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HEAD-MASTER OF ST. MARTIN'S SCIENCE SCHOOL, LEICESTER.

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1875.

TO THE REVEREND

DAVID JAMES VAUGHAN, M.A.,

HONORARY CANON OF PETERBOROUGH CATHEDRAL;

VICAR OF ST. MARTIN'S, LEICESTER;

AND FORMERLY FELLOW OF TRINITY COLLEGE, CAMBRIDGE,

This Volume is Dedicated

IN APPRECIATION OF

HIS ZEAL IN THE CAUSE OF EDUCATION,

AND IN GRATEFUL ACKNOWLEDGMENT OF

HIS PERSONAL SYMPATHY

DURING MANY YEARS.

PREFACE.

THIS Volume is intended to form, with the volume previously written, a complete treatise on Algebra and Plane Trigonometry, or at least to include as much of these subjects as is usually read.

I have, therefore, not at all confined myself to the Government Pass Syllabus for Pure Mathematics, Stage III., but have treated the subjects with sufficient fulness to enable a student to take up the Honours Papers.

The book, which is complete in itself, may be considered as an Advanced Text-Book in those subjects; and will, I presume to hope, be found useful to all who wish to pass beyond the mere rudiments of Mathematical Science.

It is unnecessary for me to say anything here as to the place Mathematics should hold as a branch of education. I may, however, remark that the recent alterations in this respect, made in the Training College Curriculum of the Education Department, show that the Government are thoroughly alive to its importance.

I have included in the volume the elementary portions of Spherical Trigonometry, for the use of those students who may prefer to take it up as an alternative subject in the Pass Examination of the Science and Art Department.

E. A.

LEICESTER, *April, 1875.*

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SECTION I.

ALGEBRA.

CHAPTER I.

THE THEORY OF QUADRATIC EQUATIONS AND EXPRESSIONS.

1. *A quadratic equation cannot have more than two roots.*

It has been shown (Vol. I., 176) that $f(x)$ is divisible by $x - a$, when it vanishes on putting a for x; that is, when a is a root of the equation $f(x) = 0$.

Hence, if α, β are roots of the quadratic equation

$$x^2 + px + q = 0,$$

$x - \alpha$ and $x - \beta$ are each of them factors of the expression $x^2 + px + q$, whatever be the value of x.

Moreover, $x - \alpha$ and $x - \beta$ are the only linear factors of $x^2 + px + q$, since their product gives an expression of two dimensions whose first term is x^2, the same as that of the expression $x^2 + px + q$. We therefore have identically

$$x^2 + px + q = (x - \alpha)(x - \beta).$$

Hence no value of x can make $x^2 + px + q$ vanish unless it make one of the factors $x - \alpha$, $x - \beta$ vanish; that is, $x^2 + px + q$ cannot vanish for any other values of x than α and β.

It follows, therefore, that α, β are the only roots of the equation

$$x^2 + px + q = 0.$$

2. *If* α, β *are the roots of the equation* $x^2 + px + q = 0$, *then* $\alpha + \beta = -p$, *and* $\alpha\beta = q$.

For (Art. 1) we have *identically*,

$$\begin{aligned} x^2 + px + q &= (x - \alpha)(x - \beta), \text{ or,} \\ &= x^2 - (\alpha + \beta)x + \alpha\beta. \end{aligned}$$

Here we have the coefficient p on the first side represented by $-(\alpha + \beta)$ on the second side, and the constant term q represented by $\alpha\beta$.

We then have

$$-(\alpha + \beta) = p, \text{ or } \alpha + \beta = -p,$$
$$\text{and } \alpha\beta = q. \quad Q.E.D.$$

COR. 1. It easily follows that

(1.) $\alpha^2 + \beta^2 = p^2 - 2q.$

(2.) $\alpha - \beta = \sqrt{p^2 - 4q}.$

(3.) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = -\dfrac{p}{q}.$

COR. 2. If the given equation be of the form $ax^2 + bx + c = 0$, we may write it thus:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

and we have

(1.) $\alpha + \beta = -\dfrac{b}{a}.$

(2.) $\alpha\beta = \dfrac{c}{a}.$

(3.) $\alpha^2 + \beta^2 = \dfrac{b^2 - 2ac}{a^2}$

(4.) $\alpha - \beta = \frac{1}{a}\sqrt{b^2 - 4ac}.$

(5.) $\frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}.$

3. *The quadratic expression* $ax^2 + bx + c$ *is a perfect square when* $b^2 = 4ac$.

When $ax^2 + bx + c$ is a perfect square, its *linear* factors must be identical; and hence the condition is that the equation

$$ax^2 + bx + c = 0$$

must have *equal roots.*

Solving this equation we get

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}).$$

Now these two values of x can be equal only when

$$b^2 - 4ac = 0, \text{ or when } b^2 = 4ac.$$

Hence the condition that $ax^2 + bx + c$ shall be a perfect square is that $b^2 = 4ac$.

COR. The roots of the equation $ax^2 + bx + c = 0$ are *rational* when $b^2 - 4ac$ *is a perfect square,* and are *impossible* when $b^2 < 4ac$.

4. *To investigate the relations between the coefficients* A, B, C, a, b, c, *in order that the expression*

$$Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy$$

shall be a perfect square.

We must have *identically*

$$\begin{aligned} &Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy \\ &= (\sqrt{A}\,.\,x + \sqrt{B}\,.\,y + \sqrt{C}\,.\,z)^2 \\ &= Ax^2 + By^2 + Cz^2 + 2\sqrt{BC}\,.\,yz + 2\sqrt{AC}\,.\,xz \\ &\quad + 2\sqrt{AB}\,.\,xy. \end{aligned}$$

Equating the coefficients of identical terms, we have

$$a = 2\sqrt{BC}, \quad b = 2\sqrt{AC}, \quad c = 2\sqrt{AB},$$

the relations required.

Cor. Solving for A, B, C, we may put these relations in the following form:

$$A = \frac{bc}{2a}, \quad B = \frac{ac}{2b}, \quad C = \frac{ab}{2c}.$$

5. *The values of* x *and* y *derived from the equations*

$$ax + by = \frac{a^3}{x} + \frac{b^3}{y} = c^2$$

are rational, when $a^2 + b^2 = c^2$.

For, solving for x, we get—

$$ac^2x^2 + (b^4 - c^4 - a^4)x + a^3c^2 = 0.$$

Hence (Art. 3, Cor.) the values of x are rational when $(b^4 - c^4 - a^4)^2 - 4\,ac^2 \,.\, a^3c^2$ is a perfect square.

The values of x are therefore rational when

$$(b^4 - c^4 - a^4)^2 = 4\,a^4c^4, \text{ or}$$
$$b^4 - c^4 - a^4 = \pm\, 2\,a^2c^2, \text{ or}$$
$$(a^2 \pm c^2)^2 = b^4, \text{ or when } a^2 \pm c^2 = \pm\, b^2.$$

And we may similarly show that the values of y are rational when

$$b^2 \pm c^2 = \pm\, a^2.$$

Now, on examination it will be found that the only condition common to these two sets is that

$$a^2 + b^2 = c^2.$$

Hence the values of x and y are both rational when

$$a^2 + b^2 = c^2. \quad Q.E.D.$$

6. *To investigate the condition that the expression*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

is resolvable into elementary factors.

If the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be solved either for x or y, the condition that either can be expressed in terms of the other in a rational form will (Art. 1) be the condition that the given expression can be broken up into elementary rational factors.

Solving for x, we find x equal to

$$-\frac{By + D \pm \sqrt{(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)}}{2A}.$$

Now, in order that this may be rational, we must have

$$(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)$$

a perfect square.

Hence (Art. 3) we must have

$$\{2(BD - 2AE)\}^2 = 4(B^2 - 4AC)(D^2 - 4AF)$$

or $$(BD - 2AE)^2 = (B^2 - 4AC)(D^2 - 4AF) \ldots (1).$$

This may be transformed into either of the following forms:

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0 \ldots\ldots\ldots (2).$$
$$(BE - 2CD)^2 = (B^2 - 4AC)(E^2 - 4CF) \ldots\ldots\ldots (3).$$
$$(DE - 2BF)^2 = (D^2 - 4AF)(E^2 - 4CF) \ldots\ldots\ldots (4).$$

Ex. I.

1. Find the difference of the squares of the roots of the equation $3x^2 - 12x - 43 = 0$.

2. Construct an equation whose roots are the reciprocals of the roots of $ax^2 + bx + c = 0$.

3. If α, β are the roots of the equation $x^2 - px + q = 0$, construct the equation whose roots are $\alpha - \gamma$, $\beta - \gamma$.

4. Find the value of a when the equation

$$ax^2 - 36x + 81 = 0$$

has equal roots.

5. Find the relation between the quantities a, b, c, a', b', c', so that the equations

$$ax^2 + bx + c = 0, \text{ and } a'x^2 + b'x + c' = 0$$

shall have a common root.

6. The equations $ax^2 + bx + c = 0$, and $2ax + b = 0$, have common roots when the former has two equal roots.

7. When is the equation

$$\sqrt{m^2x + a^2} + \sqrt{n^2x + b^2} + \sqrt{p^2x + c^2} = 0$$

reducible to a simple equation?

8. If α, β are the roots of the equation $x^2 - px + q = 0$, form the equation whose roots are $\alpha + \beta$, and $\alpha - \beta$.

9. If $x^2 + px + q$ and $x^2 + mx + n$ have a common factor, show that this common factor is $x + \frac{n - q}{m - p}$.

10. Hence show that the following relation holds, viz.,

$$(n - q)^2 + n(m - p)^2 = m(m - p)(n - q).$$

11. Show that the values of x and y derived from the equations $ax^2 + by^2 = a^2x + b^2y = c^3$ are rational when $a^3 + b^3 = c^3$.

12. Show that the expression $6x^2 + 17xy + 12y^2 + 4x + 7y - 10$ can be broken into elementary factors, and determine the factors.

CHAPTER II.

VARIATION.

7. If two quantities are connected together by some constant multiplier, either integral or fractional, one is said to *vary* as the other.

The symbol $\propto$ placed between the quantities is used to express this relation; and if we suppose m, n to be constant quantities, we may distinguish the different kinds of *variation* as follows:

(1.) If $A = mB$, then A varies *directly* as B, or $A \propto B$.

(2.) If $A = \frac{m}{B}$, then A varies *inversely* as B, or $A \propto \frac{1}{B}$.

(3.) If $A = mBC$, then A varies jointly as B and C, or $A \propto BC$.

(4.) If $A = m\frac{B}{C}$, then A varies *directly* as B, and *inversely* as C, or $A \propto \frac{B}{C}$.

(5.) If $A = mB^n$, then A varies *directly* as the nth power *of B, or* $A \propto B^n$.

(6.) If $A = \frac{m}{B^n}$, then A varies *inversely* as the nth power of B, or $A \propto \frac{1}{B^n}$.

(7.) If $A = mB + \frac{n}{C}$, then A varies as the sum of two quantities, one of which varies *directly* as B, and the other *inversely* as C.

8. *If* $A \propto B$, *and* $B \propto C$, *then* $A \propto C$.

For, let $A = mB$ .... (1),
and $B = nC$;

then, substituting in (1) for this value of B, we have

$A = m \,.\, nC = mnC$; or, since mn is constant,
$A \propto C$. *Q.E.D.*

9. *If* $A \propto B$, *then will* $A^2B + AB^2 \propto (A^2 + B^2)(A - B)$.

For, let $A = mB$, then

$A^2B + AB^2 = (mB)^2 \,.\, B + mB \,.\, B^2 = (m^2 + m)\,B^3$.... (1).

And so, $(A^2 + B^2)(A - B) = (m^2 + 1)(m - 1) \,.\, B^3$... (2).

(1) ÷ (2), then $\frac{A^2B + AB^2}{(A^2 + B^2)(A - B)} = \frac{m^2 + m}{(m^2 - 1)(m - 1)}$, a constant quantity.

$\therefore\ A^2B + AB^2 \propto (A^2 + B^2)(A - B)$. *Q.E.D.*

10. *If* $A \propto B$, *when* C *is constant,*
and $A \propto C$, *when* B *is constant;*
then $A \propto BC$, *when* BOTH *are* VARIABLE.

Let $a_{,}, b$ be corresponding values of A and B, when C is constant,

Then $A : a_{,} :: B : b$, or $a_{,} = \frac{Ab}{B}$ .... (1).

Now this value $a_{,}$ of A will, like A itself, vary as C.

Let then a, c be corresponding values of $a_{,}$, C, when B is constant.

Then $a_{,} : a :: C : c$, or $a_{,} = \frac{Ca}{c}$ .... (2).

(1) = (2), then

$$\frac{Ab}{B} = \frac{Ca}{c}, \text{ or } A = \frac{a}{bc}BC.$$

Hence A bears a constant ratio $\frac{a}{bc}$ to the product of the variable quantities B and C.

$\therefore$ A $\propto$ BC, when both B and C are variable. *Q.E.D.*

This proposition is the foundation of the so-called Double Rule of Three in Arithmetic.

We will illustrate this by an example,

Ex. If 15 men earn £180 in 6 weeks, how much will 22 men earn in 13 weeks?

We have ·

Number of pounds $\propto$ number of men, when time is not considered; that is, when number of weeks is constant.

Hence,

15 : 22 :: 180 : number of pounds required, when *time* is not considered.

Again,

Number of pounds $\propto$ number of weeks, when *men* are not considered; that is, when number of men is constant.

Hence,

6 : 13 :: 180 : number of pounds required, when *men* are not considered.

These two statements correspond to the ordinary statement of a Double Rule of Three sum.

But, by the above proposition :

Number of pounds $\propto$ number of men $\times$ number of weeks, when *both* are considered.

Hence,

15 $\times$ 6 : 22 $\times$ 13 :: 180 : number of pounds required.

This is the ordinary complete statement.

Cor. If A varies as each of the quantities B, C, D, &c.,

when all the rest are constant, then when all are variable, A varies as their product.

Ex. II.

1. If $x \propto y$, and when $x = 3$, $y = 7$; find the value of x when $y = 63$.

2. If A $\propto$ BC, and that C = 3, when A = 60, and B = 1; find the value of B, when A = 80, and C = 4.

3. If $x = my^2 + nz^2$, find m, n, when 17, 2, 1, and 23, 1, 2, are corresponding sets of values of x, y, z.

4. If a cylinder whose length is l, and diameter d, has a weight w, what will be the weight of a cylinder of the same material whose length is $l_{\prime}$, and diameter $d_{\prime}$?

5. If t be the time of a complete vibration of a pendulum of length l, then $t \propto \sqrt{l}$. Hence, find the length of a two-second pendulum, when the length of the second pendulum is 39·4 inches.

6. The time taken by a falling body varies as the square root of the space. Now, a body falls through 64 feet in 2 seconds. Hence, find the relation between time and space.

7. The strength of a beam supported at its ends varies inversely as its length, directly as its breadth, and directly as the square of its depth. Now, a beam whose length, breadth, and thickness are respectively 12 feet, 6 inches, 4 inches, will support 3 tons. Find the breaking-weight of a beam of the same material whose corresponding dimensions are 16 feet, 8 inches, 6 inches.

8. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings is as high as a pile of q sovereigns; compare the values of equal bulks of gold and silver.

9. The value of diamonds $\propto$ the square of their weight, and the square of the value of rubies $\propto$ the cube of their weight. A diamond of a carats is worth m times a ruby of b carats, and both together are worth c£. Required the value of a diamond and a ruby, each weighing n carats.

10. The heat generated by the discharge of a Leyden jar varies jointly as the *quantity* and the *tension* of the electricity; and the tension varies *directly* as the *quantity* of electricity, and *inversely* as the *surface* of the jar. Show that when the surface is constant, the quantity of electricity varies directly as the square root of the heat.

11. A rod of iron weighs 10 lbs. per foot run, what is the length in metres per kilogram?

12. British units of length (L), mass (M), and time (T) are respectively a foot, a pound, and a second; and the corresponding absolute units are a metre, a kilogram, and a second. Now, the dimensions of the units of force and work are respectively $\frac{LM}{T^2}$ and $\frac{L^2M}{T^2}$. Hence express the absolute units of force and work as British units.

CHAPTER III.

PERMUTATIONS AND COMBINATIONS.

Permutations.

11. The different arrangements that can be made of a number of things are called their **Permutations**.

Thus, *ab, ac, bc, ba, ca, cb,* are called the permutations of *a, b, c,* taken *two together;* and *abc, acb, bca, bac, cab, cba,* are the permutations of the same things taken *three together*.

In some works, the term *permutations* is limited to the arrangements of the things when taken *all together*. When they are *not taken all together*, they are called Variations. We shall include under the term "permutations" all such arrangements.

12. The *different groups* that can be made of a number of *things without* regard to the order of the things are called *Combinations*, Thus *ab*, *ba* form the same combination; so that the combinations of *a*, *b*, *c* taken two together are *ab*, *ac*, *bc*.

13. ***The number of permutations of* n *things taken* r *together is***

$$n(n-1)(n-2)\ldots\ldots(n-r+1).$$

Let *a*, *b*, *c* . . . . *k* be the *n* things. We will first take them *two* together. Now, there must be $\overline{n-1}$ things before which *a* stands first, and the same number before which *b* stands first, and so on for all the letters. Hence the total number of arrangements must be *n* times $(n-1)$, or $n(n-1)$. And hence the ***number of permutations of* n *things taken two together*** $= n(n-1)$.

Again, if we first consider the $(n-1)$ things *b*, *c*, ... *k*, we have just shown—as may be seen by putting $(n-1)$ for *n* in the above result—that

Number of permutations of $(n-1)$ things taken *two* together $= (n-1)(\overline{n-1}-1) = (n-1)(n-2)$.

Now, *a* may stand before each of these permutations of *b*, *c*, ... *k*.

Hence there are $(n-1)(n-2)$ permutations *in which* a *stands first*, when the *n* letters *a*, *b*, *c* . . . . *k* are permuted *three* together. And, of course, there are just as many in which *b*, *c*, ... *k* each stand first.

Hence, the whole number of permutations of the *n* things taken *three* together $= n$ times $(n-1)(n-2) =$

$$n(n-1)(n-2).$$

And by similar reasoning we may show that the number of permutations of *n* things taken *four* together =

$$n(n-1)(n-2)(n-3).$$

If we look at the *last term* of the *last factor* of each of these results we see that its value (neglecting the sign) is *one less than the number of things taken together*.

It is easy then to perceive *an apparent general law*, viz.: Number of permutations of *n* things taken *r* together

$$= n(n-1)(n-2)\ldots\ldots(n-\overline{r-1}).$$

By what is called *Mathematical Induction* we may assure *ourselves of the truth of this law*.

We will *assume* the law to be true when p things are taken together, and will then show that it must be true when $(p + 1)$ things are taken together.

Fixing our mind upon $(n - 1)$ things, viz., $b, c, \ldots . k$, we have by our assumption—putting therein $(n - 1)$ for n,

Number of permutations of $b, c, \ldots . k$, taken p together

$$= (n - 1)(n - 2) \ldots . (n - 1 - \overline{p - 1})$$
$$= (n - 1)(n - 2) \ldots . (n - p).$$

Now, a may stand before each of these, and hence the number of permutations of n things taken $(p + 1)$ together, where a stands first, is $(n - 1)(n - 2) \ldots . (n - p)$.

And the same may be said for each of the letters $b, c, \ldots k$.

Hence the whole number of permutations of n things taken $(p + 1)$ together is

$$n \text{ times } (n - 1)(n - 2) \ldots . (n - p),$$

or $$n(n - 1)(n - 2) \ldots . \{ n - (\overline{p + 1} - 1) \}.$$

Hence, *on the assumption that the law is true for any particular case*, we have shown it to be also true when the things are taken *one more together*.

Now, we know it to be true when the things are taken 2, 3, 4 together. It is therefore true when taken 5 together, and therefore also when taken 6, 7, &c., together. Hence, it is generally true.

We have therefore generally

Number of permutations of n things taken r together

$$= n(n - 1) \ldots . (n - \overline{r - 1}), \text{ or}$$
$$= n(n - 1) \ldots . (n - r + 1).$$

Cor. 1. Let $r = n$, then the whole number of permutations of n things taken *all* together $= n(n - 1) \ldots . (n - n + 1)$

$$= n(n - 1) \ldots . 1, \text{ or, as often written,}$$
$$= \underline{|n}, \text{ read } \textit{factorial } \text{n}.$$

Cor. 2. Every *combination* of r things will produce $\underline{|r}$ permutations, for the number is that of r things taken all together.

14. *To find the number of permutations of* n *things taken all together when they are not all different things.*

Let there be n letters, in which the letter a occurs p times, b occurs q times, c occurs r times, &c. Then the number of permutations of the n things when taken altogether is

$$\frac{\lfloor n}{\lfloor p \ \lfloor q \ \lfloor r \text{ \&c.}}.$$

For (Art. 13, Cor. 1) the number of permutations of n different things taken all together $= \lfloor n$.

Let P be the required number of permutations. Then, since every permutation contains a repeated p times, it is capable of forming $\lfloor p$ permutations on the supposition that the a's are all different letters. The whole number of permutations would then be P times $\lfloor p$, or P . $\lfloor p$. And similarly, on the supposition that all the b's are different, every permutation would form $\lfloor q$ permutations, and so on.

Hence, on the supposition that all the letters are different, the total number of permutations $= \text{P} . \lfloor p \ \lfloor q \ \lfloor r$ &c.

Hence we have $\text{P} . \lfloor p \ \lfloor q \ \lfloor r \text{ \&c.} = \lfloor n$,

And $$\therefore \text{P} = \frac{\lfloor n}{\lfloor p \ \lfloor q \ \lfloor r \text{ \&c.}}.$$

15. *To find the number of permutations of* n *things when each may be repeated once, twice, &c., times.*

Let $a, b, c, \ldots . k$ be the n quantities.

When taken *one* at a time, their number is n.

If we place a before each of these letters, we get n permutations $aa, ab, ac, \ldots . ak$, in which a stands first; and, we must also have as many where b stands first, where c stands first, &c.

Hence, on the whole, when the letters are taken *two* together, the number of permutations $= n$ times $n = n^2$.

And so we may show that, when taken *three* together, the number of permutations $= n$ times $n^2 = n^3$.

Hence it is easy to show by induction that the number of permutations of n things when each may be repeated r time $= n^r$.

Ex. 1. Find the number of permutations of 8 things taken 4 together.

Generally (Art. 13) $P = n(n-1) \ldots . (n-r+1)$.

$$\left.\begin{array}{r} \text{Here } n = 8 \\ r = 4 \end{array}\right\} \therefore P = 8(8-1) \ldots . (8-4+1) = 8 . 7 . 6 . 5.$$

Ex. 2. Find the number of permutations which can be made out of the letters of the word *Wagga-Wagga* when taken all together.

$$\text{Generally (Art. 14) } P = \frac{\underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|r}}$$

Here $n = 10$, and, if p, q, r represent respectively the number of times w, a, g appear,

$$p = 2,\ q = 4,\ r = 4.$$

$$\text{Hence } P = \frac{\underline{|10}}{\underline{|2}\ \underline{|4}\ \underline{|4}} = 5 . 7 . 9 . 10.$$

Combinations.

16. *To find the number of combinations of* n *things taken* r *together.*

Let C_r be the required number of combinations.

Now (Art. 13, Cor. 2) every combination of r things is capable of forming $\underline{|r}$ permutations.

Hence the whole number of permutations capable of being formed by C_r combinations $= \underline{|r}$ times $C_r = \underline{|r}\, C_r$.

Now this is the number of permutations of n things taken r together.

Hence, by Art. 13,

$$\underline{|r}\, C_r = n(n-1) \ldots . (n-r+1).$$

$$\therefore C_r = \frac{n(n+1) \ldots . (n-r+1)}{\underline{|r}}.$$

COR. Since

$$\frac{n(n-1)\ldots\ldots(n-r+1)}{\lfloor r}$$

$$= \frac{n(n-1)\ldots\ldots(n-r+1)\,.\,(n-r)\ldots\ldots2\,.\,1}{\lfloor r \quad . \quad \lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor r\ \lfloor n-r}, \text{ we have}$$

$$C_r = \frac{\lfloor n}{\lfloor r\ \lfloor n-r}.$$

17. *To show that* $C_r = C_{n-r}$, *or that the number of combinations of* n *things taken* r *together is equal to the number taken* (n − r) *together.*

We have

$$C_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{\lfloor r},$$

and therefore putting $n - r$ for r,

$$C_{n-r} = \frac{n(n-1)\ldots\ldots(n-\overline{n-r}+1)}{\lfloor n-r}$$

$$= \frac{n(n-1)\ldots\ldots(r+1)}{\lfloor n-r}$$

$$= \frac{n(n-1)\ldots\ldots(r+1)\,.\,r(r-1)\ldots\ldots2\,.\,1}{\lfloor n-r \quad . \quad \lfloor r}$$

$$= \frac{\lfloor n}{\lfloor n-r\,.\,\lfloor r}.$$

Hence (Art. 16, Cor.) $\quad C_r = C_{n-r}$. *Q.E.D.*

18. *To find the number of combinations that can be formed of* n *sets of things, in which there are* p *of one sort,* q *of another,* r *of another, &c., by taking one from each set.*

First, if there be *two* sets, p of one sort and q of another, we may place each of the former sort before each of the latter, and thus obtain p times q of combinations.

The number of combinations is therefore pq.

Again, if there be a third set containing r things, we may place each one of this set before all the above combinations in turn.

Hence the total number of combinations thus formed is r times pq, or pqr.

And so on for any number of sets.

Hence, number of combinations required $= pqr \dots.$

19. *To find the number of combinations of* m *things and* n *things when* p *of the former are combined with* q *of the latter.*

The number of combinations of m things taken p together

$$= \frac{m(m-1)\dots(m-p+1)}{\underline{|p}}.$$

And the number of combinations of n things taken q together

$$= \frac{n(n-1)\dots(n-q+1)}{\underline{|q}}.$$

Now each of the former combinations may be combined in turn with all of the latter. Hence the total number of combinations

$$= \frac{m(m-1)\dots(m-p+1)}{\underline{|p}} \cdot \frac{n(n-1)\dots(n-q+1)}{\underline{|q}}.$$

20. *To find when the number of combinations is greatest.*

We have $C_r = \dfrac{n(n-1)\dots(n-r+1)}{\underline{|r}}$,

and $C_{r-1} = \dfrac{n(n-1)\dots(n-\overline{r-1}+1)}{\underline{|r-1}}$

$$= \frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}}.$$

Hence, by division, $\dfrac{C_r}{C_{r-1}} = \dfrac{n-r+1}{r} = \dfrac{n+1}{r} - 1.$

Now as r continually increases, $\left(\dfrac{n+1}{r} - 1\right)$ continually diminishes, and therefore must eventually become *less than unity.*

But as long as its value is > 1, the number of combinations found must be greater than the number previously found. Hence the greatest number of combinations will be for that value of r *next before* the one which makes $\left(\frac{n+1}{r} - 1\right)$ less than unity.

When $\frac{n+1}{r} - 1 = 1$, of course $C_r = C_{r-1}$; and since the next higher value of r will make $\left(\frac{n+1}{r} - 1\right) < 1$, the value of r obtained from this equation will give the greatest number of combinations.

Solving the equation, we have $r = \left(\frac{n+1}{2}\right)$;
and, since r is integral, n must be *odd*.

Hence, when n is *odd*,

The greatest number of combinations is when taken $\frac{n+1}{2}$ *together, and also when taken* $\frac{n+1}{2} - 1$, *or* $\frac{n-1}{2}$ *together.*

When n is even, we require the value of r next before the one which makes $\frac{n+1}{r} - 1 < 1$, or $\frac{n+1}{r} < 2$, or $n + 1 < 2r$, or $r > \frac{n+1}{2}$.

The value of r required is $\therefore r = \frac{n}{2}$.

Hence, when n is *even*,

The greatest number of combinations is when they are taken $\frac{n}{2}$ *together.*

Ex. 1. Find the number of combinations of 7 things taken 3 together.

$$\text{Number required} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

Ex. 2. How many words can be formed out of 19 consonants and 7 vowels by taking 4 consonants and 2 vowels for each word?

Number of combinations formed by 19 *consonants* taken 4 together

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4}.$$

And number formed by 7 vowels taken 2 together $= \frac{7 \cdot 6}{1 \cdot 2}$.

Hence (Art. 19),

Number of *combinations* by taking 4 *consonants* and 2 *vowels.*

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{7 \cdot 6}{1 \cdot 2} = 81396.$$

Now each combination contains 6 letters, and these may (Art. 13) be permuted in $\lfloor 6$ ways.

Hence total number of words required $= 81396 \times \lfloor 6$.
$= 58605120.$

Ex. III.

