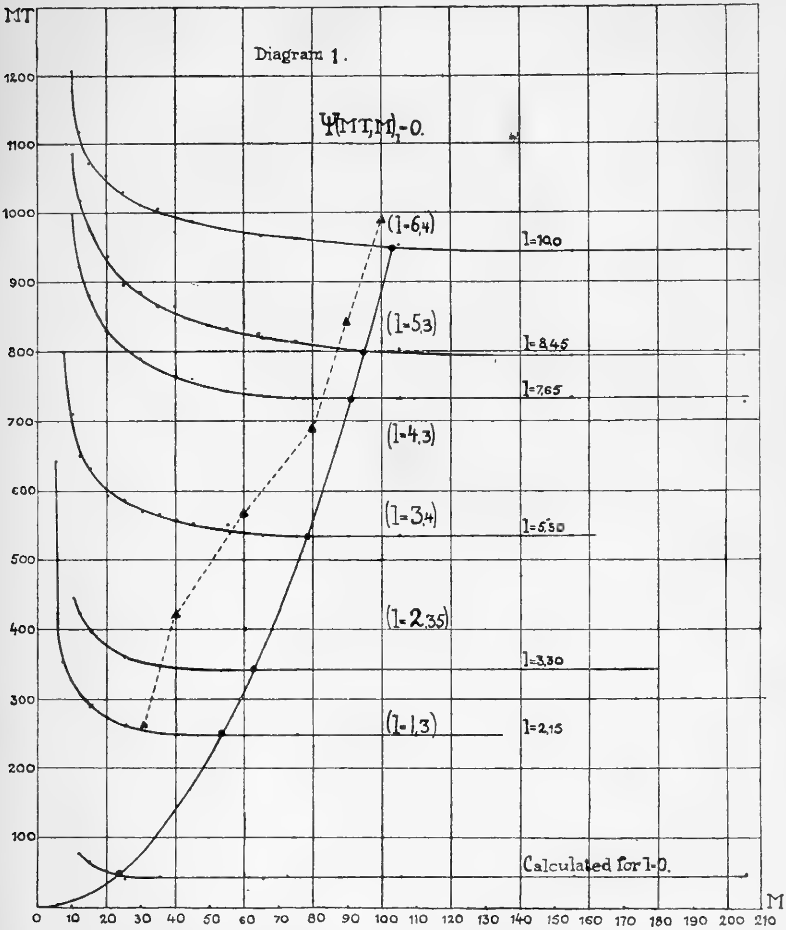
The results have been plotted in the form of six curves each for one value of l, as in Diagram 1. The curves are represented in the form $\Psi\,(MT, M)_{l=\text{const.}} = 0$.

From the diagram it is clear that the function $\Psi\,(MT, T)_l = 0$ does not represent a family of straight lines parallel to the M-axis, and that each of the six curves has a hyperbolic appearance. When M approaches a certain lower limit M_0, MT tends to infinity. The area covered by the group of curves can be divided by a parabolic boundary curve into two departments, in one of which MT is sensibly constant for a given value of l.

§ 4. I have, further, examined how MT depends upon l, when M is kept constant, and have found that the function

$$F\,(MT, l)_{M=\text{const.}} = 0$$

represents, not a single straight line*, but a family of approximately straight lines. Each line can be represented by the equation $MT = \alpha l + \beta$. For this group of curves $\partial\,(MT)/\partial l$ tends to a definite value as M increases, i.e. the curves approach a certain border line



which is comparable with Searle's straight line. The coefficients α and β have been calculated for each line by the method of least squares†, using the formulae

$$\alpha = \frac{\Sigma l\,.\,\Sigma MT - 6\Sigma l\,.\,\Sigma lMT}{(\Sigma l)^2 - 6\Sigma l^2}, \quad \beta = \frac{\Sigma l\,.\,\Sigma lMT - \Sigma MT\,.\,\Sigma l^2}{(\Sigma l)^2 - 6\Sigma l^2},$$

* G. F. C. Searle, *loc. cit.*, p. 604.

† F. Kohlrausch, *Lehrbuch d. praktischen Physik*, p. 13, 1914.

the various observations being regarded as having equal weights. The values of α and β have been thus calculated for seven different lines, and the results are given in Table 2.

TABLE 2.

Values of α, β and k.

M grm.	α	β	k cm.
12	105·86	78·40	0·740
15	102·64	65·61	0·639
20	99·29	53·54	0·539
35	108·80	47·79	0·439
65	90·78	43·83	0·482
75	90·27	43·63	0·483
105	89·79	45·47	0·506

When $l = 0$, then $MT = \beta$, and Table 2 shows how β varies with M. The curve thus extrapolated for $l = 0$ is marked "Calculated for $l = 0$" in Diagram 1.

When $MT = 0$, we have $k = | \, l \, | = | \, \beta/\alpha \, |$, where k is the correction for the lower end of the rotating cylinder.

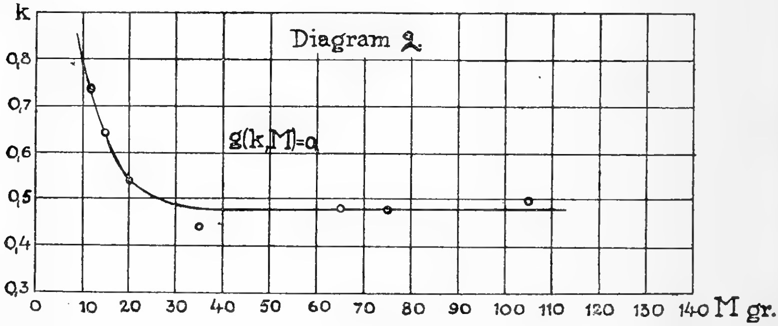


Diagram 2 shows how k depends upon M.

The facts here recorded show that equation (2) should be replaced by

$$\eta = C \frac{M_i T}{l + k_i}, \qquad \text{.........................(3)}$$

where k_i is the value of k corresponding to the load M_i.

If the value of k_i corresponding to M_i is read off from the curve of Diagram 2, the viscosity η can be calculated by equation (3). The values of k found from Diagram 2 have been used in forming Table 3.

TABLE 3.

Values of $M_iT/(l+k_i)$.

M grm.	$l=10{\cdot}0$ cm.	$l=8{\cdot}45$ cm.	$l=7{\cdot}65$ cm.	$l=5{\cdot}50$ cm.	$l=3{\cdot}30$ cm.
10	111·7	118·0	118·5	113·4	109·5
12	104·2	111·0	110·0	104·5	
15	101·2	108·0	106·8	103·2	101·0
20	99·3	104·5	102·0	98·6	
25	97·9	100·1	99·1	98·5	95·1
30	96·5	99·0	97·2	95·4	
35	95·6	97·0	95·6	94·7	92·5
40	95·0	97·0	94·5	93·7	
45	94·5	96·0	94·3	92·5	
55	93·3	94·0	91·5	92·4	91·6
65	92·4	92·0	90·7	89·7	
75	92·0	91·7	90·4	89·7	
105	91·1	90·2	90·4	90·1	
155	90·3	89·2	90·6	89·6	
205	90·0	89·3	89·3		

From Table 3 it appears that the area in Diagram 1 in which equation (3) holds good is restricted to that part of the diagram to which the parabolic boundary curve is convex. From the values of MT derived from Table 1 and plotted in Diagram 1, the equation of the parabola is found to be $M^2 = 11{\cdot}26\,(MT)$. I have not been able to give the parabola any definite physical interpretation, and it ought to be regarded as representing a diffuse limit region. But it is only when we pay regard to this, that we obtain values of η differing from each other by amounts lying within the limits of experimental error*. To make a comparison with the values of M and l which Dr Searle has used, I have, in Diagram 1, plotted (the broken line) his values of MT† (strictly speaking, MT/C, which are comparable in magnitude with my values of MT) against M.

Dr Searle has pointed out to me that the effect shown in Diagram 1 might conceivably be due to pivot friction. I have carefully considered this possibility. Before the liquid was put into the apparatus, I adjusted the pivots so that the rotation due

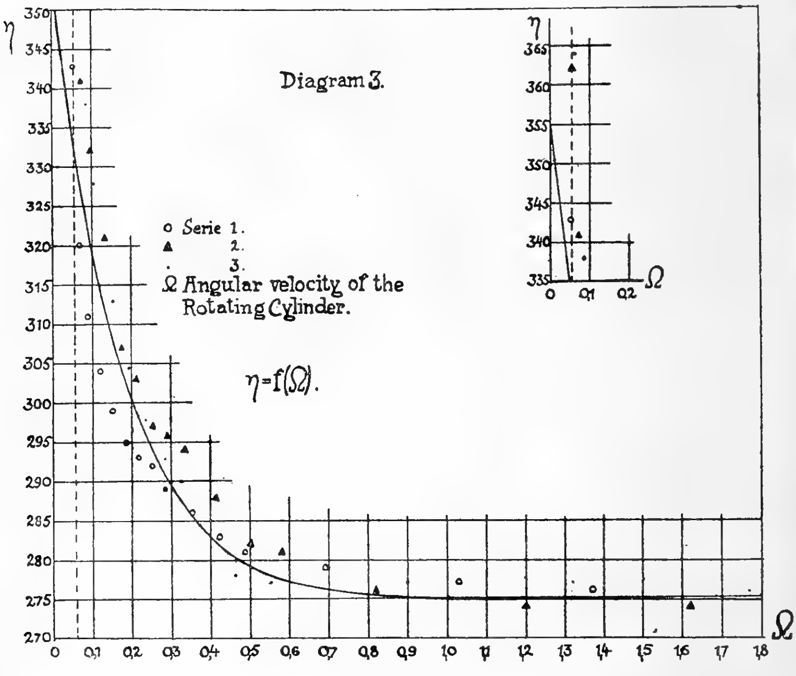
* Compare G. F. C. Searle, *loc. cit.*, Table II, p. 606.
† Calculated from Table 1, G. F. C. Searle, *loc. cit.*, p. 605.

to the weights of the two empty pans (5 grm. each) was so rapid that I was hardly able to measure, for instance, $3T$ by using a stop watch. I have, therefore, not been able to take account of any pivot friction. This cause of error would, at any rate, produce effects much smaller than those actually found.

§ 5. From the results for $M = 205$ grm. given in Table 1 we find the mean value

$$\eta = 274{\cdot}7 \text{ dyne sec. cm.}^{-2},$$

for the temperature of $+ 19{\cdot}8°$ C. To show how η depends upon



the angular velocity $\Omega = 2\pi/T$, the values of η and Ω, obtained from the first three series, have been plotted in Diagram 3. The curve drawn among the plotted points suggests that the relation between η and Ω can be expressed in the form

$$\eta = 274{\cdot}7 + \phi \exp(-\lambda\Omega^x).$$

To find the constants ϕ, λ and x, I considered the equation

$$\log_\epsilon(\eta - 274{\cdot}7) = \log_\epsilon \phi - \lambda\Omega^x. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

When the values of $\log_\epsilon(\eta - 274{\cdot}7)$ were plotted against Ω, the curve was roughly a straight line. Hence x may be taken as unity, and thus the number of constants to be found is reduced to two. By the method of least squares, I obtained $\log_\epsilon \phi = 4{\cdot}375$ and $\lambda = 5{\cdot}694$, and thus

$$\eta_{19\cdot 8} = 274{\cdot}7 + 79{\cdot}44\,\epsilon^{-5\cdot 694\,\Omega}. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Equation (5) expresses the results of the observations when Ω exceeds 0·1, but not for smaller values of Ω.

§ 6. Experiments carried out at different temperatures showed that the curves representing the function

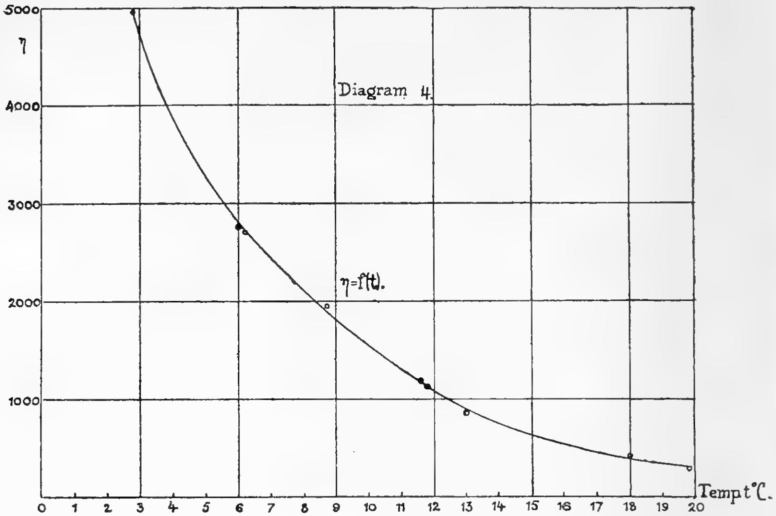
$$\Psi(MT, M)_{l=\text{const.}} = 0$$

are of the same character as those given in Diagram 1. Table 4 gives the values of η found for various temperatures. In these experiments l was 10·0 cm.; and, at each temperature, six different loads were used, in order that I might be able to decide with certainty that the values of M, used in calculating the value of η for each temperature, lay in the area to the right of the parabolic boundary line of Diagram 1. The same value of k, viz. the limiting value 0·48 cm. shown in Diagram 2, was used in calculating the various values of η. These values are not claimed to be exact. In these experiments it was very difficult to keep the temperature constant during each series of observations, and thus a determination of k for each temperature was out of the question. From the curve of the function $\eta = f(t)$, shown in Diagram 4, it follows that $|\,d\eta/dt\,|$ rises rapidly as η increases; this tallies with what was said above.

TABLE 4.

Values of η at various temperatures.

Temp. $t°$ C.	η Dyne sec. cm.$^{-2}$	Temp. $t°$ C.	η Dyne sec. cm.$^{-2}$
19·8	274·7	8·75	1950
18·0	415	6·2	2700
13·0	860	6·0	2750
11·8	1140	2·8	4970
11·6	1200		



§ 7. I thought it would be interesting to compare the results given by Searle's method with those obtained by Poiseuille's method. The utility of the latter method for very viscous liquids* is proved by the investigations of Kahlbaum and Räber† for values of η in the neighbourhood of 40, and by Ladenburg‡ for $\eta = 1{\cdot}3 \times 10^3$. Fausten§ has found that the length of the discharge tube must exceed 45 cm., if the simple Poiseuille formula

$$\eta = \pi g h R^4 \rho^2 t / 8 L m$$

is to represent actual facts. In the formula

h = Height of liquid corresponding to difference of pressure between ends of tube.

R = Internal radius of tube. L = Length of tube.

ρ = Density of liquid ($= 1{\cdot}4103 \pm 0{\cdot}0003$ grm. cm.$^{-3}$ at $19{\cdot}8°$ C.).

m = Mass of liquid discharged. t = time of discharge.

For shorter tubes, Hagenbach's* correction must be employed; otherwise the value obtained for η will be too high. As the liquid flows out into the air in an even jet, it carries kinetic energy with it; in order to allow for this, the value of η given by Poiseuille's

* H. Glaser, *Erlangen Diss.*, 1906.
† G. W. A. Kahlbaum and S. Räber, *Acta Ac. Leop.*, 84, p. 204, 1905.
‡ R. Ladenburg, *Ann. d. Phys.*, 22, p. 298, 1907.
§ A. Fausten, *Bonn. Diss.*, 1906.

formula must be multiplied, according to Hagenbach*, by a correcting factor slightly less than unity. As the thermostat could only accommodate tubes shorter than 45 cm., Hagenbach's correction was calculated, but was found to be negligible. Ladenburg† points out that both Hagenbach's and Couette's corrections to Poiseuille's formula can be entirely ignored for liquids such that η is of the magnitude $1{\cdot}3 \times 10^3$.

The discharge vessel consisted of a wide glass cylinder; through the bottom of this was bored a hole through which the discharge tube was connected with the interior of the cylinder. The whole apparatus was placed in the thermostat and the same temperature, 19·8° C., was maintained as was used in the earlier experiments. When a tube whose internal radius was about 0·26 cm. was used, the liquid did not issue in a continuous jet but in drops. The values obtained for η are given in Table 5. The mean value is $\eta = 271{\cdot}1$. The value obtained by Searle's method, viz. 274·7, differs from that obtained by Poiseuille's method by 1·3 per cent.; the agreement may be regarded as good.

TABLE 5.

Values of η by Poiseuille's method.

R cm.	l cm.	h cm.	m grm.	t sec.	η
0·3168	46·48	49·36	54·421	1790	269·9
		49·93	53·568	1757	272·3

§ 8. The influence of the base of the rotating cylinder can be eliminated, without determining k, by using the relation‡

$$\eta = C \,.\, \frac{M_1T_1 - M_2T_2}{l_1 - l_2} = C\gamma,$$

provided that the points corresponding to M_1T_1 and M_2T_2 lie to the right of the parabolic boundary line in Diagram 1. If we put $l_1 = 10{\cdot}0$ cm., we obtain the results given in Table 6.

* F. Kohlrausch, *Lehrbuch d. praktischen Physik*, pp. 264—269, 1914.
† R. Ladenburg, *loc. cit.*, p. 298.
‡ Compare C. Brodman, *loc. cit.*, p. 163.

TABLE 6.

Values of γ.

MT	944	804	734	537	342	251
l	10·0	8·45	7·65	5·50	3·30	2·15
γ		90·2	89·3	90·5	90·0	88·4

When the various values are given the same weight, the mean value of γ is 89·7, and then $\eta = 275{\cdot}4$.

§ 9. Diagram 3 and formula (5) show that η cannot be regarded as independent of Ω unless Ω exceed a certain value, in this case 0·9. Since Ω is related to the rate of shearing $r\,d\omega/dr$, according to the formula

$$r\frac{d\omega}{dr} = -\,\Omega\,.\,\frac{15{\cdot}80}{r^2},$$

it follows that η is a function of the rate of shearing. Hence, the assumption on which formula (1) is based, viz. that η is independent of the rate of shearing, seems to be unjustifiable for small values of the rate of shearing, at least in the case of the highly viscous liquid used in these experiments.

Preliminary Note on Antennal Variation in an Aphis (Myzus ribis, *Linn.*). By Maud D. Haviland, Fellow of Newnham College. (Communicated by Mr H. H. Brindley.)

[*Read* 8 March 1920.]

In 1918, during an investigation of the life-history of the Red Currant Aphis, *Myzus ribis*, Linn., it was observed that considerable variation occurred in the antennae of the winged parthenogenetic females; and the evidence pointed to the conclusion that this variation was induced by the food[3]. Antennal variation in certain Aphididae has been studied by Warren[5], Kelly[4], Ewing[2] and Agar[1]. Warren's experiments on *Hyalopterus trirhodus* showed some diminution of the correlation co-efficient in passing back from parent to grandparent. Kelly, for *Aphis rumicis*, considered that somatic variations of the parents were not inherited by the offspring. Ewing, who bred eighty-seven generations of *Aphis avenae*, concluded that the variations were not transmitted to the offspring.

Agar found some evidence of a partial inheritance of individual variations in *Macrosiphum antherini*, but he showed that this might be due to causes other than true inheritance.

Myzus ribis is a common pest of red currant bushes. The sucking of the aphides upon the leaves tends to cause red galls or blisters, within which the plant lice continue to feed and reproduce. The fifth and sixth antennal segments of the winged parthenogenetic females normally bear two sense organs of unknown function—one on the distal third of Seg. v., the other on the proximal third of Seg. vi. It was observed in 1918 that, in individuals reared on red blistered leaves, these sensoria were placed comparatively close to the articulation of Segs. v. and vi. On the other hand, if the aphides were fed upon green unblistered leaves, the sensoria were placed further away from the articulation.

For the sake of brevity, the first type of antenna will be referred to hereafter as the Red (or R) type, and the second as the Green (or G) type; but every degree of transition may exist between the two extreme types.

The experiments of 1918 were incomplete, and were conducted with a polyclonal population. They were repeated in 1919 with a monoclonal population, but the results are still far from being conclusive owing to the small numbers available in some generations. Only the winged forms show the required character. The production of these forms is probably governed by environmental factors which at present are imperfectly understood, and, for some reason, in the population used in 1919, it was unusually low. It is hoped to repeat and extend the range of the experiments in 1920.

The character chosen is the distance between the sensoria of antennal segments v. and vi. and the articulation of these two segments, expressed as the percentage of the width of the head between the eyes. The ratios are shown separately for each segment, with a dividing line to represent the articulation. Thus $\frac{19}{8}$ denotes that $\dfrac{\text{Seg. vi.} = 19\% \text{ of the head-width}}{\text{Seg. v.} = 8\% \text{ of the head-width}}$.

Each generation is designated by combinations of two letters: R (= red leaves) and G (= green leaves) and numerals, which express its complete ancestry. Thus R_2G_2 denotes the fourth generation from the fundatrix of the population, and the F_1 generation after transference to Green leaves after two consecutive generations on Red blistered leaves. In the transferred generations, the aphides were removed to the new environment when less than twelve hours old. The individuals for transference were selected wholly at haphazard. Thus, if a brood mother R_2 gave birth to four young in the day, two were transferred to red blistered leaves, and two to green leaves, and so on in equal numbers from day to day.

The pure Red (RRR, etc.) lines, and pure Green (GGG) lines were used as controls. The latter unfortunately became extinct in the third (G_3) generation. Hence for later generations the next longest unbroken line on green leaves (R_2G_2, etc.) had perforce to be taken as the control, though as it had been fed for the first two generations upon red leaves, it cannot be regarded as wholly satisfactory. In Table 1, the curves of error of the ratios of generations R_2, R_4 and R_2G_4 are shown. R_2 is the common ancestral generation. The mode of the curve of R_4 tends to shift to the left, *i.e.* the ratios of the antennal segments to the head-width are smaller. For the sake of clearness, in the graph only the curve of R_4 is shown, but those of R_3, R_5 and R_6, though with a smaller number of individuals, are almost identical with it. The curves of the ratios of R_2G_1 and R_2G_2 are very similar to their red controls. The R_2G_3 generation produced very few winged individuals, but these indicate a somewhat greater range of variation in Seg. vi. The curve of R_2G_4, as shown in the graph, has a marked tendency to shift to the right, indicating that the ratio of the antennal joints to head-width has increased, and this tendency is maintained in the succeeding generations, R_2G_5 and R_2G_6. The position in the generation series does not account for the change in the antennal structure, for the modes for the six Red generations are nearly identical.

So far we have considered only the modes. The mean ratios of the different generations are dealt with in the succeeding tables.

Table 2 shows the mean ratios of the successive generations in four lines of descent, including the red and green controls. The extinction of the green control line was unfortunate, and in future experiments it will be very desirable to obtain a pure green line. At present the explanation that suggests itself of the variation of the R_2G_2... line is that the influence of red feeding persists for at least two, and probably three generations after removal to different food, and this is somewhat confirmed by the R_4G_1... etc. line.

Tables 3, 3*a*, 4, 4*a* and 5, 5*a*, give the effect of transference upon the mean ratios of the first, second, and third generations respectively, and below each is an analysis of the ratio of each segment, indicating its increase or decrease over previous generations and the controls.

Examination of the figures seems to show that the ratios of the first generation after transference vary irrespectively of the parental ratio.

In transference to Red, the ratio of Seg. v. increases over that of the parental ratio, but in Seg. vi. it decreases (Table 3). In transference to Green, the results for both segments are quite inconclusive as regards the parental ratio (Table 3*a*). In the second generation after transference to Red, the results are likewise inconclusive for both segments (Table 4). After transference to Green, the ratio of Seg. v. shows a tendency to rise above, and Seg. vi. a tendency to fall below, the parental and grandparental ratios (Table 4*a*).

In the third generation after transference to Red, the ratio of Seg. v. rises above the ancestral ratios, and that of Seg. vi. falls (Table 5). After transference to Green, the ratio of Seg. v. rises above those of the ancestral generations, and that of Seg. vi. rises in one case and falls in the other (Table 5*a*).

These results are inconclusive, but examination of the control ratios shows that, with occasional exceptions, the ratio of a generation with a mixed ancestry tends to rise above that of the Red control, but remains below that of the Green. Many more experiments in transference are required, and a much larger number of individuals must be examined before any conclusion can be reached; but at present the evidence suggests that the antennae of *Myzus ribis* are modified according to the food supplied, and that the effect induced by feeding in one generation is discernible in the succeeding three or four generations. It is difficult otherwise to explain the difference between the ratios of R_6 and R_2G_4, and between R_6 and R_4G_3, which, translated into the terms of human relationship, would be third cousins, and first cousins once removed, respectively, for all were produced by parthenogenesis, and, except for the food, reared side by side under identical environmental conditions.

TABLE 1. *Curves showing the ratio of the distance of the sensoria from the articulation of antennal Segments V and VI to the width of head. The lower curves refer to the fifth, and the upper to the sixth segment.*

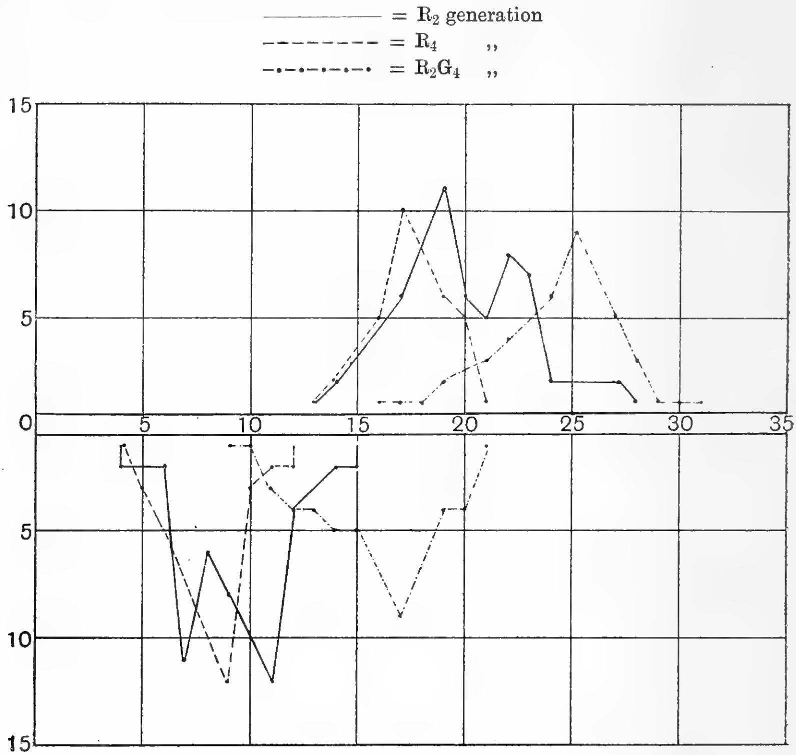


TABLE 2. *Mean ratios of the successive generations of the lines, G_2 ..., R_3 ..., R_4G_1 ..., R_2G_1 ..., and $R_2G_3R_1$ ....*

II	III	IV	V	VI	VII	VIII
G_2 $\frac{22}{8}$	G_3 $\frac{25}{19}$	—	—	—	—	—
R_2 $\frac{20}{9}$	R_3 $\frac{18}{8}$	R_4 $\frac{17}{8}$	R_5 $\frac{14}{9}$	R_6 $\frac{18}{8}$	—	—
,, ,,	,, ,,	,, ,,	R_4G_1 $\frac{22}{7}$	R_4G_2 $\frac{19}{9}$	R_4G_3 $\frac{29}{15}$	—
,, ,,	R_2G_1 $\frac{19}{7}$	R_2G_2 $\frac{18}{10}$	R_2G_3 $\frac{17}{10}$	R_2G_4 $\frac{24}{15}$	R_2G_5 $\frac{26}{18}$	R_2G_6 $\frac{25}{15}$
,, ,,	,, ,,	,, ,,	,, ,,	$R_2G_3R_1$ $\frac{20}{10}$	$R_2G_3R_2$ $\frac{17}{7}$	—

TABLE 3. *Mean ratios of the first generation transferred from Green leaves to Red blisters, with an analysis below.*

+ = increase over ancestral ratio
− = decrease from " "
0 = identical with " "

Generation Ratio	Parental Generation Ratio	Red Control Ratio	Green Control Ratio
$G_1R_1\ \frac{21}{8}$	G no winged forms	$R_2\ \frac{20}{9}$	$G_2\ \frac{22}{8}$
$G_2R_1\ \frac{21}{10}$	$G_2\ \frac{22}{8}$	$R_3\ \frac{18}{8}$	$G_3\ \frac{25}{19}$
$R_2G_1R_1\ \frac{17}{9}$	$R_2G_1\ \frac{19}{7}$	$R_4\ \frac{17}{8}$	$R_2G_2\ \frac{18}{10}$
$G_1R_2\ \frac{19}{8}$	$G_1R_1\ \frac{21}{8}$	$R_3\ \frac{18}{8}$	$G_3\ \frac{25}{19}$
$R_4G_1R_1\ \frac{19}{8}$	$R_4G_1\ \frac{22}{7}$	$R_6\ \frac{18}{8}$	$R_2G_4\ \frac{24}{15}$
$R_2G_3R_1\ \frac{20}{10}$	$R_2G_3\ \frac{17}{10}$	$R_6\ \frac{18}{8}$	$R_2G_4\ \frac{24}{15}$
$R_2G_4R_1\ \frac{21}{11}$	$R_2G_4\ \frac{24}{15}$	$R_6\ \frac{18}{8}$	$R_2G_5\ \frac{26}{18}$

Segment V

Generation	Variation from Parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_1	no winged forms	−	0
G_2R_1	+	+	−
$R_2G_1R_1$	+	+	−
G_1R_2	0	0	−
$R_4G_1R_1$	+	0	−
$R_2G_3R_1$	0	+	−
$R_2G_4R_1$	−	+	−

Segment VI

Generation	Variation from Parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_1	no winged forms	+	−
G_2R_1	−	+	−
$R_2G_1R_1$	−	0	−
G_1R_2	−	+	−
$R_4G_1R_1$	−	+	−
$R_2G_3R_1$	0	+	−
$R_2G_4R_1$	−	+	−

TABLE 3*a*. *Mean ratios of the first generation transferred from Red blisters to Green leaves, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Green Control Ratio	Red Control Ratio
$R_1G_1\ \frac{19}{8}$	R no winged forms	$G_2\ \frac{22}{8}$	$R_2\ \frac{20}{9}$
$R_2G_1\ \frac{19}{7}$	$R_2\ \frac{20}{9}$	$G_3\ \frac{25}{19}$	$R_3\ \frac{18}{8}$
$G_1R_1G_1\ \frac{19}{9}$	$G_1R_1\ \frac{21}{8}$	$G_3\ \frac{25}{19}$	$R_3\ \frac{18}{8}$
$R_3G_1\ \frac{18}{8}$	$R_3\ \frac{18}{8}$	$G_3\ \frac{25}{19}$	$R_4\ \frac{17}{8}$
$G_1R_2G_1\ \frac{19}{9}$	$G_1R_2\ \frac{19}{8}$	$R_2G_2\ \frac{18}{10}$	$R_4\ \frac{17}{8}$
$R_4G_1\ \frac{22}{7}$	$R_4\ \frac{17}{8}$	$R_2G_3\ \frac{17}{10}$	$R_5\ \frac{14}{9}$

Segment V

Generation	Variation from Parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_1G_1	no winged forms	0	–
R_2G_1	–	–	–
$G_1R_1G_1$	+	–	+
R_3G_1	0	–	0
$G_1R_2G_1$	+	–	+
R_4G_1	–	–	–

Segment VI

Generation	Variation from Parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_1G_1	no winged forms	–	–
R_2G_1	–	–	+
$G_1R_1G_1$	–	–	+
R_3G_1	0	–	+
$G_1R_2G_1$	0	+	+
R_4G_1	+	+	+

TABLE 4. *Mean ratios of the second generation after transference from Green leaves to Red blisters, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Red Control Ratio	Green Control Ratio
$G_1R_2\ \frac{19}{8}$	$G_1R_1\ \frac{21}{8}$	G_1 no winged forms	$R_3\ \frac{18}{8}$	$G_3\ \frac{25}{19}$
$G_2R_2\ \frac{18}{11}$	$G_2R_1\ \frac{21}{10}$	$G_2\ \frac{22}{8}$	$R_4\ \frac{17}{8}$	$G_3\ \frac{25}{19}$
$R_2G_3R_2\ \frac{17}{7}$	$R_2G_3R_1\ \frac{20}{10}$	$R_2G_3\ \frac{17}{10}$	$R_6\ \frac{18}{8}$	$R_2G_5\ \frac{26}{18}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_2	0	no winged forms	0	−
G_2R_2	+	+	+	−
$R_2G_3R_2$	−	−	−	−

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_2	−	no winged forms	+	−
G_2R_2	−	−	+	−
$R_2G_3R_2$	−	0	−	−

TABLE 4*a*. *Mean ratios of the second generation after transference from Red blisters to Green leaves, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Green Control Ratio	Red Control Ratio
$R_2G_2\ \frac{18}{10}$	$R_2G_1\ \frac{19}{7}$	$R_2\ \frac{20}{9}$	$G_3\ \frac{25}{19}$	$R_4\ \frac{17}{8}$
$R_4G_2\ \frac{19}{9}$	$R_4G_1\ \frac{22}{7}$	$R_4\ \frac{17}{8}$	$R_2G_4\ \frac{24}{15}$	$R_6\ \frac{18}{8}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_2	+	+	–	+
R_4G_2	+	+	–	+

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_2	–	–	–	+
R_4G_2	–	+	–	+

Table 5. *Mean ratios of the third generation after transference from Green leaves to Red blisters, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Great-grand-parental Generation Ratio	Red Control Ratio	Green Control Ratio
G_1R_3 $\frac{18}{9}$	G_1R_2 $\frac{19}{8}$	G_1R_1 $\frac{21}{8}$	no winged forms	R_4 $\frac{17}{8}$	G_3 $\frac{25}{19}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_3	+	+	no winged forms	+	–

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Red Control Ratio	Variation from Green Control Ratio
G_1R_3	–	–	no winged forms	+	–

TABLE 5a. *Mean ratios of the third generation after transference from Red blisters to Green leaves, with analysis as in Table 3.*

Generation Ratio	Parental Generation Ratio	Grand-parental Generation Ratio	Great-grand-parental Generation Ratio	Green Control Ratio	Red Control Ratio
$R_2G_3\ \frac{17}{10}$	$R_2G_2\ \frac{18}{10}$	$R_2G_1\ \frac{19}{7}$	$R_2\ \frac{20}{9}$	$G_3\ \frac{25}{19}$	$R_5\ \frac{14}{9}$
$R_4G_3\ \frac{29}{15}$	$R_4G_2\ \frac{19}{9}$	$R_4G_1\ \frac{22}{7}$	$R_4\ \frac{17}{8}$	$R_2G_5\ \frac{26}{18}$	$R_6\ \frac{18}{8}$

Segment V

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_3	0	+	+	−	+
R_4G_3	+	+	+	−	+

Segment VI

Generation	Variation from Parental Ratio	Variation from Grand-parental Ratio	Variation from Great-grand-parental Ratio	Variation from Green Control Ratio	Variation from Red Control Ratio
R_2G_3	−	−	−	−	+
R_4G_3	+	+	+	+	+

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The effect of a magnetic field on the Intensity of spectrum lines. By H. P. WARAN, M.A., Government Scholar of the University of Madras. (Communicated by Professor Sir Ernest Rutherford.)

[*Read* 8 March 1920.]

[Plates I and II.]

Since the discovery of the Zeeman effect the main attention has been directed to the detailed study of the phenomenon of the small change of wave length suffered by a monochromatic radiation in a magnetic field. The question whether a magnetic field affects the spectrum as a whole has not received much attention.

While working on the Zeeman effect with a mercury discharge tube run by an induction coil as the source, a small portion of the capillary tube being subjected to a magnetic field of about 5000 C.G.S. units as shown in Fig. 1, the light was observed to suffer a change in intensity and also in colour opposite the pole pieces when the field was thrown on. A spectroscopic examination revealed the existence of some selective changes in the spectrum in addition to the increased brilliancy of the general spectrum. It was also noticed that the changes taking place varied considerably with the pressure, at a low pressure the tube showing little change visually but greater changes in the general spectrum. Attention was concentrated on the latter.

In the case of mercury which was the first spectrum investigated, the tube, containing a trace of residual air at very low pressure, gave the principal mercury lines, viz.:

5790·66, 5769·6, 5460·7, 4916·0, 4358·34

and the principal hydrogen lines

6563, 4861·5 and 4340·7.

On applying the magnetic field, however, marked changes were observed, including a new set of lines at

5426, 5679, 5872 and 5889,

and a very strong red line at 6152, brought out prominently by the field. Mercury lines have been recorded at these wave lengths and these lines brought out are probably due to mercury. The behaviour of the line 6152 was very remarkable. It was invisible under ordinary conditions but showed up brilliantly in the magnetic field, the effect being practically instantaneous. Exhausting the tube still further and increasing the current through the tube to about 5 m.a. Four faint lines appeared at wave lengths

6234, 6152, 6123 and 6072,

and corresponding to these wave lengths mercury lines are recorded by* Stiles, Eder, Valenta, Arons and Hermann. But Arons and

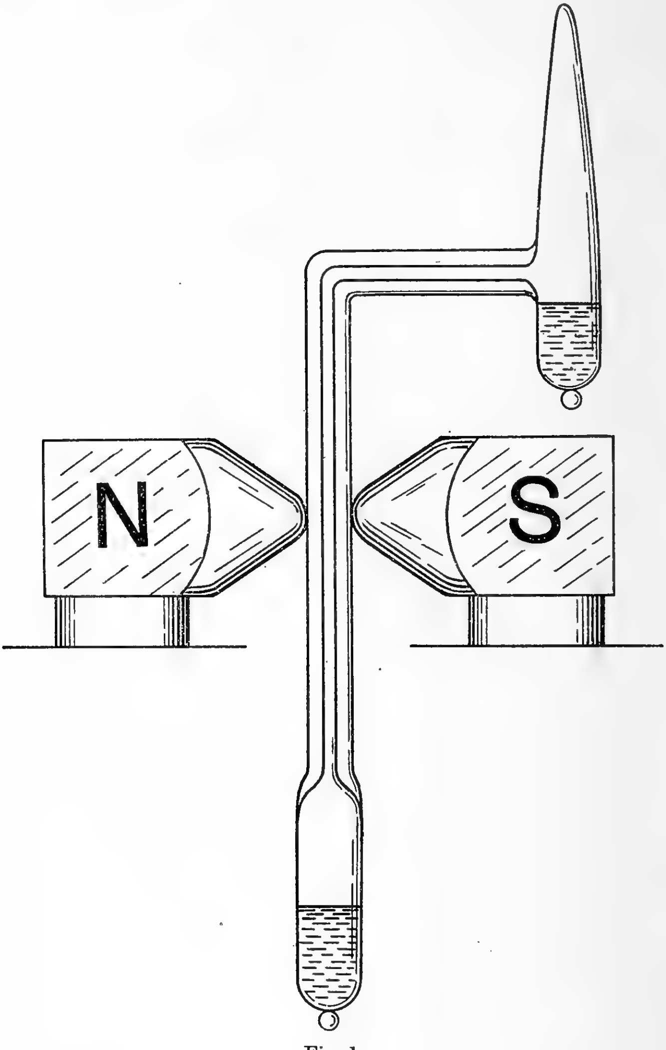


Fig. 1.

Hermann have not recorded the line 6152, while Stiles records it as of equal intensity with the line 6234. Eder and Valenta have not observed the latter lines at all, but record the line 6152 as

* Kayser, *Handbuch der Spectroscopie*, Band v. p. 538.

one of very great intensity. Examining the effect of the magnetic fields on these four lines, it is very interesting to note that the line 6152 alone increases about five times in brilliancy while the others if they suffer any change at all, decrease in intensity. It is also interesting to note that this line 6152 seems to be the same line that becomes so greatly enhanced when the tube contains a trace of helium as observed by* Collie. It seems very difficult to excite this line unless at least a trace of helium is present in the apparatus and at this stage it is not possible to suggest any explanation of its abnormal behaviour.

In addition to these very prominent changes there are also many minor changes, among which is the disappearance of a faint trace of continuous spectrum, as well as of some of the nebulous bands and lines, the remaining lines being quite sharp on a dark background.

The abnormal behaviour of the mercury spectrum in the visible region (the ultra violet spectrum has not yet been investigated) suggested the study of other spectra and the spectrum of helium was next examined.

The discharge tube contained hydrogen and a slight trace of mercury vapour as impurity and the hydrogen lines and the prominent mercury lines were also visible. The effect of the magnetic field in this case was to enhance the helium lines very considerably, leaving the hydrogen lines practically unaffected or even slightly reduced in intensity. In this spectrum there were also a few faint lines not yet identified definitely which remain quite unaffected by the magnetic field. In the further study of the helium spectrum, the gas was contained in a separate tube from which any small quantity of it could be introduced into the discharge tube. At a pressure of 1 mm. of mercury the addition of a small trace of helium produced no perceptible effect on the spectrum of residual air which showed the prominent hydrogen lines and the nitrogen bands, but no trace of any of the helium lines. But on switching on the magnetic field, the helium lines flashed out prominently and disappeared again as soon as the field was turned off. The effect is shown in the accompanying photographs (Plates I and II). In a plate taken with a greater percentage of helium the lines are visible without the magnetic field, but a great enhancement of these lines with the field is evident, and a dense new line at 4933·4 A.U. is also noticed which has not yet been definitely identified.

The spectrum of neon was also studied, and in a tube kindly lent to me by Dr Aston, there was a trace of hydrogen also present, showing the three principal hydrogen lines. Here also the effect of the field was to enhance very considerably the neon lines,

* *Proc. Roy. Soc.* 71, 25, 1902.

leaving the hydrogen lines comparatively unaffected, so that by a casual examination of the spectrum the hydrogen and the neon lines can be distinguished from one another.

The oxygen spectrum is rather difficult to excite when mixed with other gases. Yet a mixture of hydrogen, oxygen and a trace of helium was tried with success and here again the monatomic helium lines were brouglt out by the magnetic field, leaving the diatomic oxygen and hydrogen lines comparatively unaffected as shown in the photographs.

From these experiments the natural inference follows that in a mixture of the monatomic and diatomic gases, the monatomic gases alone seem to be selectively affected in a peculiar way resulting in their spectrum lines alone being very considerably enhanced or brought out prominently even when not visible at all previously. By this method minute traces of the monatomic gases when mixed with other diatomic gases can be detected. On this view we might also explain the abnormal mercury line 6152 and others as due to the radiation from the monatomic atom while the other lines may be classified as belonging to the molecule.

Examining the spectrum of the atmospheric air at low pressure in this way the effect of the magnetic field is to bring out new lines which are not present without the magnetic field, as shown in the photographs. As far as their wave lengths have been determined, though one or two of them fit in fairly well with lines catalogued as belonging to oxygen and nitrogen, yet there are others which are difficult to identify while the absence of other stronger lines of oxygen and nitrogen make even these two or three fits inconclusive.

Another interesting point noted in these experiments is the varying degrees of enhancement under the influence of the field for lines belonging to the same element helium. Preston has shown that the Zeeman effect is of the same magnitude for lines belonging to the same series, but differs in different series. Similarly we might expect the degree of enhancement of the lines in the magnetic field to depend on the series to which the line belongs.

The exact nature of this phenomena and the mechanism of the reaction that brings about these novel changes in the general spectrum is not yet definitely known and it is not desirable to attempt an explanation until the study of the spectrum has been extended to the ultra violet.

The current in the tube was usually about 3 m.a. and the effect of the field was to decrease the current by about 20 to 30 per cent. The changes of intensity observed cannot be attributed to this since the reduction of the current by a spark gap in series only brings about a proportionate decrease in brilliancy of the general spectrum.

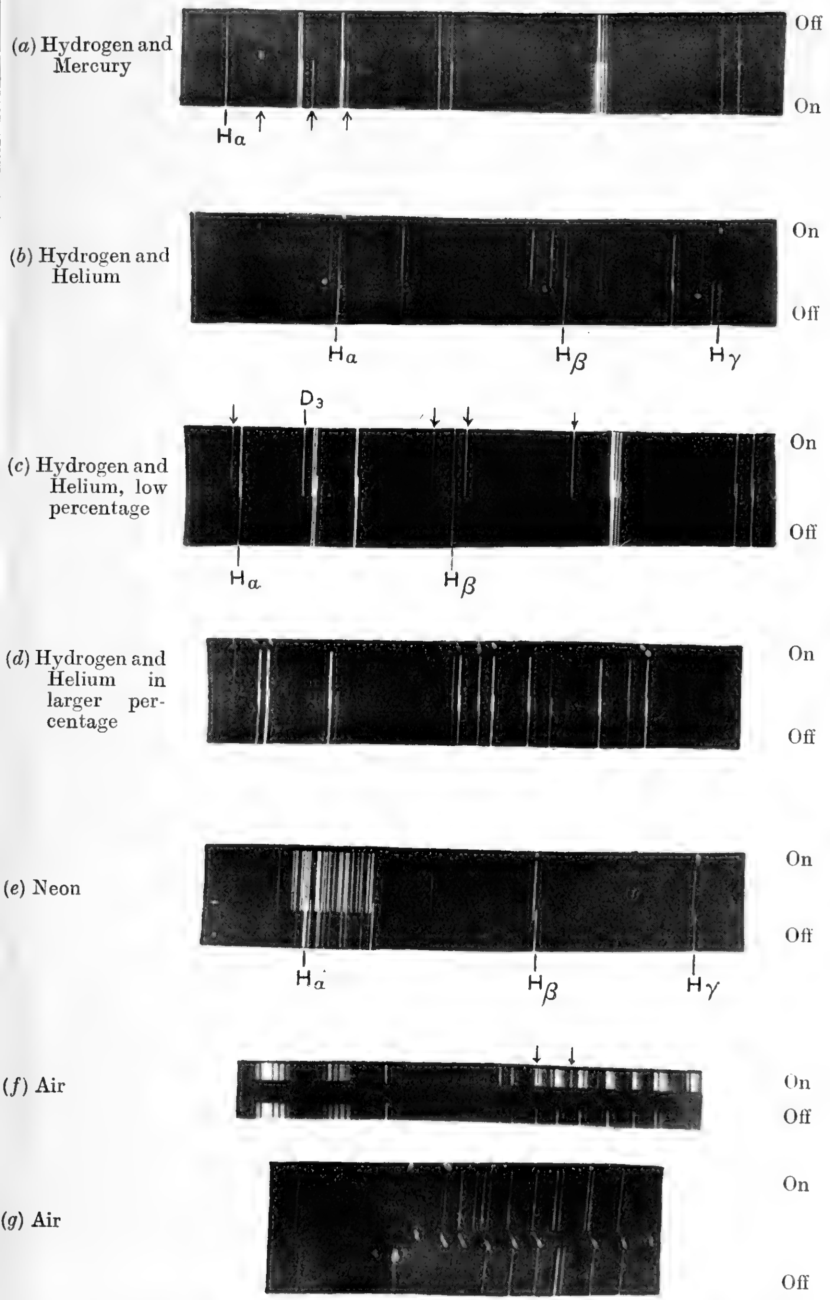
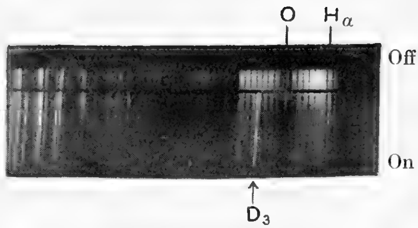
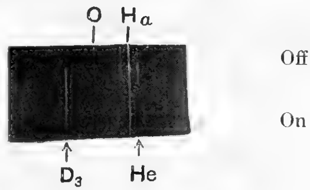


Fig. 2. Photographs showing the enhancing effect of the field. The small lateral shift is due to the camera slider, and in (a) the mercury line 6152 is indicated by the dot, while in the other cases the lines that newly turn up are indicated by the arrows.

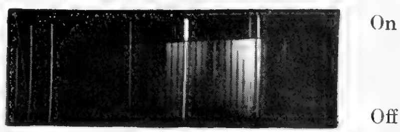
(1) Air, Oxygen, Hydrogen and trace of Helium



(2) Oxygen, Hydrogen and trace of Helium



(3) Excess of Helium and trace of Air and Oxygen



(4) Oxygen, Hydrogen and Helium and trace of Air

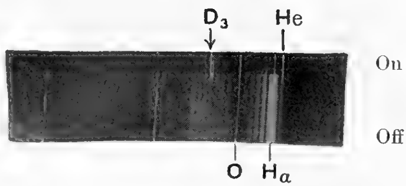


Fig. 3. Photographs showing the effect in mixtures of gases studied.

It may be of interest to note that in solar spectroscopy the spectrum of the sunspots is found to differ in many respects from that of the photosphere, considerable numbers of enhanced lines occurring in the sunspot spectrum. The existence of a powerful magnetic field in sunspots has been demonstrated by the Zeeman effect and possibly the differences in the spectrum of the sunspot and the photosphere may be attributed to this new effect of the magnetic field on the spectrum.

The further study of this effect and the examination of other spectra are in progress.

CAVENDISH LABORATORY,
CAMBRIDGE.

Further Notes on the Food Plants of the Common Earwig (Forficula auricularia). By H. H. Brindley, M.A., St John's College.

[*Read* 8 March 1920.]

In a paper published in the *Proceedings of the Cambridge Philosophical Society*, xix, Part 4, July 1918, p. 170, I recorded certain observations in August and September, 1917, on the food plants of the Common Earwig, with the view of obtaining more exact information than was then available as to the damage likely to be done by this species in a flower or kitchen garden. The paper also epitomised recent literature on the subject, a consideration of which had revealed a considerable amount of diversity and want of exact information as to the favourite food plants of earwigs in the British Isles. The observations made by myself were on earwigs kept in captivity in connection with a statistical enquiry as to the variation of the forcipes which is still in progress. The observations in 1917 were on earwigs from St Mary's, Isles of Scilly, and those recorded in the present paper were made in the second half of the year 1918 on a collection from the Bass Rock, which swarms with earwigs. The animals were all adults and were kept in large glass dishes bedded with sand slightly damped occasionally. Earwigs remain healthy in a soaked substratum if the ventilation is good, but in captivity in a warm room without circulation of air they suffer heavy mortality from fungoid attack, as I have already recorded (*Proc. Camb. Phil. Soc.*, xvii, Part 4, Feb. 1914, pp. 335–338). The fungus appears to be usually *Entomophthora forficulae* (Picard, *Bull. Soc. Étude Vulg. Zool. Agric. Bordeaux*, Jan.–Apr., 1914, pp. 25, 37, 62). The importance of ventilation and of normal temperature is well illustrated by the far fewer fungoid attacks and the low mortality when the new Insect House belonging to the Cambridge Zoological Laboratory became available in 1919. It is at present too early to say how far an improvement is obtainable in the survival of eggs and young which it is hoped to rear in the spring in normal outside temperatures in the Insect House. Earwigs offer a great contrast to cockroaches as regards desire for water; the latter thrive in captivity for months in a warm room on food which is entirely dry, while earwigs certainly visit water to drink, as I have seen in both the captive and wild conditions. I have previously recorded (*Proc. Zool. Soc. Lond.*, Nov. 1897, p. 913) how *Stylopyga orientalis* in captivity seems to pay no attention to a damp sponge when that is the only source of moisture. We have however to bear in mind that the Common Cockroach is probably an immigrant from warmer countries of the East. The earwigs under observation during the past three summers had no animal food save that afforded by those which died. In order to obtain information as to preference for one kind of plant above another they were usually given three different species, taken haphazard, at a time, for a period of two or more days.

In the following summary the observations of 1917 and 1918, with a few made in 1919, are combined. The dates when the different foods were given are noted, as in the latter part of September, when the animals tend to become lethargic, and in the succeeding two months the desire for food is much lessened, even in the artificial temperature of a laboratory. The capital letters after the names of the plants indicate those which were given at the same time, and the numbers appended indicate the preference exhibited by the earwigs: *e.g.* in food group M, M^1 was attacked more than M^2, M^2 more than M^3; in group F, F^1 after two plants indicates that they seemed to be attacked equally, and more readily than F^2: while in group Q, Q^0 indicates that the plant offered was not attacked at all. Similarly for the other groups.

24–26 Aug. '17. Alkanet, Blue (*Anchusa* sp.) C^1: *leaves* not attacked; petals gnawed considerably.

27–29 Aug. '17. Anemone, White Japanese (*Anemone japonica*) D^2: *leaves* not attacked; *petals* eaten moderately.

1–23 Sept. '17. Apple (*Pyrus Malus*) F^2: rather unripe *fruit* with skin whole was not attacked, but when cut across was gnawed moderately: 24–28 Sept. '18, *leaves* holed.

24–28 Sept. '18. Artichoke, Jerusalem (*Helianthus tuberosus*) M^1: *leaves* holed and edges gnawed down to midrib; *tuber*, cross slice attacked vigorously and its buds also devoured.

20 Sept.–5 Oct., 3–17 Nov. '18. Asparagus (*Asparagus officinalis*) O^1, T^0: *leaves* gnawed a little; *fruit* not attacked.

26–31 Aug. '18. Aster, Mauve China (*Callistephus chinensis*) H^2: *leaves* not attacked; *petals* and *flower buds* much eaten.

6–11 Sept. '18. Aster, Pink China (*Callistephus chinensis*): *leaves* slightly nibbled; *petals* much eaten; *flowers* used as a refuge.

15–20 Sept. '18. Balm, Pale Mauve (*Melissa officinalis*) J^1: *leaves* not attacked; *petals* of buds devoured.

22–23 Aug. '17. Bean, Dwarf (*Phaseolus vulgaris*) B^3: *leaves* nibbled very slightly.

30–31 Aug. '17. Bean, Scarlet Runner (*Phaseolus multiflorus*) E^3: *leaves*, *flowers* and *pods* apparently neglected: 16–18 Oct. '18, *leaves* holed a good deal and edges gnawed down to veins.

20–28 Oct. '18. Beard Tongue, Scarlet (*Pentstemon* sp.) R^0: *leaves* and *flowers* not attacked.

22–23 Aug. '17. Beet (*Beta vulgaris*) B^1: *leaves* much attacked, especially the petioles, which were opened out and their pith devoured.

20–24 Sept. '18. Bell Flower, White (*Campanula* sp.) K^1: *leaves* not touched; *petals* completely devoured.

31 Aug.–6 Sept. '18. Bindweed, Common (*Convolvulus* sp.): *leaves* much holed.

11–13 Sept. '18. Blackberry (*Rubus fruticosus*): ripe *fruit* well gnawed.

30–31 Aug. '17. Cabbage, Garden (*Brassica oleracea capitata*) E^1: *leaves* gnawed down to midrib and veins and ends of veins eaten off.

2–5 Oct. '18. Canterbury Bell, Blue (*Campanula medium*) N^1: *leaves* and *petals* well devoured.

6–7 Sept. '18. Carrot (*Daucus Carota*): *root* not attacked where covered by skin, but cut end was much gnawed.

6–11 Sept. '18. Celery (*Apium graveolens*) H^1: *leaves* holed and their edges gnawed.

29 Sept.–3 Oct. '18. Cherry (*Prunus* [*Cerasus*] sp.) M^3: *leaves* not attacked.

20–23 Oct. '18. Chickweed (*Stellaria media*) R^1: edges of *leaves* gnawed slightly.

31 Aug.–6 Sept. '18. Chrysanthemum, Garden (*Chrysanthemum indicum*): *flower buds* used as refuge, tips of *petals* apparently somewhat nibbled: 31 Aug.–6 Sept. '18, purple variety: edges of *leaves* much nibbled; *flower buds* used as refuge, tips of petals apparently somewhat nibbled: 31 Aug.–6 Sept. '18, white variety: *leaves* not attacked; *petals* much eaten.

20–24 Sept. '18. Clematis, White (*Clematis* sp.) K^1: *leaves*, a few eaten off at ends and edges gnawed here and there; *flowers* entirely devoured.

23–27 Oct. '18. *Cluvia miniata* (Natal): *leaves* not attacked; *petals* gnawed a little along edges.

15–20 Sept. '18. Cornflower (*Centaurea Cyanus*) J^1: *leaves* well eaten, only midrib left; *flowers* entirely devoured.

29 Sept.–3 Oct. '18. Cups and Saucers (*Cobaea scandens*) M^2: *petals* nibbled a little.

27 Oct.–3 Nov. '18. Dandelion (*Taraxacum officinale*): *petals* of ray florets entirely devoured.

26–31 Aug. '18. Elephant's Ear, Pink (*Begonia* sp.): *leaves* much gnawed along edges and also holed; *flowers* thoroughly devoured.

2–5 Oct. '18. Fern, Male (*Lastraea filis-mas*) O^0: *leaves* not attacked.

15–20 Sept. '18. Feverfew (*Pyrethrum* sp.) J^1: *leaves* gnawed down to midrib; *flowers* apparently not attacked.

21–28 Sept. '18. Fig (*Ficus Carica*): *leaves* not attacked; *fruit* neglected when whole, but cross section was well gnawed.

7–15 Oct. '18. Fox-glove (*Digitalis purpurea*) P^1: *leaves* holed.

6–11 Sept. '18. Fuchsia, Crimson Garden (*Fuchsia* sp.) H^3: neither *leaves* or *flowers* were attacked.

28 Sept.–2 Oct. '18. Geranium, Scarlet (*Geranium* sp.) L^2: *petals* eaten a little.

20–24 Sept. '18. Gesnera, Orange and Pink (*Gesnera* sp.) K^1: *leaves* not attacked; *petals* entirely devoured.

24–26 Aug. '17. Golden Rod (*Solidago* sp.) C^3: *leaves* gnawed at edges here and there; *flowers* apparently not attacked.

2–5 Oct. '18. Gooseberry (*Ribes grossularia*) O^0: *leaves* not attacked.

11–15 Sept. '18. Hawthorn (*Crataegus oxycantha*) I^0: neither *leaves* or *flowers* were attacked.

24–31 Aug. '18. Hollyhock, Dark Crimson (*Althaea rosea*): *leaves* not attacked; *flower buds* used as refuge, *petals* apparently eaten to some extent.

10–20 Aug. '18. Honeysuckle (*Lonicera* sp.) G^2: *leaves* not attacked; *fruit* gnawed considerably.

7–20 Oct. '18. Hydrangea, Pink (*Hydrangea* sp.) Q^0: neither *leaves* or *flowers* were attacked.

7–15 Oct. '18. Larkspur, Garden variety (*Delphinium* sp.) Q^1: *leaves* gnawed thoroughly down to midrib.

3–6 Nov. '18. Leek (*Allium porrum*) T^1: *leaves* gnawed deeply towards base.

6–15 Sept. '18. Lettuce, Cabbage (*Lactuca sativa*): *stem* abundantly gnawed and bored; *leaves* of "heart" entirely devoured.

7–27 Oct. '18. Lupin (*Lupinus polyphyllus*) S^2: *leaves* gnawed to some extent.

3–17 Nov. '18. Mallow (*Malvus ? sylvestris*): *leaves* holed and edges gnawed down to veins.

23 Oct.–17 Nov. '18. Marguerite, White-rayed (*Chrysanthemum leucanthemum*) S^1, U^2: *petals* of ray florets well gnawed.

20–21 Aug. '17. Marrow, Vegetable (*Cucurbita ovifera*) A^1: *leaves* thoroughly devoured.

20–21 Aug. '17. Michaelmas Daisy (*Aster* sp.) A^3, N^3: *leaves* hardly touched, if at all; *flowers* also neglected.

11–15 Sept. '18. Mignonette (*Reseda odorata*): *leaves* gnawed down to midrib; *flowers* attacked but slightly or not at all.

16–18 Sept. '18. Mint (*Mentha* sp.): *leaves*, edges and ends nibbled; *flowers* entirely devoured.

20–23 Oct. '18. Navew (*Brassica campestris*) R^1: *leaves* holed and edges gnawed a little; *petals* moderately attacked.

3–17 Nov. '18. Nettle (*Urtica dioica*) U^1: *leaves* well gnawed down to veins.

31 Aug.–6 Sept. '18. Onion (*Allium Cepa*) L^0: *inflorescence* used as refuge, but apparently not eaten.

7–15 Oct. '18. Pansy (*Viola tricolor*) P^1: *leaves* nibbled slightly.

10–20 Aug. '18. Parsley, Garden (*Carum Petroselinum*) G^2: *inflorescence* nibbled moderately.

29 Sept.–3 Oct. '18. Peach (*Prunus* [*Amygdalus*] sp.) N^2: *leaves* gnawed moderately.

28 Sept.–2 Oct. '18. Periwinkle, Blue (*Vinca* sp.) L^1: *leaves* and *petals* gnawed moderately.

22–23 Aug. '17. Phlox, White (*Phlox Drummondi*) B^2: *leaves* apparently not attacked; *petals* much gnawed and *pollen* found in gut of earwigs.

1–3 Sept. '17. Plum (*Prunus communis*) F^1: *fruit* well eaten.

26–31 Aug. '18. Poppy, Garden (*Papaver* sp.): dried *fruits* very popular as refuges; some were holed to obtain entrance.

1–18 Sept. '17. Potato (*Solanum tuberosum*) F^1: *tuber* in skin was neglected, but slices were thoroughly gnawed.

28–29 Aug. '17, 20–23 Oct. '18. Primrose, Evening, yellow variety (*Oenothera* sp.) D^1: *leaves* not attacked; *petals* eaten thoroughly; *pods* neglected.

7–15 Oct. '18. Privet (*Ligustrum vulgare*) Q^2: *leaves* holed and edges gnawed; *fruits* not attacked.

20–21 Aug. '17. Radish, Horse (*Raphanus sativus*) A^2: *leaves* nibbled slightly.

27–29 Aug. '17. Raspberry (*Rubus idaeus*) D^0: *leaves* not attacked, but earwigs assembled in crowds on their hairy undersides.

22–28 Sept. '18. Red hot poker (*Kniphofia* sp.) : cut end of *stem* gnawed; *leaves* and *petals* not attacked.

11–15 Sept. '18. Rest-harrow (*Ononis* sp.) I^0: apparently neither *leaves* or *flowers* were attacked.

30–31 Aug. '17. Rhubarb (*Rheum officinale*) E^2: *leaves* well gnawed.

24–26 Aug. '17. Rose, White garden variety (*Rosa* sp.) C^1: *leaves* not attacked; *petals* devoured.

7–10 Oct. '18. St John's Wort (*Hypericum* sp.) P^1: *leaves* holed and their edges gnawed; *flower buds* not attacked.

31 Aug.–6 Sept. '18. Scabious, Crimson Garden (*Scabiosa atro-purpurea*): *leaves* much holed; *flowers* apparently not attacked.

23–27 Oct. '18. Scotch Kale (*Brassica oleracea acephala*) S^3: *leaves* holed a very little; curled margins a favourite refuge.

10–24 Aug. '18. Sea Kale (*Brassica oleracea acephala*) G^1: *leaves* holed and gnawed away from edges to between veins.

6–11 Sept. '18. Snapdragon, Scarlet (*Antirrhinum* sp.): *leaves* gnawed moderately; *petals* apparently holed to some extent, also used as refuge.

23–30 Oct. '18. Sow thistle (*Sonchus oleaceus*): *leaves* holed slightly; *flower buds* not attacked.

3–17 Nov. '18. Strawberry (*Fragaria vesca*) U^3: *leaves* holed a little.

31 Aug.–6 Sept. '18. Tomato (*Lycopersicum esculentum*): *leaves* and ripe *fruit* gnawed thoroughly.

14–15 Sept. '18. Valerian, Red Garden (*Valeriana* sp.): edges of *leaves* gnawed moderately; *petals* entirely devoured.

21–24 Aug. '18. Vervain, Blue (*Verbena* sp.): *leaves* nibbled slightly, hairy undersides used for assembling; *petals* entirely devoured.

24–31 Aug. '18. Vetch, Mauve and White garden varieties (*Vicia* sp.): *leaves* attacked very slightly, if at all; *petals* entirely devoured.

23 Oct.–3 Nov. '18. Violet, Single and Double garden varieties (*Viola* sp.): *leaves* holed and edges gnawed moderately.

3–17 Nov. '19. Wartweed (*Euphorbia helioscopia*) T^2: edges of *leaves* gnawed very slightly.

15–18 Sept. '18. Wormwood (*Artemisia* sp): *leaves* not attacked.

These observations are of course subject to the drawback that in captivity animals which normally feed daily may take unusual food with apparent eagerness because no other is available; but the above record probably indicates normal preferences over a certain range of common plants, and also that some are disliked by earwigs; thus Wartweed was left entirely untouched for many days in the absence of any other food, the animals attacking potato tuber ravenously as soon as this was substituted. It seems natural that such stiff and dry foliage leaves as those of Raspberry, Hawthorn, and Cherry, should escape attack, and there is no doubt that the more succulent leaves are preferred. The list of plants affords some information which may facilitate the destruction of earwigs when they become a pest by the indications obtained as to plants which are popular as refuges, and also by the mode in which the attack on leaves is made; thus, some leaves seem to be attacked by holing as well as by gnawing along the edges, and others only by the latter method. There is no doubt that earwigs have preferences among the common plants of a flower or vegetable garden, and that if numerous they are likely to become a pest. In certain cases, as for instance, chrysanthemums, the actual damage done seems to be exaggerated by common report.

Since the epitome of recent literature on the subject in my previous paper (*Proc. Camb. Phil. Soc.*, XIX, Part 4, 1918, p. 170) was written, *The Review of Applied Entomology* has recorded attacks on beets and sugar-beets in Denmark sufficiently serious to obtain mention by Lind and others in their Report on Agricultural Pests in 1915 (*Beretning fra Statens Forsogsvirksomhed i Plantekultur*, Copenhagen, 1916, pp. 397–423).

As regards the carnivorous habit of *F. auricularia*, lean roast mutton without other food was given for several days to the earwigs under observation in 1918 and was gnawed sparingly, while mutton suet substituted for it was eaten readily and extensively. In the *Journal of the Bombay Natural History Society*, xxvi, No. 2, May, 1919, p. 688, F. P. Connor records an unnamed earwig at Amara catching moths in its forcipes and in one case nibbling its prey. F. Maxwell Lefroy (*Indian Insect Life*, p. 52) remarks: "The function of the forcipes is a mystery that will be cleared up only when their food habits and general life are better understood." They are very possibly "frightening" as well as defensive organs. Pemberton (*Hawaiian Planters' Record*, Honolulu, xxi, No. 4, Oct. 1919, pp. 194–221) mentions the benefit to cane fields arising from the destruction of the leaf-hopper parasite *Perkinsiella optabilis* by the black earwig *Chelisoches morio*.

The importance of nocturnal observations on the feeding habits of *Forficula auricularia* to a satisfactory understanding of the economic effects of this insect in gardens, urged in my previous paper, may be referred to again.

Lagrangian Methods for High Speed Motion. By C. G. Darwin.

[*Read* 8 March 1920.]

1. In the later developments of Bohr's* spectrum theory, it is necessary to calculate the orbits of electrons moving with such high velocities that there is a sensible increase of mass. The selection of the orbits permitted by the quantum theory almost necessitates the treatment of such problems by Hamiltonian methods. Working on these lines Sommerfeld† and others have calculated with a very high degree of success those spectra which involve the motion of a single electron. But the application of the Hamiltonian function involves a knowledge of the momentum corresponding to any generalized coordinate, and in the formulation of most problems the momenta are not known *a priori* but must be calculated from the corresponding velocities. In other words the formation of the Hamiltonian function must in general be preceded by that of the Lagrangian. An exception occurs in precisely the problems referred to above; for, the electromagnetic theory furnishes directly values for the momentum and kinetic energy of a moving electron in terms of its velocity, and the velocity can be eliminated between them so as to obtain the Hamiltonian function. But in even slightly more complicated cases this simple relation is destroyed—thus the problem of a single electron in a constant magnetic field can only be solved by introducing the artificial conception of rotating axes—and in general it will be necessary to follow the direct course of finding the Lagrangian function in terms of the generalized velocities, and then deducing from it the momenta and the Hamiltonian function in the usual way.

If more than one particle is in motion another difficulty enters. For the interaction of two moving particles depends on a set of retarded potentials and the effect of the retardation is readily seen to be of the same order as the increase of mass with velocity. The calculation of the retardation can only be carried out by expansion and so the results are only approximate. This is not surprising since the methods of conservative dynamics cannot apply to such effects as the dissipation of energy by radiation, effects inevitably required by the electromagnetic theory, though they do not occur in actuality. We can also see from the fact that these radiation terms are of the order of the inverse cube of the velocity of light, that it will be useless to expand beyond the inverse square.

* N. Bohr, *Kgl. Dan. Wet. Selsk.*, 1918.
† A. Sommerfeld, *Ann. Phys.*, vol. 51, p. 1, 1916.

2. We first consider the motion of a single electron in an arbitrary electric and magnetic field varying in any manner with the time and position. If m is the mass for low velocities, the momentum is known to be mv/β, where $\beta = \sqrt{1 - v^2/c^2}$. Starting from this we have quasi-Newtonian equations of motion of the type

$$\frac{d}{dt}\left\{\frac{m}{\beta}\cdot\dot{x}\right\} = F_x \qquad \text{......(2·1).}$$

The force F_x is given from the field $\mathbf{E}$, $\mathbf{H}$ as the vector $e\mathbf{E} + \frac{e}{C}[\mathbf{v}, \mathbf{H}]$, where $\mathbf{v}$ is the velocity vector of the particle's motion. $\mathbf{E}$ and $\mathbf{H}$ can be expressed in terms of the scalar and vector potentials in the form $\mathbf{E} = -\operatorname{grad}\phi - \frac{1}{C}\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{H} = \operatorname{curl}\mathbf{A}$.

Then if $\mathbf{r}_1$ is the vector x, y, z we have as the vector equation of motion

$$\frac{d}{dt}\left\{\frac{m_1}{\beta_1}\dot{\mathbf{r}}_1\right\} = -e_1\operatorname{grad}\phi - \frac{e_1}{C}\frac{\partial \mathbf{A}}{\partial t} + \frac{e_1}{C}[\dot{\mathbf{r}}_1, \operatorname{curl}\mathbf{A}] \qquad \text{...(2·2),}$$

where $\beta_1 = \sqrt{1 - \dot{\mathbf{r}}_1^2/C^2}$.

Let q be any one of three generalized coordinates representing the position of the particle. Take the scalar product of (2·2) by $\frac{\partial \mathbf{r}_1}{\partial q}$. Then since $\frac{\partial \mathbf{r}_1}{\partial q} = \frac{\partial \dot{\mathbf{r}}_1}{\partial \dot{q}}$, we have

$$\begin{aligned}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \frac{d}{dt}\left\{\frac{m_1}{\beta_1}, \dot{\mathbf{r}}_1\right\}\right) &= \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \dot{\mathbf{r}}_1\right)\right\} - \frac{m_1}{\beta_1}\left(\frac{\partial \dot{\mathbf{r}}_1}{\partial q}, \dot{\mathbf{r}}_1\right)\\ &= \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\frac{\partial}{\partial \dot{q}}(\tfrac{1}{2}\dot{\mathbf{r}}_1^2)\right\} - \frac{m_1}{\beta_1}\frac{\partial}{\partial q}(\tfrac{1}{2}\dot{\mathbf{r}}_1^2)\\ &= \mathfrak{D}_q(-m_1C^2\beta_1),\end{aligned}$$

where $\mathfrak{D}_q = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}$ the Lagrangian operator.

Again $$-e_1\left(\frac{\partial \mathbf{r}}{\partial q}, \operatorname{grad}\phi\right) = -e_1\frac{\partial \phi}{\partial q} = e_1\mathfrak{D}\phi.$$

The remainder can be reduced to

$$\frac{e_1}{C}\left(\dot{\mathbf{r}}_1, \frac{\partial \mathbf{A}}{\partial q}\right) - \frac{e_1}{C}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \frac{d\mathbf{A}}{dt}\right) \qquad \text{......(2·3),}$$

where $$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\dot{x} + \frac{\partial \mathbf{A}}{\partial y}\dot{y} + \frac{\partial \mathbf{A}}{\partial z}\dot{z}$$

and so is the total change of $\mathbf{A}$ at the moving particle. (2·3) can be reduced to $-\frac{e_1}{C}\mathfrak{D}_q(\dot{\mathbf{r}}_1, \mathbf{A})$.

Thus the whole equation of motion can be derived from a Lagrangian function

$$L = -m_1C^2\beta_1 - e_1'\phi + \frac{e_1}{C}(\dot{\mathbf{r}}_1, \mathbf{A}) \qquad \text{......(2·4)}.$$

This is valid for any fields of force including explicit dependence of ϕ and $\mathbf{A}$ on the time. The first term in L, which reduces to the kinetic energy for low velocities, differs from it in general. It is very closely connected with the "world line" of the particle.

3. To treat of the case where several moving particles interact we shall start by supposing that there is a second particle present undergoing a constrained motion so that its coordinates are imagined to be known functions of the time. The same will then be true of the potentials it generates. The motion of e_1 will then be governed by (2·4) if ϕ and $\mathbf{A}$ are expressed in terms of the motion of e_2. These potentials are given by

$$\phi = \frac{e_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/C}, \qquad \mathbf{A} = \frac{e_2}{C}\frac{\dot{\mathbf{r}}_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/C} \text{ ...(3·1)}.$$

In these expressions $r^2 = (\mathbf{r}_2 - \mathbf{r}_1)^2$ and the values are to be *retarded* values. If the time of retardation be calculated and the result substituted in (3·1) we obtain

$$\phi = \frac{e_2}{r} + \frac{e_2}{2C^2}\left\{\frac{\dot{\mathbf{r}}_2^{\,2} + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\}, \quad \mathbf{A} = \frac{e_2}{C}\frac{\dot{\mathbf{r}}_2}{r} \text{...(3·2)},$$

where now $\mathbf{r}_1$, $\mathbf{r}_2$ refer to the same instant of time. ϕ is an approximation valid to C^{-2}, but the value of $\mathbf{A}$ has only been found to the degree C^{-1} on account of the further factor C^{-1} in (2·4) which is to multiply it. Then substituting in (2·4) we obtain

$$L = -m_1C^2\beta_1 - \frac{e_1e_2}{r} - \frac{e_1e_2}{2C^2}\left\{\frac{\dot{\mathbf{r}}_2^{\,2} + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1) - 2(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\} \qquad \text{......(3·3)}.$$

The equations of motion are unaffected by adding to L the expression $-m_2C^2\beta_2 + \frac{d}{dt}\frac{e_1e_2}{2C^2}\frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r}$. The first is a pure function of the time and so contributes no terms to the equations of motion. The second contributes nothing because for any function f we have

$$\mathfrak{D}\left(\frac{d}{dt}f(q_1, q_2, \ldots, t)\right) \equiv 0.$$

The new form of L then reduces to

$$L = -m_1C^2\beta_1 - m_2C^2\beta_2 - \frac{e_1e_2}{r} + \frac{e_1e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3}\right\} \quad \text{......(3·4).}$$

From the complete symmetry of this form the roles of e_1 and e_2 may be interchanged. Further from the covariance of the operator $\mathfrak{D}$ for point transformations, both may be included in the dynamical system, so that if q is any generalized coordinate involving both $\mathbf{r}_1$ and $\mathbf{r}_2$, the equations of motion will be of the form $\mathfrak{D}_q L = 0$.

For the sake of consistency, as the last term in (3·4) is only an approximation valid to C^{-2}, the first two should be expanded only to this power. The first term will give

$$-m_1C^2 + \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^{\,2} + \frac{1}{8C^2}m_1\dot{\mathbf{r}}_1^{\,4}.$$

Generalizing our result to the case of any number of particles in any external field we have

$$L = \Sigma\tfrac{1}{2}m_1\dot{\mathbf{r}}_1^{\,2} + \Sigma\frac{1}{8C^2}m_1\dot{\mathbf{r}}_1^{\,4} - \Sigma e_1\phi + \Sigma\frac{e_1}{C}(\dot{\mathbf{r}}_1\mathbf{A}) - \Sigma\Sigma\frac{e_1e_2}{r_{12}} + \Sigma\Sigma\frac{e_1e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r_{12}} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}^{\,3}}\right\} \quad \text{...(3·5).}$$

The double summations are taken counting each pair once only.

4. The transition to the Hamiltonian now follows the ordinary rules. We find momenta $p = \dfrac{\partial L}{\partial \dot{q}}$ and solve for the $\dot{q}$'s in terms of the p's. This can be done in spite of the cubic form of the equations in the q's by use of the approximation in powers of C. The Hamiltonian function will then be $H = \Sigma p\dot{q} - L$ and the equations of motion will be the canonical equations $\dot{q} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial q}$. If $\mathbf{p}_1$ be the momentum corresponding to $\mathbf{r}_1$, the Hamiltonian in these coordinates will be

$$H = \Sigma\frac{\mathbf{p}_1^{\,2}}{2m_1} - \Sigma\Sigma\frac{\mathbf{p}_1^{\,4}}{8C^2m_1^{\,3}} + \Sigma e_1\phi - \Sigma\frac{e_1}{Cm_1}(\mathbf{p}_1, \mathbf{A}) + \Sigma\Sigma\frac{e_1e_2}{r_{12}} - \Sigma\Sigma\frac{e_1e_2}{2C^2m_1m_2}\left\{\frac{(\mathbf{p}_1, \mathbf{p}_2)}{r_{12}} + \frac{(\mathbf{p}_1, \mathbf{r}_2 - \mathbf{r}_1)(\mathbf{p}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}^{\,3}}\right\} \quad \text{...(4·1).}$$

All the applications of general dynamics, such as the Hamilton Jacobi partial differential equation, follow from this. As in ordinary dynamics, many problems can be conveniently solved in the Lagrangian form. The solution will usually depend on finding integrals corresponding to coordinates which do not occur explicitly in L and if ϕ and $\mathbf{A}$ do not involve the time explicitly there is also the energy integral. This has the form

$$\Sigma \tfrac{1}{2} m \dot{\mathbf{r}}_1^2 + \Sigma \tfrac{3}{8} \frac{m_1}{C^2} \dot{\mathbf{r}}_1^4 + \Sigma e_1 \phi + \Sigma\Sigma \frac{e_1 e_2}{r_{12}}$$
$$+ \Sigma\Sigma \frac{e_1 e_2}{2C^2} \left\{ \frac{(\dot{\mathbf{r}}_1 \dot{\mathbf{r}}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3} \right\} = \text{const.} \quad \ldots (4 \cdot 2).$$

This completes the development of the method. Its direct applications are naturally somewhat limited, since, even with the large order terms only, there are comparatively few problems that are soluble. A problem of some interest that can be solved completely is the motion of two attracting particles, where their masses have a finite ratio*.

* A discussion of this problem by the present writer will be found in *Phil. Mag.*, Vol. 39, p. 537 (1920), together with a somewhat fuller account of the general theory.

A bifilar method of measuring the rigidity of wires. By G. F. C. SEARLE, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 3 May 1920.]

§ 1. *Introduction.* In this method the couple due to the torsion of two similar wires is balanced against the couple due to the load carried by the wires and arising from bifilar action.

The method is hardly suitable for accurate measurements of rigidity, but, as an exercise in the use of a bifilar suspension, it has proved useful at the Cavendish Laboratory.

§ 2. *Bifilar couple.* We first consider two light flexible strings. Let the strings AB, CD, each l cm. in length, hang from two fixed points A, C, which are at a distance $2a_1$ cm. apart in a horizontal plane. The lower ends B, D of the strings are attached to a rigid body of mass M grm., the points B, D being $2a_2$ cm. apart. The centre of gravity of the body is symmetrical with regard to B and D and thus the tensions of the strings are equal. The line BD will then be horizontal. If, now, a couple, whose axis is vertical, is applied to the body, the body will be in equilibrium when the couple due to the obliquity of the strings balances the applied couple*.

In Fig. 1, A', B', C', D' are the projections of A, B, C, D on a horizontal plane. In our symmetrical case, $A'C'$, $B'D'$ bisect each other in O. When the body has turned through θ radians from the zero position, in which the strings are in the vertical plane through $A'C'$, then $B'D'$ will make an angle θ with $A'C'$. Let ON be the perpendicular from O on $A'B'$. Let the tension in each string be T dynes.

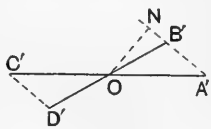


Fig. 1.

If the vertical distance of BD below AC is h cm., the vertical component of the tension is Th/l, and the horizontal component is $T.A'B'/l$. Since the weight of the body equals the sum of the vertical components,

$$Mg = 2Th/l.$$

The horizontal component of the tension at B acts along a line whose projection is $A'B'$, and hence its moment about the vertical

* For the general theory of the bifilar suspension, see Maxwell, *El. and Mag.*, Vol. II, § 459; A. Gray, *Absolute Measurements in El. and Mag.*, Vol. I, p. 242; Kohlrausch, *Physical Measurements* (1894), p. 226.

axis through O is $T.A'B'.ON/l$. Since the moment due to the two tensions equals that of the applied couple, G dyne-cm.,

$$G = 2T.A'B'.ON/l.$$

But $A'B'.ON$ is twice the area $OA'B'$ and thus is $a_1a_2 \sin\theta$. We thus obtain

$$\frac{G}{Mg} = \frac{A'B'.ON}{h} = \frac{a_1a_2 \sin\theta}{h}.$$

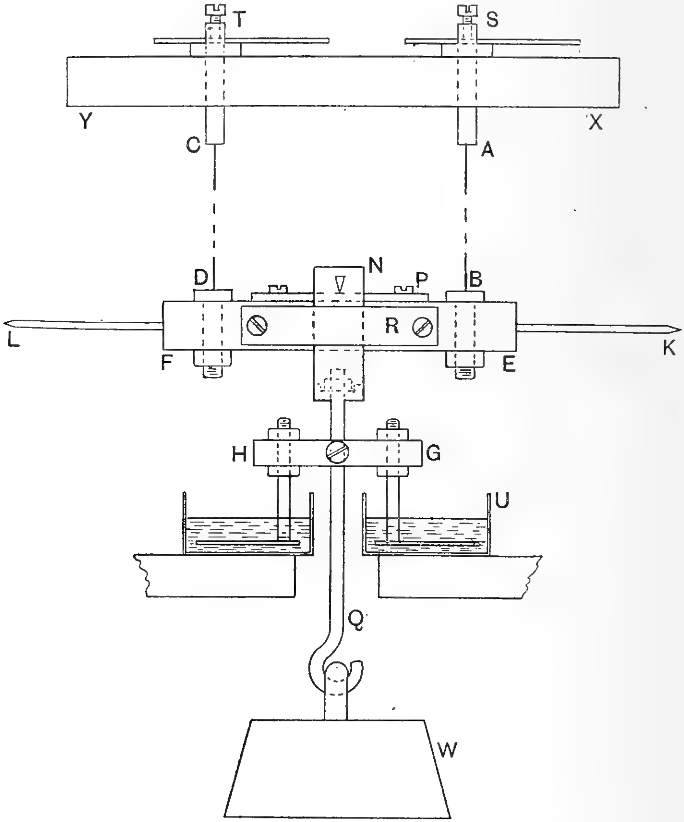


Fig. 2.

Since $h = \{l^2 - A'B'^2\}^{\frac{1}{2}}$, we see that, when $A'B'$ is small compared with l, we may put $h = l$, and so obtain

$$G = \frac{a_1a_2 \sin\theta}{l}.Mg. \qquad \text{......(1)}$$

In the examples of § 7, h never differed from l by as much as 1 in 4000.

§ 3. *Apparatus.* This is shown diagrammatically in Fig. 2. The wires are soldered into torsion heads S, T, which pass through a board XY held in a firm support.

The lower ends of the wires are soldered into screws which pass through "clearing" holes in the bar EF, and are secured with nuts. The heads of the screws are made with "flats" to fit a spanner. Before the screws are secured to EF, the torsion heads are set to zero; the screws are then secured to EF so that, when the bar is only subject to the action of the wires and of gravity, the flats on *both* screws have the same directions as when the wires hung freely.

The distance BD is, as near as may be, equal to AC.

The load is carried by a knife-edge forming part of the link N, Figs. 2, 3. The knife-edge rests in a V-groove in a plate, P, fixed to EF by screws passing through slots. By adjusting P, the tensions can be equalised; the notes emitted by the wires when plucked have the same pitch when the tensions are equal.

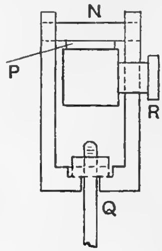


Fig. 3.

A weight W (a few kilogrammes) is suspended by the rod Q from the link N. A slot in the lower cross-piece of N allows Q to be put into place; the nut drops into a recess. The weight should be so attached to Q that it cannot turn about a vertical axis relative to Q with any freedom; otherwise it will be difficult to reduce the system to rest.

The bar may be fitted with two pointers K, L, and the readings of their *tips* are taken on two horizontal scales. These scales are adjusted to be perpendicular to KL when the torsion heads read zero. If K_0L_0 is the *straight line* through the zero positions of the *tips* and K, L are the *tips* when the bar has turned through θ, Fig. 4 shows that

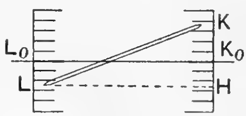


Fig. 4.

$$\sin\theta = \frac{KH}{LK} = \frac{y_1 + y_2}{p}, \qquad \text{......(2)}$$

where $y_1 = KK_0$, $y_2 = LL_0$ and $p = LK$, the whole length of the pointer system.

The deflexion of the bar is best observed optically. A metal strip R is screwed to EF, packing pieces being interposed to allow the link N free movement, and a plane mirror is fixed to R. The deflexion can be observed by aid of a telescope and scale, or of a lamp and scale. It is, however, simpler to employ a goniometer such as those which have been in constant use at the Cavendish Laboratory for several years. A description of the instrument and the method of using it for experiments of this type will be found in *Proc. Camb. Phil. Soc.*, XVIII, p. 31, or in the author's *Experimental Harmonic Motion*, p. 35. The goniometer measures the tangents of angles.

The motion of the suspended system, as so far described, being only slightly damped, it is consequently not easy to reduce the system to rest, and the vibrations of the building add to the difficulty. A simple damping device is therefore used. An annulus of thin sheet metal is carried by the bar GH, which is clamped to the rod Q. The annulus is immersed in motor lubricating oil or other highly viscous liquid contained in the annular trough U, which rests on the table. The rod Q passes through a hole in the table. By adjusting the height of GH, the annulus can be brought close to the bottom of the trough, and then the motion is so highly damped that the system is practically immune to vibrations of the floor or the table.

If the wires are overstrained by turning the heads through too large angles, the wires will no longer be vertical when the heads read zero, and it will be necessary to readjust the screws in the bar EF. To prevent overstrain, and at the same time to allow the heads to be turned through π in either direction from their zeros, a *movable* safety device is used. A metal disk, about 1 cm. in diameter, can turn freely about its centre on a screw by which it is attached to the board XY (Fig. 2). A vertical pin is fixed excentrically in the disk, the greatest distance from the pin to the axis of the head being small enough to prevent the steel wire, which forms the index of the head, from passing the pin. The torsion head can then be turned only a little more than π in either direction from zero.

Care must be taken not to bend the wires near the soldered joints. A bend at B or D will alter the effective value of a_2. If the wire AB is bent near A, the effect, when the torsion head is turned, will be the same as if the point A describes a small horizontal circle. This causes changes in a_1 as the head is turned, and, what is more serious, causes the bar EF to turn through angles which are by no means negligible, in addition to the angles directly due to the torsion of the wires. For this reason, annealed wires are more suitable for the experiment than hard drawn wires, as they are more easily straightened.

The torsion heads are read on circles divided at intervals of 45°, the dividing lines being scribed on the board XY.

§ 4. *Theory of the method.* If each torsion head is turned from its zero through ϕ radians in either direction, the bar EF will turn in the same direction until the bifilar and torsional couples are equal. If EF turns through θ, the whole twist of each wire is $\phi - \theta$.

Let the radius and the length of each wire be r cm. and l cm., and the rigidity of the metal n dyne cm.$^{-2}$. Since the wires are *nearly* vertical, the couple, due to torsion, exerted by the *pair* upon the bar is $\pi n r^4 (\phi - \theta)/l$*, to a close approximation.

The small couple due to the *bending* of the wires assists the bifilar couple; Kohlrausch† takes account of this small couple by writing in place of (1),

$$G = \frac{a_1 a_2 \sin\theta}{l'} . Mg, \qquad \text{......(3)}$$

where $$l' = l - r^2 \{2\pi E/Mg\}^{\frac{1}{2}}, \qquad \text{......(4)}$$

and E is Young's modulus.

Equating the torsional to the (corrected) bifilar couple, we have

$$\sin\theta = C(\phi - \theta), \qquad \text{......(5)}$$

where $$C = \frac{\pi n r^4 l'}{M g a_1 a_2 l}. \qquad \text{......(6)}$$

Then $$n = \frac{g a_1 a_2 l}{\pi r^4 l'} . MC. \qquad \text{......(7)}$$

§ 5. *Experimental details.* The distances $AC = 2a_1$, $BD = 2a_2$ are measured. The diameters of the wires are taken at a number of points and the mean radius is found.

The total mass, M grm., of the system carried by the wires is found. The masses of the screws are found before they are soldered to the wires.

The torsion heads are first set to zero, and the scales on which the pointers K, L are read are adjusted to be perpendicular to KL. If a goniometer is used, it is set so that its arm is in the central position when the goniometer wire coincides with its own image.

To eliminate errors due to slight bends in the wires, the readings must be taken over the range $-\pi$ to π for ϕ; the theory assumes absence of hysteresis. But in experimental work in elasticity we must realise that hysteresis effects are unavoidable, when the strains are more than infinitesimal. To ensure that the effects of hysteresis shall be orderly and not irregular, the torsion heads are taken through a *complete cycle* from π to $-\pi$ and back to π. To make the two readings for $\phi = \pi$ agree as closely as possible, a preliminary half cycle from $-\pi$ to π is done. To make the conditions uniform throughout the cycle and a half, the readings for the preliminary settings are taken and recorded; this will secure approximately constant time intervals between successive readings. Thus the heads are set in succession at the following multiples of $\pi/4$:

$$-4, -3, -2, -1, 0, 1, 2, 3,$$
$$4, 3, 2, 1, 0, -1, -2, -3, -4, -3, -2, -1, 0, 1, 2, 3, 4.$$

* G. F. C. Searle, *Experimental Elasticity*, § 39.
† Kohlrausch, *Wied. Ann.*, XVII, p. 737, 1882.

The first 8 are the preliminary readings, and only the last 17 are used.

If ϕ goes through a complete cycle, and θ is plotted against ϕ, a narrow hysteresis loop will be obtained. When readings are taken as above, there will be *two* values of θ for each value of ϕ except $\phi = -\pi$. With careful work, the two values of θ for $\phi = \pi$ will be exactly or very nearly identical; for the wires used in § 7, I have seldom found a difference between these two values as great as one minute. As a rough method of eliminating the effects of hysteresis, the mean of the two values of θ for each value of ϕ is taken as *the* value of θ for that value of ϕ.

The effect of bends in the wires near their upper ends, A, C (Fig. 2), will be the same as if these points described small horizontal circles about the centres A_0, C_0, as in Fig. 5. Let ϕ be measured

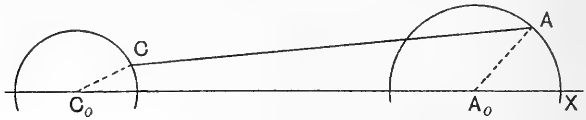


Fig. 5.

from C_0A_0X, and let $AA_0X = \phi + \alpha$, $CC_0X = \phi + \gamma$, while $A_0A = r$, $C_0C = s$, $A_0C_0 = 2a_1$. Then, if ϵ is the small angle between AC and A_0C_0,

$$\tan \epsilon = \frac{r \sin(\phi + \alpha) - s \sin(\phi + \gamma)}{2a_1 + r \cos(\phi + \alpha) - s \cos(\phi + \gamma)}.$$

When r and s are small compared with $2a_1$, $\tan\epsilon$ may be replaced by ϵ and the variable terms in the denominator may be neglected. Then, putting

$$(r \sin \alpha - s \sin \gamma)/2a_1 = P, \qquad (r \cos \alpha - s \cos \gamma)/2a_1 = Q,$$

we have

$$\epsilon = P \cos \phi + Q \sin \phi. \qquad \text{......(8)}$$

Here P and Q are the values of ϵ when $\phi = 0$ and $\phi = \frac{1}{2}\pi$.

If the line BD makes an angle θ with A_0C_0 when the heads read ϕ, the angle between BD and AC is $\theta - \epsilon$. The wires will not be quite free from torsion when the heads read zero; let η be the mean twist of the wires when $\phi = 0$. We must thus write $\sin(\theta - \epsilon)$ for $\sin\theta$ and $\phi + \eta - \theta$ for $\phi - \theta$ in the equilibrium equation (5), which thus becomes

$$\sin(\theta - \epsilon) = C(\phi + \eta - \theta). \qquad \text{......(9)}$$

To evade difficulties, θ is kept small. Then, since ϵ is also small, we may replace the sine by the angle in (9), and thus obtain

$$\theta = \frac{C}{1+C}(\phi+\eta)+\epsilon = D(\phi+\eta)+\epsilon. \qquad \text{......(10)}$$

If θ_0, ϵ_0 correspond to $\phi = 0$, we have, since $\epsilon_0 = P$,

$$\theta_0 = D\eta + \epsilon_0 = D\eta + P. \qquad \text{......(11)}$$

Thus

$$\theta - \theta_0 = D\phi + \epsilon - P = D\phi + P\cos\phi + Q\sin\phi - P \text{......(12)}$$

Since this equation is *linear* in θ, we may take θ_0 as corresponding to *any* initial position of the bar which is *near* its ideal zero position.

Thus, if β is the angle at any time between the bar and some nearly ideal zero position,

$$\beta = \theta - \theta_0. \qquad \text{......(13)}$$

Since β, though small—say less than 0·2 radian—is not infinitesimal, some correction should be made. An exact solution cannot be given, but accuracy is gained by writing $\sin\beta$ for β, and then the final formula becomes

$$\sin\beta = D\phi + P\cos\phi + Q\sin\phi - P. \qquad \text{......(14)}$$

To eliminate P and Q, we combine the observations. Let β_m correspond to $\phi = m\pi/4$. Then, putting $\phi = \pi$ and $\phi = -\pi$, so that $m = 4$ and $m = -4$, we have

$$\pi D = \tfrac{1}{2}(\sin\beta_4 - \sin\beta_{-4}). \qquad \text{......(15)}$$

A second value for πD is found by giving m the values 3, -3, 1, -1. Then

$$\pi D = \sin\beta_3 - \sin\beta_{-3} - (\sin\beta_1 - \sin\beta_{-1}). \qquad \text{......(16)}$$

The two values of D are usually in good agreement, although, when β is plotted against ϕ, the curve differs considerably from a straight line. The mean value of πD is used to find C. Thus

$$C = \frac{\pi D}{\pi - \pi D}. \qquad \text{......(17)}$$

Then n is found by (7).

The actual values of P and Q are easily found. Thus

$$P = -\tfrac{1}{4}(\sin\beta_4 + \sin\beta_{-4}), \qquad \text{......(18)}$$

$$Q = \tfrac{1}{2}(\sin\beta_2 - \sin\beta_{-2} - \pi D). \qquad \text{......(19)}$$

§ 6. *Conversion table.* A goniometer, such as those used at the Cavendish Laboratory, gives the *tangent* of the angle ψ through which the arm is turned from its zero. To find $\sin\psi$ we subtract from $\tan\psi$ the small quantity s given in the table.

$\tan\psi$	s	$\tan\psi$	s	$\tan\psi$	s
·01	·00000	·10	·00050	·19	·00334
·02	·00000	·11	·00066	·20	·00388
·03	·00001	·12	·00085	·21	·00448
·04	·00003	·13	·00108	·22	·00514
·05	·00006	·14	·00135	·23	·00585
·06	·00011	·15	·00166	·24	·00663
·07	·00017	·16	·00201	·25	·00746
·08	·00025	·17	·00240		
·09	·00036	·18	·00285		

Simple interpolation, by "proportional parts," will give s with an error not exceeding unity in the fifth place of decimals. Thus, if $\tan\psi = \cdot 124$, we find $s = \cdot 00095$, and then

$$\sin\psi = \tan\psi - s = \cdot 12305.$$

§ 7. *Practical example.* The following results were obtained for a pair of soft brass wires.

The distances AC, BD were each 6·00 cm. Hence $a_1 = a_2 = 3$ cm.
Mean radius of wires $= r =$ 0·0352 cm.
Length of each wire $= l =$ 47·30 cm.
Mass of suspended system, *excluding* the weight W (Fig. 2) = 417·6 gm.
The small correction for the buoyancy of the damper was neglected.

The deflexions were observed by a goniometer. The distance from the centre of the pivot to the scale was 40·00 cm. The central, or zero, reading is 10·00 cm. The following goniometer readings were obtained for the last 17 of the values of ϕ specified in § 5.

ϕ radians	Reading cm.	Reading cm.	Mean reading	x cm.	$\tan\beta$ $=x/40$	$\sin\beta$ obsd.	$\sin\beta$ calcd.
π	14·10	14·10	14·100	4·115	·1029	·1023	·1033
$\frac{3}{4}\pi$	12·96	13·04	13·000	3·015	·0754	·0752	·0749
$\frac{1}{2}\pi$	11·85	11·98	11·915	1·930	·0482	·0482	·0471
$\frac{1}{4}\pi$	10·80	10·96	10·880	0·895	·0224	·0224	·0220
0	9·90	10·07	**9·985**	0·000	·0000	·0000	·0000
$-\frac{1}{4}\pi$	9·16	9·33	9·245	−0·740	−·0185	−·0185	−·0204
$-\frac{1}{2}\pi$	8·30	8·43	8·365	−1·620	−·0405	−·0405	−·0415
$-\frac{3}{4}\pi$	7·37	7·44	7·405	−2·580	−·0645	−·0644	−·0653
$-\pi$	6·33		6·330	−3·655	−·0914	−·0910	−·0921

The value of x was found by subtracting from the mean reading, as given in column 4, the mean zero reading 9·985 cm. corresponding to $\phi = 0$. The differences between the readings in columns 2 and 3 are due to hysteresis. The seventh column shows that $\sin\beta$ is not proportional to ϕ.

By (15), $\pi D = \frac{1}{2}(\cdot 1023 + \cdot 0910) = \cdot 09665$,

and by (16), $\pi D = \cdot 0752 + \cdot 0644 - (\cdot 0224 + \cdot 0185) = \cdot 09870$.

Mean value of $\pi D = 0{\cdot}0977$.

Then, by (17) $$C = \frac{\pi D}{\pi - \pi D} = 0{\cdot}03210.$$

By (18), $P = -\frac{1}{4}(\cdot 1023 - \cdot 0910) = -0{\cdot}0028$,

and by (19), $Q = \frac{1}{2}(\cdot 0482 + \cdot 0405 - \cdot 0977) = -0{\cdot}0045$.

In the table, the column "sin β calcd." gives sin β as calculated by (14), using the values of πD, P and Q just found; there is fair agreement between the calculated and observed values of sin β.

The total load M was $417{\cdot}6 + 4999 = 5416{\cdot}6$ grm.

Taking $E = 10^{12}$ dyne cm.$^{-2}$, we have $r^2(2\pi E/Mg)^{\frac{1}{2}} = 1{\cdot}35$ cm., and hence, by (4), $l' = 47{\cdot}30 - 1{\cdot}35 = 45{\cdot}95$ cm.

Then, by (7),

$$n = \frac{g a_1 a_2 l}{\pi r^4 l'} . MC = \frac{981 \times 3^2 \times 47{\cdot}30}{\pi \times 0{\cdot}0352^4 \times 45{\cdot}95} \times 5416{\cdot}6 \times 0{\cdot}03210$$
$$= 3{\cdot}277 \times 10^{11} \text{ dyne cm.}^{-2}.$$

A similar set of observations, in which M was 3417·1 grm., gave the following values of sin β:

·1510, ·1123, ·0743, ·0361, ·0000, − ·0345, − ·0735, − ·1122, − ·1540.

Mean value of $\pi D = 0{\cdot}1532$. Hence $C = 0{\cdot}05127$.

Also $l' = 47{\cdot}30 - 1{\cdot}70 = 45{\cdot}60$ cm.

Then

$$n = \frac{981 \times 3^2 \times 47{\cdot}30}{\pi \times 0{\cdot}0352^4 \times 45{\cdot}60} \times 3417{\cdot}1 \times 0{\cdot}05127 = 3{\cdot}326 \times 10^{11} \text{ dyne cm.}^{-2}.$$

An independent determination of n was made by attaching a bar, of moment of inertia $K = 4{\cdot}766 \times 10^4$ grm. cm.^{2}, to each of the two wires in turn; the mean periodic time of the torsional vibrations was $T = 10{\cdot}55$ sec. Hence

$$n = \frac{8\pi K l}{T^2 r^4} = \frac{8\pi \times 4{\cdot}766 \times 10^4 \times 47{\cdot}30}{10{\cdot}55^2 \times 0{\cdot}0352^4} = 3{\cdot}316 \times 10^{11} \text{ dyne cm.}^{-2}.$$

The Rotation of the Non-Spinning Gyrostat. By G. T. BENNETT, M.A., F.R.S., Emmanuel College, Cambridge.

[*Read* 8 March 1920.]

§ 1. The following extract is taken from an old examination paper*:

> "A symmetrical wheel free to rotate about its axle is moved from rest in any position by means of the axle and is finally restored to a position in which the axle again points in the same direction as formerly. Shew that the wheel, again at rest, will have rotated through a plane angle equal to the solid angle of the cone described by the varying directions of the axle."

The proof of this result may be put briefly in a geometrical form. Translational and rotational movements being independent, the centroid of the wheel may be treated as stationary. As the gyrostat has no component rotation about its axis, the axis of rotation is at any moment some diameter of the wheel. This line has the central plane of the wheel as locus for the body-axode, and has a closed cone of arbitrary form as locus for the space-axode. The angular movement is therefore representable by the rolling of the plane on the cone. The angle of ultimate rotation of the wheel is thus (for cones of ordinary type) the excess of the four right angles of the plane surface above the total surface-angle of the cone. This difference is equal to the solid angle of the reciprocal cone described by the axis of the wheel. And hence follows the result quoted; namely, that the solid angle described by the axis of the wheel is equal to the circular measure of the plane angle of the resultant displacement of the wheel about its axis. Further, the sense of the displacement accords with the sense of circulation associated with the solid angle.

§ 2. The result may be extended to the case in which the initial and final directions of the axis are different, say a and b. For the axis may be restored to its original direction a by a subsequent movement in the plane ba; and this latter movement, which is a rotation about the normal to a and b, leaves unaltered the angle that any diameter of the wheel makes with the plane ab. Hence the original movement, shifting the axis of the wheel from a to b

* Emmanuel and other Colleges, Second Year Problems, Wed. June 8, 1898 Question 11.

by any conical movement, alters the angle between the plane *ab* and any diameter of the wheel by an angle equal to the solid angle enclosed by the cone formed by the conical surface *ab* together with the plane *ba*.

§ 3. A geometrical integration of Euler's equation leads to the same result as § 1. The axis, with its direction given by spherical polar coordinates θ and ϕ (radial and azimuthal), generates a solid angle

$$\sigma = \int (1 - \cos\theta) \, . \, d\phi. \qquad \ldots\ldots(1)$$

The equation of motion, being

$$\dot{\phi}\cos\theta + \dot{\psi} = 0, \qquad \ldots\ldots(2)$$

with zero initial values for ϕ and ψ, has as its integral

$$\phi + \psi = \sigma. \qquad \ldots\ldots(3)$$

If the axis of reference $\theta = 0$ is supposed (conveniently) external to the cone then ϕ is zero finally as well as initially, and ψ is the angle of resultant displacement of the wheel and is equal to the solid angle σ.

If, more generally, the gyrostat has a constant spin Ω about its axis, the Euler equation becomes

$$\dot{\phi}\cos\theta + \dot{\psi} = \Omega \qquad \ldots\ldots(4)$$

with

$$\phi + \psi = \sigma + \Omega t \qquad \ldots\ldots(5)$$

as its integral. And the final rotation of the gyrostat is then given by the solid angle of the cone described by the axis plus the time-integral of the spin. It may be noticed that the angle $\phi + \psi$, with a value independent of the choice of coordinates, gives in itself a natural measure of the total rotation of the wheel, as followed and estimated by projection on the plane $\theta = \pi/2$. For on that plane the circular disc shows as an ellipse, with ϕ as the azimuth of the direction of the minor axis, and ψ as the eccentric angle, measured from the minor axis, of the projection of the revolving diameter of the wheel. A distant observer on the axis $\theta = 0$, able to distinguish the two faces of the wheel, would in this way precisely reckon the amount of rotation, whole turns and fractional. He does not give merely the ultimate position, by naming a plane angle to a modulus of four right angles, but assigns the multiple of the modulus necessary for a correct account of the movement intervening between the initial and final positions.

A kinematic representation of the angle $\phi + \psi$ may be obtained by supposing the circular rim of the disc to have rolling contact with the rim of another equal disc whose plane keeps parallel to the plane $\theta = \pi/2$. The angle of rotation of this latter disc about its axis (which keeps the invariable direction $\theta = 0$) is then $\phi + \psi$.

§ 4. For the special case in which θ is constant, so that the axis of the gyrostat describes a circular cone, the rotation is stated by Sir George Greenhill* to be $2\pi -$ (conical angle described by the axle), as against the solid angle itself found above. The difference of sign of the latter can be accounted for by a reverse sign-convention: but the term 2π is unnecessary if 2π is implied as a modulus, and it appears to be wrong if the precise angle of turning is intended. If, specially, the axis of the gyrostat described only a small cone, then the angle of consequent rotation is certainly a small angle, and not an angle nearly equal to four right angles.

He adds the remark that the movement "can be shown experimentally with a penholder held between the fingers and moved round in a cone by the tip of a finger applied at the end." But the illustration is inapt; for the creep of the penholder occurs in the sense opposite to that of the conical movement. The body-axode is a circular cone and not a plane, and it rolls inside a slightly larger circular cone as space-axode; and hence the reverse movement.

§ 5. The movement of the non-spinning gyroscope here considered is not yet among those that are familiarly recognised, though it has important practical applications and deserves to rank as a dynamical commonplace. Bodies suspended from a point on an axis of symmetry behave in the same way and for the same reason. No matter how the point of suspension may be moved about, and no matter what complicated conical movement is consequently executed by the axis, the applied forces have no moment about the axis, and the spin remains zero if originally zero. The resultant rotation is then given, as above, by the solid angle of the cone described by the axis.

Aeroplane compasses, in particular, are found to keep their cards practically parallel to the floor, under the combined action of gravity and lateral acceleration, during a banked turn of the aeroplane. Hence, from inertia alone, and apart from all other sources of control or disturbance, the compass-card would be rotated, as a consequence of the turn, through an angle equal to the solid angle described by the normal to the card. For an angle of banking α and a change of course β the solid angle is not much less than $(1 - \cos \alpha)\beta$ if the banking is taken and left quickly; and for very steep banking this angle is nearly equal to the change of course itself, and the card would almost appear to "stick." As compared with considerations of magnetic disturbance due to the vertical component of the earth's field, and of mechanical disturbance due to rotation of the bowl and liquid, the pure inertia effect

* Advisory Committee for Aeronautics. Reports and Memoranda, No. 146, *Report on Gyroscopic Theory*, p. 13, § 14.

of the conical movement seems to need more emphasis than it has hitherto been awarded. It is here explicitly isolated.

The gyroscopic compass, like the magnetic compass, may at times suffer disturbance from this same source, if the compass-position in the ship and the run of the sea are such as to produce a circular or elliptical movement of the binnacle.

§ 6. It would be hard to trace to its primitive source the knowledge of the small piece of mechanics here discussed. It is really implicit in all treatises on Rigid Dynamics, but fails to emerge clearly amid the pressure of more important movements. Among empiricists it must be well-nigh prehistoric. The sailor in coiling a rope makes a winding motion of the feeding hand to remove the kinks from the overtwist of the piece which is to form the next turn of the coil. The circus clown, with the vertex of his conical cap resting on his finger-tip, or the end of a stick, easily makes it turn round and round; and the postman collecting his mail knows how to twist up the neck of his bag with a circular movement of the hand he holds it by. Later among empiricists are those who, accustomed to handle magnetic compasses, are very familiar with the rotation of the card produced so readily by giving the bowl a horizontal circular translational movement (without rotation). More lately still Mr S. G. Brown has noticed the conical motion and its effect. In the abstract of his lecture to the British Association* it is described as a "new phenomenon" and is stated as being "explainable mathematically." More fully in his lecture to the Royal Institution† he states that in virtue of the "wobbling" (*videlicet* conical) motion, "the needles and card would then have a force applied trying to carry the moving system round in the direction of the wobble." This mode of expression is of course entirely illegitimate. The rotational movement observed needs no "force" to explain it; the very essence of the inertia effect is that it occurs with no spin about the axis of rotation and no couple about that line either. Mr Brown announces also (but without demonstration) that if his compass-disc "is carried round in a horizontal circular path without any wobble the plate still goes round or tries to go round with the circular movement" and that this "should be of interest to mathematicians." It seems likely that the sheer paradox in angular momentum thus propounded will readily dissolve when all the relevant physical data are revealed: and meanwhile the interest is but that of a heresy resting on hearsay.

* British Association, Bournemouth, 1919. Evening Lecture, Fr. Sept. 12, "The Gyroscopic Compass." *Abstract*, ll. 9–14.

† *Nature*, March 11, 1920, p. 45, col. 2.

Proof of the equivalence of different mean values. By ALFRED KIENAST. (Communicated by Professor G. H. Hardy.)

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If $a_1, a_2, \ldots a_n, \ldots$ denote the terms of a sequence of complex numbers, and

$$S_n^{(0)} = a_1 + \ldots + a_n,$$
$$S_n^{(1)} = S_1^{(0)} + \ldots + S_n^{(0)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n^{(\kappa)} = S_1^{(\kappa-1)} + \ldots + S_n^{(\kappa-1)},$$

then $\lim\limits_{n\to\infty} S_n^{(\kappa)} \Big/ \binom{n+\kappa-1}{\kappa}$ is called Cesàro's κth mean* of the sequence $S_n^{(0)}$.

Putting
$$h_n^{(0)} = a_1 + \ldots + a_n,$$
$$h_n^{(1)} = \frac{1}{n}\{h_1^{(0)} + \ldots + h_n^{(0)}\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_n^{(\kappa)} = \frac{1}{n}\{h_1^{(\kappa-1)} + \ldots + h_n^{(\kappa-1)}\},$$

then $\lim\limits_{n\to\infty} h_n^{(\kappa)}$ is called Hölder's κth mean† of the sequence $h_n^{(0)}$.

In a paper "Extensions of Abel's Theorem and its converses‡" I found it convenient to introduce the expressions

$$s_n^{(0)} = a_1 + \ldots + a_n \qquad (n = 1, 2, \ldots),$$
$$s_n^{(1)} = \frac{1}{n}\{s_1^{(0)} + \ldots + s_{n-1}^{(0)}\} \qquad (n = 2, 3, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_n^{(\kappa)} = \frac{1}{n}\{s_\kappa^{(\kappa-1)} + \ldots + s_{n-1}^{(\kappa-1)}\} \qquad (n = \kappa+1, \kappa+2, \ldots),$$

and proved various theorems concerning the limits $\lim\limits_{n\to\infty} s_n^{(\kappa)}$.

Several writers have proved

Theorem 1. *Whenever Cesàro's (Hölder's) κth mean exists and is finite, then Hölder's (Cesàro's) κth mean exists too, and both have the same value.*

* Bromwich, *Infinite series*, p. 310. † *Ibid.*, p. 313.
‡ *Proc. Cambridge Phil. Soc.*, vol. XIX, 1918, pp. 129–147.

I propose to complete the researches of my above quoted paper by proving the theorem:

Theorem 2. *Whenever Hölder's (and therefore Cesàro's) κth mean exists and is finite, then* $\lim_{n\to\infty} s_n^{(\kappa)}$ *exists too, and both have the same value, and vice versa.*

The demonstration of both theorems is based upon relations between the mean values which it is possible to calculate completely, as I have found, in a most simple manner.

In §§ I to VI I determine the expression of $s_n^{(\kappa)}$ by

$$h_\lambda^{(\kappa)} \qquad (\lambda = 1, 2, \dots (n-\kappa)),$$

in § VII the expression of $h_n^{(\kappa)}$ by

$$s_{\lambda-\kappa}^{(\kappa)} \qquad (\lambda = 1, 2, \dots n),$$

in § VIII the expression of $S_n^{(\kappa)}$ by

$$h_\lambda^{(\kappa)} \qquad (\lambda = 1, 2, \dots n),$$

and finally in § IX I consider two more general mean values.

I. From the definitions follow

$$s_n^{(0)} = h_n^{(0)}, \quad h_n^{(0)} = nh_n^{(1)} - (n-1)\,h_{n-1}^{(1)},$$

$$s_n^{(1)} = \frac{1}{n}\sum_{\lambda=1}^{n-1} h_\lambda^{(0)} = \frac{1}{n}\sum_{\lambda=1}^{n-1}\{\lambda h_\lambda^{(1)} - (\lambda-1)\,h_{\lambda-1}^{(1)}\} = \frac{n-1}{n}\,h_{n-1}^{(1)},$$

$$h_n^{(1)} = nh_n^{(2)} - (n-1)\,h_{n-1}^{(2)},$$

$$s_n^{(2)} = \frac{1}{n}\sum_{\lambda=2}^{n-1}\frac{\lambda-1}{\lambda}\,h_{\lambda-1}^{(1)} = \frac{1}{n}\left[\frac{1}{2}h_1^{(2)} + \frac{2}{3}(2h_2^{(2)} - h_1^{(2)}) + \dots \right.$$

$$+ \frac{n-2}{n-1}\{(n-2)\,h_{n-2}^{(2)} - (n-3)\,h_{n-3}^{(2)}\}$$

$$\left. + \frac{n-1}{n}\{(n-2)\,h_{n-2}^{(2)} - (n-2)\,h_{n-2}^{(2)}\}\right].$$

Adding a term which is zero,

$$s_n^{(2)} = \frac{(n-1)(n-2)}{n^2}\,h_{n-2}^{(2)} - \frac{1}{n}\left\{\frac{1}{6}h_1^{(2)} + \frac{1}{6}h_2^{(2)} + \dots + \frac{n-2}{n(n-1)}h_{n-2}^{(2)}\right\},$$

etc. Now I suppose that, by continuing in this manner, I have arrived at the formula

$$s_n^{(\kappa)} = c_{n,\kappa}h_{n-\kappa}^{(\kappa)} - \frac{1}{n}\sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa}h_\lambda^{(\kappa)} \quad \dots\dots\dots\dots\dots(1),$$

where $c_{n,\kappa}$ is a function of the indices n and κ, and the coefficients $d_{\lambda,\kappa}(\lambda = 1, 2, \dots)$ are, for each κ, definite numbers which are the values of a function of λ for $\lambda = 1, 2, \dots.$

Proceeding to build up the expression for $s_n^{(\kappa+1)}$, applying the same transformations as above, we find

$$s_n^{(\kappa+1)} = \frac{1}{n}\sum_{\lambda=\kappa+1}^{n-1}\left[c_{\lambda,\kappa}\{(\lambda-\kappa)\,h_{\lambda-\kappa}^{(\kappa+1)} - (\lambda-\kappa-1)\,h_{\lambda-\kappa-1}^{(\kappa+1)}\}\right.$$
$$\left. - \frac{1}{\lambda}\sum_{\nu=1}^{\lambda-\kappa} d_{\nu,\kappa}\{\nu h_\nu^{(\kappa+1)} - (\nu-1)\,h_{\nu-1}^{(\kappa+1)}\}\right]$$
$$+ \frac{1}{n}c_{n,\kappa}\{(n-\kappa-1)\,h_{n-\kappa-1}^{(\kappa+1)} - (n-\kappa-1)\,h_{n-\kappa-1}^{(\kappa+1)}\},$$

$$s_n^{(\kappa+1)} = c_{n,\kappa}\frac{n-\kappa-1}{n}h_{n-\kappa-1}^{(\kappa+1)} - \frac{1}{n}\sum_{\lambda=1}^{n-\kappa-1} d_{\lambda,\kappa+1}h_\lambda^{(\kappa+1)} \quad \text{...........(2)};$$

from which we conclude

$$c_{n,\kappa+1} = c_{n,\kappa}\frac{n-\kappa-1}{n} \quad \text{....................(3)},$$

and a series of relations involving the numbers $d_{\lambda,\kappa}$ and $d_{\lambda,\kappa+1}$.

Equation (2) is of the same formation as (1), and therefore (1) gives the required expression of $s_n^{(\kappa)}$ by the numbers $h_\lambda^{(\kappa)}$.

II. Since $c_{n,1} = \dfrac{n-1}{n}$, (3) leads to

$$c_{n,\kappa} = n^{-\kappa}(n-\kappa)(n-\kappa+1)\dots(n-1),$$

from which follows, for $\kappa = 2$, $c_{n,2} = \dfrac{(n-1)(n-2)}{n^2}$, which is in accordance with the expression for $s_n^{(2)}$ above.

III. $d_{\lambda,\kappa}$ may be determined in the following way. Putting

$$a_1 = a_2 = \dots = a_{\lambda-1} = 0,\quad a_\lambda = 1,\quad a_{\lambda+1} = \dots = 0 \quad \text{......(4)},$$

we find

$$s_1^{(0)} = \dots = s_{\lambda-1}^{(0)} = 0, \qquad s_\lambda^{(0)} = 1, \qquad s_{\lambda+1}^{(0)} = 1, \qquad \dots,$$

$$s_2^{(1)} = \dots = s_\lambda^{(1)} = 0, \qquad s_{\lambda+1}^{(1)} = \frac{1}{\lambda+1}, \quad s_{\lambda+2}^{(1)} = \frac{2}{\lambda+2}, \quad \dots,$$

$$s_3^{(2)} = \dots = s_{\lambda+1}^{(2)} = 0, \qquad s_{\lambda+2}^{(2)} = \frac{1}{(\lambda+1)(\lambda+2)}, \quad \dots,$$

....

$$s_{\kappa+1}^{(\kappa)} = \dots = s_{\lambda+\kappa-1}^{(\kappa)} = 0, \quad s_{\lambda+\kappa}^{(\kappa)} = \frac{1}{(\lambda+1)(\lambda+2)\dots(\lambda+\kappa)}, \quad \dots;$$

and
$$h_1^{(0)} = \ldots = h_{\lambda-1}^{(0)} = 0, \quad h_\lambda^{(0)} = 1, \quad h_{\lambda+1}^{(0)} = 1, \quad \ldots,$$
$$h_1^{(1)} = \ldots = h_{\lambda-1}^{(1)} = 0, \quad h_\lambda^{(1)} = \frac{1}{\lambda}, \quad h_{\lambda+1}^{(1)} = \frac{2}{\lambda+1}, \quad \ldots,$$
$$h_1^{(2)} = \ldots = h_{\lambda-1}^{(2)} = 0, \quad h_\lambda^{(2)} = \frac{1}{\lambda^2}, \quad \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_1^{(\kappa)} = \ldots = h_{\lambda-1}^{(\kappa)} = 0, \quad h_\lambda^{(\kappa)} = \lambda^{-\kappa}, \quad \ldots.$$

Writing equation (1) for $n = \lambda + \kappa$, and substituting these special values, we obtain

$$d_{\lambda,\kappa} = (\lambda+\kappa)\left\{\frac{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)}{(\lambda+\kappa)^\kappa} - \frac{\lambda^\kappa}{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)}\right\},$$

which is, for $\kappa = 2$, in accordance with the above expression for $s_n^{(2)}$.

IV. *Lemma α. The coefficient $d_{\lambda,\kappa}$ is a positive number for* $\lambda = 1, 2, \ldots, \kappa = 2, 3, \ldots$.

It is easy to verify the inequalities

$$\frac{\lambda+\mu}{\lambda+\kappa} \geqq \frac{\lambda}{\lambda+\kappa-\mu} \qquad (\mu = 0, 1, 2, \ldots, \kappa-1),$$

from which results, by multiplication of all the left-hand and all the right-hand sides,

$$\frac{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)}{(\lambda+\kappa)^\kappa} \geqq \frac{\lambda^\kappa}{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)},$$

which demonstrates the assertion.

Lemma β. The coefficient $d_{\lambda,\kappa}$ satisfies the equation

$$\lim_{n\to\infty}\left\{d_{n,\kappa} - \frac{1}{6}(\kappa-1)\kappa(\kappa+1)\frac{1}{n}\right\} = 0\ . \quad \ldots\ldots\ldots\ldots(5).$$

To show this we expand $d_{\lambda,\kappa}$ in the form

$$d_{\lambda,\kappa} = \frac{\frac{1}{6}(\kappa-1)\kappa(\kappa+1)\lambda^{2\kappa-3} + \ldots}{\lambda^{2\kappa-2}(1+\alpha\lambda^{-1}+\ldots)} = \frac{1}{6}(\kappa-1)\kappa(\kappa+1)\frac{1}{\lambda} + \frac{c}{\lambda^2} + \ldots,$$

and the proposition follows.

A consequence of (5) is

$$\lim_{n\to\infty}\sum_{\lambda=1}^{n} d_{\lambda,\kappa} = \infty,$$

and therefore the conditions of Stolz's theorem are satisfied. Thus we can state:

Lemma γ. *If* $\lim_{n\to\infty} h_n^{(\kappa)} = H$ *exists and is finite, then*

$$\lim_{n\to\infty} \sum_{\lambda=1}^{n} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \Big/ \sum_{\lambda=1}^{n} d_{\lambda,\kappa} = H.$$

V. Let
$$a_1 = 1, \quad a_2 = a_3 = \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6);$$
then
$$s_n^{(0)} = 1 \qquad (n = 1, 2, \ldots),$$
$$s_n^{(1)} = \frac{n-1}{n} \qquad (n = 2, 3, \ldots);$$
therefore $\lim_{n\to\infty} s_n^{(1)} = 1$, and consequently
$$\lim_{n\to\infty} s_n^{(2)} = \lim_{n\to\infty} s_n^{(3)} = \ldots = 1.$$
Furthermore
$$h_n^{(0)} = h_n^{(1)} = h_n^{(2)} = \ldots = 1 \qquad (n = 1, 2, \ldots).$$
From (1) and (6) we obtain
$$\lim_{n\to\infty} s_n^{(\kappa)} = 1 = \lim_{n\to\infty} c_{n,\kappa} - \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n-\kappa} d_{\lambda,\kappa},$$
or
$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} = 0.$$

VI. Now passing in equation (1) to the limit $n \to \infty$, we find
$$\lim_{n\to\infty} s_n^{(\kappa)} = \lim_{n\to\infty} h_{n-\kappa}^{(\kappa)} - \lim_{n\to\infty} \left(\frac{1}{n} \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa}\right) \lim_{n\to\infty} \left\{\sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \Big/ \sum_{\lambda=1}^{n-\kappa} d_{\lambda,\kappa}\right\},$$
and this equation leads to the theorems:

Theorem 3. *Whenever* $\lim_{n\to\infty} h_{n-\kappa}^{(\kappa)}$ *exists and is finite, then* $\lim_{n\to\infty} s_n^{(\kappa)}$ *exists too and has the same value.*

More generally

Theorem 4. *When the function* $h_{n-\kappa}^{(\kappa)}$ *oscillates between finite limits, then* $s_n^{(\kappa)}$ *oscillates between the same limits.*

VII. The reverse propositions can be established in the same way. From the definitions follow
$$h_n^{(1)} = \frac{n+1}{n} s_{n+1}^{(1)}, \qquad s_\lambda^{(1)} = (\lambda+1) s_{\lambda+1}^{(2)} - \lambda s_\lambda^{(2)},$$
$$h_n^{(2)} = \frac{1}{n}\left[\sum_{\lambda=1}^{n} \frac{\lambda+1}{\lambda} \{(\lambda+2) s_{\lambda+2}^{(2)} - (\lambda+1) s_{\lambda+1}^{(2)}\}\right.$$
$$\left. + \frac{n+2}{n+1} \{(n+2) s_{n+2}^{(2)} - (n+2) s_{n+2}^{(2)}\}\right]$$
$$= \frac{(n+2)^2}{n(n+1)} s_{n+2}^{(2)} + \frac{1}{n} \sum_{\lambda=1}^{n} \frac{\lambda+2}{\lambda(\lambda+1)} s_{\lambda+2}^{(2)}.$$

Continuing this process we find

$$h_n^{(\kappa)} = e_{n,\kappa} s_{n+\kappa}^{(\kappa)} + \frac{1}{n} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} s_{\lambda+\kappa}^{(\kappa)} \quad \ldots\ldots\ldots\ldots\ldots(7).$$

In the same way as we determined $c_{n,\kappa}$ by (2) and (3), we obtain here

$$e_{n,\kappa+1} = e_{n+1,\kappa} \frac{n+\kappa+1}{n}.$$

Thus, since $$e_{n,1} = \frac{n+1}{n},$$

$$e_{n,\kappa} = \frac{(n+\kappa)^\kappa}{n(n+1)\ldots(n+\kappa-1)}.$$

Taking the values (4) for the numbers a_λ, we find from (7)

$$f_{\lambda,\kappa} = \lambda \left\{ \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+\kappa)}{\lambda^\kappa} - \frac{(\lambda+\kappa)^\kappa}{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)} \right\}.$$

The considerations in § IV show that

$$f_{\lambda,\kappa} \geqq 0, \qquad (\lambda = 1, 2, \ldots;\ \kappa = 2, 3, \ldots).$$

Expansion of $f_{\lambda,\kappa}$ in descending powers of λ gives

$$f_{\lambda,\kappa} = \frac{1}{6}(\kappa-1)\,\kappa\,(\kappa+1)\frac{1}{\lambda} + \frac{c}{\lambda^2} + \ldots;$$

thus $$\lim_{n\to\infty} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} = \infty.$$

Introducing in equation (7) the values (6) for the numbers a_λ, we find

$$1 = \lim_{n\to\infty} e_{n,\kappa} + \lim_{n\to\infty}\left(\frac{1}{n}\sum_{\lambda=1}^{n} f_{\lambda,\kappa}\right) \lim_{n\to\infty} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} s_{\lambda+\kappa}^{(\kappa)} \Big/ \sum_{\lambda=1}^{n} f_{\lambda,\kappa};$$

which gives, on account of Stolz's theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda=1}^{n} f_{\lambda,\kappa} = 0.$$

Thus equation (7) is completely determined and leads to

Theorem 5. *Whenever* $\lim_{n\to\infty} s_{n+\kappa}^{(\kappa)}$ *exists and is finite, then* $\lim_{n\to\infty} h_n^{(\kappa)}$ *exists too and has the same value.*

Theorem 6. *Whenever the function* $s_{n+\kappa}^{(\kappa)}$ *oscillates finitely, then* $h_n^{(\kappa)}$ *oscillates between the same limits.*

Theorems 3 and 5 together constitute theorem 2.

VIII. The relation connecting Cesàro's and Hölder's means can be deduced in the same way. We have

$$S_n^{(0)} = h_n^{(0)}, \qquad h_n^{(0)} = nh_n^{(1)} - (n-1)\, h_{n-1}^{(1)},$$

$$\frac{1}{n} S_n^{(1)} = \frac{1}{n} \sum_{\lambda=1}^{n} \{\lambda h_\lambda^{(1)} - (\lambda - 1)\, h_{\lambda-1}^{(1)}\} = h_n^{(1)},$$

$$\frac{2S_n^{(2)}}{n(n+1)} = \frac{2}{n(n+1)} \left[\sum_{\lambda=1}^{n} \lambda \{\lambda h_\lambda^{(2)} - (\lambda-1) h_{\lambda-1}^{(2)}\} + (n+1)\{nh_n^{(2)} - nh_n^{(2)}\} \right]$$

$$= 2h_n^{(2)} - \frac{2}{n(n+1)} \sum_{\lambda=1}^{n} \lambda h_\lambda^{(2)}.$$

Assuming therefore

$$\frac{S_n^{(\kappa)}}{\binom{n+\kappa-1}{\kappa}} = c_{n,\kappa} h_n^{(\kappa)} - \frac{1}{\binom{n+\kappa-1}{\kappa}} \sum_{\lambda=1}^{n} d_{\lambda,\kappa} h_\lambda^{(\kappa)} \quad \text{......(8)},$$

we find $$S_n^{(\kappa+1)} = \binom{n+\kappa}{\kappa} nc_{n+1,\kappa} h_n^{(\kappa+1)} - \sum_{\lambda=1}^{n} d_{\lambda,\kappa+1} h_\lambda^{(\kappa+1)}.$$

Hence $$c_{n,\kappa+1} = (\kappa+1)\, c_{n+1,\kappa},$$

or $$c_{n,\kappa} = \kappa!.$$

Starting from the numbers (4), we have

$$S_1^{(0)} = \ldots = S_{\lambda-1}^{(0)} = 0,\ S_\lambda^{(0)} = S_{\lambda+1}^{(0)} = \ldots = 1,$$

$$S_1^{(1)} = \ldots = S_{\lambda-1}^{(1)} = 0,\ S_\lambda^{(1)} = 1,\ S_{\lambda+1}^{(1)} = 2,\ \ldots,$$

....

$$S_1^{(\kappa)} = \ldots = S_{\lambda-1}^{(\kappa)} = 0,\ S_\lambda^{(\kappa)} = 1,\ \ldots,\ S_n^{(\kappa)} = \binom{n-\lambda+\kappa}{\kappa},\ \ldots;$$

as is easily verified by the formula

$$S_n^{(\kappa+1)} = \sum_{\nu=1}^{n} S_\nu^{(\kappa)} = \sum_{m=0}^{n-\lambda} \binom{m+\kappa}{\kappa} = \binom{n-\lambda+\kappa+1}{\kappa+1}.$$

Writing (8) for $n = \lambda$, we find

$$d_{\lambda,\kappa} = \lambda(\lambda+1)\ldots(\lambda+\kappa-1) - \lambda^\kappa \geqq 0.$$

Starting with the numbers (6), that is with the numbers (4) for $\lambda = 1$, we have

$$S_n^{(\kappa)} = \binom{n+\kappa-1}{\kappa}, \qquad h_n^{(\kappa)} = 1,$$

and from formula (8) follows

$$\sum_{\lambda=1}^{n} d_{\lambda,\kappa} = (\kappa! - 1) \binom{n+\kappa-1}{\kappa},$$

so that finally

$$\frac{S_n^{(\kappa)}}{\binom{n+\kappa-1}{\kappa}} = \kappa! h_n^{(\kappa)} - (\kappa! - 1) \frac{\sum_{\lambda=1}^{n} d_{\lambda,\kappa} h_\lambda^{(\kappa)}}{\sum_{\lambda=1}^{n} d_{\lambda,\kappa}} \quad \text{.........(9).}$$

Analogous considerations lead to

$$h_n^{(\kappa)} = \frac{1}{\kappa!} \frac{S_n^{(\kappa)}}{\binom{n+\kappa-1}{\kappa}} + \left(1 - \frac{1}{\kappa!}\right) \frac{\sum_{\lambda=1}^{n} f_{\lambda,\kappa} S_\lambda^{(\kappa)}}{\sum_{\lambda=1}^{n} f_{\lambda,\kappa} \binom{\lambda+\kappa-1}{\kappa}} \quad \text{...(10),}$$

$$f_{\lambda,\kappa} = \lambda \left\{\frac{1}{\lambda^\kappa} - \frac{1}{\lambda(\lambda+1)\ldots(\lambda+\kappa-1)}\right\} \geqq 0,$$

$$\sum_{\lambda=1}^{n} f_{\lambda,\kappa} \binom{\lambda+\kappa-1}{\kappa} = n\left(1 - \frac{1}{\kappa!}\right).$$

Formulae (9) and (10) prove theorem 1.

IX. By similar considerations it is possible to arrive at a statement about the equivalence of two means of the kind examined in Part II of my above quoted paper.

Let b_κ, c_κ denote the terms of two infinite sequences of positive real numbers, which have, when we write

$$\sum_1^n b_\kappa = B_\kappa, \qquad \sum_1^n c_\kappa = C_\kappa,$$

the properties (i) $\lim_{n\to\infty} B_n = \infty$, $\quad \lim_{n\to\infty} C_n = \infty$,

(ii) $\frac{1}{n}\sum_1^n \frac{\kappa b_\kappa}{B_\kappa}$, $\quad \frac{1}{n}\sum_1^n \frac{\kappa c_\kappa}{C_\kappa}$

tend to limits or oscillate between finite limits. Then putting as before

$$s_n^{(0)} = \sum_1^n a_n,$$

the means

$$s_n^{(1)} = \frac{1}{B_n} \sum_2^n b_\lambda s_{\lambda-1}^{(0)} \quad \text{.....................(11),}$$

$$t_n^{(1)} = \frac{1}{C_n} \sum_2^n c_\lambda s_{\lambda-1}^{(0)} \quad \text{.....................(12),}$$

are connected by two analogous relations. From (11) and (12) follow

$$b_n s_{n-1}^{(0)} = B_n s_n^{(1)} - B_{n-1} s_{n-1}^{(1)} \quad \text{...............(13),}$$

$$c_n s_{n-1}^{(0)} = C_n t_n^{(1)} - C_{n-1} t_{n-1}^{(1)} \text{..................(14).}$$

Substituting (14) in (11) we find, on adding a term which is zero,

$$s_n^{(1)} = \frac{1}{B_n}\left[\frac{b_2}{c_2} C_2 t_2^{(1)} + \frac{b_3}{c_3}\{C_3 t_3^{(1)} - C_2 t_2^{(1)}\} + \dots \right.$$

$$\left. \dots + \frac{b_n}{c_n}\{C_n t_n^{(1)} - C_{n-1} t_{n-1}^{(1)}\} + \frac{b_{n+1}}{c_{n+1}}\{C_n t_n^{(1)} - C_n t_n^{(1)}\}\right]$$

$$= \frac{1}{B_n}\sum_{\lambda=2}^{n}\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)} + \frac{b_{n+1}}{B_n}\frac{C_n}{c_{n+1}} t_n^{(1)}.$$

Since

$$\sum_1^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda = \sum_2^n \frac{b_\lambda}{c_\lambda}(C_\lambda - C_{\lambda-1}) + \frac{b_1 C_1}{c_1} - \frac{b_{n+1}}{c_{n+1}} C_n$$

$$= B_n\left[1 - \frac{b_{n+1}}{B_n}\frac{C_n}{c_n}\right],$$

we can write

$$s_n^{(1)} = t_n^{(1)} + \left(1 - \frac{b_{n+1}}{B_n}\frac{C_n}{c_{n+1}}\right)\left\{- t_n^{(1)} + \frac{\sum_2^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)}}{\sum_2^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda}\right\} \quad \dots(15).$$

Now there may be distinguished two possibilities:

Theorem 7. *If*

$$\left|\frac{\sum_2^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)}}{\sum_2^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda}\right| < K \ (\textit{fixed}),$$

$$\lim_{n\to\infty}\frac{C_n}{c_{n+1}}\frac{b_{n+1}}{B_n} = 1,$$

and if $t_n^{(1)}$ *approaches a finite limit (or oscillates between finite limits), then* $s_n^{(1)}$ *approaches the same limit (or oscillates between the same limits).*

Theorem 8. *If*

$$\lim_{n\to\infty}\frac{\sum_2^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda t_\lambda^{(1)}}{\sum_2^n\left(\frac{b_\lambda}{c_\lambda} - \frac{b_{\lambda+1}}{c_{\lambda+1}}\right) C_\lambda} = \lim_{n\to\infty} t_n^{(1)},$$

$$\frac{b_{n+1}}{B_n}\frac{C_n}{c_{n+1}} < K \ (\textit{fixed}),$$

and if $t_n^{(1)}$ *approaches a limit, then* $s_n^{(1)}$ *approaches the same limit.*

This is a known theorem*.

