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Propagation of Weak Hydromagnetic Discontinuities

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Errata MH-10

In the second column below, the unstarred integers indicate the number of lines from the top of the page, while the starred integers indicate the number of lines from the bottom.

Page No.	Line No.	Corrections
5	3,4,5	-The expressions in A_2', A_3' and A_4' should be set equal to zero.
5	4*	-Replace $\underset{\sim}{H} \times \nabla \times \underset{\sim}{H}$ by $\underset{\sim}{H} \times (\nabla \times \underset{\sim}{H})$.
7	8*	In the matrix equation (2.9), the entry $\mu^{1/2}H_y$ in the second row and third column should be $\mu^{1/2}H_z$.
9	1*	-Replace $\cos\theta$ in equation (3.8) by $\lvert\cos\theta\rvert$.
10	6,7	-Replace ... in the ranges $r < 1/2$ $1/2 \le r < 1$, $r = 1$ $1 < r < 2$, $r > 2$ ... by ... in the ranges $r \le 1/2$, $1/2 < r < 1$, $r = 1$, $1 < r < 2$, $r \ge 2$ ...
16	3*	-Replace $\chi = \tan^{-1}(1/2)$ by $\chi' = \tan^{-1}(2)$.
17	1	-Replace ... slow branch by ... slow or Alfvén branches.
17	2	-Replace ... we introduce θ_u as ... by ... we introduce θ_u, $0 \le \theta_u \le \pi$, as ...
17	14*	-Replace (c_i/u) on the right margin by $(-c_i/u)$.
18	8,9,10	-Replace $\hat{p} \cdot \nabla_{\hat{p}}$ and $\hat{p}$ in equations (4.20), (4.21) and (4.22) by $\underset{\sim}{\hat{p}} \cdot \nabla_{\hat{p}}$ and $\underset{\sim}{\hat{p}}$, respectively.
19	2,3,6	-In the $\frac{d\underset{\sim}{p}^{\pm}}{dt}$ equations of (4.23)-(4.25), replace $\underset{\sim}{p} \times \nabla \times [\quad]$ and $\underset{\sim}{p} \times \nabla \times (\quad)$ by $\underset{\sim}{p} \times \{\nabla \times [\quad]\}$ and $\underset{\sim}{p} \times [\nabla \times (\quad)]$ respectively.
21	2*	-Replace $\pm(\mu/\rho)^{1/2}\, \underset{\sim}{H}(t-t_o)$ by $\pm(\mu/\rho)^{1/2} \cdot \operatorname{sgn}(H_{ro})\, \underset{\sim}{H}(t-t_o)$.

Page No.	Line No.	Correction
21	1*	-A curly bracket is missing in the right member of (4.36). It should have the form... $\left\{ \pm c_s \underset{\sim}{n}^o \ldots\ldots (\underset{\sim}{H} - H_{n^o} \underset{\sim}{n}^o) \right\} (t - t_o)$.
21	1	-Replace $\nabla_{\hat{p}} \hat{\mathcal{H}}$ by $\nabla_{\hat{p}} \hat{\mathcal{H}} / \underset{\sim}{\hat{p}} \cdot \nabla_{\hat{p}} \mathcal{H}$.
22	1	-See above correction (p. 21, (4.36)). The term on the right side of (4.37) should have the form $\left\{ \pm c_f \underset{\sim}{n}^o \ldots (\underset{\sim}{H} - H_{n^o} \underset{\sim}{n}^o) \right\} (t - t_o)$.
23	10	-Replace $(1 + \cos^2\theta)$ in the right member of (4.38) by $\sin^2\theta$.
24	11	-Remove comma after the word " over" .
26	7*	-Replace $(2.1)' - A'$ by $(2.1) - A_1'$.
27	1	-Replace $W(x)$ by $W(\underset{\sim}{x})$ in the right member of equation (5.2).
27	3*	-Replace ... c, ρ and $\underset{\sim}{u}$ ... by ... c, ρ, $\underset{\sim}{H}$ and $\underset{\sim}{u}$ ...
28	8*,9*	-In the right member of (5.14), replace $\delta\underset{\sim}{H} \times \nabla \times \underset{\sim}{H}$ by $\delta\underset{\sim}{H} \times (\nabla \times \underset{\sim}{H})$, $\mu\underset{\sim}{H} \times \nabla \times \delta\underset{\sim}{H}$ by $\mu\underset{\sim}{H} \times (\nabla \times \delta\underset{\sim}{H})$, $\frac{\delta\rho}{\rho}$ by $-\frac{\delta\rho}{\rho}$, and $\underset{\sim}{H} \times \nabla \times \underset{\sim}{H}$ by $\mu\underset{\sim}{H} \times (\nabla \times \underset{\sim}{H})$.
29	1	-In the first row and fourth column of the matrix (in equation (5.16)), replace H_y by H_n .
36	12	-Replace δJ_z by δF_x .
38	5	-In the denominator of the right member of equation s_1 , replace $4r \sin\theta$ by $4r \sin^2\theta$.

In some reports, it will be found that some of the following corrections have already been made. (This depends upon whether the cited page came from the second or first print-lot).

Page No.	Line No.	Corrections
12	1*, 2*	-The last sentence in footnote 14 should read: The limiting expressions for the $\hat{R}$'s yield the desired results, unless r = 1 in which case they lead to linear combinations of the tabulated solutions.
27	5*	-The first two terms in the right member of (5.9) should be $-\nabla\times[(\mathbf{H}')^{\dagger}\times\mathbf{u}] - \nabla\times[\mathbf{H}\times(\mathbf{u}')^{\dagger}]$
30	7*	-Replace the 5 by 3
32	1*	-Replace "proceedure" by "procedure".
36	5	-Replace $\delta\rho_o$ by $(a\delta\rho_o)/\rho$ in the right member of (6.3).
41	2*	-Replace "propagated and to infinity" by "propagated out to infinity".
Figure 2b		-Replace χ wherever it appears by χ' and arc tan (1/2) by arc tan (2).

If in box (1,3) of the table on page (11), the expression for $\delta H^{\pm}$ is $\delta \mathbf{H}^{\pm} = -\epsilon H\hat{\mathbf{n}}$, then <u>all</u> of the following corrections should be ignored. If on the other hand, the hat, i.e., ^, is missing on the $\mathbf{n}$, then all of the following corrections should be made.

Corrections for the Table on Page 11

Box (1,1): Replace ε by $-\varepsilon$ in the expression for $\delta\mathbf{u}^{\pm}$.

Box (1,3): Replace ε by $\pm\varepsilon$ in the expressions for $\delta\mathbf{H}^{\pm}$ and $\delta\mathbf{u}^{\pm}$. Replace $\mathbf{n}$ by $\hat{\mathbf{n}}$ in the expressions for $\delta\mathbf{H}^{\pm}$ and $\delta\mathbf{u}^{\pm}$.

Box (1,4): In the expression for $\delta\mathbf{H}^{\pm}$, replace ε by $-\varepsilon$. In the expression for $\delta\mathbf{u}^{-}$, replace ε by $\pm\varepsilon$.

Box (3,1): In the expression for $\delta\mathbf{u}^{\pm}$, replace the ε in the second term by $-\varepsilon$.

Box (3,2): In the expression for $\delta\mathbf{u}^{\pm}$, replace ε by $-\varepsilon$.

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March, 1959

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ABSTRACT

A theoretical discussion is given of the propagation of weak hydromagnetic discontinuities (e.g., weak shocks) in an infinitely conducting, perfect, compressible fluid. The undisturbed flow is assumed to be steady and isentropic. It is shown that, in general, six wave fronts will evolve from a given initial manifold. As in geometrical optics, these wave fronts are constructed by means of rays. In addition, formulas are derived, describing the variation of the discontinuity "strength" of a given propagating mode along rays associated with that mode. In the special case where the undisturbed state is homogeneous, simple explicit formulas are given for the wave fronts evolving from an arbitrary initial manifold and for the strength of the disturbance on these fronts. These results are employed to solve a mixed initial boundary-value problem that has been designed to illustrate i) a method of producing hydromagnetic disturbances and ii) the fact that an initial disturbance gives rise, in general, to several propagating waves.

TABLE OF CONTENTS

I. INTRODUCTION

We shall be concerned here with the propagation of (small) hydromagnetic disturbances (e.g., weak shocks) in a perfect, compressible, infinitely conducting fluid. In treating such waves the basic hydromagnetic equations (see S. Lundquist[1] and K.O. Friedrichs[2]) may be linearized. Then, as was pointed out by Friedrichs[2], the resulting set of equations shares with Maxwell's equations and the linearized equations of gas dynamics the property of being a symmetric-hyperbolic system of linear partial differential equations. Disturbances governed by such equations are propagated with finite speeds. The common mathematical structure of the three systems suggests the possibility of treating the hydromagnetic equations by a method that has been successful in treating the others; namely, the method of R.K. Luneberg[3] and J.B. Keller[4]. In this work, we shall employ this method to study the propagation of weak hydromagnetic discontinuities. Our analysis will be patterned chiefly after the work of J.B. Keller[4].

The basic problem to be treated in this paper (Part I) is: Let $\mathcal{S}^{o}$ denote the boundary of a disturbance at time $t = t_o$. For the sake of concreteness, it may be imagined that the disturbance lies wholly within a closed region bounded by $\mathcal{S}^{o}$ and that the complementary region is undisturbed. If we assume that the state of the undisturbed medium and the 'strength' of the initial discontinuity are known, the problem is i) to determine the wave fronts evolving from $\mathcal{S}^{o}$ - there will be six of them, in general - and ii) to find how the strength of the discontinuity varies with time on each of these fronts.

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1. S. Lundquist, Arkiv f. fysik, **5**, 297(1952).
2. K.O. Friedrichs, Nonlinear Wave Motion in Magneto-hydrodynamics, Los Alamos Report No. 2105 (written 1954, distributed 1957). See also a later version of this work by K.O. Friedrichs and H. Kranzer, Report No. MH-8, AEC Computing and Applied Math. Center, Inst. of Math. Sci., N.Y.U. (1958).
3. See, for example, the lecture notes by R.K. Luneberg, Mathematical Theory of Optics, Brown University (1944); Propagation of Electromagnetic Waves, New York University (1949).
4. J.B. Keller, J. Appl. Phys. **25**, 938(1954).

A second group of problems, to be treated in Part II of this series, involves the reflection and refraction of small disturbances at curved surfaces of discontinuity in the undisturbed flow. The basic problem is to determine the hydromagnetic analogs of Snell's laws and to obtain the reflection and transmission coefficients. In general, one encounters the phenomena of multiple reflection and refraction. Conical refraction is also possible when the Alfvén speed coincides with the sound speed (see the end of Section IV).

In the last decade, work on linearized, compressible hydromagnetic flow has been confined mainly to the study of continuous time-harmonic wave motion. The investigations of N. Herlofson[5], H.C. van de Hulst[6], A. Baños, Jr.[7] - to mention just a few of many - typify this approach to the subject. The fact that compressible hydromagnetic motion admits of essentially three modes of propagation was pointed out by Herlofson and van de Hulst in these papers. The first to treat the propagation of small hydromagnetic discontinuities directly appears to have been Friedrichs[2] who catalogued the various types of disturbance waves and described the wave fronts emanating from a point disturbance. Two recent works dealing with propagation phenomena as well as with other aspects of hydromagnetic wave motion are a paper by G.B. Whitham[8] and a report by H. Grad[9].

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5. N. Herlofson, Nature 165, 1020(1950).
6. H.C. van de Hulst, Interstellar Polarization and Magneto-Hydrodynamic waves. Appears in Problems of Cosmical Aerodynamics, p. 46 (Central Air Documents Office, Dayton, Ohio, (1951)).
7. A. Baños, Jr., Phys. Rev. 97, 1435(1955); Proc. Roy. Soc. A, 233, 350(1955).
8. G.B. Whitham, Comm. Pure Appl. Math. 12, (1959).
9. H. Grad, Propagation of Magneto -hydrodynamic Waves without Radial Attenuation, Report No. NYO 2-537, Inst. of Math. Sci., N.Y.U., (1959).

Our discussion begins with a summary of the relevant hydromagnetic equations and a listing of the three basic types of propagating disturbances (Sections II and III). In Section IV the notions of rays and wave fronts are introduced. The surface of wave normals and the reciprocal surface, Fresnel's ray surface, are analyzed. The use of rays to construct wave fronts is explained. In the special case where the undisturbed flow is constant, explicit formulas are given for the wave fronts evolving from an initial manifold. A graphical method for constructing these wave fronts is described. The section ends with a description of an anomalous sort of propagation which is closely related to the phenomena of conical refraction. In Section V, we state and sketch the derivation of the formulas giving the variation of the discontinuity strength of a given propagating mode along the rays associated with that mode. Sections IV and V contain the main theoretical results of the paper. In the final section, we give an application of the theory. The problem discussed is related to a non-linear problem treated elsewhere by the K.O. Friedrichs[2] and later by J.Bazer[10]. This problem illustrates i) how weak hydromagnetic disturbances may be produced and ii) the fact that an initial discontinuity requires for its 'resolution', in general, more than one propagating mode.

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10. J. Bazer, Ap. J. 128, 686 (1958).

II. LINEARIZED EQUATIONS OF HYDROMAGNETIC MOTION

1. Linearized System of Partial Differential Equations

In dealing with the linearized system, it is necessary to distinguish between the quantities that characterize the basic flow and small variations of these quantities. If A is a basic-flow quantity (scalar, vector or dyadic), then we shall employ A' to denote its (small) variation.

The following is a list of basic-flow quantities used in this paper[11]:

ρ: mass density,

$\tau = \rho^{-1}$: specific volume (volume per unit mass),

p: pressure,

e: specific internal energy,

S: Specific entropy,

$\underset{\sim}{u}$: fluid velocity,

$\underset{\sim}{H}$: magnetic intensity,

μ: magnetic inductive capacity of free space,

$\underset{\sim}{E}$: electric intensity. $\underset{\sim}{E} = \mu \underset{\sim}{H} \times \underset{\sim}{u}$ in an infinitely conducting medium.

All basic-flow quantities are assumed to be continuous and to have continuous partial derivatives. By adding primes on ρ, τ, etc. we obtain the corresponding list for the variational quantities.

In standard dyadic notation the linearized system of partial differential equations that governs the motion of a perfect, infinitely conducting, compressible fluid is (cf. Lundquist[1] and Friedrichs[2]):

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11. The rationalized MKS Giorgi system of units is employed throughout.

$$
\begin{array}{llr}
\nabla \cdot \underset{\sim}{H}' = 0, & A_0' & \\
\underset{\sim}{H}_t' + \nabla \cdot (\underset{\sim}{u}\underset{\sim}{H} - \underset{\sim}{H}\underset{\sim}{u})' = 0, & A_1' & \\
\left[(\rho \underset{\sim}{u})'\right]_t + \nabla\left[p' + \frac{\mu}{2}(H^2)'\right] + \nabla\cdot\left\{\left[\underset{\sim}{u}(\rho\underset{\sim}{u}) - \mu\underset{\sim}{H}\underset{\sim}{H}\right]'\right\}, & A_2' & (2.1)' \\
\rho_t' + \nabla \cdot \left[(\rho\underset{\sim}{u})'\right], & A_3' & \\
\left[(\rho S)'\right]_t + \nabla \cdot \left[(\rho\underset{\sim}{u}S)'\right], & A_4' & \\
p' = p_\rho \rho' + p_S S' \quad \text{(variational equation of state)}, & A_5' &
\end{array}
$$

In these equations, the subscript 't' denotes partial differentiation with respect to time and

$$
\begin{array}{lll}
(\alpha A)' & = & \alpha A' \\
(A+B)' & = & A' + B' \\
(A*B)' & = & A' * B + A * B'.
\end{array} \qquad (2.2)
$$

Here, α is a scalar constant (i.e., is independent of space and time variables), A and B are arbitrary scalar, vector or dyadic quantities, and '$*$' is any scalar, vector or dyadic-type multiplication such that $A*B$ makes sense.

If we ignore the primes, we recover the usual hydromagnetic equations.

For the sake of simplicity it will be assumed henceforth that <u>the medium is a polytropic gas</u> and that <u>the basic flow is steady and isentropic</u>. Thus, the basic flow is governed by the equations

$$
\begin{array}{llr}
\nabla \cdot \underset{\sim}{H} = 0, & A_0 & \\
\nabla \times (\underset{\sim}{u} \times \underset{\sim}{H}) = 0, & A_1 & \\
\rho\underset{\sim}{u} \cdot \nabla\underset{\sim}{u} + \nabla p + \mu\underset{\sim}{H} \times \nabla \times \underset{\sim}{H} = 0, & A_2 & (2.1) \\
\nabla \cdot (\rho\underset{\sim}{u}) = 0, & A_3 & \\
S = \text{constant (independent of } \underset{\sim}{x} \text{ and } t), & A_4 & \\
p = \kappa\rho^\gamma, & A_5 &
\end{array}
$$

where K depends only on S, and γ is the ratio of specific heats.

