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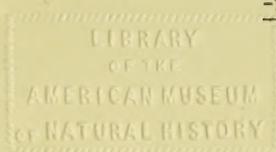
VOLUME XXII

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JOHANNES MÜLLER :—: AMSTERDAM  
:=====: SEPTEMBER 1920 :=====:



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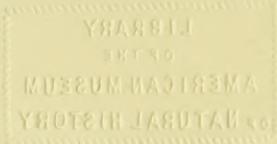
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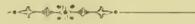
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JOHANNES MÜLLER :—: AMSTERDAM  
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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN  
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXII

N<sup>o</sup>. 1.

President: Prof. H. A. LORENTZ.

Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en  
Natuurkundige Afdeeling," Vol. XXVII).

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**Mathematics.** — “On MAC LAURIN’S *Theorem in the Functional Calculus*”. By DR. H. B. A. BOCKWINKEL. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of March 29, 1919).

In the third communication of my paper “Some observations on complete transmutation”<sup>1)</sup> I proved a restricted validity of MAC LAURIN’S theorem in the Functional Calculus for a *normal* additive transmutation. A normal transmutation was defined by me as follows:

1. There is a functional field  $F(T)$ , the functions  $u$  of which *belong to*<sup>2)</sup> the very same circle  $(\sigma)$ , and for these functions the transmutation  $T$  produces functions belonging to the very same circle  $(\alpha)$ , concentric with  $(\sigma)$ .

2. All rational integral functions are included in the functional field.

3. The transmutation  $T$  is continuous in the pair of associated fields  $F(T)$  and  $(\alpha)$ <sup>3)</sup>.

From 2 it can be derived that to any such transmutation  $T$  another transmutation  $P$  *formally* corresponds, which is given by

$$Pu = a_0 u + \frac{a_1 u'}{1} + \frac{a_2 u''}{2!} + \dots + \frac{a_m u^{(m)}}{m!} + \dots, \dots \quad (1)$$

where the quantities  $u^{(m)}$  are the derivatives of the subject of operation  $u$  and the quantities  $a_m$  functions of the numerical variable  $x$ , which by means of the formula

$$a_m = \xi_m - m_1 x \xi_{m-1} + \dots + (-1)^m x^m \xi_0 \quad (2)$$

can be derived from the functions

$$\xi_k = T(x^k) \dots \dots \dots \quad (3)$$

1) These Proc. Vol. XIX N<sup>o</sup>. 6 and 8 and Vol. XX N<sup>o</sup>. 3—7, to be quoted as l.c.

2) A function *belongs to* a circle, if it is regular *within and on* it. The symbol  $(\sigma)$  means a circle with radius  $\sigma$ .

3) See for the definition of continuity l.c. Vol. XIX N<sup>o</sup>. 8, where also definitions are given of the expressions *functional field* ( $F. F.$ ), *numerical field of operation* ( $N. F. O.$ ) (here the circle  $(\alpha)$ ) and *numerical field of functions* ( $N. F. F.$ ) (here the circle  $(\sigma)$ ).

4) By  $m_k$  the binomialcoefficient of order  $k$  of the number  $m$  is meant.

which are the transmuted of the successive positive integral powers of  $x$ : the latter functions exist according to 2 and belong to the circle  $(\alpha)$ . The above-mentioned theorem of MAC LAURIN consists in stating the equality of the transmutations  $T$  and  $P$ , in a certain numerical field  $(\alpha')$ , which is part of  $(\alpha)$  or identical with  $(\alpha)$ , if the further condition is added, that the latter transmutation is complete in  $(\alpha')$ <sup>1</sup>.

In this statement there is something unsatisfying, if we compare it with TAYLOR'S theorem for the *Theory of Functions*. The latter asserts: "if a function, in a certain circle, has some specified properties (viz. a definite differential coefficient) it can be expanded in TAYLOR'S series in that circle". It is therefore not necessary to impose further conditions on that series. Accordingly it would be desirable that also in the *Functional Calculus* the theorem might be stated in such a way that it were not necessary to impose any further condition on the series  $P$  corresponding to the given transmutation  $T$ , but that such conditions were implied in the properties of  $T$ . At the time it was our opinion that this was not the case. But now we are in a position to prove the following proposition:

*The series corresponding to a normal additive transmutation represents a complete transmutation.*

For simplicity we consider a circular domain  $(\sigma)$  round the origin as a centre and in this the infinite sequence of functions

$$1, x, x^2, \dots x^m, \dots \dots \dots (4)$$

to which, by definition, the infinite sequence of transmuted

$$\xi_0, \xi_1, \xi_2, \dots \xi_m, \dots \dots \dots (5)$$

corresponds, the latter functions being regular in a circular domain  $(\alpha)$ . If we denote by  $\epsilon$  an arbitrarily small positive quantity, then the sequence of functions

$$1, \frac{x}{\sigma + \epsilon}, \frac{x^2}{(\sigma + \epsilon)^2}, \dots \frac{x^m}{(\sigma + \epsilon)^m}, \dots \dots \dots (6)$$

derived from (4) converges uniformly to zero in the domain  $(\alpha)$ . According to a simple property of continuity (l.c. II, n°. 11) the sequence of the transmuted of the latter functions, which, by the additive property of the transmutation, is represented by

---

<sup>1</sup>) A transmutation  $P$ , represented by a series of the form (1), is called by me complete in a domain  $(\alpha)$ , if there is a certain circle  $(\rho)$ , concentric with  $(\alpha)$ , such that all functions belonging to  $(\rho)$  possess a transmuted, regular in  $(\alpha)$ . The minimum circle  $(\beta)$ , which may be taken for  $(\rho)$ , I called the domain corresponding to  $(\alpha)$  (l.c. Vol. XIX, N°. 6).

$$(\xi_0), \frac{\xi_1}{\sigma + \varepsilon}, \frac{\xi_2}{(\sigma + \varepsilon)^2}, \dots, \frac{\xi_m}{(\sigma + \varepsilon)^m}, \dots \quad (6')$$

will converge uniformly to zero in the domain  $(\alpha)$ ; because a normal transmutation is continuous in a pair of conjugate fields. For sufficiently large  $m$ -values we have therefore in all points of  $(\alpha)$

$$|\xi_m| < (\sigma + \varepsilon)^m \quad (7)$$

From the equation (2) it is now easily derived that an analogous inequality holds for the functions  $a_m$  occurring in the series (1), that is to say, that these functions, too, are less in value than the  $m^{\text{th}}$  power of a certain number independent of  $m$ . For if (7) is valid for  $m > m_0$ , we have by (2),  $|x|$  being at most equal to  $\alpha$ ,

$$\begin{aligned} |a_m| &< \sum_0^{m_0} m_k \alpha^{m-k} |\xi_k| + \sum_{m_0+1}^m m_k \alpha^{m-k} (\sigma + \varepsilon)^k \\ &< \longleftrightarrow + \sum_0^m k \longleftrightarrow \\ &< \longleftrightarrow + (\sigma + \alpha + \varepsilon)^m \end{aligned}$$

The first part of the second member of this inequality consists of a fixed number of terms, each of which is, for sufficiently large  $m$ -values, less than  $(\alpha + \varepsilon)^m$ , so that the same holds for their sum. The second part is greater than the latter quantity, hence we have for sufficiently large  $m$ -values at all points of the domain  $(\alpha)$

$$|a_m| < (\sigma + \alpha + \varepsilon)^m \quad (8)$$

and therefore also

$$\overline{\lim}_{m=\infty} |a_m|^{\frac{1}{m}} < \sigma + \alpha \quad (8')$$

Thus the upper limit in the left-hand member of this inequality is finite, and this is (Vol. XIX, N<sup>o</sup>. 6) the very condition under which the transmuting series (1) is complete in the domain  $(\alpha)$ ; moreover we infer that the corresponding domain  $(\beta)$  has a radius  $\beta$ , no greater than  $\sigma + 2\alpha$ . For all functions  $u$  belonging to the circle  $(\sigma + 2\alpha)$  the series  $P$  therefore produces a transmuted function  $Pu$  in the domain  $(\alpha)$ , and this transmuted is equal to  $Tu$ , according to the functional theorem of MAC LAURIN we gave in the form (Vol. XX, N<sup>o</sup>. 3):

*If the series  $P$ , answering to a normal additive transmutation  $T$ , is complete in the circular domain  $(\alpha)$  forming the N. F. O. of  $T$ , we have in this domain  $Tu = Pu$  for such functions of the functional field  $F(T)$  of  $T$  as belong to the circle  $(\beta)$  corresponding to  $(\alpha)$ .*

But now that we have found that the series answering to a normal additive transmutation has *necessarily* the property of representing a *complete* transmutation, our theorem can be expressed in the following form, in which it is more really a "theorem of MAC LAURIN" for the functional calculus, the unsatisfying point mentioned above having disappeared:

A *normal additive transmutation can always, in its N. F. O.* (a), *be expanded in the functional series of MAC LAURIN (1), either for all functions of its F. F., or for a certain part of it, consisting of the functions belonging to a circle* ( $\beta$ )  $>$  ( $\sigma$ ), ( $\sigma$ ) *being the N. F. F.; and the radius*  $\beta$  *is never greater than*  $\sigma + 2a$ .

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**Mathematics.** — “Observations on the expansion of a function in a series of factorials.” III. By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 3, 1919).

10. We shall now prove that the theorem of NIELSEN is *inexact*. For this we use the following lemma:

If the series  $\sum_0^{\infty} a_n$  diverges in such a way that the upper limit of the sum

$$s_n = \sum_0^n a_m \dots \dots \dots (35)$$

for  $n = \infty$  is equivalent to  $n^\theta$ , where  $\theta$  is a certain positive number, then the series

$$\sum_0^{\infty} \frac{a_n}{n^\alpha}, \dots \dots \dots (36)$$

converges or diverges according as  $\alpha > \theta$  or  $\alpha < \theta$ .

Summation by parts gives

$$\sum_0^n \frac{a_m}{m^\alpha} = \frac{s_n}{n^\alpha} + \sum_0^{n-1} s_m \left( \frac{1}{m^\alpha} - \frac{1}{(m+1)^\alpha} \right) \dots \dots (37)$$

For  $\alpha > \theta$ , the limit of the expression  $s_n/n^\alpha$  is, by the hypothesis, zero for  $n = \infty$ . Further the upper limit, for  $m = \infty$  of the general term of the series in the right-hand member is equivalent to  $m^{-(1+\delta)}$ , if  $\delta$  is a certain positive number, so that the series converges if  $n$  is made infinite. So the series in the left-hand member, too, converges for  $n = \infty$ .

Again, writing

$$s_{\alpha,n} = \sum_0^n \frac{a_m}{m^\alpha}, \dots \dots \dots (36')$$

we find on summation by parts

$$\sum_0^n a_m \equiv \sum_0^n m^\alpha \frac{a_m}{m^\alpha} = n^\alpha s_{\alpha,n} - \sum_0^{n-1} s_{\alpha,m} [(m+1)^\alpha - m^\alpha] \dots (38)$$

From this equality we may infer that, for  $\alpha < \theta$ , the upper limit

for  $n = \infty$  of  $s_{\alpha,n}$  is infinite, and is at least equivalent to  $n^{\theta-\alpha}$ . For, if this were not the case and if  $s_{\alpha,n}$  were finite for  $n = \infty$ , or equivalent to a *lower* positive power of  $n$ , say  $n^{\theta-\alpha-\delta}$ , then the first term of the right-hand member would be of order  $n^{\theta-\delta}$ , whereas the terms of the series occurring in that member would be of order  $n^{\theta-\delta-1}$ , and their sum therefore at most of order  $n^{\theta-\delta}$ . This is in contradiction to the initial hypothesis that the left-hand member of the equality is of order  $n^{\theta}$ .

The lemma has thus been proved. A corollary worthy of notice is that the series (36) for  $\alpha < \theta$  diverges in such a way that the upper limit of the sum (36') for  $n = \infty$  is *exactly* equivalent to  $n^{\theta-\alpha}$  (in the second part of our proof we found that it was *at least* equivalent to this power of  $n$ ). For, if that limit were equivalent to a higher power of  $n$ , say  $n^{\theta-\alpha+\delta}$ , then the limit between convergence and divergence of the series

$$\sum_0^{\infty} \frac{a_n}{n^{\beta}} = \sum_0^{\infty} \left( \frac{a_n}{n^{\alpha}} \right) \frac{1}{n^{\beta-\alpha}}$$

would, by the proposition proved just now, be given by  $\beta = \theta + \delta$ , and not by  $\beta = \theta$ . The same consequence may, for the rest, be deduced by observing that from the equality (37) it follows that for  $\alpha < \theta$  the left-hand member is *at most* equivalent to  $n^{\theta-\alpha}$ .

The same lemma as proved above holds for the series of factorials

$$\sum_0^{\infty} \frac{n! a_n}{\Gamma(u+n+1)}, \dots \dots \dots (39)$$

as may be proved in exactly the same way. Both lemmas are moreover a consequence of one another, because the series (36) and (39) converge and diverge for the same values of  $\alpha$ , at least if  $\alpha$  has not exactly the value  $\theta$ , which is the limit between convergence and divergence of the series.

11. We now construct a sequence of coefficients

$$a_0, a_1, \dots a_n, \dots \dots \dots (40)$$

whose upper limit for  $n = \infty$  is equivalent to  $n^{\theta}$ , whereas the upper limit of the sum (35) is equivalent to  $n^{\theta}$ ,  $\theta$  now being a positive number less than 1. It is not difficult to effect this in different manners; but we shall moreover try to secure that the *second* sum quantity

$$s_n^{(2)} = \sum_0^n s_m \dots \dots \dots (41)$$

becomes of an order which is by *less than one unity* higher than the *first* sumquantity (35), i. e. of an order less than  $n^{\theta+1}$ . We proceed in the following way. For a certain value  $n = p$  we take

$$s_p = \underline{p}^{\theta},$$

where by the notation  $\underline{p}^{\theta}$  the same thing is meant as  $E(p^{\theta})$ . Now we make the following sums decrease as soon as possible by taking a certain number of coefficients  $a_n$ , for  $n > p$ , all equal to  $-1$ . After  $\underline{p}^{\theta}$  terms the sum  $s_n$  has in this way become equal to zero. And after  $\underline{p}^{\theta}$  terms more  $s_n$  has attained the value  $-\underline{p}^{\theta}$ ; the value of  $n$  is then equal to  $p + 2\underline{p}^{\theta}$ . It may happen that for this value of  $n$  the quantity  $\underline{n}^{\theta}$  is still equal to  $\underline{p}^{\theta}$ . In this case we may say that  $|s_n|$  has again reached the required upper limit and we assume a certain number of next coefficients  $a_n$  all equal to  $+1$ . If, on the contrary,  $\underline{n}^{\theta}$  for  $n = p + 2\underline{p}^{\theta}$  is greater than  $\underline{p}^{\theta}$ , we go on taking  $a_n$  equal to  $-1$  as long as  $s_n$  has diminished to such an effect that  $|s_n|$  is again equal to  $\underline{n}^{\theta}$ . Such an  $n$ -value *must* be reached, if  $\theta < 1$ , for  $|s_n|$  would become of order  $n$ , if we never stopped taking  $a_n = -1$ . Just as in the former case we assume, after this  $n$ -value, the next coefficients  $a_n$  equal to  $+1$ , until  $s_n$  has attained the value  $\underline{n}^{\theta}$  again. And so on. The upper limit for  $n = \infty$  of the sums  $s_n$  is then equivalent to  $n^{\theta}$  and that of the coefficients  $a_n$  is equivalent to  $n^{\theta}$ . We call the values of  $n$  for which  $s_n$  is equal to  $\underline{n}^{\theta}$  *critical values* and we denote them by  $n_k$  for  $k = 1, 2, \dots$ . If we assume arbitrarily a certain value  $n = p$  as a critical value, then it follows from the preceding construction that the following critical values are uniquely determined. We might continue the same construction towards the side of the *smaller*  $n$ -values, but this is of no importance, since the *asymptotic* behaviour of the quantities in question is solely effective. The graphical representation of the coefficient  $a_n$ , considered as a function of  $n$ , consists of a number of horizontal line-segments, which are alternately above and beneath the  $n$ -axis, and are at a distance 1, the change taking place in the critical points. The graphical representation of the sums  $s_n$  consists of joined line-segments including alternately angles of  $+\frac{\pi}{4}$  and  $-\frac{\pi}{4}$  with the  $n$ -axis; and the alternation again takes place in the

1) We may write  $(p + 2\underline{p}^{\theta})^{\theta} = p^{\theta} + 2\theta p^{2\theta-1}(1-\varepsilon)$ , where  $\lim_{p \rightarrow \infty} \varepsilon = 0$ . From this it follows that for  $\theta < \frac{1}{2}$  and large  $p$ -values,  $E(p + 2\underline{p}^{\theta})^{\theta}$  is in general equal to  $E(p^{\theta})$ .

critical points, where the tops of the lines lie *within* the parabolic curve  $y = \pm n^\theta$ , at a distance less than unity from this curve.

We shall now shew that the upper limit for  $n = \infty$  of the *second* sum-quantities (41) is equivalent to  $n^{2\theta}$ . The sums

$$\sum_m^{n_k} s_m \quad \text{and} \quad \sum_m^{n_k + n_k^\theta} s_m$$

$$n_k - \underline{n_k^\theta} \qquad \qquad n_k + 1$$

are arithmetical progressions consisting of terms of equal signs only; their total value is easily verified to be exactly equal to  $\pm (\underline{n_k^\theta})^2$ . Thus we have, but for a certain constant, whose value is of no importance for the asymptotic behaviour of these sums:

$$\sum_m^{n_k + \underline{n_k^\theta}} s_m = (\underline{n_1^\theta})^2 - (\underline{n_2^\theta})^2 + \dots \pm (\underline{n_k^\theta})^2$$

Since the terms of the sum in the right-hand member increase in absolute value or remain constant over a certain number of  $n$ -values, the sum itself is, in absolute value, less than the last and the first term together. So we have

$$\left| s_{n_k + \underline{n_k^\theta}}^{(2)} \right| < (\underline{n_k^\theta})^2 + (\underline{n_1^\theta})^2 < (\underline{n_k^\theta})^2 (1 + \epsilon)$$

where  $\epsilon$  is a positive quantity which tends to zero for  $k = \infty$ . Further, for an  $n$ -value lying in the interval between  $n_k - \underline{n_k^\theta} = n_{k-1} + \underline{n_{k-1}^\theta}$  and  $n_k + \underline{n_k^\theta}$  the value of  $s_n^{(2)}$  lies *between* the two values of  $s_n^{(2)}$  for these two limits, because  $s_n$  remains of the same sign between them, and therefore  $s_n^{(2)}$  varies *monotonely*. So we have for *all*  $n$ -values of the interval considered

$$\left| s_n^{(2)} \right| < (\underline{n_k^\theta})^2 (1 + \epsilon) < n_k^{2\theta} (1 + \epsilon).$$

For the part of the  $n$ -values greater than  $n_k$  we therefore have *a fortiori*

$$\left| s_n^{(2)} \right| < n^{2\theta} (1 + \epsilon). \dots \dots \dots (42)$$

For the remaining part we have

$$n > n_k - \underline{n_k^\theta}.$$

Now

$$n_k^{2\theta} = (n_k - \underline{n_k^\theta} + \underline{n_k^\theta})^{2\theta} = (n_k - \underline{n_k^\theta})^{2\theta} (1 + \epsilon)$$

where, again,  $\epsilon$  is arbitrarily small for sufficiently great  $k$ . Thus for the latter  $n$ -values, too, we have



12. It is convenient to observe on this occasion that by means of the proposition of N<sup>o</sup>. 10 we may infer from the first part of formula (43) that never convergence of the series of factorials (6) takes place for  $R(x) < \lambda$ , a truth already shewn in another way in N<sup>o</sup>. 9 of the preceding communication.

For, if the upper limit of  $s_n$  for  $n = \infty$  is equivalent to  $n^\theta$ , then the function represented by the series

$$\sum_0^{\infty} s_n t^n$$

is, for  $t = 1$ , at most of order  $1 : (1-t)^{1+\theta}$  and hence  $\varphi(t)$  is at most of order  $1 : (1-t)^\theta$ , so that the number  $\lambda$  is not greater than  $\theta$ , whereas the limit between convergence and divergence of the series of factorials according to the lemma mentioned, is afforded by  $R(x) = \theta$ .

13. At the end of the first communication we said that, if a certain argument mentioned there and used by NIELSEN were not erroneous, then another very general case of the possibility of expanding the integral

$$\Omega(x) = \int_0^1 \varphi(t) (1-t)^{x-1} dt \dots \dots \dots (1)$$

in a series of factorials would be proved by it, viz. the case when the series of factorials *corresponding to that integral* converges. We shall now prove that this case, though not established by NIELSEN, is, indeed, exact. In order to do this we shall make use of the following lemmas:

1. *If the series of factorials converges for a certain value  $x = c$ , then it converges for any value of  $x$  whose real part  $R(x)$  is greater than that of  $c$ .*

Let

$$\sum_n^l \frac{m! a_m}{\Gamma(c + m + 1)} = s_{n,l} \dots \dots \dots (44)$$

then we find on summation by parts

$$\begin{aligned} \sum_n^l \frac{m! a_m}{\Gamma(x + m + 1)} &\equiv \sum_n^l \frac{m! a_m}{\Gamma(c + m + 1)} \cdot \frac{\Gamma(c + m + 1)}{\Gamma(x + m + 1)} = \\ &= \frac{s_{n,l} \Gamma(c + l + 1)}{\Gamma(x + l + 1)} + (x - c) \sum_n^{l-1} \frac{s_{n,m} \Gamma(c + m + 1)}{\Gamma(x + m + 2)}. \end{aligned}$$

On account of the supposed finiteness of  $s_{n,\infty}$ , the limit of the first term of the last member is for  $R(x) > R(c)$  equal to zero, if  $l$  increases indefinitely and the series in that member converges absolutely. Hence the series in the first member converges also and we have

$$\sum_n^{\infty} \frac{m! a_m}{\Gamma(x+m+1)} = (x-c) \sum_n^{\infty} \frac{s_{n,m} \Gamma(c+m+1)}{\Gamma(x+m+2)} \quad (45)$$

by which the lemma is proved. It follows from it that *the limit between the domain of convergence and the domain of divergence of a series of factorials is formed by a straight line parallel to the imaginary axis.*

Moreover we may infer from the equality (45) the following lemma:

II. *In any finite part  $S$  of the half-plane  $R(x) > R(c) + \delta$ ,  $\delta$  being a fixed, but arbitrarily small real positive quantity, the series of factorials converges uniformly.*

For  $n$  may be chosen so large, corresponding to any arbitrarily given  $\varepsilon$ , that for all  $m > n$

$$\left| \frac{s_{n,m} \Gamma(c+m+1)}{\Gamma(x+m+2)} \right| < \frac{\varepsilon}{m^{R(x-c)+1}}$$

and thus

$$\left| (x-c) \sum_n^{\infty} \frac{s_{n,m} \Gamma(c+m+1)}{\Gamma(x+m+2)} \right| < \frac{\varepsilon |x-c|}{R(x-c) n^{R(x-c)}} \quad (46)$$

Since  $R(x-c) > \delta$ , the right-hand member of this inequality is, for a sufficiently large  $n$ -value, which is independent of the value of  $x$  in the domain  $S$ , less than  $\varepsilon$ . This was to be proved.

From the inequality (46) we may infer the corollary IIa: *If the series of factorials (6) converges for a certain value  $x=c$ , then it converges uniformly on the half-line beginning at  $x=c$  and having the direction of the positive part of the real axis.*

For on this line  $R(x-c) = x-c$ ; hence the right-hand member of (46) passes into  $\varepsilon: n^{R(x-c)}$  and this is less than  $\varepsilon$  independently of the place of  $x$  on that line. The corollary is an analogon of a well-known proposition of ABEL on the uniform convergence of a power-series along the radius of a certain point of its circle of convergence, if that power-series converges at that point.

From lemma II, in connection with the fact that the terms of the series of factorials (6) are continuous functions of  $x$ , it may be inferred that such a series represents a continuous function of  $x$  in the domain

$S$ , even in case  $S$  lies wholly or partly in the domain of only *conditional* convergence; of course it must be supposed that all points of  $S$  lie at a distance greater than some *fixed* arbitrarily small quantity  $\delta$  from the straight line parallel to the axis of imaginary quantities which forms the limit between the domains of convergence and divergence of the series. Finally we shall prove that this series represents an *analytic* function in  $S$ , and for this it will be sufficient to show that the series of differential coefficients, with regard to  $x$ , of the terms of the first series converges in the same domain as this first series.

The series of derivatives may be represented by

$$- \Gamma'(x) \sum_n^{\infty} \frac{n! a_n}{\Gamma(x+n+1)} [\psi(x+n+1) - \psi(x)] \dots \quad (47)$$

where, in the notation of NIELSEN <sup>1)</sup>,  $\psi(x)$  denotes the logarithmic derivative of the Gammafunction:

$$\psi(x) = \frac{d \log \Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)} \dots \dots \dots \quad (48)$$

Summation by parts gives, if the notation (44) is used again

$$\begin{aligned} \sum_n^l \frac{m! a_m \psi(x+m+1)}{\Gamma(x+m+1)} &\equiv \sum_n^l \frac{m! a_m}{\Gamma(c+m+1)} \cdot \frac{\Gamma(c+m+1) \psi(x+m+1)}{\Gamma(x+m+1)} = \\ &= \frac{s_{n,l} \Gamma(c+l+1) \psi(x+l+1)}{\Gamma(x+l+1)} + \sum_n^{l-1} \frac{s_{n,m} \Gamma(c+m+1)}{\Gamma(x+m+2)} [(x-c)\psi(x+m+2)-1], \end{aligned}$$

where the equation of finite differences

$$\psi(x+1) - \psi(x) = \frac{1}{x}$$

satisfied by  $\psi(x)$  is taken into account. Since, for  $x = \infty$  the principal part of  $\psi(x)$  is equal to  $\log x$ , the first term of the second member has, for  $R(x-c) > \sigma$ , zero as a limit for  $l = \infty$ , and the series in that member converges absolutely. Hence the series in the first member converges also and we have

$$\sum_n^{\infty} \frac{m! a_m \psi(x+m+1)}{\Gamma(x+m+1)} = \sum_n^{\infty} \frac{s_{n,m} \Gamma(c+m+1)}{\Gamma(x+m+2)} [(x-c)\psi(x+m+2)-1].$$

From this equality it may be easily deduced that, corresponding to any  $\epsilon$ , there is a number  $N$  such that for  $n > N$  and for all  $x$  in the domain  $S$

<sup>1)</sup> Handbuch der Gammafunktion, p. 15, Leipzig, Teubner.

$$\left| \sum_n^{\infty} \frac{m! a_m \psi(x+m+1)}{\Gamma(x+m+1)} \right| < \frac{\varepsilon \log n}{\delta n^{\delta}}$$

that is  $< \varepsilon$ , since  $(\log n) : n^{\delta}$  is zero for  $n = \infty$ . By this the convergence of the first part of the series of derivatives (47) has been proved, together with the *uniform* convergence in  $S$ . The second part need not be examined more closely, since this part forms a series which, save as to the factor  $\psi(x)$  not depending on  $n$ , is equal to the series of factorials itself.

On account of its uniform convergence in the domain  $S$  and the continuity of its individual terms the series of derivatives in question represents in  $S$  the derivative of the function determined by the original series of factorials. The differential coefficient of this function is therefore determined at any point of  $S$ , and is independent of the direction in which it is taken: thus *the series of factorials represents an analytic function in the domain  $S$ .*

14. We now proceed to settle the question indicated at the beginning of the preceding section, and we first consider the case  $\lambda' = 0$ , in which the upper limit of  $a_n$  for  $n = \infty$  is equivalent to  $n^{\theta}$ . We further assume the limit of convergence and divergence of the series (6) to be given by  $R(x) = \theta$ ,  $\theta$  being a positive number less than unity. Since, as we showed, the number  $\lambda$  is in any case not greater than  $\theta$ , we have

$$\lim_{t=1} (1-t)^x \varphi(t) = 0 \quad \text{for } R(x) > \theta.$$

From this it follows that the integral (1) converges absolutely for  $R(x) > \theta$ . At the same time this convergence is *uniform* in any domain  $S$  wholly included in the finite part of the half-plane  $R(x) > \theta + \delta$ , where  $\delta$  is a certain fixed, but for the rest arbitrarily small positive number. The same holds for the integral

$$\int_0^1 \varphi(t) (1-t)^{x-1} \log(1-t) dt,$$

which may be deduced from the integral (1) by differentiating the latter under the integration-sign with regard to  $x$ . Hence this integral represents in  $S$  the derivative with regard to  $x$  of the integral (1), so that the latter is an *analytic* function of  $x$  in the domain  $S$ .<sup>1)</sup>

<sup>1)</sup> The statements mentioned here briefly have been established by PINCHERLE in his paper "Sur les fonctions déterminantes", Ann. de l'Éc. Norm. (3) XXII, 1905, p. 13—17. Their analogy with those we proved in the preceding section is manifest.



inferred from it, that sometimes *not* a certain derivative had that property. This is not the case; if the characteristic  $\lambda'$ , defined by (14), of the coefficients  $a_n$  in the power-series for the function  $q(t)$  is *finite*, this function always has a derivative of a certain order  $\omega$  possessing the property in question and we have

$$\omega = \lambda' + 1.$$

Further it is shown by HADAMARD<sup>1)</sup>, that a function  $q(t)$ , which is of order  $\omega$  with regard to the *whole* circumference, and of *lower* order  $\omega_0$  on a certain part of it containing the point  $t=1$ , can always be divided into two functions  $\varphi_1(t)$  and  $\varphi_2(t)$ , the first of which is *regular* at  $t=1$  and of order  $\omega$  on the whole circumference, and the other of order  $\omega_0$  on the whole circumference. According to the theorem proved in the same N<sup>o</sup>. 6 the integral

$$\int_0^1 \varphi_1(t) t^{x-1} dt,$$

c.q. an integral with  $\varphi_1^{(p)}(t)$  as subject of integration, can, for  $R(x) > \omega - 1 = \lambda'$ , be expanded in a series of factorials. Further the integral

$$\int_0^1 \varphi_2(t) t^{x-1} dt$$

c.q. an integral containing  $\varphi_2^{(q)}(t)$ , can be expanded in such a series for  $R(x) > \omega_0$ . Hence the integral (1), being the sum of the two preceding integrals, c.q. an integral containing  $q^{(r)}(t)$ , can be expanded in a series of factorials for

$$R(x) > \omega_0 \quad \text{or} \quad R(x) > \lambda',$$

according as  $\omega_0$  or  $\lambda'$  has a greater value. In the first case the series converges *conditionally* for

$$\omega_0 < R(x) < \lambda' + 1,$$

and in the second for

$$\lambda' < R(x) < \lambda' + 1.$$

In both cases we may state the following proposition:

*If  $\omega$  is the order of  $q(t)$  along the whole circle of convergence, and therefore  $\lambda' = \omega - 1$  the characteristic of its coefficients, and if  $\omega_0$  is the order of  $q(t)$  at the point  $t = 1$  of that circle, then the integral (1) (c. q. the integral (8)) can be expanded in a series of factorials for such values of  $x$  as satisfy at the same time the inequalities*

$$R(x) > \lambda' \quad \text{and} \quad R(x) > \omega_0.$$

<sup>1)</sup> „Essai sur l'étude des fonctions”, Journ. de Math., 1892, p. 172.

We cannot say, however, that this proposition *is a substitute* for that of NIELSEN, because a series of factorials sometimes converges for  $R(x) < \omega_0$ . Take, for instance, the function of WEIERSTRASS

$$\varphi(t) = 1 + t^2 + t^4 + t^8 + \dots + t^{2^k} + \dots$$

This function has its circle of convergence as a *natural limit*, because all parts of the circumference are equivalent, as appears from the substitution

$$t = t' e^{\frac{2\pi i h}{2^k}},$$

where  $k$  and  $h$  are arbitrary integral numbers. The order  $\omega_0$  in the point  $t=1$  cannot, therefore, differ from the order  $\omega$  along the *whole* circumference, and this order is equal to 1, the characteristic  $\lambda'$  of the coefficients of the power-series being zero. The quantity  $\lambda$  is, however, also zero and the series of factorials converges for  $R(x) > 0$  and is, for all these values of  $x$ , equal to the integral (1) <sup>1)</sup>.

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<sup>1)</sup> The convergence of the series of factorials is, in this case, *absolute*, in the strip of plane  $0 < R(x) < 1$ , owing to the large distance between the coefficients of  $\varphi(t)$  which differ from zero. If, therefore, we spoke continually in the preceding investigations of *conditional* convergence for  $R(x) < \lambda' + 1$ , we meant this *in general*.

**Mathematics.** — “On the necessary and sufficient conditions for the expansion of a function in a Binomial Series”. By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 3, 1919)

PINCHERLE has given a necessary and sufficient condition for the expansion of a function in a binomial series (*Binomialkoeffizientenreihe*)<sup>1)</sup>. It runs thus:

The necessary and sufficient condition that an analytic function  $\omega(x)$  may be expanded in a series of the form

$$\omega(x) = \sum_{n=0}^{\infty} c_n \binom{x-1}{n} \dots \dots \dots (1)$$

is that  $\omega(x)$  be coefficient-function (*fonction coefficiente*) of another analytic function  $\varphi(t)$ , which is regular and zero at infinity and whose singularities lie all within the circle  $(1,1)$ , with centre  $t=1$  and radius  $r=1$ , or on the circumference of it, provided that, in the latter case, the order of  $\varphi(t)$  on the circumference, taken in the sense defined by HADAMARD, be finite or negative infinite<sup>2)</sup>.

By a coefficientfunction  $\omega(x)$  of an analytic function  $\varphi(t)$  of the kind mentioned PINCHERLE means a function which can be deduced from  $\varphi(t)$  in a more or less simple manner, according to the order of  $\varphi(t)$ . The relation between the two functions is, however, always such that conversely  $\varphi(t)$ , called by PINCHERLE the *generating function* (*fonction génératrice*) of  $\omega(x)$ , follows from  $\omega(x)$  by the equation

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{\omega(n+1)}{t^{n+1}}, \dots \dots \dots (2)$$

This means: the *coefficients* of the series of negative integral powers of  $t$ , in which  $\varphi(t)$  may be expanded in a neighbourhood of  $t=\infty$ , are equal to the values of  $\omega(x)$  for positive integral values of  $x$ ; the name *coefficientfunction* for  $\omega(x)$  is due to this circumstance.

The question now arises, how it must be discriminated if a *given* function may be expanded in a binomial series. *This question is not*

<sup>1)</sup> S. PINCHERLE, “Sur les fonctions déterminantes”, *Annal. de l’Ecole Normale*, 1905.

<sup>2)</sup> A circle with centre  $\alpha$  and radius  $r$  will be denoted by  $(\alpha, r)$ .

answered by the theorem of PINCHERLE, at least not in a simple manner, as will appear from what follows. In order to investigate the question we should commence to deduce the series (2) from the given function  $\omega(x)$ . Next we should examine whether the function  $\varphi(t)$  represented by it has the required properties: to be regular without the circle (1,1), and on the circumference of it of finite order. For this we should try to transform the above series into another according to negative integral powers of  $t-1$

$$\varphi(t) = \sum_0^{\infty} \frac{c_n}{(t-1)^{n+1}} \dots \dots \dots (3)$$

The relation between the coefficients of the two series is given by the equations

$$c_n = \omega(n+1) - \binom{n}{1} \omega(n) + \dots + (-1)^n \omega(1) = \Delta^n [\omega(1)]^1 \dots (4)$$

and

$$\omega(n+1) = c_0 + \binom{n}{1} c_1 + \dots + c_n \dots \dots \dots (4')$$

By means of (4) we must see if the series (3) converges without the circle (1,1), and further if the characteristic  $\lambda'$  of the coefficients  $c_n$ , defined by

$$\lambda' = \lim_{n \rightarrow \infty} \frac{\log |c_n|}{\log n}, \dots \dots \dots (5)$$

is not positive infinite; the latter being the condition that  $\varphi(t)$  shall be of non-positive infinite order on the circle (1,1).

But the relation (4) is rather intricate and so it may be very difficult, if not impossible, to perform the just mentioned research. Suppose this, however, possible, and let  $\lambda'$  differ from  $+\infty$ . Then we have to examine whether the given function  $\omega(x)$  is really the coefficientfunction of  $\varphi(t)$ . For there are a great many functions  $\Omega(x)$  giving rise to the same generating function  $\varphi(t)$ , viz. all those contained in the equation

$$\Omega(x) = \omega(x) + F(x)$$

where  $F(x)$  is a function that vanishes for positive integral values of  $x$ . It is therefore necessary to consult the definition of the coefficientfunction given by PINCHERLE and to apply it to the obtained  $\varphi(t)$  in order to see if the original function  $\omega(x)$  is the result. But this is again not very easy. If the characteristic  $\lambda'$  defined by (5) is less than  $-1$ , then  $\varphi(t)$  is finite and continuous along the circum-

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<sup>1)</sup> This symbol denotes the  $n$ -th difference of  $\omega(x)$  at  $x=1$ , the increase  $\Delta x$  of the argument  $x$  being equal to unity.

ference of the circle (1,1). The coefficient-function is then defined by the integral

$$\frac{1}{2\pi i} \int_{(1,1)} \varphi(t) t^{x-1} dt : \dots \dots \dots (6)$$

taken round the just mentioned circle, and this integral is easily seen to be equal to the binomial series (1) in the half-plane of  $x$  on the right of the imaginary axis, provided definite agreements as to the value of  $t^{x-1}$  be made. It may, however, be difficult to investigate whether the integral (6) is equal to the given function  $\omega(x)$ . And in any case the investigation is intricate if  $\lambda' > -1$ , especially when the difference between  $\lambda'$  and  $-1$  is rather great. For the coefficient-function of  $\varphi(t)$  is then brought into relation with that of another generating function, with  $\lambda' < -1$ , by means of a polynomial consisting of a very large number of terms.

The question therefore naturally suggests itself, if it is possible, to find simpler tests which are *sufficient* for a function to be expanded in a binomial series. This is, indeed, the case, and we may, moreover, say that the obtained properties are about *necessary*<sup>1)</sup>. For we can prove the following theorem:

*If a function  $\omega(x)$  is regular in the finite part of the half-plane  $R(x) > \gamma$ <sup>2)</sup> ( $\gamma = \text{real}$ ), and if, in that domain, it satisfies the inequality*

$$|\omega(x)| < M |(x+b)^l a^{x-\gamma}|, \dots \dots \dots (7)$$

*where  $M$  is a positive and  $l$  and  $b$  are real numbers, the latter such that  $b + \gamma > 0$ , further  $a$  a complex number on the circumference of the circle (1,1), variable with the argument  $\psi$  of  $x - \gamma$ , the argument  $a$  of  $a$  being equal to  $-\psi$  so that*

$$a = 2 \cos \psi e^{-i\psi}, \dots \dots \dots (8)$$

*then, in the domain ( $\beta = \text{real} > \gamma$ ).*

$$R(x) > l + \beta - \frac{1}{2}, \dots \dots \dots (9)$$

<sup>1)</sup> For the sake of comparison we observe that for the expansion of a function in a series of factorials

$$\sum_0^\infty \frac{n! a_n}{x(x+1) \dots (x+n)}$$

there is a necessary and sufficient condition, stated by NIELSEN and simplified by PINCHERLE, which has some similarity with the above condition for binomial series. But it is not possible to find *simple* tests for the expansion of a function in a series of factorials. The only simple *sufficient* condition which may be given in this case, is that a function can be expanded in a series of factorials, if it is *regular and of zero value* at infinity. But this condition is far from necessary.

<sup>2)</sup>  $R(x)$  means the real part of  $x$ .

the function  $\omega(x)$  can be expanded as a binomial series

$$\omega(x) = \sum_n^{\infty} c_n \binom{x-\beta}{n} \dots \dots \dots (10)$$

if  $l + \beta - \frac{1}{2} > \gamma$ , and otherwise the expansion is possible in the domain  $R(x) > \gamma$ .

The special value  $-\psi$  of the argument  $a$  of  $a$  is such that the expression  $a^{x-\gamma}$ ,  $x-\gamma$  being given, has the greatest modulus compared to those for other values of  $a$  on the circumference of the circle (1,1). If the inequality (7) holds for fixed  $a$ -value on the circle (1,1), then expansion of  $\omega(x)$  in the series (10) is possible in the domain

$$R(x) > l + \beta - 1, \dots \dots \dots (9')$$

if  $l + \beta - 1 > \gamma$ , and otherwise in the domain  $R(x) > \gamma$ .

The sufficient condition for the expansion of a function in a binomial-series contained in the above theorem seems, indeed, very simple. If a function  $\omega(x)$  can be represented by the equality

$$\omega(x) = (x+b)^k c^x \mu(x) \dots \dots \dots (11)$$

where  $c$  is a fixed number *within* the circle (1,1) and  $\mu(x)$  a function remaining within finite limits in  $R(x) > \gamma$ , then it satisfies the inequality (7) for a value of  $l$  differing arbitrarily little from  $-\infty$ , and therefore it can be expanded in a binomial-series in the domain  $R(x) > \gamma$ . For  $c=1$  formula (11) gives an expression which shows that all functions regular in the finite part of the half-plane  $R(x) > \gamma$  and vanishing at infinity may be expanded as a binomial-series in that domain; further all functions becoming infinitely large at infinity of an order lower than a certain finite power of  $x$ ; so all irrational and logarithmic expressions.

The way in which we have arrived at our theorem is substantially the same as that followed in the ordinary theory of functions of a complex variable, in order to obtain the expansion of a function in a *power-series*; it is founded upon the fundamental theorem of CAUCHY. According to this we have

$$\omega(x) = \frac{1}{2\pi i} \int_W \frac{\omega(z) dz}{z-x} \dots \dots \dots (12)$$

where the integral is taken round a closed curve  $W$ , *within* and *upon* which  $\omega(x)$  is regular, and which contains the point  $z=x$  in its interior. If we wish to deduce from this integral an expansion according to positive integral powers of  $x-a$ ,  $a$  being a number

<sup>1)</sup> This series is taken instead of (1) for the sake of generality.

within  $W$ , then we start from the *known* expansion with *known remainder* of  $1 : (z-x)$  in such a series. In the same way we may reach our present purpose, if we use the *known* expansion with *known remainder* of the just mentioned elementary function in a binomial series, viz.

$$\frac{1}{z-x} = \sum_{m=0}^{n-1} \frac{(x-\beta)\dots(x-\beta-m+1)}{(z-\beta)\dots(z-\beta-m)} + \frac{(x-\beta)\dots(x-\beta-n+1)}{(z-\beta)\dots(z-\beta-n+1)} \cdot \frac{1}{z-x}.$$

Substituting this expression for  $1/z-x$  in the integral (12) and choosing the path of integration so as to include, besides the point  $z=x$ , the points  $z=\beta, \beta+1, \dots, \beta+n-1$ , we find <sup>1)</sup>

$$\omega(x) = \sum_{m=0}^{n-1} \binom{x-\beta}{m} \Delta^m \omega(\beta) + R_n \dots \dots (13)$$

where

$$R_n = \frac{1}{2\pi i} \int_W \omega(z) \frac{(x-\beta)\dots(x-\beta-n+1)}{(z-\beta)\dots(z-\beta-n+1)} \cdot \frac{dz}{z-x} \dots \dots (13')$$

Formula (13) is the ordinary formula of interpolation of NEWTON with a *remainder* added to it and valid for all complex  $x$ -values lying within  $W$ .

If all points  $z=\beta, \beta+1, \dots, \beta+n-1$ , are to lie within the integration-curve  $W$ , this curve will in general have to be modified with increasing  $n$ . It is required to choose  $W$  as fit as possible, that is to say: so that the remainder (13') tends to zero with indefinite increase of  $n$ , and that yet the aggregate of functions  $\omega(x)$  for which this takes place, is as extensive as possible. If, now, the form (7) is taken as majorant-value of these functions, where the number  $a$  is, as yet, left undetermined and the number  $\gamma$ , in order to have a definite case, is chosen zero (so that  $\beta > 0$ ), it is found after a rather long but principally not difficult inquirment: 1. that the most favourable integration-curve is a circle with  $z=n$  as centre and  $n$  as radius so that it passes through the origin; 2. that for  $a$  a complex number may be taken lying on the circumference of the circle (1,1), with the specifications concerning the domains of validity already mentioned in the Statement of the above theorem.

We may further observe that, in case the number  $c$  in formula

<sup>1)</sup> If a few points  $\beta, \beta+1, \dots$ , are *excluded* from the closed curve  $W$ , we obtain an expression the further examination of which leads to the so-called *zero-expansions*, which are treated in an elementary way by PINCHERLE (Rendic. d. R. Accad. d. Lincei, 1902, 2<sup>e</sup> Sem.)



values of  $\alpha$ <sup>1)</sup>, if for the derivative of any negative order the definition of RIEMANN is adopted, which in the present case, a neighbourhood of infinity being regarded, can be expressed by the identity

$$(-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{(u-t)^{\alpha-1}}{(u-1)^\alpha} \varphi(u) du. \quad (16)$$

Since as domain of  $t$  and  $u$  the part of the plane *outside* a certain circle with centre (1) is considered, it will be convenient to assume for path of integration between  $u=t$  and  $u=\infty$  the half-line which has the same direction as the vector from  $u=1$  to  $u=t$ . The quantities  $u-t$  and  $u-1$  then have the same arguments and  $(u-t)^\alpha : (u-1)^\alpha$  is real. With these agreements we have the expansion

$$(-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} = \sum_0^\infty \frac{\Gamma(n+1) c_n}{\Gamma(n+1+\alpha) (t-1)^{n+1}} \quad (17)$$

so that the derivative of negative order  $-\alpha$  of the expression  $\varphi(t) : (t-1)^\alpha$  is, as  $\varphi(t)$  itself, regular and zero at infinity. The characteristic of the derivative is, however,  $\alpha$  less and this makes it possible, by means of (15), to express the coefficientfunction  $\omega(x)$  of a generating function  $\varphi(t)$  with characteristic  $\lambda' < -1$  in terms of another generating function  $\varphi_1(t)$  whose characteristic is *any small amount* less than  $-1$ . The function  $\varphi_1(t)$  is constructed in such a way that the given function  $\varphi(t)$  is the derivative of a certain negative order  $-\alpha$  of  $\varphi_1(t) : (t-1)^\alpha$  and the number  $\alpha$  is selected from an aggregate of positive values, whose upper limit is equal to the difference between  $\lambda'$  and  $-1$ . In other words, if  $\varphi(t)$  is given by (3), we take

$$\varphi_1(t) = \sum_0^\infty \frac{c'_n}{(t-1)^{n+1}}$$

where the meaning of  $c'_n$  is given by

$$c'_n = \frac{\Gamma(n+1) c_n}{\Gamma(n+1+\alpha)}$$

with

$$\alpha = -1 - \lambda' - \delta_1 \dots \dots \dots (18)$$

$\delta_1$  being any small positive number. Then, according to (17)

$$\varphi(t) = (-1)^\alpha D^{-\alpha} \frac{\varphi_1(t)}{(t-1)^\alpha}$$

<sup>1)</sup> I have communicated the proof of this truth in the Proceedings of the meeting of September 27, 1919.

and hence by (15)

$$\omega(x) = I[\varphi(t)] = \frac{\Gamma(x)}{\Gamma(x+\alpha)} I \left[ \frac{t^\alpha \varphi_1(t)}{(t-1)^\alpha} \right]. \dots \dots (19)$$

Not only  $\varphi_1(t)$ , but also the function

$$\psi(t) = \frac{t^\alpha \varphi_1(t)}{(t-1)^\alpha}$$

has the property that the operation  $I$  applied to it gives a coefficient-function satisfying the condition (7), the inequality (14) for  $l$  being left unaltered. We only have in this case  $\gamma = -\alpha$ , instead of  $\gamma = 0$ , and the domain of validity is determined by  $R(x) > -\alpha$ , or, according to (18), by

$$R(x) > \lambda + \sigma_1 \dots \dots \dots (20)$$

where

$$\lambda = \lambda' + 1 \dots \dots \dots (21)$$

That is: the domain of validity of (7) is the domain of *absolute* convergence of the series (1) (for  $\sigma_1$  is arbitrarily small).

For the whole right-hand member of (19), that is for  $\omega(x)$  we therefore have the inequality

$$|\omega(x)| < M |(x+b)^l a^{x-(\lambda+\sigma_1)}|. \dots \dots (22)$$

where  $l$ , now, satisfies the condition

$$l < \lambda - \frac{1}{2} + \sigma_1 + \sigma \dots \dots \dots (23)$$

If, at last, the characteristic  $\lambda'$  of  $\varphi(t)$  is greater than  $-1$  or equal to  $-1$ , then, after PINCHERLE, the coefficientfunction can be expressed in terms of that of another generating function  $\varphi_1(t)$ , with a characteristic less than  $-1$ . First, let

$$-1 < \lambda' < 0,$$

then PINCHERLE considers the additional function

$$\varphi_1(t) = -D^{-1} \left( \frac{\varphi(t)}{t-1} \right),$$

having a characteristic  $\lambda' - 1$ , which, therefore, is less than  $-1$ , so that the corresponding coefficient-function  $\omega_1(x)$  satisfies, in the domain  $R(x) > \lambda - 1 + \sigma_1$ , the inequality

$$|\omega_1(x)| < M |(x+b)^l a^{x-(\lambda-1+\sigma_1)}|$$

with

$$l < \lambda - \frac{3}{2} + \sigma_1 + \sigma$$

The coefficientfunction  $\omega(x)$  of  $\varphi(t)$  is connected with the latter by the formula<sup>1)</sup>

$$\omega(x) = \Delta [(x-1)\omega_1(x-1)]$$

<sup>1)</sup> PINCHERLE, l. c., p. 64.

from which it follows that  $\omega(x)$ , precisely in the domain (20), satisfies the inequality (22) with, for  $l$ , the inequality (23). In this manner we may prove the same inequality for the intervals  $(0, 1), (1, 2) \dots$  of  $\lambda'$  in succession.

If  $\omega(x)$  satisfies the inequality (22) for a certain value of  $l$ , then, evidently, for all greater values. Thus there is a lower limit  $l_0$  for all such values, but this may possibly not be substituted for  $l$  in (22). Instead of this we may however write

$$\omega(x) \sim (x + b)^{l_0} a^{x - (\lambda + \delta_1)} \dots \dots \dots (24)$$

with the meaning that (22) holds for any  $l > l_0$ ; we may call (24) an equation of *equivalence* and say that  $\omega(x)$  is *equivalent* to the right-hand member of this equation. The exponent  $l_0$  satisfies the condition

$$l_0 < \lambda - \frac{1}{2} \dots \dots \dots (25)$$

since  $\delta$  and  $\sigma$ , were arbitrarily small. The proposition relating to the *necessary* condition for a function to be expanded in a binomial-series may thus be expressed in the following manner:

*A binomial series of the form (1) represents in any half-plane  $R(x) > \lambda + \sigma$ , differing arbitrarily little from its domain  $R(x) > \lambda$ , of absolute convergence, a function  $\omega(x)$ , which satisfies the equation of equivalence (24); the exponent  $l_0$  satisfies the inequality (25).*

If, now, this proposition is compared with that relating to the *sufficient* condition, then, to begin with, we find a complete accord between the majorant values (7) and (22). *These majorant-values are, therefore, both necessary and sufficient.* Further, as regards the domains of validity, the inequality (9) here becomes  $R(x) > l + \frac{1}{2}$ , since we had  $\beta = 1$ , or we may also write

$$R(x) > l_0 + \frac{1}{2},$$

if  $l_0$  is again the lower limit of the  $l$ -values which may be taken for the given function. From (25) the same inequality follows with regard to the domain of *absolute* convergence. Since the domain of possibly *conditional* convergence extends at most over a strip of unity-breadth on the left of the domain of absolute convergence, the investigation performed by us leaves room for the possibility that a binomial series sometimes represents a function satisfying the condition (24) also in a strip determined by

$$l_0 - \frac{1}{2} < R(x) < l_0 + \frac{1}{2}$$

or in a certain part of it. In order to come to certainty concerning this point, we should have to examine how a function represented by a binomial series behaves in the domain of conditional convergence of that series. To such an investigation we have as yet

not arrived; but we may already perceive that the result could not fill up the gap which, as regards the domains of validity, exists as yet between the necessary and the sufficient condition. First: if a function  $\omega(x)$  satisfies the equation of equivalence (24) for a *fixed* value of  $a$  on the circle (1,1) and for a certain minimum-exponent  $l_0$ , then, on account of what has been remarked on the expression  $a^{x-\gamma}$ , immediately after formula (10), that function satisfies the same inequality, when the number  $a$  *varies*, in the specified mode, together with the argument  $\psi$  of  $x$ . The index  $l_0$  cannot, however, be diminished, because it must at all events be taken for  $\psi = -\alpha$ , if  $a$  is the argument of the original *fixed* number  $a$ . The statement belonging to the inequality (9') informs us, however, that in this case expansion of  $\omega(x)$  in a binomial series is possible for  $R(x) > l_0$ . The function

$$2^x = \sum_n^{\infty} \binom{x-1}{n},$$

for which we have  $a = 2$ ,  $l_0 = 0$ , affords an illustration of this fact, for the expansion is really valid for  $R(x) > 0$ , and it is *conditionally* convergent for  $R(x) < 1$ . Therefore we can never find  $R(x) > l_0 + \frac{1}{2}$  as a *necessary* condition whereas our theorem concerning the *sufficient* condition only says that expansion is *possible* in the domain defined by the last inequality.

Secondly the last condition only holds in case  $\omega(x)$  has no singularities in the finite part of the domain  $R(x) > l_0 + \frac{1}{2}$ ; for otherwise for the latter domain the one must be substituted where  $\omega(x)$  is regular and that was defined by the inequality  $R(x) > \gamma$ .

Thus the proposition regarding the *necessary* condition states that for points in the domain of absolute convergence of the given binomial series we have  $R(x) > l_0 + \frac{1}{2}$ , but conversely it is not true that in the domain determined by this inequality there is certainly absolute convergence. A simple example is furnished by the function

$$\omega(x) = \frac{1}{x}$$

For this function  $l_0 = -\infty$ , and yet the function can only in the domain of regularity  $R(x) > 0$  be expanded in a binomial series of the form (1).

From these remarks it will be clear that in order to fill up the gap existing as yet between the necessary and sufficient conditions we must give more specified propositions for both conditions. In

other words we should have to succeed in dividing the functional aggregate of all functions satisfying the equation (24) by *special* characterising properties; and so also the aggregate of binomial series, in such a way that between the two kinds of sub-aggregates there existed a *complete* correspondence, such that functions of some sub-aggregate  $K$  could only be expanded in binomial series belonging to the sub-aggregate  $K$  and in no others. But the problem to find suchlike characterising properties will perhaps be very difficult, since it is required for it to derive the character of a function from that of the coefficients of the series representing it; a problem which already causes the greatest difficulties when it regards the more known power-series.

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**Chemistry.** — “*Urease and the radiation-theory of enzyme action*”, III.

By Dr. H. P. BARENDRECHT. (Communicated by Prof. J. BÖESEKEN).

(Communicated in the meeting of May 3, 1919).

8. *The influence of other substances on the urease activity.*

The question was raised, whether in these concentrated solutions of urea the H-ion concentration was perhaps lowered also through the slightly alkaline substance urea itself.

The electrometric measurement of the  $p_H$  in the 8% phosphate solution showed indeed a distinct increase after addition of 8% urea. It was, however, soon established, that this increase was not due to a shifting of the true reaction to the alkaline side, but to a diminution of both H-ions and OH-ions, or, in other words, to a decreased dissociation of  $H_2O$  in these concentrated solutions of urea.

This conclusion was readily arrived at by the following experiments.

The  $p_H$  was determined again in 8% phosphate solutions. Then a second time after addition of a small amount of ammonia (1 c.c. of  $\frac{1}{10}$  N). For each of these estimations 10 c.c. of 9.6% phosphate was brought with water or diluted ammonia to 12 c.c.

The results obtained are summarised as follows:

$p_H$ in:				
Phosphate + 8% urea	Difference	Phosphate alone	Difference	Phosphate + 1 c.c. $NH_3 \frac{1}{10} N$
6.79	0.11	6.68	0.03	6.71
6.97	0.08	6.89	0.03	6.92
7.59	0.10	7.49	0.07	7.56
7.93	0.10	7.83	0.18	8.01
8.20	0.10	8.10	0.34	8.44

As will be seen from SÖRENSEN'S determinations of  $p_H$  in standard phosphate mixtures, the buffer-value of these solutions becomes considerably diminished, when  $p_H$  approaches 8. Therefore the same

small quantity of alkali, which will leave a  $p_H$  of moderate value practically unchanged, will cause a distinct rise in a  $p_H$  of about 8. This is clearly borne out by the results, obtained by the addition of ammonia.

The urea, however, increases the  $p_H$  to an equal degree, whatever its original value.

This increase is therefore evidently not due to an alkalinity of urea. A substance, however, which possesses no alkalinity, can only lower the H-ion concentration, if it lowers the OH-ion concentration just as much.

This effect may be expected from all neutral substances, when added in such an amount to the solution, that the concentration of  $H_2O$  is appreciably diminished, provided their ionising power is much less than that of water. In such a case the dissociation product  $C_H \times C_{OH}$  or, as it is usually called, the dissociation-constant of water  $k_w$  will show a diminution.

Similar changes of the dissociation-constant of water have been studied before, for instance by LÖWENBERZ <sup>1)</sup> in a paper on "The influence of the addition of ethyl-alcohol on the electrolytic dissociation of water".

In view of this influence on  $k_w$  the study of urease action in solutions, mixed with some alcohol, promised interesting results.

The alcohol, employed in these experiments had been freed from carbonic acid by boiling in a flask, attached to a reflux condenser.

The same 8% phosphate buffer-mixtures and Soja extract were used as before in the determination of  $m$  in part 4 of this paper. The amount of alcohol, to be mixed beforehand with the phosphate, was calculated so as to obtain a concentration of 5% in the final solution.

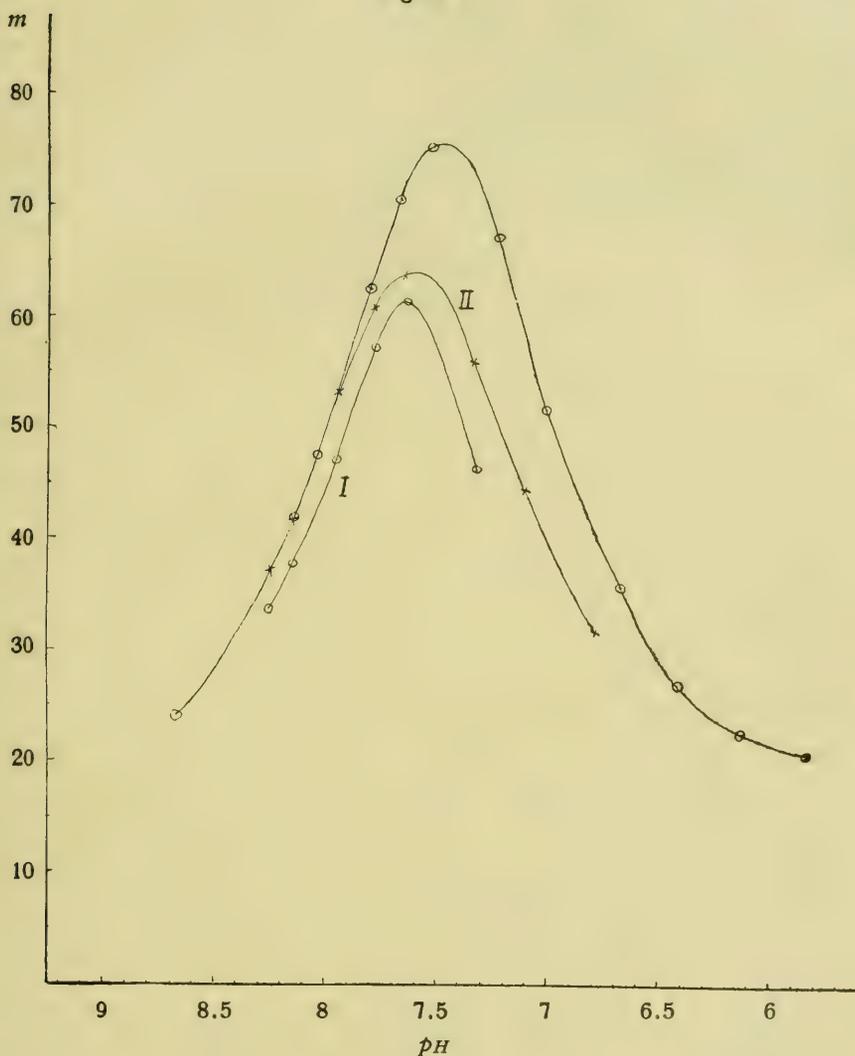
For the sake of comparison the same Soja-meal was used as in the previous experiments. Those beans had been powdered a year ago and the meal had shown already some signs of weakening. Especially in alkaline solution its constancy of activity had been found to be diminished. Therefore in the experiments with high  $p_H$  the  $m$  of the first stage had to be taken rather than the mean. And this value was multiplied by the ratio of the activities at the same  $p_H$ , found formerly and now.

The values, obtained in this way for  $m$  in the alcohol solution are plotted in figure 6 I, in which for the sake of comparison the original curve for  $m$  of fig. 3 is also reproduced.

<sup>1)</sup> Zeitschr. physik. Chem. 1896, 20, 283.

The third curve II of this figure has the following meaning:

Figure 6.



According to our interpretation of the rise of  $p_H$  of 0,10 by 5% alcohol, the  $k_w$  had increased 0,20, or the  $p_{OH}$  0,10.

As demonstrated in part 5 of this paper  $m$  as a function of  $p_H$  is:

$$m = 18 + \frac{4828}{1 + \frac{132,6 \times 10^{-8}}{(H)} + \frac{2170 \times 10^{-8}}{k_w}(H)}$$

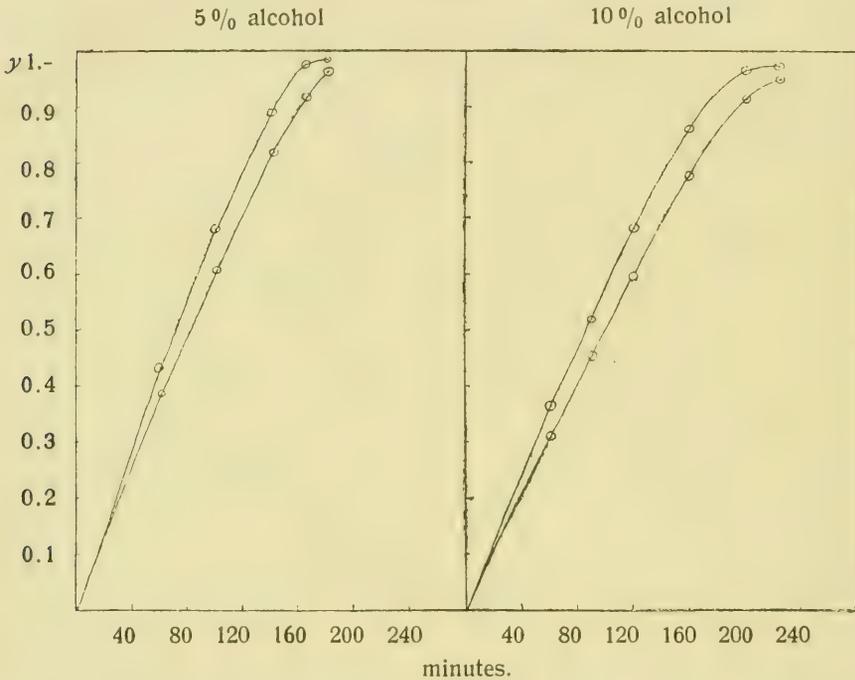
The  $k_w$  being diminished by the alcohol from  $10^{-13,78}$  to  $10^{-13,98}$ , it was now possible to calculate a new curve II for  $m$  by means of this formula. The difference between I and II may be due (at

least partly), to the experimental errors, which were very large with the old Soja-meal.

In an earlier stage of these researches the influence of neutral substances like alcohol on urease-action had been studied with the aid of the ammonium-carbonate + carbonic acid as a buffer, in experiments like those described in part 6.

For instance the experiment of May 25<sup>th</sup> 1916, in which 0,5 % of urea in ammonium carbonate (= 2 % urea) with and without 5 % of alcohol had been investigated, produced the results, represent-

Figure 7.



ed in figure 7. The mean of  $m$  without alcohol was 0,00381, with 5 % alcohol 0,00335. For the calculation of  $m$  it was assumed, that the  $p_H$  had been elevated by 5 % alcohol in this case by the same value as found above in the phosphate solutions.

From figure 6 it will be seen, that between  $p_H=7$  on the original curve and  $p_H=7,1$  on curve II, theory would predict a decrease in  $m$  from 51,3 to 44,5, that is from 1 to 0,87.

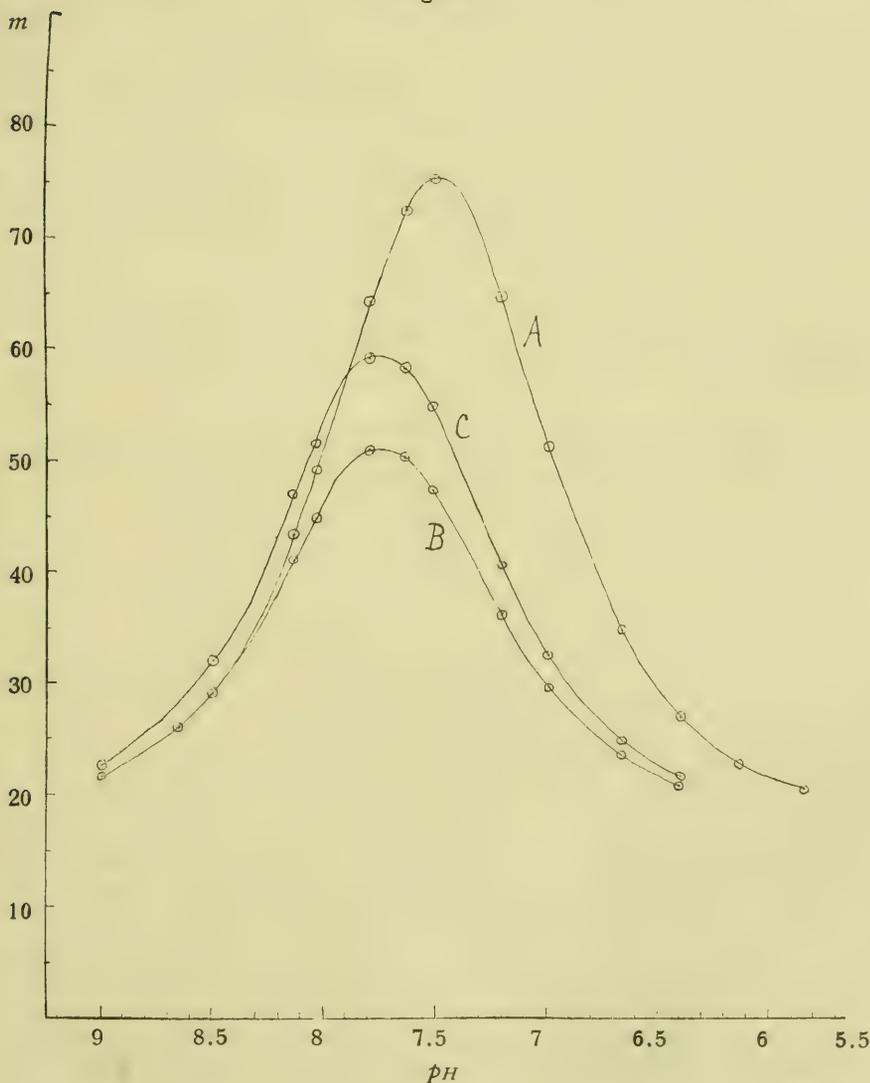
According to the above results a decrease is found from 38,1 to 33,5, that is from 1 to 0,88.

Closer consideration as well as further experimental research showed however, that the problem is more complicated than represented so far.

Alcohol not only diminishes the dissociation constant of water by diminishing the concentration  $C_{H_2O}$  in  $C_H C_{OH} = k C_{H_2O}$ , but, owing to its ionising power being far less than that of water, it will also diminish the dissociation constant of a dissolved electrolyte. Therefore the dissociation constants of the amphoteric electrolyte urease must be expected to be also diminished by alcohol.

The effects of the first influence of alcohol are, as shown in fig. 6, a lowering of the curve and a displacement of the maximum to the left; that of the second one, the diminishing of  $k_a$  and  $k_b$  at the same rate, is evidently an increase of  $m$ , the undissociated fraction, without changing the abscissa of the maximum.

Figure 8.



What we observe is therefore the superposition of a positive and negative effect, the decrease of the dissociation constant of water predominating in this case.

Repetition of the experiment of May 25<sup>th</sup> 1916, but now with 10% alcohol, gave on May 30<sup>th</sup> the result, summarised in figure 7. Without alcohol was found  $m = 0,00326$ , with 10% alcohol  $m = 0,00267$ .

The electrometric estimation of the  $p_H$  in 8% phosphate solutions with 10% alcohol had given:

with 10% alcohol	7,13	7,53
without alcohol	6,89	7,29
difference	0,24	0,24

Thus a constant rise of 0,24, which means a change in  $k_w$  from  $10^{-13,78}$  to  $10^{-14,26}$ .

Figure 8 shows, together with the original curve of fig. 1 the curve  $B$ , calculated for

$$m = .18 + \frac{4828}{1 + \frac{123.6 \times 10^{-8}}{(H)} + \frac{2170 \times 10^{-8}}{k_w}(H)}$$

with the value  $10^{-14,26}$  for  $k_w$ .

The results of this calculation are represented by column 3 of the following table.

TABLE 19.

$p_H$	$10^3 C_{OH}$	$m$	$m$ (for $k_a$ and $k_b$ 20% smaller)
8.13	74.13	41	46.8
8.03	58.88	44.8	51.4
7.80	34.67	50.8	59.—
7.64	23.99	50.3	58.3
7.52	18.20	47.4	54.7
7.21	8.90	36.1	40.7
7.—	5.495	29.8	32.7
6.67	2.57	23.7	25.—
6.40	1.38	21.—	21.8
8.5	173.8	29.2	32.—
9.—	1660.—	21.6	22.5

This table and curve *B* of Fig. 8 would predict a decrease in  $m$  from 51.3 for  $p_H = 7$  to 37.1 for  $p_H = 7.24$ , that is from 1 to 0.725, whereas according to table 42 experiment produced a decrease from 1 to 0.82.

The interpretation of this difference is, that the  $k_a$  and  $k_b$  are also appreciably diminished by 10% alcohol, that therefore the "rest-curve" of urease in this solution has to be placed above the curve *B* of Fig. 8.

The curve *C* in Fig. 8 and the fourth column in table 19 represent the values which  $m$  would have, if both  $k_a$  and  $k_b$  had increased by 20%,  $k_w$  having, as in curve *B*, the value  $10^{-14.26}$ .

Attention may here be drawn to the fact, that if a neutral body like alcohol, diminishes the dissociation constants of urease as well as of water, the new curve for  $m$  intersects the original one, which means, that in such a case a considerable decrease of activity is to be expected at low  $p_H$  (in the present case below  $p_H = 7.85$ ) and by the same agency, the addition of alcohol, an increase at high  $p_H$ .

The influence of neutral salts may be illustrated by the example of 5% KCl.

Here, as in all previous cases, 5% means 5 gram in 100 c.c. of the reacting solution.

In 8% phosphate solutions the  $p_H$  was found not to be raised, but to be diminished in a constant degree by 5% KCl.

	$p_H$		
without KCl	6.13	6.89	7.29
with 5% KCl	6.00	6.79	7.19
difference	0.13	0.10	0.10

Nevertheless in the ammonium carbonate solution KCl had produced a considerable decrease of urease activity. Without KCl  $m = 0.0031$ , with 5% KCl  $m = 0.0022$ .

The conclusion, to be drawn from these facts, is, that a neutral salt like KCl increases all dissociation constants, that of urease as well as that of water.

We thus have here again and in a marked degree, a change in the nature of urease, a general decrease in its activity, brought about by an increase of its dissociation-constants.

The raising of the dissociation-constant of water by salts had already been observed in this study in the determination of the hydroxyl-ion concentration in 8% phosphate solutions, the dissociation-constant of water having been found to be  $10^{-13.78}$ , instead of  $10^{-13.825}$ , as it is in pure water.

Moreover, the stimulating action of neutral salts on the catalytic activity of hydrogen-ions has long since been attributed to a raising of the dissociation constants of acids by salts.

Another instance of such an effect has recently been supplied by KOLTHOFF<sup>1)</sup>, who showed, that the dissociation-constants of indicators are increased by the addition of neutral salts.

Mannitol showed the same behaviour as potassium chloride.

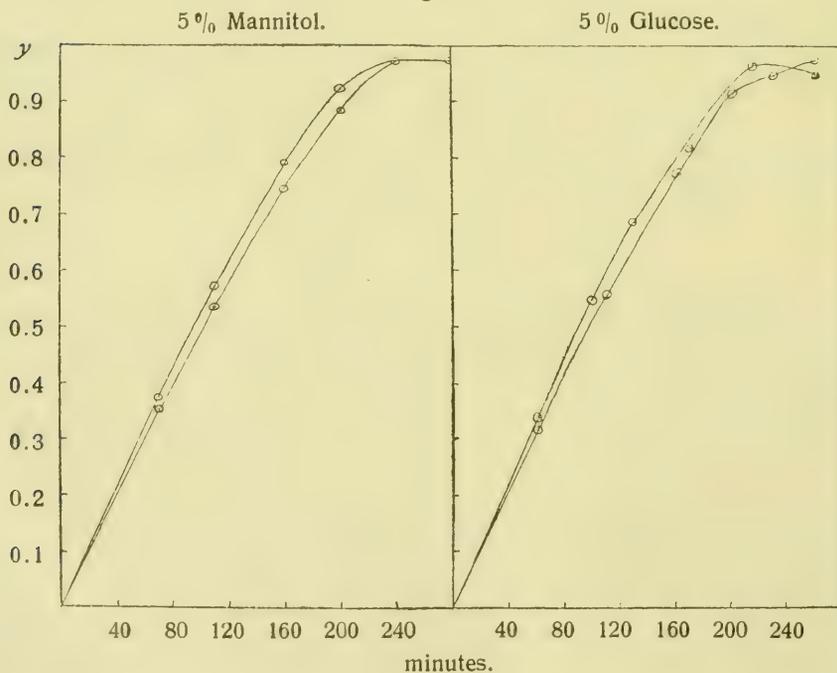
In 8% phosphate solutions the following values for  $p_H$  were determined:

without mannitol	6.89	7.29
with 5% mannitol	6.84	7.24
difference	0.05	0.05

Hence again a constant decrease of  $p_H$ , that is an increase of the dissociation-constant of water.

In ammonium carbonate solution 5% mannitol produced a small decrease of urease activity ( $m$  decreased from 0.0029 to 0.00275), from which it is to be inferred, that the dissociation-constants of urease are increased by this neutral substance also.

Figure 9.



Glucose, added to the amount of 5% to ammonium-carbonate

<sup>1)</sup> Chem. Weekblad 1918, 394.

solution, gave only rise to a very slight decrease of urease activity. The cases of mannitol and glucose are illustrated by fig. 9.

It will be seen, that the general nature of the curves is not changed by the addition of neutral substances, from which it can be inferred, that only the  $m$  in our fundamental formula

$$- dx = m \frac{x}{x + nc} dt$$

is affected.

If other substances absorbed the radiation like the hydrogen-ions, the shape of the curves would have been altered, tending with progressive change to the logarithmic curve.

The facts recorded above and the theoretical deductions might afford some explanation of at least a part of ONODERA'S observations<sup>1)</sup>, that alcohols can increase as well as decrease the action of urease. The experiments of this author, however, were made without buffer-mixtures and without any estimations of the  $p_H$ , which in the absence of buffers must have varied enormously. Since the long duration of urease action in those undefined, but certainly rather alkaline, conditions, must have had a deteriorating influence on the urease, the more so, the higher the  $p_H$ , a neat interpretation of these results is rendered impossible.

#### 9. *Reversion of the hydrolytic action of urease on urea.*

The generally accepted view of the synthetic action of enzymes, shared also by the present author, is, that one and the same enzyme is the active agent in breaking down as well as in building up its specific substrate.

The conditions, however, which cause either of the two opposite activities to predominate, have as yet not been made clear.

Some observations, made in the course of this study and a general consideration of the enzyme activity in living tissues, induced the writer to venture the following hypothesis:

Around an enzyme particle the substrate is broken down as long as the action, radiating from the enzyme, is vigorous enough. Weakened by spreading or by other causes, the same enzyme radiation produces the reverse process, synthesis.

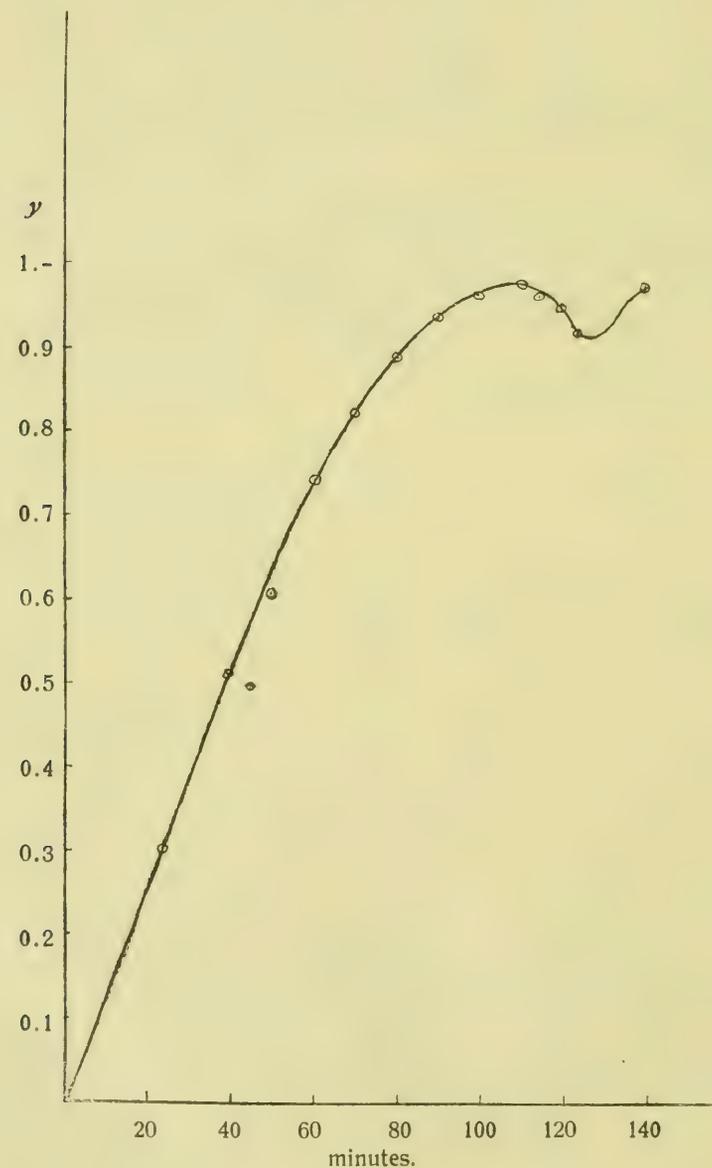
In the living plant and animal both activities of the enzyme are going on continuously under ordinary conditions as to concentration of substrate, temperature and acidity. As soon, however, as the tissue,

<sup>1)</sup> Biochem. J. 1915, 563.

by extraction and pulverisation, is made into a homogeneous mixture or solution, nearly all synthetic power of the enzyme disappears.

As is well known, in order to extract say glucosides from plants, it is necessary first to destroy the enzymes by boiling the seeds or the leaves, before the cells are broken up for extraction. The glucoside and its enzyme are, as demonstrated already by GUIGNARD, located in different adjoining cells. The hypothesis that enzymes display their synthetic effect only at a certain distance, affords at once an explanation of these facts. Through the cellwalls the radiation

Figure 10.



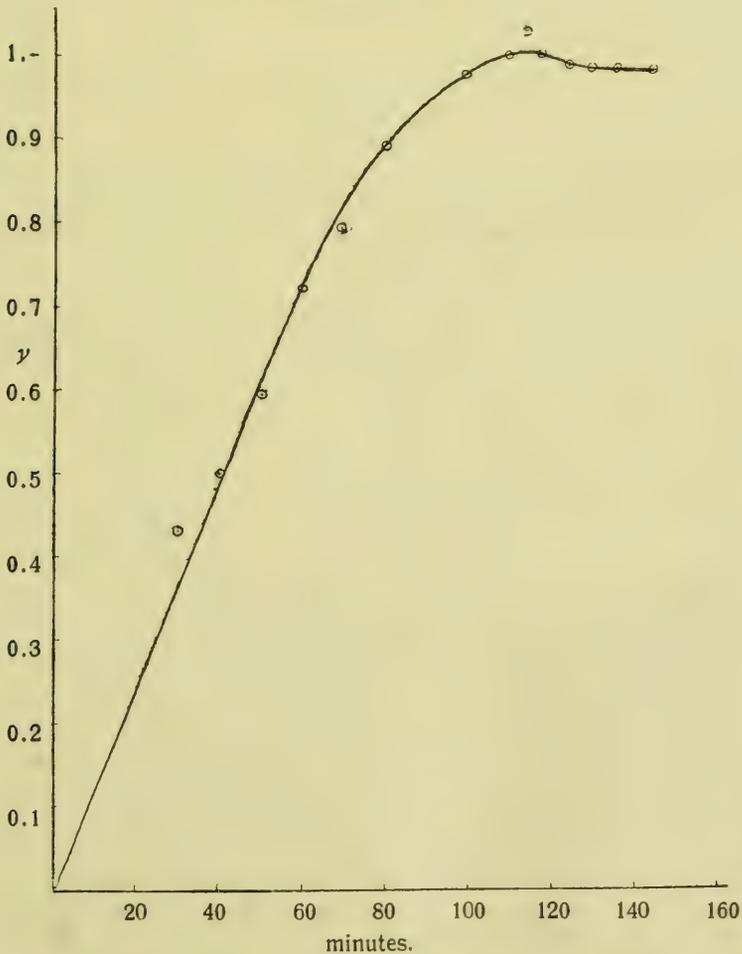
causes synthesis in the neighbouring cells; as soon as enzyme and substrate are brought together, the hydrolytic action prevails almost exclusively.

This conception of enzyme activity is borne out by experiment in different ways.

The weakening of the radiation may be occasioned, as observed above, not only by spreading, but also by other factors, e.g. by the deteriorating action of temperature and acids or alkalis in a shorter or longer time. As will be seen, in what follows, several times a reversion of the hydrolysis of urea was observed, which could be explained by a decay of the urease.

Nov. 18<sup>th</sup> 1915 an experiment was carried out with ammonium-carbonate as a buffer, in which this reversion at the end of the hydrolysis manifested itself clearly. See figure 10, p. 10.

Figure 11.



It is to be expected, that the enzyme molecules will not all decay at the same time. At the end of the hydrolysis, when the action on the rest of the urea has become very slow, and on the other hand the concentration of the reaction products has grown correspondingly, it may happen, that a sufficient proportion of the enzyme particles is in the decaying state to cause the synthetic action, having a higher concentration of substrate at its disposal, to predominate over the slow action of the unchanged enzyme on the relatively very diluted urea solution. During such a period the total effect will be reversed, returning to its normal course as soon as the unchanged enzyme particles prevail over the decaying ones, which may be occasioned by a further weakening or total decay.

Figure 11 shows the results of an experiment of Jan. 2<sup>nd</sup> 1916, like that of Nov. 28<sup>th</sup>, 1915, which shows the same phenomenon in a smaller degree.

In the phosphate solutions of high  $p_H$  also the reversion sometimes became manifest.

First a regular experiment of April 6<sup>th</sup>, 1917 may be recorded.

TABLE 20.

0.25 gr. Soja + 50 c.c. water + 3.64 gr.  $Na_2HPO_4$  2 aq. + 1.16 gr.  $KH_2PO_4$   
 10 c.c. filtrate mixed with 11.52 gr.  $Na_2HPO_4$  2 aq. + 120 c.c. water.  
 $p_H = 8.13$  0.01 % urea

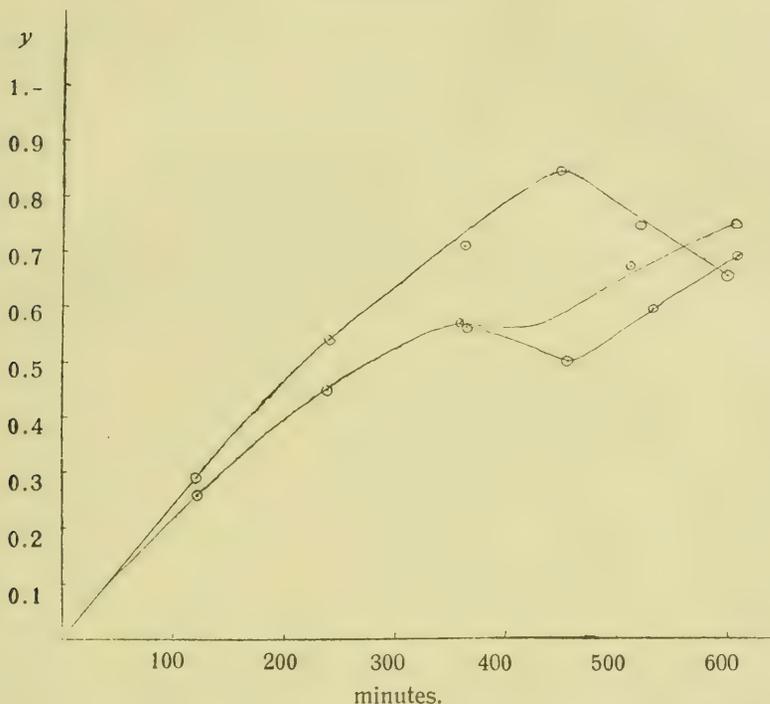
$t$ (minutes)	$y$	$m = \frac{0.0073 \log \frac{1}{1-y} + 0.01y}{t}$
80	0.17	0.000029
140	0.305	0.000030
200	0.39	0.000027
260	0.52	0.000028
321	0.625	0.000030

In this case little or no destruction had taken place during this interval; for 10 c.c. of the extract, left alone for 320 minutes in the same bath at 27° and then mixed with 2 c.c. urea (0.06 %) gave in 80 minutes 0.3 c.c.  $NH_3 \frac{1}{50}$  N, which makes  $m = 0.000026$ .

April 10<sup>th</sup> 1917. The same arrangement as on April 6<sup>th</sup> but now with longer time intervals. Part of the same mixture left in the bath at 27° till the next day, when 10 c.c. with 2 c.c. urea solution of 0.06 % gave in 260 minutes only  $y = 0.1$  (against  $y$  about 0.45 originally). Thus evidently a considerable decay of the enzyme had taken place.

April 16<sup>th</sup> and 23<sup>rd</sup> 1917. A repetition of the experiment of April 10<sup>th</sup>. The results of these three experiments are represented

Figure 12.



in fig. 12. As will be seen clearly in the curves of April 10<sup>th</sup> and 27<sup>th</sup>, there is first a decline, after which the reaction proceeds normally again. As discussed above the inference of this is, that during some time interval the decaying enzyme particles, which cause the synthesis, may prevail, and that after their total decay (or perhaps restoration) the normal enzyme action sets in again.

In order to test this conception more directly the following experiment was carried out.

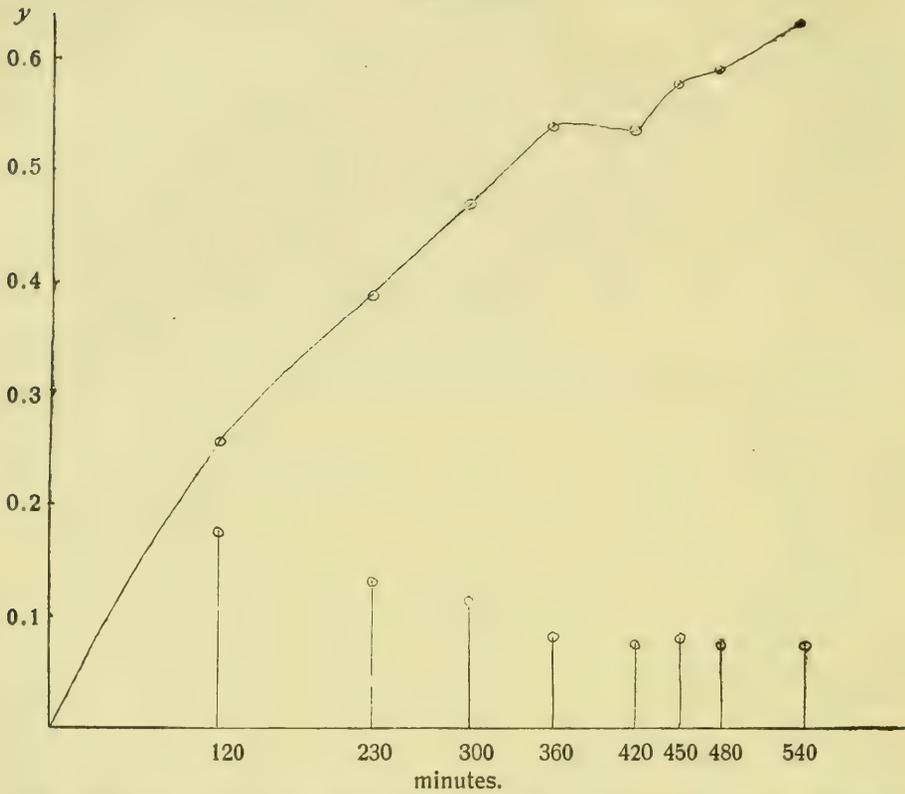
In one row of tubes the rate of hydrolysis of an 0.01% urea solution in 8% phosphate of  $p_H = 8.13$  was investigated as before.

Another row of tubes, each also with 10 c.c. of the same Soja-extract-phosphate-solution had been placed with all the other tubes at the same time in the same bath of 27°. Immediately after taking out a tube of the first row for analysis, 2 c.c. urea solution of 0.06% were introduced into a tube of the second row and the reaction in this one allowed to proceed for 120 minutes. In this way it was possible to get some information about the rate of decay of the enzyme.

From Fig. 13 it will be seen, that the nearly horizontal part of

the curve coincides with the  $T$ 's of diminishing activity according to table 21. Between  $T=120$  and  $T=230$  there seems to have

Figure 13.



been in the beginning another period of some decay, followed by a normal course from  $T=230$  to  $T=360$ .

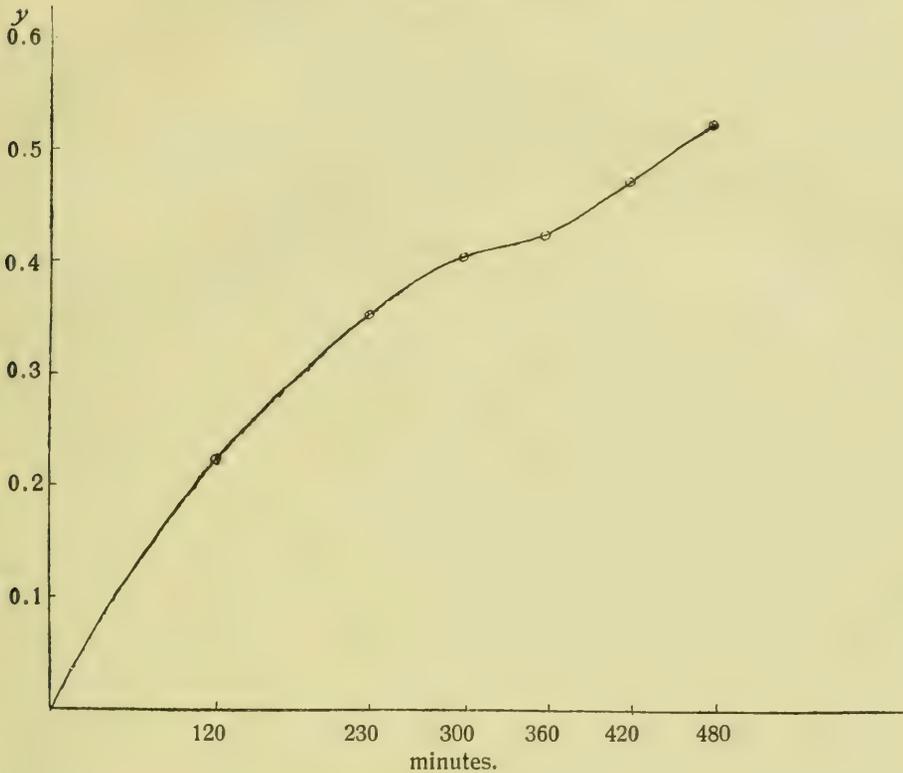
This peculiar type of curve, first a retardation and then again

TABLE 21.

$T$ (minutes)	$y$	$y$ (in 120 minutes after the time $T$ )
120	0.255	0.175
230	0.385	0.130
300	0.465	0.115
360	0.535	0.08
420	0.53	0.075
450	0.575	0.08
480	0.585	0.075
540	0.625	0.075

a normal slope, was observed more than once on repeating the same experiment at  $p_H = 8.13$ . For instance on July 19th., 1917. Fig. 14.

Figure 14.



#### 10. *Influence of the concentration of enzyme.*

In continuing the investigations of part 4 with smaller enzyme-concentrations it was first observed, that in the same conditions the same quantity of Soja-meal did not always give the same result. In preparing the necessary solutions the distilled water had been freed from carbonic acid by boiling in a tin coated copper flask. After this method of purifying the distilled water had been replaced by distilling the water once more in a Jena flask and glass condenser, rejecting the first part of the distillate, more regularity seemed to be obtained. Still, weighed off in such small amounts, the Soja-meal appeared to be not quite homogeneous. An explanation of this fact will be brought forward further on.

Nevertheless important new results could be obtained in this way.

For the sake of the necessary constancy of  $p_H$  the urea-concentration in these experiments was again 0,01 %.

Taking the same unit as before the concentration of the urease was now  $\frac{3}{26}$ .

Twice also a series of four experiments with different  $p_H$  was made simultaneously with the same Soja extract to make sure, that in all the same quantity of enzyme had been brought into solution.

Evidently the smallness of enzyme concentration involved the necessity of long reaction-time.

In order to investigate, if during such long exposures to these H-ion concentrations at  $27^\circ$  any weakening of the enzyme activity had taken place, the remaining parts of the four phosphate-enzyme-solutions were left for 24 hours in the bath at  $27^\circ$ . On the next day the required experiments were repeated with these ones.

In this way it became manifest, that the combined action of H-ion concentration, temperature and time had here, within the limits of experimental errors, no appreciable effect on urease in the acid solutions. If, however, at  $27^\circ$  the H-ion concentration was maintained beyond the neutral point, the activity of the enzyme decreased slowly, but distinctly. In alkaline solutions for the calculation of  $m$ , as before, only the measurements of the first intervals had to be used.

The acidity, used in these experiments, had itself no hydrolytic action on urea, as was ascertained by boiling samples of 10 c.c. of the liquids with  $p_H = 6,67$  and  $p_H = 7,52$ , incubating them for 24 hours at  $27^\circ$ , adding to each 2 c.c. of 0,06 % urea and maintaining these mixtures again for 24 hours at  $27^\circ$ . No ammonia was formed.

The results obtained for the specific urease-activity  $m$  of low urease concentrations are brought together in fig. 15, in which for the sake of comparison the curve of fig. 3 is also reproduced.

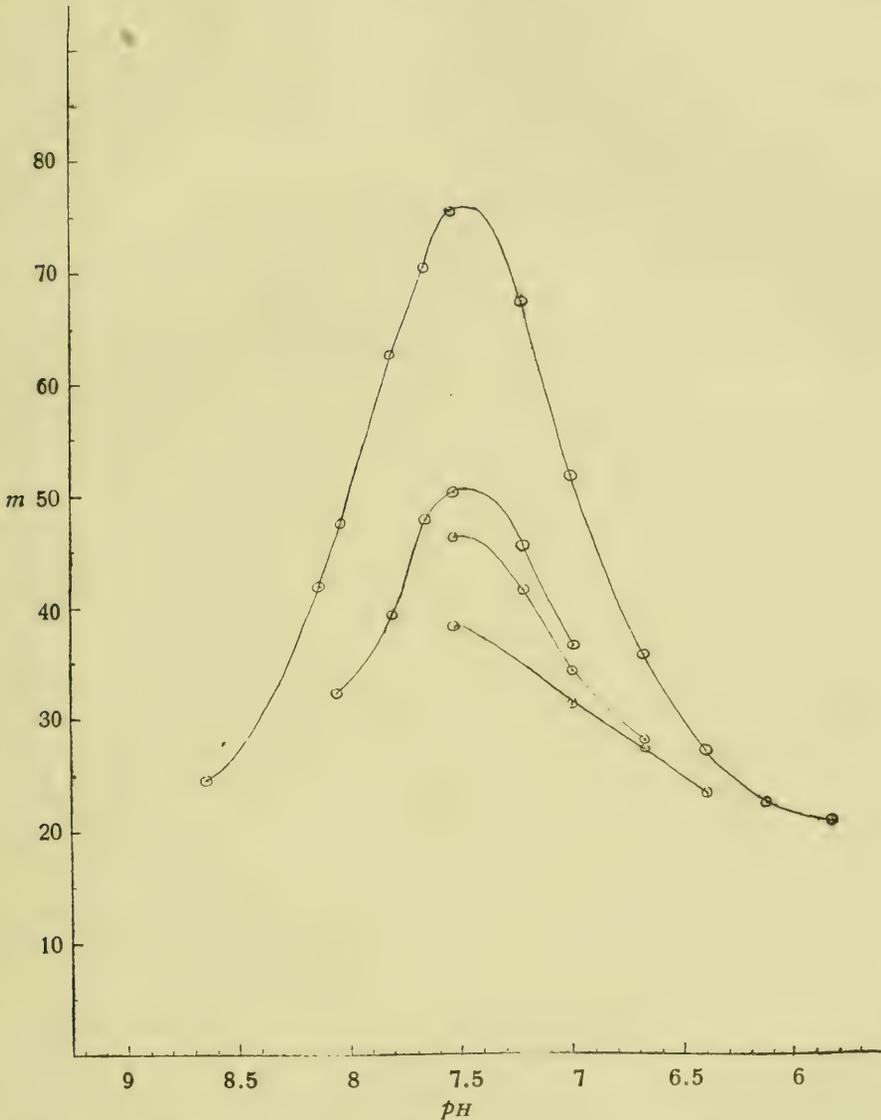
The interpretation of these results, based on the theory, put forward in part 9 is the following.

An enzyme molecule being the centre of two concentric spheres, an inner one, in which the substrate is broken down, and an outer shell, in which the reverse action is brought about, it is evident, that, if the concentration of the enzyme is large enough to make the inner spheres intersect each other, there is no place in the solution, where the condition for synthesis, that is weakened radiation, is realised; at least, so long as the enzyme has retained its original activity.

A sufficient dilution of the urease will, however, establish a distance between the enzyme-molecules, large enough, to leave the outer shell open for the reverse action. The total power of urease to convert urea into ammonium carbonate, representing the difference between the hydrolysis and the synthesis it brings about, must therefore be expected to be diminished by decreasing the urease concentration beyond a certain value.

The observed facts are clearly in accordance with this theory:

Figure 15.



above a certain value of urease concentration constancy of specific activity, below it a decrease.

The peculiarity, mentioned in the beginning of this part, that small concentrations of urease showed less constancy of power than larger ones, becomes now not only intelligible, but even adduces evidence to the theory.

For, evidently, in regions, where the specific activity is falling off, a slight change in concentration of the urease will have far more effect than at higher concentrations, where the specific activity is constant.

*To be continued.*

**Mathematics.** — “A general definition of limit with application to limit-theorems”<sup>1)</sup>. By Prof. FRED. SCHUH. (Communicated by Prof. J. CARDINAAL).

(Communicated in the meeting of May 3, 1919).

1. Let us assume an aggregate  $V$  of real or complex numbers, in which the same number may occur repeatedly. This can take place, if a mode of arising of the numbers of  $V$  has been given and various modes of arising may lead to the same number. Those equal numbers however, are considered as different elements of  $V$ , so that we distinguish between a number having arisen in the former and the same number having arisen in the latter manner.

2. Next we assume a law given by which every positive number  $\delta$  is made to correspond to a part  $V_\delta$  (consisting of at least one element) of  $V$  (covering of the aggregate of the positive numbers by the aggregate of the parts of  $V$ ), in this way, that  $V_{\delta'}$  is a part of  $V_\delta$ , if  $\delta' < \delta$ ; here  $V_\delta$  is called a part of  $V$  if each element of  $V_\delta$  is an element of  $V$ , so that the part can be identical with the whole aggregate. If now  $L$  is a (real or complex) number with the property that corresponding to every positive number  $\delta$  there exists such a positive number  $\varepsilon$  that every element  $E$  of  $V_\delta$  satisfies the inequality  $|E-L| < \varepsilon$ , then  $L$  is called the LIMIT of the aggregate  $V$  as regards the covering by which  $V_\delta$  is made to correspond to  $\delta$ . It is clear that there can be at most but one number with the before-mentioned property.

3. The covering observed in n<sup>o</sup>. 2 we call EQUIVALENT to a second covering by which a positive number  $\delta$  is made to correspond to a part  $V'_\delta$  of  $V$ , if corresponding to every positive number  $\delta$  a positive number  $\delta_1$  can be found so that  $V'_{\delta_1}$  is a part of  $V_\delta$  and  $V_\delta$  is a part of  $V'_\delta$ . It is evident that this equivalence is a transitive one. It furthermore easily appears that two equivalent coverings both give the same limit, or no limit exists in the two cases. Hence equivalent coverings can be considered as the same limittransition, so that with a MANNER OF LIMITTRANSITION we mean a set of mutually equivalent coverings of the kind mentioned in n<sup>o</sup>. 2.

4. The limit defined in n<sup>o</sup>. 2 exists, and exists only then, if corre-

<sup>1)</sup> For further particulars c.f. Hand. van het Nat. en Geneeskundig Congres 1919.

sponding to every positive number  $\varepsilon$  a positive number  $\delta$  can be found so that every pair of elements  $E$  and  $E'$  of  $V_\delta$  satisfies the inequality  $|E - E'| < \varepsilon$  (GENERAL PRINCIPLE OF CONVERGENCE). That this condition is necessary for the existence of a limit is obvious.

That this condition moreover is sufficient, appears in the case of real numbers by noticing that (the condition being fulfilled) a number satisfying the definition of a limit is furnished by the upper boundary of the numbers  $a$ , for which there exists a  $V_\delta$ , all elements of which are  $> a$ . In the case of complex numbers the theorem is further proved by applying the theorem for real numbers to the real parts and the coefficients of  $i$  of the complex numbers.

5. Let  $V$  and  $W$  be two aggregates of numbers, the elements of which being placed into correspondence. We suppose the coverings of the positive numbers by the aggregates of the parts of  $V$  resp.  $W$  to be of such a nature, that for each positive number  $\delta$  the parts  $V_\delta$  and  $W_\delta$  in the correspondence between  $V$  and  $W$  are corresponding ones.

We now form an aggregate  $U$  by adding the corresponding elements of  $V$  and  $W$ , at the same time transferring the covering to  $U$ . If now with these coverings  $V$  shows a limit  $L_v$  and  $W$  a limit  $L_w$ , then also  $U$  has a limit, viz.  $L_v + L_w$ , as may easily be deduced from the definition of limit.

Other known limit-theorems also can be stated in this manner in a general way.

6. We now suppose, that the elements of the aggregate  $V$  are real numbers. About the existence however of a limit of  $V$ , as regards the chosen covering, nothing is assumed to be known. We can then consider the lower boundary of the upper boundaries of the aggregates  $V_\delta$ , and call it the UPPER LIMIT of the aggregate  $V$  as regards the considered covering. The upper limit  $B$  is  $+\infty$ , if all aggregates  $V_\delta$  are unbounded to the right, and  $-\infty$ , if the aggregate of the upper boundaries of the aggregates  $V_\delta$  is unbounded to the left.

We likewise call the upper boundary  $O$  of the lower boundaries of the aggregates  $V_\delta$  the LOWER LIMIT of  $V$  as regards the considered covering; this lower limit can also be  $\pm\infty$ . It is easily proved, that  $B$  and  $O$  always satisfy the inequality  $O \leq B$ .

The aggregate  $V$  has a limit  $L$  then only, when  $O$  and  $B$  are equal and finite; we then have  $L = O = B$ . If  $O$  and  $B$  are both  $+\infty$ , we speak of an IMPROPER LIMIT  $+\infty$ , if  $O$  and  $B$  are both  $-\infty$ , of an IMPROPER LIMIT  $-\infty$ .

**Mathematics.** — “*A general definition of uniform convergence with application to the commutativity of limits*”. By Prof. FRED. SCHUH.  
(Communicated by Prof. CARDINAAL).

(Communicated in the meeting of May 3, 1919).

1. Let us assume two aggregates  $V$  and  $W$  with coverings of the positive numbers by the parts of  $V$  resp.  $W$ , as described in my paper “A general definition of limit”, p. 46. By these coverings to a positive number  $\delta$  a part  $V_\delta$  of  $V$  and a part  $W_\delta$  of  $W$  is made to correspond in such a way, that  $V_\delta$  and  $W_\delta$  contain at least one element,  $V_{\delta'}$  being a part of  $V_\delta$  and  $W_{\delta'}$  a part of  $W_\delta$  if  $\delta' < \delta$ .

Let  $VW$  be the productaggregate, whose elements are formed by combining an element of  $V$  with an element of  $W$  into a pair (disregarding sequence). The covering by the parts of  $VW$  is taken such, that the aggregate  $V_\delta W_\delta$  corresponds to  $\delta$ .

If we replace the coverings belonging to  $V$  and  $W$  by equivalent ones (c.f. n°. 3 of my previous paper) the covering belonging to  $VW$  is likewise replaced by one equivalent to it.

2. We assume  $VW$  covered with (real or complex) numbers, so that every element of  $VW$  is made to correspond to a number. From this arises an aggregate  $K$  of numbers, in which the same number may occur several times. We shall indicate by  $G$  an aggregate of elements of  $K$ , corresponding to a same element of  $V$ , combined with all elements of  $W$ . There likewise arises an aggregate  $H$  by considering a constant element of  $W$ .

The covering by the parts of  $VW$  can be transferred to the aggregate  $K$ . Likewise the covering by the parts of  $V$  or  $W$  is to be transferred to every aggregate  $H$  resp.  $G$ .

3. Suppose that every aggregate  $G$  (as regards the covering by the parts of  $W$ ) possesses a limit  $L_G$ ; then corresponding to every element of  $V$  there exists a number  $L_G$ . We say, that the aggregates  $G$  converge UNIFORMLY to their limits  $L_G$ , if corresponding to every positive number  $\varepsilon$  there exists such a positive number  $\delta$ , that each element  $E$  of the part  $G_\delta$  of  $G$  satisfies the inequality  $|E - L_G| < \varepsilon$ , whichever element of  $V$  (hence whichever aggregate  $G$ ) we may choose; it being required, that  $\delta$  is independent of the chosen element of  $V$ .

From the general principle of convergence (c.f. n<sup>o</sup>. 4 of my previous paper) it follows, that uniform convergence exists then and then only, if corresponding to every positive number  $\varepsilon$  a positive number  $\delta$  independent of  $G$  can be found, so that every two elements  $E$  and  $E'$  of  $G_\delta$  satisfy the inequality  $|E - E'| < \varepsilon$ .

4. Let it now be assumed, that the uniform convergence mentioned in n<sup>o</sup>. 3 does exist and moreover, that every aggregate  $H$  has a limit  $L_H$  as regards the covering by the parts of  $V$ ; in consequence of this there is a number  $L_H$  corresponding to every element of  $W$ .

We now first demonstrate, that the aggregate  $\{L_G\}$  of the numbers  $L_G$  possesses a limit  $L$  as regards the covering by the parts of  $V$ . For this purpose we must prove, that corresponding to every positive number  $\varepsilon$  such a  $V_\delta$  can be found, that every two elements of  $V_\delta$  satisfy the inequality  $|L_G - L_{G'}| < \varepsilon$ , where  $L_G$  and  $L_{G'}$  are the numbers of  $\{L_G\}$  corresponding to these elements. If  $E$  and  $E'$  are elements of  $G$  resp.  $G'$ , belonging to a same aggregate  $H$ , we have:

$$|L_G - L_{G'}| \leq |L_G - E| + |E - E'| + |E' - L_{G'}|.$$

On account of the uniform convergence of the aggregates  $G$  the aggregate  $H$  can be chosen thus, that  $|L_G - E|$  and  $|E' - L_{G'}|$  are both  $< \frac{1}{3}\varepsilon$ . As  $H$  possesses a limit, we can choose  $\delta$  thus, that every two elements  $E$  and  $E'$  of  $H_\delta$  satisfy the inequality  $|E - E'| < \frac{1}{3}\varepsilon$ . For the numbers  $L_G$  and  $L_{G'}$  corresponding to two elements of  $V_\delta$  then  $|L_G - L_{G'}| < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$  holds good.

5. We next prove, that the aggregate  $\{L_H\}$  of the numbers  $L_H$  (as regards the covering by the parts of  $W$ ) likewise has the limit  $L$ . We therefore proceed from

$$|L_H - L| \leq |L_H - E| + |E - L_G| + |L_G - L|,$$

in which  $E$  is the common element of the aggregates  $G$  and  $H$ . On account of the uniform convergence mentioned in n<sup>o</sup>. 3 we can determine the positive number  $\delta$  in such a way, that  $|E - L_G| < \frac{1}{3}\varepsilon$  holds good if  $E$  be an element of  $G_\delta$ , i.e. if  $H$  corresponds to an element of  $W_\delta$ ; here  $\varepsilon$  is an arbitrarily chosen positive number. If we have now chosen a determinate element of  $W_\delta$  (by which also  $H$  and  $L_H$  are determined), we can find the positive number  $\delta_1$  in such a way, that the inequality  $|L_H - E| < \frac{1}{3}\varepsilon$  is satisfied, if  $E$  be an element of  $H_{\delta_1}$ , i.e. corresponds to an element of  $V_{\delta_1}$ . Further we can determine  $\delta_2$  thus, that we get  $|L_G - L| < \frac{1}{3}\varepsilon$ , if  $G$  corresponds to an element of  $V_{\delta_2}$ . By making the aggregate  $G$  to

correspond to a common element of  $V_{\delta_1}$  and  $V_{\delta_2}$ , we obtain  $|L_H - L| < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$  for every aggregate  $H$  corresponding to an element of  $W_\delta$ .

6. Finally we prove, that the aggregate  $K$  (as regards the covering by the parts of  $VW$ ) likewise has the limit  $L$ . We obtain:

$$|E - L| \leq |E - L_G| + |L_G - L|,$$

$E$  being an element of  $G$ . Owing to the uniform convergence the positive number  $\sigma_1$  can be determined so, as to give  $|E - L_G| < \frac{1}{2}\varepsilon$  if  $E$  be an element of  $G_{\sigma_1}$ , i. e. corresponds to an element of  $W_{\sigma_1}$ . At the same time we can  $\sigma_2$  determine so, as to obtain  $|L_G - L| < \frac{1}{2}\varepsilon$  if  $G$  corresponds to an element of  $V_{\sigma_2}$ . If now  $\sigma$  be the smallest of the numbers  $\sigma_1$  and  $\sigma_2$ , then the inequality  $|E - L| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$  is satisfied, if  $E$  corresponds to an element of  $V_\sigma$  and to an element of  $W_\sigma$  (hence to an element of  $V_\sigma W_\sigma$ ), that is if  $E$  be an element of  $K_\sigma$ .

7. Summarizing we observe:

*If the productaggregate  $VW$  has been covered with numbers and if the aggregates  $G$  of those numbers corresponding to a same element of  $V$  converge uniformly to their limits  $L_G$  (as regards the covering of the positive numbers by the parts of  $W$ ), the aggregates  $H$  of the numbers corresponding to a same element of  $W$  having the limits  $L_H$  (as regards the covering by the parts of  $V$ ), then the aggregate of the numbers  $L_G$  (as regards the covering by the parts of  $V$ ) has a limit, which is at the same time the limit of the numbers  $L_H$  (as regards the covering by the parts of  $W$ ) and is the limit of the whole aggregate of numbers (as regards the covering by the parts of  $VW$ ).*

The equality of the two first-mentioned limits means *commutativity of two limittransitions*. One of the limittransitions relates to the covering by the parts of  $W$ , the other to the covering by the parts of  $V$ .

**Chemistry.** — “*On the Condensation of Formaldehyde with some Unsaturated Compounds.*” By H. J. PRINS. (Communicated by Prof. J. BÖESEKEN).

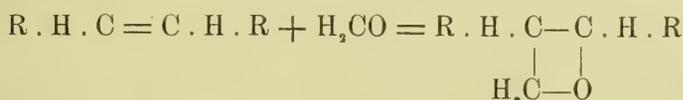
(Communicated in the meeting of May 3, 1919).

Some time ago <sup>1)</sup> the author discussed the mutual condensation of unsaturated compounds, examining the condensation of formaldehyde with styrol, anethol, isosafrol, pinene, d. limonene, camphene and cedrene. The condensation was effected either by warming the components in acetic acid solution, or by sulphuric acid in an aqueous or acetic acid solution. For the aqueous solutions we availed ourselves of the 40% solution as supplied by the trade, the acetic acid solution was obtained by warming trioxymethylene in acetic acid with or without the presence of sulphuric acid.

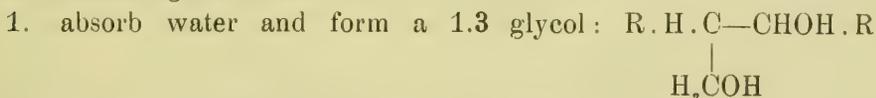
The aqueous and acetic acid solutions of sulphuric acid and formaldehyde behave differently as regards certain compounds, styrol, camphene and cedrene react but extremely slowly or not at all with the aqueous solution whereas anethol and isosafrol do react.

Contrary to isosafrol safrol does not affect an aqueous solution.

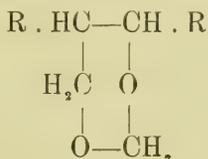
It may be assumed, that the reaction begins with an addition of the CO group to the C = C group, so that primarily a four-ring is formed.



The four-ring can now:



2. absorb formaldehyde and form a methylene ether:



<sup>1)</sup> Chem. Weekbl. 10, 1003. (1913).

ibid. 14, 932. (1917).

3. either directly by isomerisation, or indirectly via the glycol, pass into an unsaturated primary alcohol:  $R \cdot C = CH \cdot R$



If acetic acid is also present, acetates may be formed either of the glycol, or of the unsaturated primary alcohol.

In an aqueous solution we find with anethol and isosafrol a pretty exclusive and almost quantitative formation of the methylene ether.

In an acetic solution in the presence of sulphuric acid both a diacetate and a methylene ether are formed, cedrene and camphene yield in these circumstances an acetate of an unsaturated, primary alcohol.

Camphene, d. limonene and pinene yield in an acetic solution at the boiling-point of acetic acid likewise an acetate of an unsaturated alcohol besides other compounds with a higher boiling-point.

The mutual reaction of formaldehyde, pinene and limonene causing the formation of an unsaturated primary alcohol has already been observed by KRIEWITZ <sup>1)</sup>, who brought about a reaction of the components by heating them in the presence of alcohol at a high temperature.

Prior to this, the formation of a primary, unsaturated alcohol has been ascertained by LADENBURG in the reaction between formaldehyde and a tetrahydropyridine-derivate.

Except with the terpenes the condensation affords good results, ranging from 70% to 90%, of those theoretically possible; with the terpenes various products come into existence probably because the resulting unsaturated alcohol once more absorbs formaldehyde. The terpenes, containing more than one  $C = C$  group, there is besides a prior possibility of absorbing more than one molecule of formaldehyde.

The author likewise observed the setting in of suchlike reactions with anylene, citronellol, methylheptenon, citronellal, undecylenic acid, aethyl-cinnamate.

Experimental part.

Styrol and formaldehyde.

To a mixture of 33 gr. trioxymethylene and 320 gr. glacial acetic acid 32 gr. strong sulphuric acid was added, the trioxymethylene being solved by means of heating, then it was cooled down to 40° and while the mixture was shaken and occasionally refrigerated 104 gr. styrol was added in small quantities. If the temperature is allowed to rise beyond 50°, polymerization of the styrol sets in. The

<sup>1)</sup> Ber. d. deutsch. chem. Ges. 31, 57. (1899).

ibid,

32, 288, 2699 (1898).

mixture is allowed to remain overnight, then it is poured into water, taken up in ether and washed with a soda-solution. After drying and evaporating the ether, the diacetate amount can be determined by saponification. This proved to be 40 %. Then the oil is saponified with rather more than the theoretical quantity of alcoholic potassium and distilled in vacuum. By fractionation we can separate into:

Methylene-ether of 2.phenylpropylglycol. 1.3.

Colourless oil, boiling-point  $128^{\circ}$ — $130^{\circ}$ , pressure 13 mm.  $D_{40}^{180} = 1,1111$ .  $N_d^{180} = 1,53063$ .

Mol. refr. 45.64 (formula of LORENZ—LORENTZ). Calculated 45.87 (atom refr. according to EISENLOHR<sup>1</sup>).

The molecular weight determination in benzene gave: 162 and 156. Calculated for  $C_{10}H_{12}O_2$ : 164.

Analysis: % C determ. 71.6 and 71.1 Calc. for  $C_{10}H_{12}O_2$  73.2.

% H determ. 8.3 and 7.9 Calc. 7.3.

2. Phenylpropylglycol. 1.3.

Colourless, viscous liquid, boiling-point  $176^{\circ}$  pressure, 13 mm.  $D_{40}^{190} = 1,1161$ .  $N_d^{190} = 1,54267$ . Mol. refr. 42.92. Calc. 43.21.

It yields quantitatively a diacetate boiling-point  $162^{\circ}$ — $164^{\circ}$  pressure 13 mm: (chemically bound acetic acid determined by saponification with alcoholic potassium). Molec. weight of the glycol. in benzene 257 and 189. Calcul. 152. Hence it is very strongly associated in this solvent; in a weak solution a normal molecular weight is obtained<sup>2</sup>)

Analysis: found % C 70,6 and 70,9, Calc. C,  $H_{12}$ ,  $O_2$  71,0.

% H 9,2 and 8,9, Calc. 7,9.

Both methylene ether and glycol are saturated as regards a solution of bromine in  $CCl_4$ , but while the methylene ether does not react with a solution of ethylmagnesiumbromide in ether at a normal temperature, the glycol immediately reacts. If toluol is added, raising the temperature to  $100^{\circ}$ , the methylene ether reacts exceedingly violently with the GRIGNARD solution. The fact that the glycol yields a diacetate when boiled with acetic acidanhydride and sodium-acetate is a proof of its two primary alcoholgroups, the isomeric glycol, which might arise from the reaction between formaldehyde and the  $C=C$ -group of the styrol, would be a secondary-primary glycol with the phenylgroup attached to the secondary carbon atom and in the above-mentioned circumstances it would certainly lose water

<sup>1</sup>) Zeitschr. f. phys. Chem. 75, 585. (1910).

<sup>2</sup>) Chem. Weekblad 16, 929. (1919).

and yield the acetate of cinnamic alcohol. The latter could not be found.

Anethol and formaldehyde.

One grammol. anethol is stirred with a solution of two grammol. formaldehyde in 30 % sulphuric acid for three days, the product separated in the usual way yields, when distilled in vacuum:

Methylene ether of p. methoxyphenylbutylglycol.

A colourless, viscous liquid, boiling-point  $168^{\circ}$ — $170^{\circ}$  pressure 13 mm.  $D_{40}^{160} = 1.1197$ ,  $N_d^{160} = 1.53438$ . Mol. refr. 57.78 Cal. 56,74.

It does not react with a GRIGNARD-solution at a normal temperature and does not decolorize a bromine-solution.

Various efforts made to saponify the methylene ether to the correspondent glycol failed, resinification always set in or the ether remained unaltered. By leaving the ether for a fortnight in contact with 85 % formic acid, an oil was obtained, in which such an amount of formic acid was chemically bound as corresponded to 35 % di-formiate; saponification and distillation however, yielded an inconstantly boiling liquid.

Oxydation of the methylene ether.

In order to demonstrate, that with the anethol the formaldehyde had really reacted with the  $C=C$  group from the side-chain, the methylene ether was oxydized with potassium-permanganate. An acid was obtained with a meltingpoint of  $182^{\circ}$ — $183^{\circ}$ , and which revealed no depression when mixed with anisic acid.

Hence it appears, that in these circumstances formaldehyde does not react with the benzene nucleus.

Analysis of the methylene ether:

Found % C 70,1, 68,2 and 69,2. Calc. for  $C_{11}H_{16}O_2$  69,2.

% H 7,4, 7,8 Calc. 7,7.

Mol. weight in benzene: found 203 and 208. Calc. as  $C_{12}H_{16}O_2$ : 208.

Isosafrol and formaldehyde.

The condensation is effected in the same way as with the anethol.

It gives:

Methylene ether of 3,4 dioxymethylenephénylbutylglycol.

Colourless, viscous liquid, boiling-point  $182^{\circ}$ — $184^{\circ}$  pressure 13 mm.

$D_{40}^{170} = 1,2272$ .  $N_d^{170} = 1,54078$ .

Mol.refr. 56,84. Calc. 56,18.

Mol.weight in benzene: found 220 and 213. Calc. for  $C_{12}H_{14}O_4$ : 222.

Analysis:

Found % C 64,5 and 64,8. Calc. as  $C_{12}H_{14}O_4$  64,8.

% H 6,9 and 7,7. Calc. 6,3.

## Camphene and formaldehyde.

A mixture of 30 gr. trioxymethylene, 130 gr. camphene and 136 gr. glacial acetic acid is boiled for three days, the oil separated by means of water, is dried, whereupon it is converted into an acetate by boiling it with an equal weight of acetic anhydride, which is called homo-camphenol acetate owing to its being derived from a homo-camphene:

Homo-camphenolacetate:

Colourless oil, boiling-point  $124^{\circ}$ — $128^{\circ}$  pressure, 13 mm.  
 $D_{40}^{190} = 1,0013$   $N_d^{190} = 1,48209$ .

Mol.refr. 59.23. Calc. for a substance  $C_{13}H_{20}O_2$  with one  $C=C$  group: 59.02.

Homo-pinenol.

By boiling pinene with trioxymethylene and acetic acid in the way as indicated for camphene, an oil is obtained containing much chemically bound acetic acid, yielding after saponification and fractionation:

Colourless oil, boiling-point  $113^{\circ}$ — $116^{\circ}$  in vacuo, pressure 13 mm.  
 $D_{40}^{190} = 0,9720$ .  $N_d^{190} = 1,48616$ .

Mol.refr. 49,05. Calc. for  $C_{11}H_{18}O$  with one  $C=C$  group 49,66, for  $C_{11}H_{18}O$  with two  $C=C$  groups: 51,39.

Mol. weight in benzene: found 188 and 190. Calc. 166.

Homo-limonenol.

Limonene is brought into reaction with formaldehyde in the manner as described for camphene, and the alcohol is separated as indicated for the pinene.

Colourless liquid boiling-point  $122^{\circ}$ — $126^{\circ}$  in vacuo, pressure 13 mm.  $D_{40}^{190} = 0,9757$ ,  $N_d^{190} = 1,50261$ .

Mol. refr. 50,26. Calc. for  $C_{11}H_{18}O$  with two  $C=C$ -groups 51,39.

Analysis.

Found % C 78,1 and 77,7. Calc. for  $C_{11}H_{18}O$  % C 79,5.

% H 11,6 and 11,7. Calc. % H 10,9.

These results agree with those of KRIEWITZ, l.c.

Cedrene and formaldehyde.

A mixture of cedrene with the equivalent quantity of formaldehyde in a 15% solution of sulphuric acid in acetic acid, is stirred for three days. The reaction product is saponified and distilled in vacuum.

Besides other products with a considerably higher boiling-point there is obtained: Homo-cedrenol.

Colourless viscous liquid boiling-point  $168^{\circ}$ — $171^{\circ}$  in vacuo, pressure 13 mm.  $D_{40}^{19} = 1,0270$ .  $N_d^{19} = 1,51826$ .

Mol. refr. 69,08. Calc. for  $C_{16}H_{26}O$  with one  $C=C$ -group 70,55.

As an alcohol it immediately reacts with GRIGNARD-solution and with acetylchloride, it discolorizes a bromine-solution.

Analysis.

Found % C 81,6 and 80,7. Calc. for  $C_{16}H_{26}O$  % C 82,0.

% H 12,0 and 11,3. % H 11,1.

The principal aim of this investigation was: to demonstrate that formaldehyde reacts with a  $C=C$ -group in the way as indicated in the author's general scheme for the reactions between unsaturated compounds.

*Hilversum*, April 1919.

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**Botany.** — “*Photo-growth reaction and disposition to light in Avena sativa*”. By H. L. VAN DE SANDE BAKHUIJZEN. (Communicated by Prof. F. A. F. C. WENT).

(Communicated in the meeting of May 3, 1919.)

Of late years our knowledge of the influence of light on the longitudinal growth of plants has been considerably extended and deepened. While earlier investigations were generally content not to make their first observations until some hours after the beginning of illumination, attempts were made in more recent investigations to study as closely as possible from the very beginning of exposure the changes to which growth is subjected.

In 1914 BLAAUW published his investigations on the photo-growth reaction in *Phycomyces*<sup>1)</sup>, and VOGT found in 1915 a similar reaction in the coleoptiles of *Avena*<sup>2)</sup>; besides two further papers were published by BLAAUW on this reaction in seedlings of *Helianthus*<sup>3)</sup> and in certain roots<sup>4)</sup>.

All these investigations were carried out under the influence of the new points of view which BLAAUW and FRÖSCHEL had opened up in 1909 on the subject of phototropy; hence attention was not confined to the intensity, but a definite amount of light energy was administered. In some cases illumination was continuous throughout the experiment.

While VOGT does not discuss the theoretical bearing of his results, BLAAUW uses his photo-growth reaction as a basis for a discussion of the phenomena observed in phototropy. Formerly it was indeed known that phototropic curvatures arise through growth being changed somehow by unilateral illumination, but since it was not known that a brief omnilateral illumination has an effect on longitudinal growth, it was impossible to obtain any accurate conception of the changes which growth underwent immediately on unilateral

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1) A. H. BLAAUW. Licht und Wachstum. I. Zeitsch. f. Botanik. 1914.

2) E. VOGT. Über den Einfluss des Lichts auf das Wachstum der Koleoptile von *Avena sativa*. Zeitschrift f. Botanik. 1915.

3) A. H. BLAAUW. Licht und Wachstum. II. Zeitschr. f. Botanik. 1915.

4) A. H. BLAAUW. Licht und Wachstum. III. Mededeelingen van de Landbouwhoogeschool. Deel XV. 1918.

illumination. This was one of the reasons why phototropism was regarded as a thing apart.

BLAAUW once more defends the old theory of DE CANDOLLE and regards unilateral as a special case of omnilateral illumination; the former would thus have no specific action. If omnilateral illumination gives a photogrowth reaction uniform in all directions, unilateral illumination will only differ in giving an unequal growth reaction on the posterior and anterior sides. Here it is not the difference of light, but the light itself, as energy, which influences the longitudinal growth of every cell, but since the front is differently lighted from the back, the two sides will show unequal changes in growth. The result of this inequality is a curvature towards the source of light or away from it; this is the phototropic curvature. By these considerations phototropism has been saved from its isolation and has become susceptible of deeper and more exact analysis. A further important point is, that BLAAUW ascribes an influence to the posterior side also, which quite corresponds to that of the anterior. It is not the anterior nor the posterior side alone which actively causes the curvature; both are concerned: it is the difference in the change of growth of the two sides which makes the plant curve phototropically.

The possibility that unilateral illumination might be a special case of omnilateral, i. e. that every longitudinal strip of the plant might receive a photo-growth induction independently of the rest of the circumference and might execute a photo-growth reaction, suggested an investigation of the question, whether this could throw any light on various phototropic phenomena described in the literature. I found that the photo-growth reaction gives us the means of explaining satisfactorily many apparently contradictory phenomena. Since phototropism has been most fully examined in *Avena sativa* and since VOGT moreover found a photo-growth reaction here, I have made a study of the literature on this plant. While I was engaged in working up these considerations for a preliminary publication, a paper by BREMEKAMP<sup>1)</sup> appeared, which gave me no reason to change my opinion; I hope to return to this more fully in a later paper.

VOGT illuminated coleoptiles of *Avena* from above with various intensities during periods of various length, but on this account his energy numbers are not comparable with horizontal light. By the first method the plants receive much less light, since the absorbent surface is much smaller. With horizontal illumination we need apply

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<sup>1)</sup> C. E. B. BREMEKAMP. Eine Theorie des Phototropismus. Recueil des Travaux bot. néerlandais. Vol. XV. 1918.

much less light energy in order to obtain the same reaction, than with light coming vertically from above. Therefore the intensity and the energy numbers must be divided by a certain coefficient.

Now VOGT found that already after a few minutes the rate of growth began to fall off. This led after about 25 minutes to a minimum, while after 30—40 minutes the zero point was again passed. Then an acceleration of growth occurred, leading to a maximum, so that at about 60 minutes after the beginning of illumination the first rate of growth had been reestablished. After that the growth oscillated for a long time, at first with considerable but later with gradually decreasing maxima and minima.

I have now calculated from VOGT's tables<sup>1)</sup>, how many  $\mu$  the plants grew less during the first retardation period than if they had continued their growth in the dark. The following figures therefore give, in  $\mu$ , the total retardation of growth until the zero-point is again reached and the acceleration of growth begins. It should always be remembered that the intensity numbers must be divided by a coefficient before being comparable to those of horizontally incident light.

TABLE I.

Intensity.	Duration of illumination.			
	1 min.	3 min.	15 min.	continuous.
16 M.C.		39		83
64 M.C.	53			
100 M.C.	102	104	907	
500 M.C.			122	123
1000 M.C.		294	76	147
1500 M.C.			89	

Since in unilateral illumination the front is always more strongly illuminated than the back, the former is retarded more than the latter, if the retardation of growth increases with increasing intensity. Consequently a positive phototropic reaction occurs, provided the difference in growth retardation is sufficient to give a visible curvature. If the retardation of growth diminishes again with increasing intensity, a negative reaction will occur, for in this case the back side is retarded more than the front. It is seen from the table that the first and second columns will give positive curvatures. We cannot

<sup>1)</sup> VOGT l.c. Tables 8, 9 and 11.

conclude from the third column whether the numbers increase or decrease, they are not sufficiently certain. If we, however, arrange the first and the second column according to quantities of energy the figures rise, hence, we might expect much higher figures in the third column. Probably we are here beyond the maximum of growth retardation. On the other hand the figures rise again on prolonged continued illumination; this completely agrees with the fact, that illuminations of longer than 25 mins. always give positive curvatures. Now since it is known that phototropic curvature can already be detected after 25 mins. by means of a microscope<sup>1)</sup>, a sufficient difference in growth between the two sides must have occurred in this time. If the photo-growth reaction is therefore to be regarded as the basis of phototropic curvature, the numbers, giving the retardation of growth to 30 minutes after the beginning of exposure, must provide a clue<sup>2)</sup>. The occurrence of curvatures at the extreme apex within 25—30 mins. can therefore be sufficiently explained by growth retardation alone. There are however further points, which indicate that the acceleration of growth, following the retardation, has nothing to do with the establishment of the curvature.

When the acceleration of growth is over, the growth curve still shows pronounced rises and falls. These are, however, not to be considered as oscillations about a condition of equilibrium, like those of a pendulum coming to rest, but must certainly be regarded as the reactions of zones situated below. Just as in a phototropic curvature the reaction first becomes visible at the extreme apex, the first period of retardation of growth must also represent the reaction of the extreme apex; the photo-growth reaction of the inferior zones will not occur until later. This is therefore quite comparable to the progress of the phototropic curvature from the apex to the regions further below. The magnitude of the later retardations of growth cannot however be deduced from the tables, since the average rate of growth does not remain constant, which is partly attributable to the great period, which itself is moreover influenced by light.<sup>3)</sup>

1) E. PRINGSHEIM. Studien zur heliotropischen Stimmung und Präsentationszeit. Zweite Mitteilung. Cohn's Beiträge zur Biologie der Pflanzen. Bd. IX. 1909.

W. H. ARISZ. Onderzoekingen over Fototropie, Diss. Utrecht. 1914; Rec. trav. bot. Néerlandais. Vol. XII. 1915.

2) I neglect here the negative curvatures, which after a time may succeed to positive ones; these can be explained in a quite different manner, as I shall show in my detailed paper.

3) H. SIERP. Ein Beitrag zur Kenntnis des Einflusses des Lichts auf das Wachstum der Koleoptile von *Avena sativa*. Zeitschr. f. Botanik. 1918.

We are here concerned with a conduction of the photo-growth reaction; the magnitude of the growth retardation in the lower zones will likewise depend on the energy applied.

The curve, which represents the dependence of the growth retardation (ordinate) in respect of the energy applied in 5 mins. (abscissa), we call: *growth retardation curve*. From the numbers of table I we cannot very well construct this curve, as there are too few data. There is however every reason for assuming that there is an ascending portion which can bring about positive curvatures and a descending one producing negative ones. Hence somewhere there is a maximum; its existence and position will be determined later by another method. Since the rule of products will apply also to the photogrowth reaction within certain limits, there belongs to every amount of energy ( $x$ ) a definite retardation of growth ( $y$ ) after a certain time (here  $1\frac{1}{2}$  hour).

Now in unilateral illumination the front receives the full energy <sup>1)</sup>; the back receives much less. If we are on the ascending part of the curve, the retardation of growth on the anterior side ( $y_a$ ) is greater than that on the posterior side ( $y_p$ ) and only a positive curvature will occur. Now 20 M.C.S. is the practical threshold value for a readily visible curvature; on applying this amount of energy therefore, there will occur a difference of growth retardation between the front and back, which is just sufficient to produce a curvature which can be readily detected macroscopically. The threshold value of 20 M.C.S. is however only applicable to plants which have been grown in the dark, and have received a quantity of light energy = 0. It is different with plants which have had an omnilateral fore-illumination. This can be done by rotating the plants before the source of light, so that successively all sides are illuminated; in this way all parts of the circumference receive an equal amount of light. If subsequently there is a unilateral illumination, the threshold value for a positive curvature is found to be much higher. I will now consider some experiments of ARISZ <sup>2)</sup> on the so-called "disposition (German "Stimmung") to light" in oats and in the first place the omnilateral fore-illumination with less than 2000 M.C.S. applied within 3 minutes with an intensity below 25 M.C. The unilateral after-illumination always took place immediately after the omnilateral.

I find that all the changes, produced by the omnilateral

<sup>1)</sup> Reflection etc. is here left out of account.