The second relation results by substituting from (13) in (12) and proceeding in the same way. The same formula is arrived at by interchanging in (15) b_κ and c_κ, B_n and C_n, $t_n^{(1)}$ and $s_n^{(1)}$. From it we infer two theorems analogous to (7) and (8).

* Bromwich, *Infinite Series*, p. 386, Theorem V.

Notes on the Theory of Vibrations. **(1)** *Vibrations of Finite Amplitude.* (2) *A Theorem due to Routh.* By W. J. HARRISON, M.A., Fellow of Clare College.

[*Read* 3 May 1920.]

(I) Lord Rayleigh in his *Theory of Sound*, Vol. I, has considered the effect of introducing terms depending on x^2 and x^3 into the simple equation of vibratory motion $\frac{d^2x}{dt^2} + n^2x = 0$. He treats the added terms as small and employs the method of successive approximation. The object of this note is to point out that exact integrals can be obtained in the form of the series of which Rayleigh determined the first two or three terms. The solutions now obtained are valid for any relative magnitude of the added terms subject to the motion remaining vibratory.

(*a*) *The Symmetric System.*

The equation of motion is

$$\frac{d^2x}{dt^2} + n^2x \mp 2\beta x^3 = 0,$$

where β is positive, and the upper sign is taken in the first instance.

A first integral is

$$\left(\frac{dx}{dt}\right)^2 = n^2a^2 - \beta a^4 - n^2x^2 + \beta x^4,$$

where a is the amplitude of the vibration.

We have

$$\left(\frac{dx}{dt}\right)^2 = (a^2 - x^2)(n^2 - \beta a^2 - \beta x^2),$$

or
$$\left(\frac{dx_1}{dt}\right)^2 = (n^2 - \beta a^2)(1 - x_1{}^2)(1 - k^2x_1{}^2),$$

where
$$ax_1 = x, \ k^2 = \beta a^2/(n^2 - \beta a^2).$$

Hence*

$$\begin{aligned} x &= ax_1 \\ &= a \operatorname{sn}\{(n^2 - \beta a^2)^{\frac{1}{2}}t, k\}, \quad (x = 0 \text{ when } t = 0) \\ &= \frac{2\pi a}{Kk} \sum_0^\infty \frac{q^{m+\frac{1}{2}}}{1 - q^{2m+1}} \sin \frac{(2m+1)\pi(n^2 - \beta a^2)^{\frac{1}{2}}t}{2K}. \end{aligned}$$

* For the expansions of elliptic functions quoted in this paper see Whittaker and Watson, *Modern Analysis*, 1915, p. 504, and Example (5), p. 513; or Hancock, *Theory of Elliptic Functions*, Vol. I, 1910, pp. 486, 494, 495.

Let the units of length and time be chosen so that $a = 1$, $n = 1$. It is necessary that $\beta < \frac{1}{2}$, otherwise $\frac{dx}{dt}$ vanishes first for $0 < x < 1$.

The effect of the term $2\beta x^3$ on the vibrations can be exhibited by the results of numerical calculation given in the following table:

β	x	p
·01	$1{\cdot}0006 \sin pt + {\cdot}0006 \sin 3pt$	9924
·1	$1{\cdot}0074 \sin pt + {\cdot}0074 \sin 3pt$	·9214
·3	$1{\cdot}0335 \sin pt + {\cdot}0348 \sin 3pt + {\cdot}0012 \sin 5pt$	·7309
·4	$1{\cdot}0632 \sin pt + {\cdot}0676 \sin 3pt + {\cdot}0046 \sin 5pt + {\cdot}0003 \sin 7pt$	·5997
·45	$1{\cdot}0928 \sin pt + {\cdot}1028 \sin 3pt + {\cdot}0108 \sin 5pt + {\cdot}0011 \sin 7pt + {\cdot}0001 \sin 9pt$	·5063
·5	$\tanh (t/\sqrt{2})$	

We proceed to consider the equation

$$\frac{d^2x}{dt^2} + n^2x + 2\beta x^3 = 0.$$

If a is the amplitude of the motion as before, we have

$$\left(\frac{dx}{dt}\right)^2 = (a^2 - x^2)(n^2 + \beta a^2 + \beta x^2),$$

or
$$\left(\frac{dx_1}{dt}\right)^2 = (n^2 + \beta a^2)(1 - x_1^2)(1 + \mu^2 x_1^2),$$

where
$$ax_1 = x, \text{ and } \mu^2 = \beta a^2/(n^2 + \beta a^2).$$

Write $1 - x_1^2 = z^2$, so that

$$\left(\frac{dz}{dt}\right)^2 = (n^2 + 2\beta a^2)(1 - z^2)(1 - k^2z^2),$$

where
$$k^2 = \mu^2/(1 + \mu^2) = \beta a^2/(n^2 + 2\beta a^2).$$

Thus $\quad z = \operatorname{sn}\{(n^2 + 2\beta a^2)^{\frac{1}{2}}t, k\}, \quad (z = 0 \text{ when } t = 0).$

Therefore

$$\begin{aligned} x &= ax_1 \\ &= a \operatorname{cn}\{(n^2 + 2\beta a^2)^{\frac{1}{2}}t, k\} \\ &= \frac{2\pi a}{Kk} \sum_0^\infty \frac{q^{m+\frac{1}{2}}}{1 + q^{2m+1}} \cos \frac{(2m+1)\pi(n^2 + 2\beta a^2)^{\frac{1}{2}}t}{2K}. \end{aligned}$$

In this case there is no limit to the value of β, the motion remains vibratory, but the period of the gravest mode decreases as β increases. The results of calculation, with $n = 1$, $a = 1$, are as follows:

β	x	p
·1	$\cdot995 \cos pt + \cdot005 \cos 3pt$	1·072
·5	$\cdot9818 \cos pt + \cdot0179 \cos 3pt + \cdot0003 \cos 5pt$	1·318
1·0	$\cdot9742 \cos pt + \cdot0253 \cos 3pt + \cdot0006 \cos 5pt$	1·569
10·0	$\cdot9582 \cos pt + \cdot0402 \cos 3pt + \cdot0016 \cos 5pt + \cdot0001 \cos 7pt$	3·975
100·0	$\cdot9555 \cos pt + \cdot0427 \cos 3pt + \cdot0018 \cos 5pt + \cdot0001 \cos 7pt$	12·03

(b) *The Asymmetric System.*

The equation of motion is

$$\frac{d^2x}{dt^2} + n^2x + \tfrac{3}{2}\alpha x^2 = 0,$$

where α may be assumed to be positive, as changing the sign of α is equivalent to reversing the direction of the axis of x.

Let the scale of time be such that $n = 1$, and the scale of length chosen so that the amplitude of the motion measured from $x = 0$ in the direction of x positive is unity. Then

$$\left(\frac{dx}{dt}\right)^2 = (1 - x)(1 + \alpha + x + \alpha x + \alpha x^2)$$
$$= (1 - x)(b + x)(c + x),$$

where
$$b = \tfrac{1}{2}\{1 + \alpha - (1 - 2\alpha - 3\alpha^2)^{\frac{1}{2}}\}/\alpha,$$
$$c = \tfrac{1}{2}\{1 + \alpha + (1 - 2\alpha - 3\alpha^2)^{\frac{1}{2}}\}/\alpha.$$

The limits of the vibration are $x = 1$ and $x = -b$. It is necessary that α should be less than $\frac{1}{3}$, so that the greatest value of b is 2.

Writing $1 - x = (b + 1)y^2$, we have

$$\left(\frac{dy}{dt}\right)^2 = \tfrac{1}{4}\alpha(c + 1)(1 - y^2)(1 - k^2y^2),$$

where $k^2 = (b + 1)/(c + 1)$.

Hence

$$y = \operatorname{sn}\{\tfrac{1}{2}\alpha^{\frac{1}{2}}(c + 1)^{\frac{1}{2}}t, k\},$$

and $$x = 1 - (b + 1)\operatorname{sn}^2\{\tfrac{1}{2}\alpha^{\frac{1}{2}}(c + 1)^{\frac{1}{2}}t, k\}$$
$$= 1 - \frac{b + 1}{k^2}\left\{1 - \frac{E}{K} - \frac{2\pi^2}{K^2}\sum_1^\infty \frac{mq^m}{1 - q^{2m}} \cos \frac{m\pi\alpha^{\frac{1}{2}}(c + 1)^{\frac{1}{2}}t}{2K}\right\}.$$

The results of calculation are as follows:

a	b	x	p
·1	1·1125	$-\cdot 0838 + 1\cdot 0557 \cos pt + \cdot 0284 \cos 2pt + \cdot 0006 \cos 3pt$	·9855
·2	1·2680	$-\cdot 2059 + 1\cdot 1306 \cos pt + \cdot 0712 \cos 2pt + \cdot 0033 \cos 3pt$	·9477
·3	1·5657	$-\cdot 4634 + 1\cdot 2634 \cos pt + \cdot 1783 \cos 2pt + \cdot 0190 \cos 3pt + \cdot 0018 \cos 4pt + \cdot 0002 \cos 5pt$	·8152

The calculations have been performed for illustrative purposes only, and no special care has been taken to ensure the accuracy of the digits in the final decimal places.

(*c*) The solution of the equation

$$\frac{d^2x}{dt^2} + n^2x + \tfrac{3}{2}\,\alpha x^2 + 2\beta x^3 = 0$$

in the form of a Fourier Series requires rather more elaboration of the algebra.

The motion presents one novel feature which does not appear in the previous solutions. If β be positive, however small, the motion remains vibratory for any finite value of α, and if α and α/β be great, the amplitude of the motion on one side is approximately α/β times its amplitude on the other.

(II) Routh has shown (*vide Advanced Rigid Dynamics*, 1905, p. 56) that an increase in the inertia of any part of a vibrating system will increase all the periods in such a way that the modified periods are separated by the periods of the original system. This is true in general if the inertia of only one part of the system be increased, the definition of a single part being that the effect of increasing its inertia can be represented by a single term

$$\tfrac{1}{2}\,(\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \dots)^2$$

in the expression for the kinetic energy, where q_1, q_2, ... are the normal coordinates of the original system. For example, the theorem is applicable to the case of an additional mass attached at a single point of a stretched string, but not to the case of an increase of mass spread over a portion of the string, or to the case of two or more masses attached at different points.

The theorem may be simply proved as follows. Let the modified kinetic energy be

$$\tfrac{1}{2}\,(\dot{q}_1^{\,2} + \dot{q}_2^{\,2} + \dots) + \tfrac{1}{2}\,(\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \dots)^2,$$

and the potential energy be

$$\tfrac{1}{2}\,(\lambda_1^2 q_1^2 + \lambda_2^2 q_2^2 + \dots).$$

The equations of motion are typified by

$$\ddot{q}_r + \lambda_r^2 q_r + \mu_r\,(\mu_1\ddot{q}_1 + \mu_2\ddot{q}_2 + \dots) = 0.$$

The determinantal equation for the periods is

$$\begin{vmatrix} \lambda^2(1+\mu_1^2)-\lambda_1^2, & \mu_1\mu_2\lambda^2, & \mu_1\mu_3\lambda^2, \dots \\ \mu_1\mu_2\lambda^2, & \lambda^2(1+\mu_2^2)-\lambda_2^2, & \mu_2\mu_3\lambda^2, \dots \\ \dots & \dots & \dots \end{vmatrix} = 0. \quad \dots(1)$$

Let $\lambda_1^2, \lambda_2^2, \dots$ be arranged in ascending order of magnitude. If $\lambda^2 = 0$, the left-hand side of (1) is $(-1)^n$ as regards sign. If $\lambda^2 = \lambda_1^2$, the left-hand side of (1) is equal to

$$\mu_1^2 \begin{vmatrix} \lambda_1^2, & \mu_2\lambda_1^2, & \mu_3\lambda_1^2, \dots \\ \mu_2\lambda_1^2, & (\mu_2^2+1)\lambda_1^2-\lambda_2^2, & \mu_2\mu_3\lambda_1^2, \dots \\ \dots & \dots & \dots \end{vmatrix}$$

$$= \mu_1^2 \begin{vmatrix} \lambda_1^2 & \mu_2\lambda_1^2 & \mu_3\lambda_1^2 & \dots \\ 0 & \lambda_1^2-\lambda_2^2 & 0 & \dots \\ 0 & 0 & \lambda_1^2-\lambda_3^2 & \dots \\ \dots & \dots & \dots & \end{vmatrix},$$

and this is $(-1)^{n-1}$ as regards sign.

Hence all the roots in λ^2 are decreased and they are separated by $\lambda_1^2, \lambda_2^2, \dots$.

The validity of this proof depends on (1) the non-equality of any of the values of $\lambda_1^2, \lambda_2^2, \dots$, (2) the non-evanescence of any of the constants $\mu_1, \mu_2, \dots$. In case of (1) one period at least of the modified system is equal to a period of the original, but the theorem may be held to cover this case.

In case of (2) the theorem does not remain true. Suppose the μ's are all zero except $\mu_r, \mu_s, \mu_t, \dots$. Then only the periods corresponding to $q_r, q_s, q_t, \dots$ are changed. The periods belonging to these coordinates will be increased and their new values will be separated by their old values. But these new periods bear no relation to the periods belonging to the remaining coordinates and can occupy any position in regard to them except as specified above. Hence the theorem does not seem to indicate where the modified periods must lie in regard to the complete system of periods of the undisturbed system.

An example is afforded by the modification introduced into the periods of a stretched string by a load attached at a point dividing the string into two lengths which are commensurable. Rayleigh's argument (*vide Theory of Sound*, Vol. I, p. 122), which serves to maintain the validity of the theorem in this case, is acceptable owing to the strictly defined relations which exist between the periods in both states. But in an ordinary dynamical problem the theorem must be held to break down in the exceptional cases under consideration since it fails completely to indicate the position of the modified periods in relation to the original periods.

Experiments with a plane diffraction grating. By G. F. C. SEARLE, Sc.D., F.R.S., University Lecturer in Experimental Physics.

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PART I. PARALLEL LIGHT.

§ 1. *Introduction**. When a plane grating is employed in accurate measurements of wave length, the rulings are set perpendicular to the direction of the incident beam of parallel light. When these two directions are not at right angles, the diffracted beam is no longer parallel to a plane containing the directions of (*a*) the incident beam and (*b*) a line intersecting the rulings at right angles. The formulae applicable to this general case are obtained in §§ 4, 5, 7; they are tested by the experiment of §§ 8, 9 for the restricted case in which the directions (*a*) and (*b*) are at right angles.

§ 2. *The grating axes.* It is necessary to specify the three axes of a plane grating and the origin from which they start.

For a transmission grating, the origin O is a point on the centre line of one of the openings. In a reflecting grating, O would lie on the centre line of one of the reflecting portions.

The axes are

(1) The normal ON to the plane of the grating.

(2) The transverse axis OT, a line through O cutting the rulings at right angles.

(3) The longitudinal axis OL, a line parallel to the rulings.

The grating interval, i.e. the common interval measured along OT from centre to centre of the openings, will be denoted by d.

§ 3. *Diffracted wave front and ray.* At a distance of thousands of wave lengths from the grating, the wavelets due to the separate openings will merge into practically a single wave. For the mathematical purposes of this paper we shall speak of this wave as the diffracted wave front and of a normal to it as the diffracted ray. We may speak of the diffracted wave front passing through the origin O, if we understand it to be a surface through O cutting at right angles the normals to the distant wave fronts. The normal through O may be called the diffracted ray through O.

In the case of reflexion or refraction at a polished surface, the time of passage from an incident wave front to a reflected or

* I have to thank Dr J. A. Wilcken of Christ's College, and Mr C. L. Wiseman, M.A. of Peterhouse. Dr Wilcken took the observations of § 12, Part I, and assisted in other ways. Mr Wiseman gave valuable help and criticism in the mathematical parts of the paper.

refracted front is independent of the particular ray. But, in the case of a grating, the time of passage from an incident to a diffracted front increases or diminishes by $i\tau$ as the point of incidence of the "ray" is moved from the centre of one opening to the centre of the next. Here τ is the periodic time of the vibration and i is a positive integer.

§ 4. *Diffraction of a plane wave; general case.* Take the axes of x, y, z to coincide with the axes ON, OT, OL of the grating, as in Fig. 1. Let R be a point on the centre line of the qth opening and let the coordinates of R be $0, qd, h$.

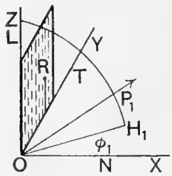


Fig. 1.

Let the direction cosines of the *forward* direction OP_1 of the incident beam be l_1, m_1, n_1, and let those of the *forward* direction OP_2 of the diffracted beam of order i be l_2, m_2, n_2.

Through O draw planes perpendicular to these two directions. The distance of R from the first plane, counted positive when the incident wave front reaches O before it reaches R, is $m_1qd + n_1h$. The distance of R from the second plane, counted positive when the diffracted front leaves O before it leaves R, is $m_2qd + n_2h$. If v_0 is the velocity of light and λ_0 the wave length in a vacuum, and if μ_1, μ_2 are the refractive indices of the media on the two sides of the grating, the times corresponding to the two distances are

$$\mu_1(m_1qd + n_1h)/v_0 \quad\text{and}\quad \mu_2(m_2qd + n_2h)/v_0,$$

and these differ by $qi\tau$. Thus, since $\tau v_0 = \lambda_0$, we have

$$\mu_2(m_2qd + n_2h) - \mu_1(m_1qd + n_1h) = \mp qi\lambda_0.$$

This result must hold good for *all* positions of R on the grating, for which q is integral. We thus obtain

$$\mu_2 m_2 = \mu_1 m_1 \mp i\lambda_0/d, \qquad\text{......(1)}$$

$$\mu_2 n_2 = \mu_1 n_1. \qquad\text{......(2)}$$

These equations completely determine the directions of the diffracted beams of order i.

Let the incident and diffracted beams make angles ϵ_1, ϵ_2 with OT and angles η_1, η_2 with OL. Then

$$\cos\epsilon_1 = m_1,\quad \cos\epsilon_2 = m_2, \qquad\text{......(3)}$$

$$\cos\eta_1 = n_1,\quad \cos\eta_2 = n_2. \qquad\text{......(4)}$$

Hence (1) and (2) may be written

$$\mu_2\cos\epsilon_2 = \mu_1\cos\epsilon_1 \mp i\lambda_0/d, \qquad\text{......(5)}$$

$$\mu_2\cos\eta_2 = \mu_1\cos\eta_1. \qquad\text{......(6)}$$

Since $m_2{}^2 + n_2{}^2$ cannot exceed unity when the direction cosines are real, the condition that a diffracted beam may exist is $m_2{}^2 + n_2{}^2 \lesseqgtr 1$, or $\cos^2 \epsilon_2 \lesseqgtr \sin^2 \eta_2$. If ϵ_2 and η_2 lie between 0 and $\frac{1}{2}\pi$, this requires that $\eta_2 + \epsilon_2 \gtreqless \frac{1}{2}\pi$.

It is noteworthy that ϵ_2 depends only upon ϵ_1 and $i\lambda_0/d$, and that η_2 depends only upon η_1.

We shall not further consider the case in which μ_1 and μ_2 are unequal, but shall confine the work to the special case of $\mu_1 = \mu_2$. The reader will find no difficulty in making the necessary modifications.

§ 5. *Diffraction of a plane wave; single medium.* In practice each medium is air, of refractive index μ relative to a vacuum. If λ is the wave length in air, $\lambda_0 = \mu\lambda$. We then obtain the simple equations

$$m_2 = m_1 \mp i\lambda/d, \quad \text{or} \quad \cos \epsilon_2 = \cos \epsilon_1 \mp i\lambda/d, \qquad \text{......(7)}$$

$$n_2 = n_1, \qquad \text{or} \quad \cos \eta_2 = \cos \eta_1. \qquad \text{..............(8)}$$

Since η may be restricted to lie between 0 and π, we have

$$\eta_2 = \eta_1 = \eta. \qquad \text{......................(9)}$$

The direction of the diffracted ray is easily constructed on a spherical diagram. Let the axes of the grating intersect a sphere about O as centre in N, T, L (Fig. 2), and let NON' be a diameter. Let the continuation of the incident ray through O meet the sphere in P_1. The great circle arc TP_1 measures ϵ_1. Calculate ϵ_2 by (7) and take $TQ = \epsilon_2$ on TP_1. About T and L as poles draw small circles through Q and P_1. Then $LP_1 = \eta$. If the small circles do not intersect, there will be no diffracted beam either by transmission or by reflexion. If the small circles intersect in the points P_2, P_2', then OP_2, OP_2' will be the directions of the two diffracted beams. Of the arcs NP_2, NP_2' of the great circle $NP_2P_2'N'$, one is greater and one less than $\frac{1}{2}\pi$. If NP_2 is *less* than $\frac{1}{2}\pi$, it corresponds to the transmitted beam, and then NP_2' corresponds to the reflected beam.

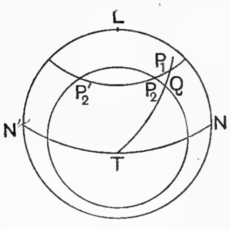


Fig. 2.

When ϵ_1 and i are given, there are *two* values of ϵ_2, and hence there are *two* points Q_- and Q_+ on TP_1. Thus there will be *two* directions $(OP_2)_-$ and $(OP_2)_+$ for the transmitted diffracted beam, and similarly for the reflected beam.

It may happen that only *one* of the two beams $(OP_2)_-$ and $(OP_2)_+$ exists. Unless P_1 is on the great circle LN, there will be two distinct values of $m_2{}^2$, and the condition $m_2{}^2 + n_2{}^2 \lesseqgtr 1$ may be satisfied by the smaller value of $m_2{}^2$ but not by the larger.

Since η_2 has the constant value η, while m_2 or $\cos \epsilon_2$ depends upon λ, it follows that, if white light is used, to each λ there will correspond a position of P_2 on the small circle through P_1 with L for pole.

§ 6. *The deviation.* If D is the angle ($< \pi$) between the forward directions of the incident and the transmitted diffracted beams, $\cos D = l_1 l_2 + m_1 m_2 + n_1 n_2$. If the plane ZOP_1 (Fig. 1) cuts OXY in OH_1, where $XOH_1 = \phi_1$, then, since $P_1OZ = \eta$,

$$l_1 = \sin \eta \cos \phi_1, \quad m_1 = \sin \eta \sin \phi_1, \quad n_1 = \cos \eta,$$

and similarly for P_2. Thus

$$1 - 2 \sin^2 \tfrac{1}{2}D = \cos D = \sin^2 \eta \cos (\phi_1 - \phi_2) + \cos^2 \eta,$$

and hence
$$\sin \tfrac{1}{2}D = \sin \eta \sin \tfrac{1}{2} (\phi_1 \sim \phi_2), \qquad \text{.........(10)}$$

as can also be shown from the isosceles spherical triangle P_1LP_2 in Fig. 2.

In the case of the transmitted diffracted beam, $l_1 l_2$ is *positive.* Noting that $n_2 = n_1 = \cos \eta$, putting $m_1 = a + b$, $m_2 = a - b$, and substituting for l_1, l_2, we find

$$2 (\sin^2 \tfrac{1}{2}D - b^2) = \sin^2 \eta - a^2 - b^2 - [(\sin^2 \eta - a^2 - b^2)^2 - 4a^2b^2]^{\frac{1}{2}}.$$

Thus $\sin^2 \frac{1}{2}D$ is greater than b^2 except when $a = 0$, and then the two are equal. When $a = 0$, $m_2 = - m_1$. If we take m_1 positive, we see, by (7), that, since i is positive, $m_1 = - m_2 = i\lambda/2d$. Hence $b = i\lambda/2d$. Thus, if D_0 is the minimum deviation,

$$\sin \tfrac{1}{2}D_0 = i\lambda/2d. \qquad \text{.....................(11)}$$

Since η does not occur in (11), $m_1 = - m_2$ gives a minimum of D for any given value of η—a minimum having the same value for *all* values of η.

§ 7. *The sloped grating.* For the experiment of §§ 8, 9 it is convenient to use axes differing from those of § 4. Now let OY (Fig. 3) coincide with OT and let OL make an angle θ with OZ. Then the direction cosines of OL are $\sin \theta$, 0, $\cos \theta$, those of OT are 0, 1, 0, and those of ON are $\cos \theta$, 0, $- \sin \theta$.

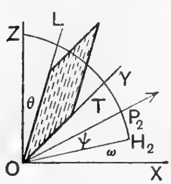


Fig. 3.

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the forward directions OP_1, OP_2 of the incident and the transmitted diffracted beams. Let $n_2 = \sin \psi$, so that the diffracted ray OP_2 makes an angle ψ with the plane OXY, counted positive when $P_2OZ < \frac{1}{2}\pi$. Let the plane ZOP_2 cut OXY in OH_2 and let $XOH_2 = \omega$. Then, if η is the common angle between OP_1 or OP_2 and OL,

$$l_2 \sin\theta + n_2 \cos\theta = \cos\eta = l_1 \sin\theta + n_1 \cos\theta. \quad \text{...(12)}$$

We also have, by (7), if $P_1OT = \epsilon_1$, $P_2OT = \epsilon_2$,

$$\cos\epsilon_2 = m_2 = m_1 \mp i\lambda/d = \cos\epsilon_1 \mp i\lambda/d. \quad \text{......(13)}$$

Hence m_2 is known at once. Using $l_2{}^2 = 1 - m_2{}^2 - n_2{}^2$, we have, by (12),

$$\sin^2\theta\,(1 - m_2{}^2 - n_2{}^2) = (\cos\eta - n_2\cos\theta)^2.$$

Solving for n_2 and taking the *negative* sign in the ambiguity, we have

$$\sin\psi = n_2 = \cos\theta\cos\eta - \sin\theta\,(\sin^2\eta - m_2{}^2)^{\frac{1}{2}}. \quad \text{...(14)}$$

Using this value of n_2 in (12), we find

$$l_2 = \sin\theta\cos\eta + \cos\theta\,(\sin^2\eta - m_2{}^2)^{\frac{1}{2}}. \quad \text{...... (15)}$$

Since $\cos NOP_2 = l_2 \cos\theta - n_2 \sin\theta$, we find from (14) and (15) that

$$\cos NOP_2 = (\sin^2\eta - m_2{}^2)^{\frac{1}{2}}. \quad \text{........... (16)}$$

Thus the *negative* sign has been correctly chosen in (14) for the *transmitted* beam, since for this $\cos NOP_2$ must be positive. If the positive sign is used in (14), $\cos NOP_2$ is negative, corresponding to the reflected diffracted beam.

In terms of ψ and ω, the direction cosines of OP_2 are $\cos\psi\cos\omega$, $\cos\psi\sin\omega$, $\sin\psi$. Hence $m_2 = \cos\psi\sin\omega$, and thus

$$\sin\omega = m_2/\cos\psi = \cos\epsilon_2/\cos\psi. \quad \text{...........(17)}$$

The two angles ψ and ω completely determine the direction of the diffracted ray.

In the experiment of §§ 8, 9 the incident rays are parallel to the axis OX of Fig. 3. Hence $l_1 = 1$, $m_1 = 0$, $n_1 = 0$. We then have

$$\cos\epsilon_2 = m_2 = \mp i\lambda/d, \quad \text{...(18)}$$

$$\cos\eta = \sin\theta, \quad \text{...(19)}$$

and thus, since $\sin^2\eta - \cos^2\epsilon_2 = \sin^2\epsilon_2 - \cos^2\eta$,

$$\sin\psi = n_2 = \sin\theta\cos\theta - \sin\theta\,(\sin^2\epsilon_2 - \sin^2\theta)^{\frac{1}{2}}, \quad \text{...(20)}$$

$$l_2 = \sin^2\theta + \cos\theta\,(\sin^2\epsilon_2 - \sin^2\theta)^{\frac{1}{2}}, \quad \text{...(21)}$$

$$\sin\omega = m_2/\cos\psi = \cos\epsilon_2/\cos\psi. \quad \text{...(22)}$$

Since ϵ_2 and θ may be restricted to be less than $\frac{1}{2}\pi$, we see that no diffracted beam will be formed if θ exceeds its critical value ϵ_2.

§ 8. *Apparatus*. The general arrangement is shown in Fig. 4. The grating G is attached to a horizontal shaft A, with its plane parallel to A and its rulings perpendicular to A. A horizontal collimator L has horizontal and vertical cross-wires intersecting at C in its focal plane; these are illuminated by a sodium flame S. The straight line joining C to the appropriate nodal point of the lens is the line of collimation, or axis, of the collimator. The parallel beam defined by C is parallel to this line. After the light has passed through the grating, it is received by a goniometer K, and an image of the collimator wires is formed in its focal plane. To fix the line of collimation, cross-wires are placed in the focal plane; they intersect in D. The goniometer is carried on a horizontal revolving shaft B, and its line of collimation is perpendicular to the shaft. One cross-wire is parallel and the other perpendicular to the shaft; the latter is also perpendicular to the shortest distance from D to the axis of the shaft. The shafts are provided with divided circles E, F, which are read by aid of the pointers U, U', V, V'. A balance weight W is attached to the circle F. The point of intersection of the line of collimation of K with the axis of B should lie approximately on the centre of the grating, and the line of collimation of L should pass through the same point.

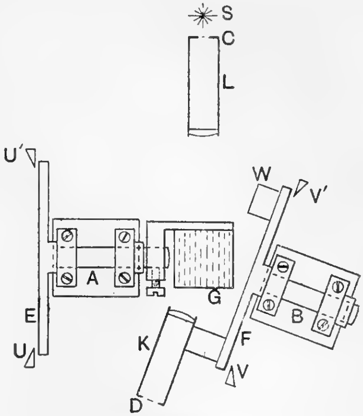


Fig. 4.

The angles θ and ψ are measured by the circles E and F.

§ 9. *Experimental details.* The shaft A is set horizontal by aid of a level. The collimator is adjusted optically by an auto-collimating method. The plane of the grating is set horizontal by a level, and the shaft is then turned through 90°, as measured by the circle E, so that the plane of the grating is vertical. Light from a flame is then reflected by a plate of glass held at 45° past the cross-wires and through the lens on to the grating, and the collimator, previously set for "infinity," is adjusted so that the image of C, the intersection of the wires, coincides with C itself. If the coincidence is recovered when the grating shaft is turned through 180°, the plane of the grating is parallel to the shaft. The line of collimation is then both horizontal and also perpendicular to the grating shaft.

The line of collimation of the goniometer is set perpendicular to the goniometer shaft B by an optical method. An auxiliary collimator, set for "infinity," is placed so that it is approximately perpendicular to the shaft. A plate of plane parallel glass is attached to the circle F near its centre so that its faces are approximately parallel to the shaft. It is convenient to make the shaft vertical; the glass plate can then be supported on a small levelling table resting on the circle. By adjusting both the collimator and the plate, the faces of the plate are made parallel to the shaft and the axis of the collimator is made perpendicular to the shaft. In this case the image of the collimator wires can, by turning the circle, be made to coincide with those wires, when *either* side of the plate faces the collimator. If the plate has been suitably placed, it will be possible, by turning the goniometer on its shaft, to receive the image of the collimator wires on the focal plane of the goniometer. The "vertical" cross-wire of the goniometer, i.e. the wire perpendicular to the shaft, is now adjusted so that it coincides with the image of the corresponding wire of the collimator. The line of collimation is then perpendicular to the shaft. The goniometer is then put into position and its shaft is levelled.

The axes of the collimator, of the grating shaft and of the goniometer shaft are adjusted to be approximately in the same horizontal plane. The plane of the grating is made vertical, and the goniometer stand is adjusted in azimuth so that one of the diffracted images of the collimator wires can be made to coincide with the goniometer wires by turning the goniometer on its shaft.

When the adjustments already described have been effected, and when the plane of the grating G is vertical, the diffracted beams are parallel to the plane OTN. If OT is inclined at an angle δ to the grating shaft, the direction of OT will be changed by 2δ if G is turned through $180°$ about the axis of the shaft from Position 1 to Position 2, when the plane is again vertical. The goniometer is turned to receive a diffracted beam when G is in Position 1. If, when G is turned into Position 2, the inclination of the beam to a horizontal plane is changed, the grating must be turned in its own plane until the inclination is the same for both positions.

Since ψ is always small, $\cos\psi$ is nearly unity and hence, by (22), $\sin\omega$ has a nearly constant value, for m_2 is independent of θ. Hence, if the image of C, the intersection of the collimator wires, lies on D, the intersection of the goniometer wires, when the plane of G is vertical, the image of C will always lie very near the "vertical" cross-wire, and one setting of the goniometer stand will suffice for all values of θ.

The plane of the grating is made horizontal and the index reading is taken. It is then turned through $90°$; it is now vertical and in its zero position. The goniometer is next adjusted so that the image of C lies on the horizontal wire of K. The goniometer is then in its zero position.

The grating is now turned through $10°$, $20°$, ... on either side of the zero, and the goniometer is turned on its shaft to bring the image of C on to the horizontal wire of K in each case. If the grating circle has *two* indices, the grating is turned through $10°$, $20°$, ... as indicated by one index. In reducing the observations the mean of the angles furnished by *both* indices is used.

Since $\epsilon_1 = \frac{1}{2}\pi$, $\cos \epsilon_2 = m_2 = i\lambda/d$. From the values of the interval d and the wave length λ, $\cos \epsilon_2$ is found and then the values of ψ corresponding to the mean values of θ are calculated by (20). These values are compared with the mean values of ψ given by the goniometer readings.

§ 10. *Distortion of the image.* As θ, and consequently ψ, increases, the observer sees that the angle between the images of the collimator wires undergoes great changes. When $\theta = 0$, the images are at right angles, but the angle diminishes rapidly as θ reaches its critical value. The theory shows that they are actually tangential one to the other when θ has its critical value, but, as no light is transmitted in the critical position, the phenomenon cannot be observed. If the collimator wires are stretched across a small circular opening, the image of the edge is distorted into an oval, which is practically an ellipse having the images of the wires as conjugate diameters. When, however, θ approaches its critical value, the oval begins to deviate from an ellipse.

In Fig. 5 let OX, OY or OT, OZ, OL, ON meet a sphere described about O as centre in X, T, Z, L, N. Let OJ be the diffracted ray corresponding to the incident ray OX; the ray OX corresponds to the line of collimation of the collimator and OJ to that of the goniometer, when the image of C is brought to the intersection of the goniometer wires. Let OP_1 be a ray nearly parallel to OX and let OP_2 be the corresponding diffracted ray. Let the great circles through Z and P_1, J, P_2 cut the great circle TXS in H, K, M. Let S be the pole of ZJK and let the great circle SJ meet ZP_2M in Q. Then $ZJQ = \frac{1}{2}\pi$.

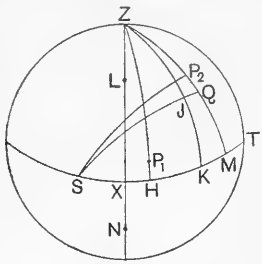


Fig. 5.

If the goniometer is mounted as described in § 8, and if its line of collimation coincides with OJ, its horizontal cross-wire will correspond to SJQ and its "vertical" wire to ZJK. The rays parallel to OP_2 will come to a focus in the focal plane of the goniometer at D', whose coordinates referred to the horizontal and vertical wires through D (Fig. 4) are $f \times$ angle QOJ and $f \times$ angle P_2OQ, where f is the focal length of the lens.

If points on a curve CC' in the focal plane of the collimator give rise to diffracted rays whose directions are shown by points on the curve JP_2 on the sphere, and if the image of CC' in the focal plane of the goniometer is DD', the angle between the "vertical" cross-wire and the tangent to DD' at D is equal to I, the angle between the great circle JZ and the tangent at J to the curve JP_2.

If $JK = \psi$, $P_2M = \psi'$, $XK = \omega$, $XM = \omega'$, then

$$\tan I = \text{limit of } \frac{JQ}{P_2Q} = \cos\psi \left(\frac{d\omega'}{d\psi'}\right)_\psi. \quad \text{............(23)}$$

If $XH = \alpha$, $P_1H = \gamma$, the direction cosines of OP_1, OP_2 are

$$l_1 = \cos\gamma\cos\alpha, \qquad m_1 = \cos\gamma\sin\alpha, \qquad n_1 = \sin\gamma,$$
$$l_2 = \cos\psi'\cos\omega', \qquad m_2 = \cos\psi'\sin\omega', \qquad n_2 = \sin\psi'.$$

Since $ZOL = \theta$, the direction cosines of OL are $\sin\theta$, 0, $\cos\theta$. But $P_1OL = P_2OL = \eta$, $P_1OT = \epsilon_1$, $P_2OT = \epsilon_2$, and thus the fundamental equations (13) and (12) become

$$\cos\psi'\sin\omega' = \cos\gamma\sin\alpha \mp i\lambda/d, \quad \text{............(24)}$$
$$\cos\psi'\cos\omega'\sin\theta + \sin\psi'\cos\theta = \cos\gamma\cos\alpha\sin\theta + \sin\gamma\cos\theta. \quad \text{......(25)}$$

The vertical collimator wire corresponds to the great circle ZX, and for this $\alpha = 0$, but γ varies. If I_V is the inclination to JZ of the corresponding path described by P_2,

$$\tan I_V = \cos\psi \left(\frac{\partial\omega'}{\partial\gamma}\right)_0 \Big/ \left(\frac{\partial\psi'}{\partial\gamma}\right)_0.$$

Differentiating (24) with respect to γ and then putting $\gamma = 0$, so that ψ', ω' become ψ, ω, we have

$$-\sin\psi\sin\omega\,(\partial\psi'/\partial\gamma)_0 + \cos\psi\cos\omega\,(\partial\omega'/\partial\gamma)_0 = 0.$$

Hence $$\tan I_V = \sin\psi\tan\omega.$$

The horizontal collimator wire corresponds to the great circle XT and for this $\gamma = 0$, but α varies. Differentiating (25) with regard to α and then putting $\alpha = 0$, we have

$$-\sin\psi\cos\omega\sin\theta\,(\partial\psi'/\partial\alpha)_0 - \cos\psi\sin\omega\sin\theta\,(\partial\omega'/\partial\alpha)_0$$
$$+ \cos\psi\cos\theta\,(\partial\psi'/\partial\alpha)_0 = 0.$$

Hence, if I_H is the inclination to JZ of the corresponding path of P_2,

$$\tan I_H = \cos\psi \left(\frac{\partial\omega'}{\partial\alpha}\right)_0 \Big/ \left(\frac{\partial\psi'}{\partial\alpha}\right)_0$$
$$= \frac{\cos\psi\cos\theta - \sin\psi\cos\omega\sin\theta}{\sin\omega\sin\theta}. \quad \text{......(26)}$$

Multiplying numerator and denominator by $\cos\omega$, and replacing $\cos^2\omega$ by $1 - \sin^2\omega$, we find

$$\tan I_H = \tan I_V + \frac{l_2\cos\theta - n_2\sin\theta}{\sin\omega\cos\omega\sin\theta}. \quad \text{.........(27)}$$

Since the direction cosines of ON are $\cos\theta$, 0, $-\sin\theta$, and l_2, n_2 in (27) refer to OJ, $l_2\cos\theta - n_2\sin\theta = \cos NOJ$. In the critical position, J lies on the great circle LT, which corresponds to the plane of the grating. Then $\cos NOJ = 0$, and the difference between the tangents vanishes, i.e. the two curves touch at J.

Since the distance of ZLX from T is constant, the curve corresponding to the vertical cross-wire is a small circle passing through J with T for pole. At J the small circle is perpendicular to the great circle TJ and the value of $\tan I_V$ can be verified by spherical trigonometry.

The horizontal cross-wire is represented by the great circle XT, but now both ϵ_1 and η vary and no simple construction is available for the *whole* curve through J corresponding to this wire. The curve touches at J the small circle passing through J and X with L as pole, and cuts at right angles the great circle LJ. Hence $\tan I_H = \cot LJZ$, and then (26) can be verified by spherical trigonometry. If we find $\cos LJZ$ and $\sin LJZ$ and divide, we obtain the alternative form

$$\tan I_H = \frac{\cos\theta - \sin\psi\sin\theta}{\cos\psi\sin\omega\sin\theta}.$$

If the angle between the two small circles which intersect in J is Δ, then Δ is the supplement of LJT. But $LJ = \tfrac{1}{2}\pi - \theta$, $JT = \epsilon_2$, $LT = \tfrac{1}{2}\pi$, and hence

$$\cos\Delta = \tan\theta\cot\epsilon_2.$$

In the experiment $\cos\epsilon_2$ has the constant value $\mp i\lambda/d$, and thus $\cos\Delta$ depends only upon θ. The angle Δ will vanish when $\cos\Delta = 1$, and this occurs when $\theta = \epsilon_2$, i.e. when θ has its critical value.

We can make visible a finite arc of the small circle with L for pole. If we illuminate with white light the *small* opening across which the wires are stretched, the position of J on this small circle will be different for the different colours. The short length of cross-wire will correspond for any colour to a small arc of a curve touching the small circle at practically its middle point. The envelope of these small arcs will be the small circle itself. The image of the horizontal wire will thus be a dark curved line running across the spectrum.

§ 11. *The critical values.* The critical position of the grating is reached when $\theta = \epsilon_2$, and we have, by (20), (22), the critical values

$$\sin\psi_c = \sin\epsilon_2\cos\epsilon_2, \qquad \sin\omega_c = \frac{\cos\epsilon_2}{(1-\sin^2\epsilon_2\cos^2\epsilon_2)^{\frac{1}{2}}},$$

$$\tan\omega_c = \frac{\cos\epsilon_2}{\sin^2\epsilon_2}, \qquad (\tan I_V)_c = \frac{\cos^2\epsilon_2}{\sin\epsilon_2}.$$

For the grating used in § 12 and with $i = 1$,

$$\cos\epsilon_2 = 0\cdot33568, \;\; \sin\epsilon_2 = 0\cdot94198, \;\; \epsilon_2 = \tfrac{1}{2}\pi - 19^\circ\,36'\,49''.$$

Then $\psi_c = 18^\circ\,26'$, $\omega_c = 20^\circ\,43'\,18''$ and $(I_V)_c = 6^\circ\,49'\,17''$.

When $\theta = 0$, and therefore $\psi = 0$,

$$\sin \omega = \cos \epsilon_2, \qquad \omega = 19^\circ\, 36'\, 49''.$$

Thus the maximum change in ω is only $1^\circ\, 6'\, 29''$.

§ 12. *Practical example.* The following results were obtained by Dr J. A. Wilcken, using a grating having 14,468 lines per inch, and, hence, an interval $d = 1{\cdot}7556 \times 10^{-4}$ cm.

The adjustments described in § 9 were either effected or tested. The plane of the grating was not *quite* parallel to the grating shaft, but as both images of the first order were observed, the mean results will be hardly affected. The calculated values of ψ were found on the assumption that the axis of the collimator is perpendicular to the transverse axis of the grating. Sodium light of mean wave length $\lambda = 5{\cdot}893 \times 10^{-5}$ cm. was used. Then, since the images were of the *first* order throughout,

$$m_2 = \frac{\lambda}{d} = \frac{5{\cdot}893 \times 10^{-5}}{1{\cdot}7556 \times 10^{-4}} = 0{\cdot}33568 = \sin 19^\circ\, 36'\, 49''.$$

Thus $\qquad \epsilon_2 = \cos^{-1} m_2 = 70^\circ\, 23'\, 11''$, and $m_2{}^2 = 0{\cdot}11268$.

Each of the observed values of ψ given in the table is the mean of four. Each of the first order images was observed, and for each image two values of θ, one on either side of zero, were taken. The grating circle, which was printed on card, was a little eccentric relative to the shaft (it was a "home-made" affair), and, consequently, although the settings were made to integral degrees by one index, the other index did not always read integral degrees. Some of the mean values of θ are, therefore, not integral degrees. The calculated values of ψ are those found from equation (20), which for convenience is written

$$\sin \psi = n_2 = \tfrac{1}{2} \sin 2\theta - \sin \theta\, [(\cos \theta + m_2)(\cos \theta - m_2)]^{\frac{1}{2}}.$$

For the sake of interest, the calculated values of ω, I_V and I_H have been added. The last line in the table gives the critical values as found by calculation.

θ mean obsd.	ψ mean obsd.	ψ calcd.	ω calcd.	I_V	I_H
° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0 0 0	0 0 0	0 0 0	19 36 49	0 0 0	90 0 0
9 54 0	0 36 0	0 34 51	19 36 52	0 12 25	86 38 28
19 54 0	1 12 45	1 12 33	19 37 7	0 25 51	83 1 5
29 50 0	1 55 30	1 55 36	19 37 34	0 41 13	78 53 41
39 51 0	2 51 45	2 50 18	19 38 19	1 0 46	73 42 30
49 48 0	4 7 30	4 7 25	19 39 58	1 28 20	66 31 55
59 49 0	6 26 15	6 22 31	19 44 27	2 16 55	54 29 53
64 44 0	8 32 15	8 29 8	19 50 24	3 2 53	44 1 14
67 45 0	10 54 0	10 51 13	19 59 10	3 55 4	33 20 15
69 44 0	14 18 0	14 10 3	20 15 19	5 9 40	20 20 49
70 23 11		18 26 0	20 43 18	6 49 17	6 49 17

The agreement between the observed and calculated values of ψ is satisfactory.

Part II. Non-parallel Light.

§ 1. *Introduction.* In the following experiments the incident light does not form a parallel beam. The diffraction now not merely changes the direction of the axial ray of the beam, but also, in general, introduces astigmatism into, or changes the astigmatism of, the incident beam. The exception is when the incident and diffracted axial rays are perpendicular to the rulings and the deviation is a minimum. The diffracted rays will, in general, pass through two focal lines when the aperture is small. If the aperture is increased, aberration will appear and all the rays will not pass accurately through the two lines. Aberration can be minimised by keeping the aperture small, but astigmatic effects are inseparable from the diffraction in the general case.

The formulae for the general case are easily obtained, but are complicated. We shall, therefore, consider only the case in which the axial ray of the incident beam is perpendicular to the rulings.

§ 2. *Diffraction of an astigmatic beam.* In Fig. 1 let OX, OY, OZ coincide with ON, OT, OL, the axes of the grating, as defined in Part I, § 2. For convenience, OZ will be taken as vertical.

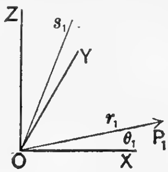


Fig. 1.

Let a beam, which started from a luminous point and therefore has a wave front, fall upon the grating near O. Let OP_1 be the continuation of the ray through O, *which has been restricted to lie in the plane* OXY, and let OP_1 be taken as the axial ray of the beam. Let $P_1OX = \theta_1$. Take OP_1 as the axis of r_1 in a new set of axes Or_1, Os_1, Ot_1, of which Os_1 is in the plane OXY and Ot_1 coincides with OZ. Let the equation to the incident wave front passing through O be

$$r_1 = \tfrac{1}{2}S_1 s_1^2 + W_1 s_1 t_1 + \tfrac{1}{2}T_1 t_1^2. \quad \text{...............(1)}$$

Let R be a point on the grating and let its coordinates referred to the grating axes be 0, qd, z, where d is the grating interval and q is an integer. Then the coordinates of R referred to the axes of the incident beam are

$$r_1 = qd \sin\theta_1,\;\; s_1 = qd\cos\theta_1,\;\; t_1 = z. \quad \text{............(2)}$$

If a line through R parallel to OP_1 cuts the wave front OF_1 in F_1, the second and third coordinates of F_1 are $qd\cos\theta_1$ and z. By (1) and (2), the distance of F_1 from the plane $r_1 = 0$, which touches the wave front at O, is $\tfrac{1}{2}S_1 q^2 d^2 \cos^2\theta_1 + W_1 qdz\cos\theta_1 + \tfrac{1}{2}T_1 z^2$, and the distance of R from the same plane is $qd\sin\theta_1$. Hence

$$F_1R = qd\sin\theta_1 - \tfrac{1}{2}S_1 q^2 d^2\cos^2\theta_1 - W_1 qdz\cos\theta_1 - \tfrac{1}{2}T_1 z^2 \text{....(3)}$$

When R and F_1 approach O, F_1R becomes more and more nearly the normal at F_1, and, for a small aperture, may be treated as the normal in the estimation of *distances*. Thus, ultimately, F_1R is the ray distance from the wave front OF_1 to R.

Let OP_2 be a diffracted ray of order i. By symmetry, OP_2 is in the plane OXY, since OP_1 is in that plane. Let $P_2OX = \theta_2$. Take the axial ray OP_2 as the axis of r_2 in a set of axes Or_2, Os_2, Ot_2, of which Os_2 is in the plane OXY and Ot_2 coincides with OZ. Let the equation to the diffracted wave front passing through O be

$$r_2 = \tfrac{1}{2}S_2 s_2^2 + W_2 s_2 t_2 + \tfrac{1}{2}T_2 t_2^2. \quad \text{..............(4)}$$

Then, if F_2R, parallel to OP_2, cuts the diffracted wave front OF_2 in F_2, the distance F_2R is ultimately the ray distance from F_2 to R. We then have

$$F_2R = qd \sin\theta_2 - \tfrac{1}{2}S_2 q^2 d^2 \cos^2\theta_2 - W_2 qdz \cos\theta_2 - \tfrac{1}{2}T_2 z^2 \text{....(5)}$$

The optical condition is that F_2R differs from F_1R by $qi\lambda$, where i is a positive integer. We thus obtain

$$F_2R = F_1R \pm qi\lambda.$$

Since this holds for all values of z and all integral values of q, we have, by (3) and (5),

$$\sin\theta_2 = \sin\theta_1 \pm i\lambda/d, \quad \text{.................(6)}$$

$$S_2 = k^2 S_1, \quad W_2 = kW_1, \quad T_2 = T_1, \quad \text{...........(7)}$$

where $k = \cos\theta_1/\cos\theta_2$. Since θ_1 and θ_2 both lie between $-\tfrac{1}{2}\pi$ and $\tfrac{1}{2}\pi$ for a transmitted beam, k is positive.

The direction of the axial ray of the diffracted beam is given by (6) and is independent of the constants S_1, W_1, T_1. Equations (7) give the form of the diffracted wave front which passes through O.

If the deviation of the axial ray is a minimum, it follows from Part I, § 6, or otherwise, that $\sin\theta_2 = -\sin\theta_1$. Since θ_1 and θ_2 both lie between $-\tfrac{1}{2}\pi$ and $\tfrac{1}{2}\pi$, $\cos\theta_2 = \cos\theta_1$, and thus $k = 1$. Hence, in the case of minimum deviation, the form of the wave front is unchanged and the diffraction merely turns it through 2θ about OZ. The restriction stated in § 2 must be noted.

If the planes of the principal sections of the incident wave front are OXY and ZOP_1, or, what is the same thing, the planes Or_1s_1, Or_1t_1, then $W_1 = 0$. It follows, from (7), that $W_2 = 0$, and thus the principal planes of the diffracted wave front are OXY and ZOP_2.

When $W_1 = 0$, the section of the incident front by the horizontal plane $t_1 = 0$ is $r_1 = \tfrac{1}{2}S_1 s_1^2$, and the distance of the centre of curvature of this section from O is S_1^{-1}. The vertical focal line of the beam passes through this centre of curvature. Similarly, the horizontal focal line is at a distance T_1^{-1} from O. The distances from O of the vertical and horizontal focal lines of the diffracted beam are S_2^{-1} and T_2^{-1}.

If the incident beam is stigmatic, $T_1 = S_1$ and $W_1 = 0$. Then $S_2 = k^2 S_1$, $T_2 = S_1$. Hence $S_2 = k^2 T_2$, and so the diffracted beam is astigmatic, unless $k = 1$, i.e. unless the deviation is a minimum.

§ 3. *The principal curvatures.* The principal curvatures of the diffracted front can be found in terms of those of the incident front.

Let the principal planes of the incident front intersect the tangent plane at O in $O\eta_1$, $O\zeta_1$ (Fig. 2). Take these, with $O\xi_1$ along OP_1, as axes for the front. Let the radii of curvature of the sections by $O\xi_1\eta_1$ and $O\xi_1\zeta_1$ be B_1^{-1} and C_1^{-1}. The equation to the incident front is then

$$\xi_1 = \tfrac{1}{2}B_1\eta_1^2 + \tfrac{1}{2}C_1\zeta_1^2. \quad \ldots\ldots\ldots(8)$$

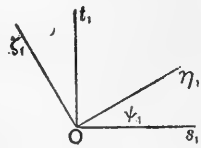


Fig. 2.

Let $O\eta_1$ make an angle ψ_1 with Os_1, as in Fig. 2. Then

$$\xi_1 = r_1, \quad \eta_1 = s_1 \cos\psi_1 + t_1 \sin\psi_1, \quad \zeta_1 = -\, s_1 \sin\psi_1 + t_1 \cos\psi_1,$$

and hence (8) is equivalent to

$$r_1 = \tfrac{1}{2}B_1 (s_1 \cos\psi_1 + t_1 \sin\psi_1)^2 + \tfrac{1}{2}C_1 (-\, s_1 \sin\psi_1 + t_1 \cos\psi_1)^2. \ldots(9)$$

Comparing (9) with (1), we find

$$\left.\begin{aligned} S_1 &= \tfrac{1}{2}(B_1 + C_1) + \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1, \\ W_1 &= \tfrac{1}{2}(B_1 - C_1)\sin 2\psi_1, \\ T_1 &= \tfrac{1}{2}(B_1 + C_1) - \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1. \end{aligned}\right\} \quad \ldots\ldots(10)$$

Then S_2, W_2, T_2 can be found by (7).

If the equation to the diffracted front referred to its principal axes is

$$\xi_2 = \tfrac{1}{2}B_2\eta_2^2 + \tfrac{1}{2}C_2\zeta_2^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

and if $\eta_2 O s_2 = \psi_2$, then B_2, C_2, ψ_2 are related to S_2, T_2, W_2 by equations similar to (10). Solving for B_2, C_2, ψ_2, we have

$$\left.\begin{matrix} B_2 \\ C_2 \end{matrix}\right\} = \tfrac{1}{2}(S_2 + T_2) \pm [\tfrac{1}{4}(S_2 - T_2)^2 + W_2^2]^{\frac{1}{2}}, \quad \ldots\ldots(12)$$

$$\tan 2\psi_2 = 2W_2/(S_2 - T_2). \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

The ambiguity in (12) has been settled so that, when $\psi_1 = 0$, $B_2 = k^2 B_1$, $C_2 = C_1$. Apart from mere reversals of direction, (13) gives two values of ψ_2 differing by $\tfrac{1}{2}\pi$, and corresponding to the axes $O\eta_2$, $O\zeta_2$. The arrangement of signs in (12) implies that when $\psi_1 = 0$, $\psi_2 = 0$. Since, by (7), W_1 and W_2 vanish together, ψ_1 and ψ_2 must reach $\tfrac{1}{2}\pi$, π, $\tfrac{3}{2}\pi$, ... together, and it follows that, for intermediate values, ψ_2 must lie in the same quadrant as ψ_1.

Equations (12), (13) with (7) give B_2, C_2, ψ_2 in terms of S_1, T_1, W_1, which are given in terms of B_1, C_1, ψ_1 by (10).

§ 4. *A simple case.* If we take $C_1 = 0$, the incident wave front is cylindrical. We then obtain

$$B_2 = \tfrac{1}{2}B_1\{k^2 + 1 + (k^2 - 1)\cos 2\psi_1\}, \quad C_2 = 0, \quad \text{...(14)}$$

$$\tan 2\psi_2 = \frac{2k \sin\psi_1 \cos\psi_1}{k^2\cos^2\psi_1 - \sin^2\psi_1} = \tan 2 \tan^{-1}\left(\frac{\tan\psi_1}{k}\right)$$

or

$$\tan\psi_2 = k^{-1} . \tan\psi_1 . \quad \text{.....................(15)}$$

The maximum difference between ψ_1 and ψ_2 occurs when $\tan\psi_1 = k^{\frac{1}{2}}$, and then $\sin(\psi_1 - \psi_2) = (k-1)/(k+1)$.

Since $C_2 = 0$, the diffracted wave front is cylindrical. There is therefore only one focal line at a finite distance from O and this distance is $B_2{}^{-1}$. If the principal planes of the incident front are turned round, ψ_1 will change, so causing B_2 to vary, and the distance of this focal line from O will vary.

§ 5. *Measurement of wave length.* The results of § 2 can be applied in the determination of the wave length of sodium light by measurements made on an optical bench. On the bench slide three carriages D, H, K, as shown in plan in Fig. 3; D carries a

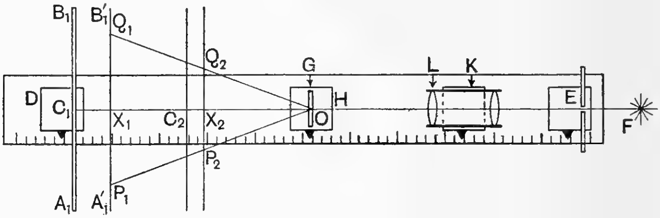


Fig. 3.

horizontal glass scale divided in mm., H carries the grating G (with vertical rulings), whose centre is O, and K carries the converging lens system L, of focal length f. At the end of the bench is a vertical slit illuminated by a sodium flame F; to identify a point E on the slit, a wire may be stretched across the slit. The divided face of the scale faces the slit and this face and the plane of the grating are perpendicular to the bench. The line through the nodal points of L is parallel to the bench and passes through E.

The scale is first placed in the position A_1B_1, at a distance from E exceeding $4f$ by about 30 cm., and the lens is adjusted to form a real undiffracted image on the scale at C_1. The axial ray of the incident beam is normal to the grating, and thus, if ϕ is the angle between the normal and the axial ray of either diffracted beam of order i, we have $k^2 = \sec^2\phi$. Since the incident beam corresponding to the point E of the slit is stigmatic, the vertical focal lines of the diffracted beams of order i will be at a distance OC_1/k^2 from O, along lines OP_1, OQ_1, each making an angle ϕ with OC_1. Sharp images of the slit will pass through P_1, Q_1 and can be focussed on the scale if it is moved to A'_1B_1'. With the grating used in § 9, the two sodium lines can easily be separated. Then $\sin\phi = \frac{1}{2}P_1Q_1/OP_1$. Since OP_1 is difficult to measure, we suppose OX_1 known, where X_1 is the mid-point of P_1Q_1. If $OX_1 = x_1$, $P_1Q_1 = 2y_1$, $\tan\phi = \frac{1}{2}P_1Q_1/OX_1 = y_1/x_1$.

The glass plate protecting the grating prevents an accurate measurement of OX_1. We therefore move the lens carriage along the bench so that the undiffracted image is focussed on the scale at C_2. If the scale is moved further towards O, the diffracted images can be focussed at P_2, Q_2. If $OX_2 = x_2$, $P_2Q_2 = 2y_2$, $\tan\phi = y_2/x_2$. Hence

$$\tan\phi = (y_1 - y_2)/(x_1 - x_2). \quad \text{..............(16)}$$

Putting $\theta_1 = 0$, $\theta_2 = \phi$ in (6), we find

$$\lambda = d \sin\phi/i, \quad \text{........................(17)}$$

where i is the order of the image and d is the grating interval. From (16) and (17), λ is determined.

Since it is an *angle* we measure, small errors of focussing will be of little account, for, in spite of them, the point in which the *axial* ray cuts the scale in each case will be correctly estimated, and this is all that is necessary.

§ 6. *Test of law of obliquity.* Let $OC_1 = u_1$, $OC_2 = u_2$, $OP_1 = v_1$, $OP_2 = v_2$. Then $v_1 - v_2 = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$. But, since $k^2 = \sec^2\phi$, we have, by § 5, $v_1 = u_1 \cos^2\phi$, $v_2 = u_2\cos^2\phi$, and thus $v_1 - v_2 = (u_1 - u_2)\cos^2\phi$. Since $u_1 - u_2$ is known from the bench readings, we can test the law for the vertical focal lines by comparing the two values of $v_1 - v_2$. As we are now concerned with the *positions* of images the focussing must be accurate.

If the slit is not too narrow, the diffracted images of order i of the horizontal wire stretched across it may be focussed on the scale. If these are at p_1, q_1 when $OC = \xi_1$ and at p_2, q_2 when $OC = \xi_2$, and if $p_1q_1 = 2\eta_1$, $p_2q_2 = 2\eta_2$, then

$$p_1p_2 = [(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2]^{\frac{1}{2}}.$$

Since, by § 2, $T_2 = T_1$, we have $p_1p_2 = u_1 - u_2$. The two values of p_1p_2 are compared.

It is difficult to obtain satisfactory readings for x and ξ. This is largely due to the fact that the diffracted rays in the horizontal plane through O do not meet in a point but touch a caustic of large radius. If Q_1R_1, drawn perpendicular to Q_1O in Fig. 3, cuts OC_1 in R_1, the radius of curvature of the caustic at Q_1 is $3Q_1R_1$. The length of the caustic between the points of contact of the tangents from M_1, N_1, where M_1N_1 is the width of grating actually used, is $3(Q_1M_1 \sim Q_1N_1)$ approximately.

§ 7. *Adjustment of the lens.* The lens, a converging system, is adjusted optically. Let its focal length be f and the distance between its nodal points be t, where t is positive when the distance between the principal foci exceeds $2f$; for a projection lens as shown in Fig. 3, t will be negative. When the distance of the luminous point E from the scale ACB exceeds $4f + t$, there are two positions of the lens for which an image of E is formed on the scale. Let M, N (Fig. 4) be the nodal points corresponding to the principal

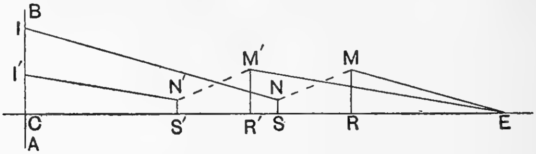


Fig. 4.

foci to the right and left of L. In Fig. 4 let EC be the horizontal line through E parallel to the bench and let the other lines be projections upon the horizontal plane through EC. Let MR, NS be the perpendiculars from M, N on EC. Let $RE = p$, $SC = q$. Then in the second position of the lens, $R'E = q$, $S'C = p$. When the angles are small, $RS = R'S' = t$. Let I, I' be the images of E in the two cases.

Take CE as axis of x and horizontal and vertical lines through C as axes of y and z. Let the second and third coordinates of M, N, I, I' be y, z, η, ζ, Y, Z, Y', Z'. Since IN is parallel to ME, and $I'N'$ to $M'E$,

$$Y = \eta + yq/p, \quad Z = \zeta + zq/p, \quad Y' = \eta + yp/q, \quad Z' = \zeta + zp/q.$$

Since $(p+q)/pq = 1/f$,

$$Y - Y' = y(q-p)/f, \quad Z - Z' = z(q-p)/f.$$

Hence, if the image has the same position for *both* cases, then $y = 0$, and $z = 0$, and thus M lies on EC. The emergent ray through N will then be *parallel* to EC for all positions of the lens carriage.

The lens is best mounted so that it can turn about a vertical pivot whose axis passes through M. If the support to which the pivot is fixed is moved through the distance MR at right angles to the bench, M will be brought on to EC. Then by turning the lens about M, N can be brought on to EC.

To identify C, a pin is mounted on a carriage so that its tip coincides with E. The carriage is moved along the bench so that the tip touches the scale AB. The point of contact is C. If L is adjusted on its carriage so that I coincides with C for both cases, then M, N lie on EC.

§ 8. *Other experimental details.* The scale AB is set perpendicular to the bench. A set square XYZ, with the right angle at X, is held with XY in contact with AB. A pin is held close to XZ. If, when the carriage D is moved along the bench, the distance from the pin to XZ is constant, AB is correctly placed on its carriage. The scale must be horizontal and the slit vertical. The plane of the grating can be set perpendicular to EC optically. The lens L is removed and a small triangle of white paper is fixed to AB so that a vertex coincides with C. The grating G is placed midway between C and E and is adjusted on its carriage so that the image of C by reflexion at G coincides with E. To allow a close test of parallax, a few grains of lycopodium may be placed on AB when the image of the slit does not fall on a dividing line.

§ 9. *Practical example.* Using a grating with $d = 1{\cdot}7526 \times 10^{-4}$ cm., the following results were obtained:

The image of first order was used; thus $i = 1$.

Bench reading 118·50 cm., glass scale readings 97·52, 73·84 cm.

" " 141·21 cm., " " " 89·64, 82·18 cm.

Hence

$y_1 = \frac{1}{2}(97{\cdot}52 - 73{\cdot}84) = 11{\cdot}84$ cm., $y_2 = \frac{1}{2}(89{\cdot}64 - 82{\cdot}18) = 3{\cdot}73$ cm.

Also $x_1 - x_2 = 141{\cdot}21 - 118{\cdot}50 = 22{\cdot}71$ cm.

Hence $\tan\phi = (y_1 - y_2)/(x_1 - x_2) = 8{\cdot}11/22{\cdot}71 = \tan 19° \, 39' \, 7''$.

Then $\lambda = d \sin\phi/i = 5{\cdot}894 \times 10^{-5}$ cm.

§ 10. *Experiment with an astigmatic incident beam.* The experimental test of the results of § 4 is a good exercise in optical manipulation. Fig. 5 is a plan of the apparatus. Two cross-wires, intersecting in E, are fitted into a tube turning about a horizontal axis

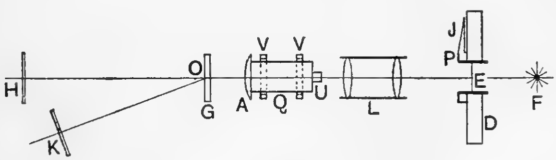


Fig. 5.

in a hole in the board D. A circular scale is attached to D and P has a pointer J which indicates its angular position. Only *one* wire is used in the measurements, but the second wire is useful as identifying E. The wires are illuminated by the sodium flame F. A lantern projection lens L is placed so that E is at its principal focus; for the best results, that end of L should face E which faces the lantern slide. Beyond L is a cylindrical tube Q, resting in two V's, V, V, and against a stop U, and thus having only one degree of freedom. A plano-cylindrical lens A is attached by its plane face to one end of Q. A lens of about $+ 2{\cdot}5$ dioptres, such as is used in spectacles, is suitable. The grating, centre O, is placed at G. A ground glass screen H can slide on the main optical bench, which also carries G, Q, L; if possible, D should be carried on the bench. On a short auxiliary bench slides a second screen K; the angle between the benches is ϕ, where $\sin\phi = i\lambda/d$. The ground sides of the screens face O.

Suppose, for a moment, that E is a luminous point. Then E, at the focus of L, gives rise to a parallel beam falling on the cylindrical lens A. This lens converts the plane wave front into a cylindrical front. If the "power" of A is $+ F$ dioptres, a "real" focal line, parallel to the generators of A's surface, will be formed $100/F$ cm. from A. This focal line can be received on the screen H. By § 4, the diffracted front is cylindrical and there is only one focal line at a finite distance. This focal line can be received on the screen K. If A is turned by turning Q on its axis, the focal line of the diffracted beam will turn about the axial ray and the distance of the focal line from O will change. The experiment tests the relation between the linear displacement of K and the angular displacement of A.

When a wire is used instead of a luminous point, images of the wire will be formed on H and K when the generators of A are parallel to the wire. If the pointer J is set in any position, a sharp image can be obtained by turning Q.

When the adjustments of § 11 have been made, H is set to receive the image of the wire, and HO is measured. As a correction we may add t/μ, where t is the thickness and μ the index of the plate covering the grating. A line ruled on H is made vertical by aid of a set square and a level, and P and Q are adjusted so that the image is vertical and $\psi_1 = 0$. Then K is set so that the diffracted image is in focus on it, and the reading of K on its bench is taken. Then P is turned by steps of $10°$ or $15°$, Q is turned in response, and K is adjusted in each case so that the image is focussed. When P has been turned through $90°$, so that $\psi_1 = \frac{1}{2}\pi$, the image is horizontal, and, by (14), since $\psi_1 = \frac{1}{2}\pi$, its distance from O is equal to the measured distance OH. When the image is vertical, $\psi_1 = 0$ or π. Since $B_1^{-1} = OH$, $B_2^{-1} = OK$, we have, by (14),

$$OK = \frac{2 \,.\, OH}{k^2 + 1 + (k^2 - 1)\cos 2\psi_1}, \qquad \text{............(18)}$$

where $k = \sec\phi$ and $\sin\phi = i\lambda/d$.

To compare theory with experiment, we may plot the value of OK given by (18) against the bench reading of K. If the zero of this bench is at the end nearest O, the points will lie about a straight line equally inclined to both axes. An alternative method is used in § 12.

§ 11. *Experimental details.* The cross-wires should be mounted so that E is as nearly as possible on the axis of P. The lines joining the nodal points of L to E are made coincident and parallel to the bench by the method of § 7. The axis of Q is set approximately parallel to the bench; optical methods are available. The cylindrical lens A is adjusted optically. For a given direction of the wire at E, there are two positions of Q, 180° apart, in which A forms a sharp image of the wire on H. If the positions of these images are not identical, the error can be corrected by moving A at right angles to its generators across the end of Q.

To set the lens L so that E is at its focus, a plane mirror is substituted for H, Q and G are removed, the cross-wires are illuminated and L and the mirror are adjusted so that E coincides with its own image. The plane of G is made perpendicular to the bench by the same method, the plate covering the grating serving as the plane mirror. The bench on which K slides is adjusted optically. First set P and Q so that a vertical image of the wire is formed on K. Then slide L along the main bench and readjust K. If the position of the image relative to K is unchanged, the auxiliary bench is correctly placed. If a micrometer eyepiece is used in place of the screen K, *two* images will be seen except when the wire is horizontal, since sodium light has a double spectrum line. Unless the wire is very fine, the images will overlap. The doubling of the images causes no inconvenience.

§ 12. *Practical example.* The following results were obtained with a grating of 14,493 lines per inch.

For this grating, $d = 1{\cdot}7526 \times 10^{-4}$ cm. The wave length was $5{\cdot}893 \times 10^{-5}$ cm. The image of first order was used; thus $i = 1$. Hence

$$\sin\phi = 0{\cdot}33625, \quad k^2 = \sec^2\phi = 1{\cdot}1275, \quad \phi = 19^\circ\, 38'\, 55''.$$

A cylindrical lens of + 2·5 dioptre was used. The corrected value of OH was 38·91 cm. The angle ψ_1 was varied from 0° to 180° by steps of 15°. The bench readings in columns 2 and 4 are theoretically identical, and their mean is given in column 5. The values of OK calculated by (18) are given in column 6. To facilitate comparison, the mean difference between columns 5 and 6 has been added to column 5, as suggested by Dr Wilcken, and the results are entered in column 7.

ψ_1	Bench reading cm.	ψ_1	Bench reading cm.	Mean reading cm.	OK calcd. cm.	OK obsd. cm.	ψ_2 calcd.
°		°					° ′
0	18·52	180	18·51	18·52	34·51	34·36	0 0
15	18·62	165	19·10	18·86	34·77	34·70	14 10
30	19·54	150	19·90	19·72	35·51	35·56	28 32
45	20·68	135	21·21	20·94	36·58	36·78	43 17
60	21·87	120	21·96	21·92	37·71	37·76	58 29
75	22·74	105	22·89	22·82	38·58	38·66	74 7
90	22·91			22·91	38·91	38·75	90 0

The observed value of OK is a little low at 0° and 90° and a little high at 45°. Probably the incident wave front was not accurately cylindrical.

The last column gives ψ_2 as calculated from $\tan\psi_2 = k^{-1}\tan\psi_1$. The difference between ψ_2 and ψ_1 is too small to admit of measurement with simple apparatus.

The Shadow Electroscope. By R. WHIDDINGTON, M.A., St John's College.

[*Received* 15 June 1920.]

A simple form of Electrostatic Voltmeter of low capacity is frequently useful in the laboratory. The instrument under description is of the gold leaf type designed primarily for class instruction and while not capable of the highest precision is yet sufficiently accurate for many purposes*.

All leaf electroscopes with which I am familiar require some sort of optical system such as a microscope to view the leaf. Attempts have been made to use a scale placed near the leaf for measuring purposes, but when too near, disturbing electrostatic effects are encountered, placed too far away parallax errors become obtrusive.

It occurred to me that the difficulties might be overcome by simply throwing a shadow of the leaf on a semitransparent scale some centimetres away, using a small 2-volt lamp as a source of light.

The first instrument made on these lines consisted of a tin cigarette box with the lamp at one end, a transparent scale at the other end and the gold leaf system with its insulation in the middle. It was found as expected that quite a sharp shadow could be obtained when the lamp filament was nearly parallel to the leaf.

The final design of electroscope is shown in section in the figure, the photographically reproduced scale, graduated in volts, being shown below. It will be seen that the scale is practically even from 100 to 500 above which the leaf becomes unstable.

The quadrant shape of metal box was chosen as being most likely to give an even scale and a constant capacity over its working range.

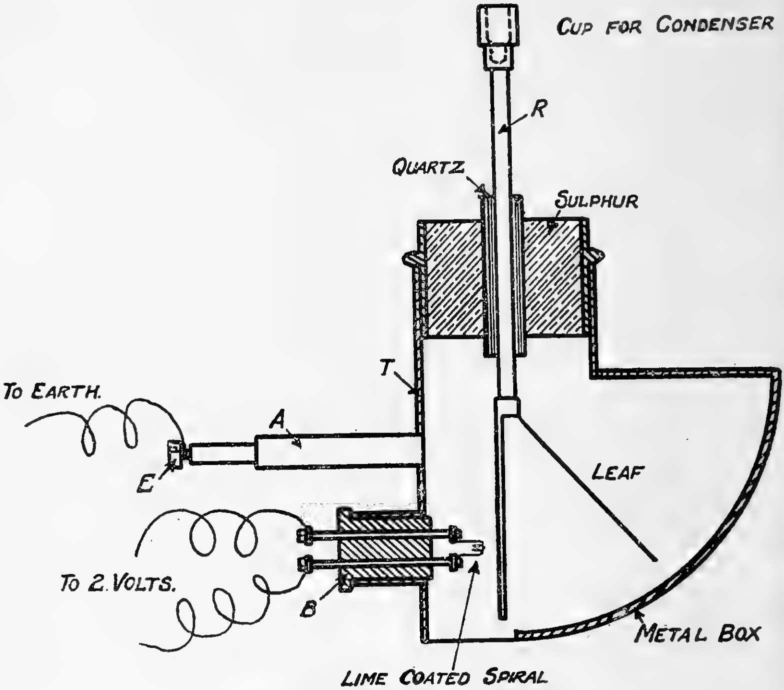
The tube (T) carries a well fitting sulphur plug fitted centrally with a quartz tube down which passes the rod (R) which carries the leaf within the case and a small cup at the top.

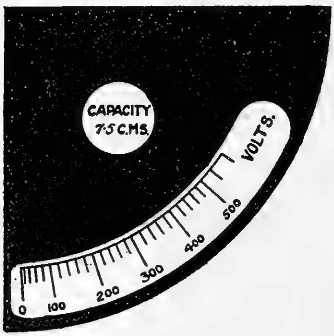
The metal arm (A) is for clamping and tilting purposes and carries an earthing terminal (E).

Just below (A), a short side tube is arranged carrying an ebonite block (B) in which a small lime coated spiral is fitted. When B is pushed home the spiral finds a place behind R. Its object is, when heated from a 2-volt cell, to provide a source of ions for

* The original instrument, of which this one is the final form, was designed in 1919 for the Naval officers under instruction in Physics at the Cavendish Laboratory.

experiments on ionization, its position behind the leaf precluding the possibility of disturbing convection currents.





The voltage range of the instrument is from 100 to 500 volts and with a good leaf it is possible to estimate to 1 volt, an accuracy sufficient for most purposes.

The scale was graduated by applying known voltages from a small direct current generator*, measuring them by a standard Weston Voltmeter.

I have found that with this instrument and the scale reproduced above, it is sufficient, when no more than approximate results are required, to register the shadow of the leaf for two positions only—zero and one other, say 200 volts. To effect this it will generally be necessary to alter the sensitiveness somewhat by adjusting the height of the sulphur block in T. This is no doubt due to the non-uniform aluminium leaf† available.

Charging the Electroscope.

After connecting E to earth, the leaf may be charged positively by induction from a rubbed ebonite rod. If a negative charge is required care should be taken not to overcharge the leaf. If an appreciable leak is observed a small piece of smooth silk rubbed lightly over Q will almost certainly cure it.

Insulation troubles are nearly always traceable to hairs and dust particles attracted under the comparatively high voltages used. It is therefore best to conduct the experiments in a dust free room.

The following are a few of the experiments which can be carried out with this instrument.

EXPERIMENT 1. *To determine the capacity* (C_e) *of the electroscope by comparison with that of a sphere of radius r cm.*

Method. Charge the leaf to a voltage V_1 as indicated by the scale reading (with the case earthed), and then share the charge on the‡ leaf with the insulated sphere thereby causing a drop in potential to V_2.

Then since $q = C_e V_1 = (C_e + r) V_2$, q being the original charge,

$$C_e = \frac{r}{V_1/V_2 - 1} \text{ cm.}$$

The following table shows a series of measurements taken on

* Kindly lent by the Electric Construction Company, Wolverhampton.

† Cut with scissors from leaf approximately ·0004 cms. thick.

‡ It is here assumed that the capacity of the sphere is equal to its radius. This is only true when the sphere is far removed from other conductors, a condition which can be approximately realised in practice if a long thin stiff vertical wire be inserted in the cup of the electroscope (or stalk of the condenser as the case may be) and the sphere touched to the top of the wire. If this precaution be neglected the results obtained will be too small.

Further, it must be remembered that when bringing up the sphere to the electroscope for charge sharing, any charge on the insulating handle will affect the leaf by induction and spoil the results. This effect may be got rid of by passing the handle through a flame occasionally, merely touching the ebonite is often sufficient to produce a charge.

these lines using an insulated brass sphere of radius 3·25 cm.*; V_1 and V_2 are the scale readings in volts.

V_1	V_2	V_1/V_2
490	330	1·485
330	235	1·405
235	160	1·470
160	110	1·452

From the above readings the mean value of $V_1/V_2 = 1{\cdot}450$, whence $C_e = 7{\cdot}2$ cm.

EXPERIMENT 2. *To determine the capacity of a parallel plate air condenser by the method of Experiment* 1.

The readings tabulated below were obtained with a specially designed circular plate air condenser†. The diameter of the central plate being 4·25 cm., and its distance from two outer earthed plates being 0·15 cm., the capacity C_a can be calculated from the formula for a parallel plate air condenser, viz. $2\dfrac{A}{4\pi d}$ cm.

Inserting the proper values for the present case leads to the value 60·2 cm.

The experimentally determined value may be expected, if anything, to be rather greater than this calculated value owing to the extra capacity of the edges of the central plate.

The method is essentially the same as in Exp. 1 but in this case the formula is

$$C_a + C_e = \frac{r}{V_1/V_2 - 1}.$$

Using the same sphere as in Exp. 1 the following results were obtained, the insulated central stalk of the condenser fitting in the electroscope cup (see figure) and the outer plates being connected to earth.

* An ordinary bedstead knob mounted on an insulating ebonite rod.

† The main point in the design is the protection of the central insulated plate from dust, small hairs etc., which under the comparatively high potentials employed would be attracted to it with resulting insulation troubles. The central plate is therefore sandwiched between two outside parallel plates, one of which is provided with a peripheral spacing ring which in butting up against the other outer plate completely encloses the inner insulated one. Insulating grooved buttons of ebonite form the insulation.

V_1	V_2	V_1/V_2
350	332	1·054
320	306	1·046
500	480	1·042
460	441	1·043
420	402	1·044

From these readings the mean value of $V_1/V_2 = 1{\cdot}045$, whence $C_a + C_e = 72{\cdot}2$ cm., and since $C_e = 7{\cdot}2$, $C_a = 65{\cdot}0$ cm.

EXPERIMENT 3. *To determine the Specific Inductive Capacity of Ebonite.*

This can be readily carried out by using a second condenser exactly similar to the one used in the previous experiment but with circular plates of ebonite separating the plates instead of air.

Then if this condenser (capacity C_b) is placed on the electroscope in the manner of the previous experiment, and charged to a potential (V_1), and the sphere is used in the manner previously described, the resulting collapse of the leaf will be so small as to be hardly readable owing to the large capacity of the ebonite condenser. It is therefore more convenient to use the air condenser of measured capacity C_a in place of the sphere. It is sufficient to hold C_a by its outer case for earthing purposes, touching its central plate momentarily to the corresponding plate of the condenser C_b on the electroscope. The potential resulting from this sharing of charge (V_2) is noted.

We then have that

$$\frac{C_a + C_b + C_e}{C_b + C_e} = \frac{V_1}{V_2},$$

whence
$$C_b + C_e = \frac{C_a}{V_1/V_2 - 1},$$

in which both C_a and C_e have been previously determined by

V_1	V_2	V_1/V_2
330	250	1·320
340	262	1·296
258	192	1·341
480	362	1·336
350	265	1·322
260	195	1·318

experiment. The above table gives the results of an experiment. From which the mean value of V_1/V_2 comes out to be 1·322.

By calculation from this value $C_b = 194{\cdot}7$ cm.

Assuming the identical dimensions of the two condensers* the Specific Inductive Capacity of Ebonite is just the ratio

$$C_b/C_a = \frac{194{\cdot}7}{65{\cdot}0}$$
$$= 2{\cdot}98.$$

A value not far removed from the accepted value which according to the table of Kaye and Laby will usually lie between 2·7 and 2·9.

EXPERIMENT 4. *The comparison of two capacities by the ionization leak method.*

It is convenient to illustrate this method by giving as an example the results of an experiment using the same two condensers as the preceding experiment.

Method. If when the lime coated spiral is glowing steadily the slow leak of the electroscope be observed firstly with C_a in position and then with C_b in position, the capacities can at once be compared, for if T_a and T_b be these times it can be shown that

$$\frac{C_a + C_e}{C_b + C_e} = \frac{T_a}{T_b}.$$

The following table gives some results obtained with this method. In order to eliminate as far as possible any variations in the amount of ionization (which depends very greatly on the temperature of the filament and therefore on the E.M.F. of the power supplying cell) the readings for T_a and T_b were taken alternately and as quickly as possible. It will be seen that under

Times in seconds	
T_a	T_b
9·2	
	25·0
9·2	
	24·6
9·6	
	25·8
9·4	
	25·4
Mean 9·35	25·2

* This can easily be tested experimentally.

the conditions of this experiment, in which a well charged 2-volt lead accumulator was used, there is very fair concordance between the various readings.

Leak observed from 400 volts to 200 volts.

If now in the above-mentioned expression we assume the previously determined values of C_a and C_e, viz. 65·0 cm. and 7·2 cm. respectively, the value of C_b comes out to 188·0 cm. leading to a value for the specific inductive capacity of ebonite of 2·90. This value is in as good agreement as is to be expected with the determination of Exp. 3.

On the Hart circle of a spherical triangle. By Professor H. F. Baker.

[*Read* 9 February 1920].

This note is concerned with the problem, given three arbitrary plane sections of any quadric, of finding a fourth section which shall be tangent to four of the tangent planes of the three given sections. If the three given sections are concurrent on the quadric they have only four tangent sections, and the fourth section is unique, the projection of the figure on to a plane (from the point of concurrence) giving rise to Feuerbach's theorem of the nine-point circle. In general the three given sections have eight common tangent planes; in fact any two of these sections lie on two quadric cones, and the six vertices of the cones so obtainable lie by threes on four coplanar lines; the three cones whose vertices are on any one of these lines have a pair of common tangent planes, which thus touch the three sections. The eight tangent planes of these are thus accounted for. There are now fourteen ways of selecting, from these eight tangent planes, four which all touch another section; six of these ways, in which the four tangent planes selected are tangent to a fourth section passing through the point of concurrence of the three given sections, are easy to recognise, and do not need further consideration. There are however eight ways of choosing four from the tangent planes which shall all touch another section lying in a plane ϖ forming with the planes of the three given sections a finite tetrahedron.

§ 1. We are thus lead to the problem of the condition necessary and sufficient in order that the sections of a quadric by the four faces of a tetrahedron should have four common tangent planes; and the main object of this note is to state this condition in a form which in fact leads to great simplification of what is generally presented as a somewhat intricate theory, and to point out several results, apparently new, which follow from this. Let the tetrahedron be O, X, Y, Z; denote the intersections of the quadric with OX by A, A', those with OY by B, B' and those with OZ by C, C'; similarly denote the intersections with YZ by U, U', those with ZX by V, V' and those with XY by W, W'. In general, if each edge of the tetrahedron be joined by planes to the two points in which the quadric meets the opposite edge, the twelve planes so obtained touch another quadric. But it may happen that this new quadric degenerates into two points, say S and S'; then, with a proper choice of notation, the four lines AU, BV, CW are concurrent in a point S, and the four lines $A'U'$, $B'V'$, $C'W'$ concurrent in another point S'. That this should be so is a necessary and sufficient condition that the four sections of the quadric by the faces of the tetrahedron should have four common tangent planes. The condition may be stated in another form; take on the edge OX, the point A_1 separated harmonically from A by O and X, and the point A_1' separated harmonically from A' by O and X; in the same way take on each edge of the tetrahedron the harmonic conjugates, with regard to the vertices of the tetrahedron lying on that edge, respectively of the intersections of the quadric with that edge. The twelve new points so obtained lie on another quadric, which we may describe as the harmonic conjugate of the original in regard to the tetrahedron. The condition in question then is that the harmonically conjugate quadric should break up into two planes, say σ and σ'; these will be the polar planes of S and S' in regard to the original quadric.

We may illustrate this condition by applying it to the (Feuerbach) case of three sections of the quadric which are concurrent on the quadric, say in O. The fourth section of the quadric touched by the four common tangent planes of the three given sections OYZ, OZX, OXY is then constructed as follows: on the plane YOZ take the line p through O, harmonically conjugate with respect to OY, OZ, to the line in which the plane YOZ is met by the tangent plane of the quadric at O; let this line p meet the quadric again in P; obtain the points Q, R of the sections ZOX, XOY in a similar way. The plane PQR is the fourth plane required. In this case one of the planes σ, σ' is the tangent plane at O.

§ 2. We may obtain a direct verification of the sufficiency of the condition in general by using it to obtain any one of the eight (Hart) sections ϖ which can be associated with three given sections YOZ, ZOX, XOY, so as to form four sections with four common tangent planes. Let the quadric, referred to YOZ, ZOX, XOY and the polar plane of O, have the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = t_1^2;$$

with an arbitrary choice of the signs of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, take

$$u = \tfrac{1}{2}(f + \sqrt{b}\sqrt{c}),\quad v = \tfrac{1}{2}(g + \sqrt{c}\sqrt{a}),\quad w = \tfrac{1}{2}(h + \sqrt{a}\sqrt{b}),$$

and then l, m, n so that

$$mn = u,\quad nl = v,\quad lm = w;$$

the eight planes required are then expressed by

$$lx + my + nz - t_1 = 0.$$

It is at once seen that this follows from the condition stated above. If we introduce λ, μ, ν so that

$$f = \sqrt{b}\sqrt{c}\cos\lambda,\quad g = \sqrt{c}\sqrt{a}\cos\mu,\quad h = \sqrt{a}\sqrt{b}\cos\nu,$$

a plane of this latter form is

$$x\sqrt{a}\,\frac{\cos\frac{1}{2}\mu\cos\frac{1}{2}\nu}{\cos\frac{1}{2}\lambda}+y\sqrt{b}\,\frac{\cos\frac{1}{2}\nu\cos\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu}+z\sqrt{c}\,\frac{\cos\frac{1}{2}\lambda\cos\frac{1}{2}\mu}{\cos\frac{1}{2}\nu}-t_1=0, \qquad \text{(H)};$$

on the other hand a common tangent plane of the three given sections in $x=0$, $y=0$, $z=0$ is at once found to be

$$x\sqrt{a}\cos(s-\lambda)+y\sqrt{b}\cos(s-\mu)+z\sqrt{c}\cos(s-\nu)-t_1=0, \quad \text{(I)}$$

where $s=\frac{1}{2}(\lambda+\mu+\nu)$; and it is easy to see that the section (I) touches the section (H) at the point of the plane (H) which lies on

$$x\sqrt{a} : y\sqrt{b} : z\sqrt{c}=p(q-r)^2 : q(r-p)^2 : r(p-q)^2,$$

where, for brevity, p, q, r stand respectively for

$$\sin(s-\lambda),\ \ \sin(s-\mu),\ \ \sin(s-\nu).$$

The four planes (I) which touch the section (H), as well as the original sections in $x=0$, $y=0$, $z=0$, are obtained from the above equation by replacing λ, μ, ν by $\pm\lambda$, $\pm\mu$, $\pm\nu$, respectively.

The eight sections (H) are obtainable from that above by replacing $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, λ, μ, ν respectively by

$$(\sqrt{a},\sqrt{b},\sqrt{c},\lambda,\mu,\nu),\ \ (-\sqrt{a},\sqrt{b},\sqrt{c},\lambda,\pi+\mu,\pi+\nu),$$
$$(\sqrt{a},-\sqrt{b},\sqrt{c},\lambda+\pi,\mu,\nu+\pi),\ \ (\sqrt{a},\sqrt{b},-\sqrt{c},\lambda+\pi,\mu+\pi,\nu)$$

together with those obtainable from these by changing the sign of t_1.

§ 3. The following result gives a construction for the position, upon the section, i, of the quadric by the plane (I), of the point in which this section is touched by the plane (H). Upon i we have three points, its contacts with the sections in $x=0$, $y=0$, $z=0$; we also have two points, namely those in which i is met by the plane from O to the intersection of the planes ABC, $A'B'C'$, which plane is at once found to have the equation

$$x\sqrt{a}+y\sqrt{b}+z\sqrt{c}=0.$$

The point to be constructed is the apolar complement of the two latter points in regard to the three former points. This result may be made clearer perhaps by stating it for a sphere in Euclidian geometry: If D, E, F be the mid-points respectively of the sides BC, CA, AB of a spherical triangle, the planes of the great circle arcs EF, BC give a diameter, and the three diameters so obtained are coplanar; let I, J denote the intersections of their plane with the inscribed circle of the triangle ABC; let P, Q, R be the points of contact of this inscribed circle with the sides BC, CA, AB. Then, upon this inscribed circle, the point of contact with the Hart circle, which touches this and certain other three tangent circles of the sides of the triangle, is the apolar complement of I, J in regard to P, Q, R. For the particular case of the nine point circle of a plane triangle the result has been remarked by Prof. F. Morley, as was pointed out to the writer by Mr J. H. Grace, *Bulletin of the American Math. Soc.*, I, 1895, 116–124 ("Apolar triangles on a conic").

§ 4. Another result may also be stated here. To introduce it and render its meaning clearer we state it first for the Hart circle of a spherical triangle in Euclidian geometry. If this circle meet the sides of the spherical triangle ABC respectively in U, U' on BC, V, V' on CA, W, W' on AB, then, with proper choice of notation, the arcs AU, BV, CW are concurrent, say in S, and the arcs $A'U'$, $B'V'$, $C'W'$ are concurrent, say in S'. The result in question is that S, S' are the centres of similitude of the circumscribed circle of the triangle ABC and the Hart circle. It is a direct generalisation of the corresponding familiar fact for the nine point circle of a plane triangle.

Stated in the more general way here adopted, which is also the more precise way, the theorem is that the lines OS, OS' are each the intersection of two planes through O which touch both the section ϖ and the section by the plane ABC. If PQR, $P'Q'R'$ be two sets of three points lying respectively on two plane sections of a quadric, such that PP', QQ', RR' are concurrent, the sections lie on a quadric cone having this point of concurrence for vertex; thus a plane through O touching the section μ by the plane ABC equally touches the section μ' by the plane $A'B'C'$. Now S, the point of concurrence of AU, BV, CW, is the vertex of one cone containing the sections μ, ϖ; and S' is similarly the vertex of one cone containing the sections μ', ϖ. The line OS', joining the vertex of one cone containing the sections μ', ϖ to the vertex of one cone containing the sections μ, μ', passes through one of the vertices of the two cones containing the sections μ, ϖ; as OSS' are not collinear, this line OS' passes through the vertex other than S of a cone containing μ and ϖ. The two cones containing μ and ϖ thus have their vertices on OS and OS'. Now to each of these cones there can be drawn from O two tangent planes, which intersect in the line joining O to the vertex of the cone; the four planes so obtained touch the sections μ and ϖ, and thus are the four common tangent planes of the cones with vertex O standing on the sections μ, ϖ. Two of these planes therefore intersect in OS and two in OS'; which is the result we desired to obtain.

There are as we have said eight sections ϖ each touched by four of the common tangent planes of the sections in YOZ, ZOX, XOY. These fall into four pairs, the planes of a pair intersecting on the polar plane of O, being harmonic conjugates in regard to this plane and O; for the pair associated as above with the two planes ABC, $A'B'C'$ the lines OS, OS' are the same. There is another pair associated similarly with the planes $AB'C'$, $A'BC$, a third pair with the planes $BC'A'$, $B'CA$ and a fourth pair associated with the planes $CA'B'$, $C'AB$. And it may be remarked that the sections by the planes ABC, $AB'C'$, $BC'A'$, $CA'B'$ are all touched by four planes, as follows at once from the fact that AA', BB', CC' are concurrent; so also the sections by the planes $A'B'C'$, $A'BC$, $B'CA$, $C'AB$ are all touched by four planes.

§ 5. Another remark may be made, relating to a property which appears in Euclidian geometry as Salmon's theorem that the tangent of the radius of the circumcircle of a spherical triangle is twice the tangent of the radius of the Hart circle.

Let P be the pole of any plane section of a quadric, upon which any point A is taken, and O be any other point; denote by ρ the Cayley separation of the lines OP, OA in regard to the quadric, and by δ the Cayley separation of P from O. It can then be shown that ρ is independent of the position of A upon the section, and is indeed symmetrical in regard to P and O, being connected with δ by an equation $\sin\delta\sin\rho = \pm 1$. Calling ρ the radius of the section in regard to the point O, it can be shown that if ρ, ρ' be the radii of any two sections α, α' whose planes intersect in a line l, and the planes joining l to O and to the vertex of one of the two cones containing α and α' be respectively denoted by ω and γ, then $\tan\rho/\tan\rho'$ is equal to the homography $(\gamma, \omega;\ \alpha, \alpha')$ or to the negative of this. In particular when the planes γ, α are harmonically separated by ω and α', this leads to $\tan\rho = 2\tan\rho'$. In our figure the plane σ, which is the polar of S in regard to the quadric, passes through the line of intersection of the planes ABC and ϖ, since S is the vertex of one of the cones containing the section by ABC and the Hart section ϖ, and this plane σ also contains the vertex of the other cone containing these sections; *it can easily be proved that the plane ω which joins O to the line of intersection of the planes ABC and ϖ is harmonically separated from ϖ by the planes ABC and σ*; thus the planes σ, ABC, ω, ϖ have the relation of the respective planes γ, α, ω, α' in the general description just given. It follows that if ρ, R be the radii of the sections ϖ, ABC, we have $\tan R = 2\tan\rho$; which is what we wished to prove.

§ 6. A last remark may be added bringing into relief the connexion between the present point of view and that of the Euclidian geometry. As hitherto, let $OXYZ$ be a tetrahedron whose faces meet a quadric in sections having four common tangent planes. Denote by $i\alpha$, $i\beta$, $i\gamma$ the Cayley separations OX, OY, OZ in regard to the quadric; by $i\alpha', i\beta', i\gamma'$ the Cayley separations YZ, ZX, XY; by (A), (B), (C) the Cayley separations of the pairs of planes meeting respectively in OX, OY, OZ; and by (A'), (B'), (C') the Cayley separation of the plane XYZ respectively from the planes YOZ, ZOX, XOY. Each of these separations is ambiguous in sign and by additive multiples of π, unless we enter into further detail. There are however equations by which all of them are deducible from α, β, γ; and these equations may be represented, when proper regard is paid to the ambiguities, by

$$\alpha' = i\pi + \beta - \gamma, \quad \beta' = i\pi + \gamma - \alpha, \quad \gamma' = i\pi + \alpha - \beta,$$

$$\tan (A) = \frac{\sinh \alpha}{\cosh (\epsilon + \alpha)}, \quad \tan (A') = \frac{\sinh (\beta - \gamma)}{\cosh (\epsilon + \beta + \gamma)}$$

where ϵ is such that

$$2 \tanh \epsilon = \tanh \alpha \tanh \beta \tanh \gamma - \tanh \alpha - \tanh \beta - \tanh \gamma.$$

And these lead to

$$(A') = (B) - (C), \quad (B') = (C) - (A), \quad (C') = (A) - (B),$$

which may be used to define the Hart section.

§ 7. In what has preceded we have stated a *sufficient* condition for the Hart section, namely that AU, BV, CW are concurrent. It can however be proved that this is also a *necessary* consequence of the existence of the four sections of the quadric all touched by four other planes, provided we exclude certain particular possibilities which are easily stated. Precisely, given three arbitrary plane sections of a quadric, no one of which degenerates into two straight lines, so that the equation of the quadric referred to these and the polar plane of their point of intersection is of the form $(abcfgh \between xyz)^2 = t_1^2$, in order that these with a fourth section (also not two straight lines) should form a set of four sections all touched by four planes, if no relations are assumed to hold among the coefficients a, b, c, f, g, h, it is necessary that the condition in question (that AU, BV, CW are concurrent) should hold.

In order that the sections by $x = 0$, $y = 0$, $z = 0$, $t = 0$ of the quadric $(abcdfghuvw \between xyzt)^2 = 0$ should have four common tangent planes, the cones enveloping the quadric along these sections must be concurrent; if Δ be the four-rowed determinantal discriminant, and A, B, ... the minors therein, it follows that the necessary and sufficient condition for this is that the equation

$$(abcdfghuvw \between \sqrt{A}, \sqrt{B}, \sqrt{C}, \sqrt{D})^2 = \Delta$$

should be satisfied for four choices of the signs of $\sqrt{A}, \sqrt{B}, \sqrt{C}, \sqrt{D}$. It proves to be possible to examine all the ways in which this can happen, and the result is as stated.

On a property of focal conics and of bicircular quartics. By Professor H. F. Baker.

[*Read* 9 February 1920.]

The property of focal conics referred to in the title is the well-known one that if P, R be any two points of the focal hyperbola of a system of confocal quadrics, and Q, S be any two points of the focal ellipse, then the distances PQ, PS have the same difference as the distances RQ, RS. The theorem remains true if every one of the distances be replaced by the Cayley separation of its end points in regard to an arbitrary quadric of the confocal system, and the original theorem is then obtainable by making the parameter of this arbitrary quadric increase without limit. It is shown that the generalised theorem is equivalent to the geometrical theorem that two enveloping cones of the arbitrary confocal exist, each of which touches the four lines PQ, QR, RS, SP. The theorem that the sum of the two focal distances of a point of an ellipse is constant may similarly be replaced by the theorem that the sum of the Cayley separations of a point of the ellipse from the foci is constant, in regard to an arbitrary confocal conic; a theorem is obtained which includes both this last result and the former. It is unnecessary to point out that this last result is equivalent with Chasles's theorem that a variable tangent plane of a quadric cone makes angles with the planes of circular section whose sum is constant (Chasles, *Géom. Supér.*, 1880, § 812, p. 517).

The property of bicircular quartics referred to is that the angles which a variable bitangent circle of one mode of generation makes with two fixed bitangent circles of another mode of generation, have a constant sum (Jessop, *Quart. Journ.*, XXIII, 1889, 375). This is shown to be equivalent to the former theorem.

There exist much more general theorems in regard to the generation of a quadric with the help of a thread of constant length, whose systematic investigation is in connexion with the theory of hyperelliptic functions (Chasles, *Liouville*, XI, 1846, 15; Darboux, *Théorie des surfaces*, Livre IV, Ch. XIV, 296–312; Staude, *Math. Ann.*, XX and XXII, 1883; Finsterwalder, *Math Ann.*, XXVI, 1886; Maxwell, *Works*, II, 156 or *Quart. Journ.*, 1867). I have added some lines in regard to this general point of view.

§ 1. If P, Q, R, S be four coplanar points of a quadric, and through the lines SP, PQ, QR, RS be drawn four arbitrary planes, respectively, α, β, γ, δ, the lines $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ meeting the quadric again respectively in P', Q', R', S', then (1) the points P', Q', R', S' are equally on a plane, (2) if by the angle between the sections of the quadric by the planes α, β be understood the Cayley separation of these planes, measured by the homography of these planes in regard to the two tangent planes to the quadric drawn from their line of intersection, then the sum of the angles at P, R, determined respectively by the sections α, β and γ, δ, is equal to the sum of the angles at Q, S, determined respectively by the sections β, γ and δ, α.

That P', Q', R', S' lie in a plane follows from the fact that the four quadrics consisting of (i) the original quadric, (ii) the planes α, γ, (iii) the planes β, δ, (iv) the planes PQR, $P'Q'R'$, have seven, and therefore eight points in common. For the relation between the angles, denote by θ the section by the plane $PQRS$, and in general by (α, β) the angle between the sections (α, β). Then we have

$$\pi = (\theta, \alpha) + (\alpha, \beta) + (\beta, \theta) = (\theta, \gamma) + (\gamma, \delta) + (\delta, \theta),$$

and therefore

$$(\alpha, \beta) + (\gamma, \delta) = 2\pi - (\theta, \alpha) - (\theta, \beta) - (\theta, \gamma) - (\theta, \delta),$$

which is also the value of $(\beta, \gamma) + (\delta, \alpha)$, the ambiguities of interpretation being properly settled in each case.

In a plane we have the theorem that if P, Q, R, S be concyclic points through which pass pairs of four circles α, β, γ, δ, namely α, β through P, β, γ through Q, γ, δ through R and δ, α through S, then the two angles (α, β), (γ, δ) have the same sum as the two angles (β, γ), (δ, α); and this, not depending on the Axiom of parallels, may well be regarded as a fundamental theorem. Further if P' be the other intersection of α and β, etc., the points P', Q', R', S' are concyclic. The connexion of this result with the theorem of the angles is incidentally remarked by Prof. W. McF. Orr, *Trans. Camb. Phil. Soc.*, XVI, 1897, 95.

§ 2. Regard the bicircular quartic in question as the projection on to an arbitrary plane of the section of a quadric by a quadric cone of general position, the centre of projection being an arbitrary point of the quadric. An arbitrary tangent plane of the cone cuts the quadric in a section projecting into a conic having two points of contact with the bicircular quartic, and this conic, passing through the nodes of the quartic, is for us a bitangent *circle*, of one mode of generation. The other three modes are obtained by considering the other three quadric cones through the intersection of the quadric and the first cone. Take then two bitangent circles of the bicircular quartic of the first mode of generation, say α and γ; their points of contact will be on another circle, say ρ, as appears from the three dimensional figure. Take also two bitangent circles of a second mode of generation, say β and δ, with points of contact on a circle, σ. We shall prove that the eight points of intersection of the pairs of circles (α, β), (β, γ), (γ, δ), (δ, α) lie on two circles θ, θ', four on each. These circles θ, θ' pass through the two intersections of the circles ρ, σ, and separate these circles harmonically; the circle ρ is orthogonal to the principal circle to which the bitangent circles of the first mode are all orthogonal, with a similar statement for σ. For the proof, let $\alpha = 0$, $\gamma = 0$ be the equations of any two tangent planes of a quadric cone, whose generators of contact lie on a plane $\rho = 0$, so that the cone has the equation $\alpha\gamma - \rho^2 = 0$. Let $\beta\delta - \sigma^2 = 0$ be another quadric cone, whereof $\beta = 0$, $\delta = 0$ are tangent planes touching the cone on $\sigma = 0$. Then a quadric $E = 0$ through the curve of intersection of the two cones has an equation of the form

$$E = \alpha\gamma - \rho^2 - m^2(\beta\delta - \sigma^2) = 0,$$

so that the four lines $\alpha = 0$, $\beta = 0$; $\beta = 0$, $\gamma = 0$; $\gamma = 0$, $\delta = 0$; $\delta = 0$, $\alpha = 0$, in which the two first planes α, γ meet the two latter planes β, δ, intersect the quadric $E = 0$ in eight points lying in the two planes $\rho + m\sigma = 0$, $\rho - m\sigma = 0$.

We have then a proof of Jessop's theorem in regard to the bicircular quartic curve*.

§ 3. Reciprocally let any two conics be taken in space, not intersecting one another. Consider a quadric touched by the common tangent planes of these two conics. Then if A, C be any two points of the first conic, and B, D any two points of the second conic, it follows from § 2 that the pairs of tangent planes to this quadric from the lines AB, BC, CD, DA touch two enveloping cones of the quadric, say F and G. Or, as a line lying in a tangent plane of a cone is a tangent line of the cone, there are two enveloping cones of the quadric which touch the lines AB, BC, CD, DA. And, comparing the equations of § 2, the vertices of these cones lie on the line joining the points R, S, in the planes of the conics, which are the poles respectively of AC, BD in regard to these conics, and separate R, S harmonically; the positions of the vertices depend on the quadric taken to touch the common tangent planes of the conics. Moreover, as the reciprocal of the

* The direct analytical proof is, of course, simple. Let the fundamental quadric be $x^2 + y^2 + z^2 + t^2 = 0$, and bitangent circles of two modes be obtained by projection of the polar sections respectively of the two points

$$[(a-d)\,x,\ (b-d)\,y,\ (c-d)\,z,\ 0],\ [(a-c)\,\xi,\ (b-c)\,\eta,\ 0,\ (d-c)\,\tau].$$

Then the angle between these circles, being the Cayley separation of these points, is the angle, in rectangular Cartesian coordinates, between the two lines

$$X/px = Y/qy,\ X/p\xi = Y/q\eta, \text{ where } p^2 = (a-d)\,(a-c),\ q^2 = (b-d)\,(b-c).$$

This generalises at once to the Cyclide; cf. Jessop, *Quartic surfaces*, 1916, p. 106.

theorem in regard to the angles, if we consider the homography of A, B in regard to the quadric, say a, and take the corresponding homographies for the pairs B, C; C, D; D, A respectively, say b, c, d, we have $ac = bd$, or $a/d = b/c$. In words, the difference of the Cayley separations of A from B and D, in regard to the quadric, is the same, for unaltered positions of B, D on the second conic, when A is replaced by any other point C of the first conic.

This result includes the particular case of the focal conics of a confocal system, for which we may also consider the further particular case of actual Euclidian distances between the points. (Cf. § 10 below, where the relation between the separation and the distance is given.)

§ 4. If we assume that the sides of the skew quadrilateral $ABCD$ in § 3 touch an enveloping cone of the quadric, we can deduce the relation between the Cayley separations in another way. In fact if the sides of a skew quadrilateral touch any quadric having ring contact with a given quadric, the sum of the Cayley separations belonging to the sides of the quadrilateral, each taken in proper sense, is zero, the separations being measured by the latter quadric. For if AT be a tangent to a quadric V, which has ring contact with a quadric U, drawn from a point A, the Cayley separation AT in regard to U is independent of T. If A be (ξ, η, ζ, τ), T be (x, y, z, t), so that, with usual notation, $V_x = 0$, $V_{x\xi} = 0$, and U be $V + P^2 = 0$, then $U_x = P_x^2$, $U_{x\xi} = P_x P_\xi$, and hence

$$U_{x\xi}/(U_x U_\xi)^{\frac{1}{2}} = P_x P_\xi / P_x\,(U_\xi)^{\frac{1}{2}} = P_\xi/(U_\xi)^{\frac{1}{2}},$$

which is independent of x, y, z, t; and $U_{x\xi}/(U_x U_\xi)^{\frac{1}{2}}$ is the cosine of the separation in question. Therefore, if the sides AB, BC, CD, DA of the skew quadrilateral touch V respectively at L, M, X, Y, we have the following relations among the separations

$$(AB) = (AL) - (BL),\ \ (BC) = (BM) - (CM),$$
$$(CD) = (CX) - (DX),\ \ (DA) = (DY) - (AY),$$
$$(AY) = (AL),\ \ (BL) = (BM),\ \ (CM) = (CX),\ \ (DX) = (DY),$$

leading to

$$(AB) + (BC) + (CD) + (DA) = 0,$$

or

$$(AB) - (AD) = (CB) - (CD).$$

In the application of this result above, V was a cone.

§ 5. We may however make an application in which U is a cone, and V not a cone, U being an enveloping cone of V. Namely, if the sides of a skew quadrilateral touch a quadric, the sum of the four Cayley separations of the vertices, each in proper sense, in regard to any enveloping cone of the quadric, is zero. The reciprocal theorem, is that if two plane sections α, γ of a quadric be both touched by each of two other sections β, δ—and if, taking a fifth arbitrary section, ω, of the quadric, we measure the angle between the planes of two sections α, β, which touch one another, in the usual way, by considering the homography of these planes in regard to the tangent planes drawn from their line of intersection to the section ω—then, with proper sense of measurement, $[\alpha, \beta]$ denoting the angle between these planes, we have

$$[\alpha, \beta] + [\beta, \gamma] + [\gamma, \delta] + [\delta, \alpha] = 0.$$

Now take one of the two quadric cones containing the sections α, γ, and regard this cone, and the section ω, as fundamental; speak of α, γ as circular sections of this cone, of opposite systems because each has two points common with the other and with ω. Then we have Chasles's theorem that a variable tangent plane of a quadric cone makes angles of constant sum with two planes of circular section of the cone, of opposite systems.

§ 6. The reciprocal theorem is that a generator of a quadric cone makes angles of constant sum with two conjugate focal lines of the cone, that is, considering the conic in which the plane of ω cuts the cone, and the quadrilateral formed by the common tangents of this conic and ω, makes angles of constant sum with the lines joining the vertex of the cone, to an opposite pair of intersections of two of these common tangents (Chasles, *loc. cit.*, § 827, p. 528). Projecting on to an arbitrary plane we have the theorem that if P be a variable point of one of two conics having S, H as common foci, the Cayley separations PS, PH in regard to the other conic have a constant sum. An elementary proof can be given depending on the fact that if PS meet the other conic in S_1, S_2, and PH meet the other conic in H_1, H_2, then, with proper notation, each of S_1H_2, S_2H_1 passes through a fixed point of the line SH.

§ 7. This theorem for conics is a particular case of the following: Two conics V, W, have both double contact with a conic U, and also both have double contact with another conic K. From a point P of K a tangent PX is drawn to V, and also a tangent PY to W; then the Cayley separations PX, PY, taken in regard to U, have a constant sum (or difference) as P varies on K. Two tangents are possible from the point P to the conic V; but the separation PX is the same for both.

If V degenerate into the pair of tangents to U from a focus S of U, and W into the pair of tangents to U from the conjugate focus H, then the conic K, touching these four tangents, will be confocal with U, and the tangents PX, PY will become the lines PS, PH. Thus the theorem includes that of § 6.

§ 8. The proof of the general theorem of § 7 is, analytically, identical with that of the following theorem, of three dimensions, which leads, in § 9, to the theorem of § 4, and may thus be regarded as summarising all the analogous theorems here obtained:—If two quadrics V, W both have ring contact with a quadric U, and also both have ring contact with a quadric K, and PX, PY be tangents respectively to V and W from a point P of K, the sum (or difference) of the Cayley separations PX, PY, in regard to U, is independent of the position of P upon K. When V and W coincide the difference of the separations is zero for all positions of P and the quadric K is unnecessary.

The theorem is easy to prove. In order that two quadrics $V=0$, $W=0$ should both have ring contact with another quadric $U=0$, they must, if $P=0$, $Q=0$ be suitable planes, be capable of the forms $V=U-P^2$, $W=U-Q^2$, and thus V, W must have two points of contact, there being an identity of the form

$$V - W = pq,$$

where $p=0$, $q=0$ are two planes. Any quadric having ring contact with both V and W is then capable of either of the identical forms

$$V + \tfrac{1}{4}(a^{-1}p - aq)^2 = 0, \quad W + \tfrac{1}{4}(a^{-1}p + aq)^2 = 0,$$

wherein a is a constant, and two such quadrics can be drawn through an arbitrary point. We may then suppose

$$U = V + \tfrac{1}{4}(a^{-1}p - aq)^2, \quad K = V + \tfrac{1}{4}(b^{-1}p - bq)^2,$$

where $K=0$ is the quadric of the enunciation, and b is a constant. Thus we have the identity

$$U - K = \tfrac{1}{4}(a^{-2} - b^{-2})(p^2 - a^2b^2q^2),$$

involving in particular that U, K have two points of contact on the line joining the points of contact of V and W. Putting

$$P = \tfrac{1}{2}(a^{-1}p - aq), \quad Q = \tfrac{1}{2}(a^{-1}p + aq),$$

this is the same as

$$(1-\sigma^2)(U-K) = P^2 + Q^2 + 2\sigma PQ,$$

where $\sigma = (a^2+b^2)/(a^2-b^2)$. This again, if U is not zero, is the same as

$$(P^2 - U)(Q^2 - U) - (PQ + \sigma U)^2 = (1-\sigma^2)UK.$$

We remarked however above (§ 4), that if θ, ϕ be the Cayley separations PX, PY, taken in regard to U,

$$\cos\theta = \frac{P}{U^{\frac{1}{2}}}, \quad \cos\phi = \frac{Q}{U^{\frac{1}{2}}},$$

where the coordinates in U, P, Q are those of the point P. If this point be on the quadric $K = 0$, but not on $U = 0$, we thus get

$$\cos\theta\cos\phi + \sigma = \pm\sin\theta\sin\phi$$

showing that $\theta \pm \phi =$ constant, as was stated.

§ 9. Now suppose a skew quadrilateral $ABCD$ of which the sides AB, BC both touch the quadric V, say in X and Y, respectively, while the sides CD, DA both touch the quadric W, say in Z and T respectively. The quadrics V, W are supposed to have two points of contact, so that quadrics can be drawn having ring contact with both. Let U be one such; let K be another such passing through C, and let A be on K. Then, considering Cayley separations in regard to U, we have (BX), (BY) equal because V has ring contact with U, and also (DZ), (DT) equal because W has ring contact with U. By § 8 we also have $(AX) - (AT)$ equal to $(CY) - (CZ)$, if a proper sense be assigned to the separations involved.

We infer therefore that

$$\begin{aligned}(AB) - (AD) = (AX) + (XB) - [(AT) + (TD)] = (AX) - (AT)\\ + (XB) - (TD) = (CY) - (CZ) + \epsilon\,(YB) - \zeta\,(ZD),\end{aligned}$$

where ϵ, ζ are each ± 1. Without making the proper detailed examination, we shall put both ϵ and ζ equal to 1, so obtaining

$$(AB) - (AD) = (CB) - (CD).$$

This is verified (§ 4) in the particular case where the quadrics V, W coincide, there being then no need for the condition that A, C lie on the same quadric K having ring contact with V and W.

§ 10. A line joining a point of one focal conic to a point of another focal conic of a confocal system of quadrics is a particular case of a line touching two confocals of the system. And such a line is part of a continuous curve which on either of these two confocals may consist partly of arcs of the line of curvature which is the intersection of these two fundamental confocals, and partly of arcs of geodesics touching this line of curvature. As was recognised by Chasles this continuous curve has everywhere the geometrical property that if we take two other confocals of the system, the homography of the tangent planes drawn to one of these from a tangent line of the curve, in respect of the tangent planes drawn to the other, is the same for every point of the curve. That the analytic formulation may equally be regarded as uniform for all parts of the curve seems often to be unnoticed; it is recognised however by Staude in the papers above referred to. Let us consider the system of confocals

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1,$$

where $a > b > c$, using λ for ellipsoids, μ for hyperboloids of one sheet, ν for hyperboloids of two sheets. Suppose that the straight portion of the curve touches the confocals for which $\lambda = p$, $\lambda = q$, and denote by dw the Cayley separation of two consecutive points of the curve taken in regard to the confocal of parameter $\lambda = \theta$. Putting

$$F(x) = 4(x+a)(x+b)(x+c)(x-p)(x-q),$$

$$L^2 = f(\lambda), \quad M^2 = f(\mu), \quad N^2 = f(\nu),$$

the curve is such that

$$\int \frac{(\lambda - p)\, d\lambda}{L} + \int \frac{(\mu - p)\, d\mu}{M} + \int \frac{(\nu - p)\, d\nu}{N} = 0,$$

$$\int \frac{(\lambda - q)\, d\lambda}{L} + \int \frac{(\mu - q)\, d\mu}{M} + \int \frac{(\nu - q)\, d\nu}{N} = 0,$$

while, with $\Theta^2 = F(\theta)$,

$$\int \frac{(\lambda - p)(\lambda - q)\, d\lambda}{(\lambda - \theta) L} + \int \frac{(\mu - p)(\mu - q)\, d\mu}{(\mu - \theta) M} + \int \frac{(\nu - p)(\nu - q)\, d\nu}{(\nu - \theta) N}$$

$$= 2w \,.\, \frac{(\theta - p)(\theta - q)}{\Theta},$$

where $w = \int dw$. By supposing θ to increase indefinitely, and replacing $\theta^{\frac{1}{2}} dw$ by ds, we have the corresponding result when Euclidian distance is used.

In the notation of hyperelliptic functions (see *Multiply-periodic functions*, Cambridge, 1907, pp. 35, 36, using the p, q as a_1, a_2 are there used), we have

$$\frac{\Theta}{2(\theta - p)(\theta - q)} \int_{x_0}^{x} \frac{(x-p)(x-q)}{x - \theta} \frac{dx}{y}$$

$$= \zeta_1 (u^{\theta, \infty}) \,.\, u_1^{x, x_0} + \zeta_2 (u^{\theta, \infty}) \,.\, u_2^{x, x_0} + \tfrac{1}{2} \log \frac{\vartheta (u^{x, \theta} + k)}{\vartheta (u^{x, \phi} + k)},$$

where (ϕ) denotes the place conjugate to (θ), k is such that $\vartheta(k)$ vanishes identically, and when θ is large the significant terms of the functions ζ_1, ζ_2 are $\theta^{-\frac{1}{2}} pq$ and $\theta^{-\frac{1}{2}}(p+q)$.

Along a straight portion of the curve, joining, suppose, the points $(\lambda_0, \mu_0, \nu_0)$, (λ, μ, ν), the places (λ), (μ), (ν) of the hyperelliptic construct are coresidual with the places (λ_0), (μ_0), (ν_0), and we can satisfy the identities

$$F(x) - [\psi(x)]^2 = 4(x-\lambda)(x-\mu)(x-\nu)(x-f_1)(x-f_2),$$

$$F(x) - [\psi_0(x)]^2 = 4(x-\lambda_0)(x-\mu_0)(x-\nu_0)(x-f_1)(x-f_2),$$

where

$$\psi(x) = 2(lx^2 + mx + n), \quad \psi_0(x) = 2(l_0x^2 + m_0x + n_0).$$

With this notation we find, for the Cayley separation of the two extreme points,

$$w = \tanh^{-1}\left(\frac{\psi(\theta)}{\Theta}\right) - \tanh^{-1}\left(\frac{\psi_0(\theta)}{\Theta}\right),$$

leading in particular, if r be the distance of these points, to $r = l - l_0$, and to

$$r = (\theta - \lambda_0)(\theta - \mu_0)(\theta - \nu_0)\frac{2\tanh w}{\Theta + \psi_0(\theta)\tanh w}.$$

The character of the symmetric functions of the places (λ), (μ), (ν), *regarded as functions of* w, along any portion of the curve, seems eminently worthy of investigation.

And it appears that the total value of w, along any closed portion of the continuous curve, is expressible by an aggregate of the periods of the integral

$$\frac{\Theta}{2(\theta - p)(\theta - q)}\int\frac{(x-p)(x-q)}{(x-\theta)\,y}\,dx,$$

where $y^2 = F(x)$, with integer coefficients; these will then be unaltered by any continuous small deformation of the arc of the curve. This remark appears to lead to all the known results.

In conclusion I should like to refer the reader to a most interesting note by Mr A. L. Dixon, *Messenger of Mathematics*, XXXII, 1903, 177.

On the construction of the ninth point of intersection of two plane cubic curves of which eight points are given. By Professor H. F. BAKER.

[*Read* 3 May 1920.]

Cayley has collected, in a paper reprinted in Vol. IV of his *Papers*, pp. 495–504 (*Quart. J.*, v, 1862), the various solutions given of this problem, regarded as a problem of plane geometry, by Plücker, Weddle, Chasles and Hart, depending for the most part on the generation of a plane cubic curve (two points at a time) by the intersection of a pencil of lines and a homographic pencil of conics. So far as I have been able to notice, geometrical conceptions present themselves to an unbiassed child in the first instance as three dimensional, and he feels it to be an abstraction to regard plane geometry as self-contained; the discussion of the most natural Axioms of geometry seems also to point in this direction; and the most valuable part of a training in geometry would seem to lie in the cultivation of a faculty for visualisation of relations in space. However these things may be, it appears to me always to be an interesting extension when a property of space is shown to follow from a property in space of higher dimensions, this being generally accompanied by the removal of some artificiality. Thus, I regard the very simple example which now follows as being logically at least as fundamental as a proof in the plane.

Let A, B, C, M, N and P, Q, R be the eight given coplanar points. Take a point D outside the plane of these. There are ∞^1 quadric surfaces containing A, B, C and the lines DM, DN; let Ω be one of these (other than that consisting of the planes ABC, DMN). Let DP, DQ, DR meet this quadric again in P_1, Q_1, R_1. A definite twisted cubic curve can be drawn through D, A, P_1, Q_1, R_1 to have BC as a chord (see below). This cubic curve, meeting Ω in D, A, P_1, Q_1, R_1, meets Ω in a further point, say O_1. If DO_1 meet the original plane in O, this is the ninth point required.

For the space cubic is the intersection of two quadric surfaces drawn through D, A, P_1, Q_1, R_1, both having the line BC as a generator; denote these by U and V. The quartic space curve of intersection of U with Ω contains D, A, B, C, P_1, Q_1, R_1, and meets the generators DM, DN of Ω; this curve then projects from D on to the original plane into a cubic curve containing the eight given points A, B, C, P, Q, R, M, N. The curve of intersection of V with Ω projects from D into another cubic through these eight points. The point O_1, on the space cubic, lies on U

and V, and on Ω, and so projects from D into a point common to the two plane cubics. This justifies the statement.

Incidentally any two cubic curves in a plane are shown to be the projections of two quartic curves in space lying on the same quadric; and the plane problem is put in connexion with the space problem of finding the remaining eighth intersection of three quadrics with seven common points.

To construct a twisted cubic curve with five given points D, A, P_1, Q_1, R_1 to have a given line BC as chord, we may for instance first construct a quadric surface by the intersection of corresponding planes of two homographic axial pencils with DA, BC as axes, three pairs of corresponding planes being those containing P_1, Q_1, R_1, and then construct a quadric surface by the intersection of corresponding planes of two homographic axial pencils with DP_1, BC as axes, three pairs of corresponding planes being those containing A, Q_1, R_1. These quadric surfaces intersect in the cubic curve required.

It is seen that analytically each step requires only the solution of linear equations. Indeed, if the conic through A, B, C, M, N be written (referred to A, B, C, D) as $Ayz + Bzx + Cxy = 0$, the line MN being $x + y + z = 0$, we may take for Ω the quadric $t(x + y + z) = Ayz + Bzx + Cxy$. The general plane cubic curve through the five points A, B, C, M, N may be taken to be

$$(Ayz + Bzx + Cxy)(lx + my + nz) + (x + y + z)\,x\,(qy + rz) = 0,$$

and two cubics through these and P, Q, R may be found by solving for the ratios of l, m, n, q, r in the three equations obtained by substituting the coordinates of P, Q, R. Corresponding to two sets of ratios $l_1 : m_1 : n_1 : q_1 : r_1$, and $l_2 : m_2 : n_2 : q_2 : r_2$ so chosen, there are two quadric surfaces

$$t(l_1x + m_1y + n_1z) + x(q_1y + r_1z) = 0,$$

$$t(l_2x + m_2y + n_2z) + x(q_2y + r_2z) = 0,$$

which intersect in a cubic curve containing D, A, P_1, Q_1, R_1 and having BC for chord. The combination of these with the equation of Ω will lead to a linear equation for O_1, from which O is found. Or the solution may be stated, naturally enough, without reference to three dimensions.

On a proof of the theorem of a double six of lines by projection from four dimensions. By Professor H. F. BAKER.

[*Read* 9 February 1920.]

The theorem in question is that if five lines in three dimensions, of which no two intersect, say a, b, c, d, e, have a common transversal, say f', and we take the five transversals other than f' of every four of these five given lines, the five new lines so obtained have also a common transversal. Namely if a' be the transversal, beside f', of b, c, d, e, and b' be the transversal, beside f', of a, c, d, e, and so on, so that we have the scheme

$$\begin{array}{cccccc} a & b & c & d & e & \\ a' & b' & c' & d' & e' & f' \end{array}$$

in which every line intersects those not occurring in the same row or column with itself, but not the others, in general, then there is a transversal f of a', b', c', d', e'.

We see that the theorem is that if we take eight lines a, b, c, d and a', b', c', d', so related that a' meets b, c, d, while b' meets a, c, d, and c' meets a, b, d and d' meets a, b, c, and if e', f' be the two transversals of a, b, c, d and e, f be the two transversals of a', b', c', d', then the meeting of one of the two former, f', with one of the two latter, e, involves the meeting of the other, e', of the two former, with the remaining one, f, of the two latter. But the original relation of the eight lines a, b, c, d, a', b', c', d' has a certain artificiality; the object of the present note is to show that there is a simple figure in four dimensions, possessing perfect naturalness, being determinate when four arbitrary lines of that space are given, from which the figure in three dimensions may be derived by projection; and that the condition for this derivation is precisely the intersection of the two transversals e and f'. The naturalness of this figure lies in the fact that three lines in four dimensions have just one transversal.

§ 1. In order to show this, it is necessary to enter into some detail in regard to the elements of the geometry of four dimensions; this appears worth while for its own sake; and in order not to over-emphasize the importance of the theorem in three dimensions which is here made the excuse for this, we first give an elementary proof of this theorem, employing only three dimensions (*Proc. Roy. Soc.* A, LXXXIV, 1911, 597).

With the notation above, denote the respective intersections (b', c), (b, c'), (c', a), (c, a'), (a', b), (a, b'), (a, f'), (b, f'), (c, f') by A, A', B, B', C, C', U', V', W'. Let f be the transversal other than e of a', b', c', d', which we may represent by $f = (a', b', c', d')/e$,

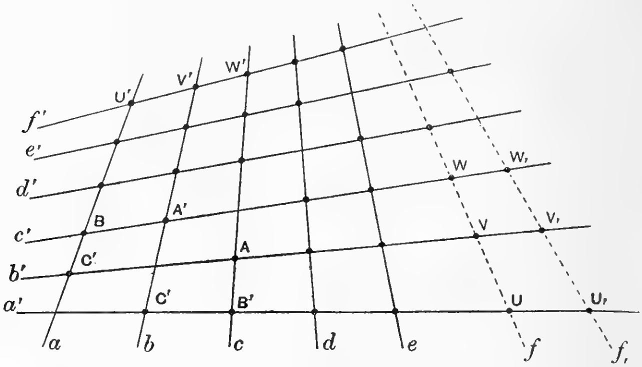


Fig. 1

and denote the points (a', f), (b', f), (c', f) respectively by U, V, W. Similarly let f_1 be the transversal other than d of (a', b', c', e'), which we may denote by $f_1 = (a', b', c', e')/d$; and let the points (a', f_1), (b', f_1), (c', f_1), be U_1, V_1, W_1.

Now take the lines

$$\left.\begin{matrix} a, b, c, f; e \\ a', b', c', f'; d' \end{matrix}\right\}.$$

The two quadric surfaces defined respectively as containing (b, c, e) and (b', c', d'), have, *both of them*, the two generators e and d', which are intersecting lines. The other common points of these two quadrics are then coplanar. Such points are A and A', respectively (b', c) and (b, c'), and U' or (a, f') and U or (a', f). Thus U lies on the plane A, A', U'). So, by considering the quadrics (c, a, e), (c', a', d'), we find that V lies on the plane (B, B', V'), and, by considering the quadrics (a, b, e), (a', b', d'), that W lies on the plane (C, C', W'). By taking the lines

$$\left.\begin{matrix} a, b, c, f_1; d \\ a', b', c', f'; e' \end{matrix}\right\},$$

and considering the pairs of quadrics

$$(b, c, d), (b', c', e'); \;\; (c, a, d), (c', a', e'); \;\; (a, b, d), (a', b', e')$$

we similarly show that U_1, V_1, W_1, lie respectively on the planes (A, A', U'), (B, B', V'), (C, C', W'), and therefore coincide respectively with U, V, W, being the intersections of these planes respectively with the lines a', b', c'. Thus $f_1 = f$ is a common transversal of the lines a', b', c', d', e'; as was to be shown.

§ 2. Now take four arbitrary lines a, b, c, d in four dimensions, of which no two intersect. Two of these lines, determined by four points, two on each, determine a threefold space, defined by the four points, and this meets a third line in the four dimensional space in a point. From this point, in the threefold space, can be drawn an unique transversal to the two lines spoken of. Thus three lines in four dimensions, of which no two intersect, have an unique transversal. Let then a' be the transversal of b, c, d, and similarly b', c', d' the transversals respectively of c, a, d; a, b, d and a, b, c. Denote the points (b', c), (b, c'), (c', a), (c, a'), (a', b), (a, b') respectively by A, B, C, A', B', C' and the points (a, d'), (b, d'), (c, d'), (a', d), (b', d), (c', d) respectively by P, Q, R, P', Q', R'.

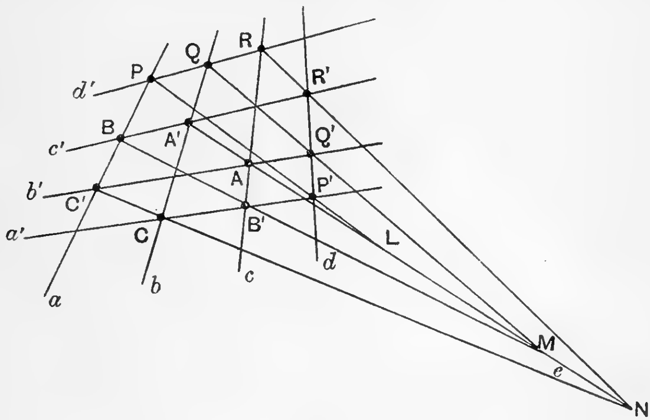


Fig. 2

In general use the word *plane* for the planar twofold space which is determined by three points, and the word *space*, or *threefold* for the planar threefold space determined by four points; as above remarked two lines determine a space, each line being determined by two points; reciprocally two spaces, in the most general case, intersect in a plane, there being a duality of properties in four dimensions wherein a space is reciprocal to a point and a plane to a line. The points A, A', being respectively on the lines $C'Q'$, BR', are in the space (a, d), and evidently are in the space (b, c); the points P, P', being on the lines QR, $B'C$ respectively, are in the space (b, c), and are evidently in the space (a, d). Thus the four points A, A', P, P' lie in a plane, which we may denote by α, namely that common to the two spaces (b, c) and (a, d). We see how much more naturally this arises than the statement, to which it is evidently analogous, in the three dimensional figure considered in § 1. It follows that the lines AA' and PP' intersect one another, say in L. Similarly the plane, β, of intersection of the spaces (c, a), (b, d), contains the lines BB' and QQ', which then intersect, say in M; and the plane, γ, of intersection of the spaces (a, b), (c, d), contains the lines CC' and RR', intersecting, say, in N. The points L, M, N are however all in each of the spaces (a, a'), (b, b'), (c, c'), and so in a line, the intersection of these spaces. For instance the line AA' joins a point (A') of the line b, to a point (A) of the line b', and so is in (b, b'); and joins a point (A) of the line c, to a point (A') of the line c', and so is in (c, c'); thus L, on the line AA', is in the spaces (b, b'), (c, c'). But the line PP' joins a point (P) of the line a to a point (P') of the line a'; thus L is equally in the space (a, a'). Similarly both M and N are in the line of intersection of the spaces (a, a'), (b, b') and (c, c'). Thus the space (d, d') passes through the line of intersection of the spaces (a, a'), (b, b'), (c, c'); for we similarly show that each of L, M, N is in the space (d, d'). We denote this line by e; evidently its relation to the lines a', b', c', d' is exactly similar with its relation to the lines a, b, c, d; the plane, α, for example, defined as that common to the spaces (b, c), (a, d), is equally the plane common to the spaces (b', c'), (a', d'); and so on. It is usual to speak of e as the line *associated with* a, b, c, d; examination of the figure of fifteen lines and fifteen points which we have constructed will show that there is entire symmetry of mutual relation, and that we may speak equally well of any one of the five lines a, b, c, d, e as being associated with the other four; further e is also associated with a', b', c', d'; and indeed, taking any line of the figure, the eight lines of the figure which do not intersect it, consist of a set of four skew lines and their transversals, and the line in question is associated with either of these two sets of four. There are then $15 \cdot 2 \div 5 = 6$ ways of regarding the figure as depending upon a set of five associated lines.

§ 3. Consider now what planes exist meeting the lines a, b, c, d. In four dimensions an arbitrary plane does not meet an arbitrary line; two such elements which meet lie in a threefold space. It can be shown that a plane meeting a, b, c, d can be drawn through two arbitrary points, one on each of any two of these four lines, so that there are ∞^2 such planes. Further that every such plane also meets the associated line e. Further that two planes meeting a, b, c, d can be drawn through an arbitrary point of the four dimensional space, and, for instance, an infinity of such planes can be drawn through any point of the line e. Also, if the two planes through an arbitrary point O, to meet a, b, c, d, meet the line e in T and U, then the two planes which can similarly be drawn through O to meet the lines a', b', c', d', meet the line e in the same two points T and U. In general two planes in four dimensions have only one point in common; when they have two points in common, the join of these points lies in both the planes which then both lie in the same threefold space. By what we have said there is a plane through OT intersecting a, b, c, d and also a plane through OT intersecting a', b', c', d', with a similar statement for planes through OU. Namely considering the two planes through O which meet a, b, c, d and also the two planes through O which meet a', b', c', d' either one of the former meets one of the latter in a line.