The assumption of isentropy implies that

$$S' = 0 , \tag{2.3}$$

$$\nabla p' = \nabla(p_\rho \rho' + p_S S') = \nabla(p_\rho \rho') \tag{2.4}$$

The relation (2.3) replaces A_4' above.

2. Linearized Discontinuity Relations

In writing down the above system of equations, we have tacitly assumed that the variational quantities are sufficiently regular - continuous with continuous space and time-derivatives. When the variational quantities are discontinuous across some surface, it is necessary to supplement these equations by an appropriate set of discontinuity relations; specifically, by the linearized hydromagnetic analogs of the Rankine-Hugoniot relations of gas dynamics. For the purpose of discussing these relations, the following notation will be employed:

$\mathcal{S}(t)$: a surface of discontinuity at time t,

$\underset{\sim}{n} = \underset{\sim}{n}(\underset{\sim}{x})$: the unit normal at points $\underset{\sim}{x}$ on $\mathcal{S}(t)$,(see equation (2.6) below),

$\underset{\sim}{U} = \underset{\sim}{U}(\underset{\sim}{x})$: the velocity of $\mathcal{S}(t)$ at $\underset{\sim}{x}$ (see (2.7))

U: the magnitude of $\underset{\sim}{U}$ (see (2.7)')

$u_n = \underset{\sim}{u}\cdot\underset{\sim}{n}$: the normal component of the fluid velocity at $\underset{\sim}{x}$ on $\mathcal{S}(t)$,

$H_n = \underset{\sim}{H}\cdot\underset{\sim}{n}$: the normal component of the magnetic intensity at $\underset{\sim}{x}$ on $\mathcal{S}(t)$,

$a = (p_\rho)^{1/2} = (\gamma p \rho^{-1})^{1/2}$: the speed of sound in a polytropic gas,

$c = U - u_n$: the velocity of $\mathcal{S}(t)$ along $\underset{\sim}{n}$ relative to the normal fluid velocity,

$\delta A = A_1' - A_o'$: the jump in the variation A' across $\mathcal{S}(t)$, A_1' is the value of A' on the side of $\mathcal{S}(t)$ into which $\underset{\sim}{n}$ points; A_o' is the value on the other side,

$\delta A_n = (\underset{\sim}{A}_1' - \underset{\sim}{A}_o')\cdot\underset{\sim}{n}$: here, $\underset{\sim}{A}'$ is a vector. δA_n is the jump of $\underset{\sim}{A}'$ in the normal direction.

If

$$\phi(\underset{\sim}{x},t) = 0, \tag{2.5}$$

is the equation of $\mathcal{S}(t)$ then $\underset{\sim}{n}$ and $\underset{\sim}{U}$ are defined, respectively, as follows:

$$\underset{\sim}{n} = \nabla\phi \,/\, |\nabla\phi|, \tag{2.6}$$

$$\underset{\sim}{U} = -(\phi_t/\ |\nabla\phi|)\underset{\sim}{n}, \tag{2.7}$$

so that

$$U = -\,\phi_t \,/\, |\nabla\phi|. \tag{2.7$'$}$$

Employing a local coordinate system on $\mathcal{S}(t)$ with the x-axis along $\underset{\sim}{n}$ we may express the linearized jump relations as follows:

$$\delta H_n = 0, \qquad \delta A_o \tag{2.8}$$

the matricial system

$$\begin{bmatrix} -\rho^{1/2}c & 0 & \mu^{1/2}H_y & -\mu^{1/2}H_n & 0 & 0 \\ 0 & -\rho^{1/2}c & \mu^{1/2}H_y & 0 & -\mu^{1/2}H_n & 0 \\ \mu^{1/2}H_y & \mu^{1/2}H_z & -\rho^{1/2}c & 0 & 0 & \rho^{1/2}a \\ -\mu^{1/2}H_n & 0 & 0 & -\rho^{1/2}c & 0 & 0 \\ 0 & -\mu^{1/2}H_n & 0 & 0 & -\rho^{1/2}c & 0 \\ 0 & 0 & \rho^{1/2}a & 0 & 0 & -\rho^{1/2}c \end{bmatrix} \begin{bmatrix} (\mu/\rho)^{1/2}\delta H_y \\ (\mu/\rho)^{1/2}\delta H_z \\ \delta u_n \\ \delta u_y \\ \delta u_z \\ a\,\delta\rho/\rho \end{bmatrix} = 0 \qquad \begin{array}{l} \delta A_{1,y} \\ \delta A_{1,z} \\ \delta A_{2,n} \\ \delta A_{2,y} \\ \delta A_{2,z} \\ \delta A_3 \end{array} \tag{2.9}$$

and the equations

$$c\,\delta S = 0 \qquad \delta A_4 \tag{2.10}$$

$$\delta p = \begin{cases} a^2\delta\rho \quad , & \text{if } c \neq 0 \\ a^2\delta\rho + p_S\delta S, & \text{if } c = 0 \end{cases} \qquad \delta A_5 \qquad (2.11)$$

To derive equations (2.8)-(2.11) from (2.1)', substitute $\nabla(a^2\rho')$ for $\nabla p'$ in (2.1)'-A_2' (see (2.4)) and then replace the operators $(\ )_t$, $\nabla(\ )$ and $\nabla\cdot(\ )$ by $-U\delta(\ )$, $\underset{\sim}{n}\delta(\ )$ and $\underset{\sim}{n}\cdot\delta(\ )$, respectively, and use δA_3 to simplify $\delta A_{2,n}$[12]. A justification of this formal procedure may be obtained by means of the usual pill-box type argument applied to moving surfaces (cf. J.B.Keller[6]). Another method would be to expand the explicit solution of the non-linear jump relations with respect to a suitable shock strength parameter and to retain only the lowest order terms. This procedure has been carried out for a polytropic gas by J. Bazer and W. Ericson[13].

III. WEAK HYDROMAGNETIC DISCONTINUITIES

Equation (2.9) may be rewritten as follows:

$$MR = \lambda R, \qquad \lambda =. \rho^{1/2}c \qquad (3.1)$$

In this equation, M represents the matrix in equation (2.9) with the diagonal terms deleted and R is the column vector $\left[(\mu/\rho)^{1/2}\delta H_y, (\mu/\rho)^{1/2}\delta H_z, \delta u_n, \delta u_y, \delta u_z, a\delta\rho/\rho\right]$. Using (3.1), we conclude that if a weak discontinuity R appears across $\mathcal{J}(t)$ then it is an eigenvector or possibly a combination of eigenvectors of the matrix M. Since M is symmetric, it has a complete set of six mutually orthogonal eigenvectors. In this section we shall tabulate these solutions and discuss some of their properties.

All solutions will be referred to as <u>modes</u>; however, only those solutions associated with non-zero eigenvalues will be called <u>waves</u> or <u>propagating</u> modes.

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12. Equation $\delta A_{1,n}$ does not appear because it is an identity.

13. J. Bazer and W. Ericson, Ap. J. to be published May (1959). See also report MH-8 of the Division of Electromagnetic Research, New York Univ., (1958).

The secular equation associated with equation (3.1) is

$$
\begin{aligned}
0 &= \mathrm{Det.}(M - \sqrt{\rho}\, cI) \\
&= (\rho c^2 - \mu H_n^2)\left\{\rho^2 c^4 - (\rho a^2 + \mu H^2)\rho c^2 + \rho a^2 \mu H_n^2\right\} \\
&= (\rho c^2 - \mu H_n^2)\left\{(\rho c^2 - \rho a^2)(\rho c^2 - \mu H_n^2) - \rho c^2 \mu (H^2 - H_n^2)\right\} .
\end{aligned}
\tag{3.2}
$$

The zeros of this equation evidently occur in pairs $\pm c'$ where $c' > 0$. Let

$$c_A \equiv b_n = (\mu H_n^2 \rho^{-1})^{1/2} \tag{3.3}$$

- the so-called Alfvén disturbance speed - be the non-negative zero of the first factor in the second (or third) line of (3.2) and let c_s (c_{slow}) and c_f (c_{fast}) be the smallest and largest zeros, respectively, of the last factor. Then from (3.2) it follows that

$$0 \leq c_s \leq \mathrm{Min.}\left\{a, b_n\right\} \leq \mathrm{Max.}\left\{a, b_n\right\} \leq c_f . \tag{3.4}$$

Moreover, setting

$$b = (\mu H^2 \rho^{-1})^{1/2}, \tag{3.5}$$

and

$$r = a^2/b^2, \tag{3.6}$$

we can easily show that

$$
\begin{aligned}
\frac{c_s}{b} &= \left\{ \frac{1}{2}(1+r) - \frac{1}{2}\left[(1+r)^2 - 4r\cos^2\theta\right]^{1/2} \right\}^{1/2}, \\
&= \left\{ \frac{1}{2}(1+r) - \frac{1}{2}\left[(1-r)^2 + 4r\sin^2\theta\right]^{1/2} \right\}^{1/2} .
\end{aligned}
\tag{3.7}
$$

$$\frac{c_A}{b} = \cos\theta, \tag{3.8}$$

$$\frac{c_f}{b} = \left\{ \frac{1}{2}(1+r) + \frac{1}{2}\left[(1+r)^2 - 4r\cos^2\theta\right]^{1/2} \right\}^{1/2} \tag{3.9}$$

$$= \left\{ \frac{1}{2}(1+r) + \frac{1}{2}\left[(1-r)^2 + 4r\sin^2\theta\right]^{1/2} \right\}^{1/2} .$$

In these equations, θ is the angle between $\underset{\sim}{n}$ and $\underset{\sim}{H}$. Figure 1 is a plot, on polar coordinate paper, of c_s/b, c_A/b and c_f/b against θ for several values of the parameter r. The curves of Figures 1a), 1b), 1c), 1d) and 1e) are typical of curves that correspond to values of r in the ranges $r < \frac{1}{2}$, $\frac{1}{2} \leq r < 1$, $r = 1$, $1 < r < 2$, and $r > 2$, respectively. By rotating each figure about the H-axis we obtain a surface in three-space which is called the surface of normal speeds.

In the table on the following page, we have listed the various types of wave-mode solutions of equation (2.9). Some contact discontinuities - i.e., solutions associated with $c = 0$ - are also listed; but only those that are connected to the wave modes by a limiting process. The reader is referred to Friedrichs[2] and Bazer and Ericson[13] for a discussion of other contact discontinuities; no use will be made of these in the sequel.

The quantity ϵ in the table is a (small) dimensionless non-zero number. The vectors $\underset{\sim}{n}^*$ and $\hat{\underset{\sim}{n}}$ are two mutually perpendicular unit vectors tangent to the surface of discontinuity associated with the given mode of propagation. The direction of $\hat{\underset{\sim}{n}}$ or of $\underset{\sim}{n}^*$ may be chosen arbitrarily thus fixing the direction of the other except for sign. Given the direction $\underset{\sim}{n}$, then as we pointed out earlier, there are two solutions associated with each of the speeds c_s, c_A and c_f, one corresponding to a normal motion of $\mathcal{S}(t)$ (with respect to the basic flow) along $\underset{\sim}{n}$ and the other in the opposite direction. This explains the presence of the pair of signs '$\pm$', '$\mp$' in the formulas of the table; the

TABLE I. SOLUTIONS OF EQUATIONS (2.9).

	(1) † $r > 0$	(2) $r < 1$	(3) $r \geq 1$	(4) $r > 0$
(1) SLOW WAVE	(1,1) $\mathbf{H}$ neither $\parallel$ nor $\perp$ to $\mathbf{n}$ $c_s/b = \left\{\frac{1}{2}(1+r) - \frac{1}{2}\left[(1+r)^2 - 4r\cos^2\theta\right]^{1/2}\right\}^{1/2}$, $\delta\mathbf{H}^{\pm} = -\epsilon\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$, $\delta\mathbf{u}^{\pm} = \mp\epsilon c_s\left(\frac{b_n^2}{c_s^2}-1\right)\mathbf{n} \pm \epsilon\frac{\mu H_n}{\rho c_s}\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$, $\delta\rho^{\pm} = -\epsilon\rho\left(\frac{b_n^2}{c_s^2}-1\right)$, $\delta S^{\pm} = 0$.	(1,2) $\mathbf{H} \parallel$ to $\mathbf{n}$ $(\mathbf{H} = H_n)$ $c_s = a$, $\delta\mathbf{H}^{\pm} = 0$, $\delta\mathbf{u}^{\pm} = \mp\epsilon\, a\left(\frac{1}{r}-1\right)\mathbf{n}$, $\delta\rho^{\pm} = -\epsilon\,\rho\left(\frac{1}{r}-1\right)$, $\delta S^{\pm} = 0$.	(1,3) $\mathbf{H} \parallel$ to $\mathbf{n}$ $(\mathbf{H} = H\mathbf{n})$ $c_s = b$, $\delta\mathbf{H}^{\pm} = -\epsilon\, H\hat{\mathbf{n}}$, $\delta\mathbf{u}^{\pm} = \pm\epsilon\, b\,\mathrm{sgn}(H_n)\hat{\mathbf{n}}$, † $\delta\rho^{\pm} = 0$, $\delta S^{\pm} = 0$,	(1,4) $\mathbf{H} \perp \mathbf{n}$ $c_s = 0$ $\delta\mathbf{H}^{\pm} = -\epsilon\,\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$ $\delta\mathbf{u}^{\pm} = \pm\epsilon\left[\frac{\mu(r+1)}{\rho r}\right]^{1/2}\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$ $\delta\rho^{\pm} = -\epsilon\rho r$ $\delta S = 0$ (Contact Discontinuity)
(2) ALFVÉN WAVE	(2,1) $c_A = b_n$, $\delta\mathbf{H}^{\pm} = \pm\epsilon\,\mathbf{n}\times\mathbf{H}$, $\delta\mathbf{u}^{\pm} = -\epsilon\frac{\mu H_n}{\rho b_n}\mathbf{n}\times\mathbf{H}$, $\delta\rho^{\pm} = \delta S^{\pm} = 0$.	(2,2) $c_A = b$, $\delta\mathbf{H}^{\pm} = \pm\epsilon\, H\mathbf{n}^{*}$, $\delta\mathbf{u}^{\pm} = -\epsilon\, b\,\mathrm{sgn}(H_n)\mathbf{n}^{*}$, †† $\delta\rho^{\pm} = \delta S = 0$,	(2,3) same as the entry in column (2)	(2,4) $c_A = 0$ $\delta\mathbf{H}^{\pm} = \pm\epsilon\,\mathbf{n}\times\mathbf{H}$ $\delta\mathbf{u}^{\pm} = -\epsilon\,\mathbf{n}\times\mathbf{H}$ $\delta\rho^{\pm} = 0, \quad \delta S^{\pm} = 0$ (Contact discontinuity)
(3) FAST WAVE	(3,1) $c_f/b = \left\{\frac{1}{2}(1+r) + \frac{1}{2}\left[(1+r)^2 - 4r\cos^2\theta\right]^{1/2}\right\}^{1/2}$ $\delta\mathbf{H}^{\pm} = -\epsilon\,\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$, $\delta\mathbf{u}^{\pm} = \mp\epsilon\, c_f\left(\frac{b_n^2}{c_f^2}-1\right)\mathbf{n} \pm \epsilon\frac{\mu H_n}{\rho c_f}\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$ $\delta\rho^{\pm} = -\epsilon\,\rho\left(\frac{b_n^2}{c_f^2}-1\right)$, $\delta S^{\pm} = 0$.	(3,2) , $c_f = b$, $\delta\mathbf{H}^{\pm} = -\epsilon\, H\hat{\mathbf{n}}$, $\delta\mathbf{u}^{\pm} = \pm\epsilon\, b\,\mathrm{sgn}(H_n)\hat{\mathbf{n}}$ †† $\delta\rho^{\pm} = 0$, $\delta S^{\pm} = 0$.	(3,3) $c_f = a$, $\delta\mathbf{H}^{\pm} = 0$ $\delta\mathbf{u}^{\pm} = \mp\epsilon\, a\left(\frac{1}{r}-1\right)\mathbf{n}$ $\delta\rho^{\pm} = -\epsilon\,\rho\left(\frac{1}{r}-1\right)\mathbf{n}$ $\delta S^{\pm} = 0$	(3,4) $c_f = (a^2+b^2)^{1/2}$ $\delta\mathbf{H}^{\pm} = -\epsilon\,\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$ $\delta\mathbf{u}^{\pm} = \pm\epsilon\left[a^2+b^2\right]^{1/2}\mathbf{n}$ $\delta\rho^{\pm} = \epsilon\rho$ $\delta S^{\pm} = 0$

† $r = a^2/b^2$

†† $\mathrm{sgn}(H_n)$ is the sign of H_n

Replace $\epsilon\left(\frac{1}{r}-1\right)$ by ϵ when $r = 1$.

upper sign belongs to a propagation with a component along $\underset{\sim}{n}$, the lower sign along $-\underset{\sim}{n}$. Thus each of the first three columns lists six mutually orthogonal propagating modes.