1. The number of permutations of some things taken 5 together : the number taken 3 together :: 42 : 1. Find the number of things.

2. How many different permutations can be made of the letters of the word *Proportion* taken all together?

3. How many can be made of each of the words *Kilimaudjaro, Hammada?*

4. Show that the number of triangles which may be formed by joining the angular points of a polygon is $\frac{1}{6} n (n - 1) (n - 2)$, n being the number of sides.

5. How many different numbers can be made by using all the figures of the number 302342044?

6. In how many ways can 10 persons be arranged about a round table?

7. The number of combinations of a number of things taken 5 together is the same as the number taken 7 together. Find the number of things.

8. The number of combinations of n things taken 2 together exceeds by 6 the number of combinations of $(n - 1)$ things taken two together. Find n.

9. If P_r represent the number of permutations of n things taken r together, show that, when $m > 2$,

$$(P_1 - 1)(P_2 - P_1)(P_3 - P_2)(P_4 - P_3)\ldots\ldots(P_m - P_{m-1}) = P_2P_3P_4\ldots\ldots P_{m-1}^2 \,.\, P_{m+1}.$$

10. The number of combinations of $2n$ things taken n together

$$\cdot\, 2^n \cdot \frac{1\,.\,3\,.\,5\ldots\ldots(2n-1)}{1\,.\,2\,.\,3\ldots\ldots\quad n}$$

11. The ratio of the combinations of $4n$ things taken $2n$ together to the combinations of $2n$ things taken n together

$$= \frac{1\,.\,3\,.\,5\ldots\ldots(4n-1)}{(1\,.\,3\,.\,5\ldots\ldots 2n-1)^2}.$$

12. Find the *total number* of signals which can be made by m needles, each of which can assume n distinct positions.

13. Show that $C_r^2 = \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{n - r}\right) C_{r-1}.C_{r+1}$.

14. If ${}^nP_r$ represent the number of permutations of n things taken r together, show that ${}^{n+1}P_r - {}^nP_r = r\,{}^nP_{r-1}$.

15. Show that $2P_2 + 3P_3 + \&c. + rP_r$

$$= (n - 1)(P_1 + P_2 + \&c. + P_r) - P_{r+1}.$$

16. At a Parliamentary election for two members there were four candidates. In how many ways can votes be recorded?

17. A firing party, consisting of one sergeant and 12 men, is to be chosen from a company of 100 men and 3 sergeants. In how many ways can this be done?

18. In taking a handful of shot from a bag, show that the chance of getting an even number is greater than the chance of getting an odd number.

(The truth of the Binomial Theorem may be assumed.)

CHAPTER IV.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

Arithmetical Progression.

21. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a common difference.

Thus, the numbers 1, 3, 5, 7, &c., are in A. P., and so are the numbers 6, $5\frac{1}{2}$, 5, $4\frac{1}{2}$, &c.

In the first series the common difference is 2, and in the second $-\frac{1}{2}$.

If a be the first term, and d the common difference, the series is evidently a, $a + d$, $a + 2d$, $a + 3d$, &c., and the nth term is $a + \overline{n - 1} . d$.

When the series contains n terms, the nth term is the *last* term. Hence, representing the last term by l, we have

$$l = a + \overline{n - 1} . d.$$

22. *To find the sum of* n *terms of an* A. P.

Let a be the first term, d the common difference, and S the required sum;

Then $S = a + \overline{a + d} + \overline{a + 2d} + \ldots . + l \ldots . (1).$

And, writing the series in reverse order, and remembering that the terms will then diminish by d, we have

$$S = l + \overline{l - d} + \overline{l - 2d} + \ldots . + a \ldots . (2).$$

Then adding together (1) and (2) we have

$$2S = \overline{a + l} + \overline{a + l} + \overline{a + l} + \ldots . + \overline{a + l};$$

and, since there are n of the terms $(a + l)$, we have

$$2S = (a + l)\,n, \text{ or } S = (a + l)\frac{n}{2}.$$

Hence we have the following practical rule to find the sum:

Add together the first and last terms, and multiply the result by half the number of terms.

COR. Since (Art. 21), $l = a + \overline{n - 1} \cdot d$, we have

$$S = (a + a + \overline{n - 1} \cdot d) \frac{n}{2} = (2a + \overline{n - 1} \cdot d) \frac{n}{2};$$

which expresses the sum of an A. P. in terms of the first term, the common difference, and the number of terms.

23. *To insert* m *arithmetic means between* a *and* b.

As there are m means, there must be altogether $m + 2$ terms.

Hence (Art. 21),

$$l = a + \overline{(m + 2) - 1} \cdot d = a + \overline{m + 1} \cdot d.$$

Here $l = b$;

and therefore $a + \overline{m + 1} \cdot d = b$, from which $d = \dfrac{b - a}{m + 1}$.

Hence the means required are

$$a + \frac{b - a}{m + 1},\ a + 2 \cdot \frac{b - a}{m + 1},\ \&c.,\ a + m \cdot \frac{b - a}{m + 1};$$

$$\text{or } \frac{ma + b}{m + 1},\ \frac{\overline{m - 1} \cdot a + 2b}{m + 1},\ \&c.,\ \frac{a + mb}{m + 1}.$$

Ex. 1. Find the sum of 16 terms of the series 2, 4, 6, &c.

Generally, $S = (2a + \overline{n - 1} \cdot d) \dfrac{n}{2}$.

Here $\left.\begin{array}{l} a = 2 \\ d = 2 \\ n = 16 \end{array}\right\}$

$$\therefore S = \left\{ 2(2) + \overline{16 - 1} \cdot 2 \right\} \frac{16}{2} = (4 + 30)\, 8 = 272.$$

Or thus, $l = a + \overline{n - 1} \cdot d = 2 + \overline{16 - 1} \cdot 2 = 32$.

Hence, by Rule (Art. 22),

$$S = (2 + 32) \frac{16}{2} = 272, \text{ as before.}$$

Ex. 2. Sum the series 12, $11\frac{1}{4}$, $10\frac{1}{2}$, &c., to 8 terms.

We have $S = (2a + \overline{n - 1} \cdot d) \dfrac{n}{2}$.

Here $\left.\begin{aligned} a &= 12 \\ d &= -\tfrac{3}{4} \\ n &= 8 \end{aligned}\right\}$

$$\therefore S = \left\{ 2\,(12) + \overline{8-1}\,(-\tfrac{3}{4}) \right\} \frac{8}{2} = \left(24 - \frac{21}{4}\right) 4 = 75.$$

Ex. 3. Insert 8 arithmetical means between 3 and 21.

We have, Art. 23, $d = \dfrac{b - a}{m + 1}.$

Here $\left.\begin{aligned} a &= 3 \\ b &= 21 \\ m &= 8 \end{aligned}\right\}$

$$\therefore d = \frac{21 - 3}{8 + 1} = \frac{18}{9} = 2, \text{ the common difference.}$$

And since the first term is 3, the 8 means are 5, 7, 9, 11, 13, 15, 17, 19.

Ex. 4. The sum of an A. P. is 140, the first term is 5, and common difference 2; find the number of terms.

We have $S = (2\,a + \overline{n-1}\,.\,d)\,\dfrac{n}{2}.$

Here

$\left.\begin{aligned} S &= 140 \\ a &= 5 \\ d &= 2 \end{aligned}\right\}$ Hence $140 = \left\{ 2\,(5) + \overline{n-1}\,.\,2 \right\} \dfrac{n}{2} = (n + 4)\,n,$

or, transposing, $n^2 + 4\,n - 140 = 0$, or

$$(n - 10)\,(n + 14) = 0; \text{ from which}$$
$$n = 10 \text{ or } -14.$$

The value $n = 10$ tells us that, commencing from the first term of the series and counting 10 terms, the sum of the terms is 140. The value $n = -14$ tells that, commencing from the *opposite end* of the series, and writing down 14 terms, the sum of the terms is also 140. The following would be the two series:

5, 7, 9, 11, 13, 15, 17, 19, 21, 23;

and − 3, − 1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23.

When the two values of n *are both* positive and integral, the series must in both cases be commenced at the first term. It will always be found that the *extra* terms which are given by the larger value will of themselves give *zero* for a sum.

Should one of the values of n be fractional, we may still give it an interpretation. This will be best understood from the following example:

Ex. 5. The sum of an A. P. is 270, the first term 9, and the common difference 4; find the number of terms.

We have $$S = (2a + \overline{n-1}\,.\,d)\,\frac{n}{2}.$$

Here

$$\left.\begin{aligned} S &= 270 \\ a &= 9 \\ d &= 4 \end{aligned}\right\}, \therefore 270 = \left\{2\,(9) + \overline{n-1}\,.\,4\right\}\frac{n}{2} = 2n^2 + 7n.$$

Transposing, then $2n^2 + 7n - 270 = 0$, or

$$(n - 10)(2n + 27) = 0; \text{ from which}$$
$$n = 10 \text{ or } -13\tfrac{1}{2}.$$

The first value $n = 10$ gives the series

9, 13, 17, 21, &c., 45.

The second value, $n = -13\frac{1}{2}$, being negative, tells us to commence with the *opposite end* of the series, viz., the term 45, and write down 13 complete terms, and *another term* which is made up of the 13th term and *half the common difference.* The 13 complete terms beginning with the term 45, and writing from right to left, are

-3, 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45.

The *incomplete term* is $-3 - \frac{1}{2}(4) = -5$.

And the sum of the numbers -5, -3, 1, 5 is *zero*, agreeably to the remark made in Ex. 4 above.

Ex. IV.

1. Find the 10th term of the series 2, 6, 10, &c., and the sum to 12 terms.

2. Find the last term of the series 3, $3\frac{3}{4}$, $4\frac{1}{2}$, &c., to 20 terms.

Sum the following series:

3. 10, 8, 6, &c., to 9 terms.

4. $5\frac{1}{6}, 6\frac{1}{2}, 7\frac{5}{6}$, &c., to 12 terms.

5. 1, 3, 5, 7, &c., to 50 terms.

6. $a + 6d$, $a + 5d$, $a + 4d$, &c., to 13 terms.

7. $x + 9y$, $x + 7y$, $x + 5y$, &c., to 10 terms.

8. $3\frac{6}{7}, 4\frac{4}{21}, 4\frac{11}{21}$, &c., to 14 terms.

9. Insert 12 arithmetic means between 2 and 41.

10. Find the sum of 12 terms of the series whose 5th term is 14, and 10th term 31.

11. The sum of 12 terms of an A. P. is 123, and the common difference is $1\frac{1}{2}$; find the first term.

12. Given m the middle term of an A. P., and n the number of terms; show that the sum of the series is mn.

13. If P, Q be the pth and qth terms respectively, find the first term and the common difference.

14. If S_r represent the sum of r terms of an A. P., show that

$$S_{2n+1} = S_{n+2} + S_{n-1} + (n^2 + n - 2)\,d.$$

15. If P, Q, R be respectively the pth, qth, rth terms, show that

$$P(q - r) + Q(r - p) + R(p - q) = 0.$$

16. There are m arithmetical progressions, the first term of each of which is 1, and the common differences respectively 1, 2, 3, &c., m, Prove that the sum of the nth terms

$$= \frac{\overline{n-1}\,.\,m^2 + \overline{n+1}\,.\,m}{2}.$$

17. If $\phi(n)$ represent the sum of n terms of an A. P., show that

$$\phi(3n) = 3\left\{\phi(2n) - \phi(n)\right\}.$$

18. Show that $\phi(n) + \phi(n+1) + \phi(n+2) + \ldots \phi(2n)$

$$= \frac{n(3n-1)}{1\,.\,2}a + \frac{n(n-1)(7n-2)}{1\,.\,2\,.\,3}d.$$

19. Show also that

$$\frac{\phi(n)}{n} + \frac{\phi(n-1)}{n-1} + \text{\&c., to } n \text{ terms} = na + \tfrac{1}{4} n(n-1) d.$$

20. Find the sum of n terms of the series whose rth term is $3r - 5$.

21. The sum of three numbers in A. P. is 15, and the sum of their squares is 83. Find the numbers.

22. The sum of four numbers in A. P. is 10, and the sum of their products taken two together is 35. Find the numbers.

23. The result in Ex. 15 may be arranged thus:

$$p(Q - R) + q(R - P) + r(P - Q) = 0.$$

What A. P. does this suggest; and what is the relation between the common difference of this series and the one in Ex. 15?

24. How many terms of the series 4, 7, 10, &c., amount to 144?

Geometrical Progression.

24. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *common ratio.*

Thus 2, 6, 18, 54, &c., is a G. P., and so is 12, 4, $1\frac{1}{3}$, &c., the common ratios being 3 and $\frac{1}{3}$ respectively.

It is evident that the common ratio may be found by dividing any term by the *preceding term.*

Let a be the first term, and r the common ratio

The G. P. would then be

$$a, ar, ar^2, ar^3, \text{\&c.}$$

If there be n terms, the last or nth term $= ar^{n-1}$.

Hence we have in G. P., $l = ar^{n-1}$ .... (1).

25. *To find the sum of a Geometrical Progression.*

Let a be the first term, r the common ratio, and S the sum of n terms; then we have

$$S = a + ar + ar^2 + \text{\&c.}, + ar^{n-1},$$

or multiplying each side by r,

$$rS = \quad ar + ar^2 + \text{\&c.}, + ar^{n-1} + ar^n.$$

Hence, subtracting the upper line from the lower, we have

$$rS - S = ar^n - a, \text{ or } S(r-1) = a(r^n - 1);$$

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

$$\text{Since } a \cdot \frac{r^n - 1}{r - 1} = \frac{ar^n - a}{r - 1} = \frac{r \cdot ar^{n-1} - a}{r - 1} = \frac{rl - a}{r - 1},$$

We have also, $$S = \frac{rl - a}{r - 1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

When r is less than unity, it is usual to write these results as follows, being obtained by changing the signs of numerator and denominator in each case and reversing the order:

$$S = a \cdot \frac{1 - r^n}{1 - r} = \frac{a - rl}{1 - r} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

26. *To find the limit of an infinite* G. P. *whose common ratio is less than unity.*

Def. The **Limit** of a series is that quantity to which the sum of the series continually approaches as the number of terms increases.

We have (Art. 25), $S = a \cdot \dfrac{1 - r^n}{1 - r}$.

But since $r < 1$, and n indefinitely large, the value of r^n is indefinitely small, and becomes smaller and smaller the larger n is. Indeed, the value of r^n continually approaches *zero*, as n increases indefinitely.

Let Σ represent the limit of the sum of the series, then we have

$$\Sigma = a \cdot \frac{1 - 0}{1 - r} = \frac{a}{1 - r}.$$

27. *To insert* m *geometrical means between* a *and* b.

Since there are m *means*, there must be $m + 2$ *terms*.

Hence (Art. 24) $l = ar^{\overline{m+2}-1} = ar^{m+1}$.

And therefore, since here $l = b$, we have

$$ar^{m+1} = b, \text{ or } r^{m+1} = \frac{b}{a}.$$

$$\therefore r = \sqrt[m+1]{\frac{b}{a}}.$$

This gives the common ratio, and the means can therefore be easily inserted.

Ex. 1. Sum the series 2, 4, 8, 16, &c., to 10 terms.

Generally, $S = a \cdot \frac{r^n - 1}{r - 1}$.

Here $\left.\begin{array}{r} a = 2 \\ r = 2 \\ n = 10 \end{array}\right\} \therefore S = 2 \cdot \frac{2^{10} - 1}{2 - 1} = 2(2^{10} - 1)$.

Ex. 2. Given the first term of a G. P. to be 5, the last term 160, and the number of terms 6. Find the common ratio.

By Art. 24, $l = ar^{n-1}$.

Here $\left.\begin{array}{r} l = 160 \\ a = 5 \\ n = 6 \end{array}\right\}$ Hence, substituting, $160 = 5\,r^{6-1} = 5\,r^5$
$\therefore r^5 = 160 \div 5 = 32$, or $r = \sqrt[5]{32} = 2$.

Ex. 3. Find the value of the recurring decimal ·666 &c.

The value required is the limit of the G. P. $\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3}$ + &c., to infinity.

Now (Art. 26) $\Sigma = \frac{a}{1 - r}$.

Here $\left.\begin{array}{r} a = \frac{6}{10} \\ r = \frac{1}{10} \end{array}\right\} \therefore \Sigma = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{9} = \frac{2}{3}$.

The meaning of this result is that $\frac{2}{3}$ is the quantity to which the value of ·666 &c. approaches as the number of decimal figures taken is increased.

Ex. 4. Find the sum to n terms, and the limit to infinity, of the series:

$$a + (a + b)\,r + (a + 2\,b)\,r^2 + \text{\&c., when } r < 1.$$

We have

$$S = a + (a+b)r + (a+2b)r^2 + \&c. + (a+\overline{n-1}.b)\,r^{n-1}$$

And $\therefore$ $$Sr = ar + (a+b)\,r^2 + \&c. + (a+\overline{n-2}.b)\,r^{n-1} + (a+\overline{n-1}.b)\,r^n.$$

Therefore, subtracting,

$$S(1-r) = a + br + br^2 + \&c. + br^{n-1} - (a+\overline{n-1}.b)r^n.$$

But (Art. 25) the sum of the G. P.

$$br + br^2 + \&c. + br^{n-1} = br.\frac{1-r^{n-1}}{1-r}.$$

$$\therefore S(1-r) = a + br.\frac{1-r^{n-1}}{1-r} - (a+\overline{n-1}.b)\,r^n.$$

$$\therefore S = \frac{a}{1-r} + br.\frac{1-r^{n-1}}{(1-r)^2} - (a+\overline{n-1}.b).\frac{r^n}{1-r} \ldots (1).$$

If r be a proper fraction, the limits of r^{n-1} and r^n, as n indefinitely increases, are evidently *zero*.

Hence we have

$$\Sigma = \frac{a}{1-r} + br.\frac{1-0}{(1-r)^2} - (a+\overline{n-1}.b).\frac{0}{1-r}$$

$$= \frac{a}{1-r} + \frac{br}{(1-r)^2}.$$

Ex. V.

1. Find the 6th term of the series 3, 6, 12, &c.
2. Find the 5th term of the series 35, 7, $1\frac{2}{5}$, &c.
3. Sum the series 81, $-$ 27, 9, &c., to 6 terms.
4. Sum the series $2\frac{4}{5}$, 7, $17\frac{1}{2}$, &c., to 12 terms.
5. Sum the series 6, $-$ 2, $\frac{2}{3}$, &c., to n terms.
6. Find the sum of 10 terms of the series whose first term is $\frac{1}{2}$, and third term $\frac{1}{8}$.

Sum to infinity the following series:

7. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$
8. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} - \&c.$
9. $10 - 9 + 8{\cdot}1 - 7{\cdot}29 + \&c.$
10. $5\frac{1}{3} + 1\frac{1}{3} + \frac{1}{3} + \frac{1}{12} + \&c.$
11. $\cdot 6 - \cdot 06 + \cdot 006 - \&c.$

12. $a^2b - a + \frac{1}{b} - \&c.$

13. $(a^2 - b^2) + (a - b) + \frac{a - b}{a + b} + \&c.$

14. $(\sqrt{5} + 2) + 1 + (\sqrt{5} - 2) + (9 - 4\sqrt{5}) + \&c.$

15. $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \&c.$

16. $\frac{3}{4} + \frac{4}{3} + \frac{1}{4} + \frac{1}{3} + \frac{1}{12} + \frac{1}{12} + \&c.$

17. $\frac{\sqrt{3} + 1}{\sqrt{3} - 1} + 1 + \frac{\sqrt{3} - 1}{\sqrt{3} + 1} + \&c.$

18. $\frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \&c.$

19. If the sum of 10 terms of G. P. is 513, and common ratio is -2, find the series.

20. Find the 5th term of the G. P. whose 3rd and 6th terms are respectively a, b.

21. If P, Q, R be respectively the pth, qth, rth terms of a geometrical series, then

$$P^{q-r} . Q^{r-p} . R^{p-q} = 1.$$

22. Show that the sum of 2^n terms of a G. P. series may be expressed in the form

$$a(r + 1)(r^2 + 1)(r^4 + 1) \ldots . (r^{2^{n-1}} + 1).$$

23. If s_1 be the sum of an infinite series, and s_2 the sum of the squares of the terms of the series, find the series.

24. If s_p be the sum of p terms of a geometric progression, show that

$$s_n - s_m = r^m s_n - r^n s_m.$$

25. If s_m be the sum of the mth powers of an infinite geometric series, then

$$\frac{1}{s_1} \pm \frac{1}{s_2} \pm \frac{1}{s_3} \pm \ldots . \textit{ad. inf.} = \frac{1}{a \pm 1} - \frac{r}{a \pm r}.$$

26. If $\frac{ma + nb}{ma - nb} = \frac{mb + nc}{mb - nc} = \frac{mc + nd}{mc - nd} = \&c.$, then $a, b, c, d, \&c.$, are in G. P.

27. If s_p be the sum of p terms of a G. P., show that the sum of the products of every two terms $= \dfrac{r}{r+1} s_p \cdot s_{p-1}$.

28. Show also that

$$ns_n + (s_1 + s_2 + s_3 + \text{\&c., to } n \text{ terms})$$
$$= \frac{s_1}{r-1}\left\{n(r^n - 2) + \frac{r}{r-1}(r^n - 1)\right\}.$$

If a, b, c, d are in geometrical progression, show that:

29. $(a^2 + b^2)(c^2 + d^2) = b^2(b + d)^2$.

30. $a^2 + b^2 + c^2 = (a + b + c)(a - b + c)$.

31. $3(a - b + c)^2 - (a^2 + b^2 + c^2) = 2(a - b + c)(a - 2b + c)$.

32. $(a + b + c + d)^2 = (a + b)^2 + (c + d)^2 + 2(b + c)^2$.

33. $a^2(a - 2b + c)^2 = (a - b)^4$.

34. $a^4(a - 3b + 3c - d)^2 = (a - b)^6$.

35. Find the sum of n terms of the series whose nth term is $2^n + n$.

36. If a, b, c, d are in geometrical progression, show that $(a - b)^2, (b - c)^2, (c - d)^2$ are in G. P.

Harmonical Progression.

28. Quantities are said to be in **Harmonical Progression** when any three consecutive terms being taken, the first is to the third as the difference between the first and second is to the difference between the second and third.

Thus, if a, b, c be in H. P., we must have

$$a : c :: a - b : b - c.$$

We cannot make a formula for the sum of n terms of an H. P., as in the cases of Arithmetical and Geometrical Progression.

29. *The reciprocals of the terms of an Harmonical Progression form a series in Arithmetical Progression.*

Let a, b, c be in H. P.

Then (Art. 28), $\quad a : c :: a - b : b - c,$

$$\therefore a(b - c) = c(a - b),$$

or dividing each side by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

that is, the difference between the first and second of the terms $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ is equal to the difference between the second and third.

Hence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

30. *To insert* m *harmonical means between* a *and* b.

Taking the reciprocals $\frac{1}{a}$ and $\frac{1}{b}$, we must first find m arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, and afterwards invert them.

Let d be the common difference of the arithmetical series.

Then (Art. 23), $$d = \frac{\frac{1}{b} - \frac{1}{a}}{m + 1} = \frac{a - b}{(m + 1)\,ab}.$$

The arithmetical means are

$$\frac{1}{a} + \frac{a - b}{(m + 1)\,ab},\ \frac{1}{a} + \frac{2(a - b)}{(m + 1)\,ab}, \text{ \&c., } \frac{1}{a} + \frac{m(a - b)}{(m + 1)\,ab}, \text{ or}$$

$$\frac{a + mb}{(m + 1)\,ab},\ \frac{2a + \overline{m - 1}\,.\,b}{(m + 1)\,ab}, \text{ \&c., } \frac{ma + b}{(m + 1)\,ab}.$$

Hence, inverting, the harmonical means are

$$\frac{(m + 1)\,ab}{a + mb},\ \frac{(m + 1)\,ab}{2a + \overline{m - 1}\,.\,b}, \text{ \&c., } \frac{(m + 1)\,ab}{ma + b}.$$

31. *If* A, G, H *respectively represent the Arithmetical, Geometrical, and Harmonical means between* a *and* c, *show that* A : G :: G : H.

We have $A - a = c - A$, and $\therefore A = \frac{1}{2}(a + c)$ ...... (1).

Again, $$\frac{G}{a} = \frac{c}{G}, \text{ and } \therefore G = \sqrt{ac} \text{ .... (2).}$$

Also, $a : c :: H - a : c - H$ and $\therefore H = \frac{2ac}{a + c}$ .... (3).

Multiplying (1) and (3), we have

$$AH = \frac{1}{2}(a + c) \cdot \frac{2\,ac}{a + c} = ac \text{..................} (4).$$

And from (2) we get $G^2 = ac$..............................(5).

$$(4) = (5), \text{ then } AH = G^2, \text{ or } A : G :: G : H.$$

COR. $A > G$, and $G > H$.

For, (1) − (2), $A - G = \frac{1}{2}(a + c) - \sqrt{ac}$

$$= \frac{1}{2}(a - 2\sqrt{ac} + c)$$

$$= \frac{1}{2}(\sqrt{a} - \sqrt{c})^2, \text{ a positive quantity.}$$

$$\therefore A > G.$$

And hence, since A, G, H form a continued proportion, it must follow that $G > H$. *Q.E.D.*

Ex. 1. The 6th term of an H. P. is $\frac{1}{13}$, and the 1st term is $\frac{1}{3}$, find the intermediate terms.

Inverting, we have 3 and 13 as the 1st and 6th terms of the corresponding A. P.

Hence $13 = 3 + 5\,d$, or $d = 2$.

Therefore the intermediate terms of the A. P. are 5, 7, 9, 11.

Hence the required terms are $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$.

Ex. 2. If the first three of four numbers are in A. P., and the last three in H.P., then the product of the means equals the product of the extremes.

Let a, b, c, d be the four numbers.

Then, since a, b, c are in A. P., we have, by Art. 31,

$$b = \frac{1}{2}(a + c) \text{ ....} (1).$$

And, since b, c, d are in H. P., we have also, by Art. 31,

$$c = \frac{2\,bd}{b + d} \text{ ....} (2).$$

Substituting in (2) for the value of b in (1), we have

$$c = \frac{(a + c)\,d}{b + d}, \text{ or } bc + cd = ad + cd;$$

$$\text{or } bc = ad. \quad Q.E.D.$$

Ex. VI.

1. Insert 7 harmonic means between 1 and $\frac{1}{64}$.

2. Insert 6 harmonic means between 2 and $\frac{3}{2}$.

3. Continue to 4 terms each way the series 2, $1\frac{1}{2}$, $1\frac{1}{5}$.

4. If P, Q, R are respectively the pth, qth, and rth terms of an H. P., show that

$$\mathrm{QR}(q - r) + \mathrm{RP}(r - p) + \mathrm{PQ}(p - q) = 0.$$

5. Given P, Q, the pth and qth terms respectively, find the nth term.

6. What quantity must be added to each of the quantities a, b, c in order to form an H. P.?

7. If s, s' be the sums of two infinite series whose first term is 1 and common ratios r and r', then will s, s', $\frac{1}{r+1-2r'}$, be in H. P.

8. If a, b, c are in H. P., show that

$$a - b : a :: a - c : a + c.$$

9. If G, H be the geometric and harmonic means between a and b, show that

$$b = \mathrm{G}\,.\,\frac{\sqrt{\mathrm{G}+\mathrm{H}} - \sqrt{\mathrm{G}-\mathrm{H}}}{\sqrt{\mathrm{G}+\mathrm{H}} + \sqrt{\mathrm{G}-\mathrm{H}}}.$$

10. The sum of three numbers in H. P. is $1\frac{2}{3}$, and the sum of their reciprocals is $8\frac{1}{5}$. Find the numbers.

11. If H be the harmonic mean between a and b, and H_1 the harmonic mean between a and H, and also H_2 the harmonic mean between H and b, show that

$$\mathrm{H}_1 : \mathrm{H}_2 :: a + 3b : 3a + b.$$

12. If A_1, A_2, be the arithmetic means corresponding to the harmonic means H_1, H_2 in the last example, show that

$$A_1 H_1 = \mathrm{A}_2\mathrm{H}_2.$$

CHAPTER V.

THE BINOMIAL THEOREM.

32. We have already shown (Art. 32, Vol. I.) how to develop by multiplication the positive integral powers of $a + b$, and indeed by *assuming* the truth of an apparent general law, shown how to develop these powers by inspection. We shall now prove this law, which goes by the name of the **Binomial Theorem,** and may be expressed as follows:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \&c.$$

where n may be integral or fractional, positive or negative.

Positive Integral Index.