<sup>2)</sup> ARISZ l.c. Tables 24 and 25. See also ARISZ: Adjustment to light in oats. Proceedings Kon. Ak. v. Wet. Amsterdam. 1913.

fore-illumination, can be satisfactorily explained by assuming that at every point of the circumference the energy of the unilateral after-illumination is simply added to that of the fore-illumination. The retardation at a given point of the circumference will therefore depend on the total energy received from the two illuminations together. If the energy difference between the anterior and posterior sides is then sufficiently great to give the difference of growth retardation required for a visible curvature, the threshold value will be reached. I have nowhere assumed a change in sensitiveness, in contradistinction to BREMEKAMP<sup>1)</sup>, who attempts to explain the increase in the threshold value by the hypothesis, that the plant has become less sensitive as a result of the unilateral fore-illumination and thus imagines a change in the perception-basis.

If we apply unilaterally  $b$  M. C. S., the front will receive  $b$ , and of this the back  $\frac{b}{m}$  M. C. S.;  $m$  is greater than 1, since the back receives less than the front. If we had, however, previously given  $a$  M. C. S. omnilaterally, the later front would have received of this  $\frac{a}{n}$  M. C. S., the later back also  $\frac{a}{n}$  M. C. S.; a unilateral after-illumination will then supply to the front in addition  $b$  M. C. S., altogether therefore  $\frac{a}{n} + b$  M. C. S.; the back receives altogether  $\frac{a}{n} + \frac{b}{m}$  M. C. S. In both cases the absolute energy difference between the front and back remains constant, i. e.  $b - \frac{b}{m} = b \frac{m-1}{m}$  M. C. S. Now the resultant curvature entirely depends on the difference of growth retardation, which corresponds to an energy difference of  $b \frac{m-1}{m}$  M.C.S. If the growth retardation curve were a straight line, there would always correspond to an energy difference of  $b \frac{m-1}{m}$  M.C. S. one and the same difference of growth retardation, whatever place of the abscissa we may consider; the threshold value would then necessarily be constant. If the growth retardation curve is, however, a curve with decreasing slope, the growth retardation difference corresponding to an abscissa difference of  $b \frac{m-1}{m}$  M. C. S. will become the smaller, the gentler

<sup>1)</sup> BREMEKAMP l.c.

the slope of the curve becomes, i.e. the greater  $x$  becomes.

We call the energy on the anterior and posterior sides respectively  $x_a$  and  $x_p$ , the retardations of growth  $y_p$  and  $y_a$ .

growth retardation

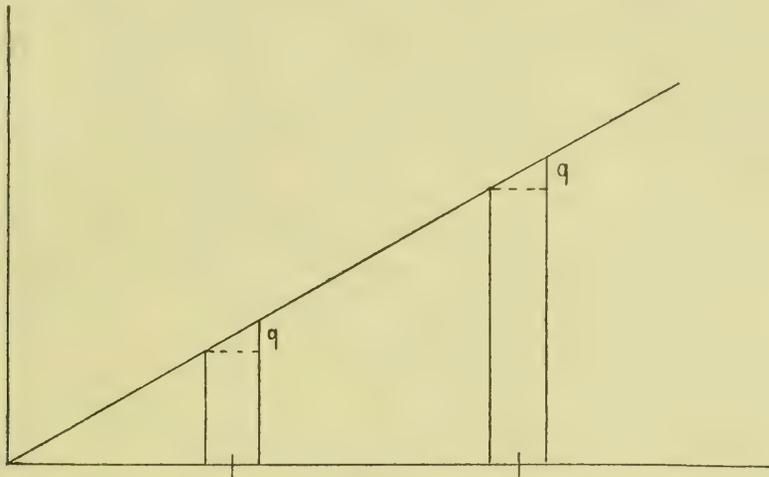


Fig. 1.

energy

In a straight with  $x_a - x_p$  constant  $= b \frac{m-1}{m}$  M.C.S. the difference in growth retardation  $y_a - y_p$  is always  $= q$  (fig. 1).

In a curve with decreasing slope, with  $x_a - x_p$  constant  $= b \frac{m-1}{m}$  M. C. S. the difference in growth retardation  $y_a - y_p$  is smaller, according as  $x_a$  is larger (fig. 2).

Now  $x_a$  will become greater by unilateral fore-illumination, for  $x_a = \frac{a}{n} + b$  M.C.S. The absolute energy difference  $b \frac{m-1}{m}$  M.C.S. or rather the energy of  $b$  M.C.S. applied unilaterally, will therefore give a smaller difference of growth retardation i. e. a smaller curvature, according as the energy of fore-illumination  $a$  is greater. In order nevertheless to obtain a constant difference of growth retardation, giving a just visible curvature,  $b \frac{m-1}{m}$  and therefore also  $b$  will have to be greater.

If the energy of fore-illumination becomes greater, the threshold value will rise, as soon as the curve of growth retardation is a curve with decreasing slope. Here we need not yet assert anything about the further relation between the magnitude of the slope and  $x$ .

In our case we can however follow the course of the growth retardation curve more closely by attempting to trace the relationship between threshold value and fore-illumination. If we calculate growth retardation

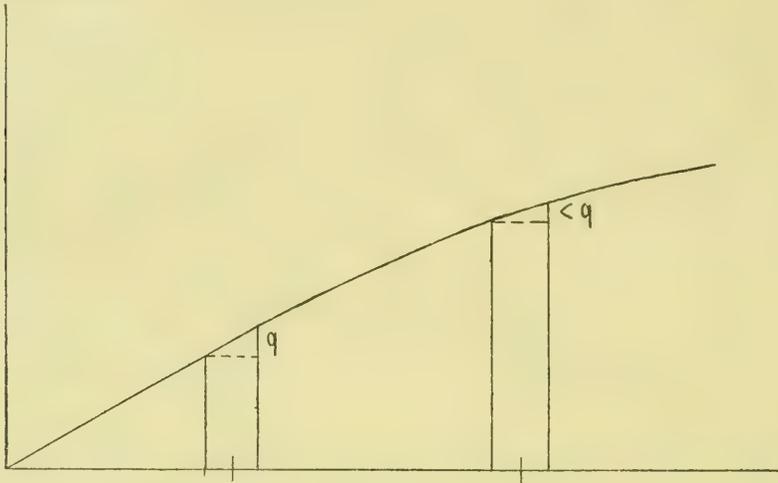


Fig. 2.

energy

from ARISZ's tables, mentioned above, the ratio of unilateral after-illumination to omnilateral fore-illumination, we find that, if after-illumination : fore-illumination = 1 : 11, there is no curvature;

with this ratio 1 : 10, 9.9 or 9.2 a few plants give a feeble positive curvature,

with the ratio 1 : 7.2 all plants curve positively.

This applies, as was already said above, only to a fore-illumination of less than 2000 M.C.S. given within 3 min. with an intensity below 25 M.C. We must, however, recalculate this for the quantities, which the anterior and posterior sides receive. Since we see, that a feeble positive curvature occurs when the ratio after-illumination : fore-illumination ( $b : a$ ) has reached a definite magnitude 1 : 9.7, the

ratio of the energy difference  $b \frac{m-1}{m}$  to the energy of the anterior side  $\frac{a}{n} + b$  must also be constant, whatever be the values of  $m$  and  $n$ .

In order to demonstrate this numerically, I propose to make certain assumptions respecting  $m$  and  $n$ ; in principle it does not matter, what values we ascribe to  $m$  and  $n$ . For  $m$  I assume 4; the back is then illuminated with  $\frac{1}{4}$  of the intensity of the front, receives therefore in unilateral after-illumination  $\frac{b}{4}$  M.C.S. Since the omnilateral energy  $a$  M.C.S. is distributed uniformly over the whole

circumference, each half, viz. the later anterior or posterior sides, will receive, to begin with, half of the energy,  $\frac{1}{2} a$  M.C.S. Moreover each point, which is turned away from the source of light, receives during rotation a further  $\frac{1}{4}$  of the energy of the illuminated side; each half receives therefore altogether  $\frac{5}{4} \times \frac{1}{2} a = \frac{5}{8} a$  M.C.S.

Here follows a table, taken from tables 24 and 25 of ARISZ; I have added the numbers representing

$$100 \times \frac{\text{energy difference between front and back}}{\text{energy of front}} \left( = 100 \times \frac{x_a - x_p}{x_a} \right).$$

TABLE II.

Energy of omnilateral fore-illumination in M. C. S.	Energy of the unilateral after-illumination in M. C. S.					
	$4 \times 5.5$ 22	$8 \times 5.5$ 44	$5 \times 12$ 60	$10 \times 12$ 120	$5 \times 100$ 500	$10 \times 100$ 1000
0 = 0	75 +	75 ++	75 ++	75 ++	75 ++	75 ++
$10 \times 5.5 = 55$	29.5 +	42.3 ++	47.9 ++	58.4 ++	70.2 ++	72.5 ++
$10 \times 12.1 = 121$	17 +	27.7 +	33.3 ++	46.2 ++	65.2 ++	69.8 ++
$36 \times 12.1 = 435$	5.6 0	10.4 +?	13.6 +	23 +	48.6 ++	58.9 ++
$100 \times 5.5 = 550$	4.5 0	8.5 0	11.1 +?	19.4 +	44.4 ++	55.8 ++
$100 \times 12.1 = 1210$	2.1 0	4.1 0	5.1 0	10.3 +?	30 ++	42.7 ++

Explanation: ++ all plants show strong positive curvature.  
 + all plants show definite positive curvature.  
 +? a few plants show slight positive curvature.  
 0 no plants curved.

$10 \times 5.5$  means: during 10 sec. omnilateral fore-illumination with 5.5 M. C.

$$29.5 = 100 \times \frac{b \frac{m-1}{m}}{\frac{a}{n} + b} = 100 \times \frac{\frac{3}{4} \times 22}{\frac{5}{8} \times 55 + 22}.$$

I here assume +? as the threshold value. We see that for this the quotient  $100 \times \frac{x_a - x_p}{x_a} = 10.4, 11.1$  and  $10.3$ , and it is therefore sufficiently constant. Then  $y_a - y_p$  has become so great that a slight positive curvature occurs. If the curvature can be represented by  $f(x_a) - f(x_p)$  and also by  $\varphi \left( \frac{x_a - x_p}{x_a} \right)$  then  $f$  is necessarily  $c \log x$  as can be readily shown by a simple mathematical consideration. Since the region of energy, about which the figures supply information, extends from about 300—900 M.C.S. we may say that this portion of the growth retardation curve has a logarithmic course. If the

quotient  $\frac{x_a - x_p}{x_a}$  for the threshold value had not been found constant, we should only have been allowed to conclude, that the growth retardation curve was a curve with decreasing slope, since the threshold value increases with increasing  $x$ . In that case also we might perhaps have obtained some further information about the function. I emphasize here that no fundamental significance should be attached to the logarithmic course of this part of the curve between 300 and 900 M.C.S. Moreover the curve from 0—100 M.C.S. is certainly not logarithmic; in a subsequent paper I hope to refer to this point in detail.

ARISZ regards the omnilateral fore-illumination followed by a unilateral after-illumination as a combination of unilateral illuminations, a short one on the posterior side (fore-illumination) and a longer one on the anterior side (fore-illumination + after-illumination). Since the later posterior side has also received energy during the fore-illumination, a tendency to curve in the opposite direction would have to be overcome. "It need cause no surprise, that the excess which must be given on one of the sides, to obtain an ipsilateral curvature, must be greater in proportion as the tendency to curvature on the other side is stronger." ARISZ therefore likewise explains the rise in the threshold value without assuming a change in the sensitiveness. This "tendency to curve in the opposite direction" however, as has been explained above, also exists in plants which have had a purely unilateral illumination, for the curvature depends on the difference of growth retardation between front and back. As in a combined omnilateral fore-illumination and unilateral after-illumination the growth retardation on the posterior side i.e. the "tendency to curve in the opposite direction", becomes relatively greater, the difference in growth retardation and accordingly, also the resulting curvature, becomes smaller (Fig. II). ARISZ regards the phototropic induction, and hence also the tendency to curve, as a primary reaction, but since according to the theory of BLAAUW, which I have here worked out further, the growth retardation is primary and the tendency to curve secondary, it is better not to employ the latter expression, but to speak of a greater or smaller growth retardation. The unilateral illumination is a special case of the omnilateral and not inversely.

We see therefore, that it is of great importance to ascertain the course of the growth retardation curve. We could not well do this from the figures of table I; moreover the energy numbers had to be divided by a coefficient in order to make them comparable with those for horizontally incident light. We will now try to ascertain the position

of the maximum of the growth retardation curve in another manner.

A second phenomenon which shows itself in the combination of an omnilateral fore-illumination with a unilateral after-illumination, is that which CLARK <sup>1)</sup> calls the increase in sensitiveness of the negative reaction. Here also we can find a simple explanation if we suppose the existence of a descending portion in the growth retardation curve. If the front receives in unilateral illumination an amount of light energy just in excess of the maximum, then the back, which receives  $\frac{1}{4}$  of this, will still be on the ascending portion. The ordinate of the front is still greater than that of the back, hence a positive curvature occurs. When we now apply more unilateral energy, we shall have to shift the points, representing the back and front energy, along the growth retardation curve, in such a manner that always  $x_a = \frac{1}{4} x_p$ . A negative curvature will then only be possible much further on, because of the great distance between  $x_a$  and  $x_p$ , i.e.  $x_a$  will have to pass far beyond the maximum before  $y_a$  becomes smaller than  $y_p$ ;  $x_p$  need not yet have passed beyond the maximum. When then the negative difference of growth retardation  $y_a - y_p$  is great enough to become visible in a curvature, a negative curvature will occur (about 5000 M.C.S. if administered in 5 mins.). This becomes different, however, if  $x_a$  and  $x_p$  both lie on the descending portion and are less remote from each other; in that case, as soon as  $x_p$  has passed the maximum, the possibility of a negative curvature will arise, the front however requires to receive much less than  $\frac{1}{4} x_p$  in order still to render a visible negative curvature possible. By means of an artifice we can arrange that  $x_a$  and  $x_p$  come near to each other, although both are on the descending portion. This artifice consists in giving an omnilateral fore-illumination with a quantity of light, which is so great, that front and back both receive a quantity of energy corresponding to the maximum of the growth retardation curve. If we then unilaterally apply an amount of energy, which by itself would have given a strongly positive curvature, there will occur a negative curvature, at least if  $x_a - x_p$  is so large that  $y_p - y_a$  can give a visible effect. Here also the magnitude of the difference of growth retardations, and hence the threshold value for the negative curvature, will depend on the slope of the growth retardation curve, as was the case with the positive curvature. Here some data from tables 25 and 26 of ARISZ are appended; I have added the numbers representing the quantities of light in M.C.S. which the front and back receive from omnilateral fore-illumination + unilateral after-illumination.

<sup>1)</sup> O. L. CLARK. Über negativen Phototropismus bei *Avena sativa*. Zeitsch. f. Botanik. 1913.

TABLE III.

Energy of unilateral after- illumination in M. C. S.	Omnilateral fore-illumination.					
	12.1 M. C. during 180 secs.			25 M. C. during 100 secs.		
	Energy to front in M. C. S.	Energy to back in M. C. S.	Reaction.	Energy to front in M. C. S.	Energy to back in M. C. S.	Reaction.
44	1405	1372	0	1606	1573	0
60	1421	1376	0	1622	1577	0
120	1481	1391	?	1682	1592	0
500	1861	1486	?	2062	1687	?
1000				2562	1812	?

Explanation:

The energy to the front, 1405 M. C. S., is  $\left(\frac{a}{n} + b\right) = \left(12.1 \times 180 \times \frac{5}{8} + 44\right)$  M. C. S.,  
calculated from

that to the back, 1372 M. C. S., from  $\left(\frac{a}{n} + \frac{b}{m}\right) = \left(12.1 \times 180 \times \frac{5}{8} + 11\right)$  M. C. S.

ARISZ means by the sign? that "some plants give a feeble positive curvature, but there are always a few which curve negatively"; this has been confirmed by clinostat experiments. He directly connects this phenomenon of "increased sensitiveness to the negative reaction" with the fact, that after 300—600 M.C.S. the strength of the maximal curvature diminishes, and considers it possible that "by combining a quantity of light, which gives a curvature in excess of the greatest maximal strength, with a quantity which is maximal or nearly so, a curvature is obtainable towards the weaker illumination". We may not connect the "decreased sensitiveness to the positive reaction" with the tendency to curve in the opposite direction and as little may we here directly connect the "increased sensitiveness to the negative reaction" with the strength of the maximal curvature, but must explain it from the course of the growth retardation curve. For the maximal curvature will be strongest in that case, where the difference between the ordinates belonging to  $x_a$  and  $\frac{1}{4} x_a$  is a maximum. The decrease in the amount of this difference is primarily connected with the decrease in slope of the growth retardation curve and it is only the rate at which the curvature diminishes at higher amounts of energy or the change to a negative curvature, which is connected with the question whether or no the growth retardation curve presents a maximum. The growth retardation curve will therefore continue to rise, although the intensity of curvature (i.e. the difference between the ordinates belonging to  $x_a$  and  $\frac{1}{4} x_a$ ) is already declining; the maximum will

therefore be situated much higher than 300—600 M. C. S. Now it results from the above table that if the back receives about 1400 M.C.S., negative curvatures may occur. In this neighbourhood therefore the maximum of the growth retardation curve must lie; there may be considerable individual variation; if the maximum lies somewhat higher, a feeble positive reaction will still be possible; if it is at or below 1400 M.C.S., negative curvatures can occur, this depends on the degree of slope of the descending portion. We can also calculate from table II that a negative curvature never occurs, if the back receives less than 1400 M.C.S. Thus 1756 M.C.S. on the front and 1006 M.C.S. on the back still give a strong positive curvature; here the  $y$  of 1756 M.C.S. must be greater than the  $y$  of 1006 M.C.S. We may therefore place the maximum of the growth retardation curve at about 1400 M.C.S.

With a unilateral illumination the posterior side will not be maximally retarded until the anterior receives  $m \times 1400$  M.C.S. This amount of energy must of course lie beyond the threshold value for the negative curvature, for otherwise  $y_p$  could never become larger than  $y_a$  and no negative curvature could occur. From this the value of  $m$  can be found approximately.

I shall indicate yet a third method by which the course of the growth retardation curve can be explored. This can be done by assuming the magnitude of the maximal curvature to be proportional to the difference between the front and back growth retardation. If the maximal strengths of curvature are then plotted against the energy values as abscissae, there results a curve of the differences of growth retardation between the anterior and posterior sides. By a simple mathematical calculation, the growth retardation curve of the front can be calculated from the curve of differences, whereby it is assumed again, that the back receives  $\frac{1}{4}$  of the energy of the front; here the magnitudes of the growth retardation do not of course represent absolute values. The points of the posterior growth retardations are found by subtracting the curve of differences from the anterior curve; these can also be found by plotting the anterior curve with abscissae four times as great (rule of products). The magnitudes of the maximal curvatures I have deduced from tables 1 and 3 of ARISZ; the energy was here always applied in 10 secs. Since the front is exposed to the full energy<sup>1)</sup>, we have here again plotted the course of the growth retardation curve. In this way we come to the following result:

<sup>1)</sup> See footnote p. 61.

from 0—75 M. C. S. the growth retardation curve is more or less rectilinear, then the slope decreases, so that it becomes approximately logarithmic from 300—700 M. C. S., while a maximum is found at about 1600 M. C. S. This value therefore agrees sufficiently well with that already found.

Finally I have also succeeded in finding theoretically the further course of the growth retardation curve, which explains both the phenomena attending more prolonged unilateral illumination and those of unilateral preceded by prolonged omnilateral illumination.

growth retardation

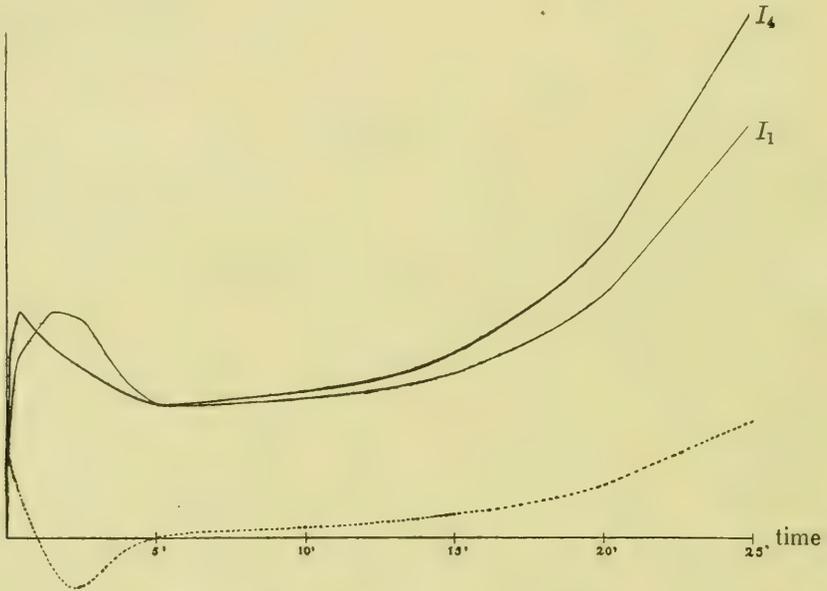


Fig. III. Diagrammatic representation of

- the growth retardation curve for an intensity 1.
- the growth retardation curve for an intensity 4.
- ..... magnitude of curvature occurring after unilateral illumination with intensity 4.

Here the growth retardation has not been represented as a function of the energy, but as a function of the time, during which there was illumination with the same intensity. Since unilateral illumination, following an omnilateral illumination of more than 5 minutes' duration, always results in a positive curvature,  $y_a$  must again be greater than  $y_p$ . This comes about through the renewed rise in the growth retardation curve, if the plant is illuminated for more than 5 minutes. Since the slope increases here, the threshold value for positive curvature must again fall. This latter fact agrees with what CLARK and ARISZ found, namely that the threshold value falls if the plants are submitted to a fore-illumination of more than 5 minutes and less

than 20 minutes. If the fore-illumination is more than 20 minutes, the growth retardation curve has become a straight line, since the threshold value now remains constant; however prolonged the fore-illumination with this intensity is, there is no further change in disposition. *Conversely there will be no question of "disposition", "change of sensitiveness", in a process where the effect increases in a rectilinear manner with increasing strength of stimulus.*

Now since also after unilateral continued illumination (for longer than about 5 mins.) a positive curvature is again obtained <sup>1)</sup>, the growth retardation curve, for an intensity  $m$  times as great, will run more steeply, i.e. for the same abscissa (time) there will be a greater ordinate (retardation of growth). If we take, however, the growth retardations of different intensities with equal duration of illumination, and plot these against the intensities, the slope of the resulting curve will of course greatly decrease at higher intensities, as a simple consideration will show. With this two facts agree: firstly that the threshold value after prolonged fore-illumination with high intensities comes to lie higher than after prolonged fore-illumination with low intensities; secondly, that prolonged unilateral illumination with a high intensity gives a feebler curvature than illumination with a low intensity during the same period.

We see therefore that the phototropic curvature is determined by the reactions of the separate longitudinal strips of the front and back respectively. Formerly the curvature was regarded as the direct result of a single condition of stimulation, the phototropic, which was considered to be induced as such. According to the view set out above, the curvature must be regarded as the resultant of the effects arising from the conditions of stimulation, which exist on the side, towards which the ultimate curvature will take place, and on the opposite side. These conditions of stimulation express themselves in photo growth reactions; the difference between the two reactions is expressed by the phototropic curvature.

The cause of that which was formerly called "disposition" lies in the peculiarities of the growth retardation curve. These peculiarities occur to some extent in every process in which the reaction is not directly proportional to the stimulus. A tangent galvanometer also becomes less "sensitive" at greater strengths of current. The "disposition" at a given point of the growth retardation curve, whether we take energy or time as abscissa, depends therefore on the magnitude and the sign (+ or —) of the angle of slope, and of

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<sup>1)</sup> See footnote 2 p. 60.

the direction in which the latter is changing. We can ascertain the sign by arranging that  $x_a$  and  $x_p$  approximate closely; then we can see from the curvature whether  $y_a - y_p$  is positive (effect: positive curvature) or whether it is negative (effect: negative curvature). In unilateral illumination  $x_a$  and  $x_p$  lie too far apart and may be on dissimilar parts of the curve, so that the latter is very difficult to draw. According as  $x_a$  and  $x_p$  approximate more closely, the angle of slope can be found with greater accuracy. Since we have found that the growth retardation curve shows an ascending, then a descending and subsequently again an ascending portion, the "disposition" must therefore have three phases; finally, after 20 mins., it is not possible to bring about a "disposition change" by further illumination with this intensity. These three phases have been repeatedly discussed in the literature and explained as processes, involving a change in the perception basis. From the above discussion it is evident, however, that disposition is a concept devoid of specific properties and is simply an expression of peculiarities of the growth retardation curve. Every growth retardation curve has a maximum at about 1400 M.C.S. and a minimum at about 5 minutes. The growth retardation curves of two intensities, which are in the ratio 1:  $m$  and with which 5000 M.C.S. can be applied within 5 minutes, intersect at two points; the first point is the energy threshold for the negative curvature ( $\pm 5000$  M.C.S.), the second is the time threshold for the second positive curvature (5 minutes). It will, however, be easier and more accurate to construct the growth retardation curves, both energy curve and time curve, from the numbers for photo growth reactions than from interpolations according to the above mentioned method, which is only qualitative, but not absolutely quantitative. Phototropism retains its value, however, since the study of photogrowth reactions presents difficulties in so far as we are here certainly concerned with the conduction of stimuli, while we only observe the reaction of the whole plant. Since in phototropic curvature the various zones curve, one after the other, we get this conduction of stimuli here separated into its successive phases. The difference in growth retardation will also be more constant as phototropic curvature, than when it is found by examining the photogrowth reactions of two different plants, in one for a quantity of energy 1, in the other for a quantity of energy  $m$ . For in unilateral illumination both experiments are carried out on the same plant, one on the front, the other on the back. The difference, the resultant of the two reactions, will thus be less influenced by individual variability.

*Utrecht, April 1919.*

**Astronomy.** — “*The Structure of the Sun’s Radiation*”. By  
P. H. VAN CITTERT. (Communicated by Prof. W. H. JULIUS).

(Communicated in the meeting of May 3, 1919).

It is known that the intensity of the light on the sun’s disc very appreciably diminishes from the centre towards the limb, and that this takes place in a different degree for the different colours: regions near the limb<sup>1)</sup> of the sun’s disc are distinctly ruddy as compared with the centre.

By the aid of a spectral photometer VOGEL<sup>2)</sup> determined this distribution of intensity for six regions of wavelength in the visible part of the spectrum. The decrease of intensity towards the limb appeared to be greater for light of shorter wavelength than for light of longer wavelength; the diminution, however, does not take place quite regularly with the wavelength, but presents an anomaly in the neighbourhood of  $5000 \text{ \AA}$ . the contrast of the limb to the centre is for this wavelength-region less than the contrasts for the other wavelength-regions would lead us to expect.

VOGEL’S observations were repeated by ABBOT<sup>3)</sup> in 1906 by the aid of a bolometer. ABBOT determined the decrease of intensity for a great number of wavelength-regions in the infra-red and the visible spectrum. He also found a strongly pronounced wavelength-effect: the contrast towards the limb increased very greatly towards the violet. Also his observations presented an anomaly in the neighbourhood of  $5000 \text{ \AA}$ , though less pronounced than that in VOGEL’S observations. In fig. 1 we have plotted as functions of the wavelength, what value the intensity has for places which are at a distance of 0.65, 0.825, and 0.95 of the radius of the sun’s disc from the centre, when in the centre the intensity is put equal to 100 for all the colours. The data are borrowed from ABBOT’S tables. It is seen that the intensity rapidly decreases towards the limb, and this the more rapidly as the wavelength is smaller, but at the same time the fact

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<sup>1)</sup> In the present paper, the phenomena at the limb itself and even in regions, more than e.g.  $\frac{9.5}{100}$  of the radius of the disc removed from the centre, are left out of consideration.

<sup>2)</sup> H. C. VOGEL, Berl. Ber., 1877.

<sup>3)</sup> C. G. ABBOT. Ann. of the Obs. of the Smiths. Inst., 2, 205, 1908; 3, 153, 1913.

strikes us that in the region  $6000\text{--}4000 \text{ \AA}$  this decrease of intensity presents an oscillation. It is noteworthy that the maximum of energy of the sun's radiation lies at the same place in the spectrum.

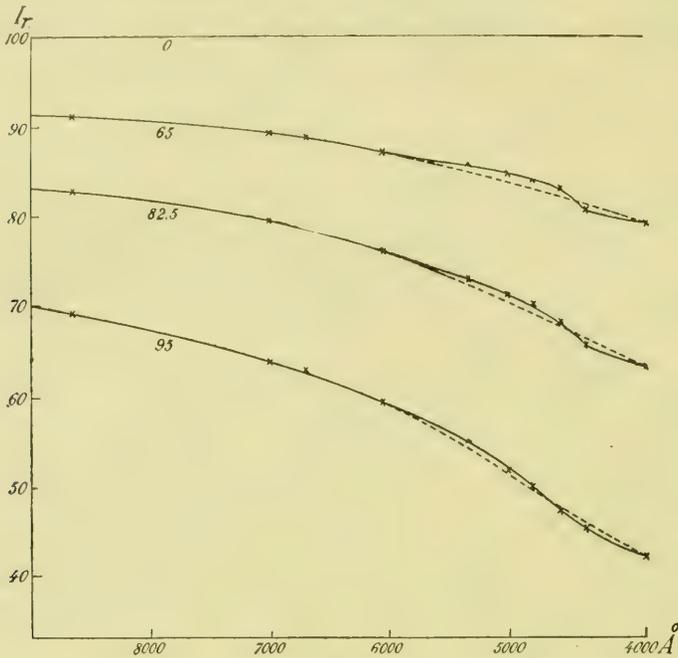


Fig. 1.

(ABBOT'S observations ranged from  $21000 \text{ \AA}$  to  $3800 \text{ \AA}$ ; the part from  $21000\text{--}9000 \text{ \AA}$  has, however, been omitted in the figure, because the curves do not present an irregular course there, but gradually approach the line  $I_r = 100$  towards  $21000 \text{ \AA}$ ).

To account for this decrease of intensity towards the limb many investigators have considered the sun as a self-luminous uniformly radiating core, surrounded by a strongly absorbing atmosphere. Now the state in the sun's atmosphere must naturally be stationary on the average: the quantity of energy that the atmosphere absorbs, must be radiated again, even though it be in another form, and half must be radiated towards the outside. Now it has appeared convincingly from the observations of the annular eclipse of April 17<sup>th</sup> 1912<sup>1)</sup> that of the total quantity of energy that the earth receives from the sun, at most one thousandth part can originate from the sun's atmosphere. It is, therefore, impossible that the absorption should be the chief cause of the diminution of intensity towards the limb.

<sup>1)</sup> W. H. JULIUS. These Proc. 15, 1451.

Continuing the investigations of RAYLEIGH <sup>1)</sup>, SCHUSTER <sup>2)</sup>, KING <sup>3)</sup> and SCHWARZSCHILD <sup>4)</sup> on molecular scattering of light, SPIJKERBOER <sup>5)</sup> has treated the problem how the distribution of light on the sun's disc would be for the different colours, if exclusively molecular scattering in a non-absorbing and not self-luminous atmosphere were the cause through which the uniform radiation of a self-luminous solar core was modified. He arrived at a distribution of light which presents close resemblance to that observed by ABBOT.

The influence of the diffusion (or molecular scattering) is determined by the product  $H = s.t$ , in which  $t =$  the thickness of the dispersive layer and  $s = \frac{32\pi^3(n-1)^2}{3N\lambda^4} =$  RAYLEIGH's coefficient of scattering. When it is now assumed that  $t$  has the same value for light of different wavelengths, that the "core" lies, therefore, equally deep for all colours, the wavelength-effect is exclusively determined by the dependence of  $s$  on  $\lambda^4$ , because, when kinds of light in the neighbourhood of the proper-frequencies are left out of account,  $(n-1)$  will vary very little along the spectrum. It appears, however, that the observed dependence of the wavelength is somewhat less great than theory would lead us to expect. This may be due to the fact that besides the *diffusion* another phenomenon appears, which has a similar influence on the distribution of light as diffusion, but which does not vary so much with the wavelength, e.g. scattering by *irregular refraction*, and possibly a very slight general *absorption*.

Now it is very probable that, particularly in the deeper layers of the sun's atmosphere, irregular refraction plays an appreciable part. The existence of a very irregular distribution of density in the solar gases can, indeed, not be doubted, the constant variations in the granulations and flocculi on the sun's disc point in any case to the existence of an intricate system of currents in that gas-mass, and these are not conceivable without differences of pressure and irregular density gradients accompanying them. The mean value of these gradients, which is small in the outmost layers of the sun, must at first increase as one gets deeper. At a certain depth the irregular density gradients must then on an average be of the same order of magnitude as e.g. the vertical gradient of our terrestrial atmosphere. A gas-mass of the dimensions of the solar atmosphere,

<sup>1)</sup> RAYLEIGH, Phil. Mag., (5). 47, 375, 1899.

<sup>2)</sup> SCHUSTER, Astrophys. Journ., 21, 1, 1905.

<sup>3)</sup> KING, Phil. Trans. R.S., A (212), 375, 1912.

<sup>4)</sup> SCHWARZSCHILD, Berl. Ber., 47. 1183, 1914.

<sup>5)</sup> J. SPIJKERBOER, Verstrooiing van licht en intensiteitsverdeling over de zonneschijf. Proefschrift. Utrecht 1917. Arch. Néerl., (3 A), 5, 1, 1918.

quite honeycombed with irregular gradients of such average magnitude, would, as JULIUS<sup>1)</sup> has demonstrated, refract, deflect and disperse the rays of light that penetrate there, so strongly to all sides, that the gas would present itself to a distant spectator as a turbid medium; the volume parts in which the density can be considered as constant, and hence the light as rectilinear, would be too small to be observed separately at such a distance. Since the degree of refractive scattering is determined by  $(n-1)$ , and accordingly varies comparatively slowly with  $\lambda$ , the co-operation of this kind of scattering with the molecular scattering will weaken the mean wavelength effect, peculiar to the latter.

Besides, the diminution of the intensity from the centre towards the limb will be greater than would be the case if only diffusion were the cause of this diminution. Irregular refraction, therefore, lessens the difference between the rates of darkening towards the limb shown by the different wavelengths, but at the same time strengthens the average contrast between limb and centre.

As another possible cause of the fact that the observed wavelength-effect is slighter than the theoretical, SPIJKERBOER suggests that the light of the longer wavelengths, as it gets less greatly weakened by diffusion, might come to us from deeper layers of the sun than the light of shorter wavelength. This supposition, evidently, excludes the older hypothesis that the sun would have to be conceived as a well-defined core surrounded by a sharply defined atmosphere. The idea that the various radiations originate at different levels is more in agreement with the conception of the sun as a glowing gas-mass, of which the density and the temperature gradually diminish towards the outside. Since the light of greater wavelengths is much less weakened than that of shorter wavelengths, it will come to us from deeper layers of the sun. The infra-red light will, accordingly, be relatively more weakened by molecular scattering than would be the case if it only came to us from equally deep layers as the violet light.

From what depth the light comes to us is, however, not only determined by the diffusion, but also by the irregular refraction. As this is little dependent on the wavelength, the differences in depth for the different colours will not be so great as in the case that only diffusion played a part. If, as is very probable, irregular refraction plays the principal part, especially in the lower layers, the difference in depth must be comparatively slight.

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<sup>1)</sup> W. H. JULIUS, These Proc. 16, 264, 1913.

Let us now consider the question how, seen from this point of view, the radiation emitted by the sun in a direction  $w$  (fig. 2) must be composed. The outer layers of the sun will emit very little energy; proceeding towards lower layers the quantity of emitted energy increases, at first slowly, then more rapidly, in consequence of the increase of temperature and density. Let us suppose  $def$  to be the layer outside which no appreciable quantity of energy is

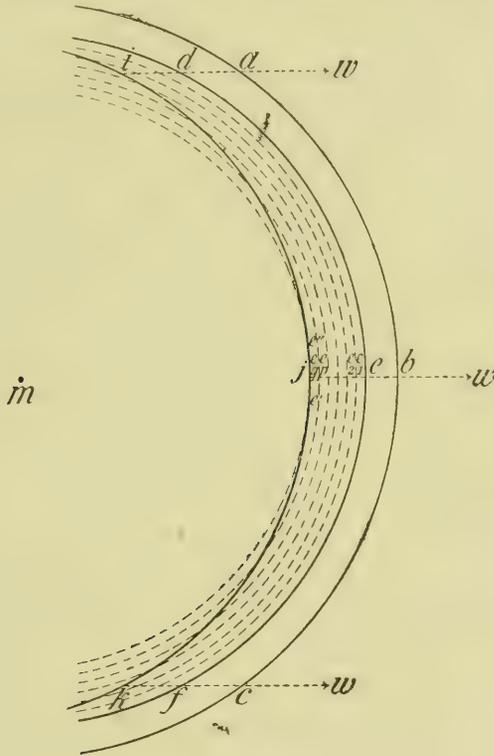


Fig. 2.

emitted. Let us first consider the radiation transmitted by the surface  $def$  in the direction  $w$ . We shall take the scattering in the atmosphere outside  $def$  later into account. From the region  $e_1$  in the centre of the sun's disc  $w$  receives a comparatively slight quantity of radiation of comparatively low temperature, from  $e_2$  a larger quantity of higher temperature, etc. At first the quantity of energy which  $w$  receives from the different depths, will increase; the deeper we get, however, the more will the emitted radiation be weakened by absorption, refractive scattering and diffusion. From  $e_p$  e.g.  $w$  will again receive less radiation than from  $e_2$ , but of higher temperature, from  $e_q$  very little, etc. Let us suppose that  $w$  does not receive an appreciable quantity of energy from layers lying deeper than  $j$ .

(It is noteworthy that  $w$  also receives radiation from  $e$  originating from regions  $e'$  and  $e''$  in the neighbourhood of the radius vector  $je$ , which radiation has finally assumed the direction  $ew$  through diffusion and irregular refraction).

Let us now consider the energy curves of the radiations coming from those different regions, as functions of the wavelength. All of them will probably have the character of the curve of radiation of the absolutely black body: slow increase of the energy from the infra-red to a maximum, and then a comparatively rapid decrease on the violet side of the maximum. The loci of the maxima of the different curves are determined not only by the temperature of the corresponding radiations, but also by the relative importance of diffusion and refractive scattering. Since the light of shorter wavelengths is much more greatly weakened by diffusion than that of longer wavelength, the maximum will be displaced by the diffusion towards the side of the longer wavelengths; the irregular refraction, on the other hand, does not displace the maximum, because it is almost independent of the wavelength. For the curves belonging to the radiation originating from deeper layers, the maximum would lie more to the *violet* than for the curves belonging to the outer layers, if the *temperature* were the only factor; more to the *red*, however, if exclusively the *diffusion* were efficient. The height of the maximum is determined by the temperature of the radiation, the density at the places of emission, and the weakening which the radiation has undergone by diffusion, irregular refraction, and absorption.

We may therefore conclude that from the centre of the sun's disc a quantity of energy is emitted in the direction  $w$  of which the energy curve, proceeding from the infra-red towards smaller wavelengths increases, at first slowly, then more rapidly to a flat maximum, and then runs very steeply down to the violet (fig. 3 I).

Not far from the limb of the disc, e.g. at  $c$  (fig. 2), we shall receive light from a longer path through the gases ( $ek > bj$ ), but it will come from less deep layers. Hence at the limb radiation is received from  $f'$  only from the regions round  $kj$  lying more outwardly. The quantity of energy that  $w$  receives from the limb of the disc is therefore for all colours smaller than that which  $w$  receives from the centre. Hence the energy curve of the limb of the sun will as a whole lie lower than that of the centre. *If the diminution of energy were the same for all the colours*, the composition of the light at the limb would be represented by a curve which had the same shape as that of the centre (fig. 3 II). Now it is directly to be

seen that this cannot be the case, for the radiation at the limb will not present such a variety of temperature, irregular refraction,

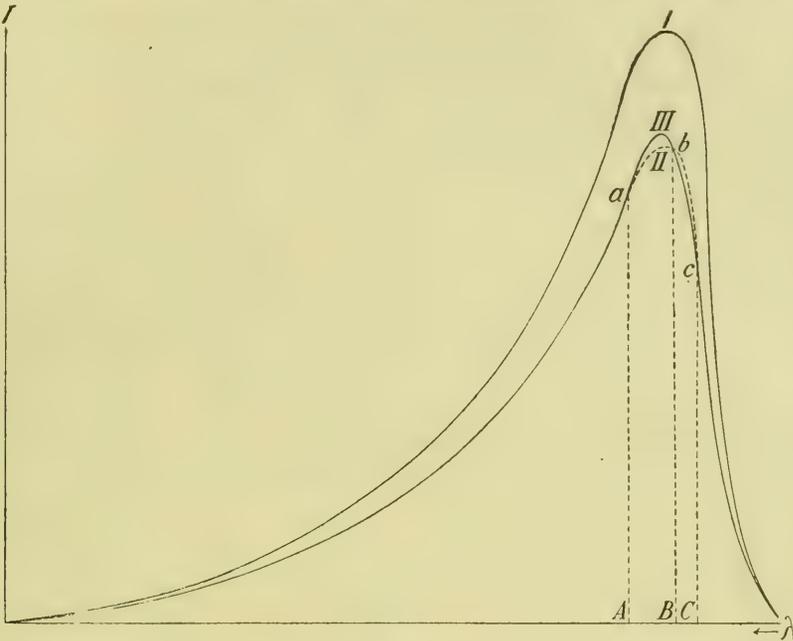


Fig. 3.

diffusion, etc. as that in the centre, because at the limb only radiation from a smaller number of layers contributes to the total radiation. The radiation at the limb will therefore have an energy curve with a maximum which is not so flat as that for the central radiation, and which, as we only receive radiation of lower temperature, lies somewhat more towards the side of the longer wavelengths (fig. 3 III).

When we now compare the curves fig. 3 I, II, and III, it appears that in the wavelength-region  $AB$  the limb radiation will have a relatively stronger, in  $BC$  a relatively slighter intensity than would be the case if the distribution of the energy were the same as in the centre.

When we now put the central radiation for all the wave-lengths  $\approx 100$ , this is graphically represented by a straight line  $C$  parallel to the wavelength-axis (fig. 4). The limb radiation can then be represented by an almost straight line  $r$ , which runs about parallel to the wave-length-axis, but which exhibits an oscillation in the region  $ABC$ .

These radiations will be weakened by the diffusion and irregular refraction in the atmosphere outside  $def$  (fig. 2), light of the shortest wavelengths most. When again we put the resulting central

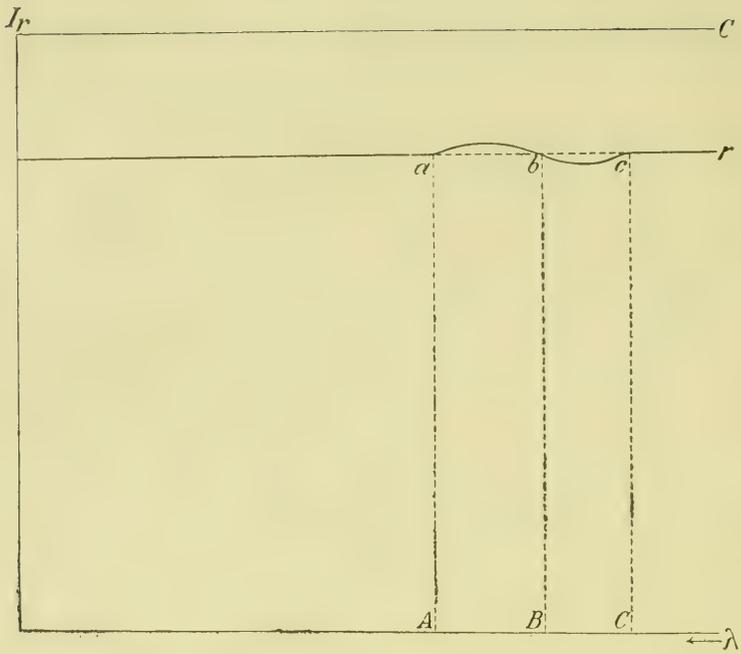


Fig. 4.

radiation for all  $= 100$ , the relative limb radiation will be represented by the line  $r$  in fig. 5. This radiation decreases therefore continu-

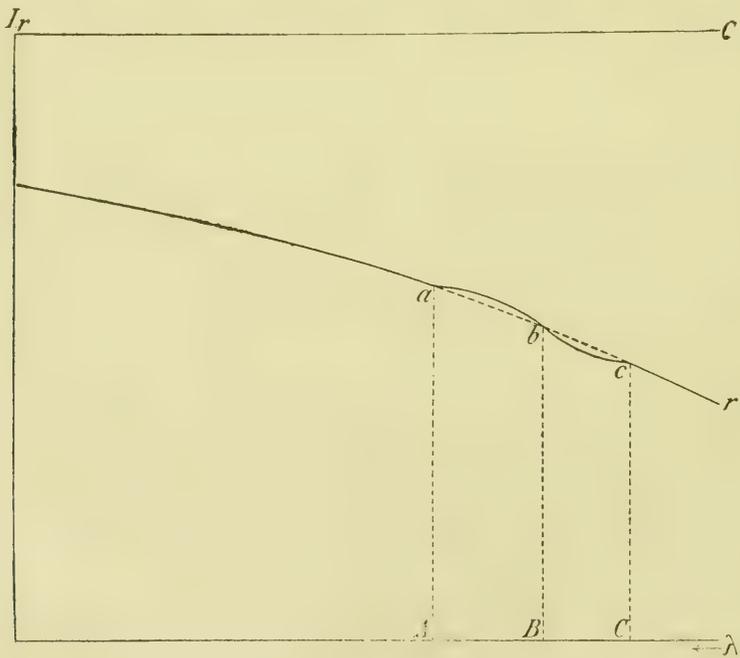


Fig. 5.

ously towards the violet, but presents the same oscillation in the region *ABC*.

Hence the relative distribution of intensity of the light on the sun's disc reckoned with respect to the centre, must present an anomaly in the neighbourhood of the maximum of energy. This anomaly bears entirely the same character and relates to the same region of the spectrum as the anomaly observed by ABBOT (c.f. fig. 5 with fig. 1; the part of the curve *r* outside *C* in fig. 5 has not been observed by ABBOT).

Summary. Starting only from the hypothesis that the sun may be conceived as a glowing gaseous body, in which the temperature and the density gradually decrease from the centre outward, and the outer layers of which consist of little luminous, little absorbing, but greatly diffusing and irregularly refracting gas-masses, we have derived an explanation of the anomaly, observed in the decrease of light of different wavelengths from the centre towards the limb of the solar disc.

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**Chemistry.** — *The Phenomenon after Anodic Polarisation.* I. By Prof. A. SMITS, G. L. C. LA BASTIDE, and J. A. VAN DEN ANDEL. (Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of March 29, 1919).

*Introduction.*

When the experimental electrical potential of iron is measured *during* the anodic polarisation, we observe that this potential becomes less negative as the density of the current increases. When this experiment is carried out in solutions containing few or no halogen ions, this phenomenon greatly increases above a definite density of current, and the iron then passes from the active into the passive state. Though this transition from the active into the passive state is not observed for all metals, all of them present, when they are anodes, a potential that is shifted more or less in the positive direction. This phenomenon of the anodic polarisation, like that of the cathodic polarisation, has been explained in a simple way by the recent views about the electromotive equilibria<sup>1)</sup>.

A new phenomenon, however, presented itself when somewhat more than two years ago SMITS and LOBRY DE BRUYN made use of a rotating commutator according to LE BLANC<sup>2)</sup> during the anodic polarisation of iron.

This commutator is constructed so that the potential difference is not measured during the passage of the current, but immediately after the current has been broken. With this method of procedure the remarkable phenomenon was discovered that when the current density was not so great that the iron became passive, the iron did not show a less negative, but a *more strongly negative* potential, which means that the iron exhibited a change of the potential *after* anodic polarisation in a direction opposite to that which was to be observed during the passage of the current.

At first it was supposed that this phenomenon had to be attributed to some mistake in the arrangement of the experiment, but it soon appeared that this was not the case, and that the same phenomenon, but slightly modified, also appeared for nickel, and as Dr. ATEN found afterwards, also for chromium<sup>3)</sup>.

<sup>1)</sup> Zeitschr. f. phys. Chem. **92**, 1 (1916).

<sup>2)</sup> Zeitschr. f. phys. Chem. **5**, 469 (1890).

<sup>3)</sup> These Proc. **XX**, 1119.

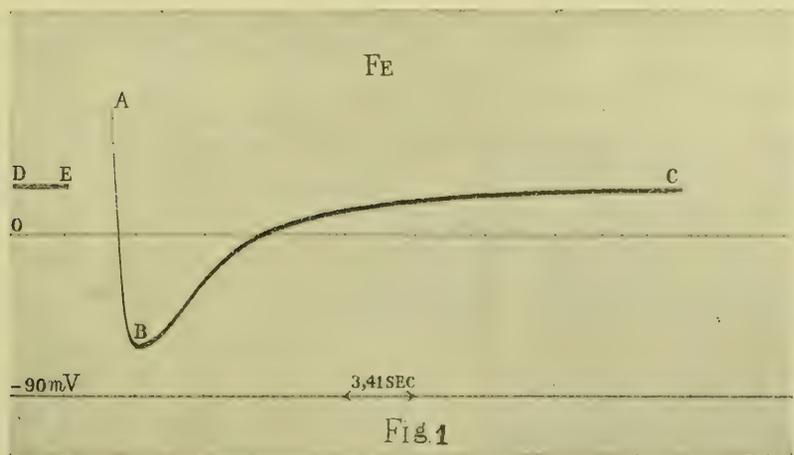
When the experiment was made without the said commutator, and the course of the potential of the iron with the time was examined immediately *after* anodic polarisation, it appeared that after having passed through a minimum, the potential again rapidly rose to the initial value.

As a fuller examination of this phenomenon would have led the investigation of SMITS and LOBRY DE BRUYN into a bye-way, this study was postponed to a later occasion, and this is the reason why the publication of this phenomenon has been withheld until now.

### *Registration of the Phenomenon.*

In order to represent the phenomenon graphically we have employed the photographic registration method, making use of MOLL's galvanometer. The potential of an iron electrode I was measured with respect to another auxiliary iron electrode II, a platinum tin leaf serving as cathode. The electrolyte was a  $\frac{1}{2}$  normal solution of ferrousulphate. The iron electrode I, which had an area of about 25 m.m<sup>2</sup>, was made anode for a short time, the density of the current being chosen so that the iron would have become passive with longer passage of the current. *Immediately* after this anodic polarisation the galvanometer was inserted into the circuit iron I-electrolyte-iron II, so that the course of the potential which iron I presented *after* the anodic polarisation, could be accurately registered.

The film now showed: first the potential of iron I with respect to iron II before the experiment, secondly the curve representing

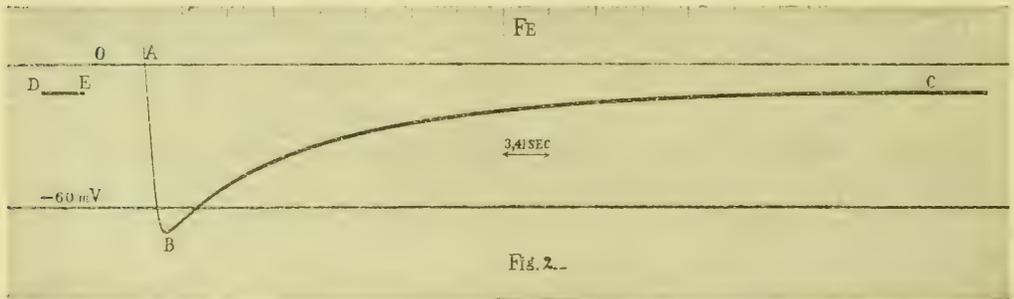


the course of the phenomenon, thirdly the zero-line, fourthly a line that corresponds to a definite potential difference, and finally time-lines given by a signalling apparatus.

Thus in our experiments with iron in  $\frac{1}{2}$  N. ferro-sulphate solution the following photographs were obtained.

The line DE represents the potential of iron I with respect to iron II before the anodic polarisation. The other two horizontal lines, which run all over the figure, are the zero-line and the line for  $-90$  milli Volt. The curve ABC represents the phenomenon observed *after* anodic polarisation of iron I with a density of current of  $\pm 0.360$  Amp/cm.<sup>2</sup>.

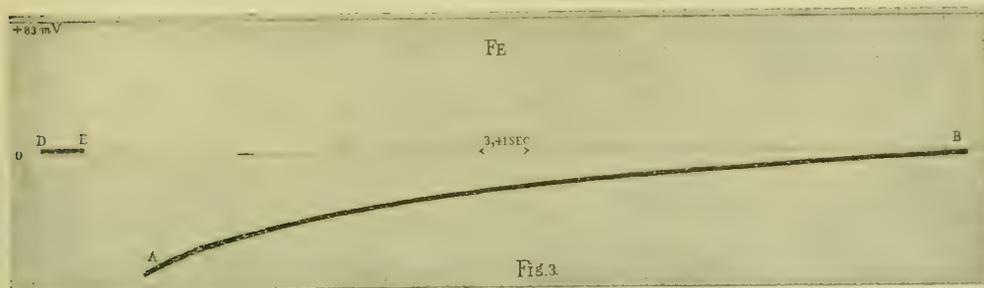
The iron electrode I is at first disturbed in noble direction after this anodic polarisation, but this disturbance ceases with great rapidity and the potential descends to a value which is about  $90$  milli Volt. more negative than the potential before the anodic polarisation. The iron keeps this negative potential only for a short



time, and rises, first with increasing, then with decreasing rapidity to the value observed before the anodic polarisation.

In fig. 2 the phenomenon has been photographed that was observed when instead of a solution of ferro-sulphate, a solution of  $\frac{1}{10}$  N. ferro-chloride was used. In a few words we may remind here that the increase of concentration in the boundary layer through the short anodic solution has brought about a slight change of the potential in positive direction. In order to find out in how far the found phenomenon is something particular that is caused by the *anodic* polarisation of iron, also the course of the potential of the iron was photographed *after cathodic* polarisation. This photo is reproduced in fig. 3, we see that the iron after cathodic polarisation with a density of current of  $\pm 0.360$  Amp/cm.<sup>2</sup>. is still pretty greatly disturbed in base direction, the potential of the iron was, namely, in this experiment at first about  $80$  milli Volt. more negative than before the cathodic polarisation, which potential difference had,

however, almost entirely disappeared again after a minute. We will point out here that a small part of the potential in negative direction



is the consequence of the decrease of concentration in the boundary layer during the cathodic deposition of the metal.

As it had appeared before, as has already been mentioned, that nickel, though much more inert, showed the same phenomenon as iron, the same experiments have been made with nickel as with iron, in which the following result was obtained.

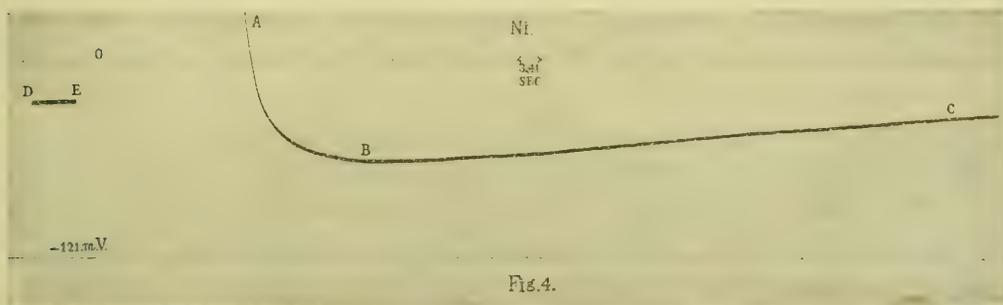


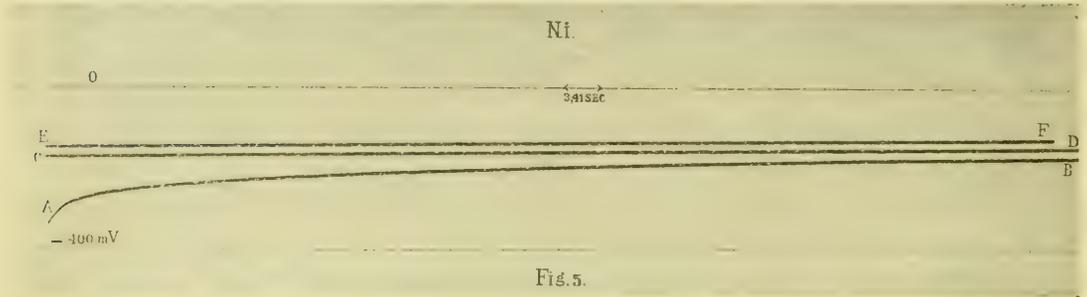
Fig. 4 gives the course of the nickel electrode with an area of 80 mm.<sup>2</sup> in a solution of  $\frac{1}{10}$  N NiSO<sub>4</sub> after anodic polarisation with a current density of  $\pm 0.075$  amp./cm., from which appears that here the same phenomenon is seen as for iron; the potential of the metal becomes more negative after anodic polarisation, and after having reached a minimum value, it slowly returns to the original value of before the anodic polarisation. The phenomenon is the same here as for iron, but it takes place here with much less rapidity.

In fig. 5 the result is recorded of the determination of the potential difference for nickel after cathodic polarisation with the same current density.

The nickel is at first pretty greatly disturbed in base direction, the potential is about 240 m.V. more negative than before the cathodic polarisation, but becomes less and less negative with decreasing

rapidity, and does not present the potential of before the cathodic polarisation again until after nearly five minutes. As the whole course occupied so much time, we have made the box with the film execute some revolutions, so that the successive pieces have come above each other on the photo.

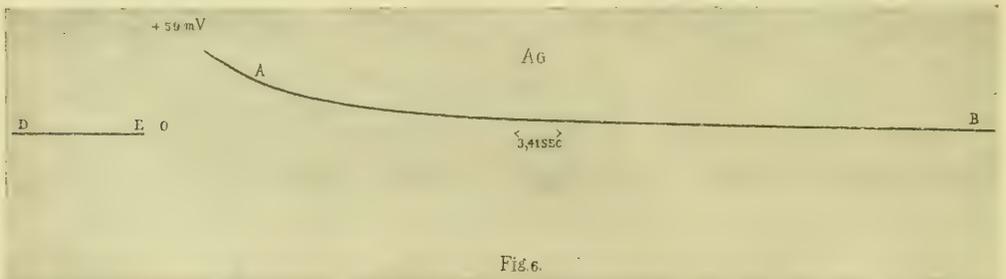
It is of importance to state here that RAHTERT<sup>1)</sup>, who has also carried out experiments on this subject, found a more negative



potential for nickel *after* anodic polarisation in agreement with us, but after cathodic polarisation a much more positive potential. This is in conflict with what was found by us, and must probably be attributed to a mistake in the arrangement in RAHTERT's investigation. For on earlier occasions the phenomenon photographed in fig. 5 was always found by SMITS and LOBY DE BRUYN and now by us.

As it was of importance to include also a few other metals in our considerations, we have also investigated silver and copper immersed in a silver-nitrate, resp. copper sulphate solution with the following result.

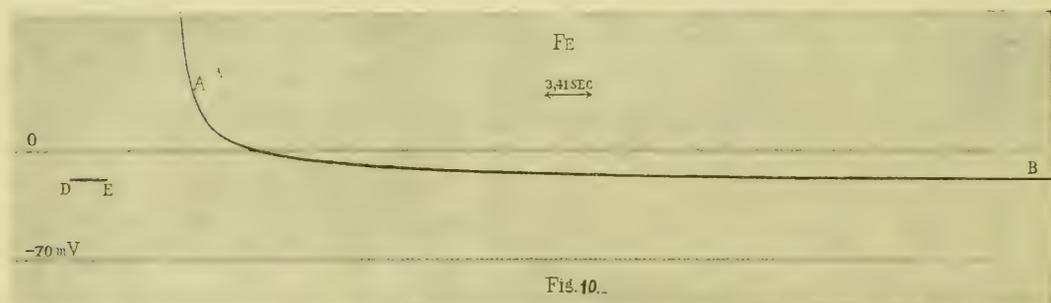
The photo fig. 6 is a reproduction of the potential course of a silver electrode with an area of 42 mm.<sup>2</sup> in a 0,1 N Ag NO<sub>3</sub>-solution after anodic polarisation with a density of current of 0,333 Amp./cm.<sup>2</sup>,



<sup>1)</sup> Zeitschr. f. phys. Chem. 86, 567 (1914).



So far a deviating behaviour has, therefore, only been found for iron, nickel, and chromium, hence for metals that have ions of different valencies, for this reason it was desirable to examine whether the phenomenon would continue to exist when the ratio of concentration between the different ions in the electrolyte is everywhere so, that the metal can coexist in state of internal equilibrium with this electrolyte. This was investigated for iron by boiling a solution of Ferro-chloride with iron-powder for a long time in the vessel in which the definitive experiment was to be made, leading over pure hydrogen at the same time. The solution thus obtained was 0.6 N. The result obtained with this solution was convincing in a high



degree. As fig. 10 shows after anodic polarisation there was nothing to be seen of a minimum, and at least qualitatively the iron behaved like silver and copper.

It is now natural to assume that also for nickel and chromium the phenomenon can be made to disappear in the same way, about which experiments are now being in progress.

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Chemistry of the University.*

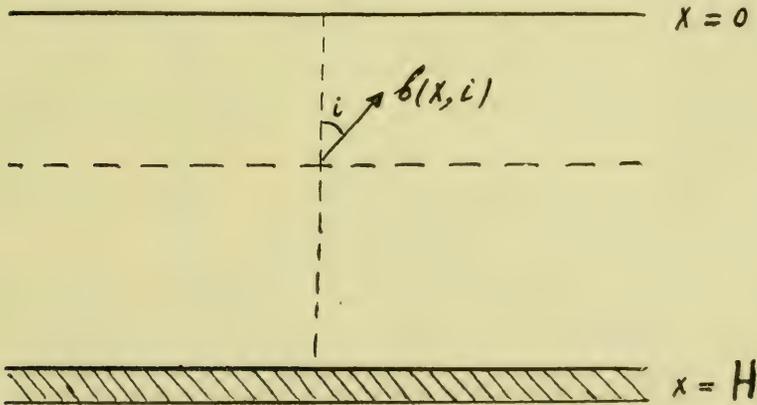
*Amsterdam, March 27<sup>th</sup> 1919.*

**Physics.** — “*On the Effective Temperature of the Sun*”. Some remarks in connection with an article by DEFANT: “*Diffusion und Absorption in der Sonnenatmosphäre.*” By H. GROOT. (Communicated by Prof. W. H. JULIUS).

(Communicated in the meeting of March 29, 1919).

In a paper “*Ueber Diffusion und Absorption in der Sonnenatmosphäre*” (Sitz. Ber. d. Berl. Akad. 1914) SCHWARZSCHILD treated the problem of the radiation of a plain layer, which must be imagined as an absolutely black body, and above which there is an absorbing and dispersive atmosphere. When on a layer (see figure 1) bounded by the planes  $x = 0$  and  $x = H$  radiation of intensity  $S$ , starting from the black body  $ZZ'$  falls from all directions, SCHWARZSCHILD denotes by  $b(x, i)$  the radiation which passes through the plane  $x$  in the same sense as the radiation  $S$ , and at an angle  $i$  with the normal to the boundary layers, and then tries to find a formula for  $b(0, i)$ .

Fig. 1.



Black body.

Accordingly  $b(0, i)$  is the total intensity of light that passes out at an angle  $i$  at the boundary of the atmosphere, and is built up of direct light and light that is dispersed once, twice etc.

SCHWARZSCHILD succeeds in solving this problem for two special cases, and finds:

a. Limiting case of exclusive absorption ( $\sigma = 0$ ):

$$b(0, i) = a + \frac{b \cos i}{k} (1 - e^{-kH \sec i}) \dots \dots \dots (1)$$

b. Limiting case of exclusive dispersion ( $k = 0$ ):

$$b(0,i) = \frac{0.5 + \cos i}{1 + \sigma H} + \frac{0.5 - \cos i}{1 + \sigma H} e^{-\sigma H \sec i} \dots \dots \dots (2)$$

Here  $k =$  coefficient of absorption,  $\sigma =$  coeff. of diffusion,  $H =$  height of the atmosphere,  $a$  and  $b$  are two numerical constants.

In his article: "Diffusion und Absorption in der Sonnenatmosphäre" (Sitz. Ber. d. K. Akad. zu Wien, Abh. II $\alpha$ . Bnd. 125 (1914)). A. DEFANT by the aid of data which he derives from ABBOT'S observations on the decrease of the intensity of radiation on the sun's disc from the centre towards the limb (Annals of the Astr. Observ. of Smithsonian Inst. Vol. III, Washington 1913, p. 158), tries to decide which of the two causes, absorption or dispersion, appears to be most active on the sun.

By means of a kind of "trial and error" method he succeeds in deriving a formula:

$$b(0,i) = \frac{0.5 + \cos i + e^{-0.0405 \lambda^{-4} \sec i} (0.5 - \cos i) - 0.3804 + 0.3136 \cos i}{1 + 0.0405 \lambda^{-4}} \dots (3)$$

which is halfway between (1) and (2) and yields numerically accurate values. This seems to point to this that the diffusion effect by far preponderates, but is yet influenced by a slight absorption.

In how far the considerations through which he arrives at formula (3), are of value, must be left undecided here. It is certain that the numerical values are pretty accurate, as table I shows convincingly.

TABLE I.

<i>cos i</i>	$\lambda = 0.433 \mu$			$\lambda = 0.604 \mu$			$\lambda = 1.031 \mu$		
	<i>b</i> (0, <i>i</i> )	<i>b</i> (0, <i>i</i> ) × 355	Observed value	<i>b</i> (0, <i>i</i> )	<i>b</i> (0, <i>i</i> ) × 375	Observed value	<i>b</i> (0, <i>i</i> )	<i>b</i> (0, <i>i</i> ) × 117	Observed value
1.0	1.2752	453	456	1.0643	399	399	0.9486	111	111
0.9	1.1906	423	419	1.0164	381	380	0.9175	107	107
0.8	1.0996	390	384	0.9656	361	360	0.8838	103	105
0.7	1.0006	355	348	0.9097	341	337	0.8491	99.4	100
0.6	0.8932	317	309	0.8476	318	313	0.8137	95.2	95.8
0.5	0.7764	276	277	0.7764	291	2·8	0.7765	90.9	90.0
0.4	0.6506	231	238	0.6917	259	265	0.7366	86.2	86.2
0.3	0.5180	184	192	0.5863	220	230	0.6912	80.9	80.9

Explanation of table I:

In this table  $b(0,i)$  calculated for the values of  $\cos i$  is given in the first column for three different values of  $\lambda$  by the aid of (3). In column 3 the found values of  $b(0,i)$  have been multiplied by a factor in order to render a comparison with ABBOT's values, recorded in the fourth column, possible.

By the aid of (3) and ABBOT's values, which I subjoin, DEFANT tries to draw a conclusion on the effective temperature of the sun.

Wavelength in $\mu$	Radiation in the centre of the sun's disc.
0.323	144
0.386	338
0.433	456
0.456	515
0.481	511
0.501	489
0.534	463
0.604	399
0.670	333
0.699	307
0.866	174
1.031	111
1.225	77.6
1.655	39.5
2.097	14.0

(ABBOT's values).

His reasoning is as follows:

For  $i = 0$  we obtain  $b(0,0)$  i.e. formula (3) then gives for every wavelength  $\lambda$  the intensity of radiation passing out in the centre of the sun's disc, when that of the area of the photosphere for this  $\lambda$  is put equal to 1. What we measure is, however, not the quantity  $b(0,0)$ , but the radiation  $i_\lambda$ , actually passing out, which is in relation with  $b(0,0)$  through the formula:

$$I_\lambda = \frac{i_\lambda}{b(0,0)} \dots \dots \dots (4)$$

in which  $I_\lambda$  is the intensity of radiation in the spectrum of the photosphere (considered as absolutely black body) for the wavelength  $\lambda$ .

By the aid of (3) and (4) and ABBOT's values the following table can, therefore, be calculated for  $I_\lambda$  (table II). According to our

supposition the photosphere radiates as an absolutely black body, so that PLANCK's formula may be applied, according to which <sup>1)</sup>:

$$I_\lambda = \frac{7.211 \times 10^8}{\lambda^5 \left( 10^{\frac{2.1563 \times 2890}{\lambda T}} - 1 \right)} \dots \dots \dots (5)$$

The quantities  $I_\lambda$  from the table are expressed in an unknown unity. When we consider this unity and  $T$  as unknown quantities, then  $T$  can be solved from two values of  $I_\lambda$  (for  $\lambda_1$  and  $\lambda_2$  e.g.).

*If our basis is correct, we must find the same temperature from all combinations in pairs of  $I_\lambda$ .*

DEFANT calculates  $T$  from the combinations

$$\left. \begin{array}{l} \lambda_1 = 0.5 \quad I_\lambda = 700 \\ \lambda_2 = 0.9 \quad I_\lambda = 180 \end{array} \right\} T = 8900^\circ$$

and

$$\left. \begin{array}{l} \lambda_1 = 0.6 \quad I_\lambda = 350 \\ \lambda_2 = 1.2 \quad I_\lambda = 90 \end{array} \right\} T = 8700^\circ$$

and considers the agreement "genügend". (loc. cit. p. 517).

TABLE II.

Wavelength $\lambda$	$1 + 0.0405 \lambda^{-4}$	$b(0,0)$	$i_\lambda$	$I_\lambda$
0.323	4.721	0.299	144	481.8
0.386	2.824	0.479	338	705.7
0.433	2.152	0.593	456	769.5
0.456	1.937	0.644	515	799.8
0.481	1.757	0.682	511	748.9
0.501	1.646	0.710	489	688.7
0.534	1.498	0.737	463	628.1
0.604	1.304	0.816	399	500.2
0.670	1.201	0.853	333	390.5
0.699	1.170	0.864	307	355.2
0.866	1.072	0.903	174	192.8
1.031	1.036	0.918	111	120.9
1.225	1.018	0.926	77.6	83.8
1.655	1.005 <sub>4</sub>	0.931	39.5	42.4
2.097	1.002 <sub>1</sub>	0.932	14.0	15.0

<sup>1)</sup> The constants are those used by DEFANT.

Unfortunately, however, a fatal error has slipped in. For to  $\lambda = 0.6$  does not correspond  $I_\lambda = 350$ , but — (interpolating graphically) —  $I_\lambda = 506$ , which yields  $T = 6600^\circ$  instead of  $8700^\circ$ , so that agreement is out of the question.

A serious objection to the whole method seems perfectly obvious to me, namely this:

The assumption that all kinds of light come to us from one photospheric surface, in other words that light of various wavelengths should come from the same depth of the sun, appears more and more untenable in the light of recent researches (see e.g. the thesis for the doctorate of J. SPIJKERBOER “Verstrooiing van licht en intensiteitsverdeeling over de zonneschijf” (1917) (Dispersion of light and Distribution of Intensity over the Sun’s Disc)). If, however, in reality light of different wavelengths originates from different parts of the sun, it becomes very questionable whether we shall be allowed to apply PLANCK’S formula, as we saw DEFANT do. For this would mean that we supposed every kind of light to have, as it were, a kind of “photosphere of its own”, which radiates as a black body, the photosphere for the greater wavelengths lying deeper than that for the smaller. It might then be expected that the temperature determined with PLANCK’S formula, becomes a function of  $\lambda$ , i. e. *T would be the greater as  $\lambda$  increases.*

In this latter remark we have a means to investigate whether the hypothesis that the photospheres overlap each other like scales can find a semblance of justification.

By graphical interpolation from the values of table II I construed table III:

TABLE III.

$\lambda$	$I_\lambda$	$\lambda$	$I_\lambda$	$\lambda$	$I_\lambda$	$\lambda$	$I_\lambda$
0.40	709	0.70	342	1.00	134	1.60	46
0.45	791	0.75	284	1.10	104	1.70	39
0.50	714	0.80	239	1.20	87	1.80	32
0.55	604	0.85	201	1.30	74	1.90	25
0.60	506	0.90	174	1.40	64	2.00	19
0.65	418	0.95	153	1.50	55		

As we do not know the unity in which  $I_\lambda$  is expressed, we require, as was already remarked before, always two values of  $I_\lambda$  ( $\alpha_1$  and  $\alpha_2$ ) to find  $T$ .

The calculation comes to this:

Let  $A = 7.210 \times 10^8$ ,  $\beta = 2.1563 \times 2890$ ,  $\alpha_1$  and  $\alpha_2$  the values of  $I_\lambda$  corresponding to  $\lambda_1$  and  $\lambda_2$ ,  $f$  an unknown factor dependent on the unities in which  $I_\lambda$  has been measured. Then the following equations hold:

$$\alpha_1 f = \frac{A}{\lambda_1^5 \left( 10^{\frac{\beta}{\lambda_1 T}} - 1 \right)} \quad \text{and} \quad \alpha_2 f = \frac{A}{\lambda_2^5 \left( 10^{\frac{\beta}{\lambda_2 T}} - 1 \right)} \quad \dots \quad (6)$$

When we choose the values of  $\lambda$  so that  $\lambda_2 = 2\lambda_1$ , and when we put  $10^{\frac{\beta}{\lambda_2 T}} = x$ , we easily get:

$$\left( \frac{\alpha_2}{\alpha_1} \right) \left( \frac{\lambda_1}{\lambda_2} \right)^5 (x^2 - 1) = x + 1 \quad \dots \quad (7)$$

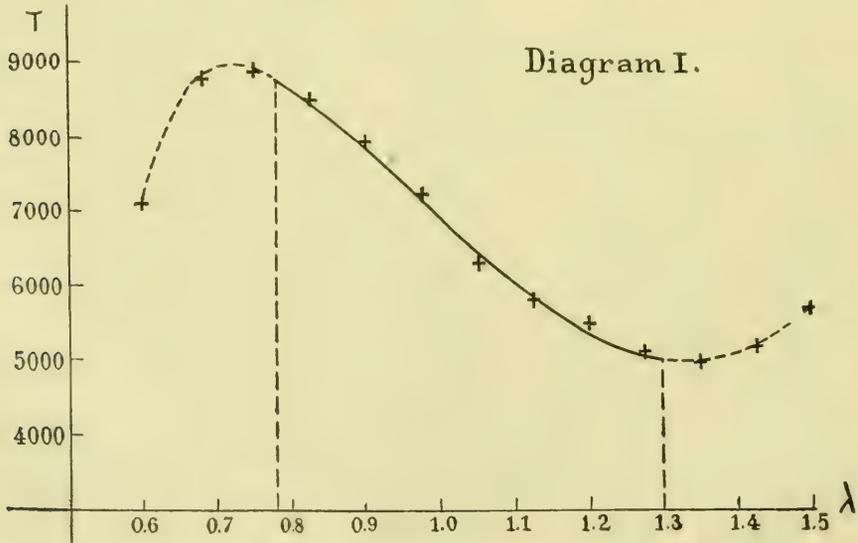
The root  $x = 1$  yields  $T = \infty$ , has therefore no physical meaning, so that we find  $T$  from:

$$\left. \begin{aligned} x &= 32 \left( \frac{\alpha_2}{\alpha_1} \right) - 1 \\ T &= \frac{\beta}{\lambda_2 \lg x} \end{aligned} \right\} \dots \quad (8)$$

In this way I found:

$\lambda_2 =$	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$T =$	7200	8800	8900	8500	7900	7200	6300	5800	5500	5100	5000	5200	5700

hence on an average a decrease of  $T$  with increase of  $\lambda$  (see diagram 1).



Nor is this manner entirely satisfactory; for now we do not know to what  $\lambda$  the found  $T$  should properly speaking belong, because the two values of  $\lambda$  ( $\lambda_1$  and  $\lambda_2$ ), which are required, can lie pretty far apart in this way of calculation. Does for ( $\lambda_1 = 0.9$ ,  $\lambda_2 = 1.8$ )  $T$  e. g. belong to  $\lambda_1$ , to  $\lambda_2$ , or to a value lying somewhere between  $\lambda_1$  and  $\lambda_2$ ?

When we want to avoid this difficulty, we may treat the equations (6) as follows:

Let  $\lambda_1 : \lambda_2$  be  $= n : m$  or

$$\left. \begin{aligned} \lambda_1 &= \frac{\rho}{m} \\ \lambda_2 &= \frac{\rho}{n} \end{aligned} \right\} \dots \dots \dots (9)$$

we find easily:

$$10^{\frac{m\beta}{\rho T}} - 1 = \left(\frac{\alpha_2}{\alpha_1}\right) \left(\frac{m}{n}\right)^5 \left(10^{\frac{n\beta}{\rho T}} - 1\right) \dots \dots (10)$$

Put:

$$10^{\frac{\beta}{\rho T}} = z \quad \left(\frac{\alpha_2}{\alpha_1}\right) \left(\frac{m}{n}\right)^5 = C \dots \dots (11)$$

then (10) passes into:

$$z^m - Cz^n + (C-1) = 0 \dots \dots (12)$$

When we take care that  $m$  is  $= n + 1$ , the shape becomes somewhat more suitable for numerical approximation, namely:

$$z^n(z - C) + (C-1) = 0 \dots \dots (12a)$$

When  $z$  has been sufficiently closely approximated,  $T$  follows from:

$$T = \frac{\beta}{\rho \lg z} \dots \dots (13)$$

In this way  $\lambda_1$  and  $\lambda_2$  can be brought close enough together to exclude indefiniteness in the choice of the  $\lambda$  to which  $T$  belongs.

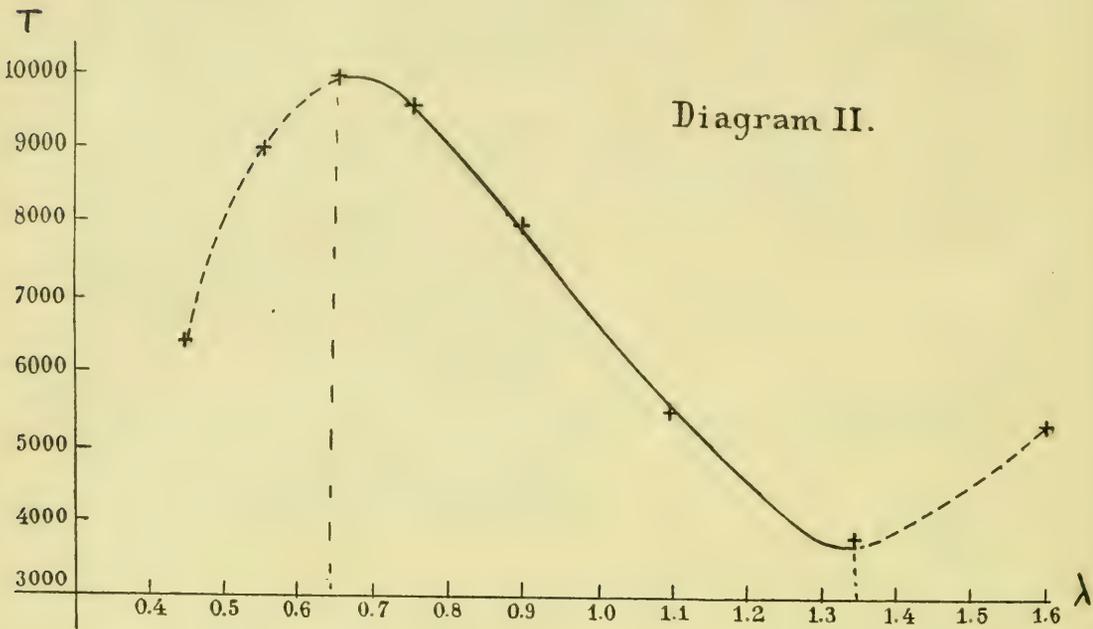
Thus we found:

$\lambda_1$	0.4	0.5	0.6	0.7	0.8	1.0	1.2	1.5	1.8
$\lambda_2$	0.5	0.6	0.7	0.8	1.0	1.2	1.5	1.8	2.0
$T$	(6400)	9000	10.000	9600	8000	5500	3800	5400	—

so that on an average:

$\lambda = 0.5-0.7$	$T = 9500$
$0.7-1.2$	6000
$1.2-1.8$	4600

hence a similar result as for the first method. (See diagram II). The deviations inter se are now much larger, as was, indeed, to be



expected, as on the small intervals  $\lambda_1 - \lambda_2$  the inevitable errors in  $I_\lambda$  (an experimental quantity!) make themselves very greatly felt.

Thus  $\left. \begin{array}{l} \lambda_1 = 1.8 \\ \lambda_2 = 2.0 \end{array} \right\}$  give an imaginary value for  $T$ , but when for  $\lambda_2 = 2.0$   $I_\lambda = 22$  is taken instead of  $I_\lambda = 19$ , then  $T$  would become  $= 18000^\circ$ .

In this manner particularly the smaller values of  $I_\lambda$  are unfavourable, hence the values for  $\lambda_1 = 1.5$  and  $\lambda_1 = 1.8$  are not much to be trusted.

The values of  $I_\lambda$  for  $\lambda < 0.5$  are strictly speaking also unreliable, because the graphical interpolation — as indeed every other too — becomes very inaccurate here.

When we leave all these doubtful values of  $T$  out of consideration we come to the result that particularly in the region of the reliable values of  $T$  (the full line in the diagram) *there is an unmistakable tendency of  $T$  to decrease on the increase of  $\lambda$ , hence exactly the reverse of what we thought we might expect a priori.*

In a following paper I propose to discuss the question to what this unexpected result is to be attributed.

Utrecht, March 1919.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN  
TE AMSTERDAM.

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PROCEEDINGS

VOLUME XXII

N<sup>o</sup>. 2.

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President: Prof. H. A. LORENTZ.

Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en  
Natuurkundige Afdeeling," Vol. XXVII).

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**Physiology.** — “*On the Index Loquelae*”. By F. ROELS and L. MOLL.  
(Communicated by Prof. H. ZWAARDEMAKER).

(Communicated in the meeting of December 28, 1918).

The idea *index vocalis* has been introduced by GRADENIGO<sup>1)</sup> and denotes the quantitative relation between whispers of moderate energy and ordinary speech. Practically this relation is best determined with the aid of the extreme distances at which they are audible. According to GRADENIGO the average index for vowels is  $\frac{1}{7}$  or  $\frac{1}{8}$ <sup>2)</sup>. According to WOLF<sup>3)</sup>, ZWAARDEMAKER<sup>4)</sup>, and REUTER<sup>5)</sup> the average index ranges from  $\frac{1}{5}$  to  $\frac{1}{7}$ . For twenty monosyllabic, aequiintensive, isozonal test-words ZWAARDEMAKER—QUIX<sup>6)</sup> and REUTER<sup>7)</sup> found the index  $\frac{1}{12}$ .

The determination of the relative energy of whispers and ordinary speech being rather difficult (as it is difficult to distinguish exactly between what is meant by the one and what by the other), ZWAARDEMAKER has had recourse to RALEIGH's mirror for sound-measurement<sup>8)</sup>. It then appeared that the normal average index

*vocalis*  $\frac{\text{reserve-air whisper}}{\text{ordinary speech}} = \frac{1}{12}$ , the normal average index *vocalis*

$\frac{\text{stage-whisper}}{\text{ordinary speech}} = \frac{1}{4}$ . These indices enable us to reduce the whisper-values obtained for vowels and words to ordinary speech-values.

In experimenting on the intensity of audition separate words are used which consist of isozonal and aequi-intensive sounds, which, for

<sup>1)</sup> GRADENIGO. Communication au congrès internat. d'otologie. Budapest 1909. GRADENIGO et STEFANINI. Sur l'acoumétrie. Propositions et études. Archives internat. de laryngologie, d'otologie et de rhinologie, 1911.

<sup>2)</sup> ZWAARDEMAKER. Over den *index vocalis* bij keuringen en bij de studie der ziektebeelden. (Handelingen van het XIVe. Ned. Nat.- en Gen. Congres te Delft 1913), p. 370.

<sup>3)</sup> WOLF. Sprache und Ohr. 1870. Frankfurt.

<sup>4)</sup> ZWAARDEMAKER *ibid.*

<sup>5)</sup> REUTER. Zeitschr. f. Ohrenheilkunde. Bd. 47, blz. 91; Onderzoekingen Physiol. Lab. Utrecht (5) Vol. 5. p. 239.

<sup>6)</sup> ZWAARDEMAKER. Over geluidmeting. Zittingsverslag dezer Academie van 23 April 1915. Vol. XXIII. p. 1405.

<sup>7)</sup> REUTER. Onderzoekingen Physiol. Lab. Utrecht (5) Vol. 5. p. 249.

<sup>8)</sup> ZWAARDEMAKER and QUIX. De studie van het spraakgehoor. Onderzoekingen Physiol. Lab. Utrecht (5) Vol. 5. p. 1.

the very reason that every component of the word is heard as distinctly as the others, excludes guessing almost altogether and facilitates the finding of a definite zone of the scale. The question arises, however, though it is not probable, whether the hearing distance extends for whispered sentences as far as for words and vowels. ZWAARDEMAKER believes on the ground of various experiences that the distance at which whispered sentences are heard in a closed space amounts to 1 to 3 m.<sup>1)</sup> according to the words that are chosen. He calls the relation between this average distance and that at which whispered sounds are heard: *index loquelae*.

In the present investigation we purposed originally only to determine the *index loquelae*. However, the combination of the whisper-method with that of the systematic or experimental introspection, procured us a number of data of some importance for the psychology of perception. We availed ourselves of these data only in so far as they influence the determination of the *index loquelae*, the subject proper of this paper.

The material used by us consisted of rather simple sentences. The shortest consisted of 3, the longest of 14 syllables. Most often the sentences were in the indicative mood, seldom in the subjunctive mood or in the imperative mood. Though the present tense was used most frequently, we sometimes also employed the past and the future. Negative sentences were rare. In selecting our material we took an adequate number of symmetrical sentences, i.e. sentences having the same number of syllables before and after the copula, which consisted of one syllable, or an even number of syllables, or we chose such sentences as were divided by a word or by appropriate punctuation into two parts of an equal number of syllables (e.g. waste not, want not; a friend in need is a friend indeed). In most sentences there was a subject, a predicate and an object in the given order; sentences without one of these parts or with a verb split up as the construction required, we term irregular sentences. Proverbs and proverbial sayings were seldom used; sometimes, but not often, a French or Latin sentence was presented to the subject. All in all we made 328 experiments with 4 subjects (R., M., D. and A.). Table I gives a survey of the material with respect to the number of syllables, the symmetry and the asymmetry, the regular or the irregular construction of the sentences. It should be noted, that some test-sentences have

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<sup>1)</sup> ZWAARDEMAKER. Ueber die Anwendung von Sig. Exners Akustik von Hörsälen auf die Theorie der medizinischen Hörapparate. Wiener Medizinische Wochenschrift. Nr. 14, 1916.

TABLE I.

Subj.	Number of Ex.	Number of Sentences										Irregular sentences.	Proverbs.
		3 s.	4 s.	5 s.	6 s.	7 s.	8 s.	9 s.	10 and more sentences.	Symm. sentences.			
R.	153	2	17	25	33	24	27	14	11	26	21	21	
M.	74	2	12	12	14	14	7	2	11	12	23	12	
D.	86	1	15	12	29	14	11	3	1	12	7	—	
A.	15	—	—	—	—	2	5	3	5	4	4	2	
Total.	328	5	44	49	76	54	50	22	28	54	55	35	

TABLE II.

Subj.	3 s.		4 s.		5 s.		6 s.		7 s.		8 s.		9 s.		10 and more s.								
	M.	M.D.	M.	M.D.																			
R.	4	—	3.88	1.12	4	3.33	1.08	3	3.18	1.10	3	3.74	0.90	4	3.50	0.89	3	4.11	0.82	4	3	0.50	3
M.	5	—	3.71	0.91	4	3.12	1.06	3	2.64	0.73	2.5	2.71	0.91	3	2.35	0.71	2	2.25	—	—	1.83	0.67	1.5
D.	5	—	3.23	1.04	3	4.37	0.73	5	3.50	1.17	4	3.82	0.79	4	3.77	1.02	4	3.67	—	—	—	—	—

contributed values for two of the last three columns; a proverb like "a friend in need is a friend indeed" contributed to the making of the column of proverbs as well as to that of the symmetrical sentences.

The material was presented in whispers with reserve air. The subject sat ready to receive the whispers with the right ear, vertical to a line from this ear to the mouth of the experimenter. The maximal distance was 5 m.; if the subject failed to correctly repeat the spoken words at that distance, it was lessened every time by 1 m. until the spoken words were reproduced correctly. When the distance was reduced to 2 m., it was lessened every time by  $\frac{1}{2}$  m. Careful introspection, at which the subject recorded minutely what he had heard, followed after each stage of the experiment. At every sitting 12 experiments were made at most, to prevent fatigue. They were all carried out in the over-furnished library of the Utrecht Clinic for Psychiatry and Neurology. Before starting the experiments proper, we determined the audition of our subjects by means of the words given by ZWAARDEMAKER and QUIX in "De Studie van het Spraakgehoor"<sup>1</sup>). It proved to be normal with all of them; the hearing-distance varied for the words employed from 6 to 30 m.

In Table II we have calculated the mean distances (M.), for every one of the three subjects R. M. and D., at which sentences of the given number of syllables were heard (the values procured by A. are left out of account as being too small in number).

As precision-indices we also give the mean deviations (M. D.) and the central values (C. V.).

It is obvious that with a larger number of syllables the sentences are heard less distinctly. For R. M. and D. this difference is resp. 1, 3.17 and 1.33 m., in the case of sentences of 3 to 10 or more syllables. Besides to the influence of factors which we cannot discuss further here, these considerable individual differences are to be ascribed chiefly to the various tasks imposed upon the subjects. It stands to reason that proverbs and proverbial sayings are generally heard at greater distances than non-proverbial expressions; symmetrical sentences farther than asymmetrical sentences; on the other hand irregular sentences are more difficult to hear than those that are constructed regularly. Table III (in which the frequency of the various kinds of sentences is tabulated in percentages of the total number of test-sentences presented) shows clearly that the intensity of audition varied considerably.

The difference of the distances at which sentences of 3 and those

<sup>1</sup>) Onderzoekingen Physiol. Lab. Utrecht (5) vol. 5 p. 1.

of 10 or more syllables are heard, is greatest with the subject M. (3.17 m.). This is owing to a considerable extent, to the high per-

TABLE III.

Subj.	Symmetrical sentences.	Proverbs.	Irregular sentences.
R.	17	13.7	13.7
M.	16.2	16.2	31.1
D.	14	—	8.1

centage of irregular sentences (31.3) with their unfavourable influence upon auditory sensation. This influence was much less on R. and D. (resp. 13.7 and 8.1 %). The comparatively small difference in the case of R. (1 m.) is to be ascribed to the circumstance that symmetrical sentences and proverbs constitute about  $\frac{1}{3}$  (30.7 %) of the total number of the test-sentences that were used. D., being a FLEMING and consequently unacquainted with Dutch proverbs (which were not presented to him) is placed between R. and M., quite in accordance with the difference in the hearing distance with 14 % favourable and 8.1 % unfavourable factors.

When we look at Table II more closely, it appears that intra-individual differences are not less conspicuous than the inter-individual ones. For, though it is true that the hearing distance with the three subjects for trisyllabic sentences is considerably greater than for sentences of 10 or more syllables, the distance decreases far from regularly. In the case of M. the decrease is gradual with only few exceptions.

Sentences of an odd number of syllables are generally heard better by R. and D. than those of an even number having one syllable more, they even perceive sentences of 9 syllables resp. 0.23 m. and 0.44 m. farther than those of 4 syllables.

The cause of this phenomenon, so strange at first, is the fact that a symmetrical construction, which largely promotes our hearing faculty, occurs more often with sentences of an odd number of syllables than with those of an even number. This is because with all symmetrical sentences the number of syllables before and after the copula is the same, while with most of them the copula is made up of one syllable (is, was, had, are etc.) Besides the agreement just alluded to there is also some difference in the shifting of the hearing distance, when the number of syllables increases. In the case of R.

the distance generally increases with an increase of the number of syllables from 4 to 9, for sentences of an odd number as well as an even number — with a few exceptions in the latter case. With *D.*, however, a diminution is observed with sentences of an odd number of syllables, an increase with those of an even number. We have seen, then, that while in the case of *M* the distance decreases regularly (with a few exceptions) from 4 to 9 syllables, with *R.* it increases for an odd as well as for an even number of syllables; with *D.* an increase is observed for the odd and the decrease for the even numbers.

What is the cause of the increase of the hearing distance with an increase of the number of syllables in the cases of the subjects *R* and *D*, especially the first; and what gives rise to the differences in the subjects? One might expect that with the increase of the number of syllables the retro-active inhibition is more keenly felt, in consequence of which the faculty of perception is weaker.

Audition, however, results from the joint action of apperception and assimilation; the latter, just as in the case of reading, plays a prominent part here. Now the process of assimilation requires a minimum of apperceived elements; it can readily be imagined that in sentences of a small number of syllables such elements are as a rule too small in number to serve as starting-points for the assimilation-process. The assertion that with shorter sentences the assimilation-process should naturally be less extensive and should therefore require fewer points of contact, does not hold good. We have observed again and again that a certain number of apperceived elements, no matter how short the sentences may be, is required to prevent assimilation from becoming mere guessing. So the number of apperceived elements required for test-sentences of various length is not proportionate to the increase of the number of syllables. It follows that the reproduction-tendencies emanating from the apperceived elements are, within certain limits, stronger than the inhibition they exert upon each other. The reverse seems to be the case with all subjects for sentences of ten or more syllables.

It is now easy to explain also the individual differences. With *M.*, for whom we observe a gradual decrease of the hearing distance with an increase of the number of syllables, the influence of inhibition is stronger than that of assimilation. With *R.*, for whom the hearing distance increases with the length of the sentences of an odd as well as an even number of syllables, the reverse obtains, viz. the unfavourable effect of inhibition is not only arrested, but with the increase of the number of points of contact, assimilation

also acts more intensely. Subject D. takes up a place between R. and M. in that for sentences of an even number of syllables assimilation gets the start of inhibition when the sentences get longer, whereas for sentences of an odd number of syllables inhibition proves the stronger. We are unable to account for this strange phenomenon.

By means of the normal average index vocalis  $\frac{\text{reserve-air whisper}}{\text{ordinary speech}} = \frac{1}{11}$ ,

which index presumably also holds for whispered sentences, we have reduced in Table IV the whisper-values for sentences of various length to ordinary-speech values. Both sorts of values in Table IV constitute the average of the hearing distances, determined for the subjects R., M., and D., with respect to the length of the various sentences.

TABEL IV.

Syllables.	Whispers.	Loud speech.
3	4.67	60.71
4	3.61	46.93
5	3.61	46.93
6	3.11	40.43
7	3.42	44.46
8	3.21	41.73
9	3.34	43.42
10 and more	2.41	31.33

Be it observed by the way that owing to the more conspicuous survey of the data in Table IV, our conclusions deduced from Table II are shown here to a greater advantage viz, the great difference between the hearing distances for sentences of 3 and for those of 10 or more syllables (loud speech nearly 30 m.); the comparatively slight difference for sentences of 4 to 9 syllables (loud speech  $\pm 3.5$  m.); the greater distance for sentences of an odd number of syllables, owing to their most often symmetrical construction; etc.

We have already pointed out that symmetrical sentences are heard at greater distance than asymmetrical sentences. Table V shows that symmetry is highly favourable to our audition. Here also great individual differences are observed; for R., M. and D. they are resp. 0.61; 1.13; and 0.24 m.

It is easy to understand that proverbs and proverbial sayings are heard at greater distance. The subjects' familiarity with them, helps

TABLE V.

Subj.	Symmetrical sentences			Asymmetrical sentences
	M.	M.D.	C.V.	M.
R.	4.11	0.79	4	3.50
M.	3.87	1	4	2.57
D.	4.21	0.96	4.5	3.97

the assimilation process consequent on the apperception of elements.

The favourable influence exerted by proverbs and proverbial sayings is, however, inferior to that of symmetrical construction, as is borne out by a comparative study of Tables V and VI. The enlargement of the hearing distance for symmetrical sentences is with R. and M. respectively 0.61 and 1.13 m., that of hearing proverbs only 0.74 m.

TABLE VI.

Subj.	Proverbs.			The other sentences.
	Mean.	M.D.	C.V.	M.
R.	3.78	1	4	3.37
M.	3.67	0.94	4	2.93

From a study of the data of Table VII it also appears that the absence of subject, predicate or object or an inversion of these parts of a sentence may be an impediment to hearing (see page 106). Irregular sentences are heard by R., M. and D. at a much shorter distance than the regular ones. The decrease of distance is resp. 0.73; 0.28; and 1.01 m. The very considerable decrease with D. is probably owing to the circumstance that as a Fleming he is less familiar with the rather uncommon construction of many of our irregular sentences.

The relation between the mean distances at which whispered sentences and whispered words are heard we term after ZWAARDEMAKER index loquelae. In otiatrics the distance at which the latter are heard is estimated at  $\pm 18$  m.

In their experiments ZWAARDEMAKER and QUIX <sup>1)</sup> arrived at the same conclusion.

TABLE VII.

Subj.	Irreg. sentences.			Reg. sentences.
	M.	M.D.	C.V.	M.
R.	2.90	0.72	3	3.63
M.	2.65	1.11	2.75	2.93
D.	2.64	0.88	2.50	3.65

The mean distance at which whispered sentences of from 3—14 syllables are heard amounts to  $\pm 3.42$  m, as shown in Table II and IV, so that the index loquelae is  $\frac{3.42}{18} = \frac{19}{100}$ .

## SUMMARY.

1. Our material consisted of sentences of 3 to 14 syllables. They were presented by means of whispers with reserve-air.

2. We determined the intensity of hearing of our subjects by means of separate words, composed of isozonal and aequi-intensive sounds. With all of them audition appeared to be normal; the distances for different words varied from 6 to 30 m.

3. The difficulty of hearing sentences increases with the number of syllables. The difference in the distances at which sentences of 3 syllables and those of 10 or more syllables are heard, averages 1.83 m. (whispers).

4. Generally the decrease of the distance in proportion to the increase of the number of syllables does not take place gradually. Sentences of an odd number of syllables are as a rule heard better than those of an even number containing one syllable more. Even sentences of 9 syllables are heard on an average 0.33 m. farther than those of 4 syllables. The reason is that symmetrical construction, which aids our hearing faculty, occurred more with an odd than with an even number of syllables.

5. Proverbs and proverbial sayings are heard at larger distances. However, the favourable influence of proverbs and proverbial sayings

<sup>1)</sup> De studie van het spraakgehoor. Onderzoekingen Physiol. Lab. Utrecht (5) dl, 5 blz. 1.

is inferior to that of mere symmetrical construction of the sentences.

6. The absence of subject, predicate or object or inversion of the order of words are an impediment to audition; the hearing distance is lessened by it.

7. Hearing is the result of the joint action of perception and assimilation. For sentences of 3 to 9 syllables the influence of the reproduction-tendencies, consequent on the apperceived elements, counterbalances that of the inhibition, caused by an increase of the number of syllables. Only for sentences of 10 and more syllables the latter seems to get the start of the former.

8. The index loquelae i.e. the relation between the mean distances at which whispered sentences and whispered words are heard is  $^{19}/_{100}$ .

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**Physics.** — “*Isothermals of monatomic substances and their binary mixtures. XX. Isothermals of neon from + 20° C. to —217° C.*”

By C. A. CROMMELIN, J. PALACIOS MARTINEZ, and H. KAMERLINGH ONNES. Communication N°. 154*a* from the Physical Laboratory at Leiden.

(Communicated in the meeting of June 29, 1918).

§ 1. *Introduction.* This paper is the continuation of a previous preliminary communication <sup>1)</sup>. The reduction of the neon-isothermals has now progressed so far, that what follows may be looked upon as a pretty nearly completed whole. The measurements refer to pressures up to about 90 atmospheres and as regards the temperature go from + 20° C. down to — 217° C.; they therefore embrace the region between the ordinary temperature down to the lowest temperature to be reached with liquid oxygen. The region from — 218° C. to —246° C., which can now also be covered by means of the hydrogen vapour cryostat <sup>2)</sup>, is here left out of account; we hope shortly to be able to continue our measurements in this region. As regards the importance of such determinations as will be communicated on this occasion and the apparatus which have been used for the purpose we may refer to previous communications on the isothermals of hydrogen and of argon. <sup>3)</sup>

For the sake of completeness and of a better survey of the whole work we have included in the table the material published in the previous communications on neon, quoted above, viz. the series I, II, III, IV, VI, VII, VIII and IX.

§ 2. *The results of the measurements* are given in table I, where  $\theta$  = the temperature on the *international* <sup>4)</sup> CELSIUS-scale, i.e. the temperature on the *international* KELVIN-scale diminished by 273.09;  $p$  = the pressure in international atmospheres (for Leiden 1 international atmosphere = 75.9488 cms mercury);

$d_A$  = the density expressed in the normal density (0° C. and 1 atm.);

$v_A$  = the volume expressed in the normal volume (0° C. and 1 atm.).

<sup>1)</sup> H. KAMERLINGH ONNES and C. A. CROMMELIN, Comm. N°. 147*d*; these Proc. XVIII (1) p. 515.

<sup>2)</sup> H. KAMERLINGH ONNES, Comm. N°. 151*a*; these Proc. XIX (2) p. 1049.

<sup>3)</sup> Comp. H. KAMERLINGH ONNES and H. H. FRANCIS HYNDMAN, Comm. N°. 69; these Proc. III p. 481; H. KAMERLINGH ONNES and C. BRAAK, Comm. N°. 97*a* these Proc. IX p. 754; C. BRAAK, Dissertation, Leiden, 1908; C. A. CROMMELIN, Dissertation, Leiden, 1910.

<sup>4)</sup> Comp. H. KAMERLINGH ONNES, Comm. Suppl. N°. 34*a*. § 5.

TABLE I.

Series.	N <sup>o</sup> .	$p$	$d_A$	$pv_A$
$\theta = + 20^{\circ}.00 \text{ C.}$				
VI	1	22.804	21.046	1.0835
VI	2	25.015	23.052	852
VI	3	26.575	24.464	863
VI	4	29.090	26.757	872
VI	5	32.572	29.891	897
VIII	1	34.887	32.002	902
VI	6	35.423	32.447	917
VI	7	37.812	34.601	928
VIII	2	39.168	35.843	928
VIII	3	44.762	40.862	955
VIII	5	54.149	49.213	1003
VIII	6	59.717	54.161	026
VIII	7	65.021	58.797	059
VIII	9	77.360	69.338	131
VIII	10	82.545	73.967	160
VIII	11	88.239	78.886	186
VIII	12	93.298	83.154	220
$\theta = 0^{\circ}.00 \text{ C.}$				
VII	1	22.064	21.869	1.0089
VII	2	23.555	23.314	103
VII	3	25.867	25.558	121
VII	4	28.468	28.089	135
VII	5	30.790	30.345	147
IX	1	39.753	39.098	168
IX	2	44.892	44.030	196
IX	5	59.777	58.234	265
IX	6	66.104	64.135	307
IX	7	74.059	71.495	359
IX	8	79.108	76.127	392
IX	9	84.662	81.347	408

TABLE I (Continued).

Series.	N <sup>o</sup> .	$\rho$	$d_A$	$\rho v_A$
$\theta = -103^{\circ}.01$ C.				
XV	1	35.558	56.40	0.6304
XV''	1	36.697	58.23	6302
XV	2	40.610	64.21	6324
XV''	2	42.107	66.53	6329
XV''	4	55.136	86.57	6369
XV	4	58.583	91.76	6384
XV	5	78.110	120.52	6481
$\theta = -141^{\circ}.22$ C.				
XVI''	1'	33.840	69.83	0.4846
XVI	2	37.707	77.71	4852
XVI''	2	38.581	79.50	4853
XVI	3	43.319	88.97	4869
XVI''	4	49.881	102.32	4875
XVI	4	51.916	106.42	4878
XVI	5	66.471	134.91	4927
XVI	6	78.558	158.06	4970
$\theta = -182^{\circ}.60$ C.				
X	2	32.067	99.89	0.3210
X''	2	32.988	102.84	3208
X	3	36.438	113.69	3205
X''	3	36.880	115.07	3205
X	4	41.371	129.44	3196
X''	4	42.533	133.15	3194
X	5	49.943	156.61	3189
X''	5	50.514	158.55	3186
X''	6	63.320	199.21	3179

TABLE I (Continued).

Series.	Nº.	$p$	$d_A$	$pv_A$
$\theta = -200^{\circ}.08$ C.				
XI	1'	26.214	105.10	0.2494
XI	2'	28.402	114.38	2483
XI	3''	31.417	127.24	2469
XI	1	34.268	139.81	2451
XI	4	34.285	139.88	2451
XI	5	39.843	164.30	2425
XI	2	39.891	164.63	2423
XI	3	46.517	194.30	2394
XI	3'	46.529	194.51	2392
XI	6	47.951	200.79	2388
III	1	61.657	263.77	2338
III	2	67.456	291.10	2317
III	3	73.850	320.35	2302
III	4	79.923	348.59	2293
$\theta = -208^{\circ}.10$ C.				
XII	1	24.071	111.90	0.2151
XII	3	28.844	136.44	2114
XII	4	31.948	153.00	2088
XII	5	37.856	185.47	2041
XII	6	41.798	207.95	2010
IV	1	58.472	308.32	1897
IV	2	64.451	345.22	1867
IV	3	69.692	377.89	1844
IV	4	74.532	409.18	1822
IV	5	79.228	439.12	1804

TABLE I (Continued).

Series.	Nº.	$p$	$d_A$	$pv_A$
$\theta = - 213^{\circ}.08 \text{ C.}$				
XIII	1	23.086	119.92	0.1925
XIII	2	24.810	129.82	1911
XIII	3	26.673	140.90	1893
XIII	4	29.365	157.70	1862
XIII	5	32.441	177.37	1829
XIII	6	37.418	210.68	1776
II	1	53.896	334.59	1611
II	2	59.769	382.03	1565
II	3	66.271	435.46	1522
II	4	72.858	484.75	1503
II	5	79.698	534.62	1491
$\theta = - 217^{\circ}.52 \text{ C.}$				
XIV	1	21.349	123.40	0.1730
XIV	2	22.997	134.72	1707
XIV	3	24.686	146.67	1683
XIV	4	26.848	162.51	1652
XIV	5	30.042	186.94	1607
XIV	6	32.795	209.68	1564
I	1	49.930	358.51	1393
I	2	53.528	395.62	1353
I	3	59.618	458.40	1301
I	4	64.975	511.85	1269
I	5	71.649	571.69	1253
I	6	79.417	632.23	1256

A graphical representation of the observations will be found in fig. 1 in a  $\left(\frac{pv_A}{T}, d_A\right)$  diagram.

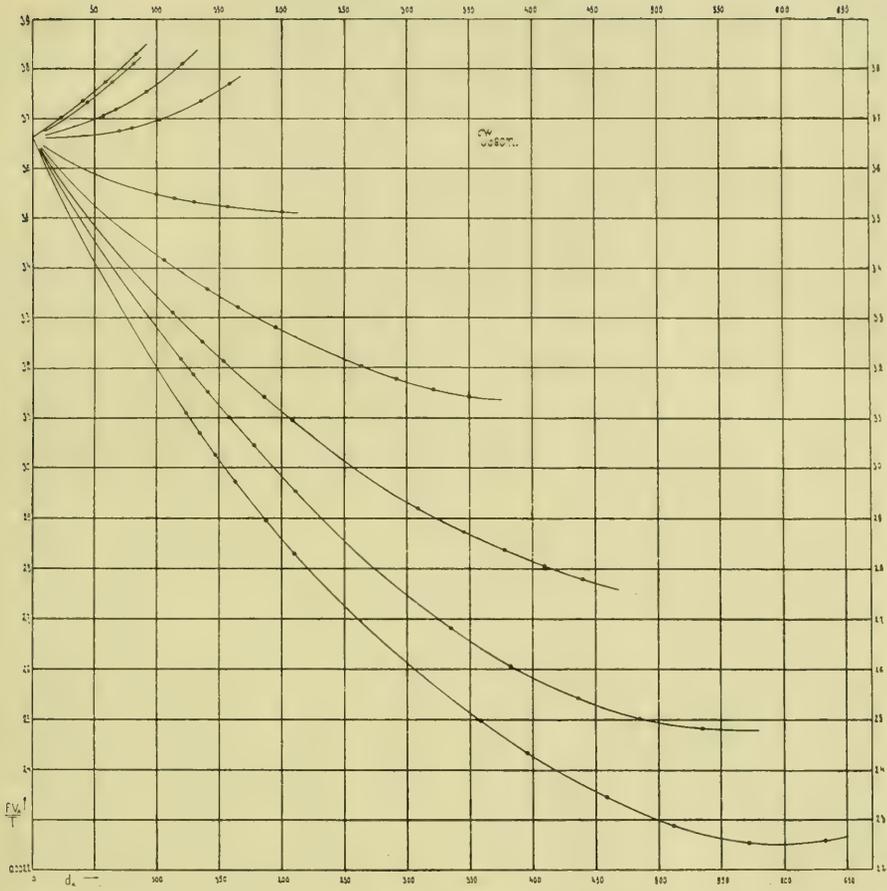


Fig. 1.

### § 3. Virialcoefficients.

By means of the above results some of the coefficients in the empirical equation of state

$$pv_A = A_A + B_A d_A + C_A d_A^2 + D_A d_A^3 + E_A d_A^4 + F_A d_A^5 \quad 1)$$

could be computed. These calculations only embrace the coefficients  $B_A$ ,  $C_A$ ,  $D_A$  and in one case  $E_A$ ; for the densities which were reached are not so high as would be necessary for the deduction of  $F_A$  and in most cases of  $E_A$  also; these were therefore found from the reduced equation of state VII. A. 3<sup>2)</sup> or VII. 1. <sup>3)</sup>, in which the coefficients  $\mathcal{C}$  and  $\mathcal{D}$  are identical. In some of the calculations,

1) H. KAMERLINGH ONNES, Comm. N<sup>o</sup>. 71; these Proc. IV p. 125.

2) H. KAMERLINGH ONNES and G. A. GROMMELIN, Comm. N<sup>o</sup>. 128; these Proc. XV (1) p. 273.

3) Suppl. N<sup>o</sup>. 19.

as will appear presently,  $D_A$ , in others  $D_A$  and  $C_A$  were assumed according to VII. A. 3. For this purpose use was made of the critical constants of neon as published on a former occasion <sup>1)</sup>

$$\theta_k = -228^{\circ}.35 C. \quad p_k = 26.86 \text{ int. atm.}$$

The calculations were conducted in three different ways:

*a.* for all temperatures only  $B_A$  was calculated from the observations, the remaining coefficients being assumed, viz.  $C_A$  and  $D_A$  according to VII. A. 3.,  $E_A$  and  $F_A$  according to VII. 1 or to VII. A. 3- (as noticed above, this comes to the same);

*b.* for the lowest 4 temperatures  $B_A$  and  $C_A$  were computed from the observations, further as under *a*;

*c.* for all temperatures  $B_A$  and  $C_A$  were deduced from the observations, for  $-200^{\circ}.08 C.$ ,  $-208^{\circ}.10 C.$  and  $-213^{\circ}.08 C.$  also  $D_A$  and for  $-217^{\circ}.52 C.$  also  $E_A$ , further as under *a*.

The calculations *c* were made first, with a view to obtaining the best possible accordance with the observations, the coefficients therefore bearing a purely empirical character. When it appeared that the values of  $C_A$  could not be connected by a smooth curve, much less those of  $D_A$ , which proved the observational material to be insufficient for the deduction of  $C_A$  and  $D_A$  as functions of the temperature, we proceeded to the methods given under *a* and *b*, in which the values of  $D_A$  and partly even those of  $C_A$  were assumed. Naturally the accordance with the observations is very much inferior with the methods *a* and *b* than with *c*.

The results of the calculations which were all conducted by the method of least squares are found in tables II and III. Table II gives the individual virial-coefficients, as calculated from the observations according to *a*, *b*, and *c*, table III the coefficients borrowed from VII. A. 3 as well as the values of  $A_A$  computed from the equation

$$A_A = A_{A_0} (1 + 0.0036618 \theta), \quad ^2)$$

where for  $A_{A_0}$  the value  $+0.99986$ , as published on a former occasion <sup>3)</sup>, was taken as a basis.

<sup>1)</sup> H. KAMERLINGH ONNES, G. A. CROMMELIN and P. G. CATH, Comm. N<sup>o</sup>. 151b, these Proc. XIX (2) p. 1058.

<sup>2)</sup> Comm. N<sup>o</sup>. 71.

<sup>3)</sup> Comm. N<sup>o</sup>. 147d.

TABLE II.

$\theta$	$B_A \times 10^3$	$B_A \times 10^3$	$C_A \times 10^6$
	According to <i>a.</i>	According to <i>b.</i>	
+ 20°.00	+ 0.54880		
0°.00	47148		
- 103°.01	16653		
- 141°.22	055249		
- 182°.60	- 0.093113		
- 200°.08	15746	- 0.18779	+ 0.21531
- 208°.10	19553	21706	18307
- 213°.08	22305	24084	18407
- 217°.52	24028	25880	19649

TABLE II (Continued).

$\theta$	$B_A \times 10^3$	$C_A \times 10^6$	$D_A \times 10^{12}$	$E_A \times 10^{18}$
	According to <i>c.</i>			
+ 20°.00	+ 0.51578	+ 0.82778		
0°.00	41334	1.1538		
- 103°.01	069193	1.1515		
- 141°.22	- 0.025378	0.71945		
- 182°.60	13435	33607		
- 200°.08	19667	27847	- 0.24096	
- 208°.10	22926	25304	0.16102	
- 213°.08	24625	21123	0.005848	
- 217°.52	29313	36427	0.46739	+ 0.57517

TABLE III.

$\theta$	$A_A$	$C_A \times 10^6$	$D_A \times 10^{12}$	$E_A \times 10^{18}$	$F_A \times 10^{27}$
		According to VII. A. 3.			
+ 20° .00	+ 1.0731	+ 0.29747			
0.00	0.99986	25440			
— 103° .01	62271	0.072156	+ 0.37445	— 0.1373	+ 40.29
— 141° .22	48281	39576	28409	0.03754	12.51
— 182° .60	33131	58524	12718	2409	— 4.190
— 200° .08	26731	96581	11124	4293	5.666
— 208° .10	23795	0.12219	0.081145	+ 0.04550	5.367
— 213° .08	21971	14073	60843	4599	4.836
— 217° .52	20345	15882	41215	4576	4.160

§ 4. *Discussion and comparison with other observations.* The differences between the  $pv_A$ -values calculated from these equations and the observed values are represented graphically in fig. 2 as functions of the densities  $d_A$ , the ordinates being the observed minus the calculated  $pv_A$ -values, expressed as percentages of the latter. In this manner the character of the deviations is more easily grasped than would be the case, if the numbers were given in the tables.

The correspondence between the new and the old series is very satisfactory on the whole; only in the isothermal for — 217° .52 C. a marked deviation may be noticed. Whereas for the isothermals of — 200° .08 C. the deviations of the observations from the most closely corresponding formula (method *c*) are within 0.1 %, differences of almost  $\frac{1}{2}$  % occur in the isothermals for — 217° .52 C.

The differences between the various sets of  $B_A$ -values obtained on this occasion from the smoothed  $B_A$ -values according to VII. A. 3, viz.  $\Delta B_A = B_A^-$  (calc.) —  $B_A$  (VII. A. 3) are represented in fig. 3; the corresponding deviations of the  $B_A$ -values obtained by CATH and one of us<sup>1)</sup> from measurements at low pressures are included in the figure.

<sup>1)</sup> P. G. CATH and H. KAMERLINGH ONNES, Comm. N<sup>o</sup>. 152e, presented to the Meeting of the Academy some time ago and shortly to be published in the Proceedings; preliminary values are given by P. G. CATH, Dissertation, Leiden 1917, p. 77.

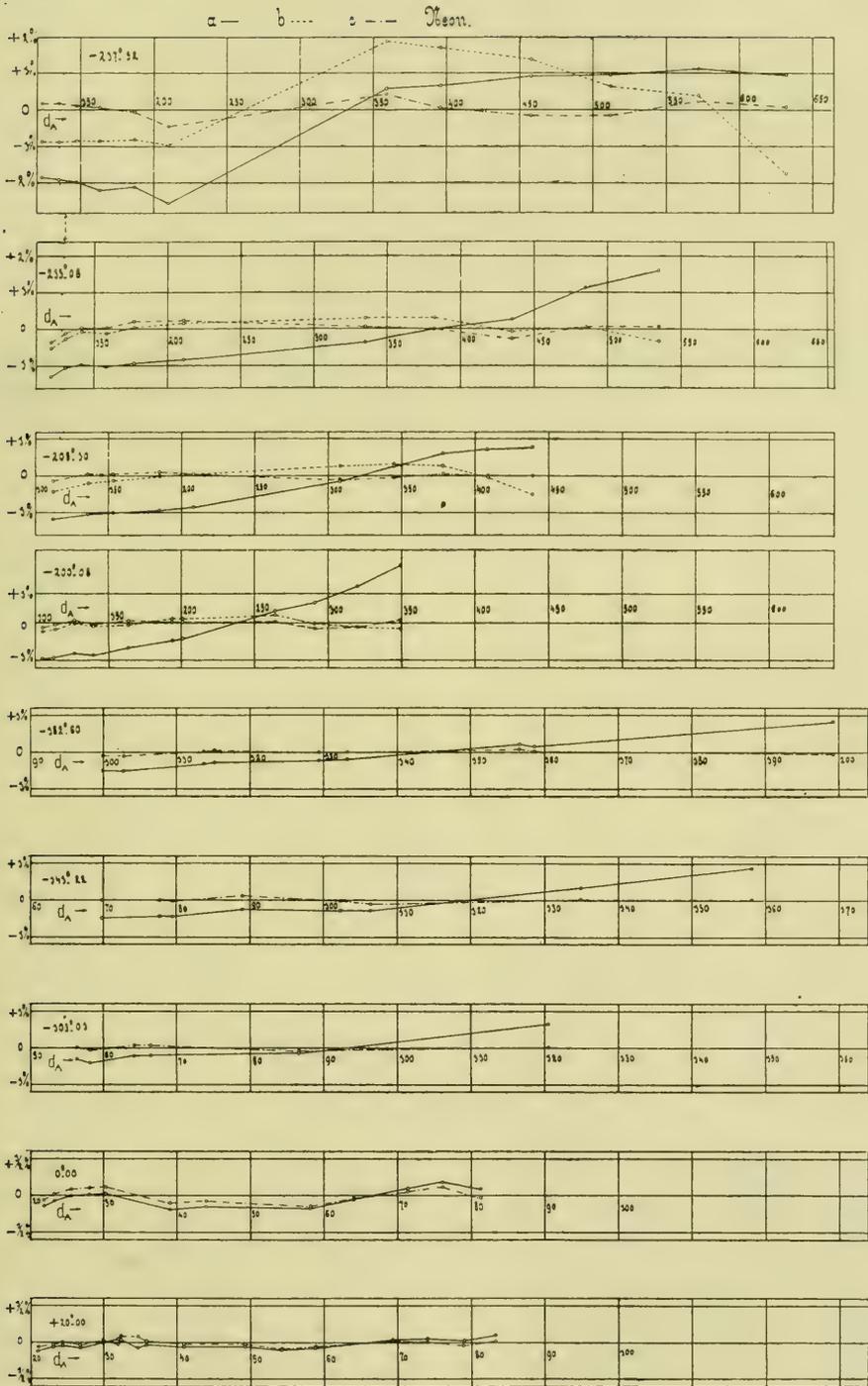


Fig. 2

It may be noticed, that the  $B_A$ -values according to *a* agree fairly

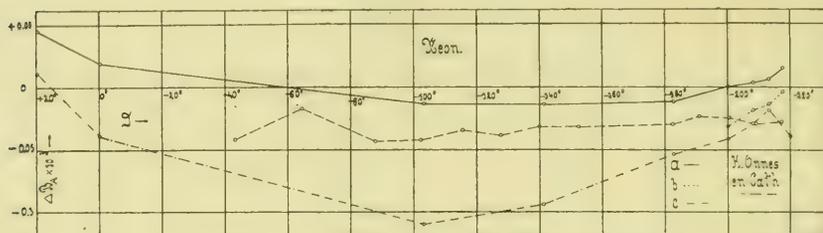


Fig. 3.

well with those according to VII. A. 3. Those obtained by method *c* deviate much more markedly, as might be expected; especially at the lower temperatures they show much smaller values.

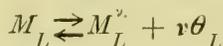
**Chemistry.** — “*Metals and Non-Metals*”. By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of May 3, 1919).

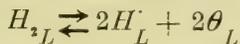
*Introduction.*

In a few previous communications I already discussed the unattackable electrodes and their efficiency as gas-electrodes. Only the hydrogen electrode was, however, discussed in detail. It was pointed out that the unattackable electrodes are among the most inert metals, being so inert that even in contact with an electrolyte they do not assume internal equilibrium, so that they are almost always in disturbed condition, and we do not even know the potential of the really unary metal.

When such a metal is immersed in the aqueous solution of an acid, while hydrogen is passed through, the electron concentration of the metal equilibrium in the electrolyte, which we represent by the equation:



becomes equal to the electron concentration of the hydrogen equilibrium:



in agreement with the pressure of the hydrogen that is passed through, which means that the hydrogen phase and the hydrogen-containing metal-phase possess the same potential, the electromotive force of the circuit metal-electrolyte-hydrogen being given by the equation:

$$E = -\frac{RT}{F} \ln \frac{(\theta_L)_M}{(\theta_L)_{H_2}} \dots \dots \dots (1)$$

Hence the hydrogen-electrode indicates the potential belonging to the three-phase equilibrium metal phase + hydrogen phase + electrolyte in accordance with the prevailing pressure, temperature, and total ion concentration ( $H'$ ) + ( $M^v$ ). As, however, as was already stated in the discussion of the  $E, X$ -fig., the said electrolyte contains an entirely negligible concentration of ions of the so-called unattackable electrode, we may substitute the words: hydrogen-ion-concentration for “total-ion-concentration”

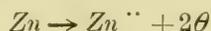
When we now consider the case that an unattackable electrode is placed in an electrolyte, chlorine being led through, we get the same thing in so far that the electron concentration of the metal equilibrium in the electrolyte is entirely dominated by the electron concentration of the chlorine equilibrium:



in agreement with the pressure of the chlorine that is led through, from which follows that the chlorine-gas phase and the chlorine-containing metal phase will possess the same potential with respect to the electrolyte. Our more recent views about the electromotive equilibria lead to the assumption of an electro-ionisation equilibrium in chlorine gas, though the concentration of ions and electrons in this gas-phase is exceedingly small. That there exists such an electro-ionisation for chlorine, is proved by the exceedingly slight electrical conductivity. Now, however, the above mentioned equilibrium (2) does not suffice, for in electrically neutral chlorine this equilibrium cannot occur *alone*, because here there is only question of particles charged negatively electrically. Besides the chlorine electrode could not assume a positive charge with respect to the electrolyte.

Undoubtedly this difficulty has also been felt in the former view about the electromotive equilibrium. The negative charge which zinc assumes on immersion into an electrolyte was explained as follows: The zinc tends to go into solution as zinc-ion, and it has evidently always been imagined that this happened through a *simultaneous* splitting up of the zinc-atom into zinc-ion + electrons, in which, however, the zinc-ions only went into solution, and the negatively charged electrons remained on the metal.

When we wish to account for the positive charge which the chlorine electrode assumes with respect to an electrolyte in an analogous way, we get into serious difficulties, which have led to the assumption by some physicists that besides the free negatively charged electrons there exist also free positively charged electrons, and that these could render important services for the chlorine electrode and for the other non-metal electrodes. For just as the splitting up



was assumed when the zinc went into solution, the process



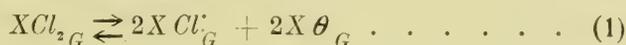
was supposed to take place at the chlorine electrode when chlorine went into solution. The chlorine ions formed go into solution and the positive electrons would be left behind in the chlorine electrode

This solution should, however, be denied all signification, because as yet free positive ions have not been met with, and everything points to the existence of only one kind of electrons, viz. the negative ones.

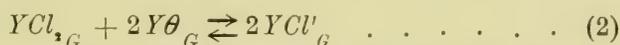
Hence in spite of a desperate attempt, the difficulty remained undiminished of force.

### *The Chlorine-Electrode.*

When considering the just-discussed difficulty, I have come to the conclusion that we must necessarily assume that the chlorine atom possesses the power to split off electrons and to absorb them, and that these two processes take place side by side, which we can represent by the following equations: <sup>1)</sup>



and



in which  $X$  and  $Y$  indicate the fractions of the original number  $Cl_2$  mol. which have undergone a positive resp. negative ionisation.

As the electrons, which are absorbed according to (2) proceed from the electron-ionisation (1) it is clear that

$$Y \leq X.$$

In the limiting case  $Y = X$  the chlorine would contain an equal number of positive and negative ions, and no electrons at all. As we have to do here with a non-metal,  $X$  and  $Y$  will be exceedingly small.

It is now the question how the positive charge of the chlorine electrode is to be explained. It is clear that for this we should have to assume that for the non-metal, chlorine, it is the negative ions that go practically exclusively into solution, and possess, therefore, a much greater solubility than the positive ones.

### *Metals and Non-Metals.*

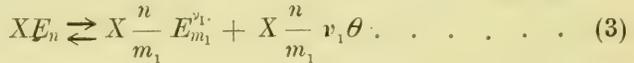
These considerations about the non-metal, chlorine, which necessarily result from the consistent application of the principles of the theory of allotropy to the electromotive equilibria, lead us to the point of view from which we can survey the metals and the non-metals in a very satisfactory way.

<sup>1)</sup> For the sake of simplicity we assume here that the positive ions are also univalent.

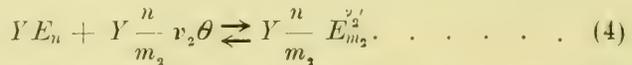
It is, indeed, clear that as the metallic and non-metallic properties in the periodic system of the elements gradually pass into each other, theory will have to make clear that between the metallic and the non-metallic state there exist only quantitative differences, and that there are therefore all kinds of intermediary states possible.

This demand can really be satisfied by an in every way plausible extension.

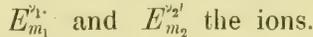
For this purpose we must assume that the atoms of all the elements, hence both metals and non-metals, can split off and absorb electrons so that the following reactions take place side by side:



and



in which  $E_n$  denotes the molecule of an element and



As is known for metals  $m_1$  is mostly = 1, and probably  $m_2$  will as a rule also be = 1. For non-metals  $m$  has often been found = 1, but several times also greater than 1. Of the factor  $m_1$  no doubt the same thing may be expected.

With perfect certainty we may only say this that

$$Y \frac{n}{m_2} v_2 \leq X \frac{n}{m_1} v_1 \quad . . . . . (5)$$

from which follows that when

$$v_1 = v_2$$

and

$$v_1 > v_2$$

the number of positive ions will be greater than the number of negative ones.

When now the question is put in what respects metals and non-metals will differ, the answer is as follows:

In the first place the factor  $X$  is comparatively great for metals and exceedingly small for non-metals, so that for non-metals also the factor  $Y$  is exceedingly small.

This is among other things in accordance with the great electric conductivity of metals and the exceedingly slight conductivity of non-metals.

In the second place for the metals the positive ions possess the greatest solubility, and for the non-metals the negative ions. This accounts among other things for the difference in electromotive

behaviour between metals and non-metals. In this it is noteworthy that the difference in solubility between the positive and the negative ions for the elements with exclusively metallic resp. exclusively non-metallic properties, is so great as to justify us in taking only positive resp. negative ions into account in the coexisting solution.

For the intermediate amphibious elements as Iodine, Sulphur, Selenium, Tellurium, Arsenic, and Antimony the existence of positive and negative ions also in solution, should certainly be taken into account to obtain a deeper insight, and here lies still an extensive field to be studied more closely, led by the more recent views about the electromotive equilibria.

*Polarisation for non-metals.*

When we indicate the non-metal by  $N$ , we may represent the formation of negative ions by the equation:



Applying the well-known thermodynamic derivation for the potential difference, we get:

$$\Delta = \frac{RT}{\nu F} \ln \frac{K' N'' N''_P}{(N''_L)} \dots \dots \dots (7)$$

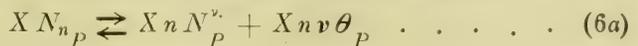
in which  $(N''_P)$  represents the concentration of the negative ions in the phase of which the electrode consists, this may be either a gas, or a liquid, or a solid phase. Starting from the electron formula:

$$\Delta = \frac{RT}{F} \ln \frac{K'_\theta (\theta_P)}{(\theta_L)} \dots \dots \dots (8)$$

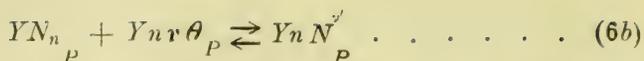
which is general, because the electron is the common constituent part of all matter, we may substitute for  $(\theta_L)$  the value that follows from (6) for the liquid phase, applying the law of the chemical mass action; thus we arrive at the already known equation:

$$\Delta = \frac{RT}{n\nu F} \ln \frac{K' (N_{nL})}{(N''_L)^\nu} \dots \dots \dots (9)$$

In order to study the polarisation phenomenon we cannot make use of equation (9), but we can use the newer equation (7). To get a good insight into this question it is desirable that we indicate the equilibria in the electrode completely; accordingly we must also express the splitting off of electrons, and write therefore:



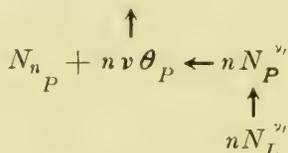
and



When we now make the non-metal electrode  $N$  anode, we withdraw the electrons. If we had to deal with a metal, hence if  $N$  was a metal, the electrode could maintain its potential in consequence of the fact that the withdrawn electrons are supplied in time by the reaction (6a), the positive ions formed going into solution. As  $N$  is however a non-metal, and the positive ions  $N''$  do not practically go into solution, the reaction (6a) will not take place to a degree of any importance.

The only way in which the non-metal  $N$  will be able to maintain its potential in the case supposed here, is this that negative ions  $N''$  from the electrolyte are deposited on the electrode, and there supply the deficit of electrons by the splitting off of electrons.

The processes that take place may be represented as follows:



As the heterogeneous equilibrium in the boundary layer between the negative ions in the electrode and in the electrolyte must set in with very great rapidity, it is clear that it will depend on the velocity of the splitting off of electrons of the negative ions whether the electrode maintains its potential or whether it does not. Above a definite current density, i.e. above a definite velocity of withdrawal of electrons this will no longer be the case, and a consequence of this will be that the electrode will contain too few electrons and too many negative ions.

It follows from equation (7) for this case that the potential difference will become more strongly positive.

This phenomenon of polarisation is called "*supertension*" in the case of gases, but it is not essentially different from the phenomenon of anodic polarisation for metals.

It is clear that when we now proceed to the cathodic polarisation for non-metals this phenomenon should be ascribed to this that the added electrons are not quickly enough absorbed by the uncharged molecules or atoms, so the electron possesses a too high concentration of electrons, and a too small concentration of negative ions. According to our equation (7) this gives rise to a less positive value of the potential difference.

As we do not measure the potential difference, but the experi-

mental electric potential it is obvious that the polarisation phenomena should be discussed in connection with the formula that holds for the "exp. electric potential" in the case that we have to do with a non-metal. This formula is as follows:

$$E = \frac{0.058}{v} \log \frac{Q_N}{(N_L^v)} + 2,8 \dots \dots \dots (10)$$

in which:

$$Q = \frac{(N_L^v)}{(\theta_L)^v}.$$

From (7) and (8) follows:

$$\frac{(N_L^v)(\theta_P)^v}{(N_P^v)(\theta_L)^v} = K \dots \dots \dots (11)$$

It has further been shown just now that with anodic polarisation ( $\theta_P$ ) decreases and ( $N_P^v$ ) increases; it follows therefore from this that in this case the *quotient of solubility* must *increase*, and the exp. electrical potential of the non-metal will therefore become more strongly positive.

For cathodic polarisation the quotient of solubility decreases, hence the experimental electrical potential of the non-metal becomes less strongly positive.

### *Amphibious Elements,*

As has already been said the amphibious elements are characterized by this that they send both positive and negative ions into solution.

If such an element is made anode, positive ions can go into solution, or negative ions can be deposited from the electrolyte on the anode, or the two processes can take place simultaneously.

If the element is made cathode, positive ions can be deposited on the electrode, or negative ions can go into solution, or the two processes can take place side by side.

The closer study of these amphibious elements, which we hope soon to undertake, following the new theory on the electromotive equilibria, will probably give a deeper insight into the character of these so remarkable elements.

*Laboratory for General and Anorganic  
Chemistry of the University.*

Amsterdam, April 10<sup>th</sup> 1919.

**Chemistry.** — "*Urease and the radiation-theory of enzyme-action*".

By Dr. H. P. BARENDRECHT.

(Communicated in the meeting of May 31, 1919).

IV.

11. *Direct synthesis of urea by urease out of ammonium carbonate.*

According to the above theory the final equilibrium in the action of urease on urea will not always be found at complete hydrolysis.

In the case of a low concentration of urease the synthetic action in the outer shell will have free play.

In an alkaline urease solution, whatever its concentration, the enzyme will partially decay in the course of time, as was shown in part 9. In this case also a reverse action will manifest itself, proportional now to the concentration of urease.

To test this inference from the hypothesis, the following experiments were carried out:

In each of six large cylinders of about 1 Litre capacity 100 c.c. of water were introduced, into which 5.786 g. of ammonium carbonate were dissolved. Each of these cylinders was closed by a rubber stopper with two borings, the one carrying a straight glass tube, provided at the bottom with a bulb with pinholes, the other a bent glass tube, allowing the air-current, which was to be blown through, to pass into a second, smaller cylinder, in which the ammonia was to be absorbed. To this purpose each of these smaller cylinders contained 186,85 g. of  $\text{H}_2\text{SO}_4$ ,  $\frac{4}{10}$  N (which is equal to 185 c.c.  $\frac{4}{10}$  N). The greater accuracy, obtained by weighing the absorbing acid, was necessary, considering that the effect looked for, was the difference of two large values and would presumably be only small.

In two of the large cylinders 3, in two others 6 g. of Soja-meal were introduced. The glass tubes of these cylinders were all closed with pieces of rubber tubing and clips. After a few hours those, making communication with the absorbing cylinders, were opened for a moment, to allow the carbonic acid, evolved by the partial change of the dissolved ammonium carbonate to ammonium carbonate, to escape through the sulphuric acid.

After leaving the six pairs of cylinders for 24 hours at room temperature, shaking through the Soja-meal from time to time, the connections with the absorbing cylinders were opened and 250 c.c. of saturated potassium carbonate was run into each of the larger cylinders through the long tubes. By then passing through for 24 hours vigorous currents of air, washed through sulphuric acid, all the ammonia was blown over into the sulphuric acid.

A few drops of octyl-alcohol, run in at the same time with the Soja-meal, prevented foaming.