Assuming that $\underset{\sim}{n}$ is neither parallel nor perpendicular to $\underset{\sim}{H}$, we have collected in column (1) all wave-mode solutions of (2.8)-(2.10). The remaining columns, (2)-(4), list solutions corresponding to other orientations of $\underset{\sim}{n}$ with respect to $\underset{\sim}{H}$. These solutions may be obtained from the corresponding solutions in column (1) by carrying out various sorts of limiting processes. Thus, assuming $r \neq 1$, the solutions in boxes (1,2) and (3,3) may be obtained simply by rotating $\underset{\sim}{n}$ into $\underset{\sim}{H}$. The solution for $r = 1$ in box (3,3) may then be obtained by replacing ϵ by $\epsilon/(r^{-1}-1)$ and then letting r approach unity.[14] However the solutions of columns (2)-(4) are derived, it is easy to verify directly that they satisfy the basic discontinuity relations (2.9)-(2.11).

Observe that two disturbance speeds in column 2 and at least two disturbance speeds in column 3 are the same. Our classification of the associated waves as slow and Alfvén and Alfvén and fast, etc. is therefore, strictly speaking, an arbitrary one - to be sure, one that is suggested by limiting processes. The fact that at least two disturbance speeds coalesce implies that λ in equation (3.1) is a multiple eigenvalue. It follows that the associated eigenvectors need not be orthogonal although we have chosen them to be so for convenience of calculation.

To obtain the solutions listed in column (1) from small-disturbance waves catalogued by Friedrichs[2], it is necessary to replace k by $\epsilon/\rho c_s^2$ and $\epsilon/\rho c_f^2$ in his slow and fast-wave formulas and by $\epsilon/\rho c_A$ in his Alfvén-wave formulas. A derivation of the solutions in Table I from the corresponding waves of finite amplitude has been given by Bazer and Ericson[13].

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14. A more systematic derivation (up to a constant factor of the solutions listed in columns (2) and (3) may be described as follows: Write each mode R (see (3.13) below) in the form $R = \epsilon \hat{R}$ where $\hat{R} \cdot \hat{R} = 1$. Then rotate $\underset{\sim}{n}$ into $\underset{\sim}{H}$ keeping the direction $\underset{\sim}{n} \times \underset{\sim}{H}$ fixed. The limiting expressions for the $\hat{R}$'s yield the desired results as long as $r \neq 1$. When $r = 1$,

One feature of the solutions of columns (2) and (3) - the solutions in which $\underset{\sim}{n}$ is parallel to $\underset{\sim}{H}$ - deserves special notice. In contrast to the corresponding solutions of column (1), the column (2)- and (3)— solutions are <u>unpolarized</u>. Specifically, the directions of the tangential components of $\delta\underset{\sim}{H}$ in the column (1)— solutions are fixed by the directions of $\underset{\sim}{n}$ and $\underset{\sim}{H}$. This is no longer true of the column (2)- and (3)— solutions; for when $\underset{\sim}{H}$ is parallel to $\underset{\sim}{n}$, a wave of any polarization can propagate along $\underset{\sim}{H}$. To prove this statement, it is necessary only to form a suitable linear combination of the solutions in boxes (2,2) and (3,2) or (1,3) and (2,3) of the table according as $r < 1$ or $r \geq 1$. The $\underset{\sim}{H}$-direction here plays a role similar to that of an optic axis in a doubly refracting crystal.

Hereafter, when we speak of a disturbance D^o on an initial surface $\mathcal{S}^o$, we shall mean a six-vector,

$$D^o = \left[(\mu/\rho)^{1/2}\delta H_y^o, (\mu/\rho)^{1/2}\delta H_z^o, \delta u_n^o, \delta u_y^o, \delta u_z^o, \; a\delta\rho_o/\rho\right], \tag{3.11}$$

that may be represented as a sum of the form

$$D^o = R_s^+ + R_s^- + R_A^+ + R_A^- + R_f^+ + R_f^-. \tag{3.12}$$

In these equations, δH_y^o, δH_z^o, δu_n^o, etc., are prescribed (small) jumps of H_y, H_z, u_n, etc., across $\mathcal{S}^o$. The R's are modes expressed in the form

$$R^{\pm} = \left[(\mu/\rho)^{1/2}\delta H_y^{\pm}, (\mu/\rho)^{1/2}\delta H_z^{\pm}, \delta u_n^{\pm}, \delta u_y^{\pm}, \delta u_z^{\pm}, a\delta\rho^{\pm}/\rho\right]. \tag{3.13}$$

The values of $\delta H_y^{\pm}$, $\delta H_z^{\pm}$, $\delta u_n^{\pm}$ are to be chosen from the appropriate row and column of the table, with proper heed being paid to the choice of the upper or lower sign. The representation (3.12) may be obtained by making use of the orthogonality properties of the R's and adjusting the ϵ's in a suitable manner. If $\epsilon = o$ for some R, that R is a absent in (3.12).

If it is desired to take into account contact discontinuities not connected with the solutions listed in column (1) by means of a limiting process, it is necessary to add a δS-component to the D^o and $R_s^{\pm}$, $R_A^{\pm}$ and $R_f^{\pm}$ - vectors and to add a contact-discontinuity mode - R_c say - to the right member of equations (3.12).

IV. RAYS AND WAVE FRONTS

1. Surface of Wave Normals

Assuming $\phi_t(\underset{\sim}{x},t) \neq 0$, we may write the equation $\phi(\underset{\sim}{x},t) = 0$ (see (2.5)) as follows:

$$\phi(\underset{\sim}{x},t) = W(\underset{\sim}{x}) - (t-t_o). \tag{4.1}$$

The surfaces $W(\underset{\sim}{x})$ = constant are called <u>wave fronts</u>.

Setting[15]

$$\underset{\sim}{p} = \nabla W, \qquad p = |\nabla W|, \tag{4.2}$$

we find, using the definition of $\underset{\sim}{n}$, U and c [see (2.6) and (2.7)'], that

$$\underset{\sim}{n} = \underset{\sim}{p}/p, \tag{4.3}$$

$$U = 1/p, \tag{4.4}$$

and

$$c = p^{-1} - (\underset{\sim}{u}\cdot\underset{\sim}{p})/p. \tag{4.5}$$

Note that U is non-negative.

In terms of this notation, we may re-express the determinantal equation (3.2) as

$$0 = p^{-6}\mathcal{H}(\underset{\sim}{x},\underset{\sim}{p}) = p^{-6}\mathcal{H}_1(\underset{\sim}{x},\underset{\sim}{p})\mathcal{H}_2(\underset{\sim}{x},\underset{\sim}{p})\mathcal{H}_3(\underset{\sim}{x},\underset{\sim}{p}), \tag{4.6}$$

where

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15. In Sections I-III the symbol p was used to denote the pressure. Hereafter, we shall use p to denote $|\nabla W|$. The quantities p and $\underset{\sim}{p}$ play roles similar to those of the total and the vector momentum variables in Mechanics - hence our notation.

$$\mathcal{H}_i(\underset{\sim}{x},\underset{\sim}{p}) = (\underset{\sim}{u}\cdot\underset{\sim}{p}-1)^2 - p^2c_i^2(\underset{\sim}{x},\underset{\sim}{n}), \qquad i = 1,2,3, \tag{4.7}$$

the quantities c_i being defined by the equations

$$c_1 = c_A(\underset{\sim}{x},\underset{\sim}{n}), \tag{4.8}$$

$$c_2 = c_s(\underset{\sim}{x},\underset{\sim}{n}), \tag{4.9}$$

$$c_3 = c_f(\underset{\sim}{x},\underset{\sim}{n}). \tag{4.10}$$

The functions $\mathcal{H}(\underset{\sim}{x},\underset{\sim}{p})$, $\mathcal{H}_1(\underset{\sim}{x},\underset{\sim}{p})$, $\mathcal{H}_2(\underset{\sim}{x},\underset{\sim}{p})$ and $\mathcal{H}_3(\underset{\sim}{x},\underset{\sim}{p})$ will be referred to as the total, Alfvén, slow and fast Hamiltonian's respectively. In equations (4.8)-(4.10), the right members are the (positive) disturbance speeds defined in equations (3.7)-(3.9) but with $\cos^2\theta$ replaced by $(\underset{\sim}{p}\cdot\underset{\sim}{H})^2/p^2H^2 \left[=(\underset{\sim}{n}\cdot\underset{\sim}{H})^2/H^2\right]$. In general, the disturbance speeds depend on $\underset{\sim}{x}$ as well as $\underset{\sim}{p}$; they are homogeneous functions of degree zero in $\underset{\sim}{p}$.

The equation

$$\mathcal{H}(\underset{\sim}{x},\underset{\sim}{p}) = 0 \tag{4.11}$$

represents for fixed $\underset{\sim}{x}$ a surface in $\underset{\sim}{p}$-space called the <u>surface of wave normals</u>. Factoring the right member of (4.7), we may easily verify that this surface consists of three branches whose equations may be obtained by setting each of the following three pairs of functions equal to zero:

$$\mathcal{H}_i^{\pm}(\underset{\sim}{x},\underset{\sim}{p}) = (\underset{\sim}{u}\cdot\underset{\sim}{p}-1) \pm pc_i(\underset{\sim}{x},\underset{\sim}{n}) \qquad i = 1,2,3. \tag{4.12}$$

Let us examine the surface of wave normals in its simplest form - when $\underset{\sim}{u} \equiv 0$. Since p is non-negative, only branches $\mathcal{H}_i^{+}(\underset{\sim}{x},\underset{\sim}{p}) = 0$, $i = 1,2,3$, are 'admissible'. Combining these equations with equations (4.8)-(4.10) we find:

Alfvén branch: $$p_A^+ = \frac{1}{c_A(\underset{\sim}{x},\underset{\sim}{n})} = \frac{1}{b|\cos\theta|}, \tag{4.13}$$

slow branch: $$p_s^+ = \frac{1}{c_s(\underset{\sim}{x},\underset{\sim}{n})} = b^{-1}\left[\left(\frac{1+r}{2}\right) - \frac{1}{2}\,C(\underset{\sim}{x},\underset{\sim}{n})\right]^{-1/2}, \tag{4.14}$$

fast branch: $$p_f^+ = \frac{1}{c_f(\underset{\sim}{x},\underset{\sim}{n})} = b^{-1}\left[\left(\frac{1+r}{2}\right) + \frac{1}{2}\, C(\underset{\sim}{x},\underset{\sim}{n})\right]^{-1/2}, \tag{4.15}$$

where $C(\underset{\sim}{x},\underset{\sim}{n})$ is defined by

$$C(\underset{\sim}{x},\underset{\sim}{n}) = \left[(1+r)^2 - 4r(\underset{\sim}{n}\cdot\underset{\sim}{H})^2/H^2\right]^{\frac{1}{2}} = \left[(1+r)^2 - 4r\cos^2\theta\right]^{\frac{1}{2}} . \tag{4.16}$$

Here, θ denotes the angle between $\underset{\sim}{H}$ and $\underset{\sim}{p}$. Evidently, the wave-normal surface is a surface of revolution, the H-axis serving as the axis of symmetry. In Figures 2a), b) and c) we have plotted cross sectional views of pb versus θ. The parameter r is chosen to be 1/2, 1 and 3/2. These curves may be obtained from the corresponding curves of Figure 1 by inverting with respect to the unit circle. The curves of Figures a) and c) are typical of the curves in the parameter classes $r < 1$ and $r > 1$, respectively. Whatever the value of r, the oval-like figures correspond to the fast branch; the solid straight lines, perpendicular to the H-axis, correspond to the Alfvén branch and the hyperbola-like curves correspond to the slow branch. The pair of dotted lines represent the traces on the cross section of the planes

$$pb|\cos\theta| = (1+r/r)^{1/2}, \quad 0 \le \theta < 2\pi, \tag{4.17}$$

to which the slow branch is asymptotic. The Alfvén branch intersects the slow branch, the fast branch or both on the H-axis according as r exceeds unity, is less than unity or equals unity. All branches have zero curvature on the H-axis, as long as $r \neq 1$. When $r = 1$, both the slow and the fast branches are singular on the H-axis —— they have points in common at $pb = 1$, $\theta = 0$ and $pb = 1$, $\theta = \pi$ and at each of these points form a cone of two nappes whose generators make an angle of $\chi = \tan^{-1}(1/2)$ radians with the H-axis (see Figure 3b).

Any ray through the origin of $\underset{\sim}{p}$-space in the direction $\underset{\sim}{n}$ intersects each of the three branches in exactly one point unless $\underset{\sim}{n}$ is perpendicular to the

H-axis in which case there is no point of intersection with the slow branch.

To generalize these results to the case in which $\underset{\sim}{u} \neq 0$, we introduce θ_u as the angle between $\underset{\sim}{u}$ and $\underset{\sim}{p}$ and rewrite the equations $\mathcal{H}_i^{\pm}(\underset{\sim}{x},\underset{\sim}{p}) = 0$, $i = 1,2,3$, as follows:

$$\left(p_i^{\pm}\right)^{-1} = U_i^{\pm} = u \cos \theta_u \pm c_i(\underset{\sim}{x},\underset{\sim}{n}) \qquad i = 1,2,3. \tag{4.18}$$

For each i, $i = 1,2,3$, three cases must be distinguished:[16] case (1), $u/c_i < 1$; case (2), $u/c_i = 1$; case (3), $u/c_i > 1$. Now we must have $p_i^{\pm} \geq 0$, $i = 1,2,3$; it follows in case (1), the 'subsonic' case, that there can be only one intersection of a ray in the direction $\underset{\sim}{n}$ with the i-th branch; namely, the one corresponding to the '+' sign in (4.18). The same applies to the case (2), the 'sonic case', except that intersections corresponding to the rays making an angle of $\theta_u = \pi$ with the $\underset{\sim}{u}$-direction are excluded. In case (3), the 'supersonic case', there are three alternatives. For all $\theta_u = \theta_u(\underset{\sim}{n})$ such that $0 \leq \theta_u < \cos^{-1}(c_i/u)$ there are two intersections; p_i in (4.18) is positive for either choice of sign. Only the positive sign leads to an intersection when $\theta_u(\underset{\sim}{n})$ is in the range $\cos^{-1}(c_i/u) \leq \theta_u < \cos^{-1}(-c_i/u)$. If $\theta_u \geq \cos^{-1}(c_i/u)$ there is no intersection. It should be emphasized that both θ_u and c_i depend on $\underset{\sim}{n}$ so that all of the above relations are implicit conditions on $\underset{\sim}{n}$. An explicit representation of the wave-normal surface could be given; but we shall not do so here. It is enough to say, that these results may be readily understood in terms of the vectorial addition of $\underset{\sim}{u}$ and $c_i(\underset{\sim}{x},\underset{\sim}{n})\underset{\sim}{n}$. The surface of normal speeds (Figure 1) is useful for this purpose.

2. Rays and Wave Fronts

Since $\underset{\sim}{p} = \nabla W_i$ each of the equations

$$\mathcal{H}_i^{\pm}(\underset{\sim}{x}_i,\underset{\sim}{p}_i) = 0, \qquad i = 1,2,3, \tag{4.19}$$

is a first-order partial differential equation. In light of the foregoing discussion, some of these equations are not 'admissible' in that they lead

16. This argument closely parallels one given by J.B. Keller[4] for the corresponding acoustic surface of wave normals.

to negative values for $p = |\nabla W|$. Here, and hereafter it will be assumed that we are dealing with admissible equations.