To prove these statements we may proceed as follows. The joining line of two points arbitrarily taken respectively, say, on the lines b and c, will meet the space (a, d) in a point, from which, in this space, a transversal can be drawn to a and d. Then the plane of the original join and this transversal is a plane, say ϖ, meeting the four lines a, b, c, d. The point of intersection of these two lines determining this plane ϖ is evidently on the plane, α, common to the spaces (b, c) and (a, d). Similarly the point of intersection of the plane ϖ with the plane, β, common to the spaces (c, a) and (b, d), is a point from which two transversals can be drawn respectively to the pairs of lines c, a and b, d; and the plane of these transversals is a plane through this point meeting the four lines a, b, c, d; conversely the join of the two points where the plane ϖ meets the lines c and a lies in the space (c, a), and so intersects the plane β, namely in the supposed unique point common to ϖ and β; this join is thus identical with the transversal drawn from the point (ϖ, β) to the lines c, a. There is thus an unique plane ϖ, meeting a, b, c, d, passing through any general point of the plane α, beside the plane α itself. It will follow from the general result enunciated above, to be proved below, that the plane ϖ', drawn through the same point of the plane α to meet the lines a', b', c', d', meets ϖ in a line intersecting the line e.

Take now any general point O, and a varying point P of the line d; a plane can be drawn through OP to meet the lines a, b, this being the plane containing OP and the common transversal of OP, a and b. Let this plane meet a, b respectively in P_1 and P_2. Thereby any position of P, on the line d, determines the position of P_1 on the line a. Conversely given O and P_1, a plane can be drawn through OP_1 to meet b and d, which, being unique, coincides with the former. Thus any position of P_1 on a determines the position of P on d. The correspondence being algebraic, it follows that P_1, P describe homographic ranges respectively on a and d. Using the line c instead of b, we obtain another range (P') on d, also homographic with (P_1). Thence the ranges (P'), (P), on d, are homographic; and, if not coincident, they will have two common points, which may coalesce. When P has a position in which it coincides with P', there is a single plane containing O, P_1, P_2, P_3, P, where P_3 is the point of c on the plane OP, P'. Thus through the point O can be drawn, either an infinity of planes meeting all of a, b, c, d, or else *two*, which may however coincide.

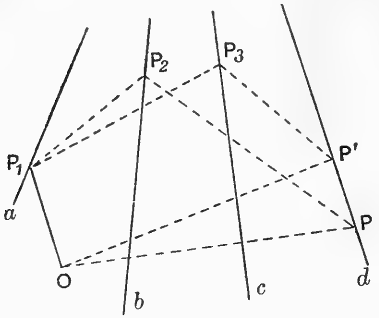


Fig. 3

When O is on the line e, the plane Od' meets a, b, c, and it also meets d because, as we have shown, e, d, d' are in a three dimensioned space. Equally the planes Oa', Ob', Oc' meet a, b, c, d. As there are thus more than two planes through O meeting a, b, c, d, it follows, by what we have shown, that there is an infinity; this is when O is anywhere on the line e. The aggregate of planes so obtained, by taking O to be every point of e, is identical with the aggregate of all planes meeting a, b, c, d, namely any plane meeting a, b, c, d can be identified with one of these; for taking O on e, and P on d, this P determines P_1, P_2, respectively on a, b, when regarded as belonging to one of the coincident ranges on e, and determines P_1, P_3, respectively on a, c, when regarded as belonging to the other range on e. Thus *every plane meeting a, b, c, d also meets e*, or more generally *five associated lines are such that every plane meeting four of them also meets the fifth**.

In general, as we have seen, from any point on a plane meeting a, b, c, d (and e), there can be drawn another such plane. If the point be on the conic through the five points in which the first

* The reader may compare the proofs of this result given by Segre, *Circolo Mat., Palermo*, II, 1888, 45, Alcune considerazioni....The elementary theorems here given for the geometry of four dimensions are of course well known; but I have thought that it was necessary for the purpose of this Note to supply demonstrations. The reader may consult Bertini, *Introduzione alla geometria projettiva degli iperspazi*, Pisa, 1907, a volume of 400 pages, p. 177. In English there is Mr Richmond's paper On the figure of six points in four dimensions, *Quart. Journ.*, XXXI, 1899; *Math. Annal.*, LIII, 1900 (see also *Trans. Camb. Phil. Soc.*, XV, 1894, 267), which deals with a diagram intimately related with that of the text, and Coolidge, *A treatise on the Circle and Sphere*, Oxford, 1916, p. 482, etc., where the lines of four dimensions are replaced by spheres. The origin of the five associated lines seems to be a result given by Stephanos, *Compt. rendus*, XCIII, 1881, p. 578. I have not seen it formally remarked that the property of the double six follows from the geometry of four dimensions; indeed the argument given in §1 was invented in ignorance of this. The fifteen points and lines of our figure (Fig. 2) are the *diagonal* points and *transversal* lines of the figure considered by Mr Richmond. See also Hudson, *Kummer's Quartic Surface* (1905), Chap. xii.

plane meets a, b, c, d, e, the second plane coincides with the first. It is not necessary for our purpose to prove this.

§ 4. The theorem that two planes can be drawn from an arbitrary point O to meet the lines a, b, c, d is obvious from the theorem in three dimensions that four skew lines have two transversals, the proof of which also depends on the fact that two homographic ranges on a line have two common points. For, if we project a, b, c, d from O, on to an arbitrary threefold space Σ, the planes joining O to the two transversals of the four lines of Σ so obtained, all meet a, b, c, d. And, we now see, e projects into a fifth line meeting these two transversals. When O is on e, the projections in Σ of a, b, c, d are all met by the projections in Σ of a', b', c', d'; for the plane Od', for example, meets a, b, c, and meets d because e, d, d' are in the same three dimensional space; thus the projections in Σ of a, b, c, d are four generators of the same system of a quadric surface of which the projections of a', b', c', d' are generators of the other system. The planes from O each meeting a, b, c, d intersect the space Σ in lines all meeting the projections of a, b, c, d; that is, in lines which are generators of this quadric of the same system as a', b', c', d'. The planes from O meeting a', b', c', d' similarly give rise to generators of the system (a, b, c, d). Thus any plane from the point O meeting a, b, c, d meets any plane from O drawn to meet a', b', c', d' in a line through O; and every line drawn from O in a plane of the former system is the intersection with this plane of a plane of the second system. If O be a point of e lying on a plane drawn from a point H, not on e, to meet a, b, c, d (which therefore also meets e), the line HO lies in a definite plane meeting a', b', c', d'. Thus *either of the two planes of the first system, those meeting a, b, c, d, drawn from a point H, not on e, meets one of the two planes of the second system, those meeting a', b', c', d', drawn from H, in a line; and the two lines so arising intersect the line e.*

§ 5. Hence we can obtain from the four dimensional figure a figure in three dimensions with the characteristics of that used in proving the double six theorem.

If, in the four dimensions, ρ, σ be the planes drawn from an arbitrary point O to meet a', b', c', d', and ρ', σ' those meeting a, b, c, d, and if ρ and σ' meet in a line, as also ρ' and σ; and if we consider the intersections with an arbitrary threefold space Σ, of these four planes, and also of the planes joining O to a, b, c, d, a', b', c', d', denoting these twelve lines respectively by $(\rho), \ldots, (a), \ldots$, then, arranged as follows:

$$\begin{matrix} (a) & (b) & (c) & (d) & (\rho) & (\sigma) \\ (a') & (b') & (c') & (d') & (\rho') & (\sigma'), \end{matrix}$$

these form a double six, any one of the lines meeting the five which do not lie in the same row or column with itself.

§ 6. Conversely we now proceed to show that if

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_1' & b_1' & c_1' & d_1' \end{matrix}$$

be eight lines in three dimensions such that no two of a_1, b_1, c_1, d_1 intersect, while d_1' intersects a_1, b_1, c_1, a_1' intersects b_1, c_1, d_1, etc., and if one of the two transversals, say l, of a_1', b_1', c_1', d_1', intersects one of the two transversals, say m', of a_1, b_1, c_1, d_1, then these lines may be obtained by projection from four dimensions; namely $a_1, b_1, c_1, d_1, a_1', b_1', c_1', d_1'$ are projections of four lines a, b, c, d in space of four dimensions and of the transversals a', b', c', d' of threes of these, respectively, while l and m' are the intersections with the original three dimensional space of planes in four dimensions meeting respectively the set a', b', c', d' and the set a, b, c, d.

We give an analytical proof of this. And for this purpose first explain an analytical view of the theorems which have been given in §§ 2, 3, 4, which indeed renders these very obvious.

It is fundamental that a point may be represented by a single symbol, say P, the same point being equally represented by any numerical multiple of this, say mP, where m is an ordinary number. Then a space of r dimensions is one in which every $r+2$ points, $P_1, P_2, \ldots, P_{r+2}$, are connected by a syzygy,

$$m_1P_1 + m_2P_2 + \ldots + m_{r+2}P_{r+2} = 0,$$

where $m_1, \ldots, m_{r+2}$ are ordinary numbers; thus the space is determined by any $r+1$ points of it, themselves not lying in a space of less than r dimensions; and, in terms of such $r+1$ points, say $A_1, \ldots, A_{r+1}$, every other point of the space may be represented by a symbol $x_1A_1 + x_2A_2 + \ldots + x_{r+1}A_{r+1}$, where $x_1, x_2, \ldots, x_{r+1}$ are ordinary numbers; whose ratios may be called the coordinates of this point, relatively to $A_1, \ldots, A_{r+1}$. Thus any point of a line determined by two points A, B, is representable by a symbol $mA + nB$, in which m, n are numbers; and any point of a plane determined by three points A, B, C, is representable by a symbol $xA + yB + zC$.

In our figure in four dimensions (§ 2), let the points

$$A = (b', c),\quad B = (c', a),\quad C = (a', b),$$
$$A' = (b, c'),\quad B' = (c, a'),\quad C' = (a, b'),$$

be regarded as fundamental. Being in four dimensions, they are connected by a syzygy; absorbing proper numerical multipliers in the symbols, this may be taken to be

$$A + B + C + A' + B' + C' = 0.$$

It is not then allowable to modify further these symbols by multiplication with numbers, except the same multiplier for all. And, the six points being supposed not to lie in a three dimensional space, there is no further syzygy connecting them.

Each of the points P, Q, R of our diagram (§ 2) is then expressible linearly by two of these six points, P by B and C', etc., while P, Q, R, being collinear, are themselves connected by a syzygy. This becomes then a syzygy for A, B, C, A', B', C', which must be the same as the fundamental syzygy. Whence it appears at once that, by absorbing proper numbers in the symbols P, Q, R, we can write

$$P = B + C', \quad Q = C + A', \quad R = A + B'.$$

By a similar argument we deduce for the points P', Q', R' of our diagram (§ 2)

$$P' = B' + C, \quad Q' = C' + A, \quad R' = A' + B.$$

From these two sets of equations we have however

$$P + P' = -(A + A');$$

there is then a point lying both on the line PP' and on the line AA', namely these lines intersect, say in the point L. So also M, N are the points $B + B'$, $C + C'$ respectively. And the identity

$$A + A' + B + B' + C + C' = 0$$

shows that L, M, N lie on a line, e.

A plane meeting the lines a, b, c can evidently be defined by three points, one on each of them, represented by symbols of the form

$$yB + zC', \quad zC + xA', \quad x_1A + y_1B'.$$

In order that the plane should also meet the line d, it is necessary and sufficient that numbers λ, μ, ν, ρ, σ exist for which there is the syzygy

$$\lambda(yB + zC') + \mu(zC + xA') + \nu(x_1A + y_1B') + \rho P' + \sigma Q' = 0;$$

if herein P', Q' be replaced respectively by $B' + C$, $C' + A$, it must reduce to the fundamental syzygy. Thus we find at once that $x_1 = x$ and $y_1 = y$, and any one of the ∞^2 planes meeting a, b, c, d is the join of points respectively on the lines a, b, c, given by

$$yB + zC', \quad zC + xA', \quad xA + yB'.$$

This plane however contains the point represented by

$$x(yB + zC') + y(zC + xA') + z(xA + yB'),$$

which is

$$yzP' + zxQ' + xyR',$$

and is therefore its intersection with the line d. And the plane contains the point represented by

$$(yB + zC') + (zC + xA') + (xA + yB'),$$

which is $$x(A+A')+y(B+B')+z(C+C')$$
and is therefore its intersection with the line e.

Similarly it can be shown that a general plane meeting the lines a', b', c', d' and e is that joining the points, respectively on a', b', c', represented by
$$yB'+zC,\quad zC'+xA,\quad xA'+yB.$$

If the plane, meeting a, b, c, d, joining the points
$$yB+zC',\quad zC+xA',\quad xA+yB',$$
pass through an arbitrary point of the four dimensional space expressed by
$$\xi A+\eta B+\zeta C+\xi' A'+\eta' B'+\zeta' C',$$
it can be shown without difficulty that
$$\frac{\xi'-\xi}{x}+\frac{\eta'-\eta}{y}+\frac{\zeta'-\zeta}{z}=0,$$
$$(\eta'-\zeta)x+(\zeta'-\xi)y+(\xi'-\eta)z=0,$$
and also
$$\frac{(\eta-\zeta')(\eta'-\zeta)}{y-z}+\frac{(\zeta-\xi')(\zeta'-\xi)}{z-x}+\frac{(\xi-\eta')(\xi'-\eta)}{x-y}=0;$$
to satisfy the condition it is necessary and sufficient, in fact, that numbers λ, μ, ν, ρ should be possible such that
$$\xi=\nu x+\rho,\qquad \eta=\lambda y+\rho,\qquad \zeta=\mu z+\rho,$$
$$\xi'=\mu x+\rho,\qquad \eta'=\nu y+\rho,\qquad \zeta'=\lambda z+\rho.$$

The first two of these equations determine the two planes meeting the lines a, b, c, d, e which pass through the arbitrary point in question. The last equation determines the two points, of the form
$$x(A+A')+y(B+B')+z(C+C'),$$
or $$(x-z)(A+A')+(y-z)(B+B'),$$
where these planes meet the line e. As this last equation is unaltered by interchanging ξ, η, ζ respectively with ξ', η', ζ', it shows that the two planes drawn through the arbitrary point to meet a, b, c, d intersect the line e in the same two points as do the planes drawn through this point to meet a', b', c', d'.

§ 7. Now consider a figure in three dimensions consisting of four lines a, b, c, d, skew to one another, and four other lines, a', b', c', d', skew to one another, each meeting three of the former. As before, with the same notation, the figure consists of a skew hexagon $BC'AB'CA'$, of which one set of three alternate sides is intersected by a line respectively in P, Q, R, and the other set of alternate sides is intersected by another line in P', Q', R' respectively. But here the six points A, B, C, A', B', C', being in three dimensions, are subject to two syzygies, which, without loss of generality, may be written

$$A + B + C + A' + B' + C' = 0,$$
$$aA + bB + cC + a'A' + b'B' + c'C' = 0.$$

The points P, Q, R, respectively in syzygy with (B, C'), with (C, A'), and with (A, B'), are themselves in syzygy. Thus, not only is P given, save for a multiplier, by an expression $\mu B + \nu' C'$, and Q by $\nu C + \lambda' A'$, and R by $\lambda A + \mu' B'$, but the multipliers μ, ν', ν, λ', λ, μ' may be chosen so that we have

$$\mu B + \nu' C' + \nu C + \lambda' A' + \lambda A + \mu' B' = 0.$$

This must then be a consequence of the two fundamental syzygies; or, for a proper value of ρ, we must have

$$\lambda : \mu : \nu : \lambda' : \mu' : \nu' = a + \rho : b + \rho : c + \rho : a' + \rho : b' + \rho : c' + \rho.$$

Thence P, Q, R may be expressed by

$$\begin{aligned} P &= bB + c'C' + \rho\,(B + C'), \\ Q &= cC + a'A' + \rho\,(C + A'), \\ R &= aA + b'B' + \rho\,(A + B'). \end{aligned}$$

Similarly P', Q', R' are expressible by symbols

$$\begin{aligned} P' &= b'B' + cC + \rho'\,(B' + C), \\ Q' &= c'C' + aA + \rho'\,(C' + A), \\ R' &= a'A' + bB + \rho'\,(A' + B). \end{aligned}$$

Suppose now that one of the two transversals, say u', of the four lines a, b, c, d, intersects one of the two transversals, say u, of the four lines a', b', c', d'. Then as before (§ 1), remarking that the two quadric surfaces determined respectively by the triads of lines (b, c, u), (b', c', u') contain respectively also the lines (a', d', u'), (a, d, u), we infer that the points A, A', P, P' are coplanar. And by similar reasoning, also because u, u' intersect one another, we infer that B, B', Q, Q' are coplanar, and also that C, C', R, R' are coplanar. Conversely let us assume only the first fact, that A, A', P, P' are coplanar. We can then prove that the numbers ρ, ρ', occurring respectively in the expressions of P, Q, R and of P', Q', R', are equal. For the fact that A, A', P, P' are coplanar involves the existence of a syzygy of the form

$$m\,[bB + c'C' + \rho\,(B + C')] + m'\,[b'B' + cC + \rho'\,(B' + C)] + pA + p'A' = 0;$$

for suitable values of m, m', p, p' and of another number k, this must be the same as

$$(a + k)\,A + (a' + k)\,A' + (b + k)\,B + (b' + k)\,B' + (c + k)\,C + (c' + k)\,C' = 0;$$

we may then take

$$p = a + k,\ p' = a' + k,\ m(b+\rho) = b + k,\ m'(b'+\rho') = b' + k,$$
$$m'(c+\rho') = c + k,\ m(c'+\rho) = c' + k;$$

these lead to $m(b - c') = b - c'$, $m'(b' - c) = b' - c$, and hence $m = m' = 1$, unless the lines d, d' have particular positions; whence $\rho = \rho' = k$.

Thus we have

$$P = bB + c'C' + \rho(B + C'),\quad P' = b'B' + cC + \rho(B' + C),\ \text{etc.,}$$

which give, in particular,

$$\begin{aligned} Q + Q' &= aA + a'A' + cC + c'C' + \rho(C + C' + A + A') \\ &= -[bB + b'B' + \rho(B + B')], \end{aligned}$$

showing that the points B, B', Q, Q' are coplanar. And similarly the points C, C', R, R' are coplanar.

§ 8. Now take a point O, not in the original space of three dimensions which we have considered; with this original space the point O determines a four dimensional space containing both. Therein take points A_1, B_1, C_1, A_1', B_1', C_1' given by

$$A_1 = (a+\rho)A + lO,\qquad B_1 = (b+\rho)B + mO,$$
$$A_1' = (a'+\rho)A' + l'O,\quad B_1' = (b'+\rho)B' + m'O,$$
$$C_1 = (c+\rho)C + nO,$$
$$C_1' = (c'+\rho)C' + n'O,$$

wherein l, m, n, l', m', n' are arbitrary numbers save for the single relation

$$l + m + n + l' + m' + n' = 0.$$

Then we have

$$A_1 + B_1 + C_1 + A_1' + B_1' + C_1' = 0.$$

Take also P_1, Q_1, R_1, P_1', Q_1', R_1' given by

$$P_1 = P + (m + n')O,\quad Q_1 = Q + (n + l')O,\quad R_1 = R + (l + m')O,$$
$$P_1' = P' + (m' + n)O,\quad Q_1' = Q' + (n' + l)O,\quad R_1' = R' + (l' + m)O;$$

then we have

$$P_1 = B_1 + C_1',\quad Q_1 = C_1 + A_1',\quad R_1 = A_1 + B_1',$$
$$P_1' = B_1' + C_1,\quad Q_1' = C_1' + A_1,\quad R_1' = A_1' + B_1.$$

The original figure in three dimensions is thus the projection from O of the figure now formed in four dimensions, and this is exactly such a figure as that we considered originally.

Geometrically, what is arbitrary in the four dimensional space is the point O, and five of the six points taken on the lines OA, OB, OC, OA', OB', OC'. These being taken, it is no doubt possible to complete the construction without use of the symbols. These seem however to add to clearness.

On transformations with an absolute quadric. By Professor H. F. Baker.

[*Read* 9 February 1920.]

We consider homographic (linear) transformations of projective space which leave unaltered a given quadric, sometimes called the Absolute. We suppose the two generators, of either system, of the quadric, which are unchanged by the transformation, to be distinct. Denoting then by DC and AB the two diagonals of the skew quadrilateral formed by these four generators, it is known that the transformation may be represented (a) by rotations about DC and AB of suitable amplitudes, whose order is indifferent, (b) by a "half turn" about an arbitrary line meeting DC and AB, followed by a half turn about another appropriately chosen line meeting DC and AB, or preceded by such a half turn of appropriate axis, (c) by a "right vector" and a "left vector" together, whose order is indifferent. The object of the present note is to mention another mode of decomposition of the transformation. For this purpose we define inversion, in regard to a point O and a plane ϖ, as the process of passing from a point P to a point P' on the line OP such that P, P' are harmonically separated by O and ϖ. We consider only cases in which ϖ is the polar plane of O in regard to the absolute quadric. We also define harmonic inversion in regard to two given skew lines, as the process of passing from any point P to a point P' on the transversal drawn from P to the lines, such that P, P' are harmonically separated by these. When the lines are polars of one another in regard to the absolute quadric, the process is the same as a half turn about either of them, and is obtainable as the sequence of two inversions about any two points taken on either of the lines so as to be conjugate to one another in regard to the quadric.

Thus if H, K be two arbitrary points respectively on AB, DC, and H_1, K_1 be two appropriately chosen points on these lines respectively, it follows from the decomposition (b) referred to above that the complete transformation can be represented by the sequence of four inversions

$$(H_1)(K_1)(H)(K),$$

wherein, since H, K_1 are conjugate points, the process represented by $(K_1)(H)$ is the same as that represented by $(H)(K_1)$. The whole is then equivalent to

$$(H_1)(H) \,.\, (K_1)(K).$$

Conversely, it is an easy matter to prove directly that the two inversions $(H_1)(H)$, about points on AB, are together equivalent to a rotation about the line DC, of amplitude equal to twice the Cayley separation of H and H_1, in regard to the quadric, with a similar fact for $(K_1)(K)$. This gives an elementary theory of the transformation.

In Euclidian space, a general movement of a rigid body is thus obtainable by a succession of four inversions, two of these about points on the central axis, and two on the line at infinity of the plane at right angles to this. Taking the central axes to be $x = 0$, $y = 0$, the two former may be taken to be reflexions in planes $z = k$, $z = k_1$, and the two latter to be reflexions in the planes $x = 0$, $x = y \tan \frac{1}{2}\theta$. By the former we obtain

$$z + z_1 = 2k, \quad z' + z_1 = 2k_1,$$

giving $z' = z + 2(k_1 - k)$; by the latter we obtain

$$x_1 = -x, \quad y_1 = y, \quad (x' - x_1) \sin \tfrac{1}{2}\theta + (y' - y_1) \cos \tfrac{1}{2}\theta = 0,$$
$$(x' + x_1) \cos \tfrac{1}{2}\theta - (y' + y_1) \sin \tfrac{1}{2}\theta = 0,$$

giving

$$x' + x_1 \cos\theta - y_1 \sin\theta = 0, \quad y' - y_1 \cos\theta - x_1 \sin\theta = 0.$$

Thus altogether

$$x' = x\cos\theta + y\sin\theta, \quad y' = -x\sin\theta + y\cos\theta, \quad z' = z + 2(k_1 - k).$$

On a set of transformations of rectangular axes. By Professor H. F. BAKER.

[*Read* 9 February 1920.]

In a paper in the *Acta Mathematica*, xxv, 1902, 291–296, Dr Burnside has greatly simplified, by geometrical considerations, results obtained in the same Journal, xxiv, 1901, 123–158, by Lipschitz, for the relations connecting the four rotations changing one system of orthogonal axes into another. In a paper, *Proceedings Lond. Math. Soc.*, ix, 1910, 197, I have incidentally noticed a theorem which is intimately connected with these results, and may be made to include them. We may associate with a rotation in Euclidian space about an axis through the origin, of direction cosines l, m, n, through an angle θ, a point of projective space, of coordinates (a, b, c, d) given by

$$a = l \sin \tfrac{1}{2}\theta, \quad b = m \sin \tfrac{1}{2}\theta, \quad c = n \sin \tfrac{1}{2}\theta, \quad d = \cos \tfrac{1}{2}\theta;$$

and we may call this point, which determines the rotation, the *representative point* of the rotation. The theorem referred to is then the following: Let Ω, Ω' be any two congruent figures upon a sphere; let Ω_1, Ω_2, Ω_3 be the figures obtained from Ω by reflexion in, or rotations of amplitude π about, the respective coordinate axes; let Ω_1', Ω_2', Ω_3' be similarly derived from Ω'; there is then a rotation changing any one of Ω_1, Ω_2, Ω_3, Ω into any one of Ω_1', Ω_2', Ω_3', Ω'. *The representative points of these sixteen rotations form a Kummer configuration.* In other words they lie in sixes upon sixteen conics, whose planes are in sixes tangent planes of sixteen quadric cones whose vertices are the points; or again, they can be arranged in twenty ways so as to form the vertices of a set of four tetrahedra of which set every two are mutually inscribed. As will be seen, the theorem is the same as that, if a point (a, b, c, d) be represented by the quaternion

$$P = ai + bj + ck + d,$$

the sixteen points

$$\begin{array}{llll} iPi, & iPj, & iPk, & iP \\ jPi, & jPj, & jPk, & jP \\ kPi, & kPj, & kPk, & kP \\ Pi, & Pj, & Pk, & P \end{array}$$

form a Kummer system, the four tetrahedra formed each by the points in any row being mutually inscribed in pairs, as are the four tetrahedra formed each by the points in any column. This last statement is easily verifiable, and it is sufficient therefore to reduce the original statement to this. By direct algebra this can be done by using the well-known fact that the rotation of representative point (a, b, c, d) is given by

$$x' = ux + hy + g_1 z, \quad y' = h_1 x + vy + fz, \quad z' = gx + f_1 y + wz,$$

where the nine coefficients are given respectively by

$$\begin{array}{llll} u,\ h,\ g_1 = & d^2 + a^2 - b^2 - c^2, & 2(ab - cd), & 2(ca + bd) \\ h_1,\ v,\ f & 2(ab + cd), & d^2 + b^2 - c^2 - a^2, & 2(bc - ad) \\ g,\ f_1,\ w & 2(ca - bd), & 2(bc + ad), & d^2 + c^2 - a^2 - b^2 \end{array}$$

each of the elements on the right being divided by $a^2 + b^2 + c^2 + d^2$, which is unity. Using the representative points given by the above scheme to calculate the rotations, it is at once seen that they have the interpretations assigned to them in the statement of the theorem.

In particular the representative points given by the forms

$$Pi,\ Pj,\ Pk,\ P$$

are respectively

$$(d, c, -b, -a), \quad (-c, d, a, -b), \quad (b, -a, d, -c), \quad (a, b, c, d),$$

and if these belong to rotations of respective amplitudes $\theta_1, \theta_2, \theta_3, \theta$ about axes of direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), (l, m, n), we have

$$\begin{array}{llll} d = l_1 \sin \tfrac{1}{2}\theta_1, & c = m_1 \sin \tfrac{1}{2}\theta_1, & -b = n_1 \sin \tfrac{1}{2}\theta_1, & -a = \cos \tfrac{1}{2}\theta_1, \\ -c = l_2 \sin \tfrac{1}{2}\theta_2, & d = m_2 \sin \tfrac{1}{2}\theta_2, & a = n_2 \sin \tfrac{1}{2}\theta_2, & -b = \cos \tfrac{1}{2}\theta_2, \\ b = l_3 \sin \tfrac{1}{2}\theta_3, & -a = m_3 \sin \tfrac{1}{2}\theta_3, & d = n_3 \sin \tfrac{1}{2}\theta_3, & -c = \cos \tfrac{1}{2}\theta_3, \\ a = l \sin \tfrac{1}{2}\theta, & b = m \sin \tfrac{1}{2}\theta, & c = n \sin \tfrac{1}{2}\theta, & d = \cos \tfrac{1}{2}\theta, \end{array}$$

from which follow, if the axes be OD_1, OD_2, OD_3, OD,

$$-bc = \cos D_2D_3 \sin \tfrac{1}{2}\theta_2 \sin \tfrac{1}{2}\theta_3 = -\cos \tfrac{1}{2}\theta_2 \cos \tfrac{1}{2}\theta_3,$$
$$\cos D_2D_3 = -\cot \tfrac{1}{2}\theta_2 \cot \tfrac{1}{2}\theta_3,$$
$$ad = \cos DD_1 \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta_1 = -\cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta_1,$$
$$\cos DD_1 = -\cot \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta_3,$$

and hence

$$\cos DD_1 \cos D_2D_3 = \cos DD_2 \cos D_3D_1 = \cos DD_3 \cos D_1D_2,$$

in accordance with the fact that the plane containing any two of these four axes is at right angles to the plane containing the other two; as also

$$\tan^2 \tfrac{1}{2}\theta_1 = -\cos D_2D_3/\cos D_1D_2 \cos D_1D_3,$$

these being equations noticed by Dr Burnside *loc. cit.*, p. 294.

But the definition of the representative point of a rotation which has been given is, like this proof of the theorem, both metrical and analytical, and it is desirable to alter it in both respects. A point of view which is not metrical is to regard the transformation

$$x' = ux + hy + g_1z,\ \ y' = h_1x + vy + fz,\ \ z' = gx + f_1y + wz, \quad \text{(I)},$$

as that unique homographic transformation of the plane of the homogeneous variables x, y, z which leaves the conic

$$x^2 + y^2 + z^2 = 0$$

unaltered, and changes the self-polar triangle $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ into the self-polar triangle (u, h_1, g), (h, v, f_1), (g_1, f, w). If the vertices A, B, C of the former be joined to the corresponding vertices A', B', C' of the latter, by lines forming a triangle D, E, F, of which E, F will be collinear with A, A', etc., and we take on the line A, A', E, F the points P, P' harmonic in regard both to A, A' and E, F; and similarly take Q, Q' on the line B, B', F, D harmonic in regard both to B, B' and F, D; and take R, R' on the line C, C', D, E harmonic in regard both to C, C' and D, E; then it can be shown that the six points P, P', Q, Q', R, R' lie in threes upon four straight lines, which are in fact

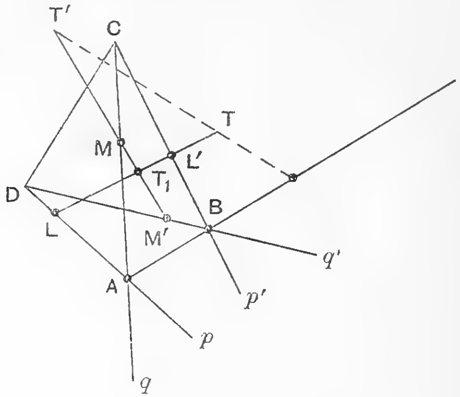
$$dx + cy - bz = 0, \quad dy + az - cx = 0,$$

$$dz + bx - ay = 0, \quad ax + by + cz = 0.$$

This gives a geometrical interpretation, which is not metrical, of a, b, c, d.

A much better point of view is however as follows. The rotation, expressed by four equations of which three are those marked (I) above, and the fourth is $t' = t$, may be regarded as a homographic transformation of projective space (x, y, z, t), leaving the quadric whose equation is $x^2 + y^2 + z^2 + t^2 = 0$ (or indeed any quadric $x^2 + y^2 + z^2 + Mt^2 = 0$) unaltered. And it may be regarded as compounded from two transformations P, Q, taken in either order, of which P is such as to leave every generator of the quadric of one system, say the q-system, unaltered, while it interchanges the generators of the other system, say the p-system, among themselves; and Q has a similar meaning with the systems of generators interchanged. In the transformation P there will be two particular generators of the p-system, say p and p', which remain unaltered; and the transformation may be described geometrically as changing any point T of space, not necessarily on the quadric, into a point T' of the transversal drawn from T to p, p', such that the homography $(T'T,\ pp')$ is constant, equal to $e^{i\theta}$ say, the transformation Q having a similar meaning in regard to two generators q, q' of the q-system and *having the same value for the corresponding homo-*

graphy; so that P, Q may be described as *equal* transformations of the p-kind and q-kind respectively (or if q' be interchanged with q as *equal* but *opposite* transformations).



If two lines p, q meet in A, two others p', q' meet in B, while p, q' meet in D and p', q meet in C, and on the transversal TLL', of p, p', be taken T_1 so that $(T_1T, LL') = \lambda$, and on the transversal T_1MM', of q, q', be taken T' so that $(T'T_1, MM') = \lambda$, it is at once shown that, referred to $ABCD$, the coordinates of T', (x', y', z', t'), are expressible in terms of (x, y, z, t), the coordinates of T, by the formulae

$$x' = x, \quad y' = \lambda^2 y, \quad z' = \lambda z, \quad t' = \lambda t.$$

These are the equations of a rotation round DC, of which every point remains unaltered, as does every plane through AB. If the planes joining DC to T', T meet AB in U', U we have

$$(U'U, AB) = \lambda^2.$$

The two processes may be taken in the reverse order. Any quadric containing the lines p, q, p', q', whose equation is $xy = Mzt$, is unaltered by the composite transformation. The transformation here taken first is $x_1 = x$, $y_1 = \lambda y$, $z_1 = \lambda z$, $t_1 = t$, which changes the p, q-generators, expressed respectively by $z = px$, $y = Mpt$ and $y = qz$, $qx = Mt$, into the generators p_1, q_1 given by $p_1 = \lambda p$, $q_1 = q$. The transformation here taken second is

$$x' = x_1, \quad y' = \lambda y_1, \quad z' = z_1, \quad t' = \lambda t_1,$$

which changes p_1, q_1 into p', q' given by $p' = p_1$, $q' = \lambda q_1$.

Conversely any homographic transformation of space which leaves every point of a line DC unchanged, and leaves also two points A, B, not on DC, both unchanged, will leave the lines joining A, B to an arbitrary point D of DC both unchanged, and also the lines joining A, B to a further arbitrary point C of this line; referred to A, B, C, D, such a transformation will be expressible by equations $x' = m\lambda^{-1}x$, $y' = m\lambda y$, $z' = z$, $t' = t$. If it be further restricted to be such as to leave any (and therefore every) quadric $xy = Mzt$ through AD, BD, AC, BC, unchanged, we may take $m = 1$. Then any conic through A, B touching the planes ADC, BDC is also unchanged, and the equations of transformation $x' = \lambda^{-1}x$, $y' = \lambda y$, $z' = z$, $t' = t$ are those just considered. By the component transformation $x_1 = x$, $y_1 = \lambda y$, $z_1 = \lambda z$, $t_1 = t$, any point O of DC is changed to a point O_1 of DC given by

$$(O_1O,\ DC) = \lambda.$$

In more general terms, if we write, using matrices,

$$P = \left|\begin{matrix} d, & -c, & b, & a \\ c, & d, & -a, & b \\ -b, & a, & d, & c \\ -a, & -b, & -c, & d \end{matrix}\right|, \quad Q = \left|\begin{matrix} d, & -c, & b, & -a \\ c, & d, & -a, & -b \\ -b, & a, & d, & -c \\ a, & b, & c, & d \end{matrix}\right|,$$

denoting by P', Q' what P, Q become by substituting a', b', c', d' for a, b, c, d, we find $PQ' = Q'P$, in particular $PQ = QP$, and

$$PQ = \left|\begin{matrix} u, & h, & g_1, & 0 \\ h_1, & v, & f, & 0 \\ g, & f_1, & w, & 0 \\ 0, & 0, & 0, & 1 \end{matrix}\right|.$$

It is then easy to verify that the transformation

$$(x', y', z', t') = P\,(x, y, z, t)$$

changes the p, q-generators of $x^2 + y^2 + z^2 + t^2 = 0$ expressed by

$$x + iy = p\,(z + it), \qquad x - iy = -\,p^{-1}\,(z - it);$$

$$x + iy = q\,(z - it), \qquad x - iy = -\,q^{-1}\,(z + it)$$

into p', q' given by

$$p' = \frac{-\,p\,(c - id) + a + ib}{p\,(a - ib) + c + id},\ q' = q;$$

this, with $a = l \sin\tfrac{1}{2}\theta$, $b = m \sin\tfrac{1}{2}\theta$, $c = n \sin\tfrac{1}{2}\theta$, $d = \cos\tfrac{1}{2}\theta$, is equivalent with

$$\frac{p' - \dfrac{l + im}{1 + n}}{p' + \dfrac{l + im}{1 - n}} = e^{i\theta}\,\frac{p - \dfrac{l + im}{1 + n}}{p + \dfrac{l + im}{1 - n}},$$

the stationary values of p being those of the generators passing through the points where the conic $t = 0$, $x^2 + y^2 + z^2 = 0$ is met by the line $ax + by + cz = 0$. The transformation

$$(x', y', z', t') = Q(x, y, z, t)$$

makes precisely the same changes in the parameters q, p respectively.

Clearly the *representative point* (a, b, c, d) *is the point to which the vertex* $(0, 0, 0, 1)$ *is changed by the transformation* P. This definition is relative to the position chosen for this vertex upon the axis of the original transformation, but this is immaterial in discussing the composition of rotations about different axes passing through the same vertex.

In particular the representative points for rotations of amplitude π respectively about the lines joining (0001) to (1000), to (0100) and to (0010), are (1000), (0100), (0010), the corresponding transformations P being

$$i = \begin{vmatrix} 0, & 0, & 0, & 1 \\ 0, & 0, & -1, & 0 \\ 0, & 1, & 0, & 0 \\ -1, & 0, & 0, & 0 \end{vmatrix}, \quad j = \begin{vmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \\ -1, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0 \end{vmatrix}, \quad k = \begin{vmatrix} 0, & -1, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & -1, & 0 \end{vmatrix},$$

by which the point (x, y, z, t) becomes changed respectively to

$$(x' = t, y' = -z, z' = y, t' = -x), \quad (x' = z, y' = t, z' = -x, t' = -y),$$
$$(x' = -y, y' = x, z' = t, t' = -z).$$

We at once find

$$i^2 = j^2 = k^2 = -\omega, \; jk = -kj = i, \; ki = -ik = j, \; ij = -ji = k,$$

and the matrix of the general transformation P can be written in terms of the matrices i, j, k in the form

$$P = ai + bj + ck + d\omega,$$

where ω is the matrix of the identical transformation. For the generators of the p-system, the transformations i, j, k lead respectively to $p' = p^{-1}$, $p' = -p^{-1}$, $p' = -p$, that is they are harmonic inversions in the pairs of generators of the p-system passing through the intersections of the conic $x^2 + y^2 + z^2 = 0, t = 0$ respectively with the lines $x = 0$, $y = 0$, $z = 0$. By the transformation P, these three pairs of generators are changed to the generators passing through the intersection of the conic respectively with the lines

$$ux + h_1 y + gz = 0, \;\; hx + vy + f_1 z = 0 \;\text{ and }\; g_1 x + fy + wz = 0,$$

as is easy to see. For instance the generators $p = 1$, $p = -1$, are changed respectively to the generators

$$p = (u + ih_1)/(1 + g), \quad p = -(u + ih_1)/(1 - g).$$

Further a succession of any number of rotations about axes through the vertex (0, 0, 0, 1), namely a transformation representable in the form

$$P_1Q_1 \,.\, P_2Q_2 \,.\, P_3Q_3 \ldots$$

is, in virtue of the commutative property of any p-transformation with any q-transformation, capable of being regarded as

$$(P_1P_2P_3 \ldots)\,(Q_1Q_2Q_3 \ldots)$$

wherein the first factor is a p-transformation, and the second factor a q-transformation. The laws of composition of the rotations are then precisely the same as those of the associated p-transformations—and the representative point for a composite transformation can be obtained by the composition of the p-transformations.

This suffices to reduce the theorem we have stated to the theorem that the representative points of the sixteen transformations referred to are those of the transformations

$$(i, j, k, \omega)\, P\, (i, j, k, \omega),$$

where P is any p-transformation. For we can pass from any figure on a sphere to any congruent figure by an appropriate rotation PQ.

But it is at once obvious that the representative point of a composite transformation PP' may be obtained by forming the product of the symbols associated therewith

$$(ai + bj + ck + d\omega)\,(a'i + b'j + c'k + d'\omega)$$

by means of the multiplication rules for i, j, k, ω given above. In fact this product gives

$$(ad' + a'd + bc' - b'c)\, i + (bd' + b'd + ca' - c'a)\, j$$
$$+ (cd' + c'd + ab' - a'b)\, k + (dd' - aa' - bb' - cc')\, \omega,$$

or say $$Ai + Bj + Ck + D\omega,$$

while the product of the two matrices P, P' is at once verified to be the same function of A, B, C, D as is P of a, b, c, d.

The theorem stated is thus proved.

Remark. Any rotation is thus associated with a quaternion, whose vector coefficients give the direction of the axis of the rotation, the amplitude of this being *twice* the angle whose cotangent is $d\,(a^2 + b^2 + c^2)^{-\frac{1}{2}}$. In particular the symbol i is associated with a rotation about the axis of x of amplitude π.

If two rotations of amplitudes θ, θ' about axes (l, m, n), (l', m', n') be equivalent to a rotation of amplitude ϕ about an axis (λ, μ, ν), we have such equations as

$$\lambda \sin \tfrac{1}{2}\phi = l \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta' + l' \sin \tfrac{1}{2}\theta' \cos \tfrac{1}{2}\theta + (mn' - m'n) \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta',$$

$$\cos \tfrac{1}{2}\phi = \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta' - \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta'\, (ll' + mm' + nn').$$

The latter of these expresses Hamilton's law that a rotation of amount $2C$ about the vertex C of a spherical triangle ABC, followed by a rotation of amount $2B$ about the vertex B, gives a rotation about the vertex A of amount $2\pi - 2A$. The former expresses that if O be the pole of the arc BC, and P be any point of the sphere,

$$\cos PO \sin a \sin B \sin C$$
$$= \cos PA \sin A - \cos PB \sin B \cos C - \cos PC \sin C \cos B.$$

The transformation here denoted by P is that called by Clifford a *right* (or *left*) vector. It can also be represented in the form

$$P = \cos \tfrac{1}{2}\theta \,.\, \omega + \sin \tfrac{1}{2}\theta \,.\, \varpi,$$

where
$$\varpi, = \begin{vmatrix} 0, & -n, & m, & l \\ n, & 0, & -l, & m \\ -m, & l, & 0, & n \\ -l, & -m, & -n, & 0 \end{vmatrix},$$

is such that $\varpi^2 = -\omega$; this is the same as $P = e^{\frac{1}{2}\varpi\theta}$.

On the generation of sets of four tetrahedra of which any two are mutually inscribed. By C. V. HANUMANTA and Professor H. F. BAKER.

[*Read* 8 March 1920.]

If from a point P the transversal be drawn to two given skew lines, the point P' of this transversal harmonically separated from P by these lines may be said to be obtained from P by harmonic inversion in regard to them. If O and ϖ be a given point and a given plane, and on the line joining O to an arbitrary point P there be taken the point P' harmonically separated from P by the point O and the plane ϖ, we may speak of P' as obtained from P by harmonic inversion in regard to O and ϖ; in particular when ϖ is the polar plane of O in regard to a given quadric it may be sufficient to speak of P' as the inverse of P in regard to O, this use of the term including the ordinary use when inversion in regard to a circle is spoken of.

When two tetrahedra $ABCD$, $A_1B_1C_1D_1$ are such that the points A, B, C, D, A_1, B_1, C_1, D_1 lie respectively on the planes $B_1C_1D_1$, $C_1A_1D_1$, $A_1B_1D_1$, $A_1B_1C_1$, BCD, CAD, ABD, ABC, they will be said to be mutually inscribed. When this is so the two transversals of the four lines AA_1, BB_1, CC_1, DD_1 are generators of a certain quadric in regard to which both the tetrahedra are self-polar (the two transversals of the four lines BC, AD, B_1C_1, A_1D_1 being generators of this quadric of the other system). If another tetrahedron $A_2B_2C_2D_2$, self-polar in regard to the same quadric, be in- and circumscribed to $ABCD$, and such that the two transversals of AA_2, BB_2, CC_2, DD_2 belong to the same system of generators of this quadric as do the transversals of $AA_1, \ldots, DD_1$, then $ABCD$, $A_2B_2C_2D_2$ may be said to be mutually inscribed *in the same sense* as are $ABCD$, $A_1B_1C_1D_1$. It is well known that there exist systems of four tetrahedra of which every two are mutually inscribed in the same sense.

The object of the present note is to point out that *any such four tetrahedra may be regarded as all derived from a single other tetrahedron, by inversion of this in the vertices of a certain further tetrahedron, taken in turn*, each of these vertices being associated with the opposite face of this tetrahedron in this process of inversion.

More precisely we may state this result thus. Let a tetrahedron $XYZT$ be self-polar in regard to a certain quadric. Denote the two systems of generators of this quadric as the p-system and the q-system. There exist linear transformations of space changing any point of the quadric into another point lying on the same q-generator; such a transformation will interchange the p-generators among themselves, save for two p-generators, each of which will be unchanged. We may call such a transformation a p-transformation. Let $ABCD$ be the tetrahedron arising from $XYZT$ by any such transformation; let $PQRS$ be the tetrahedron arising from $ABCD$ by inversion from the point T. Thus $PQRS$ may also be described as arising from $XYZT$ by the q-transformation which effects the same transformation, of the points of the section of the quadric by the plane XYZ, as does the transformation by which $ABCD$ is derived from $XYZT$; in other words if the parameters for the p, q-generators be chosen so as to be the same for the two generators which intersect at a point of the section by the plane XYZ, then $PQRS$ is derived from $XYZT$ by the q-transformation expressed by the same equation, connecting the parameters of the generators, as is the p-transformation by which $ABCD$ is derived from $XYZT$.

Then take the inverses of $PQRS$ respectively from X, Y, Z, T, and let them be $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$, $ABCD$. The tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $ABCD$ are then such that every two are mutually inscribed. And conversely given any such four tetrahedra, the transformation by which $XYZT$, $PQRS$ are determined is definite. The tetrahedron $XYZT$ is unique, being defined by the fact that the edges YZ, XT are the diagonals of the skew quadrilateral formed by (1) the two q-generators meeting the edges BC, AD, (2) the two p-generators which are the common transversals of AA_1, BB_1, CC_1, DD_1, while the other two pairs of edges are similarly obtainable. The tetrahedron $PQRS$ has however been obtained from $XYZT$ and $ABCD$ in an unsymmetrical way, and there are three other possibilities. Let $P_1Q_1R_1S_1$ be the tetrahedron obtained from $ABCD$ by inversion from X. Then $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$, $ABCD$ are obtainable from $P_1Q_1R_1S_1$ by inversion respectively from $TZYX$. So if $P_2Q_2R_2S_2$ be obtained from $ABCD$ by inversion from Y, the same four tetrads of points are obtained by inversion of $P_2Q_2R_2S_2$ respectively from $ZTXY$; and if $P_3Q_3R_3S_3$ be obtained from $ABCD$ by inversion from Z, the same four tetrads are obtained by inversion of $P_3Q_3R_3S_3$ respectively from $YXTZ$. But $P_1Q_1R_1S_1$, $P_2Q_2R_2S_2$, $P_3Q_3R_3S_3$ are obtainable from $PQRS$ by harmonic inversion in two lines in each case, respectively YZ, TX; ZX, TY and XY, TZ.

Taking then the first case, where $PQRS$ is used, since inversions in the vertices T, X, in succession, in either order, are together equivalent to harmonic inversion in the opposite edges YZ, TX, we may also say that *the tetrads* $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$ *are obtainable from* $ABCD$ *by harmonic inversion respectively in the pairs of edges* YZ, TX; ZX, TY; XY, TZ. And we have remarked above that given the four tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $ABCD$, there is a simple construction for $XYZT$.

The sets $PQRS$, $S_1R_1Q_1P_1$, $R_2S_2P_2Q_2$, $Q_3P_3S_3R_3$ are related to one another precisely as are $ABCD$, $D_1C_1B_1A_1$, $C_2D_2A_2B_2$, $B_3A_3D_3C_3$, and, with the substitution of one set of generators for the other, have the same relation to $XYZT$.

These results were obtained geometrically, and appear to constitute a simple geometrical construction for such a set of four mutually inscribed tetrahedra. But they can be readily verified analytically from the formulae by which the configuration is most usually treated.

On the reduction of homography to movement in three dimensions.
By Professor H. F. BAKER.

[*Read* 9 February 1920.]

It is known, and was assumed by Poncelet, that a self-homography of a Euclidian metrical plane is reducible, by a "movement" of one of the figures, to a perspectivity, called by Poncelet a homology, that is a transformation in which the join of two corresponding points passes through a fixed point, while corresponding lines meet on a fixed line. It is also known that this is not true for Euclidian metrical space of three dimensions. See H. J. S. Smith, *Proc. Lond. Math. Soc.*, II, 1866–1869, pp. 196–248; Chasles, *Géom. Sup.*, 1880, pp. 375–381; Salmon, *Higher Plane Curves*, 1879, p. 298.

The question arises what is the nearest analogous proposition for three dimensions. In the projective plane the reduction can also be made (instead of by a movement), by means of a transformation keeping a given arbitrary conic unaltered. In the present note we prove that in projective space of three dimensions a general homography is reducible, by means of a transformation leaving an arbitrary quadric unaltered, applied to one of the figures, to a transformation which we may call an axis-range perspectivity; namely one in which every point of a certain range is unaltered and every plane passing through a certain axis is unaltered. This seems to add something to what is known, and to be a very natural generalisation of the results in a plane.

§ 1. Let S, S_0 be two quadrics with a common self-polar tetrahedron; let the two systems of generators of S_0 be called the axes and transversals, respectively. There are four axes of S_0 which touch the curve of intersection of S and S_0, and are generators of the developable surface of common tangent planes of S and S_0; there are also four transversals of S_0 with the same property. The eight points of contact of these lines with the curve (S, S_0) are on the curve upon S where it is touched by the common tangent planes of S and S_0; they are then the common points of S, S_0, S_0', where S_0' is the polar reciprocal of S_0 in regard to S.

§ 2. We now consider a certain porismatic relation between the two quadrics. From an axis of S_0, which we distinguish by the parameter θ, two tangent planes can be drawn to S, each of which will contain a transversal of S_0, say these are distinguished by the parameters p, q. There will thus be a rational equation connecting θ and p, of the second order in p, equally satisfied by θ and q. Through the transversal p can be drawn, beside the plane (θ, p), another tangent plane to S, which, as it contains p, will contain an axis of S_0, say ϕ_1; thus the previous equation connecting (θ, p) is of the second order in θ, and is equally satisfied by ϕ_1 and p. Similarly through q can be drawn, beside (θ, q), another tangent plane to S, which, containing q, will contain an axis of S_0, say ϕ_2, and the relation connecting (θ, q) is equally satisfied by ϕ_2 and q. There is thus a single relation of the second order in each of θ, p which is equally satisfied by (θ, q), (ϕ_1, p), (ϕ_2, q). As in other cases it may happen that $\phi_1 = \phi_2$ for all values of θ, provided a certain relation hold between the quadrics.

To find a sufficient condition for this, we may consider the particular case when the axis θ coincides with the identical axis $\phi_1 = \phi_2$, or the transversal p touches S; then, this line p, being a generator of S_0, is a chord of the curve (S, S_0), and therefore, in the particular case considered, is a tangent line of the curve. Thus a sufficient condition is that the tangent planes of S at two of the four points referred to above, on the curve (S, S_0), associated with the transversals of S_0, should intersect on an axis of S_0.

More generally, if S_0 be $x^2 + y^2 + z^2 + t^2 = 0$, its generators of the two systems being of the respective forms

$$\left.\begin{aligned} x + iy &= i\theta \quad (z + it) \\ x - iy &= i\theta^{-1} (z - it) \end{aligned}\right\}, \quad \left.\begin{aligned} x + iy &= ip \quad (z - it) \\ x - iy &= ip^{-1} (z + it) \end{aligned}\right\},$$

the plane containing these,

$$x + iy + \theta p (x - iy) - \theta i (z + it) - pi (z - it) = 0,$$

touches S, say $ax^2 + by^2 + cz^2 + dt^2 = 0$, provided

$$\theta^2 p^2 (a^{-1} - b^{-1}) + \theta^2 (d^{-1} - c^{-1}) + p^2 (d^{-1} - c^{-1}) \\ + 2\theta p (a^{-1} + b^{-1} - c^{-1} - d^{-1}) + a^{-1} - b^{-1} = 0.$$

Hence, for the porismatic relation in question, by considering this equation, we see that we require that the discriminantal equation $| S_0 - \lambda S | = 0$, which with our coordinates is

$$(1 - \lambda a)(1 - \lambda b)(1 - \lambda c)(1 - \lambda d) = 0,$$

should have the sum of two of its roots equal to the sum of the other two. Writing then as usual (Salmon, *Solid Geometry*, 1882, p. 173)

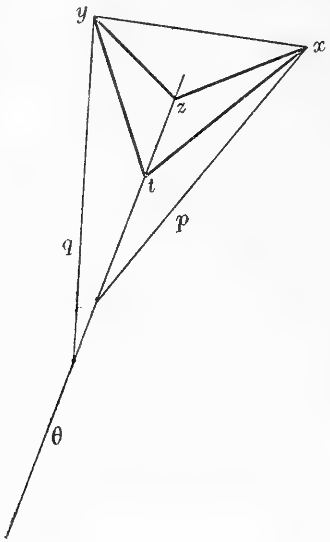
$$| S_0 - \lambda S | = \Delta_0 - \lambda \Theta_0 + \lambda^2 \Phi - \lambda^3 \Theta + \lambda^4 \Delta,$$

the condition is

$$\Theta_0{}^3 - 4\Theta_0 \Delta_0 \Phi + 8\Delta_0{}^2 \Theta = 0.$$

(Cf. Purser, *Quart. J. of Math.*, VIII, 149, and Salmon, *loc. cit.*, p. 181.)

This is then the condition for the existence of tetrahedra with faces touching S of which two pairs of opposite edges lie on S_0, or equally, for the existence of tetrahedra with vertices on S_0, of which two pairs of opposite edges lie on S. Or say, it is the condition for S to be tetrahedrally inscribed in S_0, or for S_0 to be tetrahedrally circumscribed to S.



Supposing the quadric S to be in the specified relation to S_0, there will, corresponding to each of the three ways of pairing the roots of the equation $|S_0 - \lambda S| = 0$, be two figures such as that here indicated. Here p, q are two transversals of S_0, touching S, say at the points x, y; θ is an axis of S_0 associated with p, q; and the lines xz, xt and yz, yt are the two generators of S at the points x, y. The tetrahedron of which the vertices are the points (θ, p), (θ, q), each taken doubly, of which the faces are the planes (θ, p), (θ, q), each taken doubly, of which the edges are the lines p, q and the line θ taken four times over, has its edges p, q, θ, θ as generators of S_0 while its faces touch S; the tetrahedron x, y, z, t has its vertices on S_0, and two pairs of opposite edges of it are generators of S.

Referred to (x, y, z, t) the quadric S may be taken to be

$$xy - zt = 0,$$

and the quadric S_0 to be

$$2hxy + x(gz + ut) + y(fz + vt) = 0.$$

Then the equation $|S_0 - \lambda S| = 0$ has the roots

$$\lambda_1, \lambda_2 = h \pm \{h^2 - [(fu)^{\frac{1}{2}} + (gv)^{\frac{1}{2}}]^2\}^{\frac{1}{2}},$$

$$\lambda_3, \lambda_4 = h \pm \{h^2 - [(fu)^{\frac{1}{2}} - (gv)^{\frac{1}{2}}]^2\}^{\frac{1}{2}}$$

for which $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$. The axes of S_0 are given by

$$(h - \theta)x + fz + vt = 0, \quad (h + \theta)y + gz + ut = 0,$$

and the transversals by

$$x = py, \quad hx + hpy + (f + gp)z + (v + up)t = 0;$$

the plane

$$(h - \theta)x + fz + vt + p[(h + \theta)y + gz + ut] = 0,$$

containing one of each, touches the quadric S if

$$p\theta^2 + p^2 gu + p\,(fu + gv - h^2) + fv = 0;$$

this is the form of the (2, 2) relation in this case, the two axes θ, ϕ of S_0 being θ, $-\theta$. For the axis which is the line zt we have $\theta = \infty$ and the two corresponding values of p, q are $p = 0$, $q = \infty$. It is then material to remark that the transformation

$$x_1 = \frac{h - \theta_0}{h + \theta_0}\,x, \quad y_1 = \frac{h + \theta_0}{h - \theta_0}\,y, \quad z_1 = z, \quad t_1 = t$$

changes the particular axis of S_0 determined by θ_0 into the axis determined by $-\theta_0$, and changes the plane (θ_0, p), or

$$(h - \theta_0)\,x + fz + vt + p\,[(h + \theta_0)\,y + gz + ut] = 0,$$

into the plane $(-\theta_0, p)$, for all values of p. This transformation is one which leaves the quadric S, or $xy - zt = 0$, unaltered, and leaves every point of the line zt, which is a generator of S_0, unaltered, as well as every plane through the line xy. It is thus a transformation of the kind which we have called an axis-range perspective, or, in a usual phraseology, in regard to S as the absolute quadric, it is what is called a "rotation" round the line zt.

Thus we may say: *If a quadric S be tetrahedrally inscribed in a quadric S_0, it is possible to find a self-transformation of S, which is a rotation about a generator of S_0, such as to change any axis of S_0 into its associated axis, and to change any plane through this into the plane through the associated axis containing the same transversal of S_0 as the former.*

It is easy to write down a corresponding result for the plane.

§ 3. Now consider any homography in space, say

$$x' = a_1 x + b_1 y + c_1 z + d_1 t, \;\ldots,\; t' = a_4 x + b_4 y + c_4 z + d_4 t,$$

which we denote, in matrix notation, by

$$(x', y', z', t') = \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} (x, y, z, t),$$

or also by $(x') = \vartheta\,(x)$. Take an arbitrary quadric S; this is changed by the transformation into another quadric, and is itself obtainable from another by the transformation; regarded in these two aspects let it be denoted respectively by σ and ρ', the quadric into which it is changed by the transformation being σ', and that from which it may be supposed to arise being ρ. Denote the developable surface formed by the common tangent planes of ρ and σ by (ρ, σ), and that formed by the common tangent planes of ρ' and σ'

by (ρ', σ'); the former is changed to the latter by the transformation ϑ. Let l be *the line of intersection of any two planes of the developable* (ρ, σ), and l' the line arising therefrom by the transformation, the intersection of the two planes of (ρ', σ') which correspond to the planes of (ρ, σ) taken. To the pencil of all possible planes through l will correspond a homographic pencil of corresponding planes through l'; and the locus of the line of intersection of corresponding planes of these two axial pencils, will be another quadric, say S_0. Evidently, by the construction, S is tetrahedrally inscribed in S_0, in the sense explained in § 2.

Taking then such a tetrahedron of reference as used in § 2, let the lines l, l' be respectively those there associated with the parameters θ_0 and $-\theta_0$; then the homography ϑ changes the plane (θ_0, p), of equation

$$(h - \theta_0)\, x + fz + vt + p\,[(h + \theta_0)\, y + gz + ut] = 0,$$

into the plane $(-\theta_0, p)$, whose equation is obtainable by change of the sign of θ_0, *for all values of* p. When p satisfies the equation

$$p\theta_0^2 + p^2gu + p\,(fu + gv - h^2) + fv = 0,$$

the planes (θ_0, p), $(-\theta_0, p)$ both touch S, which is σ and ρ'. Thus the plane (θ_0, p) touches the quadric ρ, and $(-\theta_0, p)$ touches σ'.

Now with the above form for the homography ϑ, the above equation for the plane (θ_0, p), which the transformation changes into $(-\theta_0, p)$, is, for all values of p, the same as

$$\begin{aligned}(h + \theta_0)\,&(a_1x + b_1y + c_1z + d_1t) + f\,(a_3x + b_3y + c_3z + d_3t)\\ &+ v\,(a_4x + b_4y + c_4z + d_4t) + p\,[(h - \theta_0)\,(a_2x + b_2y + c_2z + d_2t)\\ &+ g\,(a_3x + b_3y + c_3z + d_3t) + u\,(a_4x + b_4y + c_4z + d_4t)] = 0;\end{aligned}$$

the coefficients a_i, b_j, ... in ϑ must therefore be such that

$$\begin{aligned}&\frac{(h+\theta_0)\,a_1 + fa_3 + va_4}{h - \theta_0} = \frac{(h+\theta_0)\,b_1 + fb_3 + vb_4}{0} = \frac{(h+\theta_0)\,c_1 + fc_3 + vc_4}{f}\\ &= \frac{(h+\theta_0)\,d_1 + fd_3 + vd_4}{v} = \frac{(h-\theta_0)\,a_2 + ga_3 + ua_4}{0} = \frac{(h-\theta_0)\,b_2 + gb_3 + ub_4}{h + \theta_0}\\ &= \frac{(h-\theta_0)\,c_2 + gc_3 + uc_4}{g} = \frac{(h-\theta_0)\,d_2 + gd_3 + ud_4}{u},\end{aligned}$$

and there will be no loss of generality in supposing each of these fractions unity. With the ordinary notation of matrices these equations are then the same as

$$\begin{vmatrix} h + \theta_0, & 0, & f, & v \\ 0, & h - \theta_0, & g, & u \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = \begin{vmatrix} h - \theta_0, & 0, & f, & v \\ 0, & h + \theta_0, & g, & u \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix},$$

and enable us to infer that the matrix of the homography has the form

$$\vartheta = \begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}^{-1} \begin{vmatrix} h-\theta_0, & 0, & f, & v \\ 0, & h+\theta_0, & g, & u \\ a, & b, & c, & d \\ a', & b', & c', & d' \end{vmatrix},$$

where $a, b, c, d, a', b', c', d'$ are written respectively for

$$a_3, b_3, c_3, d_3, a_4, b_4, c_4, d_4.$$

If then we put

$$m = \frac{h-\theta_0}{h+\theta_0}, \qquad \mu = \begin{vmatrix} m, & 0, & 0, & 0 \\ 0, & m^{-1}, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix},$$

we have

$$\vartheta\mu^{-1} = \begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}^{-1} \begin{vmatrix} h+\theta_0, & 0, & f, & v \\ 0, & h-\theta_0, & g, & u \\ am^{-1}, & bm, & c, & d \\ a'm^{-1}, & b'm, & c', & d' \end{vmatrix}.$$

If the transformation associated with the matrix μ change the point (x, y, z, t) to (x_1, y_1, z_1, t_1), so that $(x_1) = \mu\,(x)$, we have from $(x') = \vartheta\,(x)$, for the change from (x_1) to (x'), the equations expressed by $(x') = \vartheta\mu^{-1}\,(x_1)$, which, by what we have just seen, are the same as

$$\begin{aligned} (h+\theta_0)\,x' + fz' + vt' &= (h+\theta_0)\,x_1 + fz_1 + vt_1, \\ (h-\theta_0)\,y' + gz' + ut' &= (h-\theta_0)\,y_1 + gz_1 + ut_1, \\ z' &= am^{-1}x_1 + bmy_1 + cz_1 + dt_1, \\ t' &= a'm^{-1}x_1 + b'my_1 + c'z_1 + d't_1. \end{aligned}$$

Without entering into general statements about the minors of a matrix, we proceed now to show in detail in an elementary way that these last are the equations of what we have called an axis-range perspective. For this purpose, write

$$F = \frac{f}{h+\theta_0}, \quad V = \frac{v}{h+\theta_0}, \quad G = \frac{g}{h-\theta_0}, \quad U = \frac{u}{h-\theta_0},$$

$$C = c - am^{-1}F - bmG, \quad D = d - am^{-1}V - bmU,$$

$$C' = c' - a'm^{-1}F - b'mG, \quad D' = d' - a'm^{-1}V - b'mU,$$

and also

$$\xi' = x' + Fz' + Vt', \quad \xi_1 = x_1 + Fz_1 + Vt_1,$$
$$\eta' = y' + Gz' + Ut', \quad \eta_1 = y_1 + Gz_1 + Ut_1;$$

then the equations of the transformation $\vartheta\mu^{-1}$ take the forms

$$\xi' = \xi_1, \ \eta' = \eta_1, \ z' = am^{-1}\xi_1 + bm\eta_1 + Cz_1 + Dt_1,$$
$$t' = a'm^{-1}\xi_1 + b'm\eta_1 + C'z_1 + D't_1;$$

assuming that the roots α, β of the equation in α,

$$(C - \alpha)(D' - \alpha) - C'D = 0,$$

are different from one another, and from unity, we can take p, p' and q, q' so that

$$\frac{pC + p'C'}{p} = \frac{pD + p'D'}{p'} = \alpha, \quad \frac{qC + q'C'}{q} = \frac{qD + q'D'}{q'} = \beta,$$

and thence, by

$$Z' = pz' + p't', \quad Z_1 = pz_1 + p't_1,$$
$$T' = qz' + q't', \quad T_1 = qz_1 + q't_1,$$

the last two equations of $\vartheta\mu^{-1}$ are replaced by

$$Z' = A\xi_1 + B\eta_1 + \alpha Z_1, \ T' = A'\xi_1 + B'\eta_1 + \beta T_1,$$

where

$$A = (pa + p'a')\, m^{-1}, \quad B = (pb + p'b')\, m,$$
$$A' = (qa + q'a')\, m^{-1}, \quad B' = (qb + q'b')\, m,$$

so that, finally, if we put

$$\zeta' = Z' - \frac{A}{1-\alpha}\xi' - \frac{B}{1-\alpha}\eta', \quad \zeta_1 = Z_1 - \frac{A}{1-\alpha}\xi_1 - \frac{B}{1-\alpha}\eta_1,$$

$$\tau' = T' - \frac{A'}{1-\beta}\xi' - \frac{B'}{1-\beta}\eta', \quad \tau_1 = T_1 - \frac{A'}{1-\beta}\xi_1 - \frac{B'}{1-\beta}\eta_1,$$

the equations of $\vartheta\mu^{-1}$ are

$$\xi' = \xi_1, \ \eta' = \eta_1, \ \zeta' = \alpha\zeta_1, \ \tau' = \beta\tau_1$$

which are the equations of an axis-range perspective in which every point of the line $\zeta = 0$, $\tau = 0$ is unchanged, and every plane through the line $\xi = 0$, $\eta = 0$ is unchanged. The latter line is the axis given in the notation first used by the parameter $-\theta_0$, belonging to the quadric S_0; the former line, we easily see, is given by

$$am^{-1}x' + bmy' + (c-1)\, z' + dt' = 0,$$
$$a'm^{-1}x' + b'my' + c'z' + (d'-1)\, t' = 0.$$

The transformation $(x_1) = \mu\,(x)$, equivalent to

$$x_1 = mx, \ y_1 = m^{-1}y, \ z_1 = z, \ t_1 = t,$$

is evidently one which leaves unaltered the fundamental quadric S whose equation was taken to be $xy - zt = 0$.

The result then is: if we apply to the figure for which the coordinates (x) have been used the transformation μ, by $(x_1) = \mu(x)$, this being a transformation leaving the arbitrary quadric S unaltered, the relation between the figure (x_1), and the figure (x'), which arose from (x) by $(x') = \vartheta(x)$, which is expressed by

$$(x') = \vartheta\mu^{-1}(x_1),$$

is that the figure (x') arises from the figure (x_1) by an axis-range perspectivity. The axis of this is an arbitrary "line in two planes" of a certain developable determined by the arbitrary quadric and the original transformation ϑ.

On the transformation of the equations of electrodynamics in the Maxwell and in the Einstein forms. By Professor H. F. BAKER.

[*Read* 9 February 1920.]

The present note, which arose from reading the paper of Mr W. J. Johnston in the *Proceedings of the Royal Society*, A, XCVI, 1919, 331, and the paper of M. Th. de Donder, *Archives du Musée Teyler* (Haarlem), III, 1917, 80–179, has a very humble purpose. (1) It is shown that the noncommutative "imaginaries" used by Mr Johnston may be interpreted as aggregates of ordinary numbers, much in the same way as the complex numbers of ordinary analysis may be interpreted as aggregates of real numbers. It is shown indeed how to interpret a system of any number of units obeying the same laws of combination. (2) By this interpretation a form of Maxwell's equations of electrodynamics is reached from which the ordinary Lorenz transformation is obvious at sight. This form of the equations is equivalent to a solution of the electrodynamic equations in terms of arbitrary functions. (3) The equations given by M. Th. de Donder, and stated by him to be the equations of electrodynamics in the Einstein field, are then considered, and their invariance under a general transformation is established. The purpose of this section is to show how simply this invariance follows by the use of notation which is familiar in other applications. The result of course includes the case of Maxwell's equations.

§ 1. It is a familiar fact that the so-called complex numbers of ordinary analysis are couplets of two *real* numbers (x, y), subject to the laws

(i) if m be a real number, $m(x, y) = (mx, my)$,

(ii) $(x, y) + (x', y') = (x + x', y + y')$,

(iii) $(x, y)(x', y') = (xx' - yy', xy' + x'y)$,

those of the numbers which we ordinarily call real being couplets $(x, 0)$, and those which we call pure imaginaries being couplets $(0, y)$. By means of these rules any number (x, y) is expressible in the form $x[1] + y[i]$, where $[1]$, $[i]$ stand for $(1, 0)$ and $(0, 1)$; and $[i][i] = -[1]$. It is this last equation which we ordinarily write $i^2 = -1$.

We may similarly have systems of more than two numbers, subject to laws of computation, addition (and subtraction), multiplication (and possibly division). In particular consider systems, built with the numbers of ordinary analysis (including complex numbers), each system consisting of n^2 numbers ($n = 2, 3, \ldots$). These we may conveniently arrange in the form of a square; and for purposes of explanation it will be sufficient to write down only two rows and columns. Suppose then

$$(a) = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix},$$

where a_{11}, a_{12}, ... are numbers of ordinary analysis, and introduce the rules (i) when m is a number of ordinary analysis $m\,(a)$ shall mean the same as (a) with each element a_{rs} replaced by ma_{rs}. (ii) If (a), (b) be squares of the same number of rows and columns $(a) + (b)$ shall mean the square of which the general element is $a_{rs} + b_{rs}$; thus $(a) + (b)$ is the same as $(b) + (a)$. (iii) $(a)\,(b)$ shall mean the square whose general element c_{rs} is formed by combining the rth row of (a) with the sth column of b, in such a way that

$$c_{rs} = a_{r1}b_{1s} + a_{r2}b_{2s};$$

thus $(a)\,(b)$ is not generally equal to $(b)\,(a)$. (iv) The zero square shall mean that in which every element is the number 0; denoting this zero square by 0, we shall then have $(a) + 0 = (a)$, and $(a)\,0 = 0\,(a) = 0$ whatever (a) may be. (v) The unit square shall mean that in which all the elements are zero except those, a_{rr}, in the principal diagonal, each of these being unity. Denoting this by E we shall then have $E\,(a) = (a)\,E = (a)$, whatever (a) may be. Addition of such squares is then not only associative, but is also commutative. It is easy to prove that multiplication is associative also; but in general it is not commutative.

Of such squares, of two rows and columns, that given by

$$\begin{pmatrix} x, & -y \\ y, & x \end{pmatrix}$$

is easily seen to obey all the laws of the complex number (x, y), the multiplication being commutative. This square may then equally be used to represent the numbers of ordinary analysis. Putting

$$(1) = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \quad (i) = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$$

this square is $x\,(1) + y\,(i)$.

Now denote the two units of ordinary analysis by 1, ϵ, where, for convenience of notation, ϵ is used instead of the ordinary i, so that $\epsilon^2 = -1$. And consider the squares of two rows and columns expressed by

$$E = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, I = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, J = \begin{pmatrix} \epsilon, & 0 \\ 0, & -\epsilon \end{pmatrix}, K = \begin{pmatrix} 0, & -\epsilon \\ -\epsilon, & 0 \end{pmatrix}.$$

By the laws above given we shall then find

$$E^2 = E, \quad EI = IE = I, \text{ etc.}$$

$$I^2 = J^2 = K^2 = -E, \quad JK = -KJ = I,$$

$$KI = -IK = J, \qquad IJ = -JI = K,$$

so that I, J, K, E behave precisely as the units of the theory of quaternions. Other binary squares can be found with the same results of composition; for instance the work below suggests that $I' = -I$, $J' = -K$, $K' = -J$ might have been taken. Adopting however the above, the usual quaternion $xI + yJ + zK + tE$ is represented by

$$\begin{pmatrix} 0, & x \\ -x, & 0 \end{pmatrix} + \begin{pmatrix} \epsilon y, & 0 \\ 0, & -\epsilon y \end{pmatrix} + \begin{pmatrix} 0, & -\epsilon z \\ -\epsilon z, & 0 \end{pmatrix} + \begin{pmatrix} t, & 0 \\ 0, & t \end{pmatrix},$$

or

$$\begin{pmatrix} t + \epsilon y, & x - \epsilon z \\ -x - \epsilon z, & t - \epsilon y \end{pmatrix},$$

and the formula of multiplication of two quaternions may be obtained by multiplying two such squares according to the rule given above.

Now consider squares of four rows and columns. But for brevity, instead of writing four rows and columns, introduce symbols to stand as above each for a square of two rows and columns. Namely, using the letter O for the square of two rows and columns of which each element is zero, let

$$e = \begin{pmatrix} O, & -E \\ E, & O \end{pmatrix}, \quad i = \begin{pmatrix} O, & I \\ I, & O \end{pmatrix}, \quad j = \begin{pmatrix} O, & J \\ J, & O \end{pmatrix}, \quad k = \begin{pmatrix} O, & K \\ K, & O \end{pmatrix}, \quad u = \begin{pmatrix} E, & O \\ O, & E \end{pmatrix},$$

where E, I, J, K are as above, E being the unit square of two rows and columns, and u the unit square of four rows and columns.

Then by the rules above given it is easy to compute that

$$e^2 = i^2 = j^2 = k^2 = -u,$$

$$ei = -ie = \begin{pmatrix} -I, & O \\ O, & I \end{pmatrix}, \quad ej = -je = \begin{pmatrix} -J, & O \\ O, & J \end{pmatrix}, \quad ek = -ke = \begin{pmatrix} -K, & O \\ O, & K \end{pmatrix},$$

$$jk = -kj = \begin{pmatrix} I, & O \\ O, & I \end{pmatrix}, \quad ki = -ik = \begin{pmatrix} J, & O \\ O, & J \end{pmatrix}, \quad ij = -ji = \begin{pmatrix} K, & O \\ O, & K \end{pmatrix},$$

together with

$$ijk = \begin{pmatrix} O, & -E \\ -E, & O \end{pmatrix}, \quad ejk = \begin{pmatrix} O, & -I \\ I, & O \end{pmatrix}, \quad eki = \begin{pmatrix} O, & -J \\ J, & O \end{pmatrix}, \quad eij = \begin{pmatrix} O, & -K \\ K, & O \end{pmatrix}$$

and

$$eijk = \begin{pmatrix} E, & O \\ O, & -E \end{pmatrix};$$

and, the multiplication being associative, it is not necessary to remark such equations as

$$ijk = -jik = jki = -ikj, \text{ etc.}$$

The squares e, i, j, k thus obey the laws of combination of the symbols occurring in Mr Johnston's paper above referred to, denoted by him by o, i, j, k.

If we consider a square of which every element is ϵ times the corresponding element in $eijk$, and denote this by m, we find at once

$$m^2 = -u, \quad em = -me, \quad im = -mi, \quad jm = -mj, \quad km = -mk,$$

and e, i, j, k, m are a system of five squares obeying the same laws of combination as the three quaternion squares I, J, K. We have $eijkm = -\epsilon u$, and save for multiplication with -1 or ϵ, or both, there are sixteen squares arising by multiplications of u, e, i, j, k, m, namely these six and those obtainable by the ten products of two of e, i, j, k, m.

As was remarked to me by Dr W. Burnside*, F.R.S., the above work is capable of generalisation.

If $e_1, e_2, \ldots, e_{2r-1}$ be squares each of m rows and columns obeying the laws $e_i^2 = -u$, $e_ie_j = -e_je_i$, where u is the unit square of m rows and columns, then the $2r+1$ squares of each $2m$ rows and columns given by

$$E = \begin{pmatrix} 0, & -u \\ u, & 0 \end{pmatrix}, \quad E_i = \begin{pmatrix} 0, & e_i \\ e_i, & 0 \end{pmatrix}, \; (i = 1, 2, \ldots, (2r-1)), \; F = \mu E E_1 \ldots E_{2r-1},$$

where $\mu, = (-1)^{\frac{1}{2}[1+(-1)^r]}$, is 1 or ϵ according as r is odd or even, obey exactly the same laws—namely if U be the unit square of $2m$ rows and columns

$$E^2 = E_i^2 = -U, \quad EE_i = -E_iE, \quad E_iE_j = -E_jE_i.$$

For instance for $r = 0$, $m = 1$, this gives

$$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \quad \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$$

as two squares of which the square of the latter is the negative of the former. These are the two units (usually denoted by 1, i) of ordinary analysis. Or if, denoting these, as we have done, by 1, ϵ, we take $r = 1$, $m = 1$, the theorem gives the binary squares

$$\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \quad \begin{pmatrix} 0, & \epsilon \\ \epsilon, & 0 \end{pmatrix}, \quad \begin{pmatrix} -\epsilon, & 0 \\ 0, & \epsilon \end{pmatrix}, \quad \text{with} \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

say, I, J, K, U, as units satisfying

$$I^2 = J^2 = K^2 = -U, \quad JK = -KJ = I,$$
$$KI = -IK = J, \quad IJ = -JI = K,$$

which are the laws of Hamilton's quaternions.

* Dr Burnside writes: "In group notation it really comes to this:—The operations $S_1, S_2, \ldots, S_n$, such that

$$S_1^2 = S_2^2 = \ldots = S_n^2 = T, \quad T^2 = E, \quad S_iS_j = S_jS_iT$$

generate a group G_n of order 2^{2n+1}. In this group E, T are the only invariant operations if n is even; when n is odd, the factor group $G_n \mid \{S_1S_2 \ldots S_n\}$ is identical in type with G_{n-1}."

§ 2. With the symbols e, i, j, k above explained, satisfying the equations

$$e^2 = i^2 = j^2 = k^2 = -u, \; ei = -ie, \; \ldots, \; jk = -kj, \; \ldots, \; eu = ue = e, \; \ldots,$$

we clearly have

$$\begin{aligned}(e\tau + i\xi + j\eta + k\zeta)\,(e\phi + iF + jG + kH) \\ = -u\,(\tau\phi + \xi F + \eta G + \zeta H) \\ + jk\,(\eta H - \zeta G) + ki\,(\zeta F - \xi H) + ij\,(\xi G - \eta F) \\ + ei\,(\tau F - \xi\phi) + ej\,(\tau G - \eta\phi) + ek\,(\tau H - \zeta\phi),\end{aligned}$$

and if we denote this by

$$uW + jkL + kiM + ijN + eiX + ejY + ekZ,$$

we further have the identity (A)

$$\begin{aligned}(e\tau + i\xi + j\eta + k\zeta)\,(uW + jkL + kiM + ijN + eiX + ejY + ekZ) \\ = e\,(\tau W + \xi X + \eta Y + \zeta Z) + i\,(\xi W + \eta N - \zeta M - \tau X) \\ + j\,(\eta W - \xi N + \zeta L - \tau Y) + k\,(\zeta W + \xi M - \eta L - \tau Z) \\ + e_1\,(\xi L + \eta M + \zeta N) + i_1\,(-\eta Z + \zeta Y + \tau L) \\ + j_1\,(-\zeta X + \xi Z + \tau M) + k_1\,(-\xi Y + \eta X + \tau N),\end{aligned}$$

where $\qquad e_1 = ijk, \;\; i_1 = ejk, \;\; j_1 = eki, \;\; k_1 = eij.$

If now we interpret ξ, η, ζ, τ as symbols of differentiation in regard respectively to x, y, z, t, the coefficients of e, i, ..., e_1, ... arising in these equations are the combinations occurring in the equations of electrodynamics considered by Mr Johnston in the paper referred to, the F, G, H, X, Y, Z being certain constant multiples of those usually denoted by these symbols, except that W, which vanishes in the electrodynamic case, is here retained, and t is a certain imaginary multiple of the time.

However, a symbol $e\tau + i\xi + j\eta + k\zeta$, if we use the interpretation of e, i, j, k above developed, is the same as

$$\tau \begin{pmatrix} O, & -E \\ E, & O \end{pmatrix} + \xi \begin{pmatrix} O, & I \\ I, & O \end{pmatrix} + \eta \begin{pmatrix} O, & J \\ J, & O \end{pmatrix} + \zeta \begin{pmatrix} O, & K \\ K, & O \end{pmatrix},$$

or
$$\begin{pmatrix} O, & -\tau E + \xi I + \eta J + \zeta K \\ \tau E + \xi I + \eta J + \zeta K, & O, \end{pmatrix},$$

or, taking E, I, J, K as above, is the same as

$$\begin{vmatrix} 0, & 0, & -\tau + \epsilon\eta, & \xi - \epsilon\zeta \\ 0, & 0, & -\xi - \epsilon\zeta, & -\tau - \epsilon\eta \\ \tau + \epsilon\eta, & \xi - \epsilon\zeta, & 0, & 0 \\ -\xi - \epsilon\zeta, & \tau - \epsilon\eta, & 0, & 0 \end{vmatrix},$$

where 0 is the zero of ordinary analysis. The above operational equations may then be obtained by multiplications of such squares of four rows.

But when ξ, η, ζ, τ stand for

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t},$$

if we take x_1, y_1, z_1, t_1 so that

$$x = x_1 + z_1, \quad \epsilon z = x_1 - z_1, \quad t = t_1 + y_1, \quad \epsilon y = t_1 - y_1,$$

we have

$$\frac{\partial}{\partial x_1} = \xi - \epsilon\zeta, \quad \frac{\partial}{\partial y_1} = \tau + \epsilon\eta, \quad \frac{\partial}{\partial z_1} = \xi + \epsilon\zeta, \quad \frac{\partial}{\partial t_1} = \tau - \epsilon\eta,$$

and, denoting these by $\xi_1, \eta_1, \zeta_1, \tau_1$, the above square is

$$\begin{vmatrix} 0, & 0, & -\tau_1, & \xi_1 \\ 0, & 0, & -\zeta_1, & -\eta_1 \\ \eta_1, & \xi_1, & 0, & 0 \\ -\zeta_1, & \tau_1, & 0, & 0 \end{vmatrix}.$$

Again

$$uW + jkL + kiM + ijN + eiX + ejY + ekZ$$

is

$$W\begin{pmatrix} E, & O \\ O, & E \end{pmatrix} + L\begin{pmatrix} I, & O \\ O, & I \end{pmatrix} + M\begin{pmatrix} J, & O \\ O, & J \end{pmatrix} + N\begin{pmatrix} K, & O \\ O, & K \end{pmatrix}$$
$$+ X\begin{pmatrix} -I, & O \\ & O, I \end{pmatrix} + Y\begin{pmatrix} -J, & O \\ & O, J \end{pmatrix} + Z\begin{pmatrix} -K, & O \\ & O, K \end{pmatrix},$$

or

$$\begin{Bmatrix} EW + (L - X)I + (M - Y)J + (N - Z)K, & O, \\ O, & WE + (L + X)I + (M + Y)J + (N+Z)K \end{Bmatrix},$$

which, if we take E, I, J, K as above, is the same as

$$\begin{vmatrix} W + (M - Y)\epsilon, & L - X - (N - Z)\epsilon, & 0, & 0 \\ -(L - X) - (N - Z)\epsilon, & W - (M - Y)\epsilon, & 0, & 0 \\ 0, & 0, & W + (M + Y)\epsilon, & L + X - (N + Z)\epsilon \\ 0, & 0, & -(L + X) - (N + Z)\epsilon, & W - (M + Y)\epsilon \end{vmatrix}.$$

Now put

$$\begin{aligned} \alpha &= L + X + \epsilon(N + Z), & \gamma &= L - X + \epsilon(N - Z), \\ \alpha' &= L + X - \epsilon(N + Z), & \gamma' &= L - X - \epsilon(N - Z), \\ \beta &= W + \epsilon(M + Y), & \delta &= W + \epsilon(M - Y), \\ \beta' &= W - \epsilon(M + Y), & \delta' &= W - \epsilon(M - Y), \end{aligned}$$

so that $$\beta + \beta' = \delta + \delta'.$$
Then the last square is

$$\begin{vmatrix} \delta, & \gamma', & 0, & 0 \\ -\gamma, & \delta', & 0, & 0 \\ 0, & 0, & \beta, & \alpha' \\ 0, & 0, & -\alpha, & \beta' \end{vmatrix},$$

and the equations of electrodynamics are expressed by the vanishing of the product

$$\begin{vmatrix} 0, & 0, & -\tau_1, & \xi_1 \\ 0, & 0, & -\zeta_1, & -\eta_1 \\ \eta_1, & \xi_1, & 0, & 0 \\ -\zeta_1, & \tau_1, & 0, & 0 \end{vmatrix} \begin{vmatrix} \delta, & \gamma', & 0, & 0 \\ -\gamma, & \delta', & 0, & 0 \\ 0, & 0, & \beta, & \alpha' \\ 0, & 0, & -\alpha, & \beta' \end{vmatrix};$$

this is equivalent to the vanishing of the two products

$$\begin{pmatrix} \tau_1, & -\xi_1 \\ \zeta_1, & \eta_1 \end{pmatrix} \begin{pmatrix} \beta, & \alpha' \\ -\alpha, & \beta' \end{pmatrix}, \quad \begin{pmatrix} \eta_1, & \xi_1 \\ -\zeta_1, & \tau_1 \end{pmatrix} \begin{pmatrix} \delta, & \gamma' \\ -\gamma, & \delta' \end{pmatrix},$$

and the equations are therefore

$$\left.\begin{aligned} &\frac{\partial \alpha}{\partial x_1} + \frac{\partial \beta}{\partial t_1} = 0, \quad \frac{\partial \alpha'}{\partial z_1} + \frac{\partial \beta'}{\partial y_1} = 0, \quad \frac{\partial \gamma}{\partial t_1} + \frac{\partial \delta}{\partial z_1} = 0, \quad \frac{\partial \gamma'}{\partial y_1} + \frac{\partial \delta'}{\partial x_1} = 0 \\ &\frac{\partial \alpha}{\partial y_1} - \frac{\partial \beta}{\partial z_1} = 0, \quad \frac{\partial \alpha'}{\partial t_1} - \frac{\partial \beta'}{\partial x_1} = 0, \quad \frac{\partial \gamma}{\partial x_1} - \frac{\partial \delta}{\partial y_1} = 0, \quad \frac{\partial \gamma'}{\partial z_1} - \frac{\partial \delta'}{\partial t_1} = 0 \end{aligned}\right\} \text{(I)}.$$

These results are easily shown directly to be the same as the ordinary forms (putting $W = 0$, $\beta' = -\beta$, $\delta' = -\delta$). They involve that each of the functions α, β, α', β', γ, δ, γ', δ' satisfies the equation

$$\frac{\partial^2 V}{\partial x_1 \partial z_1} + \frac{\partial^2 V}{\partial y_1 \partial t_1} = 0,$$

or
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial t^2} = 0;$$

and they are equivalent also to

$$\alpha = \int\left(-\frac{\partial \beta}{\partial t_1} dx_1 + \frac{\partial \beta}{\partial z_1} dy_1\right) + A(z_1, t_1),$$

$$\alpha' = \int\left(-\frac{\partial \beta'}{\partial y_1} dz_1 + \frac{\partial \beta'}{\partial x_1} dt_1\right) + A'(x_1, y_1),$$

$$\gamma = \int\left(-\frac{\partial \delta}{\partial z_1} dt_1 + \frac{\partial \delta}{\partial y_1} dx_1\right) + C(y_1, z_1),$$

$$\gamma' = \int\left(-\frac{\partial \delta'}{\partial x_1} dy_1 + \frac{\partial \delta'}{\partial t_1} dz_1\right) + C'(x_1, t_1),$$

where A, C, A', C' are arbitrary functions of their respective arguments. The expressions under the integral signs are perfect differentials in virtue of the differential equation satisfied by $\beta, \delta, \beta', \delta'$.

The equations are also equivalent to

$$U\alpha = V\beta, \qquad R\delta' = S\gamma',$$
$$U\beta' = -V\alpha', \; R\gamma = -S\delta,$$

together with those obtained by putting $-\epsilon$ for ϵ, where

$$U = \frac{\partial}{\partial x_1} + \epsilon\frac{\partial}{\partial y_1}, \quad V = \epsilon\left(\frac{\partial}{\partial z_1} + \epsilon\frac{\partial}{\partial t_1}\right),$$
$$R = \frac{\partial}{\partial x_1} + \epsilon\frac{\partial}{\partial t_1}, \quad S = \epsilon\left(\frac{\partial}{\partial z_1} + \epsilon\frac{\partial}{\partial y_1}\right).$$

What is to be remarked here however is that the equations (I) are evidently unaltered by replacing

$$\alpha, \alpha', \gamma, \gamma', y_1, t_1$$

respectively by $\quad \sigma\alpha, \sigma^{-1}\alpha', \sigma^{-1}\gamma, \sigma\gamma', \sigma y_1, \sigma^{-1}t_1,$

where σ is an arbitrary constant. These are the equations of the Lorenz transformation of the older relativity theory. If, putting $W = 0$, we equate to zero the coefficients of i, j, k, i_1, j_1, k_1, in the identity (A) at the beginning of this article (§ 2), and write

$$\tau = \frac{\sqrt{\mu K}}{ic}\frac{\partial}{\partial t}, \quad L = mP, \quad M = mQ, \quad N = mR, \quad m = -ic\sqrt{\frac{\mu}{K}},$$

where $i = \sqrt{-1}$, we obtain the six familiar equations

$$\frac{K}{c^2}\nabla(X, Y, Z) = \text{curl}\,(P, Q, R),$$
$$-\mu\nabla(P, Q, R) = \text{curl}\,(X, Y, Z),$$

and the above results furnish a general *solution* of these, in terms of two arbitrary functions satisfying the above potential equation.

§ 3. We now pass to the equations described by M. Th. de Donder as the equations of electrodynamics in the Einstein field (*loc. cit.*, p. 93). Here instead of one system of two sets each of three quantities (X, Y, Z), (L, M, N), which interchange, we have two systems each of six, connected together in a way which shall be explained, with coefficients which are quadratic functions of the coefficients in a certain Absolute quadric. These two systems, imitating the notation of the author, are denoted respectively by $(M_{23}, M_{31}, M_{12}, M_{14}, M_{24}, M_{34})$ and $(N_{23}, N_{31}, N_{12}, N_{14}, N_{24}, N_{34})$, and symbols M_{32}, M_{41}, etc., are occasionally used where

$$M_{32} = -M_{23}, \quad M_{41} = -M_{14}, \text{ etc.}$$

The differential equations in question, writing ξ, η, ζ, τ for

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t},$$

respectively, are

$$\begin{array}{rrrl} & -\eta M_{34} & +\zeta M_{24} - \tau M_{23} & = \rho V_x, \\ \xi M_{34} & & -\zeta M_{14} - \tau M_{31} & = \rho V_y, \\ -\xi M_{24} & +\eta M_{14} & -\tau M_{12} & = \rho V_z, \\ \xi M_{23} & +\eta M_{31} & +\zeta M_{12} & = \rho, \end{array}$$

$$\begin{array}{rrrl} & -\eta N_{34} & +\zeta N_{24} - \tau N_{23} & = 0, \\ \xi N_{34} & & -\zeta N_{14} - \tau N_{31} & = 0, \\ -\xi N_{24} & +\eta N_{14} & -\tau N_{12} & = 0, \\ \xi N_{23} & +\eta N_{31} & +\zeta N_{12} & = 0. \end{array}$$

If herein we replace M_{23}, M_{31}, M_{12}, M_{14}, M_{24}, M_{34} respectively by X, Y, Z, $-L$, $-M$, $-N$ and N_{23}, N_{31}, N_{12}, N_{14}, N_{24}, N_{34} by any constant multiples respectively of $-L$, $-M$, $-N$, X, Y, Z the functions on the left become, save for W, the coefficients of

$$e, i, j, k, -e_1, i_1, j_1, k_1,$$

in the identity (A) of § 2.

Our object is to show that with any transformation of x, y, z, t leaving a certain quadric differential form in dx, dy, dz, dt unaltered, and appropriate corresponding transformations of the elements M_{ij}, N_{ij} of the two systems, and of V_x, ..., ρ, the equations change into others of the same form. In the Maxwell form of the equations the differential form is simply $-(dx^2 + dy^2 + dz^2 + dt^2)$.

In the paper referred to M. Th. de Donder bases his work, after Hargreaves, *Trans. Camb. Phil. Soc.*, XXI, 25 Aug. 1908, upon the transformation of differential (integral) forms, which he compounds together by rules which appear to have a certain artificiality. In what follows this is replaced by the use of Grassman's alternate units*. For the actual transformation here considered M. Th. de Donder does not give the proof in the paper in question (*loc. cit.*, p. 147), but refers to *Mém. Acad. roy. de Belgique*, I, 1904. But more, in the present note, full use is made of the theory of matrices, which not only brings out the identity of much of the algebra with what is familiar in other work, but is also very much briefer, as the reader may easily see by comparison with the original paper. A short account of the use of matrices, under the name of squares, is given in § 1 of this note.

* Mr Bateman, *Proceedings of the Lond. Math. Soc.*, VIII, 1909, p. 245, to whom M. Th. de Donder refers, has already suggested that Grassman's units might be used.

We have quantities M_{ij}, N_{ij}, for $i, j = 1, 2, 3, 4$, such that $M_{ii} = N_{ii} = 0$, $M_{ij} = -M_{ji}$, $N_{ij} = -N_{ji}$. We consider two matrices

$$m = \begin{vmatrix} 0, & M_{12}, & M_{13}, & M_{14} \\ M_{21}, & 0, & M_{23}, & M_{24} \\ M_{31}, & M_{32}, & 0, & M_{34} \\ M_{41}, & M_{42}, & M_{43}, & 0 \end{vmatrix}, \quad n = \begin{vmatrix} 0, & N_{12}, & N_{13}, & N_{14} \\ N_{21}, & 0, & N_{23}, & N_{24} \\ N_{31}, & N_{32}, & 0, & N_{34} \\ N_{41}, & N_{42}, & N_{43}, & 0 \end{vmatrix},$$

of which the second is determined from the former by an equation given later, with the help of a certain quadric. The determinants of m, n (supposed not zero) are respectively μ^2, ν^2, where

$$\mu = M_{23}M_{14} + M_{31}M_{24} + M_{12}M_{34}, \quad \nu = N_{23}N_{14} + N_{31}N_{24} + N_{12}N_{34}.$$

It will appear later, from the equation connecting m, n, that $\nu = -\mu$. Connected with m is a certain other matrix

$$\begin{vmatrix} 0, & M_{34}, & -M_{24}, & M_{23} \\ -M_{34}, & 0, & M_{14}, & M_{31} \\ M_{24}, & -M_{14}, & 0, & M_{12} \\ -M_{23}, & -M_{31}, & -M_{12}, & 0 \end{vmatrix}.$$

Multiplying this last with m we find that in the product every element vanishes except those in the diagonal, each of which is $-\mu$. Replacing the unit matrix by unity, as usual, we therefore say that the last written matrix is equal to $-\mu m^{-1}$. The corresponding matrix derived from n is therefore equal to $-\nu n^{-1}$.

We consider a certain quadric form in four variables, whose coefficients $a, h, \ldots$ will later be regarded as functions of variables x, y, z, t, and put Δ for the matrix

$$\Delta = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}.$$

The determinant of this matrix being denoted by δ, we put

$$\epsilon = (-\delta)^{\frac{1}{2}}.$$

Then the coefficients N_{ij} are defined by the equations expressed by

$$\epsilon n = \mu \Delta m^{-1} \Delta$$

which is the same as $\qquad \epsilon m = \mu \Delta n^{-1} \Delta,$

so that M_{ij} are the same functions of N_{ij} as are N_{ij} of M_{ij}. At greater length these equations are the same as

$$\epsilon n = -\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix} \begin{vmatrix} 0, & M_{34}, & -M_{24}, & M_{23} \\ -M_{34}, & 0, & M_{14}, & M_{31} \\ M_{24}, & -M_{14}, & 0, & M_{12} \\ -M_{23}, & -M_{31}, & -M_{12}, & 0 \end{vmatrix} \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix},$$

which give for instance

$$-\epsilon N_{23} = (hw - gv)\,M_{23} + (bw - fv)\,M_{31} + (fw - cv)\,M_{12} + (bc - f^2)\,M_{14} + (fg - ch)\,M_{24} + (hf - bg)\,M_{34},$$

$$-\epsilon N_{14} = (ad - u^2)\,M_{23} + (hd - uv)\,M_{31} + (gd - wu)\,M_{12} + (hw - gv)\,M_{14} + (gu - aw)\,M_{24} + (av - hu)\,M_{34},$$

$$-\epsilon N_{12} = (av - hu)\,M_{23} + (hv - bu)\,M_{31} + (gv - fu)\,M_{12} + (hf - bg)\,M_{14} + (gh - af)\,M_{24} + (ab - h^2)\,M_{34}.$$

The reader will compare the formulae for the polar line of a line in regard to a quadric, Salmon's *Solid Geometry* (1882, p. 60), and may verify that, the coordinates of a line being l, m, n, l', m', n', where the line is given by $l't + mz - ny = 0$, $l't + m'y + n'z = 0$, the polar line $(\lambda\mu\nu\lambda'\mu'\nu')$ of $(lmnl'm'n')$ in regard to the quadric $(a, b, c, d, f, g, h, u, v, w \between x, y, z, t)^2 = 0$ is given by

$$\begin{vmatrix} 0, & -\nu, & \mu, & \lambda' \\ \nu, & 0, & -\lambda, & \mu' \\ -\mu, & \lambda, & 0, & \nu' \\ -\lambda', & -\mu', & -\nu', & 0 \end{vmatrix} = \Delta \begin{vmatrix} 0, & -n', & m', & l \\ n', & 0, & -l', & m \\ -m', & l', & 0, & n \\ -l, & -m, & -n, & 0 \end{vmatrix} \Delta.$$

Thus if $M_{23}, M_{31}, M_{12}, M_{14}, M_{24}, M_{34}$ were proportional to $l, m, n, -l', -m', -n'$, then $N_{23}, N_{31}, N_{12}, N_{14}, N_{24}, N_{34}$ would be proportional to $\lambda, \mu, \nu, -\lambda', -\mu', -\nu'$. We assume however that $M_{23}M_{14} + M_{31}M_{24} + M_{12}M_{34}$, and the corresponding expression for N_{ij}, are not zero. The above expression of n in terms of m gives, on taking the determinants of both sides, $\epsilon^4\nu^2 = \mu^4\delta^2\,\mu^{-2}$, or, from $\epsilon^4 = \delta^2, \nu^2 = \mu^2$. The explicit forms for N_{ij} show that $\nu = -\mu$.

We consider now a transformation, from the variables (x, y, z, t), upon which $a, h, g, \ldots, M_{ij}, N_{ij}$, depend to variables (x', y', z', t'); we write

$$dx = p_{11}dx' + p_{12}dy' + p_{13}dz' + p_{14}dt',$$

$$dy = p_{21}dx' + p_{22}dy' + p_{23}dz' + p_{24}dt',$$

and so on, or more briefly, with the ordinary matrix notation,

$$(dx, dy, dz, dt) = p\,(dx', dy', dz', dt'),$$

the converse equations being written

$$(dx', dy', dz', dt') = p'\,(dx, dy, dz, dt).$$

Thus the matrix p' is the inverse of p; so that if P_{rs} be the cofactor of p_{rs} in the determinant, ϖ, of p, we have $p_{sr}' = P_{rs}/\varpi$. With this transformation the quadric differential form

$$(abcduvwfgh \between dx, dy, dz, dt)^2$$

changes into another such form which is written

$$(a', \dots, h' \between dx', dy', dz', dt')^2.$$

Using Δ' for the matrix of this latter, we thus have

$$\Delta' = \bar{p}\Delta p, \quad \Delta = \bar{p}'\Delta' p',$$

where, as usual, $\bar{p}$ denotes the matrix obtained from p by interchange of rows and columns.

Corresponding to this transformation, we introduce new functions M_{ij}' .... These may be defined by the fact that if

$$(dx, dy, dz, dt) = p\,(dx', dy', dz', dt'),$$
$$(\delta x, \delta y, \delta z, \delta t) = p\,(\delta x', \delta y', \delta z', \delta t'),$$

then

$$\Sigma\,\{M_{23}\,(dy\delta z - dz\delta y) + M_{14}\,(dx\delta t - dt\delta x)\}$$
$$= \Sigma\,\{M_{23}'\,(dy'\delta z' - dz'\delta y') + M_{14}'\,(dx'\delta t' - dt'\delta x')\}.$$

Thus, in matrix notation

$$-\,m\,(dx, dy, dz, dt \between \delta x, \delta y, \delta z, \delta t)$$
$$= -\,m'\,(dx', dy', dz', dt')\,(\delta x', \delta y', \delta z', \delta t'),$$

leading to $\quad m' = \bar{p}mp, \quad m = \bar{p}'m'p'.$

If, as before, the determinants of p, m, Δ be ϖ, μ^2, $-\epsilon^2$, and the determinants of m', Δ' be similarly μ'^2, $-\epsilon'^2$, these equations, with $\Delta' = \bar{p}\Delta p$, give $\epsilon'^2 = \varpi^2\epsilon^2$, $\mu'^2 = \varpi^2\mu^2$ and hence, with a proper sign for ϵ', which is $(-\delta')^{\frac{1}{2}}$, we have $\epsilon'/\mu' = \epsilon/\mu$.

The matrix n is replaced after the transformation by a matrix n' connected with m' as was n with m, namely by

$$\begin{aligned}\epsilon' n' &= \mu'\Delta'\,(m')^{-1}\Delta' \\ &= \mu'\bar{p}\Delta p\,.\,p^{-1}m^{-1}\bar{p}^{-1}\,.\,\bar{p}\Delta p \\ &= \mu'\bar{p}\Delta m^{-1}\Delta p \\ &= \mu'\,\frac{\epsilon}{\mu}\,\bar{p}np,\end{aligned}$$

so that $\quad n' = \bar{p}np, \quad n = \bar{p}'n'p'.$

The matrix n is thus transformed by the same rule as was m.

We now introduce Grassman's units e_1, e_2, e_3, e_4, obeying the laws

$$e_i^2 = 0, \quad e_ie_j = -\,e_je_i,$$

so that, whatever $a, b, c, d, a', b', c', d'$ may be, we have

$$(e_1a + e_2b + e_3c + e_4d)^2 = 0,$$

$$(e_1a + e_2b + e_3c + e_4d)(e_1a' + e_2b' + e_3c' + e_4d')$$
$$= -(e_1a' + e_2b' + e_3c' + e_4d')(e_1a + e_2b + e_3c + e_4d),$$

while

$$(e_1a+e_2b+e_3c+e_4d)(e_1a'+e_2b'+e_3c'+e_4d')(e_1a''+e_2b''+e_3c''+e_4d'')$$
$$= e_2e_3e_4\Delta_1 + e_3e_1e_4\Delta_2 + e_1e_2e_4\Delta_3 + (-e_1e_2e_3)\Delta_4,$$

in which $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ denote the determinants obtained from the array

$$\left\| \begin{matrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{matrix} \right\|$$

by omitting in turn the first, second, third, fourth columns, and prefixing respectively the signs $+, -, +, -$.

We shall put

$$E_1 = e_2e_3e_4,\quad E_2 = e_3e_1e_4,\quad E_3 = e_1e_2e_4,\quad E_4 = -e_1e_2e_3,$$

and shall also introduce the four units given by

$$(e_1', e_2', e_3', e_4') = p'(e_1, e_2, e_3, e_4),$$

so that $\qquad e_1' = p_{11}'e_1 + p_{12}'e_2 + p_{13}'e_3 + p_{14}'e_4$, etc.

These equations are equivalent with

$$e_1 = p_{11}e_1' + p_{12}e_2' + p_{13}e_3' + p_{14}e_4', \text{ etc.},$$

and we may write the relations $e' = p'e$, $e = pe'$. The four combinations

$$E_1' = e_2'e_3'e_4',\quad E_2' = e_3'e_1'e_4',\quad E_3' = e_1'e_2'e_4',\quad E_4' = -e_1'e_2'e_3'$$

are then linear functions of E_1, E_2, E_3, E_4; we find in fact

$$(E_1', E_2', E_3', E_4') = \varpi^{-1}\bar{p}(E_1, E_2, E_3, E_4).$$

The laws of transformation from $(e_1, \ldots, e_4)$ to $(e_1', \ldots, e_4')$ are the same as those from $(dx, \ldots, dt)$ to $(dx', \ldots, dt')$. Thus the equations

$$m' = \bar{p}mp,\quad n' = \bar{p}np,$$

lead to $\qquad m'e'^2 = me^2,\quad n'e'^2 = ne^2,$

where as usual me^2 is the notation for the quadric form

$$m(e_1e_2e_3e_4)(e_1e_2e_3e_4), = -\Sigma M_{rs}e_re_s.$$

For instance

$$me^2 = m(e)(e) = m(pe')(pe') = mp(e') \,.\, p(e') = \bar{p} \,.\, mp(e')(e')$$
$$= m'(e')(e') = m'e'^2.$$

Next let $\xi, \eta, \ldots, \xi', \eta', \ldots$ denote respectively

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \ldots, \quad \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y'}, \ldots.$$

We then have

$$(\xi', \eta', \zeta', \tau') = \bar{p}\,(\xi, \eta, \zeta, \tau).$$

Hence
$$e'\xi', = e_1'\,\xi' + e_2'\eta' + e_3'\zeta' + e_4'\tau',$$
$$= \xi'e' = \bar{p}\xi\,.\,e' = pe'\xi = e\xi.$$

Now consider the differential equations. They are capable of very succinct expression. Put

$$M_1 = -\eta M_{34} + \zeta M_{24} - \tau M_{23}, \quad M_2 = -\zeta M_{14} + \xi M_{34} - \tau M_{31},$$
$$M_3 = -\xi M_{24} + \eta M_{14} - \tau M_{12}, \quad M_4 = \xi M_{23} + \eta M_{31} + \zeta M_{12},$$

these being the forms occurring on the left in these equations (p. 174). Denote the row of these four quantities by

$$m^{(1)} = (M_1, M_2, M_3, M_4).$$

The expression $e\xi\,.\,me^2$, whose meaning is

$$-\,(e_1\xi + e_2\eta + e_3\zeta + e_4\tau)$$
$$(e_2e_3M_{23} + e_3e_1M_{31} + e_1e_2M_{12} + e_1e_4M_{14} + e_2e_4M_{24} + e_3e_4M_{34}),$$

is at once found on evaluation to be equal to

$$E_1M_1 + E_2M_2 + E_3M_3 + E_4M_4,$$

which we can represent by $Em^{(1)}$. In other words we have

$$e\xi\,.\,me^2 = Em^{(1)}.$$

Thus the original differential equations are

$$\left.\begin{aligned} e\xi\,.\,me^2 &= \rho\,(E_1V_x + E_2V_y + E_3V_z + E_4) \\ e\xi\,.\,ne^2 &= 0 \end{aligned}\right\} \qquad \text{(II)}.$$

In this form it is easy to show that they are unaltered by the transformation, provided suitable values be adopted for the new values of ρ, V_x, V_y, V_z.

For we have shown that the operators $e\xi$, $e'\xi'$ are equal, and that me^2, ne^2 are equal respectively to $m'e'^2$, $n'e'^2$. Also that

$$(E_1', \ldots, E_4') = \frac{1}{\varpi}\,\bar{p}\,(E_1, \ldots, E_4).$$

Thus, as $e'\xi'\,.\,m'e'^2$ is identically equal to

$$E_1'M_1' + \ldots + E_4'M_4',$$

that is to
$$\frac{1}{\varpi}\,\bar{p}\,(E)\,(M') = \frac{1}{\varpi}\,E\,.\,pM',$$

we have $\quad (\varpi M_1, \varpi M_2, \varpi M_3, \varpi M_4) = p\,(M_1', M_2', M_3', M_4'),$

these being the equations (228), p. 146 of M. Th. de Donder's paper, in a form which to the present writer at least seems much more intelligible.

We shall then naturally take ρ', $V_{x'}'$, $V_{y'}'$, $V_{z'}'$ so that

$$\rho' (E_1' V_{x'}' + \ldots + E_4') = \rho (E_1 V_x + \ldots + E_4),$$

or $$\rho' (E') (V') = \rho (E) (V), \quad \frac{1}{\varpi} \rho' . \bar{p} (E) (V') = \rho (E) (V)$$

or $$\rho' (E) (pV') = \varpi\rho (E) (V),$$

namely, so that $$\varpi\rho (V) = \rho' (pV'),$$

or $$\rho' (p_{11} V_{x'}' + p_{12} V_{y'}' + p_{13} V_{z'}' + p_{14}) = \varpi\rho V_x,$$

$$\rho' (p_{41} V_{x'}' + p_{42} V_{y'}' + p_{43} V_{z'}' + p_{44}) = \varpi\rho,$$

which give, for instance,

$$V_x = (p_{11} V_{x'}' + \ldots + p_{14}) / (p_{41} V_{x'}' + \ldots + p_{44}),$$

and this, comparing with

$$\delta x / \delta t = (p_{11} \delta x' + \ldots + p_{14} \delta t') / (p_{41} \delta x' + \ldots + p_{44} \delta t'),$$

is in accordance with the view which regards V_x as a velocity.

Remark. We may evidently use Grassman's units in § 2 instead of the quaternion units, the Maxwell equations being a particular case of those considered here.

On the stability of periodic motions in general dynamics. By Professor H. F. BAKER.

[*Read* 9 February 1920.]

The question of the stability of periodic motions has been discussed by Lord Kelvin (*Coll. Papers*, IV, pp. 484–515), by Sir George Darwin in his paper on Periodic Orbits (*Acta Math.*, XXI, 1897), and by Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, I, 226, etc. The stability in question is what for distinction from the secular or final stability, we may call the conventional or instantaneous stability; the solution of a differential equation whose coefficients are periodic functions of the independent variable t being expressed by a sum of terms such as $e^{\alpha t}[\phi + \phi_1 t + \ldots + \phi_r t^r]$, where $\phi, \phi_1, \ldots, \phi_r$ are periodic functions of t, the motion associated therewith is called stable or not according to the character of the exponent α. In the present note I am concerned only with the development of a regular algebraic calculus for the determination of α. The method employed has already been applied to the differential equation

$$\frac{d^2x}{dt^2} + (n^2 + 4nH + 8n\lambda k_1 \cos 2t + 8n\lambda^2 k_2 \cos 4t + \ldots)\, x = 0,$$

which has so great an importance in Astronomical and other investigations. Here λ is a small number, so that the series multiplying x converges, and n is an integer. There is no difficulty when $n^2 + 4nH$ is not near an integer. If however H be small difficulty arises. The solution is a sum of terms of the form $e^{i(n+2q)t}\phi$, where ϕ is periodic in t, and the stability, in the sense considered, depends on the sign of the real quantity q^2. For $n = 1$, the value of q^2 is positive, and the motion stable, so long as H does not lie between the values

$$-k_1\lambda - \tfrac{1}{2}k_1^2\lambda^2 + (\tfrac{1}{4}k_1^3 - k_1k_2)\lambda^3 + \ldots,$$
$$k_1\lambda - \tfrac{1}{2}k_1^2\lambda^2 - (\tfrac{1}{4}k_1^3 - k_1k_2)\lambda^3 + \ldots,$$

which it is seen are two small values on either side of zero unless $k_1 = 0$. For $n = 2$, there is similarly stability so long as H does not lie between

$$-(\tfrac{2}{3}k_1^2 - k_2)\lambda^2 \quad\text{and}\quad (\tfrac{10}{3}k_1^2 - k_2)\lambda^2,$$

which again, generally include zero in their range (unless

$$\tfrac{2}{3}k_1^2 < k_2 < \tfrac{10}{3}k_1^2).$$

But for $n = 3$, there is stability unless H lie between

$$\tfrac{3}{4}k_1^2\lambda^2 - P\lambda^3, \ \tfrac{3}{4}k_1^2\lambda^2 + P\lambda^3, \ (P = \tfrac{9}{4}k_1^3 - 3k_1k_2 + k_3),$$

which limits do not include zero unless $k_1 = 0$. For greater values of n the range of values for H within which stability fails is that between two values of the form

$$\frac{2n}{n^2-1}k_1^2\lambda^2 + M\lambda^3, \quad \frac{2n}{n^2-1}k_1^2\lambda^2 + N\lambda^3.$$

(See *Royal Soc. Phil. Trans.*, A, CCXVI, 1916, 184.)

In the following we consider a system of equations

$$\frac{dx_r}{dt} = \frac{\partial F}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial F}{\partial x_r}, \qquad (r = 1, 2, \ldots, n),$$

wherein the function F is expressible as a convergent power series in a small parameter μ,

$$F = F_0 + \mu F_1 + \ldots,$$

whose coefficients are analytical functions in

$$x_1, \ldots, x_n, \ y_1, \ldots y_n,$$

having no singularities in the range of values here considered, and are periodic, of period 2π, in each of $y_1, \ldots, y_n$ separately. Thus F does not contain t explicitly. *It is further assumed that F_0 is a function of $x_1, \ldots, x_n$ only.* This is an important limitation, suggested by the use of Delaunay's variables in the problem of three bodies. We shall for the most part limit ourselves to the case when $n = 2$, so that there will be *four* differential equations.

If $x_1^0 \ldots x_n^0$ be initial values for $x_1 \ldots x_n$, the quantities

$$n_r = -\frac{\partial F_0}{\partial x_r},$$

for $x_s = x_s^0$, may be called the conventional mean motions. By Poincaré's theory there is a periodic motion provided certain restrictions for the initial circumstances are introduced, when the ratios $n_1 : n_2 : \ldots$ are commensurable—that is when there are $(n-1)$ identities $n_1\alpha_2 - n_2\alpha_1 = 0$, $n_1\alpha_3 - n_3\alpha_1 = 0, \ldots$ with integer coefficients $\alpha_1, \alpha_2, \ldots$. By appropriate linear transformation of the variables $x_1, x_2, \ldots, y_1, y_2, \ldots$, these conditions are reducible to the simpler forms $n_1 = 0$, $n_2 = 0, \ldots$, these being $n - 1$ in number. More precisely, noticing for the sake of comparison that, for $\mu = 0$, the differential equations are satisfied by

$$x_1 = x_1^0, \ x_2 = x_2^0, \ y_1 = n_1t + \varpi_1, \ y_2 = n_2t + \varpi_2,$$

where ϖ_1, ϖ_2 are arbitrary, it can be shown that, for small values of μ, there is a periodic solution, reducing for $t = 0$ to

$$x_1 = x_1^0 + A_1, \quad x_2 = x_2^0 + A_2, \quad y_1 = \varpi_1 + B_1, \quad y_2 = \varpi_2 + B_2,$$

where, corresponding to $n_1 = 0$, x_1^0, x_2^0 are such as to satisfy

$$\frac{\partial F_0}{\partial x_1} = 0, \text{ for } x_1 = x_1^0,\ x_2 = x_2^0,$$

ϖ_1 is such that the function $[F_1]$, of x_1^0, x_2^0, ϖ_1, obtained by picking out the part of F_1 depending on x_1, x_2, y_1 only and substituting x_1^0, x_2^0, ϖ, for x_1, x_2, y_1, has a maximum or minimum value, namely ϖ_1 is such that

$$\frac{\partial\,[F_1]}{\partial \varpi_1} = 0,$$

and A_1, A_2, B_1 are determined, namely by linear equations whose determinant is

$$\begin{vmatrix} \dfrac{\partial^2 F_0}{\partial x_1^{02}}, & \dfrac{\partial^2 F_0}{\partial x_1^0 \partial x_2^0} \\ \dfrac{\partial^2 F_0}{\partial x_1^0 \partial x_2^0}, & \dfrac{\partial^2 F_0}{\partial x_2^{02}} \end{vmatrix} \frac{\partial^2\,[F_1]}{\partial \varpi_1^2},$$

so that neither of these factors may be zero, while $\varpi_2 + B_2$ is arbitrary. In other words, the origin of time-reckoning is arbitrary, and either the initial value of x_2 or its corresponding mean motion.

If $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, $y_1 = \psi_1(t)$, $y_2 = \psi_2(t)$ be the equations of such a periodic motion, and we substitute in the differential equations

$$\frac{dx_r}{dt} = \frac{\partial F}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial F}{\partial x_r},$$

supposing $$F = F_0 + \mu F_1,$$

values $$x_r = \phi_r(t) + \xi_r, \quad y_r = \psi_r(t) + \eta_r,$$

and retain only first powers of the small increments ξ_r, η_r, we have a system of linear differential equations with periodic coefficients (of period $2\pi/n_2$)

$$\frac{d\xi_1}{dt} = (y_1, x_1)\,\xi_1 + (y_1, x_2)\,\xi_2 + (y_1, y_1)\,\eta_1 + (y_1, y_2)\,\eta_2,$$

$$\frac{d\xi_2}{dt} = (y_2, x_1)\,\xi_1 + (y_2, x_2)\,\xi_2 + (y_2, y_1)\,\eta_1 + (y_2, y_2)\,\eta_2,$$

$$\frac{d\eta_1}{dt} = -(x_1, x_1)\,\xi_1 - (x_1, x_2)\,\xi_2 - (x_1, y_1)\,\eta_1 - (x_1, y_2)\,\eta_2,$$

$$\frac{d\eta_2}{dt} = -(x_2, x_1)\,\xi_1 - (x_2, x_2)\,\xi_2 - (x_2, y_1)\,\eta_1 - (x_2, y_2)\,\eta_2,$$

where (y_r, x_s) denotes the value of $\dfrac{\partial^2 F}{\partial y_r \partial x_s}$ for

$$x_r = \phi_r(t), \quad y_r = \psi_r(t);$$

these values of x_r, y_r are of the forms $x_r^0 + \mu X_r$, $n_r t + \varpi_r + \mu Y_r$. Since F_0 does not contain y_1, y_2 all of the sixteen coefficients are of the form μU, where U is periodic in t, having μ as a factor, except the three

$$(x_1, x_1),\quad (x_1, x_2),\quad (x_2, x_2);$$

any one of these, say (x_r, x_s), is of the form $(x_r^0, x_s^0) + \mu\{x_r, x_s\}$, where the notation $\{\ \}$ must be observed. The matrix of four rows and columns formed by the sixteen coefficients on the right, if we retain only to the first power of μ, may then be represented by

$$\begin{pmatrix} 0\ , & 0 \\ -H_0, & 0 \end{pmatrix} + \mu \begin{pmatrix} (y, x), & (y, y) \\ -\{x, x\}, & -(x, y) \end{pmatrix},$$

where the 0 denotes a matrix of two rows and columns whose elements are all zero, H_0 denotes the value for $x_1 = x_1^0$, $x_2 = x_2^0$, of the matrix

$$H_0 = \begin{vmatrix} \dfrac{\partial^2 F_0}{\partial x_1^2}, & \dfrac{\partial^2 F_0}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 F_0}{\partial x_2 \partial x_1}, & \dfrac{\partial^2 F_0}{\partial x_2^2} \end{vmatrix},$$

(y, x) denotes for $x_r = x_r^0$, $y_r = n_r t + \varpi_r$, the matrix

$$(y, x) = \begin{vmatrix} \dfrac{\partial^2 F_1}{\partial y_1 \partial x_1}, & \dfrac{\partial^2 F_1}{\partial y_1 \partial x_2} \\ \dfrac{\partial^2 F_1}{\partial y_2 \partial x_1}, & \dfrac{\partial^2 F_1}{\partial y_2 \partial x_2} \end{vmatrix},$$

with similar meanings for (y, y) and (x, y), but $\{x, x\}$ is such that $\mu\{x, x\}$

$$= \begin{vmatrix} \dfrac{\partial^2 F_0}{\partial x_1^2} - \left(\dfrac{\partial^2 F_0}{\partial x_1^2}\right), & \dfrac{\partial^2 F_0}{\partial x_1 \partial x_2} - \left(\dfrac{\partial^2 F_0}{\partial x_1 \partial x_2}\right) \\ \dfrac{\partial^2 F_0}{\partial x_1 \partial x_2} - \left(\dfrac{\partial^2 F_0}{\partial x_1 \partial x_2}\right), & \dfrac{\partial^2 F_0}{\partial x_2^2} - \left(\dfrac{\partial^2 F_0}{\partial x_2^2}\right) \end{vmatrix} + \mu \begin{vmatrix} \dfrac{\partial^2 F_1}{\partial x_1^2}, & \dfrac{\partial^2 F_1}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 F_1}{\partial x_1 \partial x_2}, & \dfrac{\partial^2 F_1}{\partial x_2^2} \end{vmatrix},$$

wherein $\left(\dfrac{\partial^2 F_0}{\partial x_1^2}\right)$ is the value of $\dfrac{\partial^2 F_0}{\partial x_1^2}$ for $x_1 = x_1^0$, $x_2 = x_2^0$, etc., and in the first matrix we are to substitute $x_r = x_r^0 + \mu X_r$, and retain only to the first power of μ, while in the second matrix we are to put x_r^0, $n_r t + \varpi_r$ for x_r, y_r.

If the matrix of coefficients as so explained be denoted by $u + \mu v$ we require, to solve these equations, to compute what, in the notation of the paper referred to, is denoted by $\Omega(u + \mu v)$, which by a theorem there given (p. 159) is equal to

$$\Omega(u)\,\Omega[(\Omega(u))^{-1} \mu v\, \Omega(u)].$$

In case, as here,
$$u = \begin{pmatrix} 0\ , & 0 \\ -H_0, & 0 \end{pmatrix},$$

we have $u^2 = 0$, and hence

$$\Omega(u) = 1 + \begin{pmatrix} 0, & 0 \\ -H_0 t, & 0 \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix},$$

whose inverse is
$$\begin{pmatrix} 1, & 0 \\ tH_0, & 1 \end{pmatrix}.$$

Thus we require, to first power of μ,

$$\begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} \Omega \left[\mu \begin{pmatrix} 1, & 0 \\ tH_0, & 1 \end{pmatrix} \begin{pmatrix} (y, x), & (y, y) \\ -\{x, x\}, & -(x, y) \end{pmatrix} \begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} \right].$$

or
$$\begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} \Omega \left[\mu \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \right],$$

where
$$\begin{aligned} A &= (y, x) - t\,(y, y)\, H_0, \\ B &= (y, y), \\ C &= -\{x, x\} + t\,[H_0\,(y, x) + (x, y)\, H_0] - t^2 H_0\,(y, y)\, H_0, \\ D &= -(x, y) + tH_0\,(y, y). \end{aligned}$$

Here the matrizant Ω, to first power of μ, is given by the definition

$$\Omega \left\{ \mu \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \right\} = 1 + \mu \begin{pmatrix} QA, & QB \\ QC, & QD \end{pmatrix},$$

and the product is

$$\begin{pmatrix} 1, & 0 \\ -tH_0, & 1 \end{pmatrix} + \mu \begin{pmatrix} QA & , & QB \\ -tH_0 \,.\, QA + QC, & & -tH_0 \,.\, QB + QD \end{pmatrix},$$

that is
$$\begin{pmatrix} 1 + \mu QA & , & \mu QB \\ -tH_0\,(1 + \mu QA) + QC, & & 1 - \mu t H_0 \,.\, QB + \mu QD \end{pmatrix}.$$

In the paper referred to the rule reached for determining the characteristic exponents was (p. 163) to pick out the coefficient of t in the matrix such as this, as it occurs outside trigonometrical signs, then put $t = 0$, and from this form a determinantal equation. That rule was founded on the assumption (p. 162) that a certain matrix, $\Omega_0^{\omega}(u)$, had, in case of equal roots, only linear invariant factors. It will be proved below that for the application of the rule the assumption is unnecessary; this is important in the present case since two of the characteristic exponents are necessarily zero, as we shall see. Of the quantities A, B, C, D, which are given explicitly above, only parts are material for the result. Denote then provisionally the coefficient of t in the matrix last written by

$$\begin{pmatrix} \mu\alpha & , & \mu\beta \\ -H_0 + \mu\gamma, & & \mu\delta \end{pmatrix},$$

where under the trigonometrical signs t is put zero; the determinantal equation is then

$$\begin{vmatrix} \mu\alpha - \rho & , & \mu\beta \\ -H_0 + \mu\gamma, & & \mu\delta - \rho \end{vmatrix} = 0.$$

As we have computed A, B, C, D only to the first power of μ, we retain only to first power of μ in this determinant. It is convenient to multiply by the determinant

$$\begin{vmatrix} -\rho, & 0 \\ H_0, & -\rho \end{vmatrix},$$

which for n variables x and n variables y would be equal to ρ^{2n}, and for $n=2$ is equal to ρ^4. Then the above determinant becomes

$$\begin{vmatrix} -\rho\,(\mu\alpha-\rho), & -\rho\mu\beta \\ \mu\,(H_0\alpha-\rho\gamma), & \rho^2-\mu\rho\delta+\mu H_0\beta \end{vmatrix},$$

and the evaluation to the first power of μ is extremely simple, namely in general it is

$$(\rho^2)^{2n}+(\rho^2)^{2n-1}\mu M,$$

where M is the sum of the diagonal elements of the two matrices

$$-\rho\alpha,\quad -\rho\delta+H_0\beta,$$

which in general are each of n rows and columns.

Removing the factor ρ^{2n}, which was introduced, and remembering that the roots ρ are in pairs, equal, and opposite in sign (cf. p. 165 of the paper referred to), we infer that the sum of the diagonal elements of α and δ is zero, and to our approximation the equation reduces to

$$\rho^{2n}+\rho^{2n-2}\mu\Sigma\,(H_0\beta)_{ii}=0$$

which however, as was remarked above, divides by ρ^2. Thus we see that for $n=2$, ρ^2/μ is developable as a power series in μ, and further that the sign of ρ^2 is that of the negative of the sum of the diagonal elements of the matrix, of two rows and columns, expressed by $H_0\beta$.

Now consider the meaning of β. We have

$$B=\begin{pmatrix} \dfrac{\partial^2 F_1}{\partial y_1^{\,2}}, & \dfrac{\partial^2 F_1}{\partial y_1\partial y_2} \\ \dfrac{\partial^2 F_1}{\partial y_1\partial y_2}, & \dfrac{\partial^2 F_1}{\partial y_2^{\,2}} \end{pmatrix}$$

where we are to put

$$x_1=x_1^{\,0},\quad y_1=y_1^{\,0},\quad y_1=n_1t+\varpi_1,\quad y_2=n_2t+\varpi_2,$$

and, as was said, we have $n_1=0$. In general F_1 has a form

$$F_1=A_{00}+\Sigma A_{p_1p_2}\cos\,(p_1y_1+p_2y_2+h),$$

wherein A_{00}, $A_{p_1p_2}$, h are functions of x_1, x_2. Let $[F_1]$ be the part of F_1 which is a function of y_1 only,

$$[F_1]=A_{00}+\Sigma A_{p_1,0}\cos\,(p_1y_1+h),$$

which for $y_1 = \varpi_1$, $y_2 = n_2 t + \varpi_2$, does not contain t, and let F_1' be the remaining part of F_1.

We have
$$\frac{\partial^2 F_1}{\partial y_r \partial y_s} = \frac{\partial^2 [F_1]}{\partial y_r \partial y_s} + \frac{\partial^2 F_1'}{\partial y_r \partial y_s},$$
and $\dfrac{\partial^2 [F_1]}{\partial y_r \partial y_s}$ vanishes unless $r = s = 1$, while $Q \dfrac{\partial^2 F'}{\partial y_r \partial y_s}$ is an entirely periodic function not having the factor t outside trigonometrical terms. Thus in QB the only term having the factor t arises from $Q \dfrac{\partial^2 [F_1]}{\partial y_1^2}$; this is
$$-Q\Sigma A_{p_1, 0} p_1^2 \cos(p_1 \varpi_1 + h) = -t\Sigma A_{p_1, 0} p_1^2 \cos(p_1\varpi_1 + h) = t \frac{\partial^2 [F_1]}{\partial \varpi_1^2}$$
so that
$$\beta = \begin{vmatrix} \dfrac{\partial^2 [F_1]}{\partial \varpi_1^2}, & 0 \\ 0 \quad , & 0 \end{vmatrix},$$
and
$$\Sigma (H_0 \beta)_{ii} = \frac{\partial^2 F_0}{\partial x_1^{02}} \frac{\partial^2 [F_1]}{\partial \varpi_1^2},$$
which gives, for the characteristic exponents which do not vanish,
$$\rho = \pm \sqrt{\mu} \left\{ - \frac{\partial^2 F_0}{\partial x_1^{02}} \cdot \frac{\partial^2 [F_1]}{\partial \varpi_1^2} \right\}^{\frac{1}{2}}.$$
And it will be recalled that
$$\frac{\partial F_0}{\partial x_1^0} = 0, \quad \frac{\partial [F_1]}{\partial \varpi_1} = 0.$$

In order to obtain the corresponding expression when there are three variables x, and three variables y, it is necessary to carry the approximations as far as μ^2. Poincaré's corresponding work is in *Méth. Nouv.*, I, pp. 201–226.

Note. It has been remarked that two of the characteristic exponents are zero, and that the condition that, in case of equal roots, the matrix $\Omega_0^\omega(u)$ should have linear invariant factors, is not necessary for the application of the rule above. In fact as t does not occur explicitly in F, the existence of a periodic solution $x_r = \phi_r(t)$, $y_r = \psi_r(t)$ of the original equations, involves the existence also of a solution $x_r = \phi_r(t+h)$, $y_r = \psi_r(t+h)$, in which h is arbitrary. This involves the existence of a solution
$$\xi_r = \frac{\partial \phi_r(t+h)}{\partial h}, \quad \eta_r = \frac{\partial \psi_r(t+h)}{\partial h}$$
of the equations of variation, where after differentiation h is to be put zero. This is however a purely periodic solution with zero characteristic exponent. The nature of the equations of variation

however involves that the characteristic exponents are in pairs of equal values of opposite sign. There is therefore also another solution of vanishing characteristic exponent.

Now suppose that in the argument of p. 162 of the paper referred to the matrix $\Omega_0^\omega(u)$ has not linear invariant factors. It will be sufficient to take one case, and to suppose that

$$\Omega_0^\omega(u) = k \begin{vmatrix} e^{ic_1\omega}, & \cdot\,, & \cdot \\ \cdot\,, & e^{ic_2\omega}, & i\omega e^{ic_2\omega} \\ \cdot\,, & \cdot\,, & e^{ic_2\omega} \end{vmatrix} k^{-1}.$$

Noticing then, multiplying the matrices, that

$$\begin{vmatrix} X^{-1}, & \cdot\,, & \cdot \\ \cdot\,, & Y^{-1}, & -itY^{-1} \\ \cdot\,, & \cdot\,, & Y^{-1} \end{vmatrix} \begin{vmatrix} X', & \cdot\,, & \cdot \\ \cdot\,, & Y', & it'Y' \\ \cdot\,, & \cdot\,, & Y' \end{vmatrix}$$
$$= \begin{vmatrix} X'X^{-1}, & \cdot\,, & \cdot \\ \cdot\,, & Y'Y^{-1}, & i(t'-t)\,Y'Y^{-1} \\ \cdot\,, & \cdot\,, & Y'Y^{-1} \end{vmatrix},$$

of which a particular case is for $X' = X$, $Y' = Y$, $t' = t$, we have

$$\Omega_0^{\omega+t}(u) = \Omega_\omega^{\omega+t}(u)\,\Omega_0^\omega(u) = \Omega_0^t(u)\,\Omega_0^\omega(u),$$

because u is periodic, and hence, by the form assumed for $\Omega_0^\omega(u)$,

$$\Omega_0^{\omega+t}(u)\,.\,k \begin{vmatrix} e^{-ic_1(\omega+t)}, & \cdot\,, & \cdot \\ \cdot\,, & e^{-ic_2(\omega+t)}, & -i(\omega+t)\,e^{-ic_2(\omega+t)} \\ \cdot\,, & \cdot\,, & e^{-ic_2(\omega+t)} \end{vmatrix}$$

is equal to

$$\Omega_0^t(u)\,k \begin{vmatrix} e^{-ic_1t}, & \cdot\,, & \cdot \\ \cdot\,, & e^{-ic_2t}, & -ite^{-ic_2t} \\ \cdot\,, & \cdot\,, & e^{-ic_2t} \end{vmatrix},$$

showing that the matrix

$$Q_0^t(u) = \Omega_0^t(u)\,k \begin{vmatrix} e^{-ic_1t}, & \cdot\,, & \cdot \\ \cdot\,, & e^{-ic_2t}, & -ite^{-ic_2t} \\ \cdot\,, & \cdot\,, & e^{-ic_2t} \end{vmatrix} k^{-1}$$

has the period ω, and is therefore such that

$$Q_0^\omega(u) = Q_0^0(u) = 1.$$

This gives

$$\Omega_0^t(u) = Q_0^t(u)\, k \begin{vmatrix} e^{ic_1t}, & . , & . \\ . , & e^{ic_2t}, & ite^{ic_2t} \\ . , & . , & e^{ic_2t} \end{vmatrix} k^{-1}.$$

In this the terms involving t outside trigonometrical signs are

$$iQ_0^t(u)\, k \begin{vmatrix} c_1, & . , & . \\ . , & c_2, & 1 \\ . , & . , & c_2 \end{vmatrix} k^{-1},$$

and putting therein $t = 0$, we have the determinantal equation for the characteristic exponents

$$\begin{vmatrix} ic_1 - \rho, & . , & . \\ . , & ic_2 - \rho, & i \\ . , & . , & ic_2 \end{vmatrix} = 0.$$

The form of this equation establishes the result in question.

If we put $Q_0^t(u)\, k = \Phi(t)$, and, denoting the initial values of the dependent variables by x_1^0, x_2^0, x_3^0, put

$$k^{-1}(x_1^0, x_2^0, x_3^0) = (z_1^0, z_2^0, z_3^0),$$

the general solution $\Omega_0^t(u) \, . \, (x_1^0, x_2^0, x_3^0)$ becomes

$$\Phi(t) \, . \begin{vmatrix} e^{ic_1t}, & . , & . \\ . , & e^{ic_2t}, & ite^{ic_2t} \\ . , & . , & e^{ic_2t} \end{vmatrix} (z_1^0, z_2^0, z_3^0),$$

or

$$\begin{vmatrix} \Phi_{11}e^{ic_1t}, & \Phi_{12}e^{ic_2t}, & \Phi_{12}ite^{ic_2t} + \Phi_{13}e^{ic_2t} \\ \Phi_{21}e^{ic_1t}, & \Phi_{22}e^{ic_2t}, & \Phi_{22}ite^{ic_2t} + \Phi_{23}e^{ic_2t} \\ \Phi_{31}'e^{ic_1t}, & \Phi_{32}e^{ic_2t}, & \Phi_{32}ite^{ic_2t} + \Phi_{33}e^{ic_2t} \end{vmatrix} (z_1^0, z_2^0, z_3^0),$$

so that

$$\begin{aligned} x_1 &= z_1^0 e^{ic_1t}\Phi_{11} + z_2^0 e^{ic_2t}\Phi_{12} + z_3^0 e^{ic_2t}(\Phi_{13} + it\Phi_{12}), \\ x_2 &= z_1^0 e^{ic_1t}\Phi_{21} + z_2^0 e^{ic_2t}\Phi_{22} + z_3^0 e^{ic_2t}(\Phi_{23} + it\Phi_{22}), \\ x_3 &= z_1^0 e^{ic_1t}\Phi_{31} + z_2^0 e^{ic_2t}\Phi_{32} + z_3^0 e^{ic_2t}(\Phi_{33} + it\Phi_{32}), \end{aligned}$$

where the Φ's are periodic functions, and z_1^0, z_2^0, z_3^0 are arbitrary constants of integration.