33. *To prove the Binomial Theorem for a positive integral index.*

By actual multiplication, we have

$$(a + b_1)(a + b_2) = a^2 + (b_1 + b_2)a + b_1b_2.$$

$$(a + b_1)(a + b_2)(a + b_3) = a^3 + (b_1 + b_2 + b_3)a^2 + (b_1b_2 + b_1b_3 + b_2b_3)a + b_1b_2b_3.$$

$$(a + b_1)(a + b_2)(a + b_3)(a + b_4) = a^4 + (b_1 + b_2 + b_3 + b_4)a^3 + (b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)a^2 + (b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4)a + b_1b_2b_3b_4.$$

We observe—as on Page 164, (1), (2), Vol. I.—the following laws:

1. The coefficient of the *first* term is unity; that of the *second* term the sum of the letters b_1, b_2, b_3, &c., taken *singly;* that of the *third*, the sum of their products taken *two* together; that of the *fourth*, the sum of their products taken *three* together; and so on, the last term being the product of all the letters b_1, b_2, b_3, &c.

2. The index of a is, in the first term, that of the number of binomial factors, and it *sinks* one every term.

3. The number of terms is *one more* than the number of binomial factors.

We will assume this law to be true for r factors. We then have

$$(a + b_1)(a + b_2)(a + b_3) \ldots (a + b_r) = a^r + p_1 a^{r-1} + p_2 a^{r-2} + p_3 a^{r-3} + \&c. + p_r \ldots (1),$$

Where p_1 = the sum of b_1, b_2, b_3, &c., b_r taken *singly*,
p_2 = " the products taken *two* together,
p_3 = " " *three* "
&c. = &c.
p_r = the product of all the letters b_1, b_2, b_3, &c., b_r.

Multiply each side of (1) by $a + b_{r+1}$, then

$$(a + b_1)(a + b_2)(a + b_3) \ldots (a + b_r)(a + b_{r+1})$$
$$= a^{r+1} + (p_1 + b_{r+1}) a^r + (p_2 + p_1 b_{r+1}) a^{r-2} + (p_3 + p_2 b_{r+1}) a^{r-1} + \&c. + p_r b_{r+1} \ldots (2).$$

Now on the first side of (2) we have the product of $(r + 1)$ binomial factors.
And on the second side,

$p_1 + b_{r+1}$ $= b_1 + b_2 + b_3 + \&c. + b_r + b_{r+1}$
= the sum of the letters b_1, b_2, b_3, &c., b_{r+1}, taken singly;

$p_2 + p_1 b_{r+1} =$ the sum of the products of the letters b_1, b_2, b_3, &c., b_{r+1} taken *two* together.
$p_3 + p_2 b_{r+1} =$ " " " " *three* together.
&c. = &c.
$p_r b_{r+1} =$ product of all the $(r + 1)$ letters b_1, b_2, b_3, &c., b_{r+1}.

Hence in (2) we have shown that by assuming the above law to hold for r binomial factors, it also holds for *one factor more.* Now we know by actual multiplication that it holds for 2, 3, 4 binomial factors.

It therefore holds for 5, and therefore also for 6, 7, &c. Hence it is generally true for any number of binomial factors.

Let us now suppose each of the quantities b_1, b_2, b_3, &c., to be equal to b, and let there be n binomial factors $a + b$.

Then we have for their product $(a + b)(a + b)(a + b) \dots$ to n factors or $(a + b)^n$.

Hence

$$(a + b)^n = a^n + p_1 a^{n-1} + p_2 a^{n-2} + p_3 a^{n-3} + \text{\&c.} + p_n \dots\dots (3).$$

Now $p_1 = b + b + b +$ &c. to n terms $= nb$.

$p_2 = bb + bb + bb +$ &c. to as many terms as there are combinations of n things taken two together, and therefore, by Art. 25, to $\frac{n(n-1)}{1 . 2}$ terms.

$$\therefore p_2 = \frac{n(n-1)}{1 . 2} bb = \frac{n(n-1)}{1 . 2} b^2.$$

And so $p_3 = \frac{n(n-1)(n-2)}{1 . 2 . 3} b^3$ &c.

Also $p_n = b \,.\, b \,.\, b \,.\,.\,.\,.$ to n factors $= b^n$.

Hence, substituting in (3), we have $(a+b)^n$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3$$
$$+ \text{\&c.} + b^n. \quad Q.E.D.$$

For exercises upon this formula, the student is referred to Ex. X., Vol I.

COR. 1. If $a = 1, b = x$, we have, since a^n, a^{n-1}, &c., each $= 1$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3$$
$$+ \text{\&c.} + x^n.$$
$$= 1 + C_1x + C_2x^2 + C_3x^3 + \text{\&c.} + C_nx^n.$$

where C_1, C_2, C_3, &c., C_n represent respectively the number of combinations of n things taken 1, 2, 3, &c., n together.

COR. 2. Since the coefficient of the $(r+1)$th term is C_r, we have, by Art. 25,

The $(r+1)$th term of the expansion of $(1+x)^n$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r, \text{ or, as it may be written,}$$

$$= \frac{\lfloor n}{\lfloor r \; \lfloor n-r} x^r.$$

34. *The coefficients of the terms equidistant from the beginning and end are equal.*

For (Art. 33, Cor. 1) the coefficient of the $(r+1)$th term from the beginning is C_r, and that of the $(r+1)$th term from the end is C_{n-r}.

Now, in Art. 26, we have shown that $C_r = C_{n-r}$, and hence the truth of the proposition.

35. *To find the middle term of the expansion of* $(1+x)^n$.

(1.) Let n be even.

Then, as there are $(n + 1)$ terms in the expansion, the middle term will be the $\left(\frac{n}{2} + 1\right)$th term.

Hence (Art. 33, Cor. 2) putting $\frac{n}{2}$ for r, we have

Middle term

$$= \frac{n(n-1)(n-2)\ldots\ldots\left(n-\frac{n}{2}+1\right)}{1.2.3\ldots\ldots\frac{n}{2}} x^{\frac{n}{2}}.$$

$$= \frac{n(n-1)(n-2)\ldots\ldots\left(\frac{n}{2}+1\right)}{1.2.3\ldots\ldots\frac{n}{2}} x^{\frac{n}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We may transform this expression thus:

Middle term

$$= \frac{n(n-1)(n-2)\ldots\ldots\left(\frac{n}{2}+1\right).\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots\ldots 2.1}{1.2.3\ldots\ldots\frac{n}{2}.1.2\ldots\ldots\frac{n}{2}} x^{\frac{n}{2}}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots 4.3.2.1}{\left(1.2.3\ldots\ldots\frac{n}{2}\right)^2} x^{\frac{n}{2}};$$

or, separating the even and odd factors,

$$= \frac{\{n(n-2)\ldots\ldots 4.2\} \times \{(n-1)\ldots\ldots 3.1\}}{\left(1.2.3\ldots\ldots\frac{n}{2}\right)^2} x^{\frac{n}{2}}$$

$$= \frac{2^{\frac{n}{2}}\left\{\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots\ldots 2.1\right\}.\left\{1.3.5\ldots\ldots(n-1)\right\}}{\left(1.2.3\ldots\ldots\frac{n}{2}\right)^2}$$

or, striking out common factors from numerator and denominator,

$$= 2^{\frac{n}{2}} \frac{1 \cdot 3 \cdot 5 \ldots (n-1)}{1 \cdot 2 \cdot 3 \ldots \frac{n}{2}} x^{\frac{n}{2}}.$$

(2.) Let n be odd.

As there are $(n + 1)$ or an even number of terms in the expansion, there will evidently be two middle terms. These are the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2} + 1\right)$th terms

The coefficient of each may be easily shown to be

$$\frac{n(n-1)(n-2)\ldots\frac{n+1}{2}}{1 \cdot 2 \cdot 3 \ldots \frac{n+1}{2}}, \text{ or } 2^{\frac{n-1}{2}} \cdot \frac{1 \cdot 3 \cdot 5 \ldots n}{1 \cdot 2 \cdot 3 \ldots \frac{n+1}{2}}.$$

36. *To find the greatest coefficient in the expansion of* $(1 + x)^n$, n *being an integer.*

Since (Art. 31, Cor. 1) C_r represents the coefficient of the $(r + 1)$th term, and we have shown that C_r is greatest, if n be even, when $r = \frac{n}{2}$; and if n be odd, when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ (*see* Art. 20), it follows that:

(1.) When n is *even*, the greatest coefficient is the $\left(\frac{n}{2} + 1\right)$th;

(2.) When n is *odd*, the greatest coefficients are the $\left(\frac{n-1}{2} + 1\right)$th and $\left(\frac{n+1}{2} + 1\right)$th, or the $\left(\frac{n+1}{2}\right)$th and the $\left(\frac{n+3}{2}\right)$th coefficients.

These evidently correspond to the coefficients of the middle *terms.*

The Binomial Theorem for any Index.

37. Let $f(m)$ represent the algebraical expression

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \&c.,$$

where m may be either integral or fractional, positive or negative.

Hence we have

$$f(m) = 1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \&c. \ldots\ldots\ldots\ldots (1),$$

and, putting n for m,

$$f(n) = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \&c. \ldots\ldots\ldots\ldots (2).$$

Now, multiplying together (1) and (2) and arranging, we have

$$f(m)\,f(n) = 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2}x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{1 \cdot 2 \cdot 3}x^3 + \&c.$$

But, putting $(m + n)$ for m in (1), the expression on the right side of this equation is equal to $f(m + n)$.

Hence we have $$f(m) \, . \, f(n) = f(m+n) \ldots\ldots\ldots\ldots (3).$$

Hence, also,

$$f(m) \, . \, f(n) \, . \, f(p) = f(m+n) \, . \, f(p), \text{ or, by (3), } = f(m+n+p).$$

And generally,

$$f(m) \cdot f(n) \cdot f(p) \cdots f(k) = f(m + n + p + \cdots + k) \quad \text{......... (4)},$$

when $m, n, p, \ldots k$ may be either positive or negative, integral or fractional.

Let $m = n = p = \&c. = k = \frac{r}{s}$, where r and s *are positive integers.*

And suppose there are s of the quantities m, n, p, &c., k.

Then $$m + n + p + \&c. + k = s \text{ times } \frac{r}{s} = r,$$

And $$f(m) \cdot f(n) \cdot f(p) \cdots f(k) = \left\{ f\left(\frac{r}{s}\right) \right\}^s.$$

Hence (4) becomes

$$\left\{ f\left(\frac{r}{s}\right) \right\}^s = f(r), \text{ or, taking the } s\text{th root of each side,}$$

$$f\left(\frac{r}{s}\right) = \{ f(r) \}^{\frac{1}{s}} \quad \text{......... (5)}.$$

But, by (1),

$$f(r) = 1 + rx + \frac{r(r-1)}{1 \cdot 2} x^2 + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.,$$

And hence, by Art. 35, since r is a positive integer,

$$f(r) = (1 + x)^r$$

We have therefore

$$\{f(r)\}^{\frac{1}{s}} = \{(1+x)^r\}^{\frac{1}{s}} = (1+x)^{\frac{r}{s}},$$

And therefore from (5)

$$f\left(\frac{r}{s}\right) = (1+x)^{\frac{r}{s}} \quad \ldots\ldots\ldots\ldots\ldots \quad (6).$$

But from (1), putting $\frac{r}{s}$ for m,

$$f\left(\frac{r}{s}\right) = 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1.2}x^2 + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)}{1.2.3}x^3 + \&c. \quad \ldots\ldots \quad (7).$$

Hence, equating (6) and (7),

$$(1+x)^{\frac{r}{s}} = 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1.2}x^2 + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)}{1.2.3}x^3 + \&c. \quad \ldots\ldots \quad (A).$$

which proves the Binomial Theorem for a positive fractional index.

It is therefore true for a positive index, whether integral or fractional.

We shall now show it is true for a *negative* index.

We have in (3), $f(m).f(n) = f(m+n)$,

when there is no restriction as to the value of m and n.

Let $n = -m$, when m is a positive quantity.

We then have $f(m).f(-m) = f(m-m) = f(0)$.

But from (2), putting $n = 0$, we have $f(0) = 1$, and hence

$$f(m) \, . \, f(-m) = 1$$

$$\therefore f(-m) = \frac{1}{f(m)} \quad \text{..................................(8).}$$

Now we have just shown that whether m be integral or fractional, so that it be positive, $f(m) = (1 + x)^m$

And therefore $$\frac{1}{f(m)} = \frac{1}{(1+x)^m} = (1 + x)^{-m}.$$

We have therefore from (8)

$$f(-m) = (1 + x)^{-m} \quad \text{.................................... (9).}$$

But from (2), putting $-m$ for n, we have

$$f(-m) = 1 + (-m)x + \frac{-m(-m-1)}{1 \, . \, 2}x^2 + \frac{-m(-m-1)(-m-2)}{1 \, . \, 2 \, . \, 3}x^3 + \text{\&c.} \ \ldots(10).$$

Hence, equating (9) and (10), we have

$$(1 + x)^{-m} = 1 + (-m)x + \frac{-m(-m-1)}{1 \, . \, 2}x^2 + \frac{-m(-m-1)(-m-2)}{1 \, . \, 2 \, . \, 3}x^3 + \text{\&c.},$$

which proves the Binomial Theorem for any negative index.

We have therefore shown it to be true for any index.

38. *To find the sum of the coefficients of the expansion of* $(1 + x)^n$.

Allowing C_1, C_2, C_3, &c., to represent the coefficients of the *first*, *second*, *third*, &c., powers of x, we have

$$(1 + x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \&c. \ldots\ldots (1).$$

Put $x = 1$; then, since the powers of x are each $= 1$, we have

$$2^n = 1 + C_1 + C_2 + C_3 + C_4 + \&c.$$

$$\therefore 1 + C_1 + C_2 + C_3 + C_4 + \&c. = 2^n.$$

Cor. 1. Since, when n is a positive integer, C_1, C_2, C_3, &c., represent the number of combinations of n things when taken 1, 2, 3, &c., together respectively, it follows that
The total number of combinations of n things $= 2^n - 1$.

Cor. 2. Putting $x = -1$, we have

$$(1 - 1)^n \text{ or } 0 = 1 - C_1 + C_2 - C_3 + \&c, \text{ or}$$

$$C_1 + C_3 + C_5 + \&c. = 1 + (C_2 + C_4 + C_6 + \&c.)$$

Hence the number of *odd* groups which can be made of n things exceeds by *unity* the number of *even* groups.

39. *To find the* GREATEST TERM *in the expansion of* $(1 + x)^n$, *no regard being had to the sign of the term.*

By Art. 33, Cor. 2, $(r + 1)$th term $= \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots\ldots r} x^r$.

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\,.\,2\,.\,3\ \ldots\ldots\ r-1} x^{r-1} \cdot \frac{n-r+1}{r} x \ldots\ldots (1).$$

$$\text{And } r\text{th term} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\,.\,2\,.\,3\ \ldots\ldots\ r-1} x^{r-1} \ldots\ldots\ldots\ldots (2).$$

Hence, (1) ÷ (2), we have

$$\frac{(r+1)\text{th term}}{r\text{th term}} = \frac{n-r+1}{r} x = \left(\frac{n+1}{r} - 1\right) x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

We may therefore obtain the $(r+1)$th term from the rth by multiplying the latter by $\left(\frac{n+1}{r} - 1\right)x$.

And as long as $\left(\frac{n+1}{r} - 1\right)x$ continues to be greater than unity, the $(r+1)$th term is greater than the rth term.

Now, as r is *increased*, $\left(\frac{n+1}{r} - 1\right)x$ *diminishes*, and eventually *changes its sign*.

And if r be indefinitely increased, we may make the value of $\left(\frac{n+1}{r} - 1\right)x$ approach *as near to* $-x$ *as we please.*

Hence, if none of the values of $\left(\frac{n+1}{r} - 1\right)x$ is *zero*, we may, by taking r large enough, make any term as nearly equal to x times the preceding term as we please. (It must be remembered that we are not considering here the *sign* of the term).

Now $\left(\frac{n+1}{r} - 1\right)x = 0$, when $r = n + 1$, and this cannot be except when n is integral and positive. From this we learn that there are only $(n + 1)$ terms in the expansion of $(1 + x)^n$ when n is an integer and positive, and that there is no limit to the number of terms when n is negative or fractional.

It follows therefore that *when* n *is fractional or negative, there cannot possibly be a greatest term if* x *be* > 1, since by increasing r sufficiently any term may approach as nearly to x times the preceding as we please, and there is no limit to the value of r.

We will now consider separately the cases in which a greatest term is possible.

It will be necessary only to consider x as positive; for when x is negative, we may put $x = -y$, and find the greatest term in the expansion of $(1 + y)^n$, and this will be the greatest term required.

I. *Let* n *be integral and positive.*

Now when $\left(\frac{n+1}{r} - 1\right)x = 1$, we have $r = \frac{(n+1)x}{x+1}$.

(*a.*) *Let* $\frac{(n+1)x}{x+1}$ *be integral.*

Then there is a term corresponding to the value of this fraction such that it and the next succeeding term have the same magnitude, and are the greatest terms of the expansion.

Hence the rth and $(r + 1)$th terms are the greatest terms when $r = \frac{(n+1)x}{x+1}$ *an integer.*

(*b.*) *Let* $\frac{(n+1)x}{x+1}$ *be a fraction.*

Then $\left(\frac{n+1}{r} - 1\right)x < 1$, when the value of r is the *integer next greater* than $\frac{(n+1)x}{x+1}$.

Hence the rth term is the greatest term for the value of r corresponding to the integer next greater than $\frac{(n+1)x}{x+1}$.

II. *Let* n *be fractional and positive.*

It has been shown that *when* x *is greater than unity there cannot be a greatest term.*

When x *is less than unity* the conditions are exactly as in I. above.

III. *Let* n *be negative.*

We have shown that here, as in II., *there is no greatest term when* x *is greater than unity.*

When x *is less than unity,* let $n = -m$.

Then $\left(\frac{n+1}{r} - 1\right)x = -\left(\frac{m-1}{r} + 1\right)x$.

When $\left(\frac{m-1}{r} + 1\right)x = 1$, we have $r = \frac{(m-1)x}{1-x}$.

(*a.*) *Let* $\frac{(m-1)x}{1-x}$ *be integral and positive.*

Then, reasoning as before, the rth and $(r+1)$th are the greatest terms when $r = \frac{(m-1)x}{1-x}$ a positive integer.

(*b.*) *Let* $\frac{(m-1)x}{1-x}$ *be fractional and positive.*

Then, as above, the rth is the greatest term for that value of r corresponding to the integer next greater than $\frac{(m-1)x}{1-x}$.

(*c.*) *Let* $\frac{(m-1)x}{1-x}$ *be negative.*

In this case, *since* x *is* less than unity, m is also less than *unity.*

Hence $\left(\frac{m-1}{r}+1\right)x$ is always less than x, and therefore *less than unity for every value of* r.

It follows therefore that when $\frac{(m-1)x}{1-x}$ is negative—that is, when x is less than unity, and m is less than unity—*the first term of the expansion is the greatest.*

Cor. 1. Since, when $\frac{(n+1)x}{x+1}$ is < 1, we have $nx < 1$,

And when $\frac{(m-1)x}{1-x}$ is < 1, x being less than unity, we have $mx < 1$,

It follows that the conditions that the *first* shall not be the greatest term are:

(1) When n is *a positive integer*, nx must be > 1.
(2) When n is *fractional or negative.*

(*a.*) If x be > 1, this condition is sufficient.
(*b.*) If x be < 1, nx must be > 1.

Cor. 2. Hence, also, in order that the *last* term shall not be the greatest term, since we need only consider the case of n being a positive integer, the condition is that $\frac{n}{x}$ is > 1, or $n > x$.

Approximate Values.

40. *To find the* n*th root of* N, *when* N *is very nearly an* n*th power of a rational quantity.*

Let $N = a^n + b$, where b is small compared with a^n ....(1).

Then $N^{\frac{1}{n}} = (a^n + b)^{\frac{1}{n}} = a\left(1 + \frac{b}{a^n}\right)^{\frac{1}{n}}$.

$$= a\left\{1 + \frac{1}{n} \cdot \frac{b}{a^n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1 . 2} \cdot \left(\frac{b}{a^n}\right)^2 + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{1 . 2 . 3} \left(\frac{b}{a^n}\right)^3 + \&c.\right\}$$

$$= a\left\{1 + \frac{1}{n} \cdot \frac{b}{a^n} - \frac{n - 1}{1 . 2 . n^2} \cdot \frac{b^2}{a^{2n}} + \frac{(n - 1)(2n - 1)}{1 . 2 . 3 . n^3} \cdot \frac{b^3}{a^{3n}} - \&c.\right\}$$

Now, as b is small compared with a^n, the values of the successive terms of this expansion diminish rapidly, and hence we may approximate as nearly as we please to the value of $N^{\frac{1}{n}}$.

Ex. 1. Find the 7th root of 2192.

We have $N = 2192 = 2187 + 5 = 3^7 + 5 = 3^7\left(1 + \frac{5}{3^7}\right)$.

$\therefore N^{\frac{1}{7}} = 3\left(1 + \frac{5}{3^7}\right)^{\frac{1}{7}}$, from which by expansion, &c., we get

$$\sqrt[7]{2192} = 3\left\{1 + \frac{1}{7} \cdot \frac{5}{3^7} - \frac{6}{1 . 2 . 7^2} \cdot \left(\frac{5}{3^7}\right)^2 + \frac{6 . 13}{1 . 2 . 3 . 7^3} \cdot \left(\frac{5}{3^7}\right)^3 - \&c.\right\}.$$

Ex. 2. Find the 5th root of 1021.

Here $N = 1021 = 1024 - 3 = 4^5 - 3 = 4^5\left(1 - \frac{3}{4^5}\right)$.

$$\therefore N^{\frac{1}{5}} = 4\left(1 - \frac{3}{4^5}\right)^{\frac{1}{5}}$$

$$= 4\left\{1 - \frac{1}{5}\cdot\frac{3}{4^5} - \frac{4}{1\,.\,2\,.\,5^2}\cdot\left(\frac{3}{4^5}\right)^2 - \frac{4\,.\,9}{1\,.\,2\,.\,3\,.\,5^3}\cdot\left(\frac{3}{4^5}\right)^3 - \&c.\right\}$$

41. *If* c *is large compared with* a *or* b, *show that*

$$(a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}} + (a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}}$$

$$= 2c^{-(m+1)} - (m+1)c^{-(m+3)}\{a^2 - (m+2)b^2\} \text{ nearly.}$$

We have $\qquad (a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}}$

$$= c^{-(m+1)}\left\{1 - \left(\frac{2b}{c} - \frac{a^2+b^2}{c^2}\right)\right\}^{-\frac{m+1}{2}}; \text{ expanding, \&c., then}$$

$$= c^{-(m+1)}\left\{1 + \frac{m+1}{2}\left(\frac{2b}{c} - \frac{a^2+b^2}{c^2}\right) + \frac{(m+1)(m+3)}{8}\left(\frac{2b}{c} - \frac{a^2+b^2}{c^2}\right)^2 + \&c.\right\}$$

$$= c^{-(m+1)}\left\{1 + \frac{m+1}{2}\left(\frac{2b}{c} - \frac{a^2+b^2}{c^2}\right) + \frac{(m+1)(m+3)}{8}\cdot\frac{4b^2}{c^2}\right\} \quad\text{.........}\ (1),$$

where we retain no terms involving higher powers of c than the second.

And similarly

$$(a^2+b^2+2bc+c^2)^{-\frac{m+1}{2}} = c^{-(m+1)}\left\{1 - \frac{m+1}{2}\left(\frac{2b}{c} + \frac{a^2+b^2}{c^2}\right) + \frac{(m+1)(m+3)}{8}\cdot\frac{4b^2}{c^2}\right\} \ldots(2).$$

(1) + (2), then

$$(a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}} + (a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}}$$

$$= c^{-(m+1)}\left\{2 - (m+1)\frac{a^2 + b^2}{c^2} + (m+1)(m+3) \cdot \frac{b^2}{c^2}\right\}$$

$$= 2c^{-(m+1)} - (m+1)c^{-(m+3)}\{a^2 - (m+2)b^2\}. \quad Q.E.D.$$

This is an important result in the mathematical theory of magnetism.

42. *When* n + 1 *figures of a square root have been obtained by the ordinary method,* n *figures more may be found by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is* 2 n + 1.

Let N be the number whose square root is required, a the part already obtained, and b the part required.

Then $\sqrt{N} = a + b$

$\therefore N = a^2 + 2ab + b^2$ or $N - a^2 = 2ab + b^2$.

$$\therefore \frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

Now $N - a^2$ is the remainder after obtaining the part a of the square root by the ordinary method.

And we see that $N - a^2$, when divided by $2a$, gives the remaining part b of the square root, together with the quantity $\frac{b^2}{2a}$.

We have then to show that $\frac{b^2}{2a}$ is a proper fraction, and therefore has no influence upon the integral value of the remaining figures of the root.

We assume that b contains n digits; and therefore b^2 contains not more than $2n$ digits.

But a, since it contains the $n + 1$ figures of the root already found, and also n ciphers, occupying the place of those already to be found, must contain $2n + 1$ digits.

Hence $\frac{b^2}{2a}$ is less than unity, and the proposition is proved.

> This result has been assumed and illustrated by an example in Vol. I., p. 186.

43. *When* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *more figures may be found by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is* 2 n + 2.

Let N be the number whose cube root is required, a the part already obtained, and b the part required.

Then $$\sqrt[3]{N} = a + b$$

$$\therefore N = a^3 + 3a^2b + 3ab^2 + b^3,$$

or $$N - a^3 = 3a^2b + 3ab^2 + b^3$$

$$\therefore \frac{N - a^3}{3a^2} = b + \frac{b^2}{a} + \frac{b^3}{3a^2}.$$

Now $N - a^3$ represents the remainder after obtaining the part a of the cube root, and $3a^2$ is the next trial divisor.

Hence we see that on dividing the remainder by the next trial divisor we get the required quotient b, together with the quantity $\frac{b^2}{a} + \frac{b^3}{3a^2}$.

Now, in order that this may not affect the integral value of the remaining figures of the cube root, we must show that $\frac{b^2}{a} + \frac{b^3}{3a^2}$ is less than unity.

Since b contains n figures, we must have $b < 10^n$.

And since a contains $2n + 2$ figures, a cannot be less *than* 10^{2n+1}, *but* may be equal to or greater than 10^{2n+1}.

$$\therefore \frac{b^2}{a} + \frac{b^3}{3a^2} \text{ is } < \frac{(10^n)^2}{10^{2n+1}} + \frac{(10^n)^3}{3(10^{2n+1})^2},$$

$$< \frac{1}{10} + \frac{1}{3 \cdot 10^{n+2}}, \text{ and } \therefore < 1.$$

Hence the truth of the proposition.

This result has been assumed and illustrated in Vol. I, p. 193.

44. We shall now work out a few examples on the Binomial Theorem.

Ex. 1. Expand to four terms the expression $(1+3x)^{-\frac{5}{3}}$.

$$(1+3x)^{-\frac{5}{3}} = 1 + \left(-\frac{5}{3}\right) . 3x + \frac{-\frac{5}{3}\left(-\frac{5}{3}-1\right)}{1 . 2}(3x)^2$$

$$+ \frac{-\frac{5}{3}\left(-\frac{5}{3}-1\right)\left(-\frac{5}{3}-2\right)}{1 . 2 . 3}(3x)^3 + \&c.$$

$$= 1 - 5x + 20x^2 - \frac{220}{3}x^3 + \&c.$$

Ex. 2. Find the $(r+1)$th term of the expansion of $(2-5x)^{-4}$.