The quantities of ammonium carbonate and sulphuric acid were chosen such, that only a few c.c. of  $\text{NaOH } \frac{1}{10} \text{ N}$  were required to titrate the free sulphuric acid, which was left.

The results of a preliminary experiment on March 14<sup>th</sup>., 1916 were as follows:

TABLE 22.  
c.c.  $\text{NH}_3 \frac{1}{10} \text{ N}$  in:

			Mean	$\text{NH}_3$ converted into urea
Amm. carb. alone	735.8	735.9	735.85	
Amm. carb. + 3 g. of Soja	735.15	734.85	735.—	0.85
Amm. carb. + 6 g of Soja	734.85	734.75	734.8	1.05

Afterwards, however, it became evident, that an appreciable quantity of  $\text{NH}_3$  was developed from the Soja-meal<sup>1)</sup>.

In order to estimate this, in each of two large cylinders 250 c.c. of saturated potassium carbonate, 100 c.c. of water and 4 drops of octyl-alcohol were introduced. Moreover in the one 6, in the other 12 g. of Soja-meal. Each was connected with one of the small cylinders, into which 5 c.c.  $\text{H}_2\text{SO}_4 \frac{2}{5} \text{ N}$  and some water had been brought. After some 5 hours of blowing through the whole was left at room temperature till the next day and the blowing was started again and continued till the following day.

The 6 g. of Soja-meal had given 1,1 c.c. of  $\text{NH}_3 \frac{1}{10} \text{ N}$ .

The 12 g. of Soja-meal 1,65 c.c. of  $\text{NH}_3 \frac{1}{10} \text{ N}$ .

Hence in the mean for 6 g. of Soja-meal 0,97 c.c.  $\frac{1}{10} \text{ N}$ .

<sup>1)</sup> Special experiments established that no  $\text{NH}_3$  was formed out of urea by the prolonged action of saturated potassium carbonate.

Correcting the figures of March 14<sup>th</sup> by this value we get:

ammonia converted into urea by

3 g. of Soja 1,33 c.c.  $\frac{1}{10}$  N

6 g. of Soja 2.02 c.c.  $\frac{1}{10}$  N.

March 28<sup>th</sup> 1916. In the large cylinders 11.572 g. of ammonium carbonate and 100 c.c. of water. In the small cylinders 190 g. of  $H_2SO_4$  of about 0,8 N (1 gram = 7,85 c.c.  $\frac{1}{10}$  N).

In two of the six large cylinders 6 g. of Soja-meal, in two others 12 grams.

After 24 hours' standing at room temperature 250 c.c. of saturated potassium carbonate and a few drops of octyl-alcohol were introduced into each and the ammonia blown over for 48 hours.

Results:

TABLE 23  
c.c.  $NH_3$   $\frac{1}{10}$  N in:

	First cylinder	Second cylinder	Mean	Corrected for $NH_3$ out of Soja	$NH_3$ converted into urea
Amm. carb. alone	1462.4	1462.5	1462.45		
Amm. carb. + 6 g. of Soja	1460.7	1460.5	1460.6	1459.6	2.85
Amm. carb. + 12 g. of Soja	1458.5	1458.4	1458.45	1456.5	5.95

May 16<sup>th</sup> 1916. A solution of about 95 g. of ammonium carbonate in 1 Litre of water was prepared and saturated with carbonic acid.

Two large cylinders, each with 100 c.c. of this solution and 250 c.c. of saturated potassium carbonate were connected with small cylinders containing each 174 g.  $H_2SO_4$  0,8 N and all the ammonia blown over into the acid by passing an air current for 48 hours.

Four large cylinders (*A*) received each 100 c.c. of the same ammonium carbonate solution and 6 g. of Soja-meal.

Four others (*B*) each 100 c.c. of ammonium carbonate solution and 12 g. of Soja-meal.

After 2 days at room temperature one cylinder *A* and one cylinder *B* were analysed by introducing 250 c.c. of saturated potassium carbonate and blowing over the ammonia into 174 g. acid for two days. In the same way one *A* and one *B* were treated after 3 days, and two cylinders *A* and two cylinders *B* after 4 days. See Table 24.

Evidently the equilibrium is scarcely reached in about 3 days. If the activity had remained unchanged, it might have been reached much sooner, or probably no synthesis at all had been effected. According to what we have seen above, the urease had been gradually

Results :

 TABLE 24.  
 c.c.  $\text{NH}_3 \frac{1}{10}$  N in:

	First cylinder	Second cylinder	Mean	Corrected for $\text{NH}_3$ out of Soja	$\text{NH}_3$ converted into urea
Amm. carb. alone	1352.1	1352.7	1352.4		
Amm. carb. + 6 g. of Soja, 2 days	1351				
Amm. carb. + 6 g. of Soja, 3 days	1350.9				
Amm. carb. + 6 g. of Soja, 4 days	1350	1350.5	1350.3	1349.3	3.1
Amm. carb. + 12 g. of Soja, 2 days	1349.9				
Amm. carb. + 12 g. of Soja, 3 days	1348				
Amm. carb. + 12 g. of Soja, 4 days	1348.7	1348.6	1348.7	1346.7	5.7

weakened by being dissolved for such a long time in an alkaline medium and therefore a gradual displacement of the equilibrium to the side of urea was to be expected.

These experiments show clearly, that a small part of the ammonium carbonate disappears through the action of urease and that this part is proportional to the amount of urease present. Both facts are in accordance with the above theory of the synthesis of urea.

### 12. *The determination of H- and OH-ion concentration.*

Since the theory and results, communicated in this paper, will both have emphasized the dominating importance of frequent and therefore simple  $p_H$  determinations in enzyme research, it may be useful to give the following details of the apparatus used for this purpose.

It was nearly the same as that, described by the author as "A simple Hydrogen Electrode" in *Biochem. J.* 1915, 66.

The accuracy was improved by the addition of a little cup with saturated KCl solution, in which both the hydrogen-electrode and the tube of the calomel-electrode dipped. The latter was changed, as shown in Fig. 16 in order to avoid all capillaries and rubber connections. After a measurement the small flask, containing the calomel-electrode proper, was turned round and the dipping tube enclosed in a small tube with saturated KCl solution, as will be seen in the figure.

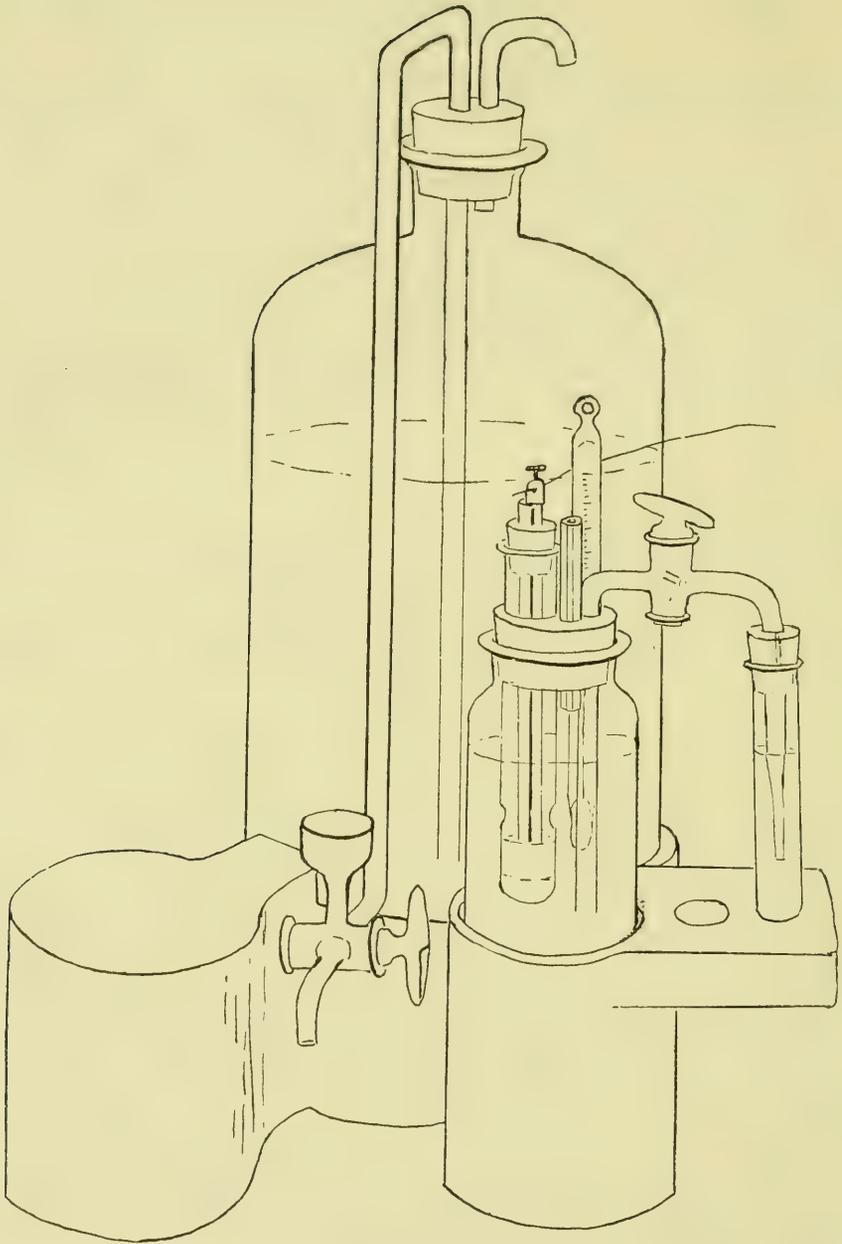


Fig. 16.

A stopcock, with a rectangular channel, allowed the small cup of about 2 c.c. capacity to be readily emptied from the large bottle with saturated KCl solution.

For the daily estimation of  $p_H$  the apparatus, represented in Fig. 17 has proved to be very satisfactory in the long run. The short perforated plunger, accurately filling up the inside of the cylinder,

when brought home, is rigidly connected to an overlapping cap, screwed on the outside of the cylinder. By turning the handle *A* backwards with cock *K* closed, cap and plunger are turned too and the latter is therefore drawn a little out of the cylinder. The liquid may thus be drawn up into the electrode tube, and by swinging the

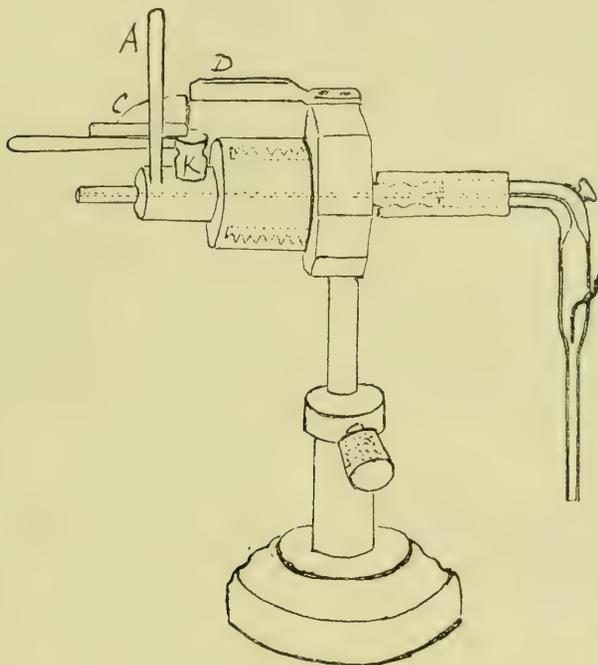


Fig. 17.

handle *A* gently to and fro the equilibrium between platinum-electrode, liquid and hydrogen can be easily established. Dead space and mistakes are avoided by the arrangements *C* and *D* on cock and frame. The protruding piece *D* allows the turning out of the plunger only, if the cock is shut and the quadrant *C* on the top of the cock is then in the position, shown in the figure. The same arrangement clearly helps to avoid the mistake of opening the cock and thus admitting the hydrogen, when the plunger is not turned home.

After shutting the cock, establishing equilibrium as described above, and adjusting the liquid in the electrode tube (by slightly turning the handle *A*) in such a way, that it just touches the point of the platinum wire, the whole apparatus is lifted up and placed on the wooden block of Fig. 16, the electrode tube dipping in the KCl-cup.

Another improvement was the carrying out of all these operations in an air-thermostat (Fig. 18), in which all the apparatus, shown

in Figures 16 and 17 as well as the hydrogen generator and wash-bottles were kept on 27°. Constancy of temperature within  $\frac{1}{4}$  of a

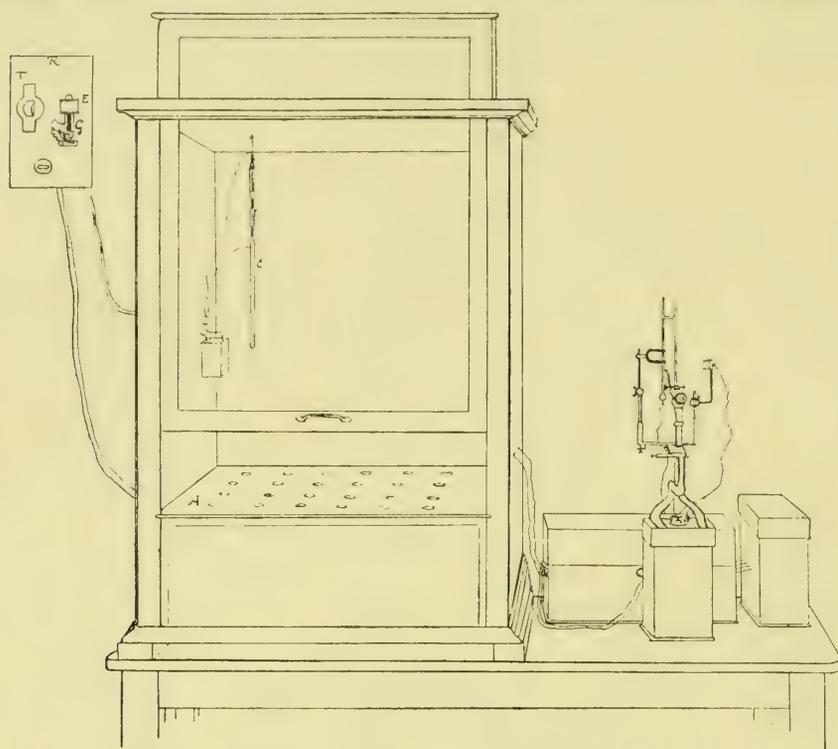


Fig. 18.

degree throughout the whole thermostat, which is amply sufficient, was attained by electrical heating in the following way.

Under the perforated bottom *A*, like the side walls and cover of the thermostat made of "eternite", a kind of concrete, some 130 metres of constantan wire of 0.4 m.m. diameter are spread out on a light frame. The current from the main of 220 Volts can be admitted to this wire by the relay *R*, an electrotechnical arrangement obtainable everywhere. A shunt of the current is reduced to some 8 Volts in the transformer *T*. This reduced current runs through the electromagnet *E* and is broken or opened by the regulator *C*. The latter is of the type, described by CLARK <sup>1)</sup>, a spiral of nickel securing contact always at the same point of the mercury meniscus in a 2 m.m. wide capillary in pure hydrogen. To reduce sparking the poles of this regulator are also connected with a small condenser.

<sup>1)</sup> J. Amer. Chem. Soc. 1913, 35, 1889.

If the current is off, the iron plunger in the evacuated glass vessel  $G$  is down and keeps the mercury in the horizontal side-tube, allowing the main current to pass between the sealed in platinum contacts. A rise of a fraction of a degree causes the large volume of mercury in the regulator  $C$  to make contact with the nickel spiral, starting the 8 volts current and thereby lifting the plunger out of the mercury in  $G$ . The running back of this mercury interrupts the main current and stops its heating effect very quickly, the heating wires being rapidly cooled, owing to their position in the open space between the table and the perforated bottom of the thermostat. Another advantage of this arrangement is the automatic stirring of the air in the thermostat by the jets of hot air rising through the holes in the bottom.

The back and front walls are double glass windows; the latter can be lifted up, allowing the necessary operations to be performed in the thermostat. These take only a few minutes. The temperature in the inside is so quickly restored after shutting the window, that the apparatus are practically maintained at the required temperature.

As will be seen in Fig. 18, the potentiometer is connected by thin wires, passing through the walls of the thermostat, with the electrode inside, with the capillary electrometer, mounted on a box, which contains an accumulator for its small lamp, and on the other side with the working accumulator and the Weston cell, contained in the third box.

As mentioned in part 5 of this paper, a direct estimation of the concentration of hydroxyl-ions, or of  $p_{OH}$ , in 8% phosphate solutions had become indispensable.

The principle, on which these determinations were based, was the following:

By saturating a blacked platinum electrode with oxygen, an OH electrode may be obtained in the same way as a H electrode is made with hydrogen.

If  $\pi_c$  be the potential of the calomel electrode with saturated KCl solution and  $\pi_{OH}$  that of the OH electrode, the electromotive force, measured in the usual way at 27° is

$$E = \pi_c + \pi_{OH} = \pi_c + 0,0595 \log \frac{C}{c},$$

where  $C$  represents the concentration of OH, corresponding to the electrolytic solution pressure of the OH electrode, and  $c$  the OHion concentration. In such a cell the oxygen electrode is positive, the calomel electrode negative.

A second solution with a different hydroxyl-ion concentration gives in the same way:

$$E' = \pi_c + \pi'_{\text{OH}} = \pi_c + 0,0595 \log \frac{C}{c'}$$

Therefore:

$$E' - E = \pi'_{\text{OH}} - \pi_{\text{OH}} = 0,0595 \log \frac{c}{c'}$$

from which, substituting  $p_{\text{OH}} = -\log c$  and  $p'_{\text{OH}} = -\log c'$ :

$$E' - E = 0,0595 (p'_{\text{OH}} - p_{\text{OH}}).$$

Thus, by measuring the electromotive force of the calomel electrode, combined, first with a solution of known  $p_{\text{OH}}$ , e.g. a SÖRENSEN'S phosphate solution, and then with an 8% phosphate solution, this equation for  $E' - E$  gives at once the value  $p'_{\text{OH}}$ .

Determinations of this kind with two SÖRENSEN'S solutions gave nearly the right value.

As is well known however, an OH electrode does not give at all as constant and accurate results as a H electrode. The potential of an OH electrode has always been found about 150 millivolts too low, which fact is commonly ascribed to the formation of some suboxide of platinum.

This constant depression is, however, eliminated in our formula. Moreover it proved possible to arrange the experiments in such a way, that the influence of the inconstancy was repressed considerably.

Some 10 c.c. of the liquid to be examined were introduced into a small tube of about 1 cm. diameter. By dipping the electrode tube into this small quantity a quick and thorough saturation with oxygen could be obtained.

The oxygen, free from hydrogen, was washed in a bottle with 3% KI solution to keep back possible traces of ozone. It was brought to the temperature of 27° by keeping the washbottle and rubber tubing as well as the other apparatus (the same as used for the  $p_{\text{H}}$  determination) in the thermostat.

After oxygen has been through for some 10 minutes, the electrode tube was lifted out of the liquid under examination and connected with the calomel electrode as usual by means of the KCl-cup; the electromotive force was read within about one minute.

This process was carried through alternately with a SÖRENSEN'S solution (usually 5,6 c.c. alkaline + 4,4 c.c. acid phosphate), then with the 8% phosphate, and then with another SÖRENSEN'S solution (9 c.c. alkaline + 1 c.c. acid phosphate). Immediately after this the same series of three observations was once or twice repeated. Every estimation of the unknown  $p_{\text{OH}}$  was thus preceded and followed at

the same interval by determination in a liquid of known  $p^{\text{OH}}$ . As a gradual change in the electromotive force of the same combination was generally noticed on returning to it after the two other estimations, the influence of this change could at any rate be eliminated for the greater part by interpolation.

The value, determined in this way for the negative logarithm of  $k_w$  in 8% phosphate solutions, was in the mean

$$13,78.$$

As to the long standing problem, why the potential of the OH electrode is usually found about 0,15 Volt too low, some information may be derived from the following observation.

If the blackened platinum electrode had been polarised cathodically in dilute sulphuric acid, the value found for its potential was too low, in accordance with the experience of previous investigators. If, however, the electrode had been the anode in sulphuric acid, the determination of its potential showed a value, by nearly the same amount too high.

The following estimations were carried out with the same 8% phosphate solution ( $p_H = 6,92$ ).

*Oxygen electrode cathodically polarised:*

$$\begin{array}{r} \rightarrow \\ \pi_H + \pi_{\text{calomel}} = 0,653 \text{ Volt} \\ \leftarrow \\ \pi_{\text{OH}} + \pi_{\text{calomel}} = -0,421 \text{ Volt} \\ \hline \pi_H + \pi_{\text{OH}} = 1,08 \text{ Volt} \end{array}$$

*Oxygen electrode anodically polarised:*

$$\begin{array}{r} \rightarrow \\ \pi_H + \pi_{\text{calomel}} = 0,653 \text{ Volt} \\ \leftarrow \\ \pi_{\text{OH}} + \pi_{\text{calomel}} = -0,736 \text{ Volt} \\ \hline \pi_H + \pi_{\text{OH}} = 1,39 \text{ Volt} \end{array}$$

The theoretical value for the electromotive force of the oxygen-hydrogen-cell at room temperature is 1,23 Volt.

This fact appears to indicate, that the difference between the potential, observed at the oxygen electrode and the theoretical value is due to a polarisation phenomenon, as it can be quantitatively reversed by reversing the state of polarisation of the electrode.

13. *General remarks.*

It will be clear from the contents of this paper, that the theoretical formula

$$\frac{nc}{0,434} \log \frac{1}{1-y} + ay = mt$$

is borne out by experiment in different ways.

The hydrogen-ions having been found to absorb the enzyme radiation, the question was considered, if the hydroxyl-ions possess this property also. If such were the case, the equation should be

$$\frac{\overset{++}{n} \overset{--}{c} + \overset{--}{n} \overset{++}{c}}{0,434} \log \frac{1}{1-y} + ay = mt.$$

By repeating the experiments of Part 3, but now with solutions of different  $p_H$ , in most cases  $p_H = 7,21$ , it was tried several times

to decide this point. Combination of the factor  $\frac{\overset{++}{n} \overset{--}{c} + \overset{--}{n} \overset{++}{c}}{0,434}$ , found for two

different  $p_H$ , should give the necessary equations to calculate both  $\overset{+}{n}$  and  $\overset{-}{n}$ . As will be clear, however, from the considerations in Part 3, the inevitable small experimental errors have an even larger influence at a  $p_H$  below or above the  $p_H = 7,52$  of maximum activity. It proved to be impossible to carry out experiments of sufficient accuracy for this purpose. Still, the values, obtained for  $\overset{-}{n}$ , though varying widely, were generally so small, sometimes even negative, that they allowed the conclusion, that the hydroxyl-ions (i.e. negative electricity) do not absorb the urease radiation.

Since the writer's first study on Enzyme-action <sup>1)</sup> in 1904, in which the probability was first pointed out, that also the catalytic action of hydrogen-ions and of many other catalysts might be due to radiation, the conception of catalysis as a radiation phenomenon has been taken up from different sides.

In a series of papers LEWIS <sup>2)</sup> has worked out the theory, that hydrogen-ions act catalytically through radiations, a molecule of the catalysed compound becoming only reactive, if its energy has been increased by absorbed radiation (ultra red heat. radiation) to the "critical" condition.

In extensive experimental researches NILRATAN DHAR <sup>3)</sup> pointed

<sup>1)</sup> Proc. K. Akad. Wetensch. Amsterdam and Zeitsch. physikal Chem. 49,4.

<sup>2)</sup> T. Chem. Soc. 1914, 2330, etc.

<sup>3)</sup> Proc. K. Akad. Wetensch. Amsterdam 1916 and T. Chem. Soc. 1917, 690.

out the analogy between chemical catalysts and light. His conclusion was, that: "probably the effects of increase of temperature, of light, and of chemical catalysts in a reaction are intimately connected and are possibly identical in nature". They all appeared to act by shifting the equilibrium between "active" and "inactive" molecules.

### SUMMARY.

1. The enzyme urease acts by a radiation, which is only absorbed by its substrate, urea, and by H-ions.

2. The mathematical formulation of this conception is

$$- dx = m \frac{x}{x + nc} dt,$$

in which  $x$  is the concentration of urea at the time  $t$ ,  $c$  the concentration and  $n$  the absorption-coefficient of the H-ions,  $m$  a constant, proportional to the concentration of urease, if H-ion concentration as well as temperature are kept constant.

Integration gives the formula

$$\frac{nc}{0,434} \log \frac{1}{1-y} + ay = mt,$$

in which  $a$  is the initial concentration of urea, and  $y$  the fraction of  $a$ , still present at the time  $t$ .

3. By numerous experiments this equation is shown to represent the kinetics of urease action at constant  $p_H$  and temperature. It explains the nearly straight lines of the hydrolysis in alkaline solutions equally well as the practically logarithmic curves in acid ones.

4. Comparing equal concentrations of urease at varying H-ion concentration, the constant  $m$  is found to be dependent on the  $p_H$ ; i. e. the activity of a given concentration of urease is a function of the  $p_H$  of the solution.

Plotting  $m$  as a function of  $p_H$ , the resulting curve is strikingly similar to the characteristic curves for the undissociated fraction of an amphoteric electrolyte as a function of  $p_H$ .

This connection can be formulated mathematically and leads to the conclusion, that urease is an amphoteric electrolyte, whose activity is greatest when undissociated. The curve obtained represents the excess of activity of undissociated over dissociated urease.

5. This mathematical formulation leads to an approximate determination of the dissociation-constants of urease, which are calculated to be not far from those of carbonic acid and ammonia.

6. The accelerating action on urease, ascribed by previous authors

to carbonic acid, is shown to be non-existent. Ammonium carbonate + carbonic acid form a powerful buffer-mixture, which can maintain constancy of  $p_H$ , the indispensable condition for constant enzyme activity in a urea solution during hydrolysis by urease.

Fresh confirmation of the above formula for the rate of hydrolysis is afforded by many experiments with this buffer-mixture.

7. The estimation of initial velocities of hydrolysis, equal concentration of urease being allowed to act on different concentrations of urea at constant  $p_H$  and  $T$ , produces results, which appear unexplainable without the radiation-theory.

The lower the  $p_H$ , the more these initial velocities increase on increase of the concentration of the urea. With high  $p_H$  there is first an increase and then a decrease on raising the urea concentration.

These facts are shown to be in perfect accordance with the radiation-theory.

8. The influence of neutral substances is investigated experimentally and theoretically. Both decrease and increase of enzyme action by the same substance are explained by the influence the neutral body has on the dissociation-constants of water or of urease or of both of them.

9. The hypothesis is put forward that urease radiation, weakened by spreading or in any other way, causes synthesis.

Experimental evidence of this is afforded by the fact, that at high  $p_H$ , where the urease is shown to be decaying, reversion of the hydrolysis is several times observed.

10. A second inference from this conception, that outside the sphere of hydrolytic action around a urease molecule there must be a region of radiation, weakened by spreading, and therefore of synthesis, explains the fact, established by a series of new experiments, that, diluting the urease concentration beyond a certain value, its specific activity is decreased. For evidently the synthetic action of undecaying urease can only be manifested, if the spheres of hydrolytic action do not intersect each other sufficiently.

11. A third inference, that in any urease solution, in which the enzyme is decaying through the combined action of alkalinity, temperature and time, a synthesis of urea from ammonium carbonate, proportional to the urease concentration, will be observed, is tested experimentally and confirmed.

12. Description of an apparatus for the simple determination H-ion concentration at constant temperature.

The determination of the hydroxyl-ion concentration, which is needed for the calculation of the dissociation-formula of urease, is

carried out with the same apparatus, after converting the hydrogen electrode into an oxygen electrode.

*Delft.*

*Laboratory of the Dutch Yeast and  
Spirit Manufactory.*

A complete account of this investigation, with all experimental results, will appear in the *Recueil des Travaux Chimiques des Pays-Bas*.

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**Zoology.** — “*On the phylogeny of the hair of mammals*”. By Prof. P. N. VAN KAMPEN. (Communicated by Prof. WEBER).

(Communicated in the meeting of May 31, 1919).

In his work on “*Die in Deutschland lebenden Arten der Saurier*” (1872) and later on <sup>1)</sup>, in a paper in which he rejects MAURER’s well-known theory on the derivation of the hairs of mammals from epidermal sense-organs, LEYDIG draws the attention to the resemblance between the structure of the hairs and the so-called thigh or femoral organs of the lizards, which he considers to be a transition form between ordinary epidermal proliferations and hairs.

Less attention has been paid to this remark than it would have deserved. For the structure of the afore-mentioned organs, whose function is not known (they probably participate in the act of copulation) closely resembles in fact the structure of a hair in a simplified form <sup>2)</sup>: they are cylindrical rods, composed of horny epidermal cells, and sunken into a follicle of the skin. They differ from hairs principally by the absence of a cutis papilla and by the fact that they do not show a differentiation in medulla, cortex and cuticle. It is true that according to MAURER <sup>3)</sup> they are composed of two kinds of cells, but the arrangement of these cells is quite different from the one of the elements of the hair.

As not one of the hypotheses which try to derive hairs from epidermal organs of lower Vertebrates and among which the afore-mentioned one of MAURER, based on a large body of facts, is best known, has been generally acknowledged (indeed, BOTEZAT <sup>4)</sup> in his review on these theories comes to the conclusion that none of them can be maintained and that the hairs in the mammals independently have taken their origin in the skin) it is desirable to examine, whether the idea uttered by LEYDIG might contain perhaps a germ of truth. Against a direct derivation of hairs from femoral organs it may be advanced that these organs among the recent reptiles

<sup>1)</sup> Biol. Centralbl., XIII, 1893.

<sup>2)</sup> LEYDIG compares them, in my opinion wrongly, to a bundle composed of hairs glued together.

<sup>3)</sup> Die Epidermis und ihre Abkömmlinge. Leipzig, 1895, p. 212 ff.

<sup>4)</sup> Anat. Anzeiger, XLVII, 1914/15.

only occur with the Lacertilians and with these far from generally, often only with the male and, besides, always in a very limited number. Elsewhere among Reptiles, organs in some degree comparable with femoral organs are only found with crocodiles, where, according to VOELTZKOW's description <sup>1)</sup>, they lie between the scales of the back.

But, though it would be difficult to assume the direct origin of the hairs from femoral organs, the question might still be raised, whether there might not be a connection between them, in so far that they have a common origin. If this were the case, their origin might probably be traced most easily with the last-mentioned, more simply constructed organs, which in this way might throw a light on the origin of the hair.

The morphological significance of the thighorgans has been elucidated by the research of SCHÄFER <sup>2)</sup>. This author not only confirms what has already been recorded by earlier investigators, viz. that with *Lacerta* the femoral organs of the male are most strongly developed in the breeding time, but he emphatically points out that in that period no keratinisation of the cells takes place. But of more importance is what he found with *Sceloporus acanthinus*: with this Iguanide no horny cells are formed in the organs, but instead of them a secretion, which is composed "aus einer völlig zerfallenen, dem Secret von Talgdrüsen ähnlich sehenden Masse". SCHÄFER comes to the conclusion that the thighorgans are glandulae celluliparae, related to those sebaceous glands, which are not connected with hairs. Keratinisation occurs only, when the secretion is slow. In connection with this conclusion the statement of MAURER <sup>3)</sup> is of importance, that with *Lacerta* the contents of part of the cells of the thighorgans is of a fatty nature.

These facts point to a close connection of the femoral organs with "holocrinous" cutaneous glands, and the conclusion that they can be derived from such glands is obvious. The difference between them is not great: if the fatty secretion in a sebaceous gland were replaced by keratinisation of the cells, then a horny rod would be formed, which would show great resemblance to the thighorgans. Now the eleidin, which appears with mammals in the process of keratinisation, according to MAURER has a fatty character <sup>4)</sup>, while on the

<sup>1)</sup> Abhandl. Senckenberg. Naturf. Ges., XXVI, H. 1, 1899.

<sup>2)</sup> Archiv f. Naturgesch., LXVIII, Bd. 1, 1902.

<sup>3)</sup> l. c., p. 220.

<sup>4)</sup> GOETTE (Arch. f. mikr. Anat., IV, 1868) also describes the occurrence of fat-globules in the young epithelial hair-germ of the sheep.

other side it is known, that the cells of the sebaceous glands of the Mammalia contain eleidin-granules and can even partly undergo keratinisation<sup>1)</sup>. One must indeed imagine the sebaceous cells to originate from ordinary epithelium-cells, which had already the capacity of keratinisation and it is not surprising that this capacity reappears now and then.

So the femoral organs of *Lacerta* have arisen from cutaneous glands; they have preserved the structure of those glands, chemically however they are modified, in connection with the strong keratinisation, which is characteristic of the skin of reptiles in general.

If the femoral organs can be derived from cutaneous glands, one can imagine the same thing in the case of the hairs of mammals. Only in this case the differentiation has become greater and the structure of the organ is more complicated, in consequence of the more important function the hair has in the life of mammals. The hair papilla is to be considered of secondary origin and to have arisen in connection with the richer nutrition, which had become necessary for the stronger growth.

But there is still another phenomenon that can be easily explained by this hypothesis. The origin of the hair as a solid epidermic thickening quite agrees with that of cutaneous glands, but also with that of the femoral organs, according to the descriptions of MAURER and SCHÄFER. As to the thighorgans, the first author already directs the attention to this similarity with cutaneous glands of the amphibians, but attaches much importance to the difference between them, which lies in the fact that the smooth muscular fibres of the glands of the amphibians are absent in the femoral organs. In this point I cannot agree with him: these muscular fibres, which in the cutaneous glands are necessary for the extrusion of the secretion, are from their very nature superfluous in the entirely horny thighorgans, and so it is perfectly clear, that they have disappeared. And the same is true for the hairs, where they are absent as well. Another point, to which MAURER attaches much importance, is the peculiar arrangement of the matrix-cells, which appears in the very first origin of the hair and of the dermal sense organ in the same manner. It seems to me however that this arrangement may be explained by the pressure of the surrounding cells upon the growing germ and so in different cases may appear in similar circumstances.

Since the researches of DE MEIJERE<sup>2)</sup> an attempt to explain the origin of the hair must take into consideration their arrangement on the skin.

<sup>1)</sup> Cf. SCHÄFER, Text-Book of Microscopic Anatomy, 1912, p. 476.

<sup>2)</sup> Morph. Jahrb., XXI, 1894.

In those places where in mammals scales occur, the hairs are inserted generally in groups behind them. In this point they therefore differ from the thighorgans, which are placed in the middle of scales. It is true, that not too much value ought to be attached to this fact, but yet I want to point out in the first place that the afore-mentioned dermal organs of the crocodiles, described by VOELTZKOW, are arranged between the scales, and further, that the similarity in location of hairs and femoral organs becomes greater, if the considerations of PINKUS<sup>1)</sup> in connection with the "hairdiscs" described by him, are right. If, as he thinks, a "Haarbezirk", that is the whole complex of scale rudiment, hair group and hairdisc, answers to the scale of reptiles, then the hairs are placed in the middle of the region of the scale, just the same as is the case with the thighorgans. PINKUS, who derives the hairdiscs from tactile spots of reptiles, cannot find an explanation for the origin of the hairs: "Das Säugetierhaar hat kein Homologen in dem Gebiet der Reptilienschuppe; sein Platz ist leer." By the hypothesis, developed before, this objection against PINKUS' theory is done away with.

MAURER has directed the attention to another arrangement of the hairs; still before the hairgroups are formed, in mammalian embryos the placing of hairs in longitudinal rows may be stated. From this fact MAURER deduces an argument for his before mentioned theory, because epidermal sense organs generally show a similar arrangement. This argument however becomes worthless by the observation

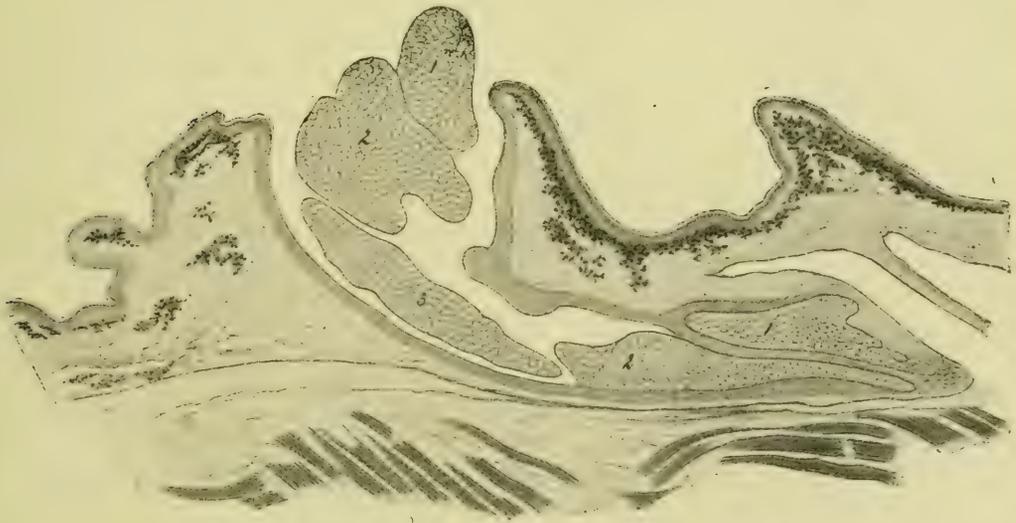


Fig. 1 *Lacerta agilis*. Bundle of three femoral organs (1—3).

<sup>1)</sup> Arch. f. mikrosk. Anat., LXV, 1905.

of MAURER himself<sup>1)</sup>, that also the first germs of the dermal glands of Triton and Anura occur in rows.

The dermal glands of Promammalia have thus developed in the Mammalia in diverging direction: the hairs, as well as the dermal glands of the mammals have arisen from them. The complex of hair and sebaceous glands is to be derived either from a compound gland, the follicles of which have taken a different direction of development, or what seems more probable to me, from the union of a number of glands into one follicle, in the same way as hairs may be united into a bundle. This last derivation may be strengthened by the fact that a number of femoral organs too, sometimes form a bundle with a common follicle. I found this in *Lacerta agilis* (see fig. p. 143).

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<sup>1)</sup> l. c., p. 159.

**Physics.** — “*Note on the circumstance that an electric charge moving in accordance with quantum-conditions does not radiate.*”

By G. NORDSTRÖM. Supplement N°. 43a to the Communications from the Physical Laboratory at Leiden. (Communicated by H. KAMERLINGH ONNES).

(Communicated in the meeting of May 31, 1919).

One often hears the remark: BOHR's atomic theory is at variance with classical electrodynamics in assuming that an electron which is moving according to quantum-conditions does not radiate energy in the form of electromagnetic waves. The assertion formulated in this way does not seem to me to state correctly where the opposition between BOHR's assumption and classical electrodynamics lies. In the sequel I shall try to substantiate this view. We shall begin by looking at the problem from a general point of view.

If an otherwise empty space contains electric charges whose motions are completely fixed, the electro-magnetic field is not singly determined by means of the MAXWELL-LORENTZ field-equations. In order to obtain a perfectly definite condition certain boundary-conditions must be fixed and it is to these that we shall give our attention. Whatever the field may be, it may be represented by the electro-dynamic potentials viz. the vector-potential  $\mathfrak{A}$  and the scalar potential  $\varphi$ , which may also be combined in a four-dimensional vector-potential with the following components:

$$\varphi_x = \mathfrak{A}_x \quad ; \quad \varphi_y = \mathfrak{A}_y \quad , \quad \varphi_z = \mathfrak{A}_z \quad , \quad \varphi_t = \varphi \quad . \quad . \quad . \quad ) \quad 1)$$

The potentials determine the field-vectors  $\mathfrak{E}$ ,  $\mathfrak{B}$  completely by means of the equations <sup>1)</sup>

$$\begin{aligned} \mathfrak{B} &= \text{rot } \mathfrak{A}, \\ \mathfrak{E} &= -\text{grad } \varphi - \frac{1}{c} \frac{\partial \mathfrak{A}}{\partial t} \quad (c = \text{velocity of light}) \end{aligned} \left\{ \begin{array}{l} . \\ . \\ . \end{array} \right. \quad . \quad . \quad . \quad ) \quad 2)$$

On the other hand the potentials  $\mathfrak{A}$ ,  $\varphi$  are not completely determined by the field. For this reason we may submit them to the condition

$$\text{div } \mathfrak{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0,$$

<sup>1)</sup> Comp. for instance M. ABRAHAM, *Theorie der Elektrizität* II, 2<sup>te</sup> Aufl., p. 36.

and we then obtain for the various potential-components (using rational units) the following differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\sigma_\alpha, \dots \dots (3)$$

( $\alpha = x, y, z, t$ ).

In this equations  $\sigma_x, \sigma_y, \sigma_z$  are the components of the electric current,  $\sigma_t$  is the density of the electricity. The motion of the charges being given,  $\sigma_x, \sigma_y, \sigma_z, \sigma_t$  are known functions of  $x, y, z, t$ . The general solution of the differential equations (3) may then be put in the following form:

Let  $x_0, y_0, z_0$  be the coordinates of a point for which  $\varphi_\alpha$  is to be found. The distance of this point to a point  $(x, y, z)$  may be called  $r$ , so that

$$r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \dots \dots (4)$$

The sign of  $r$  is not fixed by this relation, but we may leave it undetermined in the mean time.

A unit vector with components

$$\frac{x-x_0}{r}, \frac{y-y_0}{r}, \frac{z-z_0}{r} \dots \dots (5)$$

will be represented by  $r$ . The direction of  $r$  is evidently dependent upon the choice of the sign of  $r$ . Let  $F$  be an arbitrary closed surface which encloses the point  $x_0, y_0, z_0$ . A surface-element of  $F$ , considered as a vector directed outwards, will be denoted by  $d\mathfrak{S}$ . Using these symbols we may write the general integral of the differential equation (3) in the form <sup>1)</sup>:

$$\varphi_\alpha(x_0, y_0, z_0, t) = \pm \frac{1}{4\pi} \left\{ \int dV \frac{\sigma_\alpha}{r} + \int d\mathfrak{S} \left( \frac{1}{r} \text{grad } \varphi_\alpha + \frac{r}{c} \frac{\partial \varphi_\alpha}{\partial t} + \frac{r}{r^2} \rho \right) \right\} \Big|_{t-\frac{r}{c}} \dots (6)$$

The surface-integral has to be extended over the closed surface  $F$ , the space-integral over the enclosed space  $V$  containing the point  $x_0, y_0, z_0$ . The index  $t - \frac{r}{c}$  is meant to indicate, that at the

right-hand side the quantities  $\sigma_\alpha, \varphi_\alpha, \text{grad } \varphi_\alpha, \frac{\partial \varphi_\alpha}{\partial t}$  refer to the time  $t - \frac{r}{c}$ , which varies from point to point. The double sign on the right-hand

<sup>1)</sup> A proof of equation (6) may be found in Finska Vetenskaps-societets Förhandl. L. I. 1908—09, Afd. A. n<sup>o</sup>. 6. If the sign of  $r$  is not fixed beforehand, it is easily found that the right-hand side has the double sign.

side is required as long as the sign of  $r$  has not been settled, but may be chosen at will; the  $+$  sign holds for positive, the  $-$  sign for negative  $r$ . If the sign is taken positive, the moment  $t - \frac{r}{c}$  is earlier than  $t$  and  $\varphi_x$  must then be taken as a delayed potential. If the negative sign is chosen, the moment  $t - \frac{r}{c}$  is later than  $t$  and  $\varphi_x$  is to be considered as an advanced potential.

*Every function  $\varphi_x(x, y, z, t)$ , therefore, which satisfies the differential equation (3), may be considered either as a delayed or as an advanced potential, if only the contribution to the potential which is due to the boundary surface  $F$  (which may also be moved to infinity) is taken into account.*

It follows that every electro-magnetic field, i. e. every field for which the MAXWELL-LORENTZ equations hold, may be calculated for a moment  $t$  either from the condition at the time  $t - \frac{(r)}{c}$  or from that at the time  $t + \frac{(r)}{c}$ , if only the contribution by the boundary surface is taken into account. This contribution is necessarily different in the two cases.

If the surface  $F$  is made to move to infinity and if at the same time the condition is laid down, that at the boundary the surface-integral has the value zero, if the potential is considered as a delayed potential, the ordinary solution is obtained of the problem to calculate the field from the charges. But this solution is only one particular one: there are an infinite number of others.

The author may be excused for having discussed this question at some length: it seemed to him that it is not always sufficiently kept in view.

We shall now prove, that every periodical motion of electricity allows the assumption of a field such that no energy is radiated. The separate points of the electric charges will be identified by means of 3 parameters  $\xi, \eta, \zeta$ . Every set of values  $\xi, \eta, \zeta$ , therefore, denotes a definite point in the electricity sharing the motion of the latter. The motion is completely described by the equations

$$\left. \begin{aligned} x &= x(\xi, \eta, \zeta, t), \\ y &= y(\xi, \eta, \zeta, t), \\ z &= z(\xi, \eta, \zeta, t), \end{aligned} \right\} \dots \dots \dots (7)$$

that is to say, for given values of  $\xi, \eta, \zeta$  the coordinates  $x, y, z$  are

functions of the time. Let us consider the motion represented by equations (7) as being completely given. This motion we shall call *motion 1*. We therefore assume, that  $x, y, z$  are known functions of  $\xi, \eta, \zeta, t$  for all values of  $t$  from  $-\infty$  to  $+\infty$ . We then calculate the electro-magnetic field by means of *delayed* potentials and choose the boundary conditions in such a manner, that the surface-integral in (6) becomes zero for each potential-component  $\varphi_x$ , when the surface  $F$  moves to infinity. The field is then singly determined by the motion of the electricity. We shall call the field obtained *field 1*. In this case we obviously have a radiation of energy.

We shall further consider a different motion of the electricity, *motion 2*, which is obtained from motion 1 by reversing the sign of  $t$ .

$$\left. \begin{aligned} x &= x(\xi, \eta, \zeta, -t), \\ y &= y(\xi, \eta, \zeta, -t), \\ z &= z(\xi, \eta, \zeta, -t), \end{aligned} \right\} \text{motion 2.}$$

In this system all paths are evidently described in a direction opposite to that of motion 1. For motion 2 we again calculate the electro-magnetic field by means of *delayed* potentials and with the same boundary-conditions as before. We shall again obtain a field with energy-radiation, which we shall call *field 2*.

If the motion 1 is periodical, this will also be the case for motion 2 and the radiation during one period is equal for field 1 and field 2. We now pass from field 2 to a new field 3, by reversing the sign of  $t$  and at the same time the components of the magnetic field  $\mathfrak{B}_x, \mathfrak{B}_y, \mathfrak{B}_z$ . It is easily shown, that with this change of sign the MAXWELL-LORENTZ field-equations remain valid. As  $t$  changes sign, the motion of the electricity is reversed. *The motion of the electricity is therefore precisely the same in field 3 as in field 1*. Owing to the reversal of the sign of  $\mathfrak{B}_x, \mathfrak{B}_y, \mathfrak{B}_z$ , ( $\mathfrak{E}_x, \mathfrak{E}_y, \mathfrak{E}_z$ -retaining the same sign) the direction of the energy-stream is reversed, so that in field 3 we have a radiation of energy *inwards*. For a periodical motion of the electric charges the amount of energy drawn in during a period is the same as the radiation outwards in fields 1 and 2.

It is further easily found that field 3 may be calculated from *advanced* potentials, with a zero-contribution of infinity. If on the other hand the potentials are taken as *delayed*, the contribution of infinity is not equal to zero.

We now superpose field 1 and 3, which is possible since the field-equations are linear. In the two fields taken separately the electricity has the same motion, which therefore remains the same in the combined field. The density of the electricity on the other

hand is everywhere doubled by the superposition. In order to reduce this to the former value we diminish the strength of the field in the combination to half its value. The density of the electricity is thereby diminished in the same ratio, and in the new *field 4* we thus obtain precisely the same distribution and motion of the electricity as in field 1. The motion being periodical the energy-stream in field 4 evidently gives a total radiation zero during a full period. We have obtained a kind of stationary electro-magnetic waves.

Since the motion of the electricity in field 4 is identical with the motion from which we started, it has been proved that *every periodical motion of electric charges allows the assumption of an electro-magnetic field without radiation of energy*. Without further enquiry it is clear, that this proposition also holds, if the motion of the charges is not exactly periodical, but consists in, say, a planetary motion with a movement of the perihelion.

The question remains, whether BOHR's theory can draw any benefit from the result arrived at, but it seems that such is not the case. If the electrons in an atom were going round in the same orbits for all eternity, there would be nothing to prevent us assuming an electro-magnetic field such as field 4. But the sudden transitions from one allowable orbit to another cause difficulties. A simple calculation shows that in field 4 the energy-density for large distances

$r$  is proportional to  $\frac{1}{r^2}$ , the energy of the whole field, therefore,

becoming infinite. In consequence of this the change of energy associated with the transition of an electron has quite a different value to what BOHR has to assume, and it does not seem to me possible to make the two values agree. The above discussion, therefore, hardly seems to have a direct bearing on BOHR's theory, but it does seem to me to be of some use for obtaining a broader insight into the question as to where the difficulties of BOHR's theory actually lie. The result to which in my opinion it leads in this case was stated in the beginning of this note and I should like to formulate it in this way: the usual statement that it is inexplicable why an electron moving in accordance with the quantum-conditions should not radiate energy, seems to me to be based on an assumption which is not sufficiently general. A more general conception of the problem although unable to solve the difficulty, seems to me to put it in a different light.

*Leiden, April 18, 1919.*

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**Mathematics.** — “*Remark on Multiple Integrals.*” By Prof. L. E. J. BROUWER.

(Communicated in the meeting of June 28, 1919).

The object of this communication is to make two remarks in conjunction with the first part of my paper: “*Polydimensional Vectordistributions*”<sup>1)</sup> presented at the meeting of May 26, 1906.

§ 1.

I. The proof of the generalisation of STOKES’S theorem given l.c. pp. 66—70 provides this generalisation not only in the Euclidean but also in the following ametric form:

**THEOREM.** *In the  $n$ -dimensional space  $(x_1, \dots, x_n)$  let the  $(p-1)$ -tuple integral*

$$\int \Sigma F_{\alpha_1 \dots \alpha_{p-1}}(x_1, \dots, x_n) dx_{\alpha_1} \dots dx_{\alpha_{p-1}} \dots \dots \dots (1)$$

be given, where the  $F$ ’s are continuous and finitely and continuously differentiable; consider also the  $p$ -tuple integral

$$\int \Sigma f_{\alpha_1 \dots \alpha_p}(x_1, \dots, x_n) dx_{\alpha_1} \dots dx_{\alpha_p}, \dots \dots \dots (2)$$

where

$$f_{\alpha_1 \dots \alpha_p} = \sum_{\nu=1}^p \frac{\partial F_{j_\nu}}{\partial x_{\alpha_\nu}}$$

(indicatrix  $j_\nu$ ,  $\alpha_\nu$  aeq. indicatrix  $\alpha_1 \dots \alpha_p$ ).

Then, if the two-sided  $p$ -dimensional fragment<sup>2)</sup>  $G$  is bounded by the two-sided  $(p-1)$ -dimensional closed space  $g$ , the indicatrices of  $G$  and  $g$  corresponding and both  $G$  and  $g$  possessing a continuously varying plane tangent space, the value of (1) over  $g$  is equal to the value of (2) over  $G$ .

<sup>1)</sup> See Vol. IX, pp. 66—78; we take the definitions modified in accordance with note<sup>1)</sup> on p. 116 l.c. I take this opportunity of pointing out that on p. 76 l.c. lines 13 and 14 “finite sourceless current system” should be read instead of “system of finite closed currents”.

<sup>2)</sup> Math. Annalen 71, p. 306.

Of this theorem, which was enunciated by POINCARÉ<sup>1)</sup> in 1899 already, without proof however, and in a form expressing the rule of signs in a less simple manner, I shall here give the proof anew, editing it somewhat more precisely than in my quoted paper.

II. In the  $n$ -dimensional space  $(x_1, \dots, x_n)$ , which we shall denote by  $S$ , let the  $p$ -tuple integral

$$\int \Sigma \varphi_{\alpha_1 \dots \alpha_p}(x_1, \dots, x_n) dx_{\alpha_1} \dots dx_{\alpha_p} \dots \dots \dots (3)$$

be given, where the  $\varphi$ 's are continuous.

Let  $Q$  be a two-sided  $p$ -dimensional net fragment<sup>2)</sup> provided with an indicatrix and situated in  $S$ ,  $\sigma$  a base simplex of  $Q$  with the vertex indicatrix  $A_1 A_2 \dots A_p A_{p+1}$ ,  $A$  an arbitrary point of  $\sigma$ ,  $A\varphi_{\alpha_1 \dots \alpha_p}$  the value of  $\varphi_{\alpha_1 \dots \alpha_p}$  at  $A$ ,  ${}_x v_\mu$  the value of  $x_\mu$  at  $A_x$ . For every  $\sigma$  we determine the value of

$${}_x \varphi \equiv \Sigma A\varphi_{\alpha_1 \dots \alpha_p} \cdot \sigma i_{\alpha_1 \dots \alpha_p},$$

where

$${}_x i_{\alpha_1 \dots \alpha_p} \equiv \frac{1}{p!} \begin{vmatrix} {}_1 v_{\alpha_1} \dots {}_{p+1} v_{\alpha_1} \\ \dots \dots \dots \\ {}_1 v_{\alpha_p} \dots {}_{p+1} v_{\alpha_p} \\ 1 \dots \dots 1 \end{vmatrix},$$

and where, for different terms under the  $\Sigma$  sign  $A$  may be chosen differently; and we sum  ${}_x \varphi$  over the different base simplexes of  $Q$ . The upper and lower limit between which this latter sum varies on account of the free choice of the points  $A$ , we call *the upper and lower value of (3) over  $Q$* .

If we now subject  $Q$  to a sequence of indefinitely condensing simplicial divisions which give rise to a sequence  $Q', Q'', \dots$  of net fragments covering  $Q$ , then, as  $\nu$  increases indefinitely, the upper and lower value of (3) over  $Q^{(\nu)}$  converge to the same limit, which we call *the value of (3) over  $Q$* .

Let  $F$  be a two-sided  $p$ -dimensional fragment provided with an

<sup>1)</sup> Les méthodes nouvelles de la mécanique céleste III, p. 10. The significance of the rule of signs here formulated, is apparent only after comparing former publications of the same author from the Acta Mathematica and the Journal de l'École Polytechnique in which the equivalence of the identically vanishing of (2) and the vanishing of (1) over every  $g$  was pronounced.

<sup>2)</sup> Math. Annalen 71, p. 316.

indicatrix and situated in  $S$ ,  $f$  a sequence of indefinitely condensing simplicial approximations  $P', P'', \dots$  of  $F$  corresponding to a category  $\psi$  of simplicial divisions. If the values of (3) over  $P', P'', \dots$  converge to a limit which is independent of the choice of  $f$  so far as it is left free by  $\psi$ , then we call this limit *the value of (3) over  $F$  for  $\psi$* .

III. We shall now occupy ourselves with the value of (1) over the boundary  $\beta$  of a  $p$ -dimensional simplex  $\sigma$  provided with an indicatrix and situated in  $S$ . In doing so we take it that the indicatrices of  $\beta$  and  $\sigma$  correspond, that is to say, the indicatrix of an arbitrary  $(p-1)$ -dimensional side of  $\sigma$  is obtained by placing the vertex of  $\sigma$  which does not belong to this side last in the indicatrix of  $\sigma$  and subsequently omitting it. We begin by confining ourselves to the contribution of the single term

$$\int F_{\alpha_1 \dots \alpha_{p-1}} dx_{\alpha_1} \dots dx_{\alpha_{p-1}}$$

to the value of (1) over  $\beta$ . By a suitable simplicial division  $\zeta$  of the space  $(x_{\alpha_1}, \dots, x_{\alpha_{p-1}})$  we determine a simplicial division of  $\beta$ , whose base simplexes correspond in pairs to those of  $(x_{\alpha_1}, \dots, x_{\alpha_{p-1}})$ . The family of those  $(n-p+1)$ -dimensional spaces within which  $x_{\alpha_1}, \dots, x_{\alpha_{p-1}}$  are constant, cuts the plane  $p$ -dimensional space in which  $\sigma$  is contained, in a family of straight lines which connect pairs of corresponding base simplexes of  $\beta$  into  $p$ -dimensional truncated simplicial prisms. If  $e_1$  and  $e_2$  are a pair of corresponding base simplexes of  $\beta$ ,  $d$  the concomitant truncated simplicial prism,  $l$  a line segment having components  $r_{\alpha_1}, \dots, r_{\alpha_n}$  which leads from a point of  $e_1$  to the corresponding point of  $e_2$ , then the contribution of the term  $\int F_{\alpha_1 \dots \alpha_{p-1}} dx_{\alpha_1} \dots dx_{\alpha_{p-1}}$  to the value of (1) over  $e_1$  and  $e_2$  becomes

$$- e_1 i_{\alpha_1 \dots \alpha_{p-1}} \cdot \sum_{\nu=\mu}^n r_{\alpha_\nu} \cdot A \left\{ \frac{\partial F_{\alpha_1 \dots \alpha_{p-1}}}{\partial x_{\alpha_\nu}} \right\} + \epsilon,$$

where  $A$  denotes a point of  $\sigma$  which may be different for the different terms under the  $\Sigma$  sign, and  $\epsilon$  becomes indefinitely small with respect to  $e_1 i_{\alpha_1 \dots \alpha_{p-1}}$  for indefinite condensation of  $\zeta$ .

Now let  $B_1 B_2 \dots B_p$  be a vertex indicatrix of  $e_1$  and  $x_{\alpha_\mu}$  the value of  $x_{\alpha_\mu}$  at  $B_\nu$ , then the value of  $r_{\alpha_\nu} \cdot e_1 i_{\alpha_1 \dots \alpha_{p-1}}$  can be expressed as



Now let  $\chi$  be a category of simplicial divisions of  $G$  analogous to  $\psi$ . The resulting simplicial divisions of  $g$  belong to a category  $\psi$  of the kind just described. Let  $G', G'', \dots$  be a sequence of indefinitely condensing simplicial approximations of  $G$  corresponding to  $\chi$ , then, at the same time, there is hereby determined a sequence  $g', g'', \dots$  of indefinitely condensing simplicial approximations of  $g$  corresponding to  $\psi$ . Since, in virtue of III, the value of (1) over  $g^{(\nu)}$  is equal to the value of (2) over  $G^{(\nu)}$ , there exists, just as there does a value of (1) over  $g$  for  $\psi$ , a value of (2) over  $G$  for  $\chi$ , both values being equal, and not changing if some other category of the same kind is chosen in the place of either  $\psi$  or  $\chi$ .

## § 2.

In introducing l.c. p. 70 the notion of a second derivative, we have omitted to give the definition of the underlying concept of *normality* of an  $S_p$  provided with an indicatrix and an  $S_{n-p}$  provided with an indicatrix which are perpendicular to each other in an  $S_n$  provided with an indicatrix. This definition we shall here give.

Let  $T$  be the point of intersection of  $S_p$  and  $S_{n-p}$ ,  $\alpha_1 \dots \alpha_p$   $T$  the indicatrix of  $S_p$  and  $\beta_1 \dots \beta_{n-p}$   $T$  the indicatrix of  $S_{n-p}$ ; we call  $S_p$  *normal* to  $S_{n-p}$  and  $S_{n-p}$  *postnormal* to  $S_p$ , if  $\alpha_1 \dots \alpha_p T \beta_{n-p} \dots \beta_1$  is an indicatrix of  $S_n$ .

Thus, for some values of  $n$  the concepts *normal* and *postnormal* are equivalent, for other values not equivalent.

Furthermore we call a  $p$ -dimensional vector system  $V$  *normal* to an  $(n-p)$ -dimensional vector system  $W$  at the same point, and  $W$  *postnormal* to  $V$ , if, with respect to a rectangular system of coordinates the components of  $V$  are respectively normal to and of equal scalar values as the components of  $W$ .

In this terminology, the second derivative of the vector distribution  $X$  is the normal distribution of the first derivative of the postnormal distribution of  ${}^p X$ .





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**Mathematics.** — “Null systems determined by linear systems of plane algebraic curves”. By Prof. JAN DE VRIES.

(Communicated in the meeting of January 25, 1919).

1. A triply infinite system (complex)  $S^{(3)}$  of plane algebraic curves  $c^n$  contains a twofold infinity of nodal curves; for an arbitrarily chosen point  $D$  is node of a nodal curve  $\sigma^n$  belonging to  $S^{(3)}$ .

I shall now consider the null system in which the tangents  $d, d'$  of  $\sigma^n$  are associated as null rays with  $D$  as null point.

The nodal curves of a net belonging to  $S^{(3)}$  have their nodes on the Jacobian, which is a curve of order  $3(n-1)$ . It has in common with the Jacobian of a second net the  $3(n-1)^2$  nodes, which occur in the pencil common to the two nets. The remaining intersections of the two loci are critical points, i.e. nodes for the curves of a pencil. The null system, therefore, has  $6(n-1)^2$  singular null points.

2. Let  $a$  be an arbitrary straight line,  $P$  an arbitrary point. The  $\sigma^n$ , which has its node  $D$  on  $a$ , intersects the ray  $PD$ , moreover, in  $(n-2)$  points  $E$ . If  $E$  is to get into  $P$ ,  $\sigma^n$  must belong to the net that possesses a base-point in  $P$ ;  $D$  lies then on the Jacobian of that net. The locus  $(E)$  of the points  $E$  passes, therefore,  $3(n-1)$  times through  $P$ , and is consequently a curve of order  $(4n-5)$ . Each intersection of  $(E)$  with  $a$  is node of a  $\sigma^n$ , of which one of the tangents  $d$  passes through  $P$ .

There is therefore a curve  $(D)_P$  of order  $(4n-5)$  which contains the nodes of the nodal curves  $\delta^n$ , which send one of their tangents  $d$  through a given point  $P$ . It will be called the null-curve of  $P$ . For a singular point  $S$  it has in  $S$  a triple point. As  $P$  evidently is node of  $(D)$ , there lie on a ray  $d$  passing through  $P$   $(4n-7)$  points  $D$ , for which  $d$  is one of the tangents of the corresponding curve  $\sigma^n$ . From which ensues: an arbitrary straight line  $d$  has  $(4n-7)$  null-points  $D$ .

3. The null-curves  $(D)_P$  and  $(D)_Q$  have the  $6(n-1)^2$  singular points in common; for, a critical point bears  $\infty^1$  pairs  $d, d'$ .

The two curves pass further through the  $(4n-7)$  null points of the straight line  $PQ$ . Each of the remaining intersections is a point  $D$ , for which  $d$  passes through  $P$ ,  $d'$  through  $Q$ . In other words, if  $d$  revolves round  $P$ ,  $d'$  will envelop a curve of class

$(10n^2 - 32n + 26)$ . To the straight lines  $d'$ , which pass through  $P$ , belong the tangents of the  $\delta^n$ , which has its node in  $P$ . Each of the remaining  $(10n^2 - 32n + 24)$  straight lines  $d'$  evidently coincides with a ray  $d$ , and therefore contains a null-point  $D$ , for which the two null-rays have coincided. If such a straight line is called a *double null-ray*, it ensues from the above that *the double null-rays envelop a curve of class  $2(n-2)(5n-6)$* <sup>1</sup>).

4. The null-rays  $d$ , which have a null-ray  $D$  on the straight line  $p$ , envelop a curve  $(p)$  of class  $(4n-5)$ , which has  $p$  as  $(4n-7)$ -fold tangent. It, therefore, intersects  $p$  in  $(4n-5)(4n-6) - (4n-7)(4n-6)$  points, which bear each two coinciding null-rays.

*The locus of the points  $C$ , which bear a double null-ray, is, therefore, of order  $4(2n-3)$ .*

The curve  $(C)$  is evidently the locus of the cusps of the complex. As the order of  $(C)$  may also be determined in another way, it appears at the same time that the curve  $(p)$  has no other multiple tangents.

5. The case  $n = 2$  deserves a separate treatment. In the first place each line  $d$  has now only *one* null-point; this is the node of the conic, which is indicated by three points of  $d$ .

The locus  $(C)$  is now of the fourth order, and consists of four straight lines  $c_k$ . For, if the two straight lines of a nodal  $c^2$  coincide,  $c_k$  is a double line. The complex contains, therefore, *four double lines*, and they are at the same time *singular null-rays*.

The vertices  $S_{kl}$  of the complete quadrilateral formed by them are the singular points of the null-system.

The curves  $(p)_3$  and  $(q)_3$ , cf. § 4, have, besides the null-rays of the point  $pq$ , *seven* tangents in common, which have each a null-point on  $p$  and a null-point on  $q$ , and are consequently *singular null-rays*. To them belong the four straight lines  $c_k$ . Each of the remaining three singular null-rays  $s$  must belong to  $\infty^1$  nodal conics.  $S_{1,2}$  bears as singular point,  $\infty^1$  pairs of lines, which form an involution of rays; so  $S_{1,2}S_{3,4}$  belongs to two, and then to  $\infty^1$ , pairs of lines and consequently must be singular. The diagonals of the quadrilateral, which is formed by the four straight lines  $c$ , are consequently the three singular null-rays required.

<sup>1</sup> In other words, the *cuspidal tangents* of the cuspidal curves of a complex envelop a curve of class  $2(n-2)(5n-6)$ . In my paper on the characteristic numbers of a complex (These Proceedings, Vol. XVII, page 1055, § 13) the influence of the critical points in the determination of the class has been overlooked.

6. If the complex  $\{c^2\}$  has a base-point  $B$ , it is at the same time singular null-point, for two points on a ray passing through  $B$ , determine a nodal  $\sigma^2$ , with node in  $B$ . The double rays of the involution formed by the curves  $\sigma^2$  with node  $B$  are double lines of  $\{c^2\}$ , consequently singular null-rays. Other double null-rays do not exist, for if a straight line  $d$  of  $\sigma^2$  does not pass through  $B$ ,  $d'$  does. As  $B$  is node of the Jacobian of each net belonging to  $\{c^2\}$ , this point replaces four critical points. Two more singular points, therefore, lie outside  $B$ ; they are connected by a singular null-ray.

7. In a fourfold linear system  $S^{(4)}$ , each point  $D$  is node to a pencil ( $\sigma^n$ ). Two of those curves have a *cusp* in  $C \equiv D$ .

I now consider the *null-system* in which to the *null-point*  $C$  are associated the cuspidal tangents  $c, c'$  of the two cuspidal curves  $\gamma^n$ , which have their cusps in  $C$ .

The straight line  $d$  is touched in each of its points  $D$  by a nodal  $\sigma^n$ , which has its node in  $D$ . With the straight line  $PD$   $\sigma^n$  has moreover  $(n-2)$  points  $E$  in common. In order to find the locus of the points  $E$ , I shall inquire how often  $E$  gets into  $P$ . In this case  $\sigma^n$  belongs to the complex that has a base point in  $P$ ; in it occur  $(4n-7)$   $\sigma^n$ , which touch at  $d$  (§ 2). Consequently ( $E$ ) is a curve of order  $(5n-9)$ .

If  $E$  lies on  $d$ ,  $PE = d'$  touches in that point at a  $\delta^n$ , which has its node on  $D$ . Every straight line  $d$  therefore is nodal tangent of  $(5n-9)$  curves  $\sigma^n$ , of which the second tangent  $d'$  passes through  $P$ . If  $d$  is now made to revolve round a point  $Q$ , the point  $D$  describes a curve ( $D$ ) of which every point is node of a  $\delta^n$ , which sends its tangents  $d$  and  $d'$  through  $Q$  and  $P$ . In  $Q$  a  $\sigma^n$  is touched by  $QP$ , so  $Q$  and consequently  $P$  is a point of ( $D$ ), so that this curve is of order  $(5n-8)$ .

If  $C$  is one of the  $(5n-10)$  points, which ( $D'$ ) has in common with the straight line  $PQ$ , besides  $P$  and  $Q$ , the tangents  $d, d'$  fall both along  $PQ$ , so that  $C$  is a cusp of a cuspidal curve  $\gamma^n$ , which has  $c \equiv PQ$  as cuspidal tangent.

*In the above null-system a straight line therefore has  $5(n-2)$  null-points.*

If  $c$  revolves round a point  $M$ , the null-points  $C$  describe a curve of order  $(5n-8)$ , with node  $M$  (the *null curve* of  $M$ ).

8. The system  $S^{(4)}$  contains a number of curves with a triple point. If  $S^{(4)}$  is represented by the equation

$$\alpha A + \beta B + \gamma C + \delta D + \varepsilon E = 0,$$

the co-ordinates of a triple point have to satisfy the six equations:

$$\alpha A_{kl} + \beta B_{kl} + \gamma C_{kl} + \delta D_{kl} + \epsilon E_{kl} = 0,$$

in which  $A_{kl}$  etc. represent derivatives according to  $x_k$  and  $x_l$ .

The number of points has to be found, for which

$$\begin{vmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{23} & A_{31} \\ B_{11} & B_{22} & B_{33} & B_{12} & B_{23} & B_{31} \\ C_{11} & C_{22} & C_{33} & C_{12} & C_{23} & C_{31} \\ D_{11} & D_{22} & D_{33} & D_{12} & D_{23} & D_{31} \\ E_{11} & E_{22} & E_{33} & E_{12} & E_{23} & E_{31} \end{vmatrix} = 0.$$

According to a well-known rule we find for this

$$(5^2 - 4^2 + 3^2 - 2^2 + 1^2)(n-2)^2.$$

There are therefore  $15(n-2)^2$  curves  $c_n$  with a *triple point*  $S^1$ .

In such a point the nodal curves have the same tangents  $d, d'$ . Any straight line passing through  $S$  is to be considered as a cuspidal tangent  $c$ .

*The null-system therefore has  $15(n-2)^2$  singular points.*

9. I now take three points  $P, Q, R$ , arbitrarily, and consider (cf. § 7) the curves  $(D)_{PQ}$  and  $(D)_{PR}$ . To begin with they have the point  $P$  in common; for there is a  $\delta^n$ , which has  $P$  as a node, and  $PQ$  as tangent and a  $\delta^n$ , for which one of the tangents lies along  $PR$ .

Those curves have further in common the  $(5n-9)$  points  $D$ , for which  $QR$  is one of the tangents  $d$ . Another group of common points consists of the singular points  $S$ .

Let  $U$  be one of the still remaining intersections. There is in that case a  $\delta^n$  with tangents  $UP$  and  $UQ$ , and also a  $\delta^n$  with tangents  $UP$  and  $UR$ . From this it ensues that all  $\delta^n$  with node  $U$  have the straight line  $UP$  as tangent, consequently belong to a pencil in which the tangents  $d, d'$  form a parabolic involution.

The double rays of this involution have then coincided in  $UP$ , and  $U$  is cusp for only *one cuspidal*  $c^n$ . If such a point is called *unicuspidal point*, it follows from  $(5n-8)^2 - 1 - (5n-9) - 15(n-2)^2$  that  $(10n^2 - 25n + 12)$  unicuspidal curves send their tangent through  $P$ . *The cuspidal tangents of the unicuspidal points envelop a curve of class  $(10n^2 - 25n + 12)$ .*

10. In any point  $C$  of the straight line  $a$  I draw the two null-

<sup>1)</sup> If  $n = 3$ , and the system has 5 base-points, the 15 triple points are easy to indicate. One of them e.g. is the intersection of  $B_1B_2$  with  $B_3B_4$ .

rays  $c, c'$  (cuspidal tangents), and consider the correspondence between the points  $L, L'$ , which  $c, c'$  determine on the straight line  $l$ .

If  $c$  is made to revolve round  $L$ , the null-points of  $c$  describe a curve of order  $(5n-8)$ , which has a node in  $L$  (cf. § 7). To a point  $L$  therefore belong  $(5n-8)$  points  $C$  and  $(5n-8)$  points  $L'$ . The point  $al$  represents two coincidences  $L \equiv L'$ . The remaining coincidences arise from cuspidal tangents  $u$  of unicuspidal points  $U$ . *The locus of the unicuspidal points is therefore a curve of order  $2(5n-9)$ .*

This may be confirmed in the following way. If  $C$  describes the straight line  $p$ , the null-rays  $c, c'$  envelop a curve of order  $(5n-8)$  which has  $p$  as  $(5n-10)$ -fold tangent. It therefore has, not counting the points of contact,  $(5n-8)(5n-9) - (5n-10)(5n-9)$ , consequently  $2(5n-9)$  points in common with  $p$ . In each of these points the null-rays  $c$  and  $c'$  have coincided.

11. The system  $S^{(4)}$  produces in a still different way a *null-system*. Any point  $F$  is *flecnodal point* for five curves  $\varphi^n$ . In order to find this out we have only to consider the curve that arises if we make every  $\sigma^n$  that has  $F$  as node, to intersect its tangents  $d, d'$ . This  $C^{n+2}$  namely, has in  $F$  a quintuple point<sup>1)</sup>.

I now associate to each point  $F$  as *null-point* the five *null-rays*  $f$ , which are inflectional tangents for the five flecnodal curves  $\varphi^n$ .

Any point  $D$  of the straight line  $a$  is node for a  $\sigma^n$ , which touches the ray  $PD$  in  $D$ . I now determine the order of the locus of the groups of  $(n-3)$  points  $E$  which each of the curves  $\sigma^n$  has moreover in common with  $PD$ . If  $E$  lies in  $P$ ,  $\sigma^n$  belongs to a complex  $S^{(3)}$ . According to § 2 there are on  $a$   $(4n-5)$  nodes of curves  $\delta^n$  of  $S^{(3)}$  which send their tangent  $d$  through  $P$ . So  $P$  is  $(4n-5)$ -fold point of the curve  $(E)$  and the latter consequently of order  $(5n-8)$ . In each of its intersections  $F$  with  $a$  a curve  $\varphi^n$  has a flecnodal point, the inflectional tangent of which passes through  $P$ .

From this it ensues that the locus of the null-points  $F$  of the rays  $f$  out of a point  $P$  (*null-curve* of  $P$ ) is a curve of order  $(5n-8)$ . As it must have a *quintuple point* in  $P$ , an arbitrary straight line  $f$  therefore contains  $(5n-13)$  null-points.<sup>2)</sup>

<sup>1)</sup> In a point  $S$  (§ 8) the  $c^n$  with triple point replaces three of the curves  $\varphi^n$ ; for the other two the inflectional tangent lies along one of the two fixed tangents  $d, d'$ .

For a unicuspidal point (§ 9) one of the curves  $\varphi^n$  has its inflectional tangent along the fixed tangent  $d$ .

<sup>2)</sup> For  $n=3$  is  $5n-13=2$ . Each  $\varphi^3$  is then the combination of a straight line  $f$  and a  $\varphi^2$ . Each straight line  $f$  belongs in  $S^{(4)}$  to a figure  $(f, \varphi^2)$ ; its intersections with  $\varphi^2$  are the two null-points  $F$ .

12. In the null-system  $(C, c)$   $P$  has a null curve of order  $(5n-8)$  with node  $P$  (§ 7). Of its intersections with the null curve with respect to the system  $(F, f)$ , 10 lie in  $P$ . They also have the unicuspidal points  $U$  in common, for which the tangent  $u$  passes through  $P$ . In each of the remaining  $(5n-8)^2-10-(10n^2-25n+12)$  intersections  $G$ , a cuspidal curve has with its tangent  $g$  four points in common. From this it ensues that *the four-point cuspidal tangents envelop a curve of class  $(15n^2-55n+42)$ .*

If  $n$  is equal to *three*, the curves  $\gamma^3$  with four-point tangents are replaced by conics, each with one of its tangents. The null-system  $(F, f)$  then has the characteristic numbers 5 and 2; the null-curve  $(P)^7$  of  $P$  is of class 22, consequently sends 12 tangents  $f$  through  $P$ , and each of these straight lines forms with the conic touching it a  $\gamma^3$  with four-point tangent. In conformity with this, the form  $15n^2-55n+42$  produces for  $n=3$  the number 12.

13. In a quintuple infinite system  $S^{(5)}$  each point  $D$  is node for a net of nodal curves. A straight line  $d$  passing through  $D$  determines in it a pencil, of which all  $d^n$  touch at  $d$  in  $D$ . There is consequently *one* cuspidal  $\gamma^n$ , which has a straight line  $c$  passing through  $D$  as cuspidal tangent. The curves  $\gamma^n$ , with cusp  $D$ , form a system with index *two*, for the curves  $d^n$ , passing through any point  $P$ , form a pencil, which contains two curves with cusp in  $D$ . If every straight line  $c$  passing through  $D$  is made to intersect with the cuspidal  $\gamma^n$ , which it touches in  $D$ , there evidently arises a curve of order  $(n+2)$ , which has a quintuple point in  $D$ . From this it ensues that *five* cuspidal curves have in  $D$  a *cusp*, where the cuspidal tangent has a *four-point contact*.

I shall now consider the *null-system*  $(G, g)$ , in which to a point  $G$  are associated the *five* straight lines  $g$ , which are four-point cuspidal tangents for cuspidal curves  $\gamma^n$  with cusp  $G$ .

14. In each point  $C$  of the straight line  $a$  I consider the cuspidal curve  $\gamma^n$ , which sends its tangent  $c$  through  $P$ , and determine the locus of the points  $E$ , which  $\gamma^n$  has still in common with  $PC$ . If  $E$  lies in  $P$ ,  $\gamma^n$  belongs to a system  $S^{(4)}$ ; in it  $(5n-8)$  curves  $\gamma^n$  occur, which have their cusp on  $a$  (§ 7). So the curve  $(E)$  passes  $(5n-8)$  times through  $P$  and is of order  $(6n-11)$ . In each of its intersections  $G$  with  $a$ , a  $\gamma^n$  has four points in common with  $PG$ . The *null-curve* of  $P$  is therefore of order  $(6n-11)$ . As it has a *quintuple point* in  $P$ , a straight line  $g$  passing through  $P$  is *null-ray* for  $(6n-16)$  points  $G$ .

15. The system  $S^{(5)}$  contains  $\infty^1$  curves with a triple point  $T$ . If  $S^{(5)}$  is represented by

$$\alpha A + \beta B + \gamma C + \delta D + \epsilon E + \varphi F = 0,$$

the locus of the points  $T$  is determined by

$$\begin{vmatrix} A_{kl} & B_{kl} & C_{kl} & D_{kl} & E_{kl} & F_{kl} \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{vmatrix} = 0.$$

It is therefore a curve ( $T$ ) of order  $6(n-2)^1$ .

A  $\tau^n$  with triple point  $T$  determines with a nodal  $\delta^n$  which has its node in  $T$ , a pencil of nodal  $\delta^n$  with fixed tangents  $d, d'$ . The net of the curves  $\delta^n$  with node  $T$  therefore consists of  $\infty^1$  similar pencils of which the tangents  $d, d'$  form an involution. Each of the two nodal rays  $c_1, c_2$  is common cuspidal tangent for a pencil of cuspidal curves and each of these two pencils contains a  $\gamma^n$  with four-point tangent. The five null-rays  $g$  of  $T$  are therefore represented by the straight lines  $c_1, c_2$ , and the three tangents  $t_1, t_2, t_3$  of the curve  $\tau^n$ . The points  $T$  are consequently *not singular*.

16. In a sextuple infinite system  $S^{(6)}$  each point  $T$  is triple point of a  $\tau^n$ . To  $T$  as *null-point* the three tangents  $t_1, t_2, t_3$  of  $\tau^n$  are now associated as *null-rays*.

In order to find the second characteristic number of this *null-system*, I consider the curves  $\tau^n$ , of which the point  $T$  lies on the straight line  $a$  and I try to find the order of the curve, which contains the groups of  $(n-3)$  points  $E$ , in which  $\tau$  is moreover intersected by  $PT$ .

If  $E$  lies in  $P$ ,  $\tau^n$  belongs to an  $S^{(5)}$ , and  $T$  is one of the  $6(n-2)$  points which (§ 15) the curve ( $T$ ) has in common with  $a$ . So  $E$  is a  $(6n-12)$ -fold point on the curve ( $E$ ), which consequently has the order  $(7n-15)$ .

The *null-curve* of  $P$  is therefore of order  $(7n-15)$ . As it passes three times through  $P$ , a straight line  $t$  passing through  $P$  is tangent for  $(7n-18)$  curves  $\tau^n$ , which have their triple point  $T$  on  $t$ . A *null-ray*, therefore, has  $(7n-18)$  *null points*.

17. The curves ( $T$ ), which belong to two systems  $S^{(5)}$  comprised in  $S^{(6)}$ , have the  $15(n-2)^2$  points  $T$  of the system  $S^{(4)}$  in common, which forms the "intersection" of the two  $S^{(5)}$ .

The remaining intersections are *critical points*, viz. each of them is triple point for a pencil of curves  $\tau^n$ , consequently singular null-point  $S$  for  $(T, t)$ . This null system has consequently  $21(n-2)^2$  *singular null-points*.

<sup>1)</sup> If, for  $n=3$ , the system  $S^{(5)}$  has the base-points  $B_1, B_2, B_3, B_4$ , the curve ( $T$ ) consists of the straight lines  $B_k B_l$ .

As the triplets of tangents of the curves  $\tau^n$  of that pencil form an involution,  $S$  is triple point with a *cuspidal branch* for four curves  $\tau^n$ . Each singular null-point, therefore, bears *four double null rays*.

18. The null-curves of  $P$  and  $Q$  have the singular null-points  $S$  and the null-points of  $PQ$  in common. Each of the remaining intersections  $T$  sends a null-ray through  $P$ , a second through  $Q$ . From  $(7n-15)^2-21(n-2)^2-(7n-18)$  it therefore ensues that the null-rays  $t_2, t_3$  will envelop a curve of class  $(28n^2-133n+159)$ , if  $t_1$  revolves round a point  $P$ . The null-rays of  $P$  belong each twice to this envelope, each of the remaining tangents, which it sends through  $P$ , is evidently double null-ray. The *double null-rays*, therefore, envelop a curve of the class  $(28n^2-133n+153)$ .

19. In order to find the locus of the points  $T$  for which two of the null-rays coincide, I shall consider the curve  $(p)_{7n-15}$  enveloped by the null-rays of the points lying on  $p$ . It has  $p$  as  $(7n-18)$ -fold tangent, is therefore intersected by  $p$  in  $(7n-15)(7n-16)-(7n-18)(7n-17)$  points. As for each of these points two null-rays coincide, the points  $T$  with *double null-rays* lie on a curve of order  $(28n-66)$ .

It is at the same time the locus of the triple points that have a cuspidal branch.

For  $n=3$  we have a null-system  $(3,3)$ ; the curves  $\tau^3$  are three-rays in that case. An arbitrary straight line then forms figures  $c^3$  with the curves of a net of conics. The Jacobian of that net determines the three null-points of the straight line.

If the system  $S^{(6)}$  has three base-points, the three null-points of a straight line are produced by the intersection of the sides of a triangle, which has the base-points as vertices. Each base-point is the centrum of a pencil of singular null-rays.

**Mathematics.** — “On a certain point concerning the generating functions of LAPLACE.” By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 31, 1919).

1. The following remarkable proposition of the integral  $\int_0^{\infty} e^{-xr} \varphi(r) dr$ , or of the integral

$$\alpha(x) = \int_0^1 f(t)t^x dt, \dots \dots \dots (1)$$

derived from the former by the substitution  $r = -\log t$ , has been proved by LERCH<sup>1)</sup>:

If the determining function  $\alpha(x)$  vanishes for an arithmetical progression of values of  $x$  with positive common difference  $\eta$

$$x = \xi + \mu\eta, \quad (\mu = 0, 1, 2, \dots) \dots \dots (2)$$

then it vanishes for all values of  $x$ , and the generating function  $f(t)$  also vanishes.

LERCH uses for the proof a theorem of WEIERSTRASS, according to which any function which is continuous in a closed interval can be represented by a uniformly converging series of rational integral functions. Since the theorem, which is also mentioned by PINCHERLE<sup>2)</sup> and by NIELSEN<sup>3)</sup>, has a great many interesting consequences, it seems not unuseful to prove it in a manner which is independent of WEIERSTRASS's theorem. The reasoning we give in the next pages makes use of the theorem of FOURIER.

2. The following suppositions are sufficient for the purpose:

1. The function  $f(t)$  is *continuous* in the interval of integration, with possible exception as to the value  $t = 0$ .

<sup>1)</sup> Acta mathem. 27, 1903.

<sup>2)</sup> “Sur les fonctions déterminantes”, Ann. de l'Éc. Norm. 22, 1905. PINCHERLE calls  $f(t)$  “fonction génératrice” and  $\alpha(x)$  “fonction déterminante”, whereas LERCH does the reverse. We have followed the nomenclature of PINCHERLE in the text.

<sup>3)</sup> “Handbuch der Gammafunktion”, p. 118.

2. The integral (1) *exists* for a certain value  $x = c$  of  $x$ .

We put

$$g(t) = \int_0^t f(u) u^c du. \quad \dots \quad (3)$$

Then, by 2,  $g(t)$  is continuous in the *closed* interval  $(0,1)$ , and zero for  $t=0$ . Further, by 1,  $g'(t)$  is *differentiable* at all points of that interval, except, possibly, at  $t=0$ , and we have

$$g'(t) = f(t) t^c \dots \quad (4)$$

Hence, if  $\delta > 0$ , we may write

$$\int_{\delta}^1 f(t) t^x dt = \int_{\delta}^1 g'(t) t^{x-c} dt = [g(t) t^{x-c}]_{\delta}^1 - (x-c) \int_{\delta}^1 g(t) t^{x-c-1} dt$$

If, now,  $x$  is a complex number with real part  $R(x)$  *greater* than  $c$ , the number  $\delta$  in this equation may be made to approach to zero, and thus we find

$$\int_0^1 f(t) t^x dt = g(1) - (x-c) \int_0^1 g(t) t^{x-c-1} dt \quad \dots \quad (5)$$

From this it follows: If the integral (1) exists for a certain value  $x = c$  of  $x$ , it exists in the whole half-plane defined by  $R(x) > R(c)$  <sup>1)</sup>.

Further it follows from (5) that the integral in the left-hand member represents a *continuous* function of  $x$  in any domain  $S$  lying wholly in the finite part of the half-plane  $R(c) + \delta$ , where  $(\delta > 0)$ . In the same manner as above it is found that the integral

$$\int_0^1 f(t) t^x \log t dt \quad \dots \quad (6)$$

exists for  $R(x) > R(c)$  and represents the derivative of  $\alpha(x)$  at any point of this half-plane, so that  $\alpha(x)$  is also an *analytic* function. These consequences, too, are mentioned by PINCHERLE.

The proof LERCH gives of his theorem equally starts from the equation (5). In the following reasoning, however, we shall use an

<sup>1)</sup> This theorem is fundamental in the theory of generating functions. After PINCHERLE different authors have proved it, though often under less general suppositions. The reasoning in the text is due to LERCH. This reasoning is founded upon the *continuity* of  $f(t)$ , which, presumably, is forgotten by LERCH, when, at the end stating his theorem, he says that  $f(t)$  may be as well *discontinuous*. (Of course we do not mean to say that generalization is *impossible*).

equation derived from (5) by repeating once more the process which leads to the latter equation. So we put

$$h(t) = \int_0^t g(u) du \dots \dots \dots (7)$$

Then, again,  $h(t)$  is continuous and differentiable in  $(0,1)$  and we have

$$h'(t) = g(t) \dots \dots \dots (8)$$

The principal point, however, is that the latter equation is also valid at  $t=0$ . Thus the derivative of  $h(t)$  is a limited function in the closed interval  $(0,1)$ . Further, observing that

$$\lim_{t \rightarrow 0} [h(t) : t] = h'(0) = g(0) = 0,$$

we find on integrating by parts, for  $R(x) > R(c)$

$$\int_0^1 g(t) t^{x-c-1} dt = h(1) - (x-c-1) \int_0^1 h(t) t^{x-c-2} dt \dots \dots (9)$$

and hence

$$a(x) = g(1) - (x-c) h(1) + (x-c)(x-c-1) \int_0^1 h(t) t^{x-c-2} dt \dots (10)$$

3. The preceding statements are valid independently of any further hypothesis as to the character of  $f'(t)$ . Now, suppose that  $a(x)$  becomes zero for the arithmetical progression of values

$$x = \xi + \mu, \quad (\mu = 0, 1, 2, \dots) \dots \dots (11)$$

Choosing the number  $c$  in the preceding equations equal to  $\xi$ , we find  $g(1) = 0$  and the integral in the right-hand member of (5) vanishes for

$$x = \xi + 1 + \mu, \quad (\mu = 0, 1, 2, \dots) \dots \dots (12)$$

From this it follows that  $h(1) = 0$ , and, in connection with the latter result, from (10)

$$\int_0^1 h(t) t^\mu dt = 0, \quad (\mu = 0, 1, 2, \dots) \dots \dots (13)$$

Now we saw that the derivative of  $h(t)$  is limited. According to a well-known proposition  $h(t)$  can therefore be expanded in a series of FOURIER. We have

$$h(t) = \sum_0^\infty (a_n \cos 2\pi nt + b_n \sin 2\pi nt) \dots \dots (14)$$

where

$$a_0 = \int_0^1 h(t) dt$$

and, for  $n = 1, 2, 3, \dots$

$$a_n = 2 \int_0^1 h(t) \cos 2\pi nt dt, \quad b_n = 2 \int_0^1 h(t) \sin 2\pi nt dt$$

Now the functions  $\cos 2\pi nt$  and  $\sin 2\pi nt$  are for any value of  $n$  expandible in power-series

$$\cos 2\pi nt = \sum_0^\infty A_\mu t^\mu, \quad \sin 2\pi nt = \sum_0^\infty B_\mu t^\mu,$$

converging uniformly in the interval (0,1). Since  $h(t)$  is limited in that interval we may use the following reduction

$$\int_0^1 h(t) \cos 2\pi nt dt = \int_0^1 h(t) \sum_0^\infty A_\mu t^\mu dt = \sum_0^\infty A_\mu \int_0^1 h(t) t^\mu dt$$

and in a similar manner we find

$$\int_0^1 h(t) \sin 2\pi nt dt = \sum_0^\infty B_\mu \int_0^1 h(t) t^\mu dt$$

Hence by (13) all coefficients in the expansion of FOURIER are zero, and therefore  $h(t)$  is identically zero in the interval (0,1). Since, further,  $g(t) = h'(t)$ , the same thing holds for  $g(t)$ , and since  $f(t) t^\xi = g'(t)$  (except at  $t = 0$ ), the generating function  $f(t)$  itself is zero in the interval (0,1). This is the second part of LERCH'S theorem. Since the first part follows immediately from the second, the theorem has been proved in the particular case that the arithmetical progression of zeros of  $a(x)$  has 1 for its common difference.

If this difference is equal to the positive number  $\eta$  and if, therefore, the zeros are given by formula (2), we substitute

$$t^\eta = s, \quad x = \eta y, \quad \xi = \eta c$$

by which the integral passes into

$$\frac{1}{\eta} \int_0^1 f\left(s^{\frac{1}{\eta}}\right) s^{\frac{1}{\eta}-1+y} ds = \int_0^1 \varphi(s) s^y ds. \quad \dots \quad (15)$$

The function  $\frac{1}{\eta} f\left(s^{\frac{1}{\eta}}\right) s^{\frac{1}{\eta}-1} = \varphi(s)$  has the properties 1 and 2 of § 2, so that the foregoing arguments may be applied to it. The

integral (15) vanishes for the sequence of values (11), hence  $q(s)$ , and therefore also  $f(s)$ , identically vanishes in the interval (0,1). The theorem of LERCH has thus been proved completely.

4. The first part of the theorem, that  $\alpha(x)$  becomes *identically* zero, if this is the case for an arithmetical progression of  $x$ -values, may be proved in a *direct* manner, without first proving the second part; and it is an immediate consequence of the proposition:

A function  $\alpha(x)$  defined by an integral of the form (1) can, under the suppositions 1 and 2 mentioned at the beginning of § 2, be expanded in a binomial series

$$\alpha(x) = \sum_{n=0}^{\infty} c_n \binom{x-\beta}{n}. \quad \dots \quad (16)$$

where  $\beta$  is a number lying in the domain of convergence of the integral.

Suppose, for a moment, this proposition to be true. If, then,  $\alpha(x)$  becomes zero for the sequence of values (11), we take  $\beta = \xi$  in the equation (16). Substituting for  $x$  the values  $\xi, \xi + 1, \xi + 2, \dots$  in succession, we find that all coefficients  $c_n$  of the binomial expansion vanish and therefore that  $\alpha(x)$  vanishes *identically*.

The first part of LERCH's theorem is very easily proved in this manner and it would therefore be desirable that we might derive from it the second part in a short manner. But as yet we are not in a position to do this. The above demonstration is, after all, rather short, but besides, on grounds that, with a view to conciseness, we prefer not to state, we do not think it likely that the *identical* vanishing of  $\alpha(x)$  is more effective for the purpose than the vanishing for an *arithmetical progression* of values of the argument.

Nevertheless the first part of LERCH's theorem has an interest in itself, because remarkable consequences may be inferred from it. Among these LERCH mentions the truth that simple functions such as

$$\sin kx, \quad \cos kx, \quad \frac{2}{\Gamma(l-kx)}, \quad (k > 0)$$

cannot be the *determining* functions of *generating* functions, in other words that they cannot be represented by integrals of the form (1), neither can products of these functions with others which remain within finite limits in the finite part of a certain halfplane  $R(x) > c$ .

The proposition concerning the expansion of the integral (1) in a binomial series may be proved in different manners. In the first place integrals of that form belong to the general category of functions of which I showed, in an earlier communication in these

Proceedings (Vol. XXII, N<sup>o</sup>. 1) that they are expansible in series of the form (16). Consider a domain  $R(x) \geq c + \delta$ , take a positive number  $\delta_1 < \delta$  and substitute  $x = c + \delta_1 + y$  in the second integral of the right-hand member of (5), then  $R(y) > \delta - \delta_1$ , and thus positive, so that we have

$$\left| \int_0^1 t^{x-c-1} g(t) dt \right| \geq \int_0^1 t^{R(y)} \left| t^{\delta_1-1} g(t) \right| dt < \int_0^1 t^{\delta_1-1} |g(t)| dt,$$

where the latter integral exists, since  $g(t)$  is a limited function in the interval  $(0,1)$ . Hence  $\alpha(x)$  is in the whole domain considered of the form

$$\alpha(x) = (x-b) \mu(x)$$

where  $\mu(x)$  is a function remaining within finite limits and  $b$  a number lying without the domain. Suchlike functions, however, can always be expanded in series of the form in question.

A second, more direct proof, is obtained by substituting  $t = 1-u$  in the same integral as considered before, and using the following reduction

$$(1-u)^{x-c-1} = (1-u)^{\beta-c-1} (1-u)^{x-\beta} = (1-u)^{\beta-c-1} \sum_0^{\infty} (-1)^m \binom{x-\beta}{m} u^m$$

where the series for  $R(x) > R(\beta)$  converges uniformly in the interval  $0 < u < 1$ . Since, for  $R(\beta) > R(c)$  the integral

$$\int_0^1 g(1-u) (1-u)^{\beta-c-1} du$$

converges *absolutely* (on account of the continuity of  $g(1-u)$ ), we may, after performing the substitution in question, integrate term by term, and then we find (replacing again  $1-u$  by  $t$  in the partial integrals)

$$\int_0^1 g(t) t^{x-c-1} dt = \sum_0^{\infty} (-1)^m \binom{x-\beta}{m} \int_0^1 (1-t)^m t^{\beta-c-1} g(t) dt. \quad (17)$$

This expansion is, therefore, valid for  $R(x) > R(\beta) > R(c)$ . Since the product of this series with  $x-c$  can be transformed into a series of the same form, the required proposition has been proved again <sup>1)</sup>.

<sup>1)</sup> In NIELSEN'S book (l.c. p. 125) we find an analogous proof of the proposition in question; this, however, does not part from the integral in the second member of (5), but from the original integral, so that the hypothesis must be made that the latter converges *absolutely* for  $\lim t = 0$ . The reduction (5) makes this hypothesis superfluous.

A third proof has the advantage of showing that expansion of (1) according to factorials of  $x-c$  is possible when the integral only exists for  $x=c$ , even when the straight line  $R(x) = R(c)$  were the limit between the domains of convergence and divergence in the  $x$ -plane, and when the integral did not exist at all points of that line. The proof consists in repeating the process which led to the theorem of LERCH an infinite number of times. We write

$$\left. \begin{aligned} g(t) &= \int_0^t u^c f(u) du, & g_1(t) &\equiv h(t) = \int_0^t g(u) du, \\ g_2(t) &= 2 \int_0^t g_1(u) du, \dots g_n(t) &= n \int_0^t g_{n-1}(u) du, \dots \end{aligned} \right\} \dots (18)$$

Then formula (10) may be generalized in the following manner:

$$\begin{aligned} a(x) &= g(1) - g_1(1)(x-c) + g_2(1) \binom{x-c}{2} - g_3(1) \binom{x-c}{3} + \dots \\ &+ (-1)^{n-1} g_{n-1}(1) \binom{x-c}{n-1} + (-1)^n \binom{x-c}{n} \int_0^1 g'_n(t) t^{x-c-n} dt \dots \end{aligned} \quad (19)$$

The remainder has zero as a limit for  $R(x) > R(c)$ , for if  $G$  is the maximum modulus of the limited function  $g(t)$  in the interval  $(0,1)$ , we have in succession

$$|g_1(t)| < Gt \quad |g_2(t)| < Gt^2, \dots \quad |g_n(t)| < Gt^n, \dots$$

hence

$$\begin{aligned} \left| \int_0^1 g'_n(t) t^{x-c-n} dt \right| &= \left| n \int_0^1 g_{n-1}(t) t^{x-c-n} dt \right| \\ &< n \int_0^1 G t^{R(x-c)-1} dt \\ &< \frac{nG}{R(x-c)}, \text{ voor } R(x-c) > 0. \end{aligned}$$

Now  $\binom{x-c}{n}$  is for  $n = \infty$  equivalent to  $n^{-R(x-c)-1}$ , and thus the modulus of the remainder in formula (19) is for all  $n$ -values less than

$$\frac{H(x-c)}{R(x-c) n R(x-c)}, \dots \dots \dots (20)$$

where  $H$  is a certain positive number greater than  $G$ . For  $R(x) > R(c)$  the remainder has therefore zero as a limit as  $n$  increases indefinitely. Moreover the majorant-value (20) shews that on the half-line going from  $x = c$  in the direction of the positive part of the real axis, the binomial series converges *uniformly*; for  $R(x-c) = x-c$  on this line. PINCHERLE has observed (l.c.) that a similar statement, which is analogous to a known theorem of ABEL on power series, holds for the integral (1), and that it follows from the equation (5), which has been found by means of integration by parts. In the same manner the just mentioned proposition may be proved generally by means of *summation* by parts, both for series of *integral* factorials (the binomial series treated of in this note) and the series of factorials in the more restrictive sense of the word, which proceed according to *inverse* factorials. For the latter series I have shown this in a communication on those series<sup>1)</sup>. The expansion of the integral (1) in such a series is, however, as appears from investigations of NIELSEN<sup>2)</sup> and PINCHERLE<sup>3)</sup>, only possible under restricting conditions for  $f(t)$ , viz. if it is an *analytic* function, whose circle of convergence for the point  $t = 1$  passes through  $t = 0$ , and whose order on this circle is different from  $+\infty$ .

1) Proceedings XXII, N<sup>o</sup>. 1.

2) Handbuch, p. 244.

3) *Sulla svilupabilità di una funzione in serie di fattoriali*, Rendic. d. R. Acc. d. Lincei 1903 (2e Semestre).

**Bacteriology.** — “*On the so-called filtrable virus of influenza described by VON ANGERER.*” By L. K. WOLFF. (Communicated by Prof. C. EYKMAN).

(Communicated in the meeting of June 28, 1919).

Towards the end of 1918 VON ANGERER<sup>1)</sup> published communications on a virus of influenza, discovered by him. He injected rats with sputa of sufferers from influenza, filtered the blood of these rats germfree, when they were already very ill and put the filtrate into glucose-broth. After incubation at a temperature of 37° C. this broth became turbid, without bacteria distinctly being found in them. Yet VON ANGERER describes very small formations, angioplasmata, which he considers the cause of the influenza. This communication was confirmed and completed by himself and other investigators<sup>2)</sup>. The result of these researches was that the rat was no longer necessary for isolating the virus, but that it was sufficient to add blood of sufferers from influenza to the broth.

While investigating, together with DR. SNAPPER<sup>3)</sup> the secondary bacteria that are the cause of pneumonia in influenza-patients, we have sometimes observed this turbidness, without being able to find any microbe in the liquid.

Yet we were struck by the fact, that a great number of round, gram-negative granules were to be found in such a broth, but the unequal size had prevented us from considering these formations as bacteria. After the communications of VON ANGERER had been published, I have paid more attention to this turbidness, which is obtained by inoculating the blood of influenza-patients into glucose-broth and by incubating this liquid at 37° C., and I have been able to observe them in three cases of serious influenza-pneumonia. I must add at once however that I found them also in a case of endo-

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<sup>1)</sup> Münchener Med. Woch. 1918, N<sup>o</sup>. 46 and 47.

<sup>2)</sup> PRELL: *ibidem* 1918, N<sup>o</sup>. 52.

LESCHKE, Berlin. Klin. Woch. 1919, N<sup>o</sup>. 1.

See further OLSEN (Report Aertzl. Verein Hamburg Jan. 7<sup>th</sup> 1919) and KRONBERGER, Deutsche Med. Woch 1919, N<sup>o</sup>. 9, who consider the results of VON ANGERER non-specific.

<sup>3)</sup> Tijdschr. v. Geneesk. 1919, p. 1483.

carditis lenta in a child, where I did not find streptococci in the blood.

The epidemic had nearly reached its end, and I should not have been able to continue my researches if not a happy coincidence, a wrong hypothesis, as appeared afterwards, had helped me on.

Starting from the fact, observed by myself and also by other investigators, that inoculation of dead bacteria, which complicate the influenza, so pneumo- and streptococci or influenza-bacilli, on persons not only protects them against complications, but also against the influenza itself, I thought that the virus of the influenza would probably be present in the cultures in broth with blood of strepto- and pneumococci, collected by Dr. SNAPPER and myself, and so I tried to separate the virus by filtration through a Berkefeld-filter and inoculation into broth with blood. It actually succeeded the first times. I obtained liquids in which no ordinary bacteria were present, but which became turbid at 37° C. My results were however varying, at one time the liquid became turbid, at another time it did not.

After first having ascribed these varying results to the Berkefeld filters, it became evident afterwards that the presence or the absence of the turbidness was dependent on an addition of a small quantity of hemoglobin and now the riddle was soon solved. If one adds to the broth a liquid containing a small quantity of hemoglobin, this mixture remains clear at room-temperature, but it becomes turbid in the incubator after 24 hours. This turbidness is also formed in peptone, even in salt solution; the latter must be however very precisely neutral, because otherwise the turbidness is not observed. The hemoglobin solution was always made by washing erythrocytes with salt solution, then dissolving them in distilled water and filtering through a Berkefeld-filter. It is easy to give an explanation, why this turbidness is obtained in blood from serious influenza patients; in this illness a slight hemolysis of the blood arises *intra vitam* through the secondary hemolytic streptococci, and the blood we add to the broth will contain not only red blood corpuscles, but also hemoglobin, free in the plasma. And this is broken up in the incubator.

To prove this more closely I prepared the carotis of a rabbit free, let a few drops of blood flow into the broth and into a test tube (I). Then I injected distilled water in the earvein and shortly afterwards blood was drawn from the carotis and mixed in the broth and in a test tube (II). The tubes with broth were put in an incubator. I let the blood, which was received in the testtubes, coagulate;

the second contained pale red serum and spectroscopic oxyhemoglobin. After 24 hours the broth in the test tube II was decidedly turbid after slight centrifuging to remove the erythrocytes; the broth in the test-tube I was clear. Both proved to be sterile.

The question has still to be answered, what may be the cause of the turbidness. If one adds a little more hemoglobin-solution to the broth and leaves it in the incubator, the next day a turbid liquid and a red precipitate are obtained; the latter does not dissolve, or only with great difficulty in acids, but easily in diluted alkalis. The solution does not show absorptionbands in the spectroscope; in adding a little ammonium sulphide, we get directly a distinct band, characteristic of hemochromogen. If we first add potassium cyanide, and then ammonium sulphide, we get two bands, of which the left one has moved a little towards the red in comparison with the above band. All this points to the fact that we were dealing with hematin.

It is obvious that we have an autolysis of the hemoglobin. In most cases the globin will remain dissolved, as the broth is not exactly neutral; in neutral salt solution it may add however to the turbidness. If one wants to obtain the turbidness of hemoglobin in salt solution, one ought to take highly diluted hemoglobin solutions, otherwise it does not appear. This happens, because the reaction of the salt solution changes by adding a great quantity of hemoglobin solution.

The fact of getting turbid at 37° C. of tubes of broth and blood that has been drawn from the body a considerable time ago, a well-known fact to those, who experiment with this cultureliquid, depends of course on the same fact: autolysis and the formation of hematin.

Recapitulating the facts, we may say that the turbidness of broth, described by VON ANGERER after adding the filtrate of the blood of serious influenza patients, is not specific, but must appear everywhere, where in the blood an important destruction of erythrocytes has taken place. The turbidness is not a virus, but hematin (and globin) originating from the hemoglobin present.

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**Physiology.** — “*On the relation between the electrical phenomenon in cloudlike condensed odorous water vapours and smell-intensity*”. Bij H. ZWAARDEMAKER and H. ZEEHUISEN.

(Communicated in the meeting of May 31, 1919).

In earlier publications<sup>1)</sup> we set forth that all true odorous substances, a large number of saponins and antipyretica possess the property of imparting an electrical charge. This phenomenon manifests itself most distinctly with the first group, less distinctly with the second, and again less with antipyretica. However, the result is largely dependent on the solubility in water of the individual substances, as it can be of some significance only when, at the spraying, an adequate number of dissolved molecules are present in the water. Suspended particles are of themselves of no value for the phenomenon. Before proceeding we wish to call attention to a fourth group to be discussed later on, viz. the alkaloids. The phenomenon appears with many of them, but their solubility being very slight, it cannot reach a high degree of intensity. To give an idea of the great differences among the four groups, we observe that on comparison, e.g. of the charge of a saturated camphor-solution with that of a saturated quinine-solution, the former appears to be at least twice as strong as the latter. It follows then that among the organic substances of physiological activity the electrifying power is always highest in the odorous substances, so that they are most appropriate for the study of the phenomenon.

It has been established before, that in homologous series the intensities of electrifying power and olfactory capacity rise and fall concomitantly. In order to ascertain the relations of these intensities for odorous substances we selected at random 26 of them, distributed over various groups and series and we diluted their aqueous solution to such a degree that on spraying they yielded only inappreciable electrical phenomena (with an electroscope of moderate sensitiveness<sup>2)</sup> a deflection of from 0.1 to 0.2 scale-marks). When

<sup>1)</sup> K. Akad. v. Wetensch. Amst. 25 Maart 1916, 27 Mei 1916, 30 Sept. 1916, 23 Febr. 1918, 29 Juni 1918. Arch. nérl. de physiol. T. 1, p. 347; Nederl. Tijdschr. v. Geneesk. 1918 II 980—982.

<sup>2)</sup> A tension of 220 vlts made the instrument deflect 10 scalemarks. (It had a capacity of  $\pm 50$  Electrostat.-units).

examined in a simple olfactometer these solutions yielded a very weak sensation of smell, which could readily be determined by the length to which the olfactometrical cylinder had to be moved out, in order to procure a minimum perceptible.

Table Ia shows the olfaction-values of the solutions when we try to find the "Reizschwelle". Table Ib gives of the same substances the lengths to which the cylinder has to be moved out when we search for the "Erkenntnisschwelle".

TABLE 1a. The odorous substances arranged in the order of the smell-intensity of solutions yielding an approximately equal, extremely weak electrical charge (0,1—0,2 scale-marks).		TABLE 1b. The odorous substances arranged according to the "Erkenntnisschwelle" in solutions yielding an approximately equal, extremely weak electrical charge.	
Substances (arranged in ascending order of smell-intensity).	"Reizschwelle" in centims to which the cylinder is moved out.	Substances arranged in ascending order of smell-intensity.	»Erkenntnisschwelle" in centims to which the cylinder is moved out.
Caproic acid	5	Caproic acid	9
Artificial moschus	1	Artificial moschus	8
Valerianic acid	1	Valerianic acid	5
Amyl alcohol	1	Amyl alcohol	5
Cumidin	1	Cumidin	5
Allyl alcohol	1	p. Xylenol	5
Iso-amylacetate	0.5	Allyl alcohol	3
Terpineol	0.5	m. Xylidin	2.5
Skatol	0.5	o. Toluidin	1.5
Indol	0.5	p. Toluidin	1.5
Pseudocumol	0.3	Iso-amylacetate	1.3
Xylol	0.3	Terpineol	1
o. Toluidin	0.2	Skatol	1
Anilin	0.2	Pseudocumol	1
p. Xylenol	0.1	Anilin	1
m. Xylidin	0.1	Thymol	1
m. Toluidin	0.1	Benzol	1
Thymol	0.1	Toluol	1
Benzol	0.1	Xylol	0.6
Toluol	0.1	Indol	0.5
Naphthalin	0.1	Naphthalin	0.5
m. Xylenol	0.1	m. Xylenol	0.4
Guaiacol	0.1	Guaiacol	0.1
Nitrobenzol	0.1	Nitrobenzol	0.1
Pyridin	0.1	Pyridin	0.1
Vanillin	0.05	Vanillin	0.1
Average	0.54 c.m.	Average	2.16 c.m.

In the latter case, it is true, the values are based upon the intensity of the sensation, however the psychological consciousness of the quality had been previously established.

As will be seen, the average olfaction-value of our 26 substances in a dilution with which the electrifying power is next to imperceptible, is 0.54 cm (determined after the minimum perceptible without quality). The deviations vary within the tenfold of the mean values.

Apparently the liminal perceptibility of smell-capacity, concurs approximately with that of electrifying power. There is, indeed, a certain latitude of variation in the olfaction-values, while the electrifying power is considered the same for all, though probably it is not quite the same, because it is difficult to distinguish differences of subdivisious of one tenth of a scalemark.

One of us surmises that both the intensity of the smell and that of the electrical phenomenon depend in a more or less complicate way on

a. the volatility of the substances

TABLE IIa.

Arrangement of 26 odorous substances (in the stalgmometer) in a solution yielding a just noticeable electrical charge. (at 15° C.).

Water	49.3
Naphthalin	49.3
Artificial moschus	49.6
Caproic acid	49.8
Anilin	50
Indol	50
Nitrobenzol	50
p. Xylenol	50
p. Toluidin	50
Guaiacol	50
Thymol	50.1
Allylalcohol	50.2
Toluol	50.2
Pseudocumol	50.5
Amylalcohol	50.5
m. Xylidin	50.5
m. Xylenol	50.6
Benzol	50.8
Vanillin	51
Valerianic acid	51
Terpineol	51.2
o. Toluidin	52
Skatol	52.2
Cumidin	52.2
Iso-amylacetate	52.5
Pyridin	54
Xylol	55
Average	$50.9 \left( \frac{1323.2}{26} \right)$

TABLE IIb.

26 odorous substances arranged according to their boiling point.

Water	100°
Benzol	80°
Allylalcohol	97°
Toluol	111°
Iso-amylacetate	116°
Amylalcohol	116.3
Pyridin	116.7
Xylol	142°
Caproic acid	155°
Pseudocumol	169.8
Anilin	182.5
Valerianic acid	184
o. Toluidin	199.7
p. Toluidin	200.4
Nitrobenzol	205
Guaiacol	205.1
Terpineol	218
Naphthalin	218.2
p. Xylenol	220
m. Xylenol	225
m. Xylidin	226
Thymol	231.8
Cumidin	235°
Indol	253
Skatol	265
Vanillin	sublimes
Artificial moschus	??

*b.* the lowering of the surface-tension of water, which they bring about.

In view of this supposition it avails to know the boiling point of the odorous substances as well as the number of droplets of the diluted aqueous solutions. The subjoined tables (IIa and IIb) give us these data. They do not warrant the adoption of an immediate connection, though we may conclude from them that there is a more remote relation <sup>1)</sup>.

The sign of the charge may also be of some influence. Whereas for 24 out of 26 substances examined in widely different concentrations, we invariably found a positive charge, which eventually disappears with progressing dilutions, a negative charge is yielded by caproic acid and valerianic acid in highly diluted solution (positive in somewhat concentrated solutions). In the extremest dilution, in which this negative charge is just noticeable, the smell-intensity of these substances, when compared with the average of our substances, appears to be very slight.

Whether the charge increases or not, or whether it decreases, through the addition of some common salt to the solution <sup>2)</sup>, does not seem to interfere with the relative arrangement of our 26 substances. With all of them we observe an increase of the electrifying power, with the exception of naphthalin, indol and skatol, whose insignificant charge seems to remain constant, and of artificial moschus, pseudocumol, p. xylenol and thymol, whose charges are obviously getting weaker.

We conclude, therefore, merely from the facts, without attempting to find an explanation, that also of odorous substances, chosen at random, in approximately similar dilution, the smell-intensity and the electrifying power have reached their limen of perceptibility.

<sup>1)</sup> Order of the number of droplets of the saturated solution:

Naphthalin 49.3, Artificial moschus 49.6, Toluol 50.3, Pseudocumol 50.5, Benzol 50.75, Allylalcohol (1:500)51, Vanilline 51, Indol 51.5, Skatol 52.2, Xylol 55, Nitrobenzol 55.5, Pyridin (1%) 57.25, Cumidin 66.75, p. Xylenol 68.5, Anilin 69, p. Toluidin 69, Thymol 73, o. Toluidin 75.5, n. Xylidin 77, m. Xylenol 84, Caproic acid 84.5, Guaiacol 85, Terpeneol 90.5, Valerianic acid 106, Isoamylacetate 115, Amylalcohol 131.

<sup>2)</sup> E. L. BACKMAN, *Researches Physiol. Lab. Utrecht* (5). Vol. 18, p. 349; 19, p. 210.

**Chemistry.** — “*The zincates of sodium. Equilibriums in the system  $Na_2O-ZnO-H_2O$* ”. By Dr. F. GOUDRIAAN. (Communicated by Prof. J. BÖESEKEN).

(Communicated in the meeting of June 28, 1919).

The data we find in the literature about the influence of the strong bases of light metals on the insoluble weak hydroxides of heavy metals, are extremely contradictory and ill-defined. So it is generally assumed that the hydroxides of zinc, aluminium, lead and tin display an amphoteric character, so that in an excess of strong base they “dissolve” under the formation of salts. It is supposed on the other hand that the hydroxides of numerous other heavy metals do not show any propensity to form similar salts and by this the stronger electro-positive character of these metals is thought to reveal itself. And yet it is a fact that the hydroxides of the so-called strong electro-positive metals can likewise in some circumstances display an amphoteric character; nay even cuprum hydroxide can dissolve in a concentrated solution of NaOH or KOH. Hence the differences occurring with the metalhydroxides are probably only of a quantitative nature and it will be worth while to investigate to what extent the different hydroxides show this phenomenon and what compounds arise in this process. This is in the first place of importance for analytic chemistry, where numerous separations are due to the difference in solubility of the hydroxides; next investigations about this subject can give us a more definite insight into the mutual affinity of the metaloxides and metal-hydroxides.

Up till now it has not yet been ascertained what compounds are formed under the influence of hydroxides on strong bases and under what circumstances they are stable. In a pure state they are not isolated and accurately delineated. We are still quite in the dark about the nature of the hydroxides themselves. For as a rule we obtain these substances as voluminous, gelatinous products, and the question arises whether we are to consider these as solid phases of a constant or variable composition or as liquid ones of a great viscosity. The great difficulties which the procuring of these substances in a pure state, involves, are probably the cause of the very few exact data we have about the subject under discussion.

In the following lines we shall now state the results of experiments made with a view to get more closely acquainted with the compounds formed in "dissolving" zinchydroxide in NaOH i. e. the zincates of sodium, and accurately to determine their range of existence. Similar investigations concerning the correspondent compounds of some other metals are in progress.

We shall now briefly summarize the data known up till now concerning the sodium-zincates:

By means of solving ZnO in a hot NaOH-solution, followed by the addition of alcohol, COMEY and LORING JACKSON<sup>1)</sup> obtained two products, to which they ascribe respectively the formulae:

$H_2 Na_4 Zn_3 O_6 \cdot 17 H_2O$  and  $HNa ZnO_2 \cdot \frac{7}{2} H_2O$ . The first compound has a melting-point of  $100^\circ$ , the second proved not to melt even at  $300^\circ$ .

FÖRSTER and GÜNTHER<sup>2)</sup> found only one compound viz. the formula

$Zn \begin{cases} OH \\ O.Na \end{cases} 3 H_2O$ . It formed white silky-glossy needles.

HERZ<sup>3)</sup> precipitated a solution of  $Zn SO_4$  of known concentration with a KOH-solution and then investigated how much base is required once more to solve the  $Zn(OH)_2$  that is formed. It appeared that to 1 gram equivalent of Zn, 6 gram equivalents of OH were required.

HANTZSCH<sup>4)</sup> denies the existence of sodium-zincates on the ground of conductivity measurements and attributes the solution of  $Zn(OH)_2$  in NaOH to the forming of a colloidal solution.

JORDIS<sup>5)</sup> observes that in the long run, crystals of the composition:  $Zn(O.Na)_2$  arise in the cupron-element, in which zinc is found as a negative pole in a solution of NaOH.

Finally it should be mentioned, that RUBENBAUER<sup>6)</sup> and WOOD<sup>7)</sup> have determined the proportion of the number of gramatoms Zn and Na in a solution of  $Zn(OH)_2$  in NaOH; these observations can teach but little regarding the existence of definite compounds.

We see from these data of the literature how vague our knowledge about such comparatively simple compounds still is.

The equilibriums that may arise in an aqueous solution between

1) Amer. Chem. Journ. **11**, 145 (1889); Ber. d. deut. Chem. Ges. **21**, 1589 (1888)

2) Zeitschr. f. Elektroch. **6**, 301 (1899).

3) Zeitschr. f. anorg. Chem. **28**, 474 (1901).

4) Ibid. **30**, 289 (1902).

5) Zeitschr. f. Elektroch. **7**, 469 (1900).

6) Zeitschr. f. anorg. Chem. **30**, 332 (1902).

7) Journ. Chem. Soc. **97**, 886 (1910).

TABLE 1. — System  $\text{Na}_2\text{O}-\text{ZnO}-\text{H}_2\text{O}$  (Temperature  $30,0^\circ \text{C}$ .)

No.	Composition of the solution.			Composition of the rest.			Solid phases.
	% $\text{Na}_2\text{O}$	% $\text{ZnO}$	% $\text{H}_2\text{O}$	% $\text{Na}_2\text{O}$	% $\text{ZnO}$	% $\text{H}_2\text{O}$	
1	11.8	2.6	85.6	10.1	22.2	67.7	ZnO
2	17.4	5.0	77.6	15.3	16.9	67.8	ZnO
3	24.6	12.6	62.8	20.2	29.1	50.7	ZnO
4	24.9	12.9	62.2	22.0	23.5	54.5	ZnO
5	23.7	11.3	65.0	19.1	31.2	49.7	ZnO
6	27.3	16.0	56.7	22.1	32.7	45.2	ZnO
7	27.8	16.5	55.7	—	—	—	$\text{ZnO} + \text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
8	28.0	14.9	57.1	28.1	28.3	43.6	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
9	33.5	10.9	55.6	28.6	27.9	43.5	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
10	36.7	9.5	53.8	31.9	28.5	39.6	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
11	31.8	11.7	56.5	31.0	20.2	48.8	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
12	30.1	13.2	56.7	29.1	29.2	41.7	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
13	33.2	11.2	55.6	29.7	31.8	38.5	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
14	31.5	11.8	56.7	29.2	32.9	37.9	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
15	36.9	10.1	53.0	34.9	17.2	47.9	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
16	34.7	10.4	54.9	33.7	15.6	50.7	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
17	36.1	10.2	53.7	34.0	17.7	48.3	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
18	36.8	9.9	53.3	33.1	22.8	44.1	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$
19	39.2	9.7	51.1	—	—	—	$\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O} + \text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
20	39.4	9.0	51.6	42.6	7.0	50.4	$\text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
21	39.6	7.2	53.2	41.0	6.1	52.9	$\text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
22	40.7	2.0	57.3	42.0	1.8	56.2	$\text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
23	40.5	1.6	57.9	42.6	1.3	56.1	$\text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
24	40.9	1.1	58.0	42.7	0.4	56.9	$\text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
25	41.9	0.0	58.1	—	—	—	$\text{Na}_2\text{O} \cdot 3\text{H}_2\text{O}$
26	24.6	12.5	62.9	23.6	15.2	61.2	ZnO
27	19.9	15.2	64.9	16.1	27.7	56.2	$\text{Zn}(\text{OH})_2$
28	4.6	1.0	96.4	3.7	15.4	80.9	$\text{Zn}(\text{OH})_2$
29	4.5	0.4	95.1	3.7	20.3	76.0	ZnO
30	13.7	7.2	79.1	9.3	30.4	60.3	$\text{Zn}(\text{OH})_2$
31	10.1	4.7	85.2	6.2	32.7	61.1	$\text{Zn}(\text{OH})_2$

NaOH,  $Zn(OH)_2$ , and their compounds if any, must be considered as those in a ternary system with the components:  $Na_2O-ZnO-H_2O$ .

The solubility-diagram of this system could now be completely determined at a constant temperature. As starting-material for these determinations we used NaOH, prepared from sodium; the ZnO being obtained in the following manner: Pure crystallized zinc nitrate was precipitated with the required quantity of ammonia, thoroughly washed out with boiling water, the precipitate then being dried at  $140^\circ-150^\circ$ . The oxide was also obtained by glowing precipitated zinc-carbonate, this oxide being less active, owing to the strong heating, it required a longer time to reach equilibrium. Both preparations showed equal solubility. For the preparing of the hydroxide, vide infra. The water had been distilled and boiled out before use. All the determinations were executed in a thermostat of  $30,0^\circ C$ .

In table 1 and the accompanying diagram 1 a survey of the results is given. All the number-values represent weight percentages of the saturated solution. As it proved extremely difficult to accurately isolate the solid phases, we determined the composition in nearly all cases according to SCHREINEMAKERS'S rest-method.

In determining the curve AB we added ZnO as solid phase<sup>1)</sup>; the solubility of this substance appears rapidly to increase with growing NaOH-concentration. As appears from the second curve BC of the isotherm, the sodium-zincate of the formula:  $Na_2O \cdot ZnO \cdot 4H_2O$  precipitates from the solution, while constantly the NaOH-concentration is increasing. The curve CD, where solutions are saturated by the monohydrate of sodium-hydroxide:  $NaOH \cdot H_2O$  [ $Na_2O \cdot 3H_2O$ ] is immediately joined to BC. According to the melting-diagram NaOH— $H_2O$  determined by PICKERING<sup>2)</sup> this hydrate is the only stable compound of NaOH with water at  $30^\circ$ .

*Consequently only one stable zincate arises at  $30^\circ$ , having the formula:  $Na_2O \cdot ZnO \cdot 4H_2O$ . All the other zincates described in the literature, must be considered either as metastable or as not existing at all.*

The opinion of HANTZSCH<sup>3)</sup> concerning the colloidal nature of the solutions, does not hold true either, as will appear still more decidedly later on.

*Properties, preparation, etc. of the sodium-zincate*  
 $Na_2O \cdot ZnO \cdot 4H_2O$ .

The diagram referred to already shows that the sodium-zincate

1) Except in the numbers 26 and 29 (vide infra).

2) Journ. Chem. Soc. 63, 890 (1893).

3) Loc. cit.

belongs to those salts that form the so-called incongruent solutions. On adding water, these salts do not yield a simple solution, but one of the components, from which we can imagine the salt to be formed, separates on the addition of water. Many instances of this are already known. With the zincate of sodium this phenomenon is specially pronounced. Not only will the addition of water to the solid salt effect a separation of  $\text{ZnO}$ , hence a decomposition of the compound, but diluted solutions of  $\text{NaOH}$  will likewise produce the same effect. From the isotherm we may infer, that solutions below a concentration of 33,3 Gr.  $\text{NaOH}$  to 100 Gr. solution (hence 1 part  $\text{NaOH}$  to 2 parts of water) will cause a separation of  $\text{ZnO}$  from the solid salt, i.e. the concentration of the  $\text{NaOH}$ -solution will have to be raised beyond this boundary in order to obtain pure zincate crystals. The inadequate attention paid to this circumstance is probably the cause of the manifold contradictions in the literature, hardly anywhere do we find the concentration of the solutions indicated, thus obtaining in many cases mixtures of zincate crystals and  $\text{ZnO}$ , the possibility for this being very great as appears from the situation of the solubility-curves. In analysing these mixtures investigators attributed the incorrect composition of a compound to them. This occurred among others with the experiments of COMEY and LORING JACKSON<sup>1)</sup>, who prepared zincates by solving  $\text{ZnO}$  in a hot, concentrated  $\text{NaOH}$ -solution, they then added alcohol to the liquid and in this way obtained two products: one from the layer of water by shaking out so long with alcohol till crystals appeared and one from the alcoholic liquid.

I have repeated this method and in doing so I started from 50 Gr. of water, to which 50 Gr.  $\text{NaOH}$  was added. After introducing 17 Gr.  $\text{ZnO}$  in small quantities at a time, the zincate crystallized out in an appreciable quantity. According to our diagram we are in the centre of the saturation-region of the zincate. With the necessary precautions the crystal agglomerate was sucked out without any access of air and carbonic acid, and was then dried on porous earthenware. Of course the remaining liquor could not be altogether removed in this way, but washing-out without simultaneous decomposition is impossible. The analysis of the crystals produced:

$\text{Na}_2\text{O}$ ...29,8 %,  $\text{ZnO}$ ...37,8 %,  $\text{H}_2\text{O}$ ...32,4 %.

(Theoretically for  $\text{Na}_2\text{O} \cdot \text{ZnO} \cdot 4\text{H}_2\text{O}$ ...  $\text{Na}_2\text{O}$ ...28,9 %,  $\text{ZnO}$ ...37,6 %,  $\text{H}_2\text{O}$ ...33,5 %). On microscopic inspection the product appeared to be perfectly homogeneous, the crystals showed as long bars with a

<sup>1)</sup> Loc. cit. c.f. also GMEIN-KRAUT's Handbuch.



mixture will be obtained containing a greater or smaller quantity of zincate-crystals. The formulae for such products are devoid of sense and should consequently disappear from the literature.

*Stability, preparation, etc. of zinc hydroxide.*

It is generally known, that as a rule zinc hydroxide is obtained as a gelatinous, voluminous mass. Now it was essential for the present investigation to determine whether this phase has a constant or a variable composition and to find out its stability regarding ZnO. There are many indications to be found in the literature that the hydroxides obtained in varying ways do not possess the same composition. EULER<sup>1)</sup> prepared from the nitrate hydroxides which proved to have a varying solubility in ammonia. HERZ<sup>2)</sup> and HANTZSCH<sup>3)</sup> point out the great difficulties involved in obtaining Zn(OH)<sub>2</sub> in a pure state and specially the tenacity with which it retains sulphate-ions. It seems, that it is easier to obtain a pure hydroxide from the chloride or nitrate than from the sulphate. With the experiments made to confirm this statement, I proceeded from pure ZnO, which was dissolved in hydrochloric acid and nitric acid, upon which the hydroxide was precipitated by means of the addition of the quantity of ammonia calculated. If this precipitate is washed out with boiling water it almost immediately becomes more gritty, heavier and less gelatinous. After washing out a few times and following it up by desiccation at 100°, the product appeared to contain only 0,6% water. In consequence of the washing-out the hydroxide is converted into oxide already at 100°. Hence Zn(OH)<sub>2</sub> at 100° is no longer stable.

We then tried to achieve a pure hydroxide by washing out at room-temperature. It appeared however, that even after prolonged and continued extraction the products were still chloridic or nitritic. During the first hours of the extraction we observe a considerable decrease in the concentration of the adsorbed ion, afterwards it falls but very slowly. The following figures will further illustrate this statement:

Hence it is practically impossible in this way to obtain a pure hydroxide from the chloride. The products from sulphate and nitrate yielded similar results and though it is stated in the literature, that the nitrate-ion is much less strongly absorbed than the other ions, we determined that even after an extraction continued for days the

<sup>1)</sup> Ber. d. deutsch. Chem. Ges. **36**, 3400 (1903).

<sup>2)</sup> Zeitschr. f. anorg. Chem. **30**, 280 (1902); **31**, 357 (1902).

<sup>3)</sup> Loc. cit.

TABLE 2.

Duration of extraction in hours.	Cl degree in % after drying at 100°.
0	0.59
4	0.36
8	0.35
16	0.34
48	0.30

products were still nitritic. Yet I performed with the gelatinous zinc-hydroxide some measurements concerning the solubility in NaOH, not because the values determined can have an absolute significance, but exclusively with the purpose to investigate whether this hydroxide reveals a higher solubility than ZnO, and consequently must be considered as metastable towards the latter. This proves indeed to be so, as the subjoined numbers demonstrate; the points found lie considerably above the curve of solubility (*AB*) of ZnO. At the same time it appears that the solubility decreases in proportion as the hydroxide is kept for a longer time. This too is in favour of the statement, that these products must be considered *as metastable phases inclining to stabilisation to ZnO*.

TABLE 3.

Time, given in hours, since the preparation of the hydroxide.	Composition of the solution.	
	% ZnO	% Na <sub>2</sub> O
1	10.5	11.3
10	9.3	11.4
24	8.2	11.3
72	7.0	11.3

For the solubility of ZnO with a Na<sub>2</sub>O concentration of 11,3 % we find by interpolation 2,3 %, hence considerably lower.

*Crystallized zinc hydroxide.*

It may be concluded from the above that we must consider the amorphous, gelatinous hydroxides as phases of varying water-percentage, they besides being extremely difficult to purify. There now

was a possibility that under special circumstances, a crystallized hydroxide of a constant composition could be obtained. There are some intimations in the literature, that  $\text{Zn}(\text{OH})_2$  sometimes seems to arise in a crystallized form. Thus BECQUEREL<sup>1)</sup> states that he obtains the crystallized hydroxide by placing a zinc bar wound round with a copper-wire in a solution of silicic acid in caustic lye. In this process isometric octahedrons were formed, to which he ascribed the formula  $\text{Zn}(\text{OH})_2$ . Various similar indications are found in the older literature though the observers do not agree as regards the composition of the crystals<sup>2)</sup>. Of late years the zinhydroxides have been newly examined among others by KLEIN<sup>3)</sup>. He distinguishes three forms of the hydroxide; form *A* is most strongly soluble in NaOH and arises by adding drop by drop  $\text{ZnSO}_4$ -solution to a NaOH-solution. The analysis of the product dried at a normal temperature yielded:  $2\text{ZnO}\cdot\text{H}_2\text{O}$ . In course of time the forms *B* or *C* separate out from the saturated solutions of *A*; both would have the composition:  $\text{ZnO}\cdot\text{H}_2\text{O}$ , but *B* is sometimes obtained in fine crystals, whereas *C* is always amorphous.

It is clear from the preceding that we need not demonstrate that a constant composition of the amorphous hydroxides is out of the question. The water-percentage of these substances depends on all kinds of factors: preparation, duration of keeping, etc.; hence a definite formula for them is valueless.

The case is otherwise with the crystallized hydroxide. I really found it possible, to isolate *the zinhydroxide as a crystallized phase* of a constant composition. In doing so, I set about it in the following manner: to 50 c.c. of a normal solution of KOH, I added a normal solution of zinc sulphate in drops from a burette. At the outset the hydroxide forming immediately dissolves, but finally a point is reached at which the liquid remains slightly turbid when shaken. On vigorous shaking and especially on scratching the glasswall and allowing to stand for a few minutes a heavy, sandy precipitate arises. Grafting with crystals already obtained, appeared greatly to accelerate the separation. On microscopic examination the product gives an altogether homogeneous impression and it appears to consist of very small, drawn-out bar-shaped crystals. They filtrate very easily and contrary to the amorphous product, the crystals can

<sup>1)</sup> Lieb. Ann. **94**, 358 (1855).

<sup>2)</sup> Cf. among others BÖDEKER. Lieb. Ann. **94**, 358 (1855); VILLE, Comp. Rend. **101**, 375 (1885).

<sup>3)</sup> Zeitschr. f. anorg. Chem. **74**, 157 (1912). See also WOOD, Journ. Chem. Soc. **97**, 886 (1910).

be washed out very rapidly. On drying at  $40^{\circ}$ — $50^{\circ}$  the analysis yielded:

18,06 %  $\text{H}_2\text{O}$ ; 81,91 %  $\text{ZnO}$  (Theoretically for  $\text{Zn}(\text{OH})_2$ ... 18,11 %  $\text{H}_2\text{O}$   
81,89 %  $\text{ZnO}$ .)

The concentration of the  $\text{KOH}$ -solution was varied between the limits of 4,0 and 0,1 normal; the concentration of the zinc sulphate solution likewise; the crystals formed always were of the same shape and composition.

*Stability of crystallized zinc hydroxide towards zinc oxide.*

The experiments Nos 26—31 of Table 1 give an insight into the stability-relation of the crystallized  $\text{Zn}(\text{OH})_2$  and  $\text{ZnO}$ . With No. 26 the solid phase was added as crystallized  $\text{Zn}(\text{OH})_2$ ; the mass was kept for over a fortnight in the thermostat at  $30^{\circ}$ . It then appeared that the crystalline  $\text{Zn}(\text{OH})_2$  had been entirely converted and the solid phase consisted of  $\text{ZnO}$ . This was confirmed by the analysis of the solution and remainder, the point found falls on the curve  $AB$ . So it appears from this, that at  $30^{\circ}$  the crystallized zinc hydroxide is metastable towards  $\text{ZnO}$ .

With a shorter equilibrium-adjustment it proved possible to determine the metastable solubility curve of  $\text{Zn}(\text{OH})_2$ . No. 27 was set in with crystallized  $\text{Zn}(\text{OH})_2$  and after  $\pm 24$  hours the solution was analysed; the solid phase appeared to consist even then of crystallized hydroxide. Conformable to this the zinc percentage of the solution (c. f. table) was considerably higher than corresponds to the curve  $AB$ . Numbers 30 and 31 have been executed in a similar manner, here again crystallized  $\text{Zn}(\text{OH})_2$  was added as a solid phase, the solution being analysed after  $\pm 24$  hours. The determined compositions of the solution again lie considerably above curve  $AB$ . The points representing these solutions form together the metastable solubility curve  $EF$  of the crystallized zinc hydroxide.

Finally the determinations Nos 28 and 29 have been carried out in the same solution, to which crystals  $\text{Zn}(\text{OH})_2$  were added as a solid phase. After about  $\pm 24$  hours the solution yielded the composition No. 28,  $\text{Zn}(\text{OH})_2$  being present as a solid phase. Whereas after one day these crystals still appeared to be present, the solution still being of the same composition, we found three weeks later on the composition No. 29. All the  $\text{Zn}(\text{OH})_2$  crystals had disappeared; the point now found lies on  $AB$ , while the analysis of the rest indicated too, that  $\text{ZnO}$  was present as a solid phase. Other circumstances being equal the solubility of crystallized  $\text{Zn}(\text{OH})_2$  is considerably higher than that of  $\text{ZnO}$ .