As is well known, first-order partial differential equations can be solved by means of rays. Rays are simply curves in $\underset{\sim}{x}$-space that are obtained by solving a system of ordinary differential equations in $(\underset{\sim}{x},\underset{\sim}{p})$ space. If the time t is chosen as the curve parameter, this system may easily be shown to take the following form[17]:

$$\frac{d\hat{\underset{\sim}{x}}}{dt} = \frac{\nabla_{\hat{p}}\hat{\mathcal{H}}}{\hat{p}\cdot\nabla_{\hat{p}}\hat{\mathcal{H}}}, \tag{4.20}$$

$$\frac{d\hat{\underset{\sim}{p}}}{dt} = -\frac{\nabla\hat{\mathcal{H}}}{\hat{p}\cdot\nabla_{\hat{p}}\mathcal{H}}, \qquad t \geq t_o . \tag{4.21}$$

Note

$$\hat{p}\cdot\frac{d\hat{\underset{\sim}{x}}}{dt} = 1, \qquad t \geq t_o \tag{4.22}$$

with this parametrization.

In these equations, $\hat{\mathcal{H}}$ represents any one of six Hamiltonian factors introduced in (4.12) and $\hat{\underset{\sim}{x}}$, $\hat{\underset{\sim}{p}}$ denote its arguments. The symbols ∇ and $\nabla_{\hat{p}}$ are, respectively, the gradient operators in $\hat{\underset{\sim}{x}}$ and $\hat{\underset{\sim}{p}}$-space. Executing the indicated calculations for each $\hat{\mathcal{H}} = \mathcal{H}_i^{\pm}$, $i = 1,2,3$, we obtain the following pairs of ray systems:

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17. Cf. R. Courant and D. Hilbert, Mathematische Physik, (Springer, Berlin, 1937) Vol. 2, p. 82. Let $w = W(\underset{\sim}{x})$, $\underset{\sim}{p} = \nabla W$ and suppose $\mathcal{H}(\underset{\sim}{x},w,\underset{\sim}{p}) = 0$. According to the general theory, the ray system associated with this first-order partial differential equation is: $\frac{d\underset{\sim}{x}}{d\sigma} = \nabla_p\mathcal{H}$; $\frac{d\underset{\sim}{p}}{d\sigma} = -(\nabla\mathcal{H} + \mathcal{H}_w\underset{\sim}{p})$; $\frac{dw}{d\sigma} = \underset{\sim}{p}\cdot\nabla_p\mathcal{H}$. Identifying w (cf. equation (4.11)) and σ with $t-t_o$, we obtain a system of the same form as that given in (4.20) and (4.21).

± Alfvén Rays

$$\frac{d\mathbf{x}^{\pm}}{dt} = \mathbf{u} \pm (\mu/\rho)^{\frac{1}{2}} \operatorname{sgn}(\mathbf{H}\cdot\mathbf{p})\mathbf{H}, \qquad (4.23)$$

$$\frac{d\mathbf{p}^{\pm}}{dt} = -(\mathbf{p}\cdot\nabla)\left[\mathbf{u} \pm (\mu/\rho)^{\frac{1}{2}} \operatorname{sgn}(\mathbf{H}\cdot\mathbf{p})\mathbf{H}\right] - \mathbf{p}\times\nabla\times\left[\mathbf{u} \pm (\mu/\rho)^{\frac{1}{2}} \operatorname{sgn}(\mathbf{H}\cdot\mathbf{p})\mathbf{H}\right],$$

± Slow Rays

$$\frac{d\mathbf{x}^{\pm}}{dt} = \mathbf{u} \pm c_s\mathbf{n} \pm \frac{b^2 r(H_n/H^2)}{c_s(\mathbf{x},\mathbf{n})C}(\mathbf{H} - H_n\mathbf{n}) ,$$

$$\frac{d\mathbf{p}^{\pm}}{dt} = -(\mathbf{p}\cdot\nabla)\,\mathbf{u} - \mathbf{p}\times\nabla\times\mathbf{u} \mp p\,\nabla c_s , \qquad (4.24)$$

± Fast Rays

$$\frac{d\mathbf{x}^{\pm}}{dt} = \mathbf{u} \pm c_f\mathbf{n} \mp \frac{b^2 r(H_n/H^2)}{c_f(\mathbf{x},\mathbf{n})C}(\mathbf{H} - H_n\mathbf{n})$$

$$\frac{d\mathbf{p}^{\pm}}{dt} = -(\mathbf{p}\cdot\nabla)\,\mathbf{u} - \mathbf{p}\times\nabla\times\mathbf{u} \mp p\,\nabla c_f . \qquad (4.25)$$

In (4.23), $\operatorname{sgn}(\mathbf{H}\cdot\mathbf{p})$ is the sign of $\mathbf{H}\cdot\mathbf{p}$. As in the optics of crystals, the rays are in general not perpendicular to the wave fronts.

We turn now to the problem of constructing the wave fronts. Let

$$\mathbf{x}^{o} = \mathbf{f}^{o}(\xi_1,\xi_2) \qquad (4.26)$$

be a parametric representation of $\mathcal{S}^{o}$. We shall suppose that $\mathbf{f}$, $\partial\mathbf{f}^{o}/\partial\xi_1$, $\partial\mathbf{f}^{o}/\partial\xi_2$ are continuous and

$$\frac{\partial\mathbf{f}^{o}}{\partial\xi_1} \times \frac{\partial\mathbf{f}^{o}}{\partial\xi_2} \neq 0$$

so that the normal is defined at all points of $\mathcal{S}^{o}$.

A recipe for constructing the wave fronts is the following:

1) Find the wave normal

$$\hat{\mathbf{p}}^{o} = \hat{\mathbf{P}}^{o}(\xi_1,\xi_2)$$

at points, $\underset{\sim}{x}^{o} = \underset{\sim}{f}^{o}(\xi_1,\xi_2)$ of $\mathcal{S}^{o}$ by solving the system of equations

$$\hat{\underset{\sim}{p}}^{o} \cdot \frac{\partial \underset{\sim}{f}^{o}}{\partial \xi_1} = 0, \tag{4.27}$$

$$\hat{\underset{\sim}{p}}^{o} \cdot \frac{\partial \underset{\sim}{f}^{o}}{\partial \xi_2} = 0, \tag{4.28}$$

$$\hat{\mathcal{H}}(\underset{\sim}{x}^{o},\hat{\underset{\sim}{p}}^{o}) = 0. \tag{4.29}$$

Equations (4.27) and (4.28) express the fact that $\hat{\underset{\sim}{p}}^{o}$ is normal to $\mathcal{S}^{o}$ at $\underset{\sim}{x}^{o} = \underset{\sim}{f}^{o}(\xi_1,\xi_2)$. They define a line through the origin of $\underset{\sim}{p}$-space whose intersections with the branch $\hat{\mathcal{H}}(\underset{\sim}{x}^{o},\hat{\underset{\sim}{p}}) = 0$ determines two wave normals $\hat{\underset{\sim}{p}}^{o}$ and $-\hat{\underset{\sim}{p}}^{o}$. (As to whether these wave normals exist or not, see our earlier discussion in subsection 1.)

2) Next obtain a solution of each system of ray equations corresponding to each pair of initial values. Fixing on a given pair - $(\hat{\underset{\sim}{x}}^{o},\hat{\underset{\sim}{p}}^{o})$ say - we must find a solution of equations (4.20)-(4.21) of the form

$$\hat{\underset{\sim}{x}} = \hat{\underset{\sim}{F}}(\xi_1,\xi_2,t-t_o), \qquad t \geq t_o, \tag{4.30}$$

$$\hat{\underset{\sim}{p}} = \hat{\underset{\sim}{P}}(\xi_1,\xi_2,t-t_o), \qquad t \geq t_o, \tag{4.31}$$

such that

$$\hat{\underset{\sim}{F}}(\xi_1,\xi_2,0) = \underset{\sim}{f}^{o}(\xi_1,\xi_2) = \underset{\sim}{x}^{o}, \tag{4.32}$$

$$\hat{\underset{\sim}{P}}(\xi_1,\xi_2,0) = \hat{\underset{\sim}{P}}^{o}(\xi_1,\xi_2) = \hat{\underset{\sim}{p}}^{o}. \tag{4.33}$$

This is the standard initial value problem for the system (4.20)-(4.21). According to the general theory it is always possible to obtain a solution in sufficiently small neighborhoods of the point $\underset{\sim}{x}^{o} = \underset{\sim}{f}^{o}(\xi_1,\xi_2)$ if $\hat{\mathcal{H}}(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{p}})$ is continuous and has continuous partial derivatives of no less than the second order and if

$$\frac{\partial \underset{\sim}{f}^o}{\partial \xi_1} \times \frac{\partial \underset{\sim}{f}^o}{\partial \xi_2} \cdot \left(\nabla_{\hat{p}} \hat{\mathcal{H}} \Big|_{\substack{\underset{\sim}{x}=\underset{\sim}{x}^o \\ \hat{\underset{\sim}{p}}=\hat{\underset{\sim}{p}}^o}} \right) = \frac{\partial \underset{\sim}{f}^o}{\partial \xi_1} \times \frac{\partial \underset{\sim}{f}^o}{\partial \xi_2} \cdot \left(\frac{d\hat{\underset{\sim}{x}}}{dt} \Big|_{t=t_o} \right) \neq 0 \qquad (4.34)$$

- i.e., if the initial ray-direction is not parallel to $\mathcal{S}^o$ at $\underset{\sim}{x}^o$. This requirement is also a sufficient condition for the system (4.27)-(4.29) to be solvable for $\hat{\underset{\sim}{p}}^o$ (provided of course that $\hat{\mathcal{H}} = 0$ represents an admissible branch).

3) Equation (4.30) expresses at each time t the equation of the wave front in parametric form. The condition (4.34) enables us to solve for $t-t_o$, ξ_1 and ξ_2 in the right member of (4.30) in terms of the components of $\underset{\sim}{x}$ in the left member. The resulting equation for $t-t_o$ as a function of $\underset{\sim}{x}$ yields the equation of the wave front in the non-parametric form of equation (4.1).

4) The above recipe applies to the case in which the surface $\mathcal{S}^o$ shrinks to a point; but here, equations (4.27) and (4.28) must be ignored.

3. Special Cases: Graphical Construction of Wave Fronts; Fresnel's Ray Surface; Conical Propagation

3.1 Graphical Construction of Wave Fronts. In this and in the next two subsections it will be assumed that ρ, $\underset{\sim}{H}$ are constant and $\underset{\sim}{u} \equiv 0$. Let $\delta\mathcal{S}^o$ be a small neighborhood of the initial surface $\mathcal{S}^o$, located at $\underset{\sim}{x}^o$. Let $\underset{\sim}{n}^o$ denote the unit normal at $\underset{\sim}{x}^o$. Then, it follows from equations (4.23)-(4.25), that i) $\underset{\sim}{n} = \underset{\sim}{p}/p$ is constant in each of the ray systems; ii) $\underset{\sim}{n} = \pm \underset{\sim}{n}^o$ depending upon whether forward or backward propagating waves are being considered and iii) the rays are straight lines whose equations are:

$$\text{Alfvén rays:} \qquad \underset{\sim}{x} - \underset{\sim}{x}^o = \pm (\mu/\rho)^{1/2} \underset{\sim}{H}(t-t_o), \qquad (4.35)$$

$$\text{Slow rays:} \qquad \underset{\sim}{x} - \underset{\sim}{x}^o = \pm c_s \underset{\sim}{n}^o \pm \frac{b^2 r(H_{n^o}/H^2)}{c_s\, C} (\underset{\sim}{H} - H_{n^o} \underset{\sim}{n}^o)(t-t_o), \qquad (4.36)$$

$$\text{Fast rays:} \quad \underset{\sim}{x} - \underset{\sim}{x}^{o} = \pm c_f \underset{\sim}{n}^{o} \mp \frac{b^2 r(H_{n^o}/H^2)}{c_f\, C} (\underset{\sim}{H} - H_{n^o}\underset{\sim}{n}^{o})(t-t_o). \tag{4.37}$$

In these equations, c_s, c_f and C are constant on the rays; they depend on orientation of $\underset{\sim}{n}^o$ with respect to $\underset{\sim}{H}$ [see (3.7)-(3.9) and (4.16)] which, as we have seen, is constant. In geometric terms i)-iii) above tell us that the six elemental wave fronts, into which $\delta\mathcal{J}^o$ splits, are carried along their rays parallel to themselves and to $\delta\mathcal{J}^o$.

Alfvén wave fronts evolving from $\mathcal{J}^o$ are especially easy to describe. For the sake of concreteness let us assume that $\mathcal{J}^o$ is a sphere. Then $\mathcal{J}^o$ splits up into two congruent spheres propagating along $\underset{\sim}{H}$ with velocities $(\mu/\rho)^{1/2}\underset{\sim}{H}$ and $-(\mu/\rho)^{1/2}\underset{\sim}{H}$, respectively, (see Figure 3a). In short, Alfvén wave fronts propagate one-dimensionally[18].

To learn how to construct the slow and fast wave fronts, let us turn to Figure 3b. In this figure, $\delta\mathcal{J}_f$ represents the element of the wave front that, at time t, has evolved from $\delta\mathcal{J}^o$ along the fast forward ray passing through $\underset{\sim}{x}^o$. It is assumed that $\underset{\sim}{H}$ and $\underset{\sim}{n}^o$ lie in the plane of the page; hence so does the ray-segment $\underset{\sim}{x}-\underset{\sim}{x}^o$ (see equation (4.37)). The line ℓ is the trace, in the plane of the page, of the tangent plane to $\delta\mathcal{J}_f$. The point $\underset{\sim}{y}$ is the intersection of the line through $\underset{\sim}{x}^o$ and perpendicular to ℓ; - i.e., $y - \underset{\sim}{x}^o$ is directed along $\underset{\sim}{n}^o$. The basis of the construction is this: The length $\underset{\sim}{y} - \underset{\sim}{x}^o$ is simply $c_f(t-t_o)\underset{\sim}{n}^o$, where c_f is the disturbance speed associated with the direction $\underset{\sim}{n}^o$. This result follows immediately from (4.37) on projecting $\underset{\sim}{x} - x^o$ along $\underset{\sim}{n}^o$ and suggests the following graphical method: At each $\underset{\sim}{x}^o$ on $\mathcal{J}^o$, lay off a line segment of length $c_f(t-t_o)$ along $\underset{\sim}{n}^o = \underset{\sim}{n}^o(\underset{\sim}{x}^o)$. Let $\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}^o;t)$ denote the endpoint of this segment. Through each $\underset{\sim}{y}$ pass a plane normal to $\underset{\sim}{y} - \underset{\sim}{x}^o$ -

18. The propagation of elements $\delta\mathcal{J}^o$ of $\mathcal{J}^o$ having normals that are perpendicular to $\underset{\sim}{H}$ require a more careful analysis than is given here; for among other things, condition (4.34) is violated. This remark applies to the slow as well as to the Alfvén wave.

i.e., normal to $\underset{\sim}{n}^{o}$. The fast wave front (in the forward direction) is then formed by the envelope of these planes. The same construction applies to the backward propagating fast waves and forward and backward-propagating slow and Alfvén waves.

This construction has already been employed by Friedrichs[2] to determine the wave fronts emanating from a point disturbance.[†]

It should be observed that the parametric equations for the slow and fast wave fronts evolving from a sphere of radius R are relatively simple. For example, the equations for the slow wave front are (cf. (4.36)):

$$\frac{x_s}{b} = \frac{R\cos\theta}{b} \pm \cos\theta\left[\frac{c_s}{b} - \frac{r(1+\cos^2\theta)}{(c_s/b)C}\right](t-t_o), \tag{4.38}$$

$$\frac{y_s}{b} = \frac{R\sin\theta\cos\phi}{b} \pm \sin\theta\cos\phi\left[\frac{c_s}{b} - \frac{r\cos^2\theta}{(c_s/b)C}\right](t-t_o), \tag{4.39}$$

$$\frac{z_s}{b} = \frac{R\sin\theta\sin\phi}{b} \pm \sin\theta\sin\phi\left[\frac{c_s}{b} - \frac{r\cos^2\theta}{(c_s/b)C}\right](t-t_o). \tag{4.40}$$

In these equations, $\underset{\sim}{H}$ is assumed to be directed along the positive x-axis. The angles ϕ and θ are measured from the x and y-axes, respectively. The quantities c_s and C depend only on $\cos^2\theta$ (see equations (3.7) and (4.16)).

3.2 Fresnel's Ray Surface. Setting R = 0 in (4.38)-(4.40) and in the corresponding equations for the Alfvén and fast wave fronts we obtain the parametric equations of the wave fronts emanating from a point disturbance. It is easily verified that the slow and fast wave fronts are surfaces of revolution about the $\underset{\sim}{H}$-axis and that the Alfvén wave consists of two points located at $\pm(\mu/\rho)^{1/2}\underset{\sim}{H}(t-t_o)$. Figures 4a-4c are cross sections through the H-axis of these wave fronts. The wave fronts in Figures 4a and 4c are typical of those associated with values of r less than or greater than unity, respectively.