$(r+1)$th term

$$= \frac{-4(-4-1)(-4-2)\ldots\ldots(-4-r+1)}{1 . 2 . 3 \ldots\ldots r} 2^{-4-r}(-5x)^r$$

$$= \frac{(-1)^r . 4 . 5 . 6 \ldots\ldots (r+3)}{1 . 2 . 3 \ldots\ldots r} 2^{-4-r}(-1)^r . (5x)^r$$

$$= (-1)^{2r} . \frac{4 . 5 . 6 \ldots\ldots r(r+1)(r+2)(r+3)}{1 . 2 . 3 . 4 . 5 . 6 \ldots\ldots r} 2^{-4-r}(5x)^r;$$

or, since $(-1)^{2r} = 1$,

$$= \frac{(r+1)(r+2)(r+3)}{1 . 2 . 3} \cdot \frac{(5x)^r}{2^{r+4}}.$$

Ex. 3. If x be very small, show that

$$\frac{(1+2x)^{\frac{1}{2}} + (1+3x)^{\frac{1}{3}}}{2+5x-(1+4x)^{\frac{1}{2}}} = 2 - 4x, \text{ very nearly.}$$

Expanding and retaining terms containing the *first powers* of x only, we have

$$\frac{(1 + 2x)^{\frac{1}{2}} + (1 + 3x)^{\frac{1}{3}}}{2 + 5x - (1 + 4x)^{\frac{1}{2}}}$$

$$= \frac{(1 + \frac{1}{2} . 2x) + (1 + \frac{1}{3} . 3x)}{2 + 5x - (1 + \frac{1}{2} . 4x)}$$

$$= \frac{2 + 2x}{1 + 3x} = (2 + 2x) \cdot (1 + 3x)^{-1};$$

or, expanding as before,

$$= (2 + 2x)(1 - 3x);$$

or multiplying out, and neglecting the term involving x^2,

$$= 2 - 4x. \quad Q.E.D.$$

Ex. 4. If p and q be very nearly equal, and n an integer, show that

$$\left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{(n + 1)p + (n - 1)q}{(n - 1)p + (n + 1)q} \text{ nearly.}$$

We have $$\frac{p}{q} = \frac{\frac{2p}{p + q}}{\frac{2q}{p + q}} = \frac{\frac{(p + q) + (p - q)}{p + q}}{\frac{(p + q) - (p - q)}{p + q}}$$

$$= \frac{1 + \frac{p - q}{p + q}}{1 - \frac{p - q}{p + q}}.$$

$$\therefore \left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{\left(1 + \frac{p - q}{p + q}\right)^{\frac{1}{n}}}{\left(1 - \frac{p - q}{p + q}\right)^{\frac{1}{n}}};$$

expanding and retaining only two terms, since $p - q$ is very small,

$$= \frac{1 + \frac{1}{n} . \frac{p - q}{p + q}}{1 - \frac{1}{n} . \frac{p - q}{p + q}} = \frac{n(p + q) + (p - q)}{n(p + q) - (p - q)}$$

$$= \frac{(n + 1)p + (n - 1)q}{(n - 1)p + (n + 1)q}. \quad Q.E.D.$$

Ex. 5. Find the limit of $\left(1 + \frac{1}{x}\right)^x$, when x increases without limit.

Expanding by the Binomial Theorem, we have

$$\left(1 + \frac{1}{x}\right)^x = 1 + \frac{x}{1} \cdot \frac{1}{x} + \frac{x(x-1)}{1\,.\,2} \cdot \frac{1}{x^2} + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3} \cdot \frac{1}{x^3} + \&c.$$

$$= 1 + \frac{1}{1} + \frac{1 - \frac{1}{x}}{1\,.\,2} + \frac{\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right)}{1\,.\,2\,.\,3} + \&c.$$

When x increases without limit, each of the quantities $\frac{1}{x}, \frac{2}{x}, \frac{3}{x}$, &c., decreases without limit, and becomes ultimately zero.

Hence, when x is infinite,

$$\left(1 + \frac{1}{x}\right)^x = 1 + \frac{1}{1} + \frac{1 - 0}{1\,.\,2} + \frac{(1 - 0)(1 - 0)}{1\,.\,2\,.\,3} + \&c.$$

$$= 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c.$$

It is usual to denote this series by the symbol e; it is the base of the Napierian system of logarithms.

We therefore have

$$\text{Limit of } \left(1 + \frac{1}{x}\right)^x \text{ when } x \text{ is infinite} = e.$$

The approximate value of e is 2·71828.

Ex. VII.

Expand to four terms the six following expressions:

1. $(1 + x)^7$. 2. $(3 + 2x)^{\frac{1}{2}}$. 3. $(4 - 5x)^{-\frac{2}{3}}$.

4. $(a + bx^2)^{-1}$. 5. $(1 - 2x + 3x^2)^{-2}$. 6. $\left(\frac{a}{x} - \frac{b}{y}\right)^{-5}$.

Find the middle terms of:

7. $(a + b)^9$. 8. $(3 + 5x)^{10}$. 9. $(a^2 - b^2)^6$.

10. $\left(x + \frac{1}{x}\right)^{2n}$. 11. $\left(3 + \frac{x}{4}\right)^5$. 12. $(4a^{\frac{1}{2}} - 3b^{\frac{1}{5}})^7$.

Find the general or $(r + 1)$th terms of:

13. $(3-5x)^{-\frac{1}{2}}$. 14. $\left(ax + \frac{b}{x}\right)^n$. 15. $\left(a^{\frac{1}{p}} - b^{\frac{1}{q}}\right)^{-1}$.

16. If ${}^mC_p$ represent the pth coefficient of the expansion of $(a + x)^m$, show that ${}^mC_p + {}^mC_{p+1} = {}^{m+1}C_{p+1}$.

17. If P be the sum of the odd terms of the expansion of the binomial $(a + x)^m$, and Q the sum of the even terms, then will $P^2 - Q^2 = (a^2 - x^2)^m$.

18. In the last example, $4\,PQ = (a + x)^{2m} - (a - x)^{2m}$.

19. Show that $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{4^4} + \text{\&c.} = 4$.

20. If A, B are two consecutive coefficients of the expansion of $(1 + x)^n$, show that the coefficients after B are

$$B \cdot \frac{nB - A}{(n+2)A + B},\ B \cdot \frac{(nB - A)(\overline{n-1} \,.\, B - 2A)}{(\overline{n+2} \,.\, A + B)(\overline{n+3} \,.\, A + 2B)},\ \text{\&c.}$$

21. Find the greatest term of the expansion of $\left(2 + \frac{1}{3}\right)^5$.

22. Find the greatest term of the expansion of $\left(1 - \frac{1}{4}\right)^{-3}$.

23. Show that

$$\sqrt[3]{a + x} = a^{\frac{1}{3}}\left\{1 + \frac{1}{3} \cdot \frac{x}{a + x} + \frac{1 \,.\, 4}{3 \,.\, 6} \cdot \frac{x^2}{(a + x)^2} + \frac{1 \,.\, 4 \,.\, 7}{3 \,.\, 6 \,.\, 9} \cdot \frac{x^3}{(a + x)^3},\ \text{\&c.}\right\}$$

24. If n be a positive integer, prove that

$$1 - 2n + \frac{2n(2n-1)}{\underline{|2}} - \ldots + (-1)^{n-1} \cdot \frac{2n(2n-1)\ldots(n+2)}{\underline{|n-1}} = (-1)^{n-1} \cdot \frac{\underline{|2n}}{2(\underline{|n})^2}$$

25. If n be an odd number, show that

$$1 + \frac{2n(2n-1)}{\underline{|2}} + \frac{2n(n-1)(2n-2)(2n-3)}{\underline{|4}} + \text{\&c.} + \frac{2n}{\underline{|n-1} \,.\, \underline{|n+1}} = 2^{2n-2},$$

26. Show that

$$\frac{1 \cdot 2 \cdot 3 \ldots . p \cdot 2^{p+q+1}}{(q+1)(q+2) \ldots .(p+q+1)} = \frac{1}{p+1} + \frac{q}{(p+1)(p+2)}$$
$$+ \frac{q(q-1)}{(p+1)(p+2)(p+3)} + \text{\&c.} + \frac{q(q-1) \ldots . 2 \cdot 1}{(p+1) \ldots .(p+q+1)}$$
$$+ \frac{1}{q+1} + \frac{p}{(q+1)(q+2)} + \frac{p(p-1)}{(q+1)(q+2)(q+3)}$$
$$+ \text{\&c.} + \frac{p(p-1) \ldots . 2 \cdot 1}{(q+1) \ldots . (p+q+1)}.$$

27. If a, b, n are positive integers, and b less than $2a - 1$, show that the integral part of $(a + \sqrt{a^2 - b})^n$ is an odd number.

28. Show that the integral part of $(1 + \sqrt{3})^{2m+1}$ contains 2^{m+1} as a factor.

29. Find the coefficient of x^m in the expansion of

$$\frac{p + qx}{q + rx}$$

in a series of ascending powers of x, and show that if P and Q be the coefficients of x^m, according as it is expanded according to ascending or descending powers respectively; then $PQ = -\frac{(pr - q)^2}{q^2 r^2}$.

30. Prove that the coefficient of x^r in

$$(1 + 2x + 3x^2 + \text{\&c. ad inf.})^2$$

is $\frac{1}{6}(r + 1)(r + 2)(r + 3)$.

31. Show that if R is the remainder, after taking n terms of the expansion of $(1 - x)^{-2}$, then

$$R = \frac{(n + 1)x^n - nx^{n+1}}{(1 - x)^2}.$$

32. Find the middle term of $\left(x + \frac{1}{x}\right)^{2n}$, and show that the numerator contains 2^n as a factor.

33. Show that $\dfrac{(1-x)^n}{\underline{|n}} = \dfrac{1}{\underline{|n}} + \dfrac{1}{\underline{|n-1}} \cdot \dfrac{x}{1-x}$

$+ \dfrac{1}{\underline{|2} \cdot \underline{|n-2}} \cdot \dfrac{x^2}{(1-x)^2} + \text{\&c.}$

34. Prove that $\left(x + \dfrac{1}{x}\right)^{2n}$

$$= \left(x^{2n} + \frac{1}{x^{2n}}\right) + 2n\left(x^{2n-2} + \frac{1}{x^{2n-2}}\right)$$

$$+ \frac{2n(2n-1)}{1 \cdot 2}\left(x^{2n-4} + \frac{1}{x^{2n-4}}\right) + \text{\&c.} + 2^n . \frac{1.3.5\ldots(2n-1)}{\underline{|n}}$$

35. When x is small, show that $\dfrac{(1+x)^{\frac{1}{2}} + (1-2x)^{\frac{1}{2}}}{(1-3x)^{\frac{1}{3}}}$

$= 2 + \dfrac{3}{2}x$, nearly.

36. If x be greater than a, prove that the sum of all the terms of the expansion of $(x+a)^n$ but the first two is less than $(2^n - n - 1)ax^{n-1}$.

CHAPTER VI.

INDETERMINATE COEFFICIENTS AND PARTIAL FRACTIONS.

Indeterminate Coefficients.

45. *If* $a + bx + cx^2 + dx^3 + \text{\&c.} = 0$ *for all values of* x, *then each of the quantities* a, b, c, d, &c., *must be zero.*

Since for *all values* of x it is true that

$$a + bx + cx^2 + dx^3 + \text{\&c.} = 0 \ldots\ldots\ldots\ldots\ldots(1),$$

it must be true when $x = 0$.

Putting, then, $x = 0$, we have $a = 0$.

Hence, from (1), $bx + cx^2 + dx^3 + \text{\&c.} = 0$;
or, dividing each side by x,

$$b + cx + dx^2 + \text{\&c.} = 0,$$

Since this is also true for all values of x, we may put $x = 0$
We then get $b = 0$.
And so we may show that $c = 0$, $d = 0$, &c.

COR. *If for all values of* x *we have*

$$A + Bx + Cx + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.;$$

then we must have A = a, B = b, C = c, &c.

For, transposing,

$$(A - a) + (B - b)x + (C - c)x^2 + (D - d)x^3 + \&c. = 0.$$

Hence, by the above proposition, since this is true for all values of x, we have

$$\begin{aligned} A - a &= 0, \text{ or } A = a; \\ B - b &= 0, \text{ or } B = b; \\ C - c &= 0, \text{ or } C = c; \ \&c. \end{aligned}$$

In many algebraical transformations the above proposition is especially useful.

Ex. 1. Given $x = y - 3y^2$, find y in a series of ascending powers of x.

Here $\qquad x = y - 3y^2 \ldots\ldots\ldots\ldots (1).$

Assume $\qquad y = ax + bx^2 + cx^3 + dx^4 + \&c. \ldots\ldots (2).$

We have not introduced a constant term into the series for y, for from (1) we see that, when $x = 0$, $y = 0$.

Squaring (2), we have

$$\begin{aligned} y^2 = a^2x^2 &\quad + \quad b^2x^4 + \&c. \\ &+ 2abx^3 + \quad 2acx^4 + \&c. \\ = a^2x^2 &+ 2abx^3 + (b^2 + 2ac)x^4 + \&c. \ldots\ldots (3). \end{aligned}$$

Hence, substituting in (1) from (2) and (3),

$$\begin{aligned} x &= (ax + bx^2 + cx^3 + dx^4 + \&c.) \\ &\quad - 3\left\{a^2x^2 + 2abx^3 + (b^2 + 2ac)x^4 + \&c.\right\} \\ &= ax + (b - 3a^2)x^2 + (c - 6ab)x^3 + (a - 3b^2 - 6ac)x^4 + \&c. \end{aligned}$$

Equating the coefficients of like powers of x on each side of this equation, we have

$$a = 1.$$
$$b - 3a^2 = 0, \text{ or } b = 3 \times 1^2 = 3.$$
$$c - 6ab = 0, \text{ or } c = 6 \times 1 \times 3 = 18.$$
$$d - 3b^2 - 6ac = 0, \text{ or } d = 3 \times 3^2 + 6 \times 1 \times 18 = 135.$$
&c.

Substituting these values of a, b, c, d, &c., in (2), we get

$$y = x + 3x + 18x^3 + 135x^4 + \&c.$$

Ex. 2. Express the quotient of $\dfrac{1 + 5x}{1 - 2x + 7x^2}$ in ascending powers of x.

Let $\dfrac{1 + 5x}{1 - 2x + 7x^2} = a + bx + cx^2 + dx^3 + ex^4 + \&c. \ldots\ldots(1).$

Clearing of fractions, we have

$$\begin{aligned} 1 + 5x = a + bx &+ cx^2 + dx^3 + ex^4 + \&c. \\ - 2ax &- 2bx^2 - 2cx^3 - 2dx^4 - \&c. \\ &+ 7ax^2 + 7bx^3 + 7cx^4 + \&c. \end{aligned}$$

$$= a + (b - 2a)x + (c - 2b + 7a)x^2 + (d - 2c + 7b)x^3 + (e - 2d + 7c)x^4 + \&c.$$

Hence, equating coefficients of like powers of x, we have

$$a = 1$$
$$b - 2a = 5, \text{ or } b = 5 + 2 \times 1 = 5.$$
$$c - 2b + 7a = 0, \text{ or } c = 2 \times 7 - 7 \times 1 = 7.$$
$$d - 2c + 7b = 0, \text{ or } d = 2 \times 7 - 7 \times 7 = -35.$$
$$e - 2d + 7c = 0, \text{ or } e = 2(-35) - 7 \times 7 = -119.$$
&c.

Hence, by substitution in (1), we have

$$\frac{1 + 5x}{1 - 2x + 7x^2} = 1 + 7x + 7x^2 - 35x^3 - 119x^4 - \&c.$$

Ex. 3. Find the square root of $1 - x + x^2$.

Let $\sqrt{1 - x + x^2} = a + bx + cx^2 + dx^3 + ex^4 + \&c. (1).$

$$\begin{aligned} \therefore 1 - x + x^2 = a^2 &+ b^2x^2 + c^2x^4 + \&c. \\ &+ 2abx + 2acx^2 + 2adx^3 + 2aex^4 + \&c. \\ &+ 2bcx^3 + 2bdx^4 + \&c. \end{aligned}$$

Hence, equating coefficients of like powers of x, we have

$$a^2 = 1, \therefore a = 1.$$

$$2ab = -1, \therefore b = -\frac{1}{2a} = -\frac{1}{2}.$$

$$b^2 + 2ac = 1, \therefore c = \frac{1-b^2}{2a} = \frac{3}{8}.$$

$$2ad + 2bc = 0, \therefore d = -\frac{bc}{a} = -\frac{3}{16}.$$

$$c^2 + 2ae + 2bd = 0, \therefore e = -\frac{c^2 + 2bd}{2a} = -\frac{21}{64}, \text{ \&c.}$$

Hence, substituting in (1),

$$\sqrt{1 - x + x^2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 - \frac{21}{64}x^4 + \text{\&c.}$$

Partial Fractions.

46. When the denominator of an algebraical fraction can be broken up into factors it is always possible to separate the fraction into two or more fractions having denominators of lower dimensions. The limits of this work will only allow space for one or two examples. We shall not enter upon the general theory.

Ex. 1. Resolve $\dfrac{3x + 7}{x^2 + x - 30}$ into partial fractions.

Since $\quad x^2 + x - 30 = (x - 5)(x + 6)$,

assume
$$\frac{3x + 7}{x^2 + x - 30} = \frac{A}{x - 5} + \frac{B}{x + 6} \text{............(1).}$$

Clearing of fractions, then

$$3x + 7 = A(x + 6) + B(x - 5) \text{............(2).}$$

We may now proceed by either of the following methods:

I. *First Method.*

Collecting like terms, we have

$$3x + 7 = (A + B)x + (6A - 5B).$$

Equating coefficients of like powers of x, then

$$\left.\begin{array}{r} A + B = 3 \\ \text{and } 6A - 5B = 7 \end{array}\right\}, \text{ from which } A = 2,\ B = 1.$$

Hence, from (1) by substitution,

$$\frac{3x + 7}{x^2 + x - 30} = \frac{2}{x - 5} + \frac{1}{x + 6}.$$

II. *Second Method.*

To find A, we shall in (2) make the original denominator of A, viz. $x - 5$, equal to *zero*. And similarly for B.

Thus we have in (2)

$$3x + 7 = A(x + 6) + B(x - 5).$$

Put $x - 5 = 0$, or $x = 5$, then

$$3 \times 5 + 7 = A(5 + 6), \text{ or } A = 2.$$

Put $x + 6 = 0$, or $x = -6$, then

$$3(-6) + 7 = B(-6 - 5), \text{ or } B = 1.$$

Hence, as before,

$$\frac{3x + 7}{x^2 + x - 10} = \frac{2}{x - 5} + \frac{1}{x + 6}.$$

It is often more convenient to follow the second method.

Ex. 2. Resolve $\dfrac{1}{(x - 1)^2 (x + 2)(x + 3)}$ into partial fractions.

Assume

$$\frac{1}{(x-1)^2 (x+2)(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2} + \frac{D}{x+3} \ldots\ldots(1).$$

Then, clearing of fractions,

$$1 = A(x + 2)(x + 3) + B(x - 1)(x + 2)(x + 3)$$
$$+ C(x - 1)^2 (x + 3) + D(x - 1)^2 (x + 2) \ldots\ldots\ldots(2).$$

Put $x - 1 = 0$, or $x = 1$, then

$$1 = A(1 + 2)(1 + 3), \text{ or } A = \tfrac{1}{12}.$$

Put $x + 2 = 0$, or $x = -2$, then

$$1 = C(-2 - 1)^2 (-2 + 3) = 9C, \text{ or } C = \tfrac{1}{9}.$$

Put $x + 3 = 0$, or $x = -3$, then

$1 = D(-3-1)^2(-3+2) = -16D$, or $D = -\frac{1}{16}$.

We have thus determined A, C, D.

We cannot thus find B, but shall proceed as follows:

In (1) substitute the values of A, C, D just found, and transpose; then

$$\frac{B}{x-1}$$

$$= \frac{1}{(x-1)^2(x+2)(x+3)} - \frac{1}{12(x-1)^2} - \frac{1}{9(x+2)} + \frac{1}{16(x+3)};$$

or, simplifying,

$$= \frac{-7(x-1)(x+2)(x+3)}{144(x-1)^2(x+2)(x+3)} = -\frac{7}{144(x-1)}.$$

$$\therefore B = -\frac{7}{144}.$$

We have therefore from (2), by substituting in (1) the values of A, B, C, D, obtained

$$\frac{1}{(x-1)^2(x+2)(x+3)} =$$

$$\frac{1}{12(x-1)^2} - \frac{7}{144(x-1)} + \frac{1}{9(x+2)} - \frac{1}{16(x+3)}.$$

Ex. 3. Find the partial fractions of $\frac{3x+5}{x^4+x^2+1}$.

The factors of the denominator are x^2+x+1 and x^2-x+1.

We shall assume the numerators of the required partial fractions to be of the form $Ax + B$ and $Cx + D$ respectively.

We then have

$$\frac{3x+5}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \ldots\ldots(1).$$

Or $3x+5 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1) \ldots(2)$.

Let $x^2+x+1 = 0$, or $x^2 = -x-1$, then from (2)

$$\begin{aligned} 3x+5 &= (Ax+B)(-x-1-x+1) = -2Ax^2 - 2Bx \\ &= -2A(-x-1) - 2Bx \\ &= (2A-2B)x + 2A. \end{aligned}$$

Equating coefficients of like powers of x, then

$$2A = 5, \text{ or } A = \tfrac{5}{2};$$

and $2A - 2B = 3$, or $2B = 5 - 3 = 2$, or $B = 1$.

And similarly we find $C = -\frac{5}{2}$, $D = 4$.

Hence, from (1),

$$\frac{3x + 5}{x^4 + x^2 + 1} = \frac{5x + 2}{2(x^2 + x + 1)} - \frac{5x - 8}{2(x^2 - x + 1)}.$$

Ex. VIII.

1. Find x in a series of ascending powers of y, when $y = 2x - 5x^2$.

2. Find by the method of indeterminate coefficients the quotient of $\dfrac{1 - 2x}{(1 - x)^3}$.

3. Find the cube root of $1 + 3x + 4x^2$.

4. Express by a series the difference between $(1 - 3x)^{-1}$ and $(1 + 2x)^{-1}$.

5. Find the sum of the series $1^2, 2^2, 3^2$, &c., to n terms.

6. Show that $1^3 + 2^3 + 3^3 + 4^3 + \text{\&c.} + x^3 = \left\{\dfrac{n(n + 1)}{2}\right\}^2$.

Resolve into partial fractions:

7. $\dfrac{x + 9}{(x + 3)(x + 5)}$.

8. $\dfrac{1}{(x^2 - 6)(x^2 + 8)}$.

9. $\dfrac{1}{x^3 + 6x^2 + 11x + 6}$.

10. $\dfrac{1}{x(1 + x)^2(1 + x + x^2)}$.

11. $\dfrac{4x^2 + x - 1}{3x^3 - 4x^2 + x}$.

12. $\dfrac{3}{1 + x^3}$.

13. $\dfrac{1}{(x^4 + x)(x^4 - 1)}$.

14. $\dfrac{1}{(x + a)(x + b)(x + c)}$.

15. $\dfrac{x}{(x + a)(x + b)(x + c)}$.

16. $\dfrac{x^2}{(x + a)(x + b)x + c)}$.

17. $\dfrac{1}{x^3 + 6x^2 + 16x - 21}$.

18. $\dfrac{x^2 + mx + n}{(x + a)(x + b)(x + c)}$.

CHAPTER VII.

THE EXPONENTIAL THEOREM.—LOGARITHMIC SERIES.

Exponential Theorem.

47. *To expand* a^x *in a series of ascending powers of* x.

$$a^x = \{1 + (a - 1)\}^x$$
$$= 1 + x(a - 1) + \frac{x(x-1)}{1.2}(a-1)^2 + \frac{x(x-1)(x-2)}{1.2.3}(a-1)^3 + \&c.$$
$$= 1 + x(a - 1) + \frac{x^2 - x}{2}(a-1)^2 + \frac{x^3 - 3x^2 + 2x}{6}(a-1)^3 + \&c.$$

Collecting the terms involving the first power only of x, we have

$$a^x = 1 + \left\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.\right\} x + \text{terms in } x^2, x^3, \&c.$$

Assume then

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \&c. \;\ldots\ldots\ldots (1),$$

Where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$ (2).

From (1), squaring and arranging, we have

$$a^{2x} = 1 + 2Ax + (A^2 + 2B)x^2 + (2C + 2AB)x^3 + \&c. \ldots (3).$$

But from (1), putting therein $2x$ for x, we also get

$$a^{2x} = 1 + A(2x) + B(2x)^2 + C(2x)^3 + \&c.$$
$$= 1 + 2Ax + 4Bx^2 + 8Cx^3 + \&c. \;\ldots\ldots\ldots (4).$$

Hence the series in (3) and (4) are identical, and we may therefore equate the coefficients of like powers of x.

Hence $4B = A^2 + 2B$, or $2B = A^2$. Therefore $B = \frac{A^2}{1.2}$.

$8\,C = 2\,C + 2\,AB$, or $6\,C = 2\,A \cdot \frac{A^2}{1\,.\,2}$. $\therefore C = \frac{A^3}{1\,.\,2\,.\,3}$. &c.

Hence, substituting in (1),

$$a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1\,.\,2} + \frac{A^3x^3}{1\,.\,2\,.\,3} + \text{\&c.} \ldots\ldots\ldots (5).$$

Since this equation is true for all values of x, we may put

$$Ax = 1, \text{ and } \therefore x = \frac{1}{A}.$$

We then have $a^{\frac{1}{A}} = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \text{\&c.}$

Hence, by Art. 42, Ex. 5,

$$a^{\frac{1}{A}} = e, \text{ or } a = e^A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Taking the logarithm of each side of the equation to base e, we get

$$A = \log_e a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Hence, substituting in (5) for this value of A, we get

$$a^x = 1 + \frac{x \log_e a}{1} + \frac{(x \log_e a)^2}{1\,.\,2} + \frac{(x \log_e a)^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

The $(n + 1)$th term of this series is $\frac{(x \log_e a)^n}{\lfloor n}$, and the formula itself is known by the name of the **Exponential Theorem.**

Cor. 1. Put $a = e$, then since $\log_e e = 1$, we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

Cor. 2. Since in (7) we have $A = \log_e a$, it follows at once from (2) that

$$\log_e a = (a - 1) - \tfrac{1}{2}(a - 1)^2 + \tfrac{1}{3}(a - 1)^3 - \text{\&c.}$$

Logarithmic Series.

48. *To show that* $\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \text{\&c.}$

and that $\log_e (1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \text{\&c.}$

In Art. 47, Cor. 1, we have

$$\log_e a = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$$

Put $1 + x$ for a, and therefore x for $a - 1$, then

$$\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c. \ldots\ldots (1).$$

Now put $- x$, then

$$\log_e (1 - x) = - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c. \ldots\ldots (2).$$

These series are not suitable for the calculation of logarithms. We proceed to obtain from them more convergent series.

49. *To show that* $\log_e (n + 1)$

$$= \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \&c.\right\}.$$

Now $\log_e \dfrac{1 + x}{1 - x} = \log_e (1 + x) - \log_e (1 - x)$

$$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \&c.\right)$$

$$- \left(- x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \&c.\right)$$

$$= 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.\right) \ldots\ldots (1).$$

Since this is true for all values of x, we may put

$$\frac{1 + x}{1 - x} = \frac{m}{n}, \text{ and therefore also } x = \frac{m - n}{m + n}.$$

We then have $\log_e \dfrac{m}{n}$

$$= 2\left\{\frac{m - n}{m + n} + \frac{1}{3}\left(\frac{m - n}{m + n}\right)^3 + \frac{1}{5}\left(\frac{m - n}{m + n}\right)^5 + \&c.\right\} (2).$$

Now put $m = n + 1$, and therefore

$$m - n = 1, \text{ and } m + n = 2n + 1;$$

Then we have $\log_e \frac{n+1}{n}$

$$= 2\left\{ \frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \frac{1}{5} \cdot \frac{1}{(2n+1)^5} + \&c. \right\}.$$

Hence, transposing, $\log_e (n+1)$

$$= \log_e n + 2\left\{ \frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \frac{1}{5} \cdot \frac{1}{(2n+1)^5} + \&c. \right\};$$

an important series, by which the logarithm of any number may be easily computed when that of the next lower is known.

50. *To show that* $\log_e (x+1)$

$$= 2 \log_e x - \log_e (x-1) - 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

We have in (2) of the last Art., putting

$m = x^2$, and $n = x^2 - 1$, and therefore
$m - n = 1$, and $m + n = 2x^2 - 1$,

$$\log_e \frac{x^2}{x^2-1} = 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

But $\log_e \frac{x^2}{x^2-1} = \log_e \frac{x^2}{(x+1)(x-1)}$

$$= 2 \log_e x - \log_e (x+1) - (x-1).$$

Hence $2 \log_e x - \log_e (x+1) - \log_e (x-1)$

$$= 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

Or, transposing, $\log_e (x+1)$

$$= 2 \log_e x - \log_e (x-1) - 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

This is a very useful formula, by which the logarithm of any number is found from the logarithms of the two numbers next preceding.