## SUMMARY.

1. The solubility-isotherm in the system  $\text{Na}_2\text{O-ZnO-H}_2\text{O}$  was completely determined at  $30^\circ$ .

2. We found that the following substances appeared as stable, solid phases: zinc-oxide  $\text{ZnO}$ , zincate of sodium  $\text{Na}_2\text{O-ZnO-4H}_2\text{O}$ , the monohydrate of sodium hydroxide  $\text{NaOH-H}_2\text{O}$ .

3. The sodium zincate forms very strongly incongruent solutions, by solutions below a concentration of 1 part of  $\text{NaOH}$  to 2 parts of water it is decomposed while separating out  $\text{ZnO}$ .

4. The amorphous, gelatinous zinhydroxide must be considered as a phase of a varying water-percentage; it cannot possibly be cleaned of adsorbed ions. It is metastable as regards crystallized zinhydroxide.

5. Under special circumstances zinhydroxide is obtainable as a crystalline phase of the constant composition  $\text{Zn(OH)}_2$ .

6. This crystallized hydroxide is metastable at  $30^\circ$  as regards  $\text{ZnO}$ .

The cost of these investigations has been partly defrayed by a subsidy from the VAN 'T HOFF Fund which was put at my disposal. I here beg to express my cordial thanks to the Board of Administration of this Fund.

*Anorganic and Physical-chemical  
Laboratory of the Technical University.*

*Delft, June 1919.*

**Physics.** — “*The Anomalous ZEEMAN-Effect.*” By Dr T. VAN LOHUIZEN.  
(Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 31, 1919).

Different attempts have already been made to explain the ZEEMAN-Effect from the atom model of BOHR <sup>1)</sup>. As yet only the normal LORENTZ triplet has been explained, often with superfluous components, which however have disappeared in the theory of RUBINOWICZ.

The theory of the anomalous ZEEMAN-effect has not yet advanced much, nor is the explanation of the PASCHEN-BACK-effect much further.

It may, however, be tried to bring one of the parts of the problem to a solution, so that possibly the results obtained in this way might be serviceable for the complete solution of the problem.

One of the questions that presents itself here, and which I have set myself the task to answer, is this:

Does the magnetic splitting up exclusively depend on the initial and the final path in which the electron moves, or does the transition play a part in it?

In other words:

Is in the presence of a magnetic field the formula:

$$\nu = \frac{W_1 - W_2}{h} \quad . . . . . (1)$$

valid, or should it be replaced by a formula as e.g.

$$\nu = \frac{W_1 - W_2}{h} \pm \frac{eM}{4\pi mc} \quad . . . . . (2)$$

as BOHR <sup>2)</sup> thinks he has to assume.

I will put the question into still another form, because expressed in these words the solution is easiest to give.

It is known that the frequency of the vibrations of every spectrum line may be represented as the difference of two functions, so-called “sequences”, e.g.:

1) N. BOHR, Phil. Mag. 27, p. 506. 1914.  
K. HERZFELD, Phys. Zs. 15, p. 193. 1914.  
P. DEBËYE, Gött. Nachr. 1916, p. 142, Phys. Zs. 17, p. 507. 1916.  
A. SOMMERFELD, Phys. Zs. 17, p. 491. 1916.  
A. RUBINOWICZ, Phys. Zs. 19, p. 441 en 465. 1918.  
2) N. BOHR, Phil. Mag. 27, p. 506. 1914.

$$v = \psi(i) - \varphi(k) \left\{ \begin{array}{l} i = 1.2.3\dots \\ k = 2.3.4\dots \end{array} \right. \dots \dots \dots (3)$$

I have extensively set this forth in my treatise "Le Phénomène de ZEEMAN et les séries spectrales" <sup>1)</sup>, and will henceforth refer to this.

There I have demonstrated among others that for every "complex" of spectrum lines, i.e. all the spectrum lines whose frequencies satisfy equation (3) when the functions  $\psi$  and  $\varphi$  are given, and  $i$  and  $k$  each pass through the series of the whole positive values, a definite type of anomalous ZEEMAN-effect holds, provided the influence of the PASCHEN-BACK-effect be taken into account. Hence we may briefly say that every type of anomalous ZEEMAN-effect is determined by the form of the functions  $\psi$  and  $\varphi$ . When these functions have once been determined, the difference of these functions for positive whole values of the argument always yields a spectrum line with a definite type of ZEEMAN-effect.

As has been shown more at length in my cited paper <sup>2)</sup>, these functions may be indicated as:

Single	$p$		$s$			$d$			$f$			
Double	$P$	$P'$	$S$	$S'$	$D$	$D'$	$F$	$F'$				
Triple	$\Pi$	$\Pi'$	$\Pi''$	$\Sigma$	$\Sigma'$	$\Sigma''$	$\Delta$	$\Delta'$	$\Delta''$	$\Phi$	$\Phi'$	$\Phi''$

Accordingly the symbol  $\Pi\Sigma$  is the brief way of writing:

$$v = \Pi(i) - \Sigma(k) \left\{ \begin{array}{l} i = 1.2.3\dots \\ k = 2.3.4\dots \end{array} \right.$$

For the ZEEMAN-effects belonging to every complex I refer to (l.c.).

The above question may, therefore, now also be worded as follows: Is each of the above-mentioned functions ("sequences") separately changed by a magnetic field, and is, therefore, the ZEEMAN-effect that is observed the result of the change of the two sequences together?

Or could we ask when speaking of  $\Pi$ -paths,  $\Sigma$ -paths etc., by which we therefore express that an electron that jumps from an  $\Sigma$ -path to a  $\Pi$ -path gives rise to a spectrum line belonging to the complex  $\Pi\Sigma$ :

Is every  $\Pi$ -path,  $\Sigma$ -path, etc. in a magnetic field each in itself

<sup>1)</sup> T. VAN LOHUIZEN, Arch. Musée Teyler (III) 2, p. 165. 1914.

<sup>2)</sup> Henceforth to be indicated as (l.c.).

split up into different paths, each with a somewhat different energy value, so that when jumping from and to these transformed paths the electrons would emit a vibration with a somewhat changed frequency?

If this question should be answered in the affirmative, it would follow from this that given the modes of splitting up of each of the paths ( $\Pi$ ,  $\Sigma$  etc.), the observed types of ZEE MAN-effect might be found from this by simple subtraction, in other words, that the anomalous ZEE MAN-effect would follow the so-called "Kombinationsprinzip".

To answer the questions put I have done exactly the reverse. I have namely tried to determine whether in the material of observation of the anomalous ZEE MAN-effect indications were to be found of the validity or non-validity of the "Kombinationsprinzip".

In what follows I will communicate some of the results obtained by me.

As material has served what I had collected in (l.c.). In order to be able to treat a number of complexes as large as possible with a number of sequences as small as possible I have confined myself for my first investigation to the triple complexes, and added to this some single complexes with strange asymptote.

These are collected in the following table:

initial path final path	$\Sigma$	$\Delta$	$\Delta''$	$d$
$\Pi$	$\Pi \Sigma$	$\Pi \Delta$	$\Pi \Delta''$	$\Pi d$
$\Pi'$	$\Pi' \Sigma$	$\Pi' \Delta$	$\Pi' \Delta''$	$\Pi' d$
$\Pi''$	$\Pi'' \Sigma$	$\Pi'' \Delta$	$\Pi'' \Delta''$	$\Pi'' d$

For the types of ZEE MAN-effect that these complexes present, see (l.c.).

From this material some general conclusions may first of all be made.

When the electron jumps from an initial path to a  $\Pi$ -path, the ZEE MAN-effect is more complicated than when from the same initial path it jumps to a  $\Pi'$ -path, and this in its turns is again more complicated than when from the same initial path it jumps to a  $\Pi''$ -path.

It is therefore natural to assume that in a magnetic field the

$\Pi$ -path will split up into more paths than the  $\Pi'$ -path, and this again into more than the  $\Pi''$ -path.

For the double series the same thing applies to the  $P$ -path and the  $P'$ -path.

When only the  $\perp$  polarized components are considered, the same rules hold for them, in this case "more" should however be replaced by "more or equal".

The number of components polarized // is for  $\Pi'$  greater than  $\Pi''$ , and for  $P$  equal to  $P'$ .

Another general conclusion can be drawn. When the electron jumps from  $\Sigma$  (and also from  $s$  and  $S$ ) paths the components polarized  $\perp$  and // are always different.

Jumps from the other initial paths ( $\Delta$ ,  $\Delta'$ ,  $\Delta''$ , and  $d$ ) often yield coinciding  $\perp$  and // polarized components. This is e.g. the case for jumps

from $\Delta$ paths to $\Pi$ , $\Pi'$ , and $\Pi''$ paths	
„ $\Delta$ „ „ $\Pi'$ „	
„ $\Delta''$ „ „ $\Pi'$ and $\Pi''$ „	
„ $d$ „ „ $\Pi'$ „	

This peculiar behaviour of the  $\Sigma$  (resp.  $s$  and  $S$ ) paths raises the question whether this may possibly be in connection with SOMMERFELD'S<sup>1)</sup> view:

“Infolgedessen drängt sich folgende geometrische Deutung auf: die  $p$ - und  $d$ -Terme entsprechen ebenen Bahnen in der Symmetrieebene des Atoms, ähnlich den KEPLER-Ellipsen; der  $s$ -Term hat seinen Grund darin, dass die beim Wasserstoff bestehende Punktsymmetrie durch die Atomstruktur von  $Li$  und  $He$  aufgehoben ist und dass daher noch andere Bahnen als die in der Symmetrieebene möglich werden”.

The  $\Sigma$  (resp.  $s$  and  $S$ ) paths might therefore be imagined  $\perp$  to the equator, and possibly this situation outside the equator might be the reason that in its jump to a path in the equator plane the electron does not give coinciding components, whereas it might be imagined that in jumps between paths lying in the equator plane, the chance to coinciding components is much greater.

From a private conversation which I had shortly ago with Professor BOHR on this subject it appeared to me that he had grounds to suppose the  $\Sigma$  (resp.  $s$  and  $S$ ) paths to be also equatorial.

It should further be remarked that the complexes which are the

<sup>1)</sup> A. SOMMERFELD, Zur Quantentheorie der Spektrallinien, München. Ber. 4 Nov. 1916, p. 153.

subject of this investigation, occur exclusively for chemically bivalent elements. For univalent and trivalent elements there occur no Greek complexes.

After these more general remarks I will now set forth in what way I have carried out the inquiry as to the possible validity of the "Kombinationsprinzip".

On a closer examination of the different types of the abnormal ZEEEMAN-effect for the complexes mentioned it strikes us that for most of them the distances of the components from the original line are multiples of half the distance of two components from the normal LORENTZ-triplet.

When we call  $\varepsilon$  the change of energy which a path must undergo for an electron jumping from that path to an unchanged path to emit light corresponding with one of the components polarized  $\perp$  of the normal LORENTZ-triplet, while jumping from the unmodified path it emits light of a frequency of vibration equal to that of the middle component polarized  $\parallel$ , then

$$d\nu = \frac{\varepsilon}{h}$$

will indicate the difference in frequency between the two before-mentioned components.

Accordingly this  $\varepsilon$  must be proportional to the  $\mathfrak{H}$ .

I have now introduced the hypothesis that through the magnetic field each of the initial- and final paths splits up into two or more paths, which present energy-differences with the original path of  $\pm n \cdot \frac{\varepsilon}{2}$  ( $n = 0, 1, 2, 3$ ).

Then I have examined what values of  $n$  must be assigned to each of the initial and final paths to enable us to explain the observed components.

This yielded the following results:

The $\Pi$ path splits up into 7 paths with energy differences	$0, \pm \frac{\varepsilon}{2}, \pm \varepsilon, \pm \frac{3\varepsilon}{2}$
„ $\Pi'$ „ „ „ „ 4 „ „ „	$\pm \frac{\varepsilon}{2}, \pm \varepsilon$
„ $\Pi''$ „ „ „ „ 2 „ „ „	$\pm \varepsilon$
„ $\Sigma$ „ „ „ „ 2 „ „ „	$\pm \varepsilon$
„ $\Delta$ „ „ „ „ 3 „ „ „	$0, \pm \varepsilon$
„ $\Delta''$ „ „ „ „ 4 „ „ „	$\pm \frac{\varepsilon}{2}, \pm \varepsilon$
„ $d$ „ „ „ „ 3 „ „ „	$0 \pm \varepsilon$

Then we get the following types for the ZEEEMAN-effect for the different complexes.



$\Pi \Delta''$

$$\begin{array}{cccccccc}
 \frac{\overline{5\varepsilon}}{2} & (2\varepsilon) & \frac{\overline{3\varepsilon}}{2} & \varepsilon & \frac{\overline{\varepsilon}}{2} & 0 & \frac{\overline{-\varepsilon}}{2} & -\varepsilon & \frac{\overline{-3\varepsilon}}{2} & (-2\varepsilon) & \frac{\overline{-5\varepsilon}}{2} \\
 \frac{3\varepsilon}{2} - (-\varepsilon) & \frac{3\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) & \varepsilon - \left(\frac{-\varepsilon}{2}\right) & \frac{3\varepsilon}{2} - \frac{\varepsilon}{2} & \frac{3\varepsilon}{2} - \varepsilon & \varepsilon - \varepsilon & \frac{\varepsilon}{2} - \varepsilon & 0 - \varepsilon & \frac{-\varepsilon}{2} - \varepsilon & -\varepsilon - \varepsilon & \frac{-3\varepsilon}{2} - \varepsilon \\
 \varepsilon - (-\varepsilon) & \frac{\varepsilon}{2} - (-\varepsilon) & \frac{\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) & \varepsilon - \frac{\varepsilon}{2} & \frac{\varepsilon}{2} - \frac{\varepsilon}{2} & 0 - \frac{\varepsilon}{2} & \frac{-\varepsilon}{2} - \frac{\varepsilon}{2} & -\varepsilon - \frac{\varepsilon}{2} & \frac{-3\varepsilon}{2} - \frac{\varepsilon}{2} \\
 0 - (-\varepsilon) & 0 - \left(\frac{-\varepsilon}{2}\right) & \frac{-\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) & -\varepsilon - \left(\frac{-\varepsilon}{2}\right) & \frac{-3\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) \\
 \frac{-\varepsilon}{2} - (-\varepsilon) & -\varepsilon - (-\varepsilon) & \frac{-3\varepsilon}{2} - (-\varepsilon)
 \end{array}$$


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$\Pi' \Delta''$

$$\begin{array}{cccccccc}
 ? & \frac{\overline{3\varepsilon}}{2} & \varepsilon & \frac{\overline{\varepsilon}}{2} & 0 & \frac{\overline{-\varepsilon}}{2} & -\varepsilon & \frac{\overline{-3\varepsilon}}{2} & ? & -2\varepsilon \\
 \varepsilon - (-\varepsilon) & \varepsilon - \left(\frac{-\varepsilon}{2}\right) & \frac{\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) & \varepsilon - \frac{\varepsilon}{2} & \varepsilon - \varepsilon & \frac{\varepsilon}{2} - \varepsilon & \frac{-\varepsilon}{2} - \frac{\varepsilon}{2} & \frac{-\varepsilon}{2} - \varepsilon & -\varepsilon - \varepsilon \\
 \frac{\varepsilon}{2} - (-\varepsilon) & & \frac{-\varepsilon}{2} - (-\varepsilon) & \frac{\varepsilon}{2} - \frac{\varepsilon}{2} & -\varepsilon - \left(\frac{-\varepsilon}{2}\right) & -\varepsilon - \frac{\varepsilon}{2} \\
 \frac{-\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) & & & & & & & & & \\
 -\varepsilon - (-\varepsilon) & & & & & & & & & 
 \end{array}$$


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$\Pi'' \Delta''$

$$\begin{array}{ccccccc}
 (2\varepsilon) & \left(\frac{3\varepsilon}{2}\right) & \frac{\overline{\varepsilon}}{2} & 0 & \frac{\overline{-\varepsilon}}{2} & \left(\frac{-3\varepsilon}{2}\right) & (-2\varepsilon) \\
 \varepsilon - (-\varepsilon) & \varepsilon - \left(\frac{-\varepsilon}{2}\right) & \varepsilon - \frac{\varepsilon}{2} & \varepsilon - \varepsilon & \varepsilon - \left(\frac{-\varepsilon}{2}\right) & -\varepsilon - \frac{\varepsilon}{2} & -\varepsilon - \varepsilon \\
 & & & & & & -\varepsilon - (-\varepsilon)
 \end{array}$$


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$\Pi d$

$$\begin{array}{cccccccc}
 \left(\frac{5\varepsilon}{2}\right) & \frac{\overline{2\varepsilon}}{2} & \frac{\overline{3\varepsilon}}{2} & ? & \frac{\overline{\varepsilon}}{2} & (0) & \frac{\overline{-\varepsilon}}{2} & ? & \frac{\overline{-3\varepsilon}}{2} & -2\varepsilon & \left(\frac{-5\varepsilon}{2}\right) \\
 \frac{3\varepsilon}{2} - (-\varepsilon) & \varepsilon - (-\varepsilon) & \frac{3\varepsilon}{2} - 0 & \varepsilon - 0 & \frac{3\varepsilon}{2} - \varepsilon & \varepsilon - \varepsilon & \frac{\varepsilon}{2} - \varepsilon & 0 - \varepsilon & \frac{-\varepsilon}{2} - \varepsilon & -\varepsilon - \varepsilon & \frac{-3\varepsilon}{2} - \varepsilon \\
 \frac{\varepsilon}{2} - (-\varepsilon) & 0 - (-\varepsilon) & \frac{\varepsilon}{2} - 0 & & 0 - 0 & 0 - \frac{\varepsilon}{2} & -\varepsilon - 0 & \frac{-3\varepsilon}{2} - 0 \\
 \frac{-\varepsilon}{2} - (-\varepsilon) & -\varepsilon - (-\varepsilon) & \frac{-3\varepsilon}{2} - (-\varepsilon)
 \end{array}$$


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$\Pi' d$ 

$(2\varepsilon)$	$\frac{\overline{3\varepsilon}}{2}$	$\overline{\varepsilon}$	$\frac{\overline{\varepsilon}}{2}$	$0$	$\frac{\overline{-\varepsilon}}{2}$	$\overline{-\varepsilon}$	$\frac{\overline{-3\varepsilon}}{2}$	$(-2\varepsilon)$
$\varepsilon - (-\varepsilon)$	$\frac{\varepsilon}{2} - (-\varepsilon)$	$\varepsilon - 0$	$\frac{\varepsilon}{2} - 0$	$\varepsilon - \varepsilon$	$\frac{\varepsilon}{2} - \varepsilon$	$-\varepsilon - 0$	$\frac{-\varepsilon}{2} - \varepsilon$	$-\varepsilon - \varepsilon$
			$\frac{-\varepsilon}{2} - (-\varepsilon)$	$-\varepsilon - (-\varepsilon)$	$\frac{-\varepsilon}{2} - 0$			

 $\Pi'' d$ 

$(2\varepsilon)$	$\overline{\varepsilon}$	$0$	$\overline{-\varepsilon}$	$(-2\varepsilon)$
$\varepsilon - (-\varepsilon)$	$\varepsilon - 0$	$\varepsilon - \varepsilon$	$-\varepsilon - 0$	$-\varepsilon - \varepsilon$
		$-\varepsilon - (-\varepsilon)$		

Under every component it is indicated from what jump or jumps it is supposed to have arisen. A  $\frac{\text{above}}{\text{under}}$  the component expresses

that it is polarized  $\frac{\perp}{\parallel}$ . When both signs occur, the two components coincide or the polarization is incomplete. Between ( ) are placed the so-called superfluous components, which have not been found in the observation. The adjoined notes of interrogation will be discussed further on.

First a few words about the superfluous components. By far the greater part are extreme outer components. As the outer components that have been observed, are mostly very faint, it is possible that the theoretically found components are so faint that they could not be observed up to now. These components originate namely by jumps from and to the most greatly deformed paths, and according to SOMMERFELD these are less probable than the less deformed ones, so that the number of jumps of the electron from these greatly deformed paths is relatively much smaller, hence the produced component much fainter.

This explanation, is however, not applicable to the question whether the middle components of  $\Pi' \Sigma$  and  $\Pi d$  are superfluous.

It would, however, also be possible that the "Kombinationsprinzip" was dependent on a restrictive condition. Prof. BOHR was namely of opinion, as appeared to me from a conversation on this question, that in its jumping the electron should also be bound to the condition that the angular momentum in initial and final path may not differ more than  $1 \times \frac{h}{2\pi}$ . I have not succeeded as yet in introducing

this condition as restriction of the "Kombinationsprinzip". That there must somehow exist a restrictive condition, does not seem doubtful to me.

That however in some cases the non-appearance of a middle component can yet be ascribed to the observation, may appear from the following example:

In my paper (loc. cit.) the ZEEMAN-effect for  $H'\Delta''$  is given as an octet without middle component, here as a nonet with middle component.

My first statement was among other things based on the observations by MILLER<sup>1)</sup>, who has not found a middle component, whereas WENDT<sup>2)</sup> does find the middle component for the same lines: "bei dem zweiten Begleiter ist die mittlere Komponente übersehen". (p. 559).

From this it may, therefore, appear that it is by no means impossible that some of the observed types of ZEEMAN-effect are incomplete, and that possibly some of the "superfluous" components found by me will after all appear to be present on closer observation.

The outer components of  $H'\Delta''$  are marked with a note of interrogation. These components have been found both by WENDT (loc. cit.) and MILLER (loc. cit.), the distances to the middle component are, however, somewhat smaller than agrees with  $2\varepsilon$ . The state of polarisation could not be determined by MILLER on account of the slight intensity; WENDT finds  $\perp$ .

The other notes of interrogation are found beside  $Hd$ . WENDT finds for this ten components, whereas MILLER has observed twelve. The latter observer remarks here, however (loc. cit. p. 117):

"Die durch Klammern zusammengefassten Linienpaare liefen in eine Linie zusammen, wenn beide Arten von Schwingungen zugelassen waren".

When this is taken into account, the notes of interrogation may be omitted, and  $\overline{\varepsilon}$  and  $\underline{\varepsilon}$  may be substituted for them.

The 0-component has, however, not yet been accounted for.

Another factor that remains to be explained, is the state of polarisation of the components.

Though here for the anomalous ZEEMAN-effects, just as for the normal triplet, the outmost components appear always to be polarized  $\perp$ , I must confess that I have not yet succeeded as yet in

1) W. MILLER, Zeeman-effekt an Mg. Ca. u. s. w. Ann. d. Physik, 24, p. 105, 1907.

2) G. WENDT, Untersuchungen an Quecksilberlinien. Ann. d. Physik, 37, p. 535. 1912.

finding a simple rule for the state of polarisation of the components.

As regards the intensity, the supposition has already been expressed above that according as the electrons jump from or to strongly eccentric paths, the intensity of the produced components would be slighter. It is to be regretted that the material of observation does not allow us to test this supposition in every detail, because the values found by different observers are often contradictory.

MILLER (loc. cit. p. 112) gives, indeed, e.g. for the intensities of:

$$\begin{array}{rcccccccc} \Pi \Delta & 1 & 2 & 2 & 6 & 2 & 2 & 1 \\ \Pi' \Delta & 1 & 1 & 4 & 4 & 4 & 1 & 1 \\ \Pi'' \Delta & < 1 & 4 & 8 & 4 & & < 1 \end{array}$$

and from this appears a rapid decrease of intensity towards the outer components, (with which my "superfluous" components, which are still weaker, are in good agreement). Also the fact that the middle components are stronger, is in good harmony with this that each of the components can be produced by some different jumps.

This investigation can, however, not yet be universally carried through on account of the above-mentioned mutual contradictions. The causes of these differences of intensity have been investigated by ZEEMAN<sup>1)</sup>, but it can seldom be inferred from the publications of the different observers what circumstances have given rise to their differences in intensity, and which are the reliable intensities. A research as discussed above will not be possible until this has been settled with certainty. From the results of such an investigation important conclusions might be drawn as to the correctness or incorrectness of the hypotheses given by me in this paper.

From what has been found so far I think I am justified in concluding that there are indications to be found in the material of observation for the validity of the "Kombinationsprinzip", also for the anomalous ZEEMAN-effect. It is, however, not excluded that a restrictive condition in the sense as given by BOHR (see above) causes the principle not to be always clearly manifested.

At any rate I think I have shown that the "sequences" vary separately in the magnetic field, and that the observed ZEEMAN-effect is the result of the variations of the two sequences together.

In connection with this I am also of opinion that BOHR's equation:

$$v = \frac{W_1 - W_2}{h}$$

keeps its validity in the magnetic field.

*The Hague, May 7 1919.*

<sup>1)</sup> P. ZEEMAN, Proc. Amsterdam, Oct. 1912 and Researches in Magneto-Optics p. 94 et seq, Macmillan 1913

**Physics.** — “*A Theory of a Method for the Derivation of the Distribution of Energy in a Narrow Spectrum Region from the Distribution of Energy, Observed in an Interferometer*”.

By K. F. NIESSEN. (Communicated by Prof. W. H. JULIUS.)

(Communicated in the meeting of May 31, 1919).

MICHELSON has pointed out the direction in which it should be tried to find such a method <sup>1)</sup>. He first considers two absolutely monochromatic beams, which he causes to interfere with a difference of phase. If this takes place in the interferometer of MICHELSON, in which the plane of reference may lie at a distance  $l$  before the movable mirror, the difference of phase of the rays that have struck the mirrors at right angles is  $\frac{2l}{\lambda} \times 2\pi$  (or  $4\pi lm$ , when we work with the number of waves  $m$ , by which we understand the number of waves in the unity of length). The ocular is adjusted for infinity, and in the middle of the field of vision (for there the rays striking at right angles interfere) an intensity  $J(l)$  is observed, which can be calculated by the aid of the formula <sup>2)</sup>:

$$J(l) = 2J_1 + 2J_1 \cos 4\pi lm \dots \dots \dots (1)$$

from the equal intensities  $J_1$  of the interfering beams. An arbitrary beam of light may be thought to be divided into an infinite number of absolutely monochromatic beams of the intensity:

$$J_1 = \psi(\lambda) d\lambda \quad \text{or} \quad J_1 = \chi(m) dm \dots \dots \dots (2)$$

when we use the number of waves  $m$ . As intensities which are due to different frequencies join scalarly, this light admitted to the interferometer of MICHELSON, will give rise in the point under consideration to the intensity  $J(l)$ , given by:

<sup>1)</sup> Phil. Mag. (5) 31, pag. 338, 1891.

<sup>2)</sup> DRUDE, Lehrb. der Optik, p. 123.

$$\left. \begin{aligned}
 J(l) &= 2 \int_{\lambda_1}^{\lambda_2} \psi(\lambda) d\lambda + 2 \int_{\lambda_1}^{\lambda_2} \psi(\lambda) \cos \frac{4\pi l}{\lambda} d\lambda \quad \text{or} \\
 J(l) &= 2 \int_{m_2}^{m_1} \chi(m) dm + 2 \int_{m_2}^{m_1} \chi(m) \cos 4\pi lm dm \quad \text{or} \quad . . . \quad (3) \\
 J(l) &= 2 \int_{-a}^{+a} \varphi(x) dx + 2 \int_{-a}^{+a} \varphi(x) \cos 4\pi l(\bar{m} + x) dx.
 \end{aligned} \right\}$$

Here  $\lambda_1$  and  $\lambda_2$  are the lengths of the smallest and of the largest of the waves present, hence  $m_2$  and  $m_1$  are the outmost frequencies on either side,  $\bar{m}$  being the mean of them, so that we may put  $m = \bar{m} + x$ .

Further  $\varphi(x)$  has been written for  $\chi(\bar{m} + x)$ . The region of frequency  $2a$  of the beam is supposed very small by MICHELSON, so that he examines only a practically monochromatic beam. It should at once be pointed out that we shall not use this restriction in our reasoning.

$\bar{m}$  being a constant in the integration, the last equation of (3) can be put in the form:

$$J(l) = 2 \int_{-a}^{+a} \varphi(x) dx + 2 \cos 4\pi l \bar{m} \times C(l) - 2 \sin 4\pi l \bar{m} \times S(l). \quad (4)$$

in this  $C(l)$  and  $S(l)$  are the following functions of  $l$ :

$$C(l) = \int_{-a}^{+a} \varphi(x) \cos 4\pi lx dx, \quad S(l) = \int_{-a}^{+a} \varphi(x) \sin 4\pi lx dx. \quad (5)$$

Our aim is to determine the function  $\chi(m)$ , hence  $\varphi(x)$ .

According to FOURIER'S integral theorem we have generally:

$$\varphi(x) = \frac{1}{\pi} \int_0^{\infty} \cos ax da \int_{-\infty}^{+\infty} \varphi(\xi) \cos \xi a d\xi + \frac{1}{\pi} \int_0^{\infty} \sin ax da \int_{-\infty}^{+\infty} \varphi(\xi) \sin \xi a d\xi. \quad (6)$$

In this  $\int_{-a}^{+a} \varphi(\xi) \cos \xi a d\xi$  may be replaced by  $\int_{-\infty}^{+\infty} \varphi(x) \cos 4\pi l x dx$ ,

since  $\varphi(x)$  for  $-a > x > +a$  is zero.

Analogously  $\int_{-\infty}^{+\infty} \varphi(\xi) \sin \xi a d\xi$  can be replaced by  $\int_{-a}^{+a} \varphi(x) \sin 4\pi l x dx$ ,

hence  $S(l)$ . Evidently  $4\pi l$  has been taken for  $\alpha$ . Now equation (6) passes into:

$$\varphi(x) = 4 \int_0^{\infty} C(l) \cos 4\pi lx \, dl + 4 \int_0^{\infty} S(l) \sin 4\pi lx \, dl \quad . \quad (7)$$

Accordingly the function  $\varphi(x)$  will be known, when we know  $C$  and  $S$  as functions of  $l$ , hence when we can determine  $C$  and  $S$  experimentally for every value of  $l$ .

Data to attain this may be derived from "the curve of visibility" of MICHELSON, whose coordinates he defines as  $V = \frac{J_{max.} - J_{min.}}{J_{max.} + J_{min.}}$  (in which  $J_{max.}$  and  $J_{min.}$  represent the intensities in the successive maxima and minima of the system of interference fringes). It appears from (4) that  $V$  is a function of  $C(l)$  and  $S(l)$ ; when we assume  $V(l)$  to be sufficiently accurately known from photometric observations, we have at our disposal a relation between  $C(l)$  and  $S(l)$ , but without more data we cannot determine these quantities separately. Only in a few simple cases which contained a second condition concerning  $C(l)$  or  $S(l)$  has MICHELSON derived the form of  $\varphi(x)$  from that of  $V(l)$ .

The aim of our investigation is to find a means through which it is possible to find a second relation between  $C(l)$  and  $S(l)$  in any given case, and which therefore enables us to solve  $\varphi(x)$ .

In equation (4) we can think  $J(l)$  experimentally determined for a given value of  $l$ . Also  $\int_{-a}^{+a} \varphi(x) \, dx$  can be measured, this is viz.

the intensity that one of the beams causes in the middle of the field of vision, when it is not brought to interference with the other. In order to determine this quantity we have, therefore, only to cover one of the mirrors. Hence equation (4) can be considered as an equation with two unknown quantities  $C$  and  $S$ , to be taken for that value of  $l$  for which  $J$  has been measured.

We shall have to find a second equation between  $C$  and  $S$  for the same value of  $l$  to be able to solve both quantities. This means will fail, however, when in the second equation  $C$  and  $S$  appear combined in the same way as in equation (4), accordingly

when they are again derived from  $\int_{-a}^{+a} \varphi(x) \cos 4\pi l(\bar{m} + x) \, dx$ ,

in which a variable parameter is meant by  $l$ , which need not have the same physical meaning as just now.

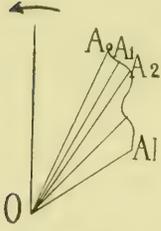
Prof. ORNSTEIN requested me to investigate whether such an equation might not be obtained from the distribution of intensity in the refraction image in the echelon. The result of this investigation is that if the intensity formula no matter of what instrument, contains  $C$  or  $S$ , it must always contain both of them. And that in the combination in which they already occurred in equation (4), unless one makes use of an artifice to be specified later on. Without this  $C$  and  $S$  cannot be solved, and  $\varphi(x)$  can, therefore, not be determined.

## 2.

We shall first prove that for absolutely monochromatic light the intensity is always an even function of the number of waves, when we premise that the light in the apparatus propagates normally, i. e. that a change of phase is accompanied with a proportional variation of path, so that e. g. reflection against a denser medium, metal and total reflection must be excluded.

A point  $Q$ , where we measure the intensity, receives its light from certain points  $P$  of the instrument, which in their turn are again illuminated by the source  $L$ . The points  $P$  are in different phases, because they receive the light from the source each by another way. As we have everywhere assumed the normal propagation of the light, the differences of phase will exclusively depend on differences of paths expressed in numbers of waves, and therefore be proportionate to  $\frac{1}{\lambda}$  or  $\bar{m}$ , in which the factors of proportionality depend on lengths, angles, indices of refraction etc. We shall leave dispersion out of consideration, because in the instrument that is to be devised later on this will be anyway excluded. Also optically the paths, which the light from the points  $P$  must still pass over before it arrives at the point of observation  $Q$ , will not be of equal length, so that other differences of phase are added to the already existing ones, which others on account of the normal propagation of the light will exclusively rest on differences of path, and will therefore be proportional to  $m$ , so that the deviations to which the points  $P$  finally give rise in  $Q$ , present mutual differences of phase which are proportional to  $m$ . When these deviations are represented as projections of vectors whose angle with the axis of projection is equal to the corresponding phase, then the points  $P$ , over which paths of light run from  $L$  to  $Q$ , which are equally long in number of waves, give vectors that overlap.

These have been joined to one vector in fig. 1. Thus the action of every instrument where only normally propagated light is used, can be determined in any point  $Q$  by the aid of a vector fan, which revolves round  $O$  as a whole in the direction of the arrow. The intensity of the square will be the result of the vectors. Suppose that for all the points  $P_0, P_1, P_2, \dots, P_l$  the phases of their disturbances in  $Q$  are compared with the phase of the disturbance given by a point  $P_0$ , the vector of which be  $AO_0$ , hence a point over which the shortest optical way passes from



the source to the point of observation, then  $\angle A_0OA_1 = c_1m$ ,  $\angle A_0OA_2 = c_2m$ , etc., in which the constants  $c_1, c_2$  etc. depend on various lengths, angles, and indices of refraction. We shall call the direction of the vector  $OA_0$  the time-direction for the sake of brevity, because it only depends on time; it will appear to play a prominent part in our reasoning, and we shall henceforth draw it always vertical, and omit the axis of projection. We have drawn here a fan whose vertex is small; in reality it will probably contain several times  $2\pi$ . Vectors namely corresponding with paths  $LQ$ , which differ a whole number of waves in length, will indeed be superposed in the figure, but will not be drawn as one vector. This happens only when the paths are optically of exactly the same length. Nor need the end points of the vectors form a continuous curve.

Let us elucidate this by constructing the fan for an echelon. It consists of a few smaller fans, which are arranged in  $O$  like spokes in a wheel, and whose number is equal to the number of plates of the echelon. Only three of them are drawn in fig. 2.

All the points  $P_1$  lying in a vertical line on the front plane of a plate lie optically equally far from the source<sup>1)</sup>, and at the same time equally far from  $Q$ . Their vectors are therefore united to a single vector  $OA_1$  the length of which is accordingly proportionate to the length of the vertical line<sup>2)</sup>. Any given point of the left-hand side of the first plate can, therefore, be considered as the point  $P_0$  just mentioned.

The indices at the points  $P$  of fig. 3 indicate which vector in fig. 2 originates from their vertical line. When the echelon is viewed at an angle  $\theta$  with the normal, the rays from  $P_1$  and  $P_0$  will

1) The light strikes in the direction  $d$  in fig. 3.

2) In this way it is also possible to find the influence of diaphragms placed before the plates, which will later on be made use of.

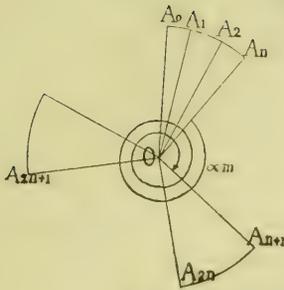


Fig. 2.

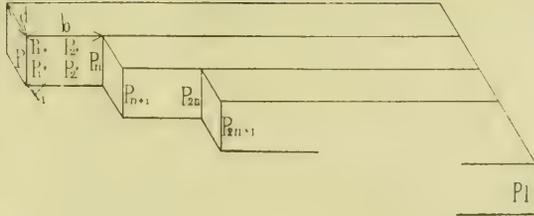


Fig. 3.

interfere under a phase difference  $\frac{x_1 \sin \theta}{\lambda} 2\pi$ . Hence  $\angle A_1 O A_0$  is  $\frac{2x_1 \sin \theta}{\lambda} \pi$ , or if in the earlier notation we put this equal to  $c_1$ , it appears that  $c_1$  is  $= 2\pi x_1 \sin \theta$ , hence that it is a constant <sup>1)</sup> that depends exclusively on distances;  $\angle A_2 O A_0 = \frac{2x_2 \sin \theta}{\lambda} \pi$ ,  $\angle A_n O A_0 =$   
 $= \frac{2b \sin \theta}{\lambda} \pi$ , when  $b$  is the width of the glass plates. The other

fans are congruent with the first, the last ray of a fan always forming the same angle  $\alpha_m$  with the first of the next;  $\angle \alpha_m$  is namely the phase difference due to the difference of two paths of light, one going from a point of the righthand edge of a plate to  $Q$ , and the other from the lefthand edge of the following plate to the same point  $Q$ . When  $\mu$  and  $d$  are the index of refraction and the thickness of every plate, then  $a$  is  $= \mu d - d \cos \theta$ , hence again a constant, depending on measures of length and physical constants.

This constant distinguishes itself from the first, because it depends on the wave length in consequence of the appearance of  $\mu$ . However  $\mu$  and hence  $a$  too, is an even function of  $m$  and for this reason we may consider  $\alpha$  as a constant in the following discussion.

Let us now return to the general case.

When the instrument is struck by light of a twice as small wavelength, the fan will be drawn out twice as far, all the angles with the time-direction being proportional to  $m$ . It appears particularly clearly with the echelon how in consequence of this the resulting vector, and together with this the intensity in the point of observation  $Q$  will change its value. The intensity is, therefore, a function of the number of waves  $m$ , and in order to prove that this is an even function of it, we shall reverse the sign of  $m$ . This

<sup>1)</sup> i. e. independent of  $m$ .

is only a mathematical artifice, to which no physical meaning must be assigned. Points  $P_1$ , which in their movement were first  $c_1m$  behind compared with  $P_0$ , and whose vector lay, therefore, at an angle  $c_1m$  on the righthand side of the time direction, will now be as it were  $c_1m$  in advance of  $P_0$ , and give a vector which again forms an angle  $c_1m$  with the time direction, but now lies on the lefthand side of it. The angle between every vector and the time-direction being proportional to  $m$ ; all the vectors will reflect in this direction, and the form of the fan, and with it the resultant, hence the intensity, will remain unchanged. Hence for any instrument in which light propagates normally, the intensity will be an even function of the number of waves. In this absolutely monochromatic light is supposed, because for compound light the intensity is no function of the number of waves on account of the integration with respect to  $m$  between definite limits.

## 3.

Now we shall, however, also admit reflections against a denser medium in the instrument. In this the phase shifts suddenly  $\pi$ ; such an abrupt change of phase we shall call a phase shifting. Let us suppose that some points  $P'$  of the instrument are illuminated by rays which have undergone such a reflection an uneven number of times. Their vectors now no longer make the angle  $c'm$ , but the angle  $c'm + \pi$  with the time-direction. Before  $P_0$  gave a vector by the aid of whose direction the time-direction was defined. When, however, the point  $P_0$  belongs also to the points  $P'$ , its vector will also turn over an angle  $\pi$ , hence obtain the opposite direction from what it had before. As the other group of points of the apparatus continues to receive normally propagated light, their vectors continue to form the angles  $cm$  with  $P'$ 's former vector.

When we now define the time-direction anew, and do so as the direction of the vector originating from that point  $P_0$  of the apparatus through which the shortest optical way passes from the source of light  $L$  to the point of observation  $Q$ , quite apart from all possible phase shiftings, the time-direction will always be the constant direction in the following considerations, with respect to which we determine the position of the vectors. We shall again think it to be always directed vertically upward. A reasoning based on this time-direction will therefore hold both for the case that this point  $P_0$  receives light that has changed its phase, and for the case that normally propagated light falls on it. We shall let the constants  $c$  keep their value, because

they only depend on distances; however  $cm$  need not be any longer the difference of phase of the interfering rays, this may have become  $cm + \pi$ .

After these extensive considerations, which as will appear later on, go to the core of the method, we proceed to the construction of the fan in the point  $Q$  in the case of two groups of points  $P$  and  $P'$ .

A point  $P_1$  and a point  $P_1'$  give vectors which form respectively the angles  $c_1m$  and  $c_1'm + \pi$  with the time-direction. When we now reverse the sign of  $m$ , the first group of vectors will again be reflected in the time-direction; a vector from the second will, however, form the angle  $\pi - c_1'm$  instead of the angle  $\pi + c_1'm$  with the time-direction, and would, therefore, be evidently reflected in the production of the time-direction. Ultimately all the vectors will, therefore, again be reflected in the same line, through which the form of the fan will remain unchanged, hence the value of the intensity will remain the same, which latter is, therefore, again an even function of  $m$ . The thesis may, therefore, be extended as follows.

In every instrument, in which the light can only undergo phase-shiftings of a whole number of times  $\pi$ , the intensity is an even function of the number of waves when absolutely monochromatic light is used.

The light fulfils these conditions naturally in all apparatus of refraction, also generally in those which are founded on interference. As will appear later on, through the suitable occurrence of phase shiftings among others in reflections against metal mirrors the intensity can also sometimes become a not-even function of the number of waves.

#### 4.

Let us now consider the case of a beam continuously composed of some frequencies. Suppose it to be possible to draw up a second equation between  $l$  and  $s$  for a given parameter  $l$  by the aid of one of the current apparatus, where we shall suppose a favourable action of the phase shifting to be absent in the metal reflections, which is actually the case with most, if not with all instruments in general use. We can now easily show that the two quantities  $C$  and  $S$  must occur in the same combination as in equation (4). For

if the new equation contains e.g.  $C$ , i.e.  $\int_{-a}^{+a} \varphi(x) \cos 4\pi lx \, dx$ , it would

contain  $J \cos 4\pi l x$  for absolutely monochromatic light, for which can also be written  $J \cos 4\pi l (m - \bar{m})$ . But for single light the intensity must be an even function in  $m$ , as we saw; hence  $J \cos 4\pi l (m - \bar{m})$  can only occur in that combination which reduces it to  $J \cos 4\pi l m$ , which is even in  $m$ , i. e. it can only be met with in forms like:

$$\begin{aligned} & \cos 4\pi \bar{l} m J \cos 4\pi l (m - \bar{m}) - \sin 4\pi \bar{l} m J \sin 4\pi l (m - \bar{m}) \quad \text{or} \\ & \cos 4\pi \bar{l} m J \cos 4\pi l x - \sin 4\pi \bar{l} m J \sin 4\pi l x. \quad . . . . . (8) \end{aligned}$$

To extend reversely the formula of intensity for single light to that for compound light we must substitute  $\varphi(x) dx$  for  $J$  in the former, and then integrate with respect to  $x$ . The grouping (8) then passes into  $\cos 4\pi \bar{l} m C - \sin 4\pi \bar{l} m S$ , exactly that of equation (4). The preceding has therefore proved that none of these instruments can yield a second equation between  $C$  and  $S$  which is independent of equation (4).  $C$  and  $S$  can, therefore, not be separated from their combination, it seems, therefore, that  $\varphi(x)$  cannot be solved. <sup>1)</sup>

## 5.

It is now obvious that the first thing to do is to examine whether the function  $\varphi(x)$  can perhaps be determined, if the light in some instrument or other can undergo a phase shifting  $\alpha$ , which differs from  $\pi$ , it being of no consequence whether it is caused by metal reflections or other phenomena. Suppose that part of these points  $P$ , now marked by double or triple accents, receives such light. The angles of the corresponding vectors with the time direction would have to be  $c_1''m, c_2''m, \dots, c_1'''m, c_2'''m, \dots$ , if the light travelled in the same directions, but everywhere in a normal way; now they are, therefore,  $c''m + \alpha$  or  $c'''m + \alpha + \pi$ , according as on the point  $P''$

<sup>1)</sup> In passing we may remark that  $C$  and  $S$  do not occur combined in another way in the formula of intensity that determines the distribution of intensity for the echelon over the whole focal plane of the telescope. This depends for the different points on a parameter, which we might call  $l'$ . I have proved by a computation which is left out here that this function of  $l'$  satisfies a differential equation of the second order, in which only combinations of  $C$  and  $S$  of the kind as in equation (4), to be taken for some values of  $l$  in connection with  $l'$ , occur as coefficients. Thus the dependence between the intensities in the image of reflection of the echelon and that in the image of interference in MICHELSON'S interferometer had been proved purely analytically. Though the formula of intensity for the echelon could be greatly modified by the supposition that diaphragms of a particular shape were placed before the glass plates, such a differential equation of the second order remained of force all the same, and so the dependence continued to exist.

or  $P'''$  under consideration light falls that has undergone besides the phase shifting  $\alpha$  resp. an even (among which zero) or an uneven number of reflections against denser media. Hence we have to deal here with four groups of points, the classification of which will be clear from the foregoing. When out of each of these groups one point is considered, resp.  $P, P', P'', P'''$ , their vectors make the angles  $cm, c'm + \pi, c''m + a, c'''m + a + \pi$  with the time-direction. The phase shifting is taken to be positive if it is a shifting back, and negative in the opposite case. When  $m$  is now made to change its sign, these vectors are now respectively reflected in the time-direction  $t$ , the direction  $t'$  forming an angle  $\pi$  with the former, the direction  $a$  forming an angle  $a$  with the time-direction, and that which forms an angle  $a + \pi$  with the time-direction, being  $a'$  in fig. 4. Accordingly the vectors can be divided into two groups, one of which is reflected in the line  $tt'$ , the other in a line  $aa'$  forming an angle  $a$ . As we have put  $a = \pi$ , the fan will now probably also be transformed, and the intensity will no longer remain the same. This is therefore no even function now of the number of waves.

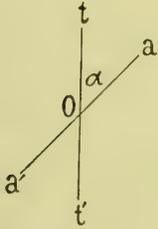


Fig. 4.

In such an instrument  $C$  and  $S$  will accordingly certainly *not* appear in the formula of intensity in the way of equation (4), which can again be proved via the intensity holding for single light. Now the separation of  $C$  and  $S$  is therefore no longer fundamentally impossible, hence we are on the right track to determine the function  $\varphi(x)$ . It is finally noteworthy, that a phase shifting in the part  $LP$  of the light path  $LPQ$  will give the same effect as

one directed in the same way and of the same extent in the part  $PQ$ , which remark will appear to be of importance for the interferometer of MICHELSON. In what precedes it has been supposed that the phase shifting  $\alpha$  preserved its value and sign in the reversal of sign of  $m$ , it must accordingly be a constant or an even function of  $m$ . Only an instrument in which it is a constant, i.e. independent of  $m$ , will be of practical use to us. Our purpose is namely to examine beams of the most "divergent" constitution, and it would be impracticable to have to take a different phase shifting into account for every kind of light. For this reason we shall e.g. not be able to make use of metal reflections. On the other hand it will hardly be possible to do without them, for in almost all interferometers there occur silvered glass-plates or metal mirrors. This difficulty is, however, not so serious as it seems. When each of the four monochromatic beams that illuminated the groups of points  $P, P', P'', P'''$

of just now, and which had already undergone the phase shifting discussed then, is besides subjected to a metal reflection, which can take place for all four of them in exactly the same way, this will mean for each of the beams a same phase shifting  $\beta$ . This latter does not deform the fan, as it revolves as a whole over an angle  $\beta$ , since all the points  $P$  have participated in the new change of phase. For some four beams of other frequency the fan will indeed be rotated through another angle  $\beta'$ , but fans drawn up for different frequencies, must yet not be composed vectorially, because two different kinds of light can never interfere. Hence also when compound light falls on the apparatus, metal reflections will have no influence on the intensity, if each of the beams undergo them an equal number of times in the identical way; this latter peculiarity is always met with in the current interferometers for reasons of a practical nature. The same thing applies for the phase-shiftings which take place with total reflections.

## 6.

We must, therefore, devise an instrument in which first of all the condition of identical metal reflections and of identical total reflections is fulfilled, but that besides has the property to be illuminated for one half by rays which apart from metal and total reflections, have undergone none or only phase shiftings of  $\pi$ , and to be illuminated for the other half by rays which apart from metal and total reflections, have been besides subjected to an extra phase shifting  $\alpha$ , which differs from  $\pi$ , and must be independent of the number of waves; the rays of the first and of the second kind must besides be coherent in order to be able to interfere.

When we now put the question what constant phase-shiftings we have now at our disposal, we come to the following answer:

1. a phase-shifting of  $\pi$  in case of reflections against a denser medium;
2. a phase-shifting of  $\pi$ , when a beam of light is narrowed to pass through a focus <sup>1)</sup>;
3. a phase-shifting  $\frac{\pi}{2}$  forward, when a beam must pass through a focal line <sup>1)</sup>.

The last two shiftings take place respectively in the focus and in the focal line; the latter alone differs from  $\pi$ , and this is the only one we can therefore use for our purpose. We shall, therefore,

<sup>1)</sup> GOUY, Ann. de Chimie et de Physique (6), 24. 1891.

make use of two cylinder lenses, whose focal lines coincide; in fig. 5 they are thought with the descriptive lines normal to the plane of the drawing, and we shall assume this position in all the following figures. Lines thought above or in the plane of the drawing will be drawn in full, whereas lines that are supposed below it, will be dotted. Rays recurring on the upper side parallel to the plane of drawing (e.g. *a*), can never be brought to the lower side

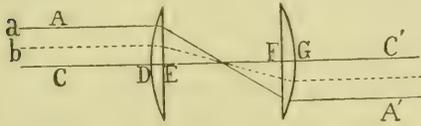


Fig. 5.

by the cylinder lenses, because the lenses have no refractive power in the direction of their descriptive lines. Analogously for rays *b* under and parallel to the plane of the drawing. The cylinder lenses will have to be of such a quality that they do not disturb the phenomenon of interference.

It is not possible to place this set of cylinder lenses without any modification into one of the arms of the MICHELSON interferometer, because the light then passes twice through the system and the resulting phase-shifting becomes again  $\pi$ . Compare fig. 6, in which the immovable and the movable mirror are the two halves of the instrument consisting resp. of the points  $P''$  and  $P$ ; the system of lenses now lies as well on the path  $LP''$  as on the path  $P''Q$ , and we saw that phase shiftings are equivalent in the two parts of the light path, when their direction is the same, e.g. when both are directed forward as here, which causes them to join to  $\pi$ . For our purpose the interferometer of MICHELSON must, therefore, be modified. Fig. 7 indicates how this can be done.

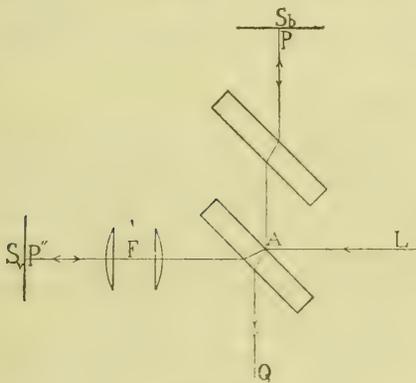


Fig. 6.

The mirrors of MICHELSON'S interferometer have been replaced by totally reflecting prisms, through which the advantage is attained that the light goes only once through the lenses. As the two beams undergo precisely the same total reflections, we need not take into account the phase-shiftings appearing in this. When a ray which has originated from a ray originating from the slit *S* on its reflection on the silvered glass plate through reflection or transmission is resp. provided with the index  $r$  or  $d$ , the figure shows how the reflected component  $1r$  of the ray 1 leaves the apparatus immediately by the side of the transmitted component  $2d$  of another ray 2,

when the rays 1 and 2 from the points  $A$  and  $B$  lie in the same horizontal plane, and  $SA$  and  $SB$  make equal angles with the radius  $SC$ , which passes through the optical centres. The rays 1 and 2 being coherent, the components  $1r$  and  $2d$  e.g. can interfere with each other. The leaving rays can therefore interfere in pairs, and as the rays which form such a pair, run immediately by the side of each other, a sharp interference image will be formed in the focal plane of the objective. Now this is not looked at, as usual, with the ocular as magnifying glass, but the ocular must form a real enlarged image of it, which falls outside the telescope, and of which only the central part will be used by us. All the light that has entered the prisms at right angles, will be focussed in one point  $Q'$ .

In order to calculate the intensity in the point  $Q'$ , it is convenient to introduce a reference plane  $RR$ , determined by the following equations:

$$ED = CO + PV \quad , \quad FG = HJ + LM \quad . \quad . \quad . \quad . \quad (9)$$

which quantities refer to geometrical lengths, and not to optical lengths.

Also the two following remarks will greatly facilitate the calculation of the phase-difference between the interfering beams.

1. Let the distance from the front plane to the opposite side be  $p$  for the immovable prism, and equal to  $q$  for the movable prism; then every ray that strikes the front plane at right angles vertically, covers a path in the prism of the length  $2p$  or  $2q$ , according as this takes place in the immovable or in the movable prism.

2. The system of lenses makes a plane wave again to a plane wave, so that in fig. 5 the optical distance from  $A$  to  $A'$  is equal to that from  $C$  to  $C'$ . Accordingly in fig. 7 the system of lenses will retard the light as much as a glass plate of the thickness  $2d$ , when the thickness of every lens in the middle amounts to  $d$ . Besides this retardation the system of lenses causes the phase-shifting  $\frac{\pi}{2}$

forward, which on the other hand means an acceleration. In order to get from  $C$  over the immovable prism to  $U$ , the light must pass three times through the glass plate, hence cover a path  $3\sigma$  in glass, and further the paths of the following lengths in the following media:

$DE$  in air,  $2p$  in glass,  $(FG - 2d)$  in air, and  $2d$  in glass.

The light that reaches the point  $U$  from  $C$  across the movable prism, must also pass three times through a glass plate, hence it must again cover a distance  $3\sigma$  in glass, and further the following

distances in the adjoined media:  $(CO + PV)$  in air,  $VT$  in air,  $2q$  in glass,  $NM$  in air,  $(ML + JH)$  in air. When the first path is compared with this last, use being made of equation (9), the path in the second (the prolongable) arm of the interferometer proves to

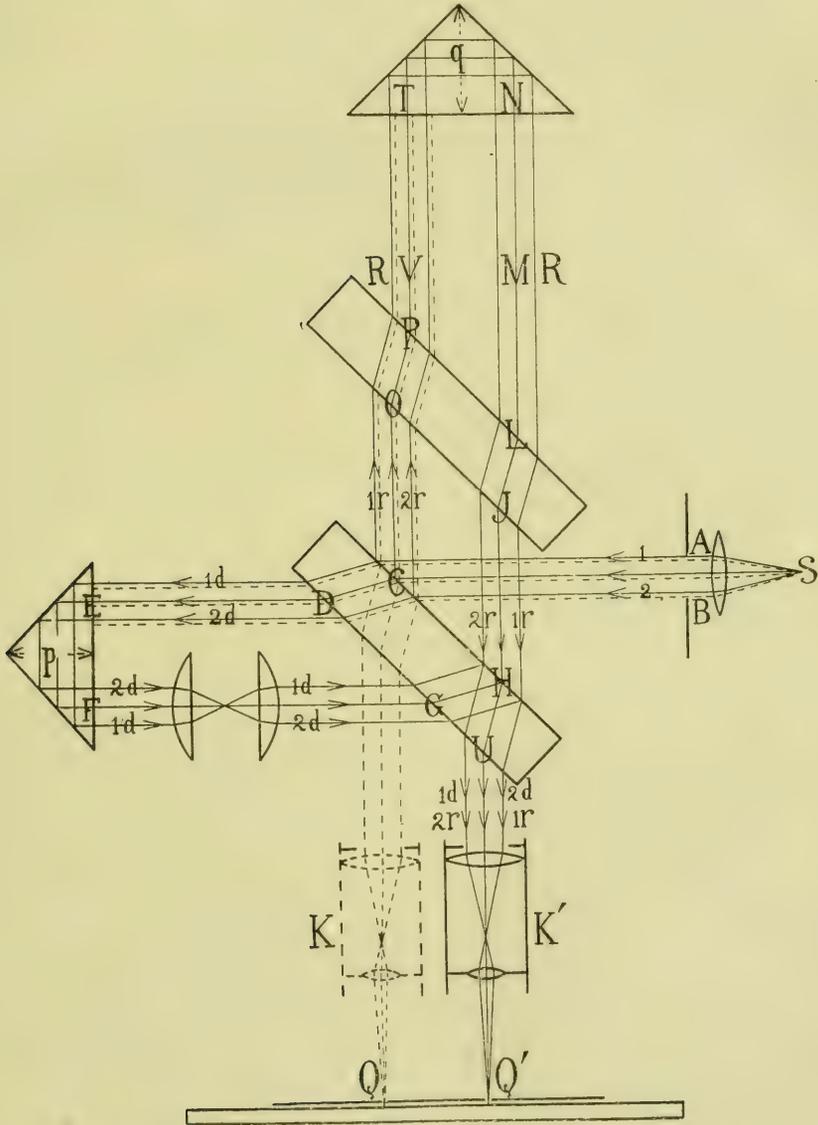


Fig. 7.

be  $2\mu(q-p) + 2VT + 2d(1-\mu)$  longer than that of the first. When we now also take into account the phase-shifting  $\frac{\pi}{2}$  forward when

the focal line is passed, (the path is seemingly shortened by this) the phase-difference of the interfering rays at  $U$  becomes:

$$4\pi VT + 4\pi \{ \pi(q-p-d) + d \} + \frac{\pi}{2} \text{ or } 4\pi l'm + \frac{\pi}{2},$$

when we put:

$$\mu(q-p-d) + d = v \dots \dots \dots (10)$$

and:

$$VT + v = l' \dots \dots \dots (11)$$

This phase-difference also exists between the rays  $1r$  and  $2l$ , which follows from the figure and from the foregoing remarks.

In analogy with equation (3) the intensity for compound light now becomes:

$$J'(l') = 2 \int_{m_2}^{m_1} \chi(m) dm + 2 \int_{m_2}^{m_1} \chi(m) \cos \left( 4\pi l'm + \frac{\pi}{2} \right) dm \text{ or}$$

$$J'(l') = 2 \int_{m_2}^{m_1} \chi(m) dn - 2 \int_{m_2}^{m_1} \chi(m) \sin 4\pi l'm dm \text{ or } \dots \dots \dots (12)$$

$$J'(l') = 2 \int_{-a}^{+a} \varphi(x) dx - 2 \sin 4\pi l'm C(l') - 2 \cos 4\pi l'm S(l') \dots \dots \dots (13)$$

$C$  and  $S$  no longer occur in the combination of equation (4). This is owing to the cylinder lenses.  $J'$  will henceforth be the intensity measured in  $Q'$ , hence with the modified interferometer,  $J$  that with the unmodified interferometer. A similar difference will also be made between  $l$  and  $l'$ :  $l$  refers to the original interferometer, and is the distance from the shiftable mirror to the reference plane;  $l'$  is the quantity that characterizes the position of the movable prism in the new interferometer, and controls the intensity. In order to calculate  $C$  and  $S$  for a given value  $l$ , from equations (4) and (13), we should have to place the mirror in one interferometer, the prism in the other in such a way that the quantities  $l$  and  $l'$  both acquire the value  $l_1$ . This would be practically unfeasible, when one worked with the two interferometers separately. We shall, therefore, try to combine them to one instrument, in which the quantities  $l$  and  $l'$  both occur, and are therefore equal. This seems to me practically no more feasible than the adjustment of the two interferometers to one and the same value  $l_1$  of the parameters. I have, however,

succeeded by the application of another slight change in the interferometer in designing a combination of the interferometers, in which the quantities  $l$  and  $l'$  always differ a same amount, which can be accurately measured. We shall further point out the way to determine this difference, and then demonstrate that the fact that this difference is constant enables us to work just as easily with this instrument and to reach just as much with it as would be the case with the ideal combination of the interferometers.

From the modified interferometer we should, namely, immediately get back the original one, when the front planes of the prisms were silvered. Accordingly the two instruments can be joined to one by leaving the upper halves of the front planes of the prisms transparent, and by silvering the lower halves. Let the line of division between the two halves lie in the plane of the drawing, then the rays that run above this plane, and have, therefore, been drawn in full, are totally reflected and are joined in a point  $Q'$ , when they struck the prisms at right angles. Rays running under the plane of the drawing, and which have therefore been indicated dotted, will be reflected by the silvered lower halves, and collected by another eyepiece  $K$  into another point  $Q$ , when they too strike the prisms at right angles. Strictly speaking we have here placed the two interferometers on top of each other, which was possible owing to the property of the cylinder lenses of never bringing a ray that runs above the plane of the drawing and parallel to it, under it. Rays that intersect the plane of the drawing could, indeed, pass from one instrument into the other, but these are not joined in the points  $Q$  or  $Q'$ , and it will appear, that we have to measure the intensity only in these points. Moreover the front plane of every prism can be divided into three regions, the top one transparent, the middle one absorbing, and the bottom one silvered, through which, as it were, a space arises between the two interferometers, depending on the height of the middle region. The upmost and the downmost regions must remain large, as all the incident light that strikes at right angles is concentrated in the points  $Q$  and  $Q'$ , and we want to have a strongly pronounced gradient of intensity in these points, when presently the movable prism is shifted.

We saw already that the intensity in the point  $Q'$  was determined by equations (12) and (13). The intensity in the point  $Q$  is represented by equation (4), in which  $l$  was the distance from the movable mirror to the plane of reference, and for which, therefore,  $VT$  of fig. 7 must be taken. We shall, therefore, call the length of  $VT$   $l$ . Summarising we find in  $Q'$  and  $Q$  resp. the intensities

$$\left. \begin{aligned}
 J'(l') &= 2 \int_{m_2}^{m_1} \chi(m) dm - 2 \int_{m_2}^{m_1} \chi(m) \sin 4\pi l' m dm \\
 J(l) &= 2 \int_{m_2}^{m_1} \chi(m) dm + 2 \int_{m_2}^{m_1} \chi(m) \cos 4\pi l m dm
 \end{aligned} \right\} \dots (14)$$

in which  $l' = l + v$  when  $v = \mu(q - p - d) + d$

A diaphragm with two fine apertures at the place of the points  $Q$  and  $Q'$  casts two beams of light of the above mentioned intensity on a photographic plate. When the movable prism is slowly shifted, the intensities of these beams change on account of the change in the parameters  $l$  and  $l'$  in (14). The difference  $l' - l$  remains, however, constant, namely equal to  $v$ , which quantity is an instrument constant, as appears from (10), if  $\mu$  does not appear in it. For that the prisms must be made of different size and so, that  $q = p + d$ . A small error in their size once for all can be neutralised by turning the plate  $PL$  over a small angle. Now  $v$  becomes  $= d$ , hence an instrument constant. When the photographic plate is shifted over large distances and in its own plane with a slight movement of the prism through coupling, and when care is taken that the shifting of the plate is always proportional to that of the prism, and further that the shifting of the prism always takes place with uniform velocity<sup>1)</sup>, the two beams will leave behind two bands on the photographic plate, where the blackening is different point for point, and which, when they are afterwards measured photometrically with the photometer and thermopile of Dr. MOIL, yield two curves, which will be of great importance for the determination of the desired function  $\varphi(x)$ .

We shall call the curves that arise when the bands are measured,  $Z(cl)$  or  $Z'(cl')$ , according as their ordinates represent the course of the blackening in the lower or the upper band, which is, therefore, made in  $Q$  or in  $Q'$ . When  $\frac{1}{c}$  denotes the ratio of the shifting of the prism, and the corresponding displacement of the photographic plate, and when in the upper band that point is assumed as zero point  $N$  that was placed in  $Q$  when the prism passed through the position  $l = 0$ , then light, the intensity of which corresponds to the position  $l$  of the prism, and is therefore indicated by equation (4), will have fallen on that band in a point whose abscissa amounts to  $cl$  measured

<sup>1)</sup> In order to keep the time of exposure the same for every position of the prism.

from the zero point. The blackening is proportional to this; hence when  $f$  is a factor of proportionality, the following equation will hold for the lower band:

$$J(l) = f Z(cl) \dots \dots \dots (15)$$

Analogously

$$J'(l') = f Z'(cl') \dots \dots \dots (16)$$

for the upper band when on it that point is chosen as zero point  $N'$  that was in  $Q'$  when the prism passed through the position  $l' = 0$ , hence when  $l = -d$  according to (14).

Now equation (14) passes into:

$$f Z(cl) = 2 \int_{m_2}^{m_1} \chi(m) dm + 2 \int_{m_2}^{m_1} \chi(m) \cos 4\pi lm dm \dots \dots (17)$$

$$f Z'(cl') = 2 \int_{m_2}^{m_1} \chi(m) dm - 2 \int_{m_2}^{m_1} \chi(m) \sin 4\pi l'm dm \dots \dots (18)$$

The functions  $fZ$  and  $fZ'$  oscillate round the same constant value:

$2 \int_{m_2}^{m_1} \chi(m) dm$ , when  $l$  and  $l'$  are increased; they will asymptotically

approach this value, for  $\int_{m_2}^{m_1} \chi(m) \cos 4\pi l m dm$  and  $\int_{m_2}^{m_1} \chi(m) \sin 4\pi l' m dm$

become zero for large values of  $l$  and  $l'$  on account of the continual reversal of the sign of the cosinus and the sinus, even when we have only to integrate over a small interval with respect to  $m$ .

According to equation (17) the ordinate  $Z$  will reach the greatest maximum for  $l = 0$ . This can be sharply determined from the course of the  $Z$  curve, when we have to do with multichromatic light. The bottom of this ordinate is the zero point  $N$ . This operation is equivalent to the adjustment at the white point in the interferometer of MICHELSON, when white light is used, and when we want to make the movable mirror coincide with the plane of reference. The adjustment can of course much more accurately be effected graphically than visually.

As  $f Z_{max.} = 4 \int_{m_2}^{m_1} \chi(m) dm$ , it follows from equations (17) and (18) that:

$$2 \int_{m_2}^{m_1} \chi(m) \cos 4\pi lm dm = f Z(cl) - \frac{1}{2} f Z_{max.} \dots \dots (19)$$

$$2 \int_{m_2}^{m_1} \chi(m) \sin 4\pi l'm \, dm = -fZ'(cl') + \frac{1}{2}fZ_{max}. \quad (20)$$

In this  $Z_{max}$  is the measured ordinate of the highest point of the  $Z$ -curve; the bottom point of this ordinate was the zero-point  $N$ , from which  $cl$  must therefore be measured along the axis of abscissae of the  $Z$ -curve. When a line is drawn parallel this axis and at a distance  $\frac{1}{2} Z_{max}$  from it, parts will remain of the ordinates the length of which will again be a function of  $cl$ , which function we shall call  $I (cl)$ .

According to (19) and (20) we now have:

$$I (cl) = Z (cl) - \frac{1}{2} Z_{max} = \frac{2}{f} \int_{m_2}^{m_1} \chi(m) \cos 4\pi lm \, dm \quad (21)$$

When the same line is drawn through the  $Z'$  curve, the remaining parts of the ordinates will represent a function  $II (cl')$  so that:

$$II (cl') = Z' (cl') - \frac{1}{2} Z_{max} = -\frac{2}{f} \int_{m_2}^{m_1} \chi(m) \sin 4\pi l'm \, dm. \quad (22)$$

The zero point  $N'$  on the axis of abscissae of the  $Z'$  curve may be determined on the following consideration. The function  $II (cl')$  is zero for  $l' = 0$ , but has several zero points, so that it is not yet known which of them is to be taken as  $N'$ ; it appears, however, from equation (22) that the function is uneven in  $l'$ . When, therefore, of the function  $II$  we determine the zero point with respect to which the other zero points lie symmetrically, we have found the point  $cl' = 0$ , hence the zero point  $N'$  on the axis of abscissae. This is the zero point, from which  $cl'$  must be measured on the axis of the  $Z'$ -curve. Especially for multichromatic light this symmetry will be very apparent. The reasoning involves that observations must have been made also for negative values of  $l'$ , i.e. that we must start with putting the movable prism somewhat nearer than would correspond with  $l' = 0$ , hence with  $l = -d$ . When we, therefore, begin with placing the prism at a distance greater than  $d$  before the plane of reference  $R$ , and moving it from there with uniform velocity, we begin with a negative value both of  $l$  and of  $l'$ , so that the initial points of the bands have negative abscissae.

We may cursorily remark that the quantity  $v = d$  can be accurately determined from the situation of the zero points  $N$  and  $N'$ .

The photographic plate was namely every moment struck by two beams of light, the intensities of which depended on two parameters

$l$  and  $l'$ , the difference of these parameters, if they refer to the same moment, being constant, viz.  $= d$ . For points of the  $Z$  and  $Z'$  curves, which have thus simultaneously arisen, the two abscissae  $cl$  and  $cl'$  will, therefore, always differ by the same amount, viz.  $cd$ .

The initial points of the curves are such points that have arisen simultaneously <sup>1)</sup>; we have, therefore, only to measure their abscissae taken to the found zero points on the axes, and to take the difference between them to find  $cd$ .

When in the graphically found functions I and II we choose the variables equal, i.e. when we measure equal portions  $cl$  and  $cl'$  on the axes from the points  $N$  and  $N'$  (hence independently of the earlier meaning of  $l$  and  $l'$ ), we find from equations (21) and (22):

$$2 \int_{m_2}^{m_1} \chi(m) \cos 4\pi lm \, dm = fI(cl) \quad . \quad . \quad . \quad (23)$$

$$2 \int_{m_2}^{m_1} \chi(m) \sin 4\pi lm \, dm = -fII(cl) \quad . \quad . \quad . \quad (24)$$

At first our intention was to solve  $C(l)$  and  $S(l)$  from equations (4) and (13), and substitute them in (7). This would come to the same thing as the introduction of  $C(l)$  and  $S(l)$  into equations (23) and (24), in order to determine them from this, and substitute them in equation (7). The same result, the formula for  $\varphi(x)$ , can be reached more quickly by the consideration that according to FOURIER'S integral theorem

$$\varphi(x) = \chi(m) = \frac{1}{\pi} \int_0^{\infty} \cos am \, da \int_{-\infty}^{+\infty} \chi(\xi) \cos \xi a \, d\xi + \frac{1}{\pi} \int_0^{\infty} \sin am \, da \int_{-\infty}^{+\infty} \chi(\xi) \sin \xi a \, d\xi.$$

When we now choose  $m$  for  $\xi$ , and put  $a = 4\pi l$ , and when we besides consider that  $\chi(m) = 0$  for  $m_1 < m < m_2$ , the equation reduces to:

$$\chi(m) = 4 \int_0^{\infty} \cos 4\pi lm \, dl \int_{m_2}^{m_1} \chi(m) \cos 4\pi lm \, dm + 4 \int_0^{\infty} \sin 4\pi lm \, dl \int_{m_2}^{m_1} \chi(m) \sin 4\pi lm \, dm.$$

Making use of equations (23) and (24), we may replace these by:

$$\chi(m) = 2f \int_0^{\infty} I(cl) \cos 4\pi lm \, dl - 2f \int_0^{\infty} II(cl) \sin 4\pi lm \, dl.$$

As we only try to find the shape of the curve  $\chi$ , we may put

<sup>1)</sup> The light may also be suddenly intercepted for a moment.

$2f = 1$ , in consequence of which the following equation is left:

$$\chi(m) = \int_0^{\infty} I(cl) \cos 4\pi lm \, dl - \int_0^{\infty} II(cl) \sin 4\pi lm \, dl \quad . \quad . \quad (25)$$

In this  $I(cl)$  and  $II(cl)$  are registered functions, which are zero for great values of  $l$  according to equations (21) and (22). When we wish to examine the finer structure of practically monochromatic light, the  $Z$  and  $Z'$  curves for white light must first be determined. The distance of the zero points  $N$  and  $N'$  to the initial points of the curves follows namely sharply from this. The prism is now again put in the same place as before at the beginning of the measurement<sup>1)</sup>, and in another photographic plate the  $Z$  and  $Z'$  curves for less compound light are now registered; now the zero points  $N$  and  $N'$  lie at the same distances from the initial points as before; without the previous measurements they would, however, hardly have been recognizable; for absolutely monochromatic light there would even have been no difference to be detected at all between the maxima or between the zero points.

The white light, however, has as it were, gauged the photographic plate. A procedure as described above is rendered possible through the fact that the zero points are known, even for the most strictly monochromatic light, and it enables us here too at least to closely approximate the required function  $\chi(m)$ . In practice this approximation cannot be carried to an extreme, because at too great differences of path the beams of light become incoherent on account of the irregular vibration of the source of light, a drawback that will adhere to every method that rests on interference.

This limit for the dissolving power of an interferential spectroscope, can of course not be exceeded here either. But all the data, with which interference phenomena can furnish us about the composition of apparently monochromatic light are used mathematically in the simplest way according to the method discussed, and lead to the drawing up of a function, which resembles the required distribution of energy  $g(x)$  most closely<sup>2)</sup>.

MICHELSON first works up his data to a curve of visibility  $V$ , and concludes from this to the value of  $C^2 + S^2$ , but has now to assume  $S = 0$ , i.e. a symmetrical distribution of energy in order to be able

<sup>1)</sup> In this use can e.g. be made of the distinct movement of the photographic plate.

<sup>2)</sup> To make the approximation still closer, we should have to make the source of light vibrate more regularly, or possess an entirely different instrument, which does not rest on interference.

to advance. He then<sup>1)</sup> gets  $C(l) = P \times V(l)$ , and will therefore find from (7):

$$\varphi(x) = 4P \int_0^{\infty} V(l) \cos 4\pi lx \, dl \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots (26)$$

in which  $P = \int_{-a}^{+a} \varphi(x) \, dx$  is therefore a constant.

We need not make this hypothesis; and it is exactly in this that the great significance of the method lies. When in spite of this we do submit to this restriction, we shall find that the curves I and II become dependent on each other. According to equation (23) and (24) we have namely:

$$2 \cos 4\pi \bar{m} C(l) - 2 \sin 4\pi \bar{m} S(l) = fI(cl)$$

$$2 \sin 4\pi \bar{m} C(l) + 2 \cos 4\pi \bar{m} S(l) = -fII(cl)$$

so that  $S=0$  would lead to:

$$-II(cl) = tg 4\pi \bar{m} \times I(cl)^2$$

Further the quantity  $V$  was a quantity that was not easy to estimate, and could, strictly speaking, only be determined from photometrical observations; it appears from its intricate definition

$$V = \frac{J_{max} - J_{min}}{J_{max} + J_{min}}$$

that it is not continuous, and accordingly can never be registered, but must be determined point for point. Equation (26), which is based on MICHELSON's visibility curve, holds therefore only for a symmetrical distribution of the energy and is not quite exact even in this case, while the  $V$ -curve is difficult to obtain, and then only very faint.

Summing up we come to the conclusion that in the discussed arrangement we have found a means to read the distribution of energy, both over a narrow and over an extensive spectral region

<sup>1)</sup> After neglecting some terms mentioned in footnote 2.

<sup>2)</sup> Substituting this in equation (25), we should have expressed  $\varphi(x)$  only by the aid of the curve  $I$ , and in this special case the cylinder lenses seem not to have been required; but in order to know  $\bar{m}$  we must know  $II$  at least for one value of  $cl$ ; hence the use of the cylinder lenses is inevitable even in this simple case, when we wish to work mathematically exactly. For the determination of  $J_{max} - J_{min}$  MICHELSON has considered  $C(l)$  and  $S(l)$  as constants in the differentiation with respect to  $l$ ; besides for  $S=0$  we find  $C(l) = P \times V(l)$  only for those values of  $l$  that satisfy  $S(l) = -tg 4\pi \bar{m} \times C(l)$ , hence for an infinite number of values of  $l$ , but not for a continuously infinite number, which would, however, be required by mathematics if equation (26) is to follow from equation (7).

from two curves that can be registered, it making no difference whether the distribution of energy is a symmetrical or an asymmetrical function; in the former case the registered curves are mutually dependent, in the latter case they are independent.

It is noteworthy that the metal reflections in the instrument of fig. 7 are not identical. The  $r$ -component has been reflected against the silver layer at an angle of  $45^\circ$  in air, and the  $d$ -component at a smaller angle in glass. Probably the phase-shiftings occurring here will be the same; otherwise MICHELSON could not have drawn up equation (4). If this supposition should be incorrect, fig. 8 represents

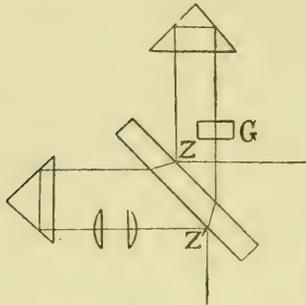


Fig. 8.

an arrangement in which this evil has been remedied; the silver layers at  $Z$  and  $Z'$  are now only struck at an angle of  $45^\circ$ . Now the front planes of the prisms must be transparent both at the top and at the bottom, and the cylinder lenses must be above the plane of the drawing. The silvering in  $Z$  (fig. 8) and in  $C$  (fig. 7) must be such that the intensities of the light of the beams entering the

telescope are equal, in spite of the fact that one has undergone reflections in the lenses. The intensities also can be kept equal by the aid of a glass plate  $G$ <sup>1)</sup>. When the intensities of the beams in the modified interferometer differ from that in the original one, one of the terms in equation (25) must be provided with a factor of proportionality, which is only to be determined experimentally.

We have scrupulously taken care that the interfering beams did not get a relative phase shifting with respect to each other in consequence of unequal metallic or total reflections, so that the phase shifting, which one beam obtains in advance of the other, has only been obtained in the focal line of the cylinder lenses ( $\frac{\pi}{2}$  large), and does, therefore, not depend on the wavelength. This renders the instrument efficacious for all possible sources of light and for spectrum regions of any given extent, supposing the lenses to be made achromatic<sup>2)</sup>. The result of the method, expressed in equation (25), is therefore perfectly exact for all possible spectrum regions

<sup>1)</sup> With such a plate in the arrangement of fig. 7 we must make  $p = q$ .

<sup>2)</sup> That the photographic plate  $Q Q'$  in fig. 7 may be adjusted once for all. A possible dispersion would not have influence on the constant phase shifting  $\frac{\pi}{2}$  either.

and distributions of energy, if the phase-shifting  $\frac{\pi}{2}$  is only brought about by means of cylinder lenses.

When an approximative method will do for the purpose, a method that is only exact for a practically monochromatic beam of one definite frequency, the phase shifting  $\frac{\pi}{2}$  can, indeed, be attained without cylinder lenses, viz. by the aid of an oblique-angled parallelepiped in which the light is totally reflected. We can now work with the unmodified interferometer of MICHELSON (we have only to put the parallelepiped in one of the arms, and one of right angles in the other, if we make the paths in glass of the same length; fig. 9), but now we must work with light that has been

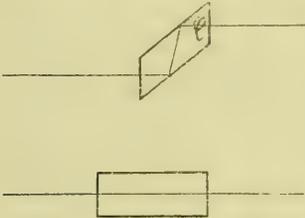


Fig. 9.

previously polarised either in the plane of drawing or normal to it. For sodium light e.g., the direction of vibration of which is normal to the plane of the drawing, we must have  $\angle \varphi = 51^\circ 20' 21''$ ; then the four phase-shiftings together amount to  $\frac{3\pi}{2}$ <sup>1)</sup>. However  $\varphi$  does not so greatly depend on the wavelength, that the same parallelepiped could not be used for neighbouring spectrum lines too. After some calculations we find namely  $\cos 2\varphi = \frac{\pm (n^2 - 1) \cos 67\frac{1}{2}^\circ - 1}{n^2}$ , in which  $n$  is the index of refraction.

Dr. ZERNIKE has pointed out another method, in which the phase-shifting keeps its value  $\frac{\pi}{2}$  only for one narrow spectrum region, just as this was the case with the totally reflecting parallelepiped: If we turn the compensator in the interferometer of MICHELSON over a small angle and if we place at the same time the movable mirror a little nearer, then in the one arm light has to cover a path in glass, which is  $d$  longer, but in air a path, that is  $d'$  shorter than in the other arm.

If we choose  $d$  and  $d'$  so, that they satisfy  $\mu d - d' = -\frac{1}{4}\lambda$  (1), even when we vary  $\lambda$  (however only over a narrow region) the phase difference of the interfering beams will be  $\frac{\pi}{2} + 4\pi lm$ , when

<sup>1)</sup> For  $\frac{\pi}{2}$  the parallelepiped would assume an impracticable form; index of refraction = 1,5153 for light crown-glass.

$l$  is the distance over which the movable mirror has been displaced from out its new position. We find  $d$  for one definite wavelength out of  $\frac{d\mu}{d\lambda} \times d = -\frac{1}{4}$  or  $d \times \left(-\lambda \frac{d\mu}{d\lambda}\right) = \frac{1}{4}\lambda$  and then  $d'$  out of (1).

Perhaps we can replace the unmovable mirror in the interferometer of MICHELSON by a concave one <sup>1)</sup>.

In conclusion I wish to express my cordial thanks to Prof. ORNSTEIN and Prof. JULIUS for their suggestions and the interest shown in this research. The research which Prof. ORNSTEIN originally asked me to undertake, was the first step on the road by which I have ultimately succeeded in finding the expedient which was to prove able to solve the problem theoretically in a perfectly satisfactory way.

I have followed the logical train of reasoning which has led to this, also in this communication, because through the considerations on the four groups of points  $P, P', P'', P'''$  it could then be naturally proved as second thesis that the required distribution of energy cannot be determined by any of the current instruments, in whatever way the observations made by them should be combined. The phase-shifting, independent of the wavelength and differing from  $\pi$ , was accordingly not only a possible, but also a necessary expedient to accomplish the task we had set ourselves.

*Utrecht, May 1919.*

*Institute for Theoretical Physics.*

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<sup>1)</sup> Gouy, Ann. de Chimie et de Physique (6) 24, pag. 198, 1891.

**Geology.** — “*Some new sedimentary boulders collected at Groningen*”.  
By Dr. P. KRUIZINGA. (Communicated by Prof. G. A. F.  
MOLENGRAAFF).

(Communicated in the meeting of May 31, 1919.)

Some ten years ago a favourable opportunity offered for collecting sedimentary boulders at Groningen, where in three different spots at the northern extremity of the Hondsrug and in the neighbourhood of the northern cemetery, which has already become known as a findingplace of erratics, important excavations were performed. First when the foundation was laid for the new tram-shed, and shortly after when new streets were being made, viz. the Tuinbouwstraat and the Koolstraat.

When trenches were dug for sewer-pipes, the Diluvium was not reached at the point, where the Tuinbouwstraat joins on to the Nieuwe Ebbingestraat. The presence of sherds of pottery at a depth of more than 1 m., however, made us suspect that the upper soil had been turned over or earthed-up. A little higher up in the Tuinbouwstraat the Diluvium emerged, and gradually rose to the surface, until about halfway it was seen half a meter above the present level of the street and was covered only by a thin layer of mould. Subsequently it first sank again below the street-level, after which it rose once more to the height just mentioned. Afterwards, on the occasion of the excursion made in 1913 by the Geol. Section of the Geol. Mijnbouwkundig Genootschap (28 p. 83), it was encountered again in the first part of the Verlengde Tuinbouwstraat, also covered with a layer of mould.

In the three localities just mentioned boulder-clay was found, which is calcareous but already oxidized. Besides this a number of bands of gravel were to be observed.

Among the large number of boulders, found by me during these excavations, there were several interesting specimens. Of the species rarely found near Groningen I mentioned already (30 p. 231) the Upper-Silurian limestone with *Pristiograptus frequens* Jaek. and the Saltholms-limestone (also the glauconitic variety, the so-called Glauconitic Terebratula-rock) from the Danian.

In the following pages I purpose to discuss three more Silurian

boulders, hitherto unknown in our country, the last two not having been met with in any other country as yet.

*Limestone with Strophomena Jentzschii Gag.*

Among the erratics found when the new tram-shed was being built, there was i. a. a plate-shaped piece of rock about 2 c.m. in thickness and 1 d.m. in length and in breadth, in which occur a large number of dorsal valves of a typical *Strophomena*. Of other fossils this boulder appeared to contain only a longitudinal section of a *Pleurotomaria* and a small pygidium of *Asaphus raniceps* Dalm. so that from this it is obvious that it belongs to the Lower Silurian.

The rock is a fine-grained, rather hard limestone, with scattered small rounded quartz-granules. I have not been able to detect glauconite. The primitive colour is undistinguishable, as through weathering it has changed into a more or less yellowish grey. Also some brownish spots still occur.

In looking up the literature I soon became aware that the very same *Strophomena* has already been described by GAGEL of East-Prussia, who termed it *Strophomena Jentzschii* (15 p. 17 44 pl. V fig. 26). One of the blocks in which this fossil has been found, consists of brownish grey hard limestone with somewhat weathered, yellowish spots. It comes from Spittelhof and contains besides numerous dorsal valves of the above mentioned fossil, also the rests of a large *Strophomena* and another irrerecognizable brachiopod. The other specimen comes from Pr. Holland and is composed of coarse-crystalline limestone in which only one dorsal valve of *Strophomena Jentzschii* Gag. occurs, beyond corals and rests of crinoids. Their petrographical character induced GAGEL to refer both erratic blocks to the Upper-Silurian series. However, it will appear presently that also these boulders, at all events the first-mentioned, have been proved to belong to the Lower-Silurian. On the other hand, according to ANDERSSON, the other may possibly originate from the Upper-Silurian and contain a closely related species.

Now in order to make assurance double sure, I begged Prof. ANDRÉE of Königsbergen to send me one of GAGEL's original samples for comparison. This request was readily complied with. A couple of well-preserved valves of Spittelhof were sent me, for which kindness I still feel greatly indebted.

My suspicion came true in every respect. The short description of this fossil by GAGEL I quote here for the sake of completeness:

Umriss querverbreitert, Schlossrand gleich der grössten Schalen-

breite. Schale anfänglich flach, dann allmählig unter einem rechten Winkel nach der Ventralseite zu gekrümmt, so dass die Dorsalschale convex wird. Oberfläche mit zahlreichen feinen, aber deutlich runden Rippen bedeckt, deren Zwischenräume durch 2—3 sehr feine Radialstreifen angefüllt sind. Ausserdem befinden sich auf dem flachen Teil der Schale noch eine Anzahl unregelmässiger, flacher, concentrisch angeordneter Runzeln, ähnlich wie bei *Strophomena rhomboidalis*. In der Mitte der Dorsalschale befindet sich oft noch eine kleine, aber deutliche Einsenkung”.

The concave ventral valve has not been discovered by GAGEL either.

In consideration of my scanty material I was not enabled to make certain about the petrographic resemblance between the boulder from Spittelhof and the one found by myself; still it seemed to be rather great.

The second and latest writer that has described erratics with *Strophomena Jentzschii* Gag. from Germany is STOLLEY (20 p. 136). Without mentioning the finding-place (only the district Schleswig-Hollstein is given), he mentions two more blocks. The one is composed of light-grey limestone, in which here and there vermiform concretions of glauconite grains occur. Beyond a number of specimens of *Strophomena Jentzschii* Gag. this boulder contains only *Orthisina plana* Pand. The other resembles in a most marked degree the preceding one, but contains only a trace of glauconite and the only fossil accompanying *Stroph. Jentzschii* Gag. was *Orthisina concava* v. d. Pahl.

To my knowledge this *Strophomena* has not been detected in erratics from Denmark.

Only a few years after GAGEL had described the species, J. G. ANDERSSON also recorded a number of erratics with the same fossil from Sweden. One of them originates from L. Brunnby in the parish of Stenasa in Oeland, one from Källunge Myr in Gotland and four from Gotska Sandön.

All these specimens differ largely from the one of Groningen as well as from the German pieces in that they are filled with a number of rolled fragments of brown phosphorite and brown to black phosphoritic sandstone. This makes them true conglomerates, which induced ANDERSSON to style them *Strophomena-Jentzschii-conglomerate*. Similar blocks have not been recorded either by STOLLEY or by GAGEL, who do not make mention either of any phosphorus-content. Neither does my specimen. What typifies ANDERSSON'S erratics, is that some-phosphorite blocks contain Upper-Cambrian fossils, viz. *Peltura scarabaeoides* Wahlb., *Sphaerophthal-*

*mus* sp. and *Agnostus pisiformis* Linn., which proves them to come from a region of Cambrian deposits, which was exposed to erosion during the early part of the Lower-Silurian period. ANDERSSON (l.c. p. 79) himself is wrong in inferring from these erratics, that they come from the very locality whence originated also the boulders they contained, and that at the very least in that place the whole Upper-Cambrian must have been eroded away. I think this need not be so at all, and I even believe that it is most likely not the case, but that the region, from which these Cambrian blocks originate, has to be looked for rather in the vicinity of the original locality of the Silurian erratics. First of all we think of the districts near the coast of the mainland of Sweden to the West and to the North of Gotland.

The cementing material also which consolidates the phosphorite blocks, varies more or less from the first-mentioned erratics, as, according to J. G. ANDERSSON it sometimes consists of grey to white spotted coarse-crystalline limestone and sometimes of grey, compact limestone, in which occur a larger or smaller number of rounded quartz-grains, as well as occasionally some glauconite.

Among the fossils in the last-mentioned erratics are *Orthisina* sp., *Platystrophia biforata* Schloth., *Strepula* sp., *Tetradella* sp., *Asaphus* sp., *Illaenus nuculus* Pomp., *Illaenus* sp., some Bryozoa and other non-descript fossils. Of all these only *Illaenus nuculus* Pomp. was known hitherto from a boulder from East-Prussia, as described by POMPECKJ (16 p. 69). The author referred it to the Lower-Silurian period. This rock consists of brownish, coarse-grained limestone with many quartz-granules.

Finally we refer to one more erratic block with *Strophomena Jentzschii* Gag. from the North-Balticum, recorded by WIMAN (23 p. 103), viz. N° 94 of Ekeby. This boulder consists of red *Asaphus*-limestone and does not contain other fossils.

The age of all these erratic blocks could be established, because *Strophomena Jentzschii* Gag. has been found in solid rock first by ANDERSSON (l.c. p. 77) in the northern part of Oeland, afterwards by LAMANSKY (22 p. 177) on the Wolchow in Russia and finally by HOLTEDAHL (29 p. 46) in South Norway near Vaekkerö and Töien.

LAMANSKY (l.c. p. 177) suspects that also the brachiopod, which is recorded by BRÖGGER (5 p. 50 pl. XI, fig. Va) as a *Strophomena rhomboidalis* Wilck. from the Expansus-shale and the lower part of the *Orthoceras*-limestone of South-Norway, is identical to *Strophomena Jentzschii* Gag. The figure alluded to, is not at all like it, as already observed by HOLTEDAHL.

This fossil is rarely but regularly found in Oeland, in the lowermost, glauconitic Asaphus-limestone, in Russia in the three divisions of the zone  $B_{III}$  ( $B_{III_\alpha}$ ,  $B_{III_\beta}$  and  $B_{III_\gamma}$ ) of LAMANSKY. On the basis of his investigations LAMANSKY parallels the lower half of  $B_{III_\alpha}$  with the Lower Asaphus limestone of Oeland, but the *Strophomena*-Jentzsch-conglomerate with the Upper Asaphus-limestone and the Gigas-limestone of Oeland, and with the upper part of LAMANSKY's zone  $B_{III_\beta}$  and with his zone  $B_{III_\gamma}$  of Russia, so that from this it follows, that *Strophomena Jentzchi* Gag. is spread over a larger vertical extent than ANDERSSON could have surmised at first. In South-Norway the fossil has been found in the zone 3 c.

If, therefore, we wish to parallel this erratic block with any of the Lower Silurian strata, it is necessary, in view of the varying petrographical character of the divisions, which deserve consideration, and in view of the different character of each of them in different regions, to find out from which region the boulder most probably originates.

According to LINDSTRÖM (11 p. 9—12) *Asaphus raniceps* Dalm. occurs already in the Lower Gray Orthoceras-limestone of Sweden and is still found in the Upper Gray Orthoceras-limestone.

According to SCHMIDT this species is observed in Russia in the zones  $B\ 2b$ — $B\ 3b$ ; according to LAMANSKY (22 p. 169) in the upper strata of the zone  $B_{III_\alpha}$  up to the lower strata of  $B_{III_\gamma}$ .

BRÖGGER asserts that it is not quite certain whether they are met with in Norway (5 p. 92).

Most probably this erratic block does not originate from the mainland of Scandinavia, *Strophomena Jentzchi* Gag being known there only in South Norway. Moreover the rocks from those zones differ from our boulder.

Likewise the Russian Silurian need not be considered although the latter fossil also occurs in Russia. It has not been observed yet to the west of Reval. In that region only  $B_{III_\gamma}$  of the zone  $B_{III}$  exists and this division consists of calcareous sandstone. Our boulder, therefore cannot come from the East-Balticum.

As has been said, only one erratic block with *Strophomena Jentzchi* Gag. from the North-Balticum is known. The petrographical character of it does not agree with this specimen. Boulders of grey limestone have been found there, indeed, which belong to the Asaphus-limestone of WIMAN and may therefore be of the same age.

In Oeland the Lower Asaphus-limestone consists of limestone partly containing glauconite and partly free from that mineral of which

the first may agree pretty well with STOLLEY's boulders, but neither of them agree with the Groningen specimen, especially as regards the amount of quartzgrains.

The presence of erratic blocks with *Strophomena Jentzschii* Gag. in Oeland, Gotland and Gotska Sandön leads us to consider also the localities of the Baltic west and north of the last two islands. It is true, the erratics found there, differ largely from the Groningen boulder; still this district is presumably to be considered as their original site. STOLLEY and ANDERSSON do the same for their blocks, while the assumption also seems warrantable of the presence of similar erratics in East-Prussia, notably the one described by POMPECKI and the Spittelhof fragment recorded by GAGEL.

Probably this specimen must be considered to originate from a narrow slip of the Baltic, a little north of Gotska Sandön and at a short distance West of Gotland.

From the foregoing it appears therefore, that the place of origin cannot be assigned more accurately, so that we cannot say for sure to which division of the zône *B<sub>III</sub>* the boulder belongs. It is therefore, like the *Strophomena-Jentzschii* conglomerate to be classed provisionally under *B<sub>III</sub>*.

#### *Calcareous Sandstone with Asaphus raniceps Dalm.*

In the Tuinbouwstraat one boulder was found among the many erratics that, judging from the fossils it contains, must be included among the Lower-Silurian. It is however of a peculiar petrographical character, as it consists of rather hard, fine-grained sandstone with a calcareous cement. The like of it appeared to be quite unknown in the literature of erratics.

This erratic block has about the size of a child's head and its primitive colour was gray to bluish-gray, as may still be observed from the inner part; the outside, however, shows a discolouration to brownish-yellow. For the rest it has suffered little from weathering. The quartz-grains are small, all but colourless and rounded. I did not encounter glauconite, but only some grains of calcite. The rock also contains a few pieces of more or less rounded, coarse-grained limestone, black at the periphery, white in the centre. These fragments, which moreover contain a large number of brown, rod-shaped bodies, are presumably little rolled boulders since they differ so much from the surrounding rock. However this is still highly problematical.

Beyond one specimen of an *Orthis*-species this block contains a

small but complete pygidium of *Asaphus raniceps* Dalm. (Length  $7\frac{1}{2}$  mm, breadth 11 mm) and numerous other indeterminable fragments of Asaphids; i.a. a fragment of an hypostome.

In the description of the previous boulder I have already communicated something about the occurrence of *Asaphus raniceps* Dalm. in the Lower Silurian deposits in Scandinavia and Russia, so that I now merely refer to it.

From the above it appears, therefore that this block is to be classed under the older strata of the Lower-Silurian, specifically under one of the divisions equivalent to the Swedish Orthoceras-limestone.

However in Scandinavia or in Bornholm no solid rock is known resembling this rock in any way. Starting from Reval,  $B_{III}$ , of LAMANSKY has developed itself as a calcareous limestone in the Western part of Estland. Fragments of this rock also occur on the beach of Odensholm, so that up to that locality at least this division retains the same petrographical character. There it has sunk already below the sea-level. Having no control-material of this rock I am unable to ascertain its similarity to this boulder.

Moreover some boulders have been discovered, which, being composed of limestone, contain a variable amount of rounded quartz-granules and agree in age with  $B_{III}$ , as may be gathered from the description of the previous species of erratics.

I therefore believe that this piece is to be considered as a quartz-rich variety of the limestone with *Strophomena Jentzschii* Gag. and of the *Strophomena-Jentzschii*-conglomerate, especially because in the previous block also occurs a pygidium that belongs to the same *Asaphus*-species.

When examining the fragment more closely with regard to a possible phosphorus-content, both the rock itself and the foreign enclosures distinctly proved to contain at least some phosphorus. The latter, however, did not give off any smell of bitumen when particles were knocked off with the hammer. Furthermore, because they are not fossiliferous, we cannot determine whether these fragments of limestone, as is the case with the erratics of the *Strophomena-Jentzschii*-conglomerate examined by ANDERSSON, are to be included under the Cambrian.

Most likely the original locality of this erratic block is that slip of the Baltic which covers the prolongation of the calcareous sandstone in Estland and continues along the North side of Gotska Sandön as far as West of Gotland, thus comprising the region, from which the *Strophomena-Jentzschii*-conglomerate originates.

*Limestone with Dinobolus transversus Salt.*

A piece of fine grained-crystalline limestone having become brownish-yellow through weathering and of about the size of a fist, contains a dorsal valve of *Dinobolus transversus* Salt. (1 p. 59 pl. V fig. 1—6), which in spite of its extreme thinness has been preserved in admirable perfection. This boulder also, which was also found in the Tuinbouwstraat at Groningen, is a completely unknown species of erratics, as the fossil mentioned just now was not met with in any other country.

The length of the valve is 3 cm., the largest breadth, across the centre, 4.2 cm. The almost straight hinge margin is 3.3 cm. long.

The dorsal valve is almost quite flat and reveals on its surface numerous, very faint, concentric lines of growth and an extremely fine radial striation. Whether there are small spines on the outer surface, as indicated by DAVIDSON (l.c. pl. V, fig. 3 and 3a) cannot be made out.

Of other fossils this boulder contains besides a number of detached portions of crinoid stems also a valve of *Pholidops implicata* Sow. (1 p. 80, pl. 8, fig. 13—17) and a valve of *Beyrichia Jonesi* Boll (13 p. 13, pl. II, fig. 10—12) and a pygidium of *Proetus concinnus* Dalm. (9 p. 78, 18 p. 41, pl. IV, fig. 1—9, 3 p. 22, pl. XVII, fig. 5).

From all this it appears, therefore, that the block belongs to the upper Silurian, the zône being undetermined yet.

*Pholidops implicata* Sow. contrary to *Pholidops antiqua* Schloth. is probably quite unknown in our upper Silurian erratics as well as in those from Germany and Denmark, which is perhaps due to the fact that various authors have considered the two as synonyms (7 p. 96, 10 p. 173). It appears however, as MOBERG and GRÖNWALL (24 p. 30) have shown, that they were used for fossils which indeed are closely allied to each other but also form a distinct contrast. Only KIESOW (6 p. 245) records that *Pholidops implicata* Sow. (= *Crania implicata* Sow.) is very abundant in West-Prussia in the boulders of the Upper-Silurian *Beyrichia* Limestone. I think however that he also refers here to *Pholidops antiqua* Schloth.

In solid rock *Pholidops implicata* Sow. is known only from the island of Gotland (from the zônes *c—h* of LINDSTRÖM (12 p. 13). MUNTKE (27 p. 12—13) mentions the fossil from the layers 2—4 distinguished by him and VAN HOEPEN (25 p. 125) from  $\gamma$  and  $\epsilon_{w3}$ . LINDSTRÖM, also, records the occurrence in Schonon (l.c. p. 26), but MOBERG and GRÖNWALL state that the species there differs distinctly

from the Gotland species and resembles *Pholidops antiqua* Schloth. It is not known as yet which of these two fossils occur in Oesel and in Estland.

*Beyrichia Jonesii* Boll. has been reported by KIESOW (13 p. 13 pl. II fig. 10—12) of Gotland from LINDSTRÖM's zônes *c—h*, by VAN HOEPEN (l. c. p. 132) from his zône  $\varepsilon^w$ . At Schonen this fossil has not been found, and nothing is known of it in Oesel and Estland.

*Proetus concinnus* Dalm. is mentioned only by SCHMIDT (l. c. p. 44) from the Lower Oesel stratum (zône *J*) of the Russian Baltic provinces, of Gotland by LINDSTRÖM (12 p. 3) from the zônes *c—e*, and by VAN HOEPEN from his zône  $\gamma$  (near Mulde) (l. c. p. 142). Moreover this fossil has been found with *Beyrichia Jonesii* Boll in boulders, associated with *Leperditia Baltica* Eichw. and *Beyrichia spinigera* Boll. (2 p. 39, 17 p. 502).

I, therefore, feel justified in assuming, that this boulder probably agrees as to its age, with the Lower Oesel stratum of the Russian Baltic provinces.

In Oesel the Lower Oesel stratum consists almost entirely of blue marl and dolomite. Limestone occurs only in the West of the peninsula Taggamois (4 p. 46). The equivalent layers in Gotland, on the other hand, are composed of marl, marly limestone and limestone, and the equivalent layers on the mainland of Sweden of grapholite-shale, so that this region cannot be considered as the place of origin.

Gotland and the part of the Baltic between this island and Oesel and of these probably, first of all, the island of Gotland together with its approximate vicinity is, therefore, in all likelihood to be looked upon as the locality from which our erratic block was derived.

Delft, May 1919.

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**Astronomy.** — “*Theory of Jupiter’s Satellites. II. The variations.*”  
By Prof. W. DE SITTER.

(Communicated in the meeting of June 28, 1919).

We still restrict ourselves to the non-periodic part  $[R_i]$  of the perturbative function, as in the determination of the intermediary orbit.<sup>1)</sup> The quantities  $h_i, k_i, v_i, \omega_i$ , which were zero in the intermediary orbit, are now determined by the equations (23),<sup>2)</sup> of which the solution is given by (24). For the determination of the five values of  $\beta_q$ <sup>2</sup> we have the determinant (28). Then  $c'_{iq}$  and  $c''_{iq}$  are determined from (27) and  $c_{iq}$  and  $c'''_{iq}$  from the first and last of (25). The expressions for the coefficients  $a_{ij}, a'_{ij}, b_{ij}, b'_{ij}, d_{ij}, d'_{ij}, e_{ij}, e'_{ij}$  are given in Vol. XII, Part I, of the Annals of the Leiden Observatory, page 31. Then we have

$$\begin{aligned} A_{ij} &= \sum_l (a'_{il} a_{lj} + b'_{il} d_{lj}), \\ B_{ij} &= \sum_l (a'_{il} b_{lj} + b'_{il} e_{lj}), \\ C_{ij} &= \sum_l (d'_{il} a_{lj} + e'_{il} d_{lj}), \\ D_{ij} &= \sum_l (d'_{il} b_{lj} + e'_{il} e_{lj}). \end{aligned}$$

The details of the computation of the quantities  $a_{ij}, a'_{ij}$ , etc. and  $A_{ij}, B_{ij}$ , etc. will be published in the Annals of the Observatory at Leiden. Here we shall only give the results. The determinant (28) is: (see formula A, next page).

The coefficients are given in units of the *eighth* decimal place. Denoting the columns by roman, and the rows by arabic numerals, we now perform the following operations:

$$\begin{array}{ll} \text{add } 4 \cdot (\text{V}) + 2 \cdot (\text{VI}) & \text{to } (\text{VII}) \\ \text{,, } 3 \cdot (\text{V}) - e_1 \cdot (\text{I}) - e_2 \cdot (\text{II}) - e_3 \cdot (\text{III}) - e_4 \cdot (\text{IV}) & \text{,, } (\text{VI}) \\ \text{,, } -2 \cdot (7) & \text{,, } (6) \\ \text{,, } 2 \cdot (7) - 3 \cdot (6) & \text{,, } (5) \\ \text{,, } e_i [(6) - 2 \cdot (7)] & \text{,, } (i) \quad (i = 1 \dots 4). \end{array}$$

The determinant then becomes: (see formula B, next page).

<sup>1)</sup> See *Outlines of a new theory of Jupiter’s Satellites*, These Proceedings, Vol. XX, p. 1289—1308, and *Theory of Jupiter’s Satellites. I. The intermediary orbit*, These Proceedings, Vol. XXI, p. 1156—1163.

<sup>2)</sup> “*Outlines*” p. 1301. The definition of  $h_i$  and  $k_i$  is slightly different here, in consequence of the introduction of  $e_i$  instead of  $\eta_i$ . We now have

$$\begin{aligned} e_i \cos g_i &= e_i + h_i \\ e_i \sin g_i &= k_i. \end{aligned}$$

+	29100.69	-	$\beta^2$	+	21.69	-	46.73	-	1.42	+	118.18	-	232.81	-	7.12	0	
+	170.79	+	22765.45	-	$\beta^2$	+	102.71	+	4.77	+	32.87	+	114.95	-	361.39	0	
-	14.41	+	37.16	+	21367.36	-	$\beta^2$	-	25.51	+	0.21	+	12.38	-	25.62	0	
+	0.36	+	1.69	-	34.51	+	21087.28	-	$\beta^2$	+	0.00	+	0.01	+	0.01	0	
A	+	82273	+	20156	-	10	+	0	+	530.06	-	$\beta^2$	-	1058.08	-	4.08	
-	97192	+	44489	+	20187	-	0	-	0	-	618.14	+	1876.26	-	$\beta^2$	-	1279.98
+	8	-	16723	-	4933	+	0	+	0	-	0.38	-	158.07	+	317.68	-	$\beta^2$
+	0	+	1	+	0	-	0	+	0	+	0.00	+	0.00	-	0.01	-	$\beta^2$

+	28694.59	-	$\beta^2$	+	347.27	+	78.82	-	1.42	+	115.60	0	0	0	0		
-	737.84	+	23493.93	-	$\beta^2$	+	383.62	+	4.76	+	27.10	0	0	0	0		
-	72.36	+	83.62	+	21385.27	-	$\beta^2$	-	25.51	-	0.15	0	0	0	0		
B	+	0.34	+	1.70	-	34.51	+	21087.28	-	$\beta^2$	+	0.00	0	0	0		
+	373866	-	146757	-	70437	+	2	+	2383.70	-	$\beta^2$	0	0	0	0		
+	97209	+	77935	+	30053	-	1	-	617.37	-	$\beta^2$	0	0	0	0		
+	8	-	16723	-	4933	+	0	+	0.38	0	-	$\beta^2$	0	0	0		
+	0	+	1	+	0	-	0	+	0.00	0	+	0.00	0	0	0	-	$\beta^2$

Consider the determinant consisting of the elements of the first five rows and columns. This can be conceived to be the result of the elimination of  $y_i$  from the linear equations

$$\sum_j F_{ij} y_j - \beta^2 y_i = 0 \quad (i, j = 1 \dots 5)$$

where  $F_{ij}$  is the element of the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column.

The new unknowns  $y_i$  are connected with the original unknowns  $c'_i$  and  $c''_i$  of the equations (27) by the relations:

$$\begin{aligned} c'_i &= y_i - e_i y_6, & (i = 1 \dots 4) \\ c_1'' &= y_6 + 3y_6 + 4y_7, \\ c_2'' &= y_6 + 2y_7, \\ c_3'' &= y_7, \\ c_4'' &= y_8. \end{aligned}$$

To each of the five roots  $\beta^2_q$  corresponds a set  $y_{iq}$  ( $i = 1 \dots 5$ ). Then  $y_6$ ,  $y_7$  and  $y_8$  are determined by

$$\sum_j F_{ij} y_j - \beta^2 y_i = 0. \quad (j = 1 \dots 5, \quad i = 6 \dots 8)$$

We now determine  $x_i$  from

$$F'_{55} = F_{55} + \sum_j x_j F_{5j}, \quad (i, j = 1 \dots 4)$$

$$F_{i5} + \sum_j x_j F_{ij} - x_i F'_{55} = 0,$$

and we put

$$F'_{ij} = F_{ij} - x_i F_{5j}, \quad (i, j = 1 \dots 4)$$

and

$$y_i = z_i + x_i y_6. \quad (i = 1 \dots 4)$$

Then  $z_i$  and  $\beta^2$  are determined by the equations

$$(a) \quad \sum_j F'_{ij} z_j - \beta^2 z_i = 0, \quad (i, j = 1 \dots 4)$$

and  $y_6$  is given by

$$\sum_j F_{5j} z_j + (F'_{55} - \beta^2) y_6 = 0. \quad (j = 1 \dots 4)$$

This determines four roots  $\beta^2_q$ . The fifth is given by

$$\beta_5^2 = F'_{55},$$

and the corresponding values of  $z_i$  and  $y_6$  are

$$y_{55} = 1, \quad z_{i5} = 0. \quad (i = 1 \dots 4)$$

To solve the equations (a) we take  $z_{qq} = 1$ . Since the coefficients of the diagonal  $F'_{ii}$  are much larger than the others, we can take

$$\begin{aligned} \beta^2_q &= F'_{qq} + \xi_q \\ z_{iq} &= \frac{F'_{iq}}{F'_{qq} - F'_{ii}} + \eta_{iq}. \end{aligned}$$

Then the quantities  $\xi_q$  and  $\eta_{iq}$  are small, and we easily find a set of equations from which they can be conveniently determined.

We find

$$F'_{56} = + 1025.13,$$

and the determinant of the four equations ( $\alpha$ ) is

+ 30250.34— $\beta^2$	— 263.42	— 214.29	— 1.41
— 235.83	+ 23296.87— $\beta^2$	+ 289.04	+ 4.76
— 71.72	+ 83.36	+ 21385.15— $\beta^2$	— 25.51
+ 0.33	+ 1.71	— 34.50	+ 21087.28— $\beta^2$

The five roots  $\beta_q^2$ , determined in this way, are (still expressed in units of the 8<sup>th</sup> decimal place):

$$\beta_1^2 = + 30261.15$$

$$\beta_2^2 = + 23299.79$$

$$\beta_3^2 = + 21374.51$$

$$\beta_4^2 = + 21084.19$$

$$\beta_5^2 = + 1025.13$$

We now put, for  $q = 1 \dots 4$ ,

$$\psi_q = \beta_q \tau + \psi_{q0} = \kappa \tau + \varpi_q + \pi_{q0} = \kappa \tau + \gamma_q \tau + \varpi_{q0} + \pi_{q0},$$

where

$$\pi_{10} = \pi_{20} = \pi_{40} = 0 \quad , \quad \pi_{30} = 180^\circ.$$

Then  $\varpi_q = \gamma_q \tau + \varpi_{q0}$  are the longitudes of the “proper” perijoves, and  $\psi_q$  for  $q = 1 \dots 4$  are the arguments of the inequalities of group II, and  $\psi_5$  is the argument of the libration.

The mean motions of these arguments are, in degrees per day <sup>1)</sup>:

El. and Masses.

$\gamma_1 = 0^\circ 148668$	$0^\circ .14407$
$\gamma_2 = 0 .039842$	$0 .039593$
$\gamma_3 = 0 .006949$	$0 .007046$
$\gamma_4 = 0 .001862$	$0 .001864$
$\beta_5 = 0 .16347$	$0 .16252$

These values are not yet final, since we have neglected:

1. the effect of the periodic part  $R_i - [R_i]$  of the perturbative function,
2. the squares and products of the quantities  $h_i, k_i, v_i, \omega_i$ ,
3. the inclinations of the orbital planes of the satellites on the plane of Jupiter's equator.

Apart from the corrections which must eventually be applied

<sup>1)</sup> The motions  $\gamma_i$  and  $\beta_5$  are sidereal ones, and do not contain precession. Accordingly the precession has been subtracted from the values of  $\gamma_i$  as given in *El. and Masses*.

later on these three accounts, the values of  $\beta_i$  and  $\gamma_i$  as given here are certainly exact and complete to the last decimal place given.

The values added for comparison are those of my theory of 1908<sup>1)</sup>, reduced to the masses adopted here. This theory is SOUILLART's with some errors corrected. The computation of the motion of the argument of the libration, however, was carried one order further than was done by SOUILLART.

The values of the coefficients  $c_{iq}$ ,  $c'_{iq}$ ,  $c''_{iq}$  and  $c'''_{iq}$  are:

$q$	1	2	3	4	5
$c_{1q}$	+ 0.96868	+ 0.02754	+ 0.02479	+ 0.00230	- 0.00054
$c_{2q}$	- .04429	+ .93485	- .17327	- .01584	+ .00032
$c_{3q}$	- .00686	+ .03313	+ .98970	+ .08804	+ .00004
$c_{4q}$	+ .00006	+ .00012	- .12098	+ 1.00000	.00000
$c'_{1q}$	+ 0.96038	+ 0.05057	+ 0.02483	+ 0.00229	- 0.00287
$c'_{2q}$	- .01944	+ .97713	- .15246	- .01400	+ .00155
$c'_{3q}$	- .00637	+ .04010	+ .99952	+ .08891	+ .00018
$c'_{4q}$	+ .00006	+ .00012	- .12098	+ 1.00000	.00000
$c''_{1q}$	+ 2.7649	+ 0.9617	- 0.07900	- 0.00718	+ 0.15668
$c''_{2q}$	- 3.3850	+ 1.8535	+ .57340	+ .05032	- .26677
$c''_{3q}$	+ 0.0300	- 0.7324	- .11736	- .01022	+ .02150
$c''_{4q}$	.0000	.0000	+ .00001	.00000	.00000
$c'''_{1q}$	+ 0.01195	+ 0.00363	- 0.00028	- 0.00002	+ 0.00012
$c'''_{2q}$	- .02934	+ .01436	+ .00405	+ .00035	- .00043
$c'''_{3q}$	+ .00052	- .01113	- .00149	- .00013	+ .00007
$c'''_{4q}$	0	0	0	0	0

If we neglect the squares and products of  $\varepsilon_q$ , (as well as products  $\varepsilon_i \varepsilon_q$ ), and if we put

$$\tau_{iq} = \pm \frac{1}{2} (c_{iq} + c'_{iq}),$$

where the lower sign must be taken if either  $i$  or  $q$ , but *not both* of them, are 2, then the effect of the variations on the radius-vector and the longitude is:

$$\frac{dr_i}{a_i} = -\frac{3}{2} \sum_q c'''_{iq} \varepsilon_q \cos \psi_q - \sum_q \tau_{iq} \varepsilon_q \cos (\lambda_i - \varpi_q) - \frac{1}{2} \sum_q (c_{iq} - c'_{iq}) \varepsilon_q \cos (l_i + \psi_q)$$

$$d\omega_i = \sum_q c''_{iq} \varepsilon_q \sin \psi_q + 2 \sum_q \tau_{iq} \varepsilon_q \sin (\lambda_i - \varpi_q) + \sum_q (c_{iq} - c'_{iq}) \varepsilon_q \sin (l_i + \psi_q),$$

where we have put

<sup>1)</sup> *On the Masses and Elements of Jupiter's satellites and the mass of the System*, these Proceedings, Vol. X, pp. 653 and 710.

$$\lambda_i = \lambda_{00} + \pi_{i0} + (c_i - \alpha) \tau,$$

$$l_i = c_i \tau.$$

The first term for  $q=1 \dots 4$  gives the inequalities of group II, and for  $q=5$  it represents the libration. The second term for  $q=1 \dots 4$  represents the equations of the centre.

We have

$q$	1	2	3	4
$\tau_{1q}$	+ 0.96453	- 0.03905	+ 0.02481	+ 0.00229
$\tau_{2q}$	+ .03186	+ .95599	+ .16287	+ 01492
$\tau_{3q}$	- .00661	- .03662	+ .99461	+ .08847
$\tau_{4q}$	+ .00006	- .00012	- .12098	+ 1.00000

For  $q=5$  this term is better written in the form

$$\begin{aligned} & - \frac{1}{2} (c_{i5} + c'_{i5}) \varepsilon_5 \cos (l_i - \psi_5) \\ & + (c_{i5} + c'_{i5}) \varepsilon_5 \sin (l_i - \psi_5). \end{aligned}$$

It then represents, like the third term for all values of  $q$ , small periodic inequalities, whose periods differ little from that of the equations of the centres.

It should be pointed out that the theory here given (intermediary orbit and variations) covers the same ground as SOUILLART's, with the exception of the small periodic perturbations and the terms of very long period arising from the action of the sun, Saturn, etc. SOUILLART does not give any term of the perturbative function, nor any interaction of two terms leading to a term of higher order, which is not taken into account here too; and he omits many terms which are included here. The above theory is *certainly complete* up to the *numerical* limit of accuracy which was fixed beforehand. This can certainly not be said of SOUILLART's theory, though it generally gives many more decimals. The new theory has proved eminently suitable for numerical computation.

**Physiology.** — “*Tonic reflexes of the labyrinth on the eye-muscles*”.

By Prof. R. MAGNUS and A. DE KLEIJN.

(Communicated in the meeting of June 28, 1919).

It is a well-known fact, that the labyrinths strongly influence the position of the eyes in the orbits. Each position of the head namely, corresponds to a special, definite position of the eyes in the orbits.

The inquiries concerning the relation between the position of the labyrinth and that of the eye is highly impeded by spontaneous movements of the eyes and it is, therefore, easy to understand that more elaborate investigations were made almost exclusively with animals, which make but few such spontaneous movements. Among the animals, which are usually experimented on, the rabbit is preferred to all others.

In 1917 an extensive publication on this subject was issued by the pharmacological Institute<sup>1)</sup>, in which the question of the relation between the positions of the labyrinth and of the eye was examined as completely as could be. Starting from a primary position of the head with a horizontal mouth-opening and the lower jaw pressed downward, other positions of the head appeared to involve both *constant vertical deviations of the eye* and *rotatory movements*; as for *side-movements*, in the direction of the eye-opening, no reliable data could be found.

The vertical deviations of the eyes always take place in opposite directions for the two eyes, whereas the principal deviations were found with the head in the two side-positions, when the upper eye deviated as much as possible downwards, the lower eye in the same degree upwards.

The rotatory movements always take place in the same direction for both eyes, the greatest deviations were found, when the head stood with its muzzle vertically upwards or downwards. During the experiments the rabbits were put immovably in various positions with regard to their surroundings, when special care was taken that any shifting of the position of the head with regard to the trunk was out of

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<sup>1)</sup> J. V. L. HOEVE und A. DE KLEIJN. Tonische Labyrinthreflexe auf die Augen. Pflügers Archiv. 169. 241. 1917.

the question (reflexes of the neck). The various positions of the eyes were determined kinematographically. Minute measuring was made possible by indicating on the photographic plates the shifting of a cross, burned into the cornea anaesthetized by cocain, with regard to a fixed system of coordinates, photographed at the same time. The results were indicated for the vertical deviations of the eye and for the rotatory movements separately, to wit for three perpendicular rotations of the head round  $360^\circ$  each.

In this way we only get to know, of course, the influence of the head, i. e. the labyrinths on the position of the *eyes*. However, for a minute analysis of the influences of the labyrinths it is desirable to know the influence of the labyrinths on each *eye-muscle*.

Now, as different positions of the head often bring about a combination of vertical deviations of the eye and rotation movements, which combination, in its turn, variously modifies the points of insertion of the eye-muscles, it stands to reason that we cannot say beforehand that the greatest deviation of the eye-ball either upwards or downwards, or the full extent of any of its rotations, necessarily implies the maximum lengthening or shortening of the eye-muscles (recti and obliqui). This made it necessary to investigate what position of the head produced the maximum and minimum shortening of the eye-muscles. The above-mentioned inquiry had clearly brought out in what way each position of the head influences the position of the eyes, so that the only thing left to be done now, was construction of a proper model of an eye, putting the eye-ball of this model in the various positions which had been found, and measuring the length of the six eye-muscles for each position accurately.

A short time ago the anatomical relations of the eye-muscles of rabbits were given in detail by WESSELY<sup>1)</sup>. However the accompanying illustrations do not give us the numbers expressing those relations.

For this reason a minute inquiry was made with various rabbits with regard to the length of the eye-muscles, size of the eye-ball, place of insertion for each muscle, etc. and in accordance with this the instrument-maker of the Institute, Mr. F. A. C. IMHOF, made a model of an eye-ball with eye-muscles to correspond.

Starting from the primary position of the eyes and guided by the information obtained before with regard to rotatory movements and vertical deviations of the eye for different positions of the head, now

<sup>1)</sup> K. WESSELY. Ueber den Einfluss der Augenbewegungen auf den Augendruck. Arch. f. Augenheilkunde. 81. 111. 1916.

the eyes of the model were placed in the corresponding position, so that now the lengths of the 6 eye-muscles for these various positions of the eyes could be measured.

In the experiments made before, three rotations had always been performed. Rotation I.

The animal originally in ventral-position with horizontal mouth-opening, Rotation of the animal round the bitemporal axis, direction of the rotation head downwards and tail upwards.

Rotation II.

The animal originally in ventral position, with horizontal mouth-opening. Rotation of the animal round the occipital-caudal axis. Direction of the rotation: Right eye downwards.

Rotation III.

The animal originally in side position, left side downward, right eye upward, vertical mouth-opening. Rotation of the animal round the venter-dorsal axis. Direction of the rotation: muzzle downwards.

For each rotation the position of the eyes was stated accurately after every 15°.

The result of the measurements of the lengths of the eye-muscles of the model for the various positions of the eyes was stated in tabular form.

However, the publication of these tabular statements must be put off for the moment, as some correction appeared to be necessary.

For when the eye, starting from the normal position, performs *rotations, unaccompanied by vertical movements*, these rotations of the eye-ball cause the points of insertion of the rectal eye-muscles on the bulb to be removed, by which the length of the rectal eye-muscles is changed passively.

However, when a rotatory movement combines with a vertical deviation of the eye, the contraction of the rectal eye-muscles does not take place with the length of those muscles of the normal position of the eye, but with the length they have got by (after) the rotation (contractions of the oblique eye-muscles). So when the eye has performed rotation the lengths of the eye-muscles must be rectified with a value, in accordance with the passive lengthening or shortening, caused by the contraction of the oblique eye-muscles.

At the same time, of course, the lengths of the m.m. oblique at different vertical deviations should be rectified with a value, in accordance with the passive lengthening or shortening of those muscles caused by the contraction of the rectal eye-muscles.

With the help of the model it was easy enough now, by first putting the eye in the normal position and stimulating either rotatory movements or vertical deviations of the eye-ball exclusively, to

state for various lengths of the oblique eye-muscles, the passive lengthening or shortening of the rectal eye-muscles, or conversely, for various lengths of the rectal eye-muscles the passive lengthening or shortening of the oblique eye-muscles.

The tabular statements, corrected according to this system, have been reproduced as curves on fig. 1—4.

So these curves represent the rectified lengths in m.m. of the 4 eye-muscles (both oblique and recti super. and infer.) for the three rotations of the head above-mentioned. For the obliq. superior only the distance from trochlea to the insertion on the bulbus has been reproduced. From all this we learn that curves of the obliq. superior and obliq. inferior form a true reflexion of each other, that is to say that, at the tonic reflex of the labyrinth these muscles act as antagonists, the lengthening of the one brings about the shortening of the other, and conversely.

At the same time the curves of the m.m. recti superior and inferior show that these muscles too are absolute antagonists.

If we compare the curves of the oblique eye muscles (fig. 1 and 2) with the curves, found at a former period for the rotatory movements of the eye, we see, that they agree with regard to the principal points. Especially the positions of the maxima and minima do not show any essential difference; the rotatory movements and the shortening of the oblique eye muscles are greatest when the head with its muzzle points vertically upward or downward. At the same time the curves for the rectal eye-muscles (fig. 3 and 4) agree with the curves, found before for the vertical deviations of the eye. Only the shape of the curve of the eye-muscles at rotation III is a little bit more pointed than the shape of the curve, found for the vertical deviations.

However the position of the maxima and minima undergo no essential change. The maximum contraction of the two rectal eye-muscles takes place when the head is almost in side position.

Now, comparing the curves of the obliqui and recti, we find the following:

At rotation I (—) the obliqui react strongly whereas the recti hardly perform any movement.

So, at this rotation we find no vertical deviations, but almost all of them are rotatory-movements.

On the other hand, at rotation II (—.—) the obliqui hardly react at all, whereas the recti superior and inferior perform strong movements; so for this rotation the vertical movements prevail, whereas rotatory movements do not take place.

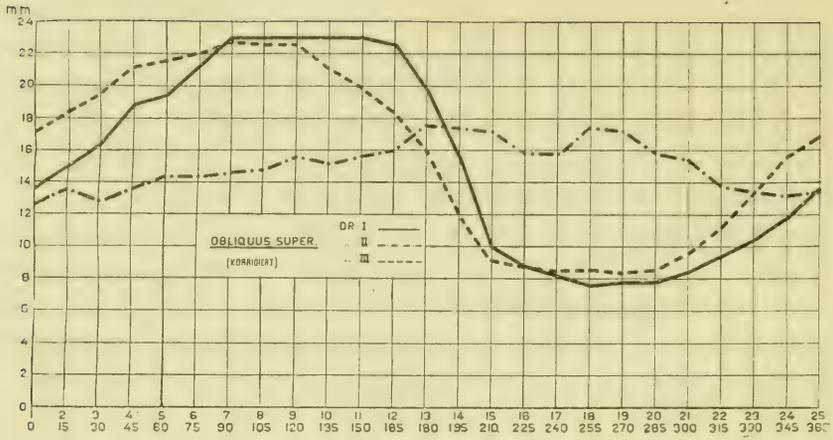


Fig. 1.

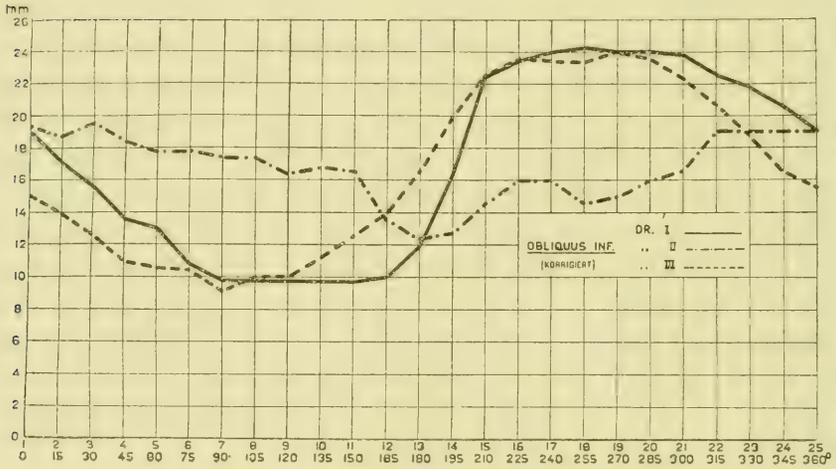


Fig. 2.

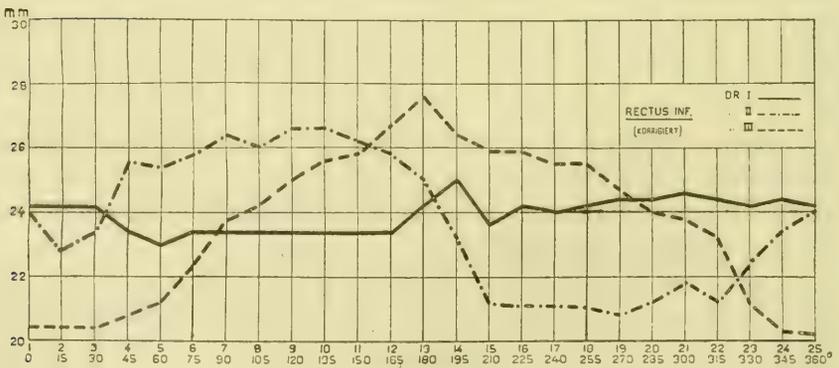


Fig. 3.

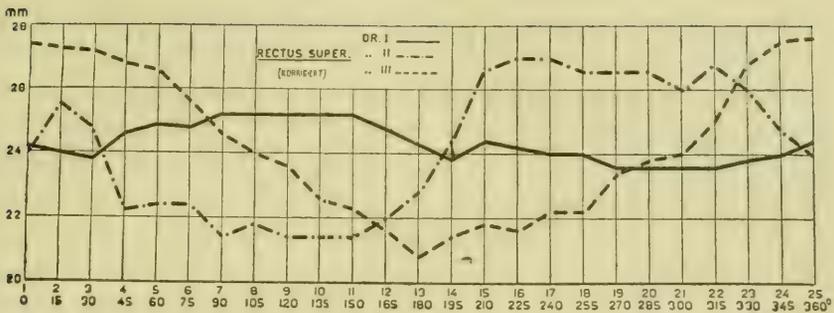


Fig. 4.

Only for rotation III (—.—.—) we find a combined reaction of the obliqui and the two recti.

From this we may safely conclude that, at rotation I the labyrinths influence almost exclusively the obliqui, at rotation II almost exclusively the rectus superior and inferior and at rotation III all four eye-muscles.

We cannot yet enter into details about the curves, but will do so afterwards, when determining how far the tonic reflexes of the labyrinth on the eye depend on definite parts of the labyrinth special of the otolithes. Be it sufficient to indicate here that for the oblique eye-muscles the curve of rotation I shows an asymmetric course, whereas for the rectal eye-strings the same thing takes place at rotation II.

The exactitude of the former definitions and the measurements now performed, may be derived from the comparison of the corresponding points on the different curves.

For the three different rotations namely, it occurred several times that the same position of the head was reached from different directions. The curve shows that, notwithstanding this, the lengths found for the eye-muscles agree wonderfully.

Corresponding points are among others:

Normal position. Rotation I No. 1 and 25. Rotation II No. 1 and 25.

Back position. Rotation I No. 13. Rotation II No. 13.

Side position (Left). Rotation II No. 19. Rotation III No. 1.

Side position (Right). Rotation II No. 7. Rotation III No. 13.

Muzzle upwards. Rotation I No. 19. Rotation III No. 19.

Muzzle downwards. Rotation I No. 7. Rotation III No. 7.

For all these positions the four eye-muscles measured have almost exactly the same length.

### Conclusions.

From this and the inquiry published before we may conclude

the following with regard to the tonic reflexes of the labyrinths on the eye-muscles:

1. With the rabbit every position of the head corresponds with a special state of contraction of the eye-muscles and therefore with a special position of the eyes, which lasts throughout the time that the head retains the same position.

2. With the rabbit for the rectus externus and internus no reliable data could be found in the bringing about of these tonic reflexes of the labyrinth. It is especially the rectus superior and inferior which cause the vertical deviations of the eye and the two obliqui which cause the rotation movements.

When this happens, the two recti, just like the two obliqui, act as antagonists, on the other hand, changes of the lengths of the recti may combine with those of the obliqui in various degrees. So these two groups of muscles act independently of each other (though of course dependent together on the labyrinths).

3. When the head stands with its muzzle vertically upwards, the two obliqui superiores (right and left) are in a state of greatest contraction, the two obliqui inferiores in a state of greatest relaxation.

The upper cornea-poles of both eyes are then rolled forward. When the head stands with its muzzle in a position vertically downwards, the two obliqui superiores are in a position of greatest relaxation, the two obliqui inferiores in a position of greatest contraction. The upper cornea-poles of both eyes are rolled backward.

For all other positions of the head we find contractions, lying between these two extremes, the two eyes always react with rollings in the same direction.

4. When the head is in side-position (left) the right rect. inf. and the left rect. superior are in a state of greatest contraction, the right rectus superior and the left rectus inferior in a state of greatest relaxation. Then the right eye has its maximum deviation downward, the left eye its maximum deviation upward. When the head is in side position (right), the left rect. inf. and the right rect. sup. are in a state of greatest contraction, the left rectus superior and the right rect. inf. in a state of greatest relaxation. For all the other positions of the head we find states of contraction of the rectus superior and inferior lying between these two extremes. Both eyes always react with opposed vertical deviations of the eye. The rectus superior of one side and the rectus inferior of the other side react in the same sense.

5. When we start from the normal head-position and we turn

it round the bitemporal axis ( $360^\circ$ ), it is principally the obliqui which react, whereas the two eyes roll in the same direction. Starting from the normal position and turning the head round the occipital-caudal axis ( $360^\circ$ ), it is especially the recti superior and inferior which react, and the eyes show opposed vertical deviations.

Starting from the side-position and turning the head round its venter-dorsal position ( $360^\circ$ ), both groups of strings react and the eye-positions are the combined results of opposed vertical deviations and rotatory movements equally directed.

6. After extirpation of the labyrinth on one side the vertical deviations of the eye and the rotatory movements continue for *both* eyes. *One* labyrinth influences the obliqui of both eyes and the rollings in the same sense; however the recti (sup. and inf.) of the two eyes and the vertical deviations of the eyes are influenced in the opposed sense.

For both eyes *one* labyrinth brings about the greatest vertical deviation of the eye with respect to its normal positions when it is lowest down, whereas the head is in side-position. Then the rectus super. of the same side and the rectus inf. of the crossed side are in a state of greatest contraction.

*One* labyrinth brings about, for both eyes, the greatest rotatory movements by contraction of the obliq. infer., when the head stands with its muzzle vertically downward.

On the other hand the greatest rotatory movements of both eyes by contraction of the obliqui superior are brought about by *one* labyrinth, when the head stands with its muzzle vertically upwards.

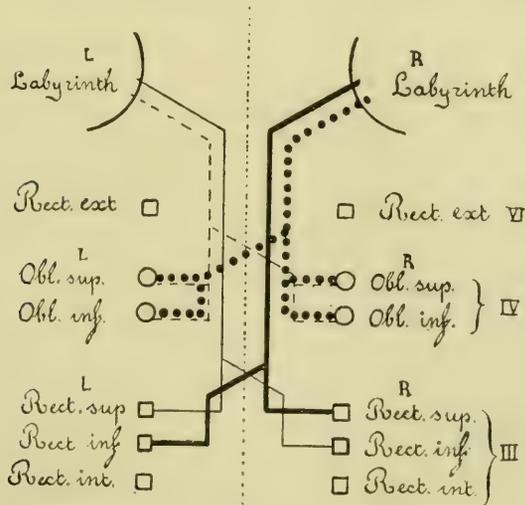


Fig. 5.

With *one* labyrinth the size of these rotatory movements is about half of that of animals with intact labyrinths.

7. For the intact animal it is possible to calculate the changes in the eye-positions by taking the sum of the influences, starting from the right and left labyrinths, on the recti super. and infer. and the obliqui super. and inf. of the two eyes.

8. After extirpation of the labyrinths on *both* sides all tonic reflexes on the eyes, mentioned above, disappear.

9. The *minimum* number of central courses, necessary for the explanation of the *tonic* reflexes of the labyrinths of the rabbit on its eyes (so *not* of the rotations-reactions and caloric reactions), have been drawn in a sketch, accompanying fig. 5.

The uninterrupted lines represent the courses of the recti super. and inf., the dotted lines those of the obliqui.