On the stability of rotating liquid ellipsoids. By Professor H. F. BAKER.

[*Read* 9 February 1920.]

The present note deals only with the paedagogic problem of reducing the *algebraic* treatment of the stability of rotating ellipsoids and spheroids, for ellipsoidal or spheroidal displacements only, to the simplest possible terms. It is a modification of what is given on pp. 69–71 of Mr Hargreaves paper on the Domains of steady motion for a liquid ellipsoid, and the oscillations of the Jacobian figure, *Camb. Phil. Trans.*, XXII, 1914. Mr Hargreaves refers to the paper of C. O. Meyer, *Crelle*, XXIV, 1842, for one of the identities he uses, but does not do himself justice in that he refrains from pointing out that Meyer's work is vitiated in an important point by a mistake of algebra.

§ 1. Supposing a, b, c to be real positive quantities in descending order of magnitude, put

$$f(x) = (x+a)(x+b)(x+c),\quad f'(x) = \frac{df(x)}{dx},\quad y^2 = f(x);$$

then we can verify without difficulty that

$$\int_0^\infty \frac{x\,dx}{y^3} = \frac{3}{4}\int_0^\infty \frac{x^2 f'(x)}{y^5}\,dx,\qquad \int_0^\infty \frac{x^2 dx}{y^3} = \int_0^\infty \frac{x^2\,[3f(x) - xf'(x)]}{y^5}\,dx.$$

Next put, y being throughout taken positive,

$$\phi = \int_0^\infty \frac{dx}{y},\quad a+b = \frac{1}{h},\quad abc = p,$$

so that

$$f(x) = (x+c)\left(x^2 + \frac{x}{h} + \frac{p}{c}\right) = x^3 + x^2\left(c + \frac{1}{h}\right) + x\left(\frac{p}{c} + \frac{c}{h}\right) + p;$$

in this regard p as a constant, so that $f(x)$ is regarded as depending on the two variable quantities h and c; so considered, denote it by P, and write

$$P_1 = \frac{\partial P}{\partial h},\quad P_2 = \frac{\partial P}{\partial c},\quad P_{11} = \frac{\partial^2 P}{\partial h^2},\text{ etc.}$$

and correspondingly write

$$\phi_1 = \frac{\partial \phi}{\partial h},\quad \phi_{12} = \frac{\partial^2 \phi}{\partial h \partial c},\text{ etc.}$$

Then if ξ, η be arbitrary we have

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2 = -\frac{1}{2}\int_0^\infty \frac{P_{11}\xi^2 + 2P_{12}\xi\eta + P_{22}\eta^2}{y^3}\,dx + \frac{3}{4}\int_0^\infty \frac{(P_1\xi + P_2\eta)^2}{y^5}\,dx;$$

substituting the forms of P_{11}, etc., and utilising the two identities remarked at starting which here take the forms

$$\int_0^\infty \frac{x\,dx}{y^3} = \frac{3}{4}\int_0^\infty x^2\left[3x^2 + 2x\left(c + \frac{1}{h}\right) + \frac{p}{c} + \frac{c}{h}\right]\frac{dx}{y^5},$$

$$\int_0^\infty \frac{x^2dx}{y^3} = \int_0^\infty x^2\left[x^2\left(c + \frac{1}{h}\right) + 2x\left(\frac{p}{c} + \frac{c}{h}\right) + 3p\right]\frac{dx}{y^5},$$

we have

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2 = -\int_0^\infty x^2Q\,\frac{dx}{y^5},$$

where $$Q = Ux^2 + 2Vx + W,$$

in which, if we put

$$t = \frac{ph}{c^2}, \quad u = ch, \quad \xi_1 = c\xi, \quad \eta_1 = h\eta,$$

the values of U, V, W are given by

$$\frac{4h^4}{3}Uc^2 = (\xi_1^2 - \xi_1\eta_1 + \eta_1^2) + \tfrac{1}{3}\xi_1^2(10u + 1) + \eta_1^2u(3t - u - 1),$$

$$\frac{4h^4}{3}Vc = t(\xi_1^2 - \xi_1\eta_1 + \eta_1^2) + \tfrac{1}{3}\xi_1^2(3u + t + 4) + \eta_1^2u(2t - 1),$$

$$\frac{4h^4}{3}W = (3t - 1)(\xi_1^2 - \xi_1\eta_1 + \eta_1^2) + \xi_1^2(2t + 1).$$

Hence as c, h, p are positive, if ξ, η *be any real quantities*, each of U, V, W is necessarily positive provided $2t > 1$. For this, being

$$\frac{2}{c} > \frac{1}{a} + \frac{1}{b}, \quad \text{or } c < 2ab/(a + b),$$

involves, because of $4a^2b^2/(a + b)^2 < ab$, also $c^2 < ab$, and hence

$$t > u, \quad 3t - u - 1 = t - u + 2t - 1 > 0.$$

Thus for all real values of ξ, η, the quadratic form in ξ, η denoted by Q is necessarily positive, and the form

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2$$

necessarily negative, the sign of y being taken positive as stated.

§ 2. Now consider a mass of homogeneous (incompressible) fluid, of mass M, whose surface has the equation

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

rotating with angular velocity ω about the axis of z. Let I, μ, $-W$ be respectively the moment of inertia, moment of momentum and potential energy of gravitation and

$$H = \frac{\mu^2}{2I} - W;$$

we have

$$I = \frac{M}{5}(a+b), \quad \mu = \frac{M}{5}(a+b)\,\omega, \quad W = \frac{3}{10} M^2\phi,$$

and if we put

$$K = \frac{10}{3M^2} \cdot H, \quad \sigma = \frac{25}{3M^3}\mu^2, \quad h = \frac{1}{a+b},$$

we have
$$K = \sigma h - \phi.$$

We are concerned to make H a minimum when μ and M, and therefore also p, $= abc$, are retained constant, and hence to make K a minimum regarded as a function of h and c, retaining σ as constant.

The necessary conditions of rotational equilibrium

$$\frac{\partial K}{\partial h} = 0, \quad \frac{\partial K}{\partial c} = 0,$$

give
$$\phi_1 = c, \quad \phi_2 = 0,$$

and the condition of stability is that

$$\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2$$

should be negative for all real values of ξ, η.

We have

$$\phi_2 = -\frac{1}{2}\int_0^\infty \frac{x\,dx}{y^3}\left(x + \frac{1}{h} - \frac{p}{c^2}\right),$$

and, in the familiar way, this vanishes for a proper value of p, which is such that

$$p > \frac{c^2}{h}.$$

As this involves also $p > \frac{1}{2}c^2/h$, or, in a previous notation $2t > 1$, we see from § 1 that *the condition of stability is satisfied for the ellipsoidal forms of rotational equilibrium when ellipsoidal displacements are considered.*

§ 3. Consider a series of ellipsoidal shapes satisfying the conditions $\phi_1 = \sigma$, $\phi_2 = 0$, for all of which abc is the same. By $\phi_2 = 0$ we can then regard c as a function of h, subject to

$$\frac{dc}{dh} = -\frac{\phi_{12}}{\phi_{22}}.$$

For increments δh, δc, p being constant, the increment of K, to the second order, with constant σ, is then

$$\begin{aligned}\delta K &= \sigma\delta h - (\phi_1\delta h + \phi_2\delta c) - \tfrac{1}{2}(\phi_{11}\delta h^2 + 2\phi_{12}\delta h\delta c + \phi_{22}\delta c^2)\\ &= -\tfrac{1}{2}\left(\phi_{11} - \frac{\phi_{12}^2}{\phi_{22}}\right)\delta h^2,\end{aligned}$$

while $$\delta\sigma = \phi_{11}\delta h + \phi_{12}\delta c = \left(\phi_{11} - \frac{\phi_{12}^2}{\phi_{22}}\right)\delta h,$$

and, from $\omega^2 = 3M\sigma h^2 = 3Mh^2\phi_1$, we have

$$\begin{aligned}\delta\left(\frac{\omega^2}{3M}\right) &= (\phi_{11}\delta h + \phi_{12}\delta c)\,h^2 + 2\phi_1 h\delta h\\ &= (\phi_{11}h^2 + 2\phi_1 h)\,\delta h - \frac{\phi_{12}^2}{\phi_{22}}\,h^2\delta h.\end{aligned}$$

But, from § 1 above, we have

$$\phi_{22} < 0,\quad \phi_{11}\phi_{22} - \phi_{12}^2 > 0,$$

while

$$\phi_{11}h^2 + 2\phi_1 h = h^2\left[-\frac{1}{2}\int_0^\infty \frac{P_{11}}{y^3}\,dx + \frac{3}{4}\int_0^\infty \frac{P_1^2}{y^5}\,dx\right] - h\int_0^\infty \frac{P_1}{y^3}\,dx,$$

wherein $$P_{11} = 2\,\frac{x^2 + xc}{h^3},\quad P_1 = -\frac{x^2 + xc}{h^2} = -\frac{1}{2}\,hP_{11},$$

so that $$\phi_{11}h^2 + 2\phi_1 h, = \frac{3}{4}\,h^2\int_0^\infty \frac{P_1^2}{y^5}\,dx,$$

is necessarily positive.

Thus we see that, *as h diminishes, for a series of shapes of rotational equilibrium, K constantly increases, σ also increases, but ω diminishes.*

§ 4. We can however show that as $a - b$ increases numerically, h does diminish, that is that $a + b$ increases. [The numerical tables show that along the series of ellipsoids a increases, but b and c both diminish.]

For

$$\tfrac{1}{2}\delta\,[(a-b)^2] = \tfrac{1}{2}\delta\left[(a+b)^2 - 4\,\frac{p}{c}\right] = -\frac{1}{h^3}\,\delta h + \frac{2p}{c^2}\,\delta c,$$

while, along the series considered, $\delta c = -\phi_{12}\delta h/\phi_{22}$; so that

$$\tfrac{1}{2}\delta\,[(a-b)^2] = -\left[\frac{\phi_{22}}{h^3} + \frac{2p}{c^2}\,\phi_{12}\right]\frac{\delta h}{\phi_{22}}.$$

We proved however (§ 1) that

$$-(\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2) = P\,(c^2\xi^2 - ch\xi\eta + h^2\eta^2) + Q\xi^2 + R\eta^2,$$

wherein P, Q, R are positive, and ξ, η are arbitrary. This gives

$$-(\phi_{22}\eta + \phi_{12}\xi) = P\,(-\tfrac{1}{2}ch\xi + h^2\eta) + R\eta;$$

hence, replacing ξ, η respectively by $2p/c^2$ and $1/h^3$, we infer

$$-\left(\frac{\phi_{22}}{h^3} + \frac{2p}{c^2}\,\phi_{12}\right) = P\left(-\frac{ph}{c} + \frac{1}{h}\right) + \frac{R}{h^3},$$

wherein $$\frac{1}{h} - \frac{ph}{c},\ = a + b - \frac{ab}{a+b},$$

is necessarily positive.

Thus, as ϕ_{22} is negative, the equation

$$\delta h = -\tfrac{1}{2}\,\phi_{22}\left(\frac{\phi_{22}}{h^3} + \frac{2p}{c^2}\,\phi_{12}\right)^{-1}\delta\,[(a-b)^2]$$

shows that δh has a sign opposite to that of $\delta\,[(a-b)^2]$.

We have thus proved that *as the axes $2\sqrt{a}$, $2\sqrt{b}$ become more unequal, the energy H, the moment of inertia $\frac{1}{5}M/h$, and the angular momentum μ, all constantly increase, while the angular velocity ω constantly diminishes.*

In Mr Hargreaves' notation (*Camb. Phil. Trans.*, XXII, 1914, 61), if momentarily m be used for the whole mass, instead of M,

$$\frac{L}{3m/2} = -h^2\phi_{11},\quad \frac{M}{3m/2} = -c^2\phi_{22},\quad \frac{N}{3m/2} = ch\phi_{12}.$$

He obtains $L > 2N$, $M > N$. The preceding work establishes $M > 2N$.

§ 5. For the series of *spheroids* of rotational equilibrium, the variables h, c are not appropriate, for a reason which will appear. Writing $a + x$, $b + x$ respectively for a, b and $h + \xi$, $c + \eta$ for h, c, we have at once, x, y being small,

$$\xi = -h^2\,(x+y) + h^3\,(x+y)^2 + \text{etc.},$$

$$\eta = -c\left(\frac{x}{a} + \frac{y}{b}\right) + c\left(\frac{x^2}{a^2} + \frac{xy}{ab} + \frac{y^2}{b^2}\right) + \text{etc.},$$

and hence

$$-\sigma\xi + \phi_1\xi + \phi_2\eta + \tfrac{1}{2}(\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2) = \Psi_1 + \Psi_2 + \text{etc.},$$

where $$\Psi_1 = (\sigma - \phi_1)\, h^2\, (x + y) - \phi_2 c \left(\frac{x}{a} + \frac{y}{b}\right),$$
and
$$\Psi_2 = (-\sigma + \phi_1)\, h^3\, (x + y)^2 + \phi_2 c \left(\frac{x^2}{a^2} + \frac{xy}{ab} + \frac{y^2}{b^2}\right)$$
$$+ \tfrac{1}{2}\left[\phi_{11} h^4 (x + y)^2 + 2\phi_{12} c h^2 (x + y)\left(\frac{x}{a} + \frac{y}{b}\right) + \phi_{22} c^2 \left(\frac{x}{a} + \frac{y}{b}\right)^2\right].$$

If herein we put $a = b$ we obtain, for an arbitrary ellipsoidal variation from the spheroid, to terms of the second order, of the function $\psi = -\sigma h + \phi$, which is a constant negative multiple of the energy H, the following
$$\delta\psi = \left[(\sigma - \phi_1)\, h^2 - \phi_2 \frac{c}{a}\right](x + y)$$
$$+ (-\sigma + \phi_1)\, h^3 (x + y)^2 + \frac{c\phi_2}{4a^2}\left[3 (x + y)^2 + (x - y)^2\right]$$
$$+ \tfrac{1}{2} (x + y)^2 \left[\phi_{11} h^4 + \frac{2ch^2}{a}\phi_{12} + \frac{c^2}{a^2}\phi_{22}\right].$$

Thus we obtain, as determining the angular velocity or momentum necessary for the rotational equilibrium of the spheroid, the single equation
$$\sigma = \phi_1 + \phi_2 \frac{c}{ah^2}$$
(explaining the reason for the change of variables), and thence
$$\delta\psi = \frac{c\phi_2}{4a^2}(x - y)^2$$
$$+ \left[\tfrac{1}{2}\phi_{11} h^4 + \phi_{12}\frac{ch^2}{a} + \tfrac{1}{2}\phi_{22}\frac{c^2}{a^2} + \frac{c}{4a^2}\phi_2\right](x + y)^2.$$

We have however shown that
$$\tfrac{1}{2}\phi_{11} h^4 + \phi_{12} h^2 \frac{c}{a} + \tfrac{1}{2}\phi_{22}\frac{c^2}{a^2}$$
is negative so long as $\frac{2}{c} > \frac{1}{a} + \frac{1}{b}$, and hence, for $a = b$, so long as $a > c$. For the value of ϕ_2, putting $a = b$, $a = c \sec^2 \alpha$, $t = \tan \alpha$, we easily compute
$$\phi_2 = \frac{1}{8t^4 c^{\frac{3}{2}}}\left\{\frac{\alpha}{t}(3 + 14t^2 + 3t^4) - (3 + 13t^2)\right\},$$
which is negative for values of α ranging from $\alpha = 0$, when the spheroid is a sphere, to the value for which

$$\frac{a}{t}(3 + 14t^2 + 3t^4) - (3 + 13t^2) = 0$$

(about 54° 22′, corresponding to a meridian eccentricity $e = \cdot 8127$, and a value of $\frac{\omega^2}{2\pi\rho}$ equal to $\cdot 187$). For this value $\phi_2 = 0$, and, by what we have previously shown, the spheroid is a particular one of the ellipsoids of rotational equilibrium.

Thus $\delta\psi$ is negative, the proof being valid for the spheroid for which $\phi_2 = 0$, and *the spheroids lying between the sphere and this ellipsoid are therefore stable for ellipsoidal displacements.*

§ 6. The coefficient of $(x + y)^2$ in the above expression for $\delta\psi$ remains negative however even after α has passed the value (about 54° 22′) for which $\phi_2 = 0$, and ϕ_2 has become positive, indeed up to $\alpha = \frac{\pi}{2}$. For from

$$\sigma = \phi_1 + \phi_2 \frac{c}{ah^2},$$

using $c = 4ph^2$, and hence $\delta c = 8ph\,\delta h$, we get

$$\delta\sigma = \frac{2\delta h}{h^4}\left[\tfrac{1}{2}\phi_{11}h^4 + \phi_{12}h^2\frac{c}{a} + \tfrac{1}{2}\phi_{22}\frac{c^2}{a^2} + \frac{c}{4a^2}\phi_2\right],$$

leading, for $a = b$, to

$$\delta\psi = \frac{(x-y)^2}{4}\frac{c\phi_2}{a^2} - \frac{(x+y)^2}{16a^2}\frac{d\sigma}{da};$$

on the other hand $\sigma = \phi_1 + \phi_2 \dfrac{c}{ah^2}$

gives $\sigma = 2a(a - c)\displaystyle\int_0^\infty \frac{x\,dx}{(x+a)^2(x+c)^{\frac{3}{2}}}$,

and hence

$$\frac{d\sigma}{da} = \int_0^\infty \left\{x(2a + c) + 3ac + 3c(a - c)\frac{x+a}{x+c}\right\}\frac{2x\,dx}{(x+a)^3(x+c)^{\frac{3}{2}}},$$

which is evidently positive.

Thus, putting $x = y$, we infer that *the spheroids are stable for spheroidal displacements for all values of α* up to $\frac{\pi}{2}$.

We can prove that

$$\sigma = 6\,[p(1+t^2)]^{\frac{1}{6}}\frac{(1+t^2)^{\frac{1}{2}}}{t^2}\left[\frac{\alpha}{t}(1 + \tfrac{1}{3}t^2) - 1\right], = 6\,[p(1+t^2)]^{\frac{1}{6}}\,U,$$

say, where $(1 + t^2)^{\frac{1}{6}}$ increases with t, while

$$\frac{dU}{dt} = \frac{d}{dt}\left\{\frac{(1+t^2)^{\frac{1}{2}}}{t^2}\left[\frac{\alpha}{t}(1+\tfrac{1}{3}t^2)-1\right]\right\} = \frac{9+7t^2}{3t^4(1+t^2)^{\frac{1}{2}}}\eta,$$

wherein

$$\eta = t\frac{9+4t^2}{9+7t^2} - \alpha,$$

leading to

$$\frac{d\eta}{dt} = \frac{4t^4}{1+t^2}\cdot\frac{6+7t^2}{(9+7t^2)^2}.$$

Hence σ, and therefore the angular momentum, constantly increases with the angle α, along the series of spheroids.

But, we may remark,

$$\frac{\omega^2}{2\pi\rho}, = 3\cos\alpha\frac{(1+t^2)^{\frac{1}{2}}}{t^2}\left[\frac{\alpha}{t}(1+\tfrac{1}{3}t^2)-1\right], = 3\cos\alpha U,$$

increases with α only until

$$\frac{\alpha}{t}(1+t^2) = \frac{9+7t^2}{9+t^2}$$

(approximately $\alpha = 68^\circ\, 26'$, $e = \cdot 93$, giving $\frac{\omega^2}{2\pi\rho} = \cdot 2247$), and afterwards diminishes.

On the general theory of the stability of rotating masses of liquid. By Professor H. F. BAKER.

[*Read* 9 February 1920.]

I venture to make some brief provisional remarks, to which I have hoped now for some years to give a more detailed examination, relating to the question why Sir George Darwin on the one hand, and Liapounoff and Mr Jeans on the other, arrive at different conclusions in regard to the stability of the so-called pear-shaped figure of equilibrium of a rotating liquid mass. These point to the conclusion that this is a case in which the empirical treatment of the convergence of an infinite series may lead to erroneous results in a concrete practical matter.

§ 1. For a mass of homogeneous incompressible liquid rotating as if solid about an axis, with angular momentum μ, moment of inertia I, and potential energy of gravitation $-W$ (where W is the volume integral of the product $dm.dm'$ of two elements of mass divided by their mutual distance), we consider the Hamiltonian function

$$H = -W + \tfrac{1}{2}\mu^2/I.$$

Let H' be the corresponding function belonging to another such mass, "sufficiently" near to the former, rotating about the same axis with the same angular momentum (and, for brevity, of the same mass, and the same centre of mass lying on the axis), but with different I and different W. We conceive that the form of this second mass can be specified, relatively to that of the former, by a certain number of parameters. In the actual problem the tale of these parameters must be unlimited; but the methods applicable when this tale is finite cannot be extended to the actual case without careful examination, and in what follows we think only of a limited tale. The difference $H' - H$ is then a function of these parameters. In the case in which a change of form of the rotating mass involves a dissipation of energy, a necessary and sufficient condition that the form of the mass first considered should be one of stable rotary equilibrium, under its own gravitation, is that $H' - H$ should be positive for all small values of the parameters. We adopt this as the condition. In the problem now being considered the first form of the mass is itself regarded as arising from another (of the same mass and centre of mass), with a different angular momentum, μ_0, and different I and W, say I_0 and W_0, of which the relative rotary equilibrium and stability have already been investigated. We have then a known form of equilibrium, to which there belongs a function H_0, then a contiguous form whose equilibrium is under examination, to which belongs the function H, and then, further, a "virtual" form, which has the same angular momentum as arises in H, to which the function H' applies. We may then suppose the parameters, above spoken of as identifying the form, to vanish for the form (H_0), may denote their values for (H) by $x, y, \ldots$, and their values for (H') by $x+\xi, y+\eta, \ldots$. We put $\frac{1}{2}\mu^2 - \frac{1}{2}\mu_0{}^2 = k$. Presuming certain conditions of continuity for the functions involved, the equations of equilibrium of the form (H), which are such as $\partial H/\partial x = 0$, $\partial H/\partial y = 0$, ..., must, for $x=0$, $y=0$, ..., together with $k=0$, be satisfied for (H_0). In general (certain conditions being introduced in the choice of the parameters) these equations determine a form (H) corresponding to every arbitrary small k; the necessary and sufficient condition however that (H_0) should be a so-called form of bifurcation, or branch form, is that these equations should lead to more than one form (H) for any given small k. A sufficient condition for the stability of the form (H), is that the quadratic form, in the arbitrary variables $\xi, \eta, \ldots$, consisting of such terms as

$$\tfrac{1}{2}\xi^2 \partial^2 H/\partial x^2 + \xi\eta\partial^2 H/\partial x\partial y + \ldots,$$

should be definite, and be positive, when, therein, the coefficients $\partial^2 H/\partial x^2$, $\partial^2 H/\partial x \partial y$, ..., are those functions of k arising by substituting the value of $x, y, \ldots$, just found from $\partial H/\partial x = 0$, $\partial H/\partial y = 0$, etc. And, in particular, if (H_0) be stable, this quadratic form must be definite and positive for $k=0$, $x=0$, $y=0$, ... Conversely this last fact, when (H_0) is known to be stable, considerably reduces the labour of considering the stability of (H).

§ 2. In our case, (H_0) is an ellipsoid, and (H) a contiguous, so-called pear-shaped, form. Sir George Darwin calculates (*Papers*, III, 349), a form for the increment of the Lagrangian function $W + \frac{1}{2}\mu^2/I$, in passing from the form (H) to the form (H'), which leads, to the same approximation, to a form for δH, or $H' - H$, which may be written thus:

$$\begin{aligned}\frac{\delta H}{3M^2/2k_0} &= k'(1 - lx^2 - my - nz)\\ &\quad + \tfrac{1}{2}ax^4 + \tfrac{1}{2}by^2 + \tfrac{1}{2}cz^2 + gx^2 z + hx^2 y + fyz\\ &\quad + \Sigma\,[h'x^2y' + \tfrac{1}{2}b'y'^2];\end{aligned}$$

for facility of comparison we may give the equations which connect the notation here employed with that used by Sir George Darwin:

$$x,\quad y,\quad z,\quad y'\,;\qquad k',\quad l,\quad m,\quad n,$$

are used respectively in place of

$$e,\quad f_2,\quad f_2^{(2)},\quad f_2^{(s)};\qquad \frac{k}{\mathrm{a}},\quad \frac{\mathrm{b}}{\mathrm{a}},\quad \frac{\mathrm{c}}{\mathrm{a}},\, -\frac{\mathrm{d}}{\mathrm{a}}\,;$$

also the coefficients $a, b, \ldots, b', h'$, expressed in Sir George Darwin's notation are, respectively, given by

$$a = \frac{\mathrm{b}^2\omega^2}{\mathrm{a}} - 2A_0, \qquad b = \frac{\mathrm{c}^2\omega^2}{\mathrm{a}} + 2C_2, \qquad c = \frac{\mathrm{d}^2\omega^2}{\mathrm{a}} + 2C_2^{(2)},$$

$$f = -\frac{\mathrm{cd}\omega^2}{\mathrm{a}}, \qquad g = -\frac{\mathrm{bd}\omega^2}{\mathrm{a}} - 2B_2^{(2)}, \qquad h = \frac{\mathrm{bc}\omega^2}{\mathrm{a}} - 2B_2,$$

$$h' = -2B_i^{(s)}, \qquad b' = 2C_i^{(s)}.$$

§ 3. For a form of possible rotary equilibrium to which such an expression for δH is appropriate, the equations for δH to be stationary are

$$\begin{aligned} 2x\,[ax^2 + hy + gz + \Sigma h'y' - k'l\,] &= 0, \\ hx^2 + by + fz \qquad - k'm &= 0, \\ gx^2 + fy + cz \qquad - k'n &= 0, \\ h'x^2 + b'y' \qquad &= 0. \end{aligned}$$

The solution $x = 0$ belongs to the series of ellipsoids; omitting this for the moment, the equations give

$$\Sigma h'y' = -\Sigma \frac{h'^2}{b'} x^2,$$

$$x^2 \begin{vmatrix} a - \Sigma(h'^2/b'), & h, & g \\ h \quad , & b, & f \\ g \quad , & f, & c \end{vmatrix} = k' \begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix}, \quad y/k' = \text{etc.}, \; z/k' = \text{etc.},$$

corresponding to two possible forms other than ellipsoids.

The stability, for displacements in which only x, y, z, y' vary, depends on the quadratic form

$$\tfrac{1}{2}\xi^2 \frac{\partial^2(\delta H)}{\partial x^2} + \xi\eta \frac{\partial^2(\delta H)}{\partial x \partial y} + \ldots + \tfrac{1}{2}\Sigma\eta'^2 \frac{\partial^2(\delta H)}{\partial y'^2} + \Sigma\eta'\eta'' \frac{\partial^2(\delta H)}{\partial y' \partial y''},$$

namely

$$2ax^2\xi^2 + 2hx\xi\eta + 2gx\xi\zeta + 2\Sigma h'x\xi\eta' + \tfrac{1}{2}b\eta^2 + f\eta\zeta + \tfrac{1}{2}c\zeta^2 + \tfrac{1}{2}\Sigma b'\eta'^2;$$

this is, however, the same as

$$\begin{aligned} &\tfrac{1}{2}\Sigma b' \left(\eta' + \frac{2h'}{b'}\xi x\right)^2 + \tfrac{1}{2}b\left(\eta + \frac{f}{b}\zeta + \frac{2h}{b}\xi x\right)^2 \\ &+ \tfrac{1}{2}\left(c - \frac{f^2}{b}\right)\left(\zeta + 2\xi x \frac{gb - hf}{bc - f^2}\right)^2 \\ &+ \frac{2\xi^2 x^2}{bc - f^2} \begin{vmatrix} a - \Sigma(h'^2/b'), & h, & g \\ h \quad , & b, & f \\ g \quad , & f, & c \end{vmatrix}, \end{aligned}$$

wherein, by what has appeared, the last term is the same as

$$\frac{2\xi^2 k'}{bc - f^2} \begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix}.$$

The solution, above omitted, arising by taking $x = 0$, requires

$$by + fz - k'm = 0, \quad fy + cz - k'n = 0, \quad y' = 0,$$

so that, if we introduce σ, given by $\sigma = hy + gz - k'l$, we obtain

$$\sigma = -\frac{k'}{bc - f^2} \begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix},$$

and the quadratic form arising in considering the stability is

$$\sigma\xi^2 + \tfrac{1}{2}b\eta^2 + f\eta\zeta + \tfrac{1}{2}c\zeta^2 + \tfrac{1}{2}\Sigma b'\eta'^2.$$

Assuming then that the ellipsoids up to the form of bifurcation, that is for $k' < 0$, are stable (as is well known), we can infer that b, $bc - f^2$, b' and σ are all positive, and hence that

$$\begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix}$$

is positive.

Hence, returning to the quadratic form above wherein x is not zero, we infer (1) that *the pear-shaped figure*, so far as the increment δH is appropriately represented by the form above, *is stable if k' is positive*. By Darwin and Jeans this conclusion is made to depend on general reasoning, due in the first place to (Liapounoff and) Schwarzschild, writing in correction of Poincaré (*Inaugural Dissertation, München*, 1896, or *Neue Annalen d. Sternwarte München*, III, 1898, 275); and (2) that a necessary and sufficient condition for this is that the determinant

$$\begin{vmatrix} a - \Sigma\,(h'^2/b'), & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

should be positive. It is however easy to verify that this determinant is, in the notation of Sir George Darwin, equal to

$$-(bc - f^2)\,N + \frac{4C_2C_2^{(2)}\omega^2}{\mathrm{a}}\left(\mathrm{b} + \mathrm{c}\,\frac{B_2}{C_2} - \mathrm{d}\,\frac{B_2^{(2)}}{C_2^{(2)}}\right)^2,$$

where $$N = 2\left[A_0 + \frac{(B_2)^2}{C_2} + \frac{(B_2^{(2)})^2}{C_2^{(2)}} + \Sigma\,\frac{(B_i^{(s)})^2}{C_i^{(s)}}\right]^2;$$

from the expression of the increment of the function H, for variation to the second order, in the case of the ellipsoids, it is known that C_2 is negative, and $C_2^{(2)}$ is positive; (these are in fact the "coefficients of stability" corresponding to Lamé functions of one root, in the former case lying between the negative squares of the greatest and mean axes of the ellipsoid, in the latter case between the negative squares of the mean and least axis). Thus (3), we infer, by *algebraic methods only*, that a sufficient condition for the *instability* of the pear-shaped figure, if we assume the preceding ellipsoids stable, is the single condition

$$\tfrac{1}{2}N, \text{ or } A_0 + \Sigma \frac{(B_m^{(n)})^2}{C_m^{(n)}} > 0, \qquad \begin{pmatrix} m = 2, 2, i \\ n = 0, 2, s \end{pmatrix}.$$

Darwin (*Works*, III, 378) computes

$$N = - \cdot 000235513.$$

It is easy to show, from what precedes, that, if $\delta\omega^2$ denote the increment of ω^2 in passing from the ellipsoid to the pear, we have

$$Nx^2 + \tfrac{1}{8}\frac{\mathrm{a}\delta\omega^2}{C_2 C_2^{(2)}} \begin{vmatrix} h, & g, & l \\ b, & f, & m \\ f, & c, & n \end{vmatrix} = 0,$$

or

$$Nx^2 + \tfrac{1}{2}\delta\omega^2 \left(\mathrm{b} + \mathrm{c}\frac{B_2}{C_2} - \mathrm{d}\frac{B_2^{(2)}}{C_2^{(2)}}\right) = 0.$$

Thus a sufficient condition for instability is $\delta\omega^2 > 0$.

This again, we see algebraically, involves that the increment δI of the moment of inertia, is negative; for, if we put D for the coefficient of $\tfrac{1}{2}\delta\omega^2$ in the last written equation, we easily find

$$\delta I = \frac{x^2}{D}\left[D^2 + N\frac{\mathrm{a}}{\omega^2}\left(1 - \frac{bc - f^2}{4C_2 C_2^{(2)}}\right)\right].$$

As $\delta\omega^2$, δI are then of opposite sign it is evidently desirable, if possible, to calculate N independently and not from these, as do Darwin (*loc. cit.*, p. 379) and Jeans (*Cosmogony*, 1919, p. 101).

Mr Jeans (*Phil. Trans.* A, CCXV, 1915, 76, 77; *Cosmogony*, 1919, p. 92, § 94) appears, if I understand him aright, to hold the view that an expression such as that above taken for δH, does not suffice to enable us to draw inferences in regard to the stability, and in particular that there should be present therein a term in x^2. And he seeks to find in this way the explanation of the difference in conclusion of Sir George Darwin and himself. I cannot agree with this view; if the expression for δH is accurate as far as it goes, it appears to suffice for forming a judgment, though hypotheses as to the relative order of smallness of x, y, ..., should clearly not form part of the process for calculating δH.

§ 4. I believe that the discrepancy of conclusion arises in another way. Sir George Darwin computes his form N only to a limited number of terms, and is satisfied with the verification (*loc. cit.*, p. 380) that the terms he would next calculate are very small in comparison. It is easy to show however that the terms of the first few orders, in Mr Jeans' expression for the normal variation from the ellipsoid to the contiguous pear, involve terms extending to infinity in Sir George Darwin's expression for this normal variation.

Taking a point (x, y, z) near to a point (x_0, y_0, z_0) of the ellipsoid

$$x^2/a + y^2/b + z^2/c = 1,$$

given by

$$x\,(a+\lambda)^{-\frac{1}{2}} = x_0 a^{-\frac{1}{2}},\quad y\,(b+\lambda)^{-\frac{1}{2}} = y_0 b^{-\frac{1}{2}},\quad z\,(c+\lambda)^{-\frac{1}{2}} = z_0 c^{-\frac{1}{2}},$$

Darwin (*loc. cit.*, p. 320) has an expansion in Lamé products

$$-\tfrac{1}{2}\lambda/p_0^2 = \text{const.} - e\Omega_{10}{}^a - f_2\Omega_{10} - f_2^{(2)}\Omega_{01} - \Sigma f_i^{(s)}\Omega_i^s,$$

where p_0 is the central perpendicular on the tangent plane at (x_0, y_0, z_0), so that, in terms of elliptic coordinates, $p_0^2 = abc/\mu\nu$.

Jeans' form of contiguous surface (*Phil. Trans.* A, CCXV, CCXVII, 1915, 1916, or *Cosmogony*, p. 88) is

$$0 = -1 + \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + exu_1 + e^2u_2 + e^3xv_2 + \ldots,$$

where u_1, u_2, v_2 are integral polynomials in $x^2, y^2, z^2, 1$, of respective orders 1, 2, 2. If herein we substitute

$$x = x_0\left(\frac{a+\lambda}{a}\right)^{\frac{1}{2}} = x_0\left(1 + \frac{\lambda}{2a} - \frac{\lambda^2}{8a^2} + \ldots\right),$$

with similar expressions for y and z, and solve for λ, we shall find a series of the form

$$\frac{\lambda}{p_0^2} = -ex_0u + e^2p_0^2v + e^3x_0p_0^4w + \ldots,$$

where $u, v, w, \ldots$, are polynomials in x_0^2, y_0^2, z_0^2, which, on examination, prove not divisible by $\dfrac{1}{p_0^2}$ or $x_0^2/a^2 + y_0^2/b^2 + z_0^2/c^2$. These are then expressible by finite series in Lamé functions. But $p_0^2 = \dfrac{abc}{\mu\nu}$, $p_0^4 = \dfrac{a^2b^2c^2}{\mu^2\nu^2}$, ..., can only be expressed by infinite series of this form.

Thus even the terms of the second order in Mr Jeans' form for λ/p_0^2 cannot be put down exactly from Darwin's results without taking account of the whole aggregate of terms in Darwin's series; and the same for terms of higher order.

§ 5. The conclusion so reached is in accordance with a note appearing in the *Compt. Rendus* (CLXX, 5 Jan. 1920, 38; "Calculs de G. H. Darwin sur la stabilité de la figure piriforme," Note de M. Pierre Humbert) long after the above was written, to which my attention was called by Mr F. P. White of St John's College. The author has calculated Darwin's series to a further approximation, and finds a result not strengthening Darwin's prevision.

But the conclusion is made almost certain by a comparison with Liapounoff's paper, "Sur un problème de Tchebychef," *St Pétersbourg Mémoires*, XVII. For the statements there given reference is made to another memoir, of which the first, the theoretical, part, appears not to be obtainable in England; indeed, were it otherwise, there might be little justification for the preceding summary remarks, save perhaps on account of the total difference of method, in view of the rigour with which Liapounoff's results in these problems are developed. As was pointed out to me by Mr S. R. U. Savoor, of Trinity College, Liapounoff makes the remark (*loc. cit.*, p. 27) that the terms of various orders in his development of what is here called λ, though presenting themselves in the first place as infinite series, can be summed, and then take the form above remarked (§ 4) as belonging to the expansion which can be deduced from Mr Jeans' work.

Liapounoff however also remarks (*loc. cit.*, p. 30) that the instability of the pear follows, when the ellipsoids have been examined, from the sign of one term only, which, in a footnote, he identifies with that above denoted by $\frac{1}{2}N$—which he states he has expressed in finite terms as an algebraic function of the axes of the ellipsoid; and gives further, also without proof, a general expression for the increase of angular momentum in passing from the ellipsoid to the pear, with the remark that the (positive) sign of this also follows from the sign of N*. (See *Péters. Mém.* XXII, 1908, 126–131.)

* The following references to Liapounoff's papers may be useful to the reader: (1) 1884, "Sur la stabilité des figures ellipsoïdales," *Toulouse Annales*, VI, 1904 (translated from the Russian); (2) 1903, "Recherches dans la théorie de la figure...," *St Pétersbourg Mémoires*, XIV; (3) 1904, "Sur l'équation de Clairaut," *ibid.* XV; (4) 1905, "Sur un problème de Tchebychef," *ibid.* XVII; (5) 1906, *Sur les figures d'équilibre peu différentes des ellipsoïdes*, Part I—published separately, unobtainable in England; (6) 1908, "Problème de minimum...," *St Pétersbourg Mémoires*, XXII; cf. *Toulouse Annales*, IX, "Problème général de la Stabilité du mouvement"; (7) 1909, second part of the memoir (5); (8) 1912, third part of the same. The second and third parts are in the British Museum (as was first discovered for me by Mr F. P. White); (9) 1916, "Sur les équations qui appartiennent aux surfaces des figures d'équilibre dérivées des ellipsoïdes d'un liquide homogène en rotation"; and "Nouvelles considérations relatives à la théorie des figures d'équilibre dérivées des ellipsoïdes dans le cas d'un liquide homogène," *St Pétersbourg Bulletin*, 1916. These last two references I take from an article by L. Lichtenstein, "Gleichgewichtsfiguren rotierender Flüssigkeiten," *Math. Zeitschrift*, VII, Berlin, 1920, 132. The same writer also gives (*ibid.* I, 1918, 232) reference to a fourth part of the memoir (8), under date 1914, and to a memoir, *Ann. de l'Éc. Norm.* XXVI, 1909, 473.

Mr S. R. U. Savoor has made a detailed application of the method of Liapounoff to the case of the rotating cylinders, of which it is hoped that a summary may be published in the *Trans. Camb. Phil. Soc.*

Sur le principe de Phragmén-Lindelöf. Par MARCEL RIESZ. (Extrait d'une lettre adressée à M. G. H. Hardy.)

[*Received* 29 July 1920.]

1. La lecture attentive de votre travail récemment paru dans les *Acta Mathematica** et de ceux de MM. Phragmén et Lindelöf†, P. Persson‡, F. Carlson¶, etc., me fait croire que le théorème suivant n'a jamais été observé.

Si la fonction $g(z)$ holomorphe dans le demi-plan $R(z) \geqq 0$ (sauf peut-être à l'infini) y satisfait à l'inégalité

(1) $$|g(z)| < C_1 e^{lr} \qquad (r = |z|)$$

et sur l'axe imaginaire à celle-ci

(2) $$|g(z)| < C_2 e^{-kr},$$

C_1, C_2, k et l désignant des constantes positives, cette fonction s'annulera identiquement.

Ce théorème rend la première démonstration dans votre travail beaucoup plus simple qu'elle ne l'était. Vous le verrez dans un instant. Mais démontrons d'abord le théorème lui-même. Il découlera immédiatement des principes généraux de MM. Phragmén et Lindelöf.

Formons la fonction

$$\psi(z) = e^{\omega z} g(z),$$

ω désignant un nombre positif. Désignons par C le plus grand des nombres C_1 et C_2. Sur l'axe réel positif on aura d'après (1)

$$|\psi(z)| < Ce^{(\omega+l)r}$$

et sur l'axe imaginaire

$$|\psi(z)| < Ce^{-kr}.$$

Il s'ensuit d'après un théorème connu de Phragmén-Lindelöf||,

* G. H. Hardy, "On two theorems of F. Carlson and S. Wigert," *Acta Mathematica*, t. 42, 1920, pp. 327–339.

† E. Phragmén et Ernst Lindelöf, "Sur une extension d'un principe classique de l'Analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier," *Acta Mathematica*, t. 31, 1908, pp. 381–406.

‡ Paul Persson, "Recherches sur une classe de fonctions entières," *Thèse pour le doctorat*, Upsal, 1908.

¶ F. Carlson, "Sur une classe de séries de Taylor," *Thèse pour le doctorat*, Upsal, 1914.

|| *Loc. cit.*, pp. 393–394; an explicit proof of the particular result required is given by H. Cramér, "Sur une classe de séries de Dirichlet," *Thèse pour le doctorat* (Stockholm), Upsal, 1917, pp. 34–36.

ϕ désignant un nombre arbitraire situé entre $-\frac{\pi}{2}$ et $\frac{\pi}{2}$,

$$|\psi(z)| < Ce^{(-k|\sin\phi| + (\omega + l)\cos\phi)r}.$$

On aura donc, δ étant choisi assez petit, pour $\phi = \pm\left(\frac{\pi}{2} - \delta\right)$,

(3) $$\psi(z) < C,$$

inégalité valable aussi sur l'axe imaginaire. En appliquant encore une fois le théorème en question de Phragmén-Lindelöf aux angles

$$\left(-\frac{\pi}{2}, -\frac{\pi}{2} + \delta\right), \ \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right) \ \text{et} \ \left(\frac{\pi}{2} - \delta, \frac{\pi}{2}\right)$$

tous moindres que π, on voit que (3) subsiste dans tout le demi-plan. C'est à dire, dans tout ce demi-plan,

(4) $$|g(z)e^{\omega z}| < C.$$

En tout point intérieur au demi-plan, on a $|e^z| > 1$. On conclut donc de (4), en faisant tendre ω vers l'infini,

$$g(z) \equiv 0.$$

2. Appliquons ce résultat à démontrer le théorème de M. Carlson.

Si $f(z) = f(re^{i\theta})$ *est holomorphe dans l'angle* $-\alpha \leqq \theta \leqq \alpha$, *où* $\alpha \geqq \frac{\pi}{2}$, *et y satisfait à l'inégalité*

$$|f(z)| < Ae^{pr} \qquad (p < \pi),$$

et
$$f(n) = 0 \qquad (n = 0, 1, 2, \ldots),$$

alors $f(z)$ *s'annule identiquement.*

En effet, la fonction $g(z) = \frac{f(z)}{\sin \pi z}$ satisfait aux hypothèses du numéro précédent, par suite elle est identiquement nulle. Quant à la condition (2), cela est évident. En ce qui concerne (1), on trouve les éléments d'une démonstration rigoureuse dans votre travail ci-mentionné (p. 329). (En réalité, votre démonstration en question du théorème de M. Carlson présente une petite lacune. Pour pouvoir appliquer le théorème de Phragmén-Lindelöf, vous auriez dû démontrer que pour tout angle $-\alpha + \delta \leqq \theta \leqq \alpha - \delta$, il y a une majorante (une certaine fonction exponentielle) qui dépend seulement de r. Vous démontrez seulement que $+ 1(\theta) \leqq k$ sur chaque vecteur issu de l'origine, sans vous préoccuper de l'uniformité.)

Voilà une autre application de ce théorème. M. Cramér a déduit d'un beau théorème qui lui est dû, le corollaire qui suit*.

* H. Cramér, "Un théorème sur les séries de Dirichlet et son application," *Arkiv för Matematik, Astronomi och Fysik*, t. 13, No. 22, 1918, pp. 1–14 (p. 12).

Si le module de la fonction entière $\phi(z)$ finit par rester inférieur à $e^{k|z|}$, où $k > 0$, il y a sur chaque vecteur issu de l'origine une infinité de points aux modules indéfiniment croissants pour lesquels

$$|\phi(z)| > e^{-(k+\epsilon)|z|}$$

et cela a lieu pour tout $\epsilon > 0$.

L'application du théorème démontré au début à la fonction $g(z) = \phi(z)\,\phi(-z)$ donne de suite le résultat de M. Cramér. D'ailleurs il est facile de déduire ce résultat directement du théorème fondamental de Phragmén-Lindelöf que nous venons d'appliquer.

3. Un corollaire immédiat de notre théorème est encore le suivant.

Une fonction $g(z)$ qui est holomorphe dans un angle d'étendue $\geqq \pi$ et y satisfait à l'inégalité

$$|g(z)| < Ce^{-kr} \qquad (k > 0)$$

s'annule identiquement.

Par une substitution de variable on en obtient immédiatement:

Soit $\Phi(x)$ une fonction analytique de la variable complexe $x = re^{i\phi}$ qui jouit des propriétés suivantes:

1°. *Elle est holomorphe à l'intérieur et sur le contour d'un domaine T renfermant l'angle $-\frac{\pi}{2\alpha} \leqq \phi - \phi_0 \leqq \frac{\pi}{2\alpha}$, sauf peut-être certaines parties de cet angle situées à l'intérieur d'un cercle autour de l'origine.*

2°. *A l'intérieur et par suite au contour de ce domaine, on a*

$$|\Phi(x)| < Ce^{-kr^{\alpha}}, \tag{5}$$

C et k étant des constantes positives.

Dans ces conditions, on a identiquement $\Phi(x) = 0$.

Le meilleur théorème de ce genre fut jusqu'ici, à ce qu'il semble, un théorème de M. Paul Persson démontré dans sa thèse (*l.c.* p. 8) et cité *in extenso* dans la thèse de M. Carlson (*l.c.* p. 36). Chez M. Persson, la condition

$$|\Phi(x)| < e^{-v(r)\,r^{\alpha}}, \quad \lim_{r=\infty} v(r) = \infty$$

figure au lieu de (5).

Note by G. H. HARDY.

I know from my own experience that it is sometimes a little difficult to pick out from Phragmén and Lindelöf's classical memoir the precise proposition of which one may have need; and Dr Cramér's dissertation, referred to by Dr Riesz, is not easily accessible to English readers. It may therefore be worth while to give an explicit statement and proof of the particular theorems used by Dr Riesz.

1. *Suppose that T is an angle of magnitude less than π, whose vertex is at the origin, that C, C_1, and K are constants, and that*

$$|f(z)| \leqq C \tag{1}$$

on the boundary of T,

$$|f(z)| \leqq C_1 e^{Kr} \tag{2}$$

throughout T. Then (1) *holds throughout T.*

We may plainly suppose, without loss of generality, that the boundaries of T are $\phi = -\alpha$ and $\phi = \alpha$, where $0 < \alpha < \frac{1}{2}\pi$. Let δ be positive, $1 < k < \dfrac{\pi}{2\alpha}$, and

$$g(z) = e^{-\delta z^k},$$

so that

$$|g(z)| = e^{-\delta r^k \cos k\phi} \leqq e^{-\delta r^k \cos k\alpha} < 1$$

throughout T. Finally, let

$$f(z)\, g(z) = h(z).$$

Since $|g(z)| \leqq 1$, we have $|h(z)| \leqq C$ on the boundary of T. Also, since $k > 1$ and $\cos k\alpha > 0$,

$$|h(z)| \leqq C_1 e^{Kr - \delta r^k \cos k\alpha}$$

tends uniformly to zero when z tends to infinity in T. Hence $|h(z)| \leqq - C$ if $-\alpha \leqq \phi \leqq \alpha$, $r = R$, and R is sufficiently large; and therefore at all points of the boundary of the region $T(R)$ defined by the inequalities just written; and therefore throughout $T(R)$.

As R is arbitrarily large:

$$|h(z)| = |e^{-\delta z^k} f(z)| \leqq C$$

throughout T; and therefore, making δ tend to zero, $|f(z)| \leqq C$ throughout T.

2. *Suppose that T is the angle $\phi_1 \leqq \phi \leqq \phi_2$, where $\phi_2 - \phi_1 < \pi$; that*

$$|f(re^{i\phi_1})| \leqq Ce^{a_1 r}, \tag{3}$$

$$|f(re^{i\phi_2})| \leqq Ce^{a_2 r}, \tag{4}$$

and that (2) *is satisfied throughout T. Then*

(5) $$|f(re^{i\phi})| \leqq Ce^{h(\phi) r}$$

where $h(\phi)$ *is the function* $A\cos\phi + B\sin\phi$ *which assumes the values* a_1 *and* a_2 *for* $\phi = \phi_1$ *and* $\phi = \phi_2$.

We may plainly suppose, without loss of generality, that $-\phi_1 = \phi_2 = \psi$, where $0 < \psi < \frac{1}{2}\pi$.

Let $$g(z) = e^{-(A-iB)z}$$
so that $$|g(z)| = e^{-h(\phi) r}$$
and $$f(z)\, g(z) = h(z).$$

Plainly $h(z)$ satisfies a condition of the type (2); and $|h(z)| \leqq C$ for $\phi = \phi_1$ and for $\phi = \phi_2$. Hence $|h(z)| \leqq C$ throughout T, which proves the theorem.

It seems worth while also to fill up explicitly the "petite lacune" in my proof in the *Acta Mathematica* signalised by Dr Riesz. It is a question simply of proving that

$$g(z) = \frac{f(z)}{\sin \pi z}$$

satisfies an inequality

(6) $$|g(z)| < Ce^{Kr}$$

for $0 \leqq \phi \leqq \frac{1}{2}\pi$. Suppose that λ is a positive constant, U the part of the positive quadrant above, and V the part below, the line $y = \lambda$. Since $|\sin \pi z|$ is greater than a constant throughout U, (6) is satisfied in U, and we need only consider V. Also

$$f'(z) = \frac{1}{2\pi i}\int \frac{f(u)}{(u-z)^2}\, du,$$

z lying in V and the contour of integration being the circle $|u - z| = 1$. Hence $f'(z)$ satisfies an inequality

$$|f'(z)| \leqq C_1 e^{Kr}$$

throughout V.

Suppose now that z lies in V and that n is the integer nearest to z (either, if there are two equidistant from z). Then

$$f(z) = f(z) - f(n) = \int_n^z f'(u)\, du,$$

the path of integration being rectilinear; and so

$$|f(z)| \leqq M\,|z - n|,$$

where M is the maximum of $|f'(u)|$ on the path of integration. Hence

$$|F(z)| = \left|\frac{f(z)}{\sin \pi z}\right| \leqq C_1 e^{K(n + \frac{1}{2} + \lambda)} \left|\frac{z-n}{\sin \pi z}\right| \leqq C_2 e^{Kn} \leqq Ce^{Kr},$$

which is the inequality required. This is substantially the argument I used before, extended however to complex values of z.

A note on the nature of the carriers of the Anode Rays. By
G. P. Thomson, M.A., Fellow of Corpus Christi College.

[*Received* 3 August 1920.]

The importance of positive rays as an instrument of research is now thoroughly established, and the success of the recent investigations of Dr Aston on isotopes makes it the more to be regretted that so far, with the exception of mercury, no metal has with certainty been identified in the photograph of the mass spectra of positive rays; and this in spite of a considerable number of attempts. In these circumstances the work of Gehrcke and Reichenheim* on anode rays suggested the possibility that these might fill the gap. From a rough measurement of e/m Gehrcke and Reichenheim showed that the anode rays probably consist of positively charged atoms of metal. They did not however photograph the particles and their method of determining e/m made use of magnetic deflection only, and so involved the assumption that the velocity was constant and known. It was therefore not capable of giving more than a rough approximation and could not lead to separation of any isotopes which may exist. In the experiments described below, which are of a preliminary nature, I have tried to improve on their method in these points.

Anode rays are formed when a mixture of various metallic chlorides and iodides with graphite is used as the anode in a discharge tube under reduced pressure. They are visible as a slightly diverging pencil of light starting from the surface of the anode and very roughly normal to it. In order to investigate the value of e/m for the particles which cause this light I placed in the path of the beam, and 2 or 3 cm. from the anode, an insulated aluminium plate pierced with a hole in which was fastened the end of a fine tube of ·3 mm. bore. The parallel beam of rays thus produced was analysed by coincident electrostatic and magnetic fields in the usual manner for positive rays†, and the deflected particles allowed to strike on a photographic plate. With exposures of about 20 minutes well-marked results have been obtained, the particles affecting the photographic plate in the same manner as positive rays.

Using a paste composed of a mixture of KI, LiCl and graphite, short parabolic arcs appeared whose magnetic displacements were

* *Verh. d. Phys. Gesell.*, 8, p. 559; 9, pp. 76, 200, 376; 10, p. 217.
† *Rays of Positive Electricity*, J. J. Thomson, p. 20.

in the ratio $\sqrt{7}$ to $\sqrt{39}$, indicating that they were due to Li and K respectively. On one plate a faint line appeared corresponding to about 140 atomic weight which is probably due to iodine, the accuracy being poor for large atomic weights.

In order to confirm the interpretation of these lines an absolute determination of e/m was made from measurements of the electric and magnetic forces. Taking e/m for the hydrogen atom as 9571 the measurements gave an atomic weight for the more deflected lines of 6·9, assuming a single charge, thus confirming the theory that they are due to single charged atoms of lithium. For accurate work a better method would be to use the apparatus, with an ordinary anode and the aluminium plate as cathode, to give positive rays, and use these to calibrate the plate; it may even be possible to get both anode and positive rays simultaneously.

Owing probably to an irregularity in the electric field the lines so far obtained are not very sharp, but one plate has been obtained where the width of the line due to lithium is no larger than the separation due to a difference of a unit in atomic weight. There is no trace of doubling, and it seems clear that if lithium consists of a mixture of isotopes with atomic weights differing by whole numbers, only that corresponding to an atomic weight of 7 occurs in any considerable quantity. Further experiments are in progress with the object of improving the definition and detecting small proportions of isotopes if these exist.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1919—1920.

ANNUAL GENERAL MEETING.

October 27, 1919.

In the Comparative Anatomy Lecture Room.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Mr C. T. R. Wilson.

Vice-Presidents:

Prof. Marr.
Prof. Sir W. J. Pope.
Prof. Sir E. Rutherford.

Treasurer:

Prof. Hobson.

Secretaries:

Mr Alex. Wood.
Mr G. H. Hardy.
Mr H. H. Brindley.

Other Members of Council:

Mr F. F. Blackman.
Prof. Sir J. Larmor.
Prof. Eddington.
Dr Marshall.
Prof. Baker.
Prof. Newall.
Dr Fenton.
Prof. Inglis.
Prof. Seward.
Dr Rivers.
Dr E. H. Griffiths.
Mr F. A. Potts.

The following were elected Fellows of the Society:

A. C. Banerji, B.A., Clare College.
W. E. H. Berwick, M.A., Clare College.
B. M. Jones, M.A., Emmanuel College, Professor of Aeronautical Engineering.
K. Prosad, B.A., St John's College.

The following were elected Associates:

T. A. Browne, Trinity College.
J. Chadwick, Gonville and Caius College.
Miss L. V. Craies, Newnham College.
H. Henderson, Gonville and Caius College.
F. P. Slater, Gonville and Caius College.
E. F. Vacin, Trinity College.

The future policy of the Society, with special reference to the form and character of its publications, and the conduct of its meetings, was discussed.

November 10, 1919.

In the Comparative Anatomy Lecture Room.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following was elected an Associate:

Donald A. MacAlister, King's College.

The following Communications were made to the Society:

1. Colourimeter Design. By H. HARTRIDGE, M.D., King's College.

2. A note on photosynthesis and hydrogen ion concentration. By J. T. SAUNDERS, M.A., Christ's College.

3. (1) The effects of some electrolytes upon spermatozoa.
(2) The effects of ions upon ciliary movement.
By J. GRAY, M.A., King's College.

4. Note on the solitary wasp, *Crabro cephalotes*. By C. WARBURTON, M.A., Christ's College.

5. Preliminary note on the life-history of a Proctotrypid (*Lygocerus sp.*) hyperparasite of *Aphidius*. By Miss M. D. HAVILAND. (Communicated by Mr H. H. Brindley.)

6. The Natural History of Rodriguez, *with Exhibits*. By H. J. SNELL and W. H. TAMS. (Communicated by Professor Stanley Gardiner.)

November 24, 1919.

In the Cavendish Laboratory.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

J. Line, M.A., Emmanuel College.
L. F. Newman, M.A., Downing College.
A. D. Ritchie, M.A., Trinity College.
J. B. Seth, B.A., Trinity Hall.
G. I. Taylor, M.A., Trinity College.
G. P. Thomson, M.A., Corpus Christi College.
S. M. Wadham, M.A., Christ's College.

The following were elected Associates:

Miss M. D. Haviland, Newnham College.
L. W. G. Malcolm, Christ's College.

The following Communications were made to the Society:

1. (1) Photographs of a Solar Prominence taken during the eclipse of 1919 May 29.

(2) The theory of relativity and recent eclipse observations.

By Professor EDDINGTON and E. T. COTTINGHAM.

2. (1) The Hydrodynamical theory of the Lubrication of a Cylindrical Bearing under Variable Load, and of a Pivot Bearing.

(2) The pressure in a viscous liquid moving through a channel with diverging boundaries.

By W. J. HARRISON, M.A., Clare College.

January 26, 1920.

In the Comparative Anatomy Lecture Room.

PROF. SIR ERNEST RUTHERFORD, VICE-PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

E. A. Milne, B.A., Trinity College.

The following were elected Associates:

A. H. Compton.
T. Kikuchi, St John's College.
D. F. Scanlan, Jesus College.
G. Shearer, Emmanuel College.

The following Communications were made to the Society:

1. Gravitation and Light. By Prof. Sir JOSEPH LARMOR.

2. Note on Mr Hardy's extension of a theorem of Mr Pólya. By E. LANDAU. (Communicated by Mr G. H. Hardy.)

3. On a Gaussian series of six elements. By L. J. ROGERS. (Communicated by Mr G. H. Hardy.)

February 9, 1920.

In the Cavendish Laboratory.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

M. D. Bhansali, B.A., St John's College.
D. C. Henry, B.A., Trinity College.

The following Communications were made to the Society:

1. The Mass Spectra of the Chemical Elements. By F. W. Aston, M.A., Trinity College.

2. An examination of Searle's method for determining the viscosity of very viscous liquids. By Kurt Molin. (Communicated by Dr G. F. C. Searle.)

3. Note on the Diophantine equation $t^3 + x^3 + y^3 + z^3 = 0$. By H. W. Richmond, M.A., King's College.

4. Mathematical notes: (1) On the stability of rotating liquid ellipsoids; (2) on the general theory of the stability of rotating masses of liquid; (3) on the stability of periodic motions in general dynamics; (4) on the invariance of the equations of electrodynamics in the Maxwell and in the Einstein forms; (5) on a property of focal conics and of bicircular quartics; (6) on the Hart circle of a spherical triangle; (7) on a proof of the theorem of a double six of lines by projection from four dimensions; (8) on a set of transformations of rectangular axes; (9) on transformations with an absolute quadric; (10) on the reduction of homography to movement in three dimensions. By Professor H. F. Baker.

February 23, 1920.

In the Botany School.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Fellows of the Society:

E. Cunningham, M.A., St John's College.
M. C. Vyvyan, B.A., Gonville and Caius College.

The following Communication was made:

The Origin of the Vegetation of the Land. By Professor Seward.

Mr Lister exhibited and described a large collection of butterflies.

Many exhibits of botanical interest were shewn.

March 8, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following was elected a Fellow of the Society:

Terry Thomas, M.A., St John's College.

The following Communications were made to the Society:

1. Further notes on the food plants of the Common Earwig (*Forficula auricularia*). By H. H. Brindley, M.A., St John's College.

2. Preliminary note on antennal variation in an aphis (*Myzus ribis*, Linn.). By Miss Maud D. Haviland. (Communicated by Mr H. H. Brindley.)

3. Studies on Cellulose Acetate. By Dr Fenton and A. J. Berry, M.A., Downing College.

4. The rotation of a non-spinning gyrostat, and its effect on the aeroplane compass. By G. T. Bennett, M.A., Emmanuel College.

5. Lagrangian methods for high-speed motion. By C. G. Darwin, M.A., Christ's College.

6. The effect of a magnetic field on the intensity of spectrum lines. By H. P. Waran. (Communicated by Professor Sir Ernest Rutherford.)

7. Generation of sets of four tetrahedra mutually inscribed and circumscribed. By C. V. Hanumanta and Professor H. F. Baker.

8. On the term by term integration of an infinite series over an infinite range and the inversion of the order of integration in repeated infinite integrals. By S. Pollard, M.A., Trinity College. (Communicated by Professor G. H. Hardy.)

9. On rotating liquid cylinders. By S. R. U. Savoor, B.A., Trinity College.

May 3, 1920.

In the Cavendish Laboratory.

Professor Sir Ernest Rutherford, Vice-President, in the Chair.

The following were elected Fellows of the Society:

N. K. Adam, M.A., Trinity College.
P. A. Buxton, M.A., Trinity College.
N. M. Shah, B.A., Trinity College.
F. P. White, M.A., St John's College.

The following was elected an Associate:

K. W. Braid, Fitzwilliam Hall.

The following Communications were made to the Society:

1. Notes on the Theory of Vibrations. By W. J. Harrison, M.A., Clare College.

2. On cylical octosection. By W. Burnside, M.A., Pembroke College.

3. (1) A bifilar method of measuring the rigidity of wires.
(2) An experiment on a piece of common string.
(3) Experiments with a plane diffraction grating, using convergent light.

By Dr G. F. C. Searle, Peterhouse.

4. Congruences with respect to composite moduli. By Major P. A. MacMahon.

5. Equivalence of different mean values. By Alfred Kienast. (Communicated by Professor G. H. Hardy.)

6. Construction of the ninth intersection of two cubic curves passing through eight given coplanar points. By Professor H. F. Baker.

7. Quintic transformations and singular invariants. By W. E. H. Berwick, M.A., Clare College.

May 17, 1920.

In the Cavendish Laboratory.

Mr C. T. R. Wilson, President, in the Chair.

The following were elected Fellows of the Society:

W. M. H. Greaves, B.A., St John's College.
W. M. Smart, M.A., Trinity College.

The following was elected an Associate:

T. Shimizu.

It was announced that the adjudicators for the Hopkins Prize had made the following awards:

For the period 1903–06 to W. Burnside, M.A., F.R.S., of Pembroke College, for investigations in Mathematical Science. For the period 1906–09 to G. H. Bryan, Sc.D., F.R.S., of Peterhouse, for investigations in Mathematical Physics, including aerodynamic stability. For the period 1909–12 to C. T. R. Wilson, M.A., F.R.S., of Sidney Sussex College, for investigations in Physics, including the paths of radioactive particles.

The following Communication was made to the Society:

The atomic nature of matter in the light of modern physics. By F. W. Aston, M.A., Trinity College. With Experiments.

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CONTENTS.

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PROCEEDINGS

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY

VOL. XX. PART II.

[MICHAELMAS TERM 1920.]

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

The Problem of Soaring Flight. By E. H. HANKIN, M.A., Sc.D., late Fellow of St John's College, Cambridge, Chemical Examiner to Government, Agra, India. (Communicated by Mr H. H. BRINDLEY.)

[*Read* 22 November 1920.]

With an introduction by

F. HANDLEY PAGE, C.B.E., F.R.Aer.S.

INTRODUCTION.

By F. HANDLEY PAGE, C.B.E.

The study of bird flight has always fascinated those who were interested in the early development of aviation, and all the original attempts at heavier-than-air flight were based on imitation of birds, both in constructive and in propelling mechanism.

Progress from such study was, however, well nigh impossible. The bird represents the finished article of millions of years of slow development to suit the difficult condition of taking the best advantage of the air structure in which it had to fly. Observers therefore had not only to study a complicated mechanism to find its basic principle of operation, but also to do so in a medium whose movements were but imperfectly understood.

The methods of observation were crude, and the observers not trained for the work of exact measurements and recording of the results. A great many of the early observers, from Leonardo da Vinci down to Weiss and others of the present day, were artists accustomed to observe with the trained eye the picture presented to their mind by bird flight, and from that they endeavoured to reproduce empirically in a mechanical form the design which they deemed the most successful in bird flight.

Directly engineering and scientific thought was directed to the study of flight problems the component parts were reduced to simple and elementary forms. Instead of the complicated supporting *plus* propelling mechanism of the wing of varying plan form and cross section the aeroplane was developed with a wing of simple geometrical shape which was used for support only and with a screw propelling mechanism quite distinct from its wings.

Great progress has been made in the design of aircraft based on mechanical methods of investigation in wind channels, where the air is driven at uniform speed with little or no turbulence against the model to be tested. There still remains a vast amount to be discovered in the actual air structure in which aircraft fly and in the aerodynamical design best suited to fit the conditions discovered. Here, without a doubt, lies the value of a careful and considered study of soaring flight. In this country, owing to the complicated and ever-changing meteorological conditions, research is difficult. In continental and particularly tropical climates conditions are more stable and investigation easier. In India Dr Hankin has been particularly fortunate in having most favourable conditions for observation and in bringing to bear on the problem observing powers trained to a high degree of accuracy by long years of practice.

The measurements made and the facts discovered relating to soaring flight are so extraordinary as to awaken suspicions at once as to their accuracy. Were they but chance observations of an observer made at odd intervals such criticisms would be just, but the careful system of investigation and re-investigation continued by Dr Hankin over a long period of years makes his results worthy of consideration by all interested in flight phenomena and open up an entirely new field for aviation and meteorological research.

The high soaring speed of 50 miles per hour over the plains of India without any discoverable wing movement brings up visions of possible motorless flight in tropical climates, where conditions of visibility and weather are ideal for this form of transport. To all those who are interested in aviation development in India and other tropical countries, Dr Hankin's discoveries must be of the greatest interest and his published results well worthy of most careful study.

The results are by no means on all fours with present aircraft design. The wing sections of the best soaring birds and of the soaring dragon-flies are characterised by ridges projecting on the under side of the wing, forming to the eye of the aeroplane designer a source of resistance entirely uncalled for. On the other hand, birds of more streamline wing cross section, where these ridges are not present to anything like the same extent, invariably fly by flapping their wings, and have little or no power to soar. Can it be that the more crude cross sections fit in with, and thus take advantage more readily of, some form of air turbulence or movement and so allow the soaring bird to draw on some at present unknown source of energy—kinetic or other—in the air?

Another observation is of interest. Birds of the smaller size are more lightly loaded than the larger birds. Thus the cheel, with a span of 51 inches, is loaded ·5 lb. per sq. ft. as compared with the vulture, with a span of 85 inches and loading of 1·5 lbs. per sq. ft. Each, however, in similar conditions of soarable air glides at approximately the same speed. According to aerodynamical theory, the speed of the vulture should be approximately $\sqrt{3}$ times as great, the loading being three times as great—if we neglect bodily resistance. Even if this is taken into account, their speeds should not be anything like the same. The explanation lies probably in "scale effect." The larger bird is, owing to its larger dimensions, able to lift more in proportion to its area, just as the full size aeroplane can compared with the model. Here nature, faced with the problem of making large birds, has avoided the increased percentage of wing weight due to increased span by concurrently increasing the lifting power of the wing with increase in area.

Apart from the study of natural soaring flight its mechanical equivalent has aroused interest and a good deal of research has been carried out on the Continent. In Germany, Gustav Lillienthal, brother of the famous Otto Lillienthal, has, according to a paper published by him in 1917*, designed a wing section of alleged similarity to that of a soaring bird, and with a glider of this wing cross section a German engineer, Friedrich Harth, claims to have flown 500 metres against a wind of 12 metres per second at a height of 40 metres.

Such attempts are, however, premature, and the results unauthenticated. Firstly, we want to know whether soaring birds can soar—as apparently they can—for indefinite distances, provided weather conditions are suitable. Secondly, if such flight is as effortless as it appears to be, from what source is the energy obtained and how the bird is able to take advantage of it. When this is discovered, then will be the time for practical application. Its importance for aviation work in tropical climates has already been referred to.

Dr Hankin's discovery of the soaring flight of dragon-flies and flying-fishes and its similarity in speed and other respects to that of the soaring bird affords a means of more closely investigating the phenomenon of soaring. Observation of birds soaring at 2000 or 3000 feet is difficult compared with the observation of insect flight at a few yards' distance. The whole subject demands most careful investigation and merits the attention of all those interested in the scientific development of the world's latest form of locomotion.

* "Der Einfluss der Flugelform auf die Flugart der Vögel" (*Sitzungsberichte der Gesellschaft naturforschender Freunde*, Berlin, 1917, No. 4).

THE PROBLEM OF SOARING FLIGHT.

By E. H. HANKIN, M.A., Sc.D.

This paper contains a short summary of existing evidence as to the nature of soaring flight.

1. *Comparison of soaring flight in birds, dragon-flies and flying-fishes.*

Birds, dragon-flies, and flying-fishes can exhibit two kinds of soaring flight—slow and fast—characterised by different wing dispositions.

As a rule, both with birds and flying-fishes (XI)*, slow soaring flight is carried out with wings dihedrally-up (*i.e.* with wing-tips at a higher level than the body). In fast soaring flight the wings are either flat or occasionally in the case of flying-fishes dihedrally-down. In dragon-flies the wings are strongly dihedrally-up in slow flight and either flat or less dihedrally-up in fast flight (XII).

In each of the three classes of soaring animals slow soaring flight is dependent, as a rule, on the presence of sunshine and fast flight is always dependent on the presence of wind (IV, pp. 52, 98, 251, 299).

With birds and flying-fishes lateral instability occurs more often late in the afternoon, when the air is becoming unsuited for soaring flight, than at other times of the day (IV, p. 294 and XI). A rare form of lateral instability in which the oscillations are just too rapid to count has been observed in each of the three classes of flying animals.

The speeds attained in soaring flight seem to be remarkably similar in the three classes. The slow speed flight is between 5 and 10 metres per second. The high speed flight has been estimated for dragon-flies at above rather than below 15 metres per second (XII). For flying-fishes, in strong winds, it may exceed 20 metres per second (XI). The air speed of vultures has been measured and found to reach a mean speed of 20 metres per second in winds of medium strength (XI).

In view of the above resemblances between the soaring flight of birds, flying-fishes and dragon-flies there seems to be no room for doubt that we are dealing with the same phenomenon in each case.

2. *The regularity of soaring flight.*

A remarkable, though familiar, phenomenon is shown by a flock of cranes in soaring flight. The birds may be seen to keep their distances from each other with marvellous exactitude and this not only when they are gliding in a straight line but also when they are

* Roman numerals in brackets refer to the appended list of my publications.

on a curved course. When watched through a binocular they resemble a number of dead birds pinned on a blue wall (IV, pp. 60 and 61).

The regularity of soaring flight thus shown conspicuously by cranes, and less strikingly by all soaring animals that fly in groups, furnishes a clear proof that the energy involved does not come from any chance or irregular currents of air.

Two alternatives appear to be suggested. Either such regular flight is due to undiscovered wing movements or its cause must be some condition widely and uniformly distributed in the atmosphere.

3. *The flight of the puttung.*

The puttung is a kite made of paper and bamboo in common use by Indian boys.

The chief peculiarity of its flight is that, in a suitable wind, it flies vertically over its string and when so flying its flight is particularly stable. To achieve this result the front limb of the bridle that attaches it to the string must be a little shorter than the hind limb.

This mode of flight can only be explained by the supposition that this kite takes energy from the air after the manner of the soaring bird.

Its flight resembles that of the soaring bird in that it is more liable to show lateral instability late in the afternoon than at other times of the day.

When struck by a gust it may, for a few seconds, fly up wind in advance of the vertical. Similarly, birds and dragon-flies soaring up wind when struck by a gust often show an increase of speed.

The structure of this kite is such that when it is exposed to wind pressure there must be a ridge on its under surface that lies transverse to the line of flight. In all the more efficient soaring animals ridges transverse to the line of flight are present on the under surfaces of the wings (IV, pp. 242 and 341).

If it is admitted that the flight of this kite is an instance of soaring then obviously the idea of undiscovered wing movements must be given up and, also, a means is indicated by which the phenomena of soaring may be submitted to an experimental investigation.

4. *Soaring flight not due to undiscovered wing movements.*

(*a*) Flying-fishes can check their speed by hanging their hind wings downwards. They habitually do so towards the end of a flight if it has been carried out at high speed. But in the highly soarable monsoon winds they often use this air brake during the whole of a flight and, as may be seen, with this brake in action they attain less speed than other fishes near them that are not using the adjustment (XI).

If their flight was due to undiscovered wing movements why should they not be able to check it by decreasing these movements?

(*b*) Dragon-flies can check speed by hanging down the abdomen and hind legs (IX and XII). Those kinds of soaring dragon-flies that habitually soar in a group over a restricted area commonly use this air brake, when in continued flight, between about 11 a.m. and 3 p.m., if there is strong sunshine. If small clouds pass over the sun the brake is taken out of use to be applied again when the sun is clear. This happens even when the clouds are too thin to throw a shadow or to cause any appreciable decrease in the intensity of the sunshine. After the sun comes out there is a latent period of about 23 seconds before the brake is reapplied (XII). These facts give a striking proof that, in fine weather, the energy for their flight is derived from sunshine and are quite inconsistent with the idea that their soaring is due to undiscovered wing movements.

(*c*) Among birds the albatross furnishes a proof that soaring flight is not due to undiscovered wing movements. Observers are agreed that this bird cannot soar in a calm when near sea level (VII). If its soaring flight was due to wing movements, why, it may be asked, should it be unable to execute these movements in the absence of wind?

5. *Soaring flight is not due to the effect of lateral gusts of wind.*

If soaring animals habitually carried their wings inclined so that the wing-tips were at a higher level than the body, then it is conceivable that soaring flight might be due to the effect of lateral gusts which, striking the underside of the wings from one side or the other, would give a succession of lifting impulses and hence keep the bird aloft.

This idea is negatived by the fact that in high speed flight the wings are placed in the flat position. When this is the case lateral gusts can have no lifting effect. Flying-fishes sometimes place their wings dihedrally-down, *i.e.* the wing-tips are at a lower level than the bases of the wings. This disposition is probably that used in flight at highest speed (XI). If lateral gusts existed and were operative their only effect in this case would be to drive the fish under water.

Though wind is favourable to the high speed flight of dragon-flies, the clearest proof exists that, in calm air in the early morning, when soarability is developing under the influence of sunshine, the coming of the lightest draught of wind causes soaring to be replaced by flapping (XII). This observation is entirely inconsistent with the idea that soaring flight is due to the effect of lateral pulsations.

6. *Soaring flight is not due to the effect of ascending currents.*

(*a*) If isolated clouds are passing over the sky soaring dragon-flies are apt to collect in the neighbourhood of a convenient ascending current and glide into it whenever the sun is observed. They glide out of it and keep out of its range as soon as sunshine returns. The behaviour of dragon-flies is such as to suggest that ascending currents are in some way inimical to their soaring flight (XII).

(*b*) Inland birds similarly avoid ascending currents so long as the sun is shining during the day time. They use them when the sun is obscured and also both in the early morning and late in the afternoon when the air is not soarable. Under a cloudy sky in disturbed weather some winds are soarable and some are un-soarable. In the latter case only, do birds make use of ascending currents (IV, pp. 20 and 283).

7. *Convection currents and soarability.*

Ample evidence exists that convection currents in the air caused by the heat of the sun, whether at ground level (IV, p. 263) or at a height (IV, p. 23) have nothing to do with soarability.

8. *The theory that turbulent motion in the air may be the source of the energy of soaring flight.*

In favour of this idea the following facts may be noticed:

(*a*) Atmospheric turbulence, in fine weather, decreases towards sunset. At this time soarability for inland birds and dragon-flies comes to an end at ground level. But soarability at this level may persist after sunset in the presence of stormy winds when turbulence must obviously continue (IV, pp. 80, 281 and 375). In the case of flying-fishes a loss of soarability has been observed shortly after sunset in the absence of appreciable wind (XI).

(*b*) The stronger the wind the more turbulence is likely to be present. The speeds attained by vultures in horizontal soaring flight have been measured and found to be greater the stronger the wind. In winds of 10 to 20 metres per second the mean air speed of the vulture has been found to be about three times the speed the same bird reaches in calm after sunset when in flapping flight. As above stated, both cheels and dragon-flies often show an increase of speed when struck by a gust (IV, pp. 250 and 377, XII).

9. *Difficulties in accepting turbulencies whose effects can be seen as an explanation.*

(*a*) Cheels have been observed catching locusts. Each locust flew in a straight line whether it was flapping, as was usually the case, or gliding downwards as sometimes occurred. The cheels were in gliding flight and always glided at a faster rate than the locusts. This was the case whether they were travelling horizontally or gliding upwards at an angle of 10 or more degrees with the horizon. Such upward gliding was often a continuation of a horizontal course. It was not due to any momentum obtained by a preceding glide in a downward direction (X).

How, it may be asked, could turbulent motion be present of sufficient force to propel the cheels and yet to have no visible effect on the course of the locusts? If, as is the fact, in virtue of its larger size, the cheel is a more efficient flying machine than the locust, one would expect the latter to be the more readily deflected or influenced by turbulence. Had dragon-flies been present, animals smaller than locusts, there is no doubt that they would have been seen flying faster than the latter.

(*b*) A dragon-fly whose flight is mostly by flapping (*Rhyothemis variegata*) is common near Calcutta. A soaring dragon-fly may sometimes be seen gliding to and fro in a group of these flapping dragon-flies. Its speed is distinctly faster than that of the latter (IV, p. 388). What form of pre-existing air movement can be imagined that propels the soaring dragon-fly and yet has no effect on the flapping dragon-fly though the latter is of lighter weight and loading?

(*c*) Cheels have been seen soaring in air containing small masses of discrete cloud material which were so numerous that the movement of any one cubic foot of air relatively to that of adjacent cubic feet could be seen if it existed. No such relative movement was visible. It is difficult to see how such an observation is consistent with the idea that turbulent movement is the cause of soaring flight (IV, p. 104).

For soaring flight to be possible under such conditions a strong glare from the sun shining through cloud is necessary. Glare due to light reflected from a cloud seems unable to furnish the energy necessary for soaring flight (IV, pp. 102 and 105, II, p. 24).

(*d*) Vultures have been seen gliding in air that was so full of aerial seeds that it looked like a snowstorm. These floating seeds were in slow equable movement that showed no turbulent motion such as one might expect to be necessary to propel a bird weighing ten or twelve pounds (IV, p. 102).

(*e*) Small feathers have often been seen floating in the air in the midst of soaring birds. These feathers in their course showed only the smallest deviations from a straight line. Sometimes a slow partial rotation of a feather may be seen but any evidence of energetic turbulent movement is conspicuously absent (IV, p. 57).

On one occasion a floating feather was seen to pass directly under the wing of a cheel. It was instantly shot out sideways to a distance of several metres from the bird, thus yielding evidence of active air movement which, so far as is yet known, must have been the effect rather than the cause of the bird's progress.

10. *Conclusions.*

This brief summary of the facts observed by me during the last ten years amply supports the view that soaring flight is inexplicable in the light of existing knowledge.

In the case of soaring flight at slow speed a proof exists that the energy involved is derived from the sun's rays. But the mode by which it becomes available to the soaring animal is, as yet, a complete mystery. Direct observation having failed to point the way to a solution, it is to be hoped that the subject will be attacked with the aid of an experimental investigation. It is only in this way that an explanation of the problem is likely to be attained.

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(IV) *Animal Flight.* Iliffe and Sons, London, 1914.

(V) "On the flight of pterodactyls" (with Dr D. M. S. Watson). *Aeronautical Journal*, October 1914, p. 324.

(VI) "Atmospheric rotary movements of small extent." *Aeronautical Journal*, Oct. 1914, p. 344.

(VII) "On the flight of albatrosses and gannets." *Aeronautical Journal*, Jan. 1915, p. 22.

(VIII) "On the flight of seagulls." *Aeronautical Journal*, July–Sept. 1915, p. 84.

(IX) "Observation of transiently visible movements." *Aeronautical Journal*, July–Sept. 1915, p. 104.

(X) "On the flight of locusts." *Aeronautical Journal*, April–June 1916, p. 56.

(XI) "Observations on the flight of flying-fishes." *Proceedings of the Zoological Society of London*, 1920, No. xxxii, p. 467.

(XII) "Observations on the flight of dragon-flies" (presented to the Royal Meteorological Society, November 1920).

Preliminary Note on the Superior Vena Cava of the Cat. By W. F. LANCHESTER, M.A., King's College, and A. G. THACKER.

[*Read* 22 November, 1920.]

The present series of observations arose out of the more or less accidental discovery of a discrepancy between the factors of the Superior Vena Cava in a cat that we were dissecting and the description of these veins given in St George Mivart's well-known text-book on the Cat. It will be remembered that in the cat the two innominate veins (or, as they are otherwise called, the brachio-cephalic veins) are formed by the union, on either side, of the External Jugulars and Subclavians, the two Innominates then uniting with each other slightly to the right of the trachea to form the single Superior Vena Cava. The right Innominate is of course the anterior portion of the surviving Superior Vena Cava. Now, according to Mivart's description, the Internal Jugular (of either side) runs into the corresponding Innominate. In our first cat we found on the contrary that the Internal Jugular ran into the External Jugular, or in other words, that the vein which unites with the Subclavian is not really an External Jugular but a Common Jugular. There are several American text-books on the cat, and we found that they agreed with our own first observations, but none of them refer to the different description given by Mivart, neither do any of them mention another vein, which does in fact usually join the Innominate (on one side only) at the point Mivart indicates for the Internal Jugular and which is usually much more conspicuous in dissection than the Internal Jugular itself. This is the vein coming from the thyroid glands, which, though it is of course a paired structure in the anterior, is usually single in the posterior part of its course. We decided to clear up this small point regarding the position of the Internal Jugular. The American text-books are correct. Out of 30 cats dissected, in 29 the Internal Jugular entered the External Jugular before the union of the latter with the Subclavian. In one small kitten the three veins—the Subclavian, the External Jugular, and the Internal Jugular—appeared to meet one another at the same point. Our observations show, therefore, that there is normally a Common Jugular in the cat, and this vein is often of considerable length. It is often longer, sometimes much longer, than the corresponding Innominate. The Thyroidean, as stated, usually falls into one or other of the Innominates; but it sometimes falls into the Common Jugular or even into the Internal Jugular. In two cases the Thyroideans were separate throughout their course.

Whilst investigating this point we began another series of observations on the Superior Vena Cava itself, which will perhaps be of more general interest than the point of anatomical detail just described. This work is still proceeding, but the results already obtained are perhaps of sufficient interest to justify a brief summary of them.

Of the large changes in the circulatory system by far the most recent is that from the condition of two Superior Venae Cavae to that of one Superior Vena Cava, and the latter condition has apparently been evolved independently in more than one order of the Mammalia. The nature of the variation in the single Vena Cava would therefore appear to have considerable interest. Cases of two Superior Venae Cavae have, we believe, been recorded for the human subject, and at least one such case is recorded for the dog. The Superior Vena Cava of the cat is not only large but long, being normally, as we shall see in a minute, almost half as long as the trachea. It is normally about four times as long as the Innominate Veins, which are, therefore, very short vessels running almost at right angles to it. We decided to measure the variation in the length of the Vena Cava, taking as our standard the length of the trachea from the posterior edge of the cricoid cartilage to its bifurcation into the two main bronchi. The trachea proved to bear a fairly constant relation to the total length of the animal, exclusive of the tail, but we checked our standard by making other measurements. The length of the trachea varied from 21% of the total length to 26·7%. To summarise the results, we found it necessary to exclude the kittens from the scheme as they showed in many respects very anomalous relations between the sizes of their different organs. This left us with only twenty-one specimens, all these being over 40 cms. in length. Taking the length of the trachea as 100, the extreme limits of variation in the length of the Vena Cava were 19 and 47·9. But the majority of the cats are by no means massed about the mean between these extremes. On the contrary, the great majority, 16 out of 21, are massed close to the upper limit. These 16 varied from 38 to 47·9. This is obviously the normal type and in it the Vena Cava is approximately four times the mean length of the Innominates. And it may be said that in this type the blood coming from the fore-limbs and head runs into a single channel at the earliest possible moment. There was one case in which the Vena Cava sank to just under 35 %, but which, nevertheless, probably belongs, we think, to the normal category; because this was the cat which had the longest trachea (26·7 %), and if the Vena Cava be judged either by the total length of the animal or by the Innominates, it comes up to the normal length. Then there are three other cases which varied from 29 % to 33 %. These are very possibly a different type, but the numbers are at present too small for us to speak with confidence on this point. In this second type the Vena Cava was approximately twice the length of the Innominates.

Finally, we had one animal which was clearly in quite a different class. In this specimen the Vena Cava was only 19 % the length of the trachea and the two Innominates were themselves swollen almost to the proportions of Venae Cavae. The mean length of the Innominates was, in fact, slightly greater than that of the Vena Cava. It is owing to the existence of this specimen that, notwithstanding the small total number, we think the curve of variation cannot be that of normal continuous variation. In view of the evolution of the single Vena Cava we think that even these preliminary results may be of some interest and, at present, they would appear to be congruous with a discontinuous method of evolution of the single Vena Cava condition.

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A Note on Vital Staining. By F. A. POTTS, M.A., Trinity Hall.

[*Read* 22 November, 1920.]

Great confusion exists as to the phenomena which are classed under the term 'vital staining' but it is established that certain stains can penetrate the living cell and enter into combination with bodies in the cell without apparently affecting the normal course of cell life. These bodies are principally granules of various kinds and some of them are undoubtedly concerned with the process of secretion. How far they are actually part of the living protoplasm is a point at issue. Very little work has been done, in the Metazoa at least, on the further history of stained granules of this kind, and it seems to offer a promising field for students of vital processes. Exceedingly interesting in this connection are the observations of Oxner* on Nemertines, the mucus cells of which contain numerous granules staining with neutral red and methylene blue but produce colourless mucus. The earlier work of Apathy† had brought to light similar effects and is more satisfactory in that the author states that he has seen in the mucus cells of Hirudinea granules stained with methylene blue actually forming blue mucus. It is not obvious that Oxner had definitely traced the genesis of his colourless mucus from stained granules.

The Nematoda are a group which offers wonderful facilities for researches of this kind, for the smaller members of the phylum are of an almost perfect transparency, are exceedingly hardy under experimental conditions, and contain many kinds of granules which take up vital stains. Cobb‡ has already indicated this extraordinary suitability of the free-living forms in particular and appealed to scientific workers to give their attention to these and other related problems. I wish in this paper to briefly summarise some observations made on a species of *Diplogaster* which was found in garden soil in Cambridge during 1919.

Neutral red was found to be by far the most effective stain. In fact no others of those used showed up the granules which I go on to describe here. The solutions employed were very concentrated especially if compared with the exceedingly dilute ones which give the best result for freshwater Crustacea (Fischel) and marine organisms. A solution made up more than ten years ago of neutral red in distilled water was especially good, its staining properties had evidently improved with age and it was almost non-toxic.

* Oxner, *Bull. de l'Institut. Oceanograph*, Monaco, 1908.

† Apathy, *Zeitschr. wiss. Mikr.*, Bd. 9, 1892. *Cf.* also Hardy, *Journ. Physiol.*, vol. XIII, 1892.

‡ Cobb, *Natural Science*, vol. 46, 1917, p. 167.

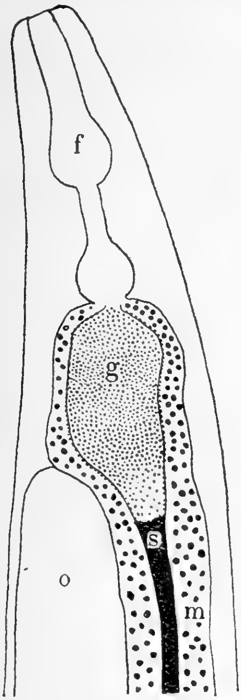
The nematode is surrounded by an elastic cuticle which is continued into the gut at the mouth and anus. Anteriorly it lines the buccal cavity and oesophagus and posteriorly the short rectum. When the animal moults this lining is detached with the external cuticle and retains its shape after detachment. The midgut or intestine consists of a single layer of cells which according to some authorities (*Cambridge Natural History*, Vol. II. p. 131) is 'coated internally and externally by a layer of cuticle.' In my experience, however, of free-living forms the cells of the midgut do not possess a firm and definite cuticle like that of the oesophagus and rectum but the cell membrane is thin and without any apparent structure.

The frequent pulsation of the second oesophageal bulb is the means by which the liquid culture medium is pumped into the gut of the nematode. Hence, when the animal is put into a solution of stain this rapidly penetrates into the cavity of the oesophagus and the anterior part of the midgut. It is some time, however, before the whole lumen of the gut is filled with the stain. But none of the neutral red ever penetrates through the external cuticle. The cuticle of the buccal cavity and oesophagus is apparently porous enough to allow the penetration of a certain amount, but the greater part passes through the cell membranes of the midgut. When a nematode has been in stain for about six hours it presents a remarkable appearance, the gut being stained pink or purple in its various parts while the remaining tissues are still perfectly transparent. The stain is taken up by the various scattered granules of the gut-cells and when their capacity for stain is exhausted, which does not occur till the experiment has lasted about twelve hours, the neutral red begins to pass through the external membrane of the gut and irregular deposits appear in the genital organs and the lateral fields of the hypodermis. Lastly, the muscle fibres take up the stain, showing a transverse banding of the individual fibres. It does not appear that the slow penetration is due to the resistance of the external membrane of the gut, but merely to the avidity with which the gut granules take up the stain.

In the midgut cells of *Diplogaster* these granules fall into two categories: (1) brownish highly refractive granules of various sizes, resistant to most reagents and scattered irregularly through the cell. Their composition is unknown, but they take up a good deal of stain. It is not proposed to deal with them in the present paper. (2) Smaller secretory granules of uniform size forming a peripheral zone round the lumen of the gut, normally colourless, but purple with neutral red. They are especially thickly developed in the first four cells between which the gut cavity is enlarged and often occupied by a bolus of living or dead bacteria.

There are, however, individuals which do not show the granular zone, but instead the gut-lumen appears to be lined with a structureless highly refractive layer which stains a brownish-red. Some Diplogasters show the granules anteriorly in the gut and the structureless layer posteriorly, but never both together. The explanation of this distribution is given by observing that the granules break down and form the substance of the structureless layer. This is a change which can be watched, beginning posteriorly and working slowly forward until the whole of the midgut is lined by the substance.

The structureless layer is not a firm cuticle as is shown by the fact that when a stained *Diplogaster* is compressed under a cover-



Diagrammatic figure of *Diplogaster* sp. to show staining of midgut with neutral red after about six hours. In the specimen figured the individual is in an intermediate state, the granules being well developed anteriorly but have broken down posteriorly to form the secretion. The cell boundaries and nuclei of the midgut are omitted.

f. oesophagus.
g. granules surrounding lumen of midgut.
s. secretion formed by breaking down of these.
m. midgut with larger resistant granules.
o. ovary.

slip to such an extent that the contents of the body are squeezed out through the anus, as the gut lining passes through the narrow aperture it changes its form and flows like a plastic material. Moreover the lining of the midgut does not appear as a definite detached layer in moulting or after the death and disintegration of the animal, as is the case in the cuticle of the oesophagus and rectum.

That the granules and the substance they produce are not artifacts or products of degeneration is shown by the fact that they are visible in unstained nematodes, taken fresh from cultures where they were growing under the most favourable conditions. Neutral red in this case only serves to differentiate pre-existing structures which are otherwise very difficult to make out.

The function of the layer is very doubtful. It might be supposed that it is digestive and the great development of the granules in the anterior part of the midgut where the lumen is enlarged and often contains an accumulation of bacteria and organic fragments would seem to support this view. But living bacteria exist embedded in or attached to the lining and appear to grow and divide freely there as in a culture medium. From this observation it seems possible that a symbiotic relationship exists between the bacteria and the nematode which is in some way furthered by the secretion of a soft lining to the gut.

While the granular phase is often seen passing into the plastic phase I have not succeeded in observing a natural evacuation of this lining preparatory to the formation of fresh granules in the gut-cells. I hope, however, that further study will reveal the relation of this phenomenon to the life of the nematode. To this end it will be necessary to find out whether it is a regularly, or at least frequently, repeated event.

Preliminary note on a Cynipid hyperparasite of Aphides. By MAUD D. HAVILAND, Fellow of Newnham College. (Communicated by Mr H. H. BRINDLEY.)

[*Read* 22 November 1920.]

Aphides are liable to parasitisation by certain Braconidae of the sub-family Aphidiidae. The larva develops in the haemocoele of the host, which dies just before the metamorphosis of the parasite, and the latter lines the empty skin with silk and pupates within it. Throughout its larval life the *Aphidius* is in its turn liable to parasitisation by certain Chalcids, Cynipids, and Proctotrypids, which are thus hyperparasites of the aphid. I described the development of one of these hyperparasites, a Proctotrypid, *Lygocerus cameroni*, Kieff., in a paper read before the Society last February. The following is a summary of some observations made on the development of certain Cynipid endo-hyperparasites of the genus *Charips*, formerly known as *Allotria*. The aphid used in the breeding experiments was *Macrosiphum urticae*, Kalt. from the nettle, and the primary, or host, parasite was *Aphidius ervi*, Hal.

These Cynipids have long been known to be hyperparasites, but at first I could not induce them to oviposit in captivity. The cause of this failure was that the material offered them was in too advanced a stage; for, unlike most of the hyperparasites of this group, which do not oviposit until the aphid is dead and the primary parasite is in metamorphosis, these Cynipidae seek out an *Aphidius* preferably in the third, or early in the fourth, instar, though a second instar larva may also be chosen. At this time the parasite is lying in the host's body cavity, and the aphis feeds as usual. Until twelve hours or so before its death there is no external sign that it contains a parasite, and yet the Cynipid unerringly recognises the presence of the latter and ignores unparasitised aphides when they are offered to it. A necessary condition for oviposition appears to be that the aphid host should be alive, and the *Aphidius* be still bathed in its body fluids. Larvae which had emptied the Aphid's skin of its contents, and had already begun to spin the cocoon, were never selected.

In captivity, ovipositions were sometimes, but not always, parthenogenetic. The female *Charips* ran among the aphides, tapping them excitedly with her antennae. When a victim was chosen, she leaped upon its back, facing the head, and clung there in spite of its struggles. Sometimes she was thrown off, but in such cases always returned to the attack, until the aphid became passive from exhaustion. The oviposition took from three to six minutes to complete, which is not surprising when it is remembered that the ovipositor must be thrust through the chitin and body wall of the aphid before probing for the *Aphidius*. Even then the organ must possess an exquisite sense of touch, for the gut of the host larva is so distended with food that the haemocoele is correspondingly reduced; and if the ovipositor were to be driven in a fraction too far the egg might be deposited in the gut cavity, and be lost at evacuation of the meconium.

The egg is oval, with a pedicel or stalk at one end and a smooth chorion. As development proceeds it becomes more spherical and the stalk disappears. It hatches about two to three days after oviposition. As in certain other internal Hymenoptera Parasitica, a 'pseudo-serosa,' or envelope of large deeply staining cells, is developed round the embryo, and presumably fulfils a trophic function. The remains of this embryonic membrane may be found in the host when the larva has ruptured and emerged from it.

The newly hatched larva is a remarkable form, armoured with dark segmental plates of chitin, which render it easily visible through the tissues of the host. There are a distinct head and thirteen body segments, the last terminating in a long tail. The mouth is produced into a proboscis, within which lie two simple mandibles. The head is furnished with four pairs of chitinous nodules, three on the ventral and one on the dorsal side. Each bears a transparent spot at the summit, possibly sensory in function. The anus, which is dorsal to the cauda, is a large conspicuous structure surrounded by a chitinous ring, and striae of chitin may be seen radiating into the lumen. At this stage, as in other Hymenoptera, there is no passage from the mid- to the hind-gut, but the bulb-shaped cavity of the latter and the wide anus, suggest that in this form it may serve some especial function in early larval life. The duration of this stage is variable. In one observed case, the chitin had been cast when the larva hatched, and was left behind in the pseudo-serosa. In other instances, it lasted from two to four days. Three or four of these larvae may be found in the same host, but, so far as is known, only one reaches maturity. In ecdysis the chitinous skin either splits down the ventral median line or else transversely across the thorax.

The second stage larva differs from the first chiefly in the absence of the chitinous plates. It is transparent, and the gut contents tinge it pale yellow. The mouth parts are less produced, but the ventral papillae on the head are more conspicuous and the first three segments bear ventral processes.

In the third stage, the tail is greatly reduced, the thoracic processes disappear, and the cephalic papillae are hardly visible. It was not ascertained whether there was actually a moult between these two stages, or whether the change of form was due merely to growth and absorption. In the fourth stage, the tail, appendages, and papillae disappear, and the anus is proportionately smaller than in the preceding instars.

All this time, the parasite, with its head orientated towards that of the host, lies in the haemocoele of the latter against the distended mesenteron, which, under the pressure, becomes much constricted. The *Aphidius* remains apparently healthy and retains some power of movement when irritated. It secretes silk as usual; but, immediately after the cocoon is woven, its development is in some way arrested, for the contents of the gut are never voided and metamorphosis does not take place. Death occurs only when the Cynipid larva is almost fully fed, and to the last the tissues remain fresh and undiscoloured. The *Charips*, which until then has been apneustic, makes its way out through the host's thorax, and its tracheal system becomes functional. Within the next twelve hours the hyperparasite devours the remains of the host, and prepares for metamorphosis within the cocoon inside the aphid's skin already woven by the *Aphidius*.

The full-grown larva is an apodous form with a well-developed head and thirteen body segments tapering somewhat posteriorly. The skin is soft and smooth, and the gut contents, seen through the white fat-body, give it a greenish colour. The buccal armature consists of labrum, mandibles, maxillae, and labium. The labrum is crescentic and bears eight small papillae. The mandibles are strongly chitinised, notched, and connected by powerful muscles to the endoskeleton of the head. The maxillae each bears a raised disc on which are three minute papillae, one of which terminates in a short seta. The labium is large and oval, and is furnished with two pairs of papillae. The salivary duct, which is conspicuously dilated immediately below its orifice, opens on the floor of the mouth under the U-shaped hypopharynx. There are six pairs of open spiracles in the full-grown larva, namely, between the first and second segments, and on segments 3, 4, 5, 7 and 9. Two larvae, out of the considerable number examined, had, in addition, a pair of spiracles on segment 8.

Pupation lasts from three to four weeks, and the total period of development seems to be from thirty to thirty-five days. The imago, when ready to emerge, gnaws a hole in the aphid's skin and creeps out. The adult insects feed on the honey dew of the aphides, which is either sipped from the leaves where it has fallen or else from the anus of the living insect.

The economic importance of this hyperparasite is probably not great, but to some extent it must be considered injurious, since it checks the *Aphidius* in its destruction of plant-lice. The inter-relations of aphides with their parasites and hyperparasites form a bionomical complex of considerable intricacy. Thus, this Cynipid hyperparasite, equally with its Braconid host, is liable to destruction by the Chalcid or Proctotrypid ecto-hyperparasites; and, from observations made in the course of this work, the evidence points to the conclusion that where the incidence of Chalcid and Proctotrypid hyperparasitism is high, few of the Cynipids survive, for they are not only dependant upon the occurrence of the host, but share its vulnerability to other parasites throughout their larval life. I attribute to this the fact that from collections of parasitised aphides, made in the field, there were proportionately more Cynipid emergences in June than in July. Most of the hyperparasites obtained from later collections were external-feeding Chalcids or Proctotrypids; and the inference is that the later broods of Cynipids suffered from a second parasitisation of their host by other hyperparasites.

A method of testing Triode Vacuum Tubes. By E. V. Appleton, M.A., St John's College.

[*Read* 22 November 1920.]

The circuit to be described affords a simple and convenient method of measuring the slope of the principal voltage-current characteristic of a triode vacuum tube. It is now well known that the effectiveness of such a tube as a relay or amplifier depends primarily on the efficiency with which the magnitude of the thermionic current can be controlled by means of the grid or intermediary electrode; that is to say, we are mainly concerned in practice with the rate of variation of anode current with grid voltage, and not with the absolute magnitudes of either quantity. The characteristic mentioned is determined if sufficient correlated values of the anode current I and the grid voltage v are known. To obtain such data accurate ammeters and voltmeters are required. Moreover, even if these values for the characteristic are known, the accurate determination of dI/dv by the ordinary graphical methods takes considerable time. As the dimensions of the slope of the characteristic are those of a conductance it seems preferable to dispense with ammeters and voltmeters and determine this quantity directly in terms of a standard resistance.

A circuit designed to determine the mean value of the mutual conductance ($\Delta I/\Delta v$) over any range of grid voltage was suggested by the writer two years ago*. With this method static currents and voltages were used. In developing the circuit for use since then a great increase in both accuracy and utility has been found to accompany the use of small alternating voltages of acoustic frequency. With the sensitive alternating current detectors now available it is possible to decrease the limits of the variables involved so that the value of $\Delta I/\Delta v$ obtained by this method differs inappreciably from the value of dI/dv obtained from a graphical analysis of the complete characteristic. We shall assume that these values are identical in the following discussion.

The circuit used is shown in Fig. 1. An alternating voltage of acoustic frequency from the alternator A is applied between the grid and the filament of the tube and also across the two non-reactive resistances R and r, the former of which is variable while the latter is small and constant. The anode and grid batteries (B_1 and B_2) are suitably fixed to give the operating conditions for

* See J. A. Fleming, *The Thermionic Valve and its Development in Radio-Telegraphy and Telephony*, p. 142.

which the conductance is required. To determine the value of dI/dv the resistance R is varied until the potential difference between the points C and D is constant. The value of dI/dv is then given simply as $1/R$. An amplifying-telephone is connected across the resistance r, the constancy of the potential difference between C and D being indicated by a note of minimum intensity in the telephone.

To determine the actual voltage amplification obtainable in any particular case (e.g. in a resistance-coupled amplifier such as has been suggested for the measurement of small ionization currents) a suitable high resistance R_0 may be included in the anode circuit while the test is made. The voltage amplification then available is given by $R_0 \dfrac{dI}{dv}$.

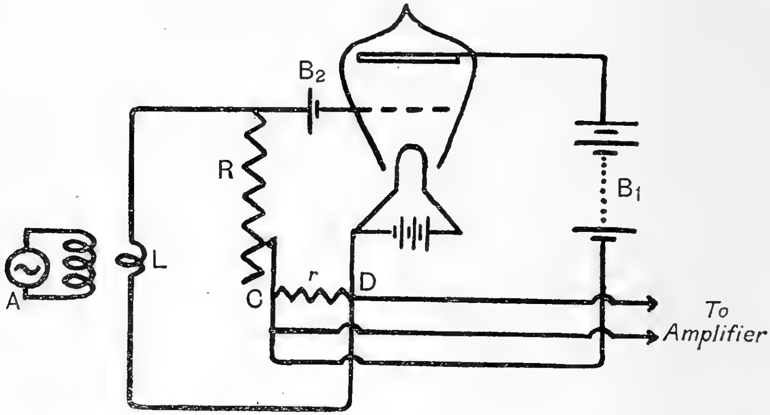


Fig. 1.

Fig. 2 shows some typical results obtained with a hard valve (Type R). For comparison the actual v-I characteristic is shown in the same figure.

Proof of Formula. Before the alternator current is switched on the potential difference between the ends of the resistance r is seen to be $I_0 \dfrac{Rr}{R+r}$ where I_0 is the stationary value of the thermionic anode current. If the alternating voltage applied between the grid and filament is $v_0 \sin pt$ the change in the potential difference between C and D may be regarded as the algebraic sum of two oppositely directed changes. The first is due to the direct effect of the applied alternating voltage and is equal to $\dfrac{rv_0 \sin pt}{r+R}$; the second may be regarded as due to the alteration of anode current from I_0

to $I_0 + \frac{dI}{dv} v_0 \sin pt$. For these two oppositely directed effects to be equal in magnitude we have

$$\frac{rv_0 \sin pt}{r+R} = \frac{Rr}{R+r}\left(I_0 + \frac{dI}{dv} v_0 \sin pt\right) - I_0 \frac{Rr}{R+r},$$

that is
$$\frac{dI}{dv} = \frac{1}{R}.$$

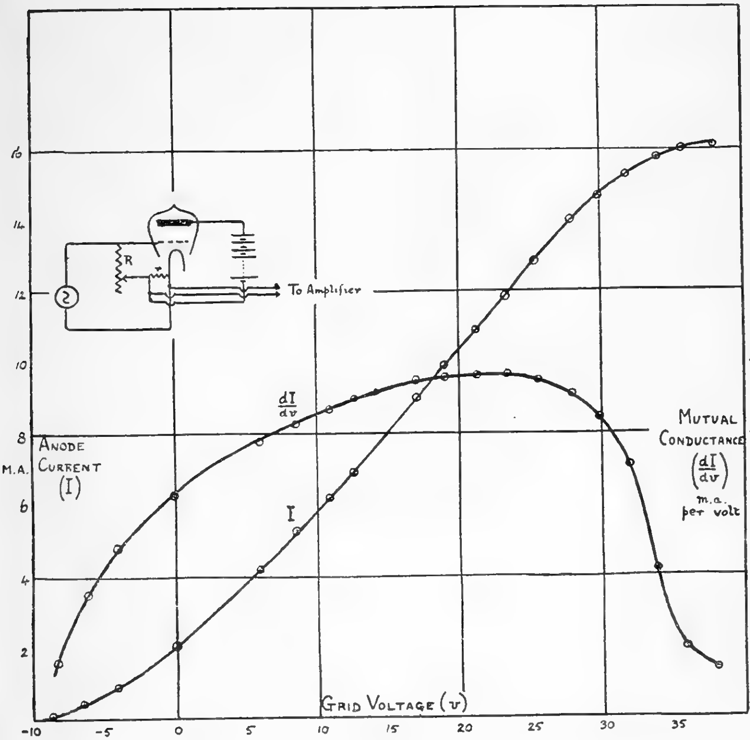


Fig. 2.

It will be seen that the method is self-compensating so far as the effect of the resistance r is concerned so long as the impedance of the coil L is small compared with R as is usually the case in practice. If, however, it occurs that the impedance of L is large compared with r the result is still a simple one, the value of dI/dv being then given by $1/(R+r)$. It will also be noticed that the result is independent of the angular frequency p of the applied alternating voltage. Thus a sinusoidal voltage is not essential and a buzzer source may be used.

The definiteness of the point of null telephone response depends on the fidelity with which the anode current changes follow the grid voltage variations. The circuit is therefore suggested as a possible method of testing whether any temporal 'lag' exists between these two quantities for any particular tube. So far as tubes of extreme exhaustion have been tested no such inertia has been found to exist. For cases of ultra-acoustic frequencies the ordinary wireless methods of testing the constancy of the potential across *CD* are applicable.

In addition to its use as an amplifier the vacuum tube has a distinct field of utility as a detector of high-frequency oscillations. When operation takes place with conditions represented by a curved portion of the characteristic a symmetrical grid voltage change produces a variation in the mean value of the anode current. As a first approximation in such a case one may consider that the magnitude of the anode current alteration is proportional to the value of d^2I/dv^2 and also to the square of the amplitude of the grid voltage variation.

It is possible to determine the value of d^2I/dv^2 directly. To do this the alternator of Fig. 1 is replaced by a small voltage battery (positive to grid) and tapping key, and the resistance *CD* by a milli-ammeter. The resistance *R* is adjusted so that the milli-ammeter deflection is unaltered when the key is closed. If the value of the resistance so obtained is R_1, we have approximately

$$\frac{1}{R_1} = \frac{dI}{dv} + \frac{1}{2}\frac{d^2I}{dv^2} \cdot \Delta v, \qquad \text{......(1)}$$

where Δv is the E.M.F. of the small battery.

If the same adjustment is now made with the battery reversed (negative to grid) we have

$$\frac{1}{R_2} = \frac{dI}{dv} - \frac{1}{2}\frac{d^2I}{dv^2} \cdot \Delta v. \qquad \text{......(2)}$$

Thus from (1) and (2) we have

$$\frac{dI}{dv} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

and

$$\frac{d^2I}{dv^2} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \cdot \frac{1}{\Delta v}.$$

With such an arrangement the results are accurate to within 3 or 4 per cent. If extreme accuracy is required the milli-ammeter may be replaced by a resistance *r* (see Fig. 1) and a potentiometric scheme and sensitive galvanometer used to indicate when the voltage drop along *CD* is constant.

The Rotation of the Non-Spinning Gyrostat. By Sir GEORGE GREENHILL and Dr G. T. BENNETT.

[*Read* 22 November 1920.]

Dr G. T. Bennett has done me the honour of criticizing a statement in § 14, p. 13, of my *Report to the Aeronautical Committee on Gyroscopic Theory*, 1914 (cited as R. G. T.), and I am pleased to have this opportunity of meeting his objections.

The diagrams of Flatland on a sheet of paper are inadequate in a discussion concerning Rotations in Space; it is advisable then to have a physical representation at hand, such as that described in Fig. 3 of R. G. T., where the displacements can be visualised, without the confusion arising from thinking of the different sides of the sheet of paper of a diagram.

In this model of Fig. 3, of an Altazimuth suspension of a stalk, as an axle carrying a gyroscopic flywheel, the three angles θ, ψ, ϕ, introduced into Dynamics by Euler in 1760, and standard to this day in mathematical treatment, may receive the corresponding astronomical names; θ, measured from the downward vertical Oz, may be called the Nadir Distance; ψ will then be called the Azimuth; and Euler's third angle ϕ, the angle the wheel has turned over the axle OZ, may then be called the Hour Angle.

The point in dispute is concerning this hour angle ϕ.

On the model, of the stalk with altazimuth suspension, carrying a flywheel moveable about the stalk as a smooth axle, the hour angle ϕ will represent the extent to which the wheel has rubbed round relatively to the axle stalk. This displacement can be shown unmistakably by chalk marks, originally in coincidence on wheel and axle, and also on the frame, and in their subsequent divergence, to show the increase in ϕ, and also in ψ.

Dr G. T. Bennett will oblige greatly if he adheres to Euler's standard notation; so I will change his equations (1), (2), (3) into

$$\frac{d\phi}{dt} + \cos\theta\frac{d\psi}{dt} = 0, \quad \frac{d\phi}{dt} + \frac{d\psi}{dt} = \operatorname{vers}\theta\frac{d\psi}{dt} = \frac{d\sigma}{dt}, \quad \phi + \psi = \sigma,$$

when the flywheel is not set spinning. Here $d\sigma$ denotes the polar element of area described on the unit sphere by Z the end of the stalk; and, in an incomplete circuit, σ must be taken to represent the solid conical angle subtended by the spherical area bounded by the arc described by Z, and the two great circles proceeding from z to the ends of the arc.

When the circuit is complete, the solid angle described will be represented by the spherical area; but there is a discrepancy of 2π to be considered, according as the area encloses the nadir z or not, when ψ increases by 2π, or oscillates and returns to its original value; but we will not delay over this here.

When the circuit of the pole Z is incomplete, the geometrical interpretation of $\phi + \psi$ is not simple, as will be seen on reference to (6), p. 71, R. G. T.; and the angle does not appear in an inspection of the model. But ϕ and ψ separately are visible in the chalk-marks and their divergence; but the angles are in different planes.

It is only in the complete circuit, where the stalk OZ is brought back to its original position, that the angle can be visualised through which the flywheel has turned relative to the frame.

In the statement of R. G. T., § 14, criticized by Dr Bennett in his § 4, where in the model the stalk is hanging vertical, $\theta = 0$, and the stalk is then revolved in azimuth, the solid angle $\sigma = 0$, $\phi + \psi = 0$. But in a complete revolution of the stalk, $\psi = 2\pi$, and $\phi = -2\pi$; that is, the wheel has rubbed once round the axle inside; and ϕ does not represent the angle the wheel has turned in space; the wheel remains stationary with respect to the frame and the stalk turns round inside the wheel; but $\phi + \psi = 0$ at any intermediate stage.

Dynamics were never studied at Cambridge as an experimental science, so it is not likely that the model of R. G. T., Fig. 3, should exist there, to place on the lecture-table between the lecturer and his class, for him to show off the variation of Euler's angles, and to handle and feel an actual state of gyroscopic motion. The non-spinning gyroscope is imitated by a plummet at the end of a thread, as a spherical pendulum, when the wheel has no rotation; and by giving the wheel a spin R, the extension is made to the most general state of the spinning gyroscope.

But a penholder is always at hand, to serve as an illustration of the angular creep in ϕ. The action is not frictional as has been objected. Butter the finger to acquire the perfect smoothness of the text-book jargon, and the creeping action is not arrested.

The experiment is not inapt, as it shows the relative motion apparently reversed, where the axle is fixed in the wheel and runs in outside bearings; here the finger and thumb, perfectly buttery. So too with the other homely familiar experiments cited by Dr Bennett.

Euler's angles require to be interpreted on the altazimuth suspension of the stalk in Fig. 3. But Prandtl suppresses the complication of the hub and vertical spindle on ball-bearings, and replaces it by the simple economical arrangement of a fixed hook, to which the stalk is hooked up, with the rubbing surfaces well greased. The motion of the stalk here is the same as with Kelvin's trunnion rings and knife-edge gimbals, or his short length of elastic wire, fixed to the stalk and the support; and it may be imitated in the swaying motion of the body seated in a chair, where one wall of the room is always faced; the conical motion does not make the body turn on the seat, not a music-stool.

It would carry us too far to discuss the modification in ϕ due to these and other modes of suspension, such as Hooke's joint and bevel-wheels.

But so long as the inertia of the stalk may be ignored, there is no modification in the θ, ψ angles of the axle of the gyroscopic wheel; but the angle ϕ will require separate consideration.

If however ϕ is suppressed by clamping the wheel to the stalk, or if the inertia of the stalk is taken into account, the motion is hyperelliptic and intractable.

The question quoted from the 1898 examination paper is a very good specimen of many such, scattered anonymously in college papers. Judging from the date, Dr Bennett ought to be able to lift for us the veil of anonymity.

A similar question by Maxwell in the Mathematical Tripos 1869, on vibration in its effect in causing a permanent deviation in a pendulum, has proved useful, nearly 50 years later, in the interpretation of compass deflection; and we have seen the deflection realised in an experiment devised by Mr C. C. Mason.

Quaternions come in useful for the geometrical interpretation of these questions on finite rotation; as for instance in the resultant rotation due to successive rotations of a spherical triangle through the exterior angles. Then there is a theorem given by Dr W. Burnside in the *Messenger of Mathematics*, XXIII., on the resultant screw displacement due to two half turns about non-intersecting axes.

The 1898 question, on the rotation that can be given to a body on a smooth axle by a conical motion, may be quoted as an answer to Aristotle's challenge—to rotate a smooth sphere—*ἥκιστα δὲ κινητικὸν ἡ σφαῖρα διὰ τὸ μηδὲν ἔχειν ὄργανον πρὸς τὴν κίνησιν.*

To the preceding remarks of Sir George Greenhill, Dr G. T. Bennett replies as follows:—

From Sir George Greenhill's comments it is happily apparent that the difference of four right angles between a result in his Report and a result in my paper is not a miscalculation of either but a discrepancy turning on a verbal ellipsis. My work has regard only to the movement of the gyroscope relatively to 'fixed space'; whereas his is limited to the movement of the gyroscope relatively to the rotating stalk of his altazimuth suspension.

In the matter of his experimental illustration with the pen-holder I was far from conjecturing that it was to be regarded as ideally smooth; and if that indeed is to be the case it must be further described as having precisely no angular velocity about its axis, if it is to imitate the gyroscope successfully.

It may be remarked that the suspension of a pendulous gyroscope by an altazimuth mounting is open to one grave objection: that the gyroscope loses one of its three degrees of freedom in its central position. And if the gyro-axis swings close past the vertical it involves in consequence excessively rapid changes in the azimuth movement of the suspension. The gunner who attempts to deal with aircraft passing overhead, using the ordinary gun-mounting, finds a corresponding difficulty with his training gear. *Διώκει παῖς ποτανὸν ὄρνιν.*

On the representation of the simple group of order 660 *as a group of linear substitutions on* 5 *symbols.* By Dr W. BURNSIDE, Honorary Fellow of Pembroke College.

[*Read* 22 November 1920.]

Except in the cases of two and of three variables, it is only few groups of linear substitutions of finite order the forms of which have been exhibited explicitly. This, it is hoped, will justify the following calculations, which seem to offer one or two points of interest. In particular the existence of a cubic three-spread in space of four dimensions, which admits a group of 660 collineations into itself, is perhaps noteworthy. The well-known cubic three-spread of Segre, defined by

$$(x_0 + x_1 + x_2 + x_3 + x_4)^3 - x_0^3 - x_1^3 - x_2^3 - x_3^3 - x_4^3 = 0,$$

admits a group of 720 collineations. That which admits the group of 660 collineations is defined by

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0.$$

The modular group, for $p = 11$, is a simple group of order 660. Its characteristics have been calculated*, and it is known to admit two representations as an irrational irreducible group on five variables. It is proposed here to set up these two representations. In one of them the multipliers of an operation P of order 11 are a, a^9, a^4, a^3, a^5; where a is a primitive 11th root of unity. Moreover the group contains an operation S of order 5 such that

$$SPS^{-1} = P^9.$$

Now when P has the above multipliers, there is only one representation of the cyclical group $\{S, P\}$ as a group of degree 5; and by suitably choosing the variables this can be brought to the form which is generated by

$$x_0' = ax_0,\quad x_1' = a^9x_1,\quad x_2' = a^4x_2,\quad x_3' = a^3x_3,\quad x_4' = a^5x_4,$$

and $x_0' = x_1,\quad x_1' = x_2,\quad x_2' = x_3,\quad x_3' = x_4,\quad x_4' = x_0.$

The variables for the required irreducible representation Γ may be therefore chosen so that Γ contains this subgroup. When the method† for finding the number of invariants of the nth degree of a group is applied to Γ it is found that the group has one cubic invariant. Now it is very easily verified that the only cubic invariant for the above metacyclical group $\{S, P\}$ is

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0,$$

and this must therefore be an invariant for Γ.

* Burnside, *Theory of Groups*, p. 502. † *loc. cit.* p. 301.

In Γ there are 5 operations A of order two, and characteristic unity, such that $ASA = S^{-1}$. Moreover Γ has no subgroup which contains an operation of order 11 and an operation of order 2. Hence P and any one of these 5 operations A will generate Γ.

If
$$\xi_i = x_0 + \omega^{-i}x_1 + \omega^{-2i}x_2 + \omega^{-3i}x_3 + \omega^{-4i}x_4,$$
$$(i = 0, 1, 2, 3, 4), \quad \omega^5 = 1,$$
the canonical form of S is
$$\xi_0' = \xi_0, \ \xi_1' = \omega\xi_1, \ \xi_2' = \omega^2\xi_2, \ \xi_3' = \omega^3\xi_3, \ \xi_4' = \omega^4\xi_4,$$
and the most general operation of order two which will transform S into its inverse, while its characteristic is unity, is
$$\xi_0' = \xi_0, \ \xi_1' = a\xi_4, \ \xi_2' = b\xi_3, \ \xi_3' = b^{-1}\xi_2, \ \xi_4' = a^{-1}\xi_1.$$

Now the above invariant, when expressed in terms of the ξ's, is a numerical multiple of
$$\xi_0^3 + 2(1 + \omega + \omega^4)\xi_0\xi_1\xi_4 + 2(1 + \omega^2 + \omega^3)\xi_0\xi_2\xi_3$$
$$+ (\omega + 2\omega^2)\xi_1\xi_2^2 + (\omega^4 + 2\omega^3)\xi_4\xi_3^2 + (\omega^3 + 2\omega)\xi_1^2\xi_3$$
$$+ (\omega^2 + 2\omega^4)\xi_4^2\xi_2.$$

The conditions that the above substitution of order 2 shall leave this unchanged are
$$ab^2(\omega + 2\omega^2) = \omega^4 + 2\omega^3,$$
$$a^2b^{-1}(\omega^3 + 2\omega) = \omega^2 + 2\omega^4,$$
and these equations clearly have 5 solutions. Now a and b must be rational functions of a and ω, for otherwise the group could not be one of finite order. They may be expressed in that form as follows.

Put
$$a + a^{-1} = \lambda_0, \ a^2 + a^{-2} = \lambda_1, \ a^4 + a^{-4} = \lambda_2,$$
$$a^3 + a^{-3} = \lambda_3, \ a^5 + a^{-5} = \lambda_4,$$
$$\mu_i = \lambda_0 + \omega^i\lambda_1 + \omega^{2i}\lambda_2 + \omega^{3i}\lambda_3 + \omega^{4i}\lambda_4,$$
$$(i = 1, 2, 3, 4).$$

Direct multiplication then gives
$$\mu_1\mu_4 = \mu_2\mu_3 = (\omega + 2\omega^2)(\omega^2 + 2\omega^4)(\omega^3 + 2\omega)(\omega^4 + 2\omega^3) = 11,$$
$$\mu_1^2 = (\omega + 2\omega^2)(\omega^2 + 2\omega^4)\mu_2,$$
$$\mu_2^2 = (\omega^2 + 2\omega^4)(\omega^4 + 2\omega^3)\mu_4,$$
$$\mu_4^2 = (\omega^4 + 2\omega^3)(\omega^3 + 2\omega)\mu_3,$$
$$\mu_3^2 = (\omega^3 + 2\omega)(\omega + 2\omega^2)\mu_1.$$
From these results it follows at once that
$$a = \frac{\mu_3}{(\omega + 2\omega^2)(\omega^3 + 2\omega)}, \quad b = \frac{\mu_1}{(\omega + 2\omega^2)(\omega^2 + 2\omega^4)},$$
involving
$$a^{-1} = \frac{\mu_2}{(\omega^2 + 2\omega^4)(\omega^4 + 2\omega^3)}, \quad b^{-1} = \frac{\mu_4}{(\omega^3 + 2\omega)(\omega^4 + 2\omega^3)}$$

is a solution of the above equations, the other solutions being $a\omega'$, $b\omega'^2$, where ω' is any primitive fifth root of unity. With these values of a and b, the operation A, when expressed in terms of the original variables is

$$\begin{aligned} 5x_i' = \sum_j \omega^{ij}\xi_j' &= \xi_0 + \omega^i a\xi_4 + \omega^{2i}b\xi_3 + \omega^{3i}b^{-1}\xi_2 + \omega^{4i}a^{-1}\xi_1 \\ &= x_0(1 + a\omega^i + b\omega^{2i} + b^{-1}\omega^{3i} + a^{-1}\omega^{4i}) \\ &\quad + x_1(1 + a\omega^{i-4} + b\omega^{2i-3} + b^{-1}\omega^{3i-2} + a^{-1}\omega^{4i-1}) \\ &\quad + x_2(1 + a\omega^{i-3} + b\omega^{2i-1} + b^{-1}\omega^{3i-4} + a^{-1}\omega^{4i-2}) \\ &\quad + x_3(1 + a\omega^{i-2} + b\omega^{2i-4} + b^{-1}\omega^{3i-1} + a^{-1}\omega^{4i-3}) \\ &\quad + x_4(1 + a\omega^{i-1} + b\omega^{2i-2} + b^{-1}\omega^{3i-3} + a^{-1}\omega^{4i-4}) \\ &\qquad (i = 0, 1, 2, 3, 4). \end{aligned}$$

On entering the values of a, b, a^{-1}, b^{-1}, it is found that there are only five distinct coefficients, viz.

$$-\tfrac{5}{11}(2\lambda_0 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4),$$

and those derived from this by cyclical permutation of indices. Finally, writing

$$\beta_i = 2\lambda_i + 4\lambda_{i+2} + 3\lambda_{i+3} + 2\lambda_{i+4},$$

the substitution A is

$$\begin{aligned} -11x_0' &= \beta_0x_0 + \beta_3x_1 + \beta_1x_2 + \beta_4x_3 + \beta_2x_4, \\ -11x_1' &= \beta_3x_0 + \beta_1x_1 + \beta_4x_2 + \beta_2x_3 + \beta_0x_4, \\ -11x_2' &= \beta_1x_0 + \beta_4x_1 + \beta_2x_2 + \beta_0x_3 + \beta_3x_4, \\ -11x_3' &= \beta_4x_0 + \beta_2x_1 + \beta_0x_2 + \beta_3x_3 + \beta_1x_4, \\ -11x_4' &= \beta_2x_0 + \beta_0x_1 + \beta_3x_2 + \beta_1x_3 + \beta_4x_4. \end{aligned}$$

This substitution and P, viz.

$$x_0' = \alpha x_0, \quad x_1' = \alpha^9x_1, \quad x_2' = \alpha^4x_2, \quad x_3' = \alpha^3x_3, \quad x_4' = \alpha^5x_4,$$

generate Γ. Since the invariant has rational coefficients, if it is unchanged by a substitution T, it must necessarily be unchanged by $\bar{T}$ derived from T by changing the sign of $\sqrt{-1}$ in each of the coefficients of T. It follows that Γ and $\bar{\Gamma}$ consist of the same set of substitutions, the correspondence between the substitutions and the operations of the abstract group being distinct for the two representations.

On the representation of algebraic numbers as a sum of four squares. By L. J. MORDELL. (Communicated by Professor H. F. BAKER.)

[*Received* 24 July 1920. *Read* 25 October.]

Professor Landau in a recent paper* entitled "Über die Zerlegung total positiver Zahlen in Quadrate" states that about twenty years ago, Professor Hilbert† gave without proof the theorem that "Every number in an algebraic field (provided that neither it nor any of its conjugate numbers are negative real quantities or if all of the conjugate fields are imaginary) can be expressed as the sum of the squares of four numbers of the field." This is an extension of the well-known theorem‡ due to Fermat and proved by Lagrange, that every positive integer§ can be expressed as the sum of the squares of four other integers. From this result it immediately follows that every positive fraction can be expressed as the sum of the squares of four other fractions, a theorem included in Hilbert's theorem, which is of course only a very special case|| in the arithmetical theory of quadratic forms with coefficients in a given algebraic field. The development of this theory however, is a matter of great difficulty, if only from the fact that it requires a knowledge of the laws of quadratic reciprocity in the field, the investigation of which, in even the simplest general cases, requires a lengthy and detailed although very interesting discussion¶.

Professor Landau gives a simple proof for the quadratic field based upon elementary algebra. He states however that he does not know whether the theorem holds universally. This indicates that a proof of Hilbert's general theorem is not easy, a view which is confirmed by considering the general theory underlying the question. The following proof for a cubic field may therefore be of interest**.

§ (1). Let then x be the root of an irreducible cubic equation

$$x^3 - ax^2 + bx - c = 0 \quad \text{.......................(1)}$$

* *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen,* 1919, pp. 392–396.

† *Grundlagen der Geometrie,* § (38).

‡ Bachmann, *Zahlentheorie,* vol. 4, p. 151.

§ The terms integers, fractions, etc. refer to rational quantities unless otherwise stated.

|| See also an account of some of A. Meyer's work on equations of the form $ax^2 + by^2 + cz^2 + dt^2 + eu^2 = 0$ in Bachmann, *Zahlentheorie,* vol. 4, pp. 259, 266.

¶ See for example Hilbert, *Ueber die Theorie des relativquadratischen Zahlkörpers, Mathematische Annalen,* vol. 51, pp. 1–127. An elementary introduction is given in Sommer "Vorlesungen über Zahlentheorie," section 5, of which there is a French translation.

** I am very greatly indebted to Professor Landau for having gone through this paper and for having suggested a number of improvements in the exposition.

where there is no loss of generality in supposing that a, b, c and all the rational numbers dealt with in this paper are integers, except when obviously otherwise. Any number f in the cubic field can be expressed in the form

$$Df = Ax^2 + Bx + C$$

where A, B, C and D are integers, of which D is positive. Hilbert's theorem then asserts that

$$f_1, f_2, f_3, f_4$$

where

$$D_1 f_1 = A_1 x^2 + B_1 x + C_1, \text{ etc.}$$

and A_1, B_1, etc. are integers, can be found so that

$$f = f_1^2 + f_2^2 + f_3^2 + f_4^2$$

provided that neither f nor any of its two conjugate numbers are negative real quantities. We shall refer shortly to this theorem by saying that H holds for f.

The case when $f = 0$ can be dismissed at once by noting that

$$0 = 0^2 + 0^2 + 0^2 + 0^2.$$

§ (2). If H holds for two numbers F_1, F_2, it also holds for their product since as is well known

$$(f_1^2 + f_2^2 + f_3^2 + f_4^2)(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2$$

where $$\psi_1 = f_1\phi_1 + f_2\phi_2 + f_3\phi_3 + f_4\phi_4 \text{ etc.};$$

and also for their quotient since

$$F_1/F_2 = F_1 F_2/F_2^2.$$

In particular if n is any positive rational fraction, H holds for nf if it holds for f.

Obviously H holds for the number $x^2 + q$ if q is a positive integer which can be expressed as the sum of the squares of three integers. This is always* the case except when q is of the form $4^\beta (8m + 7)$; and we shall say for shortness that any number, positive or negative, of the form $4^\beta (8m + 7)$ is of the form M. From the definition it is necessary but not sufficient that

$$M \equiv 0, 4, \text{ or } 7 \pmod 8 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

In particular the highest power of 2 dividing M must have an even exponent. Similar results hold for $x^2 + q$ when q is a fraction (positive of course)

$$\lambda/\mu = \lambda\mu/\mu^2$$

such that $\lambda\mu$ is not of the form M, and we can still say in this case that the fraction q is not of the form M. It is also clear that H holds for

$$ax^2 + bx + c = [(2ax + b)^2 + 4ac - b^2]/4a$$

* Bachmann, *Zahlentheorie*, vol. 4, p. 146.

if a is positive and the discriminant $4ac - b^2$ is neither negative nor of the form M.

It will be useful hereafter to prove now that if q is any integer, an even integer p can be found so that $q - p^2$ is not of the form M. This follows from (A), for if

$$q \equiv 1, 2 \pmod 4, \text{ take } p \equiv 0 \pmod 2,$$

if $\quad q \equiv 3 \pmod 8, \quad$ take $p \equiv 0 \pmod 4$,

if $\quad q \equiv 7 \pmod 8, \quad$ take $p \equiv 2 \pmod 4$.

If however $\quad q \equiv 0 \pmod 4$

put $\quad q = 4q_1,\ p = 2p_1;$

then since

$$q - p^2 = 4(q_1 - p_1^2)$$

the result follows by induction since if

$$q_1 \equiv 0 \pmod 4,$$

we can put $\quad q_1 \equiv 4q_2,\ p_1 = 2p_2$, etc.

§ (3). Hilbert's problem can be simplified by writing

$$y = Ax^2 + Bx + C$$

so that x can also be expressed rationally in terms of y by aid of equation (1), while y is also the root of an irreducible cubic*. Hence the whole question can be reduced to proving H for x where

$$x^3 - ax^2 + bx - c = 0 \quad \text{.................}(1)$$

where the real roots of the cubic and hence also c are positive. The cubic in x can be written as

$$x = \frac{(a+p)x^2 + (q-b)x + c}{x^2 + px + q} \quad \text{..............}(2)$$

where p and q are entirely arbitrary. If we can find rational values for p and q so that the discriminants

$$\rho = 4q - p^2 \text{ and } \sigma = 4c(a+p) - (q-b)^2 \quad \text{......}(3)$$

are both positive and neither of them of the form M, then H holds for both the numerator and denominator of x and hence for x.

Suppose now that a, b and c are all positive, including in particular the case when all the values of x are real. Then by considering the region common to the two parabolas

$$4\eta - \xi^2 = 0 \text{ and } 4c(a+\xi) - (\eta - b)^2 = 0 \quad \text{......}(4)$$

it is obvious that real and hence also rational values of p and q can be found to make each of the discriminants in (3) positive. Moreover, if we can find any rational values† of p and q for which neither of the

* Except in the trivial case $A = B = 0$.

† It is by no means obvious that such values exist. For example, one of the two expressions, $k + 7$, $512 - k^2$ is always of the form M. The same applies to the two expressions pq, $512q^2 - (7q - p)^2$.

discriminants in (3) is of the form M, it is clear that the same holds when p and q are replaced by

$$p + 2^N\lambda,\ \ q + 2^N\mu$$

if N is sufficiently large and λ, μ are fractions with odd denominators, and that λ, μ, N can be taken so as to bring the point

$$(p + 2^N\lambda,\ q + 2^N\mu)$$

within the common region of the parabolas (4), that is, ρ and σ will be both positive.

We can summarise the rest of this section § (3) by saying that we shall show that suitable values for p and q can be found when either a or b or c is odd, so that we must discuss the case when a, b and c are all even. It is then shown that suitable values can be found for p and q when the highest power of 2 dividing c is the first or second, and that the case when c is divisible by any power of 2 can be reduced to one of the preceding cases. It then follows that H is true for x in the case of a cubic with three real roots.

Suppose first then that a is odd and that 4^γ is the highest power of 4 contained in c. Put $c = 4^\gamma C$ and take

$$q - b = 2^{\gamma+1}\, Q,\ p = 2P$$

so that

$$\sigma/4^{\gamma+1} = C\,(a + 2P) - Q^2,\ \ \rho/4 = q - P^2.$$

Consider first the case when C is even so that it must be divisible by 2 only and not by 4. Then because a is odd, σ is not of the form M if Q is even, whatever P may be.

Also by § (2) we can find a value for P for which ρ is not of the form M so that H holds for x.

If, however, C is odd, take P even but Q even or odd according as

$$Ca \equiv 1 \text{ or } 3 \pmod 4,$$

so that σ is not of the form M. The only restriction on P is that it should be even, and from § (2) we can find even values of P for which $\rho/4 = q - P^2$ is not of the form M when q is given. Hence H holds for x if a is odd.

Writing now $x = c/y$ so that H holds for x if H holds for y and conversely, we see from

$$y^3 - by^2 + acy - c^2 = 0$$

that if b is odd, H holds for y and hence for x. Hence we need only consider the case when both a and b are even.

Should now c be odd, take in (3) p and q both odd, then

$$\rho \equiv 3 \pmod 8,\ \ \ \sigma \equiv 3 \pmod 8)$$

so that we need only prove H when a, b, c are all even.