51. *To show that, when* x *is large, whatever be the base,*

$$2 \log x = \log (x+1) + \log (x-1)$$
$$+ \frac{1}{2x}\left\{ 1 + \frac{1}{6x^2} + \frac{1}{90x^4} + \&c. \right\} \log \frac{x+1}{x-1}.$$

Now, $\log_e \dfrac{x^2}{x^2 - 1} = -\log_e \dfrac{x^2 - 1}{x^2} = -\log_e\left(1 - \dfrac{1}{x^2}\right)$

$$= -\left(-\frac{1}{x^2} - \frac{1}{2}\cdot\frac{1}{x^4} - \frac{1}{3}\cdot\frac{1}{x^6} - \&c.\right)$$

$$= \frac{1}{x^2} + \frac{1}{2}\cdot\frac{1}{x^4} + \frac{1}{3}\cdot\frac{1}{x^6}\ldots\ldots\ldots\ldots(1).$$

And $\log_e \dfrac{x+1}{x-1} = \log_e \dfrac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$, and therefore, by Art. 49 (1),

$$= 2\left\{\frac{1}{x} + \frac{1}{3}\cdot\frac{1}{x^3} + \frac{1}{5}\cdot\frac{1}{x^5} + \&c.\right\}\ldots..(2).$$

Dividing (1) by (2), we have, after performing the division,

$$\log_e \frac{x^2}{x^2-1} \div \log_e \frac{x+1}{x-1} = \frac{1}{2}\left\{\frac{1}{x} + \frac{1}{6x^3} + \frac{7}{90x^5} + \&c.\right\}\ldots(3).$$

Now, $\log_e \dfrac{x^2}{x^2 - 1} = 2\log_e x - \log_e(x+1) - \log_e(x-1)$;

Hence, multiplying each side of (3) by $\log_e \dfrac{x+1}{x-1}$, and transposing, &c.

$$2\log_e x = \log_e(x+1) + \log_e(x-1)$$
$$+ \frac{1}{2x}\left\{1 + \frac{1}{6x^2} + \frac{7}{90x^4} + \&c.\right\}\log_e\frac{x+1}{x-1}\ldots\ldots..(4).$$

Now it is shown (Art. 50) that

$$\log_e x = \log_e a \,.\, \log_a x.$$

Hence we may replace the base e in (4) by the base a, if we introduce the factor $\log_e a$ into every term. Dividing each side of the newly-formed equation by this factor, we then get the formula in (4) expressed thus:

$$2\log_a x = \log_a(x+1) + \log_a(x-1)$$
$$+ \frac{1}{2x}\left\{1 + \frac{1}{6x^2} + \frac{7}{90x^4} + \&c.\right\}\log_a\frac{x+1}{x-1}.$$

Logarithms to base e are called Napierian logarithms, from their inventor.

It is evident that, by the series of the last articles, and by similar series, Napierian logarithms may be calculated. It will be necessary to use each series however for prime numbers only, since composite numbers may be broken up into their prime factors, and their logarithms easily obtained from those of their factors.

Ex. 1. Find $\log_e 2$, $\log_e 3$.

We have

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \text{\&c.}\right\}.$$

Put $m = 2$, $n = 1$, and $\therefore \dfrac{m-n}{m+n} = \dfrac{2-1}{2+1} = \dfrac{1}{3}$.

We then have

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \text{\&c.}\right\};$$

Or, reducing to decimals and adding

$$= \cdot 6931471\ldots$$

Similarly, putting $m = 3$, $n = 2$, and $\therefore \dfrac{m-n}{m+n} = \dfrac{1}{5}$, we have

$$\log_e \frac{3}{2} = 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \text{\&c.}\right\};$$

or, since

$$\log_e \frac{3}{2} = \log_e 3 - \log_e 2,$$

we have,

$$\log_e 3 = \log_e 2 + 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \text{\&c.}\right\};$$

or, performing the necessary reductions, &c.,

$$= 1\cdot 0986122\ldots$$

Ex. 2. Find $\log_e 5$, $\log_e 10$.

Put $m = 5$, $n = 4$, and $\therefore \dfrac{m-n}{m+n} = \dfrac{1}{9}$, then since

$\log_e \frac{5}{4} = \log_e 5 - 2\log_e 2$, we have, by transposing,

$$\log_e 5 = 2\log_e 2 + 2\left\{\frac{1}{9} + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \frac{1}{5}\left(\frac{1}{9}\right)^5 + \&c.\right\};$$

or, reducing, &c.,

$$= 1\cdot6094379...$$

Again,

$$\log_e 10 = \log_e(2 \times 5) = \log_e 2 + \log_e 5$$
$$= \cdot6931471... + 1\cdot6094379...$$
$$= 2\cdot3025850...$$

Ex. 3. Find $\log_e 16205$.

$$\log_e 16205 = \log_e(2^3 \times 3^4 \times 5^2 + 5)$$
$$= \log_e\left\{2^3 \times 3^4 \times 5^2\left(1 + \frac{1}{2^3 \times 3^4 \times 5}\right)\right\}$$
$$= 3\log_e 2 + 4\log_e 3 + 2\log_e 5 + \log_e\left(1 + \frac{1}{2^3 \times 3^4 \times 5}\right).$$

By substituting the values of $\log_e 2$, $\log_e 3$, $\log_e 5$, already found, and by developing $\log_e\left(1 + \frac{1}{2^3 \times 3^4 \times 5}\right)$, the required logarithm is known.

Calculation of Common Logarithms.

52. To show that $\log_b N = \frac{1}{\log_a b} \cdot \log_a N$.

Let $N = a^x = b^y$ ....(1).

Then we have, from the definition of a logarithm,

$x = \log_a N$ ....(2).

$y = \log_b N$ ....(3).

Again, we have in (1), $a^x = b^y$, or taking logarithms to base a,

$$x = y\log_a b;$$

or, substituting from (2) and (3),

$$\log_a N = \log_b N \cdot \log_a b$$

$$\therefore \log_b N = \frac{1}{\log_a b} \cdot \log_a N.$$

COR. Put $b = 10$, and $a = e$, then

$$\log_{10} N = \frac{1}{\log_e 10} \cdot \log_e N.$$

Hence common logarithms may be found from Napierian logarithms by multiplying the latter by $\frac{1}{\log_e 10}$.

We call this quantity the *modulus* of the common system of logarithms; and if we represent the modulus by μ, we have

$$\mu = \frac{1}{\log_e 10} = \frac{1}{2\cdot 30258509\ldots} = \cdot 43429448\ldots$$

53. By multiplying by μ the series obtained in Arts. 48, 49, and putting $\mu \log_e = \log_{10}$, we have, omitting the suffix 10.

$$\log(1 + x) = \mu \left\{ x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c. \right\} \ldots (1).$$

$$\log(1 - x) = \mu \left\{ -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c. \right\} \ldots\ldots (2).$$

$$\log \frac{1 + x}{1 - x} = 2\mu \left\{ x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c. \right\} \ldots\ldots (3).$$

$$\log \frac{m}{n} = 2\mu \left\{ \frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c. \right\} (4).$$

$$\log(n + 1)$$
$$= \log n + 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \&c. \right\} (5).$$

$$\log(x + 1)$$
$$= 2\log x - \log(x-1) - 2\mu \left\{ \frac{1}{2x^2-1} + \frac{1}{3}\left(\frac{1}{2x^2+1}\right)^3 + \&c. \right\} (6).$$

Proportional Parts.

54. *To find the logarithm of a number containing* (n + 1) *digits, when the table contains the logarithm of numbers containing* n *digits.*

We have

$$\log(n+\delta) - \log n = \log\frac{n+\delta}{n} = \log\left(1 + \frac{\delta}{n}\right)$$

$$= \mu\left(\frac{\delta}{n} - \frac{1}{2}\cdot\frac{\delta^2}{n^2} + \frac{1}{3}\cdot\frac{\delta^3}{n^3} - \&c.\right).$$

Suppose n to be an integer containing 5 figures, and δ a quantity less than unity.

We then have, since μ (= ·434...) is $< \frac{1}{2}$,

$$\mu\cdot\frac{\delta^2}{2n^2} < \frac{1}{4}\cdot\frac{1}{10000^2} \text{ and } \therefore < \cdot000000003\,;$$

$$\mu\cdot\frac{\delta^3}{3n^3} < \frac{1}{6}\cdot\frac{1}{10000^3} \text{ and } \therefore < \cdot0000000000002, \&c.$$

Hence, at least as far as the *seventh* place of decimals, the omission of all the terms of the above series after the first will not affect the result.

We then have

$$\log(n+\delta) - \log n = \mu\cdot\frac{\delta}{n}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

And, similarly,

$$\log(n+1) - \log n = \mu\cdot\frac{1}{n} = d \text{ suppose, } \ldots\ldots\ (2).$$

Then we have,

$$\log(n+\delta) - \log n = \delta d\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now d is the difference between the logarithms of two consecutive numbers;

And δ is the difference (less than unity) of two numbers, the logarithm of the greater of which is required.

Hence we have the following rule:

To find the logarithm of a number which lies between two consecutive numbers, multiply the difference of the given number and the next lower in the tables by the difference of the logarithms of the numbers next above and below the given number, and add this result to the logarithm of the smaller given number.

This rule has been assumed in Vol. I. of this work, to which the student is referred for practical applications.

Remark. Since the mantissæ of all numbers, having the same digits, are the same, it follows that the above applies equally to numbers the integral part of which contains more than five digits.

55. *To prove that*

$$n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}(n-2)^r - \frac{n(n-1)(n-2)}{\underline{|3}}(n-3)^r$$
$$+ \&c. = \underline{|n}, \textit{ if } r = n, \textit{ and } = 0, \textit{ if } r < n.$$

We have

$$(e^x - 1)^n = e^{nx} - n \,.\, e^{(n-1)x} + \frac{n(n-1)}{\underline{|2}} e^{(n-2)x}$$
$$- \frac{n(n-1)(n-2)}{\underline{|3}} e^{(n-3)x} + \&c.$$

If we now expand each of the quantities e^{nx}, $e^{(n-1)x}$, &c., by the Exponential Theorem, and write down the coefficient of x^r in the whole expression, we have

Coefficient of x^r in $(e^x - 1)^n$

$$= \frac{n^r}{\underline{|r}} - n \,.\, \frac{(n-1)^r}{\underline{|r}} + \frac{n(n-1)}{\underline{|2}} \,.\, \frac{(n-2)^r}{\underline{|r}}$$
$$- \frac{n(n-1)(n-2)}{\underline{|3}} \,.\, \frac{(n-3)^r}{\underline{|r}} + \&c. \ldots\ldots\ldots\ldots\ldots (1).$$

Again, $(e^x - 1)^n$

$$= \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \&c.\right)^n$$

$$= x^n + \text{terms involving higher powers of } x \ldots\ldots\ldots (2).$$

Hence the coefficient of x^n is *unity;* and since the expansion contains no terms with lower powers of x than the nth, we may say that

$$\text{The coefficient of } x^r \text{ in } (e^x - 1)^n = 1, \text{ when } r = n, \\ \text{and } = 0, \text{ when } r < n, \quad \ldots (3).$$

Hence from (1) and (3), equating the coefficients of x^r,

$$\frac{n^r}{\lfloor r} - n\frac{(n-1)^r}{\lfloor r} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{(n-2)^r}{\lfloor r} - \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \frac{(n-3)^r}{\lfloor r} + \&c.$$

$$= 1, \text{ when } r = n,$$

$$\text{and } = 0, \text{ when } r < n.$$

$$\therefore n^r - n(n-1)^r + \frac{n(n-1)}{\lfloor 2}(n-2)^r - \frac{n(n-1)(n-2)}{\lfloor 3}(n-3)^r + \&c.$$

$$= \lfloor n, \text{ if } r = n.$$

$$\text{and } = 0, \text{ if } r < n.$$

Ex. IX.

Show that

1. $\dfrac{e^{2x} - 1}{xe^x} = 2\left\{1 + \dfrac{x^2}{\lfloor 3} + \dfrac{x^4}{\lfloor 5} + \&c.\right\}$

2. $\dfrac{e^{2x} + 1}{e^x} = 2\left\{1 + \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} + \&c.\right\}$

3. $m^n = m + \dfrac{m(m-1)}{1 \cdot 2}(2^n - 2) + \dfrac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(3^n - 3 \cdot 2^n + 3) + \&c.$

4. $n - n(n-1) + \dfrac{n(n-1)(n-2)}{2} - \&c. = 0.$

5. $n^{n-1} - n(n-1)^{n-1} + \dfrac{n(n-1)}{1 \cdot 2}(n-2)^{n-1} - \&c. = 0.$

6. $n^n - n(n-1)^n + \dfrac{n(n-1)}{1 \cdot 2}(n-2)^n - \&c. = \lfloor n.$

7. $n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1 \, . \, 2}(n-2)^{n+1} - \&c.$

$$= \frac{1}{2} n \, \underline{|n+1}.$$

8. $1^n - n \, . \, 2^n + \frac{n(n-1)}{1 \, . \, 2} 3^n - \&c. + (-1)^n \, . \, (n+1)^n$

$$= (-1)^n \, \underline{|n}.$$

9. $1^m + 2^m + 3^m + \&c. + n^m = \frac{n^{m+1}}{m+1} + \frac{n}{2} + m \, . \, \frac{n^{m-1}}{1 \, . \, 2} + \&c.$

10. If p, q are each less than 1, show that $\frac{\log(1-p)}{\log(1-q)}$ is less than $\frac{p}{q-pq}$, and greater than $\frac{p-pq}{q}$.

11. If $u_r = r^p$, show that

$$u_1 - nu_2 + \frac{n(n-1)}{1 \, . \, 2} u_3 + \&c. + (-1)^n u_{n+1} = (-1)^n \, \underline{|n}.$$

12. Show that the limit of $\left(1 + \frac{1}{nx}\right)^x$, when x is infinite, is $e^{\frac{1}{n}}$.

13. Having given that $1 \, . \, 2 \, . \, 3 \, ... \, n = \sqrt{2\pi} \, . \, n^{n+\frac{1}{2}} \, . \, e^{-n}$, when n is increased without limit, show that

$$1 \, . \, 3 \, . \, 5 \, ... \, (2n-1) = 2^{n+\frac{1}{2}} \, . \, n^n \, . \, e^{-n},$$

when n is increased without limit.

14. Hence show that the limit of

$$\frac{1 \, . \, 3 \, . \, 5 \ldots (2n-1)}{2 \, . \, 4 \, . \, 6 \ldots 2n} \cdot \frac{1}{2n-1} \text{ is } n^{-\frac{3}{2}},$$

when n is increased without limit.

15. Show that

$$(1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \&c.) \log_e (1+x) = x.$$

16. If $a = b - x$, show that $\log_e a$

$$= \log_e b - \frac{1}{2b-x}\left\{\left(2 - \frac{1}{2}\right)x + \left(\frac{2}{2} - \frac{1}{3}\right)\frac{x^2}{b} + \left(\frac{2}{3} - \frac{1}{4}\right)\frac{x^3}{b^2} + \&c.\right\}.$$

17. If $a^2 + 1 = 0$, show that

$$(1.)\quad e^{ax} + e^{-ax} = 2\left\{1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \&c.\right\}$$

$$(2.)\quad e^{ax} - e^{-ax} = 2\,a\left\{x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \&c.\right\}$$

18. If a, b, c are consecutive numbers, show that

$$\log b = \frac{1}{2}(\log a + \log c)$$

$$+ \mu\left\{\frac{1}{2\,ac+1} + \frac{1}{3}\cdot\frac{1}{(2\,ac+1)^3} + \frac{1}{5}\,\frac{1}{(2\,ac+1)^5} + \&c.\right\}$$

CHAPTER VIII.

THE MULTINOMIAL THEOREM.

56. It is often required to expand the powers of algebraical expressions containing more than two terms. This may be done gradually by means of the Binomial Theorem, but the method is often cumbersome and inconvenient. By means of the Multinomial Theorem we are able to do this more concisely.

57. *To find the general term of the expansion of*

$$(a + b + c + d + \&c.)^n.$$

Put $b_{\prime} = b + c + d + \&c.$,
and $n = p + q_{\prime}$.
Then the $(q_{\prime} + 1)$th term of $(a + b_{\prime})^n$ is

$$\frac{n(n-1)\,\ldots\,(n-q_{\prime}+1)}{\underline{|q_{\prime}}}\,.\,a^{n-q_{\prime}}b_{\prime}^{\,q_{\prime}}$$

$$= \frac{n(n-1)\,\ldots\,(p+1)}{\underline{|q_{\prime}}}\,a^p b_{\prime}^{\,q_{\prime}}\ldots\ldots\ldots\ldots(1),$$

when $q_{\prime}$ is a positive integer.
Put $c_{\prime} = c + d + e + \&c.$,
and $q_{\prime} = q + r_{\prime}$.

Then the $(r_{,} + 1)$th term of $b_{,}^{q_{,}}$, or $(b + c_{,})^{q_{,}}$ is

$$\frac{q_{,}(q_{,} - 1)\ldots\ldots(q_{,} - r_{,} + 1)}{|\underline{r_{,}}}\, b^{q_{,} - r_{,}} c_{,}^{r_{,}}$$

$$= \frac{q_{,}(q_{,} - 1)\ldots\ldots(q + 1)}{|\underline{r_{,}}}\, b^{q} c_{,}^{r_{,}} = \frac{|\underline{q_{,}}}{|\underline{q}\;|\underline{r_{,}}}\, b^{q} c_{,}^{r_{,}} \ldots\ldots\ldots(2.)$$

where $q_{,}$, q, $r_{,}$ are positive integers.

Hence, combining this with the expression in (1), where for $b_{,}^{q_{,}}$ we may substitute its general term, we find
The general term of the expansion of $(a + b + c_{,})^{n}$

$$= \frac{n(n - 1)\ldots\ldots(p + 1)}{|\underline{q_{,}}}\, a^{p} : \frac{|\underline{q_{,}}}{|\underline{q}\;|\underline{r_{,}}}\, b^{q} c_{,}^{r_{,}}$$

$$= \frac{n(n - 1)\ldots\ldots(p + 1)}{|\underline{q}\;|\underline{r}}\, a^{p} b^{q} c_{,}^{r_{,}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Proceeding in this way we shall find
The general term of the expansion of $(a + b + c + d + \&c.)^{n}$

$$= \frac{n(n - 1)\ldots\ldots(p + 1)}{|\underline{q}\;|\underline{r}\;\&c.}\,.\, a^{p} b^{q} c^{r}\ \&c.$$

where q, r, &c., *are positive integers.*

Since $n = p + q_{,}$, $q_{,} = q + r_{,}$, $r_{,} = r + s_{,}$, &c., we must have

$p + q + r + \&c. = n$, and q, r, s, &c., *positive integers.*

Cor. When n is a positive integer, we have
General term of $(a + b + c + d + \&c.)^{n}$

$$= \frac{n(n - 1)\ldots\ldots(p + 1)\,.\,p(p - 1)\ldots\ldots 1}{|\underline{p}\;|\underline{q}\;|\underline{r}\;\&c.}\, a^{p} b^{q} c^{r}\ \&c.$$

$$= \frac{|\underline{n}}{|\underline{p}\;\;|\underline{q}\;\;|\underline{r}\;\;\&c.}\, a^{p} b^{q} c^{r}\ \&c.$$

where $p + q + r + \&c. = n$.

58. *To expand* $(a_{0} + a_{1}x + a_{2}x^{2} + a_{3}x^{3} + \&c.)^{n}$.

In the result of Art. 57, put $a = a_{0}$, $b = a_{1}x$, $c = a_{2}x^{2}$, &c., then

General term of $(a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n$

$$= \frac{n(n-1)\ldots\ldots(p+1)}{\lfloor q \; \lfloor r \; \lfloor s \; \&c.} a_0^p (a_1x)^q (a_2x^2)^r (a_3x^3)^s \; \&c.$$

$$= \frac{n(n-1)\ldots\ldots(p+1)}{\lfloor q \; \lfloor r \; \lfloor s \; \&c.} a_0^p a_1^q a_2^r a_3^s \ldots x^{q+2r+3s+\&c.} \quad (A).$$

where $p + q + r + s + \&c. = n$, and q, r, s, &c., *positive integers.*

When n is a positive integer, this result, by Art. 56, Cor., may be thus expressed:

General term of $(a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n$

$$= \frac{\lfloor n}{\lfloor p \; \lfloor q \; \lfloor r \; \lfloor s \; \&c.} a_0^p a_1^q a_2^r a_3^s \ldots x^{q+2r+3s+\&c.} \ldots (B).$$

where $p + q + r + s + \&c. = n$, and p, q, r, s, &c., are positive integers.

Suppose now we are required to find the coefficient of the term involving x^m.

Taking the formula in (A) above, we have the conditions

$$p + q + r + s + \&c. = n,$$
$$\text{and } q + 2r + 3s + \&c. = m.$$

where q, r, s, &c., *are positive integers.*

Now, by assigning to q, r, s, &c., *all possible values*, subject to the condition that they are positive integers, and that

$$q + 2r + 3s + \&c. = m,$$

we may obtain sets of values of p, q, r, &c., which, when substituted in (A), will give the partial coefficients of x^m; and the sum of these will give the whole coefficient.

Ex. 1. Find the coefficient of x^6 in the expansion of $(1 - 3x + 4x^2 - 2x^3)^5$.

As $n = 5$, a positive integer, we may use the formula (B). We have

$$p + q + r + s = 5 \ldots\ldots\ldots\ldots (1).$$
$$q + 2r + 3s = 6 \ldots\ldots\ldots\ldots (2).$$

We shall find it convenient to proceed as follows:

1. Commence with the highest possible value of the letter

at the right of (2), viz., in this case, of s; and take each lower value until we arrive at $s = 0$.

2. Then with the resulting equation—here with the equation containing $q + 2r$—take first the highest possible value of r, and so on.

3. Find the value of p from (1) by means of the values of q, r, s thus obtained.

Thus :—

Take $s = 2$, and $\therefore q + 2r = 0$, where we may have
$$r = 0, q = 0, p = 3.$$
Take $s = 1$, and $\therefore q + 2r = 3$, where we may have
$$r = 1, q = 1, p = 2,$$
or,
$$r = 0, q = 3, p = 1.$$
Take $s = 0$, and $\therefore q + 2r = 6$, where we may have
$$r = 3, q = 0, p = 2,$$
$$r = 2, q = 2, p = 1,$$
$$r = 1, q = 4, p = 0.$$

These are all the possible solutions, and they are conveniently arranged thus :

p	q	r	s
3	0	0	2
2	1	1	1
1	3	0	1
2	0	3	0
1	2	2	0
0	4	1	0

Hence, from (B), since $a_0 = 1$, $a_1 = -3$, $a_2 = 4$, $a_3 = -2$, substituting the values of p, q, r, s, and omitting from the formula every term containing a letter whose value may happen to be zero, and remembering that $\lfloor 1 = 1$, we have

Coefficient of x^6

$$= \frac{\lfloor 5}{\lfloor 3 \; \lfloor 2}(1)^3(-2)^2 + \frac{\lfloor 5}{\lfloor 2}(1)^2(-3)(4)(-2)$$

$$+ \frac{\lfloor 5}{\lfloor 3}(1)(-3)^3(-2) + \frac{\lfloor 5}{\lfloor 2 \; \lfloor 3}(1)^2(4)^3$$

$$+ \frac{\lfloor 5}{\lfloor 2 \; \lfloor 2}(1)(-3)^2(4)^2 + \frac{\lfloor 5}{\lfloor 4}(-3)^4(4)$$

$$= 40 + 1440 + 1080 + 640 + 4320 + 1620 = 9140.$$

Ex. 2. Find the coefficient of x^3 in $(1 + 3x + 6x^2 + 10x^3 + \&c.)^{\frac{1}{3}}$

Here $n = \frac{1}{3}$, a fraction; we shall therefore use the formula (A).

We have

$$p + q + r + s + \&c. = \tfrac{1}{3} \quad \text{..................(1).}$$

and

$$q + 2r + 3s + \&c. = 3 \quad \text{..................(2).}$$

The equations (1) and (2) need not contain any letter beyond s; for if we take another letter, as t, we have

$$q + 2r + 3s + 4t = 3;$$

and the only value of t possible is $t = 0$, and so of any other letters. We shall therefore treat equations (1) and (2) as if they were written

$$p + q + r + s = \tfrac{1}{3}.$$
$$q + 2r + 3s = 3.$$

On trial we find the only solutions to be as in the annexed table:

p	q	r	s
$-\frac{2}{3}$	0	0	1
$-\frac{5}{3}$	1	1	0
$-\frac{8}{3}$	3	0	0

Hence the coefficient of x^3

$$= \frac{\frac{1}{3}}{1}(10) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1}(3)(6) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{\lfloor 3}(3)^3$$

$$= \frac{10}{3} - 4 + \frac{5}{3} = 1.$$

We may verify this result.

For $1 + 3x + 6x^2 + 10x^3 + \&c. = (1 - x)^{-3}$

$$\therefore (1 + 3x + 6x^2 + 10x^3 + \&c.)^{\frac{1}{3}} = \left\{(1 - x)^{-3}\right\}^{\frac{1}{3}}$$

$$= (1 - x)^{-1} = 1 + x + x^2 + x^3 + \&c.$$

Ex. X.

Find the coefficient of

1. x^3 in $(1 + x + x^2)^4$.
2. x^4 in $(1 - 2x + 3x^2 - 4x^3)^8$.
3. x^5 in $(4 - 5x + x^2)^{\frac{1}{2}}$.
4. x^3 in $(6 + 5x - 6x^2)^5$.
5. x^4 in $(a_0 + a_1x + a_2x^2 + a_3x^3)^4$.
6. x^3 in $\left(1 + \frac{1}{3}x + \frac{2}{27}x^2 + \frac{14}{81}x^3 + \&c.\right)^8$.
7. a^2bc^3d in $(a + b + c + d)^7$.

8. a^3b^2 in $(a + b + c + d)^5$.

9. abc^4d^2 in $(a + b + c + d + e)^8$.

10. x^6 in $(a_0 + a_1x + a_2x^2 + a_3x^3)^4$.

11. x^4 in $(1 + 2x + 3x^2 + 4x^3 + \&c.)^{-\frac{3}{2}}$.

12. x^5 in $(1 + 4x + 10x^2 + 20x^3 + 35x^4 + 42x^5 + \&c.)^{-4}$.

13. Expand $(1 - 3x + 4x^2)^{-\frac{1}{3}}$ to five terms.

14. Expand $(1 + 7x^2 + x^3)^{\frac{1}{2}}$ to four terms.

15. Show that the coefficient of the middle term of the expansion of $(1 + x + x^2)^m$

$$= 1 + \frac{m(m-1)}{\underline{|1}} + \frac{m(m-1)(m-2)(m-3)}{(\underline{|2})^2}$$

$$+ \frac{m(m-1)\ldots(m-5)}{(\underline{|3})^2} + \&c.$$

$$+ \frac{m(m-1)\ldots(m-2r+1)}{(\underline{|r})^2} + \&c.$$

16. If c_r denote the coefficient of x^r in the expansion of $(1 + 2x + 2x^2 + 2x^3 + 2x^4 + \&c.)^n$, show that

$$(m+1)c_{m+1} - 2nc_m - (m-1)c_{m-1} = 0.$$

17. If c_0, c_1, c_2, &c., be the successive coefficients of the expansion of $(1 + 3x + 5x^2 + \&c. + \overline{2p+1}\,x^p)^n$, show that

$$c_0 + c_1 + c_2 + \&c. + c_{np} = (p+1)^{2n}.$$

18. Show that the sum of the coefficients of the expansion of

$$\left\{1 - 3x - 5x^2 + 7x^3 + 9x^4 \ldots + (-1)^m(4m-1)x^{2m-1}\right\}^n$$

$= 0$, or $(-2)^n$, according as m is of the form $2p$ or $2p + 1$.

CHAPTER IX.

INTEREST AND ANNUITIES.

Interest.

59. *To find the amount of a given sum for a given number of years at a given rate, simple interest.*

Let P = the principal in pounds.
r = the interest of £1 for 1 year.
n = number of years.
M = the amount.

We have

Interest of P for 1 year $= Pr$.

$\therefore$,, n years $= Pnr$.

Hence $M = P + Pnr = P(1 + nr)$ ....(1).

$$\therefore \text{ also } P = \frac{M}{1 + nr} \quad\ldots\ldots(2).$$

$$\text{and } r = \frac{M - P}{nP} \quad\ldots\ldots(3).$$

COR. If interest be allowed for *fractions* of years the above formulæ hold.

60. *To find the amount of a given sum for a given number of years at a given rate, compound interest.*

Let R = the amount of £1 for 1 year.
$\therefore R = 1 + r$.

Also, the amount of P for 1 year $= PR$.

$\therefore$,, P ,, 2 ,, $= PR \cdot R = PR^2$.

,, P ,, 3 ,, $= PR^2 \cdot R = PR^3$.

&c. = &c.