Each of the four obliqui is influenced from both labyrinths, each of the two recti (super. and infer,) from only *one* labyrinth.

*One* labyrinth influences the 4 obliqui, but only the rect. super. of the same side and the rect. inf. of the crossed side. For these *tonic* reflexes of the labyrinth for the m.m. externus and internus no reliable data could be found.

**Mathematics.** — “On expansions in series of covariant and contravariant quantities of higher degree under the linear homogeneous group.” By J. A. SCHOUTEN. (Communicated by Prof. J. CARDINAAL).

(Communicated in the meeting of March 29, 1919).

*Notations*<sup>1)</sup>. A covariant affiner of degree  $p$  may be written as the general product of  $p$  ideal fundamental elements<sup>2)</sup>:

$$\mathbf{u} = \sum_{\lambda_1, \dots, \lambda_p}^{1, \dots, n} u_{\lambda_1 \dots \lambda_p} \mathbf{e}_{\lambda_1} \dots \mathbf{e}_{\lambda_p} = \mathbf{u}_1 \dots \mathbf{u}_p, \quad \dots \dots \dots (1)$$

an alternating or symmetrical one as a power of one ideal fundamental element:

$$\left. \begin{aligned} {}^p\mathbf{v} &= \sum_{\lambda_1, \dots, \lambda_p}^{1, \dots, n} {}^p v_{\lambda_1 \dots \lambda_p} \mathbf{e}_{\lambda_1} \dots \mathbf{e}_{\lambda_p} = \mathbf{v}^p, \quad p \leq n \\ {}^p\mathbf{w} &= \sum_{\lambda_1, \dots, \lambda_p}^{1, \dots, n} {}^p w_{\lambda_1 \dots \lambda_p} \mathbf{e}_{\lambda_1} \dots \mathbf{e}_{\lambda_p} = \mathbf{w}^p. \end{aligned} \right\} \dots \dots \dots (2)$$

The  $p!$  isomers of  $\mathbf{u}$ , viz. the  $p!$  products of the ideal factors in all possible orders are real rational covariants of  $\mathbf{u}$ . Each isomer is formed from  $\mathbf{u}$  by one of the  $p!$  permutations  $P_\lambda$  of the ideal factors. By a penetrating general product<sup>d</sup> of some affiners  $\mathbf{u}^p, \mathbf{v}^q, \dots$  we understand any isomer of the general product  $\mathbf{u}^p \mathbf{v}^q \dots$ . An affiner, the different isomers of which are not connected by linear relations, is called a *non special* one.

*Classes of isomers*<sup>2)</sup>. It is well-known, that the  $p$  factors of an isomer can be divided into groups of  $s_1, s_2, \dots$  factors in one single way, so that in each group the permutation is a cyclic one. The groups are called the *permutation regions* and the complex of the numbers  $s_1, s_2, \dots$  in descending order and omitting all numbers

<sup>1)</sup> See further “Die direkte Analysis der neueren Relativitätstheorie.” Verh. der Kon. Akad. v. Wet. Dl. XII N<sup>o</sup>. 6, p. 7—11.

<sup>2)</sup> Introduced firstly by E. WÆLSCH under the name of “symbolische Vektoren” in “Ueber mehrfache Vektoren und ihre Produkte, sowie deren Anwendung in der Elastizitätstheorie.” Mon. f. Math. und Ph. 17 (06) 241—280.

1 the *permutation number*. For  $a$  regions of the same extension we shall write here  $a.s_p$ . A permutation number is *higher* than an other one, when its first region is greater or, in the case of equality, when the second region is greater etc. FROBENIUS introduced the name of permutation class for all permutations with the same permutation number. In the same way we shall call the sum of all isomers with the same permutation number divided by  $p!$  the *isomer class* with that number. The number of classes is therefore equal to the number  $k$  of the whole positive solutions of the equation

$$x_1 + 2x_2 + 3x_3 + \dots = p \dots \dots \dots (3)$$

The classes are arranged in ascending order and written:

$$K_1^p \mathbf{u} = \frac{1^p}{p!} \mathbf{u} \quad , \quad K_2^p \mathbf{u} \quad , \quad \dots \quad \dots \quad , \quad K_k^p \mathbf{u}$$

The permutation number may be added as index on the left side e.g. for  $p = 6$ :

$${}_1K_1, {}_2K_2, {}_{2,2}K_3, {}_{3,2}K_4, {}_3K_5, {}_{3,2}K_6, {}_{2,2}K_7, {}_4K_8, {}_{4,2}K_9, {}_5K_{10}, {}_6K_{11}.$$

A class is called *even* or *odd* according as it consists of even or odd permutations.

*Alternations and mixings.* The affinor that is found from  $\mathbf{u}$  by replacing each of  $t$  definite groups of  $s_1, \dots, s_t$  factors (without displacing them) by the ideal factors of their alternating or symmetrical product is called a *simple alternation* resp. *mixing* of  $\mathbf{u}$  with the permutation number  $s_1, \dots, s_t$  and is written as:

$${}_{s_1, \dots, s_t}^{(\lambda)} A^p \mathbf{u} \quad \text{resp.} \quad {}_{s_1, \dots, s_t}^{(\lambda)} M^p \mathbf{u}$$

The index  $(\lambda)$  at the top on the right indicates the choice of the *permutation regions*. The affinor is called, *locally alternating* resp. *symmetrical* in these regions and in general *locally permutable*.

The sum of all simple alternations resp. mixings with the same permutation number divided by their number is called the *general alternation* resp. *mixing* with that number. The general alternations and mixings are arranged in ascending order and written as

$$\begin{aligned} \overline{A}_1^p \mathbf{u} = \frac{p}{p!} \mathbf{u} \quad , \quad \overline{A}_2^p \mathbf{u}, \dots \quad \dots, \quad \overline{A}_k^p \mathbf{u} \\ \overline{M}_1^p \mathbf{u} = \frac{p}{p!} \mathbf{u} \quad , \quad \overline{M}_2^p \mathbf{u}, \dots \quad \dots, \quad \overline{M}_k^p \mathbf{u}, \end{aligned}$$

eventually, when desirable, with the permutation number as index on the left e.g. for  $p = 6$ :

$$\begin{aligned} {}_1\overline{A}_1, {}_2\overline{A}_2, {}_{2,2}\overline{A}_3, {}_{3,2}\overline{A}_4, {}_3\overline{A}_5, {}_{3,2}\overline{A}_6, {}_{2,2}\overline{A}_7, {}_4\overline{A}_8, {}_{4,2}\overline{A}_9, {}_5\overline{A}_{10}, {}_6\overline{A}_{11} \\ {}_1\overline{M}_1, {}_2\overline{M}_2, {}_{2,2}\overline{M}_3, {}_{3,2}\overline{M}_4, {}_3\overline{M}_5, {}_{3,2}\overline{M}_6, {}_{2,2}\overline{M}_7, {}_4\overline{M}_8, {}_{4,2}\overline{M}_9, {}_5\overline{M}_{10}, {}_6\overline{M}_{11} \end{aligned}$$

When  $s_1, \dots, s_t$ , is the permutation number of  $\bar{A}_i$ ,  $\bar{A}_i$  contains

$$a_i = \binom{p}{s_1} \binom{p-s_1}{s_2} \dots \binom{p-s_1-\dots-s_{t-1}}{s_t} \quad (4)$$

simple alternations  $A_i$ . The same holds for mixings. For a non special affinor all classes and likewise all simple and general alternations resp. mixings are linearly independent.

When successively  ${}_{s_1, \dots, s_t} A$  and  ${}^{s'_1, \dots, s'_t} M$  are applied to a non special affinor, the result is then and only then zero when more than one factor of a region  $s$  belongs also to a region  $s'$ . The highest permutation number for which

$${}_{s_1, \dots, s_t} A \quad {}^{s'_1, \dots, s'_t} M \mathbf{u}^p$$

is not *always* equal to zero is called *conjugate* to  $s_1, \dots, s_t$ . From this follows that

$$s'_1, \dots, s'_t = (t+p-s_1-\dots-s_t), (s_t-1) \cdot t, (s_{t-1}-s_t) \cdot (t-1), \dots, (s_1-s_2) \cdot 2. \quad (5)$$

This relation is a reciprocal one. When the permutation regions of an alternation or mixing are numbered in such a way that a greater region has always a lower number than a smaller one, it is possible, that for *all* values of  $\varepsilon$  the  $\varepsilon^{\text{th}}$  factors of each of the regions are placed in the order of this numbering. In this case the alternation or mixing is called *ordered*. To an ordered alternation there evidently belongs only one ordered mixing with conjugate permutation number, such that these two operators do not annihilate each other. Then these two are called *conjugate*.

A general alternation and a general mixing with conjugate permutation number are called *conjugate*. Every general alternation or mixing is annihilated by all simple and general mixings resp. alternations with a permutation number that is higher than its own conjugate one. For  $p > 5$  the order of the general mixings conjugate to  $\bar{A}_2, \dots, \bar{A}_{k-1}$  is *not* the same as of  $\bar{M}_{k-1}, \dots, \bar{M}_2$ , for  $p=6$ , e.g.

$$\begin{aligned} & {}_2\bar{A}_2, {}_{2,2}\bar{A}_3, {}_{3,2}\bar{A}_4, {}_3\bar{A}_5, {}_{3,2}\bar{A}_6, {}_{2,3}\bar{A}_7, {}_4\bar{A}_8, {}_{4,2}\bar{A}_9, {}_5\bar{A}_{10} \\ & {}_5\bar{M}_{10}, {}_{4,2}\bar{M}_9, {}_{2,3}\bar{M}_7, {}_4\bar{M}_8, {}_{3,2}\bar{M}_6, {}_{3,2}\bar{M}_4, {}_3\bar{M}_5, {}_{2,2}\bar{M}_3, {}_2\bar{M}_2 \end{aligned}$$

When in  ${}_{s_1, \dots, s_t} A \mathbf{u}^p$  or  ${}^{s'_1, \dots, s'_t} M \mathbf{u}^p$  the *unaltered* factors are replaced by the ideal factors of their symmetrical resp. alternating product, we obtain a *mixed alternation* resp. an *alternated mixing* with the same permutation number, written:

$${}_{s_1, \dots, s_t} A \mathbf{u}^m \quad \text{resp.} \quad {}^{s'_1, \dots, s'_t} M \mathbf{u}^a$$

From the operators  $\overset{m}{A}$  and  $\overset{a}{M}$  we can form in the same way as above general mixed alternations  $\overset{m}{\overline{A}}$  and general alternated mixings  $\overset{a}{\overline{M}}$ . As to the order  $1, \dots, k$ , the independency and the property of being ordered, the same laws hold for these operators as for  $\overline{A}$ ,  $\overline{A}$ ,  $\overline{M}$  and  $\overline{M}$ .

When an  $A_i$  and a  $M_j$  with conjugate permutation number do not annihilate each other, the unaltered factors of  $M_j$  are alternated by  $A_i$  and those of  $A_i$  are mixed by  $M_j$ , so that:

$$A_i M_j = \overset{m}{A_i} \overset{a}{M_j} = A_i \overset{a}{M_j} = \overset{m}{A_i} \overset{a}{M_j}, \dots \dots (6)$$

and likewise:

$$\overline{A_i} \overline{M_j} = \overset{m}{\overline{A_i}} \overset{a}{\overline{M_j}} = \overline{A_i} \overset{a}{\overline{M_j}} = \overset{m}{\overline{A_i}} \overset{a}{\overline{M_j}} \dots \dots (7)$$

*Expansion of the general alternations and mixings in classes.*

*Theorem I. A general alternation resp. mixing of a non special affinor can be written in one and only one way as the sum of multiples of all classes with the same or a lower permutation number. For a mixing the coefficients of all classes are positive, for an alternation those of the even classes positive, the other ones negative. For the same permutation number their absolute values are equal:*

$$\left. \begin{aligned} \overline{A_i} &= \sum_j^{1, \dots, i} \delta_j a_{ij} K_j \\ \overline{M_i} &= \sum_j^{1, \dots, i} a_{ij} K_j \end{aligned} \right\} \delta_j = \begin{cases} +1 & \text{for } K_j \text{ even} \\ -1 & \text{,, ,, odd.} \end{cases} \quad (8)$$

The very simple proof of this theorem may be omitted. In order to determine the coefficients  $a_{ij}$  we must investigate in how many ways it is possible to choose the  $t$  permutation regions  $s_1, \dots, s_t$  of an  $A_i$  or of a  $M_i$  thus, that each of the  $t'$  regions  $s'_1, \dots, s'_t$  of a permutation of a definite  $K_j$  falls quite within one of those regions. When  $m_{ij}$  is this number, we have

$$a_{ij} = \frac{m_{ij} p!}{s'_1! \dots s'_t! \binom{s_1}{p} \binom{p-s_1}{s_2} \dots \binom{p-s_1-s_{t-1}}{s_t}} = \frac{m_{ij} p!}{s'_1! \dots s'_t! a_i} \dots \dots (9)$$

For  $p = 6$  is e.g. (see table following page)

All  $\overline{A}$ , all  $\overline{M}$  and all  $K$  being linearly independent, we have inversely:

$$\left. \begin{aligned} \delta_i K_i &= \sum_j^{1, \dots, i} \beta_{ij} \overline{A_j} \\ K_i &= \sum_j^{1, \dots, i} \beta_{ij} \overline{M_j} \end{aligned} \right\} \dots \dots \dots (11)$$



*Expansion of an affinor in alternations or mixings.*

Thus we have especially for the sum of all classes  $\overline{M}_k$  and for  $\overline{A}_k$  :

$$\left. \begin{aligned} \overline{M}_k &= - \sum_i^{1, \dots, k} \gamma_{ki} \overline{A}_i \\ \overline{A}_k &= - \sum_i^{1, \dots, k} \gamma_{ki} \overline{M}_i \end{aligned} \right\} \dots \dots \dots (12)$$

By the application of  $\overline{M}_k$  to the upper equation we learn that  $\gamma_{kk} = 1$ , so that when we consider  $\overline{M}_k$  and  $\overline{A}_k$  resp. as 0-th general alternation,  $\overline{A}_0$  and as 0-th general mixing  $\overline{M}_0$  we have,  $I$  being the identical operator, and because  $\overline{A}_1 = \overline{M}_1 = I$  :

$$I = \sum_i^{0, 2, \dots, k} \gamma_{ki} \overline{A}_i = \sum_i^{0, 2, \dots, k} \gamma_{ki} \overline{M}_i \dots \dots \dots (13)$$

In the same way we can prove:

$$I = \sum_i^{0, 2, \dots, k} \gamma'_{ki} \overline{A}_i = \sum_i^{0, 2, \dots, k} \gamma'_{ki} \overline{M}_i \dots \dots \dots (14)$$

For a permutation number  $s$  we have in these expansions :

$$\gamma_{ki} = \gamma'_{ki} = (-1)^{s-1} (p - s + 1) \dots \dots \dots (15)$$

and for a permutation number  $\alpha$  :

$$\gamma'_{ki} = \frac{\binom{p}{2\alpha} \binom{2\alpha}{\alpha}}{\binom{p-\alpha+1}{\alpha}} \dots \dots \dots (16)$$

Thus we have obtained the theorem :

*Principal theorem A: Every non special affinor can be expanded in one and only one way into general alternations, into general mixed alternations, into general mixings and into general alternated mixings with the indices 0, 2, 3, ..., k.*

So we deduce e.g. from table (10) for  $p = 6$  :

$$\left. \begin{aligned} \mathbf{u} &= ({}^6\overline{A}_{11} - 2 {}^6\overline{A}_{10} - 2 {}^4\overline{A}_9 + 3 {}^4\overline{A}_8 - {}^3\overline{A}_7 + 6 {}^{3,2}\overline{A}_6 - 4 {}^3\overline{A}_5 + {}^{3,2}\overline{A}_4 - \\ &\quad - 6 {}^{2,2}\overline{A}_3 + 5 {}^2\overline{A}_2 + \overline{A}_0) \mathbf{u} \\ \mathbf{u} &= ({}^6\overline{M}_{11} - 2 {}^6\overline{M}_{10} - 2 {}^{4,2}\overline{M}_9 + 3 {}^4\overline{M}_8 - {}^{2,3}\overline{M}_7 + 6 {}^{3,2}\overline{M}_6 - 4 {}^3\overline{M}_5 + {}^{3,2}\overline{M}_4 - \\ &\quad - 6 {}^{2,2}\overline{M}_3 + 5 {}^2\overline{M}_2 + \overline{M}_0) \mathbf{u} \end{aligned} \right\} (17)$$

Decomposing the ordinary alternations and mixings into mixed resp. alternated ones we can calculate :

$$\left. \begin{aligned} \mathbf{u} &= ({}^6\bar{A}_{11} - 2{}^2\bar{A}_{10} + 6{}^4{}_2\bar{A}_9 + 3{}^m{}_4\bar{A}_8 - 5{}^3{}_3\bar{A}_7 + 4{}^3{}_3\bar{A}_6 - 4{}^m{}_3\bar{A}_5 + \\ &\quad + 5{}^3{}_3\bar{A}_4 + 9{}^m{}_3\bar{A}_3 + 5{}^m{}_3\bar{A}_2 + \bar{A}_0) \mathbf{u} \\ \mathbf{u} &= ({}^6\bar{M}_{11} - 2{}^5\bar{M}_{10} + 6{}^4{}_2\bar{M}_9 + 3{}^4\bar{M}_8 - 5{}^3{}_2\bar{M}_7 + 4{}^3{}_2\bar{M}_6 - 4{}^3\bar{M}_5 + \\ &\quad + 5{}^3{}_2\bar{M}_4 + 9{}^2{}_2\bar{M}_3 + 5{}^2\bar{M}_2 + \bar{M}_0) \mathbf{u} \end{aligned} \right\} (18)$$

All expansions remain valid for special affinors but they are then no longer the only possible ones.

*The numbersystem of the class operators.*

FROBENIUS has shown, that the operators  $K$  are commutative both mutually and with every permutation  $P_\lambda$  and that they form a numbersystem with  $k$  units, which does not contain any nilpotent numbers viz. numbers of which some power is equal to zero. As  $k$  independent units we may choose e.g.  $K_1, \dots, K_k$  or  $\bar{A}_0, \bar{A}_2, \dots, \bar{A}_k$  or  $\bar{M}_0, \bar{M}_2, \dots, \bar{M}_k$ , which are thus all commutative both mutually and with each operator  $P, A$  or  $M$ . According to the theory of the higher complex number systems every system of this kind contains  $k$  independent numbers  $I_i, i = 1, \dots, k$ , the *idempotent principal units* (Haupteinheiten)<sup>1)</sup>, which satisfy the equations

$$I_i I_j = \begin{cases} I_i & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \dots \dots \dots (19)$$

The sum of the operators  $I_i$ , which will be called *elementary operators*, is the identical operator  $I$ . When these units are expressed in the class operators  $K$ :

$$I_i = \sum_j^{1, \dots, k} \mu_{ij} K_j \dots \dots \dots (20)$$

and when we write

$$K_i K_j = \alpha_{ijl} K_l, \dots \dots \dots (21)$$

so we have

$$\mu_{ml} = \sum_{i,j}^{1, \dots, k} \mu_{mi} \mu_{mj} \alpha_{ijl}, \dots \dots \dots (22)$$

from which we see that the coefficients  $\mu$  correspond with the group characters  $\chi$  of the symmetrical group as defined by FROBENIUS<sup>2)</sup>.

<sup>1)</sup> See e.g. the author's "Zur Klassifizierung der associativen Zahlensysteme" Math. Ann. 76 (14) 1—66.

<sup>2)</sup> Berl. Ber. (96) 985—1021, (98) 505—515, (99) 330—339, (00) 516—534; see also W. BURNSIDE, Lond. M. S. Proc. 33 (01) 146—162, 34 (02) 41—48, 1 (03) 117—123).

The general formula for this correspondence is :

$$\mu_{ml} = \chi_0^{(m)} \chi_{l-1}^{(m)} \dots \dots \dots (23)$$

from which follows for  $\mu_{m1}$  :

$$\mu_{m1} = \chi_0^{(m)^2} \dots \dots \dots (24)$$

From the tables given by FROBENIUS for the group characters we thus find directly the equations expressing the  $I$  in the  $K$  e.g. for  $p = 6$  :

	$K_1$	$K_2$	$K_3$	$K_4$	$K_5$	$K_6$	$K_7$	$K_8$	$K_9$	$K_{10}$	$K_{11}$
${}^{11}_1 I_1$	1	1	1	1	1	1	1	1	1	1	1
${}^1_{11} I_2$	1	-1	1	-1	1	-1	1	-1	1	1	-1
${}^{10}_2 I_3$	25	15	5	-5	10	0	-5	5	-5	0	-5
${}^2_{10} I_4$	25	-15	5	5	10	0	-5	-5	-5	0	5
${}^4_7 I_5$	25	-5	5	15	-5	-5	10	5	-5	0	0
${}^7_4 I_6$	25	5	5	-15	-5	5	10	-5	-5	0	0
${}^9_3 I_7$	81	27	9	27	0	0	0	-9	9	-9	0
${}^3_9 I_8$	81	-27	9	-27	0	0	0	9	9	-9	0
${}^8_5 I_9$	100	20	-20	-20	10	-10	10	0	0	0	10
${}^5_8 I_{10}$	100	-20	-20	20	10	10	10	0	0	0	-10
${}^6_6 I_{11}$	256	0	0	0	-32	0	-32	0	0	16	0

\*) The indices on the left of the operators  $I$  will be explained furtheron.

Thus we have found an expansion that is singly determined for every affinor. In fact even for a special affinor no linear relation can exist between the operators  $I$  without each coefficient separately being equal to zero, as is seen immediately by application of one of the operators  $I$ . This expansion is called the expansion in elementary affinors. Using tables as (10) and (25) we might now express the  $I$  in  $\overline{A}$ ,  $\overline{M}$ ,  $\overline{A}^m$  or  $\overline{M}^a$  and obtain thus a singly determined expansion with respect to alternations or mixings. The following way however is more simple and more instructive.

*The elementary operators as the products of two general operators.*

When  $\overline{M}_{j_1}, \dots, \overline{M}_{j_k}$  are conjugate to  $\overline{A}_0, \overline{A}_2, \dots, \overline{A}_k$ , then  $\overline{A}_2, \dots, \overline{A}_k$  are annihilated by  $\overline{M}_{j_1}$ ; therefore they cannot contain  $y_1 \geq 1$  of the principal units  $I$ . In the same way  $\overline{A}_3, \dots, \overline{A}_k$  are annihilated by  $\overline{M}_{j_1}$  and  $\overline{M}_{j_2}$ , so that they cannot contain  $y_2 \geq 1$  further principal

units etc. We thus see, that  $\overline{A}_k$  contains only  $k - \sum_{1, \dots, k-1} y_l, y_l \geq 1$  principal units, from which follows  $y_1 = y_2 = \dots = y_k = 1$ . Therefore the principal units may be arranged in such way

$${}^1I, \dots, {}^kI,$$

that

$$\overline{A} = \sum_l^{i, \dots, k} \frac{1}{d_{il}} {}^lI, \quad i > 1. \quad \dots \quad (26)$$

The same reasoning holding for the  $\overline{M}$  the principal units may be arranged in such a way

$${}^1I, \dots, {}^kI,$$

that

$$\overline{M}_i = \sum_l^{i, \dots, k} \frac{1}{d_{il}} {}^lI, \quad i > 1. \quad \dots \quad (27)$$

The coefficients of both expansions are equal, but for  $p > 5$  the operators  $I$  do not have both times the same order, e.g. for  $p = 6$ , using both indices:

$${}^{11}_1I, {}^{10}_2I, {}^9_3I, {}^7_4I, {}^8_5I, {}^6_6I, {}^4_7I, {}^5_8I, {}^3_9I, {}^2_{10}I, {}^1_{11}I \dots$$

Furtheron we shall no longer use for the  $I$  the indices on the right of p. 257.

From (26), (27) and (7) we find the relation:

$${}^jI = d_{ij}^2 \overline{A}_i \overline{M}_j = d_{ij}^2 \overline{A}_i \overline{M}_j^{\frac{m}{\alpha}} \quad \dots \quad (28)$$

As it is easily proved, that  $\overline{A}_i \overline{M}_j$  contains  $K_1$  just once, the coefficients  $d_{ij}^2$  are identical with the coefficient  $\mu_{il}$  in (20) and the  $d_{ij}$  therefore equal to the group characters in the first row of FROBENIUS. We need only know therefore this first row. For the case that  $\overline{A}_i = {}_{\alpha} \overline{A}_i$  we have

$$d_{ij} = \binom{p-1}{s-1} \quad \dots \quad (29)$$

and for  $\overline{A}_i = {}_{\alpha, 2} \overline{A}_i$ :

$$d_{ij} = \frac{\binom{p}{2\alpha} \binom{2\alpha}{\alpha}}{\binom{p-\alpha+1}{\alpha}} \quad \dots \quad (30)$$

For more general formulae the cited papers of FROBENIUS and BURNSIDE may be referred to.

So we have obtained this theorem:

*Principal theorem B. Every affinor of the  $p^{\text{th}}$  degree can be expanded*

in one and only one way in a series of  $k$  elementary affinors. An elementary affinor with the lower index  $i$  and the upper  $j$  is a quantity that is annihilated by all alternations  $A^l, l > i, l > 1$  and by all mixings  $M_h, h > j, h > 1$ . Such a quantity is also annihilated by all  $\bar{A}_i$  and  $\bar{M}_h$ ; it can arise from the application of  ${}^i l, \bar{A}_i, \bar{A}_i, \bar{M}_j$  and  $\bar{M}_j$  and it is invariant by application of  ${}^i I, \sigma_{ij}, \bar{A}_i, \sigma_{ij}, \bar{M}_j$  and  $\sigma_{ij}, \bar{M}_j$ . For a definite value of  $n$  (number of fundamental elements  $e$ ) all elementary affinors, for which the permutation number of the  $A$  contains permutation regions  $> n$ , are zero.

For  $p = 6$  e.g. the expansion is:

$$\mathbf{u} = \left( \begin{aligned} & (\bar{A}_6 \bar{M}_{11} + 25 \bar{A}_2 \bar{M}_{10} + 81 \bar{A}_3 \bar{M}_9 + 25 \bar{A}_4 \bar{M}_7 + 100 \bar{A}_5 \bar{M}_8 + 256 \bar{A}_6 \bar{M}_6 + \\ & + 25 \bar{A}_7 \bar{M}_4 + 100 \bar{A}_8 \bar{M}_5 + 81 \bar{A}_9 \bar{M}_3 + 25 \bar{A}_{10} \bar{M}_2 + \bar{A}_{11} \bar{M}_0) \end{aligned} \right) \quad (31)$$

Expansion of an elementary affinor in ordered elementary affinors of the first or of the second kind.

*Theorem II.* Every simple alternation or mixing that is annihilated by all higher ones with the same number of permutation regions of more than one factor, can be written as the sum of multiples of ordered alternations or mixings with the same permutation number that contain in each of their regions only factors from the corresponding regions of the original alternation or mixing.

Let us prove this first for an affinor:

$$\mathbf{m} = {}_{p,q} A \mathbf{u} = {}_p \mathbf{u} \circ {}_q \mathbf{v} = (\mathbf{u}_1 \dots \mathbf{u}_p) \circ (\mathbf{v}_1 \dots \mathbf{v}_q)$$

When the first factor of  $\mathbf{m}$  is no  $\mathbf{u}$ , we may apply a  ${}_{p+1} A$ , the permutation region of which contains this factor and  ${}_p \mathbf{u}$ . Then we may write  ${}_{p+1} A \mathbf{m}$  as the sum of  $p + 1$  terms, the first factor of only one of which (namely of  $\mathbf{m}$  itself) is a  $\mathbf{v}$ . Therefore  $\mathbf{m}$  may be written as a sum of terms the first factor of which is  $\mathbf{u}$ . When now in one of those terms  $\mathbf{m}_1$ , the second factor  $\mathbf{v}$  precedes the second factor  $\mathbf{u}$ , then we may apply a  ${}_{p+1} A$ , the permutation region of which contains the two first  $\mathbf{v}$  and the  $p - 1$  last  $\mathbf{u}$ . Then  ${}_{p+1} A \mathbf{m}_1$  may be written as a sum of  $p + 1$  terms in only one of which (namely in  $\mathbf{m}_1$  itself) the second and third factors are  $\mathbf{v}$  and which

have only factors of  ${}_p\mathbf{u}$  in the region  $p$ . Therefore  $\overset{P}{\mathbf{m}}_1$  may be written as a sum of terms which are all ordered with respect to the first and second factors. Proceeding in this way we obtain a sum of perfectly ordered terms. In each of these terms the region  $p$  contains only factors  $\mathbf{u}$  and the other region only factors  $\mathbf{v}$ .

Now we consider the affiner  $\overset{P}{\mathbf{m}} = {}_p\mathbf{u} \overset{d}{\circ} {}_q\mathbf{v} \overset{d}{\circ} {}_r\mathbf{w} \overset{d}{\circ} \dots$  with  $Q$  alternating regions of  $p, q, r$  etc. factors,  $p \geq q \geq r \geq \dots$ , which may be annihilated by every alternation with  $Q$  alternating regions higher than  ${}_{p,q,r,\dots}A$ . When e.g. the first factor of  $\overset{P}{\mathbf{m}}$  is  $\mathbf{w}$ , then we apply a  ${}_{p+1}A$ , the permutation region of which contains  $\mathbf{w}$  and  ${}_p\mathbf{u}$ . As this  ${}_{p+1}A$  annihilates  $\overset{P}{\mathbf{m}}$ ,  $\overset{P}{\mathbf{m}}$  is a sum of terms all beginning with a  $\mathbf{u}$ . Let now the second factor most to the left be e.g. a  $\mathbf{v}$ , then we apply a  ${}_{p+1}A$ , the permutation region of which contains the two first  $\mathbf{v}$  and the  $p-1$  last  $\mathbf{u}$ . Proceeding in this way we obtain a sum of terms which are all ordered as to the position of the  $\mathbf{u}$  with respect to the remaining factors. Now we continue with alternations  ${}_{p,q+1}A$ , the region  $p$  of which always contains the  $\mathbf{u}$  that are already ordered; thus we obtain ordering of the  $\mathbf{v}$  etc. until perfect ordering is reached. At each passage and therefore also in the final result the factors  $\mathbf{u}$  of all terms remain in the region  $p$ , the  $\mathbf{v}$  in the region  $q$  etc.

We shall apply the theorem just proved to the elementary affiner  ${}^jI \overset{P}{\mathbf{u}}$  and we shall prove that the result is singly determined and identical with

$${}^jI \overset{P}{\mathbf{u}} = \left( \sum_{\lambda}^{1, \dots, \delta_{ij}} \epsilon_{ij} A_i^{(\lambda)} M_j^{(\lambda)} \right) \overset{P}{\mathbf{u}}, \dots \dots \dots (32)$$

the summation being extended over all *ordered* alternations and mixings, the number of which is just  $\delta_{ij}$ . For this purpose we use the wellknown property that the numbersystem of the permutations  $P$  is an associative system, which may be resolved into  $k$  "original" systems with  $\delta_{ij}^2$  units. The units of such an original system may be chosen in such a way that

$$J_{pq} J_{rs} = \begin{cases} J_{ps} \text{ for } q=r \\ 0 \quad \text{,, } q \neq r. \end{cases} \dots \dots \dots (33)$$

Such a system contains at the most  $\delta_{ij}$  idempotent principal units, the sum of which is the modulus of the system  ${}^jI$ . Let now for a definite value of  $\lambda$  for any affiner  $\overset{P}{\mathbf{v}}$  be  $A_i^{(\lambda)} M_j^{(\lambda)} \overset{P}{\mathbf{v}} \neq 0$  where  $A_i^{(\lambda)}$

and  $M_j^{(\alpha)}$  are conjugate, then  $A_i^{(\alpha)} M_j^{(\alpha) \prime} \mathbf{v}$  is an elementary affinor and therefore according to theorem II we have :

$$A_i^{(\alpha)} M_j^{(\alpha) \prime} \mathbf{v} = A_i^{(\alpha)} M_j^{(\alpha)} \sum_{\beta} A_i^{(\beta) \prime} \mathbf{w}_{\beta} \dots \dots \dots (34)$$

where the  $\mathbf{w}_{\beta}^{\prime}$  are elementary affinors, while the summation has to be extended over all *ordered* alternations  $A_i^{(\beta)}$ . As  $M_j^{(\alpha)} A_i^{(\beta)} = 0$  for  $\alpha \neq \beta$  we have :

$$A_i^{(\alpha)} M_j^{(\alpha) \prime} \mathbf{v} = A_i^{(\alpha)} M_j^{(\alpha)} A_i^{(\alpha) \prime} \mathbf{w}_{\alpha} \dots \dots \dots (35)$$

Repeated application teaches thus that the operator  $A_i^{(\alpha)} M_j^{(\alpha)}$  never can be nilpotent. According to a wellknown axiom from the theory of the higher complex numbersystems we conclude from this, that there exists an idempotent number of the form

$$a_{\alpha_1} (A_i^{(\alpha)} M_j^{(\alpha)}) + \dots + a_{\alpha_{\mu}} (A_i^{(\alpha)} M_j^{(\alpha)})^{\mu} \dots \dots \dots (36)$$

For every ordered alternation there thus exists such an idempotent number and the products of these numbers being zero, they form a series of idempotent principal units. The number of ordered operators  $A_i$  and  $M$  is therefore  $\sigma_{ij}$  or less. It cannot be less, however, as a repeated application of theorem II teaches that every elementary affinor may be written in the form  $\sum_j A_i^{(\lambda)} M_j^{(\lambda) \prime} \mathbf{m}$ . If  $\mu$  were  $\neq 1$ , then the powers of  $A_i^{(\alpha)} M_j^{(\alpha)}$  would belong to the first class of PEIRCE with respect to one of the idempotent principal units and to the fourth class with respect to the other ones. As in an original number-system such numbers (nilpotent by-units (Nebeneinheiten)) cannot occur,  $\mu = 1$  and the operators

$${}_i^j I = \epsilon_{ij} A_j^{(\lambda)} M_j^{(\lambda)} \quad \lambda = 1, \dots, \sigma_{ij} \dots \dots (37)$$

are therefore idempotent principal units. In the same way

$${}_i^{\lambda} I = \epsilon'_{ij} M_j^{(\lambda)} A_i^{(\lambda)} \quad \lambda = 1, \dots, \sigma_{ij} \dots \dots (38)$$

form a similar system. These operators are called *ordered elementary operators of the first resp. second kind* and the affinors which may be derived from them, *ordered elementary affinors of the first resp. second kind*.

When  $\gamma_i$  is the number of permutations in  $A_i$  or  $M_i$  (hence for the permutation number  $s_1, \dots, s_t$  equal to  $s_1! \dots s_t!$ ) and when  $\beta_i$  is the number of operators  $A_i$  or  $M_i$  that do *not* annihilate a *definite*

operator  $M_j$  resp.  $A_j$  we may easily calculate that  $\frac{p!}{\gamma_i \gamma_j} = \frac{\alpha_i}{\beta_i} = \frac{\alpha_j}{\beta_j}$  is the coefficient of  $K_1$  in  $A_i^{(\alpha)} M_j^{(\alpha)}$  and  $M_j^{(\alpha)} A_i^{(\alpha)}$ . The coefficient of  $K_1$  in  $\overline{A_i} \overline{M_j}$  being one, this involves that

$$\epsilon_{ij} = \epsilon'_{ij} = \frac{d_{ij} \gamma_i \gamma_j}{p!} = \frac{d_{ij} \beta}{\alpha_i} = \frac{d_{ij} \beta_j}{\alpha_j} \dots \dots \dots (39)$$

Thus the expansion with regard to elementary affinars of the first kind is e.g. for  $p = 6$ :

$$\begin{aligned} \mathbf{u} = & (A_6 M_{11} + \frac{5}{3} \sum_{\lambda}^{1, \dots, 5} A_2^{(\lambda)} M_{10}^{(\lambda)} + \frac{1}{5} \sum_{\lambda}^{1, \dots, 9} A_3^{(\lambda)} M_9^{(\lambda)} + 2 \sum_{\lambda}^{1, \dots, 5} A_4^{(\lambda)} M_7^{(\lambda)} + \\ & + 2 \sum_{\lambda}^{1, \dots, 10} A_5^{(\lambda)} M_6^{(\lambda)} + \frac{1}{5} \sum_{\lambda}^{1, \dots, 16} A_6^{(\lambda)} M_6^{(\lambda)} + 2 \sum_{\lambda}^{1, \dots, 5} A_7^{(\lambda)} M_4^{(\lambda)} + \\ & + 2 \sum_{\lambda}^{1, \dots, 10} A_8^{(\lambda)} M_5^{(\lambda)} + \frac{1}{5} \sum_{\lambda}^{1, \dots, 9} A_9^{(\lambda)} M_3^{(\lambda)} + \frac{5}{3} \sum_{\lambda}^{1, \dots, 5} A_{10}^{(\lambda)} M_2^{(\lambda)} + A_{11} M_1) \mathbf{u} \end{aligned} \quad (40)$$

When an expansion of an affinor  $\mathbf{u}$  is given with regard to ordered alternations or mixings, each of which is alternated by every higher alternation resp. mixing, then this expansion is identical with the indicated one. For on application the operator  $\epsilon_{ij} A_i^{(\alpha)} M_j^{(\alpha)}$  resp.  $\epsilon_{ij} M_j^{(\alpha)} A_i^{(\alpha)}$  all terms are annihilated except those which are derived from  $A_i^{(\alpha)}$  resp.  $M_j^{(\alpha)}$ . This one term only remains unaltered and is therefore equal to  $\epsilon_{ij} A_i^{(\alpha)} M_j^{(\alpha)} \mathbf{u}$  resp.  $\epsilon_{ij} M_j^{(\alpha)} A_i^{(\alpha)} \mathbf{u}$ . The indicated expansion is therefore singly determined.

So we have obtained the following principal theorem.

*Principal theorem C. Every affinor can be written in one and only one way as a sum of ordered alternations or mixings that are annihilated by every alternation resp. mixing with a higher permutation number.*

*Expansion of an affinor with regard to reduceable covariants of different degree.*

When  $\overline{A_i}$  and  $\overline{M_j}$  are conjugate, then for  $n > 5$  it may be very well possible, that a general mixing lower than  $\overline{M_j}$  corresponds to a general alternation lower than  $\overline{A_i}$ . When however  $\overline{A_i} =_{\alpha, n} \overline{A_i}$ , then every general alternation lower than  $\overline{A_i}$  is of the form  $_{(\alpha-\beta).n, s_2, \dots, s_t} \overline{A_i}$ . The permutation number  $(\alpha + p - \alpha n), (n-1). \alpha$  is conjugate to  $\alpha, n$  and the number  $(\alpha - \beta + t + p - (\alpha - \beta)n - s_2 - \dots - s_t), (s_t - 1), (\alpha - \beta + t), \dots, (n - s_2), (\alpha - \beta)$  to  $(\alpha - \beta). n, s_2, \dots, s_t$ . The second number

is doubtlessly higher than the first, all  $s_2, \dots, s_n$  being  $< n$ . Therefore all general alternations lower than  ${}_{\alpha,n}\overline{A}_i$  are annihilated by all mixings with a permutation number higher than  $\overline{M}_j$ . When thus the whole number of times that  $p$  contains  $n$  is  $r$ , the operators  $jI$  can be arranged in  $r + 1$  sets. When the sum of operators in the  $(\alpha + 1)$ -th group is  ${}_{\alpha}I'$ , then  ${}_{\alpha}I' \mathbf{u}^p$  is a sum of quantities that may arise from the application of alternations  ${}_{\alpha,n}A$  and that are annihilated by every  ${}_{(\alpha+1),n}A$  and by all higher alternations. Each term of  ${}_{\alpha}I' \mathbf{u}^p$  is a penetrating general product of  $\alpha$  factors  $\mathbf{E}$ ,  $\mathbf{E} = \mathbf{e}_1 \widehat{\cap} \dots \widehat{\cap} \mathbf{e}_n$ , and a not-ideal affiner of degree  $p - \alpha n$ , which is a covariant of  $\mathbf{u}$  and is evidently annihilated by each operator  ${}_nA$ . Such an affiner will be called *non reduceable* under the *linear homogeneous* group and the indicated expansion the expansion with regard to *linear homogeneous non reduceable covariants*.

Now we shall prove that there exists only one expansion in non reduceable covariants. For this purpose it suffices to prove, that a penetrating general product  $\mathbf{r}$  of  $\mathbf{E}^\alpha$  and a non reduceable affiner  $\mathbf{v}$ ,  $q = p - \alpha n$ , is annihilated by all operators  $I'$  except by  ${}_{\alpha,n}I'$ . As  $\mathbf{r}$  can arise from the application of a  ${}_{\alpha,n}A$  and is therefore annihilated by every mixing with a permutation number higher than the one conjugate to  ${}_{\alpha,n}$ , this is evident for all  ${}_{\beta}I'$  for which  $\beta < \alpha$ . When  $\beta > \alpha$  we may remark that  ${}_{\beta}I'$  is a sum of multiples of operators  $\overline{M}_j \overline{A}_i$ , in which the alternations always have more than  $\alpha$  permutation regions with  $n$  factors. The desired proof will thus be given, when we have shown that  $\mathbf{r}$  is annihilated by every operator  ${}_{\beta,n}A$ ,  $\beta > \alpha$ .

There to we make use of the theorem that a non reduceable quantity possesses no linear covariants of a lower degree <sup>1)</sup>. By means of an operator  ${}_{\beta,n}A$  we may derive from  $\mathbf{r}$  a penetrating general product of  $\mathbf{E}^\beta$  and an affiner  $\mathbf{w}$ ,  $r = p - \beta n$ . However,  $\mathbf{w}$  would then be a linear covariant of  $\mathbf{v}$  of degree  $r < q$ . This is impossible, so that  $\mathbf{w}$  is zero.

Every term of  ${}_{\alpha}I' \mathbf{u}^p$  is alternating in  $\alpha$  different regions of  $n$  factors and is annihilated by every alternation with an alternation

<sup>1)</sup> The proof of this theorem will be given separately in another paper.

region of more than  $\alpha$  factors. From theorem II we conclude therefore that  ${}_{\alpha}l' \mathbf{u}^p$  can be reduced to a sum of *ordered* alternations with the permutation number  $\alpha.n$ . This decomposition is singly determined, each of these alternations being a sum of the ordered alternations from the expansion according to  $C$  with permutation numbers from  $\alpha.n$  up to the highest number below  $(\alpha + 1).n$  that have the same  $\alpha$  alternation regions of  $n$  factors.

Thus we have obtained the theorem.

*Principal theorem D. Every affinor can be written in one and only one way as a sum of terms each consisting of penetrating general products of a number  $\alpha$ , characteristic for this term, of factors  $\mathbf{E}$ , with a linear homogeneous non reduceable affinor of degree  $p - \alpha.n$  and forming an ordered alternation with the permutation number  $\alpha.n$ . This expansion may be obtained by arranging in groups the terms of the expansion with regard to elementary affinors.*

$\mathbf{u}^p$  may therefore be written

$$\mathbf{u}^p = \sum_{\alpha}^{0, \dots, p} \sum_{\lambda}^{1, \dots, \delta_{ij}} {}_{\alpha.n}L_i^{(\lambda)} \dots \dots \dots (41)$$

where  ${}_{\alpha.n}L_i^{(\lambda)}$  is the sum of the ordered elementary operators of the first kind which have the same alternation regions with  $n$  factors.

For  $n=2$  the expansion in mixed alternations according to the principal theorem  $A$  is at the same time an expansion in non reduceable covariants. It is therefore singly determined for every affinor and containing  $k$  terms it is also identical with the expansion with regard to elementary affinors. From (16) we find therefore for this case

$${}_{\alpha+1}^{k-\alpha-1}I = \frac{\binom{p}{2\alpha} \binom{2\alpha}{\alpha} \frac{m}{\alpha.2A}}{\binom{p-\alpha+1}{\alpha}} \dots \dots \dots (42)$$

From the deduced expansions in series we may derive in a simple way very general expansions in series for algebraic forms in  $m$  rows of  $n$  variables as will be shown in the next paper. At the same time we will mention how the above is connected with known expansions in series of algebraic forms.

*Expansion of the affinor of RIEMANN-CHRISTOFFEL with regard to ordered elementary affinors.*

When the derived expansion in series of elementary affinars of the first kind is applied to the affinar of RIEMANN-CHRISTOFFEL

$\overset{4}{\mathbf{K}} = \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4$ , for which is known that

$$\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 = -\mathbf{k}_2 \mathbf{k}_1 \mathbf{k}_3 \mathbf{k}_4 = -\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_4 \mathbf{k}_3 = \mathbf{k}_2 \mathbf{k}_1 \mathbf{k}_4 \mathbf{k}_3$$

$$\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 + \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_1 \mathbf{k}_4 + \mathbf{k}_3 \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_4 = 0, \quad . \quad . \quad . \quad (43)$$

we see that from all nine ordered elementary operators of the first kind  $\epsilon_{3,3} A_3^{(1)} M_3^{(1)}$  is the only one that does not give zero. Therefore

$\overset{4}{\mathbf{K}}$  itself is an example of an ordered elementary affinar.

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(Communicated in the meeting of May 3, 1919).

*Notations.*

We start from the system  $S_n^{(2)}$  with the covariant and contravariant fundamental units  $e_\lambda$  resp.  $e'^\lambda$ ,  $\lambda = a_1, \dots, a_n$ , and the fundamental multiplications  $\wedge$  (outer multiplication)  $\circ$  (general multiplication)  $\frown$  (alternating multiplication and  $\smile$  (symmetrical multiplication).

$$e_\lambda \wedge e'^\mu = \begin{cases} \kappa = (-1)^{\frac{n(n-1)}{2}} & \text{for } \lambda = \mu \\ 0 & \text{,, } \lambda \neq \mu \end{cases}$$

$$\left. \begin{aligned} e_{\lambda_1} \wedge \dots \wedge e_{\lambda_i} &= e_{\lambda_1} \frown \dots \frown e_{\lambda_i} = \text{covariant quantity of degree } i \text{ (} i\text{-vector)} \\ e'^{\lambda_1} \wedge \dots \wedge e'^{\lambda_i} &= e'^{\lambda_1} \smile \dots \smile e'^{\lambda_i} = \text{contravariant ,, ,, ,, ,, ,,} \end{aligned} \right\} i < n$$

$$e_{a_1} \frown \dots \frown e_{a_n} = E = \text{covariant scalar; } e'^{a_1} \smile \dots \smile e'^{a_n} = E' = \text{contravariant scalar}$$

$$\left. \begin{aligned} \kappa^n e_{a_2} \dots e_{a_n} E' &= e'^{a_1} \\ \kappa^n e'^{a_2} \dots e'^{a_n} E &= e_{a_1} \end{aligned} \right\} \text{cycl.}$$

By  $i$ -th (procurrent) transvection of  $\mathbf{m} = \mathbf{m}_1 \dots \mathbf{m}_P = \mathbf{m}_1 \dots \mathbf{m}_P$  and  $\mathbf{r}' = \mathbf{r}'_1 \dots \mathbf{r}'_Q$  will be understood

$$\mathbf{m}_i \cdot \mathbf{r}' = (\mathbf{m}_1 \wedge \mathbf{r}'_1) \dots (\mathbf{m}_i \wedge \mathbf{r}'_i) \mathbf{m}_{i+1} \dots \mathbf{m}_P \mathbf{r}'_{i+1} \dots \mathbf{r}'_Q \text{ } ^3)$$

1) See also: “On expansions in series of co- and contravariant quantities of higher degree etc.”, These Proceedings Vol. XXII, p. 251—266, here further cited as  $E_I$ , of which paper this communication forms the continuation and an application.

2)  $S_n^{(2)}$  is found from  $R_n^0$  by omission of all quantities that exist only under the orthogonal group. See for these systems: Over de direkte analyses der lineaire grootheden bij de rotationeele groep enz., Versl. der Kon. Akad. v. Wet. Dl. XXVI, bldz. 567—580; Ueber die Zahlensysteme der rotationalen Gruppe, Nieuw Arch. voor Wisk. Dl. XIII, 1919; Die direkte Analysis der neueren Relativitätstheorie, Verh. der Kon. Akad. v. Wet. Dl. XII N<sup>o</sup>. 6 (1919) blz. 29.

3) The sign  $\cdot$  instead of  $(,)^i$  for the transvections of the theory of invariants was first introduced by E. WAELSCH.

We have therefore, when for the sake of simplicity, is written . for . :

$$\mathbf{m} \cdot \mathbf{r}' = \mathbf{m} \cdot \mathbf{r}' = \mathbf{m} \wedge \mathbf{r}' = \varkappa (m_{a_1} r'_{a_1} + \dots + m_{a_n} r'_{a_n})$$

By *i*-th outer transvection  $\diamond_i$  of *m* affinars  $\mathbf{u} = \mathbf{u}_1 \dots \mathbf{u}_p$ ,  $\mathbf{v} = \mathbf{v}_1 \dots \mathbf{v}_q$  etc. will be understood the quantity found from  $\mathbf{u} \mathbf{v}$  by substituting *locally* for the ideal vectors  $\mathbf{u}_1, \mathbf{v}_1, \dots$ , and then for  $\mathbf{u}_2, \mathbf{v}_2, \dots$ , to  $\mathbf{u}_i, \mathbf{v}_i, \dots$  the ideal factors of their alternating product. When at the same time the other factors are *locally* substituted by the ideal factors of their alternating resp. symmetrical product, then the *i*-th outer alternating transvection  $\wedge_i$  resp. the *i*-th outer symmetrical one  $\vee_i$  is formed.

*Affinars and algebraic forms.* When the *P*-th transvection of  $\mathbf{m}$  is formed with a product  $\mathbf{r}'$  of *P* different contravariant fundamental elements  $\mathbf{r}'_1, \dots, \mathbf{r}'_P$ , we find the form :

$$\begin{aligned} F \mathbf{m} &= \mathbf{m} \cdot \mathbf{r}'_1 \dots \mathbf{r}'_P = (\mathbf{m}_1 \cdot \mathbf{r}'_1) \dots (\mathbf{m}_P \cdot \mathbf{r}'_P) = \\ &= \varkappa^P \sum_{\lambda_1 \dots \lambda_P} m_{\lambda_1 \dots \lambda_P} r'^{\lambda_1} \dots r'^{\lambda_P}. \end{aligned}$$

A special case is that  $\mathbf{r}'_1, \dots, \mathbf{r}'_P$  are all not-ideal. Then  $F \mathbf{m}$  is a form in *P* sets of *n* not-ideal variables. Thus the characteristic numbers of a covariant affinor (and therefore also of a contravariant one) may always be considered as the coefficients of such an algebraic form. When the sets  $\mathbf{r}'_1, \dots, \mathbf{r}'_P$  are given and when their order is fixed,  $F \mathbf{m}$  is singly determined by  $\mathbf{m}$ . When all sets are different, then  $\mathbf{m}$  is also singly determined by  $F \mathbf{m}$ , in the other case not, as  $\mathbf{m} + \mathbf{n}$ , where  $\mathbf{n}$  is an arbitrary affinor, alternating in two factors that correspond with two equal sets of variables, transvected with  $\mathbf{r}'$ , also forms  $F \mathbf{m}$ .

In the general case  $\mathbf{r}'_1, \dots, \mathbf{r}'_P$  are ideal,  $\mathbf{r}'$  is however equal to  $\mathbf{x}'_p \mathbf{y}'_q \dots$ , where  $p q + \dots = P$ , and where  $\mathbf{x}', \mathbf{y}', \dots$  are not-ideal. Then  $F \mathbf{m}$  is a form of the degrees  $q, \sigma, \dots$  in sets of variables that may be considered themselves as coefficients of variable forms in *p, q, \dots* sets of *n* covariant variables. Such variables will be called *variables of the degrees p, q, \dots* When the sets  $\mathbf{x}', \mathbf{y}', \dots$  and the order

of their ideal factors are given, then  $F_{\mathbf{m}}^P$  is singly determined by  $\mathbf{m}$ . For the sake of simplicity we shall choose this order in such way that  $\mathbf{r}' = \mathbf{x}'^p \mathbf{y}'^q \dots$ . In order to be able to determine also singly  $F_{\mathbf{m}}^P$ , we shall first prove the following theorem:

*Principal theorem A. Every algebraic form of the total degree  $P$ , homogeneous and of the degrees  $q, \sigma, \dots$ , in different sets of variables  $\mathbf{x}'^p, \mathbf{y}'^q, \dots$  each of which may be regarded as coefficients of variable forms  $F_{\mathbf{x}'}^p, F_{\mathbf{y}'}^q, \dots$ , linear and of the degrees  $p, q, \dots$  in sets of  $n$  different covariant variables, can be written as a product of  $P$  ideal linear forms. When for the sets of variables  $\mathbf{x}', \mathbf{y}' \dots$  is prescribed, that  $\mathbf{x}', \mathbf{y}', \dots$  are separately annihilated by definitely indicated ordered elementary operators either of the first or of the second kind <sup>1)</sup>, as for the rest the variables being able to obtain all values, then one single definite affinor of degree  $P$  belongs to the given form for a definite choice of the order of the sets.*

When the characteristic numbers of the sets are  $x'_1, \dots, x'_\alpha; y'_1, \dots, y'_\beta; \dots$ ,  $\alpha$  being  $n^p$ ,  $\beta$  being  $n^q$  etc. then every term of  $F$  has the shape:

$$n_{\rho_1 \dots \rho_\alpha, \sigma_1 \dots \sigma_\alpha}^P \dots x'_1{}^{\rho_1} \dots x'_\alpha{}^{\rho_\alpha} y'_1{}^{\sigma_1} \dots y'_\beta{}^{\sigma_\beta}.$$

$$\begin{aligned} \rho_1 + \dots + \rho_\alpha &= p \\ \sigma_1 + \dots + \sigma_\beta &= q \\ &\vdots \end{aligned}$$

When

$$\mathbf{e}'_1, \dots, \mathbf{e}'_\alpha; \mathbf{e}'_1, \dots, \mathbf{e}'_\beta; \dots$$

are the products of  $p, q, \dots$  of the fundamental units belonging to the characteristic numbers of  $\mathbf{x}', \mathbf{y}', \dots$  and

$$\mathbf{e}_1, \dots, \mathbf{e}_\alpha; \mathbf{e}_1, \dots, \mathbf{e}_\beta; \dots$$

the products formed in the same way from  $\mathbf{e}'_{a_1}, \dots, \mathbf{e}'_{a_n}$ , then the affinor

$$\mathbf{n} = \mathbf{z}^P \sum n_{\rho_1 \dots \rho_\alpha, \sigma_1 \dots \sigma_\alpha}^P \dots \mathbf{e}_1{}^{\rho_1} \dots \mathbf{e}_\alpha{}^{\rho_\alpha} \mathbf{e}_1{}^{\sigma_1} \dots \mathbf{e}_\beta{}^{\sigma_\beta} \dots$$

may be formed.

<sup>1)</sup> See  $E_I$  p. 262.

As the transvection

$$\begin{aligned} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_P} \dot{\mathbf{e}}'_{j_1} \dots \mathbf{e}'_{j_P} & \quad i_1, \dots, i_P = a_1, \dots, a_n \\ & \quad j_1, \dots, j_P = a_1, \dots, a_n \end{aligned}$$

is equal to  $\kappa^P$  for the case that  $i_e = j_e, e = 1, \dots, P$  and equal to zero in every other case, we have

$$F' = \underset{P}{\mathbf{n}} \cdot \overset{P}{\mathbf{x}'} \overset{q}{\mathbf{y}'} \dots$$

When now on one hand  $\mathbf{n}$  and on the other hand  $\overset{P}{\mathbf{x}'}, \overset{q}{\mathbf{y}'}$  are written as the products of ideal fundamental elements, then  $F'$  is really reduced to a product of  $P$  ideal linear forms. In order to derive from  $\underset{P}{\mathbf{n}}$  the affinor  $\overset{P}{\mathbf{m}}$  that is singly determined by  $F'$  we first prove the theorem:

*Theorem 1.*

When  $\overset{P}{\mathbf{q}}$  is an ordered elementary affinor of the first (second) kind and  $\overset{P}{\mathbf{r}'}$  a ditto one of the second (first) kind<sup>1)</sup>, then the  $P$ -th transvection of them is zero, when the two elementary operators  $\varepsilon_{ij} A_i^{(\alpha)} M_j^{(\alpha)}, \varepsilon_{lm} M_l^{(\beta)} A_m^{(\beta)}$ , by means of which  $\overset{P}{\mathbf{q}}$  and  $\overset{P}{\mathbf{r}'}$  ( $\overset{P}{\mathbf{r}'}$  and  $\overset{P}{\mathbf{q}}$ ) can arise, are not conjugated, i.e. when not  $l = j, m = i$  and  $\alpha = \beta$ .

As:

$$\overset{P}{\mathbf{q}} = \varepsilon_{ij} A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}}; \quad \overset{P}{\mathbf{r}'} = \varepsilon_{lm} M_l^{(\beta)} A_m^{(\beta)} \overset{P}{\mathbf{r}'},$$

we have:

$$\begin{aligned} \overset{P}{\mathbf{q}} \dot{\overset{P}{\mathbf{r}'}} &= \varepsilon_{ij} \varepsilon_{lm} \left( A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}} \right) \dot{\overset{P}{\mathbf{r}'}} = \\ &= \varepsilon_{ij} \varepsilon_{lm} \left( M_l^{(\beta)} A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}} \right) \dot{\overset{P}{\mathbf{r}'}} = \\ &= \varepsilon_{ij} \varepsilon_{lm} \left( A_m^{(\beta)} M_l^{(\beta)} A_i^{(\alpha)} M_j^{(\alpha)} \overset{P}{\mathbf{q}} \right) \dot{\overset{P}{\mathbf{r}'}}. \end{aligned}$$

Thus the transvection is in fact zero, when not  $l = j$  (therefore also  $m = i$ ) and  $\alpha = \beta$ . The same proof holds m.m. by changing the first into the second kind and vice-versa.

Let now the sums of the ordered elementary operators of the first kind that do not annihilate  $\overset{p}{\mathbf{x}'}, \overset{q}{\mathbf{y}'}, \dots$ , be  ${}_x L, {}_y L, \dots$  and the sums of the conjugate operators  ${}_x L, {}_y L, \dots$  etc. In the special case that  ${}_x L$  is a sum of elementary operators we evidently have  ${}_x L = {}_x L$ . From  $\overset{P}{\mathbf{n}}$  we first form an affinor  $\overset{P}{\mathbf{n}}_1$ , by permutating the

<sup>1)</sup> See *E<sub>I</sub>* p. 262.

$\varrho$  regions of  $p$  factors corresponding with  $\mathbf{x}'^p$  and also the  $\sigma$  regions of  $q$  factors, corresponding with  $\mathbf{y}'^\sigma$  etc. in all possible  $\varrho! \sigma! \dots$  ways, adding them and finally dividing by  $\varrho! \sigma! \dots$ . Then  $\mathbf{n}_1$  may be written

$$\mathbf{n}_1 = \mathbf{n}_x^p \mathbf{n}_y^q \dots,$$

where in the way known from symbolism of invariants  $\varrho, \sigma, \dots$  different equivalent quantities  $\mathbf{n}_x^p, \mathbf{n}_y^q \dots$  are to be introduced in order to avoid ambiguities<sup>1)</sup>. Then the given form is also obtained by

$$F = \mathbf{n}_1 \cdot \mathbf{r}' = \left( \mathbf{n}_x \cdot \mathbf{x}' \right)^p \left( \mathbf{n}_y \cdot \mathbf{y}' \right)^q \dots$$

According to theorem II we have now:

$$\begin{aligned} F &= (\mathbf{n}_x \cdot xL \mathbf{x}')^p (\mathbf{n}_y \cdot yL \mathbf{y}')^q \dots = \\ &= (xL \mathbf{n}_x \cdot \mathbf{x}')^p (yL \mathbf{n}_y \cdot \mathbf{y}')^q \dots \end{aligned}$$

When we write  $xL \mathbf{n}_x = \mathbf{u}, yL \mathbf{n}_y = \mathbf{v}$ , etc. and  $\mathbf{m} = \mathbf{u} \mathbf{v} \dots$ , then we have

$$F = \mathbf{m} \cdot \mathbf{r}' = (\mathbf{u} \cdot \mathbf{x}')^p (\mathbf{v} \cdot \mathbf{y}')^q \dots$$

$\mathbf{m}$  is the only affinor of this shape that when transvected with  $\mathbf{r}'$  gives  $F$ . In fact every affinor  $\mathbf{m}_1$  that can be written in the form

$$\mathbf{m}_1 = (xL \mathbf{m}_x) (yL \mathbf{m}_y) \dots$$

and gives zero when transvected with  $\mathbf{r}'$  is identically zero. In fact, as we have supposed that  $\mathbf{x}'$  may obtain all values that are solutions of the equation  $xL \mathbf{x}' = \mathbf{u}$ ; we thus may take for  $\mathbf{x}'$

$$\mathbf{x}' = xL \mathbf{s}'_1 \dots \mathbf{s}'_p,$$

where  $\mathbf{s}'_1 \dots, \mathbf{s}'_p$  are not-ideal *different* sets of variables and the same holds for  $\mathbf{y}'$ , etc. Then we have:

$$\begin{aligned} 0 = \mathbf{m}_1 \cdot \mathbf{r}' &= \{ (xL \mathbf{m}_x) \cdot xL \mathbf{s}'_1 \dots \mathbf{s}'_p \}^p \{ (yL \mathbf{m}_y) \cdot yL \mathbf{t}'_1 \dots \mathbf{t}'_q \}^q \dots \\ &= \{ (xL \mathbf{m}_x) \cdot \mathbf{s}'_1 \dots \mathbf{s}'_p \}^p \{ (yL \mathbf{m}_y) \cdot \mathbf{t}'_1 \dots \mathbf{t}'_q \}^q \\ &= \mathbf{m}_1 \cdot (\mathbf{s}'_1 \dots \mathbf{s}'_p)^p (\mathbf{t}'_1 \dots \mathbf{t}'_q)^q \dots \end{aligned}$$

<sup>1)</sup> Comp. Die direkte Analysis zur neueren Relativitätstheorie, p. 11, 17.

When now for  $\mathbf{s}'$ ,  $\mathbf{t}'$ , . . . are substituted here all possible combinations of  $\mathbf{e}'_{a_1}, \dots, \mathbf{e}'_{a_n}$ , we find that each characteristic number of  $\mathbf{m}^P$  is zero.

Of the proved property we make use by calling an algebraic form *non special, alternating, symmetrical, locally alternating, symmetrical* or *permutable*, an *elementary form* or an *ordered elementary form of the first or second kind*, when corresponding names are used for the corresponding affiner. By application of an operator  $K, A, M, \overline{A}, \overline{M}, \overline{A}, \overline{M}, \underset{j}{k}I, \underset{zj}{k}I$ , or  $\underset{j}{zk}I$  will be understood application of that operator to the corresponding affiner of that form. By means of the corresponding affiner we are now able to reduce a great part of the properties of forms to the formal properties of the operators  $K, A$ , etc., treated in  $E_I$ , which simplifies the treatment of forms considerably.

The characteristic numbers occurring in the linear factors are ideal identical with the symbols of ARONHOLD and CLEBSCH. When one of the sets e.g.  $\mathbf{x}'^p$  is symmetrical, then  $\mathbf{u}^p$  is also symmetrical and both may be written as the  $p$ -th power of an ideal fundamental element:

$${}^p\mathbf{u} \cdot {}^p\mathbf{x}' = \mathbf{u}^p \cdot \mathbf{x}'^p = (\mathbf{u} \cdot \mathbf{x}')^p.$$

Also in this case the occurring characteristic numbers are symbols of ARONHOLD and CLEBSCH. When  $\mathbf{x}$  is alternating, then  $\mathbf{u}^p$  too is alternating and also in this case both may be written as  $p$ -th powers:

$${}^p\mathbf{u} \cdot {}^p\mathbf{x}' = \mathbf{u}^p \cdot \mathbf{x}'^p = (\mathbf{u} \cdot \mathbf{x}')^p.$$

In this latter case the occurring characteristic numbers are identical with the complex symbols introduced by WAELSCH and WEITZENBÖCK, the multiplication of which is anticommutative. When  $\mathbf{x}'^p$  and therefore  $\mathbf{u}^p$  too is more general, then the notation in the form of powers may be still useful sometimes. Then however, the ideal roots  $\mathbf{x}'$  and  $\mathbf{u}$  do not determine any longer the isomers of  $\mathbf{x}'^p$  and  $\mathbf{u}^p$ . Both characteristic numbers are ideal numbers of complicated character in the products of which no commutation whatever is allowed any longer. By means of complex symbols WEITZENBÖCK<sup>1)</sup> has proved the first part of the principal theorem  $A$  for forms in sets of variables that are all alternating. The above proof is an extension to forms with sets of variables of more general character.

<sup>1)</sup> Beweis des ersten Fundamentalsatzes der symbolischen Methode. Sitzungsber. der Wiener Akad. 122 (13) 153—168, p. 155 etc.

*Polar operators.* Let be  $p = q \dots = 1$ . As

$$\left( y' \cdot \frac{d}{dx'} \right) \mathbf{x}' = z y',$$

we have

$$\left( y' \cdot \frac{d}{dx'} \right) \mathbf{x}'^\rho y'^\sigma \dots = z \varrho (\mathbf{x}'^{\rho-1} - y') y'^\sigma \dots$$

and

$$\begin{aligned} z^i \frac{(\varrho-i)!}{\varrho!} \left( y' \cdot \frac{d}{dx'} \right)^i \mathbf{m}_P \cdot \mathbf{x}'^\rho y'^\sigma \dots &= \mathbf{m}_P^P \cdot (\mathbf{x}'^{\rho-i} - y'^i) y'^\sigma = \\ &= \mathbf{m}_P^P \cdot \mathbf{x}'^{\rho-i} y'^{\sigma+i} \dots \end{aligned}$$

$z^i \frac{\varrho-i!}{\varrho!} \left( y' \cdot \frac{d}{dx'} \right)^i$  is therefore the  $i$ -th *polar operator* of  $\mathbf{x}'$  with

respect to  $\mathbf{y}'$ . By application of this operator the form  $F^P \mathbf{m}$  changes into a form with the sets of variables  $\mathbf{x}'^{\rho-i}, \mathbf{y}'^{\sigma+i}$ . The corresponding affiner of this form is no longer  $\mathbf{m}_P^P$ , but is derived from  $\mathbf{m}_P^P$  by application of an operator  $\sigma+iM_P$ , the permutation region of which contains the ideal factors of  $\mathbf{m}_P^P$  corresponding to  $\mathbf{y}'^{\sigma+i}$ . Application of this operator is therefore equivalent with application of  $\sigma+iM$  combined with a change of the sets of variables.

*CAPELLI'S operators*  $H^{(s)}$ . Let again be  $p = q = \dots = 1$  and let us call the sets of variables  $F^P \mathbf{m} \mathbf{x}'_1, \dots, \mathbf{x}'_m$  and the corresponding exponents  $\varrho_1, \dots, \varrho_m$ , so that  $\varrho_1 + \dots + \varrho_m = P$ , then the differential operator  $H^{(s)}$  introduced by CAPELLI is written in our notation:

$$H^{(s)} = z^s s! \sum (\mathbf{x}'_{i_1} \dots \mathbf{x}'_{i_s})_s \cdot \left( \frac{\partial}{\partial \mathbf{x}'_{i_1}} \dots \frac{\partial}{\partial \mathbf{x}'_{i_s}} \right),$$

where the summation has to be extended over all  $\binom{m}{s}$  combinations of  $s$  of the numbers  $1, \dots, m$ . By application of  $H^{(s)}$  to  $F^P \mathbf{m}$  we find:

$$\begin{aligned} H^{(s)} \mathbf{m}_P^P \cdot \mathbf{r}' &= H^{(s)} (\mathbf{u}_1 \cdot \mathbf{x}'_1)^{\varrho_1} \dots (\mathbf{u}_m \cdot \mathbf{x}'_m)^{\varrho_m} = z^s s! \sum (\mathbf{x}'_{i_1} \dots \mathbf{x}'_{i_s})_s \cdot \\ &\cdot \left\{ \frac{\partial}{\partial \mathbf{x}'_{i_1}} (\mathbf{u}_{i_1} \cdot \mathbf{x}'_{i_1})^{\varrho_{i_1}} \dots \frac{\partial}{\partial \mathbf{x}'_{i_s}} (\mathbf{u}_{i_s} \cdot \mathbf{x}'_{i_s})^{\varrho_{i_s}} \right\} (\mathbf{u}_{j_1} \cdot \mathbf{x}'_{j_1})^{\varrho_{j_1}} \dots (\mathbf{u}_{j_{m-s}} \cdot \mathbf{x}'_{j_{m-s}})^{\varrho_{j_{m-s}}} \end{aligned}$$

where  $j_1, \dots, j_{m-s}$  are the indices of  $1, \dots, m$  that do not belong to  $i_1, \dots, i_s$ . The summation has to be extended over all  $\binom{m}{s}$  possible combinations.

As

$$\frac{\partial}{\partial \mathbf{x}_i'} (\mathbf{u}_i \cdot \mathbf{x}_i') = \left( \frac{\partial}{\partial \mathbf{x}_i'} \mathbf{x}_i' \right)_1 \cdot \mathbf{u}_i = \mathbf{u}_i,$$

we have

$$H_s F \mathbf{m} = s! \varrho_{i_1} \dots \varrho_{i_s} \Sigma (\mathbf{x}'_{i_1} \dots \mathbf{x}'_{i_s}) \cdot (\mathbf{u}_{i_1} \dots \mathbf{u}_{i_s}) (\mathbf{u}_{i_1} \cdot \mathbf{x}'_{i_1})^{\rho_{i_1}-1} \dots \\ (\mathbf{u}_{i_s} \cdot \mathbf{x}'_{i_s})^{\rho_{i_s}-1} (\mathbf{u}_{j_1} \cdot \mathbf{x}'_{j_1})^{\rho_{j_1}} \dots (\mathbf{u}_{j_{m-s}} \cdot \mathbf{x}'_{j_{m-s}})^{\rho_{j_{m-s}}}.$$

As further

$${}_s A \mathbf{m} = \mathbf{u}_1^{\rho_1} \dots \mathbf{u}_{i_1-1}^{\rho_{i_1}-1} \mathbf{u}_{i_1}^{\rho_{i_1}-1} \mathbf{v}_1 \mathbf{u}_{i_1+1}^{\rho_{i_1}+1} \dots \mathbf{u}_m^{\rho_m},$$

where

$$\mathbf{v}_1 \dots \mathbf{v}_s = \mathbf{u}_{i_1} \dots \mathbf{u}_{i_s},$$

when the permutation region of  ${}_s A$  contains of  $\mathbf{u}_1^{\rho_1}, \dots, \mathbf{u}_{i_s}^{\rho_{i_s}}$  just the last factor of each, we have also:

$${}_s A \mathbf{m} = \frac{\varrho_{i_1} \dots \varrho_{i_s}}{\binom{P}{s}} \Sigma {}_s A \mathbf{m}$$

where the summation has to be extended over all  $\binom{m}{s}$  essentially different permutation regions. This infers:

$$H^{(s)} F \mathbf{m} = \binom{P}{s} s! {}_s \bar{A} F \mathbf{m}$$

viz. the CAPELLIAN operator  $H^{(s)}$  is identical with the operator  $\binom{P}{s} s! {}_s \bar{A}$ . The linear independency of the operators  $H^{(s)}$  discovered by CAPELLI and their commutativity mutually and with other operators composed of polar operators, is therefore a special case of the linear independency of the operators  $A$ , proved in  $E_1$ , and their commutativity mutually and with all operators  $\bar{M}, A, M, K$  and  $P$ . As the operators  $\bar{A}, \bar{M}, \bar{A}, \bar{M}, I, K$  may all be written as sums of multiples of products of operators  ${}_s \bar{A}, s = 1, \dots, n$  and the identical operator  $I$ , these operators have for a form the significance of definite differential operators. When the sets of variables are of higher degree, these operators have the significance of differential operators of more complicate character. The different kinds of forms mentioned on p. 273 may thus be distinguished by means of the definite differential operators by which they can be obtained and the other differential operators by which they are annihilated.

The operator  $\Omega$ . For  $m = n$ :

$$n! x^n E' : \left( \frac{d}{dx_1} \cdots \frac{d}{dx_n} \right)$$

is the well-known operator  $\Omega$  ( $\Omega$ -Prozesz). According to the preceding we have:

$$\Omega = \frac{E'}{(x_1 \cdots x_n)} H^{(n)}$$

and the application of  $\Omega$  is therefore equivalent to the application of  $n! \binom{P}{n} \bar{A}$  combined with a division by the determinant of the sets of variables. We may therefore also say that a non reducible form in  $n$  sets of  $n$  variables is a form that is annihilated by  $\Omega^1$ ). A form in  $n$  sets of  $n$  variables containing a factor  $\frac{x'_1 \cdots x'_n}{E'}$  can never be non-reducible. In fact, the corresponding affiner  $\mathbf{m}^P$  possesses a linear covariant of degree  $P-n$ . Such a form is therefore not annihilated by the operator  $\Omega$ .<sup>2)</sup>

*Expansion in series of a form in sets of two variables.*

Let  $F \mathbf{m}^P$  be a form in  $m$  sets of 2 variables and  $\mathbf{m}^P$  the corresponding affiner. As  $n = 2$ , the expansion in series of  $\mathbf{m}$  with regard to elementary affiners is identical with that with regard to non-reducible covariants<sup>3)</sup>. When we apply this expansion, we find for  $F \mathbf{m}^P$  the expansion:

$$F \mathbf{m}^P = \sum_{\alpha} \frac{\binom{P}{2\alpha} \binom{2\alpha}{\alpha}^2}{\binom{P-\alpha+1}{\alpha}} \alpha \cdot 2 A \mathbf{m}^P \cdot x'^{\rho} y'^{\sigma} \dots; \alpha' = \begin{cases} \frac{m}{2} & \text{for } m \text{ even} \\ \frac{m-1}{2} & \text{,, ,, odd,} \end{cases}$$

where each term is a sum of products of one single non-reducible form with a certain number of determinants of the form  $x'_a y'_b - x'_b y'_a$ , or shortly  $(x' y')$  as written usually (Klammerfactoren), that is characteristic of that term. That such an expansion is possible and singly determined, has first been proved by GORDAN. For the special case that there are only two sets of variables an application of permutation laws gives

<sup>1)</sup> Comp.  $E_I$  p. 264.

<sup>2)</sup> Comp. p. 279.

<sup>3)</sup> Comp.  $E_I$  p. 265.

$${}_{\alpha,2}A^m \mathbf{u}^\rho \mathbf{v}^\sigma = \frac{\binom{\rho}{\alpha} \binom{\sigma}{\alpha}}{\binom{P}{2\alpha} \binom{2\alpha}{\alpha}} 2^\alpha (\mathbf{u}^\rho \vee_\alpha \mathbf{v}^\sigma),$$

so that for this special case the expansion in series becomes:

$$F^P \mathbf{m} = \sum_{\alpha}^{0, \dots, m'} \frac{\binom{\rho}{\alpha} \binom{\sigma}{\alpha}}{\binom{P-\alpha+1}{\alpha}} 2^\alpha \binom{P}{2\alpha} \binom{2\alpha}{\alpha} (\mathbf{u}^\rho \vee_\alpha \mathbf{v}^\sigma) \cdot \mathbf{x}'^\rho \mathbf{y}'^\sigma.$$

This expansion of  $F \mathbf{u}^\rho \mathbf{v}^\sigma$  remains applicable for  $n > 2$ , because, there being only two sets of variables, only alternations of the form  ${}_{\alpha,2}A$  give not identically zero. This is the so-called second expansion in series of GORDAN<sup>1)</sup>.

The terms of the expansion with regard to non-reducible covariants may now be further decomposed in different ways. First each operator  ${}_{\alpha,2}A^m$  can be decomposed into simple mixed alternations. Then an expansion of  $F^P \mathbf{m}$  is obtained with regard to locally alternating forms for which in each term the power of a determinant of the variables is *the same* as the in the same term occurring power of the determinant of the ideal factors of  $\mathbf{m}$  corresponding to those sets of variables. That such an expansion is possible and singly determined has been first proved by A. REISSINGER<sup>2)</sup>.

Secondly each elementary affiner may be decomposed in ordered elementary affiners of the first kind. With this decomposition corresponds an expansion of the form:

$$F^P \mathbf{m} = \sum_{\alpha}^{0, \dots, m'} \sum_{\lambda}^{1, \dots, \delta_\alpha} 2^\alpha \frac{P-2\alpha+1}{P-\alpha+1} {}_{\alpha,2}A^{(\lambda)} \mathbf{u}^{\rho-\alpha, \alpha} M^{(\lambda)} \mathbf{m} \cdot \mathbf{x}'^\rho \mathbf{y}'^\sigma \dots$$

The factors of the form  $(x', y')$  occurring in each term satisfy the condition that they belong to the permutation regions of a definite *ordered* alternation  ${}_{\alpha,2}A$  acting on  $\mathbf{x}'^\sigma \mathbf{y}'^\rho$  and characteristic of that

1) STUDY, Methoden zur Theorie der ternären Formen § 3 and § 4. The so-called first expansion in series of GORDAN corresponds to an expansion in series of a *mixed* affiner and is not discussed here.

2) Ausgezeichnete Form der Polaren-Entwicklung eines symbolischen Produktes. Progr. Realsch. Kempten 1906-07.

3) See *ET* p. 263.



an expansion of the form  $F\mathbf{m}^P$  in a number of terms each of which is a sum of products of a non-reducible form with a certain number (characteristic of that term) of determinants formed by  $n$  of the sets of variables.

For  $m = n$  this expansion has been given first by CAPELLI<sup>1)</sup> and for the general case by J. DERUYTS<sup>2)</sup> and K. PETR<sup>3)</sup> 4). Both CAPELLI and PETR base their proof upon the property mentioned p. 276, that a form in  $n$  sets of  $n$  variables containing the determinant of the variables as a factor, is not annihilated by  $\Omega$ . The deduction of CAPELLI, which is most analogous to the above is based upon the theory of the differential operators  $H^{(s)}$ . DERUYTS uses his theory of the semi-invariants and -covariants and PETR makes use of differential-operators that can be constructed by means of auxiliary variables.

The terms of the expansion in non-reducible covariants may again be decomposed in different ways. Firstly each term of  $\mathbf{m}^P$  can be decomposed into general mixed alternations and these again into simple ones. To this corresponds an expansion of  $\mathbf{m}^P$  in a sum of terms consisting each of a sum of products of a number of determinants with  $s_1, \dots, s_l$  rows formed from the characteristic numbers of the sets  $\mathbf{x}', \mathbf{y}' \dots$  with one single symmetrical form. All terms are covariants, the sub-terms only then when  $s_1 = s_2 = \dots = n$ .

In each sub-term the power of a determinant of the characteristic numbers of the variables is the same as the power of the determinant of the characteristic numbers of the corresponding ideal factors of  $\mathbf{m}^P$ . Under these conditions the expansion is singly determined and an extension of the one given by REISSINGER for  $n = 2$ .

Secondly each term of  $\mathbf{m}^P$  may be decomposed into ordered alternations of the form  $z_n A$ . Then the determinants occurring in each sub-term must belong to the permutation regions of a definite ordered alternation  $z_n A$ , characteristic of that sub-term, and acting on  $\mathbf{x}'^\rho \mathbf{y}'^\sigma$ .

1) Fondamenti di una teoria generale delle forme algebriche, Mem. dei Lincei (82) § 74; Sur les opérations dans la théorie des formes algébriques, Math Ann. 37 (90) 1—37.

2) Essai d'une théorie générale des formes algébriques. Mém de Liège. 2. 17 (92) 4. 1—156; Détermination des fonctions invariantes de formes a plusieurs séries de variables. Mém. couronnés et mém. des sav. étr. de Bruxelles 53 (90—93) 2. 1—23.

3) Ueber eine Reihenentwicklung für algebraische Formen, Bull. Intern. de Prague 12 (07) 163—191.

4) The forms called here non-reduceable are called by CAPELLI: "formes propres" and by DERUYTS: "covariants de formes primaires".

This expansion is singly determined, the corresponding expansion of  $\mathbf{m}^P$  being singly determined<sup>1)</sup> and an extension of that given by GODT for  $n = 2$ . We have thus found the theorem:

*Principal theorem B. Every algebraic form, homogeneous and of the degrees  $\varrho, \sigma, \dots$  in  $m$  sets of  $n$  variables  $\mathbf{x}', \mathbf{y}' \dots$  can be expanded in one and only one way in a series of terms, each of which consists of a product of a number  $\alpha$  of determinants that are formed from the variables of  $n$  of the sets with a non-reducible form, in such a way that the determinants in each term belong to the permutation regions of a definite ordered alternation  ${}_{z,n}A$ , characteristic of that term and acting on the affiner  $\mathbf{x}'^\varrho \mathbf{y}'^\sigma \dots$ , the number  $\alpha$  being characteristic of a definite group of those terms.*

Thirdly we can proceed so far with the division, that  $\mathbf{m}^P$  becomes a sum of ordered elementary affiners all of the first or all of the second kind. With this corresponds an expansion of  $F\mathbf{m}^P$  in ordered elementary forms of the first resp. of the second kind, which may be characterized in the following way:

*Principal theorem C. Every algebraic form, homogeneous and of the degrees  $\varrho, \sigma, \dots$  in  $m$  sets of  $n$  variables  $\mathbf{x}', \mathbf{y}', \dots$ , can be expanded in one and only one way in a series of ordered forms of the first resp. of the second kind.*

Examples:

The 6-linear form

$$F\mathbf{m} = \mathbf{m}_1 \dots \mathbf{m}_6 \cdot \mathbf{x}'_1 \dots \mathbf{x}'_6$$

can be decomposed into 76 ordered elementary forms of the first kind corresponding with the affiners.

1)	${}^6A_{11}$	${}^0M^1$	$\mathbf{m}$	
2, ..., 6)	${}^5A_{10}^{(\lambda)}$	${}^2M_2^{(\lambda)}$	$\mathbf{m}$ ,	$\lambda = 1, \dots, 5$
7, ..., 15)	${}^{4,2}A_9^{(\lambda)}$	${}^2,2M_3^{(\lambda)}$	$\mathbf{m}$ ,	$\lambda = 1, \dots, 9$
16, ..., 25)	${}^4A_8^{(\lambda)}$	${}^3M_5^{(\lambda)}$	$\mathbf{m}$ ,	$\lambda = 1, \dots, 10$
26, ..., 30)	${}^{2,3}A_7^{(\lambda)}$	${}^{3,2}M_4^{(\lambda)}$	$\mathbf{m}$ ,	$\lambda = 1, \dots, 5$
31, ..., 46)	${}^{3,2}A_6^{(\lambda)}$	${}^{3,2}M_6^{(\lambda)}$	$\mathbf{m}$ ,	$\lambda = 1, \dots, 16$
47, ..., 56)	${}^3A_5^{(\lambda)}$	${}^4M_8^{(\lambda)}$	$\mathbf{m}$ ,	$\lambda = 1, \dots, 10$

<sup>1)</sup>  $E_I$  p. 265

$$\begin{array}{ll}
 57, \dots, 61) & {}_{3,2}A_4^{(\lambda)} {}_{2,3}M_7^{(\lambda)} \overset{6}{\mathbf{m}}, \quad \lambda = 1, \dots, 5 \\
 62, \dots, 70) & {}_{2,2}A_3^{(\lambda)} {}_{4,2}M_9^{(\lambda)} \overset{6}{\mathbf{m}}, \quad \lambda = 1, \dots, 9 \\
 71, \dots, 75) & {}_2A_2^{(\lambda)} {}_5M_{10}^{(\lambda)} \overset{6}{\mathbf{m}}, \\
 76) & {}_0A_1 {}_6M_{11} \overset{6}{\mathbf{m}},
 \end{array}$$

for  $n = 5$ , 1 becomes zero, for  $n = 4 : 1, \dots, 6$ , for  $n = 3 : 1, \dots, 25$  and for  $n = 2 : 1, \dots, 56$ . The expansion in elementary forms corresponds to an expansion of  $\overset{6}{\mathbf{m}}$  which is found from the preceding one by taking together the horizontal rows 1; 2, ..., 6; 7, ..., 15; etc. From this can be deduced again the expansion in non-reducible covariants. For  $n > 6 : 1, \dots, 76$ ; for  $n = 6 : 1; 2, \dots, 76$ ; for  $n = 5 : 2, \dots, 6; 7, \dots, 76$ ; for  $n = 4 : 7, \dots, 25; 26, \dots, 76$ ; for  $n = 3 : 26, \dots, 30; 31, \dots, 56; 57, \dots, 76$ , and for  $n = 2 : 57, \dots, 61; 62, \dots, 70; 71, \dots, 75; 76$ . As to the expansion of a form of the sixth degree in a number of sets of variables less than 6 e.g.

$$F \overset{6}{\mathbf{n}} = \overset{6}{n_1^2 n_2^2 n_3^2} \overset{6}{\mathbf{x}'_1 \mathbf{x}'_2 \mathbf{x}'_3}$$

we may remark, that  $\overset{6}{\mathbf{x}'_1 \mathbf{x}'_2 \mathbf{x}'_3}$ , can be obtained by a definite simple mixing  ${}_{3,2}M_7^{(\alpha)}$ . Hence, in the expansion of  $\overset{6}{\mathbf{n}}$  all ordered elementary operators of the first kind vanish, when their first factor is an alternation that is annihilated by  ${}_{3,2}M_7^{(\alpha)}$ . In the first place thus 1, ..., 25. Of 26—30 remains one term, of 31—46 there remain nine terms, among which three different ones, of 47—56 four equal terms, of 57—60 two equal terms, of 61—69 six terms, among which three different ones, of 70—75 four terms, among which two different ones, while 76 remains. In total there remain therefore for  $n \leq 3$  twelve terms and for  $n = 2$  seven terms. This last number gives also the number of terms in the expansion of WAELSCH<sup>1)</sup>.

*Expansion of a form in  $m$  sets of  $n$  variables of arbitrary degree.*

*Principal theorem D. Every algebraic form  $F \overset{P}{\mathbf{m}}$ , homogeneous and of the degrees  $\rho, \sigma, \dots$  in  $m$  different sets of variables  $\overset{p}{\mathbf{x}}, \overset{q}{\mathbf{y}}, \dots$  can be expanded in one and only one way in a series of ordered elementary forms of the first resp. of the second kind.*

<sup>1)</sup> Comp. p. 278.

The expansion may be obtained by expanding  $\mathfrak{m}^P$  in ordered elementary affinors.

Also the expansion in non-reducible covariant forms (principal theorem B) can be obtained for this most general case by expanding  $\mathfrak{m}^P$  with respect to non-reducible covariants. Then the determinants of  $n$  rows that occur in the expansion of the form generally have only an ideal meaning and therefore the non-reducible forms occurring in each term have only an ideal significance too. The terms themselves however keep their non-ideal significance and are found by taking together definite groups of terms from the expansion in elementary forms of the first kind.

**Physiology.** — “*On Cyclopia with conservation of the Rhinencephalon*”. By Prof. C. WINKLER.

(Communicated in the meeting of May 31, 1919).

In These Proceedings of February 1916 I reported the results found after examination of the brains of three cyclopic monstra.

I then pointed out, that the characteristic abnormality in all these brains, was the presence of a sack with a thin wall, formed by the roof of the third ventricle, largely extended by fluid.

I was forced to contradict a sentence, found in SCHWALBE'S “Die Morphologie der Missbildungen” where he says: “das Vorderhirn ist bei den Cyclopien nie in Hemisphären geteilt”. I must assert, that in all cases of cyclopic brain the hemispheres are well differentiated at the occipital end.

At the frontal pole however they are often united, although in one of the cases then described, there was also a sagittal fissure at the frontal pole of the hemisphere.

I therefore deny that the examination of cyclopic brains should give any support to the supposition that the terminating-time of the cyclopia must be placed in a period, preceding that in which the sagittal fissure of the telencephalon appears. Also the view that cyclopia is *inevitably* accompanied by arhinencephalia, I could only accept under certain reserve.

In fact, in all the cases which I examined, the bulbi olfactorii and the lobi olfactorii anteriores were missing. But the lobi olfactorii posteriores were present. They are found along the mesial line of the brain-base. There they were placed next to each other, because the brain-base between them was missing.

Ordinarily cyclopic brains are partially arhinencephalic, missing only the frontal part of the rhinencephalon.

Since then I have prepared the brains of several cases of cyclopia, largely differing between each other.

Now I believe that there is another monstrosity, the so-called synotia, nearly related to the cyclopic one. The former is often combined with the latter.

Through the kind cooperation of Professor KOUWER and Dr. VERMEULEN I obtained possession of such-like monstrosities.

1. a human fetus with synotia (nose, eyes and maxilla superior are existing). There is an indication of a mouth. No inferior maxilla is found. The ears are united in the mid-line in a single meatus acusticus externus (In this being the brains were unfortunately thrown away).

2. a fetus of a pig with cebocephalia (Proboscis is found above two totally separated eyes in one orbit).

3. a fetus of a lamb with incomplete cyclopia (one oblong eye with a long pupil, two optic nerves; no proboscis is visible by the naked eye, although the X-ray photo shows a little nasal bone at the os frontale).

4. a fetus of a calf with almost complete cyclopia (one oblong eye with hour-glass like pupil, two optic nerves closely situated, no proboscis.)

5. a fetus of a calf with incomplete cyclopia and synotia (two united eyes in the circle of the four eyelids, no proboscis. A mouth-opening, bordered at the upper part by a strong upper-jaw (X-ray photo). No lower jaw (X-ray photo), no tongue. The os hyoideum is completely developed (X-ray photo). The ears are united in the mid-line).

6. a fetus of a lamb with synotia. (A small proboscis, no eyes, no jaws, no mouth or tongue. There is an os hyoideum. (X-ray photo.) The animal therefore is anophthalmic, agnathic, aprosopic. The ears are united in the mid-line. Larynx and pharynx end towards the top in a blind sack. No thyroid gland.)

7. a fetus of a lamb with cyclopia and synotia, dealt with in this communication.

In all the brains of those monstra which were examined, there was found a membranous sack, which shows itself to be the roof of the third ventricle, much extended by fluid. Now and then however peculiarities were seen in this sack at its frontal or its occipital end. In No. 5 and No. 6 e.g. the sack was continued in the like-wise thin wall of the telencephalon and in N<sup>o</sup>. 7 at its distal end the mesencephalon and the cerebellum were not developed, forming a part of the thin wall of the sack covering also the IV ventricle.

The sack is not only existing in the brains of cycloplan monstra, but also in those of the synotic type.

In another paper I will describe more in details the differences of the brains in those monstra.

Here I intend to demonstrate the monstrosity, mentioned sub N<sup>o</sup>. 7. It may be considered as a sample of a cycloplan malformation, with

synotia possessing a complete rhinencephalon and therefore it already warns us that we should not rashly assert that every cyclopia must be arhinencephalic.

Regarding the drawing of this fetus, seen in the face, we notice



Fig. 1.

Drawing after a photograph of a lambfetus with cebo-cephalia and synotia. The monstrosity possesses a well-shaped and completely developed nose above the two eyes, surrounded by four eye-lids. One can discover the two medial carunculae. The ears are united together. No mouth.

that the animal possesses a complete nose with nostrils. Underneath it is found, that what appears to be an orbit, surrounded by four

eyelids and in which the two united eyes are placed with a smaller left and a larger right eye. Jaws, mouth and tongue are missing. Directly underneath the eyes the two ears lie by the side of each other in the mid-line and the two external auditory canals possess a combined opening.

Pharynx and larynx are well developed and continue in a hollow, ending upwards blind.

The X-ray photo confirms that the jaws are totally missing, but it also shows that the os hyoideum is well developed. Moreover the X-ray photo teaches that the ethmoidal bone is fully present. Crista



Fig. 2.

Drawing of the upper surface of the brains of the monstrosity reproduced in fig. 1. The sack (*a*) is visible. It adheres to the dura mater (*b*). At the distal end (*c*) it continues in the mesencephalon and in the cerebellum, represented by a thin membrane. Through this membrane shines the tela chorioidea of the fourth ventricle.

Galli, lamina cribrosa, lamina papyracea and the laryrinth of the ethmoidal bone can be distinguished. Already before removing the brains I therefore knew, that this cycloopian being could not have

been formed through the missing of the ethmoidal bone and of the mesial wall of the orbit.

On removal of the brains one is directly struck by the presence of the sack (fig. 2*b*) and the epiphysis lying free, seeming to be its point of origin. Behind it one does not find anything of a mesencephalon or of a cerebellum. The sack continues in occipital direction and the tela of the 4<sup>th</sup> ventricle shines faintly through it (fig. 2*c*). On the other hand both the hemispheres are well developed. There is a deep sagittal fissure, in which the dural septum with its sinus are found, and which has to be cut away from the crista galli in order to make its removal possible.

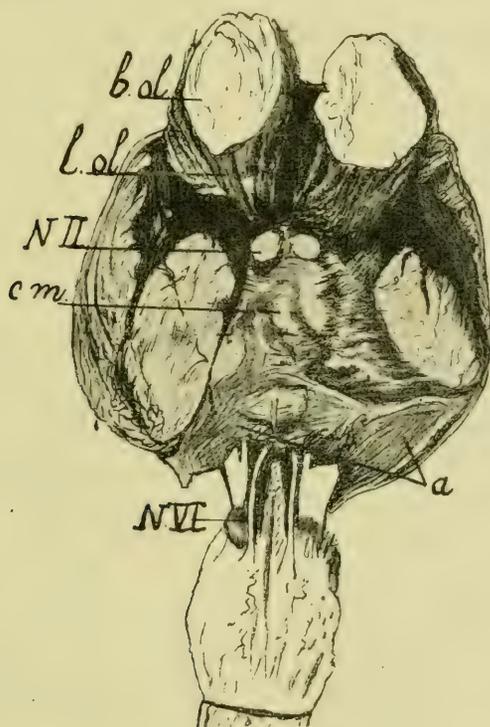


Fig. 3.

Drawing of the basal surface of the monstrosity reproduced in fig. 1. *a* = the thin membrane covering the defect in the pes pedunculi cerebri. *b. ol.* = bulbus olfactorius. *l. ol.* = lobus olfactorius. *N. II* = nervi optici. *c. m.* = corpus mammillare, shining through the membrane. *N. VI* = nervus abducens.

If we examine the basal surface of the brain (fig. 3), then we are struck by the presence of two well-developed olfactory bulbs with their tract, their lobus olfactorius anterior, posterior and cornu Ammonis, in short of a completely developed rhinencephalon.

There are two *N. optici* (fig. 3, *N. II*). The corpus mammillare

shines through the pia mater. One can surely demonstrate the N. abducens of the cranial nerves. With the nerve lying proximally from this, it is not so. Only after a microscopic examination, it is certain that it is the N. trigeminus. Between these two thick nerves the base of the brain is formed by a thin membrane, which when the brain was removed could only be spared with the utmost care.

If the sack is opened at its dorsal part and is folded backward, there appears a local defect in the brain-base, more than 1 cm. wide. There the base of the brain is formed by a membrane, 1 mm. thick, at the utmost, in which some white nerve strings diverge from the mid-line towards a proximally placed mass of nervous tissue. This nervous mass, striatum and thalamencephalon are shining through the pia mater at the base in fig. 3. There is an interrupted continuation in the brain-base at the level of the pes pedunculi.

The bony base of the brain is very remarkable. The crista galli protrudes. On both its sides the lamina cribrosa carries the bulbi olfactorii, which send their fila olfactoria through it. Moreover the optic foramina are normally formed, together with the frontal part of the os sphenoidale. Then however the sella turcica is found missing, also the hypophysis. The base of the crane is not massive, but movable, as there is a great loss of bone distally from the sella turcica.

An X-ray photo taken from the upper side makes this obvious. Here a large defect in the bony base of the skull appears. The caudal part of the os sphenoidale is missing in the mid-line and the frontal part of the clivus has fallen out as far as the arch of the Atlas. The os petrosum is intact on both sides. There are no jaws. Through the loss of the facies orbitalis of the upper jaw the eyes have sunk downwards. They are no longer lying in a bony orbit. The lamina papyracea of the ethmoidal bone is placed proximally from the double eye and therefore easily recognisable in the X-ray photo. In this case, it is not because the mesial wall of the orbit has been destroyed that the eyes have met one another in the mid-line, but because the lower wall of the orbit is absent.

This cyclopiian monstrosity possesses a complete rhinencephalon, but at the same time it becomes synotic through the loss of a lower jaw.

As to the question, what may be the cause of such a monstrosity one must acknowledge that immense difficulties arise in defending that there was an insufficiency of germ material, as far as the brain as well as the bony parts of the skull. is concerned Simpler is the view in taking the sack as point of origin.

A pathological process which calls forth the sack (and the latter is found in all the cases of this sort) is able to destroy at the dorsal brainside the mesencephalon and the cerebellum when they hinder its development at the distal end. But at the ventral side the pathological process destroys the pedunculi cerebri and the tissue on which they lie. This is the germ material out of which jaws, mouth and tongue are going to develop.

Distally bordered by the second branchial arch (the os hyoideum was intact in all the three cases of synotia) all is destroyed that is going to develop out of the first branchial arch (except occasionally of the ossicula acustica) and out of the tissue, which lies proximally from it. In this way the local defect of the pedunculi cerebri and of the tissue forming the middle part of the skull, is easily understood.

According to the view which I explained in my previous report, I think the cause of the sack to be a local process of inflammation, which by means of a mechanical influence produces a defect at the base of the brain and of the skull.

On the other hand I acknowledge the possibility that the sack and the basal defect, together may be co-effects of another more complicated pathological cause.

In debates upon this subject, held at Leiden Dr. MURK JANSEN defended the thesis that the narrowness of the amnion may perhaps produce the sack as well as the defect, by compressing the head of the embryo in a strictly defined plane. The result may be that all the germ material, which is found in this plane, may die. Should such a hypothesis be confirmed, there will arise different possibilities in the formation of these monstra, but I will not yet enter upon these.

I only wish to lay stress on the following views.

Destroying of tissue at the proximal end of the skull, so that the os ethmoidale disappears and the dorsally placed sack, the roof of the distended III<sup>rd</sup> ventricle is formed, gives rise to cyclopiian monstra. But they are not the only defect-formations which are found.

There is yet another place of predilection, where the tissue that will form parts of the skull, may be destroyed. In such cases the first branchial arch may be destroyed by pathological processes. These lead to synotia. Now and then it occurs isolated. Then we see uncomplicated synotia. Nose, eyes and upper jaw are well developed, as found in the fetus, mentioned sub I.

But also the two local destroying processes appear, independent of each other, next to one another. Then cyclopiian and synotian deformities are found together. And there may remain between

them an intact upper jaw as was the case with the monstrum, mentioned under 5.

Now and then the two local destroying processes are united. Then comes a massive fissure and the monstrosities appear, as described under 4 i.e. synotia with anophthalmia, aprosopia and agnatia.

The series of the different cyclopic deformities is joined to that of the synotian deformities. In this series one case is most remarkable, i.e. if the local destroying process, commencing at the first branchial arch spreads so far proximalwards that the upper jaw totally falls out.

Then the eyes are going to meet each other in the mid-line, while the maxillar part of the orbit is lost. Then also two eyes are found in a four eye-lid ring, but they are no longer placed in an orbit. In that case there is found a cebocephalic form of cyclopia, with a complete nose above the eyes, and a complete rhinencephalon.

To me it seems even possible that the local destroying process may spare the first branchial arch, only destroying the upper jaw. In such a case there results a cyclopia, perhaps always the cebocephalic form, but without loss of the ethmoid bone, with a complete rhinencephalon and without synotia.

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**Mathematics.** — “*The Remainder in the Binomial-series.*” By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG.)

(Communicated in the meeting of June 28, 1919).

1. We consider the binomial-series  $\sum_{j=0}^{\infty} u_j^{(m)}$ , in which  $u_j^{(m)} = \frac{x^j}{j!} \prod_{k=0}^{j-1} (m-k)$ . We suppose  $x$  real and  $m$  not zero and not a positive integral number (there otherwise the series is finite).

2. The series is convergent if  $|x| < 1$ , if  $x = 1, m > -1$  and if  $x = -1, m > 0$ , divergent in the other cases.

3. If  $|x| < 1$  and if  $|x| = 1, m > -1$  we have  $\lim_{n \rightarrow \infty} u_n^{(m)} = 0$ , as appears from N<sup>o</sup>. 2.

4. According to MAC LAURIN'S series we have:

$$(1+x)^m = \sum_{j=0}^{n-1} u_j^{(m)} + R_n,$$

in which the remainder is given by:

$$R_n = \frac{x^n}{(n-1)!} \frac{(1-\theta)^{n-p}}{p} \frac{(1-\theta)^{n-p}}{(1+\theta x)^{n-m}} \prod_{k=0}^{n-1} (m-k) \quad (p > 0),$$

hence:

$$R_n = \frac{u_n^{(m)}}{(1+\theta x)^{n-m}} = \dots \dots \dots (1)$$

$$= \frac{mx}{p} \left( \frac{1-\theta}{1+\theta x} \right)^{n-p} (1+\theta x)^{m-p} u_{n-1}^{(m-1)} \dots \dots \dots (2)$$

5. The aim of this paper is to show that  $\lim_{n \rightarrow \infty} R_n = 0$  in all cases in which the series converges, that is to demonstrate for those cases the validity of  $(1+x)^m = \sum_{j=0}^{\infty} u_j^{(m)}$ , exclusively from the remainder in the series (which is done incorrectly in a great number of hand-

1) The numbers  $\theta$  occurring in these expressions are generally unequal,  $\theta$  depending on  $n$  and on  $p$ .

books). At the same time the *advantage of the expression (2) for the remainder in the binomial-series* will be obvious.

**6.** If  $x \geq 0$ , we find according to (1) if we choose  $n > m$ :  
 $|R_n| \leq |u_n^{(m)}|$ , hence  $\lim_{n \rightarrow \infty} R_n = 0$  if the series converges.

**7.** If  $-1 < x < 0$ , it ensues from (2), if we take  $p = 1$ :  
 $|R_n| < A |m x u_{n-1}^{(m-1)}|$ , in which  $A$  is the greatest of the numbers 1 and  $(1+x)^{m-1}$ . This leads to  $\lim_{n \rightarrow \infty} R_n = 0$ .

**8.** If  $x = -1$ ,  $m > 0$ , it follows from (2) by taking  $p = m$ :

$$R_n = -u_{n-1}^{(m-1)} \dots \dots \dots (3)$$

In connexion with N<sup>o</sup>. 3 it follows that  $\lim_{n \rightarrow \infty} R_n = 0$  (the inequality  $m-1 > -1$  being satisfied), a result that cannot be obtained from LAGRANGE'S or CAUCHY'S form of the remainder if  $m < 1$ .

It can further be observed, that (3) leads to the identity

$$1 - m + \frac{m(m-1)}{2!} - \dots + (-1)^n \frac{m(m-1)\dots(m-n+1)}{n!} = \\ = (-1)^n \frac{(m-1)(m-2)\dots(m-n)}{n!};$$

the identity can also easily be demonstrated by mathematical induction, from which it appears that the identity also holds good for  $m \leq 0$ .

**9.** If we make use of ABEL'S theorem about the continuity of power-series, the examination of the remainder in the case  $|x| < 1$  suffices. In order to demonstrate, without distinguishing various cases, that then  $\lim_{n \rightarrow \infty} R_n = 0$ , in (2) we take  $p = 1$  (as in N<sup>o</sup>. 7).

**Mathematics.** — “A problem of combinatorial analysis connected with the determination of the number of different ways in which the greatest common divisor of two products can be found”. By Prof. FRED. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of June 28, 1919).

1. *The greatest common divisor (G. C. D.) of the two numbers*

$$a_1, a_2, \dots, a_m \text{ and } b_1, b_2, \dots, b_n \dots \dots \dots (1)$$

is obtained as the product of the  $mn$  greatest common divisors  $G_{ij}$  of  $a_i$  and  $b_j$  ( $i=1, 2, \dots, m$  and  $j=1, 2, \dots, n$ ). It is to be understood, that two numbers, whose G. C. D. is determined, are to be divided by that G. C. D. and that for the remaining part of the calculation the quotients are substituted for the original numbers. If such a quotient is combined with another number, we divide again by their G. C. D. etc. <sup>1)</sup>.

So the  $mn$  numbers  $G_{ij}$  are calculated in a definite order.  $G_{ij}$  is the G. C. D. of the numbers  $a_i$  and  $b_j$ ,  $a_i$  being obtained by division of  $a_i$  by all previously determined greatest common divisors that are connected with  $a_i$ , while  $b_j$  is derived in the same way from  $b_j$ .

2. We now inquire into the number of ways in which the G. C. D. of the numbers (1) can be calculated in the manner indicated in N<sup>o</sup>. 1. A different order of the  $mn$  numbers  $G_{ij}$  does not necessarily yield a different manner of calculation of the number  $G$  sought, for two manners of calculation must of course be considered as identical, if they consist of the same calculations and only differ in the order in which the separate calculations are carried out. Hence it is immaterial which of the two numbers  $G_{ij}$  and  $G_{kl}$  is calculated first, provided  $i \neq k$  and at the same time  $j \neq l$ . Consequently it is only the order of the numbers  $G_{ij}$  which agree either in the first indices or in the second, that is essential.

3. The special values of the numbers  $a_1, \dots, a_m, b_1, \dots, b_n$  may

<sup>1)</sup> The last time that one of the numbers  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$  is combined with another the division of the first-mentioned number by the G. C. D. found may of course be dispensed with, the quotient being no further involved in the calculation.

cause that permutation of two numbers  $G_{ij}$ , which agree in the first or in the second indices, does not yield a different manner of calculation of  $G$ . If  $a_1$  and  $b_1$  e.g. are prime to each other and  $a_1$  and  $b_2$  likewise, it will be immaterial whether we first determine  $G_{11}$  and then  $G_{12}$  or vice versa, both greatest common divisors being 1 and a division by  $G_{11}$  or  $G_{12}$  consequently not causing any alteration in the numbers. But we shall disregard such special cases, so that we only call two ways of calculation of  $G$  identical, if they are the same with any set of values of  $a_1, \dots, a_n, b_1, \dots, b_n$ .

4. The number  $G$  inquired into in N°. 2 then becomes equal to the number of ways, in which we can arrange the  $m n$  symbols

$$G_{ij} \text{ (} i = 1, 2, \dots, m; \text{ } j = 1, 2, \dots, n \text{)}$$

in a sequence, if we consider two sequences identical, if these can be derived from each other by some permutations of two symbols, differing both in the first and second indices.

5. If  $m = 2$  the number asked for will be the number  $A_n$  of the ways, in which we can arrange the symbols

$$G_{11}, G_{12}, G_{13}, \dots, G_{1n}, \dots \dots \dots (2)$$

$$G_{21}, G_{22}, G_{23}, \dots, G_{2n} \dots \dots \dots (3)$$

in one sequence, only paying regard to the order of two of these symbols if in the above scheme they stand in the same row or in the same column. The number  $A_n$  is the product of  $n!$  and the number  $B_n$  of the arrangements, in which the symbols (2) follow each other in the manner indicated by (2).

6. We can divide the  $B_n$  arrangements into  $n$  groups, thus, that the  $i^{\text{th}}$  group contains those arrangements, in which  $G_{2i}$  precedes the other symbols (3). That  $i^{\text{th}}$  group can be subdivided into two parts, the first part containing the arrangements in which  $G_{2i}$  precedes  $G_{1i}$ , the second the arrangements in which  $G_{2i}$  comes after  $G_{1i}$ .

The first part of the  $i^{\text{th}}$  group contains as many arrangements as when  $G_{1i}$  and  $G_{2i}$  are cancelled, hence  $B_{n-1}$  arrangements.

In the arrangements occurring in the second part of the  $i^{\text{th}}$  group the symbols  $G_{11}, G_{12}, \dots, G_{1i}$  precede all symbols (3), so that we may cancel the symbols  $G_{11}, G_{12}, \dots, G_{1i}$ . If we only pay attention to the arrangement of the symbols  $G_{21}, G_{22}, \dots, G_{2,i-1}$  mutually and towards the symbols

$$G_{2,i+1}, G_{2,i+2}, \dots, G_{2n}, \dots \dots \dots (4)$$

(considered as identical) we obtain  $\frac{(n-1)!}{(n-i)!}$  possible arrangements.

Each of these arrangements leads to  $B_{n-i}$  possible arrangements, if we also pay attention to the order in which the  $n-i$  symbols (4) are placed mutually and respectively towards  $G_{1,i+1}, G_{1,i+2}, \dots, G_{1n}$ . It appears from this, that the second part of the  $i^{\text{th}}$  group contains  $\frac{(n-1)!}{(n-i)!} B_{n-i}$  arrangements; this also holds good in the case  $i = n$ , if we interpret  $B_0$  as 1.

Hence the  $i^{\text{th}}$  group contains altogether  $B_{n-1} + \frac{(n-1)!}{(n-i)!} B_{n-i}$  arrangements. This leads to

$$B_n = \sum_{i=1}^n \left\{ B_{n-1} + \frac{(n-1)!}{(n-i)!} B_{n-i} \right\} = (n+1) B_{n-1} + (n-1)! \sum_0^{n-2} \frac{B_k}{k!}. \quad (5)$$

7. If we replace in (5)  $n$  by  $n-1$  (for which it is necessary to assume  $n > 1$ ) we shall find:

$$B_{n-1} = (n-1) B_{n-2} + (n-2)! \sum_0^{n-2} \frac{B_k}{k!},$$

from which follows according to (5):

$$B_n = 2n B_{n-1} - (n-1)^2 B_{n-2}.$$

From this equation in finite differences (which is homogeneous, linear and of the second order) and proceeding from  $B_0 = 1$  and  $B_1 = 2$ <sup>1)</sup>, we can successively compute  $B_2, B_3, B_4$ , etc. Thus we find:  $B_2 = 7, B_3 = 34, B_4 = 209, B_5 = 1546, B_6 = 13327, B_7 = 130922, B_8 = 1441729, B_9 = 17572114, B_{10} = 234662231$ .

8. Owing to  $A_n = n! B_n$  we now find for the number  $A_n$  of the ways in which we may calculate the G. C. D. of the numbers  $a_1, a_2$  and  $b_1, b_2, \dots, b_n$ :

$$A_1 = 2, A_2 = 14, A_3 = 204, A_4 = 5016, A_5 = 185520, \\ A_6 = 9595440, A_7 = 659846880, A_8 = 58130513280.$$

9. If  $m$  and  $n$  are both  $> 2$ , the determination of the number of ways, in which the G. C. D. of the two products can be found according to the method as indicated in N<sup>o</sup>. 1, has become considerably more difficult. If  $m = n = 3$  we find (by systematic finding out all the various cases) for the number sought 19164.

<sup>1)</sup> By taking  $n = 1$  the formula (5) leads, in connection with  $B_0 = 1$ , to  $B_1 = 2$ ,  $\sum_0^{n-2} \frac{B_k}{k!}$  then being 0 (as sum of zero terms).

**Chemistry.** — “*On the Phenomenon after Anodic Polarisation.*” II.

By Prof. A. SMITS, G. L. C. LA BASTIDE, and TH. DE CRAUW.

(Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of June 28, 1919).

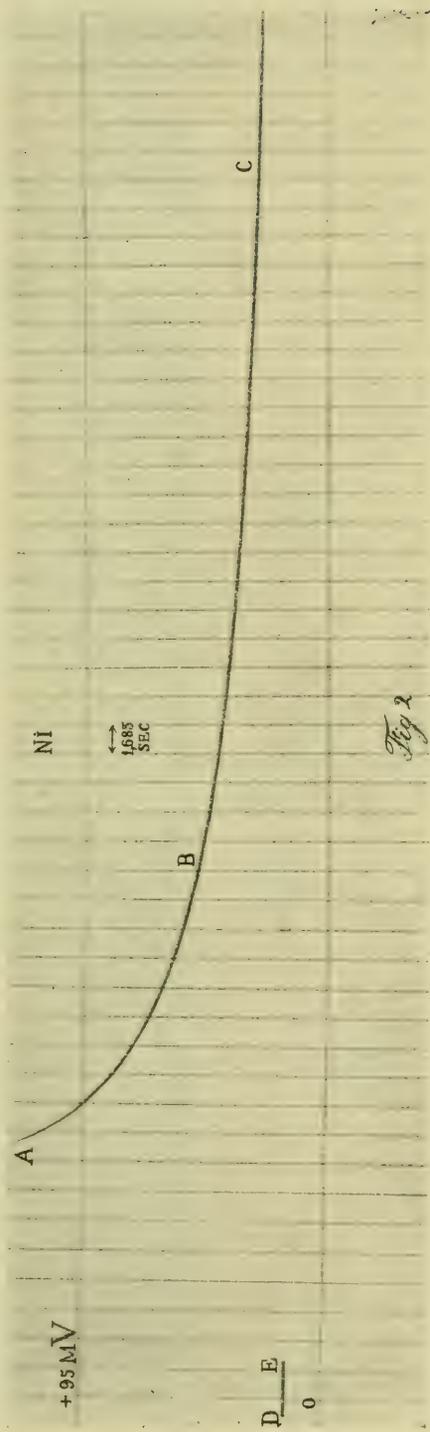
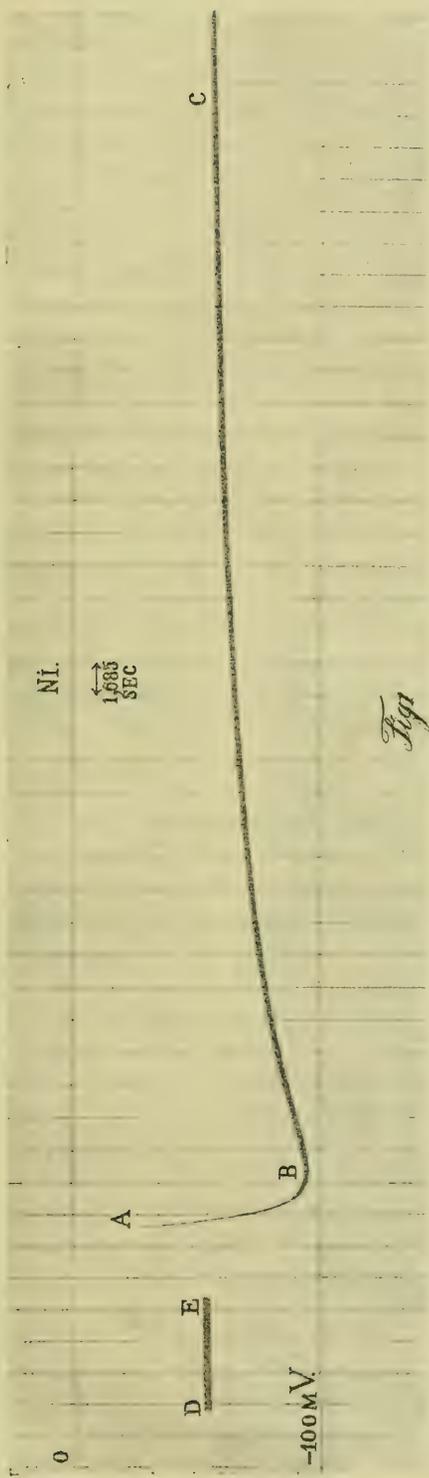
1. It was shown in a preceding communication that the phenomenon that appears after anodic polarisation of iron in an iron salt solution is owing to this that during the anodic solution the iron gets surrounded by a liquid layer which is very rich in ferro-ions. As a rule the ratio  $\frac{(Fe^{2+})}{(Fe^{3+})}$  will be greater in this liquid layer than outside it. After the current has been broken the ferri-ions will, therefore, diffuse from the surroundings into the boundary layer, whereas the ferro-ions pass outside, which causes the ratio  $\frac{(Fe^{2+})}{(Fe^{3+})}$  to become again smaller in the boundary layer. In consequence of this change the potential of the iron, which was at first less negative or positive through the disturbance of the metal, passes through a minimum value after interruption of the current.

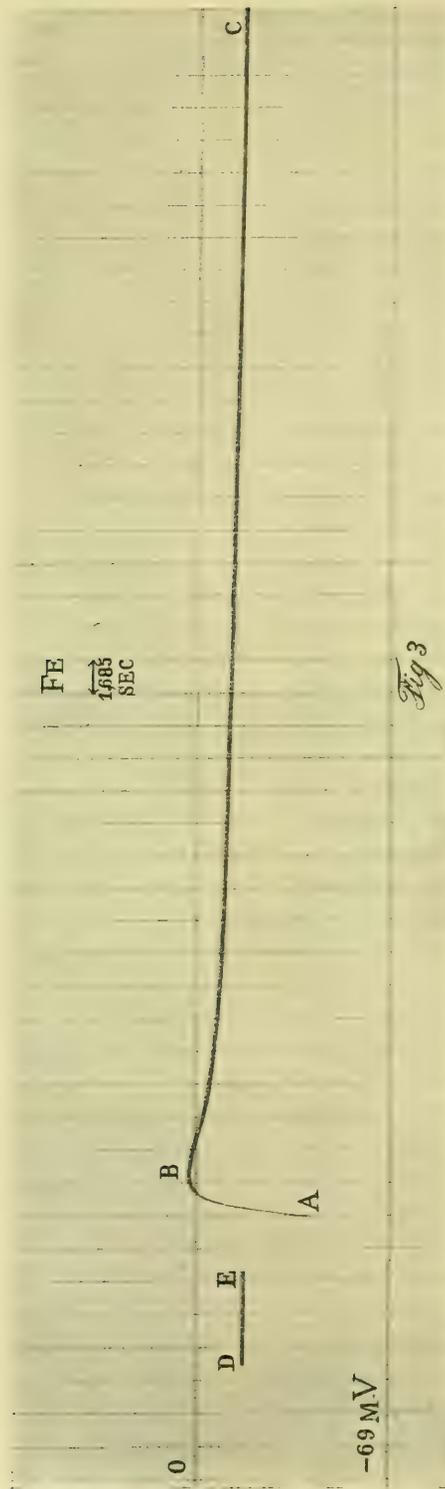
That the above explanation actually accounts for the phenomenon was proved by the fact that the phenomenon disappears altogether when the iron salt solution is previously heated in a hydrogen atmosphere with iron powder for some time. We then get a ferro salt solution which is in electromotive equilibrium with unary iron, so that in the boundary layer the ratio  $\frac{(Fe^{2+})}{(Fe^{3+})}$  practically does not change when the iron is anodically dissolved.

2. In the above mentioned communication it was pointed out that the potential of Nickel after polarisation in a solution of  $NiSO_4$  likewise passes through a minimum value, so that it was already supposed that this phenomenon would have to be explained in the same way as for iron.

To examine this the phenomenon for Nickel was first photographed when the metal was immersed in a solution of  $NiCl_2$ . In this the minimum shows itself very clearly, as the adjoined photo (fig. 1) sets forth.

The process is much quicker than when an  $NiSO_4$  solution is used, in consequence of the positively catalytic action of the chlorine ions.





*Fig 3*

Then the  $\text{NiCl}_2$  solution was heated for some time in a hydrogen atmosphere with finely divided Nickel that had been prepared by reduction of  $\text{NiO}$  at relatively low temperature, after which the experiment was repeated. As the following photo shows (fig. 2), the phenomenon had now entirely disappeared.

3. It was already pointed out in the preceding communication, that as it were, the *reflected image of the phenomenon* was to be expected, when e.g. iron immersed in a ferro-ferri-salt-solution is made cathode. The ratio  $\frac{(\text{Fe}^{2+})}{(\text{Fe}^{3+})}$  will be greatest at the iron surface, because the iron by sending  $\text{Fe}^{2+}$ -ions into solutions, strives to bring such a change in the boundary layer that in case of unary behaviour, it can be in equilibrium with it. Hence the farther the liquid layer is from the iron, the smaller will be the ratio  $\frac{(\text{Fe}^{2+})}{(\text{Fe}^{3+})}$ . When therefore iron, immersed in the supposed solution, is made cathode for a moment, the iron will be deposited from the boundary layer, and the ions from the following layer will diffuse into the boundary layer, and as the ratio  $\frac{(\text{Fe}^{2+})}{(\text{Fe}^{3+})}$  is smaller in this following layer, the iron after cathodic polarisation comes in contact with a liquid layer in which the ratio  $\frac{(\text{Fe}^{2+})}{(\text{Fe}^{3+})}$  is smaller than before the cathodic polarisation, which causes a stronger disturbance in noble direction, hence a less negative potential. After interruption of the current the said ratio will now increase again through the solution of the iron, so that the potential of the iron becomes again more negative. As on cathodic polarisation the iron gets covered with a layer of iron, which at first deviates from the unary iron in base direction, a *phenomenon after cathodic polarisation* was really to be expected also here, which would have to consist in this, that after interruption of the current the potential of the iron, after cathodic polarisation, passes through a maximum value.<sup>1)</sup>

As the photo on fig. 3 shows, this supposition is confirmed. The maximum  $B$  lies about 30 m.V. less negative than the initial potential indicated by the line  $DE$ .

It will now be examined whether the same phenomenon can also be observed for Nickel.

*Laboratory for General and Anorganic  
Amsterdam, June 27, 1919. Chemistry of the University.*

<sup>1)</sup> Through diminution of the total ion concentration the potential becomes more negative in a small degree, so that this circumstance still slightly counteracts the phenomenon under consideration.

**Physics.** — “*Researches on the Spectra of Tin, Lead, Antimony, and Bismuth in the Magnetic Field.*” By Dr. P. A. VAN DER HARST.  
(Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of June 28, 1919).

To carry out the observations of which the description will follow here, I have made use of the grating apparatus of the Physical Laboratory at Amsterdam, which had been placed at my disposal by Prof. P. ZEEMAN.

The experiments will be described more at length in my thesis for the doctorate, which will be published under the above title.

1. The spectra of tin, lead, antimony, and bismuth have been little, if at all, studied as to their ZEEMAN effect. The only scientist that made systematic observations on these elements in this respect, is PURVIS<sup>1</sup>). As is known, his results often present, however, great deviations from those of others, which is chiefly owing to an error in the measurement of the intensity of the field. Besides, PURVIS measures but few lines of these elements, viz. only those for which no longer time of exposure was required than half an hour to get them distinctly enough on the photographic plate to be measured. It seemed, therefore, desirable to me to subject the magnetic resolution for these elements to a closer examination.

2. The grating apparatus has been described by Mrs. H. B. BILDERBEEK—VAN MEURS<sup>2</sup>). The grating is a concave one of Rowland; radius 3 m., width 8 cm., 14438 lines per inch. Slit and grating are rigid with respect to each other, and with respect to an iron circular arch, on which the photographic plates are clasped, so that the exposure can be simultaneous in all orders.

The width of the slit generally amounted to about 20  $\mu$ , i.e. 8

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<sup>1</sup>) J. E. PURVIS. The Influence of a very Strong Magnetic Field on the Spark Spectra of Lead, Tin, Antimony, Bismuth and Gold. *Camb. Phil. Soc.*, **14**, 217, 1907; *Nature*, **76**, 166, 1907.

<sup>2</sup>) H. B. BILDERBEEK—V. MEURS. *Magnetische splitsing van het ultraviolette ijzerspectrum* ( $\lambda$  2300— $\lambda$  4500). *Diss.*, Amsterdam, 1909.

times the normal width of the slit<sup>1)</sup> for the arrangement used. As lens I used a double lens of quartz-calcium fluorite, transparent to ultraviolet rays, and moreover achromatic. The image of the source of light always fell at the same place of the slit.

It was generally necessary to separate the vertical and horizontal components with a calcspar rhombohedron, placed between the source of light and the lens. One of the two images formed in this way, was projected on the slit.

The magnet was a large-size Du Bois magnet, of which the end-planes of the conic pole tops were circular with a section of generally 8 mm. The distance between the poles was never greater than 4 mm. The magnetizing current was measured with a thermic Ampèremeter of HARTMANN and BRAUN, and generally amounted to 15 Ampères.

As source of light was used a spark between electrodes of the metal under examination, or alloys of it. The spark was, parallel to a condenser, in the secondary circuit of a transformer (of KOCH and STERZEL at Dresden), of which the primary circuit was fed by the municipal alternating current. Besides in the secondary circuit there was found an auxiliary spark, and an adjustable self-induction, in series with the spark. The particulars of the spark discharge (intensity of light, sharpness of the spectrum lines formed, melting of the electrodes) and the influence on this of self-induction, capacity and auxiliary spark will be fully discussed in my Thesis for the Doctorate.

As photographic plates I mostly used "Agfa Röntgen" plates, which I made previously sensitive to colour by bathing them in "Pynachrom." The plates prepared in this way were preferable in my opinion to the commercial colour-sensitive plates.

The measurements were performed with a ZEISS comparator. The results were divided into four groups according to their greater or less accuracy, which I gave in the tables the weight: 4, 3, 2, 1. The first group had a probable error of 1 % or less, the second from 1 % to 5 %, the third from 5 % tot 10 %, the fourth of more than 10 %. For the precautions taken in the measurements and the determination of the error I refer to my Thesis.

Also the preliminary experiments: the focussing of the plate-holder, the determination of the dissolving power, and the scalar value will not be treated further here, but may be found in my Thesis. The resolving power was certainly not smaller than the theoretical.

<sup>1)</sup> SCHUSTER. The Optics of the Spectroscope. *Astroph. J.*, **21**, 197, 1905.

3. *Observations.* The different intensities of the field were always determined by measuring the resolutions which were caused by them for Zn 4680, in which the degree of resolution of this line was compared with that obtained with the well-known absolute measurements of GMELIN, WEISS and COTTON <sup>1)</sup>, and FORTAT <sup>2)</sup>. As standard value for  $\frac{2d\lambda}{\lambda^2 H} \cdot 10^5$  I took the mean of the values obtained by them, viz. 9,376. I mostly worked with fields that lay in the neighbourhood of 30000 Gauss.

For tin and lead the electrodes were flexible strips of these metals, which were stretched cross-wise over the pole tops. For tin the spark was still cooled by blowing with a Föhn, because otherwise the electrodes were melted through too quickly, and the discharge passed into a continuous one. This method was discarded for lead, because with the Föhn there would often be a hitch, and an auxiliary spark was inserted in the secondary circuit in series with the illuminating spark. By regulating the distance of this auxiliary spark, we have much better control over the action of the illuminating spark. The electrodes of the auxiliary spark must not oxidise, however, because then we get there a more continuous discharge, whereas the very function of the auxiliary spark is to obviate this drawback, which is met with for the illuminating spark, and is not to be avoided there; it has, therefore, to ensure an interrupted discharge. For this purpose the bulbs of brass, of which the auxiliary spark consisted, were coated with platinum hoods. The strips of lead were kept tightly stretched over the pole tops, as otherwise they are apt to bend over towards each other, thus rendering the spark length too small.

No flexible bands could be made of antimony on account of the brittleness of this metal; I therefore used small flat rods of this metal as electrodes, which were clasped in a spark stand of brass. An advantage of this metal is that it has a pretty high melting-point, and that therefore the electrodes do not so quickly melt through. I have only used Bismuth as electrode as alloy with antimony (60 percentages by weight of bismuth) else it combined the drawbacks of tin and lead that it melted soon, and that of antimony that it was brittle. In the alloy the first drawback was eliminated, and it could further be used as antimony.

I must state further that in the tables in which SCHIPPERS <sup>3)</sup> records

<sup>1)</sup> P. ZEEMAN. Researches in Magneto-optics. Mac Millan and Co, London, 1913, p. 67. Deutsche Uebersetzung, Leipzig, J. A. BARTH, 1914.

<sup>2)</sup> R. FORTAT. Recherches de magneto-optique. Thèse, Paris, 1914.

<sup>3)</sup> H. SCHIPPERS, Messungen am Antimonspectrum. Zs. f. Wiss. Phot., 11, 235, 241.

his measurements on the antimony spectrum, and which are inserted in KAYSER'S Handbuch der Spectroscopie, I came across some lines which I never found on my plates. They are the lines 4370, 4295, 4287, 4091, 4078, 4038, 4024, 4006, 4004, 3979, 3721, 3467, 3460. From some experiments which I made with the purpose of ascertaining whether these lines existed, I think I have to conclude that SCHIPPERS was mistaken, and took lines of the third order for lines of the second order. These lines are in my opinion successively the following lines of the third order: 2913, 2863, 2858, 2727, 2719, 2692, 2683, 2671, 2670, 2653, 2480, 2311, 2306.

Besides my own results I have recorded those of PURVIS in the tables. I did not, however, use the intensity of the field which he gave, because this value is undoubtedly too high. Instead of this I have ascertained by the aid of my results what field intensity PURVIS used by comparing the average of his results with the average of mine; this separately for every element. Thus I found successively for tin, lead, antimony, and bismuth 30400, 31100, 28700, 31100 Gauss. COTTON estimates the field intensity used by PURVIS at 30800, he himself gives 39980 Gauss. For the better mutual comparison of the results those of PURVIS for  $\frac{d\lambda}{\lambda^2}$  have been divided in the sub-joined tables by the above mentioned field intensities.

The wavelengths are recorded in round values in international Ångström units.

4. *Discussion of the Tables.* On one of the photos for tin, on which the two kinds of components appeared at the same time, were further seen the quadruplets 2368 and 2762 (arc line), which had successively the values 34.6 (3), 48.0 (2), and 49. (2), 56.6 (3) as values for  $(d\lambda : \lambda^2 H) \cdot 10^{14}$ . The values between parentheses indicate the weight. 2266 and 2408 are split up. The amount of the splitting up cannot be measured, but amounts successively to less than 76 and 59. 2355 is probably split up.

For antimony the air line 3640 given by SCHIPPERS is also observed as split up. Of this  $(d\lambda : \lambda^2 H) \cdot 10^{14} = 66.2$ , weight 3.

There are some among the Bismuth lines, for which we should be cautious when judging about the splitting up. In what follows I shall indicate by an *s* everything that refers to vibrations normal to the lines of force, a *p* marks what refers to vibrations parallel to the lines of force.

3068. The *s*-figure was on some photos a triplet of asymmetric intensity. Then the corresponding *p*-figure was a doublet, which

looked exactly like a reversed line, so that I supposed the *s*-triplet to be really a doublet of a reversed line, the two middle parts of which coincided. This appeared to be true when later on, probably through the increase of the self-induction, the *s*-triplet was transformed into an ordinary doublet, the *p*-doublet into a single line. Also a photo, made with an alloy in which there was less bismuth, presented this latter form, viz. an ordinary triplet. Compare also what Purvis says about this line.

4260. I suppose that the same thing applies for this line as for 3068. When the self-induction was carried up in order to get certainty, this line became, however, too faint to allow us to draw conclusions. In favour of the supposition pleads that the splitting up of the *p*-components is as great as that of the *s*-components.

4122. This is in my opinion a different case. EXNER and HABSCHEK give two lines here, 4121.75 and 4122.08, the latter slightly heavier than the former. I too find two lines with a distance  $0.24 \text{ \AA. U.}$ , and the same ratio of strength. They do not look at all like one reversed line. The *s*-figure is a triplet which slightly changes in form when the circumstances change (see table). The difference in resolution between corresponding components in the two cases will no doubt be owing to the difficult and therefore unreliable observation. I think, however, that the difference in distance on either side of the middle component actually exists. A triplet is recorded for the *p*-figure. There is a blurred faint line, which towards red, and also but still fainter towards violet fades over some distance and then ceases more or less abruptly. I think, however, that I see a separation in some places. The measurements are of course worth little. PURVIS states 72.3 for the splitting up. This is about equal to what I found for the splitting up of the outer components of the *s*-triplet.

4723. WALI MOHAMMAD gives a description of this line as far as its behaviour is concerned for weak magnetic fields, studied by the aid of an echelon.<sup>1)</sup> I myself find on the *s* records 2 middle components, by the side of each of which there is a broad smudge, which stops pretty abruptly.

There is hardly any separation to be seen, which renders the measurement difficult. Nevertheless the different measurements of the outer components are in good harmony, those of the inner components not quite so good. The *p*-doublet is very close together, so that it is self-evident that a large comparative error arises there.

<sup>1)</sup> CH. WALI MOHAMMAD. Untersuchungen über magnetische Zerlegung feiner Spektrallinien im Vakuumlichtbogen. Ann. d. Phys. (4). 39, 225, 1912: Diss., Göttingen, 1912.

In the table the most violet component is marked by —, the most red by +. In this case the splitting up is not  $= 2 d\lambda$ , as is given in the heading of the column, but  $= d\lambda$ ; 0 does not mean that the line is at the place of the original unsplit one, but only that there is a component there.

5. *General results.* When we compare the results in what precedes, it strikes us at once that though the general course of the results may be the same, there yet occur a good many deviations, which amount to more than the given probable error. Further it occurs occasionally that a quadruplet is mistaken for a triplet, which is not astonishing with the middle components of these often blurred lines, which are so very close together. That something similar may be the case with other lines which are given as triplet, is not impossible.

An agreement as PURVIS gives between the quadruplets Pb 3740, Pb 2873, Sb 3723, Sb 3638, and possibly Sb 2668, Bi 2989, does not exist according to my measurements, except perhaps that between Sb 3638 and Bi 2989. When we examine whether there is perhaps some connection between the resolutions of the lines for which KAYSER and RUNGE <sup>1)</sup> find the known constant differences of frequency, it appears that there is no such connection. VAN LOHUIZEN <sup>2)</sup> has given series in the spectra of tin and antimony. The correctness of this is, however, doubted by SAUNDERS <sup>3)</sup> and ARNOLDS <sup>4)</sup>. I have now examined whether the laws of PRESTON were valid for these series. Of series I, which v. LOHUIZEN gives for the tin spectrum, I have found the resolutions (i.e. the values for  $d\lambda : \lambda^2 H$ , see table) 29,6 and 56,1 for the lines 3656 and 2785; in this series 2408 has a resolution smaller than 59. In series VII I find for the lines 3801, 2851, 2594, 2483, 2422 successively the splittings up 40,5, 42,5, 44,7 (56 and 46,8), 45,9. In this 2483 gives a quadruplet, the other lines give triplets. Series VIII: 3175 and 2483 give successively 69,7 (56 and 46,8) as splitting up. Antimony. Series XIII: the lines 3268 and 2574 have successively a resolution of 40,9 and 45. Series XIV: 3505 has the splitting up 63,7 and 2719 has 51,5. Series XX: 3233, 2653, 2478 have the resolutions 60,0, 63, 68,7.

<sup>1)</sup> H. KAYSER und C. RUNGE. Ueber die Spectra von Zinn, Blei, Arsen, Antimon, Wismuth. Abhandl. Berl. Akad. 1893; Wied. Ann., 52, 93, 1894.

<sup>2)</sup> T. VAN LOHUIZEN. Bijdrage tot de kennis van lijnenspectra. Diss., Amsterdam, 1912.

<sup>3)</sup> F. A. SAUNDERS. Astrophys. J., 36, 409.

<sup>4)</sup> R. ARNOLDS. Das Bogen- und Funkenspectrum von Zinn (von  $\lambda$  7800— $\lambda$  2069). Zs. f. wiss. Phot., 13, 325.

For the other tin and antimony series I have obtained no line that was split up or only one.