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† See also H. Grad's work, reference [9]

The Alfvén waves are indicated by large dots; these dots are connected to the slow wave, the fast wave or to both according as $r > 1$, $r < 1$ or $r = 1$.

The surfaces depicted in these figures are called Fresnel ray surfaces. They are related to the surfaces of wave normals as follows: First, the normal at a point of a Fresnel ray surface is directed along $\underset{\sim}{n} = \underset{\sim}{p}/p$ and the normal at a point $\underset{\sim}{p}$ of a wave-normal surface is directed along the ray associated with $\underset{\sim}{n} = \underset{\sim}{p}/p$. Second, each surface may be obtained from the pedal surface of the other by an inversion with respect to the unit circle[19]. The pedal surface of a given surface $\mathcal{S}$ is constructed as follows: Let T be the tangent plane at a point $\underset{\sim}{x}$ of $\mathcal{S}$. From the origin, draw a line perpendicular to T. Let $\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x})$ denote the foot of this perpendicular. As $\underset{\sim}{x}$ varies over, $\mathcal{S}$, $\underset{\sim}{y}$ traces out a surface called the pedal surface of $\mathcal{S}$. Thus the surface $\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}^o,t)$ in subsection 3.1 is the pedal surface of the wave front in that discussion.

3.3 Conical Propagation. When $\underset{\sim}{n}$ is directed along $\underset{\sim}{H}$ and $r \neq 1$, the slow and fast rays on Fresnel's ray surfaces are also directed along $\underset{\sim}{H}$. This follows directly from equations (4.38)-(4.40) and the corresponding fast wave equations on setting R and θ equal to zero (see also Figures 2a, 2c, 4a and 4c). On the other hand, when $r = 1$ - i.e., when the Alfvén speed is sonic - the same formulas yield two rays $\underset{\sim}{s}$ and $\underset{\sim}{s}'$ (see Figure 4b) for each ϕ. These rays are of equal length and make an angle of $\chi = \tan^{-1}(1/2)$ radians with the H-axis. In Figure 4b $\underset{\sim}{s}$ and $\underset{\sim}{s}'$ correspond to the two normals $\underset{\sim}{N}$ and $\underset{\sim}{N}'$ at the point K in Figure 2b. As ϕ is varied the endpoints of $\underset{\sim}{s}$ and $\underset{\sim}{s}'$ trace out a circle $\mathcal{C}^+$ of radius $\frac{b}{2}(t-t_o)$ on the ray surface. Similar remarks apply to the case in which $\underset{\sim}{n}$ is directed along $(-\underset{\sim}{H})$; denote by $\mathcal{C}^-$ the circle associated with this direction. The planes of $\mathcal{C}^{\pm}$ are normal to $\underset{\sim}{H}$ and their centers are located at $\pm(\mu/\rho)^{1/2}\underset{\sim}{H}(t-t_o)$.

19. Proofs of these facts are given in References 5 and 17.

This, however, is not the complete picture. It is known from the general theory of symmetric-hyperbolic partial differential equations that the domain of dependence of a general disturbance must be convex[20]. In the present case, this implies that all points of the discs whose outer edges are $\mathcal{C}^{\pm}$ must belong to the Fresnel's ray surface[21]. In Figure 4b dashed line segments PP' and QQ' represent these discs in cross section.

These facts support the existence of the following exceptional sort of propagation[22]. Let $\mathcal{S}_R^o$ be a circular disc of radius R centered at the origin and normal to the H-axis. Suppose that r = 1 and that there is an unpolarized disturbance (see Section III) on $\mathcal{S}_R^o$ at time $t = t_o$. Then $\mathcal{S}_R^o$ will give rise to two disc's $\mathcal{D}^+_{R^+}$ and $\mathcal{D}^-_{R^-}$ propagating with the velocities $(\mu/\rho)^{1/2}\underset{\sim}{H}$ and $-(\mu/\rho)^{1/2}\underset{\sim}{H}$ along the H-axis and radii of these discs increase with time by an amount equal to half the distance travelled from $\mathcal{S}_R^o$ - i.e.,

$$R^+ = R^- = R + \frac{b}{2}(t-t_o).$$

This type of propagation may be contrasted with the normal sort of propagation along $\underset{\sim}{H}$ that occurs whenever $r \neq 1$. In this case $\mathcal{S}_R^o$ gives rise to four discs, two moving along $\underset{\sim}{H}$; one with the sound speed and the other with the Alfvén disturbance speed, and two mirror images of these discs moving along $(-\underset{\sim}{H})$. Here, the radii of the discs remain fixed and equal to the radius of $\mathcal{S}_R^o$ namely R.

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20. See reference 16, p. 385.

21. Observe that this general result is suggested by the envelope construction sketched in subsection 3.1 and is consistent with fact that the wave-normal and Fresnel's ray surfaces are reciprocal.

22. We make no claim to rigor in the following discussion.

We shall henceforth refer to this type of anomalous propagation as conical propagation. A similar sort of propagation occurs in crystal optics and is intimately related with the phenomena of conical refraction. We might add that heuristic arguments analogous to those employed in optics suggest that the disturbance in $\mathcal{D}_R^+$ and $\mathcal{D}_R^-$ will be concentrated in circular discs of radius R at the centers of $\mathcal{D}_R^+$ and $\mathcal{D}_R^-$ and in rings of width 2R at the outer edges of $\mathcal{D}_R^+$ and $\mathcal{D}_R^-$.

V. VARIATION OF THE MODE STRENGTH ALONG RAYS

1. The Orthogonality Relation

Let D^o be a disturbance on the initial manifold $\mathcal{S}^o$. Then, as we showed in Section III, D^o may be expressed as the sum of propagating modes, $R_A^{\pm}$, $R_s^{\pm}$, $R_f^{\pm}$ (see (3.12) and (3.13)). This decomposition furnishes the initial conditions for each of these modes - i.e., it furnishes the values of $R_A^{\pm}$, $R_s^{\pm}$ and $R_f^{\pm}$ on $\mathcal{S}^o$. The problem we wish to study, therefore, reduces to the following one: Let R(t) denote any one of the above modes. Assume that R(t) is known at all points $\underset{\sim}{x}^o$ on $\mathcal{S}^o$. We wish to determine how R(t) varies on the appropriate ray through each $\underset{\sim}{x}^o$ on $\mathcal{S}^o$. First, a general relation (the so-called orthogonality relation) will be derived[23] whose form is independent of the mode being considered. Specialization of this relation then enables one to determine the variation of each mode along its associated system of rays.

The variation of R(t) along a ray is determined by making use of the variational equations (2.1)' and (2.2). Let us consider the variational induction equation which, according to (2.1)'-A' and (2.2), may be expressed as follows:

$$\underset{\sim}{H}'_t + \nabla \times (\underset{\sim}{H}' \times \underset{\sim}{u}) + \nabla \times (\underset{\sim}{H} \times \underset{\sim}{u}') = 0. \tag{5.1}$$

Let $A = A(\underset{\sim}{x},t)$ be any quantity, scalar, vector or dyadic and let $A^{+}(\underset{\sim}{x})$ be the value of $A(\underset{\sim}{x},t)$ on the wave front - specifically, let

23. Our derivation of this relation is patterned after J.B. Keller's derivation of the acoustic orthogonality relation.

$$A^{\dagger}(\underset{\sim}{x}) = A(\underset{\sim}{x}, t_o + W(x)) \ . \tag{5.2}$$

There exists one such function for each side of the wave front - $\mathcal{S}(t)$, say. On the wave front, (5.1) becomes

$$(\underset{\sim}{H}')^{\dagger} + \left[\nabla \times (\underset{\sim}{H}' \times \underset{\sim}{u})\right]^{\dagger} + \left[\nabla \times (\underset{\sim}{H} \times \underset{\sim}{u}')\right]^{\dagger} = 0. \tag{5.3}$$

From (5.2) it follows that

$$\nabla * A^{\dagger} = (\nabla * A)^{\dagger} + \underset{\sim}{p} * A_t^{\dagger} \ , \tag{5.4}$$

where

$$A_t^{\dagger} = \frac{\partial A(\underset{\sim}{x}, t)}{\partial t} \Bigg|_{t - t_o = W(\underset{\sim}{x})} \tag{5.5}$$

and where the operation $*$ denotes any multiplication, scalar or vector (cf.(2.2)). In particular, we may conclude that

$$\nabla \cdot (\underset{\sim}{H}')^{\dagger} = (\nabla \cdot \underset{\sim}{H}')^{\dagger} + \underset{\sim}{p} \cdot (\underset{\sim}{H}')_t^{\dagger} \tag{5.6}$$

$$\nabla \times \left[(\underset{\sim}{H}' \times \underset{\sim}{u})\right]^{\dagger} = \left[\nabla \times (\underset{\sim}{H}' \times \underset{\sim}{u})\right]^{\dagger} + \underset{\sim}{p} \times (\underset{\sim}{H}' \times \underset{\sim}{u})_t^{\dagger} \ , \tag{5.7}$$

$$\nabla \times \left[(\underset{\sim}{H} \times \underset{\sim}{u}')\right]^{\dagger} = \left[\nabla \times (\underset{\sim}{H} \times \underset{\sim}{u}')\right]^{\dagger} + \underset{\sim}{p} \times (\underset{\sim}{H} \times \underset{\sim}{u}')_t^{\dagger} \ . \tag{5.8}$$

Combining (5.3) with these equations, and utilizing the fact that the basic flow is continuous, that $c = (p^{-1} - u_n) = U - u_n$ and $\underset{\sim}{n} = \underset{\sim}{p}/p$, we find that[24]

$$p\left[c(\underset{\sim}{H}_t')^{\dagger} - \underset{\sim}{H}(\underset{\sim}{u}_t')^{\dagger} \cdot \underset{\sim}{n} + H_n(\underset{\sim}{u}_t')^{\dagger}\right] = -\nabla \left[(\underset{\sim}{H}')^{\dagger} \times \underset{\sim}{u}\right] - \nabla \times \left[\underset{\sim}{H} \times (\underset{\sim}{u}')\right]^{\dagger}$$
$$+ \underset{\sim}{u}\left[(\nabla \cdot \underset{\sim}{H}')^{\dagger} - \nabla \cdot (\underset{\sim}{H}')^{\dagger}\right] \ . \tag{5.9}$$

The daggers have been omitted over c, ρ and $\underset{\sim}{u}$ in this equation because these

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24. Here, and in the sequel c represents one of the disturbance speeds, $\pm c_s$, $\pm c_A$ or $\pm c_f$.

quantities do not depend upon time explicitly; i.e., $A^{\dagger}(\underset{\sim}{x}) = A(\underset{\sim}{x})$ for such quantities.

Let (5.9) be the equation on that side of $\mathcal{S}(t)$ into which $\underset{\sim}{n}$ points. Subtracting this equation from the corresponding equation on the opposite side and making use of the definition of δA and δA_n (see list of definitions Section II) and the fact that $\nabla \cdot \underset{\sim}{H}' = 0$ on both sides of $\mathcal{S}(t)$ (see equation (2.1)'-A_o'), we conclude that

$$- c\, \delta \underset{\sim}{H}_t + \underset{\sim}{H}\, \delta u_{t,n} - H_n \delta u_t = U\underset{\sim}{f}_1, \tag{5.10}$$

where

$$\underset{\sim}{f}_1 = \nabla \times (\delta \underset{\sim}{H} \times \underset{\sim}{u}) + \nabla \times (\underset{\sim}{H} \times \delta \underset{\sim}{u}) + \underset{\sim}{u} \nabla \cdot \delta \underset{\sim}{H}. \tag{5.11}$$

Proceeding in a similar fashion, we may express (2.1)'-A_2' and (2.1)-A_3' as follows:

$$- \rho c\, \delta \underset{\sim}{u}_t + \underset{\sim}{n}\left[a^2 \delta\rho_t + \mu \underset{\sim}{H} \cdot \delta \underset{\sim}{H}_t\right] - \mu H_n \delta \underset{\sim}{H}_t = U\underset{\sim}{f}_2, \tag{5.12}$$

$$- c\, \delta\rho_t + \rho\, \delta u_{t,n} = U f_3, \tag{5.13}$$

where $\underset{\sim}{f}_2$ and f_3 are defined by

$$\underset{\sim}{f}_2 = \nabla(a^2 \delta\rho) + \mu \delta \underset{\sim}{H} \times \nabla \times \underset{\sim}{H} + \mu \underset{\sim}{H} \times \nabla \times \delta \underset{\sim}{H} + \rho \delta \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + \rho \underset{\sim}{u} \cdot \nabla \delta \underset{\sim}{u}$$
$$+ \frac{\delta\rho}{\rho}\left[a^2 \nabla\rho + \underset{\sim}{H} \times \nabla \times \underset{\sim}{H}\right] \tag{5.14}$$

$$f_3 = \nabla \cdot (\delta\rho \underset{\sim}{u} + \rho \delta \underset{\sim}{u}). \tag{5.15}$$

In deriving the expression for $\underset{\sim}{f}_2$, we have made use of equations (2.1)-A_2 and the fact that $\delta S = 0$ for propagating modes (see equation (2.10)). Multiplying both sides of equations (5.10), (5.12) and (5.13), respectively, by $\sqrt{\mu}$, $1/\sqrt{\rho}$ and $a/\sqrt{\rho}$ and transforming to a coordinate system on $\mathcal{S}(t)$ with the x-axis directed along $\underset{\sim}{n}$ we find that the resulting equations imply the following matricial system:

$$\begin{bmatrix} -\rho^{1/2}c & 0 & \mu^{1/2}H_y & -\mu^{1/2}H_y & 0 & 0 \\ 0 & -\rho^{1/2}c & \mu^{1/2}H_z & 0 & -\mu^{1/2}H_n & 0 \\ \mu^{1/2}H_y & \mu^{1/2}H_z & -\rho^{1/2}c & 0 & 0 & \rho^{1/2}a \\ -\mu^{1/2}H_n & 0 & 0 & -\rho^{1/2}c & 0 & 0 \\ 0 & -\mu^{1/2}H_n & 0 & 0 & -\rho^{1/2}c & 0 \\ 0 & 0 & \rho^{1/2}a & 0 & 0 & -\rho^{1/2}c \end{bmatrix} \begin{bmatrix} (\mu/\rho)^{1/2}\delta H_{t,y} \\ (\mu/\rho)^{1/2}\delta H_{t,z} \\ \delta u_{t,n} \\ \delta u_{t,y} \\ \delta u_{t,z} \\ a\,\delta\rho_t/\rho \end{bmatrix} = U \times \begin{bmatrix} \mu^{1/2}f_{1,y} \\ \mu^{1/2}f_{1,z} \\ f_{2,n}/\rho^{1/2} \\ f_{2,y}/\rho^{1/2} \\ f_{2,z}/\rho^{1/2} \\ af_3/\rho^{1/2} \end{bmatrix} \qquad (5.16)$$

Observe now that the matrix in the left member of this equation is identical with that appearing in the left member of (2.9). Since this matrix is a symmetrical one, it follows that (5.16) is solvable if and only if the right member is orthogonal to the vector $R = \left[(\mu/\rho)^{1/2}H_y, (\mu/\rho)^{1/2}H_z, \delta u_n, \delta u_y, \delta u_z, a\delta\rho/\rho\right]$ associated with the disturbance speed c. This orthogonality condition may be expressed as follows:

$$\mu\, \underset{\sim}{f}_1 \cdot \delta \underset{\sim}{H} + \underset{\sim}{f}_2 \cdot \delta \underset{\sim}{u} + \frac{a^2}{\rho} f_3 = 0. \qquad (5.17)$$

Note that

$$\underset{\sim}{f}_1 \cdot \delta \underset{\sim}{H} = f_{1,y}\delta H_y + f_{1,z}\delta H_z, \qquad (5.18)$$

because $\delta H_n = 0$. Employing the definitions of the $\underset{\sim}{f}_1$, $\underset{\sim}{f}_2$ and f_3, we may re-express equation (5.18) as:

$$\nabla \cdot \left\{ \mu\delta\underset{\sim}{H} \times (\delta\underset{\sim}{u} \times \underset{\sim}{H}) + a^2\delta\rho\delta\underset{\sim}{u} \right\} + \left\{ \mu\delta H^2 + \frac{(a\delta\rho)^2}{\rho} \right\} \nabla \cdot \underset{\sim}{u}$$
$$+ \frac{\underset{\sim}{u}}{2} \cdot \left\{ \mu\, \nabla\,(\delta\underset{\sim}{H})^2 + \rho\; \nabla(\delta\underset{\sim}{u})^2 + \frac{a^2}{\rho}\nabla(\delta\rho)^2 \right\}$$
$$+ \rho\left[(\delta\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}\right] \cdot \delta\underset{\sim}{u} - \mu\left[(\delta\underset{\sim}{H} \cdot \nabla)\underset{\sim}{u}\right] \cdot \delta\underset{\sim}{H} = 0. \qquad (5.19)$$

Equation (5.19) may be derived from (5.18) by making use of i) standard vector identities and ii) the fact that $\delta\underset{\sim}{H} - (\delta\rho/\rho)\underset{\sim}{H}$ is collinear with $\delta\underset{\sim}{u}$ in all propagating modes (see δA_1 and δA_3 of equation (2.9)).