Hence also ,, P for n years $= PR^n$.

$$\left.\begin{aligned} \therefore\ M &= PR^n \\ \text{and } \therefore\ \log M &= \log P + n \log R \end{aligned}\right\} \quad\ldots\ldots(1).$$

$$\left.\begin{aligned} \text{We have also}\quad P &= \frac{M}{R^n} \\ \text{and } \log P &= \log M - n\log R \end{aligned}\right\}\text{.................(2).}$$

$$\left.\begin{aligned} \text{And again,}\quad R &= \sqrt[n]{\frac{M}{P}} \\ \text{and } \therefore \log R &= \frac{1}{n}(\log M - \log P) \end{aligned}\right\}\text{...............(3).}$$

$$\left.\begin{aligned} \text{We have also, } R^n &= \frac{M}{P} \\ \text{and } \therefore n &= \frac{\log M - \log P}{\log R} \end{aligned}\right\}\text{.....................(4).}$$

COR. If n be fractional, and interest be allowed for the fractional part of a year at the same rate, the above formulæ require modification.

For, let $n = m + \frac{1}{p}$, where $\frac{1}{p}$ is a proper fraction, and m an integer.

We have, by (1), amount for m years $= PR^m$.

Now amount of £1 for $\frac{1}{p}$ year $= 1 + \frac{r}{p}$.

Hence, amount of PR^m for $\frac{1}{p}$ year, and

$$\therefore \text{Amount of P for } \left(m + \frac{1}{p}\right) \text{ years}$$

$$= PR^m\left(1 + \frac{r}{p}\right).$$

It is easy to see here that interest is really reckoned in two ways, for whole years, and for a fraction of a year. If it be agreed that interest is to be reckoned for every mth part of a year, we obtain a different formula, as shown in the next article.

61. *To find the amount of a given sum at compound interest, interest being reckoned* m *times a year.*

In n years there are mn distinct periods, for which compound interest is to be reckoned. We therefore have to find the amount of a sum P for mn periods of time, when $\frac{r}{m}$ is the interest of £1 for *one such period.*

Hence we have simply to apply formula (1) of last article, where we must put mn for n, and $\frac{r}{m}$ for r. We then get

$$M = P\left(1 + \frac{r}{m}\right)^{mn}.$$

COR. Let m be infinite, that is, suppose interest to be considered due every instant, then

$$M = \text{limit of } P\left(1 + \frac{r}{m}\right)^{mn}, \text{ when } m = \infty : \text{ or, by Art. 42,}$$

$$M = Pe^{rn}.$$

Annuities.

62. *To find the amount of an annuity for a given time, reckoning simple interest.*

Let A be the annuity, n the number of years, r the interest of £1 for one year, M the amount.

Then, amount due at the end of

1 year $= A$.
2 „ $= A + (1 + r)A$.
3 „ $= A + (1 + r)A + (1 + 2r)A$.
&c. $=$ &c.
n „ $= A + (1 + r)A + (1 + 2r)A + \text{\&c.} + (1 + \overline{n-1}\,.\,r)A$.

$$\text{Or, } M = \left\{1 + (1 + r) + (1 + 2r) + \text{\&c.} + (1 + \overline{n-1}\,.\,r)\right\}A.$$

$$= \left\{2 \times 1 + \overline{n-1}\,.\,r\right\}\frac{n}{2}\,.\,A = nA + \tfrac{1}{2}n(n-1)rA.$$

63. *To find the present worth of an annuity to continue for a given time, reckoning simple interest.*

I. Let P be the present worth of an annuity A, which is to continue for n years.

Then if P be put out to simple interest for n years, its amount will be equal to the amount of the annuity for the *same time.*

Hence, $P(1 + nr) = nA + \frac{1}{2}n(n-1)rA$. Therefore

$$P = \frac{nA + \frac{1}{2}n(n-1)rA}{1 + nr} = \frac{nA}{2} \cdot \frac{2 + (n-1)r}{1 + nr} \text{........ (A).}$$

II. There is another way of solving this problem, which brings out a different result. It proceeds on the principle that the present worth of the annuity is the sum of the present worth of the separate annual payments.

Thus, present worth of A due 1 year hence $= \frac{A}{1 + r}$

„ „ 2 „ $= \frac{A}{1 + 2r}$

„ „ 3 „ $= \frac{A}{1 + 3r}$

&c. = &c.

„ „ n „ $= \frac{A}{1 + nr}$.

Now, taking P as the sum of these, we have

$$P = \left\{ \frac{1}{1 + r} + \frac{1}{1 + 2r} + \frac{1}{1 + 3r} + \text{\&c.} + \frac{1}{1 + nr} \right\} A\text{......(B).}$$

If this sum be put out to simple interest for n years, we shall find that its amount is not the same as that of the annuity for the same time. The reason appears to be that in reality we are reckoning both *simple* and *compound* interest.

Thus, for any specified year, as the pth, the present value of a payment $A = \frac{A}{1 + pr}$.

It is true that at *simple interest* this sum amounts to A in p years, and we may allow this sum A to remain till the n years are completed. But this is equivalent to *interest upon the interest* gained in the first p years, in addition to the interest upon the principal merely.

We shall presently find that when *compound interest* is reckoned throughout, the present worth of an annuity is the same by either method of procedure.

64. *To find the amount of an annuity, reckoning compound interest.*

We have

Amount due at the end of

$$\begin{aligned}
&1 \text{ year} &&= A \\
&2 \quad ,, &&= A + AR \\
&3 \quad ,, &&= A + AR + AR^2 \\
&\&c. &&= \&c. \\
&n \quad ,, &&= A + AR + AR^2 + \&c. + AR^{n-1} \\
& &&= A \,.\, \frac{R^n - 1}{R - 1} = \frac{A}{r}(R^n - 1).
\end{aligned}$$

Hence $M = \frac{A}{r}(R^n - 1)$.

65. *To find the present worth of an annuity, reckoning compound interest.*

I. We shall proceed on the principle that if the present worth P be put out to compound interest for n years, it ought to amount to the same as the annuity for that time.

Now, amount of P in n years $= PR^n$;

And, amount of annuity for n years $= \frac{A}{r}(R^n - 1)$.

Equating these we have

$$PR^n = \frac{A}{r}(R^n - 1),$$

$$\therefore P = \frac{A}{r}(1 - R^{-n}).$$

II. We will now proceed on the principle that the present worth P is the sum of the present worths of the respective annual payments.

We have, Art. 60,

$$\begin{aligned}
&\text{Present worth of A due 1 year hence} &&= \frac{A}{R}. \\
&\quad ,, \qquad ,, \qquad 2 \quad ,, &&= \frac{A}{R^2} \\
&\quad ,, \qquad ,, \qquad 3 \quad ,, &&= \frac{A}{R^3} \\
&\qquad\qquad\qquad \&c. &&= \&c. \\
&\quad ,, \qquad ,, \qquad n \quad ,, &&= \frac{A}{R^n}
\end{aligned}$$

$$\therefore P = \frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + \&c. + \frac{A}{R^n} = \frac{A}{R} \cdot \frac{1 - \frac{1}{R^n}}{1 - \frac{1}{R}} = \frac{A}{r}(1 - R^{-n}).$$

We thus see that when compound interest is reckoned, we arrive at the same result, by finding the present worth on either principle.

66. *To find the present worth of an annuity to continue for ever.*

I. *Reckoning compound interest.*

We have $$P = \frac{A}{r}(1 - R^{-n}).$$

When $n = \infty$, the limit of $R^{-n} = 0$; and then $P = \frac{A}{r}$.

Hence, the present worth of an annuity A to continue for ever $= \frac{A}{r}$.

II. *Reckoning simple interest.*

Taking the formula (A) in Art. 63, we have

$$P = \frac{nA}{2} \cdot \frac{2 + (n - 1)r}{1 + nr} = \frac{nA}{2} \cdot \frac{\frac{2}{n} + \left(1 - \frac{1}{n}\right)r}{\frac{1}{n} + r}.$$

Now, when $n = \infty$, the limit of

$$\frac{\frac{2}{n} + \left(1 - \frac{1}{n}\right)r}{\frac{1}{n} + r} = \frac{0 + (1 - 0)r}{0 + r} = 1.$$

Hence, the limit of P, when $n = \infty$, $= \infty \frac{A}{2} = \infty$.

This result shows that, reckoning so-called simple interest, an *infinite sum* of money is required to be left, in order to insure an equal annual payment for ever.

It indicates therefore that the only correct method of computing annuities is on the compound interest principle.

We should arrive at the same result by taking formula (B) of Art. 63.

67. *To find the present worth of an annuity to commence in* p *years, and to continue for* q *years.*

Amount of annuity A to continue for q years

$$= \frac{A}{r}(R^q - 1) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, if P be the present worth, it will amount in $(p + q)$ years to exactly as much money as the annuity, if left to accumulate for q years.

And the amount of P in $(p + q)$ years $= P \,.\, R^{p+q} \ldots$ (2).

Hence, equating (2) and (1), we have

$$P \,.\, R^{p+q} = \frac{A}{r}(R^q - 1);$$

$$\therefore P = \frac{A}{r} \cdot \frac{R^q - 1}{R^{p+q}} = \frac{A}{r}(R^{-p} - R^{-p-q}).$$

Cor. If the annuity is to continue for ever, we have $q = \infty$, and the limit of $R^{-p-q} = 0$.

Hence the present worth of a perpetual annuity which is to be entered upon in p years $= \frac{A}{r} \,.\, R^{-p}$.

Ex. 1. A sum of £a is borrowed for a period of m years, to be repaid by equal annual instalments, the first payment to be made after one year. Find the amount of the annual instalment.

Let A be the annual instalment.

Then the amount of this annual payment in m years

$$= \frac{A}{r}\{R^m - 1\}.$$

Again, if the sum a be allowed to accumulate for m years at compound interest, its amount $= a \,.\, R^m$.

Now, these two amounts ought to be equal.

Hence we have

$$\frac{A}{r}\{R^m - 1\} = a \; R^m;$$

$$\therefore A = \frac{a}{r} \cdot \frac{R^m}{R^m - 1} = \frac{a}{r} \cdot \frac{1}{1 - R^{-m}}.$$

Ex. 2. A debt a now due, is paid off in n years by a series of n payments in arithmetical progression; reckoning compound interest show that, if b be the first of the n payments, the common difference of the arithmetical progression is

$$\frac{aR^n r^2 - b(R^n - 1)r}{R(R^{n-1} - 1) - (n-1)r}.$$

If we suppose the debt and the whole of the payments to be allowed to accumulate for the n years, the two amounts should be equal.

Now, amount of the debt a in n years $= aR^n$..............(1).

And, amount of the n payments in the same time

$= b\,R^{n-1} + (b+d)\,R^{n-2} + (b+2d)\,R^{r-3} + \text{\&c.} + (b + \overline{n-1}\,d)$.

where d is the common difference of the A. P.

But (Art. 27, Ex. 4) the sum of this series

$$= \frac{bR^{n-1}}{1 - \frac{1}{R}} + \frac{dR^{n-1}}{R} \cdot \frac{1 - \frac{1}{R^{n-1}}}{\left(1 - \frac{1}{R}\right)^2} - (b + \overline{n-1}\,.\,d)R^{n-1} \cdot \frac{\frac{1}{R^n}}{1 - \frac{1}{R}}$$

$$= \frac{b}{r} \cdot R^n + \frac{d}{r^2} \cdot (R^n - R) - \frac{b + \overline{n-1}\,.\,d}{r} \;\ldots\ldots\ldots\ldots\ldots \;(2).$$

(2) = (1), then

$$\frac{b}{r} \cdot R^n + \frac{d}{r^2}(R^n - R) - \frac{b + \overline{n-1}\,.\,d}{r} = a\,.\,R^n;\ \text{or,}$$

$$\left\{\frac{R^n - R}{r^2} - \frac{n-1}{r}\right\} d = aR^n - \frac{b}{r}(R^n - 1)$$

$$\therefore d = \frac{aR^n r^2 - b\,(R^n - 1)r}{R\,(R^{n-1} - 1) - (n-1)r}.$$

Ex. XI.

1. Find the compound interest on £530 for 12 years at $3\frac{1}{2}$ per cent.

2. In what time will a sum of money double itself at 5 per cent., compound interest?

3. Find the amount of an annuity of £50 for 10 years at 4 per cent.

4. A corporation borrows £3,760 at 4 per cent., to be re-

paid in 30 years by equal annual instalments. What will be the annual payment?

5. Find the equated time for £320, £390, £450, due respectively in 6, 7, 8 months, simple interest being reckoned.

6. In how many years will £1 amount to £5 at 6 per cent. per annum, compound interest?

7. A freehold bringing in £120 a year is sold for £1,920. Find the rate of interest.

8. A debt of £1,200 is to be paid out in 20 years by annual payments, increasing in A. P. If the first payment is £70, what is the last payment?

9. If interest be payable every instant, in how many years would £1 amount to £6?

10. The present value of a freehold to be entered upon in 5 years is £1,600. Find the rent, interest being reckoned at 6 per cent.

11. A property is let out on lease for a years at an annual rental of £b, and after c years the lease is renewed on paying a fine of £d. What is the additional rent equivalent to this fine?

12. An annuity a is found to amount in n years to the same sum as when b is put out for the same time at compound interest. Express n in terms of a, b, r, where r is the interest of one pound for one year.

CHAPTER X.

CONTINUED FRACTIONS.

68. Let x be any quantity, rational or irrational; and suppose that $x = a_0 + \frac{1}{x_1}$, $x_1 = a_1 + \frac{1}{x_2}$, $x_2 = a_2 + \frac{1}{x_3}$, &c.; where a_0, a_1, a_2, a_3, &c, are respectively the greatest integers in x, x_1, x_2, x_3, &c.

We then have

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \text{\&c.}}}}, \text{ or, as often written,}$$

$$x = a_0 + \frac{1}{a_1 +}\;\frac{1}{a_2 +}\;\frac{1}{a_3 +}\;\frac{1}{\text{\&c.}}$$

Such an expression is called a *continued fraction*, and it is *rational* or *irrational* according as the quotients a_1, a_2, a_3, &c., terminate or not.

The quantities a_0, a_1, a_2, a_3, &c., are called *incomplete quotients*; the first of them, a_0, may be zero, but each of the others must at least be equal to unity.

The quantities x_1, x_2, x_3, &c., are called *complete quotients*; and when the continued fraction is made to terminate by a *complete* quotient, the resulting expression is identically equal to the given quantity x.

When, however, it is made to terminate by the successive *incomplete* quotients, we get successive approximations to the true value of x.

Thus—

(1.) We may express the true value of the given quantity x in either of the following ways:

$$a_0 + \frac{1}{x_1},\; a_0 + \frac{1}{a_1 +}\;\frac{1}{x},\; a_0 + \frac{1}{a_1 +}\;\frac{1}{a_2 +}\;\frac{1}{x_3},\; \text{\&c.}$$

(2.) We may express the corresponding approximations as follows:

$$a_0,\; a_0 + \frac{1}{a_1},\; a_0 + \frac{1}{a_1 +}\;\frac{1}{a_2},\; \text{\&c.}$$

COR. When the approximation to the given quantity x is expressed by terminating the continued fraction by an incomplete quotient, we may obtain the true value of x by substituting the corresponding *complete quotient* for the last incomplete quotient.

69. *Conversion of a given fraction into a continued fraction.*

Let $\frac{P}{Q}$ be the given fraction, and let the ordinary operation of finding the G. C. M. of P and Q be performed.

We will suppose the operation to stand thus:

$$\begin{array}{l} Q)\ P\ (a_0 \\ \quad a_0Q \\ \hline \quad p)\ Q\ (a_1 \\ \qquad a_1p \\ \hline \qquad q)\ p\ (a_2 \\ \qquad\quad a_2q \\ \hline \qquad\quad r)\ \&c. \end{array}$$

Then we have

$$\frac{P}{Q} = a_0 + \frac{p}{Q};\ \frac{Q}{p} = a_1 + \frac{q}{p};\ \frac{p}{q} = a_2 + \frac{r}{q};\ \&c$$

By successive substitutions we therefore get

$$\frac{P}{Q} = a_0 + \frac{1}{a_1 +}\ \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \cdots\cdots\ \frac{1}{a_r +}\cdots\cdots$$

It will be presently seen (Art. 72) that the approximations obtained by terminating the continued fraction by the successive *incomplete quotients* are alternately less and greater than the given fraction; and further, that they gradually approach nearer and nearer to the true value. For this latter reason these approximations are called *converging fractions*, or *convergents*.

The convergents of $\frac{P}{Q}$ may be written as follows:

$$\frac{a_0}{1},\ \frac{a_0a_1 + 1}{a_1},\ \frac{a_0a_1a_2 + a_2 + a_0}{a_1a_2 + 1},\ \&c.$$

If we look at the third of these, we notice:

(1.) Its numerator is found by multiplying the numerator

of the *second* convergent by the *third* incomplete quotient, and adding the numerator of the *first* convergent.

Thus, $a_0a_1a_2 + a_2 + a_0 = (a_0a_1 + 1)\,a_2 + a_0$.

(2.) Its *denominator* is similarly formed.

Thus, $a_1a_2 + 1 = a_1(a_2) + 1$.

The existence of a general law for forming these convergents may therefore be suspected. This law is proved in the next article, and may be expressed thus:

Law of formation of the convergents.

(1.) To find the numerator of the $(n + 1)$th convergent, multiply the numerator of the nth convergent by the next incomplete quotient, and add to the product the numerator of the $(n - 1)$th convergent.

(2.) To find the denominator pursue a similar method.

In order, however, to apply this rule practically, it is evident that we require to have found at least *two convergents* by ordinary vulgar fractions. But if we make use of a *fictitious convergent* for *the first,* we may then apply the above rule when we know only one other.

Thus, *assuming* $\frac{1}{0}$ *as the first convergent,* we may form all the convergents by the above law, if we only know $\frac{a_0}{1}$, the first approximation, and the successive incomplete quotients.

The accompanying table exhibits this to the eye at once.

Incomplete Quotients.	a_0	a_1	a_2	a_3	&c.
Convergents,....	$\frac{1}{0}$	$\frac{a_0}{1}$	$\frac{a_0a_1 + 1}{a_1}$	$\frac{a_0a_1a_2 + a_2 + a_0}{a_1a_2 + 1}$	&c.
Order,............	1st	2nd	3rd	4th	&c.

Cor. If we designate by P_0, P_1, P_2, P_3, &c., the numerators, and by Q_0, Q_1, Q_2, Q_3, &c., the denominators, of the successive convergents, we have

$$P_0 = 1,\ P_1 = a_0,\ P_2 = a_0a_1 + 1,\ P_3 = a_0a_1a_2 + a_2 + a_0, \text{ \&c.}$$
$$Q_0 = 0,\ Q_1 = 1,\ Q_2 = a_1,\qquad Q_3 = a_1a_2 + 1, \text{ \&c.}$$

70. *To show that* $P_n = a_nP_{n-1} + P_{n-2}$, *and that* $Q_n = a_nQ_{n-1} + Q_{n-2}$.

We shall *assume* the rth convergent to be formed according to this law, and then show that the $(r + 1)$th is formed according to the law.

By the last Art. we represent the rth convergent by $\dfrac{P_{r-1}}{Q_{r-1}}$.

We *assume*, then, that

$$\left.\begin{array}{l} P_{r-1} = a_{r-1}P_{r-2} + P_{r-3}, \text{ and} \\ Q_{r-1} = a_{r-1}Q_{r-2} + Q_{r-3}, \end{array}\right\} \dots\dots\dots\dots (1.)$$

$$\therefore \frac{P_{r-1}}{Q_{r-1}} = \frac{a_{r-1}P_{r-2} + P_{r-3}}{a_{r-1}Q_{r-2} + Q_{r-3}} \dots\dots\dots\dots (2.)$$

Now, Art. 68, Cor., we can from this expression obtain the true value of $\dfrac{P}{Q}$, if we replace the incomplete quotient a_{r-1} by the corresponding complete quotient x_{r-1}.

We then have $\dfrac{P}{Q} = \dfrac{x_{r-1}P_{r-2} + P_{r-3}}{x_{r-1}Q_{r-2} + Q_{r-3}}$.

But, Art. 68, $x_{r-1} = a_{r-1} + \dfrac{1}{x_r}$; we therefore have

$$\frac{P}{Q} = \frac{\left(a_{r-1} + \frac{1}{x_r}\right)P_{r-2} + P_{r-3}}{\left(a_{r-1} + \frac{1}{x_r}\right)Q_{r-2} + Q_{r-3}} = \frac{x_r(a_{r-1}P_{r-2} + P_{r-3}) + P_{r-2}}{x_r(a_{r-1}Q_{r-2} + Q_{r-3}) + Q_{r-2}},$$

or, by (1),
$$\frac{P}{Q} = \frac{x_rP_{r-1} + P_{r-2}}{x_rQ_{r-1} + Q_{r-2}}.$$

Now, replacing x_r by the corresponding incomplete quotient a_r, we get the $(r + 1)$th convergent. Thus we find—

The numerator, $P_r = a_r P_{r-1} + P_{r-2}$;
The denominator, $Q_r = a_r Q_{r-1} + Q_{r-2}$.

Hence, on the assumption that the law is true for the numerator and denominator of the rth convergent, we have shown it to be true for that of the $(r + 1)$th convergent.

Now, we knew, Art. 69, it to be true for the 3rd and 4th convergents; it is therefore true for the 5th; and hence for the 6th, &c. We conclude, then, that it is generally true.

71. *To show that* $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n$.

We have

$$P_n = a_n P_{n-1} + P_{n-2}.$$
$$Q_n = a_n Q_{n-1} + Q_{n-2}.$$

We easily get

$$\frac{P_n - P_{n-2}}{Q_n - Q_{n-2}} = \frac{P_{n-1}}{Q_{n-1}}; \text{ or,}$$

$$P_n Q_{n-1} - Q_{n-1} P_{n-2} = Q_n P_{n-1} - P_{n-1} Q_{n-2}; \text{ or,}$$
$$-(P_n Q_{n-1} - Q_n P_{n-1}) = P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}.$$

Or, multiplying each side by $(-1)^{n-1}$, we have

$$(-1)^n (P_n Q_{n-1} - Q_n P_{n-1})$$
$$= (-1)^{n-1} (P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}).$$

We learn from this equation that, *whatever be the value of* r, the quantity $(-1)^r (P_r Q_{r-1} - Q_r P_{r-1})$ has *a constant value.*

When $r = 1$, it becomes $(-1)(P_1 Q_0 - Q_1 P_0)$
$= -(a_0 \times 0 - 1 \times 1) = 1$.

We then have

Constant value of $(-1)^r (P_r Q_{r-1} - Q_r P_{r-1}) = 1$.

Put $r = n$, then we get

$$P_n Q_{n-1} - Q_n P_{n-1} = \frac{1}{(-1)^n} = \frac{(-1)^{2n}}{(-1)^n} = (-1)^n. \quad Q.E.D.$$

COR. 1. Hence, dividing each side by Q_nQ_{n-1}, we have

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = (-1)^n \cdot \frac{1}{Q_nQ_{n-1}}.$$

COR. 2. *All converging fractions are in their lowest terms.*

For we have $P_nQ_{n-1} - Q_nP_{n-1} = (-1)^n$.

Hence, if P_n and Q_n have a common measure, this measure must divide *unity*. And the same may be said of the common measure of P_{n-1} and Q_{n-1}, if any.

72. *Of two consecutive convergents, the one is greater and the other less than the true value of the continued fraction, and each convergent approaches more nearly to the true value than the preceding.*

Let $\frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be consecutive convergents to a continued fraction whose true value is x.

Then we have

$x = \frac{x_nP_n + P_{n-1}}{x_nQ_n + Q_{n-1}}$; from which we have

$$x_n = \frac{xQ_{n-1} - P_{n-1}}{P_n - xQ_n}, \text{ or } \frac{Q_n}{Q_{n-1}}x_n = \frac{x - \dfrac{P_{n-1}}{Q_{n-1}}}{\dfrac{P_n}{Q_n} - x} \quad \text{.........(1).}$$

Now Q_n, Q_{n-1}, x_n are positive quantities.

Hence, $x - \frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_n}{Q_n} - x$ are of the same sign.

Therefore x lies between $\frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_n}{Q_n}$, which proves the first part of the proposition.

Again (Art. 65), x_n (a complete quotient) is greater than unity.

Also, $\frac{Q_n}{Q_{n-1}}$ must be greater than unity.

Hence, from (1),
$$\frac{x - \dfrac{P_{n-1}}{Q_{n-1}}}{\dfrac{P_n}{Q_n} - x} > 1;$$

that is, the difference between x and $\dfrac{P_{n-1}}{Q_{n-1}}$ is greater than that between $\dfrac{P_n}{Q_n}$ and x.

Hence $\dfrac{P_n}{Q_n}$ is a nearer approximation to the true value of x than $\dfrac{P_{n-1}}{Q_{n-1}}$.

COR. Since the *first* convergent is $\dfrac{1}{0}$, and the *second* $\dfrac{a_0}{1}$, it follows that:

Convergents of an *odd* order form a decreasing series, and convergents of an *even* order form an increasing series, each series gradually approaching to the true value of the continued fraction.

73. *To find the limits of the error in taking* $\dfrac{P_{n-1}}{Q_{n-1}}$ *as an approximation to the true value of* x.

By Art. 68, we have
$$x = \frac{x_nP_n + P_{n-1}}{x_nQ_n + Q_{n-1}}; \text{ hence}$$
$$x - \frac{P_{n-1}}{Q_{n-1}} = \frac{x_nP_n + P_{n-1}}{x_nQ_n + Q_{n-1}} - \frac{P_{n-1}}{Q_{n-1}}$$
$$= \frac{x_n(P_nQ_{n-1} - Q_nP_{n-1})}{Q_{n-1}(x_nQ_n + Q_{n-1})}, \text{ or, by Art. 70,}$$
$$= (-1)^n \cdot \frac{x^n}{Q_{n-1}(x_nQ_n + Q_{n-1})}$$
$$= (-1)^n \cdot \frac{1}{Q_{n-1}\left(Q_n + \dfrac{1}{x^n}Q_{n-1}\right)}.$$

Now x_n is > 1, and therefore $\frac{1}{x_n} > 0$ and < 1.

Hence

$$x - \frac{P_{n-1}}{Q_{n-1}} < (-1)^n \cdot \frac{1}{Q_{n-1}Q_n}, \text{ and } > \frac{1}{Q_{n-1}(Q_n + Q_{n-1})}.$$

COR. 1. Since Q_{n-1} is $< Q_n$, and $\therefore Q^2_{n-1} < Q_{n-1}Q_n$, and $Q_{n-1}Q_n < Q_n^2$; we have

$$x - \frac{P_{n-1}}{Q_{n-1}} < \frac{(-1)^n}{Q^2_{n-1}}, \text{ and } > \frac{(-1)^n}{2\,Q_n^2}.$$

COR. 2. Hence if it be required to find a convergent $\frac{P_{n-1}}{Q_{n-1}}$ which differs from x by a quantity less than any given quantity $\frac{1}{a}$, we must find the successive convergents until we reach one whose denominator Q_{n-1} is $\geqq \sqrt{a}$.

74. *Any convergent* $\frac{P_n}{Q_n}$ *approaches more nearly to the value of* x *than any other fraction whose denominator is* $< Q_n$.

Let $\frac{p}{q}$ be any fraction which lies between $\frac{P_n}{Q_n}$ and x.

If $\frac{p}{q}$ be a convergent, then (Art. 72) q is necessarily $> Q_n$.

But if $\frac{p}{q}$ be not one of the convergents, then, since x lies between $\frac{P_n}{Q_n}$ and $\frac{P_{n-1}}{Q_{n-1}}$, and $\frac{p}{q}$ lies between $\frac{P_n}{Q_n}$ and x, it follows that

$$\frac{p}{q} \text{ lies between } \frac{P_n}{Q_n} \text{ and } \frac{P_{n-1}}{Q_{n-1}}.$$

Hence $\frac{p}{q} \sim \frac{P_{n-1}}{Q_{n-1}} < \frac{P_n}{Q_n} \sim \frac{P_{n-1}}{Q_{n-1}} < \frac{P_nQ_{n-1} \sim Q_nP_{n-1}}{Q_nQ_{n-1}}$.

But (Art. 68) neglecting the sign, $P_nQ_{n-1} \sim Q_nP_{n-1} = 1$;

$$\therefore \frac{p}{q} \backsim \frac{P_{n-1}}{Q_{n-1}} < \frac{1}{Q_n Q_{n-1}}, \text{ or}$$

$$pQ_{n-1} \backsim qP_{n-1} < \frac{q}{Q_n} \text{ .... (1).}$$

Now $pQ_{n-1} \backsim qP_{n-1}$ cannot be equal to *zero*, for then $\frac{p}{q} = \frac{P_{n-1}}{Q_{n-1}}$, which is contrary to our hypothesis. And hence, since p, q, P_{n-1}, Q_{n-1} are all integers,

$pQ_{n-1} \backsim qP_{n-1}$ cannot be less than *unity*.

Therefore the condition in (1) is impossible, if $q < Q_n$.