When we study the above results, we see that of the series for which it is possible to give an opinion in how far they follow PRESTON'S laws, only series XIII is not in conflict with these rules. In series VII the resolutions gradually increase; the first and the last splitting up lie farther apart than the limit of errors. Besides a quadruplet occurs there in the midst of the triplets. Even if these triplets were at bottom quadruplets, of which only the middle component was seen unsplit, which is very improbable, even then the amounts of the resolutions of the outer components would not be in agreement with each other. — The foregoing does not plead, in my opinion, in favour of v. LOHUIZEN'S results, though there remains a possibility that there are series that do not follow the rule of PRESTON. The efficacy of the ZEEMAN-effects for the discrimination of spectrum series was very apparent when RUNGE and PASCHEN found double lines in the spectra of Mg, Ca, Sr, Ba, which lines were changed in the magnetic field in the same way as double lines in principal and subordinate series in the spectra of the alkali-metals, while similar results were also found for some double lines from the Ra-spectrum. The hopes raised in 1902 and 1904 by the said researches with regard to the finding of series are accordingly not realized as far as the metals examined by me, are concerned.

I have further also tried to find regularities myself. For this purpose and also in what precedes I have made use of a graphical representation, which seemed convenient to me. I have arranged the resolutions in every element according to their amount, and then plotted them on the same scale vertically under each other, as is usually done when resolutions are to be compared inter se. Two successive resolutions on the whole differing but little in amount, there arises a curve. A vertical part represents a number of equal splittings up. When the corresponding lines are indicated by the side of the resolution figure, we have at once a survey of all the lines that possibly belong to a series. Compare the graphical representation.

In this the resolution figures are only represented half, which, however, does not give rise to difficulties, as they are symmetrical. The components are not indicated by single lines; I have blackened the whole region, where they can be found according to the probable error. The vertical lines traced in the figure, are at distances = half the normal splitting up. A sloping straight part of the curve means that between two definite amounts of splitting up the resolutions are regularly distributed. When I examined this for the resolutions



6. Tables.  
Tin.

$\lambda$	2 d $\lambda$ (H = 34930)				(d $\lambda$ : $\lambda^2$ H). 10 <sup>14</sup>				REMARKS.
	vibration ⊥ l. of f.		vibration // l. of f.		my own observations		Purvis' observ. $\frac{d \lambda}{\lambda^2} : 3,04$		
	2 d $\lambda$	g	2 d $\lambda$	g	vibrat. ⊥ l. of f.	vibrat. // l. of f.	vibrat. ⊥ l. of f.	vibrat. // l. of f.	
2335	0.322	3	0		84.6	0			
2422	0.188	3	0		45.9	0			
2429	0.18	2	0		43	0			
2483	0.24	2	0.202	3	56	46.8			
2496	0.242	3	0.14	0	55.7	32			
2547	0.227	3	0		50.2	0			
2572	0.291	3	0		63.1	0			
2594	0.210	3	0		44.7	0			
2661	0.297	3	0		59.9	0			
2707	0.357	3	0		69.9	0	72.2	0	
2780	0.322	3	0		59.7	0			
2785	0.304	3	0		56.1	0			
2813	split. up		0		split. up	0			The splitting up is about as great as that of 2814.
2814	0.270	3	0		48.9	0			
2840	0.385	3	0		68.5	0	66.1	0	
2851	0.241	3	0		42.5	0	42.6	0	
2863	0.372	3	0		65.0	0	64.5	0	
2914	0.21	2	0		35	0			
3009	0.429	4	0		67.9	0	66.1	0	
3033	0.313	3	0		48.7	0			
3034	0.463	3	0		72.1	0	70.5	0	
3142	0.368	3	0		53.2	0			
3175	0.490	3	0		69.7	0	70.4	0	
3219	0.261	3	0		36.1	0			
3262	0.357	3	0		48.2	0	49.5	0	
3283	0.30	1	0		40	0			
3331	0.462	4	0.302	3	59.6	39.1	58.8	40.0	
3352	0.39	2	0		49	0			
3656	0.276	3	0		29.6	0			
3801	0.409	4	0		40.5	0	40.0	0	
4525	0.725	3	0		50.6	0	53.6	0	
4586	0.59	2	0		40	0			
4618	1.316	3	0		88.5	0			
5563	split. up		0		split. up	0			
5589	1.02	2	0		47	0			

Lead.

$\lambda$	2 d $\lambda$ (H = 32810)				(d $\lambda$ : $\lambda^2$ H), $10^{14}$				REMARKS.
	vibration $\perp$ l. of f.		vibration // l. of f.		my own observations		Purvis' observ. $\frac{d \lambda}{\lambda^2} : 3.11$		
	2 d $\lambda$	g	2 d $\lambda$	g	vibrat. $\perp$ l. of f.	vibrat. // l. of f.	vibrat. $\perp$ l. of f.	vibrat. // l. of f.	
2476	0.294	3	0		73.2	0			
2577	0.277	3	0		63.6	0			
2613.7	0.23	1	0.13	1	51	30			<p><math>\perp</math> components: probably there are 4 lines here, 2 distinct ones and 2 faint ones; the two middle ones almost coincide. Probably the 2 faint ones belong to 2613.7, the two distinct ones to 2614. The distance middle faint one to middle distinct one = 0.45 Å.U., the distance according to Klein's table between 2613.7 and 2614 = 0.52 Å.U., which corresponds with what I measure on the photo without field. The agreement 0.45 and 0.52 is bad, but was not to be expected better in view of the components of 2613.7, which are so difficult to measure.</p> <p>// components: The splitting up 130 of 2613.7 rests on only one, very inaccurate measurement; the distance from middle to middle amounted to 0.49 Å.U.</p>
2614	0.24	2	0.21	2	53	46			
2663	0.308	3	0		66.1	0			
2698	0.414	3	0		87.0	0			
2713 <sup>1)</sup>	0.426	3	0		88.7	0			
2802	0.249	3	0.173	3	48.3	33.5	46.4	0	
2823	0.318	3	0		60.6	0	41.5	0	
2833	0.34	2	0.19	2	64	36	63.0	0	
2873	0.297	3	0.221	3	54.9	40.8	66.0	31.2	
3573	0.492	3	0		58.9	0			
3576 <sup>2)</sup>							62.1	0	
3640	0.57	2	0		66	0	66.0	0	
3672	0.481	3	0		54.2	0	57.0	0	
3683	0.609	3	0		68.3	0	69.2	0	
3740	0.576	4	0.208	3	62.8	22.6	64.4	30.6	
4020	0.501	3	0		47.3	0			
4058	0.646	3	0		57.8	0	57.6	0	
4062	0.729	4	0		67.3	0	68.2	0	
4168	0.57	2	0.468	3	50	40.8			
4245	0.537	3	0		45.3	0	42.2	0	
4386	0.543	3	0		43.0	0	37.0	0	
5609	1.706	3	0		52.0	0			

1) Arc line.

2) Should this perhaps be 3573?

## Antimony.

$\lambda$	$2 d \lambda (H = 26290)$				$(d \lambda : \lambda^2 H). 10^{14}$				REMARKS.
	vibration $\perp$ l. of f.		vibration // l. of f.		my own observations		Purvis' observ. $\frac{d \lambda}{\lambda^2} : 2.87$		
	$2 d \lambda$	g	$2 d \lambda$	g	vibrat. $\perp$ l. of f.	vibrat. // l. of f.	vibrat. $\perp$ l. of f.	vibrat. // l. of f.	
2311	0.23	2	0		81	0			
2446	0.152	3	0		48.7	0			
2478	0.220	3	0		68.7	0			
2529	0.20	2	0		60	0	55.2	0	
2574	0.15	2	0		45	0			
2598	0.19	2	0		54	0	55.5	0	
2612	0.217	3	0		60.9	0			
2617	split. up		0		split. up	0			
2653	0.23	2	0		63	0			
2669							78.4	34.7	} Are these perhaps the same?
2671	0.234	3	0.207	3	62.8	54.5			
2683	0.229	3	0		60.9	0			
2719	0.199	3	0		51.5	0			
2727	0.21	0	0		73	0			
2770							41.8	0	
2851	split. up		0		split. up	0			
2878	0.367 a. 0	3	0.160	3	$\pm 84.8$ a. 0	36.9	$\pm 77.0$ a. 0	36.4	} 3 $\perp$ and 2 // com- ponents.
3030	0.310	4	0		64.8	0	68.7	0	
3233	0.327	4	0		60.0	0	60.7	0	
3248	0.309	3	0		56.2	0			
3268	0.233	3	0		40.9	0			
3274	0.35	2	0.19	2	62	34			
3383	0.571	3	0		95.5	0			
3505	0.408	3	0		63.7	0			
3638	0.484	3	0.19	2	70.2	27	73.2	34.4	
3683	0.496	3	0		70.2	0			
3723	0.523	4	0.273	3	72.3	37.8	76.7	35.0	
4034	0.28	2	0		33	0			
4195	split. up		0		split. up	0			
4219	0.65	0	0		71	0			
4352	0.496	3	0		50.0	0			

## Bismut.

$\lambda$	$2 d \lambda (H = 26290)$				$(d \lambda : \lambda^2 H). 10^{14}$				REMARKS.	
	vibration $\perp$ l. of f.		vibration // l. of f.		my own observations		Purvis' observations $\frac{d \lambda}{\lambda^2} : 3.11$			
	$2 d \lambda$	g	$2 d \lambda$	g	vibrat. $\perp$ l. of f.	vibrat. // l. of f.	vibrat. $\perp$ l. of f.	vibrat. // l. of f.		
2628	0.175	0	0		48	0				
2898	0.247	3	0		56.0	0	56.6	0		
2938	0.286	3	0		63.2	0	59.2	0		
2989	0.314	3	0.124	2	66.7	26	64.0	26.4		
2993	0.260	3	0		55.3	0	51.8	0		
3025	0.292	3	split. up?		60.7	split. up?	59.2	21.5	p figure, i.e. I think I find the line broadened, on one plate, perhaps split up. Purvis' value is probably correct, and the resolution in this field was too small for this line to be seen separate.	
3068	0.346	3	0		69.8	0				
3077	0.242	3	0		48.7	0				
3397	0.241	3	0		39.8	0	41.5	0		
3511	0.461	3	0		71.1	0				
3596	0.321	3	0		47.2	0	47.6	0		
3793	0.285	2	0		38	0				
{ 4122	{ -0.30, 0, + 0.34 <sup>1)</sup>	2	{ -0.32, 0, + 0.31	1	{ -33, 0, + 38	{ -35, 0, + 34	72.3	0		H = 26290 Gauss; self induction 0.1 millihenry.
{ 4122	{ -0.28, 0, + 0.31 <sup>2)</sup>	2	{ too faint		{ -31, 0, + 35	{ too faint				
4260	-0.47, 0, + 0.48 <sup>3)</sup>	2	0.47	2	-49, 0, + 50	49				Is probably a triplet of a reversed line, so that the actual resolution is about 0.50.
4723	0.75 and 0.35	2	0.21	2	64 en 30	18	70.1 and 29.6	20.9		
5145	0.83	0	0		60	0				

<sup>1)</sup> The reddest component is somewhat stronger than the two others, which are equally strong. They are equally sharp.

<sup>2)</sup> The 2 outer components are about equally strong and sharp, the middle one is fainter and very vague.

<sup>3)</sup> The middle component is the strongest. The most violet component is fainter than the most red. Only when made more distinct by scratching with a needle this violet component could be measured.

occur. Hence in the search for series by the aid of equal resolutions hardly any result can be obtained by means of these curves. I have also tried it for tin and for antimony lines, which had a splitting up between 60 and 65, but likewise with no success.

I further examined whether there existed simple relations between the distances of the components for more than triple resolutions, which were rather accurately measured (weight 3 or 4). For Sb 3740, Sb 2671, and Sb 2878 I did not find simple relations. For Sn 3331 the distances of the components are in the ratio of 3 : 2, when the values 59,3 and 39,5, which lie within the limit of errors, are taken for them. For Pb 2802 the ratio is also as 3 : 2 with the values 49,1 : 32,7, Pb 2873 gives  $54,7 : 41,0 = 4 : 3$ ; Sb 3723 gives  $72,6 : 36,3 = 2 : 1$ . There is, however, no mutual connection or a simple relation to the normal resolution.

The results may be briefly summarized as follows:

The ZEEMAN-effect was measured of 35 tin lines, 23 lead lines, 27 antimony lines, and 16 bismuth lines. In this deviations were found with PURVIS' results which refer only to a few lines, probably in consequence of his less accurate measurements. No relation was found between the resolutions of those lines of the examined metals for which it was proposed to arrange them in series or those which are arranged according to the laws of the 2<sup>nd</sup> kind of KAYSER and RUNGE. The resolutions are pretty regularly distributed between values which amount to about 1 and 1,5 times the normal resolution. As an incidental result I found that a number of lines given by SCHIPPERS for antimony, are not real.

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which, by iteration, passes into

$$I [q^{(r)}(t)] = (-1)^r \frac{\Gamma(x)}{\Gamma(x-r)} I [t^{-r} q(t)]$$

It is easy to see that the latter equality is also valid for *negative* integral values of  $r$ . It is, however, remarkable that the same equality holds for not-integral values of  $r$ . This property I have made use of in the investigation of a function represented<sup>1)</sup> by a binomial series  $\sum c_n \binom{x-1}{n}$ , the most typical series in which a coefficient-function can be expanded. The object of this note is to give a proof of the general validity of the equality in question.

2. We substitute  $-\alpha$  for  $r$  and replace  $\varphi(t)$  by the expression  $\varphi(t) : (t-1)^\alpha$ , in order to have always to deal with functions which are regular for  $t = \infty$ . In accordance with RIEMANN'S<sup>2)</sup> definition of the derivative of negative order  $-\alpha$  of a function we assume as such the following one

$$(-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_1^\infty \frac{(u-t)^{\alpha-1} \varphi(u)}{(u-1)^\alpha} du \quad \dots \quad (5)$$

In this we take as path of integration the half-line beginning at  $u = t$ , whose prolonged part passes through  $u = 1$ ; and we assign the same arguments, lying between  $-\pi + \delta$  and  $+\pi - \delta$ , to  $u - 1$  and  $u - t$ . Then the so-defined derivative is also regular without the circle  $(1, 1)$  and zero for  $t = \infty$ ; and by the substitution  $u - 1 = (t - 1) : s$  we find after a slight reduction the expansion

$$(-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} = \sum_0^\infty \frac{\Gamma(n+1)g_n}{\Gamma(n+1+\alpha)(t-1)^{n+1}}, \dots \quad (6)$$

which, therefore, is related in a simple manner to the expansion for  $\varphi(t)$  itself. The order of this derivative, as will appear immediately from this series, is  $\alpha$  less than that of  $\varphi(t)$ , and therefore negative together with the latter order. On this supposition we may apply the operation  $I$  to it in the form (1), so that by this the *existence* of the first member of the equation to be proved, viz.

$$I \left[ (-1)^\alpha D^{-\alpha} \frac{\varphi(t)}{(t-1)^\alpha} \right] = \frac{\Gamma(x)}{\Gamma(x+\alpha)} I \left[ \frac{t^\alpha \varphi(t)}{(t-1)^\alpha} \right] \quad \dots \quad (7)$$

<sup>1)</sup> These Proceedings Vol. XXII, N<sup>o</sup>. 1. Nieuw Archief v. Wisk. XIII, 2e stuk (1920).

<sup>2)</sup> See BOREL, *Leçons sur les séries à termes positifs*, p. 74. The constant  $a$  occurring there is taken equal to  $\infty$  here, in connection with the regularity of  $\varphi(t)$  for  $t = \infty$ .

has been laid down. Passing to the second member we write

$$\frac{\Gamma(x)}{\Gamma(x+\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^1 s^{x-1} (1-s)^{\alpha-1} ds,$$

where the argument of  $s$  and of  $1-s$  is zero. Further

$$I \left[ \frac{t^\alpha \varphi(t)}{(t-1)^\alpha} \right] = \frac{1}{2\pi i} \int_{(1,1)} \frac{t^{\alpha+x-1} \varphi(t)}{(t-1)^\alpha} dt,$$

so that the second member in question is equal to

$$\frac{1}{2\pi i \Gamma(\alpha)} \int_{(1,1)} \frac{t^{\alpha+x-1} \varphi(t)}{(t-1)^\alpha} \left[ \int_0^1 s^{x-1} (1-s)^{\alpha-1} ds \right] dt.$$

If we substitute in the second integral  $s = u:t$ , this expression passes into

$$\frac{1}{2\pi i \Gamma(\alpha)} \int_{(1,1)} \frac{\varphi(t)}{(t-1)^\alpha} \left[ \int_0^t u^{\alpha-1} (t-u)^{\alpha-1} du \right] dt.$$

Since the argument of  $s$  was zero, the argument of  $u$  is equal to that of  $t$ ; thus the variable under the sign of integration goes along the *straight* line from  $u=0$  to  $u=t$  in the  $u$ -plane. But it may go as well from  $u=-i\delta$ , on the circumference of the circle  $(1, 1)$ , along that circumference in the positive direction to the point  $u=t$ . On this supposition we consider the system of two integrations, to be performed in succession, as a double integral. Then in the corresponding aggregate  $(u, t)$  a definite value  $u_1$  of  $u$  has to be associated with all those values of  $t$  lying, in the  $t$ -plane, on the circumference of the circle  $(1, 1)$  between  $t = u_1$  and the end-point  $t = +i\delta^1$  of that circle. Hence the double integral may be replaced by the pair of two successive integrations denoted in the expression

$$\frac{1}{2\pi i \Gamma(\alpha)} \int_{(1,1)} u^{\alpha-1} \left[ \int_u^0 \frac{(t-u)^{\alpha-1} \varphi(t)}{(t-1)^\alpha} dt \right] du,$$

where the integration according to  $t$  has to be performed in the *positive* direction from  $t = u$  to  $t = +i\delta^1$ ). On account of the properties of  $\varphi(t)$  the latter integration may be replaced by an integration

<sup>1</sup>) The here given argument is strong, in so far it is based upon *known* truths, if the functions under consideration are *finite* in the whole domain of integration. This is the case for  $R(x) > 1$  and  $R(\alpha) > 1$ , but, since both *endforms* are *analytic* functions of  $x$  [and of  $\alpha$  for  $R(x) > 0$  and  $R(\alpha) > 0$ , they must also be equal for the latter values.

from  $t = u$  to  $t = \infty$ , and one from  $t = \infty$  to  $t = 0$ . The latter gives an amount which is *independent* of  $u$ , and this amount gives zero for the *final* integration. Therefore, after changing the letters  $u$  and  $t$ , we may write for the preceding expression

$$\frac{1}{2\pi i} \int_{(1,1)} t^{x-1} \left[ \frac{1}{\Gamma(a)} \int_t^\infty \frac{(u-t)^{x-1} \varphi(u)}{(u-1)^x} du \right] dt,$$

and this, if we take (2) and (5) into account, is just equal to the first member of (7). The latter equation thus has been proved in case  $g < 0$ .

If  $g > 0$  and, to begin with,  $0 < g < 1$ , PINCHERLE defines the coefficientfunction of  $\varphi(t)$  by means of an auxiliary function

$$\varphi_1(t) = -D^{-1} \frac{\varphi(t)}{t-1} = \sum_0^\infty \frac{g_n}{(n+1)(t-1)^{n+1}} \dots \dots (8)$$

The order of  $\varphi_1(t)$  is lower by unity than that of  $\varphi(t)$  and thus negative, so that  $\omega_1(x) = I\varphi_1(t)$  is defined by (2). By (4) we have

$$I\varphi(t) = -I(t-1)\varphi_1'(t) = xI\varphi_1(t) - \theta^{-1}xI\varphi_1(t), \dots (9)$$

if  $\theta$  be the operation defined by  $\theta f(x) = f(x+1)$ . If we denote

by  $\varphi_\alpha(t)$  the result of the operation  $(\alpha) = (-1)^\alpha D^{-\alpha} \frac{1}{(t-1)^\alpha}$  applied to  $\varphi(t)$ , and by  $\varphi_{1\alpha}(t)$ , the result of the operation (1)  $= -D^{-1} \frac{1}{t-1}$ , applied to  $\varphi_\alpha(t)$ , we derive from the preceding equation

$$I\varphi_\alpha = xI\varphi_{1\alpha} - \theta^{-1}[xI\varphi_{1\alpha}]. \dots \dots (10)$$

Now, the operation  $(\alpha)$  is commutative, as will appear in the simplest manner from the expansion (6). Hence  $\varphi_{1\alpha} = \varphi_{\alpha 1}$ , if by the latter expression the result is denoted, which is obtained, if first the operation (1) and then  $(\alpha)$  is applied. But  $\varphi_1(t)$  is a function for which the equality (7) has already been proved; if this is taken into account, we may infer from (10), using the identity  $\Gamma(y+1) = y\Gamma(y)$ ,

$$I\varphi_\alpha = \frac{\Gamma(x)}{\Gamma(x+\alpha)} \left[ xI \frac{t^\alpha \varphi_1(t)}{(t-1)^\alpha} - (x+\alpha-1)I \frac{t^{\alpha-1} \varphi_1(t)}{(t-1)^\alpha} \right]. \dots (11)$$

Using again the relation (9) we may write

$$xI \frac{t^\alpha \varphi_1}{(t-1)^\alpha} - (x-1)I \frac{t^{\alpha-1} \varphi_1}{(t-1)^\alpha} = -I \left[ (t-1)D \frac{t^\alpha \varphi_1}{(t-1)^\alpha} \right]. \dots (12)$$

Further, in connection with (8)

$$-(t-1)D \frac{t^\alpha \varphi_1}{(t-1)^\alpha} = \frac{t^\alpha \varphi}{(t-1)^\alpha} + \frac{\alpha t^{\alpha-1} \varphi_1}{(t-1)^\alpha} \dots \dots \dots (13)$$

By means of (12) and (13) the equation (11) finally passes into the required identity (7).

The above argument may further be used to prove the same identity for  $h \geq g < h + 1$ , if it has already been established for  $g < h$ ,  $h$  being a positive integral number. The *general* validity has therefore been proved.

**Chemistry.** — “*In-, mono- and divariant equilibria*”. XIX. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated in the meeting of September 27, 1918).

*Equilibria of n components in n + 1 phases.*

We have seen in communication XVI, that the following equilibria may occur on a bivariant region  $E = F_1 + \dots + F_n$ :

1. the limit-curve  $E_0$ , when the quantity of one of the components approaches to zero.
2. the turning-line  $E_R$ , when a phase-reaction may occur between the  $n$  phases.
3. the critical curve  $E_K$ , when critical phenomena occur between 2 phases.

In the communications XVII and XVIII we have discussed more exactly the turning-line  $E_R$  and the region in the proximity of this line. Now we shall briefly consider the limit-line  $E_0$ .

When in an equilibrium  $E = F_1 + \dots + F_n$  the quantity of one of the components becomes zero, it then passes into the equilibrium  $E_0 = F_1 + \dots + F_n$  of  $n-1$  components in  $n$  phases, or, what comes to the same thing, of  $n$  components in  $n+1$  phases. Consequently we consider this last equilibrium, viz.

$$E = F_1 + F_2 + \dots + F_{n+1}$$

of  $n$  compounds in  $n+1$  phases.

The conditions for equilibrium are:

$$Z_i - x_i \frac{\partial Z}{\partial x_i} - y_i \frac{\partial Z}{\partial y_i} - \dots = K. \quad \dots \quad (1)$$

in which  $i = 1, 2, \dots (n+1)$

and further:

$$\left. \begin{aligned} \frac{\partial Z_1}{\partial x_1} = \frac{\partial Z_2}{\partial x_2} = \dots = \frac{\partial Z_{n+1}}{\partial x_{n+1}} = K_x \\ \frac{\partial Z_1}{\partial y_1} = \frac{\partial Z_2}{\partial y_2} = \dots = \frac{\partial Z_{n+1}}{\partial y_{n+1}} = K_y \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (2)$$

to which are still to be added the corresponding equations for  $z_1, z_2, \dots, u_1, u_2, \dots$  etc.

In (1) we have  $n+1$ , in (2)  $(n-1)(n+1)$  equations, together

$n^2 + n$  equations. Besides the  $(n-1)(n+1)$  variables  $x_1, x_2, \dots, y_1, y_2, \dots$  etc., we have still the  $n+2$  variables  $P, T, K, K_x, \dots$ , consequently in total  $n^2 + n + 1$  variables, the equilibrium has one licence, therefore, it is monovariant.

We have assumed with the deduction of (1) and (2), that each phase contains all components and that it has a variable composition, the considerations are true however also when phases occur with constant compositions or phases, which do not contain all components, provided that there is at least one variable phase, which contains all components. Later on we shall refer to other cases.

It follows from (1).

$$-V_i dP + H_i dT + x_i d \frac{\partial Z_i}{\partial x_i} + y_i d \frac{\partial Z_i}{\partial y_i} + \dots = -dK \dots \quad (3)$$

$$i = 1, 2, \dots (n + 1)$$

in which the sign  $d$  indicates, that we have to differentiate with respect to all the variables, which the function  $Z_i$  contains, consequently with respect to  $P, T, x_i, y_i, \dots$

It follows from (2):

$$\left. \begin{aligned} d \frac{\partial Z_1}{\partial x_1} &= d \frac{\partial Z_2}{\partial x_2} = \dots = d \frac{\partial Z_{n+1}}{\partial x_{n+1}} = dK_x \\ d \frac{\partial Z_1}{\partial y_1} &= d \frac{\partial Z_2}{\partial y_2} = \dots = d \frac{\partial Z_{n+1}}{\partial y_{n+1}} = dK_y \end{aligned} \right\} \dots \dots \dots (4)$$

in which the sign  $d$  has the same meaning as above.

We write the phase reaction that may occur in the equilibrium  $E = F_1 + \dots + F_{n+1}$ :

$$\lambda_1 F_1 + \lambda_2 F_2 + \dots + \lambda_{n+1} F_{n+1} = 0. \dots \dots \dots (5)$$

The  $n$  ratios between the  $n + 1$  reaction-coefficients are then defined by the  $n$  equations:

$$\left. \begin{aligned} \Sigma (\lambda) &= \lambda_1 + \lambda_2 + \dots + \lambda_{n+1} = 0 \\ \Sigma (\lambda x) &= \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{n+1} x_{n+1} = 0 \\ \Sigma (\lambda y) &= \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_{n+1} y_{n+1} = 0 \\ &\dots \dots \dots \end{aligned} \right\} \dots \dots (6)$$

When we add the equations (3), after having multiplied the first by  $\lambda_1$ , the second by  $\lambda_2$ , etc., it then follows with the aid of (4) and (6):

$$- \Sigma (\lambda V) \cdot dP + \Sigma (\lambda H) \cdot dT = 0$$

or

$$\frac{dP}{dT} = \frac{\Sigma (\lambda H)}{\Sigma (\lambda V)} \dots \dots \dots (7)$$

in which :

$$\left. \begin{aligned} \Sigma (\lambda V) &= \lambda_1 V_1 + \lambda_2 V_2 + \dots + \lambda_{n+1} V_{n+1} \\ \Sigma (\lambda H) &= \lambda_1 H_1 + \lambda_2 H_2 + \dots + \lambda_{n+1} H_{n+1} \end{aligned} \right\} \dots \dots \quad (8)$$

Herein  $\lambda_1 \dots \lambda_{n+1}$  have the values which follow from (6) consequently  $\Sigma (\lambda V)$  is the change in volume and  $\Sigma (\lambda H)$  the change in entropy, which occur at the phases-reaction (5).

The equilibrium, being monovariant, is represented in the  $P, T$ -diagram by a curve that we shall call  $E$ , its direction is defined by (7).

When we follow this curve  $E$ ,  $x_1 y_1 \dots x_2 y_2 \dots$  and consequently also  $\lambda_1 \lambda_2 \dots$  change from point to point along this curve. When in a definite point  $q$  of this curve  $\lambda_q$  becomes equal to zero, then in (8)  $\lambda_q V_q$  becomes  $= 0$  and  $\lambda_q H_q = 0$ . Then in the point  $q$  is true :

$$\left( \frac{dP}{dT} \right)_q = \frac{\Sigma (\lambda H)_q}{\Sigma (\lambda V)_q} \dots \dots \dots \quad (9)$$

Now the equilibrium  $E$  passes in point  $q$  under consideration into an equilibrium :

$$E_R = F_1 + \dots + F_{q-1} + F_{q+1} + \dots + F_{n+1} \dots \quad (10)$$

viz. into one of the equilibria  $E_R$  of  $n$  components in  $n$  phases between which a phases-reaction :

$$\lambda_1 F_1 + \dots \lambda_{q-1} F_{q-1} + \lambda_{q+1} F_{q+1} + \dots + F_{n+1} = 0 \dots \quad (11)$$

may occur. We have considered those equilibria  $E_R$  before. This equilibrium (11) is also monovariant and is represented in the  $P, T$ -diagram by a curve, the turning-line  $E_R$  of the region :

$$(F_q) = F_1 + \dots + F_{q-1} + F_{q+1} + \dots + F_{n+1} \dots \quad (12)$$

Formula (9) is also valid for this turning-line. It is apparent from our considerations :

“the curve which represents the equilibrium :

$$E = F_1 + F_2 + \dots + F_{n+1}$$

“is situated in the  $P, T$ -diagram in the common part of the  $n + 1$  “regions  $(F_1)(F_2) \dots (F_{n+1})$ ; when in a point  $q$  of this curve the “phase  $F_q$  does not participate in the reaction, the curve touches “the turning-line of the region  $(F_q)$  in this point  $q$ . In general  $n + 1$  “of those tangents may occur”.

When we apply this to the binary equilibrium  $E = F + L + G$ , in which  $F, L$  and  $G$  represent a binary compound, liquid and vapour, it then follows that the curve  $E$  must be situated within the three regions  $F + L, F + G$  and  $L + G$ . When in a point  $a$  the liquid  $L$  obtains the same composition as  $F$ , the curve  $E$  touches the limit-curve  $E_R = F + L$  (consequently the melting-line of  $F$ )

in *a*. If in *b* *G* gets the same composition as *F*, the curve *E* touches the limit-curve  $E_R = F + G$ , (consequently the sublimation-curve of *F*) in *b*. If it is possible for *L* and *G* to get the same composition in a point *c*, the curve *E* then touches the limit-curve  $E_R = L + G$  in *c*.

The known conditions, under which pressure or temperature are stationary along curve *E* (maximum or minimum) follow immediately from (7).

When one of the phases e.g.  $F_1$  is a gas, then in general  $V_1$  is very large in comparison with  $V_2, V_3, \dots$ . We now take on curve *E* a point *a*, where  $\lambda_1 = 0$ . Consequently curve *E* touches in this point *a* the limitline of the region  $F_1$ . In this point *a* is:

$$\Sigma(\lambda V)_a = (\lambda_1 V_1 + \dots + \lambda_{n+1} V_{n+1})_a \dots \dots (13)$$

and in general different from zero.

In the vicinity of point *a* however  $\lambda_1$  is no more zero and for  $\Sigma(\lambda V)$  the value from (8), not that from (13) remains true.  $V_1$  being large in comparison with  $V_2, V_3, \dots$ ,  $\Sigma(\lambda V)$  in (8) may be zero for a small value of  $\lambda_1$  already. When  $\Sigma(\lambda V) = 0$  then, according to (7)  $dT = 0$ , consequently the temperature is maximum or minimum.

Consequently we find: "when of the equilibrium

$$E = F_1 + F_2 + \dots + F_{n+1}$$

"one of the phases e.g.  $F_1$  is a gas the maximum (or the minimum) of the temperature of curve *E* is situated in general in the proximity of the point of contact of this curve with the limit-curve

$$E_R = F_1 + \dots + F_{n+1}$$

"of the region ( $F_1$ )."

When we apply this to the equilibrium  $E = F + L + G$ , discussed above, it is apparent, than this curve *E* has its point of maximum temperature in the vicinity of its point of contact with the melting line of *F*. We may also express this in the following way: the point of maximum-temperature of the equilibrium  $F + L + G$  differs only slightly from the melting-point of *F* under its own vapour pressure.

When we consider the ternary curve  $E = F_1 + F_2 + L + G$ , it follows that this curve must have its point of maximum temperature in the proximity of its point of contact with the limit-curve  $F_1 + F_2 + L$ , the common melting curve (or inversion-curve) of  $F_1 + F_2$ . We may also say: the point of maximum temperature of the equilibrium  $F_1 + F_2 + L + G$  differs only slightly from the common melting-point (or point of inversion) under its own vapour-pressure of  $F_1 + F_2$ .

When we imagine the solutions of this equilibrium *E* to be repre-

sented by a curve in a concentration diagram (e.g. a triangle), then its point of maximum temperature is situated in the proximity of its point of intersection with the line  $F_1F_2$ .

The same is true of course for the quaternary equilibrium  $E = F_1 + F_2 + F_3 + L + G$ . When we represent the solutions of this equilibrium by a curve in a concentration-diagram (e.g. a tetrahedron) its point of maximum temperature is then situated in the proximity of its point of intersection with the region  $F_1F_2F_3$ .

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*(To be continued).*

**Mathematics.** — “On LAMBERT’S series”. By Prof. J. C. KLUYVER.

(Communicated in the meeting of Sept. 27, 1919).

Transforming LAMBERT’S series

$$L(z) = \sum_{n=1}^{n=\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{n=\infty} t(n) z^n$$

SCHLÖMILCH <sup>1)</sup> deduced the asymptotic expansion

$$L(z) = \frac{C - \log \log \frac{1}{z}}{\log \frac{1}{z}} + \frac{1}{4} - \frac{B_1^2}{2 \cdot 2!} \left( \log \frac{1}{z} \right) - \frac{B_2^2}{4 \cdot 4!} \left( \log \frac{1}{z} \right)^2 - \dots$$

suitable to calculate the value of  $L(z)$  for real values of  $z$  approaching  $\pm 1$ . WIGERT <sup>2)</sup>, slightly changing the formula, obtained a somewhat more general expansion, which can be utilised, when  $z$ , tending to  $\pm 1$ , takes complex values, and LANDAU <sup>3)</sup> has simplified the proof of WIGERT’S result. HANSEN <sup>4)</sup> has shown that the circle  $|z| = 1$  is a natural limit of  $L(z)$  and in his lectures LANDAU established the same result in a simple and direct way. LANDAU’S proof is given in a paper by KNOPP <sup>5)</sup>, who in that paper, and also in his dissertation <sup>6)</sup>, discussed series of the more general type

$$N(z) = \sum_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n}$$

Assuming the coefficients  $b_n$  to fulfill certain restricting conditions, he could establish several cases in which the continuation of the function  $N(z)$  beyond the circle of convergence is impossible. In the present paper I propose to deduce a new asymptotic expansion

<sup>1)</sup> Ueber die Lambertsche Reihe. Zeitschr. f. Math. u. Phys., Bd. 6, 1861, p. 407.

<sup>2)</sup> Sur la série de Lambert et son application à la théorie des nombres. Acta Math., XLI, 1918, p. 197.

<sup>3)</sup> Ueber die Wigertsche asymptotische Funktionalgleichung für die Lambertsche Reihe. Archiv der Math. u. Phys., III. Reihe, XXVII, 1918, p. 141.

<sup>4)</sup> Démonstration de l’impossibilité du prolongement analytique de la série de Lambert et des séries analogues. Kong. Danske Vidensk. Selskabs Forth., 1907, p. 3.

<sup>5)</sup> Ueber Lambertsche Reihen. Journal f. d. reine u. ang. Math., Bd. 142, 1913, p. 283.

<sup>6)</sup> Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze. Berlin, 1907.

applying to the supposition that  $z$ , moving along a radius of the circle  $|z| = 1$ , approaches a given rational point on the circumference. Moreover the investigation will serve to add some results to those of KNOPP concerning the function  $N(z)$ .

1. Supposing  $p$  and  $q$  to be integers prime to each other,  $e^{\frac{2\pi i p}{q}} = \theta^p$  is a rational point of order  $q$  on the circle  $|z| = 1$ , and taking  $0 < x < 1$  we have

$$N(x\theta^p) = \sum_{n=1}^{n=\infty} b_{nq} \frac{x^{nq}}{1-x^{nq}} + \sum_{h=1}^{h=q-1} \sum_{n=0}^{n=\infty} b_{nq+h} \frac{x^{nq+h} \theta^{hp}}{1-x^{nq+h} \theta^{hp}}.$$

Now obviously we have

$$\frac{qx^{nq^2}}{1-x^{nq^2}} = \frac{x^{nq}}{1-x^{nq}} + \sum_{h=1}^{h=q-1} \frac{x^{nq} \theta^{hp}}{1-x^{nq} \theta^{hp}}$$

and

$$\sum_{h=1}^{h=q-1} \frac{\theta^{hp}}{1-\theta^{hp}} = -\frac{1}{2}(q-1),$$

hence we get

$$N(x\theta^p) - q \sum_{n=1}^{n=\infty} b_{nq} \frac{x^{nq^2}}{1-x^{nq^2}} + \frac{b_0}{2}(q-1) = \sum_{h=1}^{h=q-1} U_h(x, p), \quad \dots (1)$$

where the coefficient  $b_0$  is arbitrarily chosen and where

$$U_h(x, p) = \sum_{n=0}^{n=\infty} \left\{ b_{nq+h} \frac{x^{nq+h} \theta^{hp}}{1-x^{nq+h} \theta^{hp}} - b_{nq} \frac{x^{nq} \theta^{hp}}{1-x^{nq} \theta^{hp}} \right\} \dots (2)$$

The right handside can be transformed, we may write also

$$U_h(x, p) = - \left. \begin{aligned} & (1-x^h) \sum_{n=0}^{n=\infty} b_{nq+h} \frac{x^{nq} \theta^{hp}}{(1-x^{nq+h} \theta^{hp})(1-x^{nq} \theta^{hp})} + \\ & + \sum_{n=0}^{n=\infty} (b_{nq+h} - b_{nq}) \frac{x^{nq} \theta^{hp}}{1-x^{nq} \theta^{hp}} \end{aligned} \right\} (3)$$

In particular, taking  $b_n = 1$  also  $b_0 = 1$  we find for the function of LAMBERT

$$L(x\theta^p) - qL(xq^2) + \frac{1}{2}(q-1) = \sum_{h=1}^{h=q-1} T_h(x, p), \quad \dots (4)$$

where

$$\left. \begin{aligned} T_h(x, p) &= \sum_{n=0}^{n=\infty} \left\{ \frac{x^{nq+h} \theta^{hp}}{1-x^{nq+h} \theta^{hp}} - \frac{x^{nq} \theta^{hp}}{1-x^{nq} \theta^{hp}} \right\} = \\ &= - (1-x^h) \sum_{n=0}^{n=\infty} \frac{x^{nq} \theta^{hp}}{(1-x^{nq+h} \theta^{hp})(1-x^{nq} \theta^{hp})} \end{aligned} \right\} \dots (5)$$

Now we may observe that if  $0 \leq x \leq 1$ , the moduli of the factors

$(1-x^{nq+h}\theta^{hp})$  and  $(1-x^{nq}\theta^{hp})$  always exceed a fixed value, hence  $T_h(x, p)$  remains finite as  $x$  tends to unity. Accordingly the difference

$$L(x\theta^p) - qL(xq^2)$$

is always finite, therefore the point  $\theta^p$  must be a singular point of  $L(z)$  and the continuation of  $L(z)$  across the circumference must be regarded impossible.

To examine the behaviour of  $L(x\theta^p)$  if  $x \rightarrow 1$ , it will be sufficient to deduce a suitable expansion of the function  $T_h(x, p)$ .

Putting

$$x = e^{-y}, \quad \theta^p = e^{i\beta},$$

$$\varphi(u) = \frac{1}{e^{u-hi\beta}-1} = -\frac{1}{2} + \sum_{k=-\infty}^{+\infty} \frac{1}{u-hi\beta+2\pi ik},$$

we have at once

$$T_h(x, p) = \sum_{n=0}^{n=\infty} \{ \varphi(nqy+hy) - \varphi(nqy) \},$$

and from this expression it is seen, that the application of a suitable summation-formula will lead to the desired result.

I consider the set of trigonometrical series

$$g_1(t) = -\frac{1}{\pi} \sum_1^{\infty} \frac{\sin 2\pi kt}{k},$$

$$g_2(t) = +\frac{1}{2\pi^2} \sum_1^{\infty} \frac{\cos 2\pi kt}{k^2},$$

$$g_3(t) = +\frac{1}{2^2\pi^3} \sum_1^{\infty} \frac{\sin 2\pi kt}{k^3},$$

. . . . .

and the identity

$$0 = \int_0^{\infty} \left\{ g'_1\left(t - \frac{h}{q}\right) - g'_1(t) \right\} \varphi(tqy) dt \dots \dots (6)$$

Integration by parts transforms the indefinite integral into the expression

$$\left\{ g_1\left(t - \frac{h}{q}\right) - g_1(t) \right\} q(tqy) + \sum_{k=1}^{k=2m-1} (-1)^k q^{k_y} \left\{ g_{k+1}\left(t - \frac{h}{q}\right) - g_{k+1}(t) \right\} \varphi^{(k)}(tqy) +$$

$$+ q^{2m} y^{2m} \int \left\{ g_{2m}\left(t - \frac{h}{q}\right) - g_{2m}(t) \right\} \varphi^{(2m)}(tqy) dt$$

and here we have to introduce the limits 0 and  $\infty$ . In doing so we must take into account the discontinuities of  $g_1\left(t - \frac{h}{q}\right)$  and of

$g_1(t)$ . Further, we may notice that  $\varphi(tqy)$  and  $\varphi^{(k)}(tqy)$  vanish when  $t \rightarrow \infty$ , and that

$$g_{k+1}\left(-\frac{h}{q}\right) - g_{k+1}(0) = (-1)^{k+1} f_k\left(\frac{h}{q}\right)$$

where  $f_k$  denotes the Bernoullian polynomial of order  $k$ .

In this way the equation (6) leads to

$$\sum_{n=0}^{n=\infty} \left\{ \varphi(nqy + hy) - \varphi(nqy) \right\} = T_h(x, p) = - \sum_{k=0}^{k=2m-1} q^k y^k f_k\left(\frac{h}{q}\right) q^{(k)}(0) + R.$$

In this equation we have

$$\varphi(0) = \frac{i}{2} \cot \frac{\pi hp}{q} - \frac{1}{2},$$

$$\varphi^{(k)}(0) = \left(\frac{i}{2}\right)^{k+1} (D^{(k)} \cot v)_{v=\frac{\pi hp}{q}},$$

and the remainder integral  $R$  is given by

$$R = - q^{2m} y^{2m} \int_0^{\infty} \left\{ g_{2m}\left(t - \frac{h}{q}\right) - g_{2m}(t) \right\} \varphi^{(2m)}(tqy) dt.$$

Now since

$$\begin{aligned} |\varphi^{(2m)}(tqy)| &< 2m! \sum_{k=-\infty}^{k=+\infty} \frac{1}{\{t^2 q^2 y^2 + (h\beta - 2\pi k)^2\}^{m+\frac{1}{2}}} < \\ &< 2m! \sum_{k=-\infty}^{k=+\infty} \frac{1}{t^2 q^2 y^2 + (h\beta - 2\pi k)^2} \cdot \frac{1}{h\beta - \beta\pi k} \frac{1}{2m-1}, \end{aligned}$$

we infer that

$$|R| < 2 |g_{2m}(0)| 2m! q^{2m-1} y^{2m-1} \cdot \frac{\pi^{k=+\infty}}{2 \sum_{k=-\infty}^{k=+\infty} (h\beta - 2\pi k)^{2m}},$$

or that

$$|R| < 2m\pi |g_{2m}(0)| q^{2m-1} y^{2m-1} |\varphi^{(2m-1)}(0)|.$$

Thus it is shown that  $|R|$  is less than a finite multiple of the modulus of the last term in the sum that precedes, and we have found the asymptotic expansion

$$\begin{aligned} T_h(x, p) &= \frac{h}{2q} - \sum_{k=0}^{k=2m-2} \left(\frac{i}{2}\right)^{k+1} q^k \left(\log \frac{1}{x}\right)^k f_k\left(\frac{h}{q}\right) (D^{(k)} \cot v)_{v=\frac{\pi hp}{q}} + \\ &+ K q^{2m-1} \left(\log \frac{1}{x}\right)^{2m-1}, \end{aligned}$$

where the value of  $K$  is finite and independent of  $x$ .

Putting

$$A_k = \sum_{h=1}^{h=q-1} f_k\left(\frac{h}{q}\right) (D^{(k)} \cot v)_{v=\frac{\pi hp}{q}},$$

we have finally

$$\left. \begin{aligned} L(x\theta^p) - qL(xq^2) &= -\frac{1}{4}(q-1) - \sum_{k=0}^{k=2m-2} A_k \left(\frac{i}{2}\right)^{k+1} q^k \left(\log \frac{1}{x}\right)^k + \\ &+ K_1 q^{2m-1} \left(\log \frac{1}{x}\right)^{2m-1}, \end{aligned} \right\} (7)$$

and again  $K_1$  is finite and independent of  $x$ .

From (7) we conclude that, if  $x \rightarrow 1$ , the function  $L(x\theta^p)$  tends to infinity in the same manner as

$$qL(xq^2) \sim \frac{C - \log \log \frac{1}{x} - 2 \log q}{q \log \frac{1}{x}} + \frac{1}{4} q,$$

and that

$$\lim_{x \rightarrow 1} \{L(x\theta^p) - qL(xq^2)\} = -\frac{1}{4}(q-1) - \frac{i}{2q} \sum_{h=1}^{h=q-1} h \cot \frac{\pi h p}{q}. \quad (8)$$

Thus we have the rather remarkable result that it is only the real part of  $L(x\theta^p)$  that increases indefinitely in a manner quite independent of  $p$ .

2. If we only wish to shew that, when  $x \rightarrow 1$ , the function  $L(x\theta^k)$  cannot remain finite for all non zero values of  $k$ , an elementary discussion of the sum  $\sum_{k=0}^{k=q-1} L(x\theta^k)$  suffices to obtain this result.

We have at once

$$\sum_{k=0}^{k=q-1} L(x\theta^k) = q \sum_{n=1}^{n=\infty} t(nq) x^{nq},$$

and, denoting by  $D$  the greatest common measure of  $n$  and  $q$ , we may substitute

$$t(nq) = \sum_{d|D} \mu(d) t\left(\frac{n}{d}\right) t\left(\frac{q}{d}\right),$$

thus obtaining

$$\sum_{k=0}^{k=q-1} L(x\theta^k) = q \sum_{d|q} \mu(d) t\left(\frac{q}{d}\right) L(xq^d).$$

Hence, making  $x$  tend to 1, we get

$$\lim_{x \rightarrow 1} \frac{1}{L(x)} \sum_{k=0}^{k=q-1} L(x\theta^k) = \sum_{d|q} \frac{\varphi(d)}{d},$$

where  $\varphi(d)$  denotes the number of integers less than  $d$  and prime to  $d$ .

Supposing  $q$  to be equal to the product  $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ , we have

$$\sum_{d|q} \frac{\varphi(d)}{d} = \prod_{h=1}^{h=q} \left(1 + a_h \frac{p_h - 1}{p_h}\right) > 1,$$

and therefore, if  $x \rightarrow 1$ , for at least one value of  $k$  other than zero  $|L(x \theta^k)|$  must tend to infinity.

3. It will be readily seen that the method used in obtaining the asymptotic expansion of  $L(z)$  can be applied also to the series

$$N(z) = \sum_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n},$$

if only the coefficient  $b_n$  is a simple analytical function of the index  $n$ . If we choose, for instance,

$$b_n = -\frac{1}{n},$$

we have

$$M(z) = -\sum_{n=1}^{n=\infty} \frac{1}{n} \frac{z^n}{1-z^n} = \sum_{n=1}^{n=\infty} \log(1-z^n) = \log \prod_1^{\infty} (1-z^n)$$

and we can put

$$M(x \theta^p) - M(x q^2) = \log q + \sum_{h=1}^{h=q-1} V_h(x, p),$$

where

$$V_h(x, p) = \sum_{n=0}^{n=\infty} \{ \log(1-x^{nq+h} \theta^{hp}) - \log(1-x^{nq} \theta^{hp}) \}.$$

Operating as before we shall find

$$\begin{aligned} M(x \theta^p) - M(x q^2) &= \log q - \frac{1}{q} \sum_{h=1}^{h=q-1} \left\{ h \log \left| 2 \sin \frac{\pi hp}{q} \right| + \pi i h g_1 \left( \frac{hp}{q} \right) \right\} - \\ &- \frac{1}{24} (q^2 - 1) \log \frac{1}{x} - \sum_{k=1}^{k=2m-1} A'_k \left( \frac{i}{2} \right)^k q^k \left( \log \frac{1}{x} \right)^k + K_1 q^{2m} \left( \log \frac{1}{x} \right)^{2m} \end{aligned}$$

where

$$A'_k = \sum_{h=1}^{h=q-1} f_k \left( \frac{h}{q} \right) (D^{(k-1)} \cot v)_{v=\frac{\pi hp}{q}},$$

and  $K_1$  has a finite value independent of  $x$ .

Now for EULER's product  $\prod_1^{\infty} (1-x^n)$  we have, when  $0 \leq x < 1$ ,

$$M(x) = +\frac{1}{24} \log \frac{1}{x} + \frac{1}{2} \log 2\pi - \frac{1}{2} \log \log \frac{1}{x} - \frac{\pi^2}{6 \log \frac{1}{x}} + M \left( e^{-\frac{4\pi^2}{\log \frac{1}{x}}} \right),$$

hence equation (9) shows completely the behaviour of  $M(x\theta^p)$ , when  $x \rightarrow 1$ .

In fact, we may write

$$\lim_{x \rightarrow 1} \{M(x\theta^p) - M(xq^s)\} = \log q - \frac{1}{q} \sum_{h=1}^{h=q-1} \left\{ h \log \left| 2 \sin \frac{\pi h p}{q} \right| + \pi i h g_1 \left( \frac{h p}{q} \right) \right\} \quad (10)$$

and again we may notice that only the real part of  $M(x\theta^p)$  becomes infinite, the imaginary part tending to a finite limit.

Taking, for instance,  $p = 1$ ,  $q = 2$ , we shall find

$$\lim_{x \rightarrow 1} \{M(-x) - M(x^2)\} = \frac{1}{2} \log 2,$$

or

$$\lim_{x \rightarrow 1} \frac{\prod_1^{\infty} (1 + x^{2n-1})}{\prod_1^{\infty} (1 + x^{2n})} = \sqrt{2},$$

a known result in the theory of the  $\mathcal{D}$ -functions.

4. Finally, I will state that the discussion of the fundamental equations (1), (2) and (3) furnishes the proof that the function

$$N(z) = \sum_{n=1}^{n=\infty} b_n \frac{z^n}{1-z^n}$$

cannot be continued beyond the circle  $|z| = 1$  in each of the following cases:

I.  $A > b_n > B > 0$ .

In this case we shall have

$$\frac{A}{q} > \lim_{x \rightarrow 1} \frac{1-x}{\log \frac{1}{1-x}} N(x\theta^p) > \frac{B}{q}.$$

II.  $\lim_{n \rightarrow \infty} b_n = A \neq 0$ .

Now it will be seen that

$$\lim_{x \rightarrow 1} \frac{1-x}{\log \frac{1}{1-x}} N(x\theta^p) = \frac{A}{q}.$$

III.  $\lim_{n \rightarrow \infty} \frac{b_n}{n^s} = A \neq 0$ ,  $s > 0$ .

In this case the equation holds

$$\lim_{x \rightarrow 1} (1-x)^{1+s} N(x\theta^p) = \frac{A}{q^{1+s}} \Gamma(1+s) \zeta(1+s).$$

IV.  $\lim_{n \rightarrow \infty} b_n = 0$  and either  $b_n > 0$ , or the series  $\sum_{n=1}^{n=\infty} \frac{b_{nq}}{nq}$  converges for an unlimited number of integers  $q$  and the sum of this series is different from zero. In the latter supposition we have

$$\lim_{x \rightarrow 1} (1-x) N(x \theta^q) = \sum_{n=1}^{n=\infty} \frac{b_{nq}}{nq} \quad (1)$$

V.  $b_n \geq 0$  and moreover

$$\lim_{n \rightarrow \infty} \frac{1}{n} [b_{q+h} + b_{2q+h} + \dots + b_{nq+h}] = 0$$

$$(h = 0, 1, 2, \dots, q-1)$$

for an unlimited number of integers  $q$ .

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<sup>1)</sup> By a totally different method FRANCK deduced this formula in his paper: Sur la théorie des séries. Math. Annalen, Bd. 52, 1899, p. 529.



$$(Z, Z^{A_i}), \quad Z = e^{\frac{2\pi i}{m}}$$

vor durch  $S_i$ , so sind

$$S_1, S_2, \dots, S_\varphi \dots \dots \dots (2)$$

die Substitutionen des Kreiskörpers. Zur Abkürzung schreibe ich  $q$  statt  $\varphi(m)$ . Die Gruppe der Zahlklassen ( $\text{mod } m$ ) können wir also kurzweg die Gruppe des Körpers nennen. Und wenn wir Untergruppen von (2) brauchen, können wir die isomorphen Untergruppen von (1) dafür nehmen.<sup>1)</sup> Die Gruppe (1) bezeichnen wir weiter durch  $G$ .

Die Zahl  $m$  schreiben wir in der Form

$$m = 2^{h_*} l_1^{h_1} l_2^{h_2} \dots$$

wo  $l_1, l_2, \dots$  ungerade Primzahlen bezeichnen.

Um die ABEL'sche Gruppe  $G$  durch eine Basis darzustellen<sup>2)</sup> nehmen wir die primitive Wurzeln  $r_1, r_2, \dots$  resp. von  $l_1^{h_1}, l_2^{h_2}, \dots$ . Wir bestimmen die Zahlen  $A_0, A_*, A_1, \dots$  durch die Congruenzen:

$$\begin{aligned} A_0 &\equiv -1 \pmod{2^{h_*}} \equiv 1 \pmod{l_1^{h_1}} \equiv 1 \pmod{l_2^{h_2}} \dots \\ A_* &\equiv 5 \quad \text{,,} \quad \equiv 1 \quad \text{,,} \quad \equiv 1 \quad \text{,,} \quad \dots \\ A_1 &\equiv 1 \quad \text{,,} \quad \equiv r_1 \quad \text{,,} \quad \equiv 1 \quad \text{,,} \quad \dots \\ A_2 &\equiv 1 \quad \text{,,} \quad \equiv 1 \quad \text{,,} \quad \equiv r_2 \quad \text{,,} \quad \dots \\ &\dots \dots \dots \end{aligned}$$

Diese Zahlen bilden eine Basis von  $G$ . Jede Zahl aus  $G$  kann also nur auf eine Weise geschrieben werden in der Form

$$A_0^{u_0} A_*^{u_*} A_1^{u_1} \dots \pmod{m}$$

wobei die Exponenten durch die Bedingungen

$$0 \leq u_0 < 2; \quad 0 \leq u_* < \frac{1}{2} \varphi_*; \quad 0 \leq u_1 < \varphi_1; \dots$$

bestimmt sind. Zur Abkürzung ist dabei für  $\varphi(2^{h_*}), \varphi(l_1^{h_1}), \dots$  geschrieben worden:  $\varphi_*, \varphi_1, \dots$ .

Wir machen noch die folgenden Bemerkungen:

1. Wenn  $h_* = 0$  so fallen  $A_0$  und  $A_*$  weg,
2. Wenn  $h_* = 2$  so fällt  $A_*$  weg.
3. Der Fall  $h_* = 1$  kann für das Folgende ausser Acht gelassen werden<sup>3)</sup>.

§ 2. Die Zerlegungs- und die Trägheitsgruppe eines Primideals.

Satz 1. Die Trägheitsgruppe eines Primideals  $\mathfrak{p}$  das nicht teilbar ist auf  $m$ , besteht nur aus der identischen Substitution.

<sup>1)</sup> Wir sprechen weiterhin also von der Substitution  $a$ , welches bedeutet die Substitution die  $Z$  durch  $Z^a$  ersetzt.

<sup>2)</sup> „W“ § 18.

<sup>3)</sup> „W“ § 20.

Beweis: Wir wissen, dasz

$$x^m - 1 = (x-1)(x-Z) \dots (x-Z^{m-1}).$$

Es ergibt sich hieraus, nach der Division durch  $x-1$ , und nach der Substitution  $x=1$ :

$$m = (1-Z)(1-Z^2) \dots (1-Z^{m-1}) \dots \dots \dots (1) \quad (1)$$

Jeder Factor des zweiten Gliedes is also nicht teilbar durch  $\mathfrak{P}$ . Es sei nun  $S = (Z: Z^a)$  eine Substitution der Trägheitsgruppe; dann musz<sup>1)</sup>:

$$S Z \equiv Z \pmod{\mathfrak{P}}$$

also:

$$Z (Z^{a-1} - 1) \equiv \pmod{\mathfrak{P}}$$

und dies ist nur möglich wenn  $a = 1$ .

Satz 2. Die Zerlegungsgruppe eines nicht in  $m$  aufgehenden Primideals, das auf die Primzahl  $p$  teilbar ist, besteht aus den  $f$  Potenzen:

$$p, p^2, \dots, p^f \pmod{m}$$

wo  $f$  der kleinste Exponent ist, für welchen  $p^f \equiv 1 \pmod{m}$ .

Beweis: Dasselbe welches man findet in "W" S. 742 für den Fall  $m = h$ .

Bevor wir nun die Trägheitsgruppe eines Primideals bestimmen können, müszen wir Satz 146 von „H.“ vervollständigen.

Satz 3. Im Kreiskörper finden die folgenden Zerlegungen statt:

$$1. \quad 1 - Z^{\frac{n}{2^{h-h'}}} = (\xi_{01} \dots \xi_{0e_0}) 2^{h_*'}$$

für  $h_*' = 0, 1, \dots, h_* - 1$ , und  $n < 2^{h_* - h_*'}$  während  $n$  nicht teilbar ist durch 2.

$$2. \quad 1 - Z^{\frac{n}{l_i^{h_i - h_i'}}} = (\xi_{i1} \dots \xi_{ie_i}) l_i^{h_i'}$$

für  $h_i' = 0, 1, \dots, h_i - 1$ ,  $n < l_i^{h_i - h_i'}$ , während  $n$  nicht teilbar ist durch  $l_i$ .

$$3. \quad 2 = (\xi_{01} \dots \xi_{0e_0})^{f_*}$$

wo  $\xi_{0j}$  Primideale sind und  $f_*$  der kleinste Exponent bedeutet, für welchen  $2^{f_*} \equiv 1 \pmod{\frac{m}{2^{h_*}}}$  und  $e_* f_* = q \left( \frac{m}{2^{h_*}} \right)$ . Der Grad der Primideale ist  $f_*$ .

$$4. \quad l_i = (\xi_{i1} \dots \xi_{ie_i})^{f_i}$$

wo  $\xi_{ij}$  verschiedene Primideale sind des Grades  $f_i$ .  $f_i$  ist der kleinste Exponent für welchen

<sup>1)</sup> „H.“ S. 251.

$$l_i^{f_i} \equiv 1 \left( \text{mod } \frac{m}{l_i^{h_i}} \right) \text{ und } e_i f_i = g \left( \frac{m}{l_i^{h_i}} \right)$$

Alle Zahlen, welche die Form  $1-Z^g$  haben, und die nicht sich vorfinden unter  $1^\circ$  und  $2^\circ$ , sind Einheiten.

Beweis: Die unter  $3^\circ$  und  $4^\circ$  angegebenen Zerlegungen der Primzahlen  $2$  und  $l_i$  sind bekannt. <sup>1)</sup>

Aus (1) § 2 ersieht man dasz jede Zahl der Form  $1-Z^g$  eine Einheit ist, oder nur teilbar ist durch Primideale die in  $m$  aufgehen.

Im Körper  $k \left( e \frac{2\pi i}{2^{h_*-h_*'}} \right)$  gilt die Zerlegung <sup>2)</sup>:

$$2 = l_0 \cdot 2^{h_*-h_*'-1} \text{ indem } l_0 = \left( 1 - Z \frac{m}{2^{h_*-h_*'}} \right)$$

ein Prim-Hauptideal ist. Hieraus ergibt sich die unter  $1^\circ$  angegebene Zerlegung für den Fall  $n = 1$ .

Weiter ist im Körper  $k \left( e \frac{2\pi i}{l_i^{h_i-h_i'}} \right)$

$$l_i = l_i' \cdot (l_i^{h_i-h_i'}) \text{ indem } l_i = \left( 1 - Z \frac{m}{l_i^{h_i-h_i'}} \right)$$

ein Prim-Hauptideal ist. Hieraus ergibt sich die unter  $2^\circ$  angegebene Zerlegung für den Fall  $n = 1$ .

Um die übrigen Zerlegungen von  $1$ . zu beweisen bemerken wir

dasz im Körper  $k \left( e \frac{2\pi i}{2^{h_*-h_*'}} \right)$  die Zerlegungsgruppe des Primideals  $l_0$  aus allen Substitutionen dieses Körpers besteht. Diese Substitutionen

ersetzen die den Körper bestimmende Zahl  $Z \frac{m}{2^{h_*-h_*'}}$ , durch die Potenzen dieser Zahl, deren Exponenten alle Zahlen  $< 2^{h_*-h_*'}$  sind welche nicht durch  $2$  teilbar sind. Es folgt hieraus die unter  $1$ . angegebene Zerlegung für alle Fälle  $n > 1$ .

Auf gleiche Art findet man die übrigen, unter  $2$ . angegebenen Zerlegungen.

Wir müssen nun noch den letzten Teil des Satzes  $3$  beweisen. Das Produkt aller unter  $1$ . und  $2$ . genannten Zahlen der Form  $1-Z^g$  ist teilbar durch eine Potenz des Produktes  $\xi_{01} \dots \xi_{0e_0}$  deren Exponent gleich

<sup>1)</sup> „H“. Satz 125.

<sup>2)</sup> „H“. Satz 122.

$\varphi(2^{h_*}) + 2\varphi(2^{h_*-1}) + \dots + 2^{h_*-1}\varphi(2) = h_*\varphi_*$   
 ist, und auch teilbar durch eine Potenz des Produktes  $\ell_{i1} \dots \ell_{ie_i}$ ,  
 deren Exponent gleich

$$\varphi(l_i^{h_i}) + l_i\varphi(l_i^{h_i-1}) + \dots + l_i^{h_i-1}\varphi(l_i) = h_i\varphi(l_i^{h_i})$$

ist. Man findet leicht dasz auch  $m$  genau durch diese beiden Produkte  
 teilbar ist. Aus (1) ergibt sich also dasz alle Zahlen der Form  
 $1-Z^g$ , welche nicht unter 1. und 2. zerlegt sind, Einheiten darstellen.

**Satz 4.** Die Trägheitsgruppe eines in  $l_i$  aufgehenden Primideals  
 $\ell$  besteht aus den  $\varphi_i$  Substitutionen welche die Zahl  $Z$  ersetzen durch  
 Potenzen von  $Z$  deren Exponenten die Zahlen sind, welche nicht  
 teilbar sind durch  $l_i$  und prim zu  $m$ , und die Form haben:

$$1 + n \frac{m}{l_i^{h_i - h_i'}} \quad , \quad h_i' = 0, 1, \dots, h_i - 1 \quad , \quad n < l_i^{h_i - h_i'}$$

Dasselbe gilt für  $l_i = 2$  wenn man überall den Index  $i$  durch \*  
 ersetzt.

Beweis: Eine Substitution  $S$  ist dann und nur dann eine Substi-  
 tution der Trägheitsgruppe wenn für alle ganzen Zahlen  $\Omega$  des Körpers  
 $S\Omega \equiv \Omega \pmod{\ell}$  <sup>1)</sup> ist.

Nimmt man nun  $\Omega = Z$  so ergibt sich leicht der Beweis des Satzes.

**Satz 5.** Die Zerlegungsgruppe eines in  $l_i$  aufgehenden Primideals  
 $\ell$  entsteht wenn man die Trägheitsgruppe multipliziert mit den  $f_i$   
 ersten Potenzen der Zahl

$$l_i + n \frac{m}{l_i^{h_i}}$$

wo  $n$  so bestimmt ist dasz die angegebene Zahl diejenige von  $G$   
 ist, die congruent ist mit  $l_i \pmod{\frac{m}{l_i^{h_i}}}$ .  $f_i$  ist der Grad des Prim-  
 ideals  $\ell$ .

Dasselbe gilt für  $l_i = 2$  wenn man überall  $i$  durch \* ersetzt.

Beweis: Der Körper  $k = k(Z^{l_i^{h_i}})$  gehört im Körper  $K = k(Z)$   
 zur Untergruppe:

$$A_1, A_1^2, \dots, A_1^{f_i}$$

Die  $\varphi\left(\frac{m}{l_i^{h_i}}\right)$  Substitutionen:

$$A_0^{u_0} A_*^{u_*} A_1^{u_1} \dots \pmod{m}$$

$$u_0 = 0, 1; \quad u_* = 0, 1, \dots, \frac{1}{2}\varphi_* - 1; \quad u_1 = 0, 1, \dots, \varphi_1 - 1; \dots$$

1) „H.“ 251.

ergeben  $\left(\text{mod } \frac{m}{l_i^{h_i}}\right)$  eine gleiche Anzahl Substitutionen des Körpers  $k$ . Unter die Zahlen von  $G$  gibt es also nur eine Zahl, die  $\left(\text{mod } \frac{m}{l_i^{h_i}}\right)$  mit  $l_i$  congruent ist. Dies sei die im Satze genannte Zahl. Wir zeigen weiter dasz die  $f_i$  ersten Potenzen dieser Zahl genau die Gruppe formieren mit der die Trägheitsgruppe multipliziert werden musz um die Zerlegungsgruppe zu ergeben. In  $k$  findet die folgende Zerlegung statt:

$$l_i = \mathfrak{P}_1 \dots \mathfrak{P}_{e_i}^{f_i}$$

wo  $\mathfrak{P}_j$  von einander verschiedene Primideale sind. Auf Grund des Satzes 2 ist die Zerlegungsgruppe des Primideals  $\mathfrak{P}_j$

$$l_i, l_i^2, \dots, l_i^{f_i}$$

Im Körper  $K$  stimmen diese Zahlen überein mit den im Satze genannten. Aus der Zerlegung von  $l_i$  im  $K$ , dem Satze 3 zufolge, ergibt sich

$$\mathfrak{P}_i = \mathfrak{L}_i^{f_i}$$

Die Substitutionen welche  $\mathfrak{P}_i$  unverändert lassen, lassen auch  $\mathfrak{L}_i$  unverändert. Die im Satze genannten Zahlen gehören also zur Zerlegungsgruppe. Aus Satz 69 von „H“ ergibt sich nun der vollständige Beweis.

## II. Die Unterkörper des Kreiskörpers.

### § 3. Bestimmung und Eigenschaften aller Untergruppen von $G$ .

Wir deuten den Kreiskörper selbst weiter an durch  $K$ . Zu jeder Untergruppe von  $G$  gehört ein Unterkörper von  $K$ <sup>2)</sup>. Wir ziehen nur primäre Unterkörper in unsere Betrachtungen hinein. Es hat dies keine Einschränkung der Allgemeinheit zur Folge<sup>3)</sup>. Jede Untergruppe von  $G$  wird auf folgende Art bestimmt: <sup>4)</sup> Denken wir uns gegeben die Systeme der ganzen Zahlen:

$$\left. \begin{aligned} & b_{0n}, b_{*n}, b_{1n}, \dots \\ & n = 1, 2, \dots, \frac{\varphi}{r} \text{ wobei} \\ & 0 \leq b_{0n} < 2; \quad 0 \leq b_{*n} < \frac{1}{2} \varphi_*; \quad 0 \leq b_{1n} < \varphi_1; \dots \end{aligned} \right\} \dots (1)$$

1) „H.“ Satz 125.

2) „H.“ § 38.

3) „W.“ S. 77.

4) „W.“ § 14

Man konstruiert sie auf folgende Art. Zuerst kann man einige Systeme willkürlich aufschreiben. Sodann fügt man andere hinzu<sup>1)</sup>, sodass alle zusammen eine Gruppe darstellen; d. h. sodass die Summen der übereinstimmenden Zahlen zweier Systemen ein System bilden dasz man schon aufgeschrieben hat. Dabei rechnet man die Zahlen beziehungsweise zur Moduli

$$2, \frac{1}{2} \varphi_*, \varphi_1, \dots$$

Wir nehmen nun an dasz das System (1) auf diese Weise zusammengesetzt ist. Es sei nun

$$\varepsilon_0 = e^{\frac{2\pi i}{2}}; \quad \varepsilon_* = e^{\frac{2\pi i}{\frac{1}{2}\varphi_*}}; \quad \varepsilon = e^{\frac{2\pi i}{\varphi_1}}; \dots$$

Wir betrachten alle Systeme der ganzen Zahlen

$$0 \leq a_0 < 2; \quad 0 \leq a_* < \frac{1}{2} \varphi_*; \quad 0 \leq a_1 < \varphi_1; \dots$$

mit den Bedingungen:

$$\varepsilon_0^{a_0 b_{0n}} \varepsilon_*^{a_* b_{*n}} \varepsilon_1^{a_1 b_{1n}} \dots = 1.$$

$$n = 1, 2, \dots, \frac{\varphi}{r}$$

Die Zahlen  $A$ , die bestimmt sind durch die Congruenzen

$$A \equiv A_0^{a_0} A_*^{a_*} A_1^{a_1} \dots \pmod{m} \dots \dots \dots (2)$$

stellen eine Untergruppe von  $G$  dar, welche dann und nur dann primär ist, wenn nicht jede Zahl  $b_{0n}$  durch 2 teilbar ist; nicht jede Zahl  $b_{*n}$  durch 2 teilbar ist; nicht jede Zahl  $b_{1n}$  durch  $l_1$  teilbar ist wenn  $h_1 > 1$  ist, und nicht jede Zahl  $b_{1n}$  durch  $l_1 - 1$  teilbar ist wenn  $h_1 = 1$  ist; u. s. w. Der Grad der auf diese Art bestimmte Untergruppe ist gleich  $r$ . Es ist leicht ersichtlich dasz man die Bedingungen für die Zahlen  $a_0, a_*, a_1, \dots$  in folgender Form schreiben kann:

$$\frac{1}{2} \varphi(m) a_0 b_{0n} + 2 \varphi \left( \frac{m}{2h_*} \right) a_* b_{*n} + \varphi \left( \frac{m}{l_1 h_1} \right) a_1 b_{1n} + \dots \equiv 0 \pmod{\varphi} \quad (3)$$

$$n = 1, 2, \dots, \frac{\varphi}{r}$$

Jedes System  $a_0, a_*, a_1, \dots$  genügt also allen Congruenzen (3).

Die jetzt bestimmte Untergruppe der Zahlen  $A$  deuten wir ferner an durch  $g$ . Der zu dieser Untergruppe gehörigen Unterkörper durch  $k$ . Weil  $g$  primär ist, können darin die Zahlen welche

$$\equiv 1 \left( \pmod{\frac{m}{2}} \right) \text{ oder } \left( \pmod{\frac{m}{l_1}} \right) \dots$$

sind, nicht vorkommen. Auch nicht die Zahlen die durch Poten-

<sup>1)</sup> „W.“ S. 56.

zierung ( $\text{mod } m$ ) auf eine derjenigen Zahlen  $x, y, \dots$  führen, die den Bedingungen

$$x \equiv 1 \left( \text{mod } \frac{m}{2} \right), y \equiv 1 \left( \text{mod } \frac{m}{l_1} \right), \dots$$

genügen.<sup>1)</sup>

Wenn die Zahl  $m$  den Factor 2 genau einmal enthält, so gibt es keine primäre Unterkörper<sup>2)</sup>. Wir haben also nur die Fälle  $h_* = 0, 2$  of  $\geq 3$  zu betrachten. In einigen künftigen Beweisen nehmen wir nur der mindest einfachen Fall  $h_* \geq 3$ ; es ist dann leicht einzusehen welche kleine Abänderungen der Beweis für die anderen Fälle erfahren musz.

Wir machen nun noch die folgende Bemerkung, die später von groszem Nutzen sein wird. Die Charaktere einer Untergruppe  $g$  von  $G$  stellen selbst wieder eine Gruppe dar, die isomorph ist mit der Untergruppe<sup>3)</sup>. Der Charakter eines Elementes  $A$  der Untergruppe ist aber bestimmt durch das System der Zahlen

$$a_0, a_*, a_1, \dots$$

Diese Systeme bilden also eine Gruppe die isomorph ist mit der Untergruppe  $g$ . Die Zahlen  $B_n$ , welche bestimmt sind durch die Congruenzen

$$B_n \equiv A_0^{b_{0n}} A_*^{-b_{*n}} A_1^{b_{1n}} \dots \pmod{m}$$

wo die Zahlen

$$a_0, a_*, a_1, \dots \text{ und} \\ b_{0n}, b_{*n}, b_{1n}, \dots$$

verbunden sind durch die Congruenzen (3), stellen die reciproke Untergruppe von  $g$  dar.<sup>4)</sup> Die Systeme

$$b_{0n}, b_{*n}, b_{1n}, \dots$$

bilden wiederum eine Gruppe die isomorph ist mit dieser reciproken Untergruppe. Also können wir sagen: die Systeme

$$a_0, a_*, a_1, \dots$$

stellen eine Gruppe dar, deren reciproke gebildet wird von den Systemen

$$b_{0n}, b_{*n}, b_{1n}, \dots$$

wenn sie verbunden sind durch die Congruenzen (3). *Das Produkt der Grade dieser beiden Gruppen ist also stets gleich  $\varphi$* <sup>5)</sup>.

<sup>1)</sup> „W.“ § 20 und 21.

<sup>2)</sup> „W.“ § 20.

<sup>3)</sup> „W.“ § 13.

<sup>4)</sup> „W.“ S. 56, 8.

<sup>5)</sup> „W.“ S 55, 7.

## § 4. Hilfssätze.

1. In einer primären Untergruppe  $g$ , finden sich, wenn  $h_* \geq 3$  ist, keine Zahlen der Form

$1 + n \frac{m}{2^{h_* - h_*'}} \text{ wo } h_* > h_*' > 1 \text{ oder } h_*' = 0. n \text{ ist ungerade. Und wenn } h_* = 2, \text{ so finden sich keine Zahlen vor, derselben Form, wobei } h_*' = 1 \text{ ist und } n \text{ ungerade.}$

Auch finden sich in  $g$  keine Zahlen der Form

$1 + n \frac{m}{l_i^{h_i - h_i'}} \text{ wo } h_i > h_i' \geq 1, n \text{ nicht teilbar durch } l_i.$

Beweis: Die Zahlen der ersten Form sind, für den Fall dass  $h_*' = 0$  ist, gerade und kommen daher in  $G$  nicht vor; also auch nicht in  $g$ . Ist  $h_* = 2$  und  $h_*' = 1$  so kommen die Zahlen in  $g$  nicht vor wegen § 1.

Wenn nun eine der übrigen Zahlen der ersten Form in der Gruppe  $g$  vorkam, so würde es ein System ganzer Exponenten  $a_0, a_*, a_1, \dots$  geben, die den Congruenzen (3) genügen und für welche

$$A_0^{a_0} A_*^{a_*} A_1^{a_1} \dots \equiv 1 + n \frac{m}{2^{h_* - h_*'}} \pmod{m}$$

Es würde sich hieraus eine Congruenz  $(\text{mod } 2^{h_*} l_1^{h_1} \dots)$  ergeben und folglich würde, nach der Definition der Basiszahlen:

$$A_1^{a_1} \equiv 1 \pmod{l_1^{h_1}}$$

Hieraus würde folgen:  $a_1 = a_* = \dots = 0$ . Man bekäme also:

$$A_0^{a_0} A_*^{a_*} \equiv 1 \pmod{2^{h_*'}}$$

oder

$$(-1)^{a_0} 5^{a_*} \equiv 1 \pmod{2^{h_*'}}$$

und hieraus  $a_* = 2^{h_*' - 2} a_*'$  und  $a_0 = 0$ , weil  $h_*' > 1$ . Substituiert man dieses Ergebnis in den Congruenzen (3), so findet man

$$2^{h_*' - 1} a_*' b_{*n} \equiv 0 \pmod{2^{h_* - 1}}$$

Weil  $g$  primär ist, sind nicht alle Zahlen  $b_{*n}$  durch 2 teilbar; darum folgt aus der letzten Congruenz dass  $a_*'$  teilbar ist durch  $2^{h_* - h_*'}$ . Die Zahl  $a_*$  würde daher teilbar sein durch  $2^{h_* - 2}$  und dies ist unmöglich da  $a_*$  kleiner ist als diese Zahl.

Auf gleiche Art beweist man die ganze Aussage des Hilfssatzes.

2. Es sei  $l_1$  eine der in  $m$  aufgehenden ungeraden Primzahlen;  $\mathfrak{L}$  ein Primidealteiler von  $l_1$ . Der Grad der gemeinschaftlichen Unter-

gruppe von  $g$  und der Trägheitsgruppe des Primideals  $\mathfrak{L}$  ist gleich  $d_1$  wenn diese Zahl den grössten gemeinsamen Teiler der Zahlen  $b_{1n}$  bedeutet.

Ist  $h_* \geq 3$  und  $\mathfrak{L}$  ein in 2 aufgehendes Primideal so ist der Grad dieser gemeinschaftlichen Untergruppe gleich 1 oder 2 jenachdem  $b_{0n} + b_{*n}$  nicht für jeden Wert von  $n$  gerade ist oder wohl.

Ist  $h_* = 2$  so ist der Grad der gemeinschaftlichen Untergruppe = 1.

Beweis: Aus Satz 4 und Hilfssatz 1 ersieht man dasz die Zahlen der gemeinschaftlichen Untergruppe nur die Form  $1 + n \frac{m}{l_1^{h_1}}$  haben können, wo  $n$  nicht durch  $l_1$  teilbar ist. Wenn nun eine solche Zahl zu  $g$  gehört, besteht ein System ganzer Exponenten  $\alpha_0, \alpha_*, \alpha_1, \dots$  für welches

$$A_0^{\alpha_0} A_*^{\alpha_*} \dots \equiv 1 + n \frac{m}{l_1^{h_1}} \pmod{m}$$

und das den Congruenzen (3) genüge leistet. Auf gleiche Weise wie beim vorigen Beweise ergibt sich hieraus, dasz alle Exponenten, ausgenommen  $\alpha_1$ , gleich Null sind. Aus den Congruenzen (3) folgert man dann

$$\varphi \left( \frac{m}{l_1^{h_1}} \right) \alpha_1 b_{1n} \equiv 0 \pmod{\varphi}$$

oder

$$\alpha_1 b_{1n} \equiv 0 \pmod{\varphi_1} \quad n = 1, 2, \dots, \frac{\varphi}{r}.$$

Da nicht alle Zahlen  $b_{1n}$  durch  $l_1$  teilbar sind, ergibt sich hieraus dasz  $\alpha_1$  teilbar ist durch  $\frac{\varphi_1}{d_1}$ . Man beweist leicht dasz diese Bedingung auch genügend ist; denn aus

$$A_0^{\alpha_0} A_*^{\alpha_*} A_1^{\frac{\varphi_1}{d_1} \alpha_1'} A_2^{\alpha_2} \dots \equiv x \pmod{m} \dots \dots \dots (4)$$

ergibt sich:

$$1 \equiv x \pmod{\frac{m}{l_1^{h_1}}}$$

also  $x = 1 + n \frac{m}{l_1^{h_1}}$ . Es ist hier  $n$  nicht durch  $l_1$  teilbar denn im Gegensatz wäre

$$x \equiv 1 \pmod{l_1}$$

und hieraus würde sich ergeben, weil  $A_1 \equiv r_1 \pmod{l_1}$ , dasz  $\frac{\varphi_1}{d_1} \alpha_1'$

teilbar sei durch  $l_1 - 1$ ; also würde  $a_1$  teilbar sein durch  $q_1$ ; dies ist unmöglich, da  $a_1 < q_1$  ist.

Betrachten wir jetzt den zweiten Fall des Hilfssatzes. Die Zahlen der gemeinschaftlichen Untergruppe können auf Grund des Satzes 4 und des Hilfssatzes 1 nur die Form  $1 + n \frac{m}{2^{h_*-1}}$  haben, wo  $n$  ungerade ist. Wenn diese Zahl zu  $g$  gehört, so besteht ein System ganzer Exponenten  $a_0, a_*, \dots$  für welches

$$A_0^{a_0} A_*^{a_*} A_1^{a_1} \dots \equiv 1 + n \frac{m}{2^{h_*-1}} \pmod{m} \dots \dots \dots (5)$$

Es folgt hieraus wiederum dasz alle Exponenten, ausgenommen die erste Zwei, verschwinden. Die Substitution in (3) ergibt

$$2^{h_*-3} a_0 b_{0n} + a_* b_{*n} \equiv 0 \pmod{2^{h_*-2}} \dots \dots \dots (6)$$

Wir unterscheiden nun zwei Fälle:  $h_* > 3$  und  $h_* = 3$ .

Im ersten Fall ist  $a_* b_{*n} \equiv 0 \pmod{2^{h_*-3}}$ .

Weil nicht alle Zahlen  $b_{*n}$  gerade sind, folgt hieraus  $a_* = 2^{h_*-3}$

Die Congruenz geht dann über in  $a_0 b_{0n} + a_*' b_{*n} \equiv 0 \pmod{2}$ .

Nun ist weiter  $a_* < \frac{1}{2} q_*$ , also  $a_*' < 2$ . Wir brauchen daher nur zu untersuchen welche der folgenden Wertecombinationen den letzten

Congruenzen für  $n = 1, 2, \dots, \frac{q}{r}$ , genüge leisten:

$a_0 = 0, a_*' = 0$
0      1
1      0
1      1

Die zweite Combination, genügt nicht, da nicht alle  $b_{*n}$  gerade sind. Auch die dritte nicht. Die erste genügt immer, und die letzte nur wenn  $b_{0n} + b_{*n}$  für alle Werte von  $n$  gerade ist.

Auf gleiche Weise wie vorher zeigt man dasz diese Bedingungen genügend sind.

Es bleibt noch die Erledigung des Falles  $h_* = 3$  übrig. Es geschieht dies auf ganz ähnliche Art.

3. Es sei  $p$  eine nicht auf  $m$  teilbare Primzahl und  $f$  der kleinste Exponent für welchen  $p^f \equiv 1 \pmod{m}$  ist. Es sei  $d$  der Grad der gemeinschaftlichen Untergruppe von  $g$  und der Zerlegungsgruppe eines in  $p$  aufgehenden Primideals. Es sei weiter

$$p \equiv A_0^{a_0 p} A_*^{a_* p} A_1^{a_1 p} \dots \pmod{m}$$

und  $t_p$  der gröszte gemeinsame Teiler aller Zahlen

$$\frac{1}{2} \varphi(m) a_{0\rho} b_{0n} + 2 \varphi \left( \frac{m}{2^h} \right) a_{*p} b_{*n} + \varphi \left( \frac{m}{l_1^{h_1}} \right) a_{1p} b_{*n} + \dots$$

$$n = 1, 2, \dots, \frac{\varphi}{r}$$

so ist:  $\frac{f}{d} \equiv \frac{\varphi}{t_p}$ .

Beweis: Nach Satz 2 wird die im Satze genannte Untergruppe dargestellt durch die Potenzen von  $\rho$ , deren Exponenten sind

$$\frac{f}{d}, 2 \frac{f}{d}, \dots, d \frac{f}{d}$$

Also ist  $p^d$  die niedrigste Potenz von  $\rho$  welche zur Untergruppe  $g$  gehört. Es ergibt sich daher aus den Congruenzen (3) dass  $\frac{f}{d}$  die kleinste Zahl ist, welche den Congruenzen

$$\frac{1}{2} \varphi(m) a_{0\rho} \frac{f}{d} b_{0n} + 2 \varphi \left( \frac{m}{2^{h_*}} \right) a_{*p} \frac{f}{d} b_{*n} + \dots \equiv 0 \pmod{\varphi}$$

genüge leistet. Daraus folgt die Wahrheit des Satzes.

4. Es sei  $l_1$  eine in  $m$  aufgehende Primzahl, (auch 2) und  $d_1'$  der Grad der gemeinschaftlichen Untergruppe von  $g$  und der cyclischen Gruppe des Grades  $f$  aus Satz 5. Nehmen wir weiter an, dass für die in Satz 5 bestimmte Zahl die Congruenz

$$l_1 + \frac{m}{l_1^{h_1}} \equiv A_0^{a_0} A_*^{a_*} A_1^{a_1} \dots \pmod{m}$$

gilt. Es sei  $t_{l_1}$  der grösste gemeinschaftliche Teiler der Zahlen

$$\frac{1}{2} \varphi(m) a_0 b_{0n} + 2 \varphi \left( \frac{m}{2^{h_*}} \right) a_* b_{*n} + \dots$$

$$n = 1, 2, \dots, \frac{\varphi}{r}$$

so ist  $\frac{f_1}{d_1'} \equiv \frac{\varphi}{t_{l_1}}$ . (Man siehe Satz 3 für die Bedeutung der Zahl  $f_1$ ).

Beweis: Wie voriger.

### § 5. Zerlegung der Primzahlen in Primideale des Körpers $k$ .

Herr BACHMANN<sup>1)</sup> hat das Zerlegungsgesetz gegeben einer Primzahl in Primideale eines beliebigen Unterkörpers des GALOIS'schen Körpers. Seine Betrachtungen können wir hier natürlich benutzen. Sie werden eine leichte Vereinfachung erleiden können weil wir in unserem Fall mit einem ABEL'schen Körper zu machen haben: die Substitutionen haben also die commutative Eigenschaft.

<sup>1)</sup> Allgemeine Arithmetik der Zahlkörper, S. 495.

**Satz 6.** Es sei  $p$  eine nicht in  $m$  aufgehende Primzahl und  $f$  der kleinste Exponent für welchen  $pf \equiv 1 \pmod{m}$ . Dann ist  $p$  in  $k$  gleich dem Produkte von  $\frac{ed}{r}$  verschiedenen Primideale des Grades  $\frac{f}{d}$ , wo  $ef = \varphi$ .

Beweis: Die von Herrn BACHMANN bestimmte Gruppe welche  $g$  und  $sg_0s^{-1}$  gemeinschaftlich haben, wird hier die Gruppe welche  $g$  und  $g_0$  gemeinsam haben. Diese letzte ist bestimmt in § 2, 3. Im Komplex (42) von B. werden also, für jede Substitution  $s$ , immer je  $d$  einander gleich sein. Es gibt also  $\frac{rf}{d}$  verschiedene. Weil  $G$  genau  $ef$  Substitutionen enthält, musz man  $\frac{ed}{r}$  Komplexe (42) nehmen.

Die Zahl  $e$  von B. ist also hier  $= \frac{ed}{r}$  und die Zahlen  $h_i$  sind hier alle gleich  $d$ . Weiter ist auf S. 494  $s_i g_i s_i^{-1}$  hier gleich  $g_i$ . Alle Zahlen  $t_i$  von B. sind in unserem Falle  $= 1$ . Auch  $r_i = 1$  und daher alle Zahlen  $d_i$  von B. hier ebenfalls  $= 1$ . Aus dem Satze auf S. 495 von B. ergibt sich nun der Satz 6.

**Satz 7.** Es sei  $l_1$  eine in  $m$  aufgehende ungerade Primzahl und  $f_1$  der kleinste Exponent, für welchen  $l_1^{f_1} \equiv 1 \pmod{\frac{m}{l_1}}$  während  $\varphi\left(\frac{m}{l_1^{h_1}}\right) = f_1 e_1$ , dann ist die Zahl  $l_1$  im Körper  $k$  die  $\frac{\varphi_1}{d_1}$ -te Potenz eines Produktes von einander verschiedener Primideale. Die Anzahl dieser Ideale ist  $\frac{e_1 d_1 d'_1}{r}$  und ihr Grad  $\frac{f_1}{d'_1}$ .

Jedes Primideal ist im Körper  $K$  die  $d_i$ -te Potenz eines Produktes von einander verschiedener Primideale deren Anzahl  $\frac{r}{d_1 d'_1}$  ist.

Ist  $h_* \geq 3$  und  $b_{0n} + b_{*n}$  nicht für alle Werte von  $n$  gerade, so ist die Primzahl 2 im Körper  $k$  die  $\varphi_*$ -te Potenz eines Produktes von einander verschiedener Primideale des Grades  $\frac{f_*}{d_*}$  und deren Anzahl gleich  $\frac{e_* d'_*}{r}$  ist. Jedes Primideal ist im Körper  $K$  ein Produkt von einander verschiedener Primideale deren Anzahl  $\frac{r}{d_*}$  ist.

Ist  $h_* \geq 3$  und  $b_{0n} + b_{*n}$  für alle Werte  $n$  gerade, so ist die Primzahl 2 im Körper  $k$  die  $\frac{1}{2} \varphi_*$ -te Potenz eines Produktes von einander verschiedener Primideale des Grades  $\frac{f_*}{d_*}$  und deren Anzahl gleich  $\frac{2e_* d_*^l}{r}$  ist. Jedes Primideal ist im Körper  $K$  die 2-te Potenz eines Produktes von einander verschiedener Primideale deren Anzahl gleich  $\frac{r}{2d_*^1}$  ist.

Ist  $h_* = 2$  so gilt dasselbe welches oben gesagt ist für den Fall dasz  $b_{0n} + b_{*n}$  nicht für alle Werte  $n$  gerade ist.  $f'_*$  ist überall der kleinste Exponent für welchen  $2^{f'_*} \equiv 1 \pmod{\frac{m}{2^{h_*}}}$  und es ist stets

$$\varphi\left(\frac{m}{2^{h_*}}\right) = f'_* e_*.$$

Der Beweis dieses Satzes ist aufgeschlossen in den angegebenen BACHMANN'schen Betrachtungen.

#### § 6. Bestimmung der Discriminante des Unterkörpers $k$ .

Satz 8. Für die Discriminante des Körpers  $k$  gilt:

$$d = \pm 2 \varphi\left(\frac{m}{2^{h_*}}\right) \frac{(h_* - 1) 2^{h_* - 1} - 2t}{r} \prod_i l_i \varphi\left(\frac{m}{l_i^{h_i}}\right) \frac{l_i^{h_i - 1} (l_i h_i - h_i - 1) - d_i + 1}{r}$$

wo die Zahl  $t = 0$  zu setzen ist wenn  $b_{0n} + b_{*n}$  nicht für alle  $n$  gerade ist und auch wenn  $h_* = 2$  ist; andernfalls ist  $t = 1$  zu setzen.

Beweis: Es sei  $D$  die Discriminante des Körpers  $K$ ;  $\mathfrak{d}_k$  die Relativ-discriminante und  $\mathfrak{D}_k$  die Relativ-differente. Es gelten die folgenden Beziehungen: <sup>1)</sup>

$$D = d^n (\mathfrak{d}_k); \quad \mathfrak{d}_k = N_k(\mathfrak{D}_k); \quad \mathfrak{D}_k = \mathfrak{G}^{(1)} \dots \mathfrak{G}^{(r-1)}$$

$$\mathfrak{G}^{(1)} = \{(Z - Z^{A_j}), (Z^2 - Z^{2A_j}), \dots, (Z^{q-1} - Z^{(q-1)A_j})\}$$

$$j = 2, 3, \dots, r$$

$A_j$  sind die Zahlen der Untergruppe  $g$  (also nicht die Basiszahlen dieser Untergruppe) und  $A_1 = 1$ . Weil  $D$  nur durch Primzahlen teilbar ist, welche in  $m$  aufgehen, so folgt aus obigen Beziehungen dasz das Element  $\mathfrak{G}^{(j)}$  auch nur durch Primideale teilbar ist welche

<sup>1)</sup> „H“. Satz 38, 39, S. 205.

auf die Zahlen 2 oder  $l_i$  teilbar sind. Aus obiger Form des Elementes  $\mathfrak{E}^{(j)}$  folgt weiter dasz alle Zahlen dieses Ideals nur teilbar sind durch das Hauptideal  $(Z - Z^{A_j}) =$  das Hauptideal  $(1 - Z^{A_j^{-1}})$ . Ist diese letzte Zahl eine Einheit so ist das Element  $\mathfrak{E}^{(j)}$  daher identisch mit dem Ideale  $\mathfrak{G}$  aller ganzen Zahlen des Körpers  $K$ . Es wird dasz Element  $\mathfrak{E}^{(j)}$  also nur dann nicht identisch mit  $\mathfrak{G}$  sein, wenn  $A_j$  eine Zahl der Trägheitsgruppe einer der Primzahlen 2 en  $l_i$  ist. Durch Zuziehung des § 2. 2 und des Satzes 3, findet man:

$$\mathfrak{D}_k = (\xi_{01} \dots \xi_{0e_0})^{2t} \prod_i (\xi_{i1} \dots \xi_{ie_i})^{d_i^{-1}}.$$

wo das Produkt sich über alle in  $m$  aufgehenden ungeraden Primzahlen  $l_i$  erstreckt. Es folgt weiter aus Satz 7:

$$N_k(\xi_{0j}) = l_{0j}^{d_j'} \quad \text{und} \quad N_k(\xi_{ij}) = l_{ij}^{d_i'}$$

Die Zahl  $D$  is bekannt<sup>1)</sup>. Man erhält nun den Ausdruck der Zahl  $d$  wenn man die, im Anfang des Beweises niedergeschriebenen Beziehungen benutzt.

### III. Die wichtigsten Hilfssätze für die Berechnung der Klassenzahl der Ideale des Körpers $k$ .

#### § 7. Satz 9.

$$\prod_p \left( 1 - \frac{1}{n(p)^s} \right) = \prod_{n=1}^{\frac{p}{r}} \left\{ 1 - \left[ \overbrace{b_{0n}, b_{*n}, b_{1n}, \dots}^p \right] \frac{1}{p^s} \right\}$$

Das erste Produkt erstreckt sich über alle Primideale  $p$  des Körpers  $k$  welche in die Primzahl  $p$  aufgehen. Das Symbol im zweiten Produkt hat die bekannte Bedeutung<sup>2)</sup>. Wenn  $h_* = 2$  so musz  $b_{*n}$  im Symbol weggelassen werden und wenn  $h_* = 0$  ist, so musz auch  $b_{0n}$  weggelassen werden.

Beweis: Wir beweisen die beiden folgenden Behauptungen des Gliedes rechter Hand: 1. Das Symbol ist eine  $\frac{f}{d}$ -te Einheitswurzel;

2. Eine jede  $\frac{f}{d}$ -te Einheitswurzel tritt im Produkte je  $\frac{ed}{r}$  Mal auf. Hiermit wird der Beweis erbracht sein denn es ergibt sich daraus dasz das Glied rechter Hand gleich

$$\left( 1 - \frac{1}{p^{\frac{f}{d}s}} \right)^{\frac{ed}{r}}$$

<sup>1)</sup> "H". Satz 88 und 121.

<sup>2)</sup> "H". § 116.

ist, und zufolge des Satzes 6 ist dieser Ausdruck dem Gliede linker Hand gleich.

1. Es sei  $p \equiv \pm 5^{\rho_*} \pmod{2^{h_*}}$

$$p \equiv r_1^{\rho_1} \pmod{l_1^{h_1}}$$

wo  $r_1, r_2, \dots$ , die schon vorher benutzte primitive Wurzeln bedeuten. Weiter sei

$e_0$  der grösste gemeinsame Teiler von  $\frac{p-1}{2}$  und 2;  $e_0 f_0 = 2^{\rho_0}$ )

$e_*$  „ „ „ „ „  $\rho_*$  und  $\frac{1}{2} \varphi_*$ ;  $e_* f_* = \frac{1}{2} \varphi_*$

$e_1$  „ „ „ „ „  $\rho_1$  und  $\varphi_1$ ;  $e_1 f_1 = \varphi_1$

Es ergibt sich dann aus der Definition der Symbole:

$\left[ \frac{p}{2^2} \right]$  = eine  $f_0$ -te-Einheitswurzel und keine Niedrigere,

$\left[ \frac{p}{2^{h_*}} \right]$  = „  $f_*$  „ „ „ „ „ „ „ „

$\left[ \frac{p}{l_1^{h_1}} \right]$  = „  $f_1$  „ „ „ „ „ „ „ „

Es sei jetzt  $g_0$  = der grösste gemeinsch. Teiler von  $f_0$  und  $d_0$ ,

$g_*$  = „ „ „ „ „ „  $f_*$  und  $d_*$ ,

$g_1$  = „ „ „ „ „ „  $f_1$  und  $d_1$ ,

Dann ist einleuchtend dasz

$\left[ \frac{p}{2^2} \right]^{b_{0n}}$  = eine  $\frac{f_0}{g_0}$ -te-Einheitswurzel ist.

$\left[ \frac{p}{2^{h_*}} \right]^{b_{*n}}$  = „  $\frac{f_*}{g_*}$  „ „ „ „ „

$\left[ \frac{p}{l_1^{h_1}} \right]^{b_{1n}}$  = „  $\frac{f_1}{g_1}$  „ „ „ „ „

Das kleinste gemeinsame Vielfache der Zahlen  $\frac{f_0}{g_0}, \frac{f_*}{g_*}, \frac{f_1}{g_1}, \dots$  sei angedeutet durch  $v$ . Dann ist der Wert des Symbols

$$\left[ \overbrace{b_{0n}, b_{*n}, b_{1n}, \dots}^p \right]$$

eine  $v$ -te Einheitswurzel.

1) Die Buchstaben  $e_0, f_0, e_*, f_*, e_1, f_1$  u s. w. haben hier eine abweichende Bedeutung von Satz 7.

Es bleibt also noch übrig zu zeigen dasz  $\frac{f}{d}$  teilbar ist durch  $v$ .

Aus der Definition der Zahlen  $p_*, p_1, \dots$  folgt:

$$\frac{p-1}{2} = a_{0p}; p_* = a_{*p}, p_1 = a_{1p}, \dots \text{)}.$$

Aus der in Hilfssatz 3 gegebenen Definition der Zahl  $t_p$  ergibt sich dasz es der gröszte gemeinsame Teiler aller Zahlen

$$\frac{1}{2} \varphi_* \varphi_1 \varphi_2 \dots a_{0p} b_{0p} + 2 \varphi_1 \varphi_2 \dots a_{*p} b_{*n} + \varphi_* \varphi_1 \dots a_{1p} b_{1n} + \dots$$

$$n = 1, 2, \dots \frac{f}{r}.$$

ist. Alle Glieder dieser Summe sind teilbar durch  $e_1$  und weil der gröszte gemeinsame Teiler der Zahlen  $\varphi_1$  und  $a_{1p}$  auch  $= e_1$  ist, so ist einleuchtend dasz  $t_p$  teilbar ist durch  $e_1$  und daher ist  $t_p = e_1 \times$  gröszter gemeinsamer Teiler aller Zahlen

$$\frac{1}{2} \varphi_* f_1 \varphi_2 \dots a_{0p} b_{0n} + f_1 \varphi_2 \dots a_{*p} b_{*n} + \varphi_* \varphi_{*2} \dots \frac{a_{1p}}{e_1} b_{1n} \dots$$

Alle Zahlen  $b_{1n}$  sind teilbar durch  $d_1$ . Daher ist nun  $t_p = e_1 g_1 \times$  gröszter gemeinsamer Teiler aller Zahlen

$$\frac{1}{2} \varphi_* \frac{f_1}{g_1} \varphi_2 \dots a_{0p} b_{0n} + \frac{f_1}{g_1} \varphi_2 \dots a_{*p} b_{*n} + \varphi_* \varphi_2 \dots \frac{a_{1p}}{e_1} \frac{b_{1n}}{g_1} + \dots$$

Weiter ist  $\frac{f}{d} = \frac{\varphi}{t_p} = \frac{\varphi_* \varphi_2 \dots e_1 g_1}{t_p} \cdot \frac{f_1}{g_1} \dots \dots \dots (A)$

Aus dieser letzten Beziehung ergibt sich dasz sich der Faktor  $e_1 g_1$  aus Zähler und Nenner des Bruches  $\frac{\varphi_* \varphi_2 \dots e_1 g_1}{t_p}$  heraus heben lässt.

Im Nenner bleibt also nur der letztgenannte gröszte gemeinsame Teiler übrig. Wenn dieser neue Nenner nun noch einen Faktor mit

der Zahl  $\frac{f_1}{g_1}$  gemeinsam hat, so musz dieser Faktor, gemäsz der Definition des gröszten gemeinsamen Teilers, auch teilbar sein auf  $\varphi_* \varphi_2 \dots$

Denn die ganze Zahl  $\frac{a_{1p}}{e_1}$  ist prim zu  $\varphi_1$  also auch zu  $\frac{f_1}{g_1}$  und nicht

alle Zahlen  $\frac{b_{1n}}{g_1}$  sind teilbar zu  $\varphi_1$ .

Es hat sich also ergeben dasz jeder Faktor, welchen der neue Nenner noch mit  $\frac{f_1}{g_1}$  gemeinsam hat, auch sich vorfinden musz im

Zähler des Bruches  $\frac{\varphi_* \varphi_2 \dots e_1 g_1}{t_p}$ . Ein solcher etwaiger Faktor kann

1) Man siehe Hilfssatz 3.

daher herausgehoben werden, wonach der Nenner prim zu der ganzen Zahl  $\frac{f_1}{g_1}$  geworden sein wird. Nun ist weiter  $\frac{f}{d}$  eine ganze Zahl und ebenso  $\frac{f_1}{g_1}$ . Also findet man schliesslich dasz der Bruch  $\frac{g_* g_2 \dots e_1 g_1}{t_p}$  auch eine ganze Zahl sein musz; m.a.w. die ganze Zahl  $\frac{f}{d}$  ist teilbar durch  $\frac{f_1}{g_1}$ , wie sich aus (A) ergibt. Dasselbe beweist man für  $\frac{f_*}{g_*}$  u. s. w. Es ist damit bewiesen dasz  $\frac{f}{d}$  teilbar ist durch  $v$ .

2. Um die Zweite der Anfangs aufgestellten Behauptungen zu beweisen nehmen wir an dasz für zwei von einander verschiedenen Systeme

$$b_{01}; b_{*1}, b_{11}, \dots$$

und

$$b_{02}; b_{*2}, b_{12}, \dots$$

die Gleichheit

$$\left[ \overbrace{b_{01}, b_{*1}, b_{11}, \dots}^p \right] = \left[ \overbrace{b_{02}, b_{*2}, b_{12}, \dots}^p \right].$$

bestehe. Nach Einführung der Werte der Symbole findet man leicht die Congruenz

$$\frac{1}{2} \varphi(m) (b_{01} - b_{02}) a_{0p} + 2\varphi \left( \frac{m}{2^{h_*}} \right) (b_{*1} - b_{*2}) a_{*p} + \dots \equiv 0 \pmod{\varphi}.$$

Wie im Anfang des § 3 auseinander gesetzt ist, bilden die Zahlensysteme  $b_{0n}, b_{*n}, \dots$  eine Gruppe. Es werden also die in obiger Congruenz auftretenden Differenzen wiederum ein System der Zahlen  $b_{ij}$  darstellen, das zu der Gruppe gehört. Wir müssen also zeigen dasz genau  $\frac{ed}{r}$  der gegebenen Systeme der Zahlen  $b_{0n}, b_{*n}, \dots$  der Congruenz

$$\frac{1}{2} \varphi(m) b_{0j} a_{0p} + 2\varphi \left( \frac{m}{2^{h_*}} \right) b_{*j} a_{*p} + \varphi \left( \frac{m}{l_1^{h_1}} \right) b_{1j} a_{1p} + \dots \equiv 0 \pmod{\varphi} \quad (7)$$

Genüge leisten. Denn hieraus ergibt sich dann, dasz genau  $\frac{ed}{r}$  Systeme

$b_{0n}, b_{*n}, \dots$  dem Symbol  $\left[ \overbrace{b_{0n}, b_{*n}, \dots}^p \right]$  denselben Wert erteilen;

eine  $\frac{f}{d}$ -te Einheitswurzel kommt daher im Producte rechter Hand der

zu beweisenden Gleichheit, genau  $\frac{ed}{r}$  Mal vor.

Alle Systeme der Zahlen  $b_{0n}, b_{*n}, \dots$  welche der Congruenz (7) Genüge leisten, genügen auch allen Congruenzen die man aus (7) ableiten kann, wenn man der Reihe nach alle Systeme  $a_0, a_*, \dots$ , welche zur Gruppe  $g$  gehören, an der Stelle der Zahlen  $a_{0p}, a_{*p}, \dots$  setzt. Es folgt dies aus der Definition der Systeme  $a_0, a_*, \dots$  in § 3. Man bekommt dadurch genau  $r$  Congruenzen.

Die Zahlen  $p, p^2, \dots, p^{\frac{f}{d}}$  gehören, dem Beweise des Hilfssatzes 3 zufolge, nicht zur Untergruppe  $g$ . Alle Systeme der Zahlen  $b_{0n}, b_{*n}, \dots$  welche der Congruenz (7) genügen, genügen auch den Congruenzen die sich aus (7) ergeben wenn man anstatt  $a_{0p}, a_{*p}, \dots$ , der Reihe nach die Systeme

$$\begin{array}{ccc} 2a_{0p}, & 2a_{*p}, & 2a_{1p}, \dots \\ \cdot & \cdot & \cdot \\ \frac{f}{d} a_{0p}, & \frac{f}{d} a_{*p}, & \frac{f}{d} a_{1p}, \dots \end{array}$$

in (7) hineinstellt. Man bekommt also  $\frac{f}{d}$  neue Congruenzen. Durch Multiplikation der Gruppe der erstgenannten  $r$  Systeme der Zahlen  $a_0, a_*, \dots$  mit der Gruppe der zuletztgenannten  $\frac{f}{d}$  Systeme, bekommt man genau  $\frac{rf}{d}$  Systeme  $a_0, a_*, \dots$ . Setzt man diese, der Reihe nach, anstatt  $a_{0p}, a_{*p}, \dots$  in (7) hinein, so hat man im Ganzen genau  $\frac{rf}{d}$  Congruenzen welche genügt werden von den Systemen  $b_{0n}, b_{*n}, \dots$  welche eben gesucht werden. Der Bemerkung am Ende des § 3 zufolge, gibt es nun genau  $q: \frac{ed}{r} = \frac{ed}{r}$  solche Systeme der Zahlen  $b_{0n}, b_{*n}, \dots$ , was zu beweisen war.

Im Produkte des Gliedes Rechterhand, der zu beweisenden Gleichheit, finden sich  $\frac{q}{r}$  Factoren. Jede  $\frac{f}{d}$ -te Einheitswurzel tritt genau  $\frac{ed}{r}$  Mal auf. Da nun  $\frac{ed}{r} \times \frac{f}{d} = \frac{q}{r}$  so folgert man hieraus dasz auch eine jede  $\frac{f}{d}$ -te Einheitswurzel im Produkte auftritt.

Satz 10. Es ist

$$\prod_1 \left( 1 - \frac{1}{n(1)^s} \right) = \prod_n \left\{ 1 - \left[ \overbrace{b_{0n}, b_{*n}, b_{2n}, \dots}^{l_1} \right] \frac{1}{l_1^s} \right\}$$

wo  $l_1$  eine in  $m$  aufgehende ungerade Primzahl bedeutet. Das erste Produkt erstreckt sich über alle Primideale  $\mathfrak{l}$  des Körpers  $k$ , die in  $l_1$  aufgehen. Das zweite Product erstreckt sich über alle Systeme der Zahlen  $b_{ij}$  in welchen  $b_{1n} = 0$  ist. Wenn  $h_* = 2$  so musz das Symbol  $b_{*n}$  weggelassen werden; und wenn  $h_* = 0$  so musz  $b_{0n}$  weggelassen werden.

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**Physiology.** — *“Further Researches in connection with the permeability of the glomerular membrane to stereoisomeric sugars”.*

By Prof. H. J. HAMBURGER.

(Communicated in the meeting of September 27, 1919).

It has previously<sup>1)</sup> been shown that when a liquid, whose composition is efficient, is passed through the bloodvessels of the kidneys, the grape sugar dissolved in the liquid is retained by the glomerular membrane. In this concern the concentration of the Ca-ions and H-ions play the principal part. This retention of glucose is very remarkable, seeing that other crystalloids like common salt, sulphates and phosphates are not retained by the glomerular membrane. The question then arose what the cause could be of this so efficient behaviour of glucose. Had it something to do with the size of the molecule? This supposition had to be dismissed when it appeared that lactose ( $C_{12}H_{22}O_{11}$ ), which has a molecular weight almost twice as great, passed through completely, which was the case even with raffinose ( $C_{18}H_{32}O_{16}$ ). Nothing else then remained but to accept that this phenomenon was to be ascribed to the particular configuration of the glucose molecule<sup>2)</sup>.

To test this conclusion by experiment other hexoses ( $C_6H_{12}O_6$ ), isomeric with glucose, were examined. It appeared thereby that, in contrast with glucose, the fructose and mannose were allowed to pass through completely. The frog's kidney is able thus to distinguish glucose from fructose and mannose.

The cause thus indeed lay in the difference in chemical structure. Obviously the thought of the wellknown comparison with lock and key occurred: The key glucose thus does not fit in the lock glomerular membrane, while the keys fructose and mannose do.

We now asked ourselves the question: *will it perhaps be possible to point out a particular group of atoms in the glucose molecule which may be considered responsible for the retention? The attempt to answer this question forms the contents of this article.*

<sup>1)</sup> HAMBURGER and BRINKMAN, These Proc. Meeting of January 27, 1917, Vol. XIX, N<sup>o</sup>. 8.

<sup>2)</sup> HAMBURGER and BRINKMAN, These Proc. Meeting of September 28, 1918, Vol. XXI, N<sup>o</sup>. 4.

It is obvious that for this it was necessary to examine a number of isomeric and stereoisomeric sugars with regard to their behaviour towards the kidneys, and then, not only hexoses, but also pentoses. Jhr. ALBERDA VAN EKENSTEIN, Prof. BACKER, Prof. BÖSEKEN and Prof. NEUBERG were so kind as to assist me in this as much as possible. Hereby I again tender them my cordial thanks. Nevertheless the diversity of material was very limited.

The methods pursued did not differ from the previous ones<sup>1)</sup>. They were based upon the reducing powers of the secreted artificial urine, and that of the perfusion liquid. This perfusion liquid, in which the sugars were dissolved, again had the following composition: NaCl 0.5 %, KCl 0.02 %, CaCl<sub>2</sub>.6 aq 0.04 %, NaHCO<sub>3</sub> 0.285 %. The determination of the sugar was made, as in the previous case, according to BANG (1916). Lately however the newest method of this writer was used<sup>2)</sup>. This also is based upon the determination of the reduction. Here the discrepancy which can arise from differences in the intensity of boiling is eliminated, and that because hot steam is passed through the boiling liquid, whereby the cuprous oxide is instantly changed into cupric by means of iodic acid. Furthermore the oxidation by the air is avoided. The final titration is carried out with 0.01 n thiosulphate and starch solution. If  $a$  c.c. thiosulphate are used then the reduction in terms of glucose can be calculated from the formula  $(1.97 - a) : 2.8$ . If no reducing substance or sugar is present then the answer must be found to be:  $a = 1.97$ , which is indeed the case if the substances used are pure<sup>3)</sup>. This new method, it appeared, like the old one also gave very good results. The results are even nice. On the whole the figures obtained for the reduction, by the new method, are higher than those obtained when the old method is used, but, seeing that the purpose here is comparative research it is mostly not of vital importance.

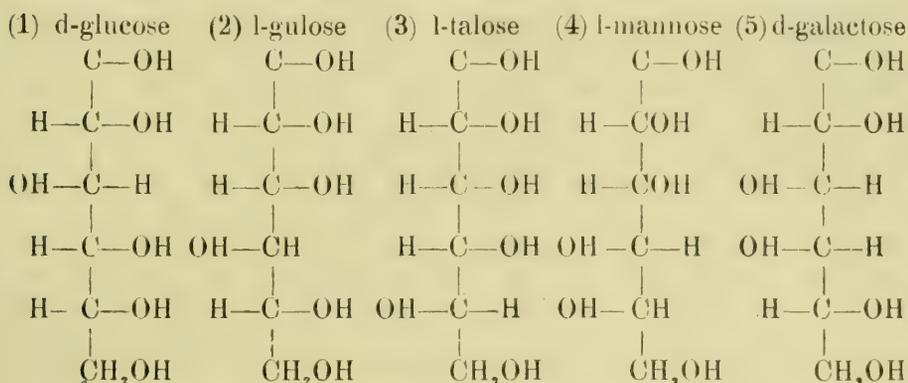
In order to have an introduction for our procedure the arbitrary hypothesis was made that the  $\begin{array}{c} C-OH \\ | \\ H-C-OH \end{array}$  group in the glucose molecule caused the retention.

<sup>1)</sup> Compare also HAMBURGER and BRINKMAN, *Biochemische Zeitschrift* **88**, 97, (1918).

<sup>2)</sup> BANG, *Biochemische Zeitschrift*, **92**, 344, (1918).

<sup>3)</sup> We wish to call special attention to that because after long searching it has appeared that the KCl "pro Analyse" of KAHLBAUM contained reducing substances. This was also the case with the KCl which was supplied by the British Drug Houses as chemically pure. After repeated crystallization the KCl was freed from reducing substances and then we obtained like BANG, a value of 1.97 for  $a$ .

If this hypothesis was true then besides the d-glucose also the l-gulose, l-talose, l-mannose and d-galactose had to be retained.

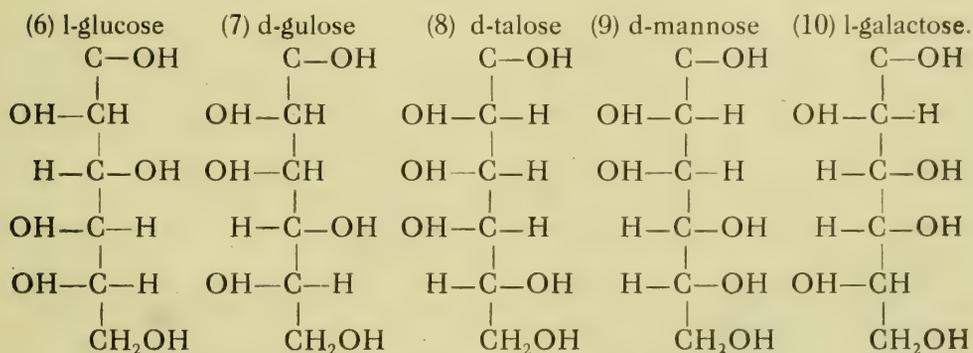


However, of these, only the d-glucose, the l-mannose and the d-galactose were at our disposal.

The experiment showed however that the l-mannose passed through completely: In 4 experiments it appeared that in the case of perfusion with a solution of 0.09%, whose reducing power corresponded to that of a glucose solution of 0.06%, the artificial urine also had a reducing power of 0.06, — there was no retention therefore. Let attention be called to the fact, by the way, that after the preparation of the l-mannose solution the rotation was + 10.5°, and, 16 hours afterwards 13.5°. In the following article we shall return to this.

Concerning the d-galactose, it was not retained completely but partly; to this also we shall return in the next article.

Our hypothesis further claimed that the following series should pass through the glomerular membrane:



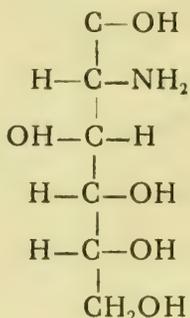
We had only l-glucose and d-mannose available.

It indeed appeared that the l-glucose in contrast with the d-glucose is allowed to pass through completely. The 0.1% glucose solution, dissolved in the customary Ringerfluid, showed a reduction of

0.075 %; in the 4 experiments the urine showed the same reduction.

In accordance with the hypothesis also of the d-mannose not a trace was retained.<sup>1)</sup>

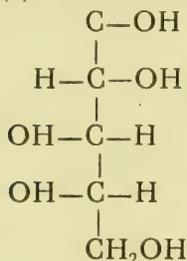
We did not have at our disposal the other abovementioned sugars except, however, the d-glukosamine



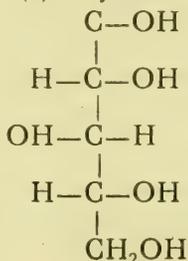
The 0.1 % solution of glukosamine as well as that of 0.08 %, passed completely through the glomerular membrane.

After this some pentoses were examined. Our hypothesis demanded that the following series should be retained:

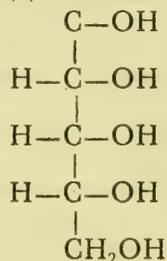
(1) l-arabinose



(2) l-xylose



(3) d-ribose



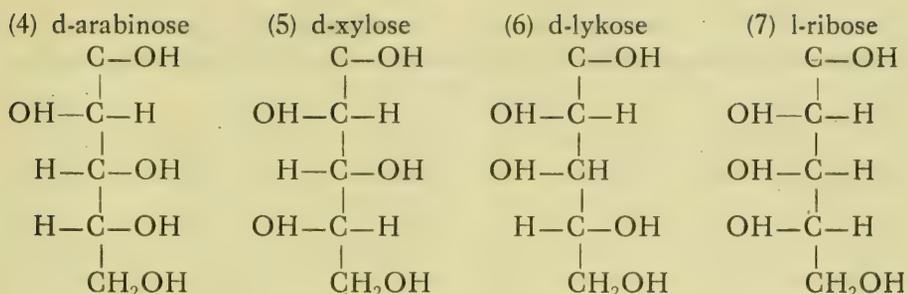
Concerning the l-arabinose, a solution of 0.07 % was perfused; it had a reduction of 0.0714 % glucose, which was also the case with the artificial urine; in contradiction to the hypothesis, therefore, nothing was retained.

In the case of l-xylose there was a retention, but this amounted only to more or less  $\frac{1}{3}$  part, while not more than  $\frac{1}{4}$  of the ribose was retained.

To this partial retention of l-xylose and d-ribose we shall return in the next article.

Lastly there follows here a series of pentoses, which, according to the hypothesis, were expected to pass through completely;

<sup>1)</sup> Compare HAMBURGER and BRINKMAN, These Proc. Meeting of 28<sup>th</sup> Sept. 1918, Vol. XXI, N<sup>o</sup>. 4.

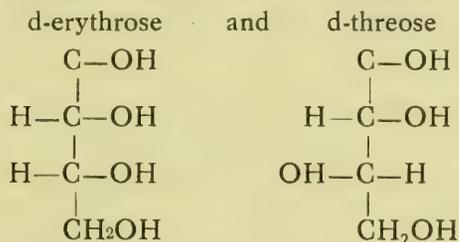


Of these 4 pentoses we were able to investigate only the first two; quite in accordance with what was to be expected there was no question of any retention of d-arabinose.

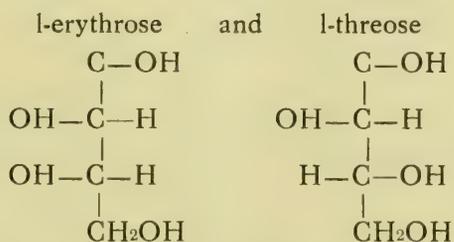
It was different in the case of d-xylose of which again a part was retained.

Let us add to this still that neither of the d-l-arabinose nor of the d- and l-arabinose anything was retained.

The tetroses could not be examined; if this had been the case it was to be expected that



would be retained and



allowed to pass through.

To simplify the survey we give here a table in which the results of the experiments are summed up. Let it be remarked here however that with d-galactose and l-xylose a much larger amount of experiments have been made. These will find a place in the following article.

Summarising the results of the above described experiments we must come to the conclusion that the hypothesis which makes

## Permeability of the kidneys to stereoisomeric sugars.

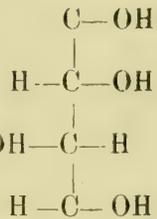
Date of experiment.	Perfusion liquid contains:	Reduction of perfusion liquid expressed in terms of glucose percent.	The urine has a reduction of:	Retention (II—III).	Percentage of retention calculated from I and IV.
15 Nov. 1918	0.09 % <i>l</i> -mannose	0.06 %	0.06 %	0	0
12 June 1919	0.05 „ <i>d</i> -galactose	0.0964 „	0.0535 „	0.0429	44 %
1 July 1919	0.25 „ „	0.193 „	0.125 „	0.0680	35 „
19 Nov. 1918	0.1 „ <i>d</i> -glucosamine	0.1125 „	0.1125 „	0	0
11 Aug. 1919	0.08 „ <i>HCl</i> -glucosamine	0.0964 „	0.0964 „	0	0
12 Aug. 1919	0.07 „ „	0.075 „	0.075 „	0	0
14 Nov. 1918	0.1 „ <i>l</i> -glucose	0.075 „	0.075 „	0	0
20 Aug. 1918	0.1 „ <i>d</i> -mannose	0.0825 „	0.0825 „	0	0
21 Aug. 1918	0.1 „ „	—	—	—	—
13 Aug. 1919	0.07 „ <i>l</i> -arabinose	0.0714 „	0.0714 „	0	0
13 Nov. 1918	0.1 „ <i>l</i> -xylose	0.0725 „	0.05 „	0.0225	31 %
5 Nov. 1918	0.05 „ <i>d</i> -ribose	0.06 „	0.0468 „	0.0132	22 „
26 Oct. 1918	0.09 „ <i>d</i> -arabinose	0.075 „	0.075 „	0	0
26 Oct. 1918	0.09 „ „	0.08 „	0.0745 „	0	0
14 Aug. 1919	0.07 „ „	0.0535 „	0.0535 „	0	0
23 Oct. 1918	0.1 „ <i>d</i> -xylose	0.125 „	0.1 „	0.025	20 %
30 Oct. 1918	0.04 „ „	0.0525 „	0.026 „	0.026	50 „
10 July 1919	0.2 „ „	0.2 „	0.1783 „	0.0217	10.5 „

the  $\begin{array}{c} \text{C—OH} \\ | \\ \text{H—C—OH} \end{array}$  group of the *d*-glucose molecule responsible for the retention, an hypothesis, which, as we have already remarked, was of a wholly arbitrary nature, cannot be correct. For if it was, then *l*-mannose and *l*-arabinose would be completely retained, but, as a

matter of fact, they are allowed to pass through. On the other hand however, in accordance with the hypothesis, d-galactose, l-xylose and d-ribose are retained, but this happens only with a part of each of the sugars which are passed through. As regards the sugars which, from the hypothesis, were expected to pass through completely, they all indeed passed through the glomerular membrane except the d-xylose, which was partially retained. To this partial retention we shall return again, as has already been said, in the next article. If we accept for the time being that even where a total retention had to take place, a partial one satisfies the hypothesis, then in any case there still remain the l-mannose and the l-arabinose of which, in contradiction to the hypothesis, nothing is retained. There will therefore be an inclination to hold a larger group of atoms of the glucose molecule responsible for the retention, for instance

$$\begin{array}{c} \text{C—OH} \\ | \\ \text{H—C—OH} \\ | \\ \text{OH—C—H} \end{array}$$

But also in this case l-arabinose had to be retained completely.



If the group of atoms is taken still larger viz.

then this could be held responsible for the retention of glucose, but then the difficulties of the partial retention still remain. For the present therefore it is risky to explain the retention by means of the image of the lock and key, unless, instead of a part of the molecule, the grouping of the whole is taken. DE GRAAFF<sup>1)</sup> came to a more or less like conclusion through his detailed experiments in connection with the behaviour of typhoid and paratyphoid bacilli towards stereoisomeric sugars.

As matters stood now there would be the inclination to get a point of contact with the specific physical properties of glucose, and we thought of surface tension, viscosity and adsorption.

As regards the *surface tension*, comparative experiments were made with d-glucose and d-fructose<sup>2)</sup>; of the latter sugar it will be

<sup>1)</sup> W. C. DE GRAAFF: The biochemical characters of paratyphoid bacilli. Leiden, S. C. VAN DOESBURGH, 1918.

<sup>2)</sup> We take these especially because, as a rule we have sufficient quantities of them at our disposal.

remembered no trace is retained by the kidney. But it appeared through stalagmometric experiments (J. TRAUBE) and through experiments where the height of the meniscus through capillary action was made use of, that there existed no difference in the surface tensions of the two sugars. Neither did the *viscosities* of the two sugars show any difference.

We now turned to the *adsorption*; would the glomerular membrane perhaps adsorb glucose, and, on the other hand, not fructose? As a matter of fact we would not have busied ourselves with these experiments had not O. COHNHEIM<sup>1)</sup> conducted similar ones. For, supposing that the glomerular membrane adsorbed d-glucose, but no laevulose, then it would be extremely difficult to explain the different behaviour of the living kidney toward these two sugars through this. But, in any case, this would at least be an indication in the direction in which the explanation of the facts has to be sought. In addition to this there remains the fact that COHNHEIM could indeed notice differences in the adsorptive power of the kidney substance of warm blooded animals towards glucose and laevulose. However there are several grounds on which this conclusion of COHNHEIM can be doubted. I shall communicate about this in another place and point out through the medium of experiments conducted with frogs' kidneys, that the methods pursued by this investigator viz., to experiment with mashed kidney substance, to determine the adsorptive power of the kidney for sugars, must lead to results which cannot be relied upon.

#### *Summary and Conclusion.*

To investigate whether a particular group of atoms can be held responsible for the retention of glucose by the kidney, a number of stereoisomeric hexoses and pentoses were examined with regard to their behaviour towards the kidneys.



The hypothesis that the group  $\text{H—}\overset{\text{C}}{\underset{|}{\text{—}}}\text{OH}$  causes the retention cannot be true seeing that l-arabinose and l-mannose which have this group pass completely through the kidney, while the d-galactose, the l-xylose and the d-ribose in which this group is also present are partly retained, it is true, but still the other part is allowed to pass through (partial retention that is).

In keeping with the hypothesis the sugars whose first

<sup>1)</sup> COHNHEIM: Zeitschr. f. Physiol. Chemie, **84**, p. 451.



**Physiology.** — “*The partial permeability of the glomerular membrane to d-galactose and some other multi-rotatory sugars*”. By Prof. H. J. HAMBURGER.

(Communicated in the meeting of September 27, 1919).

Through earlier experiments it has appeared already that d-galactose is only partially retained<sup>1)</sup> by the glomerular membrane. Upon closer examination, however, this appears to agree very little with the idea of permeability, because a filter either holds back a substance or allows it to pass through unconditionally. A mean between these two extremes can hardly be conceived when there is question of only a simple substance and not of a mixture. As a matter of fact at first we were inclined therefore to drop the idea of permeability, but, as appears from the preceding article we endeavoured in vain to clear up the matter with the help of surface tension, viscosity or adsorption. For this reason we then returned again to the idea of permeability.

On closer inspection two explanations still seemed possible to us:

1<sup>st</sup>. The concentrations of the galactose-solutions which had been used up to this (0.1 % and 0.15 %), might have been too strong that is above the toleration of the kidney for d-galactose. We were mindful here of our experience in connection with glucose; for, did we not find formerly<sup>2)</sup>, that, when the concentration of glucose in the perfusion liquid exceeded the physiological value only by 0.02 %, already a little of the glucose was suffered to pass through, and, that this quantity increased in proportion as the concentration of sugar in the perfusion liquid became stronger, so that when the concentration of glucose had become 0.2 %, practically no sugar was retained any longer? There was a possibility therefore that also the concentration of galactose might have been too strong and that it had to be ascribed to this that it was partly retained and partly not. *It was therefore desirable to make determinations of the toleration of the kidneys for d-galactose.* Should it appear then that the galactose could not, like the glucose, be *wholly* retained even in weak con-

<sup>1)</sup> HAMBURGER and BRINKMAN, These Proc. 28 Sept. 1918.

<sup>2)</sup> HAMBURGER and BRINKMAN, Biochem. Zeitschr., **88**, 97, (1918).

centrations, then the phenomenon of partial retention could not be explained by the idea of toleration and then a second explanation would per exclusionem be the correct one.

2<sup>nd</sup>. This second explanation could be sought in the fact that *the d-galactose exists in two modifications* — an  $\alpha$  and a  $\beta$  variety. In aqueous solutions these two varieties are in a state of equilibrium. Then it had to be accepted only that one of the two modifications was retained by the glomerular membrane, and the other not.

### 1. *Toleration of the kidneys for d-galactose.*

To determine the toleration, the perfusion liquid was seasoned with different quantities of d-galactose lying between 0.05 % and 0.25 %. The following table gives a survey of the results obtained. It will be clear without further explanation. Let it only be remarked that for the determination of the reduction the newest method of BANG (1918) was employed<sup>1</sup>). Each time both kidneys of two frogs were perfused at the same time (see table I).

What do these experiments teach us?

Firstly: *That none of the used galactose-solutions, whose concentrations lie, as has been said, between 0.05 % and 0.25 %, are completely retained by the kidneys.* Secondly, *that in all cases the retention amounted on an average to a half, independent of the concentration of the galactose-solution that was perfused.*

In both respects the galactose differs from the glucose; for, was there not found a *total* retention in the case of glucose when the solution was weaker than 0.05 %—0.08 % (individual differences)? In the case of galactose, on the contrary, there is no question of total retention. And, as regards the second point, in the case of galactose *the toleration remains unchanged* in spite of the increase in sugar concentration. Only when the concentration becomes as high as 0.25 % does the toleration diminish. Experiments with stronger concentrations were not made since the secretion of artificial urine then became too scanty.

We still have at our disposal a number of former experiments in which the reduction was determined by the earlier method of BANG (1916), a part of which experiments have been published already. They are found together in the following table (II). As will be noticed the results are not as uniform as those of table I, but in any case they point in the same direction.

<sup>1</sup>) Compare for this our previous article in these Proceedings.

TABLE I.  
The toleration of the glomerular membrane for d-galactose.

Date of experiments.	Frog.	I. The perfusion liquid contains of galactose:	II. Reduction of perfusion liquid expressed in glucose %.	III. The urine has a reducing power of glucose %.		IV. Retention (II-III).		V. Percentage retention of galactose. Average of the kidneys (calculated from II and IV).
				Out of the right kidney.	Out of the left kidney.	Right kidney.	Left kidney.	
12th June 1919	A.	0.05 %	0.0964	0.0535	0.0535	0.0429	0.0429	0.0429 0.0964 $\times$ 100 = 46 %
	B.	0.05 "		0.0571	0.0535	0.0393	0.0429	
14th June "	A.	0.1 "	0.125	0.0571	0.0531	0.0679	0.0715	56 "
	B.	0.1 "		0.0607	0.0571	0.0643	0.0679	53 "
3rd July "	A.	0.1 "	0.0785	0.0357	0.04	0.0428	0.0385	56 "
	B.	0.1 "		0.04	0.05	0.0385	0.0285	43 "
4th July "	A.	0.1 "	0.0714	0.0357	0.0285	0.0357	0.0429	55 "
	B.	0.1 "		0.0357	0.0321	0.0357	0.0393	53 "
1st July "	A.	0.15 "	0.112	0.05	0.057	0.062	0.055	52 "
	B.	0.15 "		0.05	0.0607	0.062	0.0513	51 "
30th June "	A.	0.2 "	0.157	0.107	0.057	0.050	0.082	42 "
	B.	0.2 "		0.0893	0.0893	0.0677	0.0677	43 "
1st July "	A.	0.25 "	0.193	0.121	0.121	0.072	0.072	37 "
	B.	0.25 "		0.13	0.063	0.13	0.063	33 "

TABLE II.  
The toleration of the glomerular membrane for d-galactose (Method of BANG 1916).

Date of experiment.	Frog.	I. The perfusion liquid contains of glucose:	II. Reduction of perfusion liquid expressed in glucose ‰.	III. The urine has a reducing power of glucose ‰.		IV. Retention (II—III).		V. Percentage retention of galactose. Average of the kidneys (calculated from I and II).
				Out of the right kidney.	Out of the left kidney.	Right kidney	Left kidney.	
31st Oct. 1918	A.	0.06 ‰	0.045 ‰	0.03	0.02	0.015	0.025	46 ‰
	B.			0.025	0.03	0.02	0.015	44 "
16th Aug. "		0.1 "	0.07 "	0.045	0.04	0.025	0.03	39 "
17th Aug. "		0.1 "	0.055 "	0.0325	0.0325	0.0225	0.0225	41 "
18th Aug. "		0.1 "	0.055 "	0.0325	0.0325	0.0225	0.0225	41 "
Sept. "	A.	0.1 "	0.0825 "	0.049	0.050	0.033	0.0325	40 "
	B.			0.078	0.0825	0.038	0 ?	46 "
	C.			0.0663	0.0675	0.0162	0.015	19 "
	A.	0.1 "	0.0725 "	0.05	0.0525	0.0225	0.02	30 "
	B.			0.045	0.048	0.0275	0.0245	36 "
	C.			0.047	0.045	0.0250	0.0275	40 "
	A.	0.15 "	0.0975 "	0.07	0.07	0.0275	0.0275	29 "
	B.			0.0525	0.05	0.045	0.0475	41 "
23rd Aug. "								

Also from this table it appears, that, in contrast with what was always found for glucose, *the galactose was not totally retained in any of the experiments, and further, that the retained part forms more or less the half — here a little less than the half.*

Consequently the first explanation mentioned on page 361 fails here and we are obliged, per exclusionem, to accept the second explanation, viz., the one based upon the fact that in aqueous solution the galactose exists in two modifications.

2. *The solution of galactose contains two forms of galactose that behave differently towards the kidneys.*

As is known a large number of sugars exhibits multirotation (DUBRUNFAUT), i.e. some time after the preparation of the aqueous solution they generally possess a slighter specific rotation than immediately after the solution is made. Several explanations, which we need not discuss here have been given for this. It is agreed upon however that under the influence of the solvent, part of the sugar changes into another form with a slighter rotatory power. The two forms are in equilibrium.

This idea is based, in the first place, upon the researches of TANRET<sup>1)</sup>, who separated in solutions, first of glucose and afterwards of galactose and other sugars, three forms with different rotatory power, and several different physical properties; these he termed  $\alpha$ ,  $\beta$  and  $\gamma$ : the  $\alpha$  modification of d-galactose with a specific rotatory power of  $+135^\circ$ , a  $\beta$  modification with  $[\alpha]_D = +81^\circ$  and a  $\gamma$  modification of  $[\alpha]_D = +53^\circ$ .

More extensive researches, especially of E. ROUX<sup>2)</sup> and further of BOURQUELOT<sup>3)</sup> have taught however that the  $\beta$  variety of TANRET is no independent sugar, but consists of a mixture of  $\alpha$  and  $\gamma$ , which are in equilibrium.

I find it useful to point out here a misunderstanding in the literature which has given me much trouble personally, and, I am informed, others also. In several scientific treaties and textbooks it is stated or taken for granted that there are two modifications only; these are called then the  $\alpha$  and  $\beta$  modifications. It is clear here that the  $\beta$  modification is actually the  $\gamma$  modification of TANRET. But, as far as I know, nobody calls attention to this fact. To what faulty reports

<sup>1)</sup> TANRET, Bulletin de la Société Chimique, [3], 13, (1895), p. 728 [3], 15. (1896), 195.

<sup>2)</sup> E. ROUX, Ann. Chim. et de Phys., VII Série, 30, p. 422.