Equation (5.19) and the following result furnish the basis for determining the variation of R(t) along its associated ray for each of the possible modes of propagation. Let $d\sigma_o$ denote a differential area on the wave front $\mathcal{S}(t)$ at time $t = t_o$. Let $\underset{\sim}{x} = \underset{\sim}{x}(t)$ be the equation of a typical ray of the bundle passing through $d\sigma_o$ at time $t = t_o$. This bundle is, of course, assumed to consist of rays belonging to the mode of propagation in question. Let $d\sigma$ be the differential area on $\mathcal{S}(t)$ at the time $t > t_o$ that corresponds to $d\sigma_o$, the correspondence being effected by the rays. Finally, set

$$\underset{\sim}{s} = \frac{d\underset{\sim}{x}}{dt} , \tag{5.20}$$

and define the quantity $E = E(t)$ - the so-called expansion ratio along the ray - by the equation

$$E(t) = \frac{d\sigma}{d\sigma_o} , \tag{5.21}$$

Then it can be shown, (see second reference in footnote 5, p. 87) that

$$\nabla \cdot \underset{\sim}{s} = \frac{d}{dt}\left[\log E(t)U(t)\right], \tag{5.22}$$

provided that the vector field defined by $\underset{\sim}{s}$ is sufficiently regular. U(t) is defined as the speed of the surface element $d\sigma$ at $\underset{\sim}{x}(t)$; specifically

$$U(t) = \frac{1}{p(t)} . \tag{5.23}$$

The quantity p(t) is obtained by solving the ray equations (see (4.23)-(4.25)). When the rays are straight lines it can be shown that

$$E(t) = K_o/K, \tag{5.24}$$

where K_o is the Gaussian curvature of the element of the wave front $\delta\mathcal{S}(t_o)$ at $\underset{\sim}{x}^o = \underset{\sim}{x}(t_o)$ and $K = K(t)$ is the Gaussian curvature of $\delta\mathcal{S}(t)$ at $\underset{\sim}{x}(t)$.

2. Variation of the Strength of the Discontinuity

It is convenient for technical reasons to employ the magnitude of $\delta\underset{\sim}{u}$ - actually $(\delta\underset{\sim}{u})^2 = \delta u^2$ - as a measure of the strength of the discontinuity of a given mode of propagation. Once the variation of $\delta\underset{\sim}{u}^2$ along the rays appropriate to the mode in question has been derived, the variation of the other discontinuities may be determined by means of the formulas collected in the Table I of Section III. Specifically, the variation of $\delta\underset{\sim}{u}^2$ along a ray fixes the variation along the ray of the 'constant' ϵ in these formulas. This, in turn, fixes the variation of the remaining discontinuities.

As in the above discussion, let $\mathcal{S}(t)$ denote the wave front of a propagating mode that at time $t = t_o$ reduces to the initial manifold $\mathcal{S}^o$. Let $\underset{\sim}{x}(t)$ denote a typical ray of the ray system that is employed in the construction of $\mathcal{S}(t)$, $t > t_o$. Then, it can be shown, whatever the propagating mode, that δu^2 satisfies an equation of the form[25]

$$E(t)U(t)\left[\rho\delta\underset{\sim}{u}^2\Big|_{\underset{\sim}{x}=\underset{\sim}{x}(t)}\right] = E(t')U(t')\left[\rho\delta\underset{\sim}{u}^2\Big|_{\underset{\sim}{x}=\underset{\sim}{x}(t')}\right]\exp\left[\int_{t'}^{t} T(t)dt\right] \tag{5.25},$$

along $\underset{\sim}{x} = \underset{\sim}{x}(t)$. In this equation $U(t)$ and $E(t)$ are the quantities defined in equations (5.23) and (5.21) above; $\rho\delta u^2\Big|_{\underset{\sim}{x}=\underset{\sim}{x}(t)}$ is the quantity $\rho\delta\underset{\sim}{u}^2$ evaluated on the ray at time t; and t' is any time such that $t_o \leq t' < t$. When $t = t_o$, $E(t')$ reduces to unity. According as the mode under consideration is an Alfvén, slow or fast mode, $T(t)$ is given by

25. This formula must not be expected to apply where the initial manifold is singular. Nor is conical propagation (see IV.3.3) covered by these results.

$$T(t) \equiv T_A(t) = \frac{1}{2}(\underset{\sim}{u} \cdot \nabla \log\rho)\Big|_{\underset{\sim}{x}=\underset{\sim}{x}(t)} , \tag{5.26}$$

$$T(t) \equiv T_s(t) = \frac{-1}{(a^2+b^2-4a^2b_n^2)^{1/2}} \Bigg\{ \frac{\mu}{\rho} \left[(\underset{\sim}{H}\cdot\nabla)\underset{\sim}{u}\right]\cdot\underset{\sim}{H} - c_s^2\left[(\underset{\sim}{n}\cdot\nabla)\underset{\sim}{u}\right]\cdot\underset{\sim}{n}$$

$$+ \frac{\underset{\sim}{u}}{2}\cdot\left[(c_s^2-a^2)\nabla\log\rho + \frac{(c_s^2-b_n^2)}{c_s^2}\nabla a^2\right]\Bigg\}\Bigg|_{\underset{\sim}{x}=\underset{\sim}{x}(t)} \tag{5.27}$$

or

$$T(t) = T_f(t) = \frac{1}{(a^2+b^2-4a^2b_n^2)^{1/2}} \Bigg\{ \frac{\mu}{\rho} \left[(\underset{\sim}{H}\cdot\nabla)\underset{\sim}{u}\right]\cdot\underset{\sim}{H} - c_f^2\left[(\underset{\sim}{n}\cdot\nabla)\underset{\sim}{u}\right]\cdot\underset{\sim}{n}$$

$$+ \frac{\underset{\sim}{u}}{2}\cdot\left[(c_f^2-a^2)\nabla\log\rho + \frac{(c_f^2-b_n^2)}{c_f^2}\nabla a^2\right]\Bigg\}\Bigg|_{\underset{\sim}{x}=\underset{\sim}{x}(t)} . \tag{5.28}$$

When ρ and $\underset{\sim}{u}$ are constant, it follows directly from (5.25)-(5.28) that

$$E(t)U(t)\left[\rho\delta u^2\Big|_{\underset{\sim}{x}=\underset{\sim}{x}(t)}\right] = E(t')U(t')\left[\rho\delta u^2\Big|_{\underset{\sim}{x}=\underset{\sim}{x}(t')}\right], \tag{5.30}$$

in each of the propagating modes. If, in addition, $\underset{\sim}{H}$ is constant, then, as we showed earlier, the rays are straight lines and $U(t) = p^{-1}(t)$ is constant along the ray. In this case equation (5.30) reduces to

$$E(t)\ \epsilon^2(t) = E(t')\ \epsilon^2(t'). \tag{5.31}$$

Furthermore, if the wave fronts are curved, we find, using equation (5.24), that

$$\frac{\epsilon^2(t)}{K(t)} = \frac{\epsilon(t')^2}{K(t')} , \tag{5.32}$$

where $K(t)$ is the Gaussian curvature of the wave front at $\underset{\sim}{x} = \underset{\sim}{x}(t)$ and $K(t')$ is the curvature at $\underset{\sim}{x} = \underset{\sim}{x}(t')$.

3. Derivation of Formula (5.25) for the Alfvén Mode

To illustrate the procedure used to obtain (5.25)-(5.28), we shall give here a detailed derivation of (5.25) for Alfvén-wave propagation. To be sure, this proceedure when applied to the slow and fast waves requires lengthier calculations, but no new ideas are involved.

We know that

$$\delta \underset{\sim}{u} = \mp \operatorname{sgn}(H_n)\, (\mu/\rho)^{1/2} \delta \underset{\sim}{H}, \tag{5.33}$$

$$\operatorname{sgn}(H_n) = \begin{cases} 1, & H_n > 0 \\ -1, & H_n < 0, \end{cases} \tag{5.34}$$

and

$$0 = \underset{\sim}{H} \cdot \delta \underset{\sim}{H} = \delta\rho = \delta u_n, \tag{5.35}$$

when the mode is an Alfvén mode (see the second row of Table I). From these relations it follows immediately that

$$\mu \delta \underset{\sim}{H}^2 + \rho \delta \underset{\sim}{u}^2 = 2\mu\delta H^2 = 2\rho\delta u^2, \tag{5.36}$$

$$\begin{aligned} \mu\delta\underset{\sim}{H} \times (\delta\underset{\sim}{u} \times \underset{\sim}{H}) &= \pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2} \left\{ \mu\delta\underset{\sim}{H}^2\underset{\sim}{H} - \mu(\delta\underset{\sim}{H}\cdot\underset{\sim}{H})\delta\underset{\sim}{H} \right\} \\ &= \pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2}(\mu\delta H^2)\underset{\sim}{H} \\ &= \pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2}\underset{\sim}{H}(\rho\delta u^2), \end{aligned} \tag{5.37}$$

and

$$\rho\left[(\delta\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}\right]\cdot\delta\underset{\sim}{u} - \mu\left[(\delta\underset{\sim}{H}\cdot\nabla)\underset{\sim}{u}\right] \cdot \delta\underset{\sim}{H} = 0. \tag{5.38}$$

Combining these equations with the orthogonality relation (5.19) we find that

$$\nabla \cdot \left[\pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2}\underset{\sim}{H}(\rho\delta\underset{\sim}{u}^2)\right] + \rho\delta\underset{\sim}{u}^2\nabla\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\nabla(\rho\delta\underset{\sim}{u}^2) - \delta\underset{\sim}{u}^2\,\frac{\underset{\sim}{u}}{2}\cdot\nabla\rho = 0 \tag{5.39}$$

or equivalently that

$$\begin{aligned} \rho\delta u^2\nabla \cdot \left[\underset{\sim}{u} \pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2}\underset{\sim}{H}\right] + \left[\underset{\sim}{u} \pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2}\underset{\sim}{H}\right]\cdot\nabla(\rho\delta\underset{\sim}{u}^2) \\ -(\rho\delta\underset{\sim}{u}^2)\,\frac{\underset{\sim}{u}}{2}\cdot\nabla\log\rho = 0. \end{aligned} \tag{5.40}$$

But in an Alfvén mode

$$\underset{\sim}{s} = \frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{u} \pm \operatorname{sgn}(H_n)(\mu/\rho)^{1/2}\underset{\sim}{H}, \tag{5.41}$$

(see equations (5.20)and (4.23)). Moreover, we have seen that (see equation (5.22))

$$\nabla \cdot \underset{\sim}{s} = \frac{d}{dt} \log \left[E(t)U(t)\right]; \tag{5.42}$$

equation (5.39) may therefore be rewritten as the following ordinary linear differential equation (in $\rho\delta\underset{\sim}{u}^2$) along the ray:

$$\frac{d(\rho\delta u^2)}{dt} + \left\{ \frac{d}{dt} \log\left[E(t)U(t)\right] + \frac{\underset{\sim}{u}}{2} \cdot \nabla \log\rho \right\} (\rho\delta u^2) = 0. \tag{5.43}$$

Integrating this equation from t' to t > t', we obtain the desired results - those stated in equations (5.25) and (5.26).

VI. RESOLUTION OF AN INITIAL DISCONTINUITY

The results of the foregoing sections enable us to construct the wave fronts evolving from an initial manifold and to find how the strengths of the initial disturbances on these fronts vary with time. The theory, as presented here, does not, in general, give information about the nature of the disturbances between or behind the wave fronts. There do exist, however, some problems with sufficiently simple geometries and initial conditions that may be solved completely and explicitly with the results at hand. One such problem will be treated now.

The geometric setup is shown in Figure 5a. In the region x < 0, we have an infinitely conducting (rigid) magnet. The region x > 0 is filled with an infinitely conducting fluid. The vectors $\underset{\sim}{x}_o$, $\underset{\sim}{y}_o$, $\underset{\sim}{z}_o$ are the unit vectors directed along the positive x, y and z-axes respectively. It is assumed that the magnetic field $\underset{\sim}{H}$ is everywhere uniform[26] and that it makes an angle θ with the unit vector $\underset{\sim}{x}_o$ which is taken normal to the face $\mathcal{S}^o$ of the magnet.

26. The magnetic inductive capacity μ is assumed that of free space everywhere.

For the sake of simplicity, it will be assumed that $0 \leq \theta \leq \pi/2$ and that the density is constant in $x > 0$.

Assume at first that there is no relative motion between the magnet and the fluid. Now suppose the magnet is set in motion with a velocity $- \delta u_y^o \underset{\sim}{y}_o$. We claim, provided $\underset{\sim}{H} \cdot \underset{\sim}{x}_o \neq 0$, that i) a flow parallel as well as perpendicular to the x-axis will result and ii) this flow obeys the boundary conditions

$$\delta u_{o,x} = 0, \tag{6.1}$$

$$\delta u_{o,y} = -\delta u_y^o, \tag{6.2}$$

on $\mathscr{S}^o$ for all $t > 0$. In these equations the subscript '0' is used to denote the region bordering on the magnet and $\delta u_{o,x}$ and $\delta u_{o,y}$ are simply $\delta \underset{\sim}{u}_o \cdot \underset{\sim}{x}_o$ and $\delta \underset{\sim}{u}_o \cdot \underset{\sim}{y}_o$, respectively (see Figure 5b).

The first of these boundary conditions follows immediately from the rigidity of the magnet. The second is a consequence of the fact that the tangential electric field must be continuous across $\mathscr{S}^o$. For, suppose that $\delta u_{o,y} \neq -\delta u_y^o$; then, an observer located at a point of $\mathscr{S}^o$ and moving with the magnet would experience no electric field within the magnet (a consequence of the infinite conductivity of the magnet) but would observe, initially, a tangential electric field

$$\underset{\sim}{E}_{tan} = \mu \underset{\sim}{H} \times \underset{\sim}{y}_o(\delta u_{o,y} + \delta u_y^o) = - \mu H \sin \theta(\delta u_{o,y} + \delta u_y^o)\underset{\sim}{z}_o$$

just within the fluid. Since the tangential electric field must be continuous across $\mathscr{S}^o$, we obtain (6.1)-(6.2) unless θ is $\pi/2$ radians; but this angle was excluded at the outset by the requirement that $\underset{\sim}{H} \cdot \underset{\sim}{x}_o \neq 0$.

This argument only explains the origin of the transverse motion or motion normal to the x-axis. It can, however, be supplemented by another argument - one that is justified in light of the final results - that explains the origin of the longitudinal motion. For this purpose, it is better to think not in terms of discontinuities, but rather in terms of relatively thin layers in which there occur abrupt but continuous transitions and to focus on the simplest case, namely, that in which H is directed along $\underset{\sim}{x}_o$. We begin with the fact that the fluid in the immediate neighborhood of the wall moves with the magnet when the latter is set in motion with the velocity $-\delta u_y^o y_o$. Since the magnetic lines of force in an infinitely conducting fluid are 'glued' to the fluid, the motion results in these lines being pulled downward in the vicinity of the magnet. This distortion of the lines of force has the effect of producing, in the neighborhood of $\mathcal{S}^o$, a transverse component of the magnetic field δH_y that varies with x. The variation of δH_y with x is accompanied by a current density of magnitude $|\delta J_z| = |\partial(\delta H_y)/\partial x|$ directed into the plane of the page. Hence, the fluid in the layer experiences a force per unit volume of magnitude $\delta J_z = \mu \delta H_y |\partial(\delta H_y)/\partial x|$ along the positive x-axis. It is this force that is responsible for the longitudinal motion.

The motion described here may also be initiated by electrical means. The idea is to produce a thin initial current layer through electrical discharge, thereby obtaining what corresponds to δJ_z above. R. M. Patrick[27] has constructed a device based on this idea.