If c is divisible by 2 but not by 4 put

$$c = 2C, \quad a = 2A, \quad b = 2B, \quad p = 2P, \quad q = 2Q$$

then $$\rho/4 = 2Q - P^2, \quad \sigma/4 = 4C(A + P) - (Q - B)^2.$$

If now A is even, take P and $Q - B$ both odd. Then since C is odd,

$$\sigma/4 \equiv 3 \pmod 8.$$

Should, however, $\rho/4$ be of the form M, in this case

$$\rho/4 \equiv 7 \pmod 8$$

for these values of P and Q, then since the only restriction on Q is that $Q - B$ should be odd, the change of Q into $Q + 2$ changes $\rho/4$ into a number $\rho_1/4$, such that

$$\rho_1/4 \equiv 3 \pmod 8$$

while σ is changed into a number σ_1 still satisfying the congruence

$$\sigma_1/4 \equiv 3 \pmod 8.$$

If however A is odd, take P even and $Q - B$ odd. Then

$$\sigma/4 \equiv 3 \pmod 8.$$

The only restriction on P is that it should be even and from § (2) we can take it so that ρ is not of the form M. Hence H holds if c is divisible by 2 but not by 4.

H holds also if c is divisible by 4 but not by 8. For putting

$$x = c/2y$$

we find

$$y^3 - \tfrac{1}{2}by^2 + \tfrac{1}{4}acy - \tfrac{1}{8}c^2 = 0,$$

where all the coefficients are integers and $c^2/8$ is divisible by 2 but not by 4. Hence H holds for y and also for x.

Suppose now that c is divisible by 8 or say 2^n but not by 2^{n+1}. Then put again

$$x = c/2y$$

so that as before

$$y^3 - \tfrac{1}{2}by^2 + \tfrac{1}{4}acy - \tfrac{1}{8}c^2 = 0.$$

Hence if $b/2$ is odd, H holds for y and hence for x. If however $b/2$ is even, put

$$a = 2A, \quad b = 4B, \quad c = 8C$$

so that the equation (1) becomes

$$x^3 - 2Ax^2 + 4Bx - 8C = 0,$$

or, putting $x = 2z$

$$z^3 - Az^2 + Bz - C = 0.$$

But C is divisible now by 2^{n-3} and not by 2^{n-2}. Hence if this process be continued, we shall arrive at an equation of the form

$$x^3 - ax^2 + bx - c = 0 \qquad \text{..................(1)}$$

where either a is odd or c is not divisible by 8. But H has already been proved for these cases so that we have proved H in the case when a, b and c are positive and in particular for the algebraic fields arising from a cubic with three real roots.

§ (4). If however a and b are not both positive, in which case equation (1) has imaginary roots, it is clear from equation (2) that if p and q are taken to satisfy the inequality $4q > p^2$ and t is a sufficiently large positive number, then $x+t$ can be expressed as the quotient of two positive definite quadratics in x. Hence as $x+t$ is also the root of a cubic of the type (1) where a, b and c are positive, it is clear that H holds for $x+t$ provided t is a sufficiently large positive number—integral or fractional.

We have now from equation (1)

$$(x+\xi)(x^2-(\xi+a)x+k) = (-\xi^2-a\xi+k-b)x+c+\xi k \quad \ldots(5)$$

where k and ξ are arbitrary rational quantities. We take k so large that the discriminant $4k-(\xi+a)^2$ is positive and not of the form M (this is always possible as k need not be an integer, take it say, a fourth of an integer), and so that $-\xi^2-a\xi+k-b$ is also positive.

Suppose ξ is not negative and put

$$\xi_1 = \frac{k\xi+c}{-\xi^2-a\xi+k-b}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

then ξ_1 is positive and also $\xi_1 > \xi$ since

$$\xi^3+a\xi^2+b\xi+c>0$$

because the real root of the cubic (1) is positive. Also H holds for $x+\xi$ if it holds for $x+\xi_1$.

Take now $\xi=0$, and find $\xi_2, \xi_3 \ldots$ from $\xi_1, \xi_2 \ldots$ with of course suitable values of $k_1, k_2 \ldots$, in the same way as ξ_1 was found from ξ. Then $\xi_1, \xi_2, \xi_3 \ldots$ form a monotonic increasing sequence whose limit is infinity. For if it were finite say L, then it follows immediately from equation (6) that

$$L^3+aL^2+bL+c=0$$

which is impossible as we have assumed that the cubic (1) has no negative root.

Hence after a certain stage (H) holds for $x+\xi_\lambda$. It is obvious then from (5) that H holds for x, so that we have also proved H for the case of the field arising from a cubic with only one real root.

NOTE.

The general law of quadratic reciprocity in any field was given without proof by Hilbert in the *Göttinger Nachrichten* for 1898, page 380 in his paper "Ueber die Theorie der relativ-Abel'schen Zahlkörper." In his paper "Mathematische Probleme" in the *Nachrichten* for 1900, the investigation of the theory of the general quadratic form with algebraic coefficients is proposed as problem 11.

The law of quadratic reciprocity above was proved by Furtwängler in the third part of his paper "Die Reziprozitätsgesetze für Potenzreste mit Primzahlexponenten in algebraischen Zahlkörpern" in the *Mathematische Annalen*, Bd. 74, 1913.

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On a Gaussian Series of Six Elements. By L. J. ROGERS. (Communicated by Prof. G. H. HARDY.)

[*Read* 26 January 1920.]

§ 1. The symbol $H(\alpha, \beta, \lambda, \mu, \gamma)$ will be used for the infinite series

$$1 + \frac{\alpha \,.\, \beta \,.\, \lambda \,.\, \mu}{1 \,.\, \gamma \,.\, (\kappa^2 - \theta^2)} + v_2 + v_3 + \dots \qquad \dots\dots\dots(1),$$

where
$$\frac{v_{n+1}}{v_n} = \frac{(\alpha + n)(\beta + n)(\lambda + n)(\mu + n)}{(1 + n)(\gamma + n)(\kappa + n - \theta)(\kappa + n + \theta)},$$

and
$$2\kappa = \alpha + \beta + \lambda + \mu + 1 - \gamma.$$

Since
$$\frac{v_{n+1}}{v_n} = 1 + \frac{1}{n}(\alpha + \beta + \lambda + \mu - 1 - \gamma - 2\kappa) + O\left(\frac{1}{n^2}\right)$$
$$= 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right),$$

it follows that the series (1) is always convergent, provided that the elements $\alpha, \beta, \dots$ are all finite, and $\gamma, \kappa \pm \theta$ are not negative integers.

When $\mu = \gamma$ the series reduces to $1 + \dfrac{\alpha \,.\, \beta \,.\, \lambda}{1 \,.\, (\kappa^2 - \theta^2)} + \dots$,

where
$$2\kappa = \alpha + \beta + \lambda + 1,$$

which series will be written

$$H(\alpha, \beta, \lambda) \dots\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

It is not necessary to introduce θ into the functional notation.

For the sake of conciseness it will be convenient to write $\alpha_n, \beta_n, \dots \kappa_n$ for $\alpha + n, \beta + n, \dots \kappa + n$.

§ 2. Corresponding to the well-known formula in hypergeometric series, it is easily seen that

$$H(\alpha, \beta, \lambda, \mu_1, \gamma_1) - H(\alpha, \beta, \lambda, \mu, \gamma)$$
$$= \frac{\alpha\beta\lambda(\gamma - \mu)}{\gamma(\gamma + 1)(\kappa^2 - \theta^2)} H(\alpha_1, \beta_1, \lambda_1, \mu_1, \gamma_2) \dots\dots\dots(1).$$

Moreover

$$\{(\kappa - \mu)^2 - \theta^2\} H(\alpha_1, \beta_1, \lambda_1, \mu, \gamma_1) - (\kappa^2 - \theta^2) H(\alpha, \beta, \lambda, \mu, \gamma)$$
$$= \frac{(\gamma - \alpha)(\gamma - \beta)(\gamma - \lambda)\mu}{\gamma(\gamma + 1)} H(\alpha_1, \beta_1, \lambda_1, \mu_1, \gamma_2) \dots\dots(2).$$

To prove this write the equation in the form

$$\{(\kappa-\mu)^2-\theta^2\}\left\{1+\frac{B_1}{\kappa_1^2-\theta^2}+\frac{B_2}{(\kappa_1^2-\theta^2)(\kappa_2^2-\theta^2)}+\dots\right\}$$
$$-(\kappa^2-\theta^2)\left\{1+\frac{A_1}{\kappa^2-\theta^2}+\frac{A_2}{(\kappa^2-\theta^2)(\kappa_1^2-\theta^2)}+\dots\right\}$$
$$=C_0+\frac{C_1}{\kappa_1^2-\theta^2}+\frac{C_2}{(\kappa_1^2-\theta^2)(\kappa_2^2-\theta^2)}+\dots,$$

where the A's, B's and C's are independent of θ, it being observed that, in the first and third series in (2), κ becomes $\kappa+1$.

The left-hand side is

$$\mu^2-2\kappa\mu+B_1\left\{1+\frac{(\kappa-\mu)^2-\kappa_1^2}{\kappa_1^2-\theta^2}\right\}$$
$$+\frac{B_2}{\kappa_1^2-\theta^2}\left\{1+\frac{(\kappa-\mu)^2-\kappa_2^2}{\kappa_2^2-\theta^2}\right\}+\dots$$
$$-A_1-\frac{A_2}{\kappa_1^2-\theta^2}-\frac{A_3}{(\kappa_1^2-\theta^2)(\kappa_2^2-\theta^2)}+\dots.$$

Hence

$$C_0=\mu^2-2\kappa\mu+B_1-A_1$$
$$=\mu(\gamma-\alpha-\beta-\lambda-1)+\frac{(\alpha+1)(\beta+1)(\lambda+1)\mu}{\gamma+1}-\frac{\alpha\beta\lambda\mu}{\gamma}$$
$$=\frac{\mu}{\gamma(\gamma+1)}(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda),$$

while $$C_n=B_{n+1}+B_n\{(\kappa-\mu)^2-(\kappa+n)^2\}-A_{n+1}.$$

Now $$A_{n+1}=\frac{\alpha\beta\lambda\mu_n}{(n+1)\gamma}B_n,\text{ and } B_{n+1}=\frac{\alpha_{n+1}\beta_{n+1}\lambda_{n+1}\mu_n}{(n+1)\gamma_{n+1}}B_n,$$

so that $$\frac{C_n}{B_n}=\frac{\alpha_{n+1}\beta_{n+1}\lambda_{n+1}\mu_n}{(n+1)\gamma_{n+1}}-(2\kappa-\mu+n)\mu_n-\frac{\alpha\beta\lambda\mu_n}{(n+1)\gamma}$$
$$=\frac{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}{\gamma(\gamma+n+1)}\mu_n \quad\dots\dots\dots\dots(3),$$

as may be easily verified, thereby giving the right-hand side of (2) and establishing the relation.

We may now obtain a continued fraction of the form

$$\frac{1}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\,\frac{e_3x}{1-} \quad\dots\dots\dots\dots(4),$$

for the ratio $H(\alpha,\beta,\lambda,\mu_1,\gamma_1)/H(\alpha,\beta,\lambda,\mu,\gamma)$.

For from (1)

$$1-\frac{H(\alpha,\beta,\lambda,\mu,\gamma)}{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}=1-\frac{e_1}{\kappa^2-\theta^2}\,\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)}{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)},$$

where $$e_1 = \frac{\alpha\beta\lambda(\gamma-\mu)}{\gamma(\gamma+1)}.$$

But from (2), by changing μ, γ into μ_1, γ_1, which does not alter κ, we have

$$(\kappa^2-\theta^2)\frac{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)} = (\kappa-\mu)^2-\theta^2-e_2\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_2,\gamma_3)}{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)},$$

where $$e_2 = \frac{(\gamma-\alpha+1)(\gamma-\beta+1)(\gamma-\lambda+1)(\mu+1)}{(\gamma+1)(\gamma+2)}.$$

These two results lead to

$$\frac{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}{H(\alpha,\beta,\lambda,\mu,\gamma)} = \frac{1}{1-}\,\frac{e_1}{(\kappa-\mu-1)^2-\theta^2-}\,\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_2,\gamma_3)}{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)}e_2.$$

The ratio of H-functions on the right-hand side differs from that on the left-hand side only in that α, β, λ, μ are increased each by unity, and γ by 2. This does not alter $\kappa-\mu$, so that we have finally

$$\frac{H(\alpha,\beta,\lambda,\mu+1,\gamma+1)}{H(\alpha,\beta,\lambda,\mu,\gamma)}$$

$$= \frac{1}{1-}\,\frac{e_1}{(\kappa-\mu-1)^2-\theta^2-}\,\frac{e_2}{1-}\,\frac{e_3}{(\kappa-\mu-1)^2-\theta^2-}\cdots$$

$$= \frac{1}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\,\frac{e_3x}{1-}\cdots,$$

$$\left.\begin{aligned} &\text{where } e_3 = \frac{\alpha_1\beta_1\lambda_1(\gamma_1-\mu)}{\gamma_2\gamma_3},\quad e_4 = \frac{(\gamma_2-\alpha)(\gamma_2-\beta)(\gamma_2-\lambda)\mu_2}{\gamma_3\gamma_4}, \text{ etc.}\\ &\text{and}\qquad \frac{1}{x} = (\kappa-\mu-1)^2-\theta^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(5).$$

It is remarkable that if we change μ into $\gamma-\mu$, the coefficients e all become symmetrical in α, β, λ, μ, and $\kappa-\mu-1$ becomes $\frac{1}{2}(\alpha+\beta+\lambda+\mu-1-2\gamma)$, so that

$$\frac{H(\alpha,\beta,\lambda,\gamma-\mu+1,\gamma+1)}{H(\alpha,\beta,\lambda,\gamma-\mu,\gamma)}$$

is symmetrical in α, β, λ, μ.

Hence

$$\frac{H(\alpha,\beta,\lambda,\gamma-\mu,\gamma)}{H(\alpha,\beta,\gamma-\lambda,\mu,\gamma)} = \frac{H(\alpha,\beta,\lambda,\gamma+1-\mu,\gamma+1)}{H(\alpha,\beta,\gamma+1-\lambda,\mu,\gamma+1)}\quad\ldots(6),$$

where the first of these fractions is the same function of γ as the second is of $\gamma+1$.

§ 3. From § 2 we have

$$\frac{(\kappa-\mu-1)^2-\theta^2}{\kappa^2-\theta^2}\,\frac{H(\alpha_1,\beta_1,\lambda_1,\mu_1,\gamma_2)}{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}$$

$$=\frac{1}{1-}\,\frac{e_2}{(\kappa-\mu-1)^2-\theta^2-}\,\frac{e_3}{1-}\ \ldots\ldots(1).$$

Now change $\alpha, \beta, \lambda, \mu, \gamma$ respectively into

$$\gamma-\alpha,\ \gamma-\beta,\ \gamma-\lambda,\ \gamma-\mu-1,\ \gamma-1.$$

It will then be seen that $\kappa-\mu-1$ becomes $-(\kappa-\mu-1)$, κ becomes $-(\kappa-\gamma-1)$, e_2 becomes e_1, and generally e_n becomes e_{n-1}.

We have in (1) then the same continued fraction as in § 2 (4), and hence

$$\frac{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}{H(\alpha,\beta,\lambda,\mu,\gamma)}$$

$$=\frac{(\kappa-\mu-1)^2-\theta^2}{(\kappa-\gamma-1)^2-\theta^2}\,\frac{H(\gamma_1-\alpha,\gamma_1-\beta,\gamma_1-\lambda_1,\gamma-\mu,\gamma_1)}{H(\gamma-\alpha,\gamma-\beta,\gamma-\lambda,\gamma-\mu,\gamma)},$$

or $$\frac{H(\gamma-\alpha,\gamma-\beta,\gamma-\lambda,\gamma-\mu,\gamma)}{H(\alpha,\beta,\lambda,\mu,\gamma)}$$

$$=\frac{(\kappa-\mu-1)^2-\theta^2}{(\kappa-\gamma-1)^2-\theta^2}\,\frac{H(\gamma_1-\alpha,\gamma_1-\beta,\gamma_1-\lambda,\gamma-\mu,\gamma_1)}{H(\alpha,\beta,\lambda,\mu_1,\gamma_1)}\ \ldots\ldots(2).$$

Let $\lambda=0$, then

$$H(\gamma-\alpha,\gamma-\beta,\gamma-\mu)$$

$$=\frac{\frac{1}{4}(\alpha+\beta-\mu-1-\gamma)^2-\theta^2}{\frac{1}{4}(\alpha+\beta+\mu-1-3\gamma)^2-\theta^2}\,H(\gamma_1-\alpha,\gamma_1-\beta,\gamma-\mu),$$

or, changing α, β, μ into $\gamma-\alpha, \gamma-\beta, \gamma-\mu$,

$$H(\alpha,\beta,\mu)=\frac{\frac{1}{4}(\alpha+\beta-\mu+1)^2-\theta^2}{\frac{1}{4}(\alpha+\beta+\mu+1)^2-\theta^2}\,H(\alpha_1,\beta_1,\mu)\ \ldots(4).$$

The series $H(\alpha,\beta,\mu)$ has been investigated by Saalschütz, *Zeitschrift f. Math.* XXXV., for the case when α is a negative integer, and generally by Dougall, *Proc. Edinburgh Math. Soc.* XXV., § 12 (20). For special cases of the continued fraction § 2 (5), see *Proc. London Math. Soc.* series 2, vol. 4, pp. 83, 394.

§ 4. *Heinean forms.*

The results of the foregoing sections have their counterparts, when every factor in the terms in H, are replaced by corresponding sines or hyperbolic sines.

If the fundamental series be taken as

$$1+\frac{\sinh m\alpha \sinh m\beta \sinh m\lambda \sinh m\mu}{\sinh m \sinh m\gamma \sinh m(\kappa-\theta) \sinh m(\kappa+\theta)}+v_2+\dots$$

then, if $e^{-2m}=q$, the factors in v_1 take the form $1-q^{\alpha}$, etc., there being an extra factor

$$\exp m(\alpha+\beta+\lambda+\mu-1-\gamma-2\kappa), \;=e^{-2m}=q.$$

Moreover

$$\frac{v_{n+1}}{v_n}=\frac{(1-q^{\alpha+n})(1-q^{\beta+n})(1-q^{\lambda+n})(1-q^{\mu+n})}{(1-q^{n+1})(1-q^{\gamma+n})(1-q^{\kappa+n-\theta})(1-q^{\kappa+n+\theta})}q,$$

so that the series is obviously convergent if $|q|<1$, and no indices of powers of q are zero.

The formula corresponding to § 2 (1) is readily obtained, giving

$$e_1=\frac{\sinh m\alpha \sinh m\beta \sinh m\gamma \sinh(\gamma-\mu)}{\sinh m\gamma \sinh m(\gamma+1)}.$$

That corresponding to § 2 (2) is more difficult, but it depends on the fact that, by employing the somewhat intricate identity

$$\begin{aligned}&\frac{\sinh m(\alpha+n+1)\sinh m(\beta+n+1)\sinh m(\lambda+n+1)}{\sinh m(n+1)\sinh m(\gamma+n+1)}\\ &\qquad-\frac{\sinh m\alpha \sinh m\beta \sinh m\lambda}{\sinh m(n+1)\sinh m\gamma}\\ &\qquad-\sinh m(\alpha+\beta+\lambda-\gamma+n+1)\\ &\qquad=\frac{\sinh m(\gamma-\alpha)\sinh m(\gamma-\beta)\sinh m(\gamma-\lambda)}{\sinh m\gamma \sinh m(\gamma+n+1)},\end{aligned}$$

we get, corresponding to $\dfrac{C_n}{B_n}$ in § 2 (3) the value

$$\frac{\sinh m(\mu+n)\sinh m(\gamma-\alpha)\sinh m(\gamma-\beta)\sinh m'(\gamma-\lambda)}{\sinh m\gamma \sinh m(\gamma+n+1)}.$$

This leads to a relation corresponding to § 2 (5), where

$$\frac{1}{x}=\sinh^2 m(\kappa-\mu-1)-\sinh^2 m\theta,$$

and the e's are altered to corresponding hyperbolic functions as explained above.

§ 5. *Convergence of the continued fraction in* § 2 (5), *when x is negative and the e's are all positive.*

A continued fraction of the type

$$\frac{1}{1+}\,\frac{e_1 t}{1+}\,\frac{e_2 t}{1+}\cdots$$

converges, if when reduced to the form

$$\frac{1}{1+}\,\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots,$$

either or both the series

$$1+d_2+d_4+\ldots,\quad d_1+d_3+d_5+\ldots$$

are divergent.

That is $$1+\frac{1}{t}\left(\frac{e_1}{e_2}+\frac{e_1 e_3}{e_2 e_4}+\frac{e_1 e_3 e_5}{e_2 e_4 e_6}+\ldots\right),$$

and $$\frac{1}{t}\left(\frac{1}{e_1}+\frac{e_2}{e_1 e_3}+\frac{e_2 e_4}{e_1 e_3 e_5}+\ldots\right),$$

must be either or both divergent.

In the first the ratio of the general term to the preceding is $\frac{e_{2n-1}}{e_{2n}}$ and in the second it is $\frac{e_{2n}}{e_{2n+1}}$.

The first ratio is

$$\frac{(\alpha+n-1)(\beta+n-1)(\lambda+n-1)(\gamma-\mu+n-1)(\gamma+2n)}{(\gamma-\alpha+n)(\gamma-\beta+n)(\gamma-\lambda+n)(\mu+n)(\gamma+2n-2)}$$

$$=1-\frac{1}{n}\{1+2(\gamma+\mu+1-\alpha-\beta-\lambda)\}+O\left(\frac{1}{n^2}\right),$$

while the second is

$$\frac{(\gamma-\alpha+n)(\gamma-\beta+n)(\gamma-\lambda+n)(\gamma+2n+1)}{(\alpha+n)(\beta+n)(\lambda+n)(\gamma-\mu+n)(\gamma+2n-1)}$$

$$=1-\frac{1}{n}\{1-2(\gamma+\mu+1-\alpha-\beta-\lambda)\}+O\left(\frac{1}{n^2}\right).$$

It is evident that one series converges and the other diverges and therefore the continued fraction converges.

In the Heinean case these ratios are approximately

$$q^{(\alpha+\beta+\lambda-\gamma-\mu-1)},\quad q^{(\gamma+\mu+1-\alpha-\lambda-\mu)},$$

one or other of which must be greater than 1. Hence one series must diverge and the continued fraction converges.

Note on Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions. By Prof. G. H. HARDY.

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1. Ramanujan's memoir 'On certain trigonometrical sums and their applications in the theory of numbers'* is devoted to the study of the function

$$c_q(n) = \sum_p e^{-2np\pi i/q} = \sum_p \cos \frac{2np\pi}{q} \quad \text{.............(1·1)},$$

where p runs through the numbers less than and prime to q. Ramanujan proves that

$$c_q(n) = \Sigma\, \delta\mu\left(\frac{q}{\delta}\right) \quad \text{.................(1·2)},$$

where $\mu(n)$ has its usual meaning† and δ is a common divisor of q and n‡. He then proceeds to express a number of the most important arithmetical functions of n as series of the form

$$\sum_{q=1}^{\infty} a_q c_q(n),$$

where a_q is independent of n. From among the many interesting results which he obtains I may quote the following:—

$$\sigma_s(n) = \frac{n^s}{\zeta(s+1)} \Sigma \frac{c_q(n)}{q^{s+1}} \quad \text{.................(1·3)},$$

and in particular

$$\sigma(n) = \frac{\pi^2 n}{6} \Sigma \frac{c_q(n)}{q^2} \quad \text{....................(1·31)},$$

$$0 = \Sigma \frac{c_q(n)}{q} \quad \text{..........................(1·32)};$$

$$d(n) = -\Sigma \frac{\log q}{q} c_q(n);$$

$$\phi(n) = \frac{6n}{\pi^2} \Sigma \frac{\mu(q)\, c_q(n)}{\phi_2(q)} \quad \text{..............(1·4)};$$

$$r(n) = \pi \Sigma \frac{(-1)^{q-1} c_{2q-1}(n)}{2q-1} \quad \text{...........(1·5)}.$$

* *Trans. Camb. Phil. Soc.*, vol. 22, 1918, pp. 259–276.

† $\mu(m)=0$ unless m is a product of ρ different primes, when $\mu(m)=(-1)^\rho$.

‡ The formula (1·2) seems to have been first stated explicitly by Ramanujan (*l.c.* p. 260). It is given for $n=1$ by Landau (*Handbuch*, 1909, p. 572) and by Jensen ('Et nyt Udtryk for den talteoretiske Funktion $\Sigma\mu(n) = M(m)$,' *Saertryk af Beretning om den 3 Skandinaviske Matematiker-Kongres*, Kristiania, 1915). The deduction of the general formula from that given by Landau is trivial; but Ramanujan makes so many beautiful applications of the sums that they may well be associated with his name.

Here $d(n)$ is the number of divisors of n, $\sigma(n)$ their sum, and $\sigma_s(n)$ the sum of their s-th powers; $\phi(n)$ is the number of numbers less than and prime to n;

$$\phi_2(n) = n^2\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p_1^{\,2}}\right)\dots$$

when $n = p^a p_1^{a_1}\dots$; and $r(n)$ is the number of representations of n as the sum of two squares.

These series have a peculiar interest because they show explicitly the source of the irregularities in the behaviour of their sums. Thus, for example, the formula (1·31) may be written in the form

$$\sigma(n) = \frac{\pi^2 n}{6}\left\{1 + \frac{(-1)^n}{2^2} + \frac{2\cos\frac{2}{3}n\pi}{3^2} + \frac{2\cos\frac{1}{2}n\pi}{4^2} \right.$$
$$\left. + \frac{2(\cos\frac{2}{5}n\pi + \cos\frac{4}{5}n\pi)}{5^2} + \frac{2\cos\frac{1}{3}n\pi}{6^2} + \dots\right\};$$

and we see at once that the most important term in $\sigma(n)$ is $\frac{1}{6}\pi^2 n$, and that irregular variations about this average value are produced by a series of harmonic oscillations of decreasing amplitude.

2. Ramanujan's proofs of his principal formulae are very interesting and ingenious, but are not, I think, the simplest or the most natural. In this note I prove a number of them by a different method. This method occurred to Mr Littlewood and me in the course of our researches connected with Waring's and Goldbach's problems, and in which Ramanujan's sums play an important part. I also include a few new results which are suggested naturally by our analysis.

The multiplicative property of $c_q(n)$.

3. The first step is to prove that

$$c_{qq'}(n) = c_q(n)\, c_{q'}(n) \quad\dots\dots\dots\dots\dots\dots(3{\cdot}1),$$

whenever q and q' are prime to one another. For this we observe that

$$c_q(n)\, c_{q'}(n) = \sum_p e^{-2np\pi i/q} \sum_{p'} e^{-2np'\pi i/q'} = \sum_{p,p'} e^{-2nP\pi i/qq'},$$

where
$$P = pq' + p'q.$$

But it is plain that every value of P is prime to both q and q', and that no two values of P are congruent to modulus qq'. Also the total number of values of P is $\phi(q)\,\phi(q') = \phi(qq')$. Hence P runs through a system of values congruent, to modulus qq', to the $\phi(qq')$ numbers less than and prime to qq'. This plainly establishes the truth of (3·1).

The value of $c_q(n)$.

4. I show next that the value of $c_q(n)$ is given by (1·2). Let us write

$$C_q(n) = \Sigma\, \delta\mu\left(\frac{q}{\delta}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4\cdot 1).$$

If q and q' are prime to one another, we have

$$C_q(n)\, C_{q'}(n) = \sum_{\delta,\,\delta'} \delta\delta'\mu\left(\frac{q}{\delta}\right)\mu\left(\frac{q'}{\delta'}\right),$$

where δ is a common divisor of q and n and δ' one of q' and n. Clearly $\delta\delta'$ runs through the common divisors of qq' and n, each once only. Also q/δ and q'/δ' are prime to one another. Hence

$$C_q(n)\, C_{q'}(n) = \sum_{D} D\mu\left(\frac{qq'}{D}\right),$$

where $D = \delta\delta'$ runs through all common divisors of qq' and n; and so

$$C_q(n)\, C_{q'}(n) = C_{qq'}(n) \quad\ldots\ldots\ldots\ldots\ldots\ldots (4\cdot 2).$$

Since $c_q(n)$ and $C_q(n)$ each possess the multiplicative property, we need only verify their identity when q is a power of a prime, say ϖ^k. Now

$$C_{\varpi^k}(n) = \sum_{p} e^{-2np\pi i/\varpi^k}$$

where p runs through the $\varpi^{k-1}(\varpi - 1)$ numbers less than and prime to ϖ^k. These may be expressed in the form

$$p = \varpi^{k-1}z + p',$$

where $z = 0, 1, \ldots, \varpi - 1$ and p' runs through the numbers less than and prime to ϖ^{k-1}. Hence

$$c_{\varpi^k}(n) = \sum_{z} e^{-2nz\pi i/\varpi} \sum_{p'} e^{-2np'\pi i/\varpi^k};$$

and the sum with respect to z is zero unless $\varpi \mid n$*. Thus

$$c_{\varpi^k}(n) = 0 \qquad (k > 1,\ \varpi \nmid n)$$

and
$$c_{\varpi^k}(\varpi\nu) = \varpi \sum_{p'} e^{-2\nu p'\pi i/\varpi^{k-1}} = \varpi c_{\varpi^{k-1}}(\nu).$$

Now plainly

$$c_{\varpi}(n) = -1 \quad (\varpi \nmid n), \qquad c_{\varpi}(n) = \varpi - 1 \quad (\varpi \mid n) \ldots (4\cdot 3).$$

Hence

$$c_{\varpi^2}(n) = 0 \quad (\varpi \nmid n), \qquad c_{\varpi^2}(n) = -\varpi \quad (\varpi \mid n,\ \varpi^2 \nmid n),$$
$$c_{\varpi^2}(n) = \varpi(\varpi - 1) \quad (\varpi^2 \mid n);$$

and generally

$$c_{\varpi^k}(n) = 0 \quad (\varpi^{k-1} \nmid n), \qquad c_{\varpi^k}(n) = -\varpi^{k-1} \quad (\varpi^{k-1} \mid n,\ \varpi^k \nmid n),$$
$$c_{\varpi^k}(n) = \varpi^{k-1}(\varpi - 1) \quad (\varpi^k \mid n) \quad\ldots\ldots\ldots\ldots (4\cdot 4).$$

* Following Landau, I write $\varpi \mid n$ for 'ϖ is a divisor of n' and $\varpi \nmid n$ for 'ϖ is not a divisor of n.'

It may be immediately verified that these are also the values of $C_{\varpi^k}(n)$. Hence $c_q(n) = C_q(n)$ when q is a power of a prime, and therefore generally.

Summation of the series (1·3).

5. It is plain that $|c_q(n)|$ does not exceed the sum of the divisors of n. It therefore follows from (3·1) that, if $s > 1$,

$$\Sigma \frac{c_q(n)}{q^s} = \prod_{\varpi} \chi_\varpi,$$

where
$$\chi_\varpi = 1 + \frac{c_\varpi(n)}{\varpi^s} + \frac{c_{\varpi^2}(n)}{\varpi^{2s}} + \dots .$$

If $\varpi \nmid n$,
$$\chi_\varpi = 1 - \varpi^{-s}.$$

If ϖ^a is the highest power of ϖ which divides n,

$$\begin{aligned}\chi_\varpi &= 1 + \frac{\varpi - 1}{\varpi^s} + \frac{\varpi(\varpi-1)}{\varpi^{2s}} + \dots + \frac{\varpi^{a-1}(\varpi-1)}{\varpi^{as}} - \frac{\varpi^a}{\varpi^{(a+1)s}} \\ &= 1 - \varpi^{a-(a+1)s} + \varpi^{-s}(\varpi - 1)\frac{1-\varpi^{a-as}}{1-\varpi^{1-s}} \\ &= (1-\varpi^{-s})\frac{1-\varpi^{(a+1)(1-s)}}{1-\varpi^{1-s}}.\end{aligned}$$

Hence
$$\begin{aligned}\Sigma \frac{c_q(n)}{q^s} &= \prod_{\varpi}(1-\varpi^{-s}) \prod_{\varpi \mid n} \frac{1-\varpi^{(a+1)(1-s)}}{1-\varpi^{1-s}} \\ &= \frac{\sigma_{1-s}(n)}{\zeta(s)} = \frac{n^{-(s-1)}\sigma_{s-1}(n)}{\zeta(s)}.\end{aligned}$$

This is Ramanujan's formula, with s in the place of $s-1$.

The formula (1·32) lies, as Ramanujan points out, somewhat deeper, since $s = 1$ does not lie inside the region of absolute convergence. I have nothing to add to Ramanujan's remarks concerning this limiting case*.

Summation of the series (1·4).

6. Let us write, with Ramanujan,

$$\phi_s(n) = n^s(1-p^{-s})(1-p_1^{-s})\dots,$$

where $p, p_1, \dots$ are the different prime factors of n, so that $\phi_1(n) = \phi(n)$. Then, if $s > 1$, we have

$$\Sigma \frac{\mu(q)\, c_q(n)}{\phi_s(q)} = \prod \chi_\varpi,$$

where
$$\chi_\varpi = 1 + \frac{\mu(\varpi)\, c_\varpi(n)}{\phi_s(\varpi)};$$

* *l.c.*, p. 265.

so that

$$\chi_{\varpi}=1+\frac{1}{\varpi^s-1}\quad(\varpi\nmid n),\qquad \chi_{\varpi}=1-\frac{\varpi-1}{\varpi^s-1}\quad(\varpi\mid n).$$

It follows that

$$\begin{aligned}\Sigma\frac{\mu(q)\,c_q(n)}{\phi_s(q)}&=\prod_{\varpi\nmid n}\left(\frac{\varpi^s}{\varpi^s-1}\right)\prod_{\varpi\mid n}\left(\frac{\varpi^s-\varpi}{\varpi^s-1}\right)\\&=\Pi\frac{1}{1-\varpi^{-s}}\prod_{\varpi\mid n}\{1-\varpi^{-(s-1)}\}\\&=\zeta(s)\,n^{-(s-1)}\,\phi_{s-1}(n).\end{aligned}$$

Thus
$$\phi_{s-1}(n)=\frac{n^{s-1}}{\zeta(s)}\,\Sigma\frac{\mu(q)\,c_q(n)}{\phi_s(q)}\quad\ldots\ldots\ldots\ldots\ldots(6\cdot1).$$

This formula is equivalent to Ramanujan's formula (9·6), and reduces to (1·4) when $s=2$.

The case $s = 1$.

7. The case in which $s=1$, which is not discussed by Ramanujan, is of particular interest.

We observe first that

$$n^{-(s-1)}\,\phi_{s-1}(n)\,\zeta(s)=\{1-p^{-(s-1)}\}\,\{1-p_1^{-(s-1)}\}\ldots\zeta(s)$$

tends to zero, when $s\to1$, unless n is a prime p or a power of a prime p, in which case its limit is $\log p$. Thus

$$\lim_{s\to1} n^{-(s-1)}\,\phi_{s-1}(n)\,\zeta(s)=\Lambda(n),$$

$\Lambda(n)$ having its usual significance in the theory of primes*. If then we suppose that the series (6·1) is convergent for $s=1$, and that its sum is continuous, we obtain

$$\Lambda(n)=\Sigma\frac{\mu(q)\,c_q(n)}{\phi(q)}\quad\ldots\ldots\ldots\ldots\ldots\ldots(7\cdot1).$$

It is not difficult to prove that this formula is incorrect.

In order to prove this, and to obtain the correct formula, I consider the function

$$f(s)=\Sigma\frac{\mu(q)\,c_q(n)}{q^{s-1}\,\phi(q)},$$

supposing first that σ, the real part of s, is greater than unity. It is plain that

$$f(s)=\prod_{\varpi}\theta_{\varpi},$$

where
$$\theta_{\varpi}=1-\frac{c_{\varpi}(n)}{\varpi^{s-1}(\varpi-1)},$$

so that

$$\theta_{\varpi}=1+\frac{1}{\varpi^{s-1}(\varpi-1)}\quad(\varpi\nmid n),\qquad \theta_{\varpi}=1-\frac{1}{\varpi^{s-1}}\quad(\varpi\mid n).$$

* $\Lambda(n)=\log\varpi$ if $n=\varpi^m$, and $\Lambda(n)=0$ otherwise.

It follows that

$$f(s) = \prod_{\varpi \nmid n} \left\{1 + \frac{1}{\varpi^{s-1}(\varpi - 1)}\right\} \prod_{\varpi \mid n} \left(1 - \frac{1}{\varpi^{s-1}}\right)$$

$$= \prod \left\{1 + \frac{1}{\varpi^{s-1}(\varpi - 1)}\right\} \prod_{\varpi \mid n} \left\{\frac{(\varpi - 1)(\varpi^{s-1} - 1)}{\varpi^s - \varpi^{s-1} + 1}\right\}$$

$$= g(s)\, h(s),$$

say. Now

$$\frac{g(s)}{\zeta(s)} = \prod \left[\left\{1 + \frac{1}{\varpi^{s-1}(\varpi - 1)}\right\} \left(1 - \frac{1}{\varpi^s}\right)\right]$$

$$= \prod \{1 + O(\varpi^{-\sigma-1})\};$$

and this product is uniformly convergent throughout any half plane $\sigma \geqslant \delta > 0$, so that $g(s)/\zeta(s)$ is regular for $\sigma > 0$. Thus $g(s)$ is regular for $\sigma > 0$, except for a simple pole at $s = 1$, with residue 1; and its properties for $\sigma > 0$ are substantially the same as those of $\zeta(s)$.

If n is a prime p, or a power of a prime p,

$$h(s) = \frac{(p-1)(p^{s-1} - 1)}{p^s - p^{s-1} + 1}$$

has a simple zero for $s = 1$, and

$$h'(1) = \frac{p-1}{p} \log p.$$

Thus in this case $f(s)$ is regular for $s = 1$, and

$$f(1) = \frac{p-1}{p} \log p.$$

In all other cases (except when $n = 1$), $h(s)$ has a zero of at least the second order, and

$$f(1) = 0.$$

We have now only to apply to $f(s)$ the arguments which, when applied to the function $1/\zeta(s)$, lead to the proof of the well known theorem expressed by the equation

$$\sum \frac{\mu(n)}{n} = 0^* \quad \text{.......................(7·2)},$$

in order to conclude that the series for $f(s)$ is convergent for $s = 1$, and that its value is $f(1)$. That is to say

$$\sum \frac{\mu(q)\, c_q(n)}{\phi(q)} = \frac{p-1}{p} \log p$$

* See for example Landau, *Handbuch*, pp. 593 *et seq.*; Hardy and Littlewood, 'New proofs of the prime-number theorem and similar theorems,' *Quarterly Journal*, vol. 46, 1915, pp. 215–219.

or zero, according as n is or is not of the form p^k; or

$$\Sigma \frac{\mu(q)\, c_q(n)}{\phi(q)} = \frac{\phi(n)\, \Lambda(n)}{n} \quad \text{..............(7·3)},$$

which contradicts (7·1).

It should be noticed that, if we are merely concerned to prove that (7·1) is false, all the difficult part of the preceding argument may be omitted. For, if (7·1) were true, $f(s)$ would certainly tend to the limit $\Lambda(n)$ when $s \to 1$, and the mere knowledge of the value of $f(1)$ is enough to show that this is not the case.

The simplest examples of the formulae (6·1) and (7·3) are obtained by taking $n = 2$, when

$$c_q(n) = \mu(q) + 2\mu(\tfrac{1}{2}q)\,^*$$

is equal to $\mu(q)$ if q is odd and to $-\mu(q)$ if q is oddly even. We then obtain

$$\Sigma \frac{(-1)^{q-1}\,|\mu(q)|}{\phi_s(q)} = (1 - 2^{1-s})\,\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - \ldots,$$

which tends to the limit log 2 when $s \to 1$; but

$$\Sigma \frac{(-1)^{q-1}\,|\mu(q)|}{\phi(q)} = \tfrac{1}{2} \log 2.$$

The last result may be written in the form

$$1 - \frac{1}{\phi(2)} + \frac{1}{\phi(3)} + \frac{1}{\phi(5)} - \frac{1}{\phi(6)} + \frac{1}{\phi(7)} - \frac{1}{\phi(10)} + \ldots = \tfrac{1}{2} \log 2.$$

The series $\Sigma \dfrac{c_q(n)}{\phi(q)}$.

8. Another very interesting series, the consideration of which is suggested naturally by the work which precedes, is the series

$$\Sigma \frac{\mu(q)}{\phi(q)} = 1 - \frac{1}{\phi(2)} - \frac{1}{\phi(3)} - \frac{1}{\phi(5)} + \frac{1}{\phi(6)} - \ldots.$$

I shall prove that this series is convergent, and has the sum 0: more generally,

$$\Sigma \frac{c_q(n)}{\phi(q)} = 0 \quad \text{......................(8·1)}.$$

Let
$$f(s) = \Sigma \frac{c_q(n)}{q^{s-1}\,\phi(q)};$$

and let us write $s - 1 = z$. Then

$$f(s) = \Pi\, \chi_\varpi,$$

where
$$\chi_\varpi = 1 + \frac{c_\varpi(n)}{\varpi^z\,\phi(\varpi)} + \frac{c_{\varpi^2}(n)}{\varpi^{2z}\,\phi(\varpi^2)} + \ldots.$$

* We agree to regard $\mu(x)$, when x is not an integer, as zero.

If $\varpi \nmid n$
$$\chi_\varpi = 1 - \frac{\varpi^{-z}}{\varpi - 1};$$
while if ϖ^a is the highest power of ϖ which divides n
$$\chi_\varpi = 1 + \varpi^{-z} + \varpi^{-2z} + \ldots + \varpi^{-az} - \frac{\varpi^{-(a+1)z}}{\varpi - 1}.$$
Thus
$$f(s) = \prod_{\varpi \nmid n} \left(1 - \frac{\varpi^{-z}}{\varpi - 1}\right) \prod_{\varpi \mid n} \left(1 + \varpi^{-z} + \ldots + \varpi^{-az} - \frac{\varpi^{-(a+1)z}}{\varpi - 1}\right)$$
$$= g(s)\, h(s),$$
say. It is plain that $h(s)$ is a finite Dirichlet's series in s. On the other hand, if
$$\zeta_n(s) = \prod_{\varpi \nmid n} \left(\frac{1}{1 - \varpi^{-s}}\right) = \zeta(s) \prod_{\varpi \mid n} (1 - \varpi^{-s}),$$
$$g(s)\, \zeta_n(s) = \prod_{\varpi \nmid n} \left\{\left(1 - \frac{\varpi^{-(s-1)}}{\varpi - 1}\right) \Big/ (1 - \varpi^{-s})\right\}$$
$$= \prod_{\varpi \nmid n} \{1 + O(\varpi^{-\sigma-1}) + O(\varpi^{-2\sigma})\}$$
is regular and bounded in any half-plane $\sigma \geqq \frac{1}{2} + \delta > \frac{1}{2}$. Thus $g(s)$ behaves, in any such half-plane, substantially like the reciprocal of $\zeta_n(s)$, or of $\zeta(s)$; and so therefore does $f(s)$. And so we can prove (8·1) by substantially the same argument as that which proves (7·2).

For $n = 1$ we obtain
$$\Sigma \frac{\mu(q)}{\phi(q)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8\cdot2).$$
It should be observed that, in this series, the terms cancel in pairs, since
$$\frac{\mu(2q)}{\phi(2q)} = -\frac{\mu(q)}{\phi(q)}$$
if q is odd. Naturally this does not in itself establish the truth of (8·2).

If $n = 2$ we find that $c_q(n)$ is equal to $\mu(q)$ if q is odd, to $-\mu(q)$ if q is an odd multiple of 2, to $2\mu(\frac{1}{2}q)$ if q is an odd multiple of 4, and to zero if $q \equiv 0 \pmod 8$. Thus we obtain
$$1 + \frac{1}{\phi(2)} - \frac{1}{\phi(3)} - \frac{2}{\phi(4)} - \frac{1}{\phi(5)} - \frac{1}{\phi(6)} - \frac{1}{\phi(7)} - \frac{1}{\phi(10)}$$
$$- \frac{1}{\phi(11)} + \frac{2}{\phi(12)} - \ldots = 0.$$

9. The following results may be proved in substantially the same way:—

$$\Sigma \frac{\mu(q)\log q}{\phi(q)} = 0,$$

$$\Sigma \frac{\mu(q)(\log q)^2}{\phi(q)} = 4A \log 2;$$

where
$$A = \Pi \left\{1 - \frac{1}{(\varpi - 1)^2}\right\},$$

ϖ running through all odd primes*;

$$\sum_{1,3,5,\ldots} \frac{\mu(q)}{\phi(q)} = 0, \qquad \sum_{1,3,5,\ldots} \frac{\mu(q)\log q}{\phi(q)} = -2A;$$

$$\Sigma \frac{(-1)^{\nu(q)}}{q} = 0,$$

where $\nu(q)$ is the number of different prime factors of q;

$$\Sigma \frac{(-1)^{\nu(q)}\log q}{q} = 0;$$

$$\sum_{1,3,5,\ldots} \frac{(-1)^{\nu(q)}}{q} = 0, \qquad \sum_{1,3,5,\ldots} \frac{(-1)^{\nu(q)}\log q}{q} = -2A;$$

$$\Sigma \frac{2^{\nu(n)}\mu(n)}{n} = \Sigma \frac{2^{\nu(n)}\mu(n)\log n}{n} = \Sigma \frac{2^{\nu(n)}\mu(n)(\log n)^2}{n} = 0;$$

$$\Sigma \frac{2^{\nu(n)}\mu(n)(\log n)^3}{n} = -24A \log 2.$$

* $A = \cdot 66016\ldots$ is the constant which appears to play a fundamental part in 'Goldbach's Problem'. See two notes, by Shah and Wilson and by Hardy and Littlewood, in vol. XIX (1919) of the *Proceedings of the Cambridge Philosophical Society* (pp. 238–244 and 245–254).

On the distribution of primes. By H. Cramér, Stockholm.

(Communicated by Prof. G. H. Hardy.)

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Throughout the whole of this paper, I shall assume the truth of the Riemann hypothesis concerning the roots of the Zeta function, viz. $\zeta(\sigma + it) \neq 0$ *for* $\sigma > \frac{1}{2}$.

This being so, it is known that

$$\pi(x) = Li(x) + O(x^{\frac{1}{2}} \log x) \qquad (1)$$

and

$$p_n \sim n \log n \qquad (2),$$

where $\pi(x)$ denotes the number of primes less than or equal to x, and p_n denotes the nth prime. The last relation is independent of the Riemann hypothesis. But very little is known as to the behaviour of the difference

$$\Delta_n = p_{n+1} - p_n$$

between two successive primes, for large values of n. It follows from the "Prime Number Theorem" (1) or (2) that

$$\frac{\Delta_1 + \Delta_2 + \dots \Delta_n}{n} \sim \log n \sim \log p_n$$

and from (1) that

$$\Delta_n = O(p_n^{\frac{1}{2}} \log^2 p_n).$$

I have recently shown* that the last relation may be replaced by

$$\Delta_n = O(p_n^{\frac{1}{2}} \log p_n).$$

So far as I know, this is all that is actually known about Δ_n. It is very probable that $\Delta_n = 2$ for an infinity of values of n; but this has not yet been proved, and it has not even been proved that $\Delta_n < \frac{1}{2} \log p_n$ or $\Delta_n > 2 \log p_n$ for an infinity of values of n. The object of the present paper is to prove the following theorem, which gives an upper limit for the frequency of certain large values of Δ_n.

Theorem. *Let* $h(x)$ *be the number of primes* $p_n \leqq x$ *satisfying the inequality*

$$p_{n+1} - p_n > p_n^k,$$

where $0 < k \leqq \frac{1}{2}$. *Then*

$$h(x) = O(x^{1 - \frac{3}{2}k + \epsilon})$$

for every positive ϵ.

* "Some theorems concerning prime numbers," *Arkiv för Matematik, Astronomi och Fysik*, Band 15 (1920), No. 5, pp. 1—32.

It is interesting to remark that we may obtain by a very trivial argument (*viz.* that the sum of all the $h(x)\,\Delta_n$'s which are greater than p_n^k must be less than $x+x^k$) the evaluation

$$h(x) = O(x^{1-k}),$$

but it seems impossible to improve this even by direct deduction from the Prime Number Theorem.

The proof of the theorem given here depends on the theory of the function $\Sigma e^{\rho z}$, which I have studied in some recent papers*. We denote here and in the sequel by $\rho = \frac{1}{2} + i\gamma$ an arbitrary zero of the function $\zeta(s)$, situated in the *upper half* of the plane of the complex variable $s = \sigma + it$.

In order to prove the theorem, we shall require a set of lemmas. It seems convenient to remark that all the sets of points we shall have to deal with in the proof consist of a finite number of finite intervals (and perhaps a finite number of isolated points). Hence their measure may be taken to be the measure in the elementary sense.

Lemma 1. *If*

$$\phi(v, t) = \sum_{\gamma \leqq v} e^{\gamma i t},$$

we have

$$\int_{x-2}^{x+2} |\phi(v,t)|\, dt = O\left(v^{\frac{1}{2}} (\log v)^{\frac{3}{2}}\right)$$

uniformly for $x > 2$.

Proof. We have

$$|\phi(v, t)|^2 = \sum_{\gamma \leqq v} \sum_{\gamma' \leqq v} e^{(\gamma-\gamma')it},$$

and thus

$$\int_{x-2}^{x+2} |\phi(v, t)|^2 dt = \sum_{\gamma \leqq v} \sum_{\gamma' \leqq v} \int_{x-2}^{x+2} e^{(\gamma-\gamma')it}\, dt$$
$$< \sum_{\gamma \leqq v} \sum_{\gamma' \leqq v} \mathrm{Min}\left(\frac{2}{|\gamma - \gamma'|}, 4\right).$$

The number of numbers γ' in the interval $(\gamma+\nu, \gamma+\nu+1)$ is $O(\log(\gamma+\nu))$†. It follows that our sum is

$$O\left(\sum_{\gamma \leqq v} \log \gamma\right) + O\left[\sum_{\gamma \leqq v-1} \left(\frac{\log(\gamma+1)}{1} + \frac{\log(\gamma+2)}{2} + \dots + \frac{\log(\gamma+[v-\gamma])}{[v-\gamma]}\right)\right]$$
$$= O(v (\log v)^3),$$

* *l.c.* (footnote [1]); *Comptes rendus*, t. 168, p. 539; and *Mathematische Zeitschrift*, Band 4, pp. 104— .

† Landau, *Handbuch*, p. 337.

uniformly in x. Hence the truth of the lemma follows by the use of Schwarz's inequality.

Lemma 2. *Let us put*

$$y = \frac{1}{t} e^{-\alpha t}$$

and consider the interval $x - 2 \leqq t \leqq x + 2$, *where* $x > 2$ *and* $\frac{1}{2} \leqq \alpha < 1$. *Then the set* S_x *of points* t *belonging to this interval such that*

$$\left| \sum_\gamma \cos \gamma t \, e^{-\gamma y} \right| \geqq \tfrac{1}{8} e^{\frac{1}{2} t}$$

is of measure M_x *such that*

$$M_x = O\left(x^2 e^{-\frac{1-\alpha}{2} x}\right).$$

Proof. We have

$$\left| \sum_\gamma \cos \gamma t \, e^{-\gamma y} \right| \leqq \left| \sum_\gamma \left(e^{i\gamma t} . e^{-\gamma y} \right) \right| = \left| y \int_0^\infty \phi(v, t) \, e^{-vy} \, dv \right|$$

$$\leqq y \int_0^\infty | \phi(v, t) | \, e^{-vy} \, dv.$$

It follows by Lemma 1 that, if we denote by y_1 and y_2 the values of y at the points $x - 2$ and $x + 2$ respectively,

$$\int_{x-2}^{x+2} \left| \sum_\gamma \cos \gamma t \, e^{-\gamma y} \right| dt = O\left(\int_0^\infty y_1 v^{\frac{1}{2}} (\log v)^{\frac{3}{2}} e^{-v y_2} \, dv \right)$$

$$= O\left(y_1 y_2^{-\frac{3}{2}} \left(\log \frac{1}{y_2} \right)^{\frac{3}{2}} \right)$$

$$= O\left(x^2 e^{\frac{1}{2} \alpha x}\right).$$

Thus the measure of the set of points t belonging to the interval of integration, such that

$$\left| \sum_\gamma \cos \gamma t \, e^{-\gamma y} \right| \geqq \tfrac{1}{8} e^{\frac{1}{2} t},$$

must be of the form

$$M_x = O\left(x^2 e^{\frac{1}{2} \alpha x} . e^{-\frac{1}{2} x}\right)$$

$$= O\left(x^2 e^{-\frac{1-\alpha}{2}}\right).$$

Lemma 3. *In the set* $\bar{S}_x$, *complementary to the set* S_x *of Lemma* 2, *we have*

$$\sum_1^\infty \frac{\eta(n)}{n((t - \log n)^2 + y^2)} = \frac{\pi + \theta}{ty},$$

where $| \theta | < 1$ *for all sufficiently large values of* x. *Here* $\eta(n)$ *denotes the arithmetical function defined for integral values of* $n \geqq 1$ *by*

$$\eta(n) = \frac{1}{m} \quad (n = p^m, \, p \text{ prime}),$$

$$\eta(n) = 0 \quad (\text{otherwise}).$$

Proof. In a recently published paper*, I have proved the formula

$$y \sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} = \frac{\pi}{t} - 2\pi \mathfrak{R} \sum_\gamma \frac{e^{(\rho-1)(t+iy)}}{t+iy} + o\left(\frac{1}{t}\right),$$

where y may denote any positive decreasing function of t which tends to zero as t tends to infinity. On the Riemann hypothesis, and assuming y to be the function of Lemma 2, this formula reduces to

$$\begin{aligned} & y \sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} \\ &= \frac{\pi}{t} - \frac{2\pi}{t} e^{-\frac{1}{2}t}\left(1+O\left(\frac{y}{t}\right)\right) \sum_\gamma \cos(\gamma t - \tfrac{1}{2}y - \arg(t+iy))\, e^{-\gamma y} \\ &= \frac{\pi}{t} - \frac{2\pi}{t} e^{-\frac{1}{2}t}\left(1+O\left(\frac{y}{t}\right)\right)\left(\sum_\gamma \cos\gamma t\, e^{-\gamma y} + O\left(y \sum_\gamma e^{-\gamma y}\right)\right) + o\left(\frac{1}{t}\right). \end{aligned}$$

But we have

$$\sum_\gamma e^{-\gamma y} = O\left(\sum_1^\infty \log n\, e^{-ny}\right) = O\left(\frac{1}{y}\log\frac{1}{y}\right).$$

Hence

$$y \sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} = \frac{\pi}{t}\left(1 - 2e^{-\frac{1}{2}t} \sum_\gamma \cos\gamma t\, e^{-\gamma y}\right) + o\left(\frac{1}{t}\right).$$

By the definition of the set S_x, we have

$$\left| 2e^{-\frac{1}{2}t} \sum_\gamma \cos\gamma t e^{-\gamma y} \right| < \tfrac{1}{4}$$

for all values of t belonging to the complementary set $\bar{S}_x$. Since $\bar{S}_x$ is contained in the interval $(x-2, x+2)$, we conclude that

$$\sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2+y^2)} = \frac{\pi+\theta}{ty},$$

where $|\theta| < 1$ for all sufficiently large values of x.

Lemma 4. *We denote by $f(v)$ the function*

$$f(v) = \sum_{n \leqq v} \eta(n)$$

introduced by Riemann, and we consider the interval $(x-2, x+2)$ of the two preceding lemmas. The set of points t belonging to this interval such that

$$\sum_{t < \log n \leqq t+y} \eta(n) = f(e^{t+y}) - f(e^t) \geqq \frac{9y}{t} e^t$$

is, for all sufficiently large values of x, a subset of the set S_x of Lemma 2, so that its measure is of the form

$$O\left(x^2 e^{-\frac{1-a}{2}x}\right).$$

* *l.c.*, footnote *, p. 272.

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Proof. To prove this lemma, it is only necessary to show that

$$\sum_{t < \log n \leqq t+y} \eta(n) < \frac{9y}{t} e^t$$

if t belongs to the complementary set $\bar{S}_x$. This follows immediately from Lemma 3, for we have, since $\eta(n) \geqq 0$,

$$\sum_{t < \log n \leqq t+y} \frac{\eta(n)}{n((t-\log n)^2+y^2)} < \frac{\pi+1}{ty}$$

if x is sufficiently great and t belongs to $\bar{S}_x$. Hence

$$\sum_{t < \log n \leqq t+y} \eta(n) < \frac{\pi+1}{ty} \cdot 2y^2 \cdot e^{t+y} < \frac{9y}{t} e^t$$

for all sufficiently large values of x.

Lemma 5. *Let S be a set of points of measure M, situated in a finite interval ab. Then it is possible to divide ab into sub-intervals of length δ (the two extreme intervals being possibly less than δ) in such a way that not more than M/δ of the points of division belong to S.*

Proof. Consider any such division of ab, and denote by $\alpha\beta$ an arbitrary sub-interval of length δ. Let x be a point in $\alpha\beta$, and denote by $\phi(x)$ the number of points of S which are "congruent" to x according to the adopted division of ab. Then it is clear that

$$\int_\alpha^\beta \phi(x)\, dx = M.$$

Thus there must be at least one point x_0 in $\alpha\beta$, such that $\phi(x_0) \leqq M/\delta$. Starting the division of ab from this point, we see that the conditions stated in the lemma are fulfilled.

Lemma 6. *Let us denote bg σ_x the set of points t, belonging to the interval $(x-1, x+1)$, such that*

$$\pi(e^{t+cy}) - \pi(e^{t-cy}) \leqq \frac{y}{t} e^t,$$

where c is a positive constant. Here $\pi(v)$ denotes as usual the number of primes less than or equal to v. Then it is possible to give such a value to c that the measure μ_x of σ_x satisfies the relation

$$\mu_x = O\left(x^4 e^{-\frac{1-\alpha}{2}x}\right).$$

Proof. We shall prove this lemma by first excluding from the interval $(x-2, x+2)$ a certain set, the measure of which satisfies the relation just stated for μ_x, and then proving that we may choose c so that

$$\pi(e^{t+cy}) - \pi(e^{t-cy}) > \frac{y}{t} e^t \quad \text{..................(3)}$$

in the part of the remaining set which belongs to $(x-1, x+1)$.

By Lemma 5, we are able to divide $(x-2, x+2)$ into sub-intervals of the length

$$y_2 = \frac{1}{x+2} e^{-\alpha(x+2)}$$

in such a way that not more than

$$\frac{M_x}{y_2}$$

of the points of division belong to the set S_x of Lemma 2.

Supposing this to have been done, we exclude from the interval $(x-2, x+2)$ the set Σ_x defined in the following way: we take first the whole set S_x and then, denoting by t_0 any of the just mentioned points of division belonging to S_x, the interval $(t_0 - 2x^2y_2, t_0 + 2x^2y_2)$. The measure of Σ_x is thus less than

$$M_x + 2x^2y_2 \cdot \frac{M_x}{y_2} = O\left(x^4 e^{-\frac{1-\alpha}{2}x}\right).$$

In order to prove the lemma, we now have to show that c may be so chosen that (3) is valid for any t in $(x-1, x+1)$ not belonging to Σ_x. It is to be noticed that the definition of Σ_x in no way involves c.

Since t does not belong to Σ_x, it does not belong to S_x. Thus we have by Lemma 3

$$\sum_1^\infty \frac{\eta(n)}{n((t-\log n)^2 + y^2)} > \frac{\pi - 1}{ty} \quad \text{...............(4)}$$

for all sufficiently large values of x. We put

$$\sum_1^\infty = \sum_{\log n \leqq t-1} + \sum_{t-1 < \log n \leqq t - x^2y_2} + \sum_{t-x^2y_2 < \log n \leqq t - cy_2} + \sum_{t-cy_2 < \log n \leqq t + cy_2}$$

$$+ \sum_{t+cy_2 < \log n \leqq t + x^2y_2} + \sum_{t+x^2y_2 < \log n \leqq t+1} + \sum_{t+1 < \log n}$$

$$= A_1 + A_2 + \ldots A_7 \quad \text{....................(5).}$$

Then we have

$$A_1 < \sum_{\log n \leqq t-1} \frac{\eta(n)}{n} < \sum_{n \leqq e^t} \frac{1}{n} = o\left(\frac{1}{ty}\right) \text{...........(6),}$$

and

$$A_7 = \sum_{\log n > t+1} \frac{\eta(n)}{n((t-\log n)^2 + y^2)} < \sum_{\log n > t+1} \frac{\eta(n)}{n(\log n - t)^2}$$

$$= \sum_{\log n > t+1} \frac{\eta(n)}{n} \left(\frac{1}{(\log n)^2 + t^2} + \frac{2t \log n}{((\log n)^2 + t^2)(\log n - t)^2} \right)$$

$$< \sum_{\log n > t+1} \frac{\eta(n)}{n} \left(\frac{1}{(\log n)^2} + \frac{2t}{\log n} \right) = o\left(\frac{1}{ty}\right) \quad \ldots\ldots\ldots\ldots (7),$$

since the series $\sum \frac{\eta(n)}{n \log n}$ is convergent.

In order to obtain similar evaluations for A_2 and A_3, we consider the division of $(x-2, x+2)$ into sub-intervals of the length y_2 which we have used for the definition of Σ_x. Since t does not belong to Σ_x, it follows from this definition that none of the points of division situated in the interval $(t - x^2 y_2, t + x^2 y_2)$ belong to the set S_x. Hence, denoting by t_0 any such point of division, and by y_0 the corresponding value of y, we obtain, by Lemma 4,

$$\sum_{t_0 < \log n \leqq t_0 + y_2} \eta(n) \leqq \sum_{t_0 < \log n \leqq t_0 + y_0} \eta(n) < \frac{9 y_0}{t_0} e^{t_0}.$$

Thus, if we consider first A_3, and group together the terms belonging to the same sub-interval (considered as interval of variation of $\log n$), we obtain

$$A_3 < K \frac{y}{t} e^t . e^{-t} \left(\frac{1}{c^2 y^2} + \frac{1}{(c+1)^2 y^2} + \frac{1}{(c+2)^2 y^2} + \ldots \right) < \frac{2K}{cty} \ldots (8),$$

supposing $c > 1$ and denoting by K a constant independent of c and t.

Grouping together the terms of A_2 in a similar way, we have

$$A_2 < e^{-(t-1)} \sum_{t-1 < \log n \leqq t - x^2 y_2} \frac{1}{(t - \log n)^2}$$

$$= O\left[e^{-t} . y e^t \left(\frac{1}{x^4 y^2} + \frac{1}{(x^2+1)^2 y^2} + \frac{1}{(x^2+2)^2 y^2} + \ldots \right) \right]$$

$$= O\left(\frac{1}{x^2 y}\right) = o\left(\frac{1}{ty}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

since the number of terms in each group is of the form $O(ye^t)$.

Of the remaining terms, A_6 may obviously be treated in the same way as A_2, and A_5 in the same way as A_3. Thus we see, from (4)—(9), that it is possible to determine two absolute constants c_0 and x_0 such that

$$A_4 = \sum_{t - cy_2 < \log n \leqq t + cy_2} \frac{\eta(n)}{n((t - \log n)^2 + y^2)} > \frac{\pi - 2}{ty}$$

for all values of t in $(x-1, x+1)$ not belonging to Σ_x, so long as $c > c_0$ and $x > x_0$. Hence, *a fortiori*,

$$\sum_{t-cy<\log n \leqq t+cy} \frac{\eta(n)}{n((t-\log n)^2+y^2)} > \frac{\pi-2}{ty}$$

and

$$\frac{e^{-(t-cy)}}{y^2} \sum_{t-cy<\log n \leqq t+cy} \eta(n) > \frac{\pi-2}{ty},$$

$$\sum_{t-cy<\log n \leqq t+cy} \eta(n) > (\pi-2)\frac{y}{t} e^{t-cy} \quad \text{.........(10).}$$

But we have

$$\sum_{t-cy<\log n \leqq t+cy} \eta(n) = \pi(e^{t+cy}) - \pi(e^{t-cy})$$

$$+ \sum_{\nu=2}^{\frac{t+cy}{\log 2}} \frac{1}{\nu}\left[\pi\left(e^{\frac{1}{\nu}(t+cy)}\right) - \pi\left(e^{\frac{1}{\nu}(t-cy)}\right)\right],$$

and

$$\sum_{\nu=2}^{\frac{t+cy}{\log 2}} \frac{1}{\nu}\left[\pi\left(e^{\frac{1}{\nu}(t+cy)}\right) - \pi\left(e^{\frac{1}{\nu}(t-cy)}\right)\right]$$

$$= O\left[\sum_2^t \frac{1}{\nu}\left(e^{\frac{1}{\nu}(t+cy)} - e^{\frac{1}{\nu}(t-cy)} + 1\right)\right] = O\left[\sum_2^t \frac{1}{\nu}\left(\frac{y}{\nu} e^{\frac{1}{2}t} + 1\right)\right]$$

$$= O(\log t) = o\left(\frac{y}{t} e^t\right).$$

Hence, if c is fixed and greater than c_0, we obtain from (10),

$$\pi(e^{t+cy}) - \pi(e^{t-cy}) > \frac{y}{t} e^t,$$

for all sufficiently large values of x, and for all values of t in $(x-1, x+1)$ not belonging to Σ_x.

Proof of the theorem.

Consider the set of points e^t, where t passes through all the points t of the set σ_x of Lemma 6. This set belongs to the interval (e^{x-1}, e^{x+1}), and its measure is less than $e^{x+1}\mu_x$. Hence if, in Lemma 6, we write x in the place of e^x, t in the place of e^t and k in the place of $1-\alpha$, this lemma will take the following form:

Let us denote by σ_x' the set of points t, belonging to the interval $\left(\frac{x}{e}, ex\right)$, such that

$$\pi\left(t + \frac{c}{\log t} t^k\right) - \pi\left(t - \frac{c}{\log t} t^k\right) \leqq \frac{t^k}{\log^2 t},$$

where c is a positive constant and $0 < k \leqq \frac{1}{2}$. Then it is possible to give such a value to c that the measure μ_x' of σ_x' satisfies the relation

$$\mu_x' = O\left(x^{1-\frac{1}{2}k} \log^4 x\right).$$

Suppose now that c has been properly fixed. Then it is clear that, if x is sufficiently large, and if p_n denotes a prime belonging to the interval $(\frac{1}{2}x, x)$ and satisfying the inequality

$$p_{n+1} - p_n > p_n^k,$$

then the interval (p_n, p_{n+1}) will contribute more than $\frac{1}{2}p_n^k$, and *a fortiori* more than $\frac{1}{2}(\frac{1}{2}x)^k$, to σ_x'. Hence we obtain

$$\tfrac{1}{2}\left(\tfrac{1}{2}x\right)^k \left(h(x) - h\left(\tfrac{1}{2}x\right)\right) = O\left(x^{1-\frac{1}{2}k+\epsilon}\right),$$

$$h(x) - h\left(\tfrac{1}{2}x\right) = O\left(x^{1-\frac{3}{2}k+\epsilon}\right).$$

If we replace in the last relation x first by $\frac{1}{2}x$, then by $\frac{1}{4}x$, and so on, and add together all the relations obtained in this way, we get

$$h(x) = O\left(x^{1-\frac{3}{2}k+\epsilon}\right).$$

Hence our theorem is proved.

CAMBRIDGE, 20 *July*, 1920.

Note on the parity of the number which enumerates the partitions of a number. By Major P. A. MacMahon.

[*Read* 25 October 1920.]

In a letter received by me from the late S. Ramanujan about a year ago he stated that it was his intention to calculate the number of the partitions of 1000 by the direct approximate formula which had been successfully used by him and G. H. Hardy in the calculation of the case $n = 200$. He enquired, at the same time, if I knew of any simple way of ascertaining whether the sought number is even or uneven as this information would be of importance to him. This note has arisen in consequence of this enquiry.

I shew, in particular, that in the case of the partitions of 1000 the parity can be found, in a few minutes, from certain congruence relations.

It is easy to derive, from the theory of the self-conjugate partitions of n, that

$$\frac{1}{\prod_1^\infty {}^m (1-q^m)} \equiv \prod_1^\infty {}^m (1+q^{2m-1}) \bmod 2.$$

Thence

$$\frac{1}{\prod_1^\infty {}^m (1-q^m)} \equiv \frac{\prod_1^\infty {}^m \{(1+q^{4m-1})(1+q^{4m-3})(1-q^{4m})\}}{\prod_1^\infty {}^m (1-q^{4m})} \bmod 2.$$

The numerator of the fraction on the right is one of Jacobi's elliptic products which has the series expression

$$1 + q + q^3 + q^6 + q^{10} + \ldots + q^{\frac{1}{2}s(+1)} + \ldots.$$

Wherefore if p_n denote the number of partitions of n

$$\Sigma p_n q^n \equiv (1 + q + q^3 + q^6 + q^{10} + \ldots) \, \Sigma p_n q^{4n} \bmod 2.$$

Put $\qquad p_n \equiv a_n \bmod 2.$

Comparison of the coefficients of like powers of q yields the four relations

$$\begin{aligned} a_{4n} \equiv a_n &+ a_{n-7} + a_{n-9} + a_{n-30} + a_{n-34} + a_{n-69} + a_{n-75} + a_{n-124} \\ &+ a_{n-132} + a_{n-195} + a_{n-205} + a_{n-282} + a_{n-294} + \ldots, \\ a_{4n+1} \equiv a_n &+ a_{n-5} + a_{n-11} + a_{n-26} + a_{n-38} + a_{n-63} + a_{n-81} + a_{n-116} \\ &+ a_{n-140} + a_{n-185} + a_{n-215} + a_{n-270} + a_{n-306} + \ldots, \\ a_{4n+2} \equiv a_{n-1} &+ a_{n-2} + a_{n-16} + a_{n-19} + a_{n-47} + a_{n-52} + a_{n-94} + a_{n-101} \\ &+ a_{n-157} + a_{n-166} + a_{n-236} + a_{n-247} + a_{n-331} + \ldots, \\ a_{4n+3} \equiv a_n &+ a_{n-3} + a_{n-13} + a_{n-22} + a_{n-42} + a_{n-57} + a_{n-87} + a_{n-108} \\ &+ a_{n-148} + a_{n-175} + a_{n-225} + a_{n-258} + a_{n-318} + \ldots. \end{aligned}$$

These relations may be written

$$\left.\begin{aligned} a_{4n} &= \sum_0^\infty {}^s\, a_{n-(8s^2\pm s)} \\ a_{4n+1} &= \sum_0^\infty {}^s\, a_{n-(8s^2\pm 3s)} \\ a_{4n+2} &\equiv \sum_0^\infty {}^s\, a_{n-1-(8s^2\pm 7s)} \\ a_{4n+3} &\equiv \sum_0^\infty {}^s\, a_{n-(8s^2\pm 5s)} \end{aligned}\right\}.$$

In fact the four relations are connected with the four elliptic products

$$\prod_1^\infty {}^m (1-q^{16m-9})(1-q^{16m-7})(1-q^{16m}),$$

$$\prod_1^\infty {}^m (1-q^{16m-11})(1-q^{16m-5})(1-q^{16m}),$$

$$q\prod_1^\infty {}^m (1-q^{16m-15})(1-q^{16m-1})(1-q^{16m}),$$

$$\prod_1^\infty {}^m (1-q^{16m-13})(1-q^{16m-3})(1-q^{16m}).$$

The formulae which have been obtained soon involve high numbers and are therefore suitable for the calculation of parity of p_n when m is large.

As an example of the use of the formulae I append the calculation of a_{1000}, the parity of p_{1000}, making use of the enumeration of the partitions of n, as far as $n = 200$, calculated by me in connection with the valuable paper by G. H. Hardy and S. Ramanujan*.

We use the first of the four relations but we first of all require the parities of $p_{250}, p_{243}, p_{241}, p_{220}, p_{216}$.

From the third relation

$$\begin{aligned} a_{250} &\equiv a_{61} + a_{60} + a_{46} + a_{43} + a_{15} + a_{10} \\ &\equiv 1+1+0+1+0+0 \equiv 1. \end{aligned}$$

From the fourth relation

$$\begin{aligned} a_{243} &\equiv a_{60} + a_{57} + a_{47} + a_{38} + a_{18} + a_3 \\ &\equiv 1+0+0+1+1+1 \\ &\equiv 0. \end{aligned}$$

From the second relation

$$\begin{aligned} a_{241} &\equiv a_{60} + a_{55} + a_{49} + a_{34} + a_{22} \\ &\equiv 1+0+1+0+0 \\ &\equiv 0. \end{aligned}$$

* *Proc. Lond. Math. Soc.* Vol. XV *et seq.*

From the first relation

$$\begin{aligned} a_{220} &\equiv a_{55} + a_{48} + a_{46} + a_{25} + a_{21} \\ &\equiv 0 + 1 + 0 + 0 + 0 \\ &\equiv 1. \end{aligned}$$

$$\begin{aligned} a_{216} &\equiv a_{54} + a_{47} + a_{45} + a_{24} + a_{20} \\ &\equiv 1 + 0 + 0 + 1 + 1 \\ &\equiv 1. \end{aligned}$$

Thence

$$\begin{aligned} a_{1000} &\equiv a_{250} + a_{243} + a_{241} + a_{220} + a_{216} + a_{181} + a_{175} + a_{126} + a_{118} + a_{55} + a_{45} \\ &\equiv 1 + 0 + 0 + 1 + 1 + 1 + 0 + 0 + 1 + 0 + 0 \\ &\equiv 1, \end{aligned}$$

establishing, in about five minutes work, that p_{1000} is an uneven number.

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PROCEEDINGS

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Note on constant volume explosion experiments. By S. LEES, M.A., St John's College.

[*Read* 24 January 1921.]

§ 1. Much work has been done in recent years in determining the values of the volumetric heats of gases at high temperatures. In the main, the experiments have been made either at constant pressure with the gas heated externally, or at constant volume with the heating produced by explosion in a closed vessel. A comprehensive collection of available data on this subject, with estimates of the probable degrees of accuracy involved, has been given by Mr D. R. Pye of Trinity College*.

Such differences as occur in the results from the two methods seem to be in one direction, the internal energy for a gas obtained by constant pressure methods being slightly less than the value obtained by constant volume methods. It seemed possible to the author that the variations of temperature experienced at any instant in different parts of an explosion vessel, might account for some part of the difference in the values of the internal energy obtained by the two methods.

In this connection, the following extract from the first *British Association Report on Gaseous Explosions* (1908) may be quoted:

> If the volumetric heat of the gas were constant, the equalisation of these temperature differences by convection and conduction, could it take place without loss of heat, would cause no change of pressure. The volumetric heat is, however, not constant, but may quite possibly be 50 per cent. greater in the hottest than in the coldest part of the mass. The attainment of thermal equilibrium must, in fact, cause a change of pressure....The amount of the change might be the subject of rough calculation, taking an assumed distribution of temperature and assuming values for the volumetric heat. Such a calculation in the present state of knowledge would only be of value as showing the possible order of magnitude of the quantity sought, and the assumptions made could therefore be of a character to make the calculation fairly simple. More accurate knowledge both of temperature distribution and of thermal capacity will enable greater accuracy to be attained in the estimation of this correction, which will be of such a kind that a method of successive approximation can be pursued, the revised values of thermal capacity resulting from its application being applied to a more accurate calculation of the correction, if necessary.

* See *Automobile Engineer*, Feb., 1920.

It may be stated at the outset that the corrections on this account to be applied to the values of total energy obtained from constant volume experiments appear to be of very small amounts, and to be well within the limits of experimental error. In spite of this, the method of computation given below may be of interest to workers on this subject.

§ 2. In discussing the state of affairs after a constant volume explosion, some simplifying assumptions will be made. The constituent gases of the exploded mixture are each assumed to follow the ideal gas law

$$pV = RT, \qquad \text{.........................(1)}$$

where p is pressure, V is volume, T is absolute temperature, and R is the same constant for a gramme-molecule of each of the constituent gases. It is further assumed that these constituent gases are intimately mixed, so that there is no variation in the chemical composition of the gas mixture at different points of the interior of the vessel. Finally it is assumed that the volumetric heat of each constituent follows a linear law of increase with temperature. Thus a given mixture will also follow a linear law of increase of volumetric heat.

§ 3. The calculations involved are made more simple if a mass M of a constituent gas be replaced by $N = M/m$, where m is its molecular weight. The mass M is thus replaced by its equivalent in gramme-molecules. In the case of any portion of a chemically homogeneous gas mixture, the weights of the constituent gases may be replaced by the number of gramme-molecules in each case, and we may speak of the number of gramme-molecules of the mixture without any difficulty.

To take into account the variation of temperature at any instant of the mixture at different parts of the interior of the explosion vessel, we may divide up the contents of the vessel as follows:

N_1 gramme-molecules of mixture at absolute temperature T_1,
N_2 ,, ,, ,, ,, ,, T_2,
N_3 ,, ,, ,, ,, ,, T_3,
etc. etc.

If N be the total number of gramme-molecules of the mixture in a vessel of volume V, the partial volumes V_1, V_2, etc. of the constituents are given by

$$pV_1 = RN_1T_1, \quad pV_2 = RN_2T_2, \quad pV_3 = RN_3T_3, \text{ etc.},$$

where p is the uniform pressure in the vessel. Thus

$$pV = p(V_1 + V_2 + V_3 + \text{etc.}) = R(N_1T_1 + N_2T_2 + N_3T_3 + \text{etc.})$$
$$= R\Sigma(N_1T_1). \qquad \text{..................(2)}$$

In most experiments of this kind, the pressure p is used as a means of measuring the mean temperature T_m. For a chemically homogeneous mixture, this mean temperature will obviously be defined as

$$T_m = \frac{\Sigma(N_1T_1)}{\Sigma N_1} = \frac{\Sigma(N_1T_1)}{N}, \quad \text{.................(3)}$$

where N is the total number of gramme-molecules in the vessel. Hence from equations (2)

$$pV = RNT_m, \quad \text{.......................(4)}$$

which since V, R, N are constants, indicates how p may be used to measure T_m.

§ 4. The total internal energy E of a gramme-molecule of the gaseous mixture at a uniform temperature T (absolute), assuming linear increase of volumetric heat with temperature, is given by

$$E = A_0T + \tfrac{1}{2}BT^2,$$

where A_0 and B are constants. If as usual we measure E from a standard temperature T_0, the internal energy E must be written

$$E = A_0(T - T_0) + \tfrac{1}{2}B(T^2 - T_0^2) = C + A_0T + \tfrac{1}{2}BT^2, \quad \text{...(5)}$$

where $C = -(A_0T_0 + \tfrac{1}{2}BT_0^2)$.

On dividing this value of E by $(T - T_0)$ we get the mean specific heat (at constant volume) per gramme-molecule, i.e. the mean volumetric heat, between T and T_0, in the form

$$A_0 + \tfrac{1}{2}B(T + T_0) = A + \tfrac{1}{2}BT, \quad \text{..............(6)}$$

where $A = A_0 + \tfrac{1}{2}BT_0$.

The true volumetric heat at temperature T is given by dE/dT and is therefore equal to

$$A_0 + BT. \quad \text{.........................(7)}$$

§ 5. Reckoned from T_0, the internal energy of the mixture defined in § 3 will be

$$C\Sigma N_1 + A_0\Sigma N_1T_1 + \tfrac{1}{2}B\Sigma N_1T_1^2 = CN + A_0NT_m + \tfrac{1}{2}B\Sigma N_1T_1^2. \quad \text{......(8)}$$

In explosion experiments, this is usually assumed to be equal to the internal energy of the whole mass taken at the mean temperature T_m, i.e. is assumed to be equal to

$$CN + A_0NT_m + \tfrac{1}{2}BNT_m^2. \quad \text{.................(9)}$$

The first two terms of expression (8) are the same as the first two terms of expression (9). We have, therefore, to compare $\Sigma N_1T_1^2$ with NT_m^2. To do this, put

$$T_1 = T_m + t_1, \quad T_2 = T_m + t_2, \quad T_3 = T_m + t_3, \text{ etc.} \quad \text{...(10)}$$

so that $NT_m = \Sigma N_1 T_1 = \Sigma N_1(T_m + t_1) = NT_m + \Sigma N_1 t_1$, from (3), i.e.

$$\Sigma N_1 t_1 = 0. \qquad \text{.......................(11)}$$

We then have

$$\Sigma N_1 T_1^2 = \Sigma N_1 (T_m + t_1)^2 = T_m^2 \Sigma N_1 + 2T_m \Sigma N_1 t_1 + \Sigma N_1 t_1^2, \text{ (12)}$$

so that from (11)

$$\Sigma N_1 T_1^2 = NT_m^2 + \Sigma N_1 t_1^2. \qquad \text{..............(13)}$$

Hence expression (8) is greater than expression (9) by the essentially positive quantity

$$\tfrac{1}{2} B \Sigma N_1 t_1^2. \qquad \text{.......................(14)}$$

To get the true value of the internal energy corresponding to uniform temperature T_m, we must therefore diminish the actually

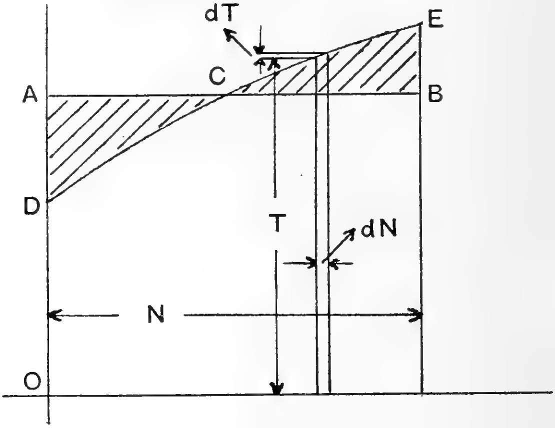

Fig. 1.

measured value (8) by expression (14). This means that the explosion experiments will give higher values, in general, for the internal energy, than constant pressure experiments, ideal accuracy in other respects being postulated for the determinations.

§ 6. There remains the problem of estimating the order of this difference of internal energy values, i.e. the order of magnitude of expression (14).

The problem of temperature distribution at any instant inside an explosion vessel requires the determination of the number dN of gramme-molecules of mixture* whose temperature lies between T and $T + dT$. If this be known, the temperature distribution can be represented by a graph like fig. 1. The graph is such that the abscissa of any point on the curve DCE represents the total number of gramme-molecules whose absolute temperature is not greater

* To emphasise the simple algebraic character of the argument, the calculus notation has been avoided in §§ 3 to 5.

than T. From its nature, therefore, the curve must always slope upwards from D to E. Referring to this diagram, it is easily seen from the definition (3) of T_m, that the shaded areas ACD and CEB, on opposite sides of the line AB, must be equal.

In the absence of more accurate knowledge as to the shape of the curve DCE, we may take it as approximately a straight line. This will, at any rate, give some kind of approximation to the truth. On this assumption, we shall apply our expressions to some results obtained by the late Prof. B. Hopkinson. In one of his explosion vessel experiments, Hopkinson* found for the instant of

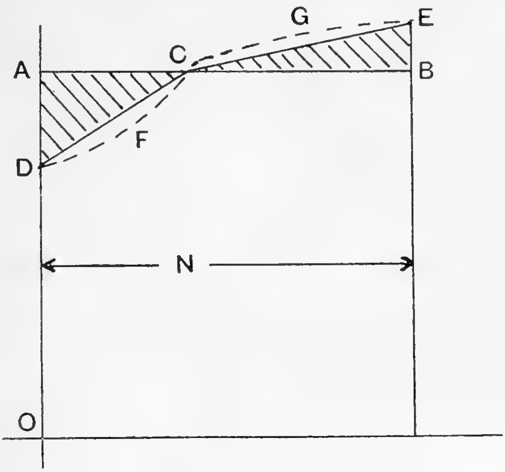

Fig. 2.

maximum pressure (corresponding in this case to a mean temperature of 1600° C.) a maximum temperature at the centre of the vessel of about 1900° C., whilst the temperature was probably as low as 1100° C. in the immediate neighbourhood of the walls. The difference between the maximum temperature and the mean being less than the difference between the minimum temperature and the mean, the curve of temperature distribution has been arbitrarily assumed as DCE in fig. 2. In this diagram, DC and CB are taken as straight lines, whilst the point C on AB has been so chosen that the area ADC equals the area CEB. It will readily be verified for the temperatures obtained by Hopkinson that $AC = 3N/8$, whilst $CE = 5N/8$.

For continuously varying temperatures throughout the gas, expression (14) must be replaced by

$$\tfrac{1}{2}B\int t^2 dN, \qquad \text{.........................(15)}$$

* *Proc. Roy. Soc.*, 1906.

the integral being taken over the shaded areas previously referred to. It is useful to note that the integral in (15) for each piece of the shaded area is exactly twice the first moment* of the area of the piece about AB. The expression (15) thus becomes in our case numerically equal to

$$B\left\{\frac{(500)^2 \times 3N}{6 \times 8} + \frac{(300)^2 \times 5N}{6 \times 8}\right\} = 25000BN. \quad \text{......(16)}$$

The mean volumetric heat per gramme-molecule for air, from 0° C. to 1600° C., is probably about 5·55 gramme-calories. Thus for air at 1600° C., the internal energy reckoned from 0° C. is about $8880N$. The increase in value of the volumetric heat with temperature does not appear to follow exactly the linear law, but in the neighbourhood of 1600° C. we may take the increase as being roughly 0·5 gramme-calories per 1000° C. Thus from expression (6) we see that B can be taken as 1×10^{-3}. Thus expression (16) becomes $25N$, which expressed in terms of $8880N$ is 0·283 %. With the assumed conditions, this is the amount by which the observed internal energy at 1600° C. should be diminished to give the corrected value corresponding to uniform temperature throughout.

It is conceivable that the true shape of the temperature distribution curve may be something like the dotted curve $DFCGE$ of fig. 2, in which case the correction would be materially increased. It is also possible that the temperature differences in the gas may diminish during the early stages of cooling at a much slower rate than the mean temperature. On the whole, however, it would seem probable that the correction ought not to exceed 1 % of the value of the internal energy, and might very well be much less.

§ 7. *Conclusion.* An effort has been made to investigate the effect of temperature variations in an explosion vessel, on the values of the total internal energy measured. Instead of approaching the problem from the point of view indicated in the *B.A. Report* †, by modifying the temperature for a given internal energy determination, the author has modified the internal energy for a given mean temperature, and has given reasons which seem to indicate that the correction is probably not more than 1 % at a temperature of 1600° C. This order of correction is within the limits of probable error of experiments at the present time ‡.

* For $\int t^2 dN = 2\int \frac{t}{2} \cdot dS$, where $dS = tdN$.

† *Loc. cit.*

‡ See Pye, *loc. cit.*

On the Latent Heats of Vaporisation. By Eric Keightley Rideal, M.A., Trinity Hall.

[*Read* 28 February 1921.]

A number of attempts have been made to associate the latent heats of evaporation (L) with the natural periods of atomic vibration (v) as calculated by the methods of Lindemann*, Einstein, Bernouilli and Nernst or determined experimentally by Rubens. There is, however, no relationship between the infra red vibration frequencies of the elements and their latent heats of evaporation. On the assumption, however, that intermolecular, chemical and physical actions take place with energy transfer in quanta and not continuously, it should be possible to derive the latent heat of evaporation of an element from some natural vibration frequency or spectral line. Although this spectral line need not necessarily be in the infra red portion of the spectrum. It is especially desirable to test this hypothesis in the case of the latent heats of evaporation, since this is a typical physical process and a conformity to theory would confirm the supposed identity of physical and chemical forces and at the same time from the thermodynamic equations connecting the latent heats of evaporation and the vapour pressure together with the molecular kinetic effusion equations of Herz†, Marcelin and Langmuir it would be possible to calculate the indetermined integration constant of the vapour pressure formula and thence the so-called Nernst chemical constants.

According to the quantum theory applied to the energy change involved in evaporation considered as a chemical process the latent heat of evaporation per gm. mol. should be given by the expression

$$L = Nh\,(v_{\text{products}} - v_{\text{reactants}}),$$

where h is Planck's constant, $v_{\text{p.}}$ the vibration frequency of the products the metal vapour and $v_{\text{r.}}$ of the reactants, the solid or liquid evaporating. The evaporation of a metal may be imagined to take place by two different processes. We may remove a complete atom from the metal to the vapour state, then the activating frequency of the metal atom in the vapour state will correspond to some very small energy transfer, since experiment has indicated that practically all vapour atoms coming in contact with the solid metal stick, *i.e.* as far as reactivity with the solid metal is concerned practically all the atoms are active, thus v products will correspond to a radiation far in the infra red and is probably not far removed

* *Physical Chemistry*, McLewis, vol. III.

† *Ann. Physik*, 17, 177 (1882).

from the natural infra red vibration frequency itself. The frequency of the radiation necessary to activate an atom in the solid will correspond to some line in the spectral series, each representing a different degree of activation of the element, which should be capable of experimental determination in the spectrum. In the attached tables are given a few of the lines in the spectrum of the various elements as observed by Rowland, Kayser and others (c/o Landolt-Bornstein Tabellen) and the latent heats calculated therefrom (L_1). It will be noted that the line corresponding to the activation frequency of the atom in the solid is generally fairly widely separated from the other lines and is thus at or near the head of a series. The choice of the particular fundamental line, however, is somewhat arbitrary and it is consequently necessary to find some alternative method for calculation of the latent heats.

If we consider that a metal is built up of space lattices of alternating valence electrons and positive nuclei, the activation of a metal atom can be assumed to take place in two distinct steps, the activation of the positive nucleus in the metal and the activation of the valence electron attached to it. The activation frequencies of the positive nucleus and the valence electron respectively are given by the $v_{\text{infra red}}$ and $v_{\text{ultra violet}}$ (or the photo electric frequency) frequencies whilst the activation frequency of the atom as a nucleus-electron complex corresponds to the so-called "radiation potential" of the element. It has already been indicated by Haber* that the transfer of a quantum of energy in a solid results in the activation of two atoms, being virtually a cleavage of a diatomic molecule linked by an electron into two active atoms. The latent heat of evaporation will therefore be given by the expression:

$$L = \tfrac{1}{2}NL\,\{v_{\text{radiation}} - \{v_{\text{infra red}} + v_{\text{ultra violet}}\}\}.$$

In the following tables are given the latent heats of evaporation calculated on this basis (L_2), the infra red radiation frequency being calculated from Lindemann's melting point formula, the ultra violet frequency from Haber's relationship

$$Mv^2_{\text{red}} = mv^2_{\text{ultra violet}},$$

where M and m are the atomic and electronic masses respectively. Where experimental values for $v_{\text{radiation}}$ (or the radiation potential) are not available there have been calculated from the approximate relationship $v_{\text{radiation}} = \frac{v_{\text{ultra violet}}}{2.3}$. In the last column are given the values of (L_3) calculated from vapour pressure data collected by Johnston (*Jour. Ind. and Eng. Chem.* 9, 876, 1917), Gebhardt (*Ann.* 40, 438, 1913) and Langmuir (*Phys. Rev.* 2, 329, 1913).

* *Ber. Deutsch. phys. ges.* 13, 1117 (1911).

Metal	Line of atomic activation with those next to it λ in $\mu\mu$.	L_1	Infra red $Nh\nu$ in calories	Ultra violet $Nh\nu$ in calories	Resonance $Nh\nu$ in calories observed	L_2	L_3 observed latent heat
K	1248, 1252, *1516·6*, 2706	18,575	225	64,100	27,480	18,420	17,900
Na	819, 1138, *1267*, 1845	23,718	382	88,640	48,090	20,466	20,500
Li	670, 8, *812*, 1868	34,264	936	104,600	—	30,000	29,000
Rb	1366, 1475, *1529*, 3581	17,644	136	53,700	—	15,240	17,780
Cs	1012, 1354, *1470*, 2931	18,511	889	57,300	—	16,180	15,900
Cd	643·8, *1039·6*, 1398	27,219	281	126,350	87,930	24,350	25,000
Zn	636·4, *1105·5*, 1305·5	25,590	408	140,600	90,000	25,500	28,800
Ag	518·4, *880·7*, 1182·9	32,000	664	141,500	—	40,300	33,700
Au	338, *405·5*, 563·3	69,800	408	187,500	—	51,700	65,500
Cu	327·4, *402·3*, 578·2	70,373	627	214,500	—	61,100	75,400
Fe	346, 357, *372*, 386	76,000	777	250,650	—	71,160	77,800
Cr	360·5, 364, 367·6, 390·3	72,230	777	240,400	—	68,370	68,200
Pb	*391·6*, 404·9, 412·9 560, *665*	42,820	180	112,600	—	32,000	45,400
Sn	*374·6*, *452*, 556	75,800 63,100	210	98,260	—	27,900	69,000
Al	*572*, 1125	49,300	702	157,000	—	44,700	52,600
Tl	377, *535*, 1151	55,130	172	105,500	—	30,000	40,500
Hg	615, 1014, *1367·4*	20,700	202	124,900	114,210	5,450	13,500
Pt	—	—	408	244,900	—	69,400	128,000
Mo	—	—	590	246,000	—	70,000	177,800
W	—	—	468	273,500	—	77,450	218,500
Ni	367, *372*, 377·5	76,140	768	250,800	—	71,230	76,500
Mn	478, *482·3*, 601·3	58,600	702	224,270	—	63,700	56,300

It will be noted that for the metals in which the natural constants have actually been determined there is a very fair agreement in the three values of (L). Marked divergencies are, however, to be noted in the case of the polyvalent elements Hg, Sn and Tl, and in those metals of high boiling points. In this latter case the fact that thermionic emission is strongly marked is an indication that the Haber relationship does not give the correct periodicity of the binding electron, but one of the electrons liberated by thermal agitation. Closer agreement is scarcely to be expected since, in many cases, the whole basis of calculation is the Lindemann melting point and the Haber vibrational relationship, both only approximations.

There is, however, a sufficiently good agreement to afford evidence for the validity of the quantum relationship in this physical molecular energy interchange. It would be of interest to determine whether the extension of this hypothesis, involving new assumptions, were also valid, *i.e.* whether the vapour pressure of a metal, *e.g.* potassium, could be raised by illumination with either monochromatic radiation of $\lambda = 1516{\cdot}6\ \mu\mu$ or with a dichromatic illumination of frequencies 2·3 10^{12} and 6·85 10^{14}, the respective infra red and ultra violet activating frequencies.

The application of the hypothesis to the calculation of the latent heat of evaporation of non-conductors is more difficult, owing to our lack of knowledge of the intermolecular forces. The latent heats are, however, all relatively small and the infra red absorption bands generally well marked, thus CCl_4 possesses a marked absorption band at 4·5 μ equivalent to the value $L = 6400$ cal. per gm. mol. which agrees with the observed value of

$$7100 - 700 = 6400 \text{ cal.}$$

In the case of naphthalene an absorption band would be anticipated at 3·36 μ, whilst actual observation indicates a strong absorption at 3·25 μ. In the case of water the complex infra red spectrum leaves little doubt as to the existence of polymers in solution. That the formation of di- and tri-hydrol takes place through the oxygen atom is indicated by the equivalence of the latent heat of evaporation to the interatomic energy provided by a vibration frequency of 3·0 μ (9,540 cal. per gm. mol.) an absorption band which is always noted in compounds containing the hydroxyl group.

The so-called chemical constants of Nernst necessary for the evaluation of the equilibrium constants and reaction affinities of reactions occurring in non-condensed systems are related to the undetermined constant of the vapour pressure formula by the relationship

$$C_0 = \frac{i + \log_e R}{2 \cdot 3023},$$

where i is the integration constant of the Clapeyron-Clausius equation

$$\frac{L_m}{RT^2} = \frac{\partial \log P}{\partial T},$$

or

$$\frac{\partial \log C}{OT} = \frac{L_m - RT}{RT^2};$$

putting

$$L_m - RT = L_0 + \alpha T + \beta T^2 + \ldots,$$

$$\frac{\partial \log_e C}{\partial T} = \frac{L_0}{RT^2} + \frac{\alpha}{RT} + \ldots,$$

$$\log_e C = \frac{-L_0}{RT} + \frac{\alpha}{R} \log T + i,$$

or

$$\log_e P = \frac{-L_0}{RT} + \left\{\frac{\alpha_0 + R}{R}\right\} \log_e T + \frac{\beta_0 T}{R} + \ldots i + \log_e R.$$

When equilibrium is established between a liquid or solid and its vapour the number of molecules striking the surface in unit time is equal to the number leaving.

According to the effusion equation developed by Herz, Knudsen and Langmuir* the number of molecules striking unit area per second in a gas at pressure p, temperature T and of molecular weight M is given by the equation

$$\mu = \frac{P}{\sqrt{2\pi RMT}},$$

where μ is the number of gm. mols. striking a square cm. per second. In a recent communication† it was shown that the rate of a unimolecular reaction

$$\frac{\partial n}{\partial t} = v \,.\, e^{-\frac{Nhv}{RT}}$$

had a physical significance in that v was the time of molecular relaxation. This relationship shown to be experimentally true by Dushman and Langmuir‡ assumes merely that intermolecular energy changes involving chemical forces obey the quantum relationship. The above data indicate that the latent energy of evaporation is the work done in overcoming forces of a nature similar if not identical with the usual chemical manifestations. We may therefore regard the vaporisation of a substance from a liquid or solid surface as a monomolecular chemical reaction.

The number of gm. molecules per square cm. of surface is

$$\frac{1}{N\pi d^2},$$

where N is the number of molecules per gm. mol. and d the molecular diameter.

The rate of evaporation in gm. mols. per square cm. per sec. is accordingly given by the expression

$$\frac{v}{N\pi d^2} e^{-\frac{Nhv}{RT}}.$$

This can be equated to the number of gm. mols. striking a square cm. per second, or

$$\frac{P}{\sqrt{2\pi RMT}} = \frac{v}{N\pi d^2} e^{-\frac{Nhv}{RT}}.$$

On taking logs of each side we obtain

$$\log_e P = -\frac{Nhv}{RT} + \tfrac{1}{2}\log T + \log_e \frac{v\sqrt{2\pi MR}}{Nd^2\pi}.$$

* *Phys. Rev.* 2, 331 (1913).
† Rideal, *Phil. Mag.* XI, 461 (1920).
‡ *Jour. Amer. Chem. Soc.* 42, 2190 (1920).

We have also seen that the expression $Nh\nu$ can be put equal to the latent heat of evaporation (L).

If p be measured in atmospheres (10^6 bars.) and L in calories the expression becomes

$$\log_e P = \frac{-L}{RT} + \tfrac{1}{2}\log T + \log_e \frac{v\sqrt{M}}{N \,.\, \pi d^2 \,.\, 43{\cdot}75}.$$

The value of the chemical constant C_0 is thus equal to

$$\frac{\log_e R + \log_e \dfrac{v\sqrt{M}}{N \,.\, \pi d^2 \,.\, 43{\cdot}75}}{2{\cdot}3023},$$

or taking the generally accepted value of $N = 6{\cdot}062\ 10^{23}$ we obtain

$$C_0 = 0{\cdot}2980 + \log_{10} \frac{v\sqrt{M}}{d^2} \,.\, 1{\cdot}200\ 10^{-26}.$$

This relationship may be tested by calculation of the chemical constant in those cases where both L and d are known, and comparing these values with those determined in the usual way from the vapour pressure curve.

In the case of CO_2

$$M = 44, \quad L_{25^\circ \text{C.}} = 3100^*, \quad \therefore \quad L = 2600, \quad v = 2{\cdot}61 \,.\, 10^{13},$$

and $$d = 4{\cdot}56\ 10^{-8}\ \text{cm.}\dagger$$

$$\therefore \quad \log_{10} \frac{v\sqrt{M}}{d^2} \,.\, 1{\cdot}200\ 10^{-26} = \log_{10} 9{\cdot}972\ 10^2,$$

or $$C_0 = 3{\cdot}29.$$

The actual value of C_0 from vapour pressure data is

$$C_0 = 3{\cdot}20.$$

For hydrogen Eucken finds $L = 229$ cal. per gm. mol.

$$d = 2{\cdot}68\ 10^{-8}\ \text{cm.}$$

Hence $$v = 2{\cdot}446\ 10^{12},$$

and $$C_0 = 2{\cdot}10 \quad \text{(observed value } 2{\cdot}20\text{)}.$$

For water, inserting the values

$$d = 4{\cdot}54\ 10^{-8}\ \text{cm.}, \quad L = 10{,}200\ \text{cal.},$$

we find $$v = 1{\cdot}09\ 10^{14},$$

we obtain $$C_0 = 3{\cdot}73 \quad \text{(observed value } 3{\cdot}70\text{)}.$$

The influence of molecular size is well exhibited in the case of benzene. Surface tension data indicate that the molecular diameter

* Landolt-Bornstein Tabellen.

† Jeans, *The dynamical theory of gases.*

across the benzene ring is 10·6 10^{-8} cm. (the distance between two carbon atoms being 6·2 10^{-8} cm., hence the diameter is

$$\frac{\sqrt{3}}{2} 2 \cdot 6 \cdot 2\ 10^{-8})$$

and that the benzene molecule lies flat upon the surface of a liquid.

Inserting the values

$$L_0 = 8000 \text{ cal.}, \quad M = 78,$$

$$C_0 = 3{\cdot}187 \quad \text{(observed value 3·20).}$$

A high value for the chemical constant is found in the case of iodine, where $C = 4{\cdot}0$. The molecular weight is

$$M = 254, \quad d = 3{\cdot}96\ 10^{-8} \text{ cm.*}$$

L per atom 2146 cal. or per mol. 4292 cal., hence

$$v = 4{\cdot}58\ 10^{13}, \quad \text{and} \quad C_0 = 4{\cdot}09.$$

Similar close agreements are to be found in the other cases where the values of L and of d are known.

The above derivation for the chemical constant leads to the dimensional expression $i = Ml^{-1} t^{-2}$, where M, l and t are the respective dimensions of mass, length and time.

If we insert temperature into the dimensional expression we obtain $i = Ml^{-1} t^{-2} \theta^{-1}$.

This can be compared with the expression derived by Nernst†,

$$i = \log \frac{(2\pi m)^{\frac{3}{2}} v^3}{k^{\frac{1}{2}}} = Ml^{-1} t^{-2} \theta,$$

and with that obtained by Sackur‡ and by Tetrode§,

$$i = \log \frac{(2\pi m)^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3} = Ml^{-1} t^{-2} \theta^{-\frac{5}{2}}.$$

It will be noted that there is a lack of agreement in these expressions as far as the temperature is concerned. Lindemann has developed from dimensional considerations the expression

$$i = \log \frac{m^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3} \cdot \theta^{\frac{5}{2} - (b' - a')/R},$$

where $b' = \frac{5}{2}R$ approximately (the atomic heat of the gas at 0° K.) and a' is the atomic heat of the solid at 0° K., which is either zero or a very small quantity. With elevation of the temperature the

* Rankine, *Proc. Roy. Soc.* A, 83, 516 (1910); *Phil. Mag.* 29, 552 (1915).
† *Grundlagen des neuen Warmesatzes,* Halle, 1918, p. 138.
‡ *Ann. der Physik,* 40, 67 (1918).
§ *Ibid.,* 434, 39, 255 (1912).

value of $b' - a'$ gradually decreases and the power of θ increases from $-\frac{5}{2}$ at 0° K. to 0 at the critical point where gas and solid become identical.

It will be noted that the above expression derived from the latent heat equation and the effusion formula gives us a general method of calculation of the thermodynamic potential differences or the difference in the fugacities of a substance distributed in two phases and is not necessarily confined to the particular case where one phase consists of the material in the zero thermodynamic environment of a perfect gas.

On the function $[x]$. By Viggo Brun (Dröbak, Norway). (Communicated by Prof. G. H. Hardy.)

[*Read* 24 January 1921.]

§ 1. In the *Proceedings of the Cambridge Philosophical Society* (Vol. XIX, Part 5, 1919) Mr Shah and Mr Wilson have discussed some formulae proposed for calculating the number of Goldbachian decompositions of an even number. They have also mentioned a formula of mine, saying: "The formula to which Brun's argument leads is ...(11)" (page 243). In reality I have not enunciated this formula, as Mr Hardy and Mr Littlewood justly remark in their "Note on Messrs Shah and Wilson's paper."

But it may appear at first sight as if my method should naturally lead to the formula (11), and I should like to explain why one will not find it so on examining the matter further.

The formula in question is deduced from the sieve method of Eratosthenes, employed twice. We will here simplify the question, examining only the common sieve of Eratosthenes. Let us determine the number of primes under 14. We write the 14 numbers

$$1 \quad \underline{2} \quad \underline{3} \quad \underline{4} \quad 5 \quad \underline{\underline{6}} \quad 7 \quad \underline{8} \quad \underline{9} \quad \underline{10} \quad 11 \quad \underline{\underline{12}} \quad 13 \quad \underline{14}$$

effacing first the numbers 2, 4, 6, 8, 10, 12, 14 and then the numbers 3, 6, 9, 12. The uneffaced numbers are the number 1 and the primes between $\sqrt{14}$ and 14. We find that

$$\pi(14) - \pi(\sqrt{14}) + 1 = 14 - \left[\frac{14}{2}\right] - \left[\frac{14}{3}\right] + \left[\frac{14}{2\cdot 3}\right]$$
$$= 14 - 7 - 4 + 2 = 5,$$

where $\pi(x)$ denotes the number of primes not exceeding x, and $[x]$ denotes the number of integers not exceeding x. This formula can easily be generalised; it is not difficult to see how*. We could now say that it was "natural" to put $[x] = x$, and we should then get the approximate formula

$$\pi(14) - \pi(\sqrt{14}) + 1 = 14 - \frac{14}{2} - \frac{14}{3} + \frac{14}{2\cdot 3} = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})\, 14;$$

and generalising, we should get

$$\pi(x) - \pi(\sqrt{x}) + 1 = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) \ldots \left(1 - \frac{1}{p_r}\right) x,$$

when p_r denotes the greatest prime under $\sqrt{x}$. But this formula

* See Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, I. p. 67.

is erroneous, the correct asymptotic formula being according to Mertens

$$\pi(x) - \pi(\sqrt{x}) + 1 = k \,.\, (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) \ldots \left(1 - \frac{1}{p_r}\right) x,$$

where
$$k = \tfrac{1}{2} e^C = 0 \,.\, 89\ldots,$$

C being the Eulerian constant. For this, and also for other reasons, I have multiplied the formula in question by an undetermined constant. I also attempted to approximate to this constant empirically; but the value 1·598 thus obtained was seriously in error. The right value 1·320 has been determined independently by Stäckel (*Sitz. der Heidelberger Akad.*, Abth. A, 1916) and by Hardy and Littlewood (*l.c. supra*).

§ 2. We have seen an example of the error of putting $[x] = x$. Let us study the function $[x]$ more closely. If we draw the curve $y = [x]$, we see that it forms a discontinuous line like the steps of a staircase. The same can be said of the curve $y = \pi(x)$, but here the steps are not regular.

Let us try to express our discontinuous function by another discontinuous, but simpler, function. We write

$$\Phi(x) = \begin{cases} 1 & (x \geqq 1), \\ 0 & (0 < x < 1). \end{cases}$$

We see that

$$[x] = \Phi(x) + \Phi\left(\frac{x}{2}\right) + \Phi\left(\frac{x}{3}\right) + \Phi\left(\frac{x}{4}\right) + \ldots;$$

e.g. $[3\,.\,5] = \Phi(3\,.\,5) + \Phi\left(\frac{3\,.\,5}{2}\right) + \Phi\left(\frac{3\,.\,5}{3}\right) = 1 + 1 + 1 = 3.$

This formula has been employed by Lipschitz in the *Comptes Rendus* of 1879. But our function will also express other discontinuous functions, such as $\pi(x)$: thus

$$\pi(x) = \Phi\left(\frac{x}{2}\right) + \Phi\left(\frac{x}{3}\right) + \Phi\left(\frac{x}{5}\right) + \Phi\left(\frac{x}{7}\right) + \Phi\left(\frac{x}{11}\right) + \ldots:$$

e.g. $\pi(6) = \Phi\left(\frac{6}{2}\right) + \Phi\left(\frac{6}{3}\right) + \Phi\left(\frac{6}{5}\right) = 1 + 1 + 1 = 3.$

§ 3. Let us try to approximate to the function Φ by a continuous function. We will choose the function

$$1 - e^{-x^s} = \frac{x^s}{1} - \frac{x^{2s}}{2!} + \frac{x^{3s}}{3!} - \ldots,$$

where s is a positive integer not less than 2. It is not difficult to see that $1 - e^{-x^s}$ is very near 0 when $x < 1$ and very near 1 when

$x > 1$. At the point of discontinuity $x = 1$, $1 - e^{-x^s} = 1 - e^{-1}$. We are now able to approximate to the function $[x]$: thus

$$[x] = \frac{x^s}{1}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots\right) - \frac{x^{2s}}{2!}\left(1 + \frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \frac{1}{4^{2s}} + \dots\right) + \dots + R,$$

or
$$[x] = \frac{\zeta(s)\, x^s}{1} - \frac{\zeta(2s)\, x^{2s}}{2!} + \frac{\zeta(3s)\, x^{3s}}{3!} - \dots + R.$$

It is not difficult to find an expression for R, valid for all values of x which are not nearer to the integers (the points of discontinuity of $[x]$) than ϵ, where $\epsilon < \frac{1}{2}$. We can give R the following form

$$R = 8\theta 2^{-s\epsilon/x + 1},$$

where $-1 < \theta < 1$ and where $s > x$, $s > 2$.

We can now give our formula the following two forms

$$[x] = \frac{\zeta(s)\, x^s}{1} - \frac{\zeta(2s)\, x^{2s}}{2!} + \frac{\zeta(3s)\, x^{3s}}{3!} - \dots + 8\theta 2^{-s\epsilon/x+1} \quad \dots(1),$$

valid for all positive values of x not in the intervals $(1 - \epsilon, 1 + \epsilon)$, $(2 - \epsilon, 2 + \epsilon)$, $(3 - \epsilon, 3 + \epsilon)$, etc.; or

$$[x] = \lim_{s \to \infty}\left(\frac{\zeta(s)\, x^s}{1} - \frac{\zeta(2s)\, x^{2s}}{2!} + \frac{\zeta(3s)\, x^{3s}}{3!} - \dots\right)$$
$$= \lim_{s \to \infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(ns)\, x^{ns}}{n!} \quad \dots(2),$$

valid for all non-integral x.

If we wish to deduce results from the method of Eratosthenes, it is advantageous to make use of these formulae. Later on I shall perhaps show some of these applications.

§ 4. We are also able to find an approximate formula for $\pi(x)$, viz.

$$\pi(x) = \frac{\eta(s)\, x^s}{1} - \frac{\eta(2s)\, x^{2s}}{2!} + \frac{\eta(3s)\, x^{3s}}{3!} - \dots + R',$$

where
$$\eta(s) = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \dots.$$

This sum can be expressed in terms of $\zeta(s)$ by employing the identity of Euler,

$$\frac{1}{\zeta(s)} = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\dots,$$

from which results

$$\log \zeta(s) = \eta(s) + \tfrac{1}{2}\eta(2s) + \tfrac{1}{3}\eta(3s) + \tfrac{1}{4}\eta(4s) + \dots.$$

We can express $\eta(s)$ in terms of $\log\zeta(s)$ by using the factors $\mu(n)$ of Möbius, but we soon discover that it is more practical to study the function

$$f(x) = \pi(x) + \tfrac{1}{2}\pi(\sqrt{x}) + \tfrac{1}{3}\pi(\sqrt[3]{x}) + \tfrac{1}{4}\pi(\sqrt[4]{x}) + \dots$$

than the function $\pi(x)$ itself; and it is not difficult to deduce the formula

$$\pi(x) + \tfrac{1}{2}\pi(\sqrt{x}) + \dots$$
$$= \frac{\log\zeta(s)\,x^s}{1} - \frac{\log\zeta(2s)\,x^{2s}}{2!} + \frac{\log\zeta(3s)\,x^{3s}}{3!} - \dots + R'' \quad \dots(3).$$

The function $f(x)$ has points of discontinuity only for integral x, and R'' can be given the same form as R, viz.

$$R'' = 8\theta 2^{-s\epsilon/x+1}.$$

It follows that

$$\pi(x) + \tfrac{1}{2}\pi(\sqrt{x}) + \dots = \lim_{s\to\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\log\zeta(ns)\,x^{ns}}{n!} \quad \dots(4)$$

for all non-integral x.

This formula has been deduced by Helge von Koch, in Vol. XXIV of the *Acta Mathematica*, but is not mentioned in the *Handbuch* of Landau nor in the *Encyclopédie des sciences mathématiques* (Pt I, Vol. III, Fasc. 4).

It is very interesting to compare the formulae (1) and (3). The only difference between them is that (1) has $\zeta(s)$ where (3) has $\log\zeta(s)$, and in spite of this the formula (1) gives an approximation for the regular function $[x]$ while the formula (3) gives an approximation for the very irregular function $f(x)$ of Riemann.

§ 5. If we use for our function $\Phi(x)$ the formula of Kronecker (see *Encycl. des sciences math.*, Pt I, Vol. III, Fasc. 3, p. 256)

$$\Phi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s}\,ds,$$

we obtain the famous formula of Riemann

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s \log\zeta(s)}{s}\,ds,$$

which is equivalent to that of Helge von Koch (4). We equally obtain the following formula, equivalent to (2),

$$[x] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s\,\zeta(s)}{s}\,ds \quad \dots\dots\dots\dots\dots\dots(5),$$

valid for all positive non-integral x.

§ 6. It is interesting to compare these two methods. It follows on comparing the two expressions employed for $\Phi(x)$ that

$$\lim_{s\to\infty}(1-e^{-x^s})=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s}\,ds \qquad (x>0,\ x\neq 1),$$

which gives a curious analogy between an integral with complex limits and a real function. The problem arises whether there is any more striking analogy between this sort of integrals and real functions, *e.g.* real integrals. And it is in reality not difficult to find a formula analogous to the well-known formula

$$\frac{1}{2\pi i}\int_C\frac{dz}{z-a}=1 \text{ (or } 0),$$

according as a is or is not contained in C.

We find that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{2}\int_A^B\frac{dx}{|x-a|^{1-\epsilon}}=1,$$

when a is contained in the interval $A-B$, while the limit is 0 if a is not contained in the interval. Here $|x-a|$ denotes the numerical value of $x-a$, and ϵ is supposed positive.

We are naturally also able to write down a formula analogous to the famous formula of Cauchy

$$f(a)=\frac{1}{2\pi i}\int_C\frac{f(z)\,dz}{z-a},$$

valid if a is contained in C. The analogue is

$$f(a)=\lim_{\epsilon\to 0}\frac{\epsilon}{2}\int_A^B\frac{f(x)\,dx}{|x-a|^{1-\epsilon}},$$

which is valid if a is contained in the interval $A-B$.

It is in the nature of the matter that the analogy can only be very limited.

A theorem concerning summable series. By Prof. G. H. HARDY.

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1. It is well known that *if the series Σa_n is summable* $(C, 1)$, that is to say if

$$s_n' \sim An \quad \text{.........................(1)},$$

where $\quad s_n = a_0 + a_1 + \ldots + a_n, \quad s_n' = s_0 + s_1 + \ldots + s_n,$

then

$$\Sigma \frac{a_n}{n+1} = \frac{a_0}{1} + \frac{a_1}{2} + \ldots \quad \text{..................(2)}$$

*is convergent**. The converse is not true, as may be seen at once from trivial instances to the contrary. It is therefore interesting to frame a theorem of this kind which embodies a *necessary and sufficient* condition for summability. Such a theorem is the following.

Theorem. *The necessary and sufficient condition that Σa_n should be summable $(C, 1)$ to sum A is that*

$$s_n + (n+1)\, b_{n+1} \to A \quad \text{....................(3)},$$

where

$$b_n = \frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \ldots \quad \text{.................(4)}.$$

2. It is plain that we may (replacing a_0 by $a_0 - A$) suppose without loss of generality that $A = 0$. If

$$s'_n = o(n), \quad s_n = s'_n - s'_{n-1} = o(n).$$

Again, (3) and (4) involve the convergence of (2), *i.e.* involve $t_n \to B$, where $a_n = (n+1)\, c_n$ and $t_n = c_0 + c_1 + \ldots + c_n$. And

$$s_n = \sum_0^n (\nu+1)\, c_\nu = (n+1)\, t_n - \sum_0^{n-1} t_\nu = o(n).$$

Hence we may suppose $s_n = o(n)$ in proving either part of the theorem.

This being so, we have

$$b_{n+1} = \sum_{n+1}^{\infty} \frac{s_\nu - s_{\nu-1}}{\nu+1} = \lim_{m\to\infty} \sum_{n+1}^{m} \frac{s_\nu - s_{\nu-1}}{\nu+1}$$

$$= \lim_{m\to\infty} \left(\frac{s_m}{m+1} - \frac{s_n}{n+2} + \sum_{n+1}^{m-1} \frac{s_\nu}{(\nu+1)(\nu+2)} \right)$$

$$= -\frac{s_n}{n+2} + \sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)},$$

$$s_n + (n+1)\, b_{n+1} = \frac{s_n}{n+2} + (n+1) \sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)}.$$

* H. Bohr, 'Bidrag til de Dirichlet'ske Rackkers Theori,' Inaugural Dissertation (Copenhagen, 1910), p. 100. The theorem follows at once from (1) and the identity

$$\sum_0^n \frac{a_\nu}{\nu+1} = 2 \sum_0^{n-2} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} + \frac{s'_{n-1}}{n(n+1)} + \frac{s'_n - s'_{n-1}}{n+1}.$$

The condition (3) is therefore equivalent to

$$\sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)} = o\left(\frac{1}{n}\right) \quad\text{.................(5).}$$

3. Suppose first that Σa_n is summable, to sum zero, *i.e.* that $s_n' = o(n)$. Then

$$\sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)} = \lim_{m\to\infty} \sum_{n+1}^{m} \frac{s'_\nu - s'_{\nu-1}}{(\nu+1)(\nu+2)}$$

$$= \lim_{m\to\infty} \left(\frac{s'_m}{(m+1)(m+2)} - \frac{s'_n}{(n+2)(n+3)} + 2\sum_{n+1}^{m-1} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)}\right)$$

$$= 2\sum_{n+1}^{\infty} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s'_n}{(n+2)(n+3)}$$

$$= 2\sum_{n+1}^{\infty} o\left(\frac{1}{\nu^2}\right) - o\left(\frac{1}{\nu}\right) = o\left(\frac{1}{\nu}\right) \quad\text{...............................(6).}$$

Hence (5), and therefore (3), is a *necessary* condition for summability.

4. Suppose now that (5) is satisfied. Then, by (6),

$$2\sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s_n'}{(n+2)(n+3)} = o\left(\frac{1}{n}\right) \quad\text{...(7).}$$

Writing

$$\phi_n = (n+2)(n+3)\sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)},$$

we obtain

$$2\phi_n - s_n' = o(n) \quad\text{......................(8).}$$

But

$$\phi_n - \phi_{n-1} = 2(n+2)\,\phi_n \sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s_n'}{n+3}$$

$$= \frac{2\phi_n - s_n'}{n+3} = o(1),$$

by (8); and therefore $\phi_n = o(n)$ and $s_n' = o(n)$, so that the series is summable to sum 0. Thus the theorem is proved.

5. In order to show that the theorem is not without application, I apply it to the deduction of two known convergence criteria*.

* See (for A) L. Fejér, 'La convergence sur son cercle de convergence d'une série de puissances effectuant une représentation conforme du cerc le sur le plan simple,' *Comptes Rendus*, 6 Jan. 1913, and 'Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene,' *H. A. Schwarz Festschrift*, 1914, pp. 42–53; G. H. Hardy and J. E. Littlewood, 'Some theorems concerning Dirichlet's series,' *Messenger of Mathematics*, vol. 43, 1914, pp. 134–147: and (for B) G. H. Hardy 'Theorems relating to the summability and convergence of slowly oscillating series,' *Proc. London Math. Soc.*, ser. 2, vol. 8, 1910, pp. 301–320; E. Landau 'Über die Bedeutung einiger neuerer Grenzwertsätze von Herrn Hardy und Axer,' *Prace Matematyczno-fizyczne*, vol. 21, 1910, pp. 97–177; M. Cipolla, 'Sul criterio di convergenza di Hardy,' *Rend. dell' Acc. di Napoli*, ser. 3, vol. 26, 1920, pp. 96–107, 151–160.

(A) *If Σa_n is summable $(C, 1)$, and $\Sigma n^p \mid a_n \mid^{p+1}$ is convergent for some positive p, then Σa_n is convergent.*

(B) *If Σa_n is summable $(C, 1)$, and either (α) a_n is real and*

$$a_n > -\frac{K}{n},$$

or (β)

$$a_n = O\left(\frac{1}{n}\right),$$

then Σa_n is convergent.

6. To prove (A) we observe that

$$b_{n+1} = \sum_{n+1}^{\infty} \frac{a_\nu}{\nu+1} = \sum_{n+1}^{\infty} \left((\nu+1)^{-\frac{2p+1}{p+1}} . (\nu+1)^{\frac{p}{p+1}} a_\nu\right),$$

and so*

$$|b_n| \leqq \left(\sum_{n+1}^{\infty} (\nu+1)^{-\frac{2p+1}{p}}\right)^{\frac{p}{p+1}} \left(\sum_{n+1}^{\infty} (\nu+1)^p |a_n|^{p+1}\right)^{-\frac{1}{p+1}}$$

$$= O\left(\left(n^{-\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\right) o(1) = o\left(\frac{1}{n}\right).$$

Hence $(n+1)\, b_{n+1} \to 0$, and so, by (3), $s_n \to A$.

7. To prove (B) we observe first that, if it is condition (β) that is given, we may suppose without loss of generality that a_n is *real*; for we may treat the real and the imaginary parts of the series separately. But then (β) becomes a special case of (α). It is therefore only necessary to consider condition (α). Further, we may plainly suppose that $A = 0$.

Suppose that $$\overline{\lim}\, s_n = \lambda > 0,$$

and choose a sequence of values of n for which

$$s_n > \tfrac{1}{2}\lambda.$$

Let us denote a value of n, belonging to this sequence, by m; and choose H so that $0 < KH < \frac{1}{4}\lambda$. Then

$$s_\nu = s_m + a_{m+1} + a_{m+2} + \ldots + a_\nu$$

$$> \tfrac{1}{2}\lambda - K\left(\frac{1}{m+1} + \ldots + \frac{1}{\nu}\right) > \tfrac{1}{2}\lambda - K\frac{\nu - m}{m+1}$$

$$> \tfrac{1}{2}\lambda - KH > \tfrac{1}{4}\lambda,$$

* By the well known inequality

$$\Sigma ab \leqq (\Sigma a^k)^{\frac{1}{k}} (\Sigma b^l)^{\frac{1}{l}} \quad \left(a \geqq 0,\ b \geqq 0,\ k > 1,\ l > 1,\ \frac{1}{k} + \frac{1}{l} = 1\right).$$

if $m < \nu \leqq m + Hm$. Hence

$$\sum_{m+1}^{m+Hm} \frac{s_\nu}{(\nu+1)(\nu+2)} > \tfrac{1}{4}\lambda \sum_{m+1}^{m+Hm} \frac{1}{(\nu+1)(\nu+2)} \sim \frac{\lambda H}{4m}$$

when $m \to \infty$. But this plainly contradicts (5). Hence

$$\varlimsup s_n \leqq 0.$$

It may be shown in just the same way* that

$$\varliminf s_n \geqq 0.$$

Hence $s_n \to 0$, and the theorem is proved.

* Using a sum $\displaystyle\sum_{m-Hm}^{m} \frac{s_\nu}{(\nu+1)(\nu+2)}$.

Standing Waves parallel to a Plane Beach. By H. C. POCKLINGTON, M.A., St John's College.

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1. The object of this paper is to investigate the standing waves parallel to the shore line in the case of an infinite fluid bounded below by a plane sloping at an angle α to the horizontal. We restrict ourselves to the case where α is a sub-multiple of a right angle and use the method of images. The conditions to be satisfied are stated in section 2 and the images formed by the boundaries (beach and free surface) are found in section 3. In section 4 we write down the velocity potential and show that it satisfies all the conditions that it should. In section 5 we find that the amplitude of oscillation at the shore is increased (in consequence of the compression of the waves as they get into shallow water) in the ratio $\sqrt{(\pi/2\alpha)}:1$.

2. Let the liquid be bounded below by a plane beach sloping at an angle $\alpha = \pi/2n$. We take the shore line as axis of z and use cylindrical coordinates, so that $\theta = 0$ is the equation of the free surface of the liquid, and $\theta = \alpha$ is that of the beach. Let the period of the standing waves be $2\pi/p$. Then the velocity potential is of the form $\phi = A\Phi \cos(pt + \epsilon)$. The conditions to be satisfied by Φ are (i) that $d^2\Phi/dr^2 + d\Phi/rdr + d^2\Phi/r^2d\theta^2 = 0$, (ii) that $d\Phi/d\theta = 0$ when $\theta = \alpha$, (iii) that $d\Phi/rd\theta = -p^2\Phi/g$ when $\theta = 0$, (iv) that at infinity Φ must have the correct form for standing waves in deep water. Let $p^2/g = \lambda$.

3. If we have a value of Φ, say Φ_1, that satisfies (i) but not (ii) we can find Φ_2 so that Φ_2 satisfies (i) and $\Phi_1 + \Phi_2$ satisfies (ii) (it will of course satisfy (i)), and may call Φ_2 the image of Φ_1 with respect to lower boundary. Similarly, if Φ_1 satisfies (i) but not (iii) and we find Φ_2 to satisfy (i) and such that $\Phi_1 + \Phi_2$ satisfies (iii) (it will of course satisfy (i)) then Φ_2 may be called the image of Φ_1 with respect to the upper boundary.

If

$$\Phi_1 = \exp\{-\lambda r \sin(\theta - \beta)\} \cos\{\eta + \lambda r \cos(\theta - \beta)\} = f(-\beta, \eta, \theta)$$

say, it is clear that its image with respect to the lower boundary may be taken to be

$$\begin{aligned}\Phi_2 &= \exp\{-\lambda r \sin(2\alpha - \beta - \theta)\} \cos\{\eta + \lambda r \cos(2\alpha - \beta - \theta)\}\\ &= f(2\alpha - \beta, \eta, -\theta) = f(\beta', \eta', -\theta) \quad \text{say.}\end{aligned}$$

The image of Φ_2 with respect to the upper boundary may be taken to be

$$\Phi_3 = \cot \beta'/2 \,.\, \exp\{-\lambda r \sin(\beta' + \theta)\} \cos\{\eta' - \pi/2 + \lambda r \sin(\beta' + \theta)\}$$
$$= \cot \beta'/2 \,.\, f(\beta', \eta' - \pi/2, \theta)$$

(and it will be found on trial that this is the only value of Φ_3 that has the same general form as that given and satisfies the conditions stated above).

4. Let

$$\Phi_1 = \exp\{-\lambda r \sin\theta\} \cos\{(n-1)\pi/4 + \lambda r \cos\theta\}$$
$$= f\{0, (n-1)\pi/4, \theta\}$$

and add to it the first $n-1$ images taken alternately with respect to the lower and upper boundaries, so that

$$\Phi = f\{0, (n-1)\pi/4, \theta\} + f\{2\alpha, (n-1)\pi/4, -\theta\}$$
$$+ \cot\alpha \,.\, f\{2\alpha, (n-3)\pi/4, \theta\} + \cot\alpha \,.\, f\{4\alpha, (n-3)\pi/4, -\theta\}$$
$$+ \cot\alpha \cot 2\alpha \,.\, f\{4\alpha, (n-5)\pi/4, \theta\} + \text{etc.},$$

the last term being

$$\cot\alpha \cot 2\alpha \ldots \cot(n-1)\alpha/2 \,.\, f\{(n-1)\alpha, 0, \theta\}$$

if n is odd and

$$\cot\alpha \cot 2\alpha \ldots \cot(n-2)\alpha/2 \,.\, f\{n\alpha, \pi/4, -\theta\}$$

if n is even.

Each term satisfies (i). Pairing the terms starting from the beginning we see that each pair satisfies (ii) and that if n is odd the odd term at the end also does so. Pairing the second term with the third, the fourth with the fifth and so on we see that each pair satisfies (iii) and that the odd term at the beginning does so, as does the odd term at the end when n is even. Also if $0 \not> \theta \not> \alpha$ the sine under the exponential sign is positive for each term except the first so that these terms vanish exponentially at the infinite part of the fluid. The first term has the correct form for standing waves. Hence Φ satisfies all the conditions and $A\Phi \cos(pt + \epsilon)$ is the velocity potential required.

5. The amplitudes at various points of the upper boundary are proportional to the values of Φ there taken positively. The value of Φ at $\theta = 0$, $r = \infty$ varies from $+1$ to -1. Hence ρ the amplitude at the origin divided by the maximum amplitude at infinity is equal to the value of Φ at the origin.

Firstly, let n be odd. Pairing the terms starting from the beginning we have

$$\rho = 2\cos(n-1)\pi/4 + 2\cos(n-3)\pi/4 \,.\, \cot\alpha$$
$$+ 2\cos(n-5)\pi/4 \,.\, \cot\alpha \cot 2\alpha + \text{etc.}$$

to $(n+1)/2$ terms, the coefficient of the last term being 1 instead of 2.

That is,

$$\rho = \exp (n-1)\, \pi i/4 + \exp (n-3)\, \pi i/4 \,.\, \cot \alpha + \text{etc.}$$

to n terms, the first term of the previous series being the sum of the first and last of this, and so on, for

$$\cot (n-r)\, \alpha = \tan r\alpha.$$

Secondly, let n be even. Pairing the terms

$$\begin{aligned} \rho &= 2 \cos (n-1)\, \pi/4 + 2 \cos (n-3)\, \pi/4 \,.\, \cot \alpha + \text{etc. to } n/2 \text{ terms} \\ &= \exp (n-1)\, \pi i/4 + \exp (n-3)\, \pi i/4 \,.\, \cot \alpha + \text{etc. to } n \text{ terms,} \end{aligned}$$

which is of the same form as in the case of n odd.

If
$$\omega = \cos \pi/n + i \sin \pi/n,$$
we have
$$\cot r\alpha = -\, i\omega^r (\omega^{n-r} - 1)/(\omega^r - 1),$$
so that

$$\begin{aligned} \rho = \exp (n-1)\, \pi i/4 &- \exp (n-3)\, \pi i/4 \,.\, i\omega (\omega^{n-1} - 1)/(\omega - 1) \\ &+ \exp (n-5)\, \pi i/4 \,.\, i^2\omega^3 (\omega^{n-1} - 1)(\omega^{n-2} - 1)/(\omega - 1)(\omega^2 - 1) \\ &- \text{etc. to } n \text{ terms,} \end{aligned}$$

the indices of ω in $i^q\omega^s$ being the triangular numbers in order. Also

$$i = \exp \pi i/2.$$

Hence*

$$\rho = (1-\omega)(1-\omega^2)(1-\omega^3) \ldots (1-\omega^{n-1}) \exp (n-1)\, \pi i/4.$$

Being real this is equal to its conjugate

$$(1+\omega^{n-1})(1+\omega^{n-2}) \ldots (1+\omega) \exp (1-n)\, \pi i/4.$$

Multiplying, we find

$$\rho^2 = (1-\omega^2)(1-\omega^4) \ldots (1-\omega^{2n-2}).$$

But ω^2, ω^4, etc. are the roots of $(x^n - 1)/(x - 1) = 0$, whence $\rho^2 = n$.

Hence the amplitude at the origin is $\sqrt{n}$ times the maximum amplitude at infinity (but this has been proved only for the case of n integral).

* *Vide* Todhunter, *Theory of Equations* (1885), p. 217 (ch. XXIV, § 292).

The Origin of the Disturbances in the Initial Motion of a Shell.
By R. H. FOWLER and C. N. H. LOCK.

[*Read* 28 February 1921.]

(1) The following paper* is an attempt to throw some light on one of the most obscure outstanding problems in gunnery, namely, the precise cause of the initial angular oscillations of the axis of a (stable) spinning shell. The factors that produce the initial oscillations have hitherto only been guessed at, and design, which aims at reducing these disturbances to a minimum, has been guided solely by empirical results. It is therefore a matter of some importance to analyse carefully the experimental evidence which has recently been acquired, for the purpose of discriminating between possible causes, and suggesting the proper lines of future research. On the scientific rather than the technical side, we point out the desirability for a solution, if possible, of the elastic vibrations of the gun under its firing stresses.

In experiments carried out by ourselves and others in January, 1919†, we succeeded in recording with reasonable accuracy the initial oscillations of the axis of a series of shells, of four different types, fired at a series of muzzle velocities from a pair of 3-inch guns of two different twists of rifling. Specimens of these observations will be found in the paper quoted. By extrapolation of the observed curves backwards to the neighbourhood of the muzzle of the gun, it is possible to deduce with some confidence rough values for the magnitude and direction of the initial angular velocity of the axis of the shell. The extrapolation is not a serious one, for the law of motion of the axis of the shell is well understood. By analysis of these values and their variation with the varying circumstances of projection, it is possible to throw light on the origin of the disturbances themselves.

The most obvious origin for these disturbances would appear to lie in random gas pressure variations during the last part of the travel of the shell down the rifled bore, and in random asymmetry of the blast which flows past the shell for the first few feet of its motion outside the barrel. The only reasonable alternative suggestion is that the initial angular velocity is primarily due to some form of barrel vibration. It is important to discriminate between

* This paper is published by permission of the Ordnance Committee, for whom the experimental work was carried out. The authors also thank the Admiralty Director of Scientific Research and Experiment, who recently propounded to one of them a technical problem on initial motions. This problem suggested the possible importance of these results.

† "The Aerodynamics of a Spinning Shell," *Trans. Roy. Soc.* A, vol. CCXXI, 1920, p. 295.

these causes, and it is the object of this paper to show by analysis of the available experimental data that it is barrel vibrations that are the dominant cause, at least in these particular experiments. Whether the barrel vibrations are the cause in general it is for future experiments to decide.

(2) *Consequences of the random pressure variation theory.* If random pressure variations and random blast disturbances are the dominant cause of the initial angular velocity of the axis of the shell, certain well-marked characteristics of the initial motion of the axis can be deduced. For shortness it is convenient to define the position of the axis OA relative to the direction of motion of the centre of gravity OP by the angles δ, ϕ of Fig. 1.

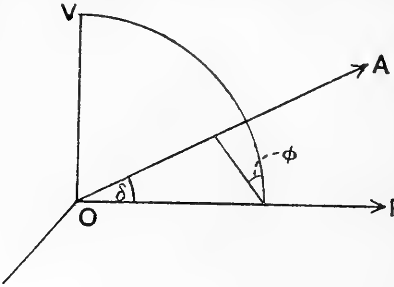


Fig. 1.

The plane POV is vertical, and the angle δ is usually called the yaw. Then the initial circumstances of the axis of the shell are defined with sufficient precision by the initial values of ϕ and $d\delta/dt$ or δ', that is to say, by ϕ_0 and δ_0'. The initial value of δ itself, obtained by extrapolation, is in practice very little different from zero, as is to be expected, and may be ignored in what follows.