<sup>3)</sup> BOURQUELOT, Journal de Pharm. et de Chem. [7], 14, (1916) 225.

this interchange of the two can lead appears for instance in the wellknown tables of LANDOLT-BORNSTEIN 4e Aufl. 1912. There we read in connection with d-galactose: "Anfangsdrehung nach 7 Minuten als  $\alpha$  Modifikation  $+117^{\circ}.5$ ; Eddrehung nach 7 Stunden als  $\beta$ -Modification  $+80^{\circ}.27$ ." In fact the final rotation which an ordinary d-galactose solution exhibits is  $80^{\circ}.27$  or simply  $81^{\circ}$ . But this is not the rotation of the second modification; that rotation is  $+53^{\circ}$ . A rotation of  $81^{\circ}$  results when the two modifications of  $117^{\circ}.5$  and  $+53^{\circ}$  are in equilibrium. And this equilibrium comes about when we start out from  $\beta$  as well as when we do from  $\alpha$ . In both cases a mixture results with a rotatory power of  $81^{\circ}$ .

It is perhaps useful that what has been brought forward for d-galactose should also be applied to d-glucose, which is so much more used.

TANRET distinguishes 3 forms of glucose: an  $\alpha$  form with  $[\alpha]_D = 106^{\circ}$ , a  $\beta$  form with  $[\alpha]_D = 53^{\circ}$  and a  $\gamma$  form with  $[\alpha]_D = 19^{\circ}$ ; Both  $\alpha$  and  $\beta$  forms when dissolved in water finally exhibit a rotation of  $53^{\circ}$ . The  $\beta$  form of TANRET, according to the researches of ROUX and others, must be considered as an equilibrium between  $\alpha$  and  $\gamma$ , and this is at present generally accepted too.  $\beta$  is therefore no independent modification, but merely a mixture. But now we read the following: There exist two forms of glucose, — the  $\alpha$  and the  $\beta$  forms and not infrequently there is added: The  $\alpha$  form is converted into the  $\beta$  form. How can those who maintain that we have to deal with a reversible reaction here, speak of an *equilibrium* between  $\alpha$  and  $\beta$ ? If it were stated that the  $\alpha$  form is *partially* converted into the  $\beta$  form, it would be clear.

We find the case put differently again by HOLLEMAN in his wellknown text book of Organic Chemistry, 5th ed., 1912, p. 300. He also speaks of an  $\alpha$  form of  $106^{\circ}$ , a  $\beta$  form of  $19^{\circ}$  and a  $\gamma$  form of  $53^{\circ}$ . This  $\gamma$  form is according to him a mixture of  $\alpha$  and  $\beta$ . The  $\gamma$  of TANRET HOLLEMAN calls thus  $\beta$ .

The question arises why TANRETS' nomenclature was not stuck to. After EMIL FISCHER had shown us how to prepare artificial glucosides<sup>1)</sup> it appeared to him that stereoisomeric modifications of each glucoside existed. One was acted upon by beer ferment (invertine), and the other by emulsine.<sup>2)</sup> By way of distinction he called the first the  $\alpha$ -glucoside and the second the  $\beta$ -glucoside and the corresponding forms of glucose  $\alpha$ - and  $\beta$ -glucose<sup>3)</sup>. It is to this nomenclature that later writers seem to have stuck. For this there was an inducement to some extent as if there existed only two forms it was not quite rational to call the second  $\gamma$ . Coincidentally that which EMIL FISCHER calls  $\beta$  corresponds with what TANRET at the same time gave the name of  $\gamma$ .

To explain now the partial retention of the galactose solution which is passed through the kidneys, we assume that *only one of the*

<sup>1)</sup> EMIL FISCHER, Ber. d. D. Chem. Ges. **26**, 2400 (1893).

<sup>2)</sup> The same, Ber. **27**, 2985, (1894); **28**, 1145, (1895).

<sup>3)</sup> C.f. also E. FISCHER, Z. f. physiol. Chemie **26**, 60, (1898).

*modifications is retained by the glomerular membrane, and the other is allowed to pass through.*

As appears from tables I and II, of not too strong concentrations about the half is retained.

It is now, upon closer inspection of what has been handled above, possible to calculate in a simple way the relative quantities of  $\alpha$ - and  $\beta$ -galactose.

If we call the amount of the  $\alpha$  variety (rotation  $+135^\circ$ ) in the galactose solution, in which there is equilibrium between  $\alpha$  and  $\beta$ ,  $x$ , then  $1-x$  is the amount of the  $\beta$  variety (rotation  $+53^\circ$ ), and then, because the rotation of the mixture  $81^\circ$  is, the following equation must hold:

$$\begin{aligned} 135x + 53(1-x) &= 81. \\ x &= 0,34. \\ 1-x &= 0,66. \end{aligned}$$

Therefore the ratio between the quantities of the  $\alpha$  and  $\beta$  forms is 34 : 66.

These figures cannot boast of great accuracy, because, in the first place, we find with other writers for the specific rotation of the  $\alpha$  variety a value of  $117^\circ$  and not  $135^\circ$ . If this value is the true one then we should get a ratio of 44 : 56 between the modifications. Further it must be remembered that the concentration and temperature of the galactose solution are not without influence on the equilibrium. In general however it can well be said that the greater half is the  $\beta$ -form ( $\gamma$ -form of TANRET).

A similar proportion can also be deduced from the researches of E. ROUX in connection with the rate of conversion of the  $\alpha$  into the  $\gamma$ -form.

It is now very remarkable also in our perfusion experiments that more or less the half of the galactose is retained. This parallellism can be considered as supporting our hypothesis. Whether it is the  $\alpha$ - or  $\gamma$ -form which is retained we cannot venture to say with any certainty at present. That might be the case, if, in the first place, the values of the rotations which we used above in deducing the relative quantities of the  $\alpha$ - and  $\beta$ -forms had been taken at the same concentrations as the physiological concentrations (0.05%—0.15%), which we used in our perfusion experiments.

In the second place the values found along chemical lines for the rotation, in concentrations of 6%—18%, leave much to be desired. And then in the next place it must not be forgotten that, as a matter of course, the degree of accuracy of our determinations of the

reduction, by our perfusion experiments is but small. To grasp this it must be remembered that the quantity of artificial urine obtained from our experiments with frogs was only 0.1 c.c. Let us take an arbitrary example to see what influence a small error has on the titration.

In table I perfusing with 0.15% galactose for obtaining the final reaction was used:

for the urine of frog A	}	right kidney	1,83	cc.	thiosulphate
		left kidney	1,81	„	„
for the urine of frog B	}	right kidney	1,83	„	„
		left kidney	1,80	„	„

From these figures the value calculated for the galactose retention for frog A is 55% and 49%, and for frog B 55% and 46%. An error of 0.02% thiosulphate, therefore, causes an error of  $55 - 49 = 6\%$  in the retentive power found.

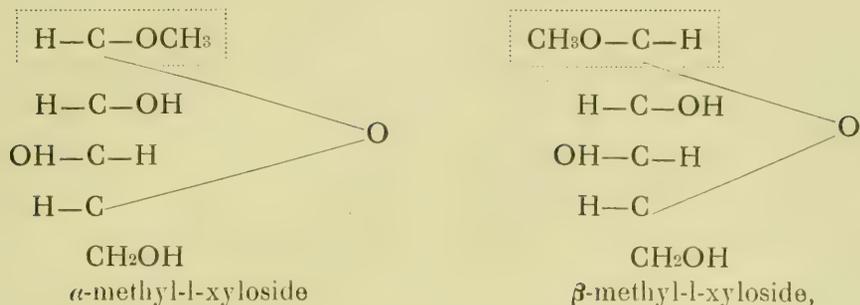
To increase the degree of accuracy it is necessary to experiment upon larger animals, which supply more urine, thus with kidneys of warm blooded animals. For this however a room is necessary which can be brought up to body temperature, which, under present circumstances, is impossible.

Be it as it may, if the differences between the retained and the not retained had been greater — for, did we not find that approximately half was retained and half not —? and, to correspond with this, the difference between the quantities of  $\alpha$  and  $\beta$  modifications in the galactose solution had also been greater, then it would at once have been obvious which form is retained and which is not. We will return to this in connection with xylose.

Before we proceed to discuss the behaviour of xylose, we wish to face an objection which may, on superficial inspection, be made against our representation. It could be remarked that when *one* of the two modifications has been removed by the kidney, from the second modification which remains in circulation, more of the first modification will be formed, and that eventually all the galactose will leave the kidney in that way. Let it be taken into consideration however that in our perfusion experiments the perfusion liquid forms only a very small quantity of urine and therefore only very little of the one kind of galactose is removed. Let it further be remembered that, unlike in the normal body, the same perfusion fluid does not remain in circulation. In our experiments the solution leaving by the renal vein does not return again by the renal artery.

But what can be the reason then that in the normal organism





the first of which is converted by invertine and the second only by emulsine, and which were distinguished by FISCHER as  $\alpha$  and  $\beta$ . Suppose the  $\text{CH}_3$  substituted by H, then  $\alpha$ - and  $\beta$ -xylose is formed again.

Similar experiments as were made with galactose were also made with xylose. We give a table which contains the experiments in question (see table III).

From this table it appears that the l-xylose, like the d-galactose, is not retained completely either. Always the greater portion passes through the glomerular membrane, a *greater portion percent, however*, than was the case with d-galactose; between  $\frac{1}{4}$ , and  $\frac{1}{4}$ , only of the xylose, used in concentrations which do not affect the production of artificial urine, on an average is retained.

We now find, according to VON LIPPMANN<sup>1)</sup> for the rotation of the l-xyloses: initial rotation  $+78^\circ$ , final rotation  $+19^\circ$ . Therefore, the  $\alpha$ -form has a rotation of  $+78^\circ$ ; the  $\beta$ -form however has up to this not been isolated, and therefore it cannot be stated, how much of this modification takes part in the final rotation of  $+19^\circ$ . Also for xylose it is therefore not possible to indicate the form, which is retained by the kidneys, and which is allowed to pass through.

That the glomerular membrane possesses the property of separating two sugars quantitatively like a sieve, retaining one and letting the other pass through, we have formerly been able to show with mixtures of glucose and fructose, and glucose and lactose<sup>2)</sup>.

Let us remark that researchers who are engaged upon distinguishing of sugars by means of microbes<sup>3)</sup> have to reckon with

<sup>1)</sup> VON LIPPMANN, Chemie der Zuckerarten.

<sup>2)</sup> HAMBURGER and BRINKMAN, These Proc. Sept. 28th 1918.

<sup>3)</sup> C.f. among others A. J. KLUYVER, Biochemische Suikerbepalingen. Diss. Delft 1914. W. C. DE GRAAFF, De biochemische eigenschappen van paratyphusbacillen, Leiden, S. C. VAN DOESBURGH, 1919.

TABLE III.  
Toleration of the kidneys for l-xylose.

Date of experiment.	Frog.	I. The perfusion liquid contains of galactose :	II. Reduction of perfusion liquid expressed in glucose %.	III. The urine has a reducing power of glucose %.		IV. Retention (II—III).	V. Percentage retention of galactose. Average of the kidneys (calculated from II and IV).
				Out of the right kidney.	Out of the left kidney.		
1st July 1919	A.	0.06 %	0.059 %	0.0357	0.0357	0.0233	39 %
	B.			0.0393	0.0393	0.0197	33 "
7th July 1919	A.	0.1 "	0.107 "	0.075	0.068	0.035	33 "
	B.			0.0721	0.075	0.033	31 "
8th July 1919	A.	0.15 "	0.168 "	0.125	0.125	0.043	25 "
	B.			0.1321	0.1469	0.028	19 "
10th July 1919	A.	0.2 "	0.2 "	2 + l =	0.171 <sup>1)</sup>	0.029	15 "
	B.			2 + l =	0.185	0.015	8 "
4th Oct. 1918	A.	0.1 " * <sup>2)</sup>	0.125 "	0.11	0.10	0.020	16 "
	B.			0.095	0.1025	0.0987	21 "
20th Nov. 1918	A.	0.1 " *	0.105 "	0.075	—	0.03	28 "
	B.			0.075	0.08	0.028	24 "

<sup>1)</sup> In this series the urine secreted from each kidney apart amounts to less than 0.1 c.c. For this reason the liquids out of the right and left kidneys were added to each other. In general it could be noticed that as the concentration of xylose became stronger less and less urine was secreted.

<sup>2)</sup> Both of the last two series marked with \* were determined with the help of BANG's earlier method (1916).

the fact that both the modifications in which a large number of sugars occur, need not behave in the same manner towards those organisms. Besides that, the circumstances in such experiments are somewhat different from those in our cases; for have we not to do with a disturbance of equilibrium in the ferment experiments, which disturbance is the result of the eventually being used up of one of the modifications, but is adjusted again? (cf. p. 367).

*The behaviour of the kidneys towards sugars other than galactose and xylose.*

It has appeared that from the experiments described up to this three cases can be distinguished:

1<sup>st</sup>. The sugar, if its concentration does not exceed the physiological border for more than to a very slight extent<sup>1)</sup>, is *completely retained*. This applies exclusively to glucose.

2<sup>nd</sup>. A *partial retention* takes place; This was the case with solutions of d-galactose, of d- and l-xylose, of d-ribose and of maltose.

3<sup>rd</sup>. *Nothing is retained*. This we found for l-glucose, l- and d-arabinose, l- and d-mannose and lactose.

But all sugars mentioned under 1, 2 and 3 exhibit multirotation and occur therefore in 2 modifications. The question is thus obvious: why do those sugars mentioned under 1 and 2 not behave like galactose, i. o. w. why do they not all exhibit partial retention? I think that the explanation must be sought herein, that, of the glucose both modifications are retained, of the galactose and other sugars mentioned under 2 only one modification, and of arabinose and the others mentioned under 3 neither of the two forms.

We will set ourselves the task to test this conjecture by experiment. We are engaged upon this; we have already obtained satisfactory results.

*Summary and conclusion.*

The experiments described above are concerned with the question what the cause can be that of a 0.1 % solution of d-galactose only a part of the sugar is retained and the other not. Two explanations were possible. The first was that the original galactose-solution which we used was of too strong a concentration viz. 0.1 %. We thought namely of our earlier researches in connection with glucose, where it appeared amongst other things that when there is passed

<sup>1)</sup> HAMBURGER and BRINKMAN, Die Toleranz der Nieren für Glukose, Bioch. Zeitschr. 94, 131, 1919.

through the blood vessels of the kidneys a glucose solution which exceeds the physiological value ( $\pm 0.07\%$ ) only by about  $0.03\%$ , already a little sugar is allowed to pass through, and, that this quantity increases according as the concentration of glucose becomes stronger, and that to the extent that with higher glucose concentrations less and less glucose is retained. The kidney cannot endure stronger glucose concentrations, i. o. w. the glomerular membrane sickens. *Towards galactose, however, the kidney behaved quite differently.* Indifferent to whether stronger or weaker concentrations were used, a portion was always allowed to pass through, and what is remarkable, always about the half (see table I and II). The first explanation could therefore not be the correct one.

Per exclusionem the second one had to be accepted then, namely this, that, of the two modifications in which glucose is present in aqueous solution, — the  $\alpha$  and the  $\beta$  modifications, — the one is retained and the other is allowed to pass through. This conception agrees with the fact which we observed previously, i. e. that the glomerular membrane is able to separate quantitatively from each other different sugars retaining one and letting pass through the other, which was demonstrated with mixtures of glucose and fructose, and glucose and lactose. The conjecture finds additional strong support in the fact that, according to our calculation, there are present practically equal quantities of the  $\alpha$  and  $\beta$  modifications in a solution of d-galactose, the same proportion thus, more or less, in which it is retained and not retained. For this very reason it cannot as yet be said with certainty, which modification is retained, the  $\alpha$  or the  $\beta$ . In the same position we are in the case of xylose, which like the d-galactose exhibits a partial retention. It has appeared namely from our perfusion experiments that on an average from  $\frac{1}{4}$  to  $\frac{1}{2}$  of the xylose is retained.

From the same point of view the partial retentions which were observed in connection with d-xylose, d-ribose and maltose may be looked at. Also these reducing sugars occur, in agreement with their multirotation, in two modifications; also these sugars exhibit partial retention.

However it has appeared that *not all* sugars that occur in two modifications, show partial retention. In the first place the d-glucose does not. If present in physiological concentration it is retained completely by the glomerular membrane, and as such it occupies a unique place; and still also the glucose occurs in two modifications. The latter applies also to l-glucose, d-mannose and l- and d-arabinose. Of these sugars nothing is retained.

By summarising the multirotatory sugars can be divided into three groups.

1<sup>st</sup> group, of which we know only one representative viz. the d-glucose of which *both modifications are retained*.

2<sup>nd</sup> group (d-galactose, d- and l-xylose, d-ribose) of which *only one modification is retained*.

3<sup>rd</sup> group (l-glucose, d-mannose, d- and l-arabinose) of which *neither of the two modifications are retained*.

In the cases of the members of the second group viz. d-galactose and l-xylose which have been subjected to more detailed examination, the retention or non-retention is governed wholly by the position which the H and HO linked to the asymmetric C-atom occupy relative to each other.

It is worthy of comment still, in the first place, that the conjecture which we have offered here, and which ought to be controlled by a large number of experiments still, gives a physiological illustration of the existence of modifications, which formerly was found along chemical lines.

In the second place it brings to light that if one desires to investigate stereoisomeric sugars with respect to lower organisms, as has already been done by several investigators, the fact has to be reckoned with, henceforth, that the sugar which is investigated is not a simple compound but a mixture, the two components of which need not behave similarly towards a microorganism.

In these investigations Mr. R. ROELINK has lent his skilled assistance.

*Physiological Laboratory.*

*Groningen, September 1919.*

ERRATUM.

p. 70 line 14 from the bottom:

for: — the growth retardation curve for an intensity 1.

read: — the growth retardation curve for an intensity 4.

line 13 from the bottom:

for: — the growth retardation curve for an intensity 4.

read: -- the growth retardation curve for an intensity 1.

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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN  
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXII

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Secretary: Prof. P. ZEEMAN.

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**Mathematics.** — “Theorem on the term by term differentiability of a series.” By Prof. F. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of June 28, 1919).

1. If

a. the functions  $u_1(x), u_2(x), \dots$  are differentiable in the interval  $a \leq x \leq b$ , denoted by  $i$ ,

b. the series  $u'_1(x) + u'_2(x) + \dots$  is uniformly convergent in the interval  $i$ ,

c. the series  $u_1(x) + u_2(x) + \dots$  is convergent for a value  $c$  of  $x$  in the interval  $i$ ,

d. the functions  $u'_1(x), u'_2(x), \dots$  are continuous for the value  $x = p$  in the interval  $i$ ,

then

a. the series  $u_1(x) + u_2(x) + \dots$  is uniformly convergent in the interval  $i$  and

$\beta$ . the function  $u_1(x) + u_2(x) + \dots$  is differentiable for  $x = p$ , its differential coefficient being  $u'_1(p) + u'_2(p) + \dots$ .

Evidently the expression “differentiable” has for  $x = a$  or  $x = b$  to be taken in the sense of differentiable on the right, resp. on the left.

For this theorem (which is usually deduced in a more restricted form <sup>1)</sup> from the term by term integrability) we shall here give a simple proof resting on the definition of differential coefficient only.

2. Proof of a. If we put  $u'_{n+1}(x) + u'_{n+2}(x) + \dots = R_n(x)$ , we have ( $x$  belonging to the interval  $i$ ):

$$\begin{aligned} u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+k}(x) &= \{u_{n+1}(c) + u_{n+2}(c) + \dots + u_{n+k}(c)\} + \\ &+ \{u'_{n+1}(\xi) + u'_{n+2}(\xi) + \dots + u'_{n+k}(\xi)\}(x - c) = \\ &= \{u_{n+1}(c) + u_{n+2}(c) + \dots + u_{n+k}(c)\} + \{R_n(\xi) - R_{n+k}(\xi)\}(x - c), \end{aligned}$$

where  $\xi = c + \theta(x - c)$ , hence:

$$\begin{aligned} |u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+k}(x)| &\leq \\ \leq |u_{n+1}(c) + u_{n+2}(c) + \dots + u_{n+k}(c)| &+ \{|R_n(\xi)| + |R_{n+k}(\xi)|\} |x - c| \leq (b - a). \end{aligned}$$

<sup>1)</sup> i.e. on the assumption, that the functions  $u'_1(x), u'_2(x)$ , etc. are continuous throughout the interval  $i$ .

$\sigma$  denoting an arbitrary positive number it follows from the assumption  $c$ , that a value  $N_1$  can be assigned such that for  $n > N_1$  we have  $|u_{n+1}(c) + u_{n+2}(c) + \dots + u_{n+k}(c)| < \frac{1}{3}\sigma$ . From the assumption  $b$  it follows moreover that a number  $N_2$  can be found, such that for  $n > N_2$ , the inequality  $|R_n(\xi)| < \frac{\sigma}{3(b-a)}$  holds good.

Let  $N$  be the greater of the two numbers  $N_1$  and  $N_2$ , then  $|u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+k}(x)| < \sigma$  for every  $n > N$  and for every  $x$  within the interval  $i$ . This establishes  $\alpha$ ; the assumption  $d$  can here be dispensed with.

**3. Proof of  $\beta$ .** If we put  $u_1(x) + u_2(x) + \dots = \varphi(x)$  and  $u'_1(x) + u'_2(x) + \dots = \psi(x)$  then the assertion  $\beta$  expresses that

$$\lim_{h \rightarrow 0} \frac{\varphi(p+h) - \varphi(p)}{h} = \psi(p). \quad \dots \quad (1)$$

If we put  $u_1(x) + u_2(x) + \dots + u_n(x) = U_n(x)$  and  $u_{n+1}(x) + u_{n+2}(x) + \dots = \varrho_n(x)$  then

$$\begin{aligned} \frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) &= \frac{U_n(p+h) - U_n(p)}{h} + \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h} - \psi(p) = \\ &= U'_n(p + \theta h) + \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h} - \psi(p) = \\ &= \psi(p + \theta h) - \psi(p) - R_n(p + \theta h) + \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h}, \end{aligned}$$

hence:

$$\left| \frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) \right| \leq \left| \psi(p + \theta h) - \psi(p) \right| + |R_n(p + \theta h)| + \left| \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h} \right|. \quad \dots \quad (2)$$

In consequence of the assumptions  $b$  and  $d$   $\psi(x)$  is continuous for  $x = p$ . Hence,  $\sigma$  denoting an arbitrary positive number, the positive number  $\varepsilon$  can be assigned such that  $|\psi(p+h) - \psi(p)| < \frac{1}{4}\sigma$  for  $|h| < \varepsilon$ ; for  $|h| < \varepsilon$  we have then also  $|\psi(p + \theta h) - \psi(p)| < \frac{1}{4}\sigma$ .

In consequence of the assumption  $b$  we may assign  $N_1$  such that for any  $n > N_1$  and any  $x$  in the interval  $i$  the inequality  $|R_n(x)| < \frac{1}{4}\sigma$  is satisfied. For  $n > N_1$  then  $|R_n(p + \theta h)| < \frac{1}{4}\sigma$ .

Let  $h$  be a definite number satisfying  $|h| < \varepsilon$  and  $h \neq 0$ , then the numbers  $N_2$  and  $N_3$  can be so determined that for  $n > N_2$  and  $n > N_3$  the inequalities  $|\varrho_n(p+h)| < \frac{1}{4}\sigma|h|$  resp.  $|\varrho_n(p)| < \frac{1}{4}\sigma|h|$  are satisfied.

Now, if  $n$  be chosen larger than the largest of the numbers  $N_1$ ,

$N_2$  and  $N_3$ , then it follows from (2) that  $\left| \frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) \right| < \delta$  for  $|h| < \varepsilon$  ( $h \neq 0$ ), whence (1) may be concluded.

**4. Remarks.** In the foregoing proof for the differentiability of the function  $\varphi(x) = u_1(x) + u_2(x) + \dots$  for  $x = p$  use has been made of the *continuity of the functions*  $u'_1(x), u'_2(x), \text{ etc. for } x = p$  only; therefore it was not necessary to suppose their continuity throughout an interval. Also there was no need to assume the integrability of the functions  $u'_1(x), u'_2(x), \text{ etc.}$ , so that the usual proof (where the series  $u'_1(x) + u'_2(x) + \dots$  is integrated term by term) does not apply to the more general formulation as given in N<sup>o</sup>. 1.

**5.** When the convergence of the series  $u_1(x) + u_2(x) + \dots$  throughout the whole interval  $i$  is assumed, then it is sufficient, in order to establish  $\beta$ , to assume the semi-uniform convergence (simple-uniform convergence of DINI) of the series  $u'_1(x) + u'_2(x) + \dots$  in the interval  $i$ . This semi-uniform convergence namely is in the first place sufficient to establish the continuity of  $\psi(x)$  for  $x = p$ . The determination of the numbers  $N_2$  and  $N_3$  presents further no difficulty, after which it is possible to attribute to  $n$  such a value  $> N_2$  and  $> N_3$ , that the inequality  $|R_n(x)| < \frac{1}{4} \delta$  holds for every  $x$  in the interval  $i$ . It is immaterial whether this inequality also is satisfied for every greater value of  $n$ .

In his "Teorica delle funzioni di variabili reali" DINI has demonstrated  $\beta$  by a more complicate transformation of  $\frac{\varphi(p+h) - \varphi(p)}{h}$ , thereby omitting the assumption  $d$  and assuming the series  $u_1(x) + u_2(x) + \dots$  to be convergent throughout the interval  $i$ .

**6.** In order to demonstrate the differentiability of  $\varphi(x)$  it was not necessary to assume the absolute convergence of the series  $u'_1(x) + u'_2(x) + \dots$ . By supposing  $|u'_n(x)| < c_n$  ( $c_n$  independent of  $x$ ) and the series  $c_1 + c_2 + \dots$  convergent (which includes the uniform absolute convergence of the series  $u'_1(x) + u'_2(x) + \dots$ ) PORTER (Ann. of Math. ser. 2, vol. 3, 1901, p. 19) has proved the differentiability of  $\varphi(x)$  for  $x = p$  in a very simple way without any supposition concerning the continuity of the functions  $u'_n(x)$  viz. by making use of the equality

$$\begin{aligned} \frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) &= \frac{U_n(p+h) - U_n(p)}{h} - U'_n(p) + \\ &+ \sum_{n+1}^{\infty} u'_j(p + \theta_j h) - \sum_{n+1}^{\infty} u'_j(p). \end{aligned}$$

**Mathematics.** — “*Involutions in a field of circles*”. By Prof.  
JAN DE VRIES.

(Communicated in the meeting of September 27, 1919).

1. In a plane are given three systems of coaxial circles  $(\alpha)$ ,  $(\beta)$ ,  $(\gamma)$  in each of which the circles are arranged in the pairs  $\alpha_1, \alpha_2$  etc. of an involution. Let  $\delta_1$  be the circle which intersects the circles  $\alpha_1, \beta_1, \gamma_1$  orthogonally,  $\delta_2$  the orthogonal circle of the corresponding circles  $\alpha_2, \beta_2, \gamma_2$ , then  $\delta_1$  and  $\delta_2$  are conjugated in an *involutionary correspondence* in the field of circles.

Since  $\alpha_1$  coincides twice with  $\alpha_2$ ,  $\beta_1$  twice with  $\beta_2$  and  $\gamma_1$  twice with  $\gamma_2$ , the involution  $(\delta_1, \delta_2)$  has *eight coincidences*.

In general an arbitrary circle  $\delta_1$  is intersected orthogonally by one circle  $\alpha$  only. However, when  $\delta_1$  belongs to the system  $(\alpha')$  of coaxial circles orthogonally intersecting the circles of  $(\alpha)$  then  $\alpha_1$ , and  $\alpha_2$  also, is an arbitrary circle from  $(\alpha)$ , whilst  $\beta_2$  and  $\gamma_2$  are perfectly defined. In this case every circle  $\delta_2$  intersecting  $\beta_2$  and  $\gamma_2$  orthogonally corresponds with  $\delta_1$ .

Hence the *orthogonal systems*  $(\alpha'), (\beta'), (\gamma')$  of  $(\alpha), (\beta), (\gamma)$  consist of *singular circles*, i.e. of circles which in the involution are conjugated each to an infinite number of circles.

There is still another way in which  $\delta_1$  may be singular. On a circle  $\alpha$  the systems  $(\beta)$  and  $(\gamma)$  determine two involutions; since these have one pair in common, on  $\alpha$  are to be found the two points of intersection of a circle  $\beta$  with a circle  $\gamma$ . Hence every circle  $\alpha$  (or  $\beta$ , or  $\gamma$ ) belongs to a triplet  $\alpha_2, \beta_2, \gamma_2$ , belonging to one system of circles and for which the orthogonal circle accordingly becomes indefinite. The circle  $\delta_1$  which intersects the corresponding circles  $\alpha_1, \beta_1, \gamma_1$  orthogonally is therefore *singular* and conjugated to every circle of a certain system of coaxial circles.

2. A further investigation of the involution  $(\delta_1, \delta_2)$  becomes comparatively simple, when we make use of a representation of the circles of the field on the points of space, to which Dr. K. W. WALSTRA has attracted attention in 1917<sup>1)</sup>.

In order to obtain this representation we take the plane of our circles as the plane of coordinates  $z = 0$ . A circle we then represent

<sup>1)</sup> These Proceedings XIX, p. 1130.

by the point on its axis with coordinate  $z$  equal to the power of the origin  $O$  with respect to the circle.

All circles with radius zero are represented by the points of a paraboloid of revolution  $\mathbf{G}$  (limiting surface) and the images of two orthogonal circles are harmonically separated by  $\mathbf{G}$ . Two reciprocal polar lines are the images of two systems of coaxial circles orthogonal to each other.

The systems  $(\alpha), (\beta), (\gamma)$  are represented by three involutions  $(A_1, A_2), (B_1, B_2), (C_1, C_2)$  situated on three straight lines  $a, b, c$ . The image  $D_1$  of the circle  $\sigma_1$ , which intersects  $\alpha_1, \beta_1, \gamma_1$  orthogonally is the pole of the plane  $A_1 B_1 C_1$ . So we have now to consider an involution  $(D_1, D_2)$  of the points of space, which involution is characterized by the property that the polar planes  $\Delta_1$  and  $\Delta_2$  of  $D_1$  and  $D_2$  meet the given lines  $a, b, c$  in the pairs  $(A_1, A_2), (B_1, B_2), (C_1, C_2)$  of three given involutions.

3. It is now easy to find the *singular* elements of the involution of circles again. In the first place we observe that  $A_1$  becomes indefinite as soon as  $\Delta_1$  passes through  $a$ ; for  $\Delta_2$  now any plane may be chosen which contains the points  $B_2$  and  $C_2$ , hence for  $D_2$  any point of the polar line  $a'$ , of the straight line  $a_2 \equiv B_2 C_2$ . If  $\Delta_1$  is made to revolve about  $a$ , then  $D_1$  moves along the polar line  $a'$  of  $a$ , and  $a_2$  describes a ruled quadric. The line  $a'$ , also describes a ruled quadric  $(a'_2)^2$  of which the polar lines  $b'$  and  $c'$  of  $b$  and  $c$  are directrices. It is obvious that to every point of  $a'$  a definite straight line of  $(a'_2)^2$  is conjugated. Similarly to the *singular lines*  $b', c'$  correspond the ruled quadrics  $(b'_2)^2, (c'_2)^2$ .

Secondly  $D_2$  becomes indefinite as soon as  $A_2, B_2$  and  $C_2$  are collinear and therefore situated on a transversal  $s$  of  $a, b, c$ . When  $s$  is made to coincide successively with the generators of the ruled quadric having  $a, b, c$  for directrices, then  $A_1, B_1$  and  $C_1$  describe three projective ranges, so that  $\Delta_1$  osculates a twisted cubic  $\sigma^3$ , of which the lines  $a', b'$  and  $c'$  are bisecants. To every point  $S \equiv D_1$  of this *singular curve*  $\sigma^3$  evidently is correlated a line  $s'$  viz. the polar line of the corresponding line  $s$ . The lines  $s'$  form a ruled quadric  $(s')^2$  with the directrices  $a', b', c'$ .

4. If  $D_1$  describes the line  $l$ , then  $\Delta_1$  revolves about the polar line  $l'$ , so that  $A_1, B_1$  and  $C_1$  describe projective ranges.  $A_2, B_2$  and  $C_2$  then also describe projective ranges; hence  $\Delta_2$  osculates a twisted cubic  $\lambda^3$ , of which  $a', b'$  and  $c'$  are bisecants. Consequently  $D_1$  and  $D_2$  are conjugated in a *cubic correspondence*. •

Since  $l$  has two points in common with  $(s')^2, \lambda^3$  rests on  $\sigma^3$  in two points. The rays of space are in this way transformed into the

fourfold infinity of twisted cubics, which intersect each of the lines  $a'$ ,  $b'$ ,  $c'$  and the curve  $\sigma^3$  twice.

A plane  $\Phi$  is transformed into a cubic surface passing through  $a'$ ,  $b'$ ,  $c'$  and  $\sigma^3$ . The images of two planes have these four lines and the image  $\lambda^3$  of their line of intersection in common.

5. A tangent plane of the limiting surface  $\mathbf{G}$  is the image of the circles which pass through a given point. The involution  $(\sigma_1, \sigma_2)$  therefore (by § 4) has the following property: *A system of coaxial circles is transformed into a class of circles with index three.*

This class contains *six circles with radius zero and three straight lines*. The *singular circles* form *three coaxial systems* (§ 1) and a class with *index three* (§ 3).

To each singular circle a system of coaxial circles is conjugated; these systems form four classes.

The image of a system of coaxial circles contains *eight* singular circles.

6. Evidently the representation of the field of circles on the points of space enables us to deduce from each involution in the latter an involution in the field of circles and vice versa.

A particularly simple involution is obtained as follows. On every ray  $h$  which meets  $OZ$  at right angles the paraboloid  $\mathbf{G}$  determines an involution of conjugated pairs  $(P, P')$ . In the field of circles the analogon hereof is the correspondence which conjugates to each other two circles intersecting orthogonally and having the same power with respect to a fixed point  $O$ .

The point  $P'$ , conjugated to  $P$ , is the intersection of the ray  $h$  with the polar plane  $\pi$  of  $P$ . If  $P$  lies on  $OZ$ , then for  $h$  may be taken any perpendicular to  $OZ$  passing through  $P$ . Since  $\pi$  now is perpendicular to  $OZ$ , to  $P$  will be conjugated every point of the line of infinity of  $z = 0$ .

A point of  $\mathbf{G}$  lies in its own polar plane and therefore constitutes a *coincidence* of the correspondence. When  $P$  reaches the vertex of  $\mathbf{G}$  or the point at infinity of  $OZ$ , then  $P'$  is an arbitrary point of  $z = 0$  or of  $z = \infty$ .

If  $P$  moves along a line  $l$ , then  $h$  describes a ruled quadric  $q^2$  and  $\pi$  a pencil of planes projective with  $q^2$ , so the locus of  $P'$  is a twisted cubic  $\lambda^3$ . The polar line  $l'$  of  $l$  meets  $q^2$  in two points  $P'$ ; each plane through  $l'$  contains besides these two points still another point  $P'$  not lying on  $l'$ . Hence  $l'$  is a chord of  $\lambda^3$ . So is  $l$ , for its points of intersection with  $\mathbf{G}$  are coincidences.

7. To the points  $P$  of a plane  $\Psi$  correspond the points  $P'$  of a cubic surface  $\Psi^3$ . Two such surfaces in the first place have the

curve  $\lambda^3$  in common, which is the image of the line of intersection of the two corresponding planes. In order to obtain a proper insight into the meaning of the figure which they have in common in addition to this, we observe that the involution  $(P, P')$  is a particular case of the following correspondence.

Let a quadric surface  $\Phi^2$  be given and the pair of polar lines  $d, d'$ . Through a point  $P$  the straight line  $t$  is drawn which meets  $d$  and  $d'$ ; the polar plane  $\pi$  of  $P$  defines on  $t$  the point  $P'$ , which we conjugate to  $P$ .

The points of intersection of  $d$  and  $\Phi^2$  we denote by  $E_1, E_2$ , those of  $d'$  and  $\Phi^2$  by  $E'_1, E'_2$ . The straight line  $E_1 E'_1$  lies in  $\Phi^2$ ; to each of its points  $P$  evidently is conjugated any of its points. To each point of  $d$  corresponds every point of  $d'$ . Thus all the edges of the tetrahedron  $E_1 E_2 E'_1 E'_2$  are singular, so that these six lines are conjugated to their points of transit through a plane  $\Psi$ . In addition to the curve  $\lambda^3$  two surfaces  $\Psi^3$  then have these six singular lines in common.

If  $\Phi^2$  now again is replaced by  $\mathbf{G}$ , then  $d$  becomes the axis  $OZ$ ,  $d'$  is the line at infinity of  $z = 0$  and the other four singular lines are to be found in the imaginary lines along which  $\mathbf{G}$  is intersected by  $z = 0$  and  $z = \infty$ .

8. If  $P$  is caused to move along a line  $l$ , which meets  $OZ$ , then  $h$  describes a system of parallel lines which is projective to the pencil constituted by the polar line of  $P$  with respect to the parabola in the plane through  $l$  and  $OZ$ . The points  $P'$  now are situated on a rectangular hyperbola which by the line at infinity of  $z = 0$  is completed to a  $\lambda^3$ .

By the correspondence of the orthogonal circles, which is alluded to in § 6 a *system of coaxial circles* is again transformed into a class *with index three*. The circles with radius zero are coincidences. The two circles of a pair are real only if they have a negative power with respect to  $O$ . When  $O$  lies without a circle, then the conjugated circle has an imaginary radius.

**Physiology.** — “*On Polonium Radiation and Recovery of Function.*”

By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of October 25, 1919).

Several organs discontinue their functions, when we remove from their environment the potassium-ions, which are always present in the circulating fluids. These functions are restored directly when potassium is replaced by other radio-active atoms in the circulating fluids to a quantity aequi-radioactive to the removed potassium<sup>1</sup>).

It does not matter whether the substitute is an  $\alpha$ -rayer or a  $\beta$ -rayer, provided its amount be such that the total radio-activity of the new constituent is about equal to that of the original one. No organ serves our purpose in this experimentation better than the heart of a cold-blooded animal, namely of the frog, because the blood flows on all sides round its cells, which are separated from it only by an endothelium.

There is a rather large number of elements that can replace potassium. Besides rubidium, which was known as such to S. RINGER, my co-workers and I found uranium, thorium, radium, ionium, emanation and actinium (as an admixture to lanthanum and cerium) to be fit substitutes, while of non-radioactive elements only caesium proved serviceable.

However, it is not only the addition of radio-active elements along the blood that can restore the lost function; this can also be effected by radiation from the outside<sup>2</sup>). We succeeded in obtaining this result with mesothorium contained in glass, with radium screened by mica, and unscreened polonium (galvanoplastic on copper). The quantity is of equal order with that which inhibits from the same distance the cultures of bacteria in their growth.  $1\frac{1}{2}$  mgr.-hour served for radiation that restored the function; 12 mgr.-hour for excitation of sterility of bacteria.

The recovery of function is, therefore, brought about by radio-activity, anyhow it is in the case of free radiations.

<sup>1</sup>) Verslag Vol. 25, p. 517 and p. 1096, p. 1282. Vol. 26, p. 555 and p. 776. Proceedings Vol. 19, p. 633 and p. 1043, p. 1161. Vol. 20, p. 768 and p. 773.

<sup>2</sup>) H. ZWAARDEMAKER, C. E. BENJAMINS and T. P. FEENSTRA, Radiumbestraling en hartswerking. Ned. Tijdschr. v. Geneesk. 1916 II, p. 1923 (10 Nov. 1916).

H. ZWAARDEMAKER and G. GRIJNS, Arch. néerland. de physiol., t. 2, p. 500, 1918

It is a moot point as yet whether this action of radiation is direct or indirect.

It may be, namely, that the radiations first liberate the potassium from the potassium-dépôts<sup>1)</sup>, which are present in the cardiac muscle and that only then this liberated potassium, diffusing to the circulating fluid, causes the function to revive.

This possibility could not be ignored a priori, it being a fact that during the radiation rather considerable quanta of potassium may quit the blood-cells and perhaps the heart-cells<sup>2)</sup>.

One of these days I was in a position to carry out an experimentum crucis.

There is namely antagonism between  $\alpha$ - and  $\beta$ -rayers. When applied coincidentally with the same activity, they counter-balance each other's action completely.

This antagonism also obtains with external polonium-radiation ( $\alpha$ -rayer) and internal appliance of potassium ( $\beta$ -rayers). This became evident when a frog's heart, which had been brought to a standstill by removing the potassium from the circulating fluid, and had recovered its beats again through polonium, ceased beating again after being given a physiological dosis of potassium, whereas it resumed its pulsations both by removal of polonium and by that of potassium.

When the polonium was removed, the potassium gradually regained its influence; when the potassium was removed, only the after-effect of the  $\alpha$ -radiation remained.

*From the existence of the antagonism polonium-potassium we must conclude that in this case there is a direct action of radiation.*

For, if the liberation of potassium-atoms (supposing it to occur)

<sup>1)</sup> In the cells of the cardiac muscle there is a rich store of potassium. It is strange that this permanent substance is of itself not competent to keep up the function. This inactivity cannot be due to incapacity of the radiation of the potassium-dépôt to reach as far as the seat of automaticity. To W. E. RINGER and to myself the radiation seemed to be too penetrating for it. Nothing less than a tissue sheet of 1 m.m. thickness is capable of lessening by half the high penetrating power of potassium. I have therefore been obliged to relinquish my original hypothesis. I am now inclined to look for the explanation in the coincident presence of iron. The cells of the cardiac muscle contain iron atoms where also the potassium-atoms are located. Consequently the miniature magnetic fields surrounding the iron atoms, will dislodge the  $\beta$ -particles of the potassium. It may, therefore, be considered whether perhaps this circumstance constitutes an obstacle for outward radiation.

Biologically, various explanations are given, starting from the inactivity of continuous causes and the stimulation of temporary ones.

<sup>2)</sup> Researches not published yet. They will be recorded elsewhere.

should have had to serve as an intermediary, it would be impossible to conceive that the addition of a small quantity of potassium, entirely within physiological limits, should have doomed the polonium-heart to a standstill. On the other hand, if the supposition had come true, the liberated and the newly added potassium-atoms would have aided each other and would have maintained the function, instead of disturbing it as was the case now in consequence of the joint action of polonium-radiation and the internal circulating potassium.

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**Physiology.** — “*Researches on scopolamin-morphin narcosis*”. By Dr. W. STORM VAN LEEUWEN and Miss M. v. D. MADE. (Communicated by Prof. MAGNUS).

(Communicated in the meeting of September 27, 1919).

Much work has been done on scopolamin-morphin narcosis ever since SCHNEIDERLIN<sup>1)</sup> introduced it into medical practice (1900). One of the principal questions that occupied the workers in this field was, whether the administration of a mixture of these two poisons brings about a “potentiated synergism”, which BÜRGI defines as being a stronger effect of a mixture of poisons than the action the component parts alone could lead us to expect. In order to ascertain whether in the scopolamin-morphin mixture “potentiation” occurs, HAUCKOLD<sup>2)</sup> has performed many experiments with rabbits, KOCHMANN<sup>3)</sup> with dogs and SCHNEIDERLIN with men.

HAUCKOLD administered subcutaneously to rabbits respectively morphin, scopolamin, and morphin-scopolamin and detected “potentiation”. He recorded that 5 mgr. of morphin + 0,5 mgr. of scopolamin per kg. can produce narcosis in a rabbit, while 10 mgr. of morphin and 200 mgr. of scopolamin injected separately did not produce a narcotic effect. From this HAUCKOLD concluded that, though scopolamin *per se* does not bring about narcosis in a rabbit, it is nevertheless capable of activating a non-narcotic morphin-dosis. This assertion, however, appeared to be based on erroneous observation, first because we have demonstrated by a method to be discussed lower down, that 0,5 mgr. of scopolamin as well as 5 mgr. of morphin produce decidedly a narcotic effect and secondly, because also with HAUCKOLD’s method, we found no potentiation, but merely simple addition of effects.

HAUCKOLD administered the scopolamin, the morphin, and the mixture scopolamin-morphin to rabbits subcutaneously and then

<sup>1)</sup> SCHNEIDERLIN. Eine neue Narkose. *Ärztl. Mitt. aus u. für Baden*. Mai 1900, quoted from HAUCKOLD SCHNEIDERLIN. Die Skopolamin-Morphin-Narkose. *Münch. Med. Wochenschr.* 1903. N<sup>o</sup>. 9, pag. 371.

<sup>2)</sup> E. HAUCKOLD. Ueber die Beeinflussung von Narkoticis durch Skopolamin. *Zeitschr. f. exp. Path. u. Ther.* Bd. 7, pag. 743, 1910.

<sup>3)</sup> M. KOCHMANN. Ueber die therapeutischen Indikationen des Skopolaminum hydrobromicum. *Die Therapie der Gegenwart*. 1903, pag. 202.

ascertained whether or no a narcosis ensued. We have repeated these experiments, but since — as also HAUCKOLD observes — the depth of the narcosis is difficult to judge in rabbits, we altered the technique by administering on the same day the various poisons to a series of about twenty rabbits, almost simultaneously. Every quarter of an hour the condition of the animals was observed and noted down, the observer not knowing what poison had been injected into the animal under observation.

In this way we made the following experiments:

6 rabbits were given 10 mgr. of morphin per kg. subcutaneously.

6 rabbits were given 1 mgr. of scopolamin per kg. subcutaneously.

6 rabbits were given 5 mgr. morphin + 0,5 mgr. scopolamin per kg. subcutaneously.

All the animals were examined regularly during 2 $\frac{1}{2}$ —3 hours.

It thereby appeared that 1 mgr. of scopolamin had only a slight narcotic effect. The action of 10 mgr. of morphin was manifest; that of 0,5 mgr. of scopolamin + 5 mgr. of morphin was less marked than that of 10 mgr. of morphin alone; consequently "potentiation" was out of the question.

After this negative result we examined the narcotic effect of scopolamin and morphin also by another method. A so-called isolated rectusfemoris preparation was made on decerebrated rabbits and the influence was recorded of morphin, of scopolamin, and of morphin + scopolamin on the homolateral contraction-reflex of the rabbit. After a slight technical correction this method, which had already often been applied to cats<sup>1)</sup>, appeared to be well-adapted for rabbits.

The reflexes elicited in decerebrated rabbits by the faradic stimulus were registered on a Kymograph; and afterwards the results of every experiment were plotted. An instance of the influence of 5 mgr. of morphin on the homolateral contraction-reflex of the rabbit is given in Fig. 1.

In these experiments series of five rabbits were given 0,5 mgr. of scopolamin, or 10 mgr. of morphin or 0,5 mgr. of scopolamin + 5 mgr. of morphin and in all these cases the injections were not given till it appeared that the reflexes elicited by equi-intense stimulation were of the same magnitude. In case scopolamin + morphin was given, first the scopolamin was injected and 20 minutes later the morphin.

The effect of the injections on the magnitude of the reflexes were

<sup>1)</sup> W. STORM VAN LEEUWEN, Quantitative pharmakologische Untersuchungen über die Reflexfunktionen des Rückenmarks bei Warmblütern. I. Mitt. Pflügers Arch. Bd. 154, page 307. 1913. III. Mitt. Pflügers Arch. Bd. 165, p. 84. 1916.

noted for the scopolamin after 40 min., for the mixture scopolamin + morphin 40 min. after the scopolamin-injection, and for the morphin after 20 minutes.

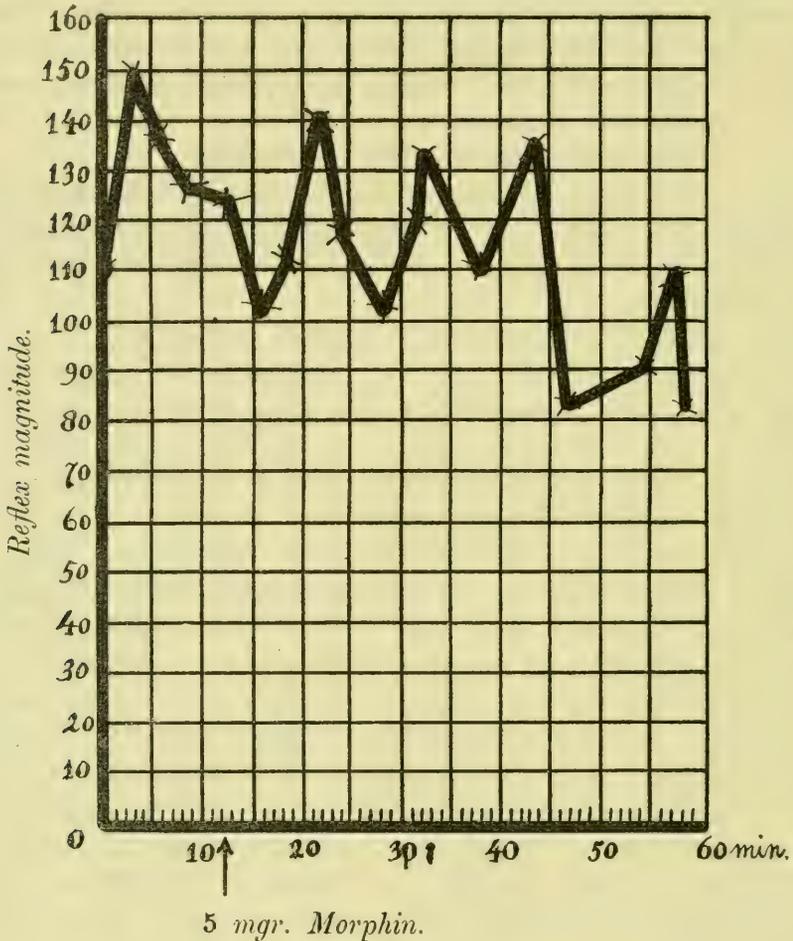


Fig. 1.

The results are as follows:

0,5 mgr. of scopolamin causes after 40 min. a decrease of 37 %  
 5 mgr. of morphin                    "        "   20   "        "        "        "   15 %  
 0,5 mgr. of scop. + 5 mgr. of morphin causes after 40 minutes a decrease of 32 %.

These percentages require correction, as it appeared from a number of control-experiments that the magnitude of the reflexes diminishes spontaneously if no poison is administered, viz. 9 % after 40 minutes and 4 % after 20 minutes, so that the above values are respectively 28 %, 11 % and 23 %.

It is evident therefore, that also if this procedure is followed there is no trace of "potentiation", the value obtained in the mixture-experiments not being higher, but lower than the sum of the actions of morphin and scopolamin separately.

HAUCKOLD and many others assume that small doses of scopolamin do not anaesthetize a rabbit. Our experiments go to show that 0.5 mgr. of scopolamin has a distinct narcotic effect upon the magnitude of the reflex. In conjunction with Dr. G. LILJESTRAND we ascertained the influence of various doses of scopolamin on a spinal reflex of the decerebrated rabbit. The result was that this narcotic effect does not increase continually with an increase of the dosis, but soon reaches an optimum and even decreases again after this (see fig. 2 firm line). It seems that with the higher doses a stimulating effect is added to the narcotic effect.

It might be generally assumed that a twice larger dosis yields a twice stronger effect. This, however, is not the case with many of the alkaloids. When plotting the relation between the dosis per kg. of the animal and the effect of such a poison, the doses along the abscissae and the (narcotic) effect along the ordinates, a curve is produced, which first ascends abruptly, and then proceeds nearly horizontally. This is seen distinctly in fig. 2 (dotted line), borrowed from a paper by LILJESTRAND, V. D. MADE and STORM VAN LEEUWEN<sup>1)</sup>, in which the full line illustrates the narcotic effect of scopolamin in various doses (concentration-effect curve of scopolamin).

Besides HAUCKOLD, KOCHMANN also studied this problem. He experimented with dogs and believed that he had detected "potentiation". Because we did not succeed in finding "potentiation" either with HAUCKOLD'S method, nor with the one commonly used at our institute (influence on reflexes of decerebrated animals), we have also put to the test KOCHMANN'S experiments.

For this purpose series of from 3 to 6 dogs were given subcutaneously morphin, or scopolamin or morph. + scop. The doses were calculated per kg. animal.

An initial experiment was performed with the same dosis KOCHMANN had used. KOCHMANN asserts that 5 mgr. of morphin and 0.5 mgr of scopolamin (we suppose his doses to be given per kg. animal, though the writer does not say so) do not of themselves produce any narcotic effect on the dog, but that their joint action puts the animal under a profound narcosis. It must be argued that this occurred only in few cases. Our initial results seemed to substantiate KOCHMANN'S findings, for it appeared that in a dog 5 mgr of morphin + 0.5 mgr

<sup>1)</sup> LILJESTRAND, V. D. MADE and STORM VAN LEEUWEN. Zur Konzentrations-Wirkungskurve des Skopolamins. Appears in Pflügers Arch. 1919.

of scopolamin exerted as strong an action as 10 mgr of morphin by itself, while 0.5 mgr of scopolamin exerts a stimulating rather

*Paralysing effect of Scopolamin on the homolateral contraction-reflex in the decerebrated rabbit.*

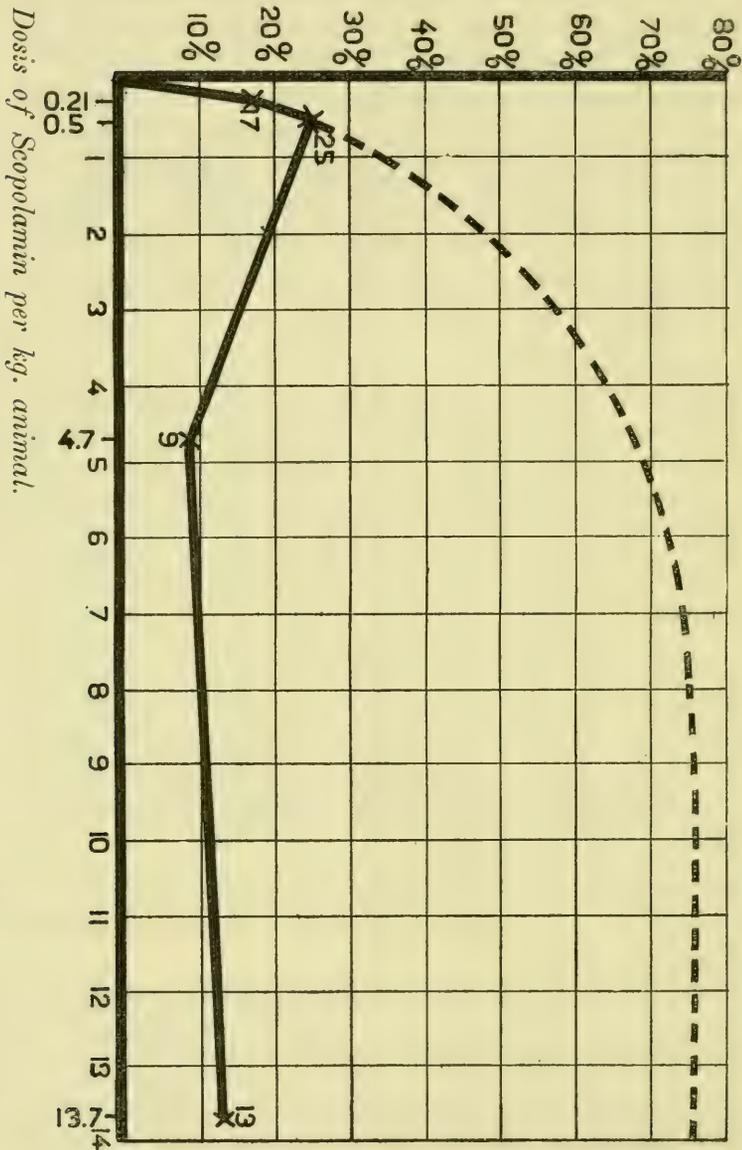


Fig. 2.

than a paralysing influence. On closer investigation it appeared, however, that the case is different, for when 5 mgr. of morphin was given alone (i.e. without scopolamin) its effect proved not to be different from that of 10 mgr. of morphin. This finding induced us

to investigate the action of different quanta of morphin on the dog. The results obtained were that doses smaller than 1,5 mgr. had but little narcotic effect or none at all; that 1,5 mgr. yields a distinct action, 2,5 mgr. a narcotic action, which is considerably stronger, while the effect of 5 mgr. differs little from that of 2,5 and finally that, as said above, the influence exerted by 10 mgr. is about equal to that of 5 mgr. If we should draw a curve of this peculiar relation between action and concentration, a so-called concentration-effect curve, it would again rise abruptly at the outset and then again proceed about horizontally, just like the curve representing the action of morphin and of scopolin on the reflexes of the rabbit.

After it had thus appeared that the action of 5 mgr. of morphin + 0,5 mgr. of scop. agrees with that of 10 mgr. of morphin, but still is not greater than that of 5 mgr. of morphin alone, we tried to find out whether the action of smaller doses of morphin was intensified by scopolamin. An injection of 2,5 mgr. of morph. per kg. was given to two dogs I and II, and to two other dogs III and IV 2,5 mgr. of morph. + 0,5 mgr. of scop. per kg. was administered. The narcosis of dog III was as profound as that of I and II, but that of IV was considerably *less* profound than that of I and II. The inevitable conclusion, therefore, is that scopolamin inhibited the action of morphin, so it did not bring about "potentiation" at all. This result was confirmed by a series of experiments with dogs, in which some received 1,5 mgr. of morphin and others 1,5 mgr. of morph. + 0,5 mgr. of scop. The narcosis with the former group was invariably more profound than that of the second.

It follows from these experiments that although narcotic symptoms are generated by scopolamin in doses of from 0,5 to 1 mgr. per kg., this poison has also a distinct stimulating effect on the dog. When small doses of morphin are mixed with scopolamin, the result is the algebraic sum of the effects of the two components and the narcotic effect of the morphin antagonizes in part the stimulating effect of the scopolamin. With larger doses of morphin the stimulating effect of the scopolamin falls back, while it would seem that the narcotic effect of morphin is sometimes reinforced in a small degree by the scopolamin, however in such a small degree that it cannot be called "potentiation", but is merely to be considered as a simple addition.

Upon the evidence founded upon very accurate experiments SCHNEIDERLIN concluded that in men the scop-morph. combination produces a real "potentiated" narcotic effect. He administered a rather

large doses of morphin to some patients, after a few days a dose of scopolamin, and again some days later half the dose of each poison. In control-experiments, some patients were first given the morph. + scopol. and afterwards the two poisons separately. In all cases the administration of the mixture resulted in a general narcosis, which did not occur after morphin or scopolamin alone. This then is a case of true "potentiation".

This finding does, however, not yield great profit for the clinic, since this result is not always obtained and, as to sensitiveness to scopolamin, patients differ too much to render a correct dosage possible.

#### *Summary.*

Morphin produces a narcotic effect in rabbits, in dogs (and also in men), but the curve representing the relation between dose and effect is a parabola, which means that with the smaller doses a small increase in the doses causes a considerable increase in the narcotic effect, whereas with the larger doses a similar increase of dose brings about a much smaller increase of the narcotic action.

Scopolamin narcotizes the rabbit. The concentration-effect curve concurs with that of morphin. Its stimulating effect is obvious in the dog, in man its action is evidently narcotic.

No "potentiation" is obtained in the rabbit with the scopolamin-morphin mixture, either by our reflex-method or by HAUCKOLD's method. Neither was "potentiation" detected in the dog.

According to SCHNEIDERLIN "potentiation" occurs in man, but it is presumably not constant.

**Physiology.** — “*On the question whether or no Darkness-nystagmus in dogs originates in the Labyrinth*”. By A. DE KLEYN M.D. and C. R. J. VERSTEEGH M.D. (Communicated by Prof. R. MAGNUS).

(Communicated in the meeting of September 27 1919).

An extensive clinical examination of miners induced OHM to believe that the nystagmus, common among this class of people, originates in the labyrinth.

RAUDNITZ's discovery that nystagmus is elicited in dogs after a prolonged sojourn in the dark has enabled OHM to adduce experimental evidence for the above hypothesis.

He starts from the consideration “dass der bei jungen Hunden durch Dunkelheit hervorgerufene Nystagmus in Bezug auf Ablauf, Ausschlag und Dauer der Zuckung, den Einfluss der Ruhe und Bewegung mit dem Augenzittern der Bergleute vollkommen übereinstimmt”.

He then proceeds by describing some experiments in which he has tried to perform bilateral labyrinth-extirpation in dogs suffering from darkness-nystagmus.

In this effort he failed, except once, when the laboratory animal did not show any more symptoms of nystagmus. However, it was so weak that it died some days afterwards.

From this doubtful success OHM concludes that the labyrinthal origin of darkness-nystagmus has been established.

By a more effectual method of labyrinth-extirpation we were so fortunate as to demonstrate that:

*Darkness-nystagmus in young dogs (RAUDNITZ, OHM) is not of labyrinthal origin.*

We found that:

1. the existing darkness-nystagmus persisted after bilateral labyrinth-extirpation.

2. darkness-nystagmus can be elicited even after bilateral labyrinth-extirpation.

Fig. 1 shows the curve of this nystagmus. We see that the typical darkness-nystagmus is interrupted by a few spontaneous greater movements. The registration was performed by attaching a wire

through the anaesthetized cornea and by thus transmitting the movements to a lever.

3. the darkness-nystagmus behaves towards a super-added vestibular nystagmus in a different way from a vestibular nystagmus, i.e. if we impart a vestibular stimulus to a dog suffering from darkness-nystagmus, thus evolving an additional vestibular nystagmus, the two forms of nyst. will be seen to persist concurrently, without exerting any influence upon each other.

If, however, we superadd to an existing vestibular nyst. (e.g. caloric) a second vestibular nyst. (e.g. rotatory nyst.) the first nyst.-deflections will be seen to be slightly irregular and of different magnitude, but the effect would seem to be rather a resultant of two movements.

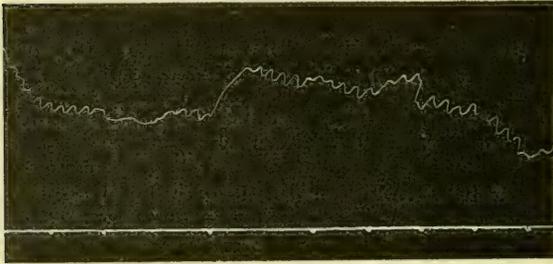


Fig 1.

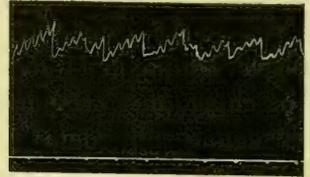


Fig 2.

Fig. 2 illustrates the typical instance of what is called the combination of darkness-nystagmus and caloric nyst. The minor deflections represent the movements of the quietly proceeding darkness-nystagmus; the larger waves show the eye-movements provoked by irrigating the ear with cold water.

We cannot say here in how far OHM is justified in looking for the origin of the miners' nystagmus in the labyrinth, but we feel confident in asserting that he is wrong in assigning the cause of the darkness-nystagmus in dogs to the labyrinth.

*(Pharmacological Institute of the  
Utrecht-University).*

**Mathematics.** — “Bestimmung der Klassenzahl der Ideale aller Unterkörper des Kreiskörpers der  $m$ -ten Einheitswurzeln, wo die Zahl  $m$  durch mehr als eine Primzahl teilbar ist”. (Zweiter Teil. 1)). By Dr. N. G. W. H. BEGER. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of October 25, 1919).

Beweis: Wir müssen zwei Behauptungen des Gliedes rechter Hand beweisen: 1°. dass der Wert des darin auftretenden Symbols eine  $\frac{f_1}{d_1}$ -te Einheitswurzel ist; 2°. dass, im Producte, jede solche Einheitswurzel  $\frac{e_1 d_1 d'_1}{r}$  Mal vorkommt. Denn wenn dies bewiesen ist, ergibt sich daraus dass das Glied rechter Hand gleich

$$\left(1 - \frac{1}{\frac{f_1}{d_1} s}\right)^{\frac{e_1 d_1 d'_1}{r}}$$

ist, und dieser Ausdruck ist, wegen Satz 7, dem Gliede linker Hand der zu beweisenden Gleichheit, gleich. Die Zahlen  $e_1, f_1$  u.s.w. haben hier dieselbe Bedeutung wie im Satz 7.

Der Beweis der ersten Behauptung folgt aus demjenigen welches im Satz 9 bewiesen ist.

Um den Beweis der zweiten Behauptung zu erbringen, nehmen wir an, dass

$$\left[ \frac{l_1}{b_{01}, b_{*1}, 0, b_{21} \dots} \right] = \left[ \frac{l_1}{b_{02}, b_{*2}, 0, b_{22} \dots} \right],$$

Nach Einführung der Werte der Symbole findet man, da die Differenzen der übereinstimmenden Zahlen zweier Systeme der  $b$  wiederum ein System der  $b$  bilden, dass dieses System der  $b$  welches wir andeuten durch  $b_0, b_*, 0, b_2, \dots$ , den Congruenzen:

$$\left. \begin{aligned} \frac{1}{2} \varphi \left( \frac{m}{l_1 h_1} \right) a_0 b_0 + 2\varphi \left( \frac{m}{2h_* l_1 h_1} \right) a_* b_* + \varphi \left( \frac{m}{l_2 h_2 l_1 h_1} \right) a_2 b_2 + \dots \\ \equiv 0 \text{ mod } \varphi \left( \frac{m}{l_1 h_1} \right) \end{aligned} \right\} \quad (8)$$

1) Fortsetzung von “Proceedings” Vol. 22. S. 331.

genüge leistet, wo  $a_0, a_*, \dots$  die Exponente bedeuten, welche in § 4 bestimmt sind. Wir müssen nun die Anzahl der Systeme der  $b$  berechnen, welche diesen Congruenzen (8) genüge leisten. Dazu bestimmen wir zunächst wieviel der gegebenen Systeme der  $b$  aus § 1, (1) eine Zahl  $b_{1n} = 0$  enthalten. Alle Zahlen  $b_{1n}$  sind teilbar durch ihren grössten gemeinsamen Teiler  $d_1$ . Es gibt also  $\frac{\varphi_1}{d_1}$  verschiedene Zahlen  $b_{1n}$ , und von diesen ist nur eine  $= 0$ . Weil es im ganzen  $\frac{\varphi}{r}$  Systeme gibt, kommen darunter  $\frac{\varphi}{r} : \frac{\varphi_1}{d_1}$  Systeme vor, welche  $b_{1n} = 0$  haben.

Alle Systeme der Zahlen  $a$ , welche die Gruppe  $g$  bilden, leisten allen Congruenzen (3) genüge, wenn man darin  $b_{1n} = 0$  setzt. Den Modulus dieser Congruenzen kann man dabei reduciren zu  $\frac{\varphi}{\varphi_1}$ . Nach der Bemerkung am Ende des Kapitels II § 3, ist die Anzahl der verschiedenen Systeme der  $a$ , wenn man  $a_1$  ausser Acht lässt, also gleich

$$\frac{\varphi}{\varphi_1} : \left( \frac{\varphi}{r} : \frac{\varphi_1}{d_1} \right) = \frac{r}{d_1}.$$

Weiter bemerken wir, dass die Systeme der  $b$  welche (8) genügen, auch den  $\frac{r}{d_1}$  Congruenzen genügen, welche man aus (8) ableiten kann, indem man darin  $a_0, a_*, \dots$  durch die, eben berechneten, Systeme der  $a$  ersetzt. Ausserdem genügen die gesuchten Systeme der  $b$  noch den  $\frac{f_1}{d_1}$  Congruenzen die man aus (8) erhält indem man die darin auftretenden  $a$  nacheinander durch ihren 2-, 3-,  $\dots$   $\frac{f_1}{d_1}$ -fachen Wert ersetzt. Denn wegen Kap. II § 4 gehören diese neuen Systeme der  $a$  nicht zu den erstgefundenen, weil die  $\frac{f_1}{d_1}$  ersten Potenzen der Zahl  $l_1 + n \frac{m}{l_1 h_1}$  nicht zu  $g$  gehören.

Wenn man nun die Gruppe der zuerstgefundenen Systeme der  $a$  mit der Gruppe der zuletztgefundenen Systeme multipliziert, so bekommt man eine Gruppe von  $\frac{r f_1}{d_1 d_1}$  Systeme der  $a$  aus welchen sich ebensoviele Congruenzen (8) ergeben, welche die gesuchten Systeme der  $b$  genüge leisten. Wegen der Bemerkung am Ende des Kap. II § 3 gibt es also

$$\varphi \left( \frac{m}{l_1^{h_1}} \right) : \frac{rf_1}{d_1 d'_1} = \frac{e_1 d_1 d'_1}{r}$$

gesuchte Systeme der  $b$ .

Hiermit ist nun gezeigt worden dasz jede  $\frac{f_1}{d'_1}$ -te Einheitswurzel welche in dem Gliede rechter Hand der, im Satze genannten, Gleichheit auftritt, darin auch  $\frac{e_1 d_1 d'_1}{r}$  Mal auftritt.

Nun ist aber die Anzahl der Factoren des Gliedes rechter Hand, wie schon berechnet ist, gleich  $\frac{\varphi}{r} : \frac{\varphi_1}{d_1}$ ; und da  $\frac{f_1}{d_1} \cdot \frac{e_1 d_1 d'_1}{r} = \frac{\varphi}{r} : \frac{\varphi}{d_1}$  ist, so wird auch *eine jede*  $\frac{f_1}{d'_1}$ -te Einheitswurzel  $\frac{e_1 d_1 d'_1}{r}$  Mal im Produkte auftreten, was zu beweisen war.

*Satz 11.* Wenn  $m$  gerade ist, so ist

$$\prod_l \left( 1 - \frac{1}{n(l)^s} \right) = \prod_n \left\{ 1 - \left[ \overbrace{b_{1n}, b_{2n}, \dots}^2 \right] \frac{1}{2^s} \right\}$$

Das erste Produkt erstreckt sich über alle Primideale  $l$  welche in 2 aufgehen im Körper  $k$ . Das zweite Produkt erstreckt sich über alle diejenigen Systeme der  $b$  in welchen  $b_{0n} = b_{*n} = 0$  ist.

Der Beweis ist ganz in Ubereinstimmung mit den beiden Vorigen.

Zuerst beweist man dasz das Symbol eine  $\frac{f}{d^*}$ -te Einheitswurzel ist.

Weiter ist es notwendig drie Fälle zu unterscheiden:

- 1°.  $h_* \geq 3$  und  $b_{0n} + b_{*n}$  nicht für alle Werte von  $n$  gerade.
- 2°.  $h_* \geq 3$  und  $b_{0n} + b_{*n}$  wohl für alle Werte von  $n$  gerade.
- 3°.  $h_* = 2$ .

#### IV. Berechnung der Klassenzahl der Ideale des Unterkörpers $k$ .

##### § 8. Hilfssatz und Ableitung der vorläufigen Formel.

1. Es ist  $\sum_{n=1}^m \left[ \overbrace{\quad}^n \right] = 0^1$ ,

Beweis: Es sei  $a$  prim zu  $m$ ; dann ist  $\left[ \overbrace{\quad}^a \right] \neq 0$  und  $\neq 1$ ; daher ist.

$$\left[ \overbrace{\quad}^a \right] \sum_{n=1}^m \left[ \overbrace{\quad}^n \right] = \sum_{a=1}^m \left[ \overbrace{\quad}^{n a} \right].$$

Es durchläuft  $na$  zugleich mit  $n$  ein vollständiges Restsystem

1) Zur Abkürzung lasse ich die Buchstaben  $b$  im Symbole weg.

(mod  $m$ ). Die letzte Summe ist daher gleich  $\sum_{n=1}^m \left[ \frac{n}{m} \right]$ . Hieraus ergibt sich leicht der Beweis.

Für die Klassenzahl  $H$  gebrauchen wir den bekannten Ausdruck<sup>1)</sup>:

$$H = \frac{1}{\varkappa} \lim_{s=1} (s-1) \prod_p \frac{1}{1-n(p)^{-s}}$$

wo  $p$  alle Primideale des Unterkörpers  $k$  durchläuft. Wenn wir nun die Sätze 9, 10 und 11 benützen und die Factoren, welche sich beziehen auf dem System  $b_{0n} = b_{*n} = b_{1n} = \dots = 0$ , von den anderen Factoren abscheiden, so findet sich:

$$H = \frac{1}{\varkappa} \lim_{s=1} (s-1) \prod_{n=2}^{\varphi/r} \prod_p \frac{1}{1-p^{-s}} \prod_p \left\{ 1 - \left[ \frac{p}{b_{0n}, b_{*n}, b_{1n}, \dots} \right] \frac{1}{p^s} \right\}^{-1}.$$

Es ist hierbei angenommen dasz  $b_{01} = b_{*1} = b_{11} = b_{21} = \dots = 0$  ist. Wir wissen nunmehr dasz:

$$\lim_{s=1} (s-1) \prod_p \frac{1}{1-p^{-s}} = 1.$$

Weiter entwickeln wir jeden Factor des dritten Productes auf bekannte Weise in einer DIRICHLET'schen Reihe und multiplizieren all' diese Reihen. Das Ergebnis ist:

$$H = \frac{1}{\varkappa} \lim_{s=1} \prod_{i=2}^{\varphi/r} \sum_{n=1}^{\infty} \left[ \frac{n}{b_{0i}, b_{*i}, b_{1i}, \dots} \right] \frac{1}{n^s}$$

Hierin setzt man

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-nx} n^{s-1} dx$$

und

$$\sum_{t=1}^m \left[ \frac{t}{m} \right] x^t = F(x)$$

und wenn man noch Gebrauch macht von der Gleichheit

$$\left[ \frac{n}{m} \right] = \left[ \frac{n'}{m} \right] \text{ wenn } n \equiv n' \pmod{m}:$$

so findet man

$$H = \frac{1}{\varkappa} \prod_{n=2}^{\varphi/r} \int_0^1 \frac{F(x)}{x(1-x^m)} dx$$

wenn man ausserdem den ersten Hilfssatz dieser § benutzt. Nach Zerlegung in Partialbrüchen kann man die Integration durchführen und findet dann:

<sup>1)</sup> „H“. Satz 55 und § 27.

$$H = \frac{1}{\alpha} \frac{\varphi/r}{n=2} - \frac{1}{m} \sum_{k=1}^m F\left(e^{\frac{2\pi ki}{m}}\right) \left\{ \log \frac{e^{\frac{k\pi i}{m}} - e^{-\frac{k\pi i}{m}}}{i} + \frac{1}{2} \pi i - \frac{k\pi i}{m} \right\}.$$

Nun ist noch

$$\sum_{k=1}^m F\left(e^{\frac{2k\pi i}{m}}\right) = \sum_{k=1}^m \sum_{n=1}^m \left[ \underbrace{\quad}_n \right] e^{\frac{2\pi kni}{m}} = \sum_{n=1}^m \left[ \underbrace{\quad}_n \right] \sum_{k=1}^m e^{\frac{2\pi kni}{m}} = 0$$

und also

$$H = \frac{1}{\alpha} \frac{\varphi/r}{n=2} - \frac{1}{m} \sum_{k=1}^m F\left(e^{\frac{2k\pi i}{m}}\right) \left\{ \log \frac{e^{\frac{k\pi i}{m}} - e^{-\frac{k\pi i}{m}}}{i} - \frac{k\pi i}{m} \right\} \dots \quad (9)$$

Um diese Form weiter zu vereinfachen, beweisen wir zuerst vier Hülfsätze.

§ 9. Hülfsätze. <sup>1)</sup>

In allen folgenden Hülfsätzen ist das System  $b_{0n} = b_{*n} = b_{1n} = \dots = 0$  ausgeschlossen.

1. 
$$F\left(e^{\frac{2k\pi i}{m}}\right) = (-1)^{b_{0n} + b_{1n} + \dots} F\left(e^{-\frac{2k\pi i}{m}}\right).$$

Der Beweis ist ganz analog mit dem des übereinstimmenden Satzes meiner Abhandlung in "Proceedings" Vol. XXI, S. 758.

2. Es sei  $2^{h'}$  die höchste Potenz von 2 die auf  $b_{*n}$  teilbar ist. Wenn  $b_{*n} = 0$  ist, so nehme man  $h'_* = h_* - 2$ . Ist aber auch  $b_{0n} = 0$  so nehme man  $h'_* = h_*$ . Ist  $h_* = 2$  so ist  $h'_* = 0$  zu setzen wenn  $b_{0n} = 1$  und  $= 2$  zu setzen wenn  $b_{0n} = 0$  ist.

Es sei weiter  $l_1^{h_1}$  die höchste Potenz von  $l_1$  welche auf  $b_{1n}$  teilbar ist. Wenn  $b_{1n} = 0$  ist, so nehme man  $h'_1 = h_1$ . u. s. w.

Es sei nun  $d = 2^{h'_*} l_1^{h'_1} \dots$  so ist: 
$$\left[ \underbrace{n + m/d}_{2^2} \right] = \left[ \underbrace{\quad}_n \right].$$

Beweis:

Wir fassen zuerst die Symbole

$$\left[ \frac{n + m/d}{2^2} \right]^{b_{0n}} \quad \text{und} \quad \left[ \frac{n}{2^2} \right]^{b_{0n}}.$$

ins Auge. Ist  $b_{0n} = 0$  so sind sie beide  $= 1$ , und also einander gleich. Ist  $b_{0n} \neq 0$  und  $n$  gerade, so ist auch  $n + \frac{m}{d}$  gerade, weil dann  $\frac{m}{d}$  teilbar ist durch 4. Die Symbole sind dann beide  $= 0$  und

<sup>1)</sup> Diese Hülfsätze musz schon KUMMER benutzt haben, wiewohl in anderer Form. Die Beweise findet man aber nirgends.

wiederum gleich. Ist, zum Schlusz,  $b_{0n} \neq 0$  und  $n$  ungerade, so ist auch  $n + \frac{m}{d}$  ungerade und

$$\left[ \frac{n + m/d}{2^2} \right]^{b_{0n}} = (-1)^{\left(n + \frac{m}{d} - 1\right) b_{0n} : 2} = (-1)^{\frac{n-1}{2} b_{0n}} = \left[ \frac{n}{2^2} \right]^{b_{0n}}$$

Für die übrigen Symbole kann man die Gleichheit auf analoge Art beweisen.

3. Wenn die Zahl  $d$  dieselbe Bedeutung hat wie oben, so ist  $F\left(e^{\frac{2\pi ki}{m}}\right) = 0$  wenn der grösste gemeinschaftliche Teiler von  $k$  und  $m \neq d$  ist, und

$$= \left[ \frac{k/d}{2} \right]^{-1} F\left(e^{\frac{2\pi di}{m}}\right)$$

wenn dieser grösste gemeinschaftliche Teiler wohl  $= d$  ist.

Beweis:

$$\begin{aligned} F &= \sum_{n=1}^m \left[ \frac{n}{2} \right] e^{\frac{2\pi nki}{m}} = \sum_{n=1}^m \left[ \frac{n + m/d}{2} \right] e^{\frac{2\pi nki}{m}} = \\ &= e^{-\frac{2k\pi i}{d}} \sum_{n=1}^m \left[ \frac{n + m/d}{2} \right] e^{\frac{2\pi(n+m/d)ki}{m}} \end{aligned}$$

Es durchläuft  $n + \frac{m}{d}$  zugleich mit  $n$  alle Zahlen für welche das Symbol  $\neq 0$  ist, weil zufolge der Definition der Zahl  $d$ , die Zahl  $m/d$  prim ist zu den Zahlen welche  $n$  durchläuft. Also findet man:

$$F = e^{\frac{2k\pi i}{d}} F$$

Ist  $k$  nicht durch  $d$  teilbar, so ergibt sich hieraus  $F = 0$ .

Nehmen wir nunmehr an dasz  $k$  durch  $d$  teilbar ist; dasz aber  $k = dtv$ ,  $m = dtm'$  so dasz  $dt$  der grösste gemeinsame Teiler ist von  $k$  und  $m$ ,  $t > 1$ . Man hat nun:

$$\begin{aligned} F\left(e^{\frac{2\pi ki}{m}}\right) &= F\left(e^{\frac{2\pi vi}{m'}}\right) = \sum_{n=1}^m \left[ \frac{n}{2} \right] e^{\frac{2\pi vni}{m'}} = \\ &= \sum_{n=1}^{m'} e^{\frac{2\pi vni}{m'}} \sum_{s=1}^t \left[ \frac{n + sm'}{2} \right] \text{ und wegen 2:} \\ &= d \sum_{n=1}^{m'} e^{\frac{2\pi vni}{m'}} \sum_{s=0}^{t-1} \left[ \frac{n + sm'}{2} \right]. \end{aligned}$$

Wir werden zeigen dasz der Wert der letzten Summe Null ist.

Es sei  $x$  eine Zahl der Form  $1 + \frac{m'}{g} y$  (wo  $g$  der grösste gemein-

same Teiler von  $n$  und  $m'$  ist) welche prim zu  $m$  ist. Eine solche Zahl  $x$  besteht, denn die Form enthält eine unendliche Anzahl Primzahlen.

Nun ist  $nx = n \left( 1 + \frac{m'}{g} y \right) = n + \frac{n}{g} y m'$  und wenn eine Zahl  $n + sm'$  mit  $x$  multipliziert wird so gibt es wiederum eine Zahl derselben Form. Wäre weiter

$$x(n + sm') \equiv x(n + s'm') \pmod{\frac{m}{d}}; \quad s, s' < t$$

so würde

$$x s m' \equiv x s' m'$$

sein, und

$$x(s - s') m' \equiv 0 \quad \text{also} \quad s - s' \equiv 0 \pmod{t}$$

Dies ist unmöglich, da  $s$  und  $s' < t$  sind. Es ist damit bewiesen dasz die Zahlen  $n + sm'$  wo  $s = 0, 1, \dots, t-1$ , nach Multiplikation mit  $x \pmod{\frac{m}{d}}$  wiederum dieselben Zahlen ergeben. Hieraus ergibt sich:

$$\left[ \underbrace{x} \right] \sum_{s=0}^{t-1} \left[ \underbrace{n + sm'} \right] = \sum_{s=0}^{t-1} \left[ \underbrace{x(n + sm')} \right] = \sum_{s=0}^{t-1} \left[ \underbrace{n + sm'} \right]$$

$$\text{Also ist: } \sum_{s=0}^{t-1} \left[ \underbrace{n + sm'} \right] = 0.$$

Um den zweiten Teil des Satzes zu beweisen, bemerken wir dasz

$$F = \left[ \underbrace{k/d} \right]^{-1} \sum_{n=1}^m \left[ \underbrace{n k/d} \right] e^{\frac{2\pi nki}{m}}$$

wenn  $k$  und  $m$  die Zahl  $d$  zum grössten gemeinsamen Teiler haben.

Die Zahl  $\frac{k}{d}$  kann mit  $m$  nur diejenigen Primfaktoren gemeinsam haben, deren zugehörige Zahlen  $b = 0$  sind. Denn wäre z.B.  $b_{1n} \neq 0$ , so ist  $d$  teilbar durch eine Potenz von  $l_1$  welche  $< l_1^{b_1}$  ist. Es ergibt sich also dasz  $n k/d$  und  $n$  zugleich alle Zahlen durchlaufen für welche  $\left[ \underbrace{n k/d} \right] \neq 0$  ist. Also ist:

$$F = \left[ \underbrace{k/d} \right]^{-1} \sum_{n=1}^m \left[ \underbrace{n} \right] e^{\frac{2\pi ndi}{m}} = \left[ \underbrace{k/d} \right]^{-1} F \left( e^{\frac{2\pi di}{m}} \right)$$

4. Wenn die Zahl  $d$  dieselbe Bedeutung hat wie oben, so ist:

$$F \left( e^{\frac{2\pi di}{m}} \right) F \left( e^{\frac{2\pi di}{m}} \right) = (-1)^{b_{0n} + b_{1n} + \dots} dm$$

$F'$  ist dieselbe Function wie  $F$ , nachdem in letzterer die Zahlen  $b$  durch  $2-b_{0n}, \frac{1}{2} \varphi_* - b_{*n}, \varphi_1 - b_{1n}, \dots$  ersetzt sind.

Beweis: Wir fassen zuerst den Fall ins Auge wo keine der Zahlen  $b$  den Wert Null hat.

$$F = \sum_{n=1}^m \left[ \frac{n}{k} \right] e^{\frac{2\pi n k d i}{m}}$$

wenn  $k$  prim. zu  $m$  ist, denn in diesem Falle durchläuft  $nk$  zugleich mit  $n$  ein vollständiges Restsystem  $(\text{mod } m)$ . Man folgert hieraus:

$$\left[ \frac{k}{m} \right]^{-1} F = \sum_{n=1}^m \left[ \frac{n}{k} \right] e^{\frac{2\pi n k d i}{m}}$$

Wir multiplizieren nun mit  $e^{\frac{2\pi n k d i}{m}}$  und nehmen die Summe über alle Werte von  $k$  welche  $< m$  und prim. zu  $m$  sind. Weil keine der Zahlen  $b = 0$  ist, findet man:

$$F F' = \sum_{n=1}^m \left[ \frac{n}{k} \right] \sum_k e^{\frac{2\pi(n+1)k d i}{m}} = d \sum_{n=1}^{m/d} \left[ \frac{n}{k} \right] \sum_k e^{\frac{2\pi(n+1)k d i}{m}}$$

Es sei  $t$  der grösste gemeinschaftliche Theiler von  $n+1$  und  $m$ . Man kann alle Zahlen  $k$ , welche  $< m$  und prim. zu  $m$  sind,  $(\text{mod } m/d)$  verteilen in  $\frac{\varphi(m)}{\varphi\left(\frac{m}{dt}\right)}$  Mal die Gruppe der Zahlen welche  $< \frac{m}{dt}$  und

prim. zu  $\frac{m}{dt}$  sind. So findet man:

$$F F' = d \sum_{n=1}^{m/d} \left[ \frac{n}{k} \right] \frac{\varphi(m)}{\varphi\left(\frac{m}{dt}\right)} \sum_k e^{\frac{2\pi(n+1)k d i}{m}}$$

in welchem Ausdruck  $k$  nun alle Zahlen durchläuft, welche  $< \frac{m}{dt}$  und prim. zu  $\frac{m}{dt}$  sind. Die letzte Summe ist daher die Summe der primitiven  $\frac{m}{dt}$ -te Einheitswurzeln; in Folge dessen erhalten wir:

$$F F' = d \varphi(m) \sum_n \left[ \frac{n}{k} \right] \frac{\mu\left(\frac{m}{dt}\right)}{\varphi\left(\frac{m}{dt}\right)}$$

Nun ist  $\mu\left(\frac{m}{dt}\right)$  nur dann nicht  $= 0$ , wenn  $\frac{m}{dt}$  nicht teilbar ist durch eine Quadratzahl. Wir brauchen in obenstehender Summe also für  $n$  nur

diejenigen Werte zu nehmen, für welche  $\frac{m}{dt}$  ein Teiler ist des Produktes  $2 l_1 l_2 \dots$ . Diese Zahlen  $n$  haben die Form  $-1 + st$  insoweit diese prim zu  $m$  sind, und wo  $s$  alle Werte annimmt welche  $< \frac{m}{dt}$  und prim zu  $\frac{m}{dt}$  sind. Setzen wir daher

$$\frac{m}{dt} = 2^a l_1^{a_1} \dots; a, a_1, \dots = 0 \text{ oder } 1$$

so ist:

$$t = 2^{h_* - h_*' - a} l_1^{h_1 - h_1' - a_1} \dots$$

Zunächst nehmen wir an, die Zahl  $t$  sei durch alle Primfaktoren von  $m$  teilbar und auch durch 8. Die Exponenten, welche in obenstehender Form von  $t$  vorkommen, sind dann grösser als Null.

Alle Zahlen  $-1 + st$ , wo  $s$  die oben genannten Werte annimmt, sind nun prim zu  $m$ . In obenstehender Summe kommen

also  $\varphi\left(\frac{m}{dt}\right)$  Glieder vor, welche  $\neq 0$  sind. Diese Summe setzen wir in der folgenden Form:

$$F F' = (-1)^{b_{0n} + b_{1n} + \dots} d\varphi(m) \sum_t \frac{\mu\left(\frac{m}{dt}\right)}{\varphi\left(\frac{m}{dt}\right)} \sum_s \left[ \frac{1-st}{\dots} \right].$$

Nun ist  $\left[ \frac{1-st}{2^2} \right] = 1$  weil  $t$  durch 8 teilbar ist.

$$\left[ \frac{1-st}{2^{h_*}} \right]^{b_{*n}} = e^{\frac{2\pi b_{*n} v_{*i}}{2^a}} \quad \text{denn } 1-st \equiv \pm 5^{s'} \pmod{2^{h_*}}$$

also

$$1 \equiv \pm 5^{s'} \pmod{2^{h_* - h_*' - a}}$$

Da die Potenz von 2, welche in den Modul auftritt,  $> 2^2$  ist, so folgt:  $s' = 2^{h_* - h_*' - a - 2} v_{*i}$ . Die Zahl  $v_{*i}$  ist ungerade, denn anderenfalls wäre  $s$  teilbar durch 2 und dies ist unmöglich weil  $t$  der grösste gemeinsame Teiler von  $n+1$  und  $m$  ist, und  $m$  gerade ist. Weiter hat man:

$$\left[ \frac{1-st}{l_1^{h_1}} \right]^{b_{1n}} = e^{\frac{2\pi b_{1n} v_1}{l_1^{a_1}}}$$

u. s. w. Aus all' diesem ersieht man dasz  $\left[ \frac{1-st}{\dots} \right] =$  eine primitive  $\frac{m}{dt}$ -te

Einheitswurzel ist. Infolge dessen besteht die Gleichheit:

$$\Sigma \left[ \frac{1-st}{2} \right] = \mu \left( \frac{m}{dt} \right).$$

$$\text{Folglich ist } F F' = (-1)^{b_{0n}+b_{1n}+\dots} d\varphi(m) \Sigma_t \frac{1}{\varphi \left( \frac{m}{dt} \right)}$$

wo  $t$  alle Teiler von  $2l_1 l_2 \dots$  durchläuft. Es ist leicht ersichtlich dasz  $\varphi$ -Mal die letzte Summe den Wert  $m$  hat, womit in unsrem besonderen Falle der Beweis erbracht ist.

Um anzugeben wie der Beweis sich gestaltet wenn nicht alle fröhergemachten Annahmen erfüllt sind, fassen wir noch den Fall ins Auge wobei  $h_1 - h'_1 - a_1 = 0$  und  $a_1 = 1$ , während alle übrigen Annahmen erfüllt sind.

In der Summe  $\Sigma \left[ \frac{1-st}{2} \right]$  nimmt die Zahl  $s$  nun nicht mehr alle Werte an, welche  $< \frac{m}{dt}$  und prim zu  $\frac{m}{dt}$  sind; denn da  $t$  nicht durch  $l_1$  teilbar ist, so kann  $1-st$ , für einige Werte von  $s$ , durch  $l_1$  teilbar sein. Die Zahlen  $z, z+l_1, \dots, z + \left( \frac{m}{dt l_1} - 1 \right) l_1$  genügen der Congruenz  $1-zt \equiv 0 \pmod{l_1}$ . Es sind  $\varphi \left( \frac{m}{dt l_1} \right)$  dieser Zahlen prim zu  $\frac{m}{dt}$ . In der Summe bleiben also

$$\varphi \left( \frac{m}{dt} \right) - \varphi \left( \frac{m}{dt l_1} \right) = (l_1 - 2) \varphi \left( \frac{m}{dt l_1} \right)$$

Zahlen der Form  $1-st$  übrig, für welche das Symbol einen von Null verschiedenen Wert erhält. Für diese Werte von  $s$  ist ebenso wie früher:

$$\left[ \frac{1-st}{2^3} \right]^{b_{0n}} = 1; \left[ \frac{1-st}{2^{h_*}} \right]^{b_{*n}} = e^{\frac{2\pi b'_{*n} v}{2^a}}; \left[ \frac{1-st}{l_2^{h_2}} \right] = e^{\frac{2\pi b'_{2n} v_2}{l_2^{a_2}}} \dots$$

$$\text{Weiter ist aber } \left[ \frac{1-st}{l_1^{h_1}} \right]^{b_{1n}} = \left[ \frac{1-st+l_1}{l_1^{h_1}} \right]^{b_{1n}}$$

Die Zahlen  $1-st$ , welche in der Summe auftreten, können wir  $(\text{mod } l_1)$  verteilen in  $\varphi \left( \frac{m}{dt l} \right)$  Mal eine Gruppe von Zahlen, welche  $< l_1$  sind. Die letzte Gruppe enthält  $l_1 - 2$  Glieder, unter welchen die Null und die Zahl 1 nicht vorkommen; denn keine Zahl  $1-st$  ist teilbar durch  $l_1$  und  $t$  und  $s$  sind nicht teilbar durch  $l_1$ , weil  $s$

prim zu  $\frac{m}{dt}$  ist und diese letzte Zahl teilbar ist durch  $l_1$ . Wir fügen nun die Glieder, in welchen das Symbol  $\left[ \frac{1-st}{l_1^{h_1}} \right]^{b_{1n}}$  denselben Wert hat, zusammen.

Es wird dieser Wert dann multipliziert mit der Summe der primitiven  $\frac{m}{dt l_1}$ -ten Einheitswurzeln. Wir sind also zu der folgenden Gleichheit gelangt:

$$\Sigma \left[ \frac{1-st}{l_1^{h_1}} \right] = \mu \left( \frac{m}{dt l_1} \right) \times \sum_{n=2}^{l_1} \left[ \frac{n}{l_1^{h_1}} \right]^{b_{1n}}.$$

Die letzte Summe hat den Wert  $-1$ , wie sich aus dem ersten Hilfssatz dieses Kap. ergibt, denn es ist:

$$0 = \sum_{u=1}^{l_1^{h_1}} \left[ \frac{n}{l_1^{h_1}} \right] = (l_1^{h_1-1} - 1) \sum_{n=1}^{l_1} \left[ \frac{n}{l_1^{h_1}} \right]^{b_{1n}}$$

da, weil  $b_{1n}$  durch  $l_1^{h_1-1}$  teilbar ist:

$$\left[ \frac{n}{l_1^{h_1}} \right]^{b_{1n}} = \left[ \frac{n'}{l_1^{h_1}} \right]^{b_{1n}} \text{ wenn } n \equiv n' \pmod{l_1}$$

$$\text{Schliesslich: } \Sigma \left[ \frac{1-st}{l_1^{h_1}} \right] = -1 \cdot \mu \left( \frac{m}{dt l_1} \right) = \mu \left( \frac{m}{dt} \right).$$

Der Beweis gestaltet sich weiter wie im vorigen Falle. Wir haben nun hinreichend angegeben wie der Beweis erbracht wird wenn man noch andere der früher gemachten Einschränkungen aufhebt; nur den Fall  $b_{1n} = 0$  werden wir noch weiter untersuchen.

In der Summe  $F$  haben in diesem Falle alle Glieder, für welche  $n$  prim zu  $\frac{m}{l_1^{h_1}}$  ist, einen von Null verschiedenen Wert. Und weiter ist

$$\left[ \frac{n}{l_1^{h_1}} \right] = \left[ \frac{n'}{l_1^{h_1}} \right] \text{ wenn } n \equiv n' \pmod{\frac{m}{l_1^{h_1}}}.$$

Ausserdem kann man alle Zahlen, welche  $< m$  und prim zu  $m$  sind,  $\left( \text{mod } \frac{m}{l_1^{h_1}} \right)$  verteilen in  $l_1^{h_1}$  Mal die Gruppe der Zahlen welche  $< \frac{m}{l_1^{h_1}}$  und prim zu  $\frac{m}{l_1^{h_1}}$  sind. Also:

$$F = \sum_n \left[ \frac{n}{l_1^{h_1}} \right] \left( e^{\frac{2\pi n d i}{m}} + e^{\frac{2\pi \left( n + \frac{m}{l_1^{h_1}} \right) d i}{m}} + \dots \right)$$

Da  $d$  teilbar ist durch  $l_1^{h_1}$ , ergibt sich hieraus

$$F = l_1^{h_1} \sum_n \left[ \underbrace{\quad}_n \right] e^{\frac{2\pi n di}{m}} = l_1^{h_1} F_1 \left( e^{\frac{2\pi n di}{m}} \right)$$

Die Function  $F_1$  leitet man aus  $F$  ab, indem man  $\frac{m}{l_1^{h_1}}$  anstatt  $m$  nimmt. Für die Function  $F_1$  haben wir schon bewiesen:

$$\begin{aligned} F F' &= l_1^{h_1} F_1 \cdot l_1^{h_1} F_1' = l_1^{2h_1} F_1 F_1' = \\ &= (-1)^{b_{0n} + b_{1n} + \dots} l_1^{2h_1} \frac{d}{l_1^{h_1}} \cdot \frac{m}{l_1^{h_1}} = (-1)^{b_{0n} + b_{1n} + \dots} dm. \end{aligned}$$

Damit ist der Beweis in unserem Falle wiederum erbracht.

§ 10. Sätze über die Realität des Unterkörpers  $k$  und Bestimmung der Zahl  $\alpha$ .

1. Wenn die Summe  $b_{0n} + b_{1n} + \dots$ , für alle Werte von  $n$ , gerade ist, so ist der Unterkörper  $k$  reell und andernfalls imaginär.

Beweis: Die den Körper  $k$  bestimmende Zahl ist

$$\eta = Z^{A^{(1)}} + Z^{A^{(2)}} + \dots + Z^{A^{(r)}}$$

wo  $A^{(i)}$  jede Zahl der Untergruppe  $g$  bedeutet.

Nun ist

$$A^{(i)} \equiv A_0^{a_{0i}} A_*^{a_{*i}} A_1^{a_{1i}} \dots \pmod{m}$$

wo die Exponenten den Congruenzen (3) genügen. Aus der Annahme dasz alle Summen  $b_{0n} + b_{1n} + \dots$  gerade sind, folgt dasz das Wertsystem  $a_0 = 1$ ;  $a_* = 0$ ;  $a_1 = \frac{1}{2} \varphi_1$ ; ... den Congruenzen (3) genügt. Dann genügt aber auch das System

$$a_{0i} + 1, a_{*i}, a_{1i} + \frac{1}{2} \varphi_1, \dots$$

Nun ist noch

$$A_0^{a_{0i}+1} A_*^{a_{*i}} A_1^{a_{1i}+\frac{1}{2}\varphi_1} \dots \equiv -A^{(i)} \pmod{m}$$

Die Zahlen  $A^{(i)}$  en  $-A^{(i)}$  sind  $\pmod{m}$  verschieden, denn wäre

$$A^{(i)} \equiv -A^{(i)} \pmod{m}$$

so würde

$$2 A^{(i)} \equiv 0 \pmod{m}$$

sein, und dies ist unmöglich, weil die  $A^{(i)}$  prim zu  $m$  sind. Wir haben also bewiesen dasz die Zahlen  $A^{(i)}$  von  $g$  in Paaren verteilt

werden können. Für jedes Paar hat  $Z^{A^{(i)}} + Z^{-A^{(i)}}$  einen reellen Wert. Man folgert hieraus leicht dasz  $\eta$  reell ist.

Nun musz noch der Beweis des zweiten Teils des Satzes erbracht werden.

Alle reellen Zahlen des Kreiskörpers  $K$  bleiben unverändert für

1) „W.“, S. 85.

die Substitution  $s = (Z: Z^{-1})$ . Denn es sei  $\Omega$  eine reelle ganze Zahl, so ist

$$\begin{aligned}\Omega &= a_0 + a_1 Z + \dots \\ a_1 \sin \frac{2\pi}{m} + a_2 \frac{\sin 4\pi}{m} + \dots &= 0\end{aligned}$$

und

$$s \Omega = a_0 + a_1 Z^{-1} + \dots = a_0 + a_1 \cos \frac{2\pi}{m} + \dots$$

Also:

$$\Omega - s \Omega = 0.$$

Ebensoleicht beweist man, umgekehrt, dass eine jede Zahl, welche durch die Substitution  $s$  nicht geändert wird, eine reelle Zahl ist. Man ersieht hieraus dass jeder reelle Unterkörper von  $K$  ein Unterkörper ist des zu der Gruppe  $s, s^2$  gehörenden Unterkörpers und dass ein Unterkörper, der zu einer Untergruppe gehört, welche die Substitution  $s$  nicht enthält, imaginär ist. Wenn nun nicht alle Summen  $b_{0n} + b_{1n} + \dots$  gerade sind, so genügt das System  $a_0 = 1; a_{*} = 0; a_1 = \frac{1}{2} \varphi_1 \dots$  den Congruenzen (3) nicht; die Untergruppe  $g$  kann daher in diesem Falle die Substitution  $s$  nicht enthalten. Der Körper  $k$  ist imaginär.

2. Wenn die Summe  $b_{0n} + b_{1n} + \dots$  nicht für alle Werte von  $n$  einen geraden Wert hat, so ist die Anzahl der Systeme der  $b$ , für welche dieser Wert ungerade ist, gleich  $\frac{\varphi}{2r}$ .

Beweis: Alle Systeme der  $b$  für welche die Summe gerade ist, bilden eine Gruppe, da  $2, \varphi_1, \varphi_2, \dots$  gerade sind. Um aus dieser Gruppe, die Gruppe aller Systeme der  $b$  zu bekommen, musz man die erstgenannte Gruppe multiplizieren mit einer Gruppe in welcher jedes System der  $b$  (ausgenommen die identische Substitution  $b_{01} = 0, b_{*1} = 0, \dots$ ) eine ungerade Summe besitzt. Wenn, in dieser letzten Gruppe, sich zwei Systeme der  $b$  vorfinden mit ungerader Summe, so würde das System, dasz man durch Anzfählung der übereinstimmenden Zahlen dieser beiden Systemen erhielt, eine gerade Summe  $b_{0n} + b_{1n} + \dots$  haben. Ein solches System kann aber in dieser Gruppe nicht auftreten. In der Gruppe können also nicht zwei Systeme vorkommen (ausser  $b_{01} = b_{*1} = \dots = 0$ ). Die Gruppe hat daher den Grad 2.

Es gibt also  $\frac{1}{2} \cdot \frac{\varphi}{r}$  Systeme der  $b$ , welche eine gerade Summe haben und natürlich gleichviel mit ungerader Summe.

*Bestimmung der Zahl  $\kappa^1$ .*

<sup>1)</sup> „H.“, S. 229.

1°. Wenn der Körper  $k$  reell ist, so ist  $w = 2$  denn  $\pm 1$  sind die einzigen reellen Einheitswurzeln.

Es ist weiter  $r_2 = 0$  und  $r_1 = \frac{\varphi}{r}$  weil der Körper ein GALOIS'scher ist.

2°. Wenn der Körper  $k$  imaginär ist, so ist  $r_1 = 0$  und  $r_2 = \frac{\varphi}{2r}$ . Nun muss noch die Zahl  $w$  bestimmt werden.

Es sei  $w_0 = 1$  wenn alle Zahlen  $a_0 = 0$  sind; andernfalls  $w_0 = 0$ .

Es sei  $2^{w_*}$  die höchste Potenz von 2 welche auf alle Zahlen  $a_*$  teilbar ist und  $w_* = h_* - 2$  wenn alle Zahlen  $a_* = 0$ .  $u_* = 0$  wenn nicht alle Zahlen  $a_*$  durch 2 teilbar sind und auch  $= 0$  wenn nicht alle  $a_0 = 0$  sind;  $u_* = 1$  wenn alle  $a_0 = 0$  und alle  $a_*$  durch 2 teilbar sind.

Es sei  $l_1^{w_1}$  die höchste Potenz von  $l_1$  welche auf alle Zahlen  $a_1$  teilbar ist, und  $w_1 = h_1 - 1$  wenn alle  $a_1 = 0$  sind.  $u_1 = 0$  wenn nicht alle Zahlen  $a_1$  durch  $l_1 - 1$  teilbar sind und  $u_1 = 1$  wenn dies wohl so ist. U. s. w.

Dann ist:

$$\text{wenn } h_* \geq 3 \text{ ist: } w = 2^{w_0 + w_* u_* + 1} l_1^{u_1(w_1 + 1)} \dots$$

$$\text{wenn } h_* = 2 \text{ ist: } w = 2^{w_0 + 1} l_1^{u_1(w_1 + 1)} \dots$$

$$\text{wenn } h_* = 0 \text{ ist: } w = 2 l_1^{u_1(w_1 + 1)} \dots$$

Beweis: Der Körper  $k$  kann nur diejenigen Einheitswurzeln enthalten, welche Potenzen von  $Z$  sind, da der Kreiskörper  $K$  nur solche enthält. Nur wenn  $m$  ungerade ist, enthält  $K$  auch die Potenzen von  $Z$  mit negativem Vorzeichen. Nehmen wir nun an dass  $Z^a$  in  $k$  liegt, dann bleibt diese Zahl ungeändert für die Substitutionen von  $g$ . Wenn wir also alle Zahlen von  $g$  durch  $A^{(i)}$  darstellen, so ist

$$Z^{aA^{(i)}} = Z^a$$

woraus sich ergibt:

$$a(A^{(i)} - 1) \equiv (\text{mod } m) \dots \dots \dots (10)$$

Wenn, umgekehrt, die Zahl  $a$  dieser Congruenz genügt, für alle Zahlen  $A^{(i)}$  von  $g$ , so enthält der Körper  $k$  die Einheitswurzel  $Z^a$ .

Sind alle Zahlen  $a_0 = 0$  so enthält der Körper die Einheitswurzeln  $\pm i$ .

Beweis: Es gilt für jede Zahl  $A^{(i)}$ :

$$A^{(i)} \equiv 5^{a_*} (\text{mod } 2^h) \text{ also } A^{(i)} \equiv 1 (\text{mod } 4).$$

Aus (10) ergibt sich daher  $a = \frac{m}{4}$ , d. h. die Einheitswurzel

$Z^{\frac{m}{4}} = i$  liegt im Körper  $k$ . Daher auch  $-i$ .

Sind alle Zahlen  $a_* = 0$  und alle Zahlen  $a_0 = 0$  so enthält  $k$  die Einheitswurzel  $e^{\frac{2\pi i}{h^*}}$ . Man beweist dies auf dieselbe Weise.

Sind alle Zahlen  $a_0 = 0$  und alle Zahlen  $a_*$  höchstens teilbar durch  $2^{w_*}$  so enthält  $k$  die Einheitswurzel  $e^{\frac{2\pi i}{2^{w_*+2}}}$ . U.s.w.

Alle Beweise werden mit Hülfe der Congruenz (10) erbracht.

Man findet nun leicht die im Satze genannte Formel wenn man beachtet dasz, wenn der Körper die Einheitswurzel  $e^{\frac{2\pi i}{h^{w_1+1}}}$  enthält, auch die  $l_1^{w_1+1}$  ersten Potenzen dieser Zahl im Körper liegen.

§ 11. *Ableitung des entgeltigen Ausdrucks für die Klassenzahl.*

Satz: Wenn, für jedes System der  $b$ , die Summe  $b_{0n} + b_{1n} + \dots$  gerade ist, wenn also der Körper  $k$  reell ist, so stellt sich die Klassenanzahl  $H$  dieses Körpers, wie folgt, dar:

$$H = \frac{\prod_{n=2}^{\varphi/r} \sum_{s=1}^{\lfloor \frac{m}{2} \rfloor} \left[ \overbrace{b_{0n}, b_{*n}, b_{1n}, \dots}^s \right] \log A_s}{R}$$

Hierin ist  $A_s = \sqrt{(1-Z^s)(1-Z^{-s})}$  und  $R$  der Regulator. Ausserdem ist angenommen:  $b_{01} = b_{*1} = b_{11} = \dots = 0$ .

Ist  $h_* = 2$  so musz  $b_{*n}$  weggelassen werden und ist  $h_* = 0$  so musz auch  $b_{0n}$  weggelassen werden.

Wenn die Summe  $b_{0n} + b_{1n} + \dots$  nicht für jedes System der  $b$  gerade ist, wenn also der Körper  $k$  imaginär ist, so stellt sich die Klassenanzahl dieses Körpers  $k$ , wie folgt, dar:

$$H = \frac{\prod_n \sum_{s=1}^m \left[ \overbrace{b_{0n}, b_{*n}, b_{1n}, \dots}^s \right] s \cdot \prod_n \sum_{s=1}^{\lfloor \frac{m}{2} \rfloor} \left[ \overbrace{b_{0n}, b_{*n}, b_{1n}, \dots}^s \right] \log A_s}{(2m)^{\varphi/2r} R'}$$

Hierin ist  $w$  die in § 10 bestimmte Zahl. Das erste Produkt ist über alle Werte von  $n$ , für welche die Summe  $b_{0n} + b_{1n} + \dots$  ungerade ist, zu erstrecken; das zweite Produkt über alle Werte von  $n$  für welche diese Summe gerade ist.

Weiter ist  $A_s = \sqrt{(1-Z^s)(1-Z^{-s})}$  und  $R'$  die Determinante der Logarithmen der absoluten Werte eines Systems von Grundeinheiten. Ist  $h_* = 2$  so ist  $b_{*n}$  wegzulassen, und ist  $h_* = 0$  so ist auch  $b_{0n}$  wegzulassen.

Beweis:

Zür Ableitung des ersten Ausdrucks benutzen wir (9) und ersetzen, in der darin auftretenden Summe, die Grösze  $k$  durch  $m-k$ . Wenn

wir dann den ersten Hilfssatz von § 9 benutzen, so finden wir  $\sum F \cdot \frac{k\pi i}{m} = 0$ . Die Gleichheit (9) lässt sich daher umformen zu:

$$H = \frac{1}{\kappa} \sum_{n=2}^{\varphi/r} - \frac{1}{m} \sum_{k=1}^m F \left( e^{\frac{2\pi ki}{m}} \right) \log A_k$$

wenn:

$$\begin{aligned} A_k &= \frac{e^{\frac{k\pi i}{m}} - e^{-\frac{k\pi i}{m}}}{i} = ie^{-\frac{k\pi i}{m}} \left( 1 - e^{\frac{2k\pi i}{m}} \right) = \\ &= -ie^{\frac{k\pi i}{m}} \left( 1 - e^{-\frac{2k\pi i}{m}} \right) = \sqrt{\left( 1 - e^{\frac{2k\pi i}{m}} \right) \left( 1 - e^{-\frac{2k\pi i}{m}} \right)} \end{aligned}$$

Dieses Resultat für  $H$  lässt sich noch verändern mittels Hilfssatz 3 und 4 von § 9. Aus Hilfssatz (3) ergibt sich, dass in der Summe nur diejenigen Glieder übrig bleiben für welche die Zahl  $k$  mit  $m$  den grössten gemeinsamen Teiler  $d$  hat. Diese Zahl  $d$  deuten wir weiter an durch  $d_n$ . Die Summe wird zu

$$F \left( e^{\frac{2\pi d_n i}{m}} \right) \sum_k \left[ \frac{k/d}{m} \right]^{-1} \log A_k$$

Es sei nun  $k = k'd_n$ , so ist, wie man leicht findet:

$$A_k = A_{k'} \cdot A_{k'+\frac{m}{d_n}} \dots A_{k'+(d_n-1)\frac{m}{d_n}}$$

Die Summe wird daher zu:

$$F \left( e^{\frac{2\pi d_n i}{m}} \right) \sum_k \left[ \frac{k/d_n}{m} \right]^{-1} (\log A_{k'} + \log A_{k'+\frac{m}{d_n}} + \dots) \quad (11a)$$

und durch Hilfssatz 2 § 9 zu:

$$F \left( e^{\frac{2\pi d_n i}{m}} \right) \sum_{s=1}^m \left[ \frac{s}{m} \right]^{-1} \log A_s \dots \dots \dots (11)$$

denn, weil  $k'$  prim zu  $m/d_n$  ist und in (11a) die Zahl  $k$  alle Werte annimmt die mit  $m$  die Zahl  $d_n$  zum grössten gemeinsamen Teiler haben, bekommt man eine Summe, in welcher  $s$  alle Zahlen  $< m$  durchläuft, welche prim zu  $m$  sind oder nur noch teilbar durch diejenigen in  $m$  aufgehenden Primzahlen, deren betreffende  $b = 0$  sind.  $s$  durchläuft also alle Zahlen für welche  $\left[ \frac{s}{m} \right] \neq 0$  ist.

In der Formel für  $H$  kommt nun das Produkt

$$\prod_{n=2}^{\varphi/r} F \left( e^{\frac{2\pi d_n i}{m}} \right) \dots \dots \dots (12)$$

zum Vorschein. Um dessen Wert zu berechnen benutzen wir Hilfs-

satz (4) von § 9. Da zu jeder Substitution einer Gruppe auch die reciproke Substitution vorkommt, so wird ins Besondere hier zu jedem System  $b_{0n}, b_{*n}, \dots$  auch das System  $2-b_{0n}, \frac{1}{2}g_*-b_{*n}, g_1-b_{1n}, \dots$  auftreten. Wenn diese Systeme von einander verschieden sind, so ist, nach dem ebengenannten Hilfssatz, das Produkt der zugehörigen Functionen  $F$  und  $F'$  gleich  $d_n m$ . Sind sie aber einander gleich so hat das Produkt den Wert  $\pm \sqrt{d_n m}$ .

Das Produkt (12) hat daher den Wert

$$m^{2r-1} \prod_{n=2}^m \sqrt{d_n} \dots \dots \dots (13)$$

wenn man das Vorzeichen ausser Acht lässt.

Bevor wir nun den Wert dieses Produktes entgültig bestimmen, formen wir die Summe aus (11) um. Zuerst können wir darin den Exponent  $-1$  weglassen, weil, wie schon bemerkt worden ist, zu jedem System der Zahlen  $b$ , auch das System  $2-b_{0n}, \dots$  vorkommt.

Weiter ist

$$\sum_{s=1}^m \left[ \underbrace{s} \right] \log A_s = \sum_{s=1}^{\frac{m}{2}} + \sum_{s=1}^{\frac{m}{2}} \left[ \underbrace{m-s} \right] \log A_{m-s}$$

Denn, für den Fall dass  $m$  ungerade ist, folgt diese Gleichheit aus der bloßen Bemerkung dass  $\left[ \underbrace{m} \right] = 0$  ist. Ist, im Gegensatz,  $m$  gerade, so ist offenbar  $\left[ \underbrace{m/2} \right] = 0$  da  $m$  durch 4 teilbar ist.

Zur weiteren Umformung benutzen wir die leicht zu erhaltende Beziehung:

$$\left[ \underbrace{-s} \right] = (-1)^{b_{0n} + b_{1n} + \dots} \left[ \underbrace{s} \right]$$

und auch die Gleichheit  $A_{m-s} = A_s$ , und erhalten so für die Summe:

$$2 \sum_{s=1}^{\frac{m}{2}} \left[ \underbrace{s} \right] \log A_s$$

Nach Einführung in die Formel (9) und nach Einsetzung des Wertes von  $x$  und der in § 6 berechneten Discriminante  $d$ , erhält man den im Satze angegebenen Endausdruck für die Klassenzahl, nachdem das Produkt aller Zahlen  $d_n$  welches in (13) auftritt berechnet ist. Weil die Berechnung dieses Produktes dieselbe ist beim zweiten Teil des Satzes, so werden wir diese Berechnung bis zum Ende des Beweises aufschieben.

*Beweis des zweiten Teils des Satzes.*

Das Produkt aus (9) zerlegen wir in zwei Produkten: das erste

ist zu erstrecken über alle Werte von  $n$  für welche  $b_{0n} + b_{1n} + \dots$  ungerade ist, während das zweite zu erstrecken ist über alle Werte von  $n$  für welche diese Summe gerade ist. Das letzte Produkt lässt sich auf dieselbe Weise umformen wie beim Beweise des ersten Teils des Satzes. Beim ersten Produkt erhalten wir, durch Benutzung des ersten Hilfssatzes § 9, nachdem in (9)  $m-k$  statt  $k$  eingeführt ist:

$$\sum_{k=1}^m F\left(e^{\frac{2\pi ki}{m}}\right) \log A_k = 0.$$

Es bleibt von der Summe aus (9) also nur übrig

$$-\frac{\pi i}{m} \sum_{k=1}^m k F\left(e^{\frac{2\pi ki}{m}}\right).$$

Wegen Hilfssatz 3 (§ 9) wird dies zu

$$-\frac{\pi i}{m} F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_k k \left[\frac{k/d}{m}\right]^{-1} = -\frac{\pi d_i}{m} F \cdot \sum_k \frac{k}{d} \left[\frac{k/d}{m}\right]^{-1}$$

wo  $k$  über alle Zahlen zu erstrecken ist, die mit  $m$  die Zahl  $d_n$  zum grössten gemeinsamen Teiler haben.

Weiter ist, wenn  $k = k'd_n$ ;  $d = d_n$ ;

$$d_n \sum_k \frac{k}{d} \left[\frac{k/d}{m}\right]^{-1} = \sum_{k'} k' \left[\frac{k'}{m}\right]^{-1} + \sum_{k'} (k' + m/d) \left[\frac{k' + m/d}{m}\right]^{-1} + \dots + \sum_{k'} (k' + (d-1)m/d) \left[\frac{k' + (d-1)m/d}{m}\right]^{-1}$$

da

$$\sum \left[\frac{k'}{m}\right]^{-1} + \sum \left[\frac{k' + m/d}{m}\right]^{-1} + \dots = \sum_s \left[\frac{s}{m}\right]^{-1} = 0$$

ist wegen Hilfssatz 1 von § 8. Wir erhalten also das Resultat:

$$-\frac{\pi i}{m} F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_{s=1}^m \left[\frac{s}{m}\right]^{-1}$$

in welchem der Exponent  $-1$  darf weggelassen werden da, wenn die Summe  $b_{0n} + b_{1n} + \dots$  ungerade ist, auch die Summe der Werte  $2 - b_{0n}, \frac{1}{2} \varphi_* - b_{*n}, \dots$  ungerade ist.

Es ist ersichtlich dass in (9) wiederum das Produkt  $\sqrt{H} d_n$  zum  $n=2$  Vorschein kommt, wenn wir die Faktoren zu zweien nehmen und jedes Produkt mittels Hilfssatz 4 (§ 9) umformen. Dabei erscheinen dann Potenzen von  $i$ , und ebensolche treten auch bei der weiteren Umformung noch auf. Man kann sie jedoch ausser Acht lassen weil die Zahl  $H$  natürlich eine positive ganze Zahl ist.

§ 12. Berechnung des Wertes des Produktes  $\prod_{n=2}^{\varphi/r} d_n$ .

Wir bestimmen die Potenz von 2 und die von  $l_1$ , welche in dem Produkte vorkommen.

Es gibt  $\frac{1}{2} \varphi_{*n}$  voneinander verschiedene Zahlen  $b_{*n}$  da die Untergruppe  $g$  primär ist. Jeder dieser Werte kommt also in allen Systeme der  $b$  genau  $\frac{\varphi}{r} : \frac{1}{2} \varphi_{*n}$  Mal vor. Nun gibt es unter den voneinander verschiedenen Werten der  $b_{*n}$  genau.

$2^{h_*-4}$  Zahlen welche genau durch 2 teilbar sind.

$2^{h_*-5}$  „ „ „ „  $2^2$  „ „

„ „ „ „ „ „

$2^0$  „ „ „ „  $2^{h_*-3}$  „ „

1 „ „ „ „  $2^{h_*-2}$  „ „

Alles zusammen genommen gibt dies die folgende Potenz von 2:

$$2^{(2^{h_*-2}-1) \frac{2\varphi}{\varphi_{*r}} - (h_*-2)}$$

wenn man bedenkt dasz das System  $b_{0n} = b_{*n} \dots = 0$  nicht mitgezählt werden musz. Man musz nun noch Rücksicht nehmen auf den Fällen worein  $b_{0n}$  und  $b_{*n}$  beide zugleich den Wert Null annehmen. Für einen jeden solchen Fall musz obenstehende Potenz von 2 noch

mit  $2^2$  multipliziert werden. Nun gibt es  $\frac{\varphi}{2r}$  Systeme in welchen

$b_{0n} = 0$  ist. Die Systeme in welchen  $b_{0n} = b_{*n} = 0$  ist, stellen eine Untergruppe der Gruppe aller Systeme  $b$  dar. Die Systeme, in welchen  $b_{*n} = 0$  ist, stellen auch eine Untergruppe dar, welche die erstgenannte Untergruppe enthält. Diese erstgenannte Untergruppe musz, wie leicht ersichtlich, multipliziert werden mit der Gruppe des Grades 2:

$$b_{0n} = 0; \quad b_{*n} = 0; \quad b_{1n} = 0, \dots$$

$$b_{0n} = 1; \quad b_{*n} = 0; \quad b_{1n} = 0, \dots$$

um die Gruppe der Systeme, in welchen  $b_{*n} = 0$  ist, zu erhalten. Man erkennt also dasz die Hälfte der Anzahl der Systeme in welchen  $b_{*n} = 0$  ist, die Anzahl der Systeme ist, in welchen  $b_{0n} = b_{*n} = 0$  ist. Dies ist also der Fall wenn nicht für jeden Wert von  $n$   $b_{0n} = b_{*n} = 0$  ist. Ist jedoch  $b_{0n} + b_{*n}$  für alle Werte von  $n$  gerade, so ist immer  $b_{0n} = b_{*n} = 0$ .

Es ist nun leicht ersichtlich dasz obenstehende Potenz von 2 noch multipliziert werden musz mit

$$2^2 \left( \frac{(t+1)\varphi}{\varphi_{*r}} - 1 \right)$$

wo  $t$  dieselbe Bedeutung hat wie in § 6 Satz 8.

Um die Potenz von  $l_1$  zu bestimmen führen wir das Folgende an:  
 Weil  $(d_1^{-1})$  der grösste gemeinsame Teiler aller Zahlen  $b_{1n}$  ist, so gibt es  $\frac{\varphi_1}{d_1}$  voneinander verschiedene Zahlen  $b_{1n}$ . Ein jeder solchen Wert kommt daher  $\frac{\varphi}{r} : \frac{\varphi_1}{d_1}$  Mal vor. Weil  $g$  primär ist, ist  $d_1$  nicht durch  $l_1$  teilbar, und, wenn  $h_1 = 1$  nicht durch  $l_1 - 1$ . Die durch  $l_1$  teilbaren Zahlen  $b_{1n}$  sind daher

$$d_1 l_1, 2 d_1 l_1, \dots, \frac{l_1 - 1}{d_1} \cdot d_1 l_1^{h_1 - 1}$$

Von diesen sind

$$\frac{(l_1 - 1)^2}{d_1} l_1^{h_1 - 3} \text{ Zahlen genau durch } l_1 \text{ teilbar.}$$

$$\frac{(l_1 - 1)^3}{d_1} l_1^{h_1 - 4} \quad ,, \quad ,, \quad ,, \quad l_1^2 \quad ,,$$

, . . . . .

$$\frac{l_1 - 1}{d_1} - 1 \text{ Zahlen genau durch } l_1^{h_1 - 1} \text{ teilbar.}$$

Wenn man noch Rücksicht nimmt auf den Fall wo alle  $b = 0$  sind, so erhält man für die Potenz von  $l_1$  welche im Produkte aller  $d_n$  vorkommt:

$$l_1 \left( \frac{l_1^{h_1 - 1} - 1}{d_1} \right)^{\frac{\varphi d_1}{\varphi_1 r} - h_1}$$

Wenn nun  $D$  die in § 6 bestimmte Discriminante des Körpers  $k$  darstellt, so gibt eine leichte Rechnung allgemein die Formel

$$\sqrt{D} \cdot \prod_{n=2}^r d_n = m^{\frac{\varphi}{2r} - \frac{1}{2}}$$

Im Nenner des Ausdrucks für  $H$  findet sich nun noch der Regulator  $R$ . Nehmen wir ein System Grundeinheiten  $\eta_i$ , so gilt:

$$\log |\eta_i^{(k)}| = \frac{1}{2} l_k (\eta_i)^2$$

weil die conjugirten Körper reell sind. Nach Einführung in  $R$  kann man die Faktoren 2 aus dem Determinante herausheben, wonach die Richtigkeit der im Satz § 11 angegebene zweite Formel bewiesen ist.

1) Diese Zahl  $d_1$  hat also dieselbe Bedeutung wie in § 4.2. Eine Verwirrung mit den in § 9.2 bestimmten Zahl  $d$ , welche im Anfang des Beweises von § 11  $d_n$  genannt ist, ist wohl nicht zu befürchten.

2) „H.“ S. 215. Im Ausdruck auf S. 376 (oben) nennt Herr HILBERT die Zahl  $R$  den Regulator. Dies ist nicht in Ubereinstimmung mit seiner Definition auf Seite 221.

**Chemistry.** — “*On the preparation of  $\alpha$ -sulphopropionic acid.*” By H. J. BACKER and J. V. DUBSKY. (Communicated by Prof. F. M. JAEGER.)

(Communicated in the meeting of September 27, 1919).

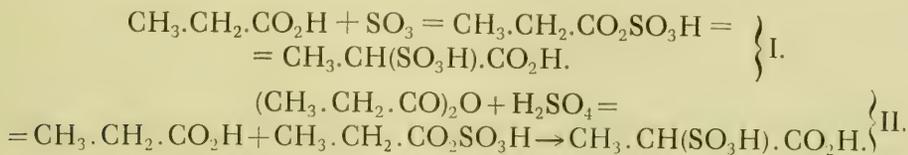
The only practical method for the preparation of the  $\alpha$ -sulphopropionic acid has hitherto been the one found by FRANCHIMONT <sup>1)</sup> i. e. the action of sulphuric acid with propionic acid anhydride.

Besides FRANCHIMONT's general method, an analogical method has been used by MEISENS <sup>2)</sup> for the preparation of the sulphonacetic acid, the first term of the sulphocarbonic acids, namely the action of sulphuric acid anhydride with acetic acid.

These two methods show much similarity. In both cases the mixed anhydride, the acetylsulphuric acid, is formed as an intermediary product, as FRANCHIMONT <sup>3)</sup> had already supposed and STILLICH <sup>4)</sup> afterwards established, and as VAN PESKI <sup>5)</sup> proved for the reaction of MEISENS.

With a view to comparing it with the method of FRANCHIMONT we have applied MEISENS' method also to propionic acid, by treating it with sulphurtrioxide.

In both cases the reaction turned out, just as in the case of the acetic acid, to be indirect, while the mixed anhydride, the propionylsulphuric acid, must be taken to be the intermediary product.



When the substances are mixed carefully whilst cooling, a colourless, very viscous liquid is formed, which yields the sulphopropionic acid only at a higher temperature, with development of heat and brown coloration.

We were able to follow the process of this reaction by titration,

<sup>1)</sup> Recueil trav. chim. **7**, 27 (1888).

<sup>2)</sup> Ann. der Chemie **52**, 276 (1844).

<sup>3)</sup> Versl. dezer Akad. **16**, 373 (1881).

<sup>4)</sup> Ber. d. dtsh. chem. Ges. **38**, 1241 (1905).

<sup>5)</sup> Versl. dezer Akad. **22**, 996 (1914).

as the propionylsulphuric acid, when hydrolyzed, neutralizes three equivalents of a basis, and the sulphonic acid only two equivalents.

The sulphopropionic acid was separated and weighed in the form of its baryumsalt. From 1 molecule propionic acid with sulphurtrioxide we got an average yield of 0.35 mol. sulphopropionic acid, and from 1 mol. propionic acid anhydride with sulphuric acid (monohydrate) 0.55 mol.

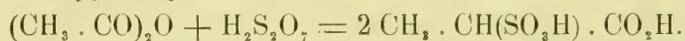
Compared with FRANCHIMONT'S method, the reaction with sulphurtrioxide is experimentally less simple. The reaction is violent, the product is more coloured, the baryumsalt likewise, and the yield is various. Only for the preparation of large quantities may the reaction be recommended.

FRANCHIMONT'S reaction, however, is very easy to carry out; but it is a drawback, that only half the propionic acid anhydride can be transformed into the sulphonic acid, the other half returning to propionic acid.

We have therefore tried to combine the advantages of both methods, the better yield of the first and the greater purity of the product obtained by the second.

For this purpose, it is possible to mix propionic acid anhydride with sulphuric acid, and to treat the product, containing one molecule free propionic acid, with the equimolecular quantity of sulphurtrioxide.

However, it is simpler to let the propionic acid anhydride react directly with *pyrosulphuric acid*.



By this method we got an average yield of 0.75 mol. sulphonic acid from 1 mol. propionic acid anhydride.

As the pyrosulphuric acid may be added in the crystallized state to the propionic acid, the method is easier than the reaction with sulphurtrioxide. The reaction proceeds more quietly and the product is purer.

We have applied this method to acetic acid anhydride also, but for such an easily accessible substance, the original method of FRANCHIMONT is simpler.

For the sulphuration of precious carbonic acid anhydrides, the method just described may, however, be recommended for its higher yield.

Groningen, Sept. 8, 1919.

Org. Chem. Lab. of the Univ.

**Mathematics.** — “Some applications of the quasi-uniform convergence on sequences of real and of holomorphic functions”. By Prof. J. WOLFF. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of September 27, 1919).

We consider a region  $G$  of the complex plane, and a sequence of functions  $f_1, f_2, \dots$  which are all analytical within  $G$ . If the sequence converges uniformly in any closed region within  $G$ , then the limiting function  $f$  is analytical within  $G$ . This theorem however, enunciated by WEIERSTRASZ, has of late been considerably extended. If it is known that the sequence converges at the points of a set having a point of the interior of  $G$  for a limiting point, and besides, that the sequence is uniformly limited in every closed region internal to  $G$ , then this is already sufficient to conclude to the uniform convergence of the sequence in any closed region within  $G$ , which involves that the limiting function  $f$  is analytical <sup>1)</sup>. This same conclusion may be drawn,  $G$  being a circle, if only the sequence of functions is supposed to be uniformly limited within  $G$  and convergent at the points  $z_1, z_2, \dots$ , such that  $\prod_1^{\infty} (z_k - z_0) = 0$ , where  $z_0$  is the centre of  $G$  and  $z_k \neq z_0$  <sup>2)</sup>; also if it is only supposed that there exist two definite numbers  $a$  and  $b$  such that nowhere in the interior of  $G$  one of these two values is assumed by a function of the sequence, and besides that the sequence converges at the points of a set having a point internal to  $G$  for a limiting point <sup>3)</sup>. None of these theorems lead to other sequences of functions than those which are embraced already by WEIERSTRASZ' theorem. The question now could be raised whether it is possible that a sequence of functions, each of which is analytical within a region  $G$ , may converge to a function, analytical within  $G$ , without this convergence being uniform in every closed region contained in  $G$ . That such sequences actually exist is shown by an example,

<sup>1)</sup> G. VITALI. *Annali di Matematica*, Serie 3<sup>a</sup>, tomo 10 (1904), p. 65.

For a simple proof *vide* a.o. *Verh. der Kon. Ac. v. W.*, dl. 27 (1918), p. 319.

<sup>2)</sup> W. BLASCHKE. *Leipz. Berichte*, Band 67 (1915), p. 194.

<sup>3)</sup> C. CARATHÉODORY and E. LANDAU. *Sitz. Ber. Ak. v. W. Berlin*, Band 32 (1911), p. 587.

given by MONTEL with the purpose of refuting the assertion of POMPEJU that it should be necessary — in order to render the limiting function  $f'$  of a within  $G$  convergent series of analytical functions analytical in  $G$  — to suppose that it is permitted to integrate the series termwise along every curve of integration in  $G$ <sup>1)</sup>. In fact MONTEL's series converges to zero throughout the whole plane whilst the termwise integration along the real axis from 0 to 1 yields not 0 but 1, so a uniform convergence in every limited region is here altogether out of the question. We state, besides, that it is neither permitted to differentiate MONTEL's series termwise for  $z = 0$ , as this would yield the result  $+\infty$  instead of zero. The example of MONTEL is embraced by none of the foregoing theorems. The want is now felt of a necessary and sufficient condition that a sequence of analytical functions, convergent in the interior of  $G$ , should have an analytical limiting function. Necessary it is that the convergence of the sequence be quasi-uniform in every closed region within  $G$ , because of the continuity of  $f'_n$  and  $f'$ . If, conversely, this quasi-uniform convergence is given, then it follows that  $f'$  is continuous within  $G$ , but not that  $f'$  is analytical, as an example of MONTEL<sup>2)</sup> shows. In the following a necessary and sufficient condition will be deduced among other things.

## I.

We consider a sequence of real functions  $f'_1(x), f'_2(x), \dots$ , each of which is continuous for  $0 \leq x \leq a$ ; the sequence converges for  $0 < x \leq a$ . Let  $f'(x)$  be the limiting function.

1. The function  $\Phi_n(x, y) = f'_n(x) - f'_n(y)$  is definite in the rectangle  $0 \leq x \leq a, 0 \leq y \leq a$  and is there continuous. The sequence of functions  $\Phi_1(x, y), \Phi_2(x, y), \dots$  converges at every point of the rectangle where not simultaneously  $x > 0$  and  $y = 0$ , or  $x = 0$  and  $y > 0$ . The limiting function is  $\Phi(x, y) = f'(x) - f'(y)$  at every point where  $x > 0$  and  $y > 0$  and vanishes at  $x = 0, y = 0$ , since there every  $\Phi_n = 0$ . Let it be given that  $f'(x)$  converges to a finite limit as  $x \rightarrow 0$ . We consider a set of points  $P_1(x_1, y_1), P_2(x_2, y_2), \dots$  where  $0 < x_k \leq a, 0 < y_k \leq a, \lim_{k \rightarrow \infty} x_k = 0, \lim_{k \rightarrow \infty} y_k = 0$ ; two arbitrary positive numbers may be given,  $\epsilon$  and  $N$ . Let  $N_0$  denote an arbitrary integer  $> N$ .

We have  $\Phi_{N_0}(0, 0) = 0$ . In consequence of the continuity of  $\Phi_{N_0}$  a number  $\delta_1$  can be assigned such that at every point  $P_k$ , where

<sup>1)</sup> P. MONTEL. Bul. des Sc. math., série 2, tome 30, part 1 (vol. 41, 1906, p. 191).

<sup>2)</sup> P. MONTEL, l. c. p. 190.

Thèse p. 98.

$x_k < \delta_1$ , and  $y_k < \delta_1$ , the inequality  $|\Phi_{N_0}(x_k, y_k)| < \frac{\varepsilon}{2}$  is satisfied.

Since  $f(x)$  tends to a finite limit as  $x \rightarrow 0$  there is a number  $\delta$ , such that for  $x_k < \delta$  and  $y_k < \delta$  the inequality  $|\Phi(x_k, y_k)| < \frac{\varepsilon}{2}$  is satisfied.

If  $\delta < \delta_1$  and at the same time  $< \delta_2$  then

$$|\Phi(x_k, y_k) - \Phi_{N_0}(x_k, y_k)| < \varepsilon \quad \text{for } x_k < \delta \text{ and } y_k < \delta.$$

There now exist only a finite number of indices for which  $x_k \geq \delta$ ,  $y_k \geq \delta$ . At each of the points  $P_k$  corresponding there to an index  $N_k > N$  can be found such that

$$|\Phi(x_k, y_k) - \Phi_{N_k}(x_k, y_k)| < \varepsilon.$$

At the points of the considered set the sequence of functions  $\Phi_n$  therefore converges quasi-uniformly.

2. Conversely, let it be given that at the points of every set  $P_1(x_1, y_1), P_2(x_2, y_2), \dots$  where  $0 < x_k \leq a$ ,  $0 < y_k \leq a$ ,  $\lim_{k \rightarrow \infty} x_k = 0$ ,  $\lim_{k \rightarrow \infty} y_k = 0$ , this sequence of functions converges quasi-uniformly.