Hence no fraction $\frac{p}{q}$ can approach more nearly to the value of x, than the convergent $\frac{P_n}{Q_n}$, if q is $< Q_n$.

75. *To solve the indeterminate equation*

$$ax + by = c$$

by means of a continued fraction.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent immediately preceding the last $\left(\text{viz., } \frac{a}{b}\right)$. Then we have (Art. 71),

$$aq - bp = \pm 1, \text{ or } a(\pm cq) - b(\pm cp) = c.$$

Or, adding and subtracting abt, we have

$$a(bt \pm cq) + b(\pm cp - at) = c.$$

Comparing this with the original equation, we see that the given equation is solved by putting

$$x = bt \pm cq$$
$$y = \mp cp - at.$$

Ex. Reduce $\frac{314159}{100000}$ to a continued fraction, and find the convergents.

100000	314159	3
	300000	
14159	100000	7
	99113	
887	14159	15
	887	
	5289	
	4435	
854	887	1
	854	
33	854	25
	66	
	194	
	165	
29	33	1
	29	
4	29	7
	28	
1	4	4
	4	

The continued fraction is

$$3 + \frac{1}{7+}\;\frac{1}{15+}\;\frac{1}{1+}\;\frac{1}{25+}\;\frac{1}{1+}\;\frac{1}{7+}\;\frac{1}{4}.$$

And the convergents

$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{9208}{2931},\ \frac{9563}{3044},\ \frac{76149}{24239},\ \frac{314159}{100000}$$

Ex. XII.

Reduce to continued fractions:

1. $\frac{328}{135}$. 2. $\frac{6371}{1395}$. 3. $\frac{7109}{1257}$. 4. $\frac{2173}{1364}$.

5. Why is $\frac{355}{113}$ a nearer approximation to π than $\frac{22}{7}$?

6. Show that $\frac{P_n}{Q_n} = \frac{P_{n+1} - P_{n-1}}{Q_{n+1} - Q_{n-1}}$.

7. If a_{n-1} be the nth incomplete quotient, prove that

$$a_n = \sqrt{\frac{(P_n - P_{n-2})(Q_n - Q_{n-2})}{P_{n-1} \cdot Q_{n-1}}}.$$

8. If $\frac{P_r}{Q_r}$ represents the $(r + 1)$th convergent, show that

$$(P_nQ_{n-1} - Q_nP_{n-1})(P_{n-1}Q_{n-2} - Q_{n-1}P_{n-2}) \ldots \text{ to } n \text{ factors}$$
$$= (-1)^{\frac{1}{2}n(n+1)}.$$

9. If $\frac{P_r}{Q_r}$ represents the $(r + 1)$th convergent to the fraction $\frac{P}{Q}$, show that the $(n + 1)$th complete quotient

$$= \frac{PQ_{n-1} - QP_{n-1}}{QP_n - PQ_n}.$$

10. Prove that $\frac{P_{2n}}{Q_{2n}} - \frac{P_n}{Q_n} =$

$$\frac{1}{Q_{2n}Q_{2n-1}} - \frac{1}{Q_{2n-1}Q_{2n-2}} + \frac{1}{Q_{2n-2}Q_{2n-3}} - \&c. + (-1)^{n-1} \cdot \frac{1}{Q_{n+1}Q_n}.$$

11. If $\frac{p}{q}$ be a convergent to a given quantity x, show that no fraction having a less denominator than q can be a nearer approximation to x than $\frac{p}{q}$.

12. Solve in positive integers the indeterminate equation $3x - 5y = 11$.

Reduction of Quadratic Irrational Quantities to Continued Fractions.

76. Every quadratic equation with rational coefficients may be reduced to the form

$$ax^2 + 2bx + c = 0,$$

where a, b, c are integers, either positive or negative.

Solving this equation, we get

$$x = \frac{\pm\sqrt{b^2 - ac} - b}{a}, \text{ which may be also}$$

written
$$x = \frac{\sqrt{b^2 - ac} \mp b}{\pm a}.$$

If we put $N = b^2 - ac$, we have

$$\frac{N - b^2}{a} = -c, \text{ and } x = \frac{\sqrt{N} \mp b}{\mp a}.$$

Hence, since for the purposes of continued fractions we need consider only the *magnitude* of these roots, and since, too, we may so choose the sign of a that x shall be always positive, we arrive at the following result:

The roots (neglecting the sign) of a quadratic equation having rational coefficients may, when irrational, be always referred to the form

$$x = \frac{\sqrt{N} + p}{q},$$

where

(1.) p and q are integers either positive or negative, the sign of q being so taken that x shall be positive;

(2.) $\frac{N - p^2}{q}$ is an integer.

Continued fractions developed from such irrational quantities are called irrational continued fractions of the second *degree.*

77. *To express* $\frac{\sqrt{N}+p_0}{q_0}$ *as a continued fraction,* p_0 *and* q_0 *being integers, and the sign of* q_0 *being so taken that the expression shall be positive, and where also* $\frac{N-p_0^2}{q_0}$ *is an integer.*

Let a_0 be the greatest integer in $\frac{\sqrt{N}+p_0}{q_0} = x$, suppose.

Then $x = a_0 + \left(\frac{\sqrt{N}+p_0}{q_0} - a_0\right) = a_0 + \frac{\sqrt{N}-(a_0q_0-p_0)}{q_0}$

$$= a_0 + \frac{N-(a_0q_0-p_0)^2}{q_0\{\sqrt{N}+(a_0q_0-p_0)\}}$$

$$= a_0 + \frac{\frac{N-(a_0q_0-p_0)^2}{q_0}}{\sqrt{N}+(a_0q_0-p_0)}$$

$$= a_0 + \frac{q_1}{\sqrt{N}+p_1} \quad\ldots\ldots\ldots\ldots\ldots(1),$$

where $q_1 = \frac{N-(a_0q_0-p_0)^2}{q_0}$,

and $p_1 = a_0q_0 - p_0$.

We then have $x = a_0 + \frac{1}{\frac{\sqrt{N}+p_1}{q_1}} = a_0 + \frac{1}{x_1}\ldots\ldots\ldots(2),$

where $x_1 = \frac{\sqrt{N}+p_1}{q_1}.$

We may notice that x_1 has the *same form* as the given quantity x; and we may therefore in the same manner obtain as complete quotients x_2, x_3, &c., expressed in the *same form* as x.

We may then write $x = \frac{\sqrt{N}+p_0}{q_0} = a_0 + \frac{1}{x_1}$,

$$x_1 = \frac{\sqrt{N}+p_1}{q_1} = a_1 + \frac{1}{x_2},$$

$$x_2 = \frac{\sqrt{N} + p_2}{q_2} = a_2 + \frac{1}{x_3},$$
$$\&c. = \&c.$$
$$x_n = \frac{\sqrt{N} + p_n}{q_n} = a_n + \frac{1}{x_{n+1}},$$
$$\&c. = \&c.$$

where a_0, a_1, a_2, &c., are all positive integers; and where x_1, x_2, x_3, &c., are also positive quantities greater than unity.

We hence have

$$x = a_0 + \frac{1}{a_1 +}\;\frac{1}{a_2 +}\;\frac{1}{a_3 +}\cdots\frac{1}{a_n +}\cdots$$

the required continued fraction.

Ex. 1. Reduce $\frac{\sqrt{17} + 3}{4}$ to a continued fraction.

The expression is positive, p_0 and q_0 are integers, and $\frac{N - p_0^2}{q_0} = \frac{17 - 9}{4} = 4.$

It therefore answers the required conditions.

We have, since *unity* is the greatest integer in the given surd,

$$\frac{\sqrt{17} + 3}{4} = 1 + \left(\frac{\sqrt{17} + 3}{4} - 1\right) = 1 + \frac{\sqrt{17} - 1}{4}$$
$$= 1 + \frac{17 - 1}{4(\sqrt{17} + 1)} = 1 + \frac{4}{\sqrt{17} + 1}$$
$$= 1 + \cfrac{1}{\cfrac{\sqrt{17} + 1}{4}} \cdots\cdots\cdots\cdots (1).$$

$$\frac{\sqrt{17} + 1}{4} = 1 + \left(\frac{\sqrt{17} + 1}{4} - 1\right) = 1 + \frac{\sqrt{17} - 3}{4}$$
$$= 1 + \frac{17 - 9}{4(\sqrt{17} + 3)} = 1 + \frac{2}{\sqrt{17} + 3}$$
$$= 1 + \cfrac{1}{\cfrac{\sqrt{17} + 3}{2}} \cdots\cdots\cdots\cdots (2).$$

$$\frac{\sqrt{17}+3}{2} = 3 + \frac{\sqrt{17}-3}{2} = 3 + \frac{1}{\frac{\sqrt{17}+3}{4}} \text{ .... (3).}$$

We have thus arrived at a quantity $\frac{\sqrt{17}+3}{4}$ exactly the same as that with which we started. The quotients will therefore recur.

We then have, by successive substitution,

$$\frac{\sqrt{17}+3}{4} = 1 + \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\cdots$$

Ex. 2. Reduce $\sqrt{26}$ to a continued fraction.

Here $p_0 = 0$, $q_0 = 1$, and the given expression answers the required conditions.

$$\sqrt{26} = 5 + (\sqrt{26} - 5) = 5 + \frac{26-25}{\sqrt{26}+5} = 5 + \frac{1}{\sqrt{26}+5}.$$

Hence $\sqrt{26} + 5 = 10 + \frac{1}{\sqrt{26}+5}$.

And so, by successive substitutions, we get

$$\sqrt{26} = 5 + \frac{1}{10+}\,\frac{1}{10+}\,\frac{1}{10+}\cdots$$

78. *To show that* (i.) $p_n = a_{n-1}q_{n-1} - p_{n-1}$
(ii.) $q_nq_{n-1} = N - p_n^2$.

We have, in the last Art.

$$x_n = \frac{\sqrt{N}+p_n}{q_n} = a_n + \frac{1}{x_{n+1}} \text{ .... (1).}$$

Putting $n - 1$ for n, then

$$\frac{\sqrt{N}+p_{n-1}}{q_{n-1}} = a_{n-1} + \frac{1}{x_n}\text{; from which}$$

$$x_n = \frac{q_{n-1}}{\sqrt{N} - (a_{n-1}q_{n-1} - p_{n-1})} \text{ .... (2).}$$

(1) = (2), then

$$\frac{\sqrt{N}+p_n}{q_n} = \frac{q_{n-1}}{\sqrt{N} - (a_{n-1}q_{n-1} - p_{n-1})}\text{; or, reducing,}$$

$$N - p_n(a_{n-1}q_{n-1} - p_{n-1}) - (a_{n-1}q_{n-1} - p_{n-1} - p_n)\sqrt{N} = q_n q_{n-1}.$$

Equating the rational and irrational terms, then $(a_{n-1}q_{n-1} - p_{n-1} - p_n)\sqrt{N} = 0$, and $\therefore p_n = a_{n-1}q_{n-1} - p_{n-1}$; and $N - p_n(a_{n-1}q_{n-1} - p_{n-1}) = q_n q_{n-1}$, or

$$q_n q_{n-1} = N - p_n \cdot p_n = N - p_n^2.$$

The expressions (i.) and (ii.) just proved contain the law of formation of the complete quotients, and hold for all positive integral values of n.

79. *The quantities* p_1, p_2, p_3, &c., *and* q_1, q_2, q_3, &c., *are all integers, and moreover, after a certain value of* n, p_n *and* q_n *are positive integers.*

We have (Art. 72) $x_n = \dfrac{a_n P_n + P_{n-1}}{a_n Q_n + Q_{n-1}}$.

Hence also, replacing the incomplete quotient a_n by the complete quotient x_n, we get

$$x = \frac{x_n P_n + P_{n-1}}{x_n Q_n + Q_{n-1}} \quad \text{.......................... (1).}$$

But $x = \dfrac{\sqrt{N} + p_0}{q_0}$, $x_n = \dfrac{\sqrt{N} + p_n}{q_n}$; then we have from (1)

$$\frac{\sqrt{N} + p_0}{q_0} = \frac{\dfrac{\sqrt{N} + p_n}{q_n} \cdot P_n + P_{n-1}}{\dfrac{\sqrt{N} + p_n}{q_n} \cdot Q_n + Q_{n-1}} = \frac{P_n \cdot \sqrt{N} + P_n p_n + P_{n-1} q_n}{Q_n \sqrt{N} + Q_n p_n + Q_{n-1} q_n}.$$

Reducing, then

$$Q_n \cdot N + p_0(Q_n p_n + Q_{n-1} q_n) + (Q_n p_0 + Q_n p_n + Q_{n-1} q_n)\sqrt{N} = P_n q_0 \cdot \sqrt{N} + q_0(P_n p_n + P_{n-1} q_n).$$

Equating the rational and irrational parts, we have

$$Q_n p_0 + Q_n p_n + Q_{n-1} q_n = P_n q_0 \quad \text{............... (2).}$$

$$Q_n \cdot N + p_0(Q_n p_n + Q_{n-1} q_n) = q_0(P_n p_n + P_{n-1} q_n) \ldots \text{ (3).}$$

From (2) we have

$Q_n p_n + Q_{n-1} q_n = P_n q_0 - Q_n p_0$; and from (3)

$$(P_n q_0 - Q_n p_0) p_n + (P_{n-1} q_0 - Q_{n-1} p_0) q_n = Q_n N.$$

Solving for q_n we get

$$(P_n Q_{n-1} - Q_n P_{n-1}) q_0 \cdot q_n = (P_n q_0 - Q_n p_0)^2 - Q_n^2 . N.$$

Hence, since (Art. 70) $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n$, we have

$$q_n = \frac{1}{(-1)^n q_0} \{ (P_n q_0 - Q_n p_0)^2 - Q_n^2 N \};$$

$$\text{but } \frac{1}{(-1)^n} = \frac{(-1)^{2n}}{(-1)^n} = (-1)^n;$$

$$\therefore q_n = \frac{(-1)^n}{q_0} \{ (P_n q_0 - Q_n p_0)^2 - Q_n^2 N \} \quad \dots\dots\dots (4).$$

And so p_n

$$= \frac{(-1)^n}{q_0} \{ Q_{n-1} Q_n N - (P_{n-1} q_0 - Q_{n-1} p_0)(P_n q_0 - Q_n p_0) \} \quad .. (5).$$

Now the expressions in (4) and (5) may be easily expressed thus:

$$q_n = (-1)^n \left\{ P_n^2 q_0 - 2 P_n Q_n p_0 - Q_n^2 . \frac{N - p_0^2}{q_0} \right\}, \text{ and}$$

$$p_n =$$

$$(-1)^n \left\{ Q_{n-1} Q_n . \frac{N - p_0^2}{q_0} - P_{n-1} P_n q_0 - P_n Q_{n-1} p_0 - Q_n P_{n-1} P_0 \right\}.$$

Now $\frac{N - p_0^2}{q_0}$ is by hypothesis integral; and hence, since all the other quantities in these expressions are integers, it follows that p_n and q_n *are integers*. Moreover, after n has reached a certain limit, they are *positive* integers.

For from (2)

$$Q_{n-1} q_n = P_n q_0 - Q_n p_0 - Q_n p_n;$$

$$\therefore \frac{Q_{n-1}}{Q_n} . q_n = \left(\frac{P_n}{Q_n} . q_0 - p_0 \right) - p_n;$$

$$\text{but } x_n = \frac{\sqrt{N} + p_n}{q_n}, \text{ or } q_n = \frac{\sqrt{N} + p_n}{x_n}.$$

Therefore, substituting,

$$\frac{Q_{n-1}}{Q_n} . \frac{\sqrt{N} + p_n}{x_n} = \left(\frac{P_n}{Q_n} q_0 - p_0 \right) - p_n.$$

$$\therefore \frac{Q_{n-1}}{Q_n} = x_n \cdot \frac{\left(\frac{P_n}{Q_n} \cdot q_0 - p_0\right) - p_n}{\sqrt{N} + p_n}.$$

Now $\frac{P_n}{Q_n}$ is the $(n+1)$th convergent to x, or $\frac{\sqrt{N}+p_0}{q_0}$; and it differs the less from x, the greater the value of n (Art. 70).

Hence by taking n large enough $\frac{P_n}{Q_n} \cdot q_0 - p_0$ may be made to differ from $\sqrt{N}$ by as small a quantity as we please.

Hence, when n has attained a certain limit,

$\frac{Q_{n-1}}{Q_n}$ differs from $x_n \cdot \frac{\sqrt{N} - p_n}{\sqrt{N} + p_n}$ by as small a quantity as we please.

Now if p_n is negative, $x_n \cdot \frac{\sqrt{N} - p_n}{\sqrt{N} + p_n}$ is greater than unity,

and therefore $\frac{Q_{n-1}}{Q_n} > 1$, or $Q_{n-1} > Q_n$.

But this is impossible, and therefore p_n cannot be negative.

Hence p_n is a *positive* integer, after n has reached a certain limit.

And so we may show that q_n is a positive integer, after n has reached a certain limit.

80. *The continued fraction is periodic after a certain number of quotients.*

For (Art. 78) we have $p_n + p_{n-1} = a_{n-1}q_{n-1}$ ....(1).

and $q_n q_{n-1} = N - p_n^2$ ....(2).

Now, by the last Art., after a certain limit, p_n and q_n are positive integers.

We shall consider only the portion of the continued fraction beyond this limit.

We have then q_n, q_{n-1}, each positive integers.

Therefore from (2) $p_n < \sqrt{N}$;

and therefore, also, $p_{n-1} < \sqrt{N}$.

Hence from (1) we have $a_{n-1}q_{n-1} < 2\sqrt{N}$.

Now a_{n-1} is a positive integer, and q_{n-1} is such.

It must therefore follow that $a_{n-1} < 2\sqrt{N}$,

and $q_{n-1} < 2\sqrt{N}$.

Or, putting n for $n-1$, we find that for all values of a_n, p_n, q_n beyond a certain limit, we must have

$$p_n < \sqrt{N},\ a_n < 2\sqrt{N},\ q_n < 2\sqrt{N}.$$

Hence the values of a_n, p_n, q_n are limited, and the complete quotient $x_n = \dfrac{\sqrt{N} + p_n}{q_n}$ must have a *finite number of different values,* as must also the incomplete quotient a_n.

Therefore commencing with the first complete quotient in which p_n and q_n are positive, the values of a_n, p_n, q_n will recur, and the number of quotients in the period cannot exceed $2\sqrt{N} \times \sqrt{N}$, or $2N$.

COR. Since from (2) we have $q_n q_{n-1} < N$,
we have also $q_{n-1} q_{n-2} < N$.

Hence, if $q_{n-1} > \sqrt{N}$, we must have $q_n < \sqrt{N}$, and $q_{n-2} < \sqrt{N}$.

Hence, if any denominator of a complete quotient is $> \sqrt{N}$, *the denominators immediately preceding and following are each* $< \sqrt{N}$.

Ex. 1. If $N = a^2 + 1$, find $\sqrt{N}$ in the form of a continued fraction.

$$\sqrt{N} = a + (\sqrt{a^2+1} - a) = a + \frac{1}{\sqrt{a^2+1}+a} = a + \frac{1}{\sqrt{N}+a}.$$

Hence, by successive substitution, we get

$$\sqrt{N} = a + \frac{1}{2a+}\ \frac{1}{2a+\ldots}.$$

Ex. 2. Develop $\dfrac{\sqrt{27}+3}{2}$ as a continued fraction.

$$\frac{\sqrt{27}+3}{2} = 4 + \frac{\sqrt{27}-5}{2} = 4 + \frac{27-25}{2(\sqrt{27}+5)} = 4 + \frac{1}{\sqrt{27}+5} \ldots (1)$$

$$\frac{\sqrt{27}+5}{1} = 10 + \frac{\sqrt{27}-5}{1} = 10 + \frac{27-25}{\sqrt{27}+5} = 10 + \frac{2}{\sqrt{27}+5} \ldots (2)$$

$$\frac{\sqrt{27}+5}{2} = 5 + \frac{\sqrt{27}-5}{2} = 5 + \frac{27-25}{2(\sqrt{27}+5)} = 5 + \frac{1}{\sqrt{27}+5} \ldots (3)$$

Hence, by successive substitutions from (1), (2), (3), we have $\dfrac{\sqrt{27}+3}{2} = 4 + \dfrac{1}{10+}\,\dfrac{1}{5+}\,\dfrac{1}{10+}\,\dfrac{1}{5+\ldots}$.

81. *Every periodic continued fraction is the development of one of the roots of a quadratic equation.*

Suppose the period to commence after the quotient x_{m-1}, and to consist of x' terms.

Then we must have $x_m = x_{m+x'}$ ....(1).

Also we have $$x = \frac{x_m P_m + P_{m-1}}{x_m Q_m + Q_{m-1}} \ldots\ldots (2.)$$

And $$x = \frac{x_{m+x'} P_{m+x'} + P_{m+x'+1}}{x_{m+x'} Q_{m+x'} + Q_{m+x'+1}},$$

which, by (1), $$= \frac{x_m P_{m+x'} + P_{m+x'-1}}{x_m Q_{m+x'} + Q_{m+x'-1}} \ldots\ldots (3).$$

From (2), solving for x_m, we get

$$x_m = \frac{xQ_m - P_m}{P_{m-1} - xQ_{m-1}}.$$

And from (3), solving for x_m, we get

$$x_m = \frac{xQ_{m+x'} - P_{m+x'}}{P_{m+x'-1} - xQ_{m+x'-1}}.$$

Equating these, we have

$$\frac{xQ_m - P_m}{P_{m-1} - xQ_{m-1}} = \frac{xQ_{m+x'} - P_{m+x'}}{P_{m+x'-1} - xQ_{m+x'-1}}$$

from which we get x as one of the roots of a quadratic equation.

Ex. If $x = 1 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{4+\ldots}$, express x as the root of a quadratic equation with integral coefficients.

Put $y = \frac{1}{3+}\,\frac{1}{4+\ldots}$, then we have

$$x = 1 + \frac{1}{2+y} \ldots\ldots (1).$$

and
$$x = 1 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+y} \ldots\ldots (2).$$

From (1) $y = \frac{3-2x}{x-1}$; and from (2) $y = \frac{43-30x}{7x-10}$.

Hence we have $\frac{3-2x}{x-1} = \frac{43-30x}{7x-10}$; or, simplifying,

$16x^2 - 32x + 13 = 0$, from which $x = \frac{4+\sqrt{3}}{4}$.

Ex. XIII.

Develop as continued fractions:

1. $\sqrt{10}$. 2. $\sqrt{17}$. 3. $\sqrt{19}$. 4. $\sqrt{29}$. 5. $\frac{\sqrt{14}+3}{5}$.

6. $\frac{\sqrt{22}+4}{3}$. 7. $\frac{\sqrt{28}+5}{3}$. 8. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

Express as continued fractions the roots of the six following equations:

9. $3x^2 - 8x + 2 = 0$. 10. $5x^2 - 6x - 7 = 0$.

11. $2x^2 - 5x + 1 = 0$. 12. $\frac{3}{x} = \frac{4}{x+2} + 1$.

13. $x + \sqrt{x+2} = 3$. 14. $x^2 - 4ax + 3a^2 = 1$.

15. Find the value of $5 + \frac{1}{10+}\,\frac{1}{10+\ldots}$.

16. Show that the product of the infinite continued fractions

$$\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{2+\ldots\ldots}, \text{ and } 4+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{4+\ldots\ldots},$$

is $1\frac{6}{7}$.

17. Show that

$$\left(\frac{1}{m+}\,\frac{1}{n+}\,\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{m+\ldots\ldots}\right)\left(q+\frac{1}{p+}\,\frac{1}{n+}\,\frac{1}{m+}\,\frac{1}{q+\ldots\ldots}\right)$$

$$=\frac{n+q+npq}{m+p+mnp}.$$

18. Show that

$$\left(a+\frac{1}{2a+}\,\frac{1}{2a+\ldots\ldots}\right)\left(a-\frac{1}{2a-}\,\frac{1}{2a-\ldots\ldots}\right)=\sqrt{a^4-1}.$$

19. Find the value of $2+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+\ldots\ldots}$.

20. Find the first four convergents to the greater roots of the equations in examples 9, 10, 11.

21. Show that the roots of a quadratic which can be expressed as a recurring continued fraction are of contrary or the same sign, according as the recurring period commences with the first quotient or not.

22. When the two roots of a quadratic are developed as a continued fraction, the quotients of the recurring period occur in reverse order.

23. If $\frac{P_{n-2}}{Q_{n-2}}, \frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be successive convergents to the continued fraction

$$\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+\ldots\ldots},$$

show that $\frac{P_n}{Q_n}=\frac{a_nP_{n-2}+b_nP_{n-1}}{a_nQ_{n-2}+b_nQ_{n-1}}$, and find whether the *successive convergents* are necessarily in their lowest terms.

24. If $\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{a+}\ \frac{1}{b+\ldots\ldots}$ be a periodic continued fraction, show that

$$\frac{P_{n+2}+P_{n-2}}{P_n}=\frac{Q_{n+2}+Q_{n-2}}{Q_n}=ab+2.$$

CHAPTER XI.

SERIES.

82. A series is said to be **convergent** or **divergent** according as the sum of n terms has or has not a *finite limit*, when n is increased indefinitely.

Thus, the series $1+\frac{1}{3}+\frac{1}{9}+$ &c. is *convergent*.

For $S_n=\dfrac{1-\frac{1}{3^n}}{1-\frac{1}{3}}=\frac{3}{2}\left(1-\frac{1}{3^n}\right)$.

And hence, when $n=\infty$, $\Sigma=\frac{3}{2}(1-0)=\frac{3}{2}$.

Again, the series $1+3+9+$ &c. is *divergent*.

For $S_n=\dfrac{3^n-1}{3-1}=\frac{1}{2}(3^n-1)$.

And, when $n=\infty$, we have $\frac{1}{2}(3^n-1)=\infty$.

83. *If all the terms of an infinite series are of the same sign, and each term is always greater than some given finite quantity, the series is divergent.*

Let each of the terms be greater than a given quantity k.
Then, the *sum of n terms* $>nk$.

Now, if n be indefinitely increased, nk is greater than any assignable quantity.

Hence there is no limit to the sum of the series when n is infinite, and therefore the series is divergent.

84. *If the term of an infinite series be alternately positive and negative, and the terms continually decrease, the series is convergent.*

Let $a - b + c - d + e - f +$ &c. be the series, where the quantities $a, b, c, d,$ &c., continually decrease.

We then have $\Sigma = (a-b)+(c-d)+(e-f)+\text{\&c.}, \ldots\ (1),$
and $\Sigma = a-(b-c)-(d-e)-\text{\&c.}, \ldots\ (2).$

From (1), since $a > b$, $c > d$, &c., we learn that $\Sigma > a - b$.

And from (2), since $b > c$, $d > e$, &c., we learn that $\Sigma < a$.

Hence Σ lies between a and $a - b$, and therefore is finite; and the series is convergent.

85. *When the terms of an infinite series are such that after some finite number of terms, the ratio of each term to the preceding is numerically less than some proper fraction, the series is convergent.*

Let $a_1 + a_2 + a_3 +$ &c. be the portion of the infinite series, after a finite number of terms, in which the ratio of each term to the preceding is numerically less than a given proper fraction k.

Then we have $a_2 < ka_1$, $a_3 < ka_2$, &c.

Hence $\Sigma < a_1 + ka_1 + k^2a_1 + \text{\&c.} < \dfrac{a_1}{1-k}$.

This portion therefore of the infinite series has a finite limit. Now the sum of the finite number of terms before the terms $a_1, a_2, a_3,$ &c. is necessarily finite. Hence the sum of the whole series is finite, and the series is convergent.

Cor. 1. In the same manner it may be shown that

When the terms of an infinite series are such that after some finite number of terms, the ratio of each term to the preceding is numerically greater than unity, the series is divergent.

COR. 2. So also we may prove that

When after some finite number, the ratio of each term to the preceding is unity, and the terms have all the same sign, the series is divergent.

Case when the Ratio of each Term to the Preceding approaches Unity.

86. *When the ratio of each term to the preceding is less than unity, but still approaches unity, the series* may *be convergent or divergent.*

87. *To show that the series* $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \&c.$ *is convergent when* m *is greater than unity, and divergent when* m *is equal to or less than unity.*

The ratio of the nth to the $(n-1)$th term $= \frac{1}{n^m} \div \frac{1}{(n-1)^m}$

$$= \left(\frac{n-1}{n}\right)^m.$$

This ratio is less than unity, but continually approaches unity as n is indefinitely increased.

(1). Let m be *less than unity.*

Now, $S_n = \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \dots + \frac{1}{n^m}$.

And the *least* of these terms is $\frac{1}{n^m}$.