Now random variations of gas pressure *across* the base of the shell and random asymmetry of the blast will result in a disturbing couple (practically impulsive) acting on the shell, whose plane of action must be expected to vary from round to round in an entirely arbitrary manner. That is to say, this theory demands that all values of ϕ_0 should be equally probable. Again, the impulsive disturbing couple which acts on the shell can hardly depend in any way on the axial spin of the shell or the twist of the rifling. The mean value of the disturbing impulsive couple, and therefore of δ_0', should thus be the same for similar shells fired at similar velocities from similar guns with different twists of rifling. These are two deductions from the theory which can be tested by the experimental results. We may say at once that neither deduction is sufficiently fulfilled. We may be certain that, though no doubt such random pressure effects occur and are appreciable, the main cause of the disturbance lies elsewhere.

(3) *Consequences of the barrel vibration theory.* On the other hand, if the main cause of the disturbances lies in the vibration of the barrel, which constrains the base of the shell to follow its movements so long as the driving band is engaged in the rifling, we can draw at once very different deductions. For a gun is not a figure of revolution, and its vibrations may well be expected to take place in or nearly in the same plane from round to round, and to be in or nearly in the same phase each time when the driving band disengages*. This effect might be expected to be all the more marked and give rise to larger disturbances when the gun has a considerable curvature (technically *droop*). It appears in such cases that the gun temporarily straightens more or less while the shell is travelling down the bore.

It follows then as a consequence of this theory that ϕ_0 should be roughly constant from round to round fired under similar conditions, or at least that the values of ϕ_0 should be highly correlated. Again, in this case the twist of the rifling may play a fundamental part in the phenomenon. For there will at least be stresses in the barrel proportional to the twist of the rifling, and we cannot say that the theory demands a value of δ_0' independent of the twist of the rifling. The further discussion of this case is unfortunately somewhat speculative at present and requires a little mathematical analysis. It is postponed until we have described the experimental evidence.

(4) *The experimental evidence.* The evidence available is deduced almost entirely from the experiments previously referred to. The following table gives the mean values of δ_0' and δ_0'/Ω† which can be deduced from that experiment, for groups fired under similar conditions from guns of two different riflings. The other entries are the reference numbers of the observations‡, Ω itself, the muzzle velocity and the twist of the rifling. The values of δ_0' for the various rounds in any one group are reasonably consistent.

It can be seen at once, by inspection of the columns for δ_0' and δ_0'/Ω, that δ_0' is distinctly greater for the gun rifled 1 turn in 30 diameters—that is, with the sharper twist of rifling—than for the gun rifled 1 in 40. On the other hand δ_0'/Ω appears to be practically the same (to the accuracy of the experiment) for both twists of rifling in each group. *The disturbing impulse, other things being*

* The case of a rifle is well-known, in which the direction of departure of the bullet is very largely affected by barrel vibrations, which are themselves affected by the presence or absence of the bayonet.

† The notation is that of our previous paper *loc. cit.* $\Omega = AN/B$, where A and B are respectively the axial and transverse moments of inertia of the shell and N is the axial spin in radians per sec. The twist of the rifling is specified as 1 complete turn in n diameters of the bore. Ω and N are inversely proportional to n.

‡ *Loc. cit.*

equal, is roughly proportional to the spin of the shell. We can even go somewhat further on these figures, for we notice that the values of δ_0'/Ω show a well-marked constancy through the whole table.

Table I.

Muzzle velocity, f.s.	Nos. of rounds	Ω radians per sec.	Mean value of δ_0' radians per sec.	Mean value of δ_0'/Ω	Twist of rifling
2130	IV 1, 2, 3, 5	150	1·7	0·014	1/40
2120	IV 24–26	200	3·0	0·015	1/30
1553	IV 7–9	110	1·5	0·014	1/40
1547	IV 16–18	146	2·1	0·015	1/30
1565	I 5–7	125	2·6	0·021	1/40
1563	I 27, 28	167	3·1	0·019	1/30
1585	II 5–7	124	1·0	0·008	1/40
1589	II 22, 23	165	2·3	0·014	1/30
1583	III 5–7	145	2·0	0·014	1/40
1567	III 22, 23	192	3·2	0·017	1/30
1084	I 9, 10	87	1·1	0·013	1/40
1119	I 22–24	120	1·9	0·016	1/30

The next table, Table II, also contains values of δ_0'/Ω for each group for which they have been deduced from the observations. The mean values of δ_0'/Ω for all groups for each gun are the same, confirming the more precise comparison of Table I.

The main purpose of Table II is to exhibit the correlation (if it exists) between the values of ϕ_0 for the various rounds of a group. It contains the mean value of ϕ_0 for each group, and the "spread" of the values of ϕ_0 in the group, the spread being the smallest angle which will include all members of the group.

The entries of the table show unmistakable regularities. In the first place the spread is in general far too small for one to suppose that the observed values of ϕ_0 represent a random distribution over the whole 360°. The groups of two rounds each provide only slight evidence on this point. But even here there is some evidence of correlation, for it is obviously equally probable that the spread of a pair will or will not exceed 90° for random distributions of ϕ_0. Of the seven such groups, the spread of one only (III 22, 23) exceeds 90° and the average spread of the seven is only 50°.

There are twenty-one groups in the table containing three or more rounds. In a group of three rounds distributed at random, the probability of a spread of less than θ radians can easily be shown to be

$$\frac{3\theta^2}{4\pi^2}, \quad (\theta \leqslant \pi),$$

$$\frac{3\theta - 3\pi}{\pi} + \frac{3\theta(4\pi - 3\theta)}{4\pi^2}, \quad (\pi \leqslant \theta \leqslant \tfrac{4}{3}\pi).$$

Table II.

Mean muzzle velocity, f.s.	Nos. of rounds	Mean value of ϕ_0, degrees	Spread degrees	Mean of δ_0'/Ω
		Gun rifled 1 in 40		
2346	I 20, 21	179	20	0·022
2167	I 1–4	99	100	0·020
1565	I 5–7	30	61	0·021
1312	I 17, 18	− 55	51	—
1072	I 8–10	−105	104	—
922	I 11–14	− 83	30	0·011
2024	II 1–4	83	51	0·011
1584	II 5–7	− 3	100	0·008
1334	II 14–16	− 24	202	—
934	II 8–10	−128	46	0·014
2025	III 1–4	152	109	0·008
1583	III 5–7	157	147	0·014
1312	III 14–16	−107	40	—
1077	III 11–13	−135	53	—
931	III 8–10	−128	5	0·015
884	IV 10–12	−137	98	—
		Gun rifled 1 in 30		
1563	I 27, 28	20	19	0·019
1326	I 25, 26	45	29	0·015
1119	I 22–24	29	51	0·015
1589	II 22, 23	36	72	0·014
1119	II 17–19	6	49	0·018
1567	III 22, 23	55	110	0·017
1292	III 20, 21	37	49	0·019
1119	III 17–19	− 5	10	0·009
2121	IV 24–26	113	172	0·016
1547	IV 16–18	17	49	0·015
1078	IV 13–15	7	12	0·008
900	IV 21–23	81	72	0·010

The greatest possible value of the spread is of course $\frac{4}{3}\pi$ which occurs when the three values of ϕ_0 are of the form

$$\alpha, \quad \alpha + \tfrac{2}{3}\pi, \quad \alpha + \tfrac{4}{3}\pi.$$

It is thus equally probable that the spread of a group of three will or will not exceed 147° on a basis of random distribution. In the twenty-one groups the spread only twice exceeds and once equals 147°, and the average spread is only 70°. The probability with a random distribution that any *individual* spread should be less than or equal to 70° is only 0·113. The probability that the average of twenty-one spreads shall be as small as 70° will be excessively small.

We may therefore be fairly confident on the evidence of the spread that the main part of the angular velocity of the axis of the shell is due to some cause which operates in or nearly in the same plane for all rounds fired under the same conditions. This conclusion is supported by the results of two five-round groups of similar shells fired from another type of gun*. In each case the spread of the five values of ϕ_0 was less than 30°.

We are, however, justified in concluding that there are other regularities in Table II. It is arranged in order of muzzle velocities for the shells of each type, fired from the gun of each twist of rifling. Thus the table consists of eight sets of such arrangements separated by spaces. In each such set which is sufficiently complete there is a well-marked progressive change noticeable in the mean value of ϕ_0 as the velocity changes through the set.

Finally, compare together the mean values of ϕ_0 for all groups at the same muzzle velocity from the same gun, irrespective of the type of shell. We obtain the following sets of numbers. For the gun rifled 1 in 40,

(2000–2170)	99, 83, 152
(1560)	30, −3, 157
(1320)	−55, −24, −107
(1070)	−105, −135
(920)	−83, −128, −128, −137.

For the gun rifled 1 in 30,

(1570)	20, 36, 55, 17
(1320)	45, 37
(1110)	29, 6, −5, 7.

The numbers in brackets on the left specify the muzzle velocity to which the group belongs. These mean values of ϕ_0 are obviously, with one exception, heavily correlated. We may say in fact that the main part of the initial disturbance consists of an angular velocity of the axis of the shell located in a plane which varies little from round to round or even with changes in the centre of gravity and moments of inertia of the shell, but which changes

* Gun: 16-pdr, 9-cwt.

progressively with the muzzle velocity and also with the twist of rifling. To this we may add on the evidence of Table I that the size of the disturbance appears to be roughly proportional to Ω.

On referring to the characteristics of the disturbances due to the two possible causes described in sections 2 and 3, we see that the evidence is fairly conclusively against the theory of random gas pressure variations which demands random ϕ_0's and δ_0' independent of n. On the other hand there is nothing to prevent its fitting in with the theory of barrel vibrations. We must examine this theory now in greater detail, though, as we have already said, all we can say is speculative and intended only to suggest lines of further work on the problem.

(5) *The motion of a shell with the base constrained.* Let us consider the circumstances under which a shell emerges from the muzzle of a gun. It seems probable that even for fairly small values of the clearance* the shell will cease to touch the bore, except at the driving band, from the moment when its shoulder reaches the lip of the muzzle. This is a direct consequence of the actual numerical quantities concerned—it is unnecessary to give numerical details here. Thus from the moment at which the shoulder emerges the shell may be regarded as practically free to swing about the driving band†.

Let us suppose in the first instance that owing to the firing stresses the muzzle end of the gun vibrates, so that the shell is constrained to take up the sideways velocity of the end of the gun just before it leaves the muzzle. We will assume first that the axis of the bore is unaltered in direction by this vibration. To work out the motion of the shell under such conditions we refer it to fixed rectangular axes Ox, Oy, Oz, such that Ox is parallel to the axis of the bore, Oy vertical, and Oz horizontal and to the right as viewed from behind the gun. That point in the shell whose motion is constrained (the base point) may be taken to be the centre of a section through the middle of the driving band. Let $\mathbf{\Lambda}$ be a unit vector representing the direction of the axis of the shell (components l, m, n) and $\mathbf{V}$ a vector representing the total constrained velocity of the base point (components u, v, w). Then at the moment when the driving band disengages, u is the muzzle velocity and v, w are the components of the barrel vibration.

The resultant action on the shell may be taken to be (1) a force $\mathbf{F}$ acting through the base point, whose position is defined by the vector $-d\mathbf{\Lambda}$ relative to the centre of gravity of the shell. The axial

* The excess of the minimum internal diameter of the bore over the maximum external diameter of the shell (driving band ignored).

† If the size of the initial disturbance is found to vary decidedly with the clearance, this paragraph will need modification. But the succeeding arguments hold unaltered provided less time is taken to refer to the moment at which the shell is first free to swing about the band.

distance of the centre of gravity from the base point is taken to be d; (2) a constraining couple acting about Ox.

The velocity of the centre of gravity is therefore $\mathbf{V} + d\mathbf{\Lambda}'$, and

$$\mathbf{F} = m^* \frac{d}{dt}\{\mathbf{V} + d\mathbf{\Lambda}'\}^*.$$

The moment of this force about the centre of gravity is $d\,[\mathbf{F}\,.\,\mathbf{\Lambda}]$. The moment of momentum about the centre of gravity is† $AN\mathbf{\Lambda} + B\,[\mathbf{\Lambda}\,.\,\mathbf{\Lambda}']$. The equation of angular motion is therefore (*omitting the constraining couple* which we assume to act about the axis of the bore, Ox)

$$\frac{d}{dt}\{AN\mathbf{\Lambda} + B\,[\mathbf{\Lambda}\,.\,\mathbf{\Lambda}']\} = m^*d\left[\frac{d}{dt}\{\mathbf{V} + d\mathbf{\Lambda}'\}\,.\,\mathbf{\Lambda}\right]$$

$$= m^*d\,\frac{d}{dt}\,[\{\mathbf{V} + d\mathbf{\Lambda}'\}\,.\,\mathbf{\Lambda}] - m^*d\,[\mathbf{V}\,.\,\mathbf{\Lambda}'],$$

or

$$\frac{d}{dt}\{AN\mathbf{\Lambda} + B_1\,[\mathbf{\Lambda}\,.\,\mathbf{\Lambda}']\} = m^*d\,[\mathbf{V}'\,.\,\mathbf{\Lambda}], \qquad \text{......(1)}$$

where $B_1 = B + m^*d^2$. The y- and z-components of equation (1) are unaffected by the ignored constraining couple. When written out at length they are

$$\frac{d}{dt}\{ANm + B_1\,(nl' - ln')\} = m^*d\,(w'l - u'n), \qquad \text{......(2)}$$

$$\frac{d}{dt}\{ANn + B_1\,(lm' - ml')\} = m^*d\,(u'm - v'l). \qquad \text{......(3)}$$

Equations (2) and (3) are exact. If, now, we recall that the axis of the shell is only slightly inclined to the axis Ox, we may approximate by assuming that m and n are small, $l = 1$, and $l' = 0$. Writing $\zeta = m + in$ we can combine these equations into the single one

$$\frac{d}{dt}\{AN\zeta + iB_1\zeta'\} = -\,im^*d\,(v' + iw') + im^*du'\zeta. \qquad \text{...(4)}$$

The approximations made in obtaining (4) are certainly legitimate. To interpret (4) we must approximate further in a more speculative manner.

If barrel vibrations are of primary importance, a rough calculation, for the 3-inch shells used in these experiments, shows that $|\,v' + iw'\,|$ must not be less than about $\frac{1}{30}u'$; u' is the ordinary linear acceleration of the shell at the muzzle. Also $|\,\zeta\,|$ must be of the order of $|\,\zeta'\,|\,\tau$, where τ is the total time of emergence of the shell from the barrel (about 0·0005 secs.) so that $|\,\zeta\,| < 0{\cdot}0015$. It follows that the last term in equation 4 is in absolute value at

* The mass of the shell is m^*. Vector and scalar products are denoted by [.] and (.) respectively. The notation is that of our previous paper, pp. 326, 327.

† *Loc. cit.* p. 327.

most $\frac{1}{20}$ of the last but one, and may be omitted*; the equation is then integrable as it stands. If we take as zero time the moment at which the shell begins to be able to rotate freely about the driving band (the moment of emergence of the shoulder), we may take $t = 0$ for the lower limit of integration and (practically) $\zeta_0 = 0$, $\zeta_0{}' = 0$. Then we find

$$iB_1\zeta' + AN\zeta = -im^*d\,\{\delta\,(v + iw)\}, \qquad \ldots\ldots(5)$$

where $\delta\,(v + iw)$ denotes the change in $v + iw$ in the interval $(0, t)$. As the interval in which the shell disengages is small, the action is practically impulsive, so that the term in ζ' in equation 5 may be neglected to the present approximation. We thus obtain as a first rough value for ζ', at the moment τ when the band clears the muzzle,

$$\zeta'\,(\tau) = -\frac{m^*d}{B_1}\,\{\delta\,(v + iw)\}. \qquad \ldots\ldots(6)$$

This can of course be obtained more directly if we are content to ignore the nature of the approximations made at each stage. Since τ is of the order of 0·0005 sec. the value of $\zeta\,(\tau)$ is negligible by comparison. There are of course random gas pressure and blast effects to be superposed on the disturbance resulting from (6), but we are not concerned with these here. The alteration in $\zeta'\,(\tau)$ produced by including the term in ζ can easily be calculated and is found to be of the order of 5 %.

Thus we see that on this theory the main part of the disturbance will be the acquisition of an angular velocity of the axis $\zeta'\,(\tau)$ given by (6) whose nature depends essentially on

$$(v + iw)_\tau - (v + iw)_0.$$

This change of velocity of the free end of the muzzle in the interval $(0, \tau)$ may *a priori* be expected to be fairly constant in direction from round to round, thus satisfying the main requirement of the observations. Whether it could be really proportional to the twist of rifling it is more difficult to say; owing to the torsional strains, such proportionality is not *a priori* impossible. We would content ourselves with pointing out here that

$$(v + iw)_\tau - (v + iw)_0$$

is directly observable; the principal object of this discussion is to urge the importance of its proper determination. The foregoing discussion also suggests problems in the theory of elasticity. It would be of great interest if any sort of approximation to the elastic vibrations of a gun under firing stresses could be obtained theoretically.

* This omitted term represents the disturbing effect of the gas pressure acting through the centre of the base. It is only effective *after* a disturbance has already been set up.

Tides in the Bristol Channel. By G. I. Taylor, F.R.S.

[*Read* 24 January 1921.]

It is well known that tidal waves coming from an open ocean increase in amplitude as they reach the shallower water which usually surrounds the land. This increase is specially marked when the tidal wave enters a contracting channel. The cause of the increase is well understood, and in certain simple cases its amount has been discussed mathematically.

Unfortunately it is very seldom that the nature of the bottom or of the coast line of a channel permits us to apply these results with any hope of getting even an approximate representation of the actual state of affairs in any existing tidal basin. At any rate I do not know of any case in which it has been attempted.

One of the most striking examples of the effect of a contracting channel in increasing the height of the tidal wave is that of the Bristol Channel where there is one of the largest tides in the world. On looking at a chart of that region, I was struck by the way in which both the depth and the breadth of the channel at low water appear to contract almost uniformly from the entrance to the head. Under these circumstances it seemed worth while to work out theoretically the increase in tide which might be expected in a channel whose breadth and mean depth both decrease uniformly from the open end to the head, and both vanish there. The results might then be comparable with the observed tides in the Bristol Channel.

Theoretical Calculation.

The differential equation which represents the variation in amplitude of the tides in a channel as the mean depth and breadth vary, is given in Lamb's *Hydrodynamics**. It is

$$\frac{g}{b}\frac{d}{dx}\left(hb\frac{d\eta}{dx}\right)+\sigma^2\eta=0 \quad\quad (1),$$

where η represents the rise and fall of tide, b the breadth, and h the mean depth of a section of the channel taken at a distance x from the head. $2\pi/\sigma$ is the period of the tidal oscillation and g is the acceleration due to gravity.

Two cases have been solved by Lamb, namely (1) h constant, b proportional to x, and (2) h proportional to x, b constant. For the Bristol Channel neither of these is suitable, we require both b and h to be proportional to x.

* P. 267, 4th edition.

Let h_0 be the mean depth and b_0 the breadth of a section at distance x_0 from the head of the channel. Then

$$h = h_0 x/x_0, \quad b = b_0 x/x_0.$$

Inserting these values in (1) the equation becomes

$$\frac{d}{dx}\left(x^2 \frac{d\eta}{dx}\right) + k\eta x = 0 \quad \text{.....................(2)},$$

where

$$k = \sigma^2 x_0/h_0 g \quad \text{.......................(3)}.$$

Putting $z^2 = kx$, $\xi = \eta z$, this equation becomes

$$\frac{d^2\xi}{dz^2} + \tfrac{1}{2}\frac{d\xi}{dz} + \left(4 - \frac{1}{z^2}\right)\xi = 0.$$

The solution of this is $\xi = AJ_1(2z)$, where J_1 represents a Bessel's Function of the first order. Replacing the original variables the solution becomes

$$\eta = KJ_1\{2\sqrt{(kx)}\}/\sqrt{(kx)} \quad \text{..................(4)},$$

where K is a constant.

Comparison with tidal observations in the Bristol Channel.

For this purpose a chart of the Bristol Channel was taken and a curved line was drawn down the middle of the channel (see fig. 1).

Sections A, B, C, D, E, F, G were taken at convenient points and roughly at right angles to the centre line. These sections are shown on the sketch chart (fig. 1). The breadth b, and mean depth h, of the channel at low water at each of these sections was found. The distance x of its mid-point from the head of the channel down the curved line was also measured. The figures so obtained are given in Table I. The head of the channel was taken as being at Portishead, near Bristol.

TABLE I. *Dimensions of Bristol Channel at various sections.*

Section	Distance from Portishead, miles	Mean depth, fathoms	Breadth, miles
A	61·7	20	22
B	49·7	15	20
C	42	11	14
D	25	8	12
E	15·5	$4\frac{1}{2}$	7
F	8	3	4
G	0	3	1

Two diagrams were then drawn showing the relationship between b and x, and between h and x. These are shown in figs. 2

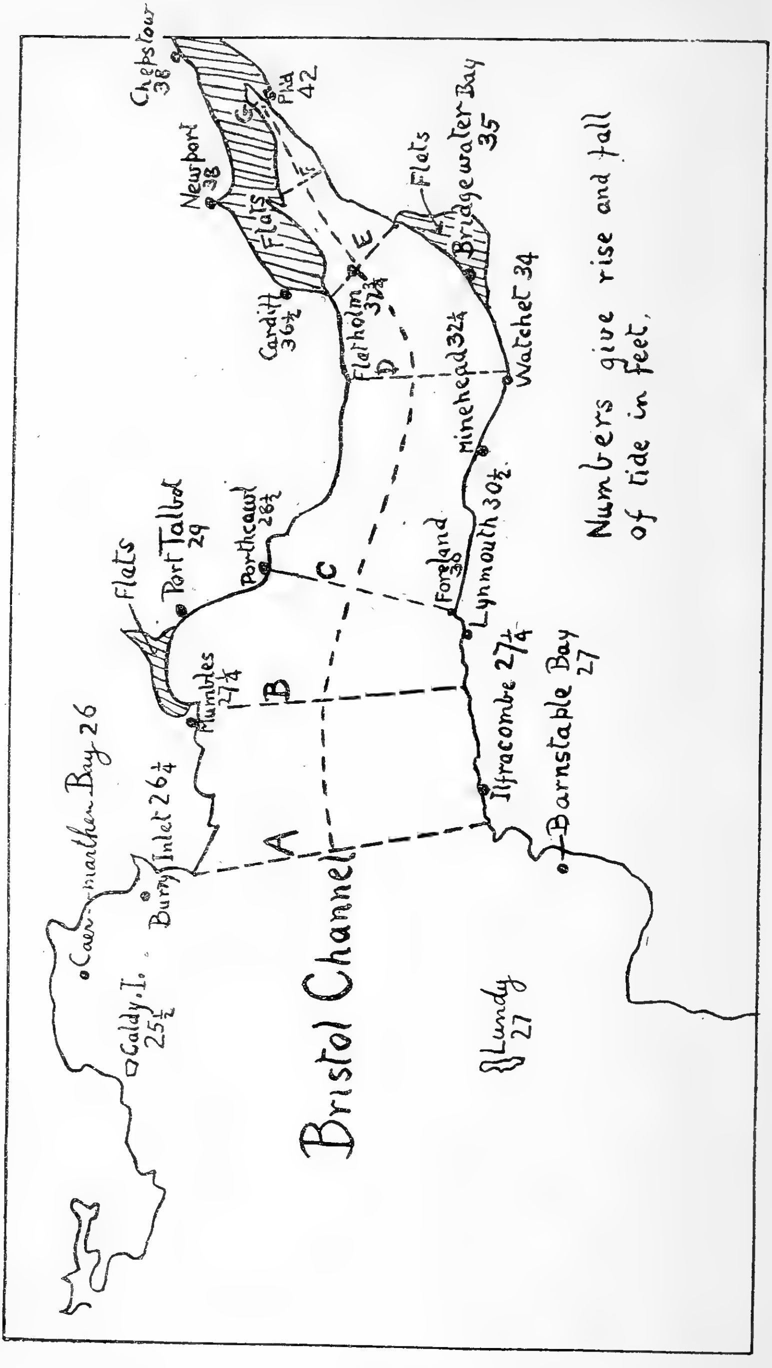
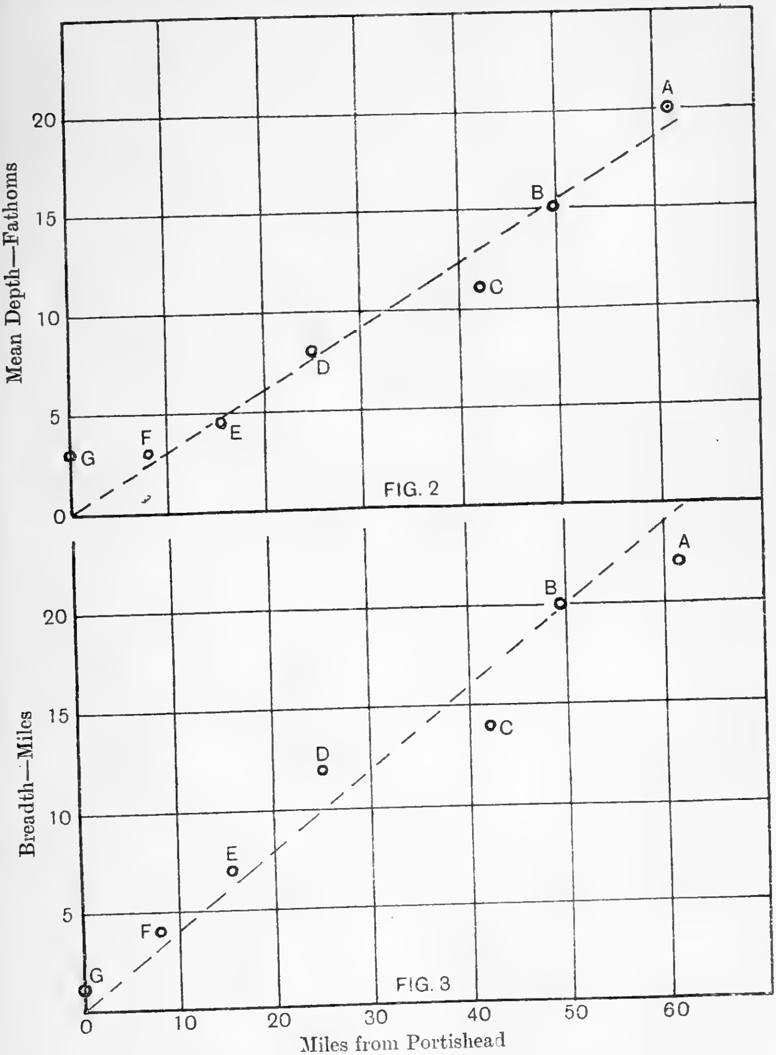

Chepstow 38
Pild 42
Newport 38
Flats
Bridgewater Bay 35
Flats
Cardiff 36¼
Flatholm 32¾
Watchet 34
Minehead 32¼
Numbers give rise and fall of tide in feet.
Port Talbot 29
Porthcawl 28½
Flats
Foreland 35
Lynmouth 30½
Mumbles 27¼
Ilfracombe 27¼
Barnstaple Bay 27
Caermarthen Bay 26
Burry Inlet 26¼
Bristol Channel
Caldy. I. 25½
Lundy 27
A
B
C
D
E
F
G

Fig. 1.

and 3. It will be seen that the assumption that the breadth and the mean depth of the channel increase uniformly from the head towards the mouth is a fairly good approximation to the truth.



Figs. 2 and 3. Dimensions of Bristol Channel.

On looking at the sketch chart (fig. 1) it will be seen that the breadth increases westward of section A, but that owing to two deep bays, Barnstaple Bay and Caermarthen Bay, it ceases to increase even approximately uniformly. For this reason the mouth of the channel has been taken at section A.

Straight lines were next drawn in figs. 2 and 3 to represent the best values to take for the uniform rates of increase in breadth and depth of the channel. These are shown as dotted lines there. The values of h_0 and b_0 in equation 3 were taken in this way as $x_0 = 80$ nautical miles, $h_0 = 25$ fathoms. Since the period of the semi-diurnal tide is 12·4 hours, $\sigma = 2\pi/12{\cdot}4$. From these data k is found to be $\cdot 0118$ (miles)$^{-1}$.

If the rise and fall of tide is known at one point of the channel, equation 4 enables us to find it at all other points. In order to compare the theoretical and observed increase in tide due to the contracting walls of the channel, the simplest method appears to be to use the observed rise and fall of tide at the mouth of the Bristol Channel to determine the constant K, in equation 4, and then to apply equation 4 to calculate theoretically the tides at places up the channel at which tidal measurements have been made.

The rise and fall of tide at the time of spring tides has been measured at a number of places on the shores of the Bristol Channel. These are shown on the sketch chart (fig. 1) by means of numbers placed against the names of the places in question*.

The rise and fall of tide at section A has been taken as 27 feet, a figure which appears to agree with the measured tides in the neighbourhood. Using this figure the theoretical rise and fall of tide has been calculated for all values of x, and a curve has been drawn (see fig. 4) to show the relationship between them. On looking at this curve it will be seen that in the range with which we are concerned the curve is very nearly a straight line.

The observed tides, η, and distances, x, for the places shown on the sketch chart (fig. 1) are given in columns 3 and 2 of Table II. They are represented by means of the dots in fig. 4. It will be seen that the agreement between the theoretical tides and the observed tides is very close, much closer indeed than one might have expected when it is remembered that the rise and fall of tide in the upper part of the channel is as great, or even greater than the depth at low water. The calculated tides at the places mentioned in Table II are taken from the curve (fig. 4), and are given in column 4.

In order to compare the predicted with the observed increase in tide due to the contracting channel, 27 feet has been subtracted from the figures in each of columns 3 and 4 of Table II. The results are given in columns 5 and 6.

It appears from these results that the usual hydrodynamical theory of tides accounts quantitatively as well as qualitatively for the abnormally high tides which exist at the head of the Bristol Channel.

* These figures are obtained from the Admiralty publication *West Coast of England Pilot*, 6th edition, 1910.

TABLE II. *Comparison between observed and calculated increases in tides at various distances up the Bristol Channel over those at the entrance.*

Station	Distance, x, from Portishead, miles	Rise and fall of tide, η		Increase in tide over that at section A	
		Observed feet	Calculated feet	Observed feet	Calculated feet
Section A	61·7	27	27	0	0
Ilfracombe	58	$27\frac{1}{4}$	27·7	$\frac{1}{4}$	0·7
Mumbles	51·6	$27\frac{1}{4}$	28·6	$\frac{1}{4}$	1·6
Port Talbot	47	29	29·7	2	2·7
Porthcawl	42	$28\frac{1}{2}$	30·7	$1\frac{1}{2}$	3·7
Foreland	42	30	30·7	3	3·7
Minehead	30	$32\frac{1}{4}$	33·1	$5\frac{1}{4}$	6·1
Watchet	25	34	34·2	7	7·2
Bridgewater	$19\frac{1}{2}$	35	35·3	8	8·3
Cardiff	$15\frac{1}{2}$	$36\frac{1}{2}$	36·2	$9\frac{1}{2}$	9·2
Flatholm	$15\frac{1}{2}$	$37\frac{3}{4}$	36·2	$10\frac{3}{4}$	9·2
Newport	8	38	37·9	11	10·9
Portishead	0	42	39·8	15	12·8

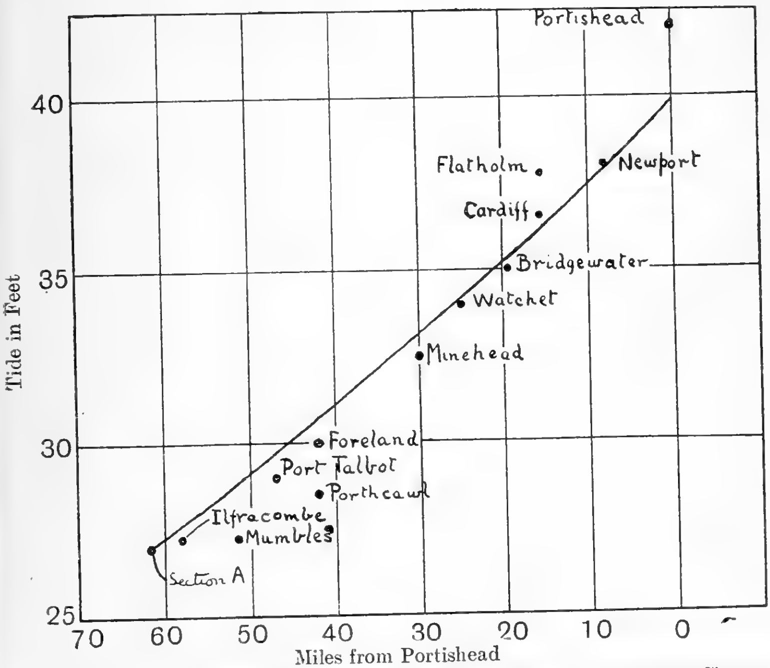


Fig. 4. Comparison between observed and calculated tides in the Bristol Channel.

Experiments with Rotating Fluids. By G. I. Taylor, F.R.S.

[*Read* 24 January 1921.]

The present communication contains a summary of results on three subjects connected with the dynamics of rotating fluids.

1. *Experiments to illustrate the difference between two and three dimensional fluid motion.*

The first experiment to be shown consists in towing a solid circular cylinder of the same density as water through a tank containing water, the whole system rotating at a uniform speed. The tank is a circular glass crystallising dish which is filled three-quarters full with water and floats in another dish of slightly larger diameter. The inner dish is rotated by means of a jet of water which strikes it tangentially. In this way a uniform speed is obtained.

The solid cylinder made of box-wood or wax is placed with its generators vertical, *i.e.* parallel to the axis of rotation. It is held in a position close to the side of the dish by means of a device which releases it at an appropriate moment. It is towed horizontally through the tank by means of two threads which pass through two small rings fixed to the dish on the opposite side to the point to which the cylinder is initially attached. It is found that the cylinder moves straight through the liquid, moving relatively to the rotating system in the direction in which the thread is pulling it, so that it passes through the centre of the dish.

When the same experiment is performed with a solid sphere instead of a solid cylinder, it is found that the motion is very different. The sphere is deflected and moves through the tank in a curved path, leaving the centre of the dish well on its right if the system is rotating in the clockwise direction when seen from above. These experiments confirm a theoretical prediction given by the author*.

Another prediction which is verified with remarkable accuracy is that all small steady motions of a rotating fluid relative to the rotating system must be two dimensional. A consequence of this is that if a spot of coloured water is placed in the rotating water, and if any small motion is communicated to it, the colouring matter is drawn out into sheets which are always parallel to the axis of rotation. If these sheets are observed along the axis of rotation they appear as thin lines. This property is so strikingly

* *Proc. Roy. Soc.* A. 1917, p. 99.

verified that after a time the sheets may become so thin and closely wound round one another that it is only possible to see that the colouring matter is not uniformly diffused through the liquid by placing one eye directly over the rotating basin. The sheets then suddenly reveal themselves as they pass vertically under the eye, and disappear as soon as they get into a part of the basin which is not exactly under the eye.

2. *Motion of a sphere in a rotating fluid.*

The steady motion of a sphere in a rotating fluid along the axis of rotation is discussed mathematically. The velocity of the fluid at any point is expressed by means of Stokes' stream function. So far as the present writer is aware the Stokes' stream function has hitherto only been used in problems where the motion is symmetrical about an axis and is confined to axial planes. It is equally applicable however to cases in which only the first of these conditions holds, and it is used in the present instance. The expression obtained which represents the stream lines when the whole system is given a uniform vertical velocity so as to bring the sphere to rest is

$$\psi = f \sin^2 \theta,$$

where $f = z^2 + \sqrt{\mu^4 + 3\mu^2 + 9}\left\{\cos(z+\epsilon) - \frac{\sin(z+\epsilon)}{z}\right\}$

and $z = kr,\ k = 2\Omega/v,\ \mu = ka,\ \tan(\mu+\epsilon) = 3\mu/(3-\mu^2)$.

Ω is the angular velocity of rotation of the fluid.

v is the velocity of the sphere along the axis.

r, θ are the polar coordinates of a point referred to the centre of the sphere as origin, and a is the radius of the sphere.

The components of velocity of the fluid at any point are found from this expression by the formulae

$$u = -2f\cos\theta/r^2,\ v = \frac{1}{r}\frac{df}{dr}\sin\theta,\ w = kf\sin\theta/r.$$

In this expression the axes of reference are not rotating. It is found that although the solution allows slip to take place at the surface of the sphere, the actual solution obtained involves no slip. This is a point of considerable importance because it is the assumption that there is a slip at the surface of solids which vitiates all the ordinary hydrodynamical theories of the motion of fluids. The stream lines due to the motion of a sphere along the axis of a rotating fluid may therefore be expected to be more like the theoretical stream lines than they are in the case when the fluid is not rotating. This is found to be the case.

One consequence of the fact that the velocity of the fluid at the surface of the sphere is zero, relative to fixed axes, is that as a sphere moves up the axis of a rotating fluid the liquid streaming past it will not tend to rotate it. This is found to be true. It is shown experimentally that a light sphere initially rotating with the liquid in a tall rotating jar of water, stops rotating directly it is moved along the axis of the jar, but that it starts rotating again as soon as the motion along the axis ceases.

3. *Stability of fluid motion between two concentric cylinders.*

The late Lord Rayleigh has stated, though without formal proof, that for three dimensional symmetrical disturbances the steady motion of a perfect liquid between two cylinders which rotate with different speeds is stable if the square of the circulation round circular paths concentric with the cylinders increases on passing from the inner cylinder to the outer one. But that it is unstable otherwise. This conclusion is now proved to be correct by calculating the actual motion in a normal disturbance.

All calculations about the stability of liquid between two rotating cylinders have assumed two dimensional motion. In the case of two dimensional motion a rotation of the whole system makes no difference to the type of motion. Its stability or instability are determined only by the relative motion of the two cylinders.

Experiments made by Mallock* and Conette† showed that if the inner cylinder is fixed while the outer one rotates the motion only becomes unstable at a very much higher relative velocity than if the inner one is fixed and the outer one rotates. This evidently suggests that the instability observed is not two dimensional. According to Rayleigh's criterion for the stability of the symmetrical disturbance of an inviscid fluid between rotating cylinders, the case when the outer cylinder is fixed and the inner one rotates should always be unstable. If it is observed to be stable this must be some effect due to viscosity.

In the case when the inner cylinder is fixed and the outer one rotates symmetrical disturbances should be on the limit between stability and instability. The slightest rotation of the inner cylinder in the same direction as the outer one should make the disturbances stable, while the slightest rotation in the opposite direction should make them unstable.

It appears therefore that the method adopted by previous experimenters in which one or other of the cylinders was fixed is unfortunate.

* Mallock. *Phil. Trans.* A, 1896, p. 41.

† Conette. *Annales de Chimie et de Physique*, [6], XXI, p. 433.

Preliminary experiments made with a pair of concentric cylinders which are both of them capable of being rotated, show that the Rayleigh condition appears to be verified for high speed rotation, but that at low speed it is very considerably modified, presumably by viscosity, the motion being stable beyond the limits prescribed by the Rayleigh theory.

Calculations of the effect of viscosity on the stability of symmetrical disturbances are very difficult, but an equation has been obtained which can probably be solved graphically; and the author hopes shortly to be able to say whether viscosity should increase or decrease the stability of symmetrical disturbances.

Experiments on focal lines formed by a zone plate. By G. F. C. SEARLE, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 28 February 1921.]

§ 1. *Introduction**. In the usual theoretical investigation of the properties of a zone plate, the luminous point is taken to be on the axis of the zone plate, and in the practical measurements, such as those which have been made at the Cavendish Laboratory for many years, the incident rays are not inclined at more than small angles to the axis of the zone plate. When, however, the luminous point is not on the axis, the zone plate gives rise to two focal lines, as a thin lens does under similar circumstances; the positions of these focal lines are investigated in the present paper. The theory has been extended to the case in which any non-spherical wave front falls upon the zone plate at any angle of incidence, and the positions and directions of the focal lines of the emergent wave front have been found. The experiments illustrating the theory were made with the kind assistance of Mr G. S. Clark-Maxwell of King's College.

§ 2. *Theory of zone plate.* Let OG_1G_2, ... (Fig. 1) be the section by the plane of the figure of an infinitely thin plane opaque screen, and let $X'OX$ be the normal to the screen at O. Let the spaces to the left and right of the screen in Fig. 1 be called the object and image spaces respectively. On the screen take a point G_1 near O and let $OG_1 = \rho_1$. Let G_2, G_3, ... be other points on the screen such that $OG_n{}^2 = \rho_n{}^2 = \rho_1{}^2 + (n-1)k^2$. Let narrow circular slits be cut in the screen with O as centre and passing through G_1, G_2, ... This system of screen and slits forms a theoretical zone plate†.

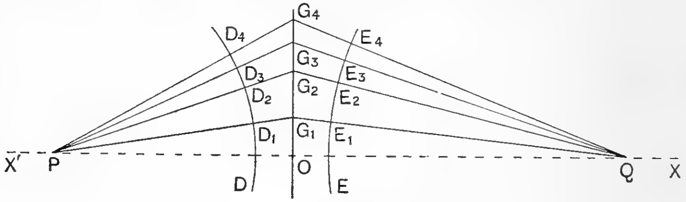


Fig. 1.

* When this paper was read, I did not know that the general theory of the action of a diffraction grating or of a zone plate upon a wave front of any form had been indicated by Sir J. Larmor in "The Dioptrics of Gratings," *Proc. Lond. Math. Soc.* Vol. XXIV. p. 166 (1893).

† To obtain strong images, the widths of the slits through $G_1, G_2, \ldots$ are increased, so that the edges of the nth slit have the radii σ_n, τ_n, where $\sigma_n{}^2 = \rho_n{}^2 - z^2$, $\tau_n{}^2 = \rho_n{}^2 + z^2$. Zone plates are made by photography from large scale drawings, and the attempt is generally made to make $z^2 = \frac{1}{4}k^2$. If this were *accurately* done, no images of *even* order would be formed. The zone plate used in the experiments is a "phase-reversal" plate made by Prof. R. W. Wood and given by him to the Cavendish Laboratory.

On $X'OX$ take P, Q at distances u, v from O, and let u and v be positive when P and Q are in the object and image spaces respectively. Then, when ρ_n/u, ρ_n/v are small,

$$\begin{aligned} PG_n + QG_n &= (u^2 + \rho_n{}^2)^{\frac{1}{2}} + (v^2 + \rho_n{}^2)^{\frac{1}{2}} \\ &= u + v + \tfrac{1}{2}\{\rho_1{}^2 + (n-1)\,k^2\}\{1/u + 1/v\}. \end{aligned}$$

Hence, for all values of the positive integer n greater than 1,

$$PG_n + QG_n = PG_{n-1} + QG_{n-1} + \tfrac{1}{2}k^2\,(1/u + 1/v).$$

Thus, the paths PG_1Q, PG_2Q, ... increase by equal steps of h, where

$$h = \tfrac{1}{2}k^2\,(1/u + 1/v).$$

Let a train of spherical waves with centre P and wave length λ fall upon the zone plate from the object space, and let D be one of the wave fronts. Let PG_1, PG_2, ... meet D in D_1, D_2, ... Waves will travel out from the slits into the image space. Let E be a sphere about Q as centre, and let G_1Q, G_2Q, ... meet E in E_1, E_2, ... Then the disturbances at E_1, E_2, ... will have the same phase, if the distances $D_1G_1E_1$, $D_2G_2E_2$, ... increase by steps of $p\lambda$, where p is a positive or negative integer. When the distance of E from O is some thousands of wave lengths, the separate wavelets due to the ring slits will merge into a single wave indistinguishable from E. We may thus speak of E as the emergent wave front. The wave D will thus give rise to the wave E, if $PG_n + QG_n$ exceeds $PG_1 + QG_1$ by $(n-1)\,p\lambda$. An image of P will then be formed at Q.

Hence, Q will be an image of P, if $h = p\lambda$. If f_p be the corresponding focal length, and F_p the corresponding "power," we have

$$1/u + 1/v = F_p = 1/f_p = 2p\lambda/k^2. \qquad \text{......(1)}$$

Thus, the power is proportional to p and is positive or negative with p. We here follow the custom of practical opticians, who treat the power of a thin converging lens as *positive*.

The zone plate thus acts as a lens with a number of positive and negative focal lengths, and, for a given position of a real or virtual luminous point P on $X'OX$, there will be a number of images, some real and some virtual.

If k^2 is found by measuring the rings with a travelling microscope, λ can be found, when f_p and p are known.

§ 3. *Oblique incidence.* Let the luminous point now lie off the axis $X'OX$ at P (Fig. 2) in the plane of the figure. Let $PO = u$, and let u be positive when P is in the object space. Let the acute angle between PO and $X'OX$ be θ. Let Q be a point in the plane of the figure, let $OQ = b$, and let b be positive when Q is in the image space. Let the acute angle between OQ and $X'OX$ be ϕ. Then, since $OG_n = \rho_n$,

$$PG_n{}^2 = u^2 + 2u\rho_n \sin\theta + \rho_n{}^2, \quad QG_n{}^2 = b^2 - 2b\rho_n \sin\phi + \rho_n{}^2.$$

Hence, by expanding,

$$PG_n = u\,\{1 + \tfrac{1}{2}(2\rho_n \sin\theta/u + \rho_n{}^2/u^2) - \tfrac{1}{8}(2\rho_n \sin\theta/u + \rho_n{}^2/u^2)^2 + \ldots\}.$$

Thus, as far as terms in $1/u$,

$$PG_n = u + \rho_n \sin\theta + \tfrac{1}{2}\rho_n{}^2(1 - \sin^2\theta)/u = u + \rho_n \sin\theta + \tfrac{1}{2}\rho_n{}^2 \cos^2\theta/u.$$

Similarly, $\quad QG_n = b - \rho_n \sin\phi + \tfrac{1}{2}\rho_n{}^2 \cos^2\phi/b.$

Hence

$$PG_n + QG_n - (PG_1 + QG_1) = (\rho_n - \rho_1)(\sin\theta - \sin\phi) + \tfrac{1}{2}(\rho_n{}^2 - \rho_1{}^2)(\cos^2\theta/u + \cos^2\phi/b).$$

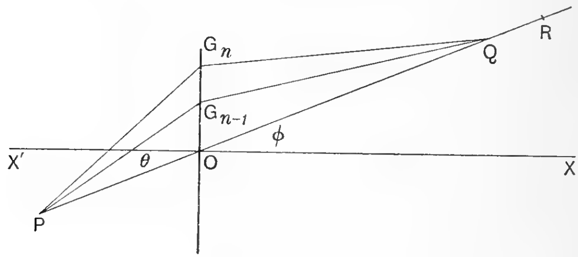


Fig. 2.

If this difference is $(n-1)\,p\lambda$, there will be concentration of light at Q. Now $\rho_n{}^2 - \rho_1{}^2 = (n-1)\,k^2$, but $\rho_n - \rho_1$ is not proportional to $n-1$. Hence, if the equation is to hold for all integral values of n, we must have $\sin\phi = \sin\theta$. Thus $\cos^2\phi = \cos^2\theta$, and then,

$$1/u + 1/b = 2p\lambda \sec^2\theta/k^2 = \sec^2\theta/f_p = F_p \sec^2\theta. \quad \ldots(2)$$

Let H_1, H_2, ... lie on a straight line through O perpendicular to the plane of Fig. 2, and let $OH_n = \rho_n$. Then, if R is a point on OP, the path PH_nR is, for given values of $OP = u$ and $OR = c$, equal to the path PH_nR when P and R lie on the axis. Hence, if c be positive when R lies in the image space, there will, by (1), be a concentration of light at R, when

$$1/u + 1/c = 1/f_p = F_p. \quad \ldots\ldots(3)$$

As in § 2, we can speak of the emergent wave front, but in this case the front will not be spherical. By symmetry, the plane of Fig. 2 and a plane through PO and perpendicular to the latter plane are the principal planes of the emergent wave front. The radii of curvature of the principal sections of this front are b and c, as given by (2) and (3).

The emergent rays of order p, i.e. the normals to the emergent wave front of order p, do not pass through a single point but through two focal lines. The primary line passes through Q and is perpendicular to the plane of Fig. 2, and the secondary line through R is in that plane.

If a slit illuminated with sodium light is placed at P, and if the slit is perpendicular to the plane of Fig. 2, there will be a focal line image of the slit at Q, and the image will be perpendicular to that plane. If the slit is in the plane of Fig. 2 and perpendicular to OP, there will be a focal line image at R, the image lying in the plane of Fig. 2.

§ 4. *General case.* Let OX (Fig. 3) be the axis of the zone plate and OP_1 the forward direction of the chief ray of the incident beam. Take OY perpendicular to OX in the plane P_1OX, and OZ perpendicular to that plane. Then the zone plate lies in the plane YOZ. Let $P_1OX = \theta_1$.

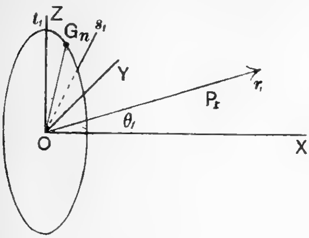


Fig. 3.

Let the refractive indices of the object and image spaces be μ_1, μ_2. Let v_0 be the velocity of light, and λ_0 the wave length, in a vacuum, and let τ be the periodic time of the vibration. Then $\tau v_0 = \lambda_0$.

Take OP_1 as the axis of r_1 in a new set of axes Or_1, Os_1, Ot_1, such that Os_1 is in the plane XOY and Ot_1 coincides with OZ. Let the equation, referred to these axes, of the incident wave front, when passing through O, be

$$r_1 = \tfrac{1}{2}S_1s_1^2 + W_1s_1t_1 + \tfrac{1}{2}T_1t_1^2. \qquad \text{......(4)}$$

Let G_n be a point on the zone plate on the circle of radius ρ_n, and let $G_nOY = \omega$. Then the x, y, z coordinates of G_n are 0, $\rho_n \cos\omega$, $\rho_n \sin\omega$, and its r_1, s_1, t_1 coordinates are

$$r_1 = \rho_n \cos\omega \sin\theta_1, \quad s_1 = \rho_n \cos\omega \cos\theta_1, \quad t_1 = \rho_n \sin\omega.$$

If a straight line through G_n parallel to OP_1 cuts the incident wave front OD_1D_2 ... in D_n, the second and third coordinates of D_n, referred to the axes of r_1, s_1, t_1, are $\rho_n \cos\omega \cos\theta_1$ and $\rho_n \sin\omega$. Hence, by (4), the distance of D_n from the plane $r_1 = 0$, which touches the wave front at O, is

$$\rho_n^2 (\tfrac{1}{2}S_1 \cos^2\omega \cos^2\theta_1 + W_1 \cos\omega \sin\omega \cos\theta_1 + \tfrac{1}{2}T_1 \sin^2\omega),$$

and the distance of G_n from the same plane is $\rho_n \cos\omega \sin\theta_1$. Hence

$$D_nG_n = \rho_n \cos\omega \sin\theta_1 - \rho_n^2 (\tfrac{1}{2}S_1 \cos^2\omega \cos^2\theta_1 + W_1 \cos\omega \sin\omega \cos\theta_1 + \tfrac{1}{2}T_1 \sin^2\omega).$$

When n diminishes, D_nG_n becomes more and more nearly the normal to the wave front, and, for a small aperture, may be treated as the normal in the estimation of *distances*. Thus, when n is not great, D_nG_n may be taken as the ray distance from the wave front to G_n.

Let OP_2 be the forward direction of the chief ray of the emergent beam, which, by symmetry, must lie in the plane XOY. Let $P_2OX = \theta_2$. When referred to axes Or_2, Os_2, Ot_2, chosen similarly to Or_1, Os_1, Ot_1, let the equation to the emergent wave front, as it passes through O, be

$$r_2 = \tfrac{1}{2}S_2 s_2^2 + W_2 s_2 t_2 + \tfrac{1}{2}T_2 t_2^2. \qquad \text{......(5)}$$

If a line through G_n parallel to OP_2 cuts the emergent wave front OE_1E_2 ... in E_n, the distance E_nG_n is ultimately the ray distance from E_n to G_n. We then have

$$E_nG_n = \rho_n \cos\omega \sin\theta_2 - \rho_n^2 \left(\tfrac{1}{2}S_2 \cos^2\omega \cos^2\theta_2 + W_2 \cos\omega \sin\omega \cos\theta_2 + \tfrac{1}{2}T_2 \sin^2\omega\right).$$

The optical condition is that the time of passage of light over the distance $E_nG_n - E_1G_1$ in the second medium is less than the time of passage over the distance $D_nG_n - D_1G_1$ in the first medium by $(n-1)p\tau$. Hence

$$\mu_2 (E_nG_n - E_1G_1)/v_0 - \mu_1 (D_nG_n - D_1G_1)/v_0 = -(n-1)p\tau.$$

Thus, since $v_0\tau = \lambda_0$, we have

$$\begin{aligned}(\rho_n - \rho_1) \cos\omega\, (\mu_2 \sin\theta_2 - \mu_1 \sin\theta_1) & \\ - (\rho_n^2 - \rho_1^2)\left[\tfrac{1}{2}(\mu_2 S_2 \cos^2\theta_2 - \mu_1 S_1 \cos^2\theta_1)\cos^2\omega\right. & \\ + (\mu_2 W_2 \cos\theta_2 - \mu_1 W_1 \cos\theta_1)\cos\omega \sin\omega & \\ \left. + \tfrac{1}{2}(\mu_2 T_2 - \mu_1 T_1)\sin^2\omega\right] &= -(n-1)p\lambda_0.\end{aligned}$$

Now $\rho_n^2 - \rho_1^2 = (n-1)k^2$, but $\rho_n - \rho_1$ is not proportional to $n-1$. Hence, if the equation is to hold for all integral values of n greater than 1, we must have

$$\mu_2 \sin\theta_2 = \mu_1 \sin\theta_1. \qquad \text{......(6)}$$

Hence, the chief ray obeys the ordinary law of refraction, and θ_2 is known when θ_1 is given. Since $\rho_n^2 - \rho_1^2 = (n-1)k^2$, we have

$$\begin{aligned}\tfrac{1}{2}(\mu_2 S_2 \cos^2\theta_2 - \mu_1 S_1 \cos^2\theta_1)\cos^2\omega & \\ + (\mu_2 W_2 \cos\theta_2 - \mu_1 W_1 \cos\theta_1)\cos\omega \sin\omega & \\ + \tfrac{1}{2}(\mu_2 T_2 - \mu_1 T_1)\sin^2\omega &= p\lambda_0/k^2. \qquad \text{......(7)}\end{aligned}$$

Equation (7) holds for all values of ω. Putting $\omega = 0$, we have

$$\mu_2 S_2 \cos^2\theta_2 - \mu_1 S_1 \cos^2\theta_1 = 2p\lambda_0/k^2. \qquad \text{......(8)}$$

Putting $\omega = \tfrac{1}{2}\pi$, we have

$$\mu_2 T_2 - \mu_1 T_1 = 2p\lambda_0/k^2. \qquad \text{......(9)}$$

Hence, since $\cos^2 \omega + \sin^2 \omega = 1$,

$$\tfrac{1}{2}(\mu_2 S_2 \cos^2\theta_2 - \mu_1 S_1 \cos^2\theta_1)\cos^2\omega + \tfrac{1}{2}(\mu_2 T_2 - \mu_1 T_1)\sin^2\omega = p\lambda_0/k^2,$$

and thus, by (7), $\mu_2 W_2 \cos\theta_2 - \mu_1 W_1 \cos\theta_1 = 0.$ ......(10)

Since θ_2 is known by (6), the last three equations determine S_2, W_2 and T_2.

In the ordinary use of the zone plate, the medium on either side of the zone plate is air of refractive index μ, and thus $\mu_2 = \mu_1 = \mu$, and $\theta_2 = \theta_1 = \theta$. If the wave length of the light in air is λ, we have $\mu\lambda = \lambda_0$. If the "power" of the zone plate in air is F_p, $2p\lambda/k^2 = F_p$, and then the equations giving S_2, W_2, T_2 become

$$S_2 = F_p \sec^2\theta + S_1, \qquad W_2 = W_1, \qquad T_2 = F_p + T_1. \quad \text{...(11)}$$

If the incident beam is due to a luminous point at a distance u from O in the object space, $S_1 = T_1 = -1/u$, $W_1 = 0$. Then

$$S_2 = \sec^2\theta/f_p - 1/u, \qquad W_2 = 0, \qquad T_2 = 1/f_p - 1/u. \quad \text{...(12)}$$

Since $W_2 = 0$, the focal lines of the emergent beam are in and perpendicular to the plane XOY. If their distances from O are c and b respectively, where b, c are positive when the focal lines are in the image space, we have $S_2 = 1/b$, $T_2 = 1/c$, and then

$$1/u + 1/b = \sec^2\theta/f_p, \qquad 1/u + 1/c = 1/f_p, \quad \text{......(13)}$$

as was found in § 3.

§ 5. *The principal curvatures of the emergent wave front.* Let the principal planes of the incident front at O (Fig. 4) intersect the tangent plane at O in $O\eta_1$, $O\zeta_1$. Take these lines, with $O\xi_1$ along OP_1, as axes for the front. Let the radii of curvature of the sections of the front by $O\xi_1\eta_1$, $O\xi_1\zeta_1$ be B_1^{-1} and C_1^{-1}, counted positive when the sections are concave towards P_1. The equation to the incident front referred to these axes is then

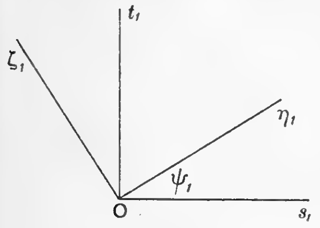


Fig. 4.

$$\xi_1 = \tfrac{1}{2}B_1\eta_1^2 + \tfrac{1}{2}C_1\zeta_1^2. \quad \text{...(14)}$$

Let $O\eta_1$ make an angle ψ_1 with Os_1, as in Fig. 4. Then

$$\xi_1 = r_1, \quad \eta_1 = s_1\cos\psi_1 + t_1\sin\psi_1, \quad \zeta_1 = -s_1\sin\psi_1 + t_1\cos\psi_1,$$

and hence (14) is equivalent to

$$r_1 = \tfrac{1}{2}B_1(s_1\cos\psi_1 + t_1\sin\psi_1)^2 + \tfrac{1}{2}C_1(-s_1\sin\psi_1 + t_1\cos\psi_1)^2. \quad \text{...(15)}$$

Comparing (15) with (4), we find

$$S_1 = \tfrac{1}{2}(B_1 + C_1) + \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1, \qquad \text{......(16)}$$

$$W_1 = \tfrac{1}{2}(B_1 - C_1)\sin 2\psi_1, \qquad \text{......(17)}$$

$$T_1 = \tfrac{1}{2}(B_1 + C_1) - \tfrac{1}{2}(B_1 - C_1)\cos 2\psi_1. \qquad \text{......(18)}$$

If the equation to the emergent front, referred to its own principal axes, is

$$\xi_2 = \tfrac{1}{2}B_2\eta_2^2 + \tfrac{1}{2}C_2\zeta_2^2, \qquad \text{......(19)}$$

then B_2, C_2 are the principal curvatures. If $\eta_2 O s_2 = \psi_2$, then B_2, C_2, ψ_2 are related to S_2, W_2, T_2 by

$$S_2 = \tfrac{1}{2}(B_2 + C_2) + \tfrac{1}{2}(B_2 - C_2)\cos 2\psi_2, \qquad \text{......(20)}$$

$$W_2 = \tfrac{1}{2}(B_2 - C_2)\sin 2\psi_2, \qquad \text{......(21)}$$

$$T_2 = \tfrac{1}{2}(B_2 + C_2) - \tfrac{1}{2}(B_2 - C_2)\cos 2\psi_2. \qquad \text{......(22)}$$

Solving for B_2, C_2 and using (11), we have

$$B_2 = X + Y, \qquad C_2 = X - Y, \qquad \text{......(23)}$$

where

$$X = \tfrac{1}{2}(S_2 + T_2) = \tfrac{1}{2}[F_p(1 + \sec^2\theta) + B_1 + C_1], \qquad \text{......(24)}$$

$$\begin{aligned} Y &= \tfrac{1}{2}[(S_2 - T_2)^2 + 4W_2^2]^{\frac{1}{2}} \\ &= \tfrac{1}{2}[F_p^2\tan^4\theta + 2F_p\tan^2\theta\,(B_1 - C_1)\cos 2\psi_1 + (B_1 - C_1)^2]^{\frac{1}{2}}. \end{aligned} \qquad \text{......(25)}$$

These equations, with (23), determine B_2 and C_2. But, as either sign can be given to the square root, we are left in doubt as to which of the two focal lines corresponds to B_2 and which to C_2. To avoid confusion we must write

$$Y^0 = \tfrac{1}{2}(F_p\tan^2\theta + B_1 - C_1), \quad Y^{\pi/2} = \tfrac{1}{2}(F_p\tan^2\theta - B_1 + C_1), \qquad \text{......(26)}$$

where Y^0, $Y^{\pi/2}$ are the values to be assigned to Y for $\psi_1 = 0$ and for $\psi_1 = \tfrac{1}{2}\pi$. For intermediate values of ψ_1, we take Y intermediate between Y^0 and $Y^{\pi/2}$. Since $4Y^2$ is a sum of squares, Y cannot vanish unless $W_2 = 0$, or, what is the same thing, unless $W_1 = 0$. When $B_1 - C_1$ is not zero, W_1 vanishes only when $\psi_1 = 0$ or $\psi_1 = \tfrac{1}{2}\pi$. Hence Y does not change from positive to negative or *vice versa* as ψ_1 goes from 0 to $\tfrac{1}{2}\pi$.

In practical work it is convenient to determine the quantities appearing in X and Y by observing B_2^0, C_2^0, $B_2^{\pi/2}$, $C_2^{\pi/2}$, the values of B_2 and C_2 for $\psi_1 = 0$ and $\psi_1 = \tfrac{1}{2}\pi$. We have, in accordance with (26),

$$\left.\begin{aligned} B_2^0 &= F_p\sec^2\theta + B_1, & C_2^0 &= F_p + C_1, \\ B_2^{\pi/2} &= F_p\sec^2\theta + C_1, & C_2^{\pi/2} &= F_p + B_1. \end{aligned}\right\} \qquad \text{...(27)}$$

Hence

$$\left.\begin{aligned} X &= \tfrac{1}{2}(B_2{}^0 + C_2{}^0) = \tfrac{1}{2}(B_2{}^{\pi/2} + C_2{}^{\pi/2}), \\ F_p \tan^2\theta &= B_2{}^0 - C_2{}^{\pi/2} = B_2{}^{\pi/2} - C_2{}^0, \\ B_1 - C_1 &= B_2{}^0 - B_2{}^{\pi/2} = C_2{}^{\pi/2} - C_2{}^0 . \end{aligned}\right\} \quad \text{.........(28)}$$

The equation $\tan 2\psi_2 = 2W_2/(S_2 - T_2)$ leads, by (11), to

$$\tan 2\psi_2 = \frac{2W_1}{S_1 - T_1 + F_p \tan^2\theta}$$

$$= \frac{(B_1 - C_1)\sin 2\psi_1}{(B_1 - C_1)\cos 2\psi_1 + F_p \tan^2\theta},$$

which gives two values of ψ_2 corresponding to the two focal lines. In practical work it is more convenient to use $\sin 2\psi_2$. We have

$$\sin 2\psi_2 = \frac{2W_2}{B_2 - C_2} = \frac{2W_1}{2Y} = \frac{B_1 - C_1}{2Y}\sin 2\psi_1, \quad \text{......(29)}$$

where, as we may restrict ψ_1 to range from 0 to $\frac{1}{2}\pi$, we may specify that $|\psi_2| \gtreqless \frac{1}{2}\pi$. Equation (29) gives the value of ψ_2 corresponding to B_2; the value corresponding to C_2 differs by $\frac{1}{2}\pi$ from that given by (29).

When $Y = 0$, the two focal lines coalesce into a point image of the luminous point. Unless $B_1 = C_1$, this can only occur when $\psi_1 = 0$ or $\psi_1 = \frac{1}{2}\pi$. When this is satisfied, we must further have $S_2 - T_2 = 0$, or, by (11), $F_p \tan^2\theta = -(B_1 - C_1)\cos 2\psi_1$, where $\psi_1 = 0$ or $\psi_1 = \frac{1}{2}\pi$. If θ^0 and $\theta^{\pi/2}$ are the required values of θ, we have

$$\tan^2\theta^0 = -(B_1 - C_1)/F_p, \quad \tan^2\theta^{\pi/2} = (B_1 - C_1)/F_p.$$

For real values of θ, p must have the values 1, 2, ... in one case and $-1, -2$, ... in the other. If $B_1 - C_1$ is positive, p is $-1, -2$, ... for θ^0, and 1, 2, ... for $\theta^{\pi/2}$. If, however, $B_2 - C_2$ is negative, p is 1, 2, ... for θ^0, and $-1, -2$, ... for $\theta^{\pi/2}$. The value of $\tan^2\theta$ will be the same in each case, viz.

$$\tan^2\theta = |(B_1 - C_1)/F_p| = |(B_1 - C_1)/p| \,.\, F_1^{-1}. \quad \text{...(30)}$$

The image will be real or virtual according as the values of θ and p make X positive or negative.

§ 6. *Experimental details.* In the experiments the apparatus shown diagrammatically in plan in Fig. 5 was used. The zone plate G is fixed, with its axis ON horizontal, to a table turning about a vertical axis and carrying a horizontal divided circle H, which is read by the pointers J, J. The adjustment of the zone plate so that its centre O lies on the vertical axis of the table may be effected by aid of a long focus microscope or by simple mechanical devices. It is convenient to mount the base of the revolving table

H upon a carriage sliding on a graduated track R at right angles to the optical bench S. The zone plate can then be moved aside and be replaced at will. The focal lines are observed by aid of a low-power eye-piece or of a telescope T mounted on a carriage sliding along the optical bench S. The length of the telescope, which must remain unchanged during any set of observations, should be adjusted so that an object about one metre from the objective may be seen in focus. A cross-wire which has been properly focussed will enable the observer to secure more definite settings of the telescope or eye-piece. If a telescope is used, the positions of virtual as well as of real images and focal lines can be observed.

In testing the results of § 3, a small circular hole P in a sheet of metal, illuminated by a sodium flame Z, may form the object. The hole P lies on the line OX passing through O and parallel to the length of the bench. The distance PO may be about $2f_1$.

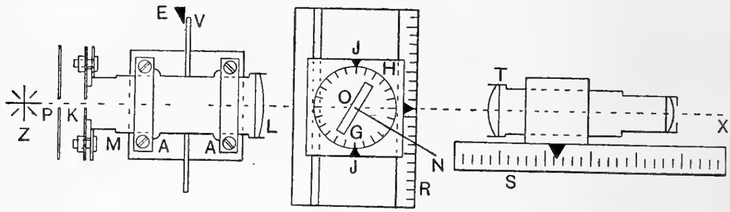


Fig. 5.

A string, stretched parallel to the length of the bench, may be used in setting P on to the line OX and in making the axis of the telescope coincide with OX. The zero reading of the circle H is found by holding a set square against the zone plate and adjusting the table so that the appropriate edge of the square is parallel to OX; suitable optical appliances would give greater accuracy of adjustment. The angle NOX is θ. When $\theta = 0$, the images of P will be found on moving the telescope along the bench. By comparing the bench reading of the telescope for any image with its reading when focussed on the zone plate, the distance of the image from O is found. The distance is called positive when the image is real.

If θ is now increased from zero, the circular image of P will be drawn out into a horizontal *focal line* at a point R on OX, and the position of R will coincide with that of the *image* of P for $\theta = 0$. There will also be a vertical focal line at Q on OX, but this will move towards O as θ is increased, the "power" of the zone plate changing from F_p to $F_p \sec^2 \theta$ for this line. If $PO = u$, and if, for any value of θ, and for any value of p, the distances QO, RO are

b, c respectively, while v is the distance from O of the image when $\theta = 0$, we have

$$\frac{1}{u}+\frac{1}{v}=F_p,\quad \frac{1}{u}+\frac{1}{b}=F_p\sec^2\theta,\quad \frac{1}{u}+\frac{1}{c}=F_p. \quad \ldots(31)$$

The distances u, v or c, and b are measured. The accuracy of the second formula is tested by finding the value of $(1/u + 1/b)\cos^2\theta$ for various values of θ.

The experiment is improved by substituting for the small hole a pair of slits cut in a metal plate and intersecting accurately at right angles. To obtain sharp "images" of the slits formed by focal lines, one slit must be vertical, the other horizontal, since the focal line at P due to a luminous point at Q is vertical, and the focal line at P due to a luminous point at R is horizontal.

When $\theta = 0$, the zone plate forms images of the crossed slits. We can use the multiplicity of focal lengths to produce an *apparent* image of the crossed slits when θ is not zero. With $\theta = 0$, the telescope is set on the *image* of the second order ($p = 2$). Then $1/u + 1/v = F_2$. If θ is now made $\frac{1}{4}\pi$, the horizontal focal line of the second order is given by $1/u + 1/c = F_2$. The vertical focal line of the *first* order for $\theta = \frac{1}{4}\pi$ is given by

$$1/u + 1/b = F_1 \sec^2 \tfrac{1}{4}\pi = 2F_1.$$

But $F_2 = 2F_1$, and hence b for the first order equals c for the second order. The two focal lines will thus be at the same distance from O, and the observer will have the impression that he sees a true image of the slits. If, however, a small hole is used in place of the crossed slits, two focal lines will be seen.

For testing the results of § 5, additional apparatus is required. A horizontal tube M, arranged with a draw tube for adjustment of length, turns in bearings A, A (Fig. 5) and carries a vertical divided circle V, which is read by the index E. A lens L is fixed to the end of the tube nearer O. At the other end is a plate K pierced by a small circular hole, by one slit or by a pair of slits crossed at right angles. This plate can be turned in its own plane about the axis of M and thus a slit can be given any desired direction. The plate K is held against a flange at the end of the tube by three nuts working on three studs carried by a plate on the other side of the flange. The tube passes through a hole in the latter plate with an easy fit. (The plate with the hole P is removed.)

If L has spherical faces, the length of M can be adjusted so that K is in the focal plane of L. The system then forms an ordinary collimator. If L is an astigmatic lens, spherical on one face and cylindrical on the other, and if M is adjusted so that KL equals one of the two focal lengths of L, one of the focal lines formed when a small hole is used in K will be at infinity, and the arrangement may be called an astigmatic collimator. The emergent wave front in this case will be cylindrical.

By rotating M, the principal planes of the emergent wave front emerging from an astigmatic lens L can be turned about the axis of M*.

The principal planes of the emergent wave front due to a luminous point at K are perpendicular and parallel to the axis of the cylindrical surface of L. We will denote by ψ_1 the inclination of the first of these planes to XON, the horizontal plane of incidence of the chief ray upon the zone plate; then the inclination of the other principal plane is $\psi_1 + \frac{1}{2}\pi$. If, in place of a small hole, a slit is used, its inclination to the horizontal plane through OX must be ψ_1 or $\psi_1 + \frac{1}{2}\pi$, when $\theta = 0$. When θ is not zero and ψ_1 is not zero, no sharp focal line will be formed unless the slit is inclined to XON at one of the two angles δ and $\delta + \frac{1}{2}\pi$. To find the angle δ, first find the focal lines formed at Q and R on OX by the system of lens and zone plate, when a luminous point is placed at K. Next place a luminous point at Q; the direction of that focal line which lies in the plane of K gives one of the required directions of the slit. When the slit is placed in this position and is illuminated, there will be a long sharp focal line at Q. If the slit is turned through $\frac{1}{2}\pi$, there will be a long sharp focal line at R. If we carry out this process mathematically we can calculate δ. In practice it is easy to adjust the direction of the slit so that the images formed by focal lines are sharp.

§ 7. *First practical example.* In this experiment the results of § 3 were tested.

A vertical slit was used in the collimator M (Fig. 5), and the instrument was adjusted so that the slit was accurately at the focus of a converging lens of 3·5 dioptre power. The rays falling on the zone plate came, in effect, from an infinitely distant slit, and thus $1/u = 0$. The real images for $p = +1$ and the virtual images for $p = -1$ were observed by aid of a telescope. The real and virtual horizontal focal lines did not change their positions when θ was increased from zero. Since b, the distance of the vertical focal line from the zone plate is positive or negative with p, the product bp is always positive. The values of bp given for $\theta = 15°$ are the means of those found for $\theta = 15°$ and $\theta = -15°$, and similarly for the other angles. By (2), since $1/u = 0$, $bp \sec^2 \theta$ is constant.

* The appliance (Fig. 5) for rotating the wave front may be used with advantage in the experiment described in § 10 of "Experiments with a plane diffraction grating" (*Proc. Camb. Phil. Soc.* Vol. xx. p. 105). In the method there described, the angle between the vertical and the *cross-wire* is erroneously taken to be ψ_1, whereas, in the theory, the angle between the vertical and the generators of the cylindrical surface of the lens is ψ_1. These two angles are not generally identical. In the actual example given, the difference was small. When the experiment was done, I had not the appliance shown in Fig. 5 above.

θ	Real image bp	Virtual image bp	Mean	$bp\sec^2\theta$
0°	65·96 cm.	65·84 cm.	65·90 cm.	65·90 cm.
15°	62·02	61·26	61·64	66·06
30°	50·38	48·94	49·66	66·21
45°	33·80	32·69	33·24	66·48
60°	16·71	16·18	16·44	65·76

The mean value of $bp\sec^2\theta$ is 66·08 cm., and the corresponding "power" of the zone plate for images of the first order is

$$F_1 = 1/66{\cdot}08 \text{ cm.}^{-1} = {\cdot}015133 \text{ cm.}^{-1} = 1{\cdot}5133 \text{ dioptre.}$$

§ 8. *Second practical example.* The results of § 5 were tested in these experiments.

The lens L (Fig. 5) was built up of a spherical lens of 2·5 dioptres in power (2·5 D) and a plano-cylindrical lens of powers 0 and 1 D. The axis of the cylindrical surface will be called the axis of L. When this axis is vertical, $\psi_1 = 0$. In each set of observations, θ was made 45° and −45°, and the mean results for the two positions are given. In the first set, the slit was vertical when $\psi_1 = 0$. Then ψ_1 was gradually changed from 0 to $\frac{1}{2}\pi$, and at the same time the direction of the slit and the position of the telescope were changed so as to keep the slit in view throughout. The image of the slit, formed by focal lines, was very sharp if the direction of the slit was very carefully adjusted. The table records B_2, the *reciprocal* of the distance of the focal line from the zone plate, expressed in dioptres. Thus the reciprocal of 40 cm. would appear as 2·5 D.

The axis of L was again made vertical and the slit was made horizontal. Then ψ_1 was changed step by step from 0 to $\frac{1}{2}\pi$. The reciprocal of the distance of this focal line from the zone plate is C_2.

In each case the virtual focal line of the first order was used, and thus B_2 and C_2 were both negative and $p = -1$. From the values given in the table we have, in dioptres,

$$B_2{}^0 = -2{\cdot}9976, \quad B_2{}^{\pi/2} = -3{\cdot}9002, \quad C_2{}^0 = -2{\cdot}4201, \quad C_2{}^{\pi/2} = -1{\cdot}5006.$$

Since (28) gives two expressions for X, $F_p\tan^2\theta$ and $B_1 - C_1$, we take the mean in each case. Thus

$$X = -\tfrac{1}{4}(5{\cdot}4177 + 5{\cdot}4008) = -2{\cdot}7046,$$

$$F_p\tan^2\theta = -\tfrac{1}{2}(1{\cdot}4970 + 1{\cdot}4801) = -1{\cdot}4886,$$

$$B_1 - C_1 = \tfrac{1}{2}({\cdot}9026 + {\cdot}9195) = {\cdot}9110.$$

Hence $Y^0 = -0{\cdot}2888$, $Y^{\pi/2} = -1{\cdot}1998$, and Y is, therefore, taken as negative for all values of ψ_1. Using the values of $F_p\tan^2\theta$ and of $B_1 - C_1$ just found, Y was calculated for $\psi_1 = 15°, \ldots$ by (25). There is good agreement between $X + Y$ and the observed value of B_2 and between $X - Y$ and the observed value of C_2.

The distance OL was approximately 10 cm.

$$X = -2{\cdot}7046\ D.$$

ψ_1	B_2 obsd.	C_2 obsd.	Y	$X+Y$	$X-Y$
°	D	D	D	D	D
0	−2·9976	−2·4201	− ·2888	−2·9934	−2·4158
15	−3·1397	−2·2594	− ·4174	−3·1220	−2·2872
30	−3·3580	−2·0325	− ·6499	−3·3545	−2·0547
45	−3·6127	−1·8037	− ·8726	−3·5772	−1·8320
60	−3·7821	−1·6375	−1·0490	−3·7536	−1·6556
75	−3·8790	−1·5347	−1·1613	−3·8659	−1·5433
90	−3·9002	−1·5006	−1·1998	−3·9044	−1·5048

§ 9. *Third practical example.* In this experiment the directions of the focal lines were observed.

The apparatus was the same as in § 8, and θ was 45°. For each value of ψ_1, the direction of the slit was adjusted so that the image formed by focal lines was as sharp as possible, and the cross-wire of the telescope was set parallel to the edges of the image. The setting of the cross-wire thus depends upon the accuracy of adjustment of the slit, for a change of direction of the slit changes the direction of the "image," in addition to changing the sharpness of its edges. The zone plate carriage was then moved aside on its slide R (Fig. 5) and the telescope was moved back, until a fine wire stretched across the face of L was in focus. The reading of the circle V, when this wire was vertical, was known. The tube M was turned so that this wire was parallel to the cross-wire of the telescope; the direction of the cross-wire was then given by the reading of V. In each case the focal line corresponding to $p = -1$ was used; this focal line was virtual.

Two sets of measurements were made. In the first set, when $\psi_1 = 0$, the slit was vertical. Then ψ_1 was changed by steps of 15° to 90°, and the image, which corresponds to B_2, was followed up. The inclination of this image to the vertical is ψ_2. In the second set, when $\psi_1 = 0$, the slit was horizontal, and this image corresponds to C_2. The inclination of this image to the horizontal is also ψ_2. The table below gives the two values found for ψ_2 and their mean. In calculating ψ_2 from (29), we take $B_1 - C_1 = 0{\cdot}9110$, as found in § 8, and use Y as given in the table in § 8. Since Y is negative when $p = -1$, and since $B_1 - C_1$ is positive, it follows that ψ_2 is negative when ψ_1 is positive.

ψ_1	ψ_2 B_2 image	ψ_2 C_2 image	ψ_2 Mean obsd.	ψ_2 Calcd.
0°	0° 0′	0° 0′	0° 0′	0° 0′
15	−14 30	−16 15	−15 22	−16 32
30	−18 30	−18 45	−18 38	−18 51
45	−12 45	−15 30	−14 8	−15 44
60	−10 0	−11 30	−10 45	−11 3
75	− 5 23	− 5 18	− 5 20	− 5 39
90	0 0	0 0	0 0	0 0

Since $F_p \tan^2 \theta = 1{\cdot}4886$, when $p = +1$ and $\theta = 45°$, and since $B_1 - C_1 = {\cdot}9110$, $Y_0{}^0$ and $Y^{\pi/2}$ are positive, and thus ψ_2 has the same sign as ψ_1. Using the *real* focal lines corresponding to $p = 1$, we verified that ψ_2/ψ_1 is *positive.*

§ 10. *Fourth practical example.* In these experiments a true image of the object at K (Fig. 5) was produced; the direction of the axis of the zone plate was so adjusted that the astigmatism due to its obliquity balanced the astigmatism due to the lens.

The mean value found for F_1 in § 7 is 1·5133 dioptre. Hence $F_1 (1 + \sec^2 \theta)$ is at least equal to $3{\cdot}0266D$ and $F_2 (1 + \sec^2 \theta)$ and $F_3 (1 + \sec^2 \theta)$ are still larger. Since $B_1 - C_1 = {\cdot}9110$, X and p are positive or negative together.

In the first set of observations, $\psi_1 = 0$. The slit was made horizontal and, with $\theta = 0$, the telescope was focussed on an image for which $p = -1$, -2 or -3; each image was virtual. This image is in the same place as the image which is formed by horizontal focal lines when θ is changed from zero. The slit is then made approximately vertical and the zone plate is turned until θ is such that the image is again in focus. The image is then in focus for all directions of the slit.

In the second set of observations, $\psi_1 = \frac{1}{2}\pi$, and the images for which $p = 1$, 2 or 3 were used; these images were real.

The calculated values of θ were found from (30), using $F_1 = 1{\cdot}5133$ and $B_1 - C_1 = {\cdot}9110$ dioptre.

Real images		Virtual images		Mean obsd.	Calcd.
Order	Obsd.	Order	Obsd.		
p	θ	p	θ	θ	θ
1	37° 45′	−1	37° 30′	37° 38′	37° 48′
2	30 30	−2	28 30	29 30	28 45
3	23 45	−3	22 30	23 8	24 8

Corrections in Papers by Dr G. F. C. Searle.

(i) "A bifilar method of measuring the rigidity of wires" (*Proc. Camb. Phil. Soc.* Vol. XX. p. 61). Equation (10) on p. 67 should read

$$\theta = \frac{C}{1+C}(\phi + \eta) + \frac{\epsilon}{1+C} = D(\phi + \eta) + \frac{\epsilon}{1+C}.$$

When this correction is made, the quantity C remains unchanged, but the values given for P and Q in (18) and (19) must be multiplied by $1 + C$. In the practical example (p. 69) P is changed from $-{\cdot}0028$ to $-{\cdot}0029$ and Q from $-{\cdot}0045$ to $-{\cdot}0046$. There will be corresponding small changes in the last column of the table of results on p. 68.

(ii) "Experiments with a plane diffraction grating" (*Proc. Camb. Phil. Soc.* Vol. XX. p. 88). An error affecting §§ 10, 12 of this paper is noticed in a footnote to § 6 of "Experiments on focal lines formed by a zone plate" (*Proc. Camb. Phil. Soc.* Vol. XX. p. 340).

The Tensor Form of the Equations of Viscous Motion. By
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In the general theory of relativity the Principle of Equivalence asserts that all laws relating to phenomena in a geometrical field of force which depend on the g's and their first derivatives only will also hold in a permanent gravitational field. Eddington comments on this* that "it would be quite consistent with the general idea of relativity if the true expression of such laws involved the Riemann-Christoffel tensor, which vanishes in the artificial field and would have to be replaced before the equations were applied to the gravitational field. But were we to admit that, the principle of equivalence would become absolutely useless." The following example from three dimensions illustrates the significance of this point by analogy.

The equations of motion of a viscous fluid in terms of the velocities may be obtained in tensor form either by generalising the corresponding Cartesian equations, or by first generalising the equations of motion involving the pressures and then substituting for the pressures in terms of the velocities. The two forms are found to differ by a term involving $G_{\mu\nu}$, which is of course zero in Galilean space, so that the two forms are in fact equivalent. The explicit emergence of $G_{\mu\nu}$ in such a simple case is however interesting; although there would in any case be no field for the application of an analogue of the principle of equivalence since the second derivatives of the g's are elsewhere involved in both forms.

The stress system $(p_{xx}, p_{xy}, \ldots)$ in rectangular three-dimensional co-ordinates is a symmetrical tensor of the second rank. The precise generalised definition of p_{xx}, etc., in general co-ordinates is to some extent arbitrary; let us assume they are defined so as to constitute a contravariant tensor $P^{\mu\nu}$. It is to be noted that in this case the contravariant vector expressing the force across the element of surface dS is

$$\tfrac{1}{6}\epsilon_{\nu\rho\sigma} P^{\mu\nu} dS^{\rho\sigma}; \qquad \ldots\ldots(1)$$

here the element of surface is represented by the antisymmetrical contravariant tensor $dS^{\rho\sigma}$ (such that $dS^{\rho\sigma}=0$ when $\rho=\sigma$ and $dS^{\rho\sigma} = -dS^{\sigma\rho}$ when $\rho \neq \sigma$), and $\epsilon_{\nu\rho\sigma}$ denotes the covariant tensor†

* "Report on the Relativity Theory of Gravitation," *Phys. Soc. Lond.* (1918), p. 43.

† For the properties of the ϵ-tensors see J. E. Wright, "Invariants of quadratic differential forms," *Camb. Math. Tracts*, No. 9, p. 21. They may be used for converting a tensor of rank p into one of rank $|n-p|$, n being the number of dimensions; for example, by the use of them it is easy to show how it is that antisymmetrical tensors of the second rank in three dimensions (such as the vector product) degenerate into vectors.

of the third rank whose components are zero when any two of the suffixes are equal, and equal to $\pm\sqrt{g}$ when all three are unequal, the sign being given by the parity of the number of inversions in ν, ρ, σ. If x_1, x_2, x_3 are the co-ordinates, and if the element of surface is denoted in the more usual way by dS_1, dS_2, dS_3, then the x_1-component of (1) reduces to

$$\sqrt{g}\,(P^{11}dS_1 + P^{12}dS_2 + P^{13}dS_3).$$

The "mean pressure" p is the invariant $\frac{1}{3}g_{\mu\nu}P^{\mu\nu}$.

If k is the coefficient of viscosity, the usual equations* for the pressures

$$\left.\begin{aligned} p_{xx} &= -p - \tfrac{2}{3}k\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2k\frac{\partial u}{\partial x} \\ p_{xy} &= p_{yx} = k\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \end{aligned}\right\} \qquad \ldots\ldots(2)$$

generalise into

$$P^{\mu\nu} = g^{\mu\nu}\left[-p - \tfrac{2}{3}k\,(u^\tau)_\sigma\right] + k\left[g^{\nu\tau}(u^\mu)_\sigma + g^{\mu\sigma}(u^\nu)_\sigma\right] \ldots(3)$$

the notation $(u^\mu)_\sigma$ denoting the covariant derivative; and the equations of motion in terms of the pressures†

$$\rho\frac{Du}{Dt} = \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z} \qquad \ldots\ldots(4)$$

generalise into
$$\rho\left(\frac{Du^\mu}{Dt} - X^\mu\right) = (P^{\mu\nu})_\nu \qquad \ldots\ldots(5)$$

where the generalisation of "differentiation following the motion" is given by

$$\frac{Du^\mu}{Dt} = \frac{\partial u^\mu}{\partial t} + u^\rho\,(u^\mu)_\rho. \qquad \ldots\ldots(6)$$

Substituting for $P^{\mu\nu}$ in (5) and remembering that the covariant derivatives of the g's are identically zero we find

$$\begin{aligned} \rho\left(\frac{Du^\mu}{Dt} - X^\mu\right) &= -g^{\mu\nu}\frac{\partial}{\partial x_\nu}\left(p + \tfrac{2}{3}k\,(u^\tau)_\sigma\right) + k\left[g^{\nu\sigma}(u^\mu)_{\sigma\nu} + g^{\mu\sigma}(u^\nu)_{\sigma\nu}\right] \\ &= -g^{\mu\nu}\frac{\partial}{\partial x_\nu}\left(p - \tfrac{1}{3}k\,(u^\tau)_\sigma\right) + kg^{\nu\tau}(u^\mu)_{\sigma\nu} + kg^{\mu\sigma}\left[(u^\nu)_{\sigma\nu} - (u^\nu)_{\nu\sigma}\right], \end{aligned} \qquad \ldots\ldots(7)$$

where in the last term ν and σ have been interchanged as being dummies. These are the differential equations satisfied by the velocities.

On the other hand the usual equations for the velocities obtained by combining (2) and (4) directly, namely,

$$\rho\frac{Du}{Dt} = \rho X - \frac{\partial p}{\partial x} + \tfrac{1}{3}k\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + k\nabla^2 u \quad \ldots(8)$$

* Lamb, *Hydrodynamics*, fourth edition, p. 570. † *Loc. cit.* p. 572.

generalise into

$$\rho\left(\frac{Du^{\mu}}{Dt} - X\right) = - g^{\mu\nu}\frac{\partial}{\partial x_{\nu}}\left(p - \tfrac{1}{3}k\,(u^{\sigma})_{\sigma}\right) + kg^{\tau\nu}\,(u^{\mu})_{\sigma\nu} \quad \text{...(9)}$$

which differ from (7) by the absence of the term

$$kg^{\mu\sigma}\left[(u^{\nu})_{\sigma\nu} - (u^{\nu})_{\nu\sigma}\right]. \qquad \text{......(10)}$$

Now the Riemann-Christoffel tensor $B^{\rho}{}_{\mu\nu\sigma}$ is conveniently defined by the identity

$$(A_{\mu})_{\nu\sigma} - (A_{\mu})_{\sigma\nu} = A_{\rho}\,B^{\rho}{}_{\mu\nu\sigma},$$

where A_{μ} is any covariant vector; but it is easy to prove also that if A^{μ} is any contravariant vector, then

$$(A^{\mu})_{\nu\sigma} - (A^{\mu})_{\sigma\nu} = - A^{\rho}\,B^{\mu}{}_{\rho\nu\sigma}.$$

Contracting this by putting $\sigma = \mu$ and summing, we have

$$(A^{\mu})_{\nu\mu} - (A^{\mu})_{\mu\nu} = - A^{\rho}\,G_{\rho\nu}.$$

It follows that the term in question, (10), is simply

$$- kg^{\mu\sigma}\,G_{\rho\sigma}\,u^{\rho},$$

which vanishes, the space being Galilean. Were one attempting, however, to discuss viscous motion in non-Galilean space, with the generalisations of (2) and (4) as a dynamical basis, one would be led to an incorrect result by hastily generalising (8), although this is merely a combination of (2) and (4); and the interest lies in the circumstance that it is precisely the contracted Riemann-Christoffel tensor that appears as an error term.

Insect Oases. By C. G. LAMB, M.A.

[*Read* 7 March 1921.]

Cases of extremely limited distribution are familiar to all collectors, the limitation being sometimes so great as to amount to a single tree or a few square yards of ground, but it is nearly always possible to correlate the distribution with the presence of the necessary pabulum or with the environment. The establishment of a satisfactory case of a persistent isolated colony requires not only a careful search of the locality in respect to space, but observation over a considerable period of time in order to eliminate possible secular disturbances. It happens that the author has visited a particular locality at the same period of the year for many years and has investigated it with much care, so that the required time and space conditions may be taken to be well satisfied. Further, the part of the district to be dealt with has the additional advantage of being singularly homogeneous in its flora. It consists of a tract of "towans" or sandy waste in the parish of St Merryn, N. Cornwall, known as "Constantine Commons": this waste is of fair extent and is characterised by great uniformity in its flora which includes an exceptional number of the Boraginaceae, *Echium* at times forming a perfectly astonishing spectacle: it also bears the spotted hemlock, the henbane and the opium poppy in fair plenty. The subsoil is clay, so that there is always permanent water in parts, and it is intersected in places with ancient slate slab walls bearing very old tamarisk bushes which indeed form the only shelter against the gales. The district is full of archaeological interest and would repay investigation, and the neighbourhood has yielded several other insects of much interest, though not exhibiting the localised habit of those to be considered. In this perfectly homogeneous area, and in places which careful investigation shows to be in no way different from the surroundings, certain species of Diptera appear to inhabit quite definite oases or islands. One may proceed to such a spot at the proper season with the certainty that the insect will be found, and that in fair or even great numbers, while the rest of the area may yield one or two specimens at most, and that with extreme rarity. In spite of much investigation the author has not been able to find any circumstances whatever correlated to the distribution in the examples given below: whether it be due to a "herd" instinct or not is of course unanswerable: it is more probable that they are cases of approaching extinction. It would be of much interest to have similar cases recorded, but unfortunately the observations necessitate a very long and close acquaintance with a given district. All the insects mentioned below are quite capable of flight, but are usually very sedentary in habit, being only obtainable by "sweeping."

Lucina fasciata Meigen. This South European species was first recorded many years ago from Ireland, and was rediscovered in England on the coast near Weston, Somerset, by the author. It occurs very sporadically in the locality under consideration, but in one very restricted area it is quite common, the boundary of its distribution being reasonably definite, though no observable physical boundary is to be seen.

Oxyna flavipennis Loew. This species is apparently extremely rare although it is said to breed in *Achillea millifolium*; the late Mr G. H. Verrall in 40 years of most careful investigation only found two specimens. In the present locality it occurs in great numbers on the sheltered side of one out of many of the tamarisk hedges that intersect the commons, but is practically restricted to some 40 yards of the hedge.

Aphaniosoma quadrinotatum Becker. This is a species first described from the Canary Islands and afterwards found in Spain. It is confined to a similarly restricted spot associated with *Matricaria*, which is, however, quite abundant elsewhere where the insect is absent.

Syntormon mikii Strobl. This is another Spanish species which occurs only in a very small marshy hollow near the commons. That same hollow produces three other species of the genus, namely *pumilus*, *monilis* and the ubiquitous *pallipes*.

A second type of homogeneous locality is afforded by the spots where small streamlets run on to the sands of the various bays that break the coast line: these localities are all practically alike in character and flora. On the banks of one of the streams, and in an area of but a few square feet, occurred our last example.

Ochthera mantispa Loew, an insect first described from Rhodes, and found round the Mediterranean littoral. This singularly isolated colony persisted for some years, but is now probably extinct owing to disturbance of the natural conditions by visitors to the bay. In spite of assiduous search no other colony has been found.

It will be noticed that all the species referred to, with the exception of the *Oxyna*, are of southern distribution. Other southern insects have been found in the district, but are not so striking in their isolation. Two other rare flies are also found there, and in somewhat similar isolation, namely the dolichopids *Dolichopus signifer* and *Acropsilus niger*, but these are so far only known to be of Central European distribution, though the occurrence of the former in Ireland many years ago makes it possible that it is another of the former class.

The most probable reason for the extremely restricted distribution of these insects is that we are in the presence of the last stage of the extinction of a species. With creatures of such comparatively sedentary dispositions, once an island is formed it has little chance of increase. There must be a minimum population density which is such that when the density approaches that minimum the chance of reproduction is practically zero; as there is no evidence that any of the species have sex attracting powers, this minimum density might well be attained at a comparatively small distance from the centre of the colony.

A Note on the Hydrogen Ion Concentration of some Natural Waters. By J. T. SAUNDERS, M.A., Christ's College.

[*Read* 7 March 1921.]

The natural waters referred to in this preliminary note are all fresh and all occur in districts where the soil or sub-soil contains chalk, gault or lime in some form or other. In such districts the natural waters have a fairly constant hydrogen ion concentration. When the water issues from the ground as in the case of springs and wells the P_H is found to vary only within the limits 7·1–7·2. The following table shows the values of P_H for water issuing from the ground:

Locality	P_H	Date
Well at Cherryhinton, Cambs....	7·1	March 1921
Springs at Shelford ("Nine Wells"), Cambs....	7·2	,, 1921
Siphon spring at Warlingham, Surrey ...	7·2	,, 1919
Cambridge tap water (supplied from wells)...	7·1	various dates

These four cases are all very different yet the P_H of the water is remarkably constant. The well at Cherryhinton is 8 ft. deep with 2 ft. of water at the bottom. The water comes through the gault. At Shelford the "Nine Wells" are a number of springs bubbling up through fissures in the chalk. The siphon spring at Warlingham is intermittent, running freely for eight to ten weeks at intervals varying from two to seven years. Here, as at Shelford, the water comes up through fissures in the chalk. The Cambridge Town supply is derived from deep wells running through the gault to water-bearing strata.

As the water leaves the source and flows along in the stream that arises from the spring the value of P_H increases gradually until it reaches a value varying only within the limits 8·25–8·5, at which value it remains constant. The siphon spring water at Warlingham flowed along a wide grassy ditch. Within half-a-mile of the source the P_H of the water had risen to 8·4 at which value it remained constant for, at any rate, another mile, when I was unable to trace it further. The water flowing from the "Nine Wells" at Shelford behaves in much the same fashion, but here I was able to trace the water further and found that the P_H, after reaching the value 8·3 within half-a-mile of the source, fell again at a mile and a half from the source to 8·05. This lowering of the P_H is undoubtedly due to the stream, which has now become almost sluggish, mixing with its waters the acid products of decomposition from the bottom. At first the stream flows over clean ground, there is no débris and the bottom is swept clean by the force of running water. But further on the stream expands, its pace slackens and débris accumulates at the bottom. In rivers and streams, however sluggish, there is sufficient stream to mix the waters thoroughly and to bring up the acid waters from the bottom. Hence the P_H of rivers and streams is variable and not constant, depending on the pace of the stream and the amount of disturbance of the bottom.

In ponds and lakes that are large and deep, no disturbance of the bottom will occur and there is no general mixing of the waters. Here again we find the P_H to be constant within the limits 8·25–8·5 as the following table shows:

Locality	P_H	Date
Upton Broad, Norfolk	8·4	March, April 1919
S. Walsham Broad, Norfolk	8·35	„ „
Railway Ballast Pits, Chesterton, Cambridge...	8·4	August 1920
Large Brick Pit, Cambridge	8·5	March 1921
„ „ Madingley	8·25	March 1921

The two Broads in Norfolk are large shallow waters not more than 6–8 ft. deep. The pits at Cambridge and Madingley are 20–30 ft. deep. The area in all cases exceeds an acre. In small shallow ponds no such constancy occurs. Here the influence of the bottom will cause a lowering of the P_H; the presence of masses of vegetation, however, may increase the P_H, for plants, during photosynthesis, extract CO_2 from bicarbonates and render the water alkaline. If, on a sunny day, water be taken from a small pond containing masses of Spirogyra the value of the P_H will be found to be as much as 9·0.

On shaking in a test tube well or spring waters from these districts where lime in some form or other is abundant in the soil or sub-soil, it is found that P_H rises to a value varying from 8·25–8·5 and remains constant within these limits. Bubbling air through the water produces the same effect. If the effects of decomposition and photosynthesis be avoided it will be found that the P_H of waters in these districts when saturated with air in solution has a value that is constant within the limits 8·25–8·5.

The method used for the determination of the value of P_H has been that developed by Clark and Lubs.

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Osterhout and Haas. "The Dynamic of Photosynthesis." *J. Gen. Physiol.* 1918, I. 1.

The Mechanism of Ciliary Movement. By J. GRAY, M.A., Balfour Student, and Fellow of King's College, Cambridge.

[*Read* 7 March 1921.]

Ciliary movement is a well-known phenomenon in the animal kingdom, and although there is general agreement concerning the morphological structure of the ciliary apparatus, yet our knowledge of its physiology is small. Most of the theories which have been advanced have been based upon the morphological structure of the cilium, and different authors have attributed different functions to the various structures which become visible in stained preparations. The present paper is a preliminary account of an attempt to throw light on this vexed problem by the application of experimental methods.

The material used has been the gills of *Mytilus edulis*, which, as explained elsewhere*, form an admirable subject for study. The main cilia on the gills can be divided into three classes (*a*) the frontal and terminal cilia, (*b*) the lateral cilia, (*c*) the latero-frontal cilia. Of these groups, the first two resemble each other in the fact that they perform work by creating a current of water or bring about the active transport of food and mucus. The function of the latero-frontal cilia is rather difficult to determine, but it seems certain that they do not perform work as do the other types of cilium. These latero-frontal cilia may either serve to keep the filaments of the gill apart, or to direct the water currents formed by the lateral cilia on to the face of the gill.

There is no doubt that the large latero-frontal cilia are composed of a number of cilia or separate fibres fused together. The frontal cilia also have probably the same constitution. There is, further, a wealth of morphological evidence to show that in a very large number of ciliated cells, the cilium itself is composed of fibres fused together, or that part of essential ciliary apparatus is fibrous. The work-performing cilia are remarkably efficient, and are capable of producing a rapid flow of water or food particles. All the cilia possess a very considerable degree of elasticity, since, when deformed by any external agent, they regain their normal shape when the disturbing element is removed.

A considerable number of workers have taken cognisance of the fact that cilia are essentially elastic bodies, but it is of interest to consider the problem afresh.

* J. Gray, *Quart. Journ. Micros. Science*, vol. 64, 1920, p. 345.

Consider a simple strip of steel wire ABC attached at one end C (Fig. 1). If a stress is applied to the wire so as to distort it to A_1B_1C, then it is clear that in passing from the position of rest ABC to its new position A_1B_1C, a considerable part of the energy used to disturb the wire is stored in the wire, in other words when in the position A_1B_1C the strip of wire possesses a definite amount of potential energy. If we wish to make the wire do work in the direction from A to A_1, it is obvious that such a mechanism would be extremely inefficient since the whole of the potential energy taken up by the wire itself would be unavailable for work. Consider now the same wire distorted to abC. Again potential energy is stored. On releasing the wire *the whole* of this energy is set free and is available for work.

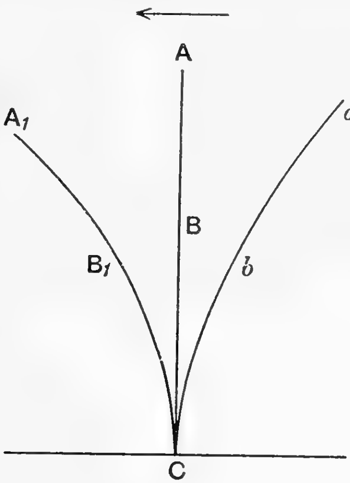


Fig. 1.

Heidenhain and numerous other workers have regarded cilia as comparable to the wire which performs work in moving from A to A_1. We therefore reach the paradoxical conclusion that cilia perform a surprising amount of work*, and yet are exceedingly inefficient machines.

Let us now consider the actual movement of the frontal or terminal cilia on the gills of *Mytilus*.

Under normal conditions the rate of beat of these cilia is so great, that it is only possible to observe the movement in detail when the rate of beat is reduced by the addition of some viscous but non-toxic substance (*e.g.* gum arabic) to the external medium.

During the forward or effective beat (Fig. 2) the cilium behaves as

* Bowditch calculated that each cell of the mucous membrane of the Frog is capable of lifting its own weight to a height of 14 feet in 1 min.

an elastic rod which moves forward from a pivot at its base, and which exposes the maximum amount of surface to the water. During the recovery stroke (Fig. 3), the movement is slower, and the shape of the cilium is different. The cilium is drawn back as a limp string or piece of unstretched rubber in which a stress is set up which starts at the base and is transmitted to the free end. The path followed by the cilium is essentially the same as that of a fishing line during the backward movement of the cast, whereas during the forward effective beat the cilium resembles a stretched spring which is suddenly released. The essential difference is that during the forward effective stroke the cilium is expending energy in the form

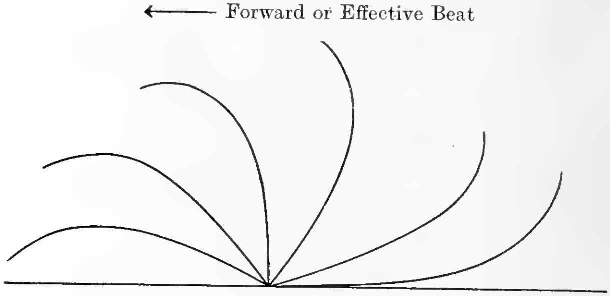


Fig. 2.

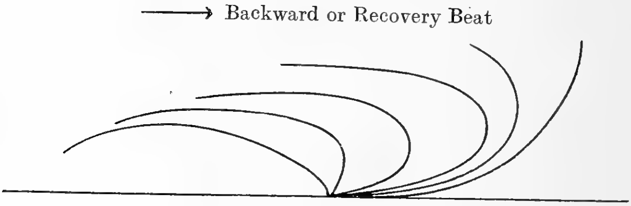


Fig. 3.

of work, whereas during the recovery stroke the cilium is storing potential energy and is performing a minimum of external work during its change of position.

The simplest conception of such a mechanism is to regard the cilium at the end of its effective stroke as a relaxed elastic body. The recovery stroke is brought about by the setting up of a tension in this body: the tension begins at the base of the cilium and passes on to its apex. The direction of the tension is approximately along the line which the cilium occupies at the end of the recovery stroke. In other words the position at the end of the recovery stroke represents the equilibrium between the force which is distorting the cilium and its own elasticity. If the force applied be removed, the potential energy stored in the cilium will be released, and the cilium will fly forward: in doing so the potential energy will be expended in the form of work done on the water.

There are, however, certain cilia (*e.g.* in Ctenophores) which differ from the cilia of *Mytilus* in that they can be stopped in the position normally occupied at the end of the recovery beat. We must regard such cilia as possessing potential energy when at rest, in the same way as a stretched spring which is held back by a mechanical catch. Such cilia appear to resemble muscle fibres very closely.

Now we may enquire the origin of the potential energy stored in the cilium during the recovery stroke. It must be derived from the chemical energy either in the cilium itself or in some other part of the cell. The problem, therefore, narrows itself to a determination of the possible methods whereby chemical energy can be converted into kinetic energy by means of a fibrous and elastic mechanism. At this point the similarity of the ciliary apparatus to a muscle fibre becomes obvious, and it is convenient to summarize the mechanism of the muscle fibre as analysed by A. V. Hill*.

When at rest a muscle fibre may represent a stretched elastic body which possesses potential energy stored as tension energy in some substance *A* (which may be inactive connective tissue fibres). Within the muscle, however, are certain fibres *B* which are capable of developing a tension when in contact with some chemical substance which is liberated at the time of excitation. We can regard this chemical substance as lactic acid, so that the muscle fibres are capable of developing a tension (just as a piece of catgut develops a tension) when in contact with lactic acid—by the absorption of water. In the resting muscle the substance *A* is kept stretched possibly by the osmotic properties of the liquid in its interstices. When stimulated, lactic acid is set free from some carbohydrate compound, the fibres *B* take up the energy thus set free and develop a tension by taking up water from the interstices of the substance *A*. Consequently the muscle contracts and utilises the energy in both *A* and *B*. After shortening the lactic acid diffuses away, *i.e.*, is removed from the fibres *B*. They therefore give up their water which passes back into the interstices of *A*, and the muscle lengthens. It is clear that there is no *a priori* objection to applying such a hypothesis to the cilia of Ctenophores—but as far as these cilia are concerned it is a hypothesis and nothing more.

In the case of the cilia of *Mytilus* we can analyse the system as follows. At the end of the effective stroke a tension is set up in the fibres by the liberation of some chemical substance (which may be an acid) from the interior of the cell. The equilibrium between this tension and the elasticity of the cilium draws the latter to the position occupied at the end of the recovery stroke, and in doing so

* A. V. Hill and W. Hartree, *Phil. Trans. Roy. Soc.* vol. 210 B, 1920, p. 153.

some of the tension energy of the fibres is stored in the cilium as a whole. The supply of the chemically active substance now ceases, and that portion already located in the fibres diffuses away, so that the cilium flies forward by virtue of its stored potential energy.

In order to test this hypothesis the first obvious line of enquiry is to locate the essential parts of the ciliary mechanism. This problem cannot now be discussed at length: it may suffice to say that the cilium is not of itself automatic; when separated from the cell it does not move. The essential portions of the mechanism lie towards the free edge of the cell. The nucleus does not appear to play an essential rôle.

Owing to the small size of ciliated cells, it is impossible to analyse the movement by such mechanical methods as are applicable to a muscle; ciliated cells have, however, one great advantage in that each individual cell can be observed. Thus, when the amplitude of the contraction of the heart is gradually abolished, it is impossible to say whether this is due to the partial reduction of the contraction of all the cells, or to the total abolition of contraction in some cells, while the amplitude of the others remains unaltered. In the case of cilia this difficulty does not exist, as the beat of a single cilium can be observed throughout the whole experiment.

During the present work an attempt has been made to analyse ciliary movement by a determination of possible means whereby the movement can be influenced or abolished in a reversible manner, *i.e.* without serious derangement of the mechanism itself.

1. *The effect of acids and alkalis.*

As shown in a previous publication* cilia are extremely sensitive to acids. A certain degree of acidity in the external medium causes a cessation of movement. There is no reduction in the amplitude of the beat: the speed of both the effective and recovery beats becomes gradually slower, often with prolonged pauses at the end of each stroke. Eventually the cilia come to rest at the end of the effective stroke. Since there is no diminution in the actual amount of contraction, we are forced to conclude that the cilia do not stop because the actual contractile or elastic mechanisms are deranged but because the rate at which the transformation of chemical into potential energy is gradually reduced and finally ceases. Further, the rate at which potential energy is converted into kinetic energy is also reduced. This conclusion is confirmed by the study of cells in which we know that the amount of convertible chemical energy is small, *viz.*, spermatozoa. The work of Cohn† has shown that in

* J. Gray, *Quart. Journ. Micros. Science*, vol. 64, 1920, p. 345.
† E. J. Cohn, *Biological Bulletin*, vol. 34, 1918, p. 167.

such cells the cessation of movement in an acid solution involves no wastage of energy: the acid simply prevents the chemical energy being converted into kinetic energy.

It is important to notice that the efficiency of acids to cause cessation of movement depends upon the ease with which they penetrate the cell surface, so we may conclude that the liberation of chemical energy into potential energy takes place inside the cell, and not at its surface.

The effect of all moderate strengths of acid (*i.e.* those strengths which rapidly stop movement but do not kill the cell) is entirely reversible by means of alkalis. Again, we find that the efficiency of alkalis depends upon the ease with which they penetrate into the cell, *e.g.* ammonia is much more efficient than the strong alkalis. Within fairly wide limits the rate of ciliary movement depends upon the alkalinity of some area with the cell—from about $P_H 5$ to $P_H 10$ there is a progressive increase in the rate of beat.

The reduction in the rate of transformation of chemical energy into potential energy, by an increase in the acidity of the cell interior, is in accord with Kondo's* investigations on the rate of production of lactic acid from muscle extract. The production of lactic acid from muscle extract is a self-limited reaction which is checked by the formation of the lactic acid—or by another acid in the medium. The fact that the effective beat of a cilium is slowed when the cell interior is more acid than normal is clearly explicable if we assume that the liberation of the potential energy into kinetic energy is dependent upon the rate at which an acid, like lactic acid, diffuses away from some special structure or fibre—the more acid the cell or medium the slower will the potential energy be liberated.

It is clear that the observed facts so far described, place no obstacle in the way of accepting our provisional hypothesis. There is, however, one point which should be mentioned. If potential energy can be stored in the cilium by the liberation of lactic acid at the surface of certain fibres, then one would expect that if lactic acid or any other acid is used as an experimental means of stopping ciliary action, the fibres should remain in the shortened condition, and not in the relaxed state. If the concentration of the experimental acid is raised above the minimum value to cause stoppage of movement, it is true that the cilia move away from the fully relaxed position, but they never approach the end of the recovery beat. The same phenomenon occurs with muscle fibres which are rendered inexcitable by acid. As pointed out by Mines†, however, the shortening of the muscle fibre (or the distortion of a cilium) is brought about by a *local* concentration of acid at the surface of

* K. Kondo, *Biochem. Zeit.* vol. 45, 1912, p. 63.
† G. R. Mines, *Journ. of Phys.* vol. 46, 1913, p. 188.

the fibre, and so the degree of tension set up in the fibres may be due to the *difference* in the concentration of hydrogen ions at the surface of the fibres and in some other part of the cell.

On the other hand, it is possible that if the acidity of the whole cell is raised to the value which normally only exists at the surface of the fibres, there may be a breakdown in the whole colloidal structure of the cell.

2. *The effect of metallic ions.*

It has already been shown* that an artificial solution containing NaCl, KCl, $MgCl_2$, and $CaCl_2$ whose P_H is about 7·8 forms a satisfactory medium for ciliary activity. If a similar solution be prepared in which KCl is omitted the different cilia on the gill react in different ways. The frontal cilia and terminal cilia are practically unaffected within two or three hours; the lateral cilia, however, quickly stop. On adding KCl to the solution they rapidly recover. Recovery can also be brought about by making the solution slightly more alkaline. It is interesting to note that the lateral cilia beat in a definite rhythm, and it is possible that potassium is necessary for this rhythm. If the concentration of potassium be increased above the normal value, the beat of these cilia is well maintained even in solutions in which the whole of the NaCl is replaced by KCl; the frontal and terminal cilia are not appreciably affected; the latero-frontal cilia however go into a state of prolonged contraction. After some time they gradually recover. The effect of potassium on the latero-frontals is antagonised by alkali. Further work is required for an elucidation of these facts, but it is interesting to note the different effects of potassium on different types of cilia.

If an artificial solution be prepared which contains all the normal constituents with the exception of $Ca^{\cdot\cdot}$, all the cilia come to rest within $1\frac{1}{2}$–2 hours. If $Ca^{\cdot\cdot}$ is added as soon as the cilia have ceased to beat, complete recovery takes place: if, however, the addition of $Ca^{\cdot\cdot}$ is delayed for some time, the recovery is much less perfect, the rate of beat is slow and there is often a marked pause at the end of the recovery stroke. After stoppage in the absence of $Ca^{\cdot\cdot}$, rapid and complete recovery takes place on the addition of alkali, even when $Ca^{\cdot\cdot}$ continues to be absent. It is noticeable that the effect of the absence of $Ca^{\cdot\cdot}$ upon cilia is similar to the action of such solutions upon the heart. It is not clear whether the effect of alkali mobilises further stores of $Ca^{\cdot\cdot}$, or whether the absence of $Ca^{\cdot\cdot}$ causes a reduction in the alkalinity of the cell interior. Mines† concluded that in the absence of $Ca^{\cdot\cdot}$ the heart ceases to beat because the actual contractile mechanism is deranged, while the

* J. Gray, *Quart. Journ. Micros. Science*, vol. 64, 1920, p. 345.
† G. R. Mines, *Journ. of Phys.* vol. 46, 1913, p. 188.

supply of potential energy from chemical energy is fully maintained*. It is hoped that further work will throw more light on this problem.

The action of Mg·· is interesting, although the details of the experiments cannot be given here. It seems probable that the presence of Mg·· stabilises the cell—probably the surface of the cell—to the other ions in the medium. It regulates the rate at which other ions can enter the cell, and the rate at which intracellular ions leave the cell. In this respect it can usually be replaced by Ca··.

During the course of these experiments a fairly close parallel is visible between the effect of ions—both anions and kations—upon muscle fibres and the ciliary mechanism. There is, however, one respect in which the cilia of *Mytilus* differ from cardiac muscle—*viz.* they are remarkably insensitive to the salts of the rare earths. On the other hand spermatozoa and the cilia on the blastulae of *Echinus* are just as sensitive to these salts as is cardiac muscle. Possibly the difference depends upon the position of the sensitive surface within the cell: if it lies near the surface, the trivalent ions can reach it, whereas if it lies deeper in the cell they may never penetrate. It is interesting to notice that those cilia which are sensitive to trivalent positive ions are also more sensitive to hydrogen ions than other types of cilia.

3. *Effect of osmotic pressure.*

When cilia are exposed to any solution whose osmotic pressure is above a certain critical value, all movement ceases: the cilia remain in a position between the beginning and end of the effective beat, and are consequently very obvious. On reducing the osmotic pressure normal movement is at once resumed.

Although the details in connection with osmotic stoppage of cilia require further investigation, it may be pointed out that the known facts fit in with our initial hypothesis. The tension set up in a fibre by exposure to an acid depends upon the uptake of water: if the amount of water in the cell is reduced below a certain critical amount, it is obvious that this will affect the tension set up in the fibre, and consequently the beat is affected and at a critical point will be abolished altogether.

The general conclusion which may now be drawn is that the mechanism of ciliary movement and muscular activity may be of essentially the same nature.

* The work of other authors does not appear to agree with this conclusion.

A note on the biology of the 'Crown-Gall' fungus of Lucerne. By J. LINE, M.A., Emmanuel College.

[*Read* 7 March 1921.]

'Crown-Gall' of Lucerne has been investigated and described by several workers since the first report upon it (published) in 1898 by von Lagerheim*.

The earlier accounts merely describe the external appearance of the diseased plants, without giving any details of the fungus causing the disease. More recently (1920) two important papers by Wilson†, and Jones and Drechsler‡, have appeared describing the disease and its causative fungus, *Urophlyctis Alfalfae* (Lagerh.), P. Magnus, in great detail.

Before these papers appeared, a detailed investigation of the fungus had been for some time in progress at Cambridge, and was in fact approaching completion at the time of their publication. In view of this it has been thought desirable to publish a brief account of the work, which is in the main confirmatory of the paper by Jones and Drechsler, and like that, is in considerable disagreement with that of Wilson.

External features of the disease.

Diseased plants are found to bear wart-like masses of tissue at about the level of the soil (fig. 1). In advanced cases these may be as much as six inches across, but they are rarely found to extend more than an inch or so below the surface of the ground.

When these masses are cut across, they show characteristic dark brown areas, the spore cavities in section, among the white tissue composing the gall, giving a marbled appearance. The name 'marbled gall' has been suggested to distinguish this type of gall from true bacterial crown-gall. The disease is reported to be fairly common on Lucerne in certain areas west of the Rocky Mountains in the United States; in this country it has so far been reported from three areas only: in 1906 Salmon§ observed it in Kent; it was reported from Bedfordshire by Mr Amos (University Lecturer in Agriculture) in 1917, and was found in two fields near Cambridge in 1919 by the writer. It has again been found in Kent (1920), and in another field, adjoining the first, near Cambridge. It is probable that it is much more common than these reports would suggest,

* Lagerheim, G. (1898). *Bihang K. Svenska Vet. Akad. Handl.* Bd. 24, Afd. 3, No. 4.

† Wilson, O. T. (1920), *Bot. Gaz.* v. 70, No. 1, pp. 51–68.

‡ Jones, F. R. and Drechsler, C. (1920). *Journ. Agric. Res.* (U.S.A.), vol. 20, No. 4, p. 295.

§ Salmon, E. S. (1907). *Journ. S.E. Agric. Coll.* Wye, No. 16, p. 296.

as the galls are not easily observed until the plant is removed from the soil. A number of normal, leafy shoots may develop on the plant close to the galls, and in hot weather it is often noticed that these shoots become yellow and show signs of wilting: in this way it is possible to pick out infected plants in a field. The most convenient starting point for a description of the parasite is the resting spore and its germination. Mature spores are globular, flattened at one pole, the average dimensions being 30μ by 45μ, with an extremely brittle wall nearly 2μ thick, of a rich golden brown colour, lined with a thin colourless membrane. Only a very small percentage of the spores examined were induced to germinate; hanging drop cultures were started with different liquid media containing

Fig. 1.

spores from galls of various ages, some of which were more or less completely rotted. Some spores were treated previously with lactic acid, pepsin and other reagents which have been found to induce germination in other cases; others again were exposed to low temperatures, but consistent results were not obtained. It was however found that very slight pressure was often sufficient to start germination in certain cases, and that the most easily germinated spores were obtained from galls which had become rotted owing to the action of mould fungi (often *Fusarium* sp.) and bacteria. The development of external zoosporangia as described by C. E. Scott* was never observed; the first sign of germination being a vibratory motion of the spore contents visible through the wall. This may continue for half an hour, but as a rule the escape of zoospores begins almost at once. Irregular cracks appear in the spore wall; a portion

* Scott, C. E. (1920). *Science*, N.S., No. 1340, pp. 225, 226.

of the inner membrane is sometimes extruded, and through the cracks the zoospores, accompanied by and often entangled in, drops of oil, are seen to escape; they were found to remain active for several hours: no fusion of the zoospores was ever observed. The rapid growth of bacteria in almost all the drop cultures greatly hindered the observations on the behaviour of the zoospores.

Reaction of the host plant to infection.

The zoospores can apparently only penetrate the host at points where the tissues are relatively unprotected by either cuticle or cork. By far the most common starting points of naturally occurring galls are the numerous adventitious buds arising in continuous succession from the woody rootstock of the Lucerne plant. Many of these buds are developed some distance below the ground; they consist of a small axis and a number of leaf rudiments. The zoospores appear to penetrate between the outer scale leaves, and to enter the cells of the young leaves and of the growing point itself. Wherever penetration is effected the host is stimulated to locally increased cell division, the mass of tissue resulting bearing greater or less resemblance to the normal bud, according to the degree of infection. An extensive branching vascular system develops with the gall in direct communication with the vascular system of the host stem; the galls are thus hypertrophied buds or parts of buds. When first observable they appear as minute, white, shining projections from the rootstock or from a bud.

The cell originally entered by the zoospore could often be traced, although it is as a rule rapidly covered in by the division of the surrounding cells. The actual penetration of the host cell by the zoospore has not yet been observed.

From each point of infection the fungus spreads out radially into the host tissues, invading particularly the thin walled cells which have been developed as a result of the presence of the fungus: their contents are absorbed, and their cavities linked up by the absorption of certain of the walls to form an irregularly branching central cavity inhabited by the fungus.

Active living hyphae of the fungus are only found in the peripheral regions, the older portions of the gall being occupied by developing resting spores and degenerated mycelium. The persistent walls of all cells entered become thickened and often curiously pitted; the inner side, in contact with the fungus, appearing somewhat mucilaginous.

In sections of growing galls it is noticeable that part of the walls in the path of the hyphae disappear before the hyphae come in contact with them, but rim-like projections of these walls persist long after the cell is incorporated into the main fungal cavity.

Growth of the fungus is by no means regular in all directions, and the branching cavity formed by the invaded cells may be of any shape. Once the wall is thickened the fungus never grows through into the adjoining cells, and cell division ceases in the tissues surrounding the older parts of the cavity. It is sometimes observed that a portion of the host tissue is completely isolated by the fungus, and slowly dissolved without actual invasion of the cells composing it. In sectioned material it is not at all easy to make out the method of growth of the hyphae or of the resting spores, but if young and actively growing galls are dissected out and the tissues stained after fixing in bulk, it is possible to reconstruct the fungus thallus with great certainty.

Development of the fungus.

Figs. 2 to 7 are camera-lucida sketches from preparations made in this way. The youngest hyphae (*Hy*, figs. 3 and 5) are seen to possess a very narrow lumen and thin wall, their diameter being about $\cdot 5\mu$. They are terminated by a swollen portion (*C*, figs. 2, 5 and 6) containing rather dense protoplasm, and at first one nucleus only. This swollen portion will be referred to as the 'collecting cell,' since similar terms have been employed by other writers for analogous structures.

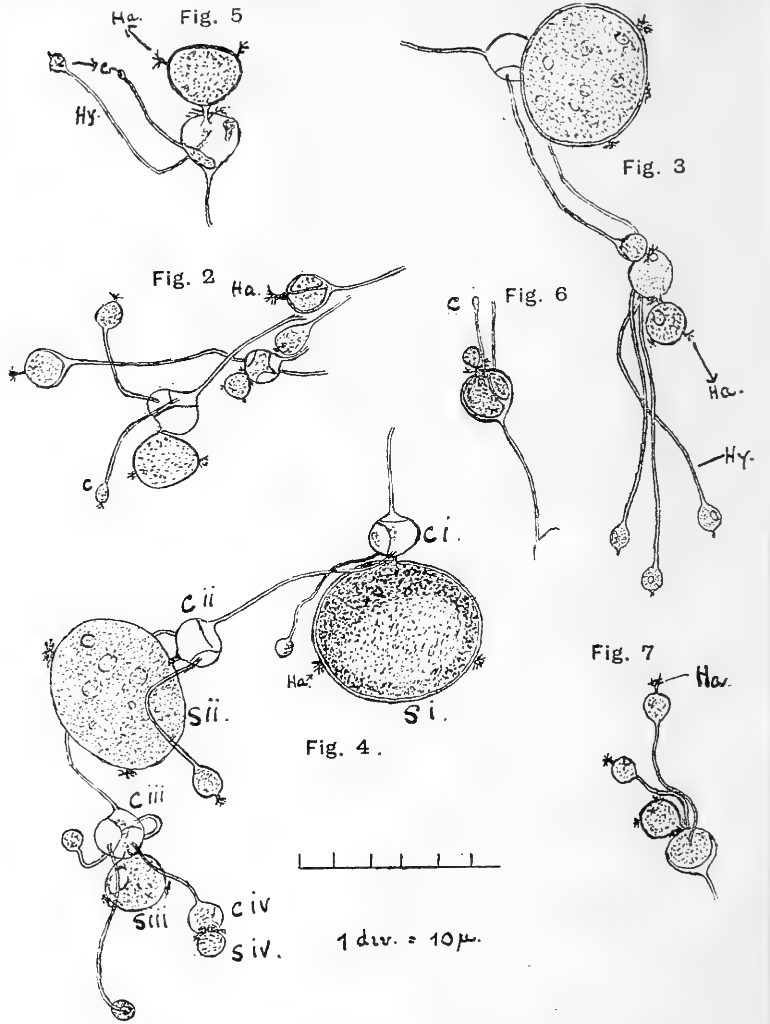
The extreme end of each of these collecting cells develops a short, very delicate and much branched process, which is considered by Jones and Drechsler to have an absorptive function (*Ha*, figs. 2 and 7). It could never be determined with certainty whether this process was a branching hypha or an outgrowth of the wall only: a similar structure is described by Büsgen* for *Cladochytrium Butomi*, and by Schroeter† for *Physoderma* (= *Urophlyctis*) *pulposa*.

The collecting cell increases in size, developing from 10 to 15 nuclei, until it is about 10μ in thickness. Details of nuclear division have not been made out, the resting nucleus shows one deeply staining mass of chromatinic material, but very little normal reticulum. Fine cross walls are then laid down, oblique to the axis of the collecting cell, cutting off 2 to 4 uninucleate masses of protoplasm, peripheral in position, from a central, multinucleate portion. The former give rise to branch hyphae, the latter to one resting spore; in each case by a process of proliferation. From each of the peripheral cells a papilla arises, the end of which enlarges; into this the single nucleus and contents of the cell are passed; it rapidly elongates to form another hypha of limited growth exactly like the one first considered. The resting spore arises from the centre of the apical haustorial process, as a rounded cell on a short stalk, simultaneously with the branch hyphae. Into this cell the

* Büsgen, M. (1887). Beitrag zur Kenntniss der Cladochytrien. Cohn's *Beitr. Biol. Pflanzen*, Bd 4, pl. 15.

† Schroeter, J. (1882). *Bot. Centbl.* Bd 11, Nos. 31, 32, p. 219.

entire contents of the central portion of the collecting cell are passed; it rapidly increases in size, the absorption of food material being probably assisted by a zone of haustorial processes exactly similar



C i, the original collecting cell. *S i*, the spore proliferated from it.
C ii, S ii; C iii, S iii; C iv, S iv; similar structures proliferated in succession from *C i*.
Ha, haustorial processes.

to those developed apically on the young collecting cells, but in this case arising about midway between the equatorial plane and the pole of the spore remote from the collecting cell (*Ha*, figs. 3, 4 and 5).

More than 100 nuclei are formed in the spore as it matures; they are on the whole larger, and show a reticulum better than those in the collecting cells.

Figs. 3 and 4 show this method of proliferation, spores of four different ages developed in succession on one portion of a thallus being shown in fig. 4.

The branched (haustorial) processes are not found to persist on the ripe spores, but as a rule the small depressions from which they arise can be made out.

From this description of the fungus it would appear that it can no longer be regarded as forming its resting spores as a result of the conjugation of two hyphae, in the manner described by Magnus* and Schroeter†; this is the conclusion also reached by Jones and Drechsler ‡; their description of the spore formation agrees exactly with that in this paper. It should be noted that the fungus even at the earliest stage in the host plant is bounded by a thin wall, forming a definite mycelium. No trace of a plasmodial stage as described by Wilson§ has been observed.

Host Plants

Lucerne (*Medicago sativa*) has been the only host plant observed associated with the fungus in this country; *M. falcata* is reported to be about equally attacked in the United States with *M. sativa*, under the same conditions, and *M. denticulata* to be immune. Several attempts to infect *M. falcata* and *M. lupulina* have failed, but are being repeated.

Unsuccessful attempts to bring about infection have also been made with all commonly cultivated leguminous crops, and a number of common leguminous pasture plants and weeds.

Infection has been induced at all seasons of the year and with Lucerne plants of all ages from six months old upwards. It is found however that under normal conditions actual infection of the host tissue does not take place during the summer months, the most favourable time being from September to February. From observations made in the field it seems probable that a very wet condition of the soil is favourable for infection, though actual flooding is not necessary. All attempts to cause infection of the youngest seedling stages have so far failed.

I should like to take this opportunity of expressing my thanks to Professor Biffen for suggesting the work, and for supplying some material for investigation, to Professor Seward for laboratory accommodation and to Mr F. T. Brooks for directing the work and for much stimulating criticism.

* Magnus, P. *Ber. Deut. Bot. Gesell.* Bd 20, Heft 5, pp. 291–296.
† Schroeter, J. (1882). *Bot. Centbl.* Bd 11, Nos. 31, 32. p. 219.
‡ Jones, F. R. and Drechsler, C. (1920). *Journ. Agric. Res.* (U.S.A.), vol. 20, No. 4, p. 295.
§ Wilson, O. T. (1920). *Bot. Gaz.* v. 70, No. 1, pp. 51–68.

On some Alcyonaria in the Cambridge Museum. By Sydney J. Hickson, M.A., F.R.S., Professor of Zoology in the University of Manchester.

[*Read* 7 March 1921.]

Clavularia dura n.sp.

A very small specimen of a creeping Clavularia was found in the collection made by Dr J. C. Verco in 20–30 fathoms off Adelaide, S. Australia. The stolon consists of a few flat strands about 5 mm. in width attached to the horny tubes of a Gymnoblastic hydrozoon. On this stolon there stand at considerable distances apart five calices in the shape of inverted cones 1·5 mm. in height with a diameter of ·8 mm. at the distal end and 0·3 mm. at the end where it is attached to the stolon. At the free base of each cone (*i.e.* the distal end of the calyx) there are eight grooves radiating from the centre but the circumference of the cone is quite smooth.

The body wall of the stolon and of the calices is rendered perfectly rigid by a dense amalgamation of calcareous spicules, as in Tubipora, *Telesto rubra* and Sarcodictyon. So hard are the calices that I was unable to break them open with a pair of needles and it required a sharp blow on the cover-glass to crush them. When a calyx had been thus crushed the tentacles were seen to be armed with numerous spicules in the form of curved rods 0·1 mm. in length with a few small tubercles. The whole colony was pure white in colour. In form and habit this new species approaches very closely the *Clavularia ramosa* from the coast of Victoria* but differs from it in the hardness of the walls and the restriction of the eight grooves to the distal end of the calyx. Moreover in the new species I have not been able to find any of the double club spicules which are so characteristic of *C. ramosa*.

Sarcodictyon catenata Forbes.

An Alcyonarian having very much the same appearance as the British *Sarcodictyon catenata* of Forbes was sent to the Museum by Dr J. C. Verco from 20–35 fathoms off Adelaide, S. Australia, in 1904.

The interest of this specimen lies in the fact that the genus Sarcodictyon has been found hitherto only within the British sea area.

In a former paper† I expressed agreement with Sars in suggesting that the genus should be merged with Clavularia and I

* Hickson, *Trans. Zool. Soc.* Lond. XIII, 1894, p. 340.
† *Op. cit.* p. 332.

said that the genus had been only imperfectly described. In doing so I did not do justice to the excellent description of *Sarcodictyon catenata* by Herdman* and I wish to make a sincere although belated apology. The submergence of the genus in Clavularia has unfortunately been accepted by May, Kükenthal and other writers, but since 1894 I have examined several species of Clavularia and by the kindness of Prof. Herdman several specimens of *Sarcodictyon catenata*, and I have come to the conclusion that it is desirable to retain the generic name Sarcodictyon. Sarcodictyon differs from all the species of Clavularia I have examined—except *Clavularia dura*—in having a stolon protected by hard inflexible walls of fused calcareous spicules and in having long retractile anthocodiae which can be withdrawn into shallow convex calices situated on the strands of the stolon. The stolon consists of flattened strands about 1·5–2 mm. in width, forming a network slightly expanded at the nodes and in the places where the zooids are situated. The colour of the stolon seems to be almost invariably red, pink or yellow.

Clavularia dura appears to be a connecting link between the two genera in having hard inflexible walls but differs from Sarcodictyon in the large conical calices and in the absence of colour.

The specimen from Australia is growing on a dead cockle shell about 35 mm. in length and breadth. The stolon is of a pale pink colour and forms a very irregular network of strands with meshes 5 or 6 mm. across, and the strands are about 1 mm. in width. As the zooids are all completely retracted it is very difficult to determine the exact distribution of the spicules in the anthocodiae but in these parts of the colonies free spicules can be found up to 0·2 mm. in length which have the form of irregular spindles provided with numerous irregular tubercles. The wall of the calices and of the stolon can be seen to be formed of spicules of the same form which in further growth have become jammed together to form a solid structure.

On comparing the specimen with specimens from the British sea area I can find no characters to separate it from *S. catenata*. It is true that the pale pink colour distinguishes it from all the other specimens I have seen; but as *S. catenata* is known to vary from red to yellow the colour character is obviously not reliable. There seems to me, therefore, to be no other course to adopt than to name this specimen *S. catenata* notwithstanding the enormous distance that separates the Australian from the British habitat.

* *Roy. Phys. Soc. Edin.* VIII, 1883, p. 31.

Pseudocladochonus hicksoni Versluys.

This very interesting Alcyonarian was first described by Versluys from the collections of the Siboga Expedition near Ceram and Halmaheira*. It exhibits a curious similarity in its method of gemmation and general form to the fossils belonging to the family Auloporidae and particularly to the genus Cladochonus from the carboniferous strata; but according to Versluys, who has made an elaborate study of the genus and of the Auloporidae, it affords us an example of convergence rather than one of genetic affinity with the fossil family.

In a small collection of Alcyonaria from the Uraga Channel near Tokyo, Japan 40–200 fathoms, I found a few small specimens of this species which exhibit all the principal characters described in detail by Versluys.

The size of the retracted zooids 2 mm. × 1 mm. and the diameter of the stems from which they arise 1 mm. are the same as in the type.

The yellow bands where the stem is slightly constricted just above the origin of the zooids can be clearly seen in some of the older fragments but are obscure in the younger branches. Sections through the stem show an arrangement of solenia divided by septa similar to that of the type and the precise arrangement also seems to vary in different parts of the stem. In one of my sections there are four septa meeting in a central column, in another eight septa which do not meet in the centre, but show some fusions of their free borders. There can be little doubt that the 4-septate condition of the first-named section has been derived from an 8-septate condition by an increase in the thickness of the septal walls and fusion at the centre.

The small amount of the material at my disposal has prevented me from making a further investigation of this structure but sufficient has been done to prove that the identification of the genus is correct and that the specimens do not represent an aberrant species of the genus Telesto.

The only difference that I can observe between the Japanese and the Molluccan specimens is in the character of the spicules. In all the preparations I have made which show clearly isolated spicules or spicules which have not yet become firmly joined with their neighbours to form the solid wall or septa of the tubes, the spicules of the Japanese specimens possess more numerous and larger spines and tubercles than they do in the type specimens. The length of the larger spicules (c. 0·12 mm.) does not show any material difference but there is such a wide range of variation both in size and shape of the spicules in any one preparation, that

* Siboga Expedition. Monograph XIII *c*, 1907.

obviously it would be absurd to suggest a specific difference on this single very variable character.