We are dealing here with a mixed initial boundary-value problem; the face of the magnet $\mathcal{S}^o$ serves both as an initial manifold and a boundary. Let D^o represent the disturbance on $\mathcal{S}^o$ for all $t \geq 0$. Without loss of generality, it may be assumed that D^o has the following form (cf. equation (3.12)):

$$D^o = \left[(\mu/\rho)^{1/2}\delta H_{o,y}, 0, 0, -\delta u_y^o, 0, \delta\rho_o\right] \qquad (6.3)$$

That $\delta u_{o,x} = 0$ and $\delta u_{o,y} = -\delta u_y^o$ follows from the boundary conditions (6.1) and (6.2). That the terms involving $\delta H_{o,z}$ and $\delta u_{o,z}$ may be assumed to vanish is a consequence of symmetry considerations. At this point the

27. R. V. Patrick, Avco Research Laboratory Report No. 28 (1958).

components $(\mu/\rho)^{1/2}\delta H_{o,y}$ and $\delta\rho_o$ must be regarded as unknowns to be determined after the boundary conditions have been met.

We begin our analysis by expressing D^o as

$$D^o = R_A^+ + R_s^+ + R_f^+, \tag{6.4}$$

(see equations (3.11)-(3.13)) where

$$R_s^+ = \epsilon_s\left[(\mu/\rho)^{1/2}H\sin\theta,\ 0, -c_s\left(\frac{b^2\cos\theta^2}{c_s^2}-1\right), \frac{-b^2\sin\theta\cos\theta}{c_s}, 0, a\left(1-\frac{b^2\cos^2\theta}{c_s^2}\right)\right] \tag{6.5}$$

$$R_A^+ = \epsilon_A\left[0, -H\sin\theta, 0, 0, b\cos\theta\sin\theta, 0\right], \tag{6.6}$$

$$R_f^+ = \epsilon_f\left[(\mu/\rho)^{1/2}H\sin\theta,\ 0, -c_f\left(\frac{b^2\cos\theta^2}{c_f^2}-1\right), \frac{-b^2\sin\theta\cos\theta}{c_f}, 0, a\left(1-\frac{b^2\cos^2\theta}{c_f}\right)\right] \tag{6.7}$$

Since the magnet is rigid, propagation along $(-\underset{\sim}{x}_o)$ is excluded; this explains the absence of R_A^-, R_s^-, R_f^- in the right member of equation (6.4). The expressions for R_s^+, R_A^+ and R_f^+ are obtained from equation (3.13) and the entries in the first column of the Table I, in Section III. It is here assumed, in addition to the other requirements on θ, that $\theta \neq 0$. The results for $\theta = 0$ will be derived by taking limits. The values of ϵ_s, ϵ_A, and ϵ_f could now be obtained by making use of the orthogonality of the vectors R_s^+, R_A^+ and R_f^+; but it is simpler to equate components. This proceedure leads to the following equations[28]:

$$\epsilon_A = 0, \tag{6.9}$$

$$\frac{\epsilon_s}{c_s}(b^2\cos^2\theta - c_s^2) + \frac{\epsilon_f}{c_f}(b^2\cos^2\theta - c_f^2) = 0, \tag{6.10}$$

$$\frac{\epsilon_s}{c_s} + \frac{\epsilon_f}{c_f} = \frac{\delta u_y^o}{b^2\sin\theta\cos\theta} \quad . \tag{6.11}$$

---

28. $\delta H_{o,y}$ and $\delta\rho_o$ can be calculated once ϵ_s and ϵ_f have been found simply by adding $\delta H_{s,y}$ to $\delta H_{f,y}$ and $\delta\rho_s$ to $\delta\rho_f$ in the formulas listed below.

Evidently, the Alfvén mode plays no role in the resolution of the initial discontinuity. Solving equations (6.10) and (6.11) for ϵ_s and ϵ_f and substituting for ϵ_s and ϵ_f in (6.6) and (6.7), we find the following formulas for the components of R_s and R_f

$$
\begin{aligned}
\delta H_{y,s} &= \frac{(c_f^2/b^2-\cos^2\theta)}{[(1-r)^2+4r\sin\theta]^{1/2}} \; \frac{H}{\cos\theta} \left(\frac{c_s}{b}\right) \frac{\delta u_y^o}{b}, && s_1 \\
\delta u_{x,s} &= \frac{-\sin\theta\cos\theta}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \; \delta u_y^o, && s_{2,x} \\
\delta u_{y,s} &= -\frac{(c_f^2/b^2-\cos^2\theta)}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \; \delta u_y^o, && s_{2,y} \qquad (6.12) \\
\delta \rho_s &= -\rho \frac{\sin\theta\cos\theta}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \left(\frac{\delta u_y^o}{c_s}\right), && s_3
\end{aligned}
$$

and

$$
\begin{aligned}
\delta H_{y,f} &= \frac{(\cos^2\theta-c_s^2/b^2)}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \; \frac{H}{\cos\theta} \left(\frac{c_f}{b}\right) \frac{\delta u_y^o}{b}, && f_1 \\
\delta u_{x,f} &= \frac{\sin\theta\cos\theta}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \; \delta u_y^o, && f_{2,x} \\
\delta u_{y,f} &= \frac{-(\cos^2\theta-c_s^2/b^2)}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \; \delta u_y^o, && f_{2,y} \qquad (6.13) \\
\delta \rho_f &= \rho \frac{\sin\theta\cos\theta}{[(1-r)^2+4r\sin^2\theta]^{1/2}} \left(\frac{\delta u_y^o}{c_f}\right). && f_3
\end{aligned}
$$

When $r = 1$, equations (6.12) and (6.13) simplify to:

$$
\begin{aligned}
\delta H_{y,s} &= H(1+\sin\theta)^{1/2}(\delta u_y^o/2b), && s_1' \\
\delta u_{x,s} &= -\cos\theta\,(\delta u_y^o/2), && s_{2,x}' \qquad (6.14)
\end{aligned}
$$

$$\delta u_{y,s} = -(1+\sin\theta)(\delta u_y^o/2), \qquad s'_{2,y}$$

$$\delta\rho_o = -\rho\frac{\cos\theta}{(1-\sin\theta)^{1/2}}(\delta u_y^o/2b), \qquad s'_3 \qquad (6.14\text{ cont'd})$$

and

$$\delta H_{y,f} = H(1-\sin\theta)^{1/2}(\delta u_y^o/2b), \qquad f'_1$$

$$\delta u_{x,f} = \cos\theta(\delta u_y^o/2), \qquad f'_{2,x}$$

$$\delta u_{y,f} = (1-\sin\theta)(\delta u_y^o/2), \qquad f'_{2,y} \qquad (6.15)$$

$$\delta\rho_f = \rho\frac{\cos\theta}{(1+\sin\theta)^{1/2}}(\delta u_y^o/2b). \qquad f'_3$$

In equations (6.12)-(6.15), components of R_s^+ and R_f^+ not mentioned are zero. Equations (6.12)-(6.15) determine R_s^+ and R_f^+ on the initial manifold $\mathscr{S}^o$. But the rays associated with each mode are evidently parallel and the wave fronts evolving from $\mathscr{S}^o$ planar. R_s^+ and R_f^+, therefore, remain constant (see end of Section V, 2) as the waves propagate out from $\mathscr{S}^o$. The disturbance is also constant between and behind the wave fronts.

In Figures 5b-5f we have sketched the wave forms. The ratio r is assumed to exceed unity and we have supposed, in addition, that

$$(\cos^2\theta - c_s^2/b^2)(c_f/b) < (c_f^2/b^2 - \cos^2\theta)(c_s/b); \qquad (6.16)$$

this relation can always be satisfied by choosing θ sufficiently small. The regions labelled (2), (1) and (0) are, respectively, the undisturbed region, the region between the slow and fast wave fronts, and finally the region behind the slow wave front and $\mathscr{S}^o$. In the (x,t)-plane (see Figure 5b) the paths of the slow and fast waves are the lines whose equations are

$$x_s = c_s t, \qquad (6.17)$$

$$x_f = c_f t, \qquad t > 0, \qquad (6.18)$$

and the width $\Delta_1(t)$ of region (1) is

$$\Delta_1(t) = (c_f - c_s)t, \qquad t > 0. \tag{6.19}$$

No attempt has been made to depict accurately the relative sizes of the jumps in the various wave forms. We have merely tried to indicate when a jump δA of a quantity A is positive or negative and when $|\delta A_s|$ exceeds, equals, or is less than $|\delta A_f|$. In this regard, we observe that the relations $c_s \leq c_f$ and (6.16) in combination with equations (6.12) and (6.13) imply that

$$\begin{aligned} |\delta H_{y,s}| &> |\delta H_{y,f}|, \\ |\delta u_{x,s}| &= |\delta u_{x,f}|, \\ |\delta u_{y,s}| &> |\delta u_{y,f}|, \\ |\delta \rho_s| &> |\delta \rho_f|. \end{aligned} \tag{6.20}$$

It is easy to verify that the result of letting sin Θ approach zero in equations (6.12) and (6.13) is a single disturbance of Alfvén-wave type - specifically the limit is

$$\begin{aligned} \delta H_y &= (\rho/\mu)^{1/2}\, \delta u_{y,o} \qquad\qquad \Theta = 0 \text{ or } \pi, \\ \delta u_y &= -\delta u_y^o, \\ \delta\rho &= \delta u_x = 0. \end{aligned} \tag{6.21}$$

These equations represent the limit of the slow mode or the fast mode according as $r > 1$ or $r < 1$[29]; the disturbance on the remaining wave of the pair vanishes

29. When $r < 1$, c_s and c_f approach a and b respectively as Θ approaches zero. When $r > 1$, c_s and c_f approach b and a respectively as Θ approaches zero.

in the limit. When $r = 1$, neither the fast nor the slow disturbances vanishes entirely; however, in this case, the wave fronts coalesce since $\Delta_1(t) = t(c_f - c_s)$ vanishes. Superposing the slow and fast disturbances, we then arrive at equations (6.21). This result is in complete agreement with that obtained by Friedrichs[2] and by one of us[10] in a closely related problem.

The general effect of the wave motion is to adjust the motion of the fluid to that of the wall. In the final steady state, which is achieved in region (0) after the wave has propagated out to infinity, the fluid is at rest with respect to the wall and the tangential magnetic field is increased.

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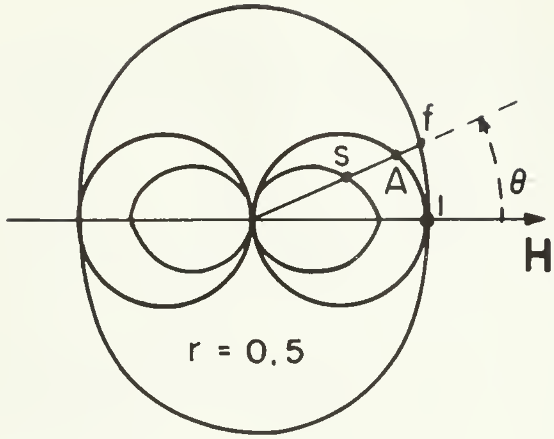
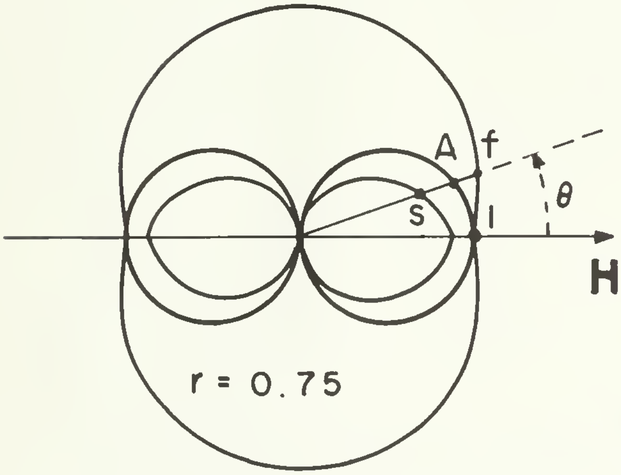


(a)



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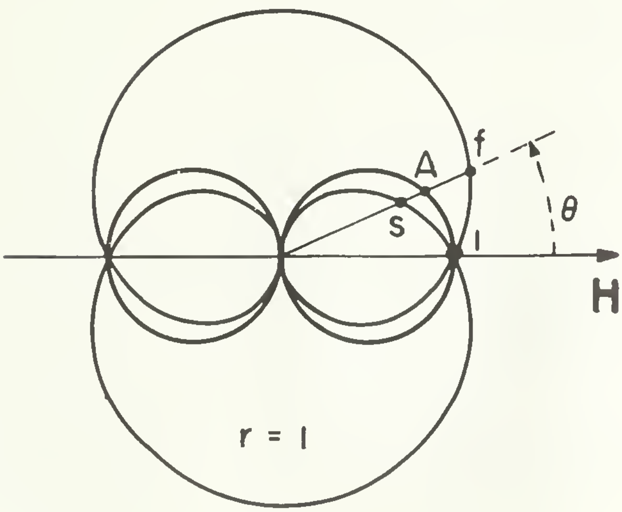

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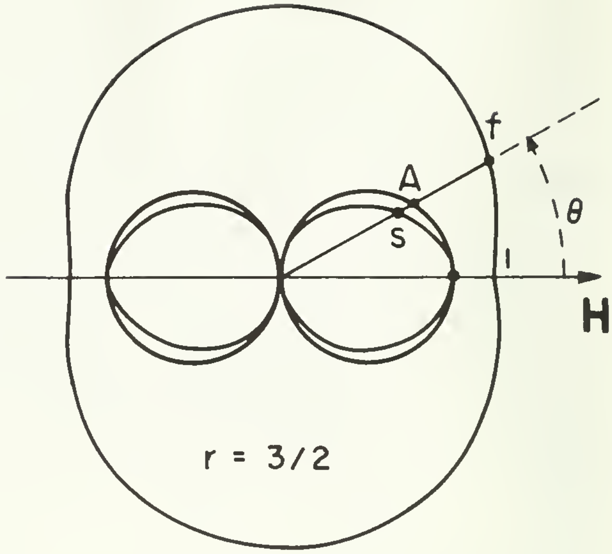

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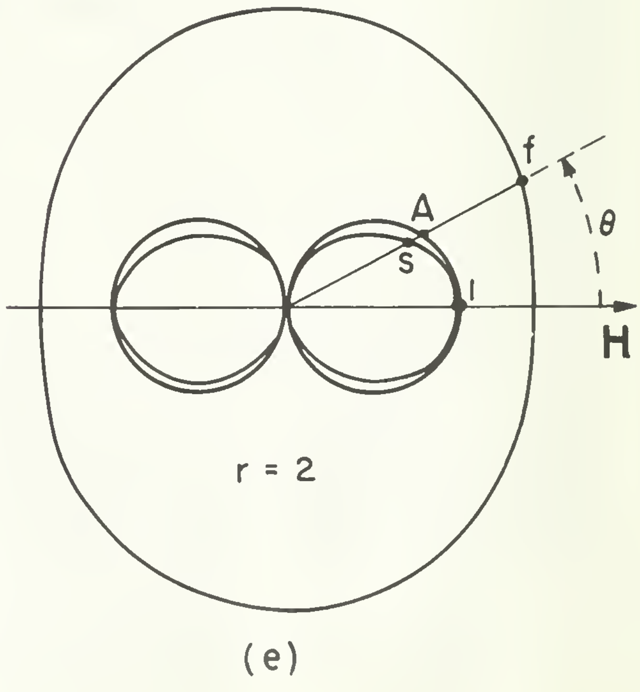


(e)

Figure 1. Surfaces of normal speeds for several values of the parameter r.