Hence, $S_n > n$ times $\frac{1}{n^m} > n^{1-m}$.

But since m is less than unity, $1 - m$ is positive, and the value of n^{1-m} increases indefinitely as n is increased.

Hence, as n is indefinitely increased, the sum of n terms becomes greater than any assignable quantity.

The series is therefore divergent.

(2.) Let m be *equal to unity*.

$$\text{Then } \Sigma = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$+ \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \&c.$$

$$\therefore \quad \Sigma > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 8\left(\frac{1}{16}\right) + \&c.$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c.$$

Hence Σ is greater than any assignable quantity, and the series is *divergent.*

(3) Let m be *greater than unity.*

$$\text{We have } \Sigma = 1 + \left(\frac{1}{2^m} + \frac{1}{3^m}\right) + \left(\frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} + \frac{1}{7^m}\right)$$
$$+ \left(\frac{1}{8^m} + \frac{1}{9^m} + \dots + \frac{1}{15^m}\right) + \&c.$$

$$\therefore \Sigma < 1 + 2\left(\frac{1}{2^m}\right) + 4\left(\frac{1}{4^m}\right) + 8\left(\frac{1}{8^m}\right) + \&c.$$
$$< 1 + \frac{1}{2^{m-1}} + \frac{1}{2^{2m-2}} + \frac{1}{2^{3m-3}} + \&c.$$

Now this is a geometric series whose common ratio is $\frac{1}{2^{m-1}}$. It has therefore a finite sum. Hence the given series has also a finite sum, and is convergent.

88. *If two series,* $u_0 + u_1 + u_2 + \dots, v_0 + v_1 + v_2 + \dots,$ *be such that the limit of* $\frac{u_n}{v_n}$ *when* n *becomes infinite is a finite quantity, then the two series are both convergent or both divergent.*

Let $\frac{u_0}{v_0} = k_0, \frac{u_1}{v_1} = k_1, \dots, \frac{u_n}{v_n} = k_n$, &c.

And let k be the limit to which k_n continually approaches *as n is increased* indefinitely.

We have

$$u_n + u_{n+1} + u_{n+1} + \ldots\ldots = k_n v_n + k_{n+1} v_{n+1}$$
$$+ k_{n+2} v_{n+2} + \ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Now, as k is the limit to which k_n continually approaches, all the ratios $k_n, k_{n+1}, \ldots\ldots$, must lie between $k + h$ and $k - h$ where h may be as small a quantity as we please.

Hence, $k_n v_n + k_{n+1} v_{n+1} + k_{n+2} v_{n+2} + \ldots\ldots$ lies between

$$(k + h)(v_n + v_{n+1} + v_{n+2} + \ldots\ldots) \text{ and}$$
$$(k - h)(v_n + v_{n+1} + v_{n+2} + \ldots\ldots),$$

where h is as small a quantity as we please.

Hence, from (1), $u_n + u_{n+1} + u_{n+2} + \ldots\ldots$ lies between

$$(k + h)(v_n + v_{n+1} + v_{n+2} + \ldots\ldots) \text{ and}$$
$$(k - h)(v_n + v_{n+1} + v_{n+2} + \ldots\ldots),$$

where h is as small as we please.

Hence the series $u_n + u_{n+1} + u_{n+2} + \ldots\ldots$ will have a finite limit according as the series $v_n + v_{n+1} + v_{n+2} + \ldots\ldots$ has a finite limit.

The two series, $u_n + u_{n+1} + u_{n+2} + \ldots\ldots$, and $v_n + v_{n+1} + v_{n+2} + \ldots\ldots$, are therefore both convergent or both divergent.

And since the finite number of terms preceding u_n and v_n cannot affect this conclusion, it follows that the given series are either both convergent or both divergent.

89. *The series* $\phi(1) + \phi(2) + \phi(3) + \phi(4) + \&c.$, *and*

$$\phi(1) + m\,\phi(m) + m^2\,\phi(m^2) + m^3\,\phi(m^3) + \&c.,$$

are convergent and divergent together, when $\phi(x)$ *is positive, and continually diminishes as* x *increases,* m *being a positive integer.*

By extending the first series we may write it thus

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \&c.$$
$$= \phi(1)$$
$$+ \{\phi(2) + \phi(3) + \ldots\ldots + \phi(m)\}$$
$$+ \{\phi(m + 1) + \phi(m + 2) + \ldots\ldots \phi(m^2)\}$$
$$+ \{\phi(m^2 + 1) + \phi(m^2 + 2) + \ldots\ldots \phi(m^3)\} + \&c.\ldots\ldots (1).$$

But since the quantities $\phi(1)$, $\phi(2)$, $\phi(3)$, &c. are positive, and continually diminish, we have

$$\phi(2) + \phi(3) + \dots + \phi(m) > (m-1)\,\phi(m), \text{ and } < (m-1)\,\phi(1),$$

$$\phi(m+1) + \phi(m+2) + \dots + \phi(m^2) > (m^2 - m)\,\phi(m^2), \text{ and } < (m^2 - m)\,\phi(m),$$

$$\phi(m^2+1) + \phi(m^2+2) + \dots + \phi(m^3) > (m^3 - m^2)\,\phi(m^3), \text{ and } < (m^3 - m^2)\,\phi(m^2),$$

&c., &c.

Hence, adding these inequalities, we have, by (1),

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{\&c.}$$
$$> \phi(1) + (m-1)\left\{\phi(m) + m\phi(m^2) + m^2\phi(m^3) + \text{\&c.}\right\},$$
$$\text{and} > \phi(1) + (m-1)\left\{\phi(1) + m\phi(m) + m^2\phi(m^2) + \text{\&c.}\right\}.$$

Or, arranging, we have

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{\&c.}$$
$$> \frac{1}{m}\phi(1)$$
$$+ \frac{m-1}{m}\left\{\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \text{\&c.}\right\} \dots (2).$$
$$\text{And} < \phi(1) + (m-1)\left\{\phi(1) + m\phi(m) + m^2\phi(m^2) + \text{\&c.}\right\} \dots (3).$$

From (2), we learn that when the second series is divergent, the first is divergent; and from (3), that the first is convergent when the second is convergent.

Ex. The series $u_1 + u_2 + u_3 + \dots$ is convergent when the limit of $\frac{-\log u_n}{\log n}$ is greater than unity, and divergent when the limit is less than, or equal to, unity.

By what has just been proved,

The two series $u_1 + u_2 + u_3 + \dots$, and $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \dots$ are both convergent or both divergent, if the limit, *when n is infinite*, of the ratio $u_n : \frac{1}{n^m}$ is a finite quantity.

Let k be this limit.

Then, when n is infinite, we have

$$u_n \cdot n^m = k, \text{ or } \log u_n + m \log n = \log k;$$

$$\text{or } \frac{-\log u_n}{\log n} = m - \frac{\log k}{\log n};$$

or, since the limit of $\frac{\log k}{\log n} = 0$, the limit of $\frac{-\log u_n}{\log n} = m$.

Now, by Art. 86, the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \ldots$ is convergent when m is greater than unity, and divergent when m is less than, or equal to, unity.

Hence, since m is the limit of $\frac{-\log u_n}{\log n}$, and in that case the convergency or divergency of the series $u_1 + u_2 + u_3 + \ldots$ depends upon that of the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} +$ &c., it follows that the series $u_1 + u_2 + u_3 + \ldots$ is convergent when the limit of $\frac{-\log u_n}{\log n}$ is greater than unity, and divergent when equal to, or less than unity.

Ex. XIV.

Test the convergency or divergency of the following series:—

1. $\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \ldots$
2. $\frac{1}{\lfloor 2} + \frac{2}{\lfloor 3} + \frac{3}{\lfloor 4} + \ldots$
3. $\frac{1}{\lfloor 2} + \frac{1 \cdot 3}{\lfloor 3} + \frac{1 \cdot 3 \cdot 5}{\lfloor 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{\lfloor 5} + \ldots$
4. $\frac{a}{\lfloor a+1} + \frac{2a^2}{\lfloor a+2} + \frac{3a^3}{\lfloor a+3} + \ldots$
5. $\frac{a}{a+b} + \frac{a^2}{(a+b)(a+2b)} + \frac{a^3}{(a+b)(a+2b)(a+3b)} + \ldots$

6. $\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \ldots$

7. $1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots$

8. $1 + \frac{2n(2n-1)}{1 \cdot 2} + \frac{2n(2n-1)(2n-2)(2n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$

9. $\frac{x}{2} + \frac{2^3x^2}{2^2+1} + \frac{3^3x^3}{3^2+1} + \ldots + \frac{n^3x^n}{n^2+1} + \ldots$

10. $1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \ldots$

11. Show that the series whose nth term is $\phi(n)x^n$ is convergent or divergent according as the limit of $\frac{\phi(n)}{\phi(n-1)}$, when n is infinite, is less or greater than unity.

If the limit be negative, or unity, what is the test you apply?

12. Show that the $\frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots$ is convergent, and that the error in taking the first n terms for the series is less than $\frac{1}{(n-1)\,\underline{|n-1}}$.

Recurring Series.

90. The series $a_0 + a_1x + a_2x + \ldots$ is called **a recurring series**, when the relation existing between every set of (r) consecutive terms can be expressed by the equation

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \ldots + p_{r-1}a_{n-r+1} = 0;$$

where $p_1, p_2, p_3, \ldots p_{r-1}$ are constant quantities, and where n may have any integral value.

It is usual to call $1 + p_1x + p_2x^2 + \ldots + p_{r-1}x^{r-1}$ *the scale of relation*.

A simple case is an ordinary geometric series.

Thus, in the series $1 + 2x + 4x^2 + 8x^3 + \dots$, we have

$$a_0 = 1,\ a_1 = 2,\ a_2 = 4, \text{ \&c., } a_n = 2^n, \text{ \&c.}$$

Hence $\dfrac{a_n}{a_{n-1}} = \dfrac{2^n}{2^{n-1}} = 2$, or $a_n - 2a_{n-1} = 0$.

This equation expresses generally the relation existing between any *two* consecutive terms.

So we have $a_1 - 2a_0 = 0$, $a_2 - 2a_1 = 0$, $a_3 - 2a_2 = 0$, &c. And *the scale of relation* of the series is $1 - 2x$.

The following examples will illustrate the method of finding the scale of relation of a given series.

Ex. 1. Given the series $2 + 2x - 4x^2 - 22x^3 - 46x^4 -$ &c., find the scale of relation.

We can see that the given series is not an ordinary G. P., and therefore its scale of relation will consist of more than *two* terms.

Assume it to be $1 + p_1x + p_2x^2$.

Then we have

$$-4 + 2p_1 + 2p_2 = 0 \dots\dots\dots\dots (1),$$
$$-22 - 4p_1 + 2p_2 = 0 \dots\dots\dots\dots (2),$$
$$-46 - 22p_1 - 4p_2 = 0 \dots\dots\dots\dots (3).$$

From (1) and (2) we get $p_1 = -3$, $p_2 = 5$, and these values also satisfy (3). Hence we conclude that our *assumption* is correct, and that the scale of relation is $1 - 3x + 5x^2$.

We *might have found* that the value of p_1 and p_2 in (1) and (2) would not satisfy (3), and we must then have made another assumption.

For instance, we should have assumed the scale of relation to be $1 + p_1x + p_2x^2 + p_3x^3$.

In that case we must have had given us another term in the series; for we should have required *three* equations to determine the values of p_1, p_2, p_3, and a *fourth* to verify our results. In the next example we shall assume the scale of relation to *consist of four* terms, as on trial it would be *found* $1 + p_1x + p_2x^2$ would not answer the conditions.

Ex. 2. Find the scale of relation of the series

$$1 + 10x + 34x^2 + 67x^3 + 115x^4 + 247x^5 + 716x^6 + \&c.$$

Assume the scale of relation to be $1 + p_1x + p_2x^2 + p_3x^3$.

Then
$$\begin{aligned} 67 + 34p_1 + 10p_2 + p_3 &= 0. \\ 115 + 67p_1 + 34p_2 + 10p_3 &= 0. \\ 247 + 115p_1 + 67p_2 + 34p_3 &= 0. \\ 716 + 247p_1 + 115p_2 + 67p_3 &= 0. \end{aligned}$$

From the first three equations we find $p_1 = -3$, $p_2 = 4$, $p_3 = -5$; and these values satisfy the *fourth* equation.

Hence the scale of relation is $1 - 3x + 4x^2 - 5x^3$.

91. *To find the sum of a recurring series.*

Let $a_0 + a_1x + a_2x^2 + \ldots.$ be the series, whose scale of relation is $1 + p_1x + p_2x^2 + \ldots. + p_{r-1}x^{r-1}$.

We have then

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \ldots. + p_{r-1}a_{n-r+1} = 0 \ldots\ldots (1),$$

where n has any positive integral value.

Let $\Sigma = a_0 + a_1x + a_2x^2 + \ldots. + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + \ldots.$

Multiplying each side of the equation by $1 + p_1x + p_2x^2 + \ldots. p_{r-1}x^{r-1}$, we have

$$\begin{aligned} &(1 + p_1x + p_2x^2 + \ldots. + p_{r-1}x^{r-1})\Sigma \\ &= a_0 + a_1x + a_2x^2 + \ldots. + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + \ldots. \\ &\quad + p_1a_0x + p_1a_1x^2 + \ldots. + p_1a_{r-3}x^{r-2} + p_1a_{r-2}x^{r-1} + \ldots. \\ &\quad + p_2a_0x^2 + \ldots. + p_2a_{r-4}x^{r-2} + p_2a_{r-3}x^{r-1} + \ldots. \\ &\quad + \ldots\ldots\ldots\ldots\ldots \\ &\quad + p_{r-2}a_0x^{r-2} + p_{r-2}a_1x^{r-1} + \ldots. \\ &\quad + p_{r-1}a_0x^{r-1} + \ldots. \end{aligned}$$

Collecting the coefficients of like powers of x, and remembering that the coefficient of x^{r-1}, viz.,

$$a_{r-1} + p_1a_{r-2} + p_2a_{r-3} + \ldots. + p_{r-2}a_1 + p_{r-1}a_0,$$

and all succeeding coefficients are each equal to zero, as we *learn from (1)* above, we get

$$(1 + p_1x + p_2x^2 + \ldots . + p_{r-1}x^{r-1})\,\Sigma$$

$$= a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2 + \ldots . + (a_{r-2} + p_1a_{r-3} + \ldots . + p_{r-2}a_0)x^{r-2}$$

$$\therefore \Sigma = \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2 + \ldots . + (a_{r-2} + p_1a_{r-3} + \ldots . + p_{r-2}a_0)x^{r-2}}{1 + p_1x + p_2x^2 + \ldots . + p_{r-1}x^{r-1}} \quad \ldots\ldots\ldots (A).$$

Cor. When $r = 3$, or the scale of relation is $1 + p_1x + p_2x^2$, we have

$$\Sigma = \frac{a_0 + (a_1 + p_1a_0)x}{1 + p_1x + p_2x^2} \quad \ldots\ldots\ldots (2).$$

And so, when $r = 4$, or the scale of relation is $1 + p_1x + p_2x^2 + p_3x^3$,

$$\Sigma = \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2}{1 + p_1x + p_2x^2 + p_3x^3} \quad \ldots\ldots\ldots (3).$$

Ex. 1. Sum the infinite series $2 + 2x - 4x^2 - 22x^3 - 46x^4 -$ &c.

By Art. 90, Ex. 1, the scale of relation is $1 - 3x + 5x^2$.

Hence from (3) above, remembering that a_0, a_1, &c., represent the coefficients of the given series, and that p_1, p_2 are the coefficients of the powers of x in the scale of relation, we have

$$\Sigma = \frac{2 + \{2 + (-3)\,2\}\,x}{1 - 3x + 5x^2} = \frac{2 - 4x}{1 - 3x + 5x^2}.$$

Ex. 2. Find the sum of the infinite series

$$1 + 10x + 34x^2 + 67x^3 + 115x^4 + 247x^5 + 716x^6 + \&c.$$

We found the scale of relation to be $1 - 3x + 4x^2 - 5x^3$.

Hence, by (3) above,

$$\Sigma = \frac{1 + \{10 - 3(1)\}x + \{34 - 3(10) + 4(1)\}x^2}{1 - 3x + 4x^2 - 5x^3}$$

$$= \frac{1 + 7x + 8x^2}{1 - 3x + 4x^2 - 5x^3}.$$

Since, by the process of long division, or otherwise, the expression found in (A) above will give the original series, we give to the value of Σ there found the name of the *generating function* of the series.

92. *When the scale of relation can be broken up into simple factors, it is easy to find the general term of the series and the sum to* n *terms.*

Let the scale of relation be capable of being broken up into the $(r - 1)$ factors, $(1 - b_1x)$, $(1 - b_2x)$, &c.

Then the generating function can be resolved into partial fractions having these factors for denominators.

Thus we have

$$\Sigma = \frac{P_1}{1 - b_1x} + \frac{P_2}{1 - b_2x} + \dots + \frac{P_{r-1}}{1 - b_{r-1}x}$$

$$= P_1(1 + b_1x + b_1^2x^2 + \dots + b_1^{n-1}x^{n-1} + \dots)$$

$$+ P_2(1 + b_2x + b_2^2x^2 + \dots + b_2^{n-1}x^{n-1} + \dots)$$

$$+ \&c.$$

$$+ P_{r-1}(1 + b_{r-1}x + b_{r-1}^2x^2 + \dots + b_{r-1}^{n-1}x^{n-1} + \dots)$$

We thus find the nth term of the series to be

$$(P_1b_1^{n-1} + P_2b_2^{n-1} + \dots + P_{r-1}b_{r-1}^{n-1})x^{n-1}.$$

Also, the sum to n terms,

$$= P_1 \cdot \frac{1 - (b_1x)^n}{1 - b_1x} + P_2 \cdot \frac{1 - (b_2x)^n}{1 - b_2x} + \dots$$

$$+ P_{r-1} \cdot \frac{1 - (b_{r-1}x)^n}{1 - b_{r-1}x}.$$

Ex. Find the sum to n terms and to infinity of the series

$$8 - 29x + 257x^2 - 1691x^3 + 12053x^4 - \&c.$$

The scale of relation is easily found to be $1 + 5x - 14x^2$.

$$\therefore \Sigma = \frac{8 + \{-29 + 5(8)\}x}{1 + 5x - 14x^2} = \frac{8 + 11x}{(1 - 2x)(1 + 7x)}.$$

Resolving into partial fractions, we have

$$\Sigma = \frac{3}{1 - 2x} + \frac{5}{1 + 7x}$$

$$= 3\{1 + 2x + (2x)^2 + \dots + (2x)^{n-1} + \dots\}$$

$$+ 5\{1 - 7x + (7x)^2 + \dots + (-7x)^{n-1} + \dots\}.$$

Therefore sum to n terms, or,

$$S = 3 \cdot \frac{1 - (2x)^n}{1 - 2x} + 5 \cdot \frac{1 - (-7x)^n}{1 + 7x}.$$

When the given recurring term does not contain the powers of x, but is of the form $a_0 + a_1 + a_2 + \dots$, we may first find the scale of relation and the sum of the series $a_0 + a_1x + a_2x^2 + \dots$, and put $x = 1$ in the result.

Summation of Series.

93. Certain series may be summed by particular artifices.

In the series $u_1 + u_2 + \dots + u_n$, suppose u_n to be broken up into the difference of two terms, one of which is the same function of $(n - 1)$ as the other is of n.

I. Thus, suppose $u_n = U_n - U_{n-1}$.

Then, putting for n all values from 1 to n, we have

$$S = (U_1 - U_0) + (U_2 - U_1) + \dots + (U_n - U_{n-1})$$
$$= U_n - U_0.$$

II. Suppose $u_n = U_{n-1} - U_n$.

We have, similarly,

$$S = (U_0 - U_1) + (U_1 - U_2) + \dots + (U_{n-1} - U_n)$$
$$= U_0 - U_n.$$

Ex. 1. Sum the series

$$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \dots + n(n+1).$$

We have

$$u_n = n(n+1) = \tfrac{1}{3} n(n+1) \{(n+2) - (n-1)\}$$
$$= \tfrac{1}{3} n(n+1)(n+2) - \tfrac{1}{3}(n-1) n (n+1)$$
$$= U_n - U_{n-1},$$

where $\quad U_n = \tfrac{1}{3} n(n+1)(n+2),$

and therefore $\quad U_0 = \tfrac{1}{3}(0)(1)(2) = 0.$

Hence, S or $U_n - U_0 = \tfrac{1}{3} n(n+1)(n+2) - 0.$

$$\therefore S = \tfrac{1}{3} n(n+1)(n+2).$$

Ex. 2. Sum the series

$$\frac{1}{1 \,.\, 3} + \frac{1}{3 \,.\, 5} + \frac{1}{5 \,.\, 7} + \dots \frac{1}{(2n-1)(2n+1)}.$$

Here $u_n = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2} \cdot \dfrac{(2n+1) - (2n-1)}{(2n-1)(2n+1)}$

$$= \frac{1}{2} \cdot \frac{1}{2n-1} - \frac{1}{2} \cdot \frac{1}{2n+1} = U_{n-1} - U_n,$$

where $U_n = \dfrac{1}{2} \cdot \dfrac{1}{2n+1}$, and $\therefore U_0 = \dfrac{1}{2} \cdot \dfrac{1}{2(0)+1} = \dfrac{1}{2}.$

Hence, by II. above

$$S = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2n+1} = \frac{n}{2n+1}.$$

94. *When the* n*th term of a series is of the form*

$$u_n = (an + b)(a \,.\, \overline{n+1} + b) \dots (a \,.\, \overline{n+m-1} + b),$$

where a, b *are constants, and* m *the number of factors, then* $S = U_n - U_0$, *where*

$$U_n = \frac{(an + b)(a . \overline{n+1} + b) \dots (a . \overline{n+m} + b)}{(m+1)a};$$

that is, U_n *is found by taking one factor more, and dividing by as many times the common difference as there are then factors.*

$$\text{For } u_n = \frac{u_n}{(m+1)a}\left\{(a . \overline{n+m} + b) - (a . \overline{n-1} + b)\right\}$$

$$= \frac{u_n(a . \overline{n+m} + b)}{(m+1)a} - \frac{(a . \overline{n-1} + b)u_n}{(m+1)a}$$

$$= \frac{(an + b)(a . \overline{n+1} + b) \dots (a . \overline{n+m} + b)}{(m+1)a}$$

$$- \frac{(a . \overline{n-1} + b)(an + b) \dots (a . \overline{n+m-1} + b)}{(m+1)a}$$

$$= U_n - U_{n-1},$$

$$\text{where } U_n = \frac{(an+b)(a . \overline{n+1}+b) \dots (a . \overline{n+m}+b)}{(m+1)a} \dots (1).$$

Hence, by Art. 93, $S = U_n - U_0$, where U_n has the value in (1).

Ex. 1. Sum the series

$3 . 5 . 7 + 5 . 7 . 9 + \dots + (2n + 1)(2n + 3)(2n + 5)$.

We have $u_n = (2n + 1)(2n + 3)(2n + 5)$.

Here the common difference is 2, and with one more factor, the number of factors will be 4.

$$\text{Hence } U_n = \frac{(2n + 1)(2n + 3)(2n + 5)(2n + 7)}{4(2)}$$

$$= \frac{1}{8}(2n + 1)(2n + 3)(2n + 5)(2n + 7).$$

$$\text{Therefore } U_0 = \frac{1}{8}(1)(3)(5)(7) = \frac{105}{8}.$$

$$\therefore\ S = \frac{1}{8}\left\{(2n+1)(2n+3)(2n+5)(2n+7) - 105\right\}$$

$$= 2n^4 + 16n^3 + 43n^2 + 44n.$$

When u_n consists of the *sum of several terms* of the form in Art. 94, it is evident that U_n must also consist of the sum of an equal number of terms, each obtained according to the law there proved. The next example is an illustration.

Ex. 2. Sum the series $1^3 + 2^3 + 3^3 + \ldots + n^3$.

We have $u_n = n^3 = n\{(n-1)(n+1)+1\}$

$$= (n-1)n(n+1) + n.$$

Hence $$U_n = \frac{(n-1)n(n+1)(n+2)}{4(1)} + \frac{n(n+1)}{2(1)}$$

$$= \frac{1}{4}n^2(n+1)^2.$$

Therefore $U_0 = 0$.

Hence S or $U_n - U_0 = \frac{1}{4}n^2(n+1)^2 = \left\{\frac{1}{2}n(n+1)\right\}^2$.

95. *When the* n*th term of a series is of the form*

$$u_n = \frac{1}{(an+b)(a.\overline{n+1}+b)\ldots(a.\overline{n+m-1}+b)},$$

where a, b *are constants, and* m *the number of factors, then* $S = U_0 - U_n$, *where*

$$U_n = \frac{1}{(m-1)a.(a.\overline{n+1}+b)(a.\overline{n+2}.b)\ldots(a.\overline{n+m-1}.b)};$$

that is, where U_n *is found by omitting the first factor, and dividing by as many times the common difference as there are factors remaining.*

For u_n

$$= \frac{1}{(m-1)a}\cdot\frac{(a.\overline{n+m-1}+b)-(an+b)}{(an+b)(a.\overline{n+1}+b)\ldots(a.\overline{n+m-1}+b)}$$

$$= \frac{1}{(m-1)a}\left\{\frac{1}{(an+b)(a.\overline{n+1}+b)\ldots.(a.\overline{n+m-2}+b)} - \frac{1}{(a.\overline{n+1}+b)\ldots.(a.\overline{n+m-1}+b)}\right\}$$

$$= U_{n-1} - U_n,$$

where $$U_n = \frac{1}{(m-1)\,a\,.\,(a\,.\,\overline{n+1}+b)\ldots.(a\,.\,\overline{n+m-1}+b)} \qquad (1).$$

Hence, by Art. 93, $S = U_0 - U_n$, where U_n has the value in (1).

Ex. 1. Sum the series $\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \ldots. + \frac{1}{n(n+1)(n+2)}$.

Here $u_n = \frac{1}{n(n+1)(n+2)}$.

Therefore $U_n = \frac{1}{2(1)(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)}$.

Hence also, $U_0 = \frac{1}{2(1)(2)} = \frac{1}{4}$.

Therefore $U_0 - U_n$, or $S = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$.

Hence also, since the limit of $\frac{1}{2(n+1)(n+2)}$ is 0, when $n = \infty$, we have $\Sigma = \frac{1}{4}$.

Ex. 2. Sum the series $\frac{1}{1\,.\,2\,.\,3} + \frac{3}{2\,.\,3\,.\,4} + \ldots. + \frac{2n-1}{n(n+1)(n+2)}$.

$$\text{Here } u_n = \frac{2n - 1}{n(n+1)(n+2)}$$

$$= \frac{2}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}.$$

$$\text{Therefore } U_n = \frac{2}{1(1)(n+2)} - \frac{1}{2(1).(n+1)(n+2)}$$

$$= \frac{2}{n+2} - \frac{1}{2(n+1)(n+2)}$$

$$= \frac{4n+3}{2(n+1)(n+2)}.$$

$$\text{Hence } U_0 = \frac{3}{2(1)(2)} = \frac{3}{4}.$$

$$\text{Therefore } U_0 - U_n, \text{ or } S = \frac{3}{4} - \frac{4n+3}{2(n+1)(n+2)}.$$

Hence also, since $\dfrac{4n+3}{2(n+1)(n+2)} = \dfrac{4 + \frac{3}{n}}{2(n+1)(1+\frac{2}{n})}$

$= 0$, when $n = \infty$, we have $\Sigma = \dfrac{3}{4}$.

Figurate Numbers.

96. We give the name **Figurate Numbers** to those of the following series, where the nth term of any series is the sum of n terms of the preceding series :—

1st order,	1, 1, 1, 1, &c.
2nd ,,	1, 2, 3, 4, &c.
3rd ,,	1, 3, 6, 10, &c.
	&c.

For the first order, we have $S_n = n$.

And for the second, $S_n = \frac{1}{2} n(n+1)$.

This is the nth term of the 3rd order.

Hence, to find the sum of n terms of the third order, we have

$$u_n = \tfrac{1}{2} n(n+1).$$

Therefore (Art. 94),

$$U_n = \tfrac{1}{2}\cdot\frac{n(n+1)(n+2)}{3(1)} = \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}.$$

And therefore $U_0 = 0$.

Hence $U_n - U_0$ or $S_n = \dfrac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$.

To find S_n for the rth order,

Assume that for the pth order, $S_n = \dfrac{n(n+1)\ldots\ldots(n+p-1)}{1\,.\,2\ldots\ldots p}$.

Then for the $(p+1)$th order, $u_n = \dfrac{n(n+1)\ldots\ldots(n+p-1)}{1\,.\,2\ldots\