In conclusion it may be said that the comments of Versluys on the relation of this interesting genus to the family Telestidae appear to be sound as also is his conclusion that there is no evidence at present as to any direct relationship with the fossil genus Cladochonus and the Auloporidae.

Telesto trichostemma Dana.

A few branches of a Telesto that must be attributed to the widely distributed species *T. trichostemma* were found in the Uraga Channel off Tokyo, Japan, in 40–200 fathoms. The walls of the main and lateral zooids are so densely crowded with large spindle-shaped and profusely tuberculated spicules that they are quite rigid and they show eight shallow longitudinal ridges as in other specimens of the species. These walls, however, become soft and flexible on prolonged boiling in potash and then the outline of the individual spicules can be clearly distinguished. In this respect the species differs from *T. rubra* in which the walls are also rigid, but do not soften or show the outlines of spicules clearly after prolonged boiling in potash. The colour is pale red. An interesting feature of these specimens is that they support a number of specimens of the rare entoproctous polyzoon *Barentsia discreta*.

Leptogorgia sp.?

The specimen was obtained by Charles Darwin in the Galapagos Islands during the voyage of the "Beagle" in 1835. It is evidently only a small fragment of a much larger specimen.

The branches freely anastomose in one plane forming meshes of 6 sq. mm. or less but of very variable size. The branches are about 2 mm. in diameter and almost cylindrical in shape. They are of a dark red brown colour spotted on both sides with flat yellow calices.

The spicules are spindle-shaped with five or six encircling rows of prominent compound tubercles and have a length of about 0·1 mm. and a breadth, including the tubercles, of 0·04 mm.

The axis is composed of a horny substance without any deposit of calcium carbonate and is perforated longitudinally by a series of chambers filled with a light transparent spongy substance. In a branch of the axis 0·4 mm. in diameter 20 of these chambers can be counted in a millimeter of length.

No attempt has been made to identify the species of this gorgonid because the specimen is only a fragment and because I have not had the opportunity of studying the systematic part of Kükenthal's monograph on the Gorgonaria in the "Valdivia" series of publications. Until this monograph or some other monograph that gives a critical examination of the most unsatisfactory and confusing literature of the family, comes to hand, the attempt to identify the species is little better than a waste of time.

Virgularia mirabilis sp.? O. F. Müller.

In 1889 I recorded the occurrence of specimens of the genus Virgularia from the coast of Victoria under the name *Virgularia löwenii*, but as recent researches on the stages of growth of the northern species of the genus have shown that the type of this species is but a growth form of *V. mirabilis*, it is clear that the name I gave to the Australian specimen must be changed.

The question is whether these specimens, however, are correctly identified with a species that has hitherto been recorded only from the N. Atlantic and Mediterranean waters, and I have therefore re-examined the specimens in order to compare them with examples of *V. mirabilis* from our own coast. The result of this examination has been the failure to discover any satisfactory characters to distinguish them. In the number of the autozooids in the leaves (about 30), in the characters of the calices, and in the number and position of the single row of siphonozooids the Australian specimens resemble *V. mirabilis* and correspond with the description and figures of *V. löwenii* as given by Kölliker.

But before assuming that this identification is sound and that we have in this case an example of a species with bipolar distribution, it is really necessary to examine a large series of growth forms of the Australian Virgularias.

Many years ago specimens of Virgularia from Australia were recorded by Gray and named *V. elegans*, and A. Thomson has recently recorded a Virgularia under the same specific name from the Ceylon seas*. It is difficult to determine from the published descriptions of these specimens what are the specific differences between *V. elegans* and *V. mirabilis*, as no account is given of the position and number of the siphonozooids. It seems possible, however, that they are all representatives of a species that has a very wide distribution where a suitable habitat occurs. For the present, therefore, I am content to leave the name *V. mirabilis* for the Victoria specimens.

Cavernularia chuni Kükenthal and Broch†.

Seven specimens of a species of Cavernularia were found by Dr C. Hose washed ashore after a storm on the beach at Miri, Sarawak, Borneo. As this is, I believe, only the second record of a pennatulid being "washed ashore"—the other being *Cavernu-*

* *Report Pearl Fish.*, Suppl. XXVIII, 1905.
† "Valdivia," Pennatulacea, XIII, p. 190.

laria malabarica recorded by Fowler*—the fact is of some interest in the natural history of the sea-pens, and I was surprised to find that the species of the Bornean Cavernularia is not the same as that which is washed ashore in the Bay of Bengal.

The principal measurements of the five perfect specimens were as follows:

	1	2	3	4	5
Length of rachis	25	35	40	48	50 mm.
„ stalk	7	15	8	16	10 mm.

From these figures it is clear that the stalk is relatively short, the rachis-stalk ratio varying from 2·5 : 1 in specimen 2 to 5 : 1 in specimens 3 and 5.

As in the type specimen, there is no sharp distinction in diameter between the stalk and rachis, the stalk passing abruptly into the rachis by the appearance of the zooids only. This feature is in marked contrast to that of *C. malabarica*—in which the passage from the stalk to rachis is marked by a great increase in diameter.

The specimens are so much contracted and distorted that any figures that might be given of the diameter would be untrustworthy. As a guide, however, to the proportions in the species, I may say that I estimated the greatest diameter of the rachis of specimen 5 to be about 15 mm. There is no axis in the two specimens that were dissected.

The spicules of the rachis are needles 0·3–0·5 mm. in length by 0·04 mm. in breadth arranged vertically to the surface and penetrating down almost to the centre of the rachis. There are no spicules that are divided at the extremities.

In the stalk the spicules at the surface are small rods and oval in shape, ·05–0·1 mm. in length, but in the depths there are numerous rod-shaped spicules of the same type as those that occur in the rachis.

The character and arrangement of the spicules are like those described for *Cavernularia chuni* by Kükenthal and Broch, with which species the Bornean specimens also agree in the total absence of an axis.

In the type, however, the stalk is relatively much longer, the stalk-rachis ratio being 1–1·4.

As the species was founded on a single specimen and this stalk-rachis ratio may be variable, it seems probable that the specimens from Borneo should be called *Cavernularia chuni.*

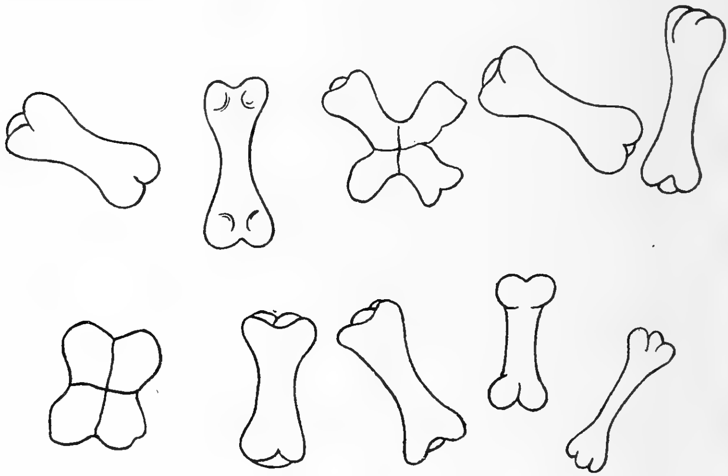
The locality given for the type which is deposited in the Vienna Museum is Coamong (?). (I do not know where Coamong is and cannot find such a place mentioned in Stieler's Atlas.)

* *P.Z.S.* 1894.

Cavernularia darwinii n.sp.

In the stores of the Cambridge Zoological Museum there is a specimen of the genus Cavernularia from C. Darwin's "Beagle" collection labelled Chatham Island, Galapagos Islands, September 1835. As I can find no record of any species of this genus in the Eastern Pacific Ocean and as this specimen is of special interest from its association with the great English naturalist and his memorable voyage in the "Beagle" I decided to examine it carefully with a view to giving it a definite specific name.

The specimen is unfortunately not very well preserved and is strongly contracted and bent, but there are three autozooids killed expanded, one of which has been mounted as a preparation for the microscope.



Spicules of *Cavernularia darwinii.* [× 200.]

Allowing for the contraction and bends the specimen is about 90 mm. in length, the rachis being 38 mm. and the stalk 12 mm. and therefore the stalk-rachis ratio about 1–3.

It is quite impossible to make any accurate statement about the arrangement of the zooids on the rachis or of the relative number of autozooids and siphonozooids owing to the extremely contracted and convoluted condition of the surface. The only statement that can be made without having recourse to a large series of sections is that apparently there are relatively very few autozooids.

The presence of an axis can be proved by probing with a needle and so far as can be judged by this method it extends about half-way up the rachis and about half-way down the stalk. The general statement may be made therefore that the specimen has an incomplete axis.

The spicules of the rachis, although very variable in shape, are at the same time, very characteristic and can be easily distinguished from the rachis spicules of any other species I have examined. The most prevalent type is that of a short rod 01–0·13 mm. in length, terminating in swollen extremities divided into two, three, four or sometimes five convex facets having an appearance which is extraordinarily like that of a metacarpal bone of a mammal. The variations even in this type of spicule are numerous as the number of the facets varies at each end independently. Thus, there may be one facet at one end, two at the other or two at each end, three at one end and four at the other and so on; but generally speaking if the number of these facets is not the same at each end the excess at one end is not greater than one over that at the other end. This may be represented as follows in figures:

1–1, 1–2, 2–2, 2–3, 3–3, 3–4, 4–4, 4–5.

In addition to the spicules of this type there are some quadruplets which are either simple crosses with rounded ends or crosses with two or three convexities at the end of each of the branches or, in a few cases, simple square plates with very rounded angles. The most noteworthy thing about these rachis spicules, however, is the absence of plain rods of full size or of oval spicules. Apart from a few small spicules, which are probably growth stages, all the spicules are swollen at the extremities and most of them show divided lines of growth.

In the outer layer of the stalk there is a dense armature of spicules of the same type as those prevalent in the rachis; in the inner structures of the stalk there are apparently very few spicules, but several of these which are found lying vertically to the surface in the fleshy septa are longer than those in the rachis, being 0·2 mm. in length.

Cavernularia malabarica Fowler.

Two specimens of this species were obtained by Dr Imms from Puri, Orissa Coast, Bay of Bengal. Their principal measurements are:

Length of rachis	27·25 mm.	Diameter of rachis	35·22 mm.
,, stalk	13·10 mm.	,, stalk	10·8 mm.

The sharp distinction in diameter between rachis and stalk which is a character of this species and of *Cavernularia glans* was well marked in the specimens. The stalk seems to be a little longer in proportion to the rachis than in the type specimens, but it is nevertheless a short stalk.

The species is of some interest as it afforded the first known examples of pennatulids to be "washed ashore."

The Influence of Function on the Conformation of Bones. By A. B. Appleton, M.A., Downing College.

[*Read* 7 March 1921.]

Inspection in a museum of a series of mammalian skeletons is sufficient to indicate some sort of relationship between the osseous details and the locomotor abilities of the animal, whether on plains, in the trees, or in water.

The femur is a bone which, in association with neighbouring bones and muscles, repays detailed study.

We find something in common between the femora of jumpers belonging to quite different animal groups, even though all may not jump in exactly the same way. The same is true of runners (cursorials). It is true in spite of the fact that each mammalian group of living forms tends to exhibit its own characteristic musculature, skeletal features and probably, too, characteristic nervous mechanisms. And it is to be presumed in the first place that tendencies exhibited by cursorials, say, belonging to various groups, may be legitimately regarded as *adaptations*. How far their peculiar musculature is really of advantage to them will be discussed in the sequel.

A short summary will be here given of some muscular peculiarities of specialised cursorials, jumpers and arboreal mammals, and their relationship to peculiarities of the femur discussed.

To what extent such peculiarities are determined by "environmental" influences acting in the individual requires not only a study of ontogeny, but experimental investigation of the effect of modifying the conditions of growth. In fact, a study of human variations demands from us an answer to the question: Are all of these variations of hereditary origin, as Pearson and Lee assert*?

The great plasticity of bone under mechanical influences suggests these influences as possible modifying circumstances during ontogeny. In what manner bone will react, is, however, at present ill-understood; though it seems unlikely that Manouvrier's supposition can be true, *viz.* that under transmitted pressure during youth, a femur will *bend* as a *vital* process or reaction. We only know of bending as a twig is bent when the pressure is too great for a femur softened by rickets or osteomalacia.

Bone, however, does react to stress†. If too little stress falls upon an adult bone, it undergoes partial atrophy; it becomes im-

* Pearson and Lee, 'Long Bones of the English Skeleton,' *Drapers' Company Memoirs* (*Biometric Series*), x and xi, pp. 267, 287.

† Murk Jansen, 'On Bone Formation,' 1920.

possible to draw the line between pathology and physiology. Certain observations from human orthopaedic surgery throw light on this question.

Human bone-grafts grow in thickness under certain conditions of stress. Els* has recently stated that a thick bone-graft diminishes in girth till it resembles the bone it replaces. The repair, again, of a fracture is assisted by the transmitted pressure obtainable when a calliper-splint is used. The *internal* structure of an astragalus changes under gross alterations of mechanical stress as occur in flat-foot. The observations of Dendy and Nicholson† on the spicules of sponges suggest also the continual living reaction of skeletal elements to mechanical factors.

When therefore we recall Hunter's observation that during growth there must be a continual remodelling of the neck of the human femur, we find ample grounds for supposing that one's habit of walking, or peculiarities of musculature, or of the methods of employing it, will probably play an important part in the production of individual variations.

Much valuable information is provided by Pearson and Bell in the vast array of statistics in their recent monograph on the femur of mediaeval Londoners and of Primates. They show that while there is a considerable correlation between dimensions of epiphyses of the femur, and between these and its total (and shaft) length, this is much less evident in the female, in whom also the (upper) epiphysis shows less variation in absolute size. On the other hand, and this seems very important, the correlation of coronal and sagittal diameters of the shaft is quite small, and of the neck diameters smaller still. This appears to me just what one would expect from the incidence of "environmental" factors, affecting diameters in different ways. The assumption that such are hereditary features, evolved by selection, is unnecessary. The different correlations and variabilities on the two sides may be accounted for by muscular peculiarities, asymmetrical locomotion, and unusual habits.

Could we but anticipate the differences of right from left which may appear in the rising generation of to-day with its "scooters" and differentiation in function of lower limbs!

Musculature and Cursorial Specialisation.

By some means many groups have evolved cursorial types with similar muscular characteristics. The **Primitive** type of thigh musculature was doubtless of the pattern found in the living tree-shrew (Tupaia) and the lemurs. Matthew, in fact, suggests an arboreal ancestry for mammalia.

* Els, *Anatom.* Hefte, 1 Abt. 176 Heft, 58 Band, Heft 3, 1920.
† Dendy and Nicholson, *Proc. Royal Soc.* LXXXIX, B, 1917, p. 573.

Certainly, eocene mammalia had femora of the same pattern; and probably also musculature. The M. femorococcygeus (Fig. 1) may be first considered. In primitive types this muscle passes down to be attached to the outer border of the femur below the third trochanter (Fig. 2). From the latter the M. gluteus superficialis passes upwards and forwards. These muscles are often, and were probably originally, continuous, as also with the M. biceps and M. tenuissimus. Tupaia presents this general arrangement with little modification.

A **Cursorial** life leads in all groups to a downward migration of the attachment of the femorococcygeus, while this muscle tends at the same time to gain a new upper attachment to the tuber ischii

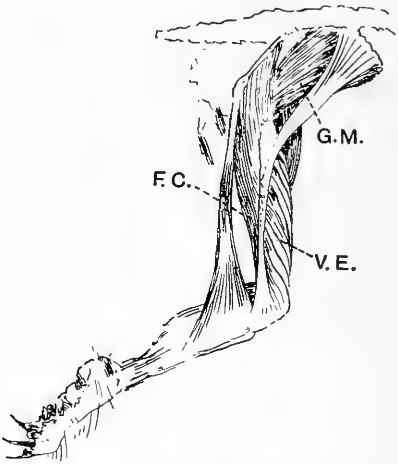


Fig. 1. Thigh musculature of hedgehog. A primitive condition. *F.C.* = femorococcygeus continuous with superficial gluteus above. *G.M.* = gluteus medius. *V.E.* = vastus externus. (From a preparation by Dr W. L. H. Duckworth)

of the pelvis. The femorococcygeus may even extend downwards till it reaches the patella and then it gains a new function—knee extension. The Felidae show various degrees of specialisation. Even within the single species *Felis pardalis*, the muscle may be attached to either femur or patella (Parsons). The "long vastus" of veterinary surgeons is none other than this transformed femorococcygeus with (in Artiodactyla) accession of the superficial gluteus which has also lost its femoral attachment. This muscle in the horse plays an important part in bracing up the thigh during a stride (Stillman). The superficial gluteus, if not thus incorporated in the long vastus, undergoes, first, proximal migration, then regression—as in Carnivora. Whether its persistence in Equidae is of some functional value is not clear. Certainly in the rabbit it may experimentally be deprived of its femoral attachment without noticeable difference in the animal's locomotion; though it is possible that minute differences would be found in its skill at turning rapidly or in the placing of its feet when at full speed.

A hyrax (*H. capensis*), which Prof. Keith kindly placed at my disposal, presented an interesting intermediate stage, foreshadowing the "long vastus." The superficial gluteus was attached not only to the femur, but its anterior fibres were adherent to the femorococcygeus, which passed down to the patella as in the cursorials.

Most jumping animals present very similar changes in the femorococcygeus. Whatever the group of origin, whether a cat, a cow, or a kangaroo, the line of specialisation is similar.

In the Primates the femorococcygeus tends to disappear or become incorporated in the superficial gluteus and to lose its lower femoral attachment by regression or migration of its lower fibres. The superficial gluteus (*i.e.* gluteus maximus) is attached just below the great trochanter to a more or less rudimentary third trochanter. The lemurs, however, and the giant apes retain the femorococcygeus attachment along the femoral shaft; they are not specialised, as are the monkeys, for a modified arboreal life, and man for a new mode of progression. That the gibbon presents in this respect to some extent a *parallel* evolution with monkeys and man is supported by the fossil Paidopithex whose femoral shaft is more like that seen in lemurs and the gorilla (*vide infra*).

In man traces of the femorococcygeus are still to be found in the external intermuscular septum as low down as the condyle—just in the position which the muscle occupies in the gorilla. The primitive condition found in the gorilla, chimpanzee, and orang, is probably truly primitive. There is no reason to suppose that this condition is ever acquired secondarily after the specialised features found in cursorials, jumpers, and monkeys have arisen. There is in fact reason to believe that *Irreversibility* of evolution of a muscle like the femorococcygeus is as true as that of many other organs, such as teeth. Various kinds of mammalian specialisation alter its primitive arrangement; to this it will not revert, but would more probably merely atrophy, with lethargic habits.

To prove the condition in the gorilla to be secondary to that in catarrhine monkeys requires proof that the "primitive" femorococcygeus is more suited to him than, say, the catarrhine arrangement with a biceps which has extended its fascial attachment upwards to the middle of the thigh, and with complete absence (as an independent muscle) of the femorococcygeus.

The plasticity of muscle and of bone is of the same order; they both show phylogenetic as well as ontogenetic response to function.

But reversion to slow-moving habits will no more resuscitate an absent muscle than a lost third trochanter.

The adductor group of musculature (with which must be considered the MM. semimembranosus and caudofemoralis [Leche] from their frequent synergism) shows important modifications characteristic of various specialisations.

Attached in the primitive type along the whole femoral shaft and down to the head of the tibia, the bulk of this musculature in cursorials comes to be concentrated in the region of the knee. The value of this will be pointed out subsequently. At the same time it shows great increase in size.

Carnivora deviate far less from the primitive type than do Artiodactyla and Perissodactyla, though the specialised types, such as Canidae, show great increase in size of the adductors.

TABLE I. Weight* of Adductor Group Musculature† (including Semimembranosus).

	Choloepus didactylus	Civet	Nasua	Ursus	Felis (cat)	Canis (dog)	Ovis	Rhinoceros	Equus	Macropus	Lepus cuniculus	Gorilla	Homo (at birth)	Cyno-cephalus
A Attached to shaft of femur	2·7	2·8 ?	3·0	2·3	1·6	3·0	1·3	1·2	0·5	1·3	1·5	5·3	3·0	0·9
B Attached to region of knee	1·2	1·2 ?	1·5	0·8	1·7	2·6	2·8	1·5	5·4	1·0	2·1	1·6	2·1‡	1·3
Total weight $A+B$	3·9	4·0	4·6	3·1	3·3	5·6	4·1	2·7	5·9	2·4	3·6§	6·9	5·0	2·2
Ratio $\frac{A}{B}$	2·2	2·3[1]	2·0[1]	3·1	1·0	1·1	0·5	0·8	0·1	1·2	0·7	3·3	1·4	0·7
Author providing muscle-weights	Macintosh	Macalister		Haughton	Appleton	Appleton	Appleton	Haughton	Appleton	Haughton	Appleton	Appleton	Appleton	Appleton

[1] Approximate.

It is seen that there is a reduction in the adductor musculature attached to the shaft in just those specialised types, cursorials and saltatorials, which exhibit the specialised femorococcygeus; but that the total mass may become very large, as in the horse, dog (and lion). The adductor mass in the gorilla is also of great size. A resemblance of the baboon to the cursorials is noticeable in this as in other respects.

* Unit of weight is M. rectus femoris; the best unit yet found for comparative purposes; little affected by specialisation, and containing a factor depending on femoral length.

† Muscles concerned: all adductors, pectineus, presemimembranosus (ischio-condyloideus), semimembranosus, caudofemoralis.

‡ Includes condylar part of adductor magnus. The figures of Dursy for the adult are similar.

§ The rectus femoris is exceptionally small (partly replaced in function by tensor fasciae femoris).

The knee extensors show in cursorials and saltatorials a considerable development in size, the vastus externus being particularly affected. Its method of attachment to the femur is influenced by its internal structure, the penniform pattern, as in man, being relatively more powerful than the simple form of the hedgehog; it is largely determined by the attachments of the neighbouring femorococcygeus and gluteus superficialis. Where these lose their attachment to the lateral margin of the femur, the vastus externus and crureus are able to spread around the outer aspect (as in the baboon, man, and sheep) of the shaft. In primitive types, however, it is unable so to spread backwards on account of the

Fig. 2. Left femur of *Priodontes giganteus*, showing the descending extensor ridge ("frontal pilaster") and the flange-like third trochanter. M. vastus externus envelops the extensor ridge and occupies the hollow in front of the third trochanter.

femorococcygeus and the third trochanter. Now the vasti muscles become large in other animals beside cursorials and saltatorials. In edentates, which present a primitive type of femorococcygeus, gluteus, and adductor, the vasti are, compared with the rectus femoris, of unusually large size. As will be seen below, a special outgrowth of bone, the "descending extensor ridge" (Fig. 2), appears on the front below the great trochanter for the attachment and accommodation of the vastus externus. This long outgrowth would be unnecessary and does not occur when the superficial gluteus (and femorococcygeus) have no attachment to the shaft as in the specialised runners and jumpers; in them the vastus externus extends backwards round the shaft. The ridge occurs in some edentates, rodents, and the Lemuroidea (including Tarsius). That it does not appear in gorilla, where the superficial gluteus is attached close up to the great trochanter and the third trochanter has disappeared (as in recent Carnivora), suggests the close relationship to this structure.

TABLE II. Weights* of Knee Extensors.

	Badger	Dog	Sheep	Opossum	Kangaroo	Choloepus	Armadillo	Lemur	Macaque	Gorilla	Tarsius	Man
Total weight of vasti and crureus	2·0	2·8	2·0	2·1	2·7	1·2	—	—	3·0	—	27·4	7·7
Weight of vastus externus only	1·5	1·8	1·2	—	2·3	—	3·3†	4·2†	2·0	3·4	17·6†	2·2

The special tendency to development of the knee-extensors in jumpers is to be seen, notably in Tarsius with its very long femur. That the M. rectus femoris (the ambiens of reptiles) is not included in this development is illustrated in the above figures.

Correlation of changes in the Femur with Muscular Specialisation.

Special attention will be here directed to the form of the shaft in cross-section. Much interest attaches thereto owing to the great contrast in form presented by the gorilla to the human genus.

Unspecialised mammalian types generally present a transversally oval cross-section at mid-shaft, and from the lateral margin somewhere near its middle a third trochanter projects: a common type in early tertiary mammalia (Pantolestes, Meniscotherium, Nesodon). Cursorials and saltatorials tend to exhibit a rounded shaft, and the third trochanter shifts up the femur and becomes reduced or disappears.

There is in these more **active** types an increase in the antero-posterior diameter of the femur. Does this appear in adaptation to the increased bending moment, tending to break the shaft across, or is it merely a means of providing attachment for muscles, *viz.* the vasti and crureus, the knee extensors?

The more active animals do throw an enormous strain on the femur in springing from the limb when at the gallop or jump; far greater than the animal's dead weight. An increase in this sagittal diameter is indeed very necessary, and is the more necessary in that these animals have angulated limbs, unlike the graviportal types (Gregory) with straight pillar-like limbs. The increased

* Weights are given in terms of the M. rectus femoris as the unit.
† A descending extensor ridge is present; it is no test of blood-relationship.

antero-posterior diameter of the condyles, so remarkable a feature of running and jumping types, especially in certain extinct artiodactyla (see Fig. 3), must be a part of the same evolutionary response to requirements. The internal structure of the deep condyles illustrates their true nature. The data of Pearson and Lee* provide an excellent illustration of this effect in the lower Primates, though not so interpreted by them. Tarsius, Indris, and Nyctipithecus, all jumpers, contrast strongly with Apes, Loris, Mycetes and others†. Man again shows the deeper and narrower condyles of active hind-limbs; how precarious must be an argument as to his ancestry from a study of indices based on these measurements,

Fig. 3. Lower end of femur of Anthracotherium, showing deep condyles of a cursorial type. From a specimen in the possession of Mr Forster Cooper, reproduced with his kind permission.

without further knowledge of the knee-joint movements in various animals.

The importance of the shaft of a long bone as a surface for muscular attachment is suggested by an examination of animals in which transmitted propulsive forces are not great. They exhibit flattening of the shaft, the width far exceeding the sagittal diameter; this is conspicuous in the sloth. The great relative width is necessary for provision of muscular attachments; there can hardly be lateral strain sufficient to justify such width. The slow loris and apes are broad in the shaft (Pearson and Lee). And when width of shaft is

* Pearson and Lee, 'Long Bones of the English Skeleton.' *Drapers' Company Memoirs* (*Biometric Series*), X and XI, pp. 329, 337, 412.

† The length of the femur is a poor standard for comparison because of its great variability with function.

influenced by the presence of a long flange-like third trochanter the requirements of muscular attachments are obvious. In fact the shaft in the small primitive types may be regarded as consisting of a pillar for strength and an external flange for muscular attachment, as in the eocene Meniscotherium and others. The great width in the extinct Megatherium is probably of a similar nature: X-ray investigation of the primitive femur of Priodontes supports this view of its nature.

In larger and more active animals increase of stress requires a stouter femur; there is now less need for flanges. Also, as Scott has shown, all heavy types, despite their relative want of activity, show a general tendency to reduction of the external flange and third trochanter *in situ* (*e.g.* elephant). Data do not appear available to show whether reduction of the superficial gluteus, or of the vasti, can account for such loss in the elephant.

The descending extensor ridge already mentioned is developed in association with a peculiar muscular combination. That it has also the advantage of withstanding stress from the forwardly projecting great trochanter which in these animals (*e.g.* Tarsius, Priodontes) forms a conspicuous attachment for the vastus ex-externus, is suggested by X-ray examination of the bone*.

The actual bulk of the vastus externus is probably closely related to the degree of projection of the third trochanter, from which the fibres of gluteus superficialis pass upwards and forwards across it. A third trochanter placed lower down the shaft will tend to be longer.

The term "frontal pilaster" applied to this ridge by Pearson and Bell suggests a comparison with the pilaster on the back of a human femur: only so long as both are regarded as adaptations for muscular attachment does the term seem justifiable.

The human "pilaster," to which the adductors are attached, was regarded by Manouvrier as being due to the backward extension of the crureus muscle, on the outer side; and to expansion of the front of the bone. How is it that in Tarsius, with far larger knee extensors, this backward encroachment does not occur? It is an example of the influence exerted by the M. femorococcygeus, which in primitive types is attached to the outer margin of the flattened femur. A distinct line is produced which marks also the attachment of the crureus, vastus externus and adductors, the attachment in fact of the external intermuscular septum. A similar line for attachment of the vastus internus is found along the inner margin. These lines may for convenience be termed the External and Internal **septal lines.** While the M. femorococcygeus is attached to the femoral shaft, the external septal line never gets displaced to the back of the femur.

* Bone trabeculae may be considered indicative of lines of Pressure (Murk Jansen).

Hence a "linea aspera," formed by approximation of the two septal lines on the back of the femur cannot be formed.

The rounded shaft with linea aspera, which some artiodactyles possess (*e.g.* sheep), is rendered possible by the specialisation of the femorococcygeus, and by the downward displacement of adductors. The shaft is rounded in Tarsius from considerations of strength; but the septal lines are on the lateral margins of the bone—a femorococcygeus is attached. Since the "interseptal" space provides attachment for adductors, this if broad, as in the elephant, tends to flattening—merely to enable the adductors to pass to their more lateral insertion. Thus the posterior surface of types with septal lines remains flattened, even where the anterior surface may become very convex. Heavy types, such as the elephant and Toxodon, show this peculiarity to advantage.

The obliquity of the posterior surface in Carnivora has a similar origin; the adductors here have a presumably important lateral attachment; the backward external projection permits of the accommodation of the large adductors.

Actual production of a narrow interseptal space, and finally of a "linea aspera," as in man, is associated with expansion of the knee extensors, and downward migration* of the bulk of the adductor musculature: these are characteristic of cursorials and saltatorials. Considerations of accommodation would make it necessary, when vasti muscles are attached behind the femur, that flattening of the inner and outer margins of the bone should occur. For the vasti ultimately gain attachment to the patella. It is a similar problem to the accommodation of the enormous vastus externus of Tarsius. The surface below the great trochanter is hollowed out for its accommodation.

Why is it that carnivora, and some Artiodactyla (*e.g.* Cervus), show such a tendency to a wider interseptal space and a prominent *lateral* position for the external septal line? It is not merely due to the bulk of the adductors attached to the shaft, though large for cursorials; for these are large also in catarrhine monkeys, and still larger in man, but in them there is a "linea aspera." There is probably some functional significance in an extreme lateral attachment for the adductors in the cursorial Carnivora; their external rotatory effect will counterbalance the internal rotatory effect of the synergic gluteus medius. This attachment is not found in the catarrhine monkeys and man; it is sufficient in them for the adductors to be attached largely to intermuscular septa; and the septal lines become placed close together. The structure of the quadriceps extensor appears to be closely associated with the backward displacement of the *internal* septal line in man, and perhaps monkeys too; a penniform arrangement of the fibres of the vastus

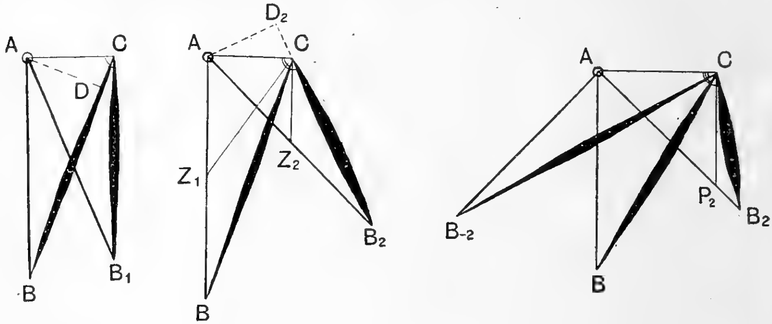
* Except in monkeys and man.

internus is rendered possible, increasing power and diminishing range of movement.

There appears then abundant reason to believe that peculiarities of shape of the femur cannot be properly understood without reference to the muscles attached and adjacent thereto. And further, that the muscles themselves have been evolved in accordance with the habits of an animal and of its forerunners.

Effective Leverage as a factor in Muscular Specialisations.

A consideration of the mode of action of the **Adductor*** group shows that in cursorials and saltatorials their function is largely one of hip-extension. The adductor function proper is more characteristic of arboreal types with a wide range of abduction.



A CAVIA. B LEPUS. C BOS.

Fig. 4. Lengths AB, AC, are proportional to lengths of femur and postacetabular pelvis in various types. (Used merely as a diagram)

A. (Cavia porcellus) illustrates in AB_1 position of thigh in which semimembranosus obtains maximum effective leverage, measured by perpendicular AC to joint A. AC is greater than AD, which measures leverage when thigh occupies position AB.

B. (Lepus cuniculus) when extension proceeds further till thigh is in position AB_2, leverage is again less, AD_2 being less than AC. At this time, a shaft adductor CZ_2 is acting to the best advantage.

C (Guernsey cow) all positions of flexion in front of the vertical AB require use of the lowest, viz. the knee adductors, which only attain their maximum power at AB_1 (ACB_1 being a right angle). At same point of extension AB_2 at which rabbit employs shaft conductor CZ_2, the cow is still employing CP_2 adductors attached close to the knee.

Considering extension alone in the first place, the maximum effective leverage can be easily shown (*e.g.* for a semimembranosus or ischio-condyloideus) to obtain when the muscle is at right angles (Fig. 4 A) to the line joining acetabulum with tuber ischii. If the thigh is at right angles to the pelvis, the most effective

* Including semimembranosus.

muscles of the whole adductor group will be those attached in the region of the knee; so also in flexed positions of the knee. It is thus that the perpendicular from *muscle* to *joint* is the longest—and this is a measure of the leverage. The femur is of course practically always longer than the postacetabular pelvis; if shorter, the muscle would have to be placed at right angles to the femur for optimum leverage effect.

In more extended positions of the thigh (Fig. 5) such "adductor" muscles as pass to the knee-joint will be at some disadvantage; the most effective muscles will now be attached further up the shaft; and the greater the extension, the higher up the shaft will hip extensors be required.

A relatively longer femur will after a smaller movement of extension involve the employment of these shaft "adductors," rather than those of the knee. Application of these principles to the cursorial, saltatorial, and arboreal animals shows some correspondence with actual life.

Arboreal animals require a greater range of extension at the hip than do cursorials; and the mass of their adductors is attached to the femoral shaft (Fig. 5). Ungulata present a much smaller range of extension (Fig. 4 C). Carnivora are intermediate and so is their adductor musculature (*vide* Table I).

Again, cursorials present a shortened femur, not only as compared with other limb-segments (Gregory), but as compared with the postacetabular pelvis (Fig. 4 C, cow). This will in itself involve the continued use of the semimembranosus at a range of extension at which in longer-limbed animals shaft adductors might be employed. Hence the concentration of "adductors" at the knee in Artiodactyla and other cursorials and jumpers; and, along with this, changes in the form of the femur.

That shortening of the femur, alone, will not result in the cursorial pattern of adductors, *viz.* with attachment largely to the knee-region, is shown in the gorilla (Fig. 5); in him, a large part of the adductors is attached to the middle of the shaft, in association with the habitual employment of the thigh in the almost fully-extended position. In man, the ratio of shaft to knee adductors is large also, larger than the figure in Table I would suggest; his semimembranosus is in fact reduced far below its size in the gorilla; extension from full flexion is not a frequent or habitual movement.

The conditions governing adductor attachment, in so far as true adduction is concerned, are probably similar; in man the extensor function of the group is largely taken over by the enlarged gluteus maximus (*seu* superficialis) owing to the unsatisfactory position of the adductors; and the adductor longus in particular has become an adductor *par excellence*. The relationship of the form of the preacetabular part of the pelvis to the functions of the adductors should prove full of interest. The postacetabular part of the pelvis shows in its modifications in various groups an attempt to provide the semimembranosus, etc., with more effective leverage (notably in Artiodactyla), as judged on these principles. Lowering (or lengthening) the postacetabular pelvis vastly increases the effective leverage of hamstring muscles. The changes in the femorococcygeus, its transference to the tuber ischii and knee in cursorials, the migration upwards of the biceps in the catarrhines, are all explainable on these lines.

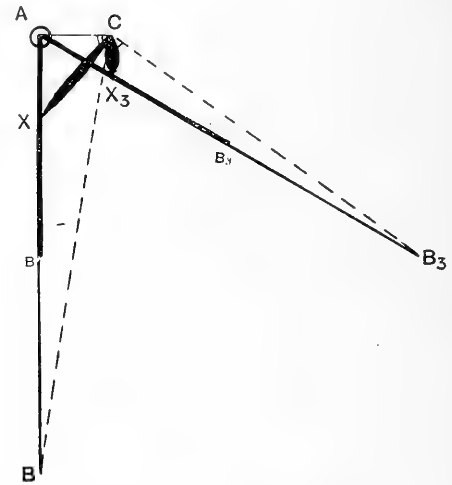


Fig. 5. AB and AB_3 = femur of Semnopithecus. $A\textsc{b}$ and $A\textsc{b}_3$ = femur of Gorilla, reduced to same AC. AC = postacetabular pelvis. In positions of great extension, AB_3, $A\textsc{b}_3$, semimembranosus to knee is almost useless as an extensor in any but flexed positions or positions near AB. The most effective position for an extending adductor lies up near head of femur, represented by a muscle CX_3 (relatively lower on the shaft of the gorilla). The muscle CX has to shorten itself CX_3 to 0·4 of its length at CX, but muscle CB is CB_3 only shortened 0·1 of its length.

Shortening of the femur in cursorials is explained by Gregory* as an adaptation which allows of more "open angles of insertion" of hip-extensors and other muscles. As a matter of fact, in ordinary locomotion, the femoral attachments are the fixed points or origins of the muscles; and their effective leverage on the hip-joint will be *lessened* by shortening of the femur. The shortening appears to me, considered only from this point of view, to be a disadvantage. Gregory is in difficulties over explaining the lengthening of the femur in the recent greyhound and racehorse. Perhaps they lengthen to increase the effective leverage.

* Gregory, *Annals of the New York Academy of Sciences*, vol. XXII, p. 291.

What, then, has brought about the shortening of the typical cursorial femur (*e.g.*, Artiodactyla [Gregory])? I think it is to be found in the enormous strain thrown upon the knee extensors in the active animals with long femora. The longer femur when the knee is flexed puts an almost impossible burden on the vasti during a violent thrust from the tibia. In large animals it is inadmissible; the femur shortens. Confirmation of this is found in the enormous size of the vasti in Tarsius (see Table II), a small-sized jumper with long limb-segments. The larger jumpers, the kangaroos, exhibit distinct shortening of the femur. It is better to develop the necessarily powerful hip muscles than to depend on powerful knee muscles; the centre of gravity is, in fact, thus placed so much the higher up the limb.

Considerations of the actual shortening (Fig. 5) undergone by muscles passing to the higher and lower parts of the femur respectively in fully extended positions, would suggest that muscles do not all present the same percentage shortening over the complete range of hip movement. If all shortened by the same fraction of their length they would effect different ranges of hip movement, distally attached muscles causing extensive angular movements, proximally attached muscles small ones. Comparative study of the internal structure of the Great Adductor will, it is hoped, prove enlightening.

Animal Oecology in Deserts. By P. A. Buxton, M.A., Fellow of Trinity College, Cambridge.

[*Read* 7 March, 1921.]

The following notes are the outcome of a somewhat extended sojourn in Mesopotamia and N.W. Persia during the war. Conditions were very unfavourable for consistent investigations. I take this opportunity of publishing my unfinished observations, in the hope that they will furnish raw material for others. Aspects of the subject which are already well known have been entirely omitted from this short paper in which comprehensiveness is not aimed at.

Climatic conditions.

It is of course well known that deserts are dry, and many of them hot as well. It must be clearly recognized that well-nigh all the hottest places in the world are in desert or semi-desert country, and that cold is almost as characteristic of deserts as is heat. Wind too must be considered as a factor with which the desert fauna has to contend. At Menjil in N.W. Persia the wind sweeps through a pass in the Elburz mountains with such velocity that one can barely stand against it. This wind blows with great regularity from about 9 a.m. till sundown throughout the warm weather. It is caused by the daily heating of the N. Persian plateau under the sun's rays; the heated air rises and is replaced by an inrush from the sides: at Menjil, this inrush is concentrated in a rocky defile. Such a terrific but steady wind must be an important spreader of small organisms. A large camp at Ruz in the valley of the Diyala in E. Mesopotamia was smitten by an exaggerated dust-devil early in November, 1918. The wind carved a lane through the camp ripping every tent that lay in its path: heavy articles of kit were blown through the air and deposited on the opposite bank of the Ruz canal: an officer of my acquaintance was pulled out of his tent with all his camp furniture and dropped twice, with such violence that three of his ribs were broken. Such winds are rare, but in most desert countries small dust-devils are common; they wander about the desert in a somewhat aimless manner, sweep débris, bushes, etc. from the ground, and drop them later in a different place. The fact that so many desert animals live beneath rocks and excavate burrows is possibly to be explained as an attempt to reach an equable temperature, and to avoid wind and dust: the generally accepted explanation is, I believe, that the scanty vegetation affords no cover and necessitates an underground life. Some of the meteorological conditions of deserts have received, I believe, no attention so far as their effect on the fauna is concerned. The extremely low relative humidity, the sharp spell of heat by day and of cold by night, the great heat of the stones and soil on which some insects crouch and the effect of direct sunlight on animals are among the problems which call for study. We know indeed that they must have their influence on the desert fauna, but we have no accurate knowledge of what that influence is. Much might be learnt by exposing animals to one or other of these conditions, using an adequate number of controls.

Colouration of animals.

We have all of us known from childhood that in the desert the animals, the birds and the insects are coloured like the desert: any desert fauna furnishes countless examples of this. There are, however, a few very disturbing facts, which, I believe, are not widely known. The Gerbilles (*Gerbillus*, *Meriones*, *Dipodillus*, *Tatera*, etc.) and the Jerboas (*Jaculus*, *Alactaga*) all appear to be perfect examples of protective coloration, but they are strictly nocturnal. This is also true, I believe, of the Cape Jumping Hare (*Pedetes*) and the jumping mice of the deserts of southern North America (*Perodipus*, *Zapus*) and the Spiny Mice (*Acomys*) of Sinai, etc. It is difficult to explain the coloration of these animals unless we suppose that it is of some protective value by moonlight. One does not know from what foe the Gerbilles need to escape, unless it is foxes: certainly in Mesopotamia and Persia owls were almost non-existent in the desert, except the resident Little Owl (*Athene noctua* subsp.), and in winter the Short-eared Owl; both of these owls feed almost entirely by day. As I have said, the Gerbilles and Jerboas are strictly nocturnal, not appearing at dusk or at dawn: at Qazvin in N.W. Persia I lived for four months close to colonies of *Meriones erythrourus* Gray and *M. blackleri lycaon* Thos. Every night I was out in the desert between sundown and 9 p.m., and I was often about just when the dawn was breaking, but I never once saw a Gerbille except in the hours of darkness, when they were extremely abundant, crossing the light cast by my lantern. Among the birds the Cream-coloured Courser (*Cursorius gallicus*) is efficiently protected by its colour and disruptive pattern so long as it crouches; but when it runs the long legs raise the body from the ground and cause it to cast an extremely conspicuous black shadow. The See See Partridge (*Ammoperdix griseogularis*) [*bonhami*] is another example of imperfect protection by coloration: the bird is crepuscular, hiding by day in holes under rocks. In the evening its movements are betrayed by the long shadow it casts as the suns rays decline, and this in spite of its short legs. It is a curious fact that most desert birds are protectively coloured as chicks and as adults, but their eggs are in no way specially coloured and in general resemble the eggs of related birds which do not breed in deserts.

It appears to have been overlooked that black animals form a definite element in the fauna of the great palaearctic desert belt which stretches from Marocco to the Gobi. Examples are a number of the Wheatears (*Saxicola melanoleuca*, *S. lugens*, *S. leucopyga*, *S. monacha*, *S. morio*, etc., all of them predominantly black, with a greater or lesser amount of white). One might also mention the ravens *Corvus umbrinus* and *C. rhipidurus* (*affinis*). Among the insects the coleopterous family Tenebrionidae is characteristic of deserts in many parts of the world. Black tenebrionids, belonging to no less than five sub-families (Erodiinae, Zophosinae, Tentyrinae, Adesmiinae and Pimeliinae) are a very conspicuous feature of the deserts of the palaearctic region from Marocco eastwards to Turkestan, and to Sind. The great majority of these black forms are diurnal: in these same deserts there occur a number of Tenebrionidae which are not black: the majority of these are buff, or grey, or brown, and these species are mainly nocturnal. Other sub-families of Tenebrionidae occur in the deserts of Australia and America, and many of these insects are black, but I do not know whether they are diurnal. In the Orthoptera there is an example in the Phasgonuridae (Locustidae): in Algeria I took that remarkable insect *Eugaster guyoni*; this is a large stout locust, black and highly polished, with some red prominences on the thorax. It is unable to leap, for its hind legs are barely stronger than its forelegs, an unusual condition in this family. This species also is diurnal and of course extremely conspicuous: it is probably protected by its copious secretion of blood, or at least one is tempted to suppose that this is the case. It is probably the case that none of these black animals are preyed upon by larger animals: wheatears are extremely wary birds, which perch on the very summit of some upstanding rock and are always ready to dive under a stone at the approach of danger: ravens, so far as one knows, act more often on the offensive than the defensive: the tenebrionids are covered with an intensely hard exoskeleton, and are probably often attacked by birds or lizards without suffering harm: I say this because I have frequently taken specimens of *Adesmia* and *Pimelia* with several legs or antennae missing, or with dinted but unpierced elytrae. Granted that the black creatures are in some way protected, and therefore not in need of protective coloration, we do not in the least know why they are black, a colour which must render them extremely hot in the desert while the sun shines. To sum up, of course I admit that the majority of animals which live in deserts are coloured protectively, but find the protection much less efficient than I had supposed. There is a small element in the fauna which is not protectively coloured, and these animals are all black.

Concentration.

One is justified in saying that concentration in point of time and of place is a characteristic of desert animals. The rain falls, the plants blossom, the animals appear and breed and are no more seen. All that is most apparent in the life of the desert is concentrated in a couple of months. Concentration in place is just as noticeable. The great leaf-bases of the date palms in Lower Mesopotamia give cover, especially after rain, to a host of animals, grass-hoppers, beetles, bugs, ants, termites, centipedes, millipedes, woodlice, scorpions, spiders, snails, oligochaets, lizards and others. In deserts generally plants do not cover the ground and every bush or patch of scrub is an oasis in itself, full of specialized forms of animal life. In some places in the Algerian deserts for instance, nearly every large stone shelters a collection of insects, myriapods, arachnids, isopods, and often also lizards, snakes, and even small mammals and birds. The hordes of migratory birds which suddenly appear and pack every bush in an oasis are yet another example of the concentration of life in the desert.

Winter and summer.

In the palaearctic desert region the butterflies for the most part are on the wing in spring, and a few appear also in the autumn. In cold weather they are not seen and presumably they pass the winter as they do in more northerly climates: that is to say some of them as eggs, some as hibernating larvae, some as pupae. In the hot weather also the majority of species are not in the imago stage, and we do not yet know what they are doing. One might suppose that the rise of temperature would cause the stages to be passed through more and more rapidly, but this is not the case. In Lower Mesopotamia (at Amara) *Colias croceus* (*edusa*) is on the wing from March to May, and in November and December; *Pieris rapae* February to May and October to December, and as a great rarity in summer. Both these species certainly produce a succession of broods in spring, and probably in autumn also. They are in fact continuous-brooded while the temperature is neither very hot nor very cold. The three lycaenids, *Zizera lysimmon*, *Tarucus mediterraneae* and *T. balcanicus*, have, I think, similar periods of emergence, but I have not a sufficient number of records for each species to enable me to state definitely that this is the case. We do not know in what stage these five species pass the summer, nor what is the factor which retards their development during the hot season. It is not lack of food for the pabula of *C. croceus* and *P. rapae* are grown as crops and irrigated all through the summer, and the Zizyphus, the food-plant of the two Taruci, is evergreen. Possibly development is retarded or inhibited when the relative humidity falls below a certain figure. The lycaenid *Chilades galba* is in quite a different biological group: at Amara it is on the wing from June to August abundantly, and rarely during early September; I have examined 80 specimens, the earliest taken 8 June. It appeared that a large number of pyralid moths were only on the wing in the heat of summer in Mesopotamia.

Belenois mesentina is a vigorous species able to maintain itself in deserts, but found in other terrains also. Grosvenor informs me that it breeds in great numbers continuously through the summer at Tank and Bojikhel, Waziristan; it is the only butterfly on the wing at that time of year. The food-plant (Capparis) is green and succulent all the summer. Possibly the butterfly's ability to continue breeding through a period of great heat is one of the factors which make it an abundant species over a very wide extent of country.

Venational Abnormalities in the Diptera. By C. G. Lamb, M.A.

[*Read* 7 March 1921.]

The published records on this subject are very scanty, only two or three notes having appeared for many years past. The most remarkable case on record is described and figured by F. W. Edwards (*Entomologist's Monthly Magazine*, 1914, p. 59). It is that of a tachydromid in which the simple venation natural to that genus suddenly flourished out into a highly complex and irregular network of cross- and accessory-veins, which show nearly every form of abnormality that the wing can be afflicted with; this physiological explosion remains unique. Dr Keilin records a case of true malformation, in which the distal part of both wings is greatly abbreviated and deformed, in the *Bull. Soc. Ent. France*, 1917, p. 194, and suggests that the condition has been brought about by pressure. The abnormalities which are now referred to are usually smaller and more regular, and are scarcely striking enough to be called teratological, but as there is apparently no definite boundary between the two sets of cases, that term will be used for convenience.

The only recently published matter bearing on the abnormalities to be considered is contained in three short papers by Kröber published in the *Zeitschrift für Insekten Biologie* in 1910, where several cases similar to the following were figured, these will be referred to in passing. Apparently the above include all that has been published on the subject of wing teratology though there are several scattered notes on antennal teratologies.

Abnormal venation is naturally more probable in those flies which have an approach to the generalized Panorpa-like venation, both from the greater number of the veins and from the presence of two possible stress systems at right angles, and consequently it is not surprising that Mr F. W. Edwards of the British Museum, who has had an exceptionally intimate knowledge of the Nematocera, informs me that he has frequently seen aberrations in the venation of that section, and that they are particularly common in the tipulid genera Eriocera and Tricyphona. On the other hand Mr J. E. Collin considers flies to be remarkably free from wing abnormalities, and that they occur only in certain species such as the one to be referred to later on. The author's experience is practically confined to the non-nematocerous families and agrees with that of Mr Collin. Neglecting one exceptional species, among the many thousands of individuals that have passed before him, there were less than 20 cases of teratological conditions, most of which are here figured. The families the author has principally dealt with are all characterised by simple venation, roughly of the syrphid or muscid type, in which the cross-veins are very few in number.

These teratological conditions are either those of deficit or of excess. The first is shown in Figs. 1 and 2; the former is the tip of the wing of *Platypeza dorsalis* showing the absence of the lower part of the fork to the fourth vein; such a break is sometimes definitely specific as in the genus Cryptophleps (Dolichopids), and on the other hand the presence or absence of such a fork may be a matter of indifference; thus in the tachinid *Rhacodineura antiqua*, the upward bend so characteristic of the family is normally absent or represented only by a small fragment of the upper portion: very rarely, as in one specimen in the Cambridge series, it is fully developed on both sides. Mr Colbran Wainwright drew the author's attention to the fact that this bend is also normally absent in the tachinids *Actia frontalis* and *Phytomyptera nitidiventris*. Fig. 2 shows the wing of *Oscinis nana*, in which the second vein definitely falls short of the costa, a very rare condition in flies. Such a non-attainment of the wing margin is not uncommon for the parts other than the costa, and may even be such a constant character as to be helpful in determination of the family. Teratology by deficit is however rare except as a definite character. In the above cases it is normally bilateral, and one of the remarkable things is that bilateral occurrence is quite usual and occurs in the next series of cases, especially in those shown in Figs. 9 to 12.

Teratology by excess takes three principal forms; the commonest is that of little hang-veins ("anhangs") which occur in many places, for example on the second long vein as in Fig. 10 or on the discal cell as in Fig. 3 which shows one on the discal cell of a specimen of *Ocydromia glabricula* (Empidae) in which family the only other recorded case of abnormal veins is that of the Tachydromia mentioned above. These hang-veins are often present normally as for example in *Ernoneura argus* (Cordylurids) where the second vein has a regular row of them hanging from it, each carrying a spot of pigment. It is also always present in some species of Parhydra (Ephydridae) at the end of the second vein, where it forms a little hook which is often quite constant for a given species, though in others it is rather variable, or, in rare instances, absent. The size of the hang-veins differs greatly; in Fig. 4 is shown one in an agromyzid which is doubled at the base, though simple on the other wing; here it is long; that in Fig. 3 is smaller, while those in Fig. 14, the case to be referred to later of *Ptilonota guttata* (Ortalidae), show various forms; the limit is when the hang-vein degenerates into a little hard spot on the vein, a sort of local breakdown in its continuity.

The second class consists in an extra cross-vein between the third and the fourth long veins as shown in Fig. 5, *Sapromyza pallidicornis* (Sapromyzidae), Fig. 6, *Minettia plumicornis* (Sapromy-

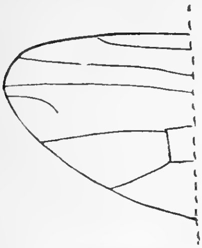


Fig. 1. *Platypeza dorsalis.*

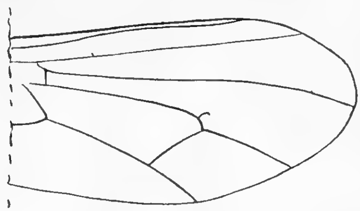


Fig. 3. *Ocydromia glabricula.*

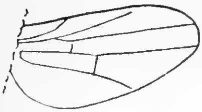


Fig. 2. *Oscinis nana.*

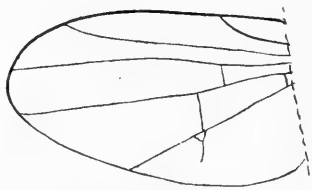


Fig. 4. *Agromyza* sp.

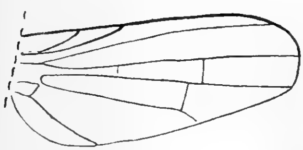


Fig. 5. *Sapromyza pallidicornis.*

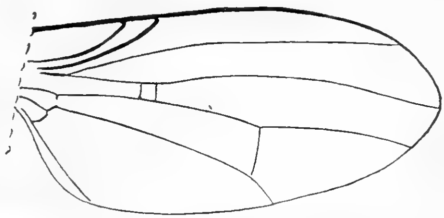


Fig. 6. *Sapromyza plumicornis.*

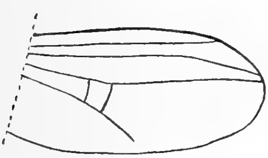


Fig. 8. *Orthochile nigrocerulea.*

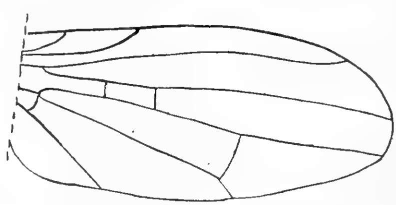


Fig. 7. *Ditaenia grisescens.*

zidae) and Fig. 7, *Ditaenia grisescens* (Sciomyzidae). The only known case in which the cross-vein at the end of the discal cell is duplicated is shown in Fig. 8, *Orthochile nigrocerulea* (Dolichopodidae).

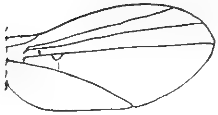


Fig. 9. *Agromyza* sp.

Fig. 10. *Cyrtoneura stabulans.*

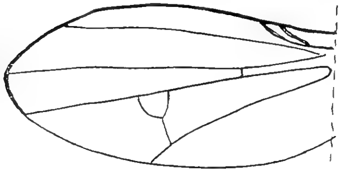


Fig. 11. *Sapromyza dedecor.*

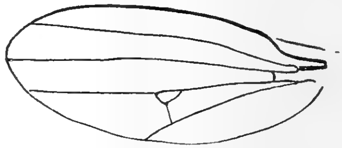


Fig. 12. *Noterophila glabra.*

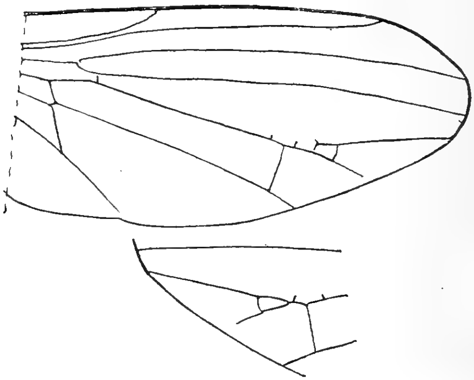


Fig. 13. *Platypeza modesta.*

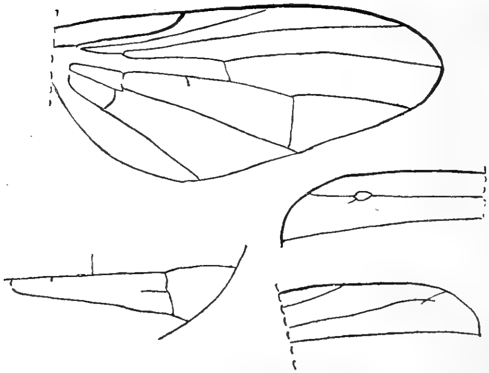


Fig. 14. *Ptilonota guttata.*

The third class is the most interesting and is shown in Fig. 9, a species of Agromyza, Fig. 10 *Cyrtoneura stabulans* (Muscidae), Fig. 11, *Sapromyza dedecor* (Sapromyzidae), and Fig. 12, *Noterophila glabra* (Drosophilidae). Kröber also figures two similar cases, in *Mydaea urbana* and *Polietes lardaria*, both in the Anthomyiidae. The small triangle formed looks just as if the cross-vein in its development had torn the fourth vein into its two constituent parts M_1 and M_2: this condition is often bilateral.

A similar formation of a triangle, but taking the form of a cross-vein between the two branches M_1 and M_2 beyond the discal cell, is shown in Fig. 13, *Platypeza modesta* (Platypezidae), this is accompanied in both wings by small hang veins. A noticeable feature is that the teratologies very frequently occur in the neighbourhood of the discal cell, that is to say, in that part of the wing where one would expect stress phenomena to occur owing to the presence of veins at right angles.

The above examples constitute almost all those that the author has met with, and they occur in single specimens of various species spread over many families: in some of those species hundreds of specimens have been seen. This remarkable degree of uniformity breaks down in a species of ortalid, *Ptilonota guttata* (Fig. 14). Here the teratological diathesis is extraordinary, taking as a rule the form of hang-veins, which vary from dots near the second vein to simple veinlets or complex forms as shown in the figure: these hang-veins are also common in the discal cell itself. Out of the 40 specimens in the Cambridge collection no fewer than 15 exhibit one at least of these abnormalities, many of them more than one. Mr Edwards has kindly informed the author that in the British Museum set, about 5 out of 21 show small defects of the nature of hang-veins or dots, and Mr J. E. Collin tells him that 9 out of his 23 specimens exhibit the same phenomenon. The species also shows much inconstancy in the acrostichal bristles; further, the author figured in the *E.M.M.* 1911, p. 216 a unique case of true Batesonian teratology occurring in the same species, the specimen having on one of the antennae a small accessory third joint bearing two extra aristas. The species seems to be endowed with extreme natural instability, and it would be of interest to hear of similar occurrences in other families of insects.

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The Cooling of a Solid Sphere with a Concentric Core of a Different Material. By Professor H. S. CARSLAW.

[*Read* 2 May 1921.]

1. Fourier himself remarked* that the present temperature gradient near the surface might be used to obtain an estimate of the time that has elapsed since the earth began to cool from its molten state. And in a paper† which attracted much attention at the time of its publication, and to which, even towards the end of his life, Kelvin attached considerable weight, he based his estimate of the Age of the Earth upon the simple mathematical problem of the distribution of temperature in a solid bounded by the plane $x = 0$ and extending to infinity in the direction of x positive; the initial temperature is constant and the boundary $x = 0$ is kept at zero.

In this case, with the usual notation, the temperature v at the time t is given by

$$v = \frac{2v_0}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{(\kappa t)}}} e^{-\alpha^2}d\alpha,$$

and

$$\frac{\partial v}{\partial x} = \frac{v_0}{\sqrt{(\pi\kappa t)}}\, e^{-\frac{x^2}{4\kappa t}},$$

the initial temperature being v_0.

In 1895 Perry reopened the question in a series of papers‡ the aim of which was to show that other possible internal conditions would give greater ages than Kelvin's estimate of 10^8 years, which was regarded by the geologists as quite inadequate. Heaviside§ made important contributions to this discussion and the problem solved in this paper is one to which he refers. He mentions that he had obtained its solution by his "operational method," but his work has not been published.

* Cf. *Œuvres de Fourier* (Darboux's Edition), T. 2, p. 284.
† *Trans. R. Soc., Edinburgh*, **23**, p. 157, 1864.
‡ *Nature*, **51**, pp. 224, 341 and 582, 1895.
§ Heaviside, *Electromagnetic Theory*, vol. 2, ch. v. 1899.

Though the discovery of radioactivity has definitely closed the controversy as to the reliability or otherwise of the results obtained by Kelvin's method, or similar methods, the mathematical problem treated in this paper seems of sufficient interest to justify the publication of a solution on lines which I have found useful in dealing with other questions of the conduction of heat.

2. It is more convenient to start with the case of a sphere whose surface is kept at a constant temperature, the initial temperature of the whole being zero.

Let the sphere be of radius b, the core being of radius a. The surface $r = b$ is kept at a constant temperature v_0.

Let the temperature, conductivity, specific heat and density of the core from $r = 0$ to $r = a$ be v_1, K_1, c_1 and ρ_1: and the corresponding quantities from $r = a$ to $r = b$ be v_2, K_2, c_2 and ρ_2.

Also let $\kappa_1 = K_1/c_1\rho_1$ and $\kappa_2 = K_2/c_2\rho_2$.

Then if we write $u_1 = v_1 r$ and $u_2 = v_2 r$, we have the equations:

$$\frac{\partial u_1}{\partial t} = \kappa_1 \frac{\partial^2 u_1}{\partial r^2}, \quad 0 < r < a \ldots(1), \qquad \frac{\partial u_2}{\partial t} = \kappa_2 \frac{\partial^2 u_2}{\partial r^2}, \quad a < r < b \ldots(1'),$$

$$u_1 = 0, \text{ when } r = 0 \ldots\ldots(2), \qquad u_2 = bv_0, \text{ when } r = b \ \ldots(2'),$$

$$u_1 = 0, \text{ when } t = 0 \ldots\ldots(3), \qquad u_2 = 0, \text{ when } t = 0 \ \ldots(3'),$$

$$u_1 = u_2, \text{ when } r = a \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$K_1\left(a\frac{\partial u_1}{\partial r} - u_1\right) = K_2\left(a\frac{\partial u_2}{\partial r} - u_2\right), \text{ when } r = a \ldots(5).$$

It is clear that

$$u_1 = A_1 \sin \alpha r \, e^{-\kappa_1\alpha^2 t},$$

$$u_2 = (A_2 \sin \mu\alpha\,(r - a) + B_2 \sin \mu\alpha\,(b - r))\, e^{-\kappa_1\alpha^2 t}$$

satisfy (1) and (1′), when $\mu = \sqrt{(\kappa_1/\kappa_2)}$.

They also satisfy (4) and (5), provided that

$$A_1 \sin \alpha a = B_2 \sin \mu\alpha\,(b - a),$$

$$K_1 A_1 [\alpha a \cos \alpha a - \sin \alpha a] = K_2 [\mu\alpha a\,(A_2 - B_2 \cos \mu\alpha\,(b - a)) - B_2 \sin \mu\alpha\,(b - a)].$$

Thus we have

$$A_2 = \frac{\sigma \cos \alpha a \sin \mu\alpha\,(b-a) + \sin \alpha a \cos \mu\alpha\,(b-a) + \dfrac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha\,(b-a)}{\sin \mu\alpha\,(b-a)} A_1,$$

$$B_2 = \frac{\sin \alpha a}{\sin \mu\alpha\,(b - a)} A_1,$$

where $$K_1 = \mu\sigma K_2.$$

Now introduce the path (P) of Fig. 1*. In this standard path the argument of α lies between 0 and $\frac{\pi}{4}$ at infinity on the right, and between $\frac{3\pi}{4}$ and π on the left. The condition (2′) at $r = b$ suggests the suitable value for A_1. Then we are led to the solutions

$$u_1 = \frac{bv_0}{\iota\pi}\int \frac{\sin \alpha r}{F(\alpha)} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha \quad \text{...(6)},$$

$$u_2 = \frac{bv_0}{\iota\pi}\int \left\{\frac{\sin \mu\alpha (r-a)}{\sin \mu\alpha (b-a)} + \frac{\sin \alpha a \sin \mu\alpha (b-r)}{F(\alpha) \sin \mu\alpha (b-a)}\right\} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha \text{...(7)},$$

where

$$F(\alpha) = \sigma \cos \alpha a \sin \mu\alpha (b-a) + \sin \alpha a \cos \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b-a),$$

and the integrals are taken over the path (P) of Fig. 1 in the α-plane.

Fig. 1.

The value of u_2 given by (7) reduces to

$$u_2 = \frac{bv_0}{\iota\pi}\int \frac{G(\alpha)}{F(\alpha)} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha \quad \text{..................(8)},$$

where

$$G(\alpha) = \sigma \cos \alpha a \sin \mu\alpha (r-a) + \sin \alpha a \cos \mu\alpha (r-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (r-a),$$

and the integrals are taken over the path (P).

The values of u_1 and u_2 given in (6) and (8), from the way in which they have been obtained, satisfy the differential equations (1) and (1′), and the conditions (2), (4) and (5), which hold when $r = 0$ and $r = a$.

Putting $r = b$ in (8), we have $\frac{bv_0}{\iota\pi}\int \frac{e^{-\kappa_1\alpha^2 t}}{\alpha} d\alpha$, over the path (P). Introduce the path (Q) of Fig. 2 formed by the path (P), the image of this path in the real axis, and the circular arcs dotted in the

* A similar path was used by me in Chapter XVIII of my book on *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, 1906. See also, for the method of this paper, *Phil. Mag., London* (Ser. 6), **39**, p. 603, 1920.

diagram, joining the ends of these two curves. Then use Cauchy's Theorem and we see that $u_2 = bv_0$ when $r = b$.

Thus the condition (2′) is satisfied.

There remains only the initial condition, u_1 and u_2 are to vanish when $t = 0$.

Now the equation

$$F(\alpha) \equiv \sigma \cos \alpha a \sin \mu\alpha (b-a) + \sin \alpha a \cos \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b-a) = 0 \quad \ldots(9)$$

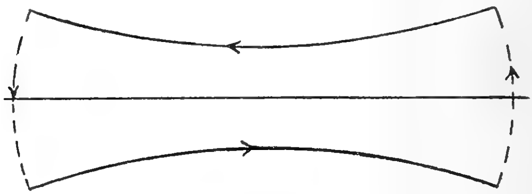

Fig. 2.

has no imaginary roots, or repeated roots, and it has an infinite number of real roots symmetrically placed with regard to the origin*.

Putting $t = 0$ in (6) and (8), we have the integrals

$$\frac{bv_0}{\iota\pi}\int \frac{\sin \alpha r}{F(\alpha)} \frac{d\alpha}{\alpha} \quad \text{and} \quad \frac{bv_0}{\iota\pi}\int \frac{G(\alpha)}{F(\alpha)} \frac{d\alpha}{\alpha}.$$

Consider the closed circuit of Fig. 3, consisting of the path (P) and the part of a circle, centre at the origin, lying above the path (P). There are no poles of these integrands within this circuit and, when the radius of the circle tends to infinity, the integral over the circular arc vanishes.

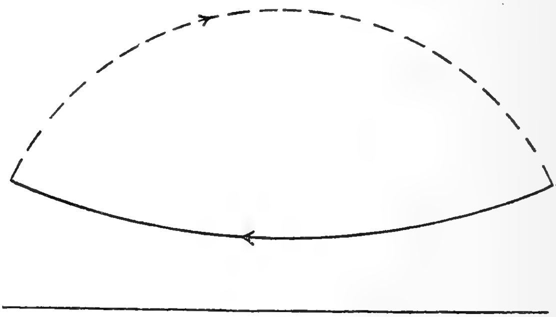

Fig. 3.

It follows that both integrals vanish over the path (P) and the initial conditions (3) and (3′) are satisfied.

* See below § 5.

Finally the solutions in (6) and (8) are obtained as infinite series. For we have

$$u_1 = \frac{bv_0}{2\iota\pi}\int \frac{\sin \alpha r}{F(\alpha)} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha}\, d\alpha,$$

and
$$u_2 = \frac{bv_0}{2\iota\pi}\int \frac{G(\alpha)}{F(\alpha)} \frac{e^{-\kappa_1\alpha^2 t}}{\alpha}\, d\alpha,$$

over the path (Q) of Fig. 2.

Then, by Cauchy's Theorem, we have

$$u_1 = rv_0 + 2bv_0 \sum_1^\infty \frac{\sin \alpha_n r}{F'(\alpha_n)} \frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n} \quad \text{...........(10)},$$

$$rv_0 + 2bv_0 \sum_1^\infty \frac{\sigma\cos\alpha_n a \sin\mu\alpha_n(r-a) + \sin\alpha_n a\cos\mu\alpha_n(r-a) + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin\alpha_n a\sin\mu\alpha_n(r-a)}{F'(\alpha_n)} \frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n} \quad \text{......(11)},$$

the summation being taken over the positive roots of (9).

3. In the discussion of § 2, put $V_1 = v_0 - v_1$ and $V_2 = v_0 - v_2$. Then V_1 and V_2 are the temperatures in the core and surrounding sphere, when the surface $r = b$ is kept at temperature zero and the initial temperature of the whole is v_0.

Also the gradient, when $r = b$, namely $\dfrac{\partial V_2}{\partial r}$, is given by the equation

$$\Big)_{r=b} = -2\mu v_0 \sum_1^\infty \frac{\sigma\cos\alpha_n a\cos\mu\alpha_n(b-a) + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin\alpha_n a\cos\mu\alpha_n(b-a) - \sin\alpha_n a\sin\mu\alpha_n(b-a)}{F'(\alpha_n)} e^{-\kappa_1\alpha_n^2 t} \quad \text{......(12)},$$

the summation extending over the positive roots of (9).

4. With the constants which Perry and Heaviside adopted,

$$a = 6{\cdot}38 \times 10^8, \quad b - a = 4 \times 10^5, \quad v_0 = 4 \times 10^3,$$
$$K_1 = {\cdot}47, \qquad K_2 = {\cdot}00595,$$
$$c_1\rho_1 = 2{\cdot}86, \qquad c_2\rho_2 = {\cdot}507,$$
$$\kappa_1 = K_1/c_1\rho_1 = {\cdot}1643, \qquad \kappa_2 = K_2/c_2\rho_2 = {\cdot}0117^*,$$
$$\mu = \sqrt{(\kappa_1/\kappa_2)} = 3{\cdot}742, \qquad \sigma = \sqrt{(K_1c_1\rho_1/K_2c_2\rho_2)} = 21{\cdot}1.$$

Thus $\mu\sigma = 79$ and $\mu\dfrac{(b-a)}{a} = 2{\cdot}35 \times 10^{-3}$.

Also the gradient of 1° in 50 ft. is 1/2743 degrees per cm.

* This corresponds to Kelvin's value of 400 for κ in foot-year units (*loc. cit.* § 15 and *Mathematical and Physical Papers*, vol. 3, p. 302).

The equation (9) is

$$F(\alpha) \equiv \sigma \cos \alpha a \sin \mu\alpha (b-a) + \sin \alpha a \cos \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b-a) = 0.$$

The roots of this equation will be the common roots, if any, of

$$\left.\begin{aligned} \sin \alpha a &= 0 \\ \sin \mu\alpha (b-a) &= 0 \end{aligned}\right\} \quad \text{.........................(13)},$$

and the roots of

$$\sigma \cot \alpha a + \cot \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} = 0 \quad \text{......(14)}.$$

Since $(b-a)/a$ is small, the values of αa, if any, given by (13) will be large. Thus for our solution we require only the earlier roots of (14), which, with the above constants, reduces to

$$21{\cdot}1 \cot x + \cot (2{\cdot}35 \times 10^{-3} x) = \frac{20{\cdot}8}{x} \quad \text{........(15)},$$

where $x = \alpha a$.

It will be found* that the first two roots of (15) are

$$x_1 = 2{\cdot}9871 \text{ or } 180^\circ - 8^\circ 51',$$

$$x_2 = 5{\cdot}980 \text{ or } 360^\circ - 17^\circ 22',$$

and that x_n lies between $n\pi - \frac{1}{2}\pi$ and $n\pi$.

Taking the first term only, the value of t is required for which

$$\frac{1}{2743} = -2\mu v_0 \left(\frac{\sigma \cot \alpha_1 a \cot \mu\alpha_1 (b-a) + \dfrac{1-\mu\sigma}{\mu\alpha_1 a} \cot \mu\alpha_1 (b-a) - 1}{\sigma a \operatorname{cosec}^2 \alpha_1 a + \mu (b-a) \operatorname{cosec}^2 \mu\alpha_1 (b-a) + \dfrac{1-\mu\sigma}{\mu\alpha_1^2 a}} \right) e^{-\kappa_1 \alpha_1^2 t} \text{...(16)}.$$

This gives the equation

$$\frac{1}{2743} = 8 \times 3{\cdot}742 \times 10^{-4} \times \frac{2{\cdot}0298}{5{\cdot}9763} \times e^{-{\cdot}1643 \times \left(\frac{2{\cdot}9871}{6{\cdot}38 \times 10^8}\right)^2 t},$$

since the numerator and denominator of the fraction on the right of (16) are, respectively, $-2{\cdot}0298 \times 10^4$ and $5{\cdot}9763 \times 10^{11}$.

Reducing the answer to years, this leads to $9{\cdot}02 \times 10^9$ years†.

* I am indebted to Mr R. J. Lyons for the solution of this equation.

† Heaviside (*loc. cit.* p. 19) gives $9{\cdot}03 \times 10^9$ but adds that he has not taken special pains to get the third figure right.

As Heaviside compares his result with Perry's for the problem when the capacity of the skin is neglected, it may be worth while to point out that some arithmetical errors have crept into Perry's solution. (Cf. *Nature*, **51**, p. 225.)

The first two roots of the equation $19 \tan \alpha + \alpha = 0$ are

$$\alpha_1 = 2{\cdot}985676 \text{ or } 180^\circ - 8^\circ 56',$$

$$\alpha_2 = 5{\cdot}9783345 \text{ or } 360^\circ - 17^\circ 28'.$$

The first term in the series for the temperature is 138·13 instead of 142·7, and the second term is 4·82 instead of 5·65 given by Perry.

5. In § 2 it has been assumed that the equation (9) has no imaginary roots or repeated roots, and that it has an infinite number of real roots, symmetrically placed with regard to the origin.

There is no difficulty at all in proving that the equation has no pure imaginary roots, and that its real roots are not repeated; and, since $F(\alpha)$ is an odd function of α, to every positive root α, there corresponds a negative root $-\alpha$.

It is harder to prove that there is no root of the form $\xi \pm \iota\eta$, where ξ and η do not vanish. To prove this, we need to show that the equation

$$\sigma \cot \alpha a + \cot \mu\alpha (b-a) + \frac{1-\mu\sigma}{\mu\alpha a} = 0$$

has no root of this form, when μ, σ, a and b are real and positive and $b > a$.

$$\left.\begin{aligned} &\text{Let} && U_1 = \sin \alpha r,\ 0 < r < a \\ &\text{and} && U_2 = \frac{\sin \mu\alpha (b-r)}{\sin \mu\alpha (b-a)} \sin \alpha a,\ \ a < r < b \end{aligned}\right\} \quad \ldots\ldots\ldots(17).$$

$$\text{Then} \quad \frac{d^2U_1}{dr^2} + \alpha^2 U_1 = 0 \text{ and } \frac{d^2U_2}{dr^2} + \mu^2\alpha^2 U_2 = 0 \quad \ldots\ldots(18).$$

Also

$$U_1 = 0, \text{ when } r = 0 : U_1 = U_2, \text{ when } r = a : U_2 = 0, \text{ when } r = b \quad \ldots\ldots\ldots\ldots\ldots(19).$$

Further, if α is a root of our equation (14),

$$a\frac{dU_2}{dr} - U_2 = \mu\sigma\left(a\frac{dU_1}{dr} - U_1\right), \text{ when } r = a \quad \ldots(20).$$

Now let α, β be two roots of equation (14).

Also let U_1, U_2 be as above, and let V_1, V_2 be the corresponding expressions, when β is substituted for α.

Then

$$(\alpha^2-\beta^2)\left(\int_0^a U_1V_1\,dr + \frac{\mu}{\sigma}\int_a^b U_2V_2\,dr\right)$$

$$= \int_0^a (U_1V_1'' - V_1U_1'')\,dr + \frac{1}{\mu\sigma}\int_a^b (U_2V_2'' - V_2U_2'')\,dr, \text{ by (18)}$$

$$= \Big[U_1V_1' - V_1U_1'\Big]_0^a + \frac{1}{\mu\sigma}\Big[U_2V_2' - V_2U_2'\Big]_a^b$$

$$= (U_1V_1' - V_1U_1') - \frac{1}{\mu\sigma}(U_1V_2' - V_1U_2'), \text{ when } r = a, \text{ by (19)}$$

$$= 0, \text{ by (20)}.$$

But if α and β are conjugate imaginaries $\xi \pm \iota\eta$, U_1 and V_1 are conjugate imaginaries; also U_2 and V_2 are conjugate.

Thus $\int_0^a U_1 V_1\, dr$ and $\int_a^b U_2 V_2\, dr$ are both positive.

But we have just shown that

$$(\alpha^2 - \beta^2)\left(\int_0^a U_1 V_1 dr + \frac{\mu}{\sigma}\int_a^b U_2 V_2\, dr\right) = 0.$$

If follows that our equation cannot have roots of the form $\xi \pm \iota\eta$, when neither ξ nor η vanish.

[*Added* June 3, 1921].

6. My attention has been called to the fact that the problem with which this paper deals was set in the Mathematical Tripos, Part II, 1904, 4 June, 2–5 p.m. In that paper Question 7 reads as follows:

A solid sphere of conductivity k and diffusivity a^2 and of radius b is enclosed in a spherical shell of conductivity k' and diffusivity a'^2 and of internal and external radii b and c. Initially the whole system is at uniform temperature u_0 and from the epoch $t = 0$ onwards the surface $r = c$ is kept at zero temperature. Prove that at any subsequent time t the temperature at a distance r from the centre is given by equations of the form

$$u = \sum_s A_s \frac{\sin \lambda_s r}{\sin \lambda_s b}\frac{1}{r} e^{-\lambda_s^2 a^2 t}, \text{ when } b > r > 0,$$

$$u = \sum_s A_s \frac{\sin \lambda_s' (c-r)}{\sin \lambda_s' (c-b)}\frac{1}{r} e^{-\lambda_s^2 a^2 t}, \text{ when } c > r > b,$$

where $\lambda_s' a' = \lambda_s a$, and λ_s is the sth root in order of increasing magnitude of the equation

$$k'a \tan \lambda b + ka' \tan \lambda' (c-b) = 0,$$

further

$$A_s = \frac{2u_0}{\lambda_s^2} \frac{\{(k-k') \sin \lambda_s' (c-b) + k'c\lambda_s'\} \sin^2 \lambda_s b \sin^2 \lambda_s' (c-b)}{kb \sin^2 \lambda_s' (c-b) + (a^2/a'^2)\, k' (c-b) \sin^2 \lambda_s b}.$$

Explain the bearing of a numerical solution of this problem on the calculation of the age of the Earth.

This solution of our problem is wrong. The equations which the temperatures (u, u') must satisfy at $r = b$ are

$$\left.\begin{aligned} u &= u' \\ k\frac{\partial u}{\partial r} &= k'\frac{\partial u'}{\partial r} \end{aligned}\right\},$$

so that, if we take terms of the type given in the summation, the second of these equations leads to

$$b\,(k\lambda_s \cot \lambda_s b + k'\lambda_s' \cot \lambda_s' (c-b)) = k - k'.$$

It is clear that the mistake arose from taking

$$k\frac{\partial}{\partial r}(ur) = k'\frac{\partial}{\partial r}(u'r),$$

instead of the proper equation at this surface.

As a matter of fact the equation for λ given in the Tripos question arises in the corresponding problem for linear flow of heat, when the equations for the temperature, with this notation, are as follows:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < b: \quad \frac{\partial u'}{\partial t} = a'^2 \frac{\partial^2 u'}{\partial x^2}, \quad b < x < c.$$

$$u = 0, \text{ when } x = 0 : u' = 0, \text{ when } x = c.$$

$$u = u_0, \text{ when } t = 0 : u' = u_0, \text{ when } t = 0.$$

$$\left.\begin{aligned} u &= u' \\ k\frac{\partial u}{\partial x} &= k'\frac{\partial u'}{\partial x}\end{aligned}\right\}, \text{ when } x = b.$$

And this problem* can be solved by Fourier's method on the same lines as the solution of the sphere problem which is given in the next sections, a solution suggested by this Tripos question.

7. With the notation of § 2, the sphere problem reduces, on substituting $vr = u$, to the solution of the following equations:

$$\frac{\partial u_1}{\partial t} = \kappa_1 \frac{\partial^2 u_1}{\partial r^2}, \quad 0 < r < a \ldots(21), \quad \frac{\partial u_2}{\partial t} = \kappa_2 \frac{\partial^2 u_2}{\partial r^2}, \quad a < r < b \ldots(21'),$$

$$u_1 = 0, \text{ when } r = 0 \ldots\ldots(22), \quad u_2 = 0, \text{ when } r = b \ \ldots(22'),$$

$$u_1 = v_0 r, \text{ when } t = 0 \ldots\ldots(23), \quad u_2 = v_0 r, \text{ when } t = 0 \ \ldots(23'),$$

$$u_1 = u_2, \text{ when } r = a \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

$$K_1\left(a\frac{\partial u_1}{\partial r} - u_1\right) = K_2\left(a\frac{\partial u_2}{\partial r} - u_2\right), \text{ when } r = a \ldots(25).$$

As before

$$\left.\begin{aligned} u_1 &= \sin \mu\alpha (b - a) \sin \alpha r \, e^{-\kappa_1 \alpha^2 t} \\ u_2 &= \sin \alpha a \sin \mu\alpha (b - r) \, e^{-\kappa_1 \alpha^2 t}\end{aligned}\right\} \ldots\ldots\ldots\ldots(26)$$

satisfy (21), (21′), (22), (22′) and (24), provided that $\mu^2 = \kappa_1/\kappa_2$.

Further (25) is satisfied, if

$$F(\alpha) \equiv \sigma \sin \mu\alpha (b - a) \cos \alpha a + \sin \alpha a \cos \mu\alpha (b - a)$$
$$+ \frac{1 - \mu\sigma}{\mu\alpha a} \sin \alpha a \sin \mu\alpha (b - a) = 0 \ldots(27),$$

where $K_1 = K_2 \mu\sigma$.

This is the equation in α given in § 2 (9).

Unless $\mu (b - a)/a$ is rational, the only roots of $F(\alpha) = 0$ are those of

$$\sigma \cot \alpha a + \cot \mu\alpha (b - a) + \frac{1 - \mu\sigma}{\mu\alpha a} = 0 \ldots\ldots\ldots(28).$$

* See also §§ 5, 6 of my paper cited above.

In the first place we confine ourselves to the case when $\mu(b-a)/a$ is not rational.

As in § 5, let α, β be two different positive roots of (28).

Also let

$$\left.\begin{aligned} U_1 &= \sin\mu\alpha(b-a)\sin\alpha r, \quad 0<r<a \\ U_2 &= \sin\alpha a\sin\mu\alpha(b-r), \quad a<r<b \end{aligned}\right\} \quad \text{.........(29).}$$

And let V_1, V_2 be the corresponding expressions when β is substituted for α.

We know from § 5 that

$$\int_0^a U_1V_1\,dr + \frac{\mu}{\sigma}\int_a^b U_2V_2\,dr = 0 \quad \text{...........(30).}$$

Following Fourier's method, our solution is obtained by expanding $v_0 r$ in an infinite series of these terms:

$$\left.\begin{aligned} v_0 r &= \sum_n A_n \sin\mu\alpha_n(b-a)\sin\alpha_n r, \quad 0<r<a \\ &= \sum_n A_n \sin\alpha_n a\sin\mu\alpha_n(b-r), \quad a<r<b \end{aligned}\right\} \quad \text{...(31).}$$

Then with the usual assumptions as to the possibility of this expansion and of term by term integration of the series, we have from (30)

$$A_n\left[\sin^2\mu\alpha_n(b-a)\int_0^a \sin^2\alpha_n r\,dr + \frac{\mu}{\sigma}\sin^2\alpha_n a\int_a^b \sin^2\mu\alpha_n(b-r)\,dr\right]$$

$$= v_0\left[\sin\mu\alpha_n(b-a)\int_0^a r\sin\alpha_n r\,dr\right.$$

$$\left.+\frac{\mu}{\sigma}\sin\alpha_n a\int_a^b r\sin\mu\alpha_n(b-r)\,dr\right].$$

Evaluating these integrals and using (28), it will be found that

$$A_n = \frac{2bv_0\sin\alpha_n a}{\alpha_n\left(a\sigma\sin^2\mu\alpha_n(b-a)+\mu(b-a)\sin^2\alpha_n a+\dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin^2\alpha_n a\sin^2\mu\alpha_n(b-a)\right)} \quad \text{...(32).}$$

Also, from (26), it will be seen that the solution of the problem is

$$u_1 = 2bv_0\sum\frac{\sin\alpha_n a\sin\mu\alpha_n(b-a)\sin\alpha_n r}{a\sigma\sin^2\mu\alpha_n(b-a)+\mu(b-a)\sin^2\alpha_n a+\dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin^2\alpha_n a\sin^2\mu\alpha_n(b-a)}\,\frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n} \quad \text{...(33),}$$

$$u_2 = 2bv_0\sum\frac{\sin^2\alpha_n a\sin\mu\alpha_n(b-r)}{a\sigma\sin^2\mu\alpha_n(b-a)+\mu(b-a)\sin^2\alpha_n a+\dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin^2\alpha_n a\sin^2\mu\alpha_n(b-a)}\,\frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n} \quad \text{...(34).}$$

This agrees with the results given in § 2 (10) and (11), if we remember that u_1 and u_2 now correspond to (rv_0-u_1) and (rv_0-u_2) of that section.

8. The case when $\mu\,(b-a)/a$ is rational remains to be discussed. Suppose that it is equal to p/q, a positive fraction in its lowest terms.

Then the equation $F\,(\alpha) = 0$ [cf. § 7 (27)] is satisfied by

$$a\alpha = q\pi,\ 2q\pi,\ \text{etc.}$$

as well as by the roots of (28).

When $a\alpha = sq\pi$, it will be seen that $\mu\alpha\,(b-a) = sp\pi$.

Thus, in addition to the terms of (26),

$$\left.\begin{aligned} u_1 &= \sin \mu\alpha_n\,(b-a)\sin\alpha_n r\, e^{-\kappa_1\alpha_n^2 t} \\ u_2 &= \sin\alpha_n a \sin\mu\alpha_n\,(b-r)\, e^{-\kappa_1\alpha_n^2 t}\end{aligned}\right\},$$

where α_n is a root of (28), we have

$$\left.\begin{aligned} u_1 &= \mu K_2 \cos ps\pi \sin qs\pi\frac{r}{a}\, e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t} \\ u_2 &= -\,K_1\cos qs\pi\sin ps\pi\left(\frac{b-r}{b-a}\right) e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t}\end{aligned}\right\} \quad \ldots(35),$$

where s is any positive integer.

The theorem of § 5 [cf. § 7 (30)] applies to all the solutions

$$\left.\begin{aligned} &\sin\mu\alpha_n\,(b-a)\sin\alpha_n r \\ &\sin\alpha_n a\sin\mu\alpha_n\,(b-r)\end{aligned}\right\},$$

and

$$\left.\begin{aligned} &\mu K_2\cos ps\pi\sin qs\pi\frac{r}{a} \\ -\,&K_1\cos qs\pi\sin ps\pi\left(\frac{b-r}{b-a}\right)\end{aligned}\right\}.$$

Thus we assume that $v_0 r$ can be expanded in an infinite series with terms of the type

$$\left.\begin{aligned} &A_n\sin\mu\alpha_n\,(b-a)\sin\alpha_n r, && 0<r<a \\ &A_n\sin\alpha_n a\sin\mu\alpha_n\,(b-r), && a<r<b\end{aligned}\right\},$$

and

$$\left.\begin{aligned} &A_s\,\mu K_2\cos ps\pi\sin qs\pi\frac{r}{a}, && 0<r<a \\ -\,&A_s\,K_1\cos qs\pi\sin ps\pi\left(\frac{b-r}{b-a}\right), && a<r<b\end{aligned}\right\}.$$

The coefficient A_n has been found above in (32). The coefficient A_s is given by

$$\begin{aligned} A_s\Big[\mu^2K_2^2\cos^2 ps\pi\int_0^a \sin^2 qs\pi\frac{r}{a}\,dr \\ +\frac{\mu}{\sigma}K_1^2\cos^2 qs\pi\int_a^b\sin^2 ps\pi\left(\frac{b-r}{b-a}\right)dr\Big] \\ = v_0\Big[\mu K_2\cos ps\pi\int_0^a r\sin qs\pi\frac{r}{a}\,dr \\ -\frac{\mu}{\sigma}K_1\cos qs\pi\int_a^b r\sin ps\pi\left(\frac{b-r}{b-a}\right)dr\Big].\end{aligned}$$

It follows that

$$A_s = -\frac{2bv_0}{ps\pi}\frac{(b-a)\cos qs\pi}{K_2\left(a+\dfrac{K_1}{K_2}(b-a)\right)}.$$

Thus

$$A_s = -\frac{2bv_0 \cos qs\pi}{s\pi\mu K_2(\sigma p+q)},\ \text{since } K_1 = \mu\sigma K_2 \quad \ldots\ldots(36).$$

Also these terms in our solution are as follows:

$$\left.\begin{aligned} &-2bv_0\frac{\cos ps\pi\cos qs\pi\sin qs\pi\dfrac{r}{a}}{s\pi(\sigma p+q)}e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t} \\ \text{and}\qquad &2bv_0\frac{\sigma\sin ps\pi\left(\dfrac{b-r}{b-a}\right)}{s\pi(\sigma p+q)}e^{-\kappa_1\frac{q^2s^2\pi^2}{a^2}t}\end{aligned}\right\} \quad \ldots\ldots(37).$$

On referring to § 2 (10) and (11), it will be seen that this solution agrees with the result obtained in § 2 by the method of contour integrals. And it may be noted that the assumptions involved in Fourier's method were not necessary in the discussion of § 2.

Symbolical methods in the theory of Conduction of Heat. By Dr T. J. I'a. Bromwich, F.R.S.

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In 1914, I communicated a paper "Normal Coordinates in Dynamical Systems" to the London Mathematical Society*; and I explained there (§§ 2, 3, 4) the relation of my methods to the symbolical methods used by Heaviside for various Electrical problems. But I reserved† any general application of the corresponding methods in Conduction of Heat and Diffusion; owing to the pressure of the War and other difficulties I have not had leisure to arrange my results for publication until now.

I have given here only the special points which are suggested in connexion with problems arising out of the question of the "Age of the Earth" as handled by Heaviside‡ and Perry; reference is also made to the paper by Prof. H. S. Carslaw on the same topic, communicated at the same time as this paper.

Not much importance is attached to the estimate made here ($9{\cdot}37 \times 10^9$ years) for the age of the Earth; but the same *data* have been adopted as in the original suggestion of Perry, that the internal conductivity and heat capacity affected the estimate more than the corresponding constants of the skin. It would be easy to estimate new values of κ and k with which the formula of § 2 (for g/v_0) would yield almost any value of t from 10^9 to 10^{10} years§. The ease of manipulating the constants in the formula of § 2 is in marked contrast to the labour involved in solving the same problem by means of a Fourier-expansion; and mistakes are far more quickly detected in the numerical work. Some comparisons of the work will be found in § 3.

The contents of the paper are as follows:

§ 1. General consideration of a method for solving Conduction of Heat problems.

§ 2. Application of Heaviside's method to the problem of a sphere surrounded by a shell of different material‖.

* *Proc. Lond. Math. Soc.* ser. 2, vol. 15, 1916, p. 401.

† *L.c.* p. 402.

‡ *Electromagnetic Theory*, vol. 2, §§ 227–237.

§ Compare Heaviside's estimates (*l.c.* §§ 232, 236). On radio-active grounds, the most recent estimates appear to be from 3×10^9 to 5×10^9 years (H. N. Russell, *Proc. Roy. Soc.* A, vol. XCIX. (1921), p. 84).

‖ This problem appears to have been solved originally by Heaviside himself (see his remarks in *Electromagnetic Theory*, vol. 2, § 230), but his solution was not published. A solution with an erroneous result was set as a question in Part II of the Mathematical Tripos, 1904.

§ 1. *General consideration of a method for solving Conduction of Heat problems.*

Using the notation explained in §§ 4, 5 of my paper quoted above, it is evident that the typical solution of a Conduction of Heat problem, with a solid originally at zero temperature, and the surface of the solid maintained at constant temperature v_0, is of the form

$$v = \frac{v_0}{2\pi\iota}\int_{a-\iota\infty}^{a+\iota\infty} e^{\lambda t} u \frac{d\lambda}{\lambda}. \qquad (a > 0)$$

Here u is to reduce to unity at the surface and is to satisfy the differential equation

$$\kappa\Delta_2 u - \lambda u = 0$$

at other points in the solid, where Δ_2 is Laplace's operator and $\kappa = k/c$ is the fundamental constant of the heat-equation in the solid; k is the conductivity and c the heat capacity (per unit *volume*).

To connect this formula with Prof. Carslaw's (*l.c.* § 2) is easy; if we write

$$\lambda = -\kappa\theta^2,$$

u will become U say, where U satisfies the equation

$$\Delta_2 U + \theta^2 U = 0$$

and also reduces to unity at the surface. The integral for v becomes

$$v = \frac{v_0}{\pi\iota}\int e^{-\kappa\theta^2 t} U \frac{d\theta}{\theta},$$

where the path of integration is given by

$$-\kappa(\xi^2 - \eta^2) = a, \quad \text{or} \quad \eta^2 - \xi^2 = a/\kappa, \quad \text{if} \quad \theta = \xi + \iota\eta.$$

The beginning and end of the path correspond to the two points given by

$$-2\kappa\xi\eta \to -\infty, \quad -2\kappa\xi\eta \to +\infty;$$

thus, choosing the upper half of the rectangular hyperbola

$$\eta^2 - \xi^2 = c/\kappa,$$

the path will start at infinity in the first quadrant and will end at infinity in the second quadrant as sketched.

This path agrees with Carslaw's path (P), and so the integral v is identical with his, when U is written out at length.

To explain the connexion of these complex integrals with Heaviside's symbolical treatment is also easy; Heaviside writes symbolically

$$p = \frac{\partial}{\partial t} = \kappa q^2,$$

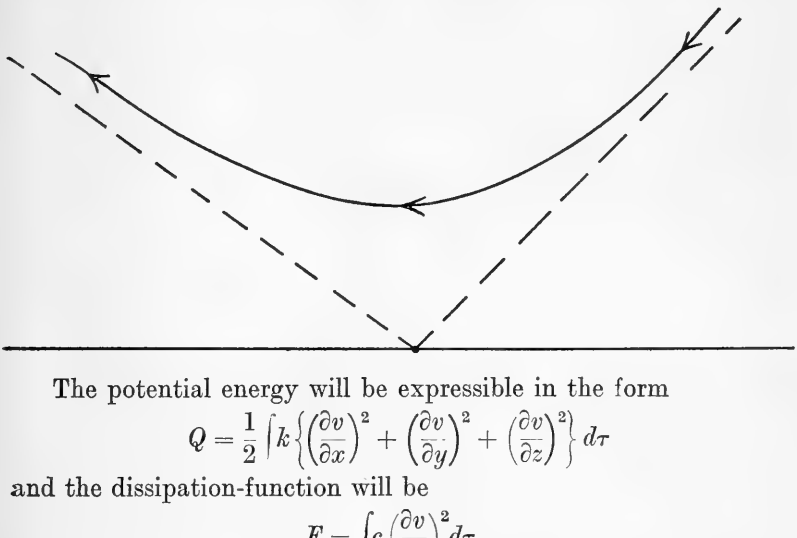
and then solves the equation

$$\Delta_2 V - q^2 V = 0$$

subject to the condition $V = 1$ at the surface.

It is evident that the function V will be equivalent to U on writing $\theta = \iota q$, or $q = -\iota\theta$, and thus Heaviside's forms of the solutions can be translated at once into complex integrals, if desired*; and it has been proved (see § 4 of my L.M.S. paper) that in general Heaviside's standard method of interpreting his symbolical solutions is equivalent to the evaluation of the original complex integral as a sum of residues (taken for all the poles of the function u/λ).

Before leaving these general considerations it will be convenient to note briefly the theorems relating to the special differential equation of Diffusion which follow immediately on the lines of §§ 8, 9 in my paper previously quoted.



The potential energy will be expressible in the form

$$Q = \frac{1}{2}\int k\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\} d\tau$$

and the dissipation-function will be

$$F = \int c\left(\frac{\partial v}{\partial t}\right)^2 d\tau,$$

both integrals being taken through the volume of the solid; and here there is no kinetic energy function.

Thus, if $\lambda = \alpha_1, \alpha_2$ are two distinct poles of the function u, and if ϕ_1, ϕ_2 are the corresponding residues of the function, we find (as in formula (75) in § 9 of my L.M.S. paper)

$$\int c\phi_1\phi_2 d\tau = 0 \text{ and } \int k\left(\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x} + \frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial y} + \frac{\partial\phi_1}{\partial z}\frac{\partial\phi_2}{\partial z}\right) d\tau = 0.$$

* Since this relation gives $q = \eta - \iota\xi$, it follows that the path selected above corresponds to one in which *the real part of q is positive*; and this agrees with the convention adopted in § 4 below.

Thus *it follows that the poles α_1, α_2 are purely real*; for if α_1, α_2 are supposed to be conjugate complexes, so also are ϕ_1, ϕ_2; and thus the product $c\phi_1\phi_2$ is essentially positive, so that the integral $\int c\phi_1\phi_2\, d\tau$ could not vanish.

Further we have $F = -\dfrac{\partial Q}{\partial t}$, and taking the special value ϕ_1 for v, we find that

$$\alpha_1^2 \int c\phi_1^2 d\tau = -\alpha_1 \int k \left\{ \left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2 \right\} d\tau,$$

and (since α_1 is not zero) it is evident that α_1 must be *negative**.

The corresponding poles for the functions U will be *real* (since $\kappa\theta^2 = -\lambda$) and will occur in pairs given by $\theta = \pm\sqrt{(-\alpha_1/\kappa)}$.

There is no need for the thermal coefficients c and k to be constants in these theorems, provided that at any surface of discontinuity, the functions v and $k\dfrac{\partial v}{\partial \nu}$ are continuous, where $\partial/\partial\nu$ implies differentiation along the normal to the surface of discontinuity.

It will be noticed that the theorem in § 5 of Prof. Carslaw's paper follows at once, because c has one of two constant values; and the ratio of these constants is equal to

$$\frac{k_2}{\kappa_2}\Big/\frac{k_1}{\kappa_1} = \frac{\kappa_1}{\kappa_2}\Big/\frac{k_1}{k_2} = \frac{\mu^2}{\mu\sigma} = \frac{\mu}{\sigma}.$$

Hence
$$\int_0^a r^2\phi_1\phi_2\, dr + \frac{\mu}{\sigma}\int_a^b r^2\phi_1\phi_2 dr = 0,$$

which is equivalent to the result obtained by Prof. Carslaw.

§ 2. *Application of Heaviside's method.*

We can illustrate Heaviside's method conveniently by solving symbolically the problem of a sphere of radius a and heat-constants k, c, surrounded by a thin shell of thickness $l = b - a$ and heat-constants k_1, c_1. The solid so formed is initially at zero temperature, and at a certain instant ($t = 0$) the outer surface is raised to temperature v_0 and is maintained at this temperature afterwards.

We shall write
$$\kappa = k/c, \quad \kappa_1 = k_1/c_1,$$

and
$$p = \frac{\partial}{\partial t} = \kappa q^2 = \kappa_1 q_1^2.$$

Then suppose that V reduces to the value A at $r = a$; the differential equations of the problem reduce to

$$\frac{\partial^2}{\partial r^2}(rV) = q^2(rV), \quad \frac{\partial^2}{\partial r^2}(rV_1) = q_1^2(rV_1),$$

* It should be noticed that we cannot draw this conclusion immediately from the corresponding theorem of § 9 of my L.M.S. paper.

which yield the formulae

$$V = \frac{Aa \sinh qr}{r \sinh qa}, \quad V_1 = \frac{Aa \sinh q_1 (b - r)}{r \sinh q_1 l} + \frac{b \sinh q_1 (r - a)}{r \sinh q_1 l},$$

because $V = V_1 = A$ at $r = a$, and $V_1 = 1$ at $r = b$.

Further we are to have

$$k \frac{\partial V}{\partial r} = k_1 \frac{\partial V_1}{\partial r} \quad \text{at } r = a,$$

and so

$$kA \left(q \coth qa - \frac{1}{a}\right) = - k_1 A \left(q_1 \coth q_1 l + \frac{1}{a}\right) + \frac{k_1 q_1 b}{a \sinh q_1 l},$$

giving* $$A = \frac{q_1 b/\sinh q_1 l}{(k/k_1)(qa \coth qa - 1) + (q_1 a \coth q_1 l + 1)}.$$

If we write $kl/k_1 b = s$, this formula can be written

$$A = \frac{q_1 l/\sinh q_1 l}{s (qa \coth qa - 1) + (a/b)(q_1 l \coth q_1 l + l/a)}.$$

The problem actually amounts to evaluating

$$\frac{g}{v_0} = \frac{\partial V_1}{\partial r} \quad \text{at } r = b,$$

where g is the gradient of temperature at the outer surface of the shell at time t. It will be found at once that

$$\frac{g}{v_0} = - \frac{Aa}{b} \frac{q_1}{\sinh q_1 l} + q_1 \coth q_1 l - \frac{1}{b}.$$

An expression for g as a Fourier-sum can be found by Heaviside's general process; but as this should be equivalent to the result calculated by Prof. Carslaw, I do not stop to write out the result and proceed to simplify the above formulae by means of approximations suitable to the data of this particular problem. These approximations correspond to (i) treating qa as having a sufficiently large real part to allow $\coth qa$ to be replaced by unity and (ii) treating $q_1 l$ as small.

The above formulae reduce then to the following, on rejecting $(q_1 l)^4$ and higher powers,

$$A = \frac{1 - \frac{1}{6}(q_1 l)^2}{1 - s + sqa + \frac{1}{3}(a/b)(q_1 l)^2},$$

$$\frac{g}{v_0} = - \frac{Aa}{bl} \{1 - \tfrac{1}{6}(q_1 l)^2\} + \frac{1}{l} - \frac{1}{b};$$

* On writing $q = \iota\alpha$, $q_1 = \iota\mu\alpha$, $k/k_1 = \mu\sigma$, it will be found that

$$Aa = b \sin \alpha a / F(\alpha),$$

where $F(\alpha)$ is the function defined in Prof. Carslaw's paper. Then V_1 is easily seen to lead to formula (7) of that paper.

where in the last formula the relation* $q_1{}^2 = 0$ has been used in simplifying $q_1 \coth q_1 l$.

Hence
$$\frac{g}{v_0} = \frac{a}{bl}\{1 - A + \tfrac{1}{6}A(q_1 l)^2\}$$
$$= \frac{a}{bl}\left\{1 - \frac{1 - \frac{1}{3}(q_1 l)^2}{1 - s + sqa + \frac{1}{3}(q_1 l)^2}\right\},$$
where we write 1 for a/b in the small term containing $(q_1 l)^2$ in the denominator.

Write for brevity $\gamma = 1 - s + sqa$ and then
$$\frac{g}{v_0} = \frac{a}{bl}\left\{1 - \frac{1 - \frac{1}{3}(q_1 l)^2}{\gamma} + \frac{\frac{1}{3}(q_1 l)^2}{\gamma^2}\right\}$$
$$= \frac{a}{bl}\left\{1 - \frac{1}{\gamma} + \tfrac{1}{3}(q_1 l)^2\left(\frac{1}{\gamma} + \frac{1}{\gamma^2}\right)\right\}.$$

Following Heaviside, we proceed to expand $1/\gamma$ in powers of $sqa/(1-s)$, and then retaining only terms in q, q^3 (the terms in q^2, $q_1{}^2$ being identically zero), we find
$$\frac{g}{v_0} = \frac{a}{bl}\left[-n + \frac{1}{1-s}\left\{nqa + n^3q^3a^3 - \tfrac{1}{3}(q_1 l)^2\left(nqa + \frac{2nqa}{1-s}\right)\right\}\right],$$
where, for brevity, we write $n = s/(1-s)$.

In the actual problem s is small (about $\frac{1}{20}$) and so in the last term of the formula we may put
$$2n/(1-s) = 2n$$
without sensible error, and then the term in l^2 reduces to
$$-\tfrac{1}{3}(q_1 l)^2(3nqa) = -nl^2aq_1{}^2q.$$

To interpret these formulae we use the results†
$$q = \frac{1}{\sqrt{(\pi\kappa t)}}, \quad q^3 = -\frac{1}{2\kappa t}\frac{1}{\sqrt{(\pi\kappa t)}};$$
then
$$q_1{}^2q = \frac{\kappa}{\kappa_1}q^3 = -\frac{1}{2\kappa_1 t}\frac{1}{\sqrt{(\pi\kappa t)}}.$$

Hence we find that
$$\frac{g}{v_0} = \frac{na}{bl}\left[-1 + \frac{a}{(1-s)\sqrt{(\pi\kappa t)}}\left(1 - \frac{n^2a^2}{2\kappa t} + \frac{l^2}{2\kappa_1 t}\right)\right],$$
which is a slight extension of Heaviside's formula, given for the case when the thickness of the shell is negligible‡.

* See § 4 (iv) below.

† § 4 below (i) and (ii).

‡ *Electromagnetic Theory*, vol. 2, § 236, formula (39). The extension consists merely in the addition of the last two terms in the bracket: that terms of this type would be present could be foreseen from Heaviside's formula (27) in § 229 (for the corresponding *plane* problem). There is also the external factor a/b in the present formula.

To estimate the numerical values of these terms is easy; take the *data* adopted in Prof. Carslaw's paper (following Perry and Heaviside),

$$l = 4 \times 10^5, \quad a = 6{\cdot}38 \times 10^8, \quad \kappa = {\cdot}1643, \quad \kappa_1 = {\cdot}0117.$$

Then $$s = {\cdot}04953, \quad n = {\cdot}05211.$$

After a little trial it was found that a value of t about 3×10^{17} (that is, about $9{\cdot}36 \times 10^9$ years) would fit the *data* for g and v_0 used by Lord Kelvin (see below): and then it appears that

$$\tfrac{1}{2}l^2/\kappa_1 t = 2{\cdot}3 \times 10^{-5},$$

so that this term will be negligible in our calculations (and naturally the same inference can be made with reference to the terms already neglected in l^4, l^6, ...).

On the other hand it is found that

$$\tfrac{1}{2}n^2a^2/\kappa t = {\cdot}0114;$$

and accordingly this term, and other terms in n^4a^4/κ^2t^2, will probably affect the conclusion.

We shall accordingly complete the formula for g, by including higher powers of nqa; on rejecting q^2, q^4, q^6, ..., the result is

$$\frac{g}{v_0} = \frac{a}{bl}\left\{-n + \frac{1}{1-s}(nqa + n^3q^3a^3 + n^5q^5a^5 + \ldots)\right\}$$

$$= \frac{na}{bl}\left\{-1 + \frac{a}{(1-s)\sqrt{(\pi\kappa t)}}\left(1 - \frac{1}{2}\frac{n^2a^2}{\kappa t} + \frac{1.3}{2^2}\frac{n^4a^4}{\kappa^2t^2} - \frac{1.3.5}{2^3}\frac{n^6a^6}{\kappa^3t^3} + \ldots\right)\right\}.$$

As already explained, we have rejected the terms in l^2, l^4, ...; and the interpretation of q, q^3, q^5, ... follows from § 4 (i)–(iii) below.

The series now obtained is not convergent; but it possesses the asymptotic property that the error in stopping at any stage in the series is less than the following term of the series (see § 4 (ix) below).

Inserting the values given above for a, κ, t, the series in the bracket ( ) becomes

$$1 - {\cdot}0114 + {\cdot}0004 = {\cdot}9890,$$

the error being less than the following term (roughly ·00002).

Thus our formula becomes (approximately)

$$\frac{g}{v_0} = \frac{na}{bl}\left\{-1 + \frac{a\,({\cdot}9890)}{(1-s)\sqrt{(\pi\kappa t)}}\right\}.$$

If we now assume the values assigned by Lord Kelvin

$$g = 1/2743, \quad v_0 = 4000,$$

the corresponding value of t can be estimated by writing our formula in the shape

$$\frac{a}{\sqrt{(\pi\kappa t)}}({\cdot}9890) = (1-s)\left(1 + \frac{g}{v_0}\frac{bl}{na}\right) = {\cdot}9505 + {\cdot}6653 = 1{\cdot}6158.$$

On reduction this formula gives

$$t = 2{\cdot}95 \times 10^{17}$$

or in *years** $\quad t = 9{\cdot}36 \times 10^{9}.$

Finally we shall estimate the error which may be due to replacing $\coth qa$ by 1 in the formula for A and g used above. It is easy to see that

$$1 - \frac{1}{1 - s + sqa \coth qa} = 1 - \frac{1}{1-s}\left\{1 + nqa\left(\frac{1+\rho}{1-\rho}\right)\right\}^{-1}$$

where $\quad \rho = e^{-2qa}.$

Thus on expansion we obtain

$$1 - \frac{1}{1-s}\left\{1 - nqa\left(\frac{1+\rho}{1-\rho}\right) + n^2q^2a^2\left(\frac{1+\rho}{1-\rho}\right)^2 - \ldots\right\}$$

$$= -\frac{s}{1-s} + \frac{nqa}{1-s}(1 + 2\rho + 2\rho^2 + \ldots) - \frac{n^2q^2a^2}{1-s}(1 + 4\rho + \ldots)$$

$$+ \frac{n^3q^3a^3}{1-s}(1 + 6\rho + \ldots) - \ldots$$

$$= -\frac{s}{1-s} + \frac{1}{1-s}(nqa - n^2q^2a^2 + n^3q^3a^3 \ldots)$$

$$+ \frac{2}{1-s}(nqa - 2n^2q^2a^2 + 3n^3q^3a^3 - \ldots)\,\rho + \ldots$$

$$= P_0 + P_1\rho + P_2\rho^2 + P_3\rho^3 + \ldots$$

where the first term in P_1 is $2nqa/(1-s)$.

The series P_0 leads to the asymptotic series already used; and similarly we can obtain series for $P_1\rho$, $P_2\rho^2$, etc. Clearly the most important term (with the numerical values under consideration) is the first term in $P_1\rho$, which is

$$\frac{2nqa}{1-s}e^{-2qa} = \frac{2n}{1-s}\frac{a}{\sqrt{(\pi\kappa t)}}e^{-a^2/\kappa t}$$

by § 4 (vii) below. Thus in comparison with the term $nqa/(1-s)$ in P_0, the relative order of this term is $2e^{-a^2/\kappa t}$.

* On comparing this result with the value $9{\cdot}02 \times 10^9$ found by Prof. Carslaw from the first term of the Fourier-expansion, I was led (by comparison with the numbers given in the first example of § 3 below) to the conjecture that the discrepancy must be due to the neglect of the second and higher terms in the Fourier-formula.

Prof. Carslaw has kindly re-calculated his formula for the above value of t, and obtains (using the first and second terms) a gradient

$$g = {\cdot}00035092 + {\cdot}00001348 = {\cdot}0003644.$$

This is equal to 1/2744; and so agrees with the value assumed for g to one part in 3000, which is roughly the same order as the correction on account of neglecting $e^{-a^2/\kappa t}$.

Now here $a^2/\kappa t = 8{\cdot}4$ roughly, and so

$$2e^{-a^2/\kappa t} = 4{\cdot}5 \times 10^{-4} \text{ nearly.}$$

The corresponding correction to the value of t will be of the relative order 9×10^{-4}, and so will change t to about $9{\cdot}37 \times 10^9$ (in years).

The correction on account of the terms in $P_2\rho^2$, $P_3\rho^3$, ... will lead similarly to an estimate of the relative orders $2e^{-4a^2/\kappa t}$, $2e^{-9a^2/\kappa t}$, which are entirely unimportant in the present problems.

§ 3. *Numerical tests of the formulae of* § 2.

In view of the difference in form between the series of § 2 and the corresponding Fourier-expansions, it seemed desirable to compare the results of numerical calculation with values not very different from those of § 2. Prof. J. Perry* had given the results of some calculations in connexion with a problem which may be regarded as the limit of that of § 2, when the thickness of the shell (l) tends to zero, the value of s remaining fixed. However some discrepancies were found (see below) and I decided to recalculate with slightly different constants so as to reduce the labour of calculating the Fourier-expansion; I had not then† the advantage of Prof. Carslaw's results with which to compare my work.

If we make l tend to zero in § 2 we find that

$$A = 1/(1 - s + sqa \coth qa),$$

and then if we replace V by $(v_0 - v)/v_0$ we obtain the formula

$$\frac{v_0 - v}{v_0} = \frac{1}{(1 - s + sqa \coth qa)} \frac{a \sinh qr}{r \sinh qa}.$$

This represents the symbolical solution for the temperature v of a sphere initially at temperature v_0, radiating into a medium at zero temperature; this problem was solved by Fourier‡ in the form

$$\frac{v}{v_0} = \frac{2a}{rs} \Sigma \frac{\sin\theta \sin(\theta r/a)\, e^{-\kappa\theta^2 t/a^2}}{\theta(\theta - \sin\theta\cos\theta)},$$

where the summation refers to the roots of the equation

$$1 - s + s\theta \cot\theta = 0.$$

* *Nature*, vol. 51, 1895, p. 225.

† These calculations were made at intervals in the latter part of 1916 and in 1917.

‡ *Theory of Heat* (Freeman's translation), § 293.

It is easy to confirm the Fourier-expansion from the symbolical formula above, by using Heaviside's general rule*.

However, in the actual problem we want the value of v at $r = a$; and for this purpose, from the considerations already given in § 2, it will be sufficient to replace $\coth qa$ by unity to obtain the series

$$\frac{v}{v_0} = 1 - \frac{1}{1 - s + sqa} = -n + \frac{1}{1-s}(nqa + n^2q^2a^2 + n^3q^3a^3 + \ldots)$$

where $n = s/(1-s)$.

Then (as in § 2) we obtain the asymptotic series

$$\frac{v}{v_0} = n\left\{-1 + \frac{a}{(1-s)\sqrt{(\pi\kappa t)}}\left(1 - \frac{1}{2}\frac{n^2a^2}{\kappa t} + \frac{1\,.\,3}{2\,.\,4}\frac{n^4a^4}{\kappa^2t^2} - \ldots\right)\right\}.$$

As explained in § 2, the formula is valid only if $e^{-a^2/\kappa t}$ is negligible; and to obtain four-figure accuracy, this requires $a^2/\kappa t$ to be not less than about 8·5. Hence to obtain good results from the asymptotic series, n must be small, of about the order 1/15 to 1/20.

In the Fourier-expansion, to avoid the labour of the actual calculation of the roots of the equation for θ, I decided to adopt a simple value for θ_1 (which is the root requiring the greatest accuracy), and to deduce the corresponding value of s.

The value $\theta_1 = 170\pi/180 = 2{\cdot}9671$ was selected; and this gave $1/s = 17{\cdot}827$. Then θ_2 was not very different from $2\theta_1$, and the value $\theta_2 = 5{\cdot}944$ was comparatively easy to calculate. A further simplification in the arithmetic was made by taking

$$a^2/\kappa t = \theta_1{}^2 = 8{\cdot}8 \quad \text{(nearly)}.$$

Then the first and second terms in the Fourier-expansion for v were found to be

$$v_0\,({\cdot}04248 + {\cdot}00191) = ({\cdot}04439)\,v_0.$$

The corresponding asymptotic series is found to be

$$1 - {\cdot}01546 + {\cdot}000725 - {\cdot}000075 + {\cdot}000008 = {\cdot}98520.$$

Then on substitution we find

$$v = v_0\,({\cdot}04441);$$

* This rule may be written in the form

$$\frac{F(p)}{\Delta(p)} = \frac{F(0)}{\Delta(0)} + \Sigma\frac{F(\alpha)\,e^{\alpha t}}{\alpha\Delta'(\alpha)},$$

where the summation refers to all the roots $p = \alpha$ of $\Delta(p) = 0$.

Proofs of this equation (from different points of view) have been given in *Phil. Mag.* vol. 37, 1919 (see pp. 417, 418), and *Proc. Lond. Math. Soc.* vol. 15, 1917 (see pp. 419, 420); Heaviside's own discussion will be found in his *Electrical Papers*, vol. 2, pp. 226, 373.

In the present problem, the term $F(0)/\Delta(0)$ reduces to 1 and cancels a term from the other side of the equation; and the values of α are $-\kappa\theta^2/a^2$ (found by writing $qa = \iota\theta$).

so that the two formulae agree as well as could be expected with four-figure accuracy.

In Prof. Perry's calculation, the value $1/s = 20$ was taken, and the value of $\kappa = k/c$ as in § 2 above, while t was taken as 96×10^8 years; then with $v_0 = 4000$ the first and second terms are stated to be equal to

$$142{\cdot}7 \quad + 5{\cdot}65 = 148{\cdot}4.$$

I did not succeed in confirming this value; and Prof. Carslaw has recalculated the Fourier-formula with the above *data*. His result is

$$138{\cdot}13 + 4{\cdot}83 = 142{\cdot}96, \quad \text{or say } 143{\cdot}0.$$

The corresponding value of the asymptotic series is found to be

$$1 - {\cdot}01131 + {\cdot}00038 - {\cdot}00002 = {\cdot}98905.$$

Hence $$\frac{v}{v_0} = \tfrac{1}{19}(-1 + 1{\cdot}6783) = {\cdot}03570,$$

and so $$v = 142{\cdot}8,$$

which agrees sufficiently closely with Prof. Carslaw's result*.

Perry has also given numerical results for the same Fourier-expansion when the constants are adjusted so that $s = 1$; the asymptotic series used above will clearly fail under this condition, and a fresh formula becomes necessary.

When $s = 1$, the equation for θ becomes

$$\cot \theta = 0,$$

thus the values of θ are $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, ...; and the Fourier-expansion simplifies to

$$\frac{v}{v_0} = \Sigma \frac{2}{\theta^2} e^{-\kappa\theta^2 t/a^2} = \frac{8}{\pi^2}\left(e^{-\omega} + \tfrac{1}{9}e^{-9\omega} + \tfrac{1}{25}e^{-25\omega} + \ldots\right)$$

where $$\omega = \pi^2 \kappa t/4a^2.$$

In Perry's actual calculation, the value of ω is nearly equal to 1; and so the corresponding value of

$$a^2/\kappa t = \tfrac{1}{4}\pi^2 = 2{\cdot}47 \quad \text{nearly}.$$

Thus the method of approximation adopted in § 2 needs reconsideration; and it turns out that the new formulae are not very convenient for numerical work in this special case (see below). However it is easy to recalculate this simple Fourier-expansion for other values of ω; and to select values which are suitable for purposes of comparison.

* The correction on account of the first term in $e^{-a^2/\kappa t}$ is of the same relative order of magnitude as in the calculation of § 2; and this gives

$$\left(\frac{1{\cdot}6782}{19}\right)(4{\cdot}5 \times 10^{-4})(4000) = {\cdot}17,$$

which accounts for the small residual difference.

Putting $s = 1$ the symbolical formula for v at $r = a$ becomes

$$\frac{v}{v_0} = 1 - \frac{\tanh qa}{qa} = 1 - \frac{1}{qa}(1 - 2e^{-2qa} + 2e^{-4qa} - \ldots).$$

Thus the first approximation (analogous to § 2) is given by

$$\frac{v}{v_0} = 1 - \frac{1}{qa} = 1 - 2\sqrt{\frac{\kappa t}{\pi a^2}} = 1 - \frac{4}{\pi}\sqrt{\frac{\omega}{\pi}},$$

where the value of $1/q$ is given by § 4 (vi). The value of the next term in this series is

$$\frac{2}{qa} e^{-2qa} = \frac{4}{\sqrt{\pi}} \int_{a/\sqrt{(\kappa t)}}^{\infty} e^{-w^2} \frac{dw}{w^2},$$

which requires the error-function integral to obtain a formula suitable for actual numerical calculation. However, the numerical value is less than

$$(\kappa t/a^2)\, e^{-a^2/\kappa t} \times 2\sqrt{\frac{\kappa t}{\pi a^2}},$$

or

$$\left(\frac{4\omega}{\pi^2} e^{-\pi^2/4\omega}\right) \times \frac{4}{\pi}\sqrt{\frac{\omega}{\pi}}.$$

The following terms in the series will be negligible if $a^2/\kappa t$ exceeds 2.

It is easy to test the accuracy of our results by taking say $\omega = \frac{1}{4}$, with $a^2/\kappa t = \pi^2 = 9{\cdot}87$; so that $(\kappa t/a^2)\, e^{-a^2/\kappa t}$ is of order $\frac{1}{2} \times 10^{-5}$, and so is negligible.

The Fourier-expansion is then

$$\frac{8v_0}{\pi^2}(e^{-\frac{1}{4}} + \tfrac{1}{9}e^{-\frac{9}{4}} + \tfrac{1}{25}e^{-\frac{25}{4}} + \ldots) = \frac{8v_0}{\pi^2}({\cdot}7789 + {\cdot}0117 + {\cdot}0001)$$

$$= \frac{8v_0}{\pi^2}({\cdot}7907) = v_0\,({\cdot}6409).$$

The symbolical formula gives (for $\omega = \frac{1}{4}$)

$$v = v_0\left(1 - \frac{2}{\pi\sqrt{\pi}}\right) = v_0\,(1 - {\cdot}3592) = v_0\,({\cdot}6408)$$

and thus the agreement is as close as could be hoped for.

Even for $\omega = \frac{1}{2}$, when $(\kappa t/a^2)\, e^{-a^2/\kappa t}$ is of the order 1/700, the two formulae agree to three significant figures; the Fourier-expansion is

$$\frac{8v_0}{\pi^2}(e^{-\frac{1}{2}} + \tfrac{1}{9}e^{-\frac{9}{2}} + \ldots) = \frac{8v_0}{\pi^2}({\cdot}6065 + {\cdot}0012) = v_0\,({\cdot}4926)$$

while the other gives

$$v = v_0\left(1 - \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\right) = v_0\,(1 - {\cdot}5079) = v_0\,({\cdot}4921).$$

It is therefore clear that, under proper conditions, the Fourier-expansion is numerically equivalent to the simpler formula obtained by the symbolical method*.

§ 4. *Evaluation of certain symbolical expressions used in the foregoing sections.*

As explained already in § 1, the fundamental meanings of the symbolical formulae can be obtained by translation into complex integrals. In fact the meaning of the function $f(q)$, where

$$\kappa q^2 = p = \partial/\partial t,$$

is given by the complex integral

$$\frac{1}{2\pi\iota}\int_{\alpha-\iota\infty}^{\alpha+\iota\infty} f(\nu)\, e^{\lambda t}\,\frac{d\lambda}{\lambda}$$

where $$\kappa\nu^2 = \lambda.$$

In the first instance we are concerned only with functions of q, which are in reality even functions†; or functions which are (in theory) such that $f(\nu)$ is expressible as a one-valued function of λ. But in the symbolical transformations of $f(q)$, it is generally convenient to manipulate algebraically, without restricting the functions used; and then we must adopt some definite convention as to the interpretations. First we make ν single-valued by means of a cut along the negative real axis in the λ-plane; and we select that value for ν which has its real part positive‡.

Further the functions $f(q)$ are (in their original forms) such as to tend to zero when $q \to \infty$; and thus the complex integral above can be replaced by an integral along the path indicated in the λ-plane.

When the path has been modified, we can suppose the algebraic transformations of the function $f(\nu)$ carried out so as to correspond

* For Perry's case ($\omega = 1$) the two formulae work out as v_0 (·2982) and v_0 (·2816); but, as already remarked, the value of $(\kappa t/a^2)\, e^{-a^2/\kappa t}$ is then about $\frac{1}{50}$, so that a discrepancy is to be anticipated unless the definite integral given is evaluated numerically.

† For instance $qa \coth qa$ can be written in the form

$$\left\{1+\frac{1}{2!}(qa)^2+\frac{1}{4!}(qa)^4+\dots\right\} \Big/ \left\{1+\frac{1}{3!}(qa)^2+\frac{1}{5!}(qa)^4+\dots\right\}.$$

Then the corresponding function of ν is equal to

$$\left(1+\frac{a^2\lambda}{2!\kappa}+\frac{a^4\lambda^2}{4!\kappa^2}+\dots\right) \Big/ \left(1+\frac{a^2\lambda}{3!\kappa}+\frac{a^4\lambda^2}{5!\kappa^2}+\dots\right).$$

‡ This has practically the same effect as if we regard q as *real and positive*, in our algebraic work.

to the symbolical manipulation of $f(q)$. For, on this path we can write

$$\nu = +\iota\theta \text{ (on the upper straight part),}$$

or

$$-\iota\theta \text{ (on the lower straight part)}$$

and

$$\lambda = -\kappa\theta^2;$$

thus the convergence is always ensured by the presence of the exponential $e^{-\kappa\theta^2 t}$.

With these general remarks we proceed now to evaluate the special functions used.

(i) *Interpretation of* q.

This becomes

$$\frac{1}{2\pi\iota}\int \nu e^{\lambda t}\frac{d\lambda}{\lambda} = \frac{1}{2\pi\iota}\int_\infty^0 (-\iota\theta)\, e^{-\kappa t\theta^2}\frac{2d\theta}{\theta} + \frac{1}{2\pi\iota}\int_0^\infty (+\iota\theta)\, e^{-\kappa t\theta^2}\frac{2d\theta}{\theta}$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\kappa t\theta^2}\,d\theta = \frac{2}{\pi}\frac{1}{\sqrt{(\kappa t)}}\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{(\pi\kappa t)}};$$

the contribution from the small circle there tends to zero with the radius (its value is in fact proportional to the square-root of the radius).

Hence

$$q = 1/\sqrt{(\pi\kappa t)}.$$

(ii) *Interpretation of* q^3.

Repeating the foregoing transformations, we obtain

$$-\frac{2}{\pi}\int_0^\infty \theta^2 e^{-\kappa t\theta^2}\,d\theta = \frac{1}{\kappa}\frac{\partial}{\partial t}\frac{1}{\sqrt{(\pi\kappa t)}},$$

and so

$$q^3 = -\frac{1}{2\sqrt{\pi}}\frac{1}{(\kappa t)^{\frac{3}{2}}}.$$

(iii) *Interpretation of* q^{2m+1}.

Similarly

$$q^{2m+1} = \frac{1}{\kappa^m}\left(\frac{\partial}{\partial t}\right)^m \frac{1}{\sqrt{(\pi\kappa t)}} = (-1)^m \frac{1\,.\,3\,\ldots\,(2m-1)}{(2\kappa t)^m\sqrt{(\pi\kappa t)}}.$$

It should be noticed that symbolically the relation

$$q^{2m+1} = \frac{1}{\kappa^m}\left(\frac{\partial}{\partial t}\right)^m q$$

is obvious, since $\kappa q^2 = p$.

(iv) *Interpretation of* $q^2, q^4, \ldots$.

It is here easy to verify that the two integrals from the upper and lower paths cancel; so that $q^2 = 0$, $q^4 = 0$, etc. This again is obvious symbolically, since $q^2 = \frac{1}{\kappa}\frac{\partial}{\partial t}(1) = 0$.

(v) *Interpretation of* $1/q$.

This is

$$\frac{1}{2\pi\iota}\int_{a-\iota\infty}^{a+\iota\infty}\left(\frac{\kappa}{\lambda}\right)^{\frac{1}{2}} e^{\lambda t}\frac{d\lambda}{\lambda},$$

and by a known result, due to Cauchy,

$$\frac{1}{2\pi\iota}\int_{a-\iota\infty}^{a+\iota\infty}\frac{e^{\lambda t}\,d\lambda}{\lambda^{1+m}}=\frac{t^m}{\Gamma(1+m)},$$

if $$1+m>0 \quad\text{and}\quad t>0.$$

Hence $$\frac{1}{q}=\frac{\sqrt{(\kappa t)}}{\Gamma(\frac{3}{2})}=\frac{\sqrt{(\kappa t)}}{\frac{1}{2}\Gamma(\frac{1}{2})}=2\sqrt{\left(\frac{\kappa t}{\pi}\right)},$$

a result which follows symbolically from (i) by observing that

$$\frac{1}{q}=\frac{\kappa}{p}q=\kappa\int_0^t\frac{dt}{\sqrt{(\pi\kappa t)}}=\frac{2}{\sqrt{\pi}}\sqrt{(\kappa t)}.$$

(vi) *Interpretation of* $1/q^n$.

The method of (v) shows at once that

$$\frac{1}{q^{2m+1}}=\frac{(\kappa t)^{m+\frac{1}{2}}}{\Gamma(m+\frac{3}{2})},\qquad \frac{1}{q^{2m}}=\frac{(\kappa t)^m}{\Gamma(m+1)},$$

and we can sum up all the preceding formulae in the single statement

$$\frac{1}{q^n}=\frac{(\kappa t)^{\frac{n}{2}}}{\Gamma(1+\frac{1}{2}n)};$$

which may be stated even more simply in the shape

$$\frac{1}{p^n}=\frac{t^n}{\Gamma(1+n)},$$

where n may be taken as having any (positive or negative) value. The last result is obvious by successive integration when n is an integer: and it is natural to conjecture that the same formula will be valid generally.

(vii) *Interpretation of* qe^{-qx}, *where* x *is positive.*

This is readily obtained by expansion in powers of x; and the result is

$$q-q^2x+q^3\frac{x^2}{2!}-q^4\frac{x^3}{3!}+q^5\frac{x^4}{4!}-\dots.$$

Using the results found above we obtain the series

$$\frac{1}{\sqrt{(\pi\kappa t)}}\left\{1-\frac{1}{2\kappa t}\frac{x^2}{2!}+\frac{1.3}{(2\kappa t)^2}\frac{x^4}{4!}-\frac{1.3.5}{(2\kappa t)^3}\frac{x^6}{6!}\dots\right\}$$

$$=\frac{1}{\sqrt{(\pi\kappa t)}}\left\{1-\frac{x^2}{4\kappa t}+\frac{1}{2!}\left(\frac{x^2}{4\kappa t}\right)^2-\frac{1}{3!}\left(\frac{x^2}{4\kappa t}\right)^3+\dots\right\}=\frac{1}{\sqrt{(\pi\kappa t)}}e^{-x^2/4\kappa t}.$$

If these operations appear to need further justification, it is easy to see that the direct expression of qe^{-qx} by means of a complex integral leads to the formula

$$\frac{2}{\pi}\int_0^\infty e^{-\kappa t\theta^2}\cos(\theta x)\,d\theta=\frac{1}{\sqrt{(\pi\kappa t)}}e^{-x^2/4\kappa t}.$$

(viii) *Interpretation of e^{-qx} and $\frac{1}{q}e^{-qx}$, x being positive.*

We have proved that

$$qe^{-qx} = \frac{1}{\sqrt{(\pi\kappa t)}} e^{-x^2/4\kappa t}.$$

Thus, on integrating with respect to x, we see that

$$e^{-qx} = -\int \frac{dx}{\sqrt{(\pi\kappa t)}} e^{-x^2/4\kappa t} + \text{const.}$$

$$= -\frac{2}{\sqrt{\pi}} \int^{z} e^{-v^2} dv + \text{const.}$$

where $$z = \tfrac{1}{2}x/\sqrt{(\kappa t)}.$$

Now when $x \to 0$, $e^{-qx} \to 1$, and so

$$e^{-qx} = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-v^2} dv = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-v^2} dv.$$

We have now a verification of the work; for as x tends to $+\infty$, e^{-qx} tends to 0 (since the real part of q is positive); and this property is seen to hold for the integral just found.

The value of $(1/q)\,e^{-qx}$ is easily found by observing that

$$\left(\frac{1}{q} + x\right) e^{-qx} = \frac{1}{q} - \frac{qx^2}{2\,!} - \frac{3q^3x^4}{4\,!} - \frac{5q^5x^6}{6\,!} - \dots$$

$$= 2\sqrt{\left(\frac{\kappa t}{\pi}\right)} \left\{1 - \frac{x^2}{4\kappa t} + \frac{1}{2\,!}\left(\frac{x^2}{4\kappa t}\right)^2 - \dots\right\} = 2\sqrt{\left(\frac{\kappa t}{\pi}\right)} e^{-x^2/4\kappa t}.$$

Thus $$\frac{1}{q} e^{-qx} = 2\sqrt{\left(\frac{\kappa t}{\pi}\right)} e^{-x^2/4\kappa t} - \frac{2x}{\sqrt{\pi}} \int_z^\infty e^{-v^2} dv,$$

where as before $$z = \tfrac{1}{2}x/\sqrt{(\kappa t)}.$$

This is a form suited for numerical work; but a more useful result is the simpler formula

$$\frac{1}{q} e^{-qx} = \frac{x}{\sqrt{\pi}} \int_z^\infty e^{-v^2} \frac{dv}{v^2}.$$

For large values of z this integral can be converted into an asymptotic series of which the first term is $\dfrac{x}{2\sqrt{\pi}} \dfrac{e^{-z^2}}{z^3}$.

(ix) The only point remaining is to establish the asymptotic property for the series in powers of q, used in §§ 2, 3 above; this series is derived from the function

$$f(q) = 1 - \frac{1}{1 - s + sqa},$$

and suppose that as a matter of algebraic expansion we obtain

$$f(q) = A_0 + A_1 q + A_2 q^2 + \dots.$$

When $f(q)$ is converted into a complex integral along the path indicated above, there is a contribution from the small circle round the origin, which tends to the value A_0, when the radius of the circle tends to zero; and the straight paths contribute the integral

$$\frac{1}{\pi\iota}\int_0^\infty \{f(\iota\theta) - f(-\iota\theta)\}\, e^{-\kappa\theta^2 t}\,\frac{d\theta}{\theta} = \frac{2}{\pi}\int_0^\infty \phi(\theta)\, e^{-\kappa\theta^2 t}\, d\theta,$$

where $$\phi(\theta) = \frac{f(\iota\theta) - f(-\iota\theta)}{2\iota\theta} = \frac{sa}{(1-s)^2 + s^2a^2\theta^2}.$$

Thus $$f(q) = A_0 + \frac{2}{\pi}\int_0^\infty \phi(\theta)\, e^{-\kappa\theta^2 t}\, d\theta$$

gives the interpretation of the function $f(q)$.