Let  $M$  denote the maximum of  $f(x)$  at 0,  $m$  the minimum. It is possible to construct a set of points  $x_1, x_2, \dots$  where  $x_k > 0$  and  $\lim_{k \rightarrow \infty} x_k = 0$ , such that  $\lim_{k \rightarrow \infty} f(x_k) = M$ ; similarly we can construct a set  $y_1, y_2, \dots$ , where  $y_k > 0$  and  $\lim_{k \rightarrow \infty} y_k = 0$ , such that  $\lim_{k \rightarrow \infty} f(y_k) = m$ .

If we now consider the set of points

$$P_1(x_1, y_1), P_2(x_2, y_2), \dots, \quad \text{then } \lim_{k \rightarrow \infty} \Phi(x_k, y_k) = M - m, \quad (1)$$

unless  $m = +\infty$  or  $M = -\infty$ . Now it is possible, by hypothesis, to choose at each of the points  $P_k$  from a finite number of indices  $1, 2, \dots, n(\varepsilon)$  an index  $n_k$  such that

$$|\Phi(x_k, y_k) - \Phi_{n_k}(x_k, y_k)| < \frac{\varepsilon}{2}. \quad (2)$$

where  $\varepsilon$  denotes a given arbitrary positive number. In consequence of the continuity of each of the functions  $\Phi_{n_k}(x, y)$  and since the number of indices here considered is finite, a number  $\delta$  exists such that at every point  $P_k$ , where  $x_k < \delta$  and  $y_k < \delta$  the relation

$$|\Phi_{n_k}(0, 0) - \Phi_{n_k}(x_k, y_k)| < \frac{\varepsilon}{2},$$

is satisfied.

Hence since

$$\Phi_{n_k}(0, 0) = 0, \quad \text{we have } |\Phi_{n_k}(x_k, y_k)| < \frac{\varepsilon}{2}. \quad (3)$$

From (2) and (3) it follows that  $|\Phi(x_k, y_k)| < \varepsilon$  for  $x_k < \delta$ ,  $y_k < \delta$  and from this, since  $\lim_{k \rightarrow \infty} x_k = 0$  and  $\lim_{k \rightarrow \infty} y_k = 0$ ,

$$\lim_{k \rightarrow \infty} \Phi(x_k, y_k) = 0 \dots \dots \dots (4)$$

From (1) and (4) it follows that  $M = m$ , hence  $\lim_{x \rightarrow 0} f(x)$  exists and is finite, unless  $m = +\infty$  or  $M = -\infty$ .

If however  $m = +\infty$ , then at the points of every set  $x_1, x_2, \dots$  where  $x_k > 0$  and  $\lim_{k \rightarrow \infty} x_k = 0$ , the  $\lim_{k \rightarrow \infty} f(x_k)$  is  $+\infty$ .

To every  $x_k$  of such a set a  $y_k > 0$  can be found such that  $f(y_k) > f(x_k) + k$  and  $\lim_{k \rightarrow \infty} y_k = 0$ . Then  $\lim_{k \rightarrow \infty} \Phi(x_k, y_k) = -\infty$ , which is incompatible with the formula (4). Similarly it appears that  $M$  cannot possibly be  $-\infty$ . Collecting our results we have the following theorem:

*If a sequence of functions  $f_n(x)$  be given each of which is continuous for  $0 \leq x \leq a$ , and if  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for  $0 < x \leq a$ , then a necessary and sufficient condition that  $f(x)$  should tend to a finite limit at 0 is: the quasi-uniform convergence of the sequence of functions  $\Phi_n(x, y) = f_n(x) - f_n(y)$  at the points of every set  $(x_1, y_1), (x_2, y_2), \dots$  where  $0 < x_k \leq a$ ,  $0 < y_k \leq a$  and  $\lim_{k \rightarrow \infty} x_k = 0$ ,  $\lim_{k \rightarrow \infty} y_k = 0$ .*

## II.

3. Let  $f_1(x), f_2(x), \dots$  be continuous for  $0 \leq x \leq a$  with finite derivatives  $f'_n(0)$  at 0. Let the series be convergent for  $0 < x \leq a$ .

The functions  $f_n^*(x) = \frac{f_n(x) - f_n(0)}{x}$  for  $x > 0$  and  $f'_n(0)$  for  $x = 0$  are in consequence of the suppositions continuous at  $0 \leq x \leq a$ ; at  $0 \leq x \leq a$  the sequence of functions  $f_n^*(x)$  converges to the limiting function  $f^*(x) = \frac{f(x) - f(0)}{x}$ , where  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ .

If  $f(x)$  has a finite derivative at 0, then  $f^*(x)$  tends to a finite limit for  $x \rightarrow 0$ , and *vice versa*.

Hence, by applying the theorem demonstrated in I, we find, if we take into consideration that

$$f_n^*(x) - f_n^*(y) = \frac{y f_n(x) - x f_n(y) + (x - y) f_n(0)}{xy};$$

If a sequence of functions  $f_n(x)$  is given each of which is continuous for  $0 \leq x \leq a$  and has a finite derivative  $f'_n(0)$  at 0, and if, for  $0 \leq x \leq a$ ,  $\lim_{n=\infty} f_n(x) = f(x)$ , then a necessary and sufficient condition that  $f(x)$  should have a finite derivative at 0 is: the quasi-uniform convergence of the sequence of functions

$$\psi_n(x,y) = \frac{yf'_n(x) - xf'_n(y) + (x-y)f'_n(0)}{xy}$$

at the points of every set  $(x_1, y_1), (x_2, y_2), \dots$  where  $0 < x_k \leq a, 0 < y_k \leq a$ , and  $\lim_{k=\infty} x_k = 0, \lim_{k=\infty} y_k = 0$ .

4. We observe that in this theorem nothing is supposed about the convergence or divergence of the sequence  $f'_n(0)$ . Let this sequence be convergent. Then the sequence of continuous functions  $f_n^{**}(x)$  is convergent for  $0 \leq x \leq a$  and converges to  $f^{**}(x)$  for  $x > 0$ , to  $\lambda = \lim_{n=\infty} f'_n(0)$  for  $x = 0$ .

If  $f'(0) = \lim_{n=\infty} f'_n(0)$  then

$$\lim_{x=0} f^{**}(x) = \lambda \dots \dots \dots (5)$$

We now consider a set of points  $x_1, x_2, \dots$ , where  $\lim_{k=\infty} x_k = 0$ . By  $\epsilon, N$  we denote two given positive numbers. Then there exists an index  $N_0 > N$  such that

$$|f'_{N_0}(0) - \lambda| < \epsilon \dots \dots \dots (6)$$

Since  $f'_{N_0}(0) = f^{**}_{N_0}(0)$  and  $f^{**}_{N_0}(x)$  is continuous and because of (5), there exists a certain number  $k_0$  such that for every  $k > k_0$

$$|f^{**}_{N_0}(x_k) - f^{**}(x_k)| < \epsilon.$$

Since at each of the finite number of points where  $k \leq k_0$  an index  $N_k > N$  can be found such that

$$|f^{**}_{N_k}(x_k) - f^{**}(x_k)| < \epsilon,$$

it appears that the sequence of functions  $f_n^{**}(x)$  converges quasi-uniformly at the points of the considered set.

If this sequence of functions converges quasi-uniformly at the points of every set of the above-considered kind, then it results from an argument as held in § 2 that

$$f'(0) = \lim_{n=\infty} f'_n(0).$$

We have therefore the following theorem:

If a series of functions  $f_n(x)$  be given, each of which is continuous

for  $0 \leq x \leq a$  and has a finite derivative  $f'_n(0)$  at 0, and if  $f'_n(x)$  for  $0 \leq x \leq a$  converges to  $f'(x)$  and the sequence  $f'_n(0)$  is convergent, then a necessary and sufficient condition that it should be permitted to "differentiate term-wise at 0" i.e. that  $\lim_{n \rightarrow \infty} f'_n(0)$  should be  $f'(0)$ , is: the quasi-uniform convergence of the sequence of functions

$$f_n^*(x) = \frac{f'_n(x) - f'_n(0)}{x}$$

at the points of every set  $x_1, x_2, \dots$  where  $\lim_{k \rightarrow \infty} x_k = 0$ .

The function  $f_n(x) = \frac{x}{1+n^2x^2}$  converges everywhere to zero, hence  $f'_n(0) = 0$ ,

$$f_n^*(x) = \frac{1}{1+n^2x^2}, \quad f'_n(0) = 1, \quad \psi_n(x, y) = \frac{1}{1+n^2x^2} - \frac{1}{1+n^2y^2}.$$

$\psi_n(x, y)$  everywhere converges to a continuous function, i. e. quasi-uniformly in every finite region so that the criterion given in § 3 is satisfied. Accordingly  $f'(0)$  is finite. However, if we choose  $\varepsilon < 1/2$  and the number of indices is finite, then for every  $x$  which is small enough we have for each of these indices  $f_n^*(x) > 1/2$ , whilst for  $x \neq 0$  we have  $\lim_{n \rightarrow \infty} f_n^*(x) = 0$ . The criterion, enunciated in this § is therefore not satisfied. Accordingly we have

$$\lim_{n \rightarrow \infty} f'_n(0) = 1 \neq f'(0).$$

The foregoing theorems can be immediately extended to the complex plane and are capable of analogous proofs.

### III.

5. Let there be given a series of functions  $f_1(z), f_2(z), \dots$ , each of which is analytical in the interior of the circle  $|z| < a$ , and convergent within this region. We shall now demonstrate the following theorem:

*A necessary and sufficient condition that the limiting function  $f$  should be analytical within  $|z| < a$  is the quasi-uniform convergence of the sequence of functions*

$$\psi_n(x, y, z) = \frac{(y-z)f_n(x) + (x-y)f_n(z) + (z-x)f_n(y)}{(x-z)(y-z)} \quad (1)$$

at the points of every closed and limited set  $V(x, y, z)$  having no point in common with the sets  $(x = z, y \neq z)$  and  $(x \neq z, y = z)$ .

For  $\psi_n(x, y, z)$  may be written  $\Phi_n(x, z) - \Phi_n(y, z)$ , where

$$\Phi_n(u, v) = \frac{f_n(u) - f_n(v)}{u - v} \dots \dots \dots (2)$$

Since the functions  $f_n$  are for  $|z| < a$  analytical  $\psi_n(x, y, z)$  is an analytical function of  $x, y$  and  $z$  in the region  $|x| < a, |y| < a, |z| < a$ , if only for  $u = v$  we put  $\Phi_n(u, v) = \Phi_n'(u)$ , which has been tacitly assumed in the above.

6. Let it be given that  $f'$  is analytical. Let  $(x_0, y_0, z_0)$  denote a point of  $V$ , and  $x_0 \neq z_0, y_0 \neq z_0$ .

We then have

$$\lim_{n \rightarrow \infty} \psi_n(x_0, y_0, z_0) = \frac{(y_0 - z_0)f'(x_0) + (x_0 - y_0)f'(z_0) + (z_0 - x_0)f'(z_0)}{(x_0 - z_0)(y_0 - z_0)} = \psi(x_0, y_0, z_0).$$

In a sufficiently small neighbourhood  $\Omega'$  of  $(x_0, y_0, z_0)$  we have also

$$\lim_{n \rightarrow \infty} \psi_n(x, y, z) = \frac{(y - z)f'(x) + (x - y)f'(z) + (z - x)f'(y)}{(x - z)(y - z)} = \psi(x, y, z).$$

The being analytical of  $f'$  involves the same of  $\psi(x, y, z)$  within  $\Omega'$  and in particular the continuity of  $\psi(x, y, z)$ . If  $\varepsilon$  and  $N$  are given then an index  $N_0 > N$  exists such that

$$|\psi_{N_0}(x_0, y_0, z_0) - \psi(x_0, y_0, z_0)| < \varepsilon.$$

From the continuity of  $\psi_{N_0}$  and  $\psi$  within  $\Omega'$  it follows that in  $\Omega'$  is contained a region  $\Omega$  in which everywhere

$$|\psi_{N_0}(x, y, z) - \psi(x, y, z)| < \varepsilon \dots \dots \dots (3)$$

If  $x_0 = y_0 = z_0$  then  $\psi_n(x_0, y_0, z_0) = 0$  for every  $n$ . Also  $\psi_n$  is continuous in a neighbourhood  $\Omega'$  of  $(x_0, y_0, z_0)$  and moreover we have  $\lim \psi(x, y, z) = 0$ , so that here too the inequality (3) is satisfied

$$\begin{aligned} x &= x_0 \\ y &= y_0 \\ z &= z_0 \end{aligned}$$

in a neighbourhood  $\Omega$  of  $(x_0, y_0, z_0)$ .

Every point of  $V$  therefore lies in a neighbourhood  $\Omega$  such that throughout  $\Omega V$  by one and the same index  $N_0$  the relation (3) is satisfied. Since  $V$  is limited and closed  $V$  can be covered by a finite number of these neighbourhoods<sup>1)</sup>, whence it follows that  $\psi_n$  converges quasi-uniformly at the points of  $V$ .

7. Conversely, let be given the quasi-uniform convergence of  $\psi_n$  at the points of every  $V$ . Let  $z_0$  denote an arbitrary point where

<sup>1)</sup> E. BOREL. Leçons sur les Fonctions monogènes, p. 11.

$|z_0| < a$ . For  $V$  we now choose an arbitrary set of points  $P_0(z_0, z_0, z_0)$ ,  $P_1(x_1, y_1, z_0)$ ,  $P_2(x_2, y_2, z_0), \dots$  where  $x_k \neq z_0$ ,  $y_k \neq z_0$ ,  $\lim_{k \rightarrow \infty} x_k = z_0$ ,  $\lim_{k \rightarrow \infty} y_k = z_0$ .

*Ex hypothesi* it is possible at every point  $P_k$  of  $V$  to choose  $n_k$  such that

$$|\psi_{n_k}(x_k, y_k, z_0) - \psi(x_k, y_k, z_0)| < \frac{\varepsilon}{2} \quad \dots \quad (4)$$

Since  $\psi_n$  is continuous and the number of indices  $n_k$  is finite a positive number  $\delta$  exists such that for  $|x_k - z_0| < \delta$  and  $|y_k - z_0| < \delta$  we have

$$\psi_{n_k}(x_k, y_k, z_0) - \psi_{n_k}(z_0, z_0, z_0) < \frac{\varepsilon}{2} \quad \dots \quad (5)$$

Since  $\psi_{n_k}(z_0, z_0, z_0) = 0$ , it follows from (4) and (5) that

$$|\psi(x_k, y_k, z_0)| < \varepsilon \text{ for } |x_k - z_0| < \delta \text{ and } |y_k - z_0| < \delta,$$

$$\text{i. e. } \left| \frac{f(x_k) - f(z_0)}{x_k - z_0} - \frac{f(y_k) - f(z_0)}{y_k - z_0} \right| < \varepsilon.$$

It results from this, that  $\frac{f(x_k) - f(z_0)}{x_k - z_0}$  tends to a finite limit as  $x_k \neq z_0$  coincides successively with the points of an arbitrary set of which  $z_0$  is the sole limiting point, and from this again that a finite  $\lim_{x \rightarrow z_0} \frac{f(x) - f(z_0)}{x - z_0}$  exists, hence  $f(z)$  has a finite derivative

at  $z_0$ . Since this holds good for any  $|z| < a$  it follows from a theorem enunciated by GOURSAT that  $f(z)$  is analytical within  $|z| < a$ .

It may be stated that here only a part of the supposition has been made use of.

8. *If the sequence of functions  $f_1(z), f_2(z), \dots$ , each of which is analytical for  $|z| < a$ , converges at a point  $z_0$  internal to this region and the function  $f_n^*(x, y) = \frac{f_n(x) - f_n(y)}{x - y}$  converges quasi-uniformly at the points of every closed set internal to  $|x| < a, |y| < a$ , then  $f_n(z)$  converges for every  $|z| < a$ , the limiting function  $f$  is analytical and everywhere there is*

$$f'(z) = \lim_{n \rightarrow \infty} f_n'(z).$$

That  $f_n(z)$  converges everywhere is involved in the convergence

of  $\frac{f_n(z) - f_n(z_0)}{z - z_0}$ . For  $x = y = z$   $f_n^*(x, y)$  coincides with  $f'_n(z)$ , hence  $f'_n(z)$  converges also. If  $C$  denotes a circle situated within  $|z| < a$ , with centre  $z_0$ , then  $f_n^*(z_0, z)$  converges quasi-uniformly within  $C$ . This function is continuous in  $C$  and for  $z = z_0$  coincides with  $f'_n(z)$ . In consequence of the quasi-uniform convergence the limiting function is also continuous within  $C$ . This limiting function is  $\frac{f(z) - f(z_0)}{z - z_0}$

for  $z \neq z_0$ .

Hence

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{n \rightarrow \infty} f'_n(z_0),$$

whence it follows that  $f'(z_0)$  does exist and  $f'(z_0) = \lim_{n \rightarrow \infty} f'_n(z_0)$ . Since  $z_0$  may be chosen arbitrarily within  $|z| < a$  the theorem is hereby established.

9. If the sequence of functions  $f_1(z), f_2(z), \dots$ , each of which is analytical for  $|z| < a$ , in this region converges everywhere to an analytical function  $f(z)$  and if besides for any  $z$  the following relation holds

$$f'(z) = \lim_{n \rightarrow \infty} f'_n(z),$$

then the function  $f_n^*(x, y) = \frac{f_n(x) - f_n(y)}{x - y}$  converges quasi-uniformly

at the points of every closed set internal to  $|x| < a, |y| < a$ .

In consequence of  $f_n$  being convergent  $f_n^*(x, y)$  converges to  $\frac{f(x) - f(y)}{x - y}$  for  $x \neq y$  and to  $f'(z)$  for  $x = y = z$ , since there  $f_n^*(x, y) = f'_n(z)$ . The function  $f_n^*(x, y)$ , which is everywhere continuous therefore converges to a function which, since  $f$  is analytical, is also everywhere continuous. This involves the quasi-uniform convergence of  $f_n^*(x, y) = f'_n(z)$  at the points of every closed set within  $|x| < a, |y| < a$ .

10. The theorems of § 8 and 9 may be resumed concisely as follows:

A necessary and sufficient condition that it should be permitted to differentiate termwise a convergent series of analytical functions is the quasi-uniform convergence of the series  $\sum f_n^*(x, y)$ .

11. If within  $|z| < a$  a convergent series of analytical functions

$f_n(z)$  is given, such that the limiting function  $f(z)$  has a finite derivative at 0, and if moreover the series  $f_1^*(0, z) + f_2^*(0, z) + \dots$  converges quasi-uniformly at  $0 < |z| \leq b < a$ , then it is possible to assemble the terms of the series in groups in such a way that the new series may be differentiated termwise at 0.

If we put  $\sum_1^n f_n(z) = S_n(z)$  and  $\sum_1^n f_n^*(0, z) = S_n^*(0, z)$ , then  $S_n^*(0, z)$  is continuous for  $0 \leq |z| \leq b$  and  $S_n^*(0, 0) = S_n'(0)$ .

Besides, for  $|z| > 0$  we have

$$\lim_{n \rightarrow \infty} S_n^*(0, z) = \frac{f(z) - f(0)}{z} = f^*(0, z).$$

By hypothesis this function tends to a finite limit  $f'(0)$  as  $z \rightarrow 0$ .

Now let an arbitrary number  $\epsilon_1 > 0$  be given. We construct a set of points  $z_1, z_2, \dots$  where  $\lim_{k \rightarrow \infty} z_k = 0$ . From a limited number of indices an index can be chosen at each of these points such that

$$|S_{n_k}^*(0, z_k) - f^*(0, z_k)| < \epsilon_1$$

Hence an infinite number of points  $z_k$  exists where one and the same index can be used which we denote by  $n_1$ . As  $z_k$  tends to zero  $S_{n_1}^*(0, z_k)$  tends to  $S'_{n_1}(0)$  and  $f^*(0, z_k)$  to  $f'(0)$ . Thence it follows that  $|S'_{n_1}(0) - f'(0)| \leq \epsilon_1$ . Let  $\epsilon_1, \epsilon_2, \dots$  be a decreasing sequence of positive numbers, having zero as limit. It is again possible to choose for every  $z_k$  out of a limited number of indices an index  $n_k > n_1$  such that  $|S_{n_k}^*(0, z_k) - f^*(0, z_k)| < \epsilon_2$ , whence may be concluded, as before, to the existence of an index  $n_2 > n_1$  such that  $|S'_{n_2}(0) - f'(0)| \leq \epsilon_2$ .

Thus pursuing we find that there is a partial series of functions:  $S'_{n_1}(z), S'_{n_2}(z), \dots$ , such that  $\lim_{p \rightarrow \infty} S'_{n_p}(0) = f'(0)$ , which establishes the theorem <sup>1)</sup>.

12. The theorem of the foregoing § may be reversed as follows:

If within  $|z| < a$  a convergent sequence of analytical functions  $f_n(z)$  is given such that the limiting function  $f(z)$  is continuous for  $z \neq 0$  and has a finite derivative  $f'(0)$  at 0, and if moreover a partial sequence  $f_{n_1}(z), f_{n_2}(z), \dots$  can be found where  $\lim_{p \rightarrow \infty} f'_{n_p}(0) = f'(0)$ , then the sequence of functions  $f_n^*(0, z)$  converges quasi-uniformly for  $0 < |z| \leq b$ , where  $b$  is an arbitrary number  $< a$ .

We denote by  $\epsilon$  and  $N$  two given positive numbers. Then a number  $n_p > N$  can be determined such that

$$|f'(0) - f'_{n_p}(0)| < \epsilon.$$

<sup>1)</sup> Verh. Ac. v. W., vol. 27 (1919), p. 1102.

Since  $\lim_{z=0} f^*(0, z) = f'(0)$  and  $\lim_{z=0} f_{n,p}^{**}(0, z) = f_{n,p}'(0)$ , it is possible to determine a number  $\delta$  such that

$$|f^*(0, z) - f_{n,p}^{**}(0, z)| < \varepsilon \quad \text{for } |z| < \delta.$$

Since at the points of the annular region  $\delta \leq |z| \leq b$  the function  $f_{n,p}^{**}(0, z)$  converges to the continuous function  $f^*(0, z)$ , this convergence is quasi-uniform there *i. e.*: from a finite number of indices  $> N$  at every point  $z$  of the annular region an index  $n_z$  may be chosen which satisfies

$$|f^*(0, z) - f_{n_z}^{**}(0, z)| < \varepsilon.$$

At every point  $0 < |z| \leq b$  therefore from a finite number of indices  $> N$  such a choice can be made, which establishes the theorem.

13. Some of the results here obtained we shall apply to MONTEL'S example, which was cited in the introduction and is reproduced below:

The function  $\varphi_n(z) = n^2 z e^{-n^2 z^2}$  tends to zero as  $n \rightarrow \infty$  for all real values of  $z$ . Now consider the three rectangles  $I_n \left( -n \leq x \leq n, -\frac{1}{2n} \leq y \leq \frac{1}{2n} \right)$ ,  $II_n \left( -n \leq x \leq n, \frac{1}{n} \leq y \leq n \right)$  and  $III_n \left( -n \leq x \leq n, -n \leq y \leq -\frac{1}{n} \right)$ .

There exists a polynomial  $P_n(z)$  which within  $I_n$  differs less than  $\frac{1}{4n}$  from  $\varphi_n(z)$  and within  $II_n$  and  $III_n$  less than  $\frac{1}{n}$  from zero.

Evidently the function  $P_n(z)$  tends to zero for  $n \rightarrow \infty$  throughout the whole plane. According to the theorem established in § 5 the function  $\psi_n(x, y, z)$  must converge quasi-uniformly at the points of every set  $V: P_1(x_1, y_1, 0), P_2(x_2, y_2, 0), \dots$  where  $x_k \neq 0, y_k \neq 0, \lim_{k \rightarrow \infty} x_k = 0$  and  $\lim_{k \rightarrow \infty} y_k = 0$ . That this is really the case may be proved as follows: Choose an arbitrary pair of numbers  $\varepsilon, N$ . Let

$N_0$  be  $> N$  and at the same time  $> \frac{1}{2\varepsilon}$ . From a certain index  $k_0$  onwards  $x_k$  and  $y_k$  are both internal to  $I_{N_0}$ , so that

$$\left| \psi_{N_0}(x_k, y_k, 0) - N_0^2 (e^{-N_0^2 x_k^2} - e^{-N_0^2 y_k^2}) \right| < \frac{1}{4N_0} < \frac{\varepsilon}{2}$$

Hence a number  $\delta$  exists such that for  $|x_k| \leq \delta$  and  $|y_k| \leq \delta$  we have

$$|\psi_{N_0}(x_k, y_k, 0)| < \varepsilon$$

This, we remind, is the case from a certain index  $k_1$  onwards.

Besides, a number  $N_1 > N$  can be found satisfying  $N_1 > \frac{4M}{m^2\varepsilon}$ ,

where  $M$  denotes the maximum of  $|x_k|$  and  $|y_k|$ ,  $m$  the minimum for  $k = 1, 2, \dots, k_1 - 1$ ) and such that at the same time  $\frac{1}{N_1} < |x_k| < N_1$  and  $\frac{1}{N_1} < |y_k| < N_1$  for  $k = 1, 2, \dots, k_1 - 1$ . The points  $P_1, P_2, \dots, P_{k_1-1}$  then are contained in  $II_{N_1} + III_{N_1}$ , and we have

$$\left| \psi_{N_1}(x_k, y_k, 0) \right| < \frac{4M}{N_1 m^2} < \epsilon.$$

Since now at every point of the plane  $\lim \psi_n = 0$ , at each point of  $V$  one of the two indices  $N_0$  and  $N_1$  satisfies the relation  $|\psi_n - \lim_{n=\infty} \psi_n| < \epsilon$ , which proves the quasi-uniform convergence. Besides at 0 the derivative of the limiting function is zero, and  $\lim_{n=\infty} P'_n(0) = +\infty$ <sup>1)</sup>.

It follows from § 8 that the function  $P_n^*(x, y)$  cannot be quasi-uniformly convergent at the points of every set  $V$ . For  $V$  we choose:  $x = 0, 0 \leq y \leq a$ . Since the continuous function  $P_n^*(x, y)$  for  $y \neq 0$  converges to zero and for  $y = 0$  to  $+\infty$ , the convergence is not quasi-uniform.

14. *If in a region  $G$  a series of analytical functions converges to an analytical function and if, besides, the series of derived functions converges quasi-uniformly at the points of every closed set contained in  $G$ , then it is permitted to differentiate the series termwise everywhere within  $G$ .*

In the first place we observe that every region within  $G$  contains another region where the convergence is uniform<sup>2)</sup>, so that the termwise differentiation is permitted in this last region.

Since  $f'_n(z)$  converges to a function which is continuous within  $G$  and coincides on an everywhere dense set of points with  $f'(z)$ , we have everywhere  $f'(z) = \lim_{n=\infty} f'_n(z)$ .

<sup>1)</sup> For any  $n$ , indeed,

$$|P'_n(0) - \varphi'_n(0)| = \frac{1}{2\pi} \left| \int_{I_n} \frac{\{P_n(t) - \varphi_n(t)\} dt}{t^2} \right| < \frac{4n^2 + 2}{2\pi} < \frac{2n^2 + 1}{3}$$

Since  $\varphi'_n(0) = n^2$ , we have  $|P'_n(0)| > \frac{n^2 - 1}{3}$ .

<sup>2)</sup> P. MONTEL. Thèse, p. 83.

**Physiology.** — “*On Spray-Electricity and Waterfall-Electricity*”.

By Prof. H. ZWAARDEMAKER and Dr. F. HOGEWIND.

(Communicated in the meeting of October 25, 1919).

The generation of Spray-electricity and that of Waterfall-electricity are no doubt cognate processes; still they are by no means identical.

It may perhaps be useful, therefore, that we should here enlarge upon their congruency and their difference.

Spray-electricity is generated when the air causes waterdrops to break up and diffuse; waterfall-electricity is evolved when existing waterdrops strike against a boundary plane of air-liquid or air-solid substance. This induces electrification of the spray-nebula at the very spot where the cloud arises, whereas the electrical charge of the waterfall does not take its beginning before the water reaches the bottom. In either case small and large drops are formed with opposite charges. Both with spray-electricity and with waterfall-electricity the surrounding air is laden to a large distance with those diminutive droplets, driven off in all directions.

With spraying the large drops follow their primitive course till they strike on some impediment or other. These drops have become electrified long before they encounter this impediment. In the case of waterfall-electricity, however, large drops as well as small ones form at the very moment when the electric charge begins, i.e. the moment when the jet collides with the impediment.

In either case the conditions of

“pressure”  
and “temperature”

largely reinforce the electrical effect. An overpressure of two atmospheres yields notably more electricity than one atmosphere. To obtain a considerable reinforcement of spray-electricity it is only necessary to store up the nebula in a space, whose temperature is 10° higher. Likewise waterfall-electricity will be considerably increased by heating the reservoir from which the waterfall proceeds. The presence of an electric field will augment either in a marked degree.

The distance at which the disc, doing duty for an impediment, is arranged, has an influence upon either.

In the case of spray-electricity there is an optimal and a critical distance. According to LENARD<sup>1)</sup> waterfall-electricity augments with the distance of the disc.

However, the liquid medium and the small chemical additions are of consequence. In most experiments water was the medium, but paraffinum liquidum will also generate spray-electricity, while waterfall-electricity can be obtained also with mercury. We have chiefly noted the influence of small additions of known chemical nature.

Spray-electricity is markedly raised by the addition of substances that lower the surface-tension and are moreover volatile (odorous matter, antipyretica, narcotica, alkaloids). It may thereby rise to an amount which waterfall-electricity cannot approach by far.

*Perfectly pure water does not yield spray-electricity that is distinguishable with an ordinary electrocope.<sup>2)</sup>*

*Traces of odorous matter are capable of rendering it excessive.*

*Spray-electricity, therefore is a means to detect the presence of small amounts of odorous matter otherwise than by the sense of smell.*

Pure water of itself generates waterfall-electricity, as the water becomes positively electric at the moment of its collision.

*Minute additions of odorous matter, gustatory substances, colloidal substances, modify the charge of the water considerably, now in a positive, now in a negative sense.*

LENARD was the first who studied this problem in 1892, and immediately put forward an interpretation<sup>3)</sup>, which he altered a little and extended in a subsequent publication.<sup>4)</sup>

In our experiments we made use of LENARD's apparatus, with slight alterations.

A strong metal cylinder of  $\pm 2\frac{1}{2}$  liter capacity, with the lid attached to it hermetically by three screws, is provided at the bottom with a metal pipe with a tap. The pipe terminates in a glass tube with a fine outlet (1 mm. in diameter). In the lid there are two apertures, one of which is connected with the supply-pipe of the compressor with a gas-chamber of 2 m.<sup>3</sup> capacity, while the other which is closed with a screw-stopper, subserves the filling of the cylinder.

<sup>1)</sup> P. LENARD: "Ueber die Elektrizität der Wasserfälle". Wied. Ann. 46 p. 584. 1892

<sup>2)</sup> Unless in the presence of an electric field.

<sup>3)</sup> P. LENARD: "Ueber die Elektrizität der Wasserfälle". Wied. Ann. 46—1892.

<sup>4)</sup> Id. id. "Ueber Wasserfallelektrizität". Ann. der Physik. Bd. 47—1915.

Inside the supply-pipe of the gas-chamber an amber tube of  $\pm 2\frac{1}{2}$ , cm. is fitted for isolation, besides two taps: the one close to the cylinder, the other near the main-pipe leading to the gas-chamber.

The whole cylinder is suspended in a trivet from which it is insulated by amber pins.

The outlet is placed at about 1 m. over a large glass receiving-reservoir, in which a zinc plate rests on two wooden blocks. This tank is connected by a conducting-wire with the metal cylinder.

In its turn the reservoir is isolated from the environment by a paraffin-plate supported on four amber feet.

The whole apparatus is connected by a conducting-wire to an earthed electroscope.

Thus the cylinder, the receiving-reservoir and the electroscope are connected inter se by an electric circuit; they are at the same time insulated from the environment.

The pressure in the air-pump and the gas-chamber, registered by a manometer, is brought up to two atmospheres, the cylinder is filled with 1 Liter of the liquid to be examined, and the two taps in the supply-pipe are opened, so that the liquid in the cylinder is subjected to a pressure of 2 atmospheres. Now when the lower tap is turned on, the fluid flows under a high pressure out of the glass tube and strikes at an angle of  $90^\circ$  against the zinc plate below it. This produces positive or negative electricity according to the nature of the liquid and causes a deflection of the electroscope.

The deflection, registered by the electroscope after 1 minute's perfusion, is taken as the index for waterfall-electricity.

The entire apparatus being of a rather large capacity the electroscope takes some time before deflecting, which does not occur before the whole capacity is electrified. This takes more time with some liquids than with others.

For this reason the stopwatch is not put in operation before the electroscope begins to deflect and the liquid then continues flowing for a full minute after this.

During the experiment the room is well aired, because the air in the room is also charged and that in a sense opposite to the charge of the liquid. It is obvious that this would greatly interfere with the electrification of the liquid in the subsequent experiments.

Moreover, the receiving reservoir is covered with a close-mesh iron gauze, provided with a circular opening in the middle, through which the jet passes. This gauze serves to keep back the migrating droplets and possible foam, and to allow the extremely hazy nebula to spread in the surrounding air.

Our standard-liquid was pure tapwater, which yielded a mean deflection of 50 scale-marks. The Utrecht tapwater is very pure and contains few salts. In addition the temperature is pretty constant, which is of great importance, since temperature very much influences water-electricity.

A control-test with water was inserted between every set of two experiments, in order to ascertain the accuracy of the apparatus.

Furthermore the cylinder was washed out with tapwater after each experiment to remove small quanta of lingering electrifying substances, which might render the results of the following experiments less reliable.

In the case of water-electricity we found that odorous substances did not all act in the same way. Most of them reinforced the positive charge of the water; others hardly modified it or did not do so at all; a few again even weakened it so as to excite a negative charge. All this occurred seemingly without any special method. It is true, stronger concentrations (which are insignificant with the almost always slightly soluble odorous substances) generally give a greater increase or decrease. Besides, in the homologous series we found an augmentation of the deflection according as we passed from the lower to the higher terms.

For the present it seems utterly impossible to draw a hard-and-fast line, separating the reinforcing from the weakening odorous substances.

That waterfall-electricity is not identical with spray-electricity, may appear e. g. from the behaviour of indol, which markedly increases the charge in the former, but is almost inoperative with the second.

Conversely thymol e. g. gives a strong nebula-charge and hardly any waterfall-electricity.

Another instance is that of *fresh*-distilled water, which very distinctly intensified waterfall-electricity, whereas it remains inactive in spraying.

We subjoin a list of some odorous substances with their charges in the numerator of the fraction and the deflection of tapwater in the denominator, as we noted them down directly after the reading. The sign of the charge is also given:

Phenol  $\frac{1}{1000}$  N. sol. :  $+\frac{100}{80}$

Cressol  $\frac{1}{1000}$  N. sol. :  $+\frac{110}{80}$

Xylenol  $\frac{1}{1000}$  N. sol. :  $+\frac{120}{80}$

Amylacetate $1/1000$ N. sol.:	$+ 120/35$	1 % aethylalcohol.	$+ 100/50$
Thymol sol. (sat.)	$+ 40/85$	5 % „	$+ 90/50$
Toluol (dil. sol.)	$+ 100/50$	10 % „	$+ 20/50$
Artificial Moschus (dil.)	$+ 66/50$	20 % „	$- 20/50$
1 % Amylalcohol	$+ 100/50$		
Indol. sol. (dil.)	$+ 100/50$	Bornylacetate (sat.)	$+ 40/50$
Encalyptol (sat.)	$+ 120/50$	Camphor sol. ( „ )	$- 20/50$
Safrol (sat.)	$+ 100/50$	Capronic acid sol.	$+ 10/50$
Citrol (sat.)	$+ 20/50$	Acetic acid $1/8$ % sol.	$+ 20/50$
		„ 1 % „	$- 10/50$

From this it appears that most odorous substances increase the positive charge of the water. Only bornylacetate and camphor two substances closely allied as to smell, lower, resp. reverse, the charge.

The odorous substances that belong to the acids, lessen the charge of the water, just as all other acids do.

Finally very strong concentrations of odorous substances, such as we can procure with aethylalcohol, will lessen the charge or will even produce a negative charge, just as is the case with spray-electricity.

When passing on to *gustatory substances* we found, that on the one hand all sweet substances are more or less reinforcing, though the concentration must be stronger than in the case of odorous substances, and that, on the other hand, all salts and acids lessen water-electricity, while such bitter substances as belong to the electrolytes, also lessened it.

Bitter substances, however, that must be classed under the colloidal substances again raised the positive charge of water, just as all the other colloidal substances examined.

#### *Sweet substances.*

Saccharose	$1 \text{ ‰} : + 70/35$
Laevulose	$1 \text{ ‰} : + 100/50$
Glucose	$1/4 \text{ ‰} : + 85/50$
„	$1 \text{ ‰} : + 100/50$
Sacch. lactis	$1 \text{ ‰} : + 120/50$
Glucocoll	$1/4 \text{ ‰} : + 120/50$
Glycerin	$1 \text{ ‰} : + 100/50$

#### *Bitter substances.*

Chinin pur (sat. sol.)	$- 20/50$
Bisulj. chin.	$1 \text{ ‰} - 20/65$
Chloret. magnes	$1 \text{ ‰} - 15/50$
Sulfas natrius	$1 \text{ ‰} - 15/50$
Chloret. plumb.	$1/2 \text{ ‰} - 5/50$

#### *Colloidal Bitter substances.*

Fel Tauri	$1/1000 : + 100/50$
Extr. quassiae sicc.	$1/800 : + 100/75$
Glucocol. acid natr.	$1/1000 : + 80/50$

<i>Salts.</i>		<i>Acids.</i>
Sulfas natric.	$1 \frac{0}{0} : -1^{\frac{6}{50}}$	Ac. lactic $1 \frac{0}{8} \frac{0}{0} : +2^{\frac{0}{50}}$
Sulf. kalic.	$1 \frac{0}{0} : -2^{\frac{6}{50}}$	„ acet. $1 \frac{0}{8} \frac{0}{0} : +3^{\frac{0}{50}}$
Sulf. ammon.	$1 \frac{0}{0} : -3^{\frac{0}{50}}$	„ „ $1 \frac{0}{0} : -1^{\frac{0}{50}}$
Chloret magn.	$1 \frac{0}{0} : -1^{\frac{6}{50}}$	„ hydrochl. $1 \frac{0}{8} \frac{0}{0} : -2^{\frac{0}{50}}$
„ natr.	$1 \frac{0}{0} : -1^{\frac{2}{50}}$	„ citric. $1 \frac{0}{0} : -2^{\frac{0}{50}}$
„ kal.	$1 \frac{0}{0} : -3^{\frac{0}{50}}$	„ amygdal. $1 \frac{0}{2} \frac{0}{0} : -4^{\frac{0}{50}}$
„ ammon.	$1 \frac{0}{0} : -1^{\frac{1}{50}}$	„ tartaric. $1 \frac{0}{2} \frac{0}{0} : -7^{\frac{0}{100}}$
„ plumbic.	$1 \frac{0}{0} : -6^{\frac{0}{50}}$	
„ calcic.	$1 \frac{0}{0} : -4^{\frac{0}{50}}$	
Nitras kalic.	$1 \frac{0}{0} : -6^{\frac{0}{50}}$	

*Bases.*

Sol. NaOH	$1 \frac{0}{0} = -3^{\frac{0}{50}}$
„ KOH	$1 \frac{0}{0} : -4^{\frac{0}{50}}$
„ NH <sub>4</sub> OH	$1 \frac{0}{4} \frac{0}{0} : +2^{\frac{0}{50}}$

Here again we notice the lyotrope series, as with spray-electricity, at all events for the anions series:

Sulfas	Kalic.	$1 \frac{0}{2}$	N.:	-30
Phosph.	„	$1 \frac{0}{3}$	N.:	-30
Citras	„	$1 \frac{0}{2}$	N.:	-25
Chloret.	„	$1 \frac{0}{2}$	N.:	-22
Nitras	„	$1 \frac{0}{3}$	N.:	-15
Acetas	„	$1 \frac{0}{2}$	N.:	-13
Rhodan.	„	$1 \frac{0}{2}$	N.:	-12
Iodet.	„	$1 \frac{0}{2}$	N.:	-10

In this list only acetate occupies too low a place, as compared with the lyotrope series in the case of spray-electricity. Of the other substances that proved to be active in spraying, we investigated some glucosides and saponins, antipyretica and alkaloids, which, if not examined as a salt, invariably heightened the charge.

*Glucosides and Saponins.*

Aesculin	$1 \frac{0}{800} : +7^{\frac{6}{50}}$
Saponin	$1 \frac{0}{800} : +7^{\frac{0}{50}}$
Digitalin	sol.: $+6^{\frac{6}{50}}$

*Alkaloids.*

Caffein (dil.)	$+1^{\frac{00}{50}}$
Theophylin (dil.)	$+6^{\frac{0}{50}}$
Hydrochl. morphini	$+5^{\frac{0}{20}}$

*Antipyretica.*

Pyramidon	$+9^{\frac{0}{50}}$
Antipyrin	$+1^{\frac{00}{50}}$

Lastly, all colloidal substances examined, appeared to increase waterfall-electricity, even an albumin-solution, which always contains salts, provided the solution be dialysed.

*Colloidal substances.*

Gelatin $\frac{1}{6}$ ‰ sol.:	+ $\frac{70}{50}$	Extr. quassica $\frac{1}{800}$ :	+ $\frac{100}{75}$
Tragacanth (very dil.):	+ $\frac{65}{50}$	Glucoholz. natr. $\frac{1}{1000}$	+ $\frac{80}{50}$
Traganth (dil. sol.):	+ $\frac{70}{50}$	Albumin ov. sicc. $\frac{1}{4}$ ‰	+ $\frac{70}{30}$
Amyl. oryzae $\frac{1}{3}$ ‰:	+ $\frac{75}{50}$	„ „ dialysed	+ $\frac{25}{30}$
Dextrin $\frac{1}{5}$ ‰:	+ $\frac{75}{50}$		
Gummi arabic $\frac{1}{600}$ :	+ $\frac{40}{25}$		
Fel Tauri $\frac{1}{1000}$ :	+ $\frac{100}{40}$		

It goes without saying that with stronger concentrations the deflection diminishes in consequence of an increased viscosity.

As said, the influence of the temperature on waterfall-electricity is great.

Tapwater of  $8^{\circ}$  : 40

„ „  $35^{\circ}$  : 100.

Rise of temperature, therefore, increases the positive charge of water exceedingly.

The influence of addition of a salt-solution is also a fact that cannot be denied. Alcohol, which, without salt, increases the positive charge in water considerably, partly loses this faculty at first in the presence of common salt, and even loses it altogether with a concentrated salt-solution. Then the negative charge of the salt predominates. Salt added to the negatively electrifying camphor heightens this negative charge. Camphor and salt co-operate. It should seem then that in the case of waterfall-electricity a simple summation takes place of the effects of water, salt and the volatile addition<sup>1)</sup>.

In endeavouring to account for these phenomena we might look upon spray-electricity as well as upon waterfall-electricity as a form of frictional electricity. In either case the friction, between the liquid and the air in the outflow of the spray, between liquid and zinc plate in the waterfall, would set free electrons that are scattered about in the surrounding air. But then the liquid were invariably to be charged positively, which turns out differently in a majority of cases, as shown by the experiments. We presume, therefore, that a more intricate process is at work, in which larger corpuscles arise as carriers of the electric charge. Such formations might perhaps arise from the so-called ions, an equal number<sup>2)</sup> of positively- and negatively-charged ions, round

<sup>1)</sup> In the case of spray-electricity the process is a much more complicate one. (See E. L. BACKMAN, *Researches Physiol. Lab. Utrecht* (5) XIX p. 210.

<sup>2)</sup> H. ZWAARDEMAKER. "Le phénomène de la charge des brouillards de substances odorantes. *Arch. Neerl. Physiol. de l'homme et des animaux*" Tome I 1917 p. 347.

which the water vapour may condense into droplets and with which salt-droplets may combine afterwards.

LENARD<sup>1)</sup> believes that *in* the superficial layers of every di-electric liquid there is not only an electric double-layer, generated by the molecular forces of the liquid itself, the negative layer being situated on the outside, but also that these layers differ as to material.

These differences, which vary with the substances dissolved in the liquid (electrolytes, volatile substances, complex molecules) affect the thickness and the strength of the electric double-layer.

It appears then that LENARD reduces the problem of the origin of waterfall-electricity and of spray-electricity to his hypothesis regarding the specific condition of the surface of every di-electric liquid.

Strictly the origin of the electrification would then be, not an emission of electrons, but a discharge of extremely fine droplets, the so-called "carriers", which, varying with the surface condition of the liquid, are either very small and charged negatively, because they take their origin entirely from the outer negatively charged layer of the liquid, or they are somewhat larger and may be positively charged, since the majority of them arise from the interior positive layer of the liquid.

For further particulars we refer to LENARD's article itself.

Suffice it to state that most of our results with waterfall-electricity are sufficiently explained by this theory.

*Not* however the intensifying influence of rise of temperature (LENARD's private opinion, founded on theoretical considerations, was that a lessening influence was to be looked for).

Neither does this theory explain why camphor and bornylacetate diminish waterfall-electricity; no more is the question of the intensifying action of the sweet substances and the colloidal substances settled by it.

The results obtained before in the Utrecht Physiological Laboratory in experiments on spray-electricity<sup>2)</sup> are much less easy to explain with the aid of this theory.

First of all pure tapwater (Utrecht Water Company) and *fresh*-distilled water (old-distilled water *is* active) give no or an inappreciable charge in spraying.

Secondly the intensification of the charge in consequence of addition

1) P. LENARD. Ueber Wasserfallelektricität. Ann. der Physik. Bd. 47—1915.

2) H. ZWAARDEMAKER. Het in overmaat geladen zijn van reukstofhoudende nevels. Verslagen K. A. v. Wetensch. Deel XIX N<sup>o</sup>. 1.

H. ZWAARDEMAKER. Specifieke reukkracht en odoroscopisch ladingsverschijnsel in homologe reeksen. Id. Deel XIX N<sup>o</sup>. 2.

of an active substance, is generally much greater with spraying than with the waterfall.

Thirdly, according to LENARD's theory, salt-solutions yield a distinct negative waterfall electricity; on the other hand they did not give a charge with spraying<sup>1)</sup>.

Finally LENARD's theory proves the bigger positive carriers of a common salt solution to contain sodium; the smaller negative carriers on the contrary consist of pure water molecules. This he demonstrated by bringing the carriers between the plates of a condenser, in which process the bigger positive Na-containing carriers went over to the negatively charged condenser-plate where the presence of sodium could be proved. The smaller negatively charged carriers went over to the positive plate, where no sodium could be found.

A similar experiment with *sprayed* salt-solution<sup>2)</sup> shows that the big, positive carriers as well as the small negative ones contain sodium.

We conclude, therefore, that though there is a correlation between waterfall-electricity and spray-electricity, they are obviously not quite identical.

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H. ZWAARDEMAKER. Reukstofmengsels en hun laadvermogen door nevelectriciteit Id. Deel XXV 30 Sept. 1916.

H. ZWAARDEMAKER. Le sens de l'adsorption des Subst. Volatiles Acta Oto-laryngologica.

H. ZWAARDEMAKER en H. ZEEHUISEN. Over het teeken v. h. ladingverschijnsel en de bij dit verschijnsel waargenomen invloed op de lyotrope reeksen. Verslagen K. A. v. Wetensch. deel XXVII 1918.

E. L. BACKMAN. De olfactologie der methylbenzolreeks. Id. Deel XXV 27 Jan. '17.

E. L. BACKMAN. Ueber die Verstäubungselektricität der Riechstoffe. Arch. f. d. ges. Phys. Bd. 168 S. 351.

C. HUYER. De olfactologie v. aniline en homologen. Diss. "Onderz." (5) Deel XVIII, p. 1, 89.

<sup>1)</sup> Afterwards we succeeded in demonstrating a slight negative charge also with spraying a 1% common salt solution by lessening the capacity of the receiving disc. This charge, however, is not nearly so great as the one evoked in the waterfall and is far inferior to the spray-electricity generated by additions of volatile substances and of substances that activate the surface.

<sup>2)</sup> A. STEFANINI and G. GRADENIGO. Inhalazione di Nebbië Salina Secche. Lucca 1914, p. 22.

**Physiology.** — “*The Action of Atropin on the Intestine depending on its amount of Cholin*”. By Dr. J. W. LE HEUX. (Communicated by Prof. R. MAGNUS).

(Communicated in the meeting of September 27, 1919).

The action of atropin on the intact, isolated small intestine of mammals has been the subject of a considerable amount of experiments, but no explanation, could be found for the various and widely different results achieved by the several authors.<sup>1)</sup>

While MAGNUS<sup>2)</sup> established that the isolated intestine of the cat, in RINGER's solution, is paralysed in large doses of atropin (0,3 %), but mostly reacts on moderate quanta of atropin (0,025 — 0,075 %) with symptoms of stimulation, the pendulum movements, executed by the intestine, getting more regular, especially when the movements of the intestine were previously insignificant, also an inhibiting effect of very small atropin-doses (0,005—0,05 %) was demonstrated by UNGER<sup>3)</sup>. Other researchers also sometimes found this (paralysing) effect, sometimes they did not. In the isolated small intestine of rabbits and dogs KRESS<sup>4)</sup> noted stimulation consequent on moderate amounts of atropin, paralysis through large quanta, whereas others again obtained widely varying results. In most cases small quantities of atropin caused inhibition in the small intestine of rabbits, whereas moderate atropin-doses alternately stimulated and paralysed the gut without any regularity.

According to P. TRENDELENBURG<sup>5)</sup> the intact intestine of the rabbit reacts regularly on small quanta of atropin with paralysis, but on moderate quantities now with symptoms of stimulation, now with paralysis.

According to the same writer<sup>5)</sup> an exception to irregular behaviour

<sup>1)</sup> An extensive survey of the literature is given by G. LILJESTRAND, Pflüger's Archiv, Bd. 175, p. 111, 1919.

<sup>2)</sup> R. MAGNUS, Versuche am überlebenden Dünndarm von Säugetieren. I. Mitt. Pflüger's Archiv. Bd. 108, pag. 1, 1905.

<sup>3)</sup> M. UNGER, Beiträge zur Kenntnis der Wirkungsweise des Atropins und Physostigmins auf den Dünndarm von Katzen. Pflüger's Archiv. Bd. 119, pag. 373, 1907.

<sup>4)</sup> K. KRESS, Wirkungsweise einiger Gifte auf den isolierten Dünndarm von Kaninchen und Hunden. Pflüger's Archiv. Bd. 109, pag. 608, 1905.

<sup>5)</sup> P. TRENDELENBURG, Physiol. u. Pharmacol. Untersuchungen über Dünndarm-peristaltik. SCHMIEDEBERG's Archiv. Bd. 81, pag. 55, 1907.

of the gut of various species of animals towards atropin is afforded by the small intestine of the guinea-pig, which is regularly inhibited by atropin, a question to which we will revert in this paper.

It is evident from this short survey that the behaviour of the intestine of different mammals towards small and moderate quanta of atropin is varying and inconstant. No doubt the researchers who obtained these various results, have been working under incongruous circumstances. As yet no one has succeeded in accounting for their conflicting results.

From experiments by v. LIDTH DE JEUDE<sup>1)</sup> it appeared distinctly that the explanation is not to be looked for in the different composition of the salt-solution, in which the isolated intestine was examined.

LILJESTRAND<sup>2)</sup> showed that the various results could neither be ascribed to the different composition of the atropin-preparations; he is rather inclined to believe that the explanation can be found in the gut itself. The object of the present paper is to verify this conception.

WEILAND<sup>3)</sup> has demonstrated that from the stomach, the small intestine and the large intestine a coctastable substance can be abstracted through extraction with water, which has the property of urging on the movement of the gut, and that this stimulating action can be arrested antagonistically by small quantities of atropin. It afterwards turned out<sup>4)</sup> that this substance consists for the greater part of cholin and that it can be obtained from the small intestine of the rabbit to such a quantity that it must act a prominent part in evolving the automatic intestinal movements.

Now cholin belongs pharmacologically to the group of pilocarpin; the stimulating effect exercised by cholin, just as by pilocarpin, on the gut is antagonised by slight quantities of atropin.

VAN LIDTH DE JEUDE<sup>5)</sup> showed by his experiments that the quanta of atropin required to check the pilocarpin-action upon the gut, are very slight; already a concentration of atropin of 1 to 10—50 million will do. Now I found that the atropin-concentrations, necessary

1) A. P. v. LIDTH DE JEUDE, *Quantitatieve onderzoekingen over het antagonisme van sulfas atropini enz.* Thesis. Utrecht 1916.

2) G. LILJESTRAND, l.c.

3) WEILAND, *Zur Kenntnis der Entstehung der Darmbewegung.* Pflüger's Archiv. Bd. 147, pag. 171, 1912.

4) J. W. LE HEUX, *Choline als Hormon der Darmbewegung.* Pflüger's Archiv. Bd. 173, pag. 8, 1918.

5) A. P. v. LIDTH DE JEUDE, l.c.

to induce a temporary inhibition of the intestinal movements, fall within the same limits.

When combining these facts, the question arises whether perhaps the inhibition of small atropin-doses is to be considered as an antagonism for the cholin present in the intestinal wall.

If this is the case, the inhibition of small quantities of atropin, will not appear when the cholin has been previously removed from the intestinal wall, but it will come forth again after the addition of cholin and subsequent administration of atropin.

Aside from this inhibition of small quantities of atropin, its actual influence is, according to MAGNUS's <sup>1)</sup> experiments, one that stimulates AUERBACHS's plexus. It is only large quantities that paralyse the nerve centra and muscles of the intestinal wall.

The action of these latter quantities we leave out of consideration.

Experimental evidence in support of the above hypothesis may be obtained in the following way:

1<sup>st</sup> A gut, inhibited originally by small quantities of atropin, is to be brought into a condition in which a small quantity of atropin is without effect, through repeated washings, so that cholin is removed from the intestinal wall.

2<sup>nd</sup> The atropin-effect is to reappear in this gut after giving cholin.

3<sup>rd</sup> A gut, which is originally inhibited by moderate quantities of atropin, is to be brought, through repeated washing, into a condition, in which the same quantity of atropin has only a stimu-appeared lating effect.

The experiments made to prove this, were performed with the isolated small intestine of rabbits and guinea-pigs, which, as had before, are provided with rich quantities of cholin and — as may be expected, readily give them off to the environing fluid.

#### *Experiments with the small intestine of the rabbit.*

In the experiments with the small intestine of the rabbit a difficulty arose in that after some days the spontaneous movements diminished with the washing out of the cholin, which made the results less clear.

Turning to account LAQUEUR's <sup>2)</sup> experience that loops of intestine

<sup>1)</sup> R. MAGNUS, Versuche am überlebenden Dünndarm von Säugetieren. V. Mitt. Pflüger's Archiv. Bd. 108, pag. 1, 1905.

<sup>2)</sup> E. LAQUEUR, Over den levensduur van geïsoleerde zoogdier-organen met automatische functie. Verslagen Kon. Akademie v. Wetenschappen te Amsterdam. 24 April 1914, XXII, p. 1318.

kept in horseserum at a low temperature, retain mobility for days, we now proceeded as follows:

A certain number of loops were severed from the fresh small intestine, which had been cautiously cleaned with Tyrode solution; their movements were registered by MAGNUS's method. The vessels containing the loops were filled with 75 c.c. Tyrode solution of 38° C.

When small quantities of atropin (0,002—0,01) were added, the pendulum movements of the loops got invariably smaller.

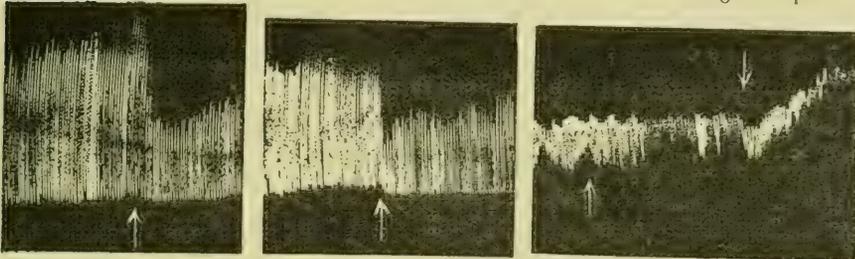
This is illustrated in Fig. 1.

Fig. 1:

Fig. 2.

Fig. 3.

15 mg. Atropin



0.01 Atropin

15 mg. Atropin

0.01 Atropin

Fig. 1.

Pendulum movements of the rabbit's small intestine suspended directly after killing the animal. By administering 0,01 mgr. of atropin the magnitude of the movement is reduced by half.

Fig. 2.

The loop whose movements are registered here is the same as in fig. 1. The previous action of the atropin (0.01 mgr) is entirely eliminated through washing three times with Tyrode solution. 15 mgr. of atropin now yields a strong inhibition on the movements while the tonus is only slightly lessened.

Fig. 3.

A loop of the same gut, standing for 3 × 24 hrs. in the refrigerator in horseserum that was repeatedly refreshed. Subsequently the loop is washed out eight times with warm Tyrode solution and no longer reacts on 0.01 mgr. of atropin. After 15 mgr. of atropin a marked stimulation appears together with a considerable increase of tonus.

After this the loops of intestine were washed out with fresh Tyrode solution three times every 10 minutes, by which, as we know from our own experience, the effect brought about by the preceding small atropin-dosis was again completely eliminated.

When thereupon a moderate dosis (15 mgr.) of atropin was added, the pendulum movements became considerably smaller in far and

away most cases, while at the same time the tonus was more or less lowered.

This is exemplified in Fig. 2. In a few cases the initial inhibition, caused by this moderate quantity of atropin, was followed by a rather strong stimulation.

This occurred, when the loops had already been cleaned *several* times, so that the gut had reached a stage, in which the inhibition of moderate atropin doses passes into a stimulating action. The next day some loops were again severed from the gut which had been kept standing during the night in horse-serum at a low temperature; after having been carefully washed free from the adherent serum they were suspended as before. Every ten minutes these loops were cleaned with fresh Tyrode solution. One of them (we always took the same) was experimented on to ascertain whether 0.01 mgr. of atropin still evolved inhibition on the movements. If it did not, the other loops of intestine were cleaned again some times and examined with regard to their behaviour towards atropin.

In the great majority of cases it appeared again that a small amount of atropin (0.01 mgr.) did not cause the slightest change in movements or tonus, but that after administering 15 mgr. of atropin a stimulation of the intestinal movements together with a large increase of tonus was noticeable. See Fig. 3. Over and beyond all this the primitive inhibition of the atropin could be elicited again in this stage of the experiment, if a small amount of cholin (1—2 mgr.) had previously been added to the loops.

We did not always succeed in reaching this stage already on the second day, so that it proved necessary to keep the gut in the repeatedly refreshed horse-serum some days longer, in order to arrive at a condition in which small doses of atropin do not affect the gut, which again had been cleaned repeatedly with Tyrode solution.

We also succeeded in obtaining this condition by merely cleaning the gut with Tyrode solution, i.e. without the appliance of horse-serum. It is true, though, that, as mentioned before, the movements will become smaller then, and the results less clear. This proves, however, that the results are not influenced by horse-serum.

In the foregoing we have thus shown for the rabbit's small intestine:

- 1<sup>st</sup>. that repeated washing, which, as demonstrated before, deprives the intestinal wall of cholin, evolves a condition in which the initial inhibition of small amounts of atropin, is arrested.

- 2<sup>nd</sup>. that by administering cholin this inhibition of atropin may be elicited again.

3<sup>rd</sup>. that the effect of moderate quantities of atropin, which was variable at first and in my experiments was mostly inhibitory, may be altered in a constant, stimulating effect by repeated washing.

*Experiments with the small intestine of the guinea-pig.*

We now took the small intestine of the guinea-pig as the object of our investigation.

Recently TRENDELENBURG<sup>1)</sup> has suggested an effective method to register graphically the peristaltic movements of the surviving small intestine of the guinea-pig and to determine to a certain extent in numerical values the action exercised on these movements by various poisons.

TRENDELENBURG records that atropin (acting on the small intestine of the guinea-pig) is *invariably* inhibiting peristalsis.

The great thing in our experimentation was to ascertain whether here also, as with the rabbit's small intestine, its behaviour towards atropin is governed by its condition. We used TRENDELENBURG's method and proceeded as follows:

The guinea-pig was killed by a blow on the neck, the small intestine was cautiously severed from the mesentery, and cleaned several times, with a warm fluid<sup>2)</sup> after LOCKE. Subsequently the intestine was cut into 5 parts, one of which was suspended immediately, the other pieces were put in separate dishes with LOCKE's solution, which was refreshed every now and then.

The loop of intestine which was suspended in a vessel of 150 c.c. capacity, was first left to itself with an interior pressure of 0 mm. H<sub>2</sub>O.; then the pressure was gradually heightened and we determined at what pressure peristalsis first appeared (critical pressure). Then the interior pressure was lowered to 0 and after 3 minutes the critical pressure was again determined.

This determination was repeated after 0,1, 1, and 5 mgr. of atropin had been added respectively. In accordance with TRENDELENBURG's report, arrest of peristalsis took place, so that no peristalsis occurred any more even when the pressure was made considerably higher.

After being carefully cleaned, a second loop was suspended, which was kept standing for some hours in LOCKE's solution; the same determinations were made prior to and posterior to the administration of atropin. In most cases it appeared already now that the

<sup>1)</sup> TRENDELENBURG, l.c.

<sup>2)</sup> It is essential that the fluid should be prepared from pure salts and with pure water distilled from glass apparatus.

small atropin-dosis yielded a much weaker inhibition on the peristalsis than with the first loop.

Subsequently another loop was examined, which had been kept standing in LOCKE'S solution some hours longer again, and had been washed a few times more etc.

Table I comprises the results of the complete experiment.

From it we see that the gut, which is arrested by quantities of 0,1, 1, and 5 mgr. of atropin, is brought after a 2½ hours' washing with LOCKE'S solution into a condition in which 0,1 and 1 mgr of atropin produces a *much weaker* inhibition.

TABLE I.

Number of the loop of intestine.	Time of washing.	Critical interior pressure in mm H <sub>2</sub> O.			
		normal.	after 0.01 mgr. of atropin.	after 1 mgr. of atropin.	after 5 mgr. of atropin.
1	0	7 .	Inhibition	Inhibition	Inhibition
2	2.5	20	25	30	Inhibition
3	6	20	16	20	20 pendulum movements.
4	8	12—16	13	—	—
5	11	10	7	—	—

After the process of washing had been prolonged for 6 hours the inhibition of these atropin quantities had entirely stopped, but now peristalsis appears after 0,1 mgr. of atropin already at a lower interior pressure as in the normal period. Also with loops of intestine that have been washed 8 and 11 hours atropin causes distinct stimulation, so that peristalsis comes forth already at a lower interior pressure.

With loop N<sup>o</sup>. 3 a very considerable increase of the pendulum movements was also perceptible after 5 mgr. of atropin.

It has thus been proved also for the small intestine of the guinea-pig that the inhibition of atropin, already established by TRENDELENBURG, can be arrested by washing and that the atropin-action proper, stimulation of AUERBACH'S plexus, can be elicited by small quantities of atropin.

Now it is obvious also why TRENDELENBURG noted only the inhibition of atropin. It was because he did not allow the loops to stand in a solution, but always experimented with fresh ones taken from the guinea-pig under urethannarcosis.

The results obtained by our experiments put us in a position to view the variable action of atropin on the gut in a new light and to interpret the conflicting results of the various researchers.

In the living animal cholin is present in the intestinal wall in such quantities that they stimulate AUERBACH's plexus. On removal, to a certain extent, of the cholin from the surviving gut by a prolonged washing, the real action of atropin manifests itself distinctly. According to the earlier experiments of MAGNUS<sup>1)</sup> it consists in a stimulation of AUERBACH's plexus by moderate quanta, whereas only very large doses paralyse the centra, the nerve, and the muscle. Originally the intestinal movements are not affected by small quantities of atropin.

It is, therefore, upon the presence of more or less cholin in the intestinal wall that the atropin-action depends.

Cholin has a stimulating effect upon AUERBACH's plexus, which is antagonised by atropin. So long as an adequate quantity of cholin is present in the intestinal wall, a small dosis of atropin will inhibit the cholin action and consequently inhibit the intestinal movements.

The result of the action of moderate quantities of atropin will depend on the circumstance whether the immediate stimulating action on the plexus, or the antagonism for cholin preponderates.

In case the gut contains little cholin the stimulating action comes to the front, in case it contains much cholin the antagonism (inhibition) predominates. With a moderate cholin-content a stimulation will succeed an initial inhibition.

Likewise we are now enabled to account for the results of earlier researches.

The fact that with the cat's small intestine the stimulating effect of moderate quanta of atropin occurs more often than with the rabbit's or the guinea-pig's, tallies with our experience that the former contains less cholin than the latter two.

On the other hand it stands to reason that with the guinea-pig gut, which was always found to be rich in cholin, the inhibition of atropin appears regularly.

It is obvious now why the isolated rabbit's gut, according to the previous treatment, is now inhibited by atropin, now again is stimulated, while on the other hand the intact gut, which could not be liberated from cholin by washing, is according to TRENDELENBURG<sup>2)</sup> inhibited regularly.

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<sup>1)</sup> R. MAGNUS, l.c.

<sup>2)</sup> P. TRENDELENBURG, l.c.

*Summary.*

We have demonstrated in an earlier paper that the isolated small intestine yields to salt-solutions quantities of cholin, which are capable of stimulating AUERBACH's plexus.

This loss of cholin results in a changed behaviour of the gut towards atropin.

The rabbit's small intestine that is inhibited previous to washing by small doses of atropin, no longer reacts on them; on the other hand it is now stimulated by moderate doses.

The normal guinea-pig gut is invariably inhibited by atropin. This effect also here disappears after washing and is substituted by a stimulation through moderate quanta of atropin.

This is to be interpreted as follows: the real action of moderate quanta of atropin on the gut is stimulation of AUERBACH's plexus; if the gut contains much cholin, so that the plexus is readily stimulated, this stimulation is arrested through the antagonism of a small amount of atropin, occasionally weakened, and the result is inhibition. Moderate quanta of atropin are also inhibitory when this antagonism is strong enough, but in the presence of small quanta of cholin in the gut the latter will be stimulated.

It is clear, therefore, that here we have to do with a case in which the presence of a well known chemical substance (cholin) in the tissue determines the manner in which this substance reacts on a poison (antropin).

Sept. 1919.

*The Pharmacological Institute of the  
Utrecht University.*

**Zoology.** — “*The interrelations of the species belonging to the genus Saturnia, judged by the colour-pattern of their wings.*”

By Prof. J. F. VAN BEMMELEN.

(Communicated in the meeting of May 3, 1919).

In his handbook of Palaearctic Macrolepidoptera STANFUSS says on p. 106, at the end of his passage on the relative age of the three species *Saturnia spini*, *pavonia* and *pyri*: “If we exclusively paid attention to the imagines, much might be said in favour of the opinion, that *spini* is older than *pavonia*. For both sexes of *spini*, and likewise those of the new species from Kasikoparan (*S. Cephalariae* Ch.) discovered as late as 1882, possess a remarkably uniform type. From this type the male of *S. pavonia* sharply deviates, this species thereby appearing as recently changed in its imaginal dress. But as to the question if we should place the origin of *Sat. pyri* before the evolution of these minor forms or after it, the imago of this latter species does not seem fit to allow of a really certain conclusion.

In regard to the caterpillar- and pupal-stage, however, things are different. For these it can be clearly proved that *S. spini*, *pavonia* and *pyri* form in many instances three different degrees of protective resemblance against certain hostile factors of the outer world. In this scale everywhere *spini* occupies the lowest, *pyri* the highest degree. Keeping in view the excessively near relation and the great similarity of biological conditions between the three species, we are obliged to assume that *spini* came into existence before *pavonia*, and *pavonia* before *pyri*, or, using the scientific expression for these relations: that phylogenetically *spini* is the oldest, *pavonia* a younger, *pyri* the youngest form. For it evidently would be absurd to assume that in a series of so intimately related forms, the more perfect living being should have originated at an earlier date than the less perfect one”.

These considerations of STANFUSS induced me to compare the wing-markings of Saturnine-species, as well among themselves as with those of related genera, to see if this line of investigation did or did not lead to corresponding results as the inspection of the caterpillars.

In accordance with the general rules for the character of the wing-markings, which I thought myself justified in proclaiming, and which I tested as to their applicability to such families as Hepialids, Cossids, Aretids and Sphingids, I came to the conclusion that not *pyri* but *pavonia* should be considered as the oldest form. For in *pavonia* the festooned submarginal transverse lines and bands deviate less from the outer wing-margin and also show a smaller difference between their anterior and posterior extremity, the submarginal band therefore having the simplest and least irregular type. In the same way the difference between fore- and hindwing, as well on their upper- as on their underside — and therefore also between the superior and inferior surface of each of the wings for itself —, is smaller in *pavonia* than in the two other species.

In comparison to other Bombycids and to the remaining families of Heterocera, the colour-pattern of *pavonia* shows a greater similarity to the general primordial pattern of seven dark transversal bars, which I deduced from the comparison of all these forms, than the two other Saturnids. It might seem that this assertion is contradicted by the fact, that in *pavonia* the male at first sight looks entirely different from the female by its colour as well as by its inferior size, a difference which has apparently made a deep impression on STANDFUSS. But on nearer inspection and consideration the difference is by no means so important as it looks, and need not be regarded as of high importance. For the difference in hues is evidently connected with the frequently occurring feature of discoloration (i. e. partial self-colour) by which on the superior surface the hindwing has partially turned into yellow, while at the underside the same hue has spread over the proximal part of the forewing. On both wings this discoloration is accompanied by a slight and incomplete fading of the pattern.

Moreover it may be doubted, on very sound arguments, if the yellow hue — apart from its spreading over the dominion of spots and stripes, which are rendered more or less invisible by it — should be considered as a secondary modification of an older and more original hue, which latter therefore should have persisted on the upperside of the forewing and on the underside of the hind one. For this yellow-brown hue is characteristic for quite a number of Bombycine moths belonging to different genera, and in so far impresses us as a very original colour. It might therefore be assumed, that its occurrence in the male of *pavonia* should be considered as a reversion to an older condition, instead of being the appearance of a new hue.

But we may safely leave these questions unheeded, for the hues in which a pattern is executed, need not be taken into consideration, neither in this case nor in others, when judging the character of the pattern itself.

Now, if we carefully compare the markings of the male with those of the female and pay due attention to the half-vanished stripes and spots under the overspreading yellow, it becomes clear, that down to the minutest details, male and female agree in pattern. No more is there any necessity to consider the sexual dimorphism as an important progressive feature, by which *pavonia* should distinguish itself from *spini* and *pyri*, and be characterised as a recently and strongly modified form, in contrast to the other two, which would have remained more conservative. For secondary and tertiary sexual differences occur in all kinds of Lepidoptera, as well as in other insects. In the Bombycine moths this feature shows itself in a remarkably high number of forms. Now have we really to consider each of these cases as a separate and independent deviation from a common original condition, in which male and female were alike in shape, size, hues and pattern? Or did the phenomenon of sexual dimorphism already show itself amongst primitive Bombycidae, in the days when the difference between them was less considerable than at present, and they still counted fewer specific and varietal forms; dimorphic species therefore then existing side by side with monomorphic as well as now. In this case the sexual dimorphism of *pavonia* might repose on the manifestation or the permanency of an old hereditary disposition.

But even apart from this question, which can hardly be solved with certainty, it remains doubtful, if the species *pyri* and *spini* should really be considered older than *pavonia*, on account of the similarity of their sexes. For I by no means consider it as proved, that in *pavonia* the male, which deviates from the general hue of the genus, may be considered as the modified form, while the female, which seems to show so much more similarity to *spini* and *pyri*, may be regarded as the unchanged form.

When the general rules for the colour-pattern are blindly applied the solution of this question might seem easy enough. In the female of *pavonia* as well as in both sexes of *spini* and *pyri* the fore- and hindwings on their upper- as well as on their underside, show the same clear whitish hue, here and there overspread with a black sprinkling, and subdivided into fragments by nervural and transversal straight or undulating lines. But this

general clear tint, reigning over the whole extension of the wings, as well as the similarity in design between fore- and hindwings, upper- and underside, impresses me as a secondarily acquired uniformity, rather than as a really original feature.

In my opinion we meet here with a similar case as in the wholly selfcoloured butterflies e. g. *Gonepteryx rhamni* or *Aporia crataegi*, in which without the least doubt the uniform hue is the consequence of the simplification of the shades, hand in hand with the total or partial regression of the markings. Still more striking is the similarity with the Parnassine butterflies (which, it need hardly be said, is of course wholly superficial and occasional). For in these as well as in Saturninae a set of highly differentiated eye-spots on the forewings, but still more pronounced on the hindwings, form the most conspicuous part of the pattern, though it may undoubtedly be taken for granted that the spotted design of the Parnassines has developed from a far more complete array of simpler and more uniform markings, such as are seen in *Thais polyxena* and its consorts.

As soon as we consider the similarity of fore- and hindwing, upper- and underside of the female of *pavonia* and of both sexes of *pyri* and *spini* as secondarily acquired characteristics, there need no more be any objection against the supposition that the male of the first-named species is more original than the female.

From this point of view we may further remark, that the existing contrast between the yellow upperside of the hindwing and the greyish one of the forewing, on which the markings are less hidden under the ground-colour, corresponds in a higher degree to the general type of butterfly-design than does the exact similarity of fore- and hindwing; the overwhelming majority of Lepidoptera showing a similar difference between fore- and hindwing.

In the fact that the discoloration (in this case yellowing) occurs on part of the *upperside* of the *hindwing* and of the *underside* of the *forewing*, the male of *pavonia* is in harmony with both sexes of *Smerinthus ocellata* and of many other Lepidoptera. Up to a certain degree the frequent occurrence and similar extension of this discoloration can be attributed to the influence of the resting attitude of these moths, the discoloured areas being exactly those which during this attitude remain covered by similar parts of the adjoining wing. In his paper: "Sur la position de repos des Lépidoptères", J. T. OUDEMANS has directed our attention to this circumstance and expressed his opinion that during the development and modification of the colour-pattern in the course of time, the



Fig. 1. SATURIPAO CREPUSCULARIS ♀.

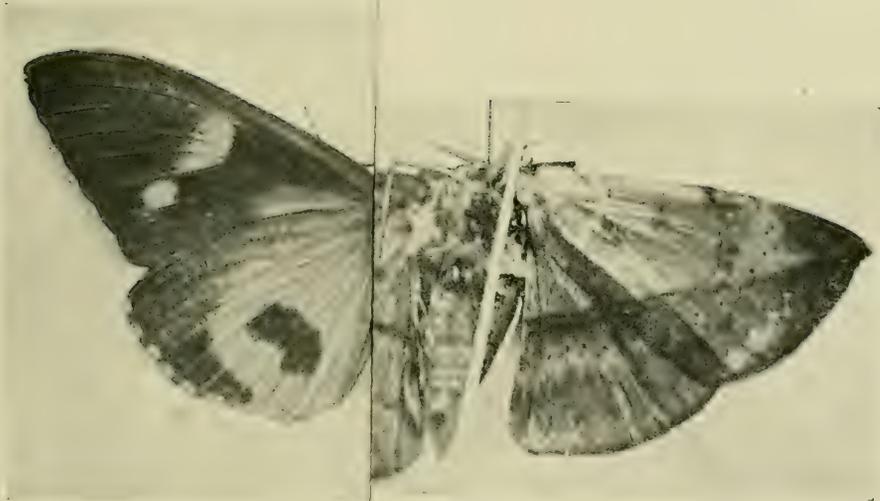


Fig. 2. OPHIMONDIA PUNCTATA ♀.

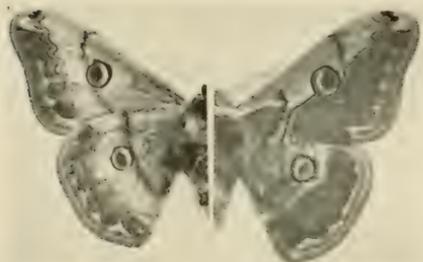


Fig. 1. SATURNIA BOISDUVALII ♂.



Fig. 3. NYCTIPAO CREPUSCULARIS ♀.



Fig. 2. OPHIDERES DIVIDENS ♂.



Fig. 4. EMMONDIA PUNCTATA ♀.

hidden and the exposed parts had independently proceeded each along its own course (p. 81—83), on which the one as well as the other could get on at a higher speed and so reach a stage more remote from the original common condition.

From a general point of view I feel inclined to join this opinion, but in the case at hand it brings us little light. The difference between the covered and the exposed wing-areas in the attitude of rest, already slight in the male of *pavonia*, is quite insignificant in the female, as well as in both sexes of *pyri* and *spini*. The wing-markings on these areas seem not to be influenced to any notable degree by the habit of passing the forewings over the greater part of the hindwings during day-time. That traces of such an influence are still visible in the male of *Sat. pavonia*, might be taken as an indication that the original influence of the said habit is now gradually losing its force.

The highly conspicuous eye-spots in my opinion must have evolved from simpler discoidal marks, hand in hand with the above mentioned change in the influence of the resting attitude. This is already proved by their very different degree of differentiation in the several species of the genus *Saturnia* and of kindred genera. Eye-spots moreover always are special differentiations, secondarily developed on the base of a more primitive and simple colour-pattern, whose elements have occasioned them by modifications in the original shape, colour, size and direction.

In my previous paper, on the wing-markings of Sphingides, I tried to prove this assertion for the case of *Smerinthus ocellata*, in the same way as I formerly did for the Hepialid moth *Zelotypia stacyi*. It must however be possible to prove it as well in other cases, e.g. for *Vanessa io* and many other Nymphalids, and likewise for Satyrids and Lycaenids. *Cynthia* for instance shows in what way eye-spots and simple spots alternate in the row of submarginal markings, and also often how an eye-spot on the superior surface is represented by a common one on the underside. The difference between the two seasonal forms in the first place depends on the contrast in the differentiation of the eye-spots: in the dry-monsoon-generation they are scantily developed, in the wet-dito highly so.

Now, do the hybridisations of *STANDFUSS* throw any light on these questions? As far as I can see not much; in general the hybrids are intermediate forms, but as to the width of the submarginal dark seam, they agree more with *pavonia* than with *spini*, and assuredly far more than with *pyri*.

But in my opinion in order to judge about the interrelations of the colour-patterns of Saturnia, it is absolutely necessary to compare with each other as many different forms as possible, just as in the case of other groups of Lepidoptera. It certainly cannot stand criticism to draw consequences from the exclusive consideration of three intimately connected species as to the relative age of their colour-patterns.

When pursuing this broader way, the well-foundedness of the above-mentioned assertion, viz. that the colour-pattern of Saturnids is a special case of that of Bombycids, and the latter again of that of Heterocera in general, is clearly proved. To begin with: next to the three above mentioned species stands Saturnia (*Caligula* Jordan) *boisduvalii*, on whose upper side the submarginal dark seam broadens from before backwards in still higher degree than in *pyri*, which reduces the somewhat median clearer area under the eye-spot, (broadest in *pavonia*) to almost nothing by forcing it back in proximal direction.

The two dark borders by which this area is limited (and which I suppose to be V and VI) are in one place locally connected by a black transversal link. This part of the pattern of *boisduvalii* therefore shows the greatest similarity with the corresponding area of the pattern of the male *pavonia*, but in the latter the connecting link seems to run between IV and V. Generally speaking, it is not easy accurately to make out the exact consecutive number of the bands for each separate form, yet the comparison of the superior surface of the different species leaves the general impression, that the posterior broadening of the dark submarginal area is brought about by the progressive darkening of the colour in a proximal direction, which consecutively incorporates the dominion of a more proximally situated transversal bar. In the female of *pavonia* this darkening process is restricted to the area of Bar II, in the male it has advanced unto III, in the same way as in both sexes of *spini*, in *pyri* it has reached IV, in *boisduvalii* V.

On the inferior surface the broadening of the dark area proceeds more slowly and more equally over the whole extension of the dark submarginal field. Consequently the underside of *spini* e.g. resembles both the upper- and the underside of *pavonia* to a higher degree than its own superior surface. On the hindwing the backward broadening of the dark submarginal border is less pronounced than on the forewing, and this edge is there separated by a light-coloured band over its whole length, from the dark festooned line, that runs along the distal side of the eye-spot.

On the underside however (which is never represented even in large illustrated works) we again meet with the same feature, as mentioned just now for *pavonia*, *spini* and *pyri*, viz. that the course of the transversal bars is far more regular and original than on the upperside, this bringing about a much greater similarity between fore- and hindwing on the first mentioned surface.

I want to draw your attention to a simple dark transverse stripe on the middle of both wing-pairs, almost devoid of incurvations and rather faint. On the forewings this stripe runs along the distal border of the eye-spot, on the hindwings along the proximal one. Regardless of this difference, I think we have to deal in both cases with vestiges of bar IV. To this assumption I am especially led by the comparison of the upperside. On that of the forewing the anterior part of this bar is quite apparent up to the eye-spot, which constitutes a marked difference between *boisduvalii* and the three first-mentioned species. Past the eye-spot however the posterior part of the bar is wanting, but from the postero-interior border of that spot a black stripe runs across to Bar VI, turning sharply at an angle in the middle of its course and then running parallel to Bar VI, perpendicularly to the posterior wing-border and as far as this latter. The lastnamed part belongs to Bar V, as is proved by its comparison with the upperside of the hindwing, on which the Bars IV, V and VI may be perceived in their full extension, though faint and half-hidden under the hairy coating.

On the underside of *boisduvalii* therefore the dark submarginal border of both wing-pairs is located between Bars II and III; on the upperside however this is only the case on the hindwing. Bar III may even remain separated from the dark seam as an independent isolated stripe, as is shown in the illustrations of SETZ (Vol. II Pl. 31<sup>a</sup>), in contrast with the specimen at my disposition, where the black internal border was not free from the much broader marginal seam.

In some respects *Saturnia (Neoris) Moorei schenckii* corresponds with *boisduvalii*, e. g. in the presence on its underside of an extremely faint yet complete transversal stripe, which takes its course along the eye-spots. In this species however no difference exists as to the situation of this stripe on fore- and on hindwing, while the stripe is also present on the upper-side, though in an incomplete and unequal way.

Bar V is here the most pronounced and regular, VI and VII are hardly visible.

The eye-spots have been removed outward, and in consequence

the dark submarginal seam has been reduced, while Bar II and III cannot be distinguished from each other. They are represented by a single dark double-line, which in its anterior part is very much festooned. At the outer side of this double-line a white bar runs along, followed by a particularly broad margin of a light-manilla brown hue.

The comparison with the female *pavonia* makes it probable, that the dark submarginal edge ought to be situated between the white bar and the dark double-line, and that therefore we may assert that it is absent. Yet this is true only to a certain degree: the broadening of the dark submarginal border is, as we remarked before, a consequence of the advancing of the obscuration over the areas between the succeeding transversal bars in a proximal direction, and this identical process is also seen to take place in *schenckii*, viz. on the fore-wing between double-line II + III and the dark stripe representing IV.

At the underside of *schenckii* the pattern is simple, and moreover pale and reduced.

A.o. the eye-spots on the hindwings are much smaller, paler and less complete than at the upper side.

Amongst the many genera near-akin, that are arranged around the Saturnids, a great number of additional arguments may be found for the above-mentioned assertion, that their colour-pattern may be derived from the same scheme of seven dark transversal bars, which proved applicable to *Arctiids*.

We only need point out forms like *Rhodinia fugax*, probably showing II, III, and V or VI, or *Laepadamartis* (SERTZ II, Pl. 32<sup>d</sup>), which on its forewing wears I, II, III and V or VI, on its hind-wing I, II, III, IV and V (the last two only in part). Even in such a complicated and special pattern as that of *Brahmaeids* it is comparatively easy, with a little attention, to find again the seven primary transversal bars.

However it is not only the ground-plan of the wing-design that may be shown with great probability to be common to numerous and various groups of moths, also the modifications of this plan seem to take place after the same rules among the different families of *Lepidoptera*. In the present case of Saturnine moths it is the broadening of the submarginal dark border in the direction of the hind margin, that constitutes the principal difference in pattern between the various species mentioned and leads to their arrangement in the sequel: *pavonia*, *spini*, *pyri*, *boisduvalii*, while the pattern of *schenckii* seems to be due to a secondary regression of the dark band on its forewings.

This broadening in a caudal direction makes the impression, that an oblique line of separation runs across the forewing from tip to root, dividing it into an antero-interior lighter field and a postero-exterior darker one. Calling this line the  $V$ -diagonal (because the lines of the two wings in expanded attitude form the letter V) we may state, that this  $V$ -diagonal-design occurs in numerous Lepidoptera of various families, though always as a secondary modification of the original pattern of transversal bars. In the group of Chaerocampine Sphingides, which I hope to discuss in a future communication, this feature is particularly striking.

We find it however in the same way in the families of Hepialidae, Noctuidae and Geometridae, and likewise in numerous Microlepidoptera. In the overwhelming majority of these cases the  $V$ -diagonal-pattern is restricted to the upper side of the fore-wing; the underside showing the primitive pattern of transversal bars.

As specially striking examples may be mentioned the Noctuid genera *Ophideres*, *Nyctipao* and *Emmondia*, the Bombycidgenus *Eupterote*, as well as many Geometrids.

The species of *Ophideres* give rise to the remark that the separation of the forewing-area in an antero-internal and a postero-external part, brought about by the  $V$ -diagonal-design, in many species e.g. *O. tyrannus*, *salamia* and *fullonica* (comp. Seitz III, Pl. 66) seems to be connected with the resemblance of the entire forewing-design to a withered leaf, this likeness bringing the said species under the category of the leaf-imitating Lepidoptera.

We have to deal here once more with such a feature, as can show itself in many different shapes, and therefore in its real character is evidently independent of the importance it may in some particular cases possess for the establishment of protective resemblance.

It further results from the mutual comparison of different species, that the contrast between the anterior and the posterior part of the wing-surface may be very different in quality as well as in quantity. In some species the forepart is light, the hindpart dark, in others they are nearly alike. In *O. materna* a transversal design of Cossid-markings (traits effilochés BOTKE, Rieselung EIMER) spreads over both parts; in the male the  $V$ -diagonal is present, in the female absent, while on the contrary an  $A$ -diagonal-design is partly developed as a light-coloured-streak. For besides the  $V$ -diagonal an  $A$ -diagonal can be distinguished, running from before and inside to behind and outside in an oblique direction over the wing-surface, often it seems to possess some connection with the division of the

wing into a beam- and a fan-part (Spreiten- und Faltenteil as SPULER has termed them).

In the species *Miniodes discolor*, near akin to the genus *Ophideres*, the dark *A*-diagonal separates the orange-coloured forewing-area into two parts: both of them well showing the Cossid markings. A remarkable detail in this case is the uniform pink colouring of the hindwing on its upperside, the under one showing just the reverse: self-coloured forewing, Cossid markings on the hind one.

On the upper-side of *Nyctipao crepuscularis* (SEITZ III, Pl. 58) markings in three different directions coöperate to form the pattern: 1. transversal markings, especially the white line running from the anterior to the posterior margin through the centre of fore- and hindwing, but also the dark bar near to the wing-root, representing Bar VI or VII, 2. the *V*-diagonal designs of the forewing, 3. a white stripe, forming an obtuse angle with the *V*-diagonal, and running parallel to the *A*-diagonal. Inside the angle, made by the *V*- and the *A*-diagonals, an eye-spot has been differentiated from elements of the transversal markings.

Still sharper these three directions of wing-markings stand out in dark bars against a creamy-white fond on the fore-wings of the arctiid *Areas galactina*. So numerous are the cases of *V*- and *A*-diagonal-design, that I will not even venture to summarize them and compare them to each other. As a general result however of my comparative investigations, I feel justified to assert, that everywhere the secondary character of this pattern may be stated with surety, and that in the great majority of cases the underside shows no trace of one of these oblique lines, whereas clear vestiges of the seven original transversal bars nearly always occur, and are also frequently present on the upperside, though generally incomplete. In *Rhopalocera* the *V*-diagonal-design, when it occurs, is restricted to the underside, and here serves to establish the leaf-imitating character.

*Groningen*, April 1919.

**Botany.** — “*The Influence of Light on the Cell-increase in the Roots of Allium Cepa*”. By H. W. BERINSOHN. (Communicated by Prof F. A. F. C. WENT).

(Communicated in the meeting of October 25, 1919).

MRS. DROOGLEEVER FORTUYN—VAN LEYDEN<sup>1)</sup> has found that the cells of young cats increase periodically in such a way, that during the night, the number of karyokineses reaches its maximum, and in the later morning hours and the early afternoon a minimum number is reached. KARSTEN<sup>2)</sup> stated likewise that a periodical karyokinesis takes place in the young buds of Zeamais, which reaches its maximum also during the night. However in the roots of Vicia Faba he did not find any periodicity and so he concludes: “Das Wurzelwachstum entbehrt der Periodizität.” During these experiments the plants remained in the dark. Now he tried to influence the periodicity by exposing the young plants to the light of an electric lamp, in which he succeeded. On the other hand he did not trace the influence of light and dark on the cell-increase in the roots. As the root growth is evidently not a periodical one, the influence of light and dark will be most obvious here.

I chose *Allium Cepa* to experiment upon, because the *Allium* cells are easily fixed and stained; because one finds a great number of karyokineses in the roottips and because there are a great many roots, so that it is possible to examine parts of one and the same individual under different circumstances.

At 8 a. m., 11 a. m. and 3 p. m. I took a few roottips away from a germinating onion, which was exposed to full daylight. After that I put the same onion in the dark and left it alone until the next day, then I took a few tips off at 6 a. m., at 12 a. m. and at 6.30 p. m., while the onion remained in the dark. During these two days the temperature differed  $\frac{1}{2}^{\circ}$  C. (registered with a maximum and a minimum thermometer).

I always took care to take roottips shorter than 25 mm. and of about the same length. The roottips were fixed in sublimate

<sup>1)</sup> Mrs. DROOGLEEVER FORTUYN—VAN LEYDEN. Proceedings Konink. Akad. Amst. Vol. 19. 1916, p. 38.

<sup>2)</sup> KARSTEN. Zeitschr. f. Botanik 1915, p. 1.

sodium chloride, after passing the different alcohols, they were enclosed in paraffin and then they were cut into series of  $10\mu$  and the sections were stained according to HEIDENHAIN'S ironhaematoxylin method. I counted the number of nuclei of some of these sections over a length of 1 mm. from the roottop and I fixed the number of mitoses. I took care to count only in the central sections. Table I and II give my results.

TABLE I (in daylight).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
8 a.m.	4000	—	—	—	—	—
11 a.m.	4345	139	53	53	17	16
3 p.m.	2290	47	26	14	5	2

TABLE II (in the dark).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
6 a.m.	4702	210	125	66	19	0
12 a.m.	4204	180	147	21	20	4
6.30 p.m.	4043	124	65	29	10	20

In order to compare these figures, I expressed them in percentages in the following tables. KARSTEN takes the average of his countings. In my opinion it is more exact to express these facts in percentages, just as MRS. DROOGLEVER FORTUYN-VAN LEYDEN does, for it is most improbable that KARSTEN always examined the same number of cells.

TABLE III (in daylight).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
8 a.m.	4000	0.00 %	0.00 %	0.00 %	0.00 %	0.00 %
11 a.m.	4345	3.19 "	1.01 "	1.01 "	0.38 "	0.37 "
3 p.m.	2290	2.05 "	1.13 "	0.61 "	0.21 "	0.08 "

TABLE IV (in the dark).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
6 a.m.	4702	4.46 ‰	3.19 ‰	1.4 ‰	0.4 ‰	0.00 ‰
12 a.m.	4204	4.28 „	3.49 „	0.49 „	0.48 „	0.09 „
6.30 p.m.	4043	3.06 „	1.60 „	0.71 „	0.24 „	0.49 „

As is evident, the number of karyokineses reaches its maximum between 8 a. m. and 11 a. m. (solar time), which agrees with the well-known fact that good cell-divisions in Hyacinth and Allium are found between 10 a. m. and 11 a. m. From 11 a. m. the number of cell-divisions decreases to 2.05 ‰ at 3 p. m.

I found the greatest number of karyokineses with the onion in the dark at 6 a. m., 4.46 ‰. At 12 a. m. this had slightly decreased to 4.28 ‰, and at 6.30 p. m. the decrease was still greater. At 6 a. m. and at 12 a. m. the maximum number of karyokineses in the dark exceeded the maximum number of cell-divisions found in the daylight, while the maximum number of karyokineses in the light surpassed the minimum number of cell-divisions in the dark only by a slight degree; so that the conclusion seems justified: The number of karyokineses in the rootcells of Allium Cepa increases in the dark, which is stated by KARSTEN<sup>1)</sup> for Spirogyra and other plants.

When we compare in the tables III and IV the number of spirema, loose chromosome stages with the number of monaster stages, then we see in the first table from 11 till 3 o'clock an increase of the number of spirema and loose chromosome stages and a decrease of the monaster stages. This would point to an increase in the number of cell-divisions and nevertheless the total number of mitoses has diminished. We see the same phenomenon on table IV from 6 a. m. to 12 a. m.

By considering spirema, loose chromosomes and monaster as one stage (prophase) the number of nuclei in prophase, in table III at 11 a. m. is 2.02 ‰ and at 3 p. m. 1.74 ‰, which points to a decrease. The same can be applied to table IV. At 6 a. m. 4.59 ‰ and at 12 a. m. 3.98 ‰ and at 6.30 p. m. 2.31 ‰ is in prophase, so there is a total decrease. In my opinion this fact is a confirmation of the general conception to consider spirema, loose chromosomes and monaster as one stage.<sup>2)</sup>

<sup>1)</sup> KARSTEN. Zeitschr. f. Botanik 1918.

<sup>2)</sup> PEKELHARING. Weefselleer, p. 67.

From similar facts, as are contained in table III and IV, it seems also possible to me, to conclude something about the rapidity from prophase to anaphase and from anaphase to telophase. Let us consider table III for that purpose. At 11 a. m. 2.02 % were in prophase and 0.38 % in anaphase. The number of karyokinesis figures in prophase has decreased with 13.8 % at 3 p. m. and the number of cell-divisions in anaphase has decreased with 44.7 %, so the decrease is intenser, that is to say, the transition from anaphase to telophase is quicker than the transition from prophase to anaphase. The same holds good for the onion in the dark during the whole day, but during the day an inversion takes place in such a way that from 6 a. m. to 12 a. m. the transition from prophase to anaphase is quicker than from anaphase to telophase.

Of course these facts are too scanty to draw such far-going conclusions, but the aim of this calculation was only to show that it is possible to learn the relative rapidity. If one wants to undertake such experiments it is necessary in the first place to fix the time of observation much shorter, i. e. one hour or one hour and a half. It is also possible to derive the duration of one cell-division from such tables. When we consider table I we do not find karyokineses at 8 a. m., and at 11 a. m. we find 16 nuclei in telophase. So the cell-division would take about 3—4 hours with *Allium Cepa*. JOLLY found with *Triton* the duration of the kariokinesis  $2\frac{1}{2}$  hours in the erythrocytes at a temperature of 20° C.

From the table of MRS. DROOGLEEVER FORTUYN—VAN LEYDEN I think I may conclude the duration of a cell-division being 12 hours with a cat, because at 2 p. m.  $\pm 0.23\%$  nuclei were in prophase and no telophases were stated. Only at 2 a. m. 0.20% nuclei were seen in telophase for the first time.

When we summarize the results, we see that the roottips of the onion show more cell-divisions in the dark than in the light. Evidently light has a retaining influence. Besides it is probable that the transition process from prophase to anaphase is a slower one than the transition process from prophase to telophase.

By lack of time I could not control these facts any further. To attain this, it would be necessary to make an investigation into the daily oscillations in the number of karyokineses with the onion, if possible the time of observation ought to be as long as possible (3 to 4 days). At the same time the above-mentioned experiment ought to be repeated. One onion suffices for these two experiments. The bulb is cut into two halves and one half is used for the first series of experiments and the other half voor the second experiment.

The two series of experiments are made with parts of one and the same individual. Neither MRS. DROOGLEEVER FORTUYN—VAN LEYDEN, nor MR. KARSTEN have done this, so the results lose reliability.

Notwithstanding the incompleteness of my investigation, I thought the facts I found, of sufficient importance to be examined further, and for this reason I published this communication. At the same time I make use of the opportunity of thanking MR. M. W. WOERDEMAN as well for the incitement to this research, as for the kind assistance lent to me.

*Amsterdam.*

*Laboratory of Histology.*

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**Physics.** — “*The Propagation of Light in Moving Transparent Solid Substances. I. Apparatus for the Observation of the FIZEAU-Effect in Solid Substances.*” By Prof. P. ZEEMAN.

(Communicated in the meeting of May 3, 1919).

1. As a result of an experiment by ARAGO with a *glass* prism FRESNEL drew up his bold hypothesis on the convection coefficient in 1818. When in 1851 FIZEAU wanted to put FRESNEL's hypothesis to the test, he experimented, however, with *water*, and examined whether or not the velocity of light in standing water differs from that in moving water.

There are many reasons to be adduced for carrying out an experiment, so exceedingly difficult as that of FIZEAU, in the first place with water; it is, however, also interesting to examine the motion of light in *solid*, transparent, rapidly moving substances. In this connection experiments with rapidly moving quartz and glass have been made by Miss SNETHLAGE and myself. In this communication I will give the description of the apparatus with which these experiments have been made. It may be well to call attention to a few points referring to FIZEAU's experiment with water.

Let  $c$  be the velocity of light in vacuo,  $\mu$  the index of refraction of the water,  $w$  the velocity of the water with respect to the tube in which it moves; then the velocity of propagation of the light with respect to the tube is according to FRESNEL:

$$\frac{c}{\mu} \pm \left(1 - \frac{1}{\mu^2}\right)w \dots \dots \dots (1)$$

In this the upper or the lower sign is to be taken according as the water and the light move in the same or in opposed directions.

In 1895 LORENTZ demonstrated that FRESNEL's convection coefficient in a dispersive medium must be replaced by:

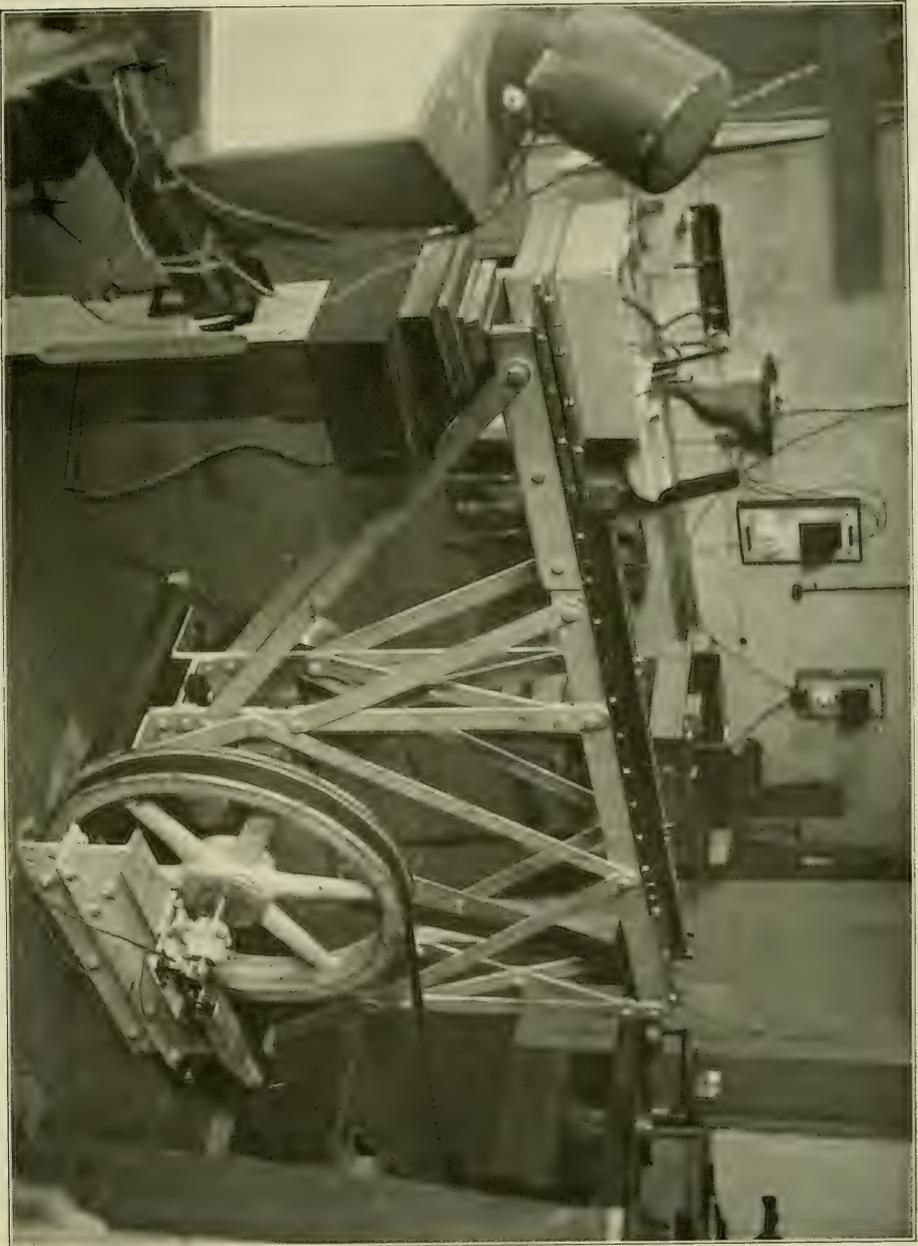
$$1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu} \frac{d\mu}{d\lambda}$$

This changes formula (1) into:

$$\frac{c}{\mu} \pm \left(1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu} \frac{d\mu}{d\lambda}\right)w \dots \dots \dots (2)$$

The experiments made by FIZEAU in 1851, plead in favour of

**P. ZEEMAN: "The Propagation of Light in Moving Transparent Solid Substances. I. Apparatus for the Observation of the Fizeau-Effect in Solid Substances".**





formula (1). MICHELSON and MORLEY's investigation of 1886, performed with MICHELSON's interferometer in one of the numerous forms into which as a real PROTEUS this wonderful instrument is capable of being changed, gave with white light a value of the convection coefficient which was in excellent agreement with the coefficient that follows for yellow sodium light from formula (1).

Experiments that have been carried out by me with different colours ranging from violet to red, and in which the axial velocity of water in the tube was directly measured have been communicated by me in different papers to this Academy <sup>1)</sup>. The validity of the formula (2) with the term of dispersion could be demonstrated with an accuracy exceeding 0.5 %. The optical effect that is measured in these experiments, is a displacement of interference bands, which is given in parts of the distance of two bands by the formula:

$$\frac{4l}{\lambda \cdot c} \left( 1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu} \frac{d\mu}{d\lambda} \right) \mu^2 w, \quad . . . . . (3)$$

in which  $l$  represents the length of the whole liquid column which is in motion.

2. The apparatus that has been used for the investigation of the motion of light in solid substances, is shown diagrammatically in

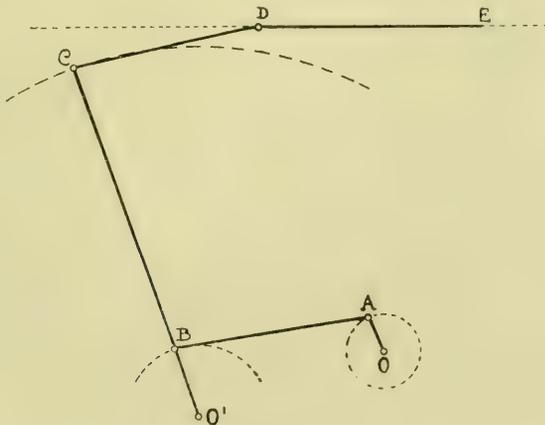


Fig. 1.

fig. 1 on a scale of  $\frac{1}{20}$ , and might also be used with simple modifications for the investigation of costly liquids and compressed gases.

The moving, transparent substance is rigidly connected with a piece  $DE$ , and can therefore rapidly move to and fro parallel to

<sup>1)</sup> ZEEMAN, These Proc. 17, 445, 1914; 18, 398, 1915; 18, 711, 1915; 18, 1240, 1916; 19, 125, 1916.

the dotted line, while a beam of light traverses the substance parallel to  $DE$ .

The piece  $DE$  is moved to and fro, as it is coupled with the rods  $DC$  and  $O'C$ . Normal to the plane of the drawing, axes have been fixed at  $O$  and  $O'$  in a very strongly constructed frame, on which the bed is fastened, along which  $DE$  moves. The axis in  $O$  is rotated by a 3 H.P. motor, so that point  $A$  describes a circle;  $B$ , connected with  $A$  by the rod  $AB$ , acquires a movement backward and forward, which is transferred to  $C$  enlarged.

The piece  $DE$  of fig. 1 is shown diagrammatically in fig. 2 seen from above on a scale of  $\frac{1}{10}$ . In  $A$  and  $B$  there are bronze shoes which can slide along steel guides. All these have been constructed with great care, so that a rectilinear, horizontal motion of the shoes can be obtained. The rods of the transparent substance, which

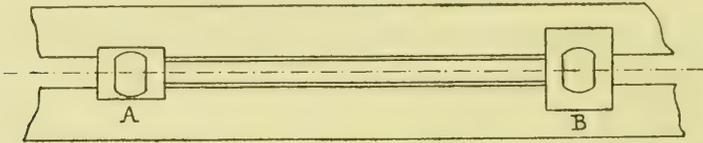


Fig. 2.

rest on a wooden block which is connected with 4 screws to  $A$  and  $B$ , participate in this motion.

The Plate, which is annexed to this communication, gives a general survey of the apparatus. The thickness of the guides amounts to 9 mm., the width to 70 mm., the length to 1.84 m. They rest on heavy rectangularly bent pieces, which constitute the sides of the bed, and which for greater firmness are connected by very solid pieces about half a meter long, which are bent twice rectangularly. These pieces are arranged on the lefthand at the bottom side, on the righthand on the upper side of the bed, as is to be seen in the Plate. The entire length of the upper part of the apparatus is 2.30 m., the length of the stroke is about 1 meter.

In order to ensure the regular movement of the apparatus it appeared to be necessary to provide it with two fly-wheels, a large one seen in the foreground, and a smaller one fastened on the other side of the axis of rotation, and just visible on the Plate. The whole apparatus is fastened with solid bolts to a granite slab, cemented to the large pillar of the laboratory.

As appears from a consideration of fig. 1, the rate of motion of the shoe is variable, with two, slightly differing maxima of velocity,

one in going, and one in returning along the guides. The maximum velocity is practically constant over a distance of about 20 cm. When the fly-wheel performs 184 revolutions a minute, the maximum velocity rises to somewhat more than 10 meters per second. This is the highest value that can be reached.

The driving apparatus was constructed by the works Werkspoor at Amsterdam. The execution of the mechanical parts had to be adapted to the optical requirements in the Laboratory.

With regard to the optical arrangement of the experiment we may refer to my former communications on the FIZEAU-effect. The quickly moving transparent, solid substance takes the place of the running water of the earlier experiments. The length of the moving column of quartz or glass ranged between 100 and 140 cm. in different experiments. After the successive application of numerous improvements it was possible to cause the beams of light to interfere through the moving quartz or glass, and to obtain pure interference lines at the greatest velocity which the apparatus admits.

The experiment comes to this, that the interference bands are photographed twice, first with a movement of the column to the right, and then with a movement to the left. These photos should be taken by admitting light during a time of the order of one hundredth of a second and at the moment of the maximum velocity.

The optical effect to be expected, when  $l$  represents the length of the moving transparent substance is:

$$\frac{4l}{\lambda \cdot c} \left( \frac{1}{\mu} - \frac{1}{\mu^2} - \frac{\lambda}{\mu^2} \frac{d\mu}{d\lambda} \right) \mu^2 v \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It appears from this formula, which will be proved later on (see II), that the optical effect is approximately proportional to  $\mu - 1$ . In FIZEAU's experiment the optical effect is proportional to  $\mu^2 - 1$  according to (3). This difference is connected with the fact that in FIZEAU's experiment in its usual form, the velocity in a definite point of space is always the same, whereas in the experiment now considered the light must overtake the moving bar.

As regards the optical effect observed, the method considered now will accordingly be two and a half times less favourable for a value of  $\mu = 1,5$  than FIZEAU's usual method, because  $\frac{\mu^2 - 1}{\mu - 1} = \mu + 1$ .

This is more or less compensated by an advantage with regard to the dispersion term. As follows from formulae (3) and (4), the ratio of the dispersion term to the principal term is in the second case 1,6 times that in the former experiment.

We shall now discuss a few more particulars of the arrangement and the use of the apparatus.

3. *Determination of velocity.* In order to get an insight into the course of the velocity in the movement along the guides the position of  $DE$  (see fig. 1) was determined corresponding with 16 different, equidistant positions of the fly-wheel. The graph indicating the connection between the positions of the fly-wheel and the deviations, has about the shape of a sinusoid, but the two halves of the curve are not symmetrical, and in particular, the course of the graph in the neighbourhood of the two boundary values is not exactly the same, as already appears from a consideration of fig. 1, when  $A$  is imagined to move along the dotted circle.

The velocity-time curve can be graphically derived from the path-time curve. At the maxima the velocity is practically constant over a distance of 20 cm., of which only 10 cm. are used. As was already stated the maximum velocity amounts to  $1000 \frac{\text{cm.}}{\text{sec.}}$  for 184 revolutions per minute, and proportionally the calculated velocity can be derived for another number of revolutions. Whether really the maximum velocity should be taken into account in the calculation, depends further also on the position of the moving column at the moment that the shutter, before the objective of the telescope, transmits light. In some cases this position did not correspond to that of the maximum velocity, which circumstance was of course taken into account in the interpretation of the photos.

In the most accurate experiments the maximum velocity of the column was directly measured (for the method used see one of the following communications), which renders us independent of the supposition that the fly-wheel possesses a constant angular velocity.

It appeared in the experiments that the machine ran more uniformly when the fly-wheel rotates clockwise (seen from the side of the larger fly-wheel) than in the opposite direction. Of course this favourable direction was always used.

4. *Shutter.* Only at the moment that the machine has its greatest velocity may the light be admitted to the photographic plate. The following arrangement was made for this purpose. The axis of the fly-wheel is provided with a toothed wheel, which engages with a second toothed wheel with double the number of teeth. An insulated brass ring with cams is fitted on each side of this second wheel. The cams on the two rings are placed

diametrically opposite one another so that they take each other's places as regards level after every whole revolution of the fly-wheel, and can make contact with a suitably fixed sliding contact.

By means of the cam an electric current is closed in a circuit containing also windings of a coil that acts electromagnetically on a shutter or light interrupter. When the second cam makes contact, the current passes through a second coil, which closes the shutter. Every time the fly-wheel, and consequently the moving column, have arrived at the *same position* and move *in the same direction*, the shutter is opened, and light is allowed to pass for a moment. The intensity of the interference figure not being strong enough to give a satisfactory photo with light that has been admitted once, a photo is taken e.g. thirty times successively on the same plate. As the light is let through three times a second, this takes only ten seconds.

To take the second photo, i.e. when the column of quartz moves in the opposite sense, a duplicate arrangement is used, placed symmetrically with respect to the one sketched. By means of a double-pole throw-switch it becomes possible to make one series of photos succeed the other immediately without loss of time and without stopping the machine. Only the photographic plate must be moved a little. The two large toothed wheels, and between them the small one can be distinguished on the Plate near the bearing at the bottom on the righthand side.

As some time passes between the moment that contact is effected by the cam and the opening of the shutter, this time must be taken into account. To do so it is necessary to perform a phase determination.

5. *Phase determination*, i.e. to ascertain by a separate experiment, if really at the moment of the greatest velocity the light passes through the shutter. For this purpose the shutter is put at the place of the greatest velocity by the side of the bed. The wooden beam containing the transparent substance, is provided with a black screen with an opening. A glowlamp is placed in the line: shutter-place of the greatest velocity, a line which is normal to the longitudinal direction of the apparatus. When the machine is running and when the shutter is in action, the lamp must be observed through it. When it does not work at the right moment, the field of view remains dark. Then the phase can be improved, and can at last be made accurate by gradually shifting the large toothed wheel with respect to the small one. This causes the contact to be formed at another moment.

6. *Observation and Photography of the Interference-bands.* The shutter is placed before the objective of the telescope, which was also used before to record the FIZEAU-effect<sup>1)</sup>). In the focal plane of the telescope, which is provided with a negative achromatic system of lenses to increase the effective focal distance, a system of wires has been placed, which is photographed at the same time with the interference bands.

The position of the interference fringes with respect to the wires is determined. Immediately behind the wires the photographic plate is in a plateholder mounted quite independently of the telescope with the cross wires. The photographic plate can be put in the required position without the telescope being touched, and be shifted to take the successive photos.

The telescope, the plate-holder, the interferometer, and the glass rectangular prism, in which the interfering beams are reversed, are mounted on separate freestone piers, which are cemented to the large pier. This last mentioned prism, which also served in the earlier experiments, is visible on the righthand side of the Plate.

7. *Measurement of the time that the shutter or interrupter is opened.* This time, which is of the order of 1 or 2 hundredths of a second, is dependent on the current in the coils of the circuit of the interrupter and can be regulated by this and by the change of the width of the opening in the moving screen of the interrupter. For the determination of the time the interrupter is placed before the lens of a camera, with which a small lamp is photographed, which revolves on a disk with known velocity. During the time that the interrupter is opened, the lamp describes part of a circle, the length of which is measured.

8. *Checking and regulating of the apparatus.* After the interference fringes have been made as distinct as possible, the beams passing only through the air, a compensator is placed in one beam, consisting of a plane parallel glass plate 7 m.m. thick and with a diameter of 25 m.m., made by Hilger. This plate can be rotated round a vertical and horizontal axis, and enables us to change the slope and the distance of the interference fringes in a simple way. In many cases it was unnecessary to insert this compensator, as the desired interference fringes were already obtained with the interferometer alone. Then the column of quartz and glass

<sup>1)</sup> Cf. ZEEMAN. These Proc. 18, 400, 1915.

is introduced into the beam of light, precautions being taken which will be mentioned in communication II. It must then first of all be ascertained whether the apparatus and particularly the guides satisfy the high requirements on which the efficacy of the whole arrangement depends.

With a slow movement of the shoes with the quartz column along the guides the interference fringes did not remain stationary, but changed with regard to distance and slope. When the apparatus had first been put together, this movement of the bands was very great. It is clear that it must also occur for a perfectly homogeneous column bounded by parallel planes, when the movement does not take place along a perfectly straight line. For then the whole column of a length of more than 100 cm. acts as a compensator of exceedingly great thickness. In order to make the circumstances as favourable as possible the steel plates of the guides were laid on the supporting plates about in the correct position. The bolts which were used to keep the apparatus in place, are then screwed down till a sensitive level, which could be placed in longitudinal and in transverse direction, indicated a plane as much as possible horizontal, determined by the upper planes of the steel guides. It was then examined if the inner edges of the guides were as perfectly straight and parallel as possible, and improvements were made in this respect by filing and grinding. At last the free play of the bronze shoes in their movement along the guides was removed as much as possible. Great improvements were successively made to the apparatus in this direction, so that rotation and change of distance of the fringes became comparatively slight. It was, however, impossible to have the interference fringes quite steady when the apparatus was slowly moved. This is, however, not necessary; what is required after all is that the same positions of the interference bands are found again when the shoes have returned to the same point of the guides. The results prove that this is actually the case, and that the occasional deviations fall now in one sense, now in another.

The excellent definition of the interference fringes, recorded with the quickly moving apparatus in itself proves already, that every time about the same position of the bands is obtained, as 20 or 30 images are superposed (see above § 4), which could never produce a definite image, when the single light impressions were not almost identical. Sometimes the system of fringes proved to be rotated, and then the photo had to be rejected. Of course care had also to be taken that the guides were well oiled, and there is one more *dynamic particularity* that had to be seen to. When the motor has

been started for the first time, then the apparatus hardly ever gains the maximum velocity, which corresponds to 184 revolutions of the fly-wheel per minute. It gives the impression that the apparatus is hampered by a resistance, e.g. only 140 revolutions are made. The starting is then repeated a few times, and at the third or fourth attempt the machine suddenly runs very smoothly without jerking. Then the feeling of uneasiness of the operator at the exceedingly rapid motion of the large apparatus so close to the delicate optical parts of the interferometer, has abated somewhat and the experiment can begin. <sup>1)</sup>

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<sup>1)</sup> The experiment is not entirely without danger. When the experiment with glass was to begin, four beautifully finished glass cylinders 20 cm. long and 2.5 cm. thick were placed in the wooden shoe, and optically adjusted. In the very first experiment with this glass column one of the glass cylinders, which evidently had not been properly fastened, got loose, while the apparatus moved at full speed; it smashed all the other pieces and knocked the brass end-pieces off the shoe. The glass cylinders were entirely smashed, the work of months was destroyed. It was a wonder that the interferometer and the glass rectangular prisms remained undamaged.

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**Mathematics.** — “*Ueber die Struktur der perfekten Punktmengen*”  
(dritte Mitteilung <sup>1)</sup>). By Prof. L. E. J. BROUWER.

(Communicated in the meeting of October 25, 1919).

§ 1.

Ein *Flächensystem* bzw. eine *Fläche* ist im folgenden definiert mittels eines solchen zweidimensionalen Fragmentes <sup>2)</sup> bzw. zusammenhängenden zweidimensionalen Fragmentes, in welchem die in einem willkürlich gewählten Elementeckpunkte mündenden Elementeseiten entweder alle oder alle bis auf zwei je zwei Elementen gemeinsam sind. Im ersteren Falle sprechen wir von einem *gewöhnlichen*, im letzteren von einem *aussergewöhnlichen* Elementeckpunkt, während wir eine Elementeseite gewöhnlich oder aussergewöhnlich nennen, je nachdem sie zu zwei oder zu einem einzigen Elemente gehört. Das Flächensystem bzw., die Fläche besteht aus dem Fragmente *mit Ausnahme* der aussergewöhnlichen Elementeseiten und Elementeckpunkte, welche zusammen die *Grenze* des Flächensystems bzw. der Fläche bilden.

Ein Flächensystem bzw. eine Fläche, deren Elemente Grundsimplexe einer simplizialen Zerlegung des Flächensystems oder der Fläche  $\omega$  sind, wird ein *Teilflächensystem* bzw. eine *Teilfläche* von  $\omega$  genannt werden.

Sei  $\alpha$  ein auf der Fläche  $\omega$  nicht-negativer Charakteristik gelegenes und auf  $\omega$  kein elementares <sup>1)</sup> Restgebiet bestimmendes, abgeschlossenes Kontinuum,  $\alpha' = \omega, \alpha'', \alpha''', \dots$  eine  $\alpha$  approximierende Folge von Teilflächen von  $\omega$ , durch welche also  $\alpha$  als  $\mathfrak{D}(\alpha', \alpha'', \dots)$  bestimmt ist, so dürfen wir für jedes  $v$  annehmen, dass  $\alpha^{(v)}$  auf  $\omega$  kein elementares Restgebiet bestimmt, dass  $\alpha^{(v+1)}$  eine Teilfläche von  $\alpha^{(v)}$  ist und dass die Grenzen von  $\alpha^{(v)}$  und  $\alpha^{(v+1)}$  keinen Punkt gemeinsam haben. Sei  $k^{(v)}$  die Charakteristik von  $\alpha^{(v)}$ ,  $m^{(v)}$  die maximale Anzahl von einander nicht treffenden, zusammen nicht zerstückelnden einfachen geschlossenen Kurven von  $\alpha^{(v)}$ , so kann für wachsendes  $v$

<sup>1)</sup> Für die erste und zweite Mitteilung vgl. diese Proceedings XII, S. 785; XIV, S. 137.

<sup>2)</sup> Vgl. Math. Annalen 71, S. 306.

<sup>3)</sup> Ein Teilgebiet von  $\omega$  heisst elementar, wenn es nur auf  $\omega$  zusammenziehbare einfache geschlossene Kurven enthält.

weder  $k^{(g)}$  noch  $m^{(g)}$  zunehmen, so dass eine endliche positive Zahl  $g$  existiert mit der Eigenschaft, dass  $k^{(g+\mu)} = k^{(g)}$  und  $m^{(g+\mu)} = m^{(g)}$  für jedes nicht negative  $\mu$ . Hieraus folgt, dass die von  $\alpha^{(g+\mu+1)}$  in  $\alpha^{(g+\mu)}$  bestimmten Restflächen alle Zylinderflächen sind, dass die topologische Gestalt von  $\alpha^{(g+\mu)}$  für jedes  $\mu$  gleich derjenigen von  $\alpha^{(g)}$  ist und dass  $\alpha^{(g+\mu+1)}$  aus  $\alpha^{(g+\mu)}$  durch Zurücknahme der Ränder nach Innen entsteht, so dass zwischen den  $\alpha^{(g+\mu)}$  und  $\alpha$  die folgenden Beziehungen existieren:

1. Zu jeder Kombination  $(\varepsilon, \varepsilon_1, \mu)$  gibt es ein solches nur von  $\varepsilon$  und  $\mu$  abhängendes und für festes  $\mu$  mit  $\varepsilon$  gegen 0 konvergierendes  $\varepsilon_1$ , dass jede  $\varepsilon$ -Kette<sup>1)</sup> von  $\alpha^{(g+\mu)}$  mittels einer endlichen Folge von  $\varepsilon_1$ -Abänderungen<sup>1)</sup> innerhalb  $\alpha^{(g+\mu)}$  in eine  $\varepsilon_1$ -Kette von  $\alpha$  übergeführt werden kann.

2. Zu jedem  $\varepsilon$  gibt es ein solches mit  $\varepsilon$  gegen 0 konvergierendes  $\varepsilon_1$ , dass jede  $\varepsilon$ -Kette eines  $\alpha^{(g+\mu)}$  mittels einer endlichen Folge von  $\varepsilon_1$ -Abänderungen innerhalb  $\alpha^{(g+\mu)}$  in eine Kette von  $\alpha$  übergeführt werden kann.

3. Jedes  $\alpha^{(g+\mu)}$  besitzt die gleiche minimale Multiplizität der Basis der Zyklosis<sup>2)</sup>, wie  $\alpha$ .

4. Zu jeder Kombination eines hinreichend kleinen  $\varepsilon'$ , eines  $\varepsilon'_1$  und eines  $\mu$  gibt es ein solches nur von  $\varepsilon'$  und  $\mu$  abhängendes und für festes  $\mu$  mit  $\varepsilon'$  gegen 0 konvergierendes  $\varepsilon_1$ , dass jedes System von  $(\varepsilon, \varepsilon')$ -Fundamentalketten<sup>2)</sup> von  $\alpha^{(g+\mu)}$  mittels einer endlichen Folge von  $\varepsilon_1$ -Abänderungen innerhalb  $\alpha^{(g+\mu)}$  in ein System von  $(\varepsilon_1, \varepsilon'_1)$ -Fundamentalketten von  $\alpha$  übergeführt werden kann.

5. Zu jedem  $\varepsilon'$  gibt es ein solches mit  $\varepsilon'$  gegen 0 konvergierendes  $\varepsilon''$ , dass jedes System von  $(\varepsilon, \varepsilon')$ -Fundamentalketten eines  $\alpha^{(g+\mu)}$  mittels einer endlichen Folge von  $\varepsilon''$ -Abänderungen innerhalb  $\alpha^{(g+\mu)}$  in ein System von  $(\varepsilon, \varepsilon'')$ -Fundamentalketten von  $\alpha$  übergeführt werden kann.

6. Zu jedem  $\varepsilon$  gibt es ein solches  $\varepsilon^0$ , dass jedes System von  $(\varepsilon, \varepsilon')$ -Fundamentalketten von  $\alpha$  gleichzeitig ein System von  $(\varepsilon^0, \varepsilon')$ -Fundamentalketten eines jeden  $\alpha^{(g+\mu)}$  ist.

Wir bringen diese Beziehungen zum Ausdruck, indem wir  $\alpha$  als *zyklomatisches Extrakt* von  $\alpha^{(g)}$  bezeichnen.

Sei nunmehr  $\omega$  ein willkürliches auf  $\omega$  gelegenes abgeschlossenes Kontinuum,  $\alpha_1$  die Vereinigung von  $\alpha$  und den elementaren Restgebieten von  $\alpha$ ,  $\alpha_1^{(g_1)}$  eine  $\alpha_1$  als zyklomatisches Extrakt enthaltende Teilfläche von  $\omega$ . Alsdann gibt es zu jeder Kombination  $(\varepsilon, \varepsilon_1)$  ein

<sup>1)</sup> Vgl. Math. Annalen 72, S. 422.

<sup>2)</sup> Ibid., S. 424.

solches mit  $\varepsilon$  gegen 0 konvergierendes  $\varepsilon_1$ , dass jede  $\varepsilon$ -Kette von  $\alpha_1^{(g_1)}$  mittels einer endlichen Folge von  $\varepsilon_1$ -Abänderungen innerhalb  $\omega$  in eine  $\varepsilon_1$ -Kette von  $\alpha$  übergeführt werden kann. Wir werden auch  $\alpha$  ein *zyklomatisches Extrakt* von  $\alpha_1^{(g_1)}$  nennen.

Insbesondere werden wir ein zyklomatisches Extrakt eines Elementes als *Elementarkontinuum*, ein zyklomatisches Extrakt einer Zylinderfläche als *Zylinderkontinuum* bezeichnen.

## § 2.

Sei  $\alpha$  eine auf der Fläche  $\omega$  nicht-negativer Charakteristik gelegene, auf  $\omega$  kein elementares Restgebiet bestimmende, abgeschlossene Punktmenge,  $\alpha' = \omega, \alpha'', \alpha''', \dots$  eine  $\alpha$  approximierende Folge von Teilflächensystemen von  $\omega$ , so dürfen wir für jedes  $\nu$  annehmen, dass  $\alpha^{(\nu)}$  auf  $\omega$  kein elementares Restgebiet bestimmt, dass  $\alpha^{(\nu+1)}$  ein Teilflächensystem von  $\alpha^{(\nu)}$  ist und dass die Grenzen von  $\alpha^{(\nu)}$  und  $\alpha^{(\nu+1)}$  keinen Punkt gemeinsam haben. Sei  $\beta^{(\nu)}$  das aus  $\alpha^{(\nu)}$  durch Tilgung derjenigen Stücke, welche die topologische Gestalt eines Elementes oder eines Zylinders besitzen, hervorgehende Flächensystem,  $S$  ein willkürliches Stück von  $\beta^{(\nu)}$ ,  $k(S)$  die Charakteristik von  $S$ ,  $m(S)$  die maximale Anzahl von einander nicht treffenden, zusammen nicht zerstückelnden einfachen geschlossenen Kurven von  $S$ , so sind die Zahlen  $k(S)$  und  $m(S)$  beide nicht-negativ und nicht beide gleich Null. Mithin existieren zwei ganze nicht-negative Zahlen  $g$  und  $h$  mit der Eigenschaft, dass jedes  $\beta^{(g+\mu)}$  sich aus  $h$  Stücken  $\beta_1^{(g+\mu)}, \beta_2^{(g+\mu)}, \dots, \beta_h^{(g+\mu)}$  zusammensetzt, dass die topologische Gestalt von  $\beta_s^{(g+\mu)}$  für jedes  $\mu$  und jedes  $s$  gleich derjenigen von  $\beta_s^{(g)}$  ist, dass  $\beta_s^{(g+\mu+1)}$  aus  $\beta_s^{(g+\mu)}$  durch Zurücknahme der Ränder nach Innen entsteht und dass das als  $\mathfrak{D}(\beta_s^{(g)}, \beta_s^{(g+1)}, \dots)$  bestimmte Stück von  $\alpha$  ein zyklomatisches Extrakt von  $\beta_s^{(g)}$  ist.

Das hiermit erhaltene Resultat kann in der folgenden Form, welche, wie eine triviale Ueberlegung zeigt, für eine Fläche  $\omega$  negativer Charakteristik ihre Gültigkeit behält, ausgedrückt werden:

*Zu jeder auf einer Fläche  $\omega$  gelegenen abgeschlossenen Punktmenge  $\alpha$  existiert ein Teilflächensystem  $\psi$  von  $\omega$  mit der Eigenschaft, dass  $\alpha$  sich aus erstens von den Stücken von  $\psi$  je einem zyklomatischen Extrakt, zweitens einer Menge von Elementarkontinuen und Zylinderkontinuen zusammensetzt.*

Ist insbesondere  $\omega$  eine Kugel, so sind alle Stücke von  $\alpha$  Elementarkontinua.

Ist  $\omega$  eine projektive Ebene, so sind entweder alle Stücke von  $\alpha$

Elementarkontinua, oder alle bis auf eines, das ein zyklomatisches Extrakt von  $\omega$  ist.

Ist  $\omega$  ein Torus, so sind entweder alle Stücke von  $\alpha$  Elementarkontinua, oder alle bis auf eines, das ein zyklomatisches Extrakt von  $\omega$  ist, oder endlich alle bis auf eine abgeschlossene zyklisch geordnete Menge von Zylinderkontinuen; die Teilzylinder von  $\omega$ , von denen diese Zylinderkontinua zyklomatische Extrakte sind, sind alle auf  $\omega$  stetig ineinander überführbar.

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**Mathematics.** — “*Ueber Transformationen ebener Bereiche*”. By  
B. VON KERÉKJÁRTÓ. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of October 25, 1919).

In der vorliegenden Note wird eine Anwendung <sup>1)</sup> gemacht vom folgenden BROUWERSCHEN Fixpunktsatze:

*Eine eindeutige stetige Abbildung der abgeschlossenen Kreisscheibe auf einen Teilbereich derselben lässt wenigstens einen Punkt invariant.*

Mit Hilfe dieses Satzes beweisen wir nämlich das folgende

**THEOREM.** *Eine eindeutige stetige Abbildung eines von endlichvielen Jordanschen Kurven begrenzten abgeschlossenen ebenen Bereiches auf einen Teilbereich desselben, bei welcher die Grenzkurven des ursprünglichen und des Bildbereiches paarweise äquivalent sind, jedoch eine und nur eine Grenzkurve in eine äquivalente übergeht, lässt wenigstens einen Punkt invariant.*

(Hierbei sollen zwei einander nicht kreuzende Kurven *äquivalent* genannt werden, wenn in ihrem Zwischengebiete keine Grenzkurve liegt).

$C_1$  sei die äussere Grenzkurve des gegebenen Bereiches, ihr Bild  $C_1'$  sei mit ihr äquivalent; die übrigen Grenzkurven seien  $C_2, C_3, \dots, C_n$ , ihre Bilder  $C_2', C_3', \dots, C_n'$ . Man erweitere die gegebene Abbildung durch eine an sie anschliessende Abbildung des Innern von  $C_\alpha$  auf das Innere von  $C_\alpha'$  ( $\alpha = 2, 3, \dots, n$ ). Hiermit erhält man eine eindeutige stetige Abbildung des Innern von  $C_1$  auf das Innere von  $C_1'$ , welche nach dem obigen BROUWERSCHEN Satze wenigstens einen Punkt invariant lässt; dieser Fixpunkt kann aber nicht im Innern von  $C_\alpha$  ( $\alpha \neq 1$ ) liegen, gehört somit zum ursprünglichen abgeschlossenen Bereich.

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<sup>1)</sup> Für eine analoge Anwendung nebst daraus gezogenen Konsequenzen vgl. Math. Annalen 80, S. 34.



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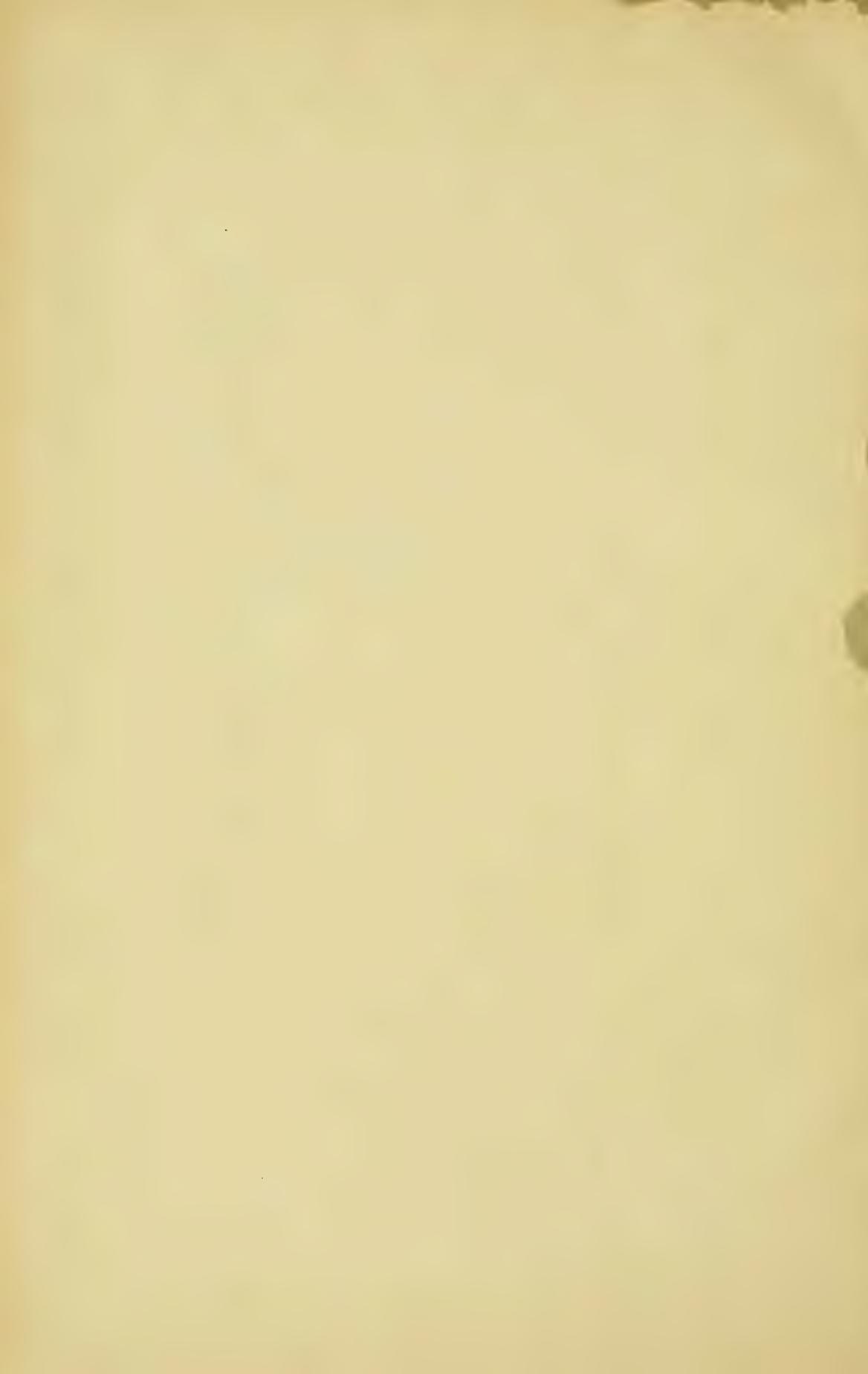
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