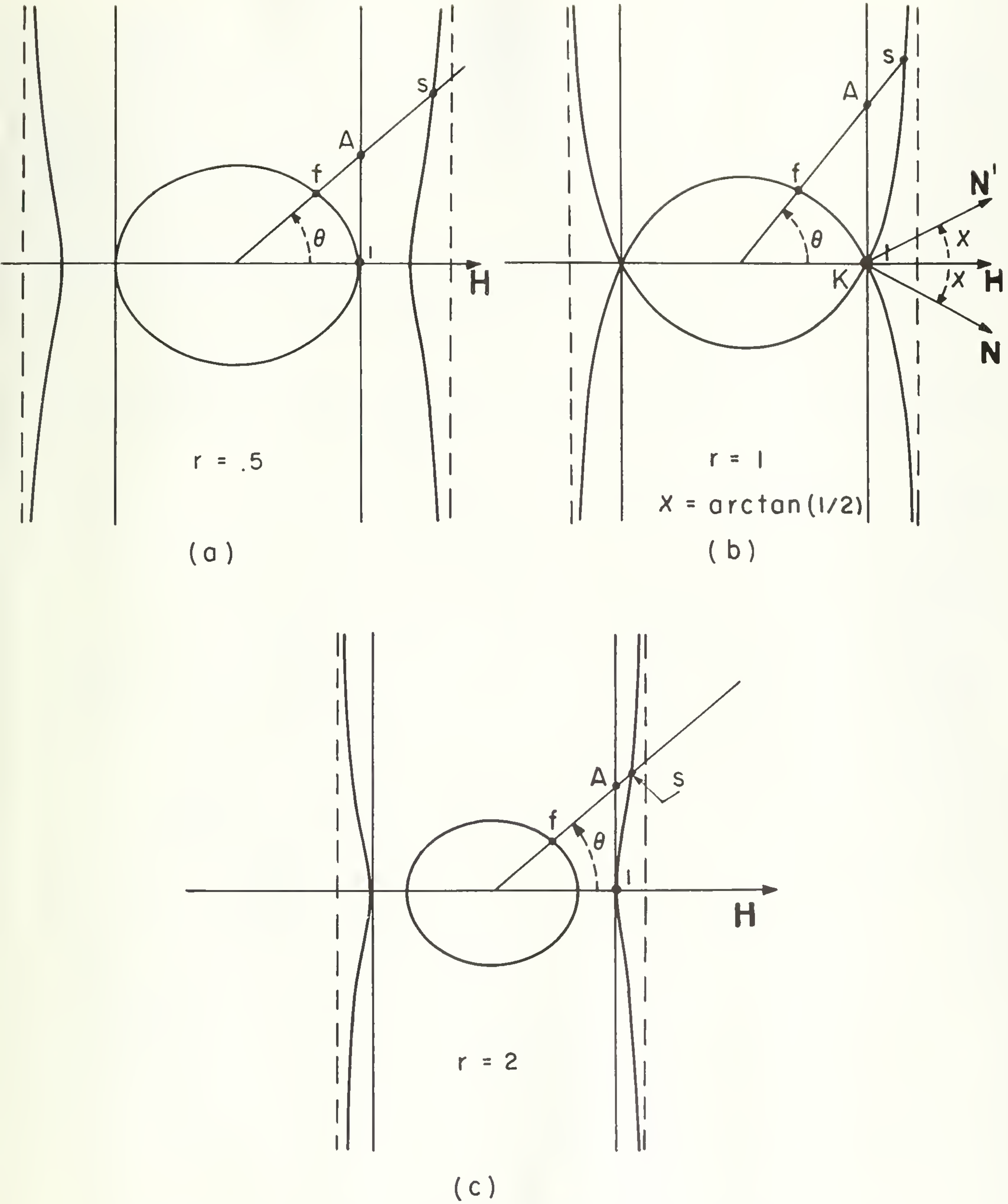
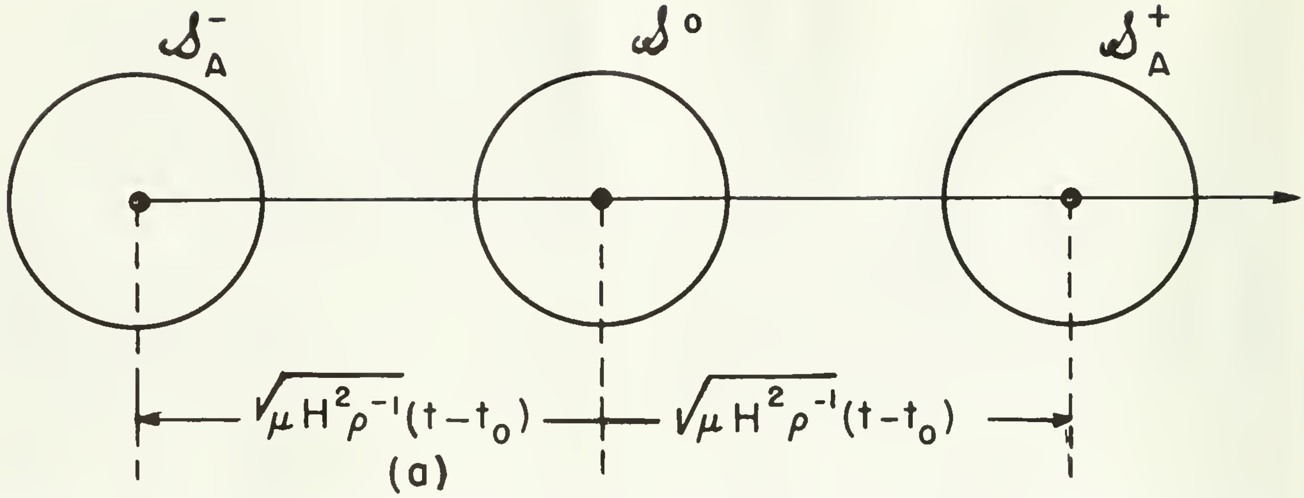
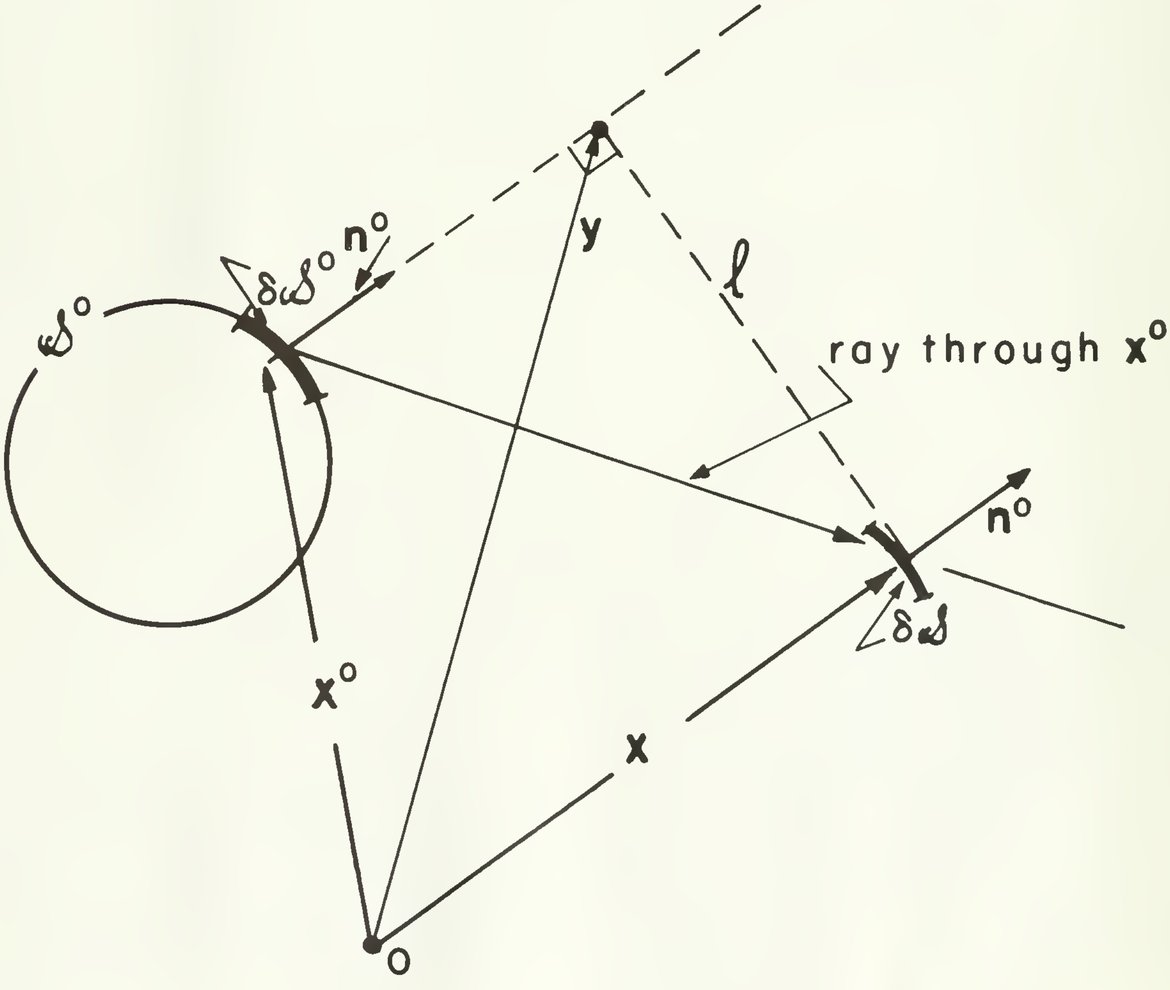


Figure 2. Surfaces of wave normals for several values of the parameter r.



(a)

Figure 3a. Depicting the one-dimensional propagation of Alfvén disturbances waves.



(b)

Figure 3b. Depicting the propagation of the element $\delta\mathcal{S}^0$ at $\underset{\sim}{x}^0$ along the forward ray through $\underset{\sim}{x}^0$.

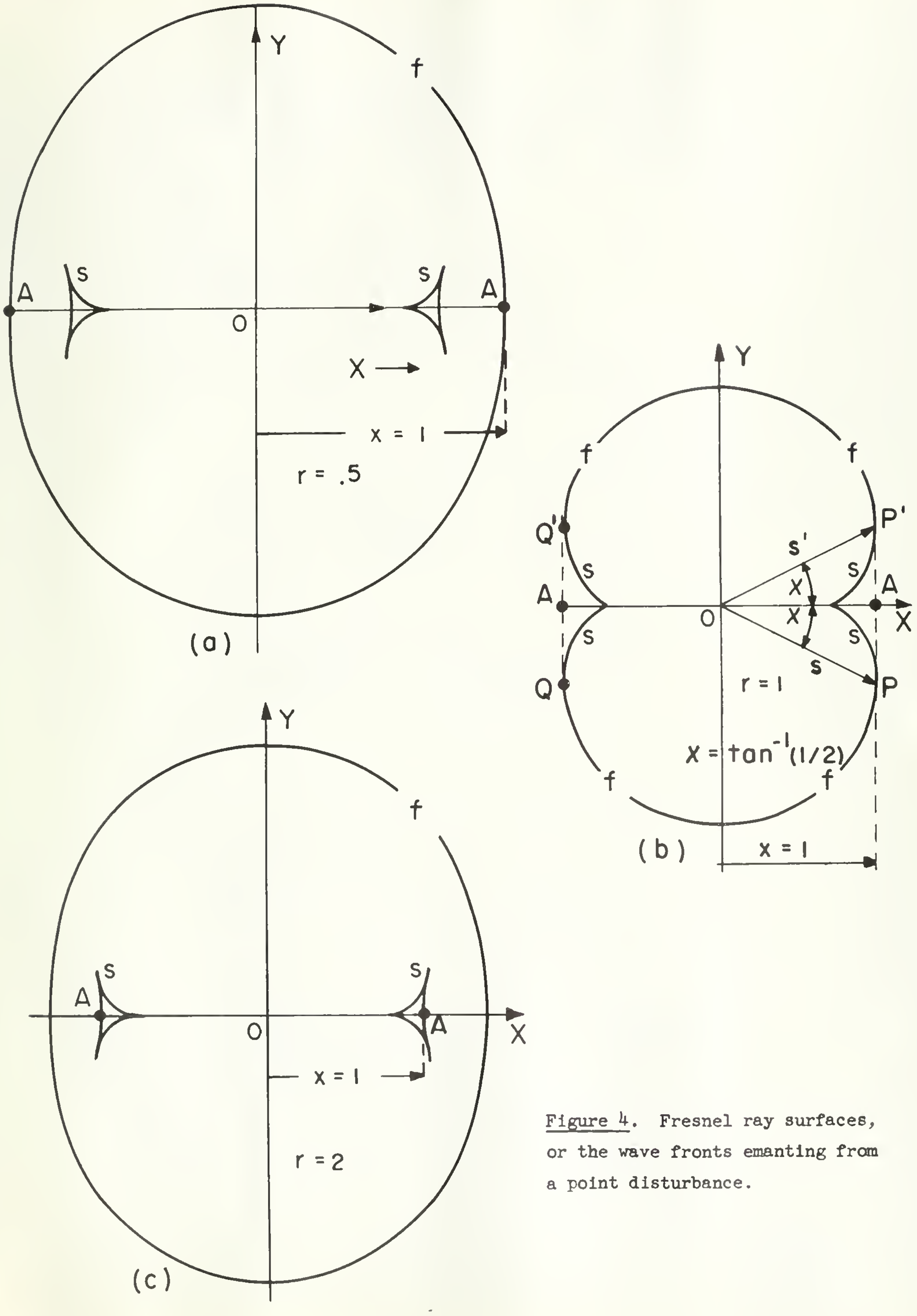


Figure 4. Fresnel ray surfaces, or the wave fronts emanting from a point disturbance.

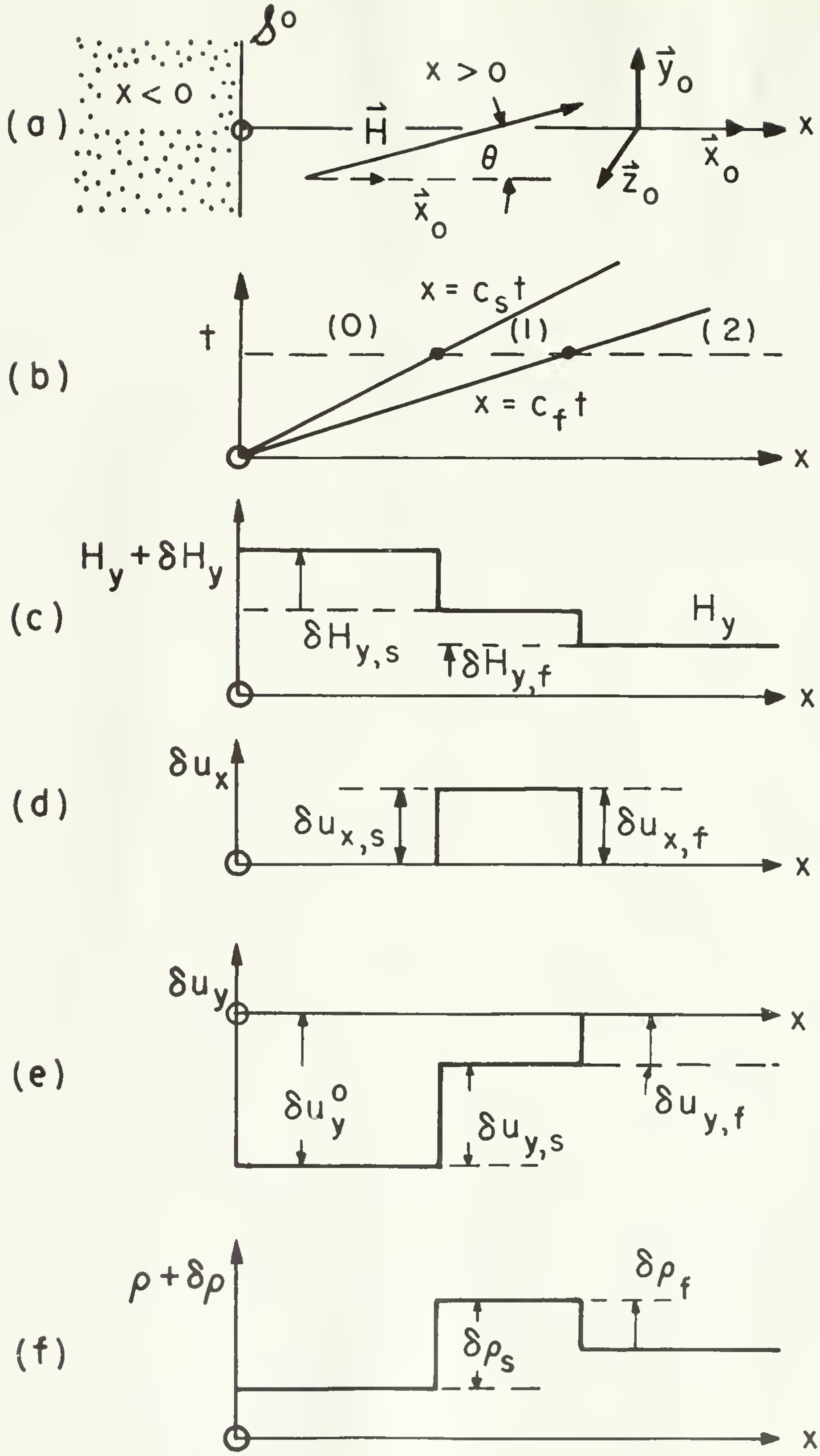


Figure 5. The wave forms at time t that result when the magnet in the region $x < 0$ (see (a)) is set in motion with a velocity $(-\delta u_y^0)$ along the y-axis. It has been assumed that r exceeds unity and that θ is small (cf. equation (6.16)).

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