Now $\phi(\theta)$ can be expanded in powers of θ, when θ is sufficiently small; and, having regard to the connexion between $\phi(\theta)$ and $f(q)$, it is clear that the expansion will be of the form

$$\phi(\theta) = A_1 - A_3\theta^2 + A_5\theta^4 - \dots.$$

Denote the sum of the first n terms of this expansion by $S_n(\theta)$; then it is easy to see that we have the algebraic identity*

$$\phi(\theta) - S_n(\theta) = (-1)^n \frac{(1-s)^2 A_{2n+1}\theta^{2n}}{(1-s)^2 + s^2a^2\theta^2}.$$

Thus for all real values of θ, the difference between $\phi(\theta)$ and $S_n(\theta)$ is numerically less than $(-1)^n A_{2n+1}\theta^{2n}$; and accordingly the interpretation of $f(q)$ may be taken as

$$A_0 + \frac{2}{\pi}\int_0^\infty S_n(\theta)\, e^{-\kappa\theta^2 t}\, d\theta,$$

with an error which is numerically less than

$$\frac{2}{\pi}\int_0^\infty (-1)^n A_{2n+1}\theta^{2n} e^{-\kappa\theta^2 t}\, d\theta.$$

Thus we may say that the error in replacing $f(q)$ by

$$A_0 + A_1 q + A_3 q^3 + \dots + A_{2n-1} q^{2n-1}$$

(or by $A_0 + A_1 q + A_2 q^2 + \dots + A_{2n} q^{2n}$) is less than the numerical value of the following term $A_{2n+1} q^{2n+1}$ (when interpreted according to the rules already given). This establishes the asymptotic property assumed in regard to the series for $f(q)$.

* On multiplying up, we see that

$$\{\phi(\theta) - S_n(\theta)\}\{(1-s)^2 + s^2a^2\theta^2\}$$

is a polynomial of degree $2n$ in θ; further when θ is small it can be expanded in powers of θ, the first term being $(-1)^n(1-s)^2 A_{2n+1}\theta^{2n}$. Accordingly the product must reduce to this single term.

On the effect of a magnetic field on the intensity of spectrum lines. By H. P. Waran, M.A., Government of India Scholar of the University of Madras. (Communicated by Professor Sir Ernest Rutherford, F.R.S.)

(Plate III.)

[*Read* 16 May 1921.]

Since my last communication to the society* on the phenomenon of the changes in the general spectrum brought about by the influence of the magnetic field, I have come to know that the phenomenon has been noticed as early as 1858† and studied to a certain extent by various observers. Yet the complex changes taking place in the source when the radiation is emitted from

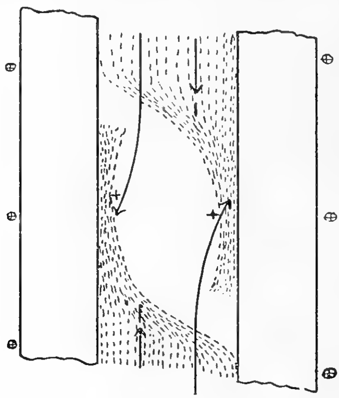


Fig. 1.

inside a magnetic field, does not seem to have had the attention it deserves. But in 1913‡ Messrs Kent and Frye made a study of the phenomenon in some particular cases with a view to its complete elucidation. They do not seem to have come to any definite conclusions and from their experiments they seem to be inclined to attribute the effects observed mainly to the chemical disintegration of the walls of the discharge tube brought about by the disruptive action of the blast of ions deflected to the sides by the magnetic field.

Considering a section of the discharge transverse to the magnetic field, as in Fig. 1, with the lines of force entering the plane of

* *Proc. Camb. Phil. Soc.* vol. 20, Part I, p. 45.
† A. de la Rive, *Annales de chimie et de physique* (3), 54, p. 238, 1858.
‡ *Astrophysical Journal*, 37, pp. 183–189, 1913.

the paper normally, it is seen that the discharge current through the tube which consists of the unrectified current from the secondary of an Induction Coil is split into two streams and deflected to the opposite sides of the walls of the tube. The brighter stream corresponding to the break current is arranged to be on the side facing the spectroscope. Since they are travelling in opposite directions both the ions forming the discharge are deflected in the same direction on to the glass and with a strong discharge in an intense field the magnitude of the corrosion of the sides of the glass under this sand blast action of the ions is considerable. Where this is predominant the real effect gets mixed up with the effects of the chemical disintegration and the phenomenon may get attributed entirely to the latter cause. This conclusion is inevitable from the intense fields and strong currents Messrs Kent and Frye used without taking adequate precautions to exclude the complications of the attendant chemical disintegration of the tube.

To examine the extent of such disintegration heavy currents and strong fields were employed for a few minutes and Fig. 2 shows an enlarged view of a side of the capillary of the glass discharge tube in the region opposite the pole pieces. The deep channels found to have been cut in the glass show that the disintegration of the glass is considerable. The consequent complications are sure to lead one to erroneous conclusions in case proper precautions are not taken to exclude them and make allowances for such effects. To study the phenomenon free of such complications, this disruptive action of the ions should be reduced to a minimum by the employment of a moderate field and current and even then the tube must be of a material like quartz not liable to such easy decomposition, arising from any local heating.

In my experiments I used a quartz discharge tube with aluminium electrodes the diameter of whose capillary was of the order of about 2 mm. The current generally used was about 2 milliamperes and the field was of the order of about 5000 C.G.S. units. Further the tube was in permanent communication with a mercury pump fitted with McLeod's gauge and drying tube of phosphorous pentoxide. Thus owing to the large volume in connection with the discharge tube the pressure of gas in the tube remained constant throughout the experiment and the disturbing influence of the gradual absorption or emission of gas by the electrodes or walls of the tube during the course of an experiment was considerably compensated. Under such circumstances the results obtained with the quartz tube are identical with those obtained with the glass tube and hence the phenomenon cannot be attributed to the chemical disintegration of the walls of the tube.

Another possible view of the phenomenon put forward by Messrs Kent and Frye and others is that the change is due to the decrease in cross-section of the discharge brought about by the deflecting action of the field. From Fig. 1 it was seen that the discharge is pushed on either side into two thin filaments leaving the centre of the tube more or less free, thus reducing the effective cross-section of the discharge. They regard it as tantamount to a change to a narrower capillary. But that the effect is not entirely due to this can be shown by the following considerations.

Though a narrower capillary does generally tend to enhance the weaker lines and bring about a trace of continuous spectrum, its effect is not of such a selective nature tending to the enhancement of some lines previously almost invisible, at the same time decreasing the intensity of some others as observed in the case of mercury and many other gaseous spectra. Further the effect of the field is to weaken the continuous spectra in many cases though under the same circumstances a narrower capillary tends to intensify the continuous spectrum.

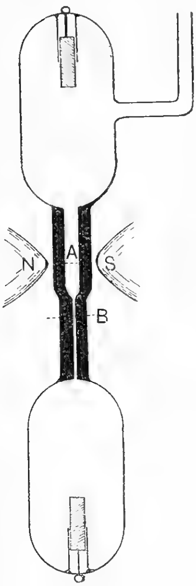


Fig. 3.

To test this point further a tube of the form shown in Fig. 3, in which the capillary portion was made of two sections in series, with sectional areas in the ratio of about 10 : 1, was made and placed transverse to the field as indicated. An image of the portion of the tube between *A* and *B* was thrown on the slit of the spectroscope and the spectra given by these two capillaries were compared. There was very little change except that the portion of the spectrum corresponding to the narrower capillary was distinctly brighter throughout the whole spectrum. A magnetic field was now applied transversely at *A* to see how far the narrowed section of the discharge through *A* gave a spectrum resembling that given by *B*. The change introduced by the field was quite different. With hydrogen at low pressure, while the effect of the narrowed capillary *B* was to increase the intensity of the hydrogen lines the effect of the field on *A* was to decrease the intensity of the hydrogen to a slight extent and increase the intensity of the mercury lines as shown in Fig. 4 (1). Such results clearly show that though the decrease in the cross-section may be the cause of some of the changes observed, yet there are others that cannot go under this simple explanation and they point to the existence of some other influence exerted by the magnetic field.

Further, experimenting with a tube of the form illustrated below, Fig. 5, in which the discharge is in the direction of the field effects are observed similar to those obtained with the discharge going transverse to the field. With a leyden jar in parallel with the coil and a spark gap in series with the tube shunted across the condenser, at low pressures the stream-like character of the discharge is lost and the discharge passes through the whole section of the tube without seeming to suffer any visible change in the cross-section. But the spectroscopic changes observed are similar. The fact that in many cases the change over the spectrum is brought about practically instantaneously on the application of the field is also against the possibility of any progressive changes

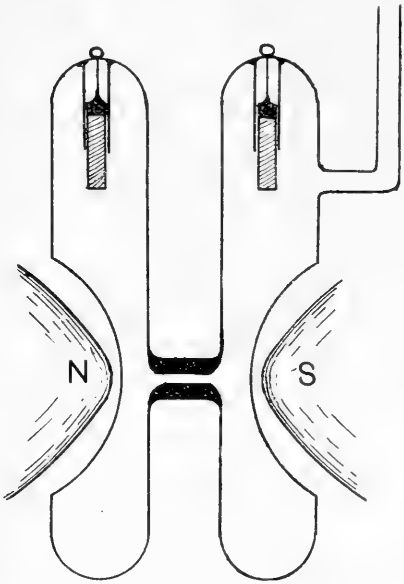


Fig. 5.

resulting from the absorption or liberation of some of the component gases.

There is at times a great similarity between the effect produced by the condensed discharge through the tube and that brought about by the magnetic field, and from that, though we may conclude that the magnetic field brings about a similar violent excitation of the spectrum in some way, we cannot as yet go so far as to say that their effects are identical. For there are important differences between the effects produced by the two. Experimenting with hydrogen the effect of a condensed discharge is found to be mainly to broaden the hydrogen Balmer series lines, especially those towards the violet, where as the magnetic field under the same conditions leaving the Balmer series lines unaffected brings about the secondary spectrum very prominently as shown in Fig. 4 (2).

It is possible that the change in the spectrum is purely mechanical in origin. In Fig. 1 we saw that under the influence of the magnetic field the discharge was deflected to either side. In each of these streams the electrons being much more easily deflected out of their path than the positive ions, they might be going in a layer much closer to the walls than the positive ions as shown in Fig. 1. In such a case we may regard the massive positive particles as bombarding the layer of electrons adjacent to the walls, thus giving rise to mechanical and electrical reactions that cause a peculiar excitation of their spectra. Orbits and frequencies previously not natural might then become possible and this would account for the change observed.

The spectrum of hydrogen with its simple structure is only very slightly susceptible to the influence of the field, excepting of course the secondary spectrum which it brings about. When the gas is pure and at a low pressure the Balmer series lines undergo very little change in their intensity. But when it is present mixed with other gases or at higher pressures it is brought out intensely by the field though not prominent without it. From our observation of the enhancement of the lines of the monatomic gases, now confirmed independently by the study of Messrs Kent and Frye on Argon, if we are to attribute it to the enhancing effect of the magnetic field on the atoms, this would mean that at higher pressures atoms predominate giving the Balmer series, while at lower pressures the molecules predominate giving the secondary spectrum, a conclusion in accordance with that arrived at by G. P. Thomson in his study of the spectrum of the hydrogen positive rays. The effect of the field in enhancing the secondary spectrum shows however that this division of share in the radiation is by no means clear cut between the atom and the molecule.

From the enhancement of the Balmer series lines observable at higher pressures it is possible that the atoms are mainly responsible for them and by the employment of a magnetic field in conjunction with Prof. Wood's† long discharge tube the lines of the Balmer series obtainable in the laboratory could probably be still further increased.

Experimenting with nitrogen having a constitution and spectrum considerably more complex than hydrogen the effects observed are very complicated. When a condensed spark is not employed to excite the tube, the effect of the field is to enhance the band spectrum and to bring in some of the lines belonging to the line spectrum of nitrogen with a different intensity distribution

* G. P. Thomson, *Phil. Mag.* vol. 40 (1920), p. 240.

† R. W. Wood, *Proc. Roy. Soc.* A, vol. 97, p. 455.

Fig. 2.

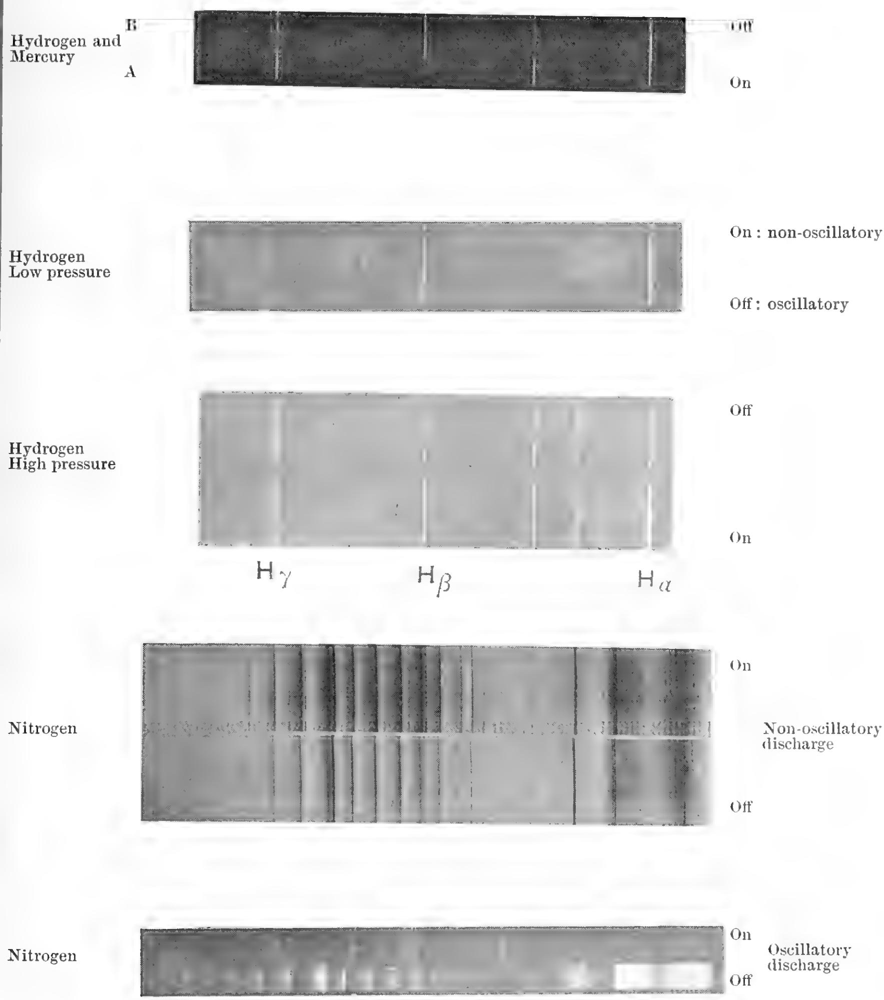

B
Hydrogen and Mercury
A
Off
On
On : non-oscillatory
Hydrogen Low pressure
Off : oscillatory
Off
Hydrogen High pressure
On
$H\gamma$
$H\beta$
$H\alpha$
On
Nitrogen
Non-oscillatory discharge
Off
On
Nitrogen
Oscillatory discharge
Off

Fig. 4.

from the line spectra obtained with a condensed spark. The dense line at 4277·7 A.U. brought out very strongly by the field is generally absent without it, especially when the gas is not quite pure. With a condensed discharge it is rather faint. It has been catalogued as a negative pole band by Exner and Haschek*.

The phenomena attending the employment of the magnetic field are many and varied and complications arising from local variations of potential current, temperature and pressure known to occur there are difficult to eliminate entirely. But by independent variations of some of these factors it has not been possible to reproduce the phenomenon observed and hence the phenomena cannot be attributed entirely to these disturbing causes. It is possible that it might be due to some other influence brought about by the field. Further study of the phenomenon is in progress.

In conclusion I beg to express my indebtedness to the kind and sympathetic help of Professor Sir Ernest Rutherford throughout this work.

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CAMBRIDGE.

* Kayser, *Handbuch der Spectroscopie*, vol. 5, p. 826.

On a property of focal conics and of bicircular quartics. By C. V. Hanumanta Rao, University Professor, Lahore. (Communicated by Prof. H. F. Baker.)

[*Received* 20 April. *Read* 2 May 1921.]

The present note arises directly out of a note with the same title by Prof. H. F. Baker in the *Proceedings* (vol. XX. pp. 122–130). The property referred to in the title is that a varying circle, of one mode of generation, makes with two fixed circles, of a second mode, angles with a constant sum; and this result is here deduced from a particular case of it, viz. that where the curve consists of two circles. A preliminary series of results is inserted of some interest in themselves. By distances and angles will be meant throughout the Cayley separations, and the quadric or conic with respect to which the homographies are considered is indicated in each case.

1. Given two conics α, β in a plane, they have four common tangents meeting in three pairs of points. Indicate by V_1, V_2 one such pair of points. Then the sum of the distances, with respect to α, of V_1P and V_2P, is constant as P moves on β; and this constant remains the same if α, β are interchanged.

The proof of this result depends on the existence of two fixed points on the line V_1V_2, and this fact in turn is an easy consequence of the space figure of two conics with two common points.

Conversely, given a conic α and two fixed points V_1, V_2 in its plane, the locus of a point P, which moves so that the sum of the distances with respect to α of V_1P and V_2P is constant, is a conic β touching the four tangents from V_1, V_2 to α.

In particular, when one of the three pairs of points such as V_1, V_2 is projected into the circular points at infinity, the other two pairs are the foci and the conics are confocal. This leads to a slightly more general definition of confocal conics than the usual one, namely taking a fundamental conic Σ the system of conics touching any four arbitrary fixed tangents of Σ may be called a confocal system. Or again when the conic Σ is made to degenerate tangentially into the two circular points at infinity, we have the usual definition of confocal conics; it was this idea in fact which suggested the theorems of this note.

2. Precisely similar results hold in space of three dimensions. Given a quadric α and two points V_1, V_2, the locus of a point P such that the sum of the distances with respect to α of V_1P and V_2P is constant, is a quadric β which is enveloped by the enveloping

cones of α from V_1, V_2. In fact taking α as $\Sigma x^2 = 0$, the equation to β is of the form

$$\frac{(\Sigma x x_1)^2}{\Sigma x_1^2} - 2\cos C \frac{(\Sigma x x_1)(\Sigma x x_2)}{\sqrt{(\Sigma x_1^2)(\Sigma x_2^2)}} + \frac{(\Sigma x x_2)^2}{\Sigma x_2^2} = (\Sigma x^2)\sin^2 C,$$

where C is a constant, and this is clearly a quadric enveloped by the two cones like $(\Sigma x^2)(\Sigma x_1^2) = (\Sigma x x_1)^2$.

Conversely, given two quadrics α, β having two common enveloping cones from V_1, V_2, either quadric may be thought of as the locus of a point P such that the sum of the distances with respect to the other quadric, of V_1P and V_2P, is constant; and the constant is the same whichever quadric is considered as the locus.

In particular, for some definite value of the constant the latter quadric will degenerate into a pair of planes, viz. the two planes of intersection of the two enveloping cones. Thus, given two conics β_1, β_2 intersecting in two points, V_1, V_2 indicating the vertices of the two cones through both of them, the locus of a point P such that the sum of the distances with respect to β_1, β_2, of V_1P and V_2P, is constant, is a quadric enveloped by the two cones.

Taking the two points V_1, V_2 in the first result of this article to coincide, we find that if two quadrics have ring contact with V for the pole of the plane of contact, and if P be an arbitrary point on either quadric, then the distance VP with respect to the other quadric is a constant.

3. Reciprocally, take two quadrics having two common conics. Then an arbitrary tangent plane to either quadric makes with the planes of the two conics angles (measured with respect to the other quadric) whose sum is constant. Indicating the quadrics by

$$\Sigma x^2 + 2\lambda zt = 0, \quad \Sigma x^2 + 2\mu zt = 0,$$

the constant referred to is found to be arc tan $\dfrac{\sqrt{(1-\lambda^2)(\mu^2-1)}}{1-\lambda\mu}$.

In particular taking the two quadrics as a cone and a sphere, we have the well-known theorem that a varying tangent plane to a cone makes, with two circular sections of opposite systems, angles with a constant sum.

Conversely, given a quadric α and two arbitrary planes z, t, the envelope of a plane which makes with them two planes angles with a constant sum, is a quadric β intersecting the given quadric along the two given planes z, t.

We observe that in the above results the second quadric consists of two planes, and two conics having two common points represent the elliptic quartic curve which by projection is to yield the bicircular quartic. From this particular result we shall deduce the theorem in the general case.

For this purpose consider a sub-group of the possible positions of the moving plane, namely the planes all of which pass through a fixed point. They are then tangent planes to an enveloping cone of β, and we have the result that through the curve of intersection of this enveloping cone and the quadric α there passes another cone touched by the planes z, t. This is so in virtue of the following simple result:

Given two quadrics λ, μ with two common conics z, t, through the common curve of λ and any enveloping cone of μ there passes another cone touched by the planes of z, t. For if $S = 0$ and $S = zt$ be the two quadrics, any enveloping cone of the first is $SS' = P^2$ and meets the second in a curve lying on the cone $P^2 = S'zt$, which clearly has z, t for tangent planes.

We have thus established the well-known result for the common curve of two arbitrary quadrics, viz. that a varying tangent plane to one of the cones through this curve makes with two fixed tangent planes to a second such cone, angles with a constant sum.

4. Quadrics having ring contact are just concentric spheres, and two quadrics with two common conics are equivalent to two spheres. But no such complete projective reduction can be effected in the case of two arbitrary quadrics, and what we have done in Art. 3 is to establish a general result for the common curve of two arbitrary quadrics by deducing it from the particular case where one quadric degenerates into two planes.

Convex Solids in Higher Space. By Dr W. Burnside, Honorary Fellow of Pembroke College.

[*Received* 14 July 1921.]

Definitions. A set of linear $(n-1)$-spreads in n-dimensional space is said to be "general" when no $n+1$ of them meet in a point, no n in a line, no $n-1$ in a 2-spread, ..., and no three in an $(n-2)$-spread.

A set of points in n-dimensional space are said to be the internal points of a convex polyhedron, when each pair A and B of them satisfy the following conditions: (i) no point of the finite line AB lies on any one of a certain set of m $(n-1)$-spreads; (ii) the line AB produced from B meets at least one of the m $(n-1)$-spreads at a finite distance; (iii) the line BA produced from A meets at least one of the m $(n-1)$-spreads at a finite distance. If, in addition, it is always possible to choose B so that AB produced from B meets any assigned one of the m $(n-1)$-spreads before it meets any of the others, then each of the m $(n-1)$-spreads is said to form part of the boundary of the convex polyhedron.

Consider five 3-spreads A, B, C, D, E in 4-dimensional space of "general" position and such that no one of their five points of intersection is at infinity. Denote by a, b, c, d, e the points of intersection of B, C, D and E; ....; A, B, C and D. If e and q are on opposite sides of E, then eq produced from q does not meet A, B, C, D or E; and therefore q cannot be an internal point of a convex polyhedron bounded by the five 3-spreads. If e and q are on the same side of E, let eq meet E in p. In E the four points a, b, c, d are the vertices of a tetrahedron. If p is outside this tetrahedron, it must be separated from one of the vertices, say a, by the plane through the other three. Hence p and a, and therefore also q and a, are on opposite sides of A: and q cannot be an internal point of a convex polyhedron bounded by the five 3-spreads. It follows that the only points that can be internal points of a convex polyhedron bounded by the five 3-spreads are the points of the finite lines joining e to every internal point of the tetrahedron $abcd$: and these points clearly satisfy all the conditions.

Hence five 3-spreads of general position in 4-dimensional space, whose intersections are all finite points, form the boundary of just one convex polyhedron. An obvious extension of this reasoning shows that $n+1$ $(n-1)$-spreads in n-dimensional space, of general position, no one of whose intersections is at infinity, form the boundary of just one convex polyhedron.

There are just $\frac{1}{2}(n+1)(n+2)$ points of intersection and $\frac{1}{6}n(n+1)(n+2)$ lines of intersection of a general set of $(n-1)$-spreads, $n+2$ in number, in n-dimensional space. The lines pass n by n through the points and the points lie 3 by 3 on the lines. If each $(n-1)$-spread is denoted by a single symbol i, the point in which all the $(n-1)$-spreads except i and j meet may be denoted by ij and the line of intersection of all the $(n-1)$-spreads except i, j and k by ijk. The three points ij, ik, jk lie on the line ijk. If this configuration is projected from an arbitrary point of the n-dimensional space upon an arbitrary $(n-1)$-spread in it, the configuration becomes a like one in $(n-1)$-dimensional space. If the points of the original configuration are all finite points, the projection may clearly be carried out so that if ij is between ik and jk in the original configuration, the same is true after projection. Taking again an arbitrary point and an arbitrary $(n-2)$-spread in the $(n-1)$-dimensional space, the configuration may be projected into a like one in $(n-2)$-dimensional space; and the process may be continued. Moreover if all the points of the original configuration are finite points (so that for each set of three such as ij, ik, jk one is actually between the other two) and if each projection is carried out as suggested above, then in the final two-dimensional figure ij will be between ik and jk if it was so in the original configuration.

It will be said that ij and ik are opposite or adjacent according as jk is or is not between them. If ij, ik are adjacent and also ij and il, then ik and il are adjacent. Hence, with m single symbols, the $m-1$ points $i1, i2, \ldots . im$ may be divided into two sets such that all those of either set are adjacent, while any two taken one from each set are opposite. A suitable symbol to indicate the separation is 12 13 | 14 15 16, all those on either side of the bar being adjacent.

It will now be shown that, apart from permutation of the single symbols, there is just one scheme for the separations of the set of $\frac{1}{2}m(m-1)$ points arising from m symbols.

When n is odd, 7 for example, the typical separation is

$$\begin{array}{l}
|\ 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \\
|\ 21 \quad 23 \quad 24 \quad 25 \quad 26 \quad 27 \\
13 \ |\ 23 \quad 34 \quad 35 \quad 36 \quad 37 \\
24 \ |\ 14 \quad 34 \quad 45 \quad 46 \quad 47 \\
15 \quad 35 \ |\ 25 \quad 45 \quad 56 \quad 57 \\
26 \quad 46 \ |\ 16 \quad 36 \quad 56 \quad 67 \\
17 \quad 37 \quad 57 \ |\ 27 \quad 47 \quad 67
\end{array} \qquad \ldots\ldots(\mathrm{i}),$$

which is associated in an obvious way with the symbol

$$\{12\}\{34\}\{56\}\{7\}.$$

When n is even, say 8, the typical separation is

$$\begin{array}{ccccccc}
\mid 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\mid 21 & 23 & 24 & 25 & 26 & 27 & 28 \\
13 & \mid 23 & 34 & 35 & 36 & 37 & 38 \\
24 & \mid 14 & 34 & 45 & 46 & 47 & 48 \\
15 & 35 & \mid 25 & 45 & 56 & 57 & 58 \\
26 & 46 & \mid 16 & 36 & 56 & 67 & 68 \\
17 & 37 & 57 & \mid 27 & 47 & 67 & 78 \\
28 & 48 & 68 & \mid 18 & 38 & 58 & 78
\end{array} \qquad \ldots\ldots\text{(ii)},$$

which is associated in a similar way with the symbol

$$\{12\}\{34\}\{56\}\{78\}.$$

There is no difficulty in verifying that for three symbols and for four symbols, all possible separations arise from these by permutations of the single symbols. It will be shown here that if the scheme is general for 7 it is general for 8. If the proof is examined it will be quite clear that the same method may be used for any two consecutive numbers. Assuming that the separation of 21 points is given by the scheme (i), the question to be settled is how the seven points 18, 28, ..., 78 fit into it and how they are separated among themselves.

Suppose for instance that 48 is the first of the new symbols that occurs to the left of the bar. Then 48 is opposite to 45, 46 and 47; so that 45 and 58, 46 and 68, and 47 and 78 are adjacent. It follows that 68 occurs to the left of the bar and 58, 78 to the right. Since 18, 28 are both adjacent to 12, 28 is opposite to 18. So 23, 28 and 23, 38 being adjacent, 28 and 38 are opposite: 24, 28 and 24, 48 being adjacent, 28 and 48 are opposite. In a similar way 28 is shown to be opposite to 58, 68, 78. The scheme that thus arises for the 28 points is

$$\begin{array}{ccccccc}
\mid 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\mid 21 & 23 & 24 & 25 & 26 & 27 & 28 \\
13 & \mid 23 & 34 & 35 & 36 & 37 & 38 \\
24 & 48 & \mid 14 & 34 & 45 & 46 & 47 \\
15 & 35 & \mid 25 & 45 & 56 & 57 & 58 \\
26 & 46 & 68 & \mid 16 & 36 & 56 & 67 \\
17 & 37 & 57 & \mid 27 & 47 & 67 & 78 \\
28 & \mid 18 & 38 & 48 & 58 & 68 & 78
\end{array}$$

Comparing this with scheme (ii) it is changed into the latter by the permutation (846), the other single letters remaining unchanged. Similarly if 38 is the first to occur to the left of the bar the scheme that arises is changed in (ii) by the permutation (8357).

The general statement is that if, in forming a scheme for $2n$ from the standard scheme for $2n-1$, the first of the new symbols that enters to the left of the bar is $2r+1, 2n$, then the permutation

$$(2n, 2r+1, 2r+3, \ldots, 2n-1)$$

will change the resulting scheme for $2n$ into the standard scheme for $2n$: and that if $2r, 2n$ is the first that enters to the left of the bar, the permutation

$$(2n, 2r, 2r+2, \ldots, 2n-2)$$

has a similar effect. There is no difficulty in establishing similar results for building up the scheme for $2n+1$ from that for $2n$.

Returning now to the configuration of $n+2$ $(n-1)$-spreads in space of n dimensions, of general position, whose $\frac{1}{2}(n+1)(n+2)$ points of intersection are all finite points, the separation

$$ia, ib, ic, \ldots \mid id, ie, \ldots$$

implies that of the $n+1$ points which do not lie on the ith $(n-1)$-spread, the set $ia, ib, ic, \ldots$ are separated by the ith $(n-1)$-spread from the set $id, ie, \ldots$.

In particular for five planes in ordinary space the scheme is

$$\begin{array}{llll} \mid 12 & 13 & 14 & 15 \\ \mid 21 & 23 & 24 & 25 \\ 13 \mid & 23 & 34 & 35 \\ 24 \mid & 14 & 34 & 45 \\ 15 & 35 \mid & 25 & 45 \end{array}$$

The first line implies that plane 1 does not divide the tetrahedron 2345 into two parts; and the second line implies that plane 2 does not divide the tetrahedron 1345 into two parts.

The third line implies that plane 3 does divide the tetrahedron 1245 into two parts, one of which is the tetrahedron 2345, while the other is a polyhedron with vertices 12, 14, 15, 23, 34, 45 bounded by each of the five planes. Similarly plane 4 divides the tetrahedron 1235 into the tetrahedron 1345 and a polyhedron with vertices 12, 13, 25, 14, 35, 45 bounded by each of the five planes. Plane 5 divides the tetrahedron 1234 into two polyhedra, each of which is bounded by all the five planes, 12, 14, 23, 34, 15, 35 being the vertices of one and 12, 14, 23, 34, 25, 45 those of the other.

Hence any five planes of general position in space, whose points of intersection are all finite, form the five faces of just two distinct convex polyhedra each of which has six vertices. The scheme for six 3-spreads in 4-dimensional space may be similarly dealt with. Any five of the 3-spreads bound a convex polyhedron, and the scheme shows how this polyhedron is divided by the remaining 3-spread. The result shows that there are just three distinct convex polyhedra each of which is bounded by all the six 3-spreads. Two of these have eight vertices, viz.

12	23	25	26	14	34	45	46
12	14	15	16	23	34	35	36

It is obvious that two of the 3-dimensional faces of these are tetrahedra and the other four polyhedra with five faces and six vertices.

The remaining one has nine vertices, viz.

12 14 16 23 25 34 36 45 56

Its 3-dimensional faces are all polyhedra with six vertices and five faces.

A similar examination of the next scheme shows that seven 4-spreads in 5-dimensional space bound just four distinct convex polyhedra. The vertices are

23	24	37	47	12	17	25	26	57	67		
23	24	37	47	13	14	35	45	36	46		
13	14	16	23	24	26	37	47	67	15	25	57
13	14	16	23	24	26	37	47	67	35	45	56

Note on the Velocity of X-ray Electrons. By R. WHIDDINGTON, M.A.

[*Received* 5 August 1921.]

It has been known for many years that X-rays have the power of ejecting high speed electrons from the surface of materials on which they are incident.

The earliest attempts to determine the velocities of these electrons were made by Dorn in 1900 and Innes in 1907. Their general results being that the velocity, as measured by photographic records of the curvature in a known magnetic field, lay between 6 and 8×10^9 cm./sec. and was independent of the intensity of the exciting radiation.

In 1912 the problem was attacked from a different angle, by interpreting the absorption experiments of Beatty. It had previously been shown that a fourth power law of velocity absorption was quite fairly accurately true for fast moving electrons such as those ejected by X-rays*.

Beatty having measured, in essence, the range in air of the electrons ejected by various qualities of X-ray it was therefore possible to deduce their velocity*.

In the paper cited it was shown that the velocity of the *fastest* electrons ejected by X-rays was very nearly equal to $10^8 \times A$, where A is the atomic weight of the radiator supplying the X-rays. It was further predicted that future work should show the existence of two sets of electrons of definite speeds.

In the early months of this year de Broglie† published a short account of some highly important results in which by using a Coolidge X-ray tube he was able to take magnetic photographs of X-ray electrons with only an hour's exposure.

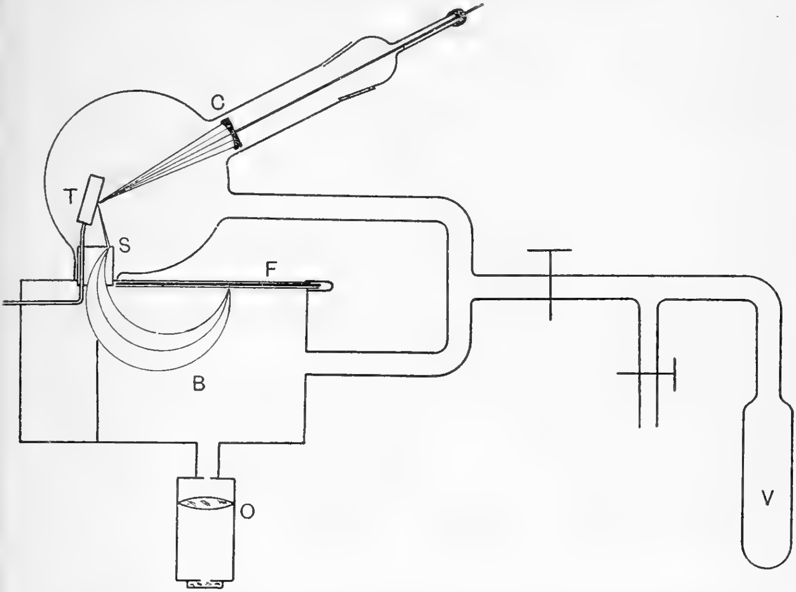
The present experiments were then in progress and it seems worth while now to publish a preliminary account of the apparatus finally adopted and initial results obtained.

The problem resolves itself into obtaining as strong a source of X-rays as possible, causing them to pass through a thin sheet of solid matter and observing the magnetic spectrum on the emergent side. De Broglie used a Coolidge tube to provide a powerful source of X-rays, a method which commends itself on the grounds of simplicity and ease of working.

The first attempts I made were with an apparatus consisting of a fine slit covered with thin metal foil on which Cathode-rays were focussed, the electrons ejected from the emergent side being subject to magnetic deviation on a photographic film. Photographs

* Whiddington, *Proc. Roy. Soc.* 1912.
† De Broglie, *Comptes Rendus*, 1921.

were obtained after considerable difficulty, but the method was abandoned since it was found that in spite of all precautions the thin foil rapidly disintegrated, wearing thin and eventually breaking. The apparatus finally adopted is shown in the figure.



Cathode-rays from the concave cathode C are focussed on the water-cooled target T. Immediately below T is the fine slit S (·5 mm. wide and 5 mm. long). About 2 cm. below this slit is another wider slit leading the rays into the evacuated box B on the under surface of which a photographic film F can be placed. The collimator O projects a little spot of light on the film for reference purposes. The whole system of X-ray bulb and camera is evacuated by a liquid air charcoal bulb V.

At right angles to the plane of the figure a nearly uniform magnetic field is applied so that the X-ray electrons streaming into the box are focussed on the film. This method was originally used by Rutherford for the determination of the speed of β-rays, was later used by Rawlinson and Robinson* in an experiment on X-ray electrons and was also used by de Broglie† in the experiments just cited.

Under good conditions as much as five milliamperes can be passed through the bulb and it is interesting to compare the efficiency of the arrangement above illustrated with that of de Broglie using a Coolidge tube.

* *Phil. Mag.* 1913. † *Loc. cit.*

If the distance between T and S is 1 cm., and we assume that the distance between the target of a Coolidge tube and the slit is (say) 8 cm., it follows that with the arrangement above, there is an available intensity 64 times as great.

Even if only 1 milliampere is used the available intensity will be more than ten times as great.

One disadvantage of the arrangement is that the coils producing a field in B, produce a small but appreciable magnetic field in the region STC.

The result is that the cathode-rays from C are deflected. This deflexion must be balanced out by an additional compensating coil in series with the main coils. It is fortunate in this connexion that it is the fastest rays arriving at T which are the most effective X-ray parents and which are least affected by the stray field*.

The photographic film used was 10 cm. long and the current in the field coils was adjusted to give a range of velocities from 3×10^9 to 10^{10} cm./sec. approximately, yielding a dispersion of $0{\cdot}071 \times 10^9$ cm./sec. per millimetre.

It is possible to take a photograph with the apparatus in half an hour although longer exposures are desirable. Using a platinum or rhodium target, for example, and a copper foil over the slit, the strongest lines on the film correspond to velocities $6{\cdot}02 \times 10^9$ and $5{\cdot}74 \times 10^9$ cm./sec.

If we apply the quantum relation to the X-ray K doublet of copper we get the corresponding velocities of electrons carrying the same energy to be $5{\cdot}61 \times 10^9$ and $5{\cdot}33 \times 10^9$ cm./sec.

It is interesting to note that while the differences are precisely the same in both experimental and calculated cases, the actual values differ by about 10 per cent. Whether this difference is real or due to a defect somewhere in the apparatus must be determined by further investigation.

It is worth remembering, however, that both the present writer and Hull† found that velocities of parent electrons distinctly in excess of the value demanded by the quantum relation were required for the production of a fluorescent radiation of any definite wave length.

It seems possible that the difference if it be real may be accounted for by taking into account the sums of the individual energies required for the simultaneous excitation of the radiations of both K and L series.

It is hoped to publish a full and extended account of this work very shortly.

* See also Whiddington, *Proc. Roy. Soc.* 1911.
† *Physical Review*, 1916.

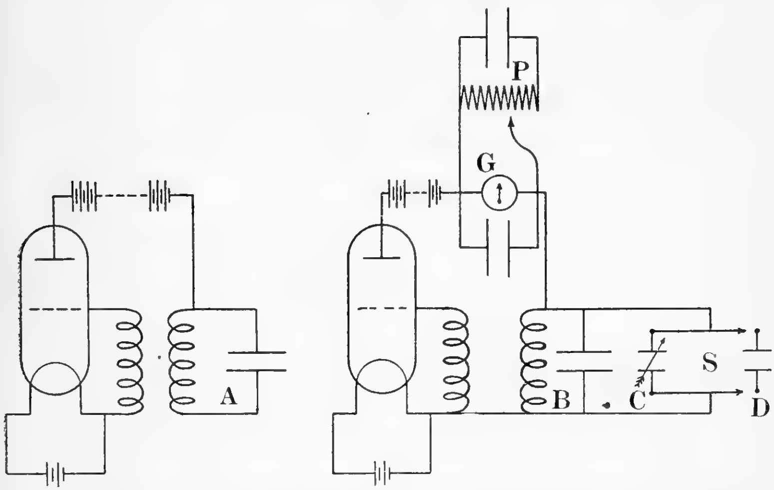
A Laboratory Valve method for determining the Specific Inductive Capacities of Liquids. By R. WHIDDINGTON, M.A.

[*Received* 30 July 1921.]

The method outlined below has been used successfully by students as a laboratory exercise in the Physics Laboratories of Leeds University.

The method employs alternating electro-motive forces of low frequency generated by a thermionic valve linked with the usual reacting circuits.

The apparatus is shown diagrammatically below. On the left is shown a standard circuit generating oscillations in the closed circuit *A* consisting of a fixed condenser of about ·5 m.f.d. and a large air core coil. The natural frequency of this circuit is about 1000.



Very loosely linked to this circuit is a similar one *B* also comprising an air core coil and condenser box, but in parallel with the latter is arranged a small variable condenser *C* and a mercury cup switch *S* so that an additional small condenser *D* can be switched in at will. *C* is a moving vane air condenser fitted with pointer and scale, *D* is a small parallel plate condenser the dielectric medium of which can be easily changed without altering the distance between the plates. Included in the anode circuit of this arrangement is a condenser shunted, and an aperiodic needle galvanometer *G* connected as shown to a potentiometer *P* to balance the steady current flowing.

This galvanometer indicates in a very convenient manner the slow beats set up when A and B are suitably adjusted.

To determine the capacity of D it is merely necessary to adjust C until a definite beat rate between A and B is established and then switch in D. A new beat rate will be set up which can be restored to the original rate by decreasing C, until the decrease in C is equal to the additional capacity of D. The experiment is then repeated with a liquid dielectric in D, the ratio of the two capacities being the Specific Inductive Capacity required.

It is to be noticed that the actual capacity change indicated by C is not required so long as the scale is known to be uniform by previous calibration.

It was found in the case of the particular condenser used in the position C that the scale was for all practical purposes uniform over the central portion of its scale, each division representing ·0000065 m.f.d.

In an experiment carried out by Mr L. G. Stanton the following results were obtained using olive oil as the dielectric.

Dielectric Air.

D in	D out	Difference	
116	160	44	Mean difference 44
118	162	44	

Dielectric Olive Oil.

D in	D out	Difference	
36	166	130	Mean difference 131
32	164	132	

The Specific Inductive Capacity of the oil is therefore 2·98 which is very close to that usually given in Physical Tables.

The Theoretical Value of Sutherland's Constant in the Kinetic Theory of Gases. By C. G. F. JAMES, Trinity College, Cambridge. (Communicated by Mr R. H. FOWLER.)

[*Received* 28 April: *read* 2 May 1921.]

§ 1. In any attempt to deduce, from observations on viscosity and diffusion for gases, facts as to the nature of the intermolecular forces, it is necessary to find the theoretical relation that holds between the so-called Sutherland's constants, and any assumed intermolecular force. It is assumed that the molecule behaves as a perfectly elastic sphere surrounded by a field of attractive force. This is in fact the only known model, capable of satisfactorily predicting the observed laws of variation with temperature of these quantities, at least at ordinary temperatures. The relations in question have been worked out by Professor Chapman in various papers*. His formulae for Sutherland's constants are given on p. 459 of his first paper on the subject.

It appears however that the formulae in question are affected by a certain error explained below. This mistake affects the relations between Sutherland's constant S for a single gas, and analogous quantities, and the potential of the intermolecular field. It must be definitely understood however that this in no way affects the rest of his theory, or the numerical values of σ, the molecular diameter, obtained. These are in fact deduced directly from observed values of S. It is only when theoretical values of S become important, that this mistake is of any significance.

It was suggested to me by Mr Fowler that the correct theoretical determination of this constant is of considerable importance. This, then, forms the subject of this paper.

It is known, that with fair accuracy the relation between the viscosity μ of a single gas and the temperature T is, when T is sufficiently large,

$$\mu \propto T^{\frac{1}{2}}/(1 + S/T),$$

where S is Sutherland's constant; and can be calculated in terms of the forces in action. Thus if $\phi(r)$ is the potential of the force between two molecules whose distance apart is r, Prof. Chapman's result was

$$S = \phi(\sigma)/3R,$$

R being the usual gas constant for one molecule ($1{\cdot}372 \times 10^{-6}$).

* *Phil. Trans.* (A), vol. 211 (1912), p. 432; vol. 216 (1915), p. 276; vol. 217 (1917), p. 115.

The correct result is not so simple as this, and cannot be expressed so generally.

In the same way, the coefficient of diffusion for a mixture of two gases satisfies a relation

$$D_{12} \propto T^{\frac{3}{2}}/(1 + S_{12}/T),$$

where if σ_{12} denotes the mean of the two molecular diameters,

$$S_{12} = \lambda\phi\,(\sigma_{12})/R,$$

where λ is a number depending on the law of force assumed. Prof. Chapman gives $\lambda = \frac{1}{2}$ for all laws of force.

Prof. Chapman informs me that Enskog* has already pointed out the necessity for the corrections here referred to, and that his results agree with those found in § 6. The results of the other sections have not, I understand, been given by him.

§ 2. *A note on the approximations employed.* It is necessary to regard the absolute temperature T as large, since the theory gives for the denominators of μ and D_{12} series in descending powers of T. We shall also neglect squares and higher powers of $\phi\,(\sigma)$. Actually it will be seen that all terms involving $\phi\,(\sigma)$ to any power are multiplied by $1/T$ to the same power. Thus the assumption is really that $\phi\,(\sigma)/T$ is negligible in higher powers than the first.

As regards the terms involving $\{\phi(\sigma)/T\}^2$ it will be found that Prof. Chapman's statement, that these are positive in each case, is not affected.

§ 3. *Statement of the Problem.* The first step is the determination of the deflection of a typical molecule, relative to a selected molecule which is conveniently supposed reduced to rest. In the diffusion problem these molecules will be of opposite kinds. In the viscosity case we will for simplicity consider a single gas only.

To the order of approximation explained above it is only necessary to consider molecules that actually suffer impact with the selected molecule. Let A and B be the centres of the respective molecules at impact, so that in the general case:

$$AB = \sigma_{12} = \tfrac{1}{2}\,(\sigma_1 + \sigma_2).$$

Let TN, NT' be the asymptotes of the initial and final relative paths; QB, BQ' the directions of motion immediately before and immediately after impact. Let V be the relative velocity at a great distance, p the perpendicular onto the asymptotes. The molecule B is typified by the direction and magnitude of V, by p, and by an azimuthal angle ϵ, determining the direction of p, in a plane at right angles to V. The true deflection required is

$$2\chi = T_1\hat{N}T',$$

* Inaugural Dissertation, Uppsala, 1917.

while the angle taken by Prof. Chapman is

$$2\chi' = Q_1\hat{B}Q'.$$

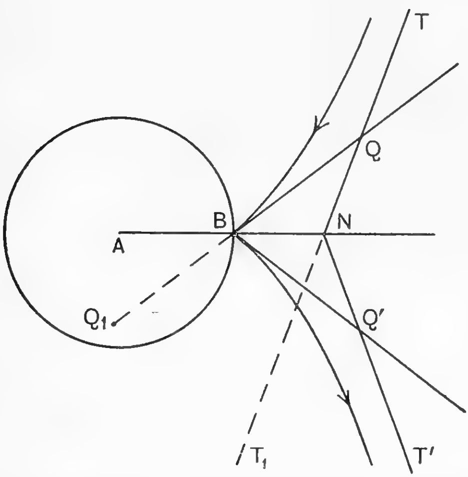
We propose to calculate the effect of this difference on the viscosity and diffusion problems. Two laws of force will be considered:

(1) The inverse power law.

(2) A shell of constant force from $r = \sigma_{12}$ to $r = d = \alpha\sigma_{12}$, where α is a moderately small number.

§ 4. *Calculation of the Correction in the angle* χ. This correction is seen from the figure to be negative and equal in magnitude to

$$\psi = B\hat{Q}N.$$



Let $\{-f(r)\}$ be the acceleration of B in the relative orbit. Thus $\phi(r)$ being the potential of the intermolecular field,

$$f(r) = -\frac{d\Phi(r)}{dr} = -\frac{m_1 + m_2}{m_1 m_2}\frac{d\phi(r)}{dr},$$

$\Phi(r)$ being introduced for convenience merely. In our case, the forces being attractive, $f(r)$ and $\Phi(r)$ are positive.

Let ϖ be the perpendicular onto the tangent at any point. Then we have

$$\frac{d}{dr}\left(\frac{1}{\varpi^2}\right) = -\frac{2}{c^2}f(r),$$

where c is the angular momentum pV. Integrating:

$$\frac{1}{\varpi^2} = \frac{1}{p^2}\left(1 + \frac{2\Phi(r)}{V^2}\right).$$

Now, if ρ is the radius of curvature at any point of the path, and μ the inclination of the radius vector to the tangent,

$$-f(r) = \frac{v^2}{\rho \sin \mu} = \frac{V^2 p^2 r}{\varpi^3 \rho} = \frac{V^2 r}{\rho p}\left\{1 + \frac{2}{V^2}\Phi(r)\right\}^{\frac{3}{2}},$$

or to the first order

$$\frac{1}{\rho} = -\frac{pf(r)}{V^2 r}.$$

If s is the arc of the path this is

$$\frac{d\psi}{ds} = -\frac{pf(r)}{V^2 r},$$

leading to

$$\psi = -\frac{p}{V^2}\int_\infty \frac{f(r)}{r}\frac{ds}{dr}\,dr.$$

Also, from the equation

$$\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2 = \frac{2r^2}{p^2V^2}\{\tfrac{1}{2}V^2 + \Phi(r)\} - 1,$$

we obtain

$$\frac{ds}{dr} = \left\{\frac{\{V^2 + 2\Phi(r)\}\,r^2}{r^2\{V^2 + 2\Phi(r)\} - p^2V^2}\right\}^{\frac{1}{2}},$$

which to the first order is

$$\frac{ds}{dr} = \frac{r}{(r^2 - p^2)^{\frac{1}{2}}},$$

whence

$$\psi(\sigma) = \frac{p}{V^2}\int_\sigma^\infty \frac{f(r)\,dr}{(r^2 - p^2)^{\frac{1}{2}}}.$$

It will be observed that $f(r)$ may be of any order of magnitude, provided only $\Phi(r)$ remains small.

§ 5. *The Viscosity of a Single Gas. General Analysis.* On reference to the papers cited* it will be seen that we have now to evaluate

$$\Omega''(V) = \pi V \int_0^{p_\nu} \sin^2 2\chi\, p\,dp,$$

where p_ν is the greatest value of p for orbits involving collision, so that $p_\nu = \sigma\{1 + 2\Phi(\sigma)/V^2\}^{\frac{1}{2}}$. To our order we may put $p_\nu = \sigma$. Putting $\chi = \chi' - \psi$, and expanding in ψ to the first power only,

$$\Delta\Omega''(V) = -\pi V \int_0^\sigma 4\psi \sin 2\chi' \cos 2\chi' p\,dp.$$

* First Paper, p. 453: $\Omega''(V)$ is denoted there by $\Omega''_{11}(V_0)$. See also pp. 457, 458.

Now $\cos\chi' = p/p_v = p/\sigma = t$ (say). This substitution gives

$$\Delta\Omega''(V) = -8\sigma^2\pi V\int_0^1 \psi t^2(1-t^2)^{\frac{1}{2}}(2t^2-1)\,dt.$$

It is now necessary to specialize the law of force.

§ 6. *The Inverse Power Law.* We take

$$f(r) = k/r^n,\quad \Phi(r) = k/(n-1)r^{n-1},$$

giving

$$\psi = \frac{kp}{V^2}\int_\sigma^\infty\left\{\sum_0^\infty s_r\left(\frac{p}{r}\right)^{2r}\right\}\frac{dr}{r^{n+1}},$$

where s_r is the coefficient of ξ^r in the expansion of $(1-\xi)^{-\frac{1}{2}}$. Thus

$$\psi = \frac{kp}{V^2}\sum_0^\infty\frac{s_r}{n+2r}\,\frac{p^{2r}}{\sigma^{n+2r}} = \frac{k}{V^2\sigma^{n-1}}\sum_0^\infty\frac{s_r}{n+2r}\,t^{2r+1},$$

$$\Delta\Omega''(V) = -\frac{8\pi kV}{V^2\sigma^{n-3}}\int_0^1\sum_0^\infty t^{2r+3}\frac{s_r}{n+2r}(1-t^2)^{\frac{1}{2}}(2t^2-1)\,dt,$$

which readily evaluates to

$$\Delta\Omega''(V) = -\frac{16\pi k}{V\sigma^{n-3}}h_n,$$

where h_n is the sum of the series $\displaystyle\sum_0^\infty\frac{r+1}{(2r+7)(2r+5)(2r+3)(2r+n)}$.

The next step is the evaluation of

$$\begin{aligned}\Delta R'' &= \int_0^\infty\Delta\Omega''(V)\,V^6e^{-\frac{1}{2}hmV^2}\,dV\\ &= -\frac{16k\pi h_n}{\sigma^{n-3}}\int_0^\infty V^5e^{-\frac{1}{2}hmV^2}\,dV\\ &= -\frac{16k\pi h_n}{\sigma^{n-3}}\left(\frac{2}{hm}\right)^3.\end{aligned}$$

Now $\Phi(\sigma) = k/(n-1)\sigma^{n-1}$. Substituting,

$$\Delta R'' = -\frac{16\pi\sigma^2}{(hm)^4}\left\{\frac{hm\Phi(\sigma)}{3}h_n'\right\},$$

where $\displaystyle h_n' = 24(n-1)\sum_0^\infty\frac{r+1}{(2r+7)(2r+5)(2r+3)(2r+n)}.$

This will be found to lead to

$$\mu = \frac{5m}{16\sigma^2\pi^2}\left(\frac{RT}{m}\right)^{\frac{1}{2}}\Big/\left\{1+\frac{hm\Phi(\sigma)}{3}(1-h_n')\right\}.$$

Now $h = 1/2RT$, $\Phi(\sigma) = 2\phi(\sigma)/m$. Thus

$$S = \frac{\phi(\sigma)}{R}\,\frac{1-h_n'}{3}.$$

The numerical values of $RS/\phi(\sigma)$ are as follows:

n	4	5	6	7	8	9
$RS/\phi(\sigma)$	0·213	0·196	0·183	0·172	0·163	0·156

Prof. Chapman's value is 0·333 in all cases. The calculation is very simple for n odd, from $n = 9$ (inclusive). The other values are more troublesome. The value for $n = 8$ was interpolated merely.

§ 7. We proceed now to the same calculation for the law of force:

$$f(r) = A, \quad \sigma \leqslant r \leqslant \alpha\sigma,$$
$$f(r) = 0, \quad r > \alpha\sigma.$$

Thus
$$\Phi(\sigma) = A\sigma(\alpha - 1).$$

Hence
$$\psi = \frac{Ap}{V^2}\int_\sigma^{\alpha\sigma}\frac{dr}{(r^2-p^2)^{\frac{1}{2}}} = \frac{Ap}{V^2}\int_\sigma^{\alpha\sigma}\sum_0^\infty s_r\left(\frac{p}{r}\right)^{2r}\frac{dr}{r},$$
$$\psi = \frac{Ap}{V^2}\left\{\log\alpha - \left[\sum_1^\infty \frac{s_r}{2r}\frac{p^{2r}}{r^{2r}}\right]_\sigma^{\alpha\sigma}\right\}$$
$$= \frac{A\sigma}{V^2}\left\{t\log\alpha + \sum_1^\infty \frac{s_r}{2r}t^{2r+1}\left[1 - \frac{1}{\alpha^{2r}}\right]\right\}.$$

This leads to
$$\Delta\Omega''(V) = 16A\sigma^3\pi h_a/V,$$
where
$$h_a = \sum_1^\infty \frac{(r+1)}{2r(2r+3)(2r+5)(2r+7)}\left\{1 - \frac{1}{\alpha^{2r}}\right\} + \frac{1}{105}\log\alpha.$$

By comparison with § 6 we can at once write down the number corresponding to h_a', namely
$$h_a' = 24h_a/(\alpha - 1).$$
In this way we draw up the table

a	1·25	1·50	1·75	2·00	2·50	3	4
$RS/\phi(\sigma)$	0·166	0·206	0·227	0·241	0·259	0·270	0·283

a	5	6	7	8	9	10
$RS/\phi(\sigma)$	0·291	0·297	0·301	0·304	0·307	0·309

§ 8. For a given value of $\phi(\sigma)$, Sutherland's constant S decreases, so that finally for a thin shell of intense force it vanishes. The result is thus, to this order at any rate, the same as for elastic spheres exerting no force.

Thus suppose $f(r) = 0$ for $r > \sigma + \epsilon$, while $\Phi(\sigma) = \int_{\sigma+\epsilon}^{\sigma} f(r)\,dr$ remains finite. This gives without approximation

$$\psi = \int_\sigma^{\sigma+\epsilon} \frac{p}{V^2} \frac{f(r)}{r} \left\{1 + \frac{2\Phi(\sigma)}{V^2}\right\}^{-\frac{3}{2}} \frac{ds}{dr} dr$$

$$= \frac{p\sigma^2}{V^2 p_\nu^3} \operatorname{cosec} \chi',$$

in the limit $\epsilon \to 0$. Approximately

$$\Delta\Omega''(V) = -8\pi\sigma^2 \Phi(\sigma)/V \int_0^1 (2t^2 - 1)\, t^3 dt$$

$$= -2\sigma^2 \Phi(\sigma)/3V,$$

whence
$$\Delta R'' = -\frac{16\pi\sigma^2}{(hm)^4} \left\{\frac{hm\,\Phi(\sigma)}{3}\right\},$$

and it will be seen that this leads to $S = 0$. The approximation consists only in the neglect of higher terms in the expansion of $\sin^2 2(\chi' - \psi)$. These terms will not disappear.

§ 9. *The Second Order Terms.* It will be sufficiently typical to consider only the inverse power law. These terms arise in five ways.

I. From the second term in the expansion of $\psi(\sigma)$.

II. From the first term in the expansion of $\psi(\sigma)$; and the second term in the expansion of $\left(\frac{ds}{dr}\right)_{r=\sigma}$.

III. From the first term in $\psi(\sigma)$ arising from the approximate value of the upper limit p_ν. This is conveniently dealt with simultaneously with the last.

IV. From the ψ^2 term in the expansion of $\sin^2 2(\chi' - \psi)$.

V. From the deflection of molecules, which do not actually suffer collision.

These and these only lead to terms of type $1/T^2$ in the denominator of the expression for μ. I have verified by actual calculation that taken together they give a term of the type $K\{\phi(\sigma)/RT\}^2$, where K is a positive number.

§ 10. *Diffusion of a Mixture of Two Gases.* We have to calculate*

$$\Omega'(V) = 4\pi V \int_0^{p_\nu} \sin^2 \chi\, p\, dp,$$

and $\sin^2 \chi = \sin^2 \chi' - 2\psi \sin \chi' \cos \chi'$, so that

$$\Delta\Omega'(V) = 8\pi V \int_0^{p_\nu} \psi \sin \chi' \cos \chi'\, p\, dp.$$

For the inverse power law we find

$$\Delta\Omega'(V) = -8\pi k H_n / V_\sigma^{m-3},$$

* $\Omega'(V)$ is Prof. Chapman's $\Omega'_{12}(V_0)$.

where $$H_n = \sum_0^\infty \frac{2(r+1)}{(2r+1)(2r+3)(2r+5)(2r+n)}.$$

The next step in the integration is to calculate

$$\Delta P'_{12} = -\int_0^\infty V^4 \Delta\Omega'(V) e^{-\xi V^2} dV, \quad \text{where } \xi = -\frac{hm_1 m_2}{m_1+m_2},$$

or $$\Delta P'_{12} = -\frac{4\pi k H_n}{\sigma^{n-3}} \left\{\frac{hm_1 m_2}{m_1+m_2}\right\}^{-2}.$$

The coefficient of diffusion D_{12} contains a factor $\left(1+\frac{S_{12}}{T}\right)^{-1}$; namely

$$D_{12} = \frac{3}{16}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sigma^2_{12}} \left(\frac{RT}{h}\right)^{\frac{3}{2}} \left(\frac{1}{m_1}+\frac{1}{m_2}\right)(1+S_{12}/T)^{-1},$$

and on reference to the papers cited, together with the above analysis, it will be seen that

$$S_{12} = \phi(\sigma)\{1-4(n-1)H_n\}/2R.$$

By comparison with the above, and with the viscosity calculations, we can at once write down, for the shell of force case

$$S_{12} = \phi(\sigma)\{1-4H_a/(\alpha-1)\}/2R,$$

where

$$H_a = \left[\sum_1^\infty \frac{2(r+1)}{(2r+1)(2r+3)(2r+5)}\left\{1-\frac{1}{\alpha^{2r}}\right\} + \frac{2}{105}\log\alpha\right].$$

The values of $RS_{12}/\phi(\sigma)$ are given in the following tables:

Inverse Power Law.

n	4	5	6	7	8	9
$RS_{12}/\phi(\sigma)$	0·227	0·201	0·182	0·167	0·154	0·144

The numerical calculation becomes simple for n odd, from $n = 7$ inclusive. $n = 8$ was interpolated merely.

The Shell of Constant Force.

a	1·25	1·50	1·75	2	2·5	3	4
$RS_{12}/\phi(\sigma)$	0·153	0·208	0·243	0·268	0·303	0·327	0·358

a	5	6	7	8	9	10
$RS_{12}/\phi(\sigma)$	0·378	0·393	0·404	0·413	0·420	0·425

Prof. Chapman's value was 0·5 in all cases. That this sequence tends to zero as α tends to unity can be verified as before.

It can also be verified, in the same way, that the next term in the series in inverse powers of T is still positive.

On the Stability of the Steady Motion of viscous liquid contained between two rotating coaxal circular cylinders. By W. J. HARRISON, M.A., Fellow of Clare College, Cambridge.

[*Received* 29 August 1921.]

In the previous paper* on this subject an error has been made which invalidates the results given in Part I. The solution given in equation (26) satisfies equation (25), but is not sufficiently general to provide a solution of the problem, as it makes two values of m identical. This error has been pointed out by Prof. W. McF. Orr. Between equations (4) and (12) there are also various errors and misprints which are misleading, although they do not affect the subsequent work. A brief statement of these and a slight modification of the method of obtaining equation (17) will be given first.

The equation following (4) is correct. Integrating the last three integrals by parts we obtain the equation preceding (5), except that the final expression in it should be written

$$-\int\left[p_{xx}\frac{\partial u}{\partial x}+p_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+p_{yy}\frac{\partial v}{\partial y}\right]d\tau.$$

Equation (5) is correct after making the same alteration. Equations (6) should be written

$$p_{xx}=-p-\tfrac{2}{3}\mu\operatorname{div}q+2\mu\frac{\partial u}{\partial x},\text{ etc.,}$$

$$p_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

In equation (7), $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2$ should be replaced by $\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$, and (8) should be written $p'=p+\tfrac{2}{3}\mu\operatorname{div}q$.

The equation giving the critical value of μ for a given disturbance is obtained by equating the left-hand side of equation (7) to zero. Varying u, v, μ in this equation, putting $\delta\mu=0$, and integrating by parts where necessary, we obtain equations determining a state of disturbed motion such that the relative kinetic energy is stationary, μ being at the same time stationary and a maximum. After performing these operations the fluid is treated as incompressible.

Thus we arrive at equations determining the mode of disturbance which is most likely to cause the steady motion to change to turbulent motion. Hence for all greater values of μ than that which

* Tamaki and Harrison, *Trans. Camb. Phil. Soc.*, vol. XXII, No. 22.

gives stationary relative kinetic energy for this mode of disturbance, the steady motion is bound to be stable whatever the nature of the disturbance.

These equations are (compare with (9) in the previous paper)

$$2\mu\nabla^2 u - \rho\left(2u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + v\frac{\partial V}{\partial x}\right) = \frac{\partial p}{\partial x},$$

$$2\mu\nabla^2 v - \rho\left(u\frac{\partial V}{\partial x} + u\frac{\partial U}{\partial y} + 2v\frac{\partial V}{\partial y}\right) = \frac{\partial p}{\partial y}.$$

Writing $$u = -\frac{\partial\psi}{\partial y},\quad v = \frac{\partial\psi}{\partial x},$$

$$U = -\frac{\partial\Psi}{\partial y},\quad V = \frac{\partial\Psi}{\partial x},$$

eliminating p, and transforming to polar coordinates (r, θ), noticing that Ψ is a function of r only, we find

$$2\mu\nabla^4\psi + 2\rho\left(\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r}\right)\frac{1}{r}\left(\frac{\partial^2\psi}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)$$
$$+\rho\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = 0.$$

Now $$\Psi = A\log r + \tfrac{1}{2}Br^2.$$

Hence $$\nabla^4\psi - \frac{2A\rho}{\mu r^3}\left(\frac{\partial^2\psi}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) = 0.$$

Assume that ψ varies as $e^{\iota\lambda\theta}$, where λ can take integral values, and we have equation (17).

Let the notation be changed by writing

$$k = 2A\lambda\rho/\mu,$$
$$\psi = r^{\iota m+1}e^{\iota\lambda\theta},$$

so that (21) becomes

$$m^4 + 2(1+\lambda^2)m^2 + km + (1-\lambda^2)^2 = 0.$$

Write $\log_e(b/a) = n$, then equation (25) becomes

$$2\sigma s\cos\sigma n\cos sn - (4p^2 - \sigma^2 - s^2)\sin\sigma n\sin sn - 2\sigma s\cos 2pn = 0 \quad \ldots\ldots(1).$$

This equation is identical with equation (15) on page 125 of Orr's paper in the *Proceedings of the Royal Irish Society*, vol. XXVII, Section A. He shows that either s or σ must be imaginary.

Write $\sigma = \iota\sigma'$, and (1) becomes

$$2\sigma' s\cosh\sigma' n\cos sn - (4p^2 + \sigma'^2 - s^2)\sinh\sigma' n\sin sn$$
$$- 2\sigma' s\cos 2pn = 0 \quad \ldots\ldots(2).$$

Thus equations (23) are replaced by

$$\left.\begin{aligned} \sigma'^2 - s^2 - 2p^2 &= 2\,(1+\lambda^2), \\ (p^2+\sigma'^2)\,(p^2-s^2) &= (1-\lambda^2)^2, \\ 2p\,(\sigma'^2+s^2) &= -\,k. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

Expressing σ' and s in terms of p, we have

$$s^2 = 2\sqrt{\{\lambda^2+(1+\lambda^2)\,p^2+p^4\}} - (1+\lambda^2+p^2) \quad \ldots(4),$$

$$\sigma'^2 = 2\sqrt{\{\lambda^2+(1+\lambda^2)\,p^2+p^4\}} + (1+\lambda^2+p^2) \quad \ldots(5).$$

Thus $$\frac{\lambda\rho A}{\mu} = -\,4p\sqrt{\{\lambda^2+(1+\lambda^2)\,p^2+p^4\}}.$$

Now p may be taken either positive or negative, and A is either positive or negative according to the nature of the motion of the cylinders. Putting

$$A = \frac{a^2b^2\omega}{b^2-a^2}, \quad (b>a)$$

and assuming p and ω to be treated as positive, we have

$$\frac{\rho\omega a^2}{\mu} = \left(1-\frac{a^2}{b^2}\right)\frac{p}{\lambda}\sqrt{\{\lambda^2+(1+\lambda^2)\,p^2+p^4\}} \quad \ldots\ldots(7).$$

The discussion of equation (2) proceeds exactly as in Orr's paper, using the values for s and σ' obtained above, and it is easily shown that there is no solution for which sn is less than π except one for which sn is zero.

Putting $s^2 = p^2 + \epsilon$ in (4), we see that

$$(1-\lambda^2)^2 + 2\epsilon\,(2p^2+1+\lambda^2) = 0.$$

Hence ϵ is negative, and p is always numerically greater than s. Thus from (5) $\sigma' > \sqrt{3}p > \sqrt{3}s$, and, therefore, $\sinh \sigma' n$ and $\cosh \sigma' n$ each exceed 100 and are approximately equal. Thus (2) becomes

$$\tan sn = \frac{2\sigma' s}{4p^2+\sigma'^2-s^2}$$

$$= \frac{\sqrt{\{3p^4+2\,(1+\lambda^2)\,p^2-(1-\lambda^2)^2\}}}{3p^2+1+\lambda^2} \quad \ldots\ldots(8).$$

We have to solve the equations (4) and (8) for p and s, using integral values of λ. The result of the substitution in (7) gives the corresponding critical values of μ. Now for a given λ, $\tan sn$ given by (8) has no real stationary value, and thus $\tan sn$ must be less than $\sqrt{\tfrac{1}{3}}$. Hence the smallest value of sn, except the zero value, satisfies the inequality $\pi \leqslant sn \leqslant 7\pi/6$. The corresponding value of p is also the smallest of the series of roots in p, and therefore gives the left-hand side of (7) its minimum value for given values of b/a and λ.

The calculated values of $\rho\omega a^2/\mu$ for series of values of b/a and λ are shown below.

$\rho\omega a^2/\mu$.

λ	b/a		
	1·1	2	10
1	39480	459	22
2	19760	242	17*
3	13200	176	22
4	—	149	—
5	—	140	—
6	—	137*	—
7	—	146	—
40	424	—	—
41	423*	—	—
42	424	—	—
43	425	—	—
45	428	—	—
50	443	—	—

The values marked * give the critical values of μ for the corresponding values of b/a.

Write $b - a = D$, then we have

$b/a =$	1·1	2	10
$D/a =$	·1	1	9
Critical value of $\rho\omega a^2/\mu =$	423	137	17
Critical value of $\rho\omega D^2/\mu =$	4·23	137	1380

The critical values of $\rho\omega D^2/\mu$ may be compared with Orr's result $B\rho D^2/\mu = 177$ for the shearing motion between parallel planes, showing that the shearing motion between circular cylinders is relatively more stable than the shearing motion between parallel planes if b/a is small, and relatively less stable if b/a is large.

There are some approximations which are of interest.

(*a*) *λ large.*

Since s^2 is positive, it follows that p^2 is greater than the positive root of

$$3p^4 + 2(1 + \lambda^2)p^2 - (1 - \lambda^2)^2 = 0,$$

considered as a quadratic equation in p^2. The same result follows by considering the reality of tan sn. Hence

$$p^2 > \tfrac{1}{3}\{2(\lambda^4 - \lambda^2 + 1)^{\frac{1}{2}} - (1 + \lambda^2)\}.$$

Therefore p is of the same order as λ, if λ is large. Accordingly

$$\frac{\rho\omega a^2}{\mu} = \left(1 - \frac{a^2}{b^2}\right) \frac{4p^2\sqrt{(\lambda^2+p^2)}}{\lambda} \quad \text{.................(9)},$$

$$\tan sn = \sqrt{(3p^4 + 2\lambda^2 p^2 - \lambda^4)}/(3p^2+\lambda) \quad \text{...........(10)},$$

$$s^2 = 2p\sqrt{(\lambda^2+p^2)} - \lambda^2 - p^2 \quad \text{..................(11)}.$$

(10) and (11) are the same as Orr's (30) and (21).

(*b*) *b/a nearly unity; λ not large.*

We find $$s = p, \quad sn = 7\pi/6.$$

(*c*) *a and b large; $b - a$ finite; λ not large.*

We have $$s = p, \quad sn = pn = 7\pi/6.$$

Write $$b - a = d.$$

Then $$n = \log_e\left(\frac{b}{a}\right) = \log_e\left(1 + \frac{d}{a}\right) = \frac{d}{a},$$

and $$p = \frac{7\pi a}{6d}.$$

Therefore $$\frac{\rho\omega d^2}{\mu} = \frac{8}{\lambda}\left(\frac{7\pi}{6}\right)^3$$
$$= 394/\lambda.$$

This approximation may be compared with the calculations given above for $b/a = \cdot 1$. If, however, we make a and b infinite, the restriction of λ to finite integral values is meaningless, and therefore we cannot expect to deduce the results for shearing motion between two parallel planes.

The soaring flight of dragon-flies. By E. H. Hankin, M.A., Sc.D., Agra, India.

[*Read* 16 May 1921.]

I. *Introduction.*

Dragon-flies of the larger species to be found in Agra, such as *Hemianax ephippiger*, frequently spend the day gliding at a height of several metres above tree-top level. At sunset they come down to low levels in search of prey, and hence have been described as of crepuscular habits. When flying at a height during the day they may be watched through a binocular till one's arms are tired without a single flap of the wings being observed. Proofs that such flight is not due to undiscovered wing movements will be found in a later paragraph, thus leading to the conclusion that it is an instance of soaring flight. Smaller species, such as *Pantala flavescens* and *Tramea burmeisteri*, that habitually fly nearer to ground level are more suitable for detailed observation and on these most of my work has been done.

II. *Soaring flight at low speed.*

The mode of flight of *Pantala flavescens* during the months of July and August and in fine weather is as follows.

If, in the early morning, one watches a group of these dragon-flies, it will be seen that their flight consists of alternate periods of flapping and gliding. The flaps at this time are of the hind wings only. As the sun gets stronger the amount of flapping is seen to decrease and to be more and more replaced by gliding. By about nine o'clock the dragon-flies are showing two methods of flight; either they glide with the abdomen horizontal, aided by two or three flaps of the hind wings every two or three yards of their course (Fig. 1), or, on the other hand, they glide with the abdomen elevated, travelling horizontally for the most part, and apparently may proceed in this way for indefinite distances (Fig. 2). Glides of 10 or 15 seconds without a flap may be observed. Flaps are apt to occur when two dragon-flies happen to approach each other. In the case of another species (*Rhyothemis variegata*), observed in Calcutta, proofs were obtained that such occasional flapping of their brilliantly coloured hind wings is used as a signal to other individuals (*Animal Flight*, p. 388).

The mode of flight continues to vary with the time of day. If there is strong sunshine and but little wind, then, at about eleven o'clock, the dragon-flies may be seen gliding with the abdomen and hind legs hanging down as shown in Fig. 3. This adjustment is maintained till about three o'clock when the abdomen is seen to be again elevated above the level of the thorax. Then, later in the afternoon as the sun gets weaker, the gliding is interrupted by periods of flapping of the hind wings. The amount of flapping gradually increases till, towards twilight, the dragon-flies descend to within a few inches of the ground in search of prey. They are then in fast jolting flight caused by flapping of all four wings.

Fig. 1. Dragon-fly in flap-gliding flight seen from the side and below. The hind legs are parallel with the abdomen. The latter is horizontal.

Fig. 2. Dragon-fly in slow speed soaring flight seen from the side. The abdomen is elevated above the level of the thorax.

Fig. 3. Dragon-fly in soaring flight with hind legs and abdomen hanging down.

III. *Object of flight with abdomen hanging down.*

A clue to the meaning of the adjustment was obtained by observing instances in which it occurred in single individuals while others near by were not using it.

On several occasions it has been noticed that dragon-flies with abdomens down show less speed than others whose abdomens are up. With various species of dragon-fly gliding over water it has been repeatedly noticed that the abdomen is lowered when they need to check speed, either to avoid another dragon-fly or on turning as they reach some obstruction. In a light wind dragon-flies gliding to and fro over a restricted course, as is usual with *Pantala*, have been seen to have abdomens down when going with the wind and to raise their abdomens each time they go in the opposite direction.

A dragon-fly has been observed to lower the abdomen for a moment, apparently to check speed, while catching an insect.

We are thus led to the conclusion that lowering the abdomen and hind legs is an adjustment intended to check speed. It is obvious that in the lowered position the abdomen and legs must present more resistance to speed ahead than occurs when these organs are in the up position.

IV. *Conditions under which continued flight occurs with abdomen down.*

From the diary of my observations it soon appeared that this adjustment did not occur on days when the presence of thin cirrus cloud had been noted.

On one occasion it was noticed that the dragon-flies kept changing the position of their abdomens from up to down every few minutes. Suspecting the cause of this I obtained several pieces of coloured glass through which the sun could be observed without inconvenience. It was found that small thin clouds were passing over it which were too small to cast a shadow or even to cause any appreciable decrease in the intensity of the sunshine. Whenever one of these clouds was over the sun the dragon-flies held their abdomens up, downwards whenever the sun was clear.

It was further found that if small cumulus clouds are rapidly crossing the sky, and if there is not much wind at ground level, dragon-flies glide with abdomens up when there is shadow and with abdomens down when there is sunshine. After the sun comes out there is a short interval before the abdomens are lowered. If the edges of the cloud shadow are sharp cut this interval may be measured and this was done on three occasions the times being 22, 23 and 27 seconds.

The depressed position of the abdomen in continued flight is seen only in the hotter months of the year and then only when the sun is shining in full strength.

V. *Significance of the use of a brake in continued flight.*

The interest of the above observations lies in their bearing on the question whether the soaring flight of dragon-flies is due to undiscovered wing movements. If this were the case, it may be asked, why could not their flight be checked by cessation of such movements and why should the use of a brake in continued flight be necessary?

These observations also furnish a proof of the dependence of soaring flight in light winds on sunshine. It is remarkable that the presence of an amount of thin cloud far too small to have any known effect on the flight of birds should be able to influence the flight of dragon-flies.

VI. *High speed soaring flight of dragon-flies.*

The first observation of high speed flight of dragon-flies was made by Leeuwenhoek about 200 years ago. In discussing the compound eyes of insects and as a proof of the quickness of sight compatible with eyes of this description, he relates how he watched a swallow chasing a dragon-fly over the surface of a large pond and how the swallow was baffled by the speed and the unexpected turns of the insect which kept it always several feet in front of the enemy*.

In my experience such high speed flight over water is due to a combination of soaring with occasional flapping. This form of rapid flight in which a dragon-fly can easily outdistance a swallow, so far as I have seen, occurs only in the presence of sunshine and wind.

On rare occasions it has been my fortune to observe dragon-flies in a strong wind and to be able to form a definite opinion as to the absence or presence of flapping in their flight. The following instance appears to be of sufficient interest to be worth transcribing from my diary in detail:

> August 10th, 1915, at 10.05 a.m. on the Bharatpur Road, 22 miles from Agra.
>
> A strong wind that tended to blow off my hat. A cluster of dragon-flies (*Pantala*) were gliding to leeward of a small tree but generally remaining a little to one side of it so that they met the full force of the wind. They were in continued gliding flight. Long grass about 2 metres high was below

* Miall, *The Early Naturalists, Their Lives and Works* (Macmillan and Co., 1912, p. 206).

them. They kept at a distance of 1 to 3 metres above it. The grass was waving and showed, where the dragon-flies were gliding, no sign of its being sheltered by the tree. Sometimes, in a stronger gust, a few of the dragon-flies came up to leeward of one of the branches. Rarely a few went further to leeward than usual. The group generally extended from near the tree to about 35 metres to leeward of it. The dragon-flies showed lateral swaying, and one of them showed lateral instability for an instant just after a gust.

10.20. When quite near the tree the dragon-flies occasionally flapped. Away from the tree they appeared to glide only. Glides of at least 10 metres up wind without a flap were observed.

In view of the probable speed of the wind on this occasion, it is likely that the speed of these dragon-flies through the air was above rather than below 15 metres per second.

When in low speed flight dragon-flies appear to travel a little faster than a locust. The speed of locusts, whose flight depends on flapping, has been measured by me on different occasions and found to be 4 metres per second.

VII. *The relation of ascending currents to the soaring flight of dragon-flies.*

The facts of the case are simple. Whenever the air is soarable or, in other words, whenever there is sunshine, dragon-flies avoid such currents.

For instance, on one occasion, observations were being made at a time when the sky was partially clouded with short intervals of sunshine. Dragon-flies of two species were gliding, so long as there was cloud shadow, in the current reflected up from the windward side of a stable. Whenever there was slight sunshine the dragon-flies elevated their abdomens and glided away, generally only for a short distance from the ascending current. But twice during the period of observation the sun came out so strongly that the dragon-flies put on a brake, that is to say they were gliding with abdomens depressed. On each of these occasions every dragon-fly went a long distance, 50 yards or more, from the ascending current.

Dragon-flies appear to remember the position of an ascending current. At Futteypur-Sikri they have been seen collecting on the windward side of the hill towards sunset, coming from distances of half a mile or more to get there.

Thus observations on dragon-flies yield valuable evidence that soaring flight is not due to the use of ascending currents. As happens with soaring birds, such currents are avoided so long as the air is capable of supporting soaring flight.

VIII. *Comparison of low and high speed flight.*

As above stated, low speed flight is dependent on the presence of sunshine. In the early morning, before the sun has attained its full strength, it is clearly favoured by the absence of wind. On occasions when the air was uncomfortably hot and very dry and when, therefore, the slightest movement of the air could be recognised by its cooling effect on the skin and when the dragon-flies were gliding often within two or three feet of my head, it has been noticed that the coming of the slightest draught of wind caused gliding to be replaced by flapping.

Sometimes during the daytime, if the air is nearly calm, wind seems to be unwelcome to the dragon-flies. On such occasions they retire to the shelter of the leeward side of a tree on the coming of a puff so light that it causes only a gentle movement of the leaves.

On the other hand, if conditions are suitable for high speed flight the presence of wind appears to be helpful rather than harmful. The speed of flight appears to increase with the strength of the wind. In such "soarable" winds, dragon-flies travelling up wind may be seen to glide ahead with a distinct increase of speed whenever they are struck by a gust. A similar increase of speed on entering a gust occurs in the case of cheels and other soaring birds.

In low speed flight the wings of dragon-flies are in the "up" position, the wing tips being on a higher level than the body. The tips of the front pair of wings are more elevated than those of the hind wings. Hence the gliding dragon-fly has a distant resemblance to a staggered biplane. The abdomen is elevated above the level of the thorax (Fig. 2).

In high speed flight the abdomen appears to be generally if not always horizontal and the wings appear to be less elevated than when flight is at low speed.

IX. *Conclusions.*

Various species of dragon-flies have been found to possess the power of soaring flight.

Facts have been adduced that are incompatible with the idea that this soaring flight is due either to undiscovered wing movements or to the use of ascending currents.

Soaring flight is now known to occur in three different classes of animals, namely dragon-flies, flying-fishes and birds. It is remarkable that, despite their widely different structure, size, and weight, and the very different conditions under which they soar, there should be such similarity of the flight in these three classes of animals. In each class evidence is available that low speed flight depends on the presence of sunshine and high speed flight on the presence of wind. The speeds attained in the three classes are comparable if not identical.

The Gluteal Region of Tarsius spectrum. By A. B. Appleton.

[Plate IV.]

[*Read* 16 May 1921.]

Attention will be here called to certain anatomical features which have come to light in the course of a special examination of the thigh musculature of *Tarsius*, an animal interesting not only for its peculiar saltatory method of locomotion in the trees, but also for the varied Primate features exhibited in its anatomy. It combines features of the Insectivora, of Lemuroidea and of Anthropoidea, and it is in comparison with these animals that its musculature must be chiefly considered.

The dissections on which the following account is based were made upon two Tarsii kindly provided by Dr W. L. H. Duckworth*.

Burmeister's classical description of 1846 provides a careful and accurate description, but he had not the advantage of that masterly study by Leche† in 1883 of the pelvic region of Insectivora for help in a determination of the identity of the various muscles found in *Tarsius*.

Attention in this paper will be centred on two muscles which are absent in some Primates and in many other Mammalia; but which are not uncommonly found in the more primitive of living Mammalia. These muscles are the femorococcygeus and the caudofemoralis—employing Leche's nomenclature. They apparently have their counterparts among Monotremes (Leche) and the latter among Reptilia‡.

The superficial gluteal musculature of *Tarsius* consists of a thin sheet attached to the 3rd trochanter of the femur, and a thicker band passing from the caudal vertebrae to the back and outer margin of the femur.

The latter of these we here describe as the femorococcygeus muscle.

The thin muscle sheet attached to the 3rd trochanter, which, with Burmeister, we regard as the conjoined tensor fasciae latae and superficial gluteus, arises from the crest of the ilium, and, through the lumbodorsal aponeurosis, from sacral spines, extending as far back as the 1st caudal vertebra.

* References are made to dissections of specimens of *Tupaia, Lemur, Simia* and *Troglodytes*, also so provided.

† Leche, *K. Svensk. Vet. Akad. Handl.* 1883.

‡ Caudifemoralis of Gadow, *Morph. Jahrb.* 1882.

A thicker band of musculature forms its cephalad margin and this has a special attachment to the 3rd trochanter by a small tendon. Supplied by the superior (cephalad) gluteal nerve, it is thus distinguishable from the superficial gluteus muscle which forms the hinder part of the muscle sheet and is supplied by the inferior (caudad) gluteal nerve. The latter nerve reaches the hinder edge of the muscle from behind the gluteus medius and enters its deep surface.

The cephalad gluteal nerve enters the deep surface of the tensor fasciae latae after passing through the gluteus minimus.

A complete deficiency of muscular tissue occurs behind the tensor fasciae latae, a thick fascia only being present; and through this the fibres of the superficial gluteus gain attachment to the 3rd trochanter. In front of the tensor fasciae latae the fascia lata of the thigh is thinner than is the case in those numerous Mammalia where it receives the pull of the "tensor" muscle. This muscle in fact acts in *Tarsius* as a flexor and rotator of the thigh on the trunk, and not as an accessory extensor of the knee (which occurs in the cursorial groups, *e.g.* Artiodactyla and Carnivora).

Our description differs from that of Burmeister who states that the tensor fasciae latae of *Tarsius* loses itself below in the fascia lata. He mentions no tendinous attachment to the 3rd trochanter. The gluteus medius and gluteus minimus are described by Burmeister. They are little larger than in *Tupaia* (Table I)*.

The **femorococcygeus** muscle claims our next attention.

Burmeister evidently saw this muscle, but, erroneously, as we think, described it as a "pyriformis" muscle; moreover, he apparently regarded the caudofemoralis as a second or deep pyriformis. These two muscles in *Tarsius* so closely resemble the muscles described by Leche in *Tupaia*, and are still more like the muscles of *Lemur*, that we cannot regard them as different structures. Nerve supply is also similar in all these cases. A representative of the femorococcygeus is even found in the anthropoid apes, supplied by a branch from the nerve to biceps (Orang): and these animals possess a separate pyriformis homologous with the human one.

The femorococcygeus of *Tarsius* arises from the deep aspect of the transverse processes of the 1st, 2nd, and 3rd caudal vertebrae and from the intervening intertransverse ligaments. Its origin is therefore quite distinct from that of the superficial gluteus. Below the tail and immediately medial to its place of origin is situated the sacrococcygeus; and it passes out from under cover of the intertransversarius caudae (of Burmeister) and dorsal caudal

* In *Macropus* the gluteus medius is, compared with the rectus femoris, twice as large as in *Tarsius* (data by Haughton, *Pr. R. Ir. Acad.* IX). In cursorials it is also much increased in size.

musculature. It is inserted by muscular fibres along the outer side of the back of the femur for four-sevenths of its total length, covering a broader area above, at the back of the great and 3rd trochanters, than along the shaft. There is a delamination into strips of muscle attached behind the 3rd trochanter, close to the centre of rotation of the hip-joint; the hinder and larger part of the muscle, attached to the femoral shaft, is quite distinct from the remainder.

The nerve-supply of the femorococcygeus is provided by a special branch passing from the great sciatic nerve* caudad to gluteus medius; the nerve enters it behind the great trochanter. The femorococcygeus is superficial except at its insertion, where the bulky vastus externus overlaps it. It covers the great sciatic nerve; also the obturator internus and gemelli, quadratus femoris, adductor magnus and caudofemoralis from above downwards. The muscle is relatively smaller than in *Lemur*, and the insertion is less extensive. It closely resembles the muscle of Simiidae, except in being readily separable from the superficial gluteus. A further difference from Simiidae lies in its (primitive) origin from coccygeal vertebrae. In Simiidae its origin is from the tuber ischii.

The **caudofemoralis** muscle of *Tarsius* arises from the tuber ischii cephalad to the biceps tendon. A flat muscular band, it passes down to the middle of the thigh where it gains insertion to the back of the femur, a rough line marking the place, for a distance equal to one-fifth of the femoral length, equidistant from either end. It is immediately medial to the insertion of the femorococcygeus.

It receives its nerve supply from the nerve to the hamstring muscles (from the great sciatic).

It is situated in its whole length on the deep aspect of the femorococcygeus, except when the great sciatic nerve intervenes between them [*v.* Plate IV, fig. 2].

In *Tupaia* this muscle is a far thinner sheet, with two places of insertion†: (1) to the back of the shaft (one-fourth of the length of the femur); (2) just proximally to the internal condyle. In *Lemur* it is thin and the length of its insertion one-tenth of the length of the femur. In Simiidae it is indistinguishable.

The nerve to the hamstring muscles (*i.e.* to biceps, semitendinosus and semimembranosus) passes deep to the caudofemoralis muscle.

These nerve-relationships are characteristic of the muscle‡ also in *Tupaia* among Insectivora, and in *Lemur* among Primates, in specimens I have dissected. They, together with the attachments of the caudofemoralis to bone, are sufficient to establish the identity of

* As also in *Tupaia* and *Lemur*.

† Cf. the corresponding caudifemoralis of *Reptilia*—Gadow, *loc. cit.* 1882.

‡ Cunningham's figure of the thigh of *Phalangista* shows (as the "ischiofemoral muscle") agreement in that animal also. '*Challenger*' *Report*, 1882, vol. v, Plate III.

the muscle. It appears from the description of Burmeister that his "deep" or "second pyriformis," apparently (*partim*) our caudofemoralis, is represented to be superficial to the great sciatic nerve, not deep to it as we have stated. That error is here a possibility is suggested by the following consideration.

The femorococcygeus and the caudofemoralis muscles are very closely apposed close to their insertions. It is here that the great sciatic nerve passes through, sandwiched between them*; and by its sheath it is so intimately bound up with the muscles that every movement of the muscles must carry the nerve with them. This may possibly be an arrangement for steadying the nerve during rapid and extensive hip extension. In *Lemur*, and in *Tupaia*, such a close association of the caudofemoralis with the femorococcygeus around the great sciatic nerve does not occur.

The tenuissimus† is absent in *Tarsius*, as also in *Lemur*. Klaatsch has shown it to be absent also in other species of Prosimiae. I find, with Leche, that it is present in *Tupaia*.

The quadratus femoris is triangular in shape, as is characteristic of Insectivora and Lemuroidea, and is inserted to the broad space behind and between the lesser (2nd) and 3rd trochanters. It does not occupy the extensive area of insertion found in *Lemur*, and is thus more like *Tupaia* and Simiidae. The bulk of the various muscles, in comparison with *Tupaia* and *Lemur*, forms an instructive study.

As a standard of comparison the weight of the rectus femoris is employed.

All specimens were long preserved in spirit.

Table I.

	Tarsius		*Tupaia*	*Lemur*
Muscle	Actual weight in grains	Weight Rectus femoris = 1	Weights Rectus femoris = 1	
Gluteus supfl. and tensor fasciae latae	3·7	1·3	1·2	0·98 ‡
Femorococcygeus	4·0	1·4	1·2	2·18
Caudofemoralis	3·0	1·0	0·20	0·28
Gluteus medius	3·5	1·2	1·00	1·94
Gluteus minimus	1·8	0·6	0·25	—
Biceps	2·6	0·9	1·05	1·10
Semitendinosus	2·9	1·0	0·58	—
Tenuissimus	—	—	?	—
Semimembranosus, Presemimembranosus	9·8	3·4	2·30	—
Quadratus femoris	0·7	0·2	—	0·71

* See Plate IV, fig. 1.

† The iliofibularis (possibly) of Reptilia; Gadow, *loc. cit.* 1882.

‡ An arbitrary line of division had to be made in this specimen.

The similarity of *Lemur* and *Tupaia* (except as to the gluteus medius) is noteworthy. In *Tarsius* the caudofemoralis is large, while the gluteus medius and quadratus femoralis are little developed.

Table II.

Actual weights of thigh muscles (supplied from lumbar plexus) in a spirit specimen of *Tarsius*—and of certain shank muscles.

	Gracilis	Adductores and pectineus	Obturatores externus and intermedius	Sartorius	Iliacus	Psoas minor	Psoas major	**Rectus femoris**	Vastus externus	Vastus internus	Crurei	Gastrocnemius	Plantaris	Soleus	Popliteus	**Femur** (from spirit)	Femur (dried)
Actual weights in grains	1·6	6·7	1·5	3·7	3·5	1·4	1·8	2·9	51·2	16·5	9·0	5·4	1·8	1·7	1·1	18·0	12·3

It will be noticed that the hip extensors of *Tarsius* are not developed to the huge extent of the quadriceps extensor. Their weight compared with that of the quadriceps is in the ratio $1/5\frac{1}{2}$, viz.:

$$\left.\begin{array}{l}\text{Gluteus medius}\\ \text{Femorococcygeus}\\ \text{Caudofemoralis}\\ \text{Adductor magnus}\end{array}\right\}\ 14{\cdot}5 \text{ grains.}\quad \text{Quadriceps extensor } 79{\cdot}6 \text{ grains.}$$

Now this ratio shows curiously little variation among Mammalia. It is convenient for the present purpose to employ the same set of four hip-extensors [it would, therefore, prove of very doubtful utility in comparisons with Sauropsida or Monotremes (see also footnote ‡ p. 471)].

In the less specialized forms the ratio as thus defined of $\frac{\text{hip extensors}}{\text{knee extensors}}$ is rather less than unity. It tends to rise somewhat in certain cursorial groups, viz. Artiodactyls and Perissodactyls.

A large leaping animal like the Kangaroo shows no marked disturbance of this ratio. And while the ratio in *Lemur* resembles that of most Mammalia, it is found in *Tarsius* to be extraordinarily small.

Table III.

	Tarsius spectrum	*Lemur*	Average of four Suffolk Sheep, Nos. I, II, IV, V	New born Horse	Rhinoceros*		Kangaroo*	Opossum*	Phalanger*	Badger*	Dog* (dingo)
Hip extensors / Rectus femoris	5·0	5·67	2·74	2·70	4·6		3·17 †	2·4	—	1·8	3·0
Quadriceps ext. / Rect. fem.	27·5	7·64	2·91	2·82	3·2		4·0	3·1	—	3·0	4·2
Hip extensors / Quadriceps ext.	0·18	0·74	0·90 ‡	0·97 ‡	1·4		0·79	0·77	0·83	0·62	0·73

How then shall we account for the great difference in *Tarsius*? The length of the femur relative to other parts of the body appears to be a matter of great importance. Cursorial animals and jumpers require the action of powerful knee extensors at the moment of springing from a foot, whether in galloping or leaping; for the line of thrust from the foot is placed approximately at a right angle to the long axis of the femur. The quadriceps itself is situated disadvantageously for powerful action when the knee is fully flexed. And the longer the femur, the greater the moment of the animal's weight about the knee-joint. It is as a mitigation of the strain so placed upon the knee extensors that I regard the shortening§ of the femur in cursorial animals; and these muscles are not of disproportionate development in them. In the Macropodidae also, shortening of the femur occurs with a saving of knee-extensor musculature. In the *small* jumpers, however, such as *Pedetes*, *Macroscelides*, *Dipus* and *Tarsius*, this compensatory shortening is not a feature, and a powerful quadriceps is essential. This muscle-group is here shown to be of extraordinary size in *Tarsius*: it would prove of interest to ascertain the muscle-weights in the other saltatory mammals: I am aware of no records.

It appears that in the production of such a form as *Tarsius*, the rectus femoris took no part in the great development of the quadriceps extensor; the remaining muscles (forming the "triceps extensor") alone have been concerned,—most of all the vastus externus.

* Haughton, assuming his identification of muscles to be the same.

† Haughton, *P. R. Ir. Acad.* IX, fails to distinguish a caudofemoralis. It is, however, in *Marsupialia* a distinct muscle from the quadratus femoris as Carlsson (*K. Sv. Vet. Akad. Handl.* 1915) (*Hypsiprymnodon*), Parsons (*Petrogale*), McCormick (*Dasyurus*) have shown. Cunningham, *loc. cit.* 1882, saw both muscles in *Thylacinus* and *Phalangista*.

‡ Inclusion of tensor fasciae latae as a knee extensor would lower this ratio slightly, while admission of semimembranosus as a hip extensor raises this ratio considerably.

§ Appleton, *Proc. Camb. Phil. Soc.* 1921, p. 386.

The musculature of the gluteal region is not only little developed in point of size: it is, moreover, comparatively little specialized as regards muscle attachments.

The changes in *Tarsius* from a *Tupaia*-like form have comprised:

*(1) Attachment of the caudofemoralis (of Leche) to the tuber ischii instead of to the caudal vertebrae†.

*(2) Loss of the condylar attachment of the caudofemoralis (of Leche).

*(3) Increase in size of the remaining part of the caudofemoralis, and concentration of its insertion to the middle of the femur.

(4) Disappearance of the muscular part of the upper part of biceps.

(5) Loss of the tenuissimus.

(6) Close envelopment of the great sciatic nerve by the muscle fibres of femorococcygeus and caudofemoralis.

(7) Concentration of the pull of the tensor fasciae latae on to the 3rd trochanter.

Primitive features, retained in *Tarsius*, but lost in anthropoid Apes comprise the following:

(1) Persistent caudofemoralis muscle (of Leche).

(2) Independent femorococcygeus (more or less fused with gluteus maximus in Apes: Gorilla, Chimpanzee, Orang).

(3) Caudal attachment of femorococcygeus. (Origin is ischial in Apes *mihi* = Duvernoy's ischiofemoralis, and Fick's tuberofemoral muscle).

‡(4) Persistent 3rd trochanter, associated with the attitude of a flexed hip, and the passage of a superficial gluteus muscle-sheet and tensor fasciae latae across a large vastus externus muscle.

In the last four points, *Tarsius* is in agreement with *Lemur*. The most striking difference between these two in the gluteal region is found in the extensive attachment to the femur of quadratus femoris in *Lemur*; the smaller size of the caudofemoralis in *Lemur*, and its "more primitive" origin from coccygeal vertebrae.

The *caudofemoralis* (Leche) has been confused with the *femorococcygeus* (Leche). It appears to be merely a verbal confusion, owing partly to the description by Windle and Parsons of the *femorococcygeus* of Carnivora and certain Rodentia under the name of the *caudofemoralis* §. In *Petrogale* a similar muscle is included by Parsons under the term *biceps* ||. He describes what appears to be the *caudofemoralis* of Leche in *Petrogale* as the

* Certain Marsupials resemble *Tarsius* and differ from *Tupaia* in these features; a parallel development.

† *Hypsiprymnodon* is the only Marsupial found by Carlsson, *loc. cit.* 1915, to retain a coccygeal origin for the caudofemoralis.

‡ Appleton, *loc. cit.* p. 380.

§ Parsons, F. G., *Proc. Zool. Soc.* 1898.

|| Parsons, F. G., *op cit.* 1896.

*ischiofemoralis**; Carlsson† identifies the corresponding muscle of *Hypsiprymnodon* as the *caudofemoralis*.

Gregory‡ has fallen into the error of comparing the *caudifemoralis* of Reighard and Jennings (the Cat), in Parsons' meaning, with the *caudofemoralis* of *Hypsiprymnodon* in Leche's sense of the term.

In the present state of the literature, it is indeed not possible to compare muscles of similar name described by different authors without further enquiry into the attachments and nerve-relationships of these muscles. The position of the great sciatic nerve is a ready method of distinguishing the *femorococcygeus* (caudofemoralis of Parsons) from the *caudofemoralis* (ischiofemoralis of Parsons).

The nomenclature of Leche has some claim from usage, and is here employed. Ischiofemoralis is an excellent *descriptive* name in certain more specialised mammals: but in the more primitive animals (some Reptilia and Insectivora)§ a *caudal*|| attachment rather than *ischial* is found. If it is agreed to use different names for what is practically the same muscle, according to interspecial variations in its attachments, then there must be no verbal homologisation of muscles. The caudofemoralis (or caudifemoralis) muscles of certain authors are no more comparable with the caudofemoralis of Leche and Carlsson than are the various muscles which go by the name of pectineus in various mammalia homologous with one another.

Some correlated changes in bones, associated with muscular peculiarities of *Tarsius*, may be mentioned:

(1) Large transverse processes to the first three caudal vertebrae¶, associated with well developed femorococcygeus.

(2) 3rd trochanter and descending extensor ridge on femur (*v. supra*).

(3) Marking on the posterior aspect of the femoral shaft for attachment of caudofemoralis (paralled in Marsupialia).

(4) Femoral shaft with high pilastric index—associated with the powerful quadriceps extensor.

The peculiar features of *Tarsius* deserve comparison with those of other jumping animals. Some have been already considered.

(1) *Length of femur*. In large jumpers (Macropodidae) shortening occurs for a similar reason in cursorial types**; viz. diminution of the necessary mass of knee-extensors. In small jumpers

* Cunningham, *loc. cit.* 1882, also uses this term.

† Carlsson, A., *Kungl. Svensk. Vetensk. Handl.* 1915.

‡ Gregory, *Bull. Amer. Mus. Nat. Hist.* 1918.

§ Also *Hypsiprymnodon* (Carlsson) and Monotremata (Westling, *Bihang t. K. Sv. Vet.-Akad. Handl.* 1889).

|| For *M. caudofemoralis* (Leche).

¶ Cf. Duckworth, *Morph. and Anthrop.* vol. I, 1915, p. 106.

** Appleton, *loc. cit.* 1921

this does not appear to be necessary. For this musculature in jumping animals other than *Tarsius*, and the Kangaroo, verification is needed.

(2) Attachment of muscles to tuber ischii instead of to tail.

In Reptilia and Insectivora, both femorococcygeus and caudofemoralis (or their precursors) are attached to caudal vertebrae. But in two Insectivora *Macroscelides* and *Erinaceus*), one of them a jumper, the femorococcygeus is recorded as attached to the tuber ischii. A new type of hip-extensor is produced, acting as far from the hip-joint as possible, and therefore exerting the greater leverage.

The same principle is seen in cursorial types where the long vastus (= gluteus maximus + ? femorococcygeus) gains attachment to the tuber ischii, while at the same time the tuber ischii shifts away from the hip-joint downwards and backwards (Artiodactyla).

In *Tarsius* it is the caudofemoralis alone which gains an ischial attachment; but in giant apes we see the femorococcygeus attached here. An ischial attachment for hip-extensors appears to be no prerogative of jumpers, but is shared alike by the active Metatheria and cursorial Eutheria, and also by the giant apes.

To sum up, *Tarsius* exhibits, in the musculature of the gluteal region, primitive features like the genus *Lemur*, recalling *Tupaia* and *Hypsiprymnodon*, and less closely reproducing the conditions of Monotremes and Reptiles. Along with these features we find certain modifications, such as the ischial attachment and development of the caudofemoralis, in which *Tarsius* has progressed beyond the conditions in *Lemur* and is paralleled by many Marsupialia. Its small size is held accountable for a certain lack of parallelism between *Tarsius* and the Kangaroo.

While certain of the primitive features of *Tarsius* and *Lemur* are lost in Simiidae, others, such as a well developed femorococcygeus, are retained in the latter.

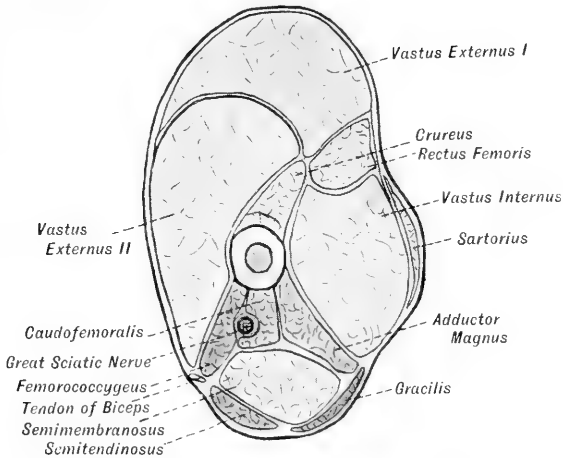


Fig. 1. *Tarsius spectrum.* Cross section of thigh. × 3. Junction of upper and middle thirds.

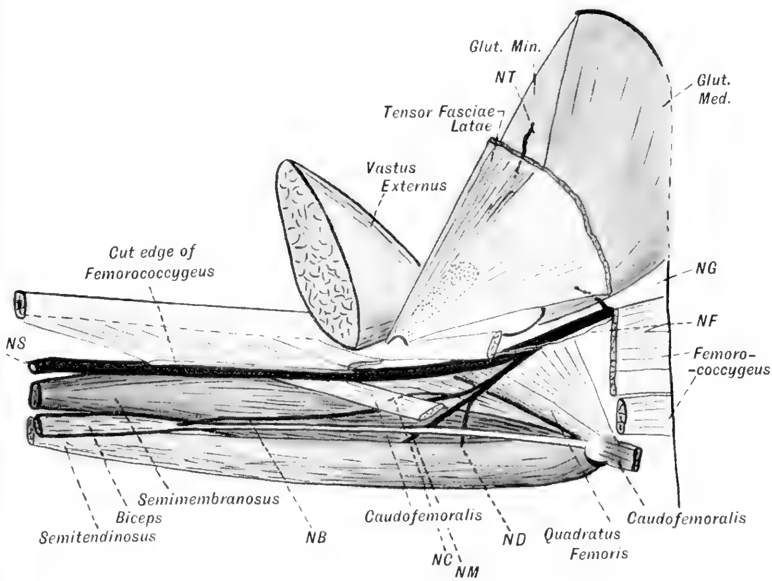


Fig. 2. *Tarsius spectrum.* Outer view left thigh. Parts of Femorococcygeus, Caudofemoralis, Vastus Externus, Tensor Fasciae Latae and Superficial Gluteus removed.

NB = Nerve to Biceps; *NC* = Nerve to Caudofemoralis; *ND* = Nerve to Semitendinosus; *NF* = Nerve to Femorococcygeus; *NG* = Nerve to Gluteus Superficialis (*seu* Maximus); *NM* = Nerve to Semimembranosus; *NS* = Great Sciatic Nerve; *NT* = Nerve to Tensor Fasciae Latae.

An unusual type of male secondary characters in the Diptera. By C. G. Lamb, M.A.

[*Read* 16 May 1921.]

Secondary sexual characters may be classed as persistent or sporadic: the former occur with much regularity throughout a genus or family, the latter crop up suddenly in isolated species of a genus, and are often similar in form in quite unrelated families. Examples of the persistent type are the antennal dimorphism of so many nematocerous and other families, and the male eye holopticism in many families. The sporadic types are such as: (1) frontal processes as in *Ceratitis* sp., (2) very varied types of leg adornment, which is a common form, probably connected with courting habits, (3) replacement of head bristles by hair tufts as on some Chrysosomatinae, (4) costal lumps as in the two quite unrelated species, *Pemphigonotus mirabilis* (Chloropids) and *Ommatius Lema* (Asilids), (5) differences in the palpal structure, as in many Dolichopids and others. The list could be extended, but one thing is apparent, all the characters of either type are peripheral in their position on the insect.

The author has recently completed the study of certain families of flies contained in the collections made by Dr H. Scott in the Seychelles, the descriptions of which will appear in a forthcoming volume of the Percy Sladen Expeditions Reports, *Trans. Linn. Soc.* London, Ser. 2, Zool., vol. 18. Amongst these are some exceedingly interesting forms belonging to the Dolichopodidae. These belong to a new genus, *Craterophorus*, which is probably to be referred to the subfamily Chrysosomatinae; these differ from other known species of fly in having centrally placed secondary characters; further each of the three species constituting the genus bears the same set of characters, one of which is nearly unique, the others being absolutely so. The genus is apparently endemic, being found in the native forests of the Island of Mahé. There are no tarsal or antennal differences between the sexes. The special characters under consideration are four in number: (1) the wing of the male (Fig. 1) is quite different in shape from that of the female (Fig. 2) having a sharply angled posterior margin; this peculiar form of wing is almost if not quite unique. The other three characters are more strictly central; they are shown in the photograph (Fig. 3) and diagram (Fig. 4), these characters are as follows: (2) the Dolichopodidae are usually devoid of alulae, but in these species one is developed though it is of a totally exceptional form.

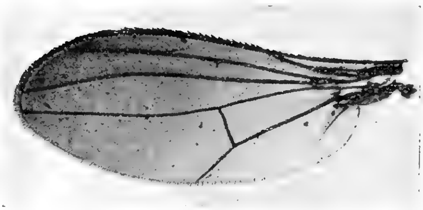


Fig. 1. *Craterophorus mirus* ♀ wing.

Fig. 2. *Craterophorus mirus* ♂ wing

Fig. 3.

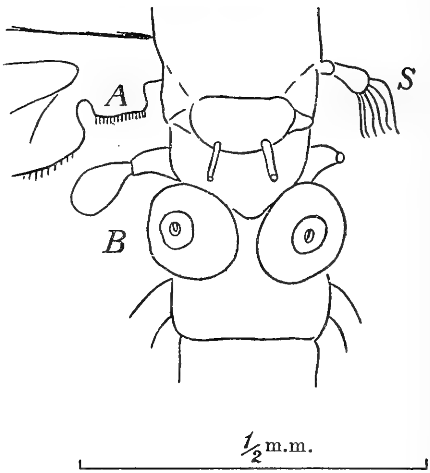


Fig. 4 *Craterophorus mirus* ♂.

It consists of a straight edged lobe (*A*, Fig. 4) which is highly chitinized and has the margin beset with a very close even row of tiny stiff bristles, co-planar with the lobe, the bristles having the utmost regularity of size and shape: they form a perfect comb with the teeth nearly touching. (3) In most dolichopids the squama consists of a more or less pronounced knob bearing a characteristic row of bristles, which are frequently spread out into an elegant fan. In the present species the squama (*S*, Fig. 4) is long stalked, apparently mobile, and instead of the usual fan of long hairs, it carries a set of parallel stout bristles hooked at the tip. The relative position of the last two structures almost irresistibly calls to mind the relation of a hand to a harp. In the figure only one structure is shown on each side to avoid confusion. (4) The fourth structure (*B*, Fig. 4) is even more remarkable: it consists of a pair of large spheroidal bodies, one on each side of the basal segment of the abdomen and confluent below with the hind epimeron. Each is hollow and has a round orifice on the dorsal surface; from the bottom springs a rod which nearly reaches the surface of the sphere. The significance of the whole complex is entirely problematic, though it is certain that the several parts are in some way duly correlated. The general appearance, as said above, is that of some musical instrument, although the function of the bulbs is quite problematic, to suggest 'resonators' is too fanciful, nor can one see the relation of the whole to the remarkable wing form.

It is much to be regretted that the paucity of specimens and the fact that none but pinned individuals were available rendered it impossible to submit the bulbs to a competent histologist, as a suitable examination might have helped to throw some light on this assemblage of mysteries.

A Note on the Mouth-parts of certain Decapod Crustaceans. By
L. A. Borradaile, M.A., Fellow and Tutor of Selwyn College, Cambridge, and Lecturer in Zoology in the University.

[*Read* 16 May 1921.]

In that close study of the relation of aquatic organisms to their surroundings which is now becoming an important part of Biology, much attention is given to their modes of feeding; and one of the most interesting branches of the subject concerns those organisms which obtain their food from organic particles, alive or dead, in suspension in the water. Orton has investigated this habit in various animals (Molluscs, Tunicates, etc.), and Potts has shown that it is practised by the Coral-gall Crab *Hapalocarcinus*, and that the mouth-parts of this animal are correspondingly modified, the endopodite of the third maxilliped and the exopodites of the second and first being provided with fringes of bristles which are used for gathering the food, while the inner jaws are reduced in the absence of the need for powerful organs to masticate it. These adaptations are analogous to those that appear in Cirripedia and Branchiopoda, which are also feeders upon suspended matter.

Sundry other crustaceans which live in the mantle-chamber or pharynx of sessile or subsessile organisms must be presumed to get food of the same kind at second hand. Some time ago, in the course of a study of the prawns of the subfamily Pontoniinae, of which most members are commensal and a number live in the mantle cavity or pharynx of bivalve molluscs and ascidians, I endeavoured to discover some correspondence between the habitat of the animals and the structure of their mouth-parts; but was disappointed to find practically no trace of this. The jaws of the crab *Pinnotheres*, which lives between the valves of the shell of Lamellibranchs, show the same absence of specialisation in the direction in which *Hapalocarcinus* is specialised. They have some interesting peculiarities, but not those that might be expected from an animal which was nourished upon finely divided food. Probably these cases are to be explained by the fact that the crustaceans feed, not upon food in suspension, but upon the strings of mucus in which it is entangled by their host. Orton has recently shown that this is done by *Pinnotheres*.

A like explanation, however, cannot be given for *Porcellana platycheles*. This animal is not a commensal, and gathers for itself suspended food, taking it by means of long fringes upon the third maxillipeds, but is provided with well-developed inner mouth-parts. Possibly it feeds, not, as *Hapalocarcinus* does, only upon the very minute organisms which make up the nannoplankton, but also upon suspended particles of greater size and toughness. Or it may be that it uses the chelae for seizing food, though I have not seen it do this. Another case of the same kind would seem to be presented by the prawn *Paratypton*, which was found by Potts to live in a coral gall somewhat in the same way as *Hapalocarcinus*. Details of its habits are not known, but it does not at present appear likely that it can obtain any but finely divided food. Yet its jaws show neither any provision for gathering such food nor any marked reduction in masticatory structures.

All these organisms deserve a good deal more investigation than they have received.

An Apparatus for Projecting Spectra. By H. Hartridge.

[*Read* 2 May 1921.]

If a Thorpe replica celluloid diffraction grating of about 14,000 lines to the inch be mounted in optical contact with the hypotenuse of a right angled glass prism, and if now a parallel beam of white light be caused to enter normally at one of the other faces, then it is found, while the direct beam is totally internally reflected and occupies the same position as it would if the prism had no grating mounted on it, that the 1st and 2nd order spectra of one side are found approximately in their normal relationship in regard to the direct beam, but that the spectra of the other side pass out through the hypotenuse of the prism without suffering reflection. If the prism be made of crown glass the yellow region of the 1st order spectrum is in a straight line with the incident beam, and the dispersion of this spectrum is about 54 % greater than that of the diffraction grating alone. It is also found that (1) the resolving power for a given aperture is increased about 41 %, (2) the purity factor for a given collimator slit width is also increased by about half, (3) the intensity of this spectrum for a given purity and dispersion is also considerably greater than that of the grating alone. It is to this latter feature that the arrangement owes its value. To make use of this spectrum for projection purposes the green, blue and violet regions of the 2nd order spectrum are screened off by a deep orange colour filter.

The transmission of the spectra through the hypotenuse of the glass prism, in apparent contradiction to the laws of geometrical optics, is almost certainly due to the inclination of the sides of the separate strip elements of the replica, which when the cast was made filled the grooves of the original grating. These sides are then roughly normal to the incident light rays and a number of elementary wavelets would therefore, on Huyghen's hypothesis of diffraction, pass out of the surface, but since they would retain the phase relationships of the original beam they would be in a condition to interfere with one another, and thus form the spectra that are observed.

This prism grating would therefore seem to be in closer theoretical relationship with the echelon of Michelson, or the echelette of R. W. Wood, than with the ordinary diffraction grating.

Note on true and apparent hermaphroditism in sea-urchins. By J. Gray, M.A., Balfour Student, Cambridge University.

[*Read* 16 May 1921.]

During the winter of 1913 I had occasion to examine at Naples the gonads of a large number of specimens of the sea-urchin *Arbacia pustulosa*. It is exceedingly easy to distinguish the two sexes of this animal; the eggs and ovarian tissue of the female contain a dark red pigment, whereas the testes of the male are devoid of any pigment. The gonad of the female is brick-red in colour from the very early stages of its development. That of the male is yellowish white.

On opening one individual, a unique condition of the gonads was observed: four of the gonads were apparently female—being of the usual colour, and of considerable size. Part of the fifth gonad was also in the same condition, but the majority of the gland was typically *male* in appearance. On examining this portion of the abnormal gland, spermatozoa were found, which, though normal in appearance, were either motionless or only capable of feeble movement.

On sectioning the whole of the gonads, I was most surprised to find no trace of ova or ovarian tissue. On the other hand, those parts of the gonads which were female in appearance were found to be full of a mass of degenerating spermatogonial cells, which failed to take up any of the usual stains. The walls of the gonad were perfectly normal.

This animal is of interest as it would appear that a derangement of the sex-cells has been attended by an inversion of the secondary sexual characters. Although no parasite was observed, the condition of the gonad recalls the well-known effects of parasitic castration in the Crustacea. If this view is correct, the male of *Arbacia* appears to be heterozygous for sex.

Owing to the courtesy of Mr H. M. Fox, I have been able to examine the gonads of a true case of hermaphroditism in the urchin *Strongylocentrotus lividus*. In this case three of the gonads were completely female, while the other two contained both ripe eggs and spermatozoa, which were fertile *inter se*. In one of the abnormal gonads most of the tubules were female. It appears that this animal was essentially a female—possibly an "intersex." No abnormality was found in either the male or female portions of the gonads.

As far as I am aware the only other description of an hermaphrodite sea-urchin is that given by Gadd* of an individual of *Strongylocentrotus droebachiensii*, in which one gonad was male and the remaining four female.

* *Zool. Anz.* vol. XXXI, 1906–07, p. 635.

On Certain Simply-Transitive Permutation-Groups. By Dr W. Burnside, Honorary Fellow of Pembroke College.

[*Received* 19 September 1921.]

In 1900 I proved (*Proc. L. M. S.* vol. XXXIII, p. 177) that a simply-transitive permutation-group of prime degree p must contain a self-conjugate subgroup of prime order. In the second edition of my *Theory of Groups* (1911) it was shown that a simply-transitive permutation-group degree p^m, which contains a permutation P of order p^m, necessarily has a self-conjugate subgroup containing $P^{p^{m-1}}$. I ventured then to express an opinion that a similar result was true for any simply-transitive permutation-group which contained a transitive Abelian subgroup. Quite recently I have succeeded, with a single exception, in justifying this expression of opinion in a remarkably simple way.

Denote by $x_{a,b}$ a set of mn variables, where the first suffix takes all values from 0 to $m-1$, and the second all values from 0 to $n-1$.

The permutation
$$x'_{a,b} = x_{a+1,b}$$
is a regular permutation M, of order m, in the mn variables, and
$$x'_{a,b} = x_{a,b+1}$$
is a regular permutation N, permutable with M. The two generate a regular Abelian group $\{M, N\}$ simply-transitive in the mn variables.

If ϵ, η are primitive mth and nth roots of unity, and if
$$\xi_{i,j} = \sum_{ab} \epsilon^{-ia}\eta^{-jb}x_{a,b},$$
then
$$M\xi_{i,j} = \epsilon^i\xi_{i,j}, \qquad N\xi_{i,j} = \eta^j\xi_{i,j},$$
so that the mn quantities $\xi_{i,j}$ are the reduced variables for the Abelian group $\{M, N\}$ and each gives a distinct representation of the group. Also
$$\sum_{i,j} \epsilon^{ui}\eta^{vj}\xi_{i,j} = \sum_{i,j,a,b} \epsilon^{i(u-a)}\eta^{j(v-b)}x_{a,b}$$
$$= mnx_{u,v}.$$

Suppose now that a simply-transitive group G in the mn x's contains $\{M, N\}$. Since no irreducible representation of $\{M, N\}$ occurs more than once, no irreducible representation of G, when it is completely reduced, can occur more than once; and therefore* G_0, the subgroup that leaves $x_{0,0}$ unchanged has just one linear invariant in each irreducible representation of G.

For each irreducible representation of G, the reduced variables must be expressible as a set of ξ's. Let
$$\xi_{\alpha_1,\beta_1}, \xi_{\alpha_2,\beta_2}, \ldots, \xi_{\alpha_\mu,\beta_\mu}$$
be the reduced variables for an irreducible representation Γ.

* *Theory of Groups*, Second Edition, p. 275.

Since $mn\, x_{0,0} = \sum_{i,j} \xi_{i,j}$, the one linear invariant that G_0 has in Γ is

$$\sum_{i=1}^{\mu} \xi_{a_i, \beta_i}.$$

Suppose now that

$$x_{a_1, b_1},\ x_{a_2, b_2},\ \ldots,\ x_{a_p, b_p},$$

is a set of variables which are permuted transitively by G_0. Then

$$\sum_{j=1}^{j=p} x_{a_j, b_j}$$

is a linear invariant for G_0.

Now $$\sum_{i=1}^{\mu} \xi_{a_i, \beta_i} = \sum_{i=1}^{\mu} \sum_{a,b} \epsilon^{-a_i a} \eta^{-\beta_i b} x_{a,b},$$

and since the right-hand side is invariant for G_0 which permutes x_{a_j, b_j} $(j = 1, 2, \ldots p)$, transitively,

$$\sum_{i=1}^{\mu} \epsilon^{-a_i a_j} \eta^{-\beta_i b_j}$$

is independent of j.

Also $$mn \sum_{j=1}^{p} x_{a_j, b_j} = \sum_{j=1}^{p} \epsilon^{a_j u} \eta^{b_j v} \xi_{u,v},$$

and since the only linear invariant that G_0 has in the symbols ξ_{a_i, β_i} $(i = 1, 2, \ldots \mu)$ is their sum

$$\sum_{j=1}^{p} \epsilon^{a_j a_i} \eta^{b_j \beta_i}$$

is independent of i.

Moreover the immediately preceding result may be expressed in the form that

$$\sum_{i=1}^{\mu} \epsilon^{a_i a_j} \eta^{\beta_i b_j}$$

is independent of j.

Hence

$$\sum_{i=1}^{\mu} \sum_{j=1}^{p} \epsilon^{a_j a_i} \eta^{b_j \beta_i} = \mu \sum_{j=1}^{p} \epsilon^{a_j a_i} \eta^{b_j \beta_i} = p \sum_{i=1}^{\mu} \epsilon^{a_j a_i} \eta^{b_j \beta_i}.$$

Now $$\sum_{i=1}^{\mu} \epsilon^{a_j a_i} \eta^{b_j \beta_i}$$

is the sum of the multipliers of the operation, $M^{a_j} N^{b_j}$ is the irreducible representation Γ. Hence in Γ

$$\chi_{M^{a_j} N^{b_j}} = \frac{\mu}{p} \sum_{j=1}^{p} \epsilon^{a_j a_i} \eta^{b_j \beta_i}$$

for each i.

Suppose now that Γ is that irreducible representation to which $\xi_{k,0}$ belongs. Then

$$\chi_{M^{a_j}N^{b_j}} = \frac{\mu}{p} \sum_{j=1}^{p} \epsilon^{ka_j}.$$

Now choose the set

$$x_{a_1, b_1}, x_{a_2, b_2}, \ldots, x_{a_p, b_p},$$

so that it contains the symbol $x_{1,0}$. Then

$$\chi_M = \frac{\mu}{p} \sum_{j=1}^{p} \epsilon^{ka_j}.$$

If m is not a prime k may be chosen so that ϵ^k is a primitive qth root of unity, where q is a prime factor of m; and when k is so chosen,

$$\chi_{M^q} = \mu.$$

Hence, if m is not a prime, G has a self-conjugate subgroup containing M^q.

If m and n are different primes, and Γ is that irreducible representation to which $\xi_{1,0}$ belongs, so that

$$\chi_{M^{a_j}N^{b_j}} = \frac{\mu}{p} \sum_{j=1}^{p} \epsilon^{a_j},$$

then $$\frac{\mu}{p} \sum_{j=1}^{p} \epsilon^{a_j} = \sum_{i=1}^{\mu} \epsilon^{a_j\alpha_i} \eta^{b_j\beta_i}.$$

Unless each β_i is zero, in which case the group has a self-conjugate subgroup containing N, this equation actually contains powers of η on the right. Hence when the indices of the powers of η are reduced (mod. n) each power must occur with the same coefficient. This shews that μ must be a multiple of n, and that the reduced variables for Γ must be

$$\xi_{a_1, i}, \xi_{a_2, i}, \ldots, \xi_{a_r, i} \quad (i = 0, 1, 2, \ldots, n-1).$$

The family of representations to which Γ belongs accounts therefore for $(m-1)n$ of the reduced variables. The remaining reduced variables are

$$\xi_{0,0}, \xi_{0,1}, \ldots, \xi_{0,n-1},$$

and in each irreducible representation whose reduced variables belong to this set

$$\chi_M = \chi_E.$$

Hence when m, n are different primes G has a self-conjugate subgroup containing either M or N. It is clear that the same method of proof will apply, when the transitive Abelian subgroup has three or more independent generators. Hence:—

A simply-transitive permutation-group, which contains a regular transitive Abelian subgroup, always has a self-conjugate subgroup, except possibly when the operations of the Abelian subgroup are all of the same prime order.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1920—1921.

ANNUAL GENERAL MEETING.

October 25, 1920.

In the Cavendish Laboratory.

MR C. T. R. WILSON, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:
Prof. Seward.

Vice-Presidents:
Prof. Sir E. Rutherford.
Mr C. T. R. Wilson.
Dr E. H. Griffiths.

Treasurer:
Prof. Hobson.

Secretaries:
Mr H. H. Brindley.
Prof. Baker.
Mr F. W. Aston.

Other Members of Council:

Dr Marshall.	Prof. Marr.
Prof. Newall.	Mr C. T. Heycock.
Dr Fenton.	Mr H. Lamb.
Prof. Inglis.	Prof. Hopkins.
Mr Rivers.	Dr Bennett.
Mr F. A. Potts.	Dr Hartridge.

The following were elected Fellows of the Society:

T. M. Lowry, M.A., Professor of Physical Chemistry.
S. Pollard, M.A., Trinity College.

The following were elected Associates of the Society:

E. S. Bieler, Gonville and Caius College.
T. M. Cherry, Trinity College.
L. Harding, Gonville and Caius College.
W. W. Hurst, Jesus College.
I. Jones, Emmanuel College.
E. B. Ludlam, Trinity College.
J. K. Roberts, Trinity College.

The following Communications were made to the Society:

1. On the stability of the steady motion of viscous liquid contained between two rotating co-axal circular cylinders. By K. TAMAKI and W. J. HARRISON, M.A., Clare College.

2. Sur le principe de Phragmén-Lindelöf. By M. MARCEL RIESZ and Professor G. H. HARDY.

3. A note on the nature of the carriers of the Anode Rays. By G. P. THOMSON, M.A., Corpus Christi College.

4. On the distribution of primes. By M. H. CRAMÉR. (Communicated by Professor G. H. Hardy.)

5. Note on Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions. By Professor G. H. HARDY.

6. On the representation of an algebraic number as a sum of four squares. By L. J. MORDELL, M.A., St John's College. (Communicated by Professor H. F. Baker.)

7. The parity of the number which enumerates the Partitions of a number. By Major P. A. MACMAHON.

November 8, 1920.

In the Cavendish Laboratory.

MR C. T. R. WILSON, VICE-PRESIDENT, IN THE CHAIR.

The following was elected an Associate of the Society:

M. H. Belz, Gonville and Caius College.

The following communication was made to the Society:

The Inner Structure of Atoms. By Professor Sir ERNEST RUTHERFORD.

November 22, 1920.

In the Comparative Anatomy Lecture Room.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

A. B. Appleton, M.A., Downing College.
R. H. Atkinson, M.A., Queens' College.
E. S. Dewing, B.A., Gonville and Caius College.
C. D. Ellis, B.A., Trinity College.
U. R. Evans, M.A., King's College.
J. E. G. Harris, B.A., Jesus College.
L. J. Mordell, M.A., St John's College.
E. K. Rideal, M.A., Trinity Hall.
R. C. Staples-Browne, M.A., Emmanuel College.
B. M. Wilson, B.A., Trinity College.

The following were elected Associates of the Society:

J. Humphries, Queens' College.
A. C. G. Menzies, Christ's College.
Miss Lorna M. Swain, Newnham College.

The following Communications were made to the Society:

1. A note on Vital Staining. By F. A. Potts, M.A., Trinity Hall.

2. Preliminary note on the Superior Vena Cava of the Cat. By W. F. Lanchester, M.A., King's College, and A. G. Thacker.

3. Preliminary note on a cynipid hyperparasite of Aphides. By Miss M. D. Haviland. (Communicated by Mr H. H. Brindley.)

4. The Problem of Soaring Flight. By E. H. Hankin, Sc.D., St John's College. (Communicated by Mr H. H. Brindley.)

5. On the rotation of a non-spinning gyrostat. By Sir George Greenhill.

6. A method of testing Triode Vacuum Tubes. By E. V. Appleton, M.A., St John's College.

January 24, 1921.

In the Cavendish Laboratory.

Professor Seward, President, in the Chair.

The following were elected Fellows of the Society:

E. D. Adrian, M.D., Trinity College.
A. Munro, M.A., Queens' College.
E. G. D. Murray, M.A., Christ's College.
H. W. C. Vines, M.B., Christ's College.

The following were elected Associates of the Society:

Miss M. Barker, Newnham College.
J. Burtt-Davy.
D. A. Keys, Corpus Christi College.

The following Communications were made to the Society:

1. (*a*) Experiments with Rotating Fluids.

(*b*) Tides in the Bristol Channel. By G. I. Taylor, M.A., Trinity College.

2. The deterioration of fabric under the action of Light and its physical explanation. By F. W. Aston, M.A., Trinity College.

3. Note on Constant Volume Explosion Experiments. By S. Lees, M.A., St John's College.

4. On the function $[x]$. By Viggo Brun. (Communicated by Professor G. H. Hardy.)

February 7, 1921.

In the Botany School.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following was re-elected a Fellow of the Society:

E. H. Hankin, Sc.D., St John's College.

The following were elected Associates of the Society:

E. N. Hewitt, Trinity College.
W. Schlundt.

The following Communications were made to the Society:

1. The Development of Photosynthetic Activity during Germination. By G. E. BRIGGS, M.A., St John's College. (Communicated by Professor Seward.)

2. A theorem concerning summable series. By Professor G. H. HARDY.

3. Vectors and Tensors. By E. A. MILNE, M.A., Trinity College.

4. (*a*) Standing waves parallel to a plane beach.

(*b*) A kinetic theory of the Universe. By H. C. POCKLINGTON, M.A., St John's College.

5. (*a*) A configuration in four dimensions.

(*b*) The representation of a cubic surface on a quadric surface.

(*c*) On Delaunay's method in planetary theory.

(*d*) A periodic motion in dynamics. By Professor H. F. BAKER.

February 21, 1921.

In the Anatomy School.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

G. S. Adair, B.A., King's College.
H. S. Carslaw, Sc.D., Emmanuel College.
G. N. Nicklin, M.A., St John's College.
D. F. W. Scanlan, B.A., Jesus College.
J. T. Wilson, M.A., St John's College, Professor of Anatomy.

The following was elected an Associate of the Society:

B. N. Banerji, Clare College.

The following Communication was made to the Society:

The present position of the Helmholtz theory of hearing. By W. HARTRIDGE, M.D., King's College.

February 28, 1921.

In the Cavendish Laboratory.

PROF. SIR ERNEST RUTHERFORD, VICE-PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

G. E. Briggs, M.A., St John's College.
H. T. H. Piaggio, Sc.D., St John's College.

The following Communications were made to the Society:

1. On the nature of crystal-reflection of X-rays. By Professor Sir JOSEPH LARMOR.

2. An experiment on focal lines formed by a Zone plate. By G. F. C. SEARLE, Sc.D., Peterhouse.

3. The origin of the disturbances in the initial motion of a shell. By R. H. FOWLER, M.A., Trinity College, and C. N. H. LOCK, B.A., Gonville and Caius College.

4. On the latent heats of Vaporisation. By E. K. RIDEAL, M.A., Trinity Hall.

March 7, 1921.

In the Comparative Anatomy Lecture Room.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

R. E. Holthum, B.A., St John's College.

The following was elected an Associate of the Society:

A. Harrison White.

The following Communications were made to the Society:

1. A peculiar case of heredity in the Sweet Pea. By Professor PUNNETT.

2. (*a*) Insect Oases.

(*b*) Wing Teratologies in Diptera. By C. G. LAMB, M.A., Clare College.

3. Some Alcyonaria in the Cambridge Museum. By S. J. HICKSON, M.A., Downing College. (Communicated by Mr H. H. Brindley.)

4. The Mechanism of Ciliary Movement. By J. GRAY, M.A., King's College.

5. The influence of function on the conformation of bones. By A. B. APPLETON, M.A., Downing College.

6. A Note on the Hydrogen Ion Concentration of some Natural Waters. By J. T. SAUNDERS, M.A., Christ's College.

7. Animal oecology in deserts. By P. A. Buxton, M.A., Trinity College.

8. The Biology of the Crown Gall Fungus of Lucerne. By J. Line, M.A., Emmanuel College.

May 2, 1921.

In the Cavendish Laboratory.

Professor Seward, President, in the Chair.

The following Communications were made to the Society:

1. On active molecules in physical and chemical reactions. By E. K. Rideal, M.A., Trinity Hall.

2. (*a*) An experiment which favours the resonance theory of hearing.
   (*b*) A criticism of Wrightson's theory of hearing.
   (*c*) A method of projecting interference bands.
   (*d*) A method of projecting absorption spectra.
   (*e*) The shift of absorption bands with change of temperature. By H. Hartridge, M.D., King's College.

3. The cooling of a solid sphere with a concentric core of a different material. By H. S. Carslaw, Sc.D., Emmanuel College.

4. An alignment chart for thermodynamical problems. By C. R. G. Cosens, B.A. (Communicated by Professor Inglis.)

5. Symbolical methods in the theory of Conduction of Heat. By T. J. I'A. Bromwich, Sc.D., St John's College.

6. On a property of focal conics and of bicircular quartics. By C. V. Hanumanta-Rao, M.A., Trinity College. (Communicated by Professor H. F. Baker.)

May 16, 1921.

In the Comparative Anatomy Lecture Room.

Professor Seward, President, in the Chair.

The following was elected an Associate of the Society:

A. Pratt, Magdalene College.

The following Communications were made to the Society:

1. The soaring flight of Dragon-flies. By E. H. Hankin, Sc.D., St John's College.

2. An unusual type of secondary male characters in the Diptera. By C. G. Lamb, M.A., Clare College.

3. A note on the mouth-parts of certain Decapod Crustaceans By L. A. Borradaile, M.A., Selwyn College.

4. Hermaphrodite Sea-urchins. By J. Gray, M.A., King's College.

5. (*a*) Preliminary note on the development of muscle, bone and body-weight in Sheep.

(*b*) On the alleged Inheritance of an acquired Character in Man.

(*c*) On the so-called Gluteus Maximus of *Tarsius*. By A. B. Appleton, M.A., Downing College.

6. On the effect of a magnetic field on the intensity of spectrum lines (second paper). By H. P. Waran, M.A., Christ's College. (Communicated by Professor Sir Ernest Rutherford.)

7. The theoretical value of Sutherland's constant in the kinetic theory of Gases. By C. G. F. James, Trinity College. (Communicated by Mr R. H. Fowler.)

8. Orthogonal Systems and the moving trihedral. By T. S. Yang. (Communicated by Professor H. F. Baker.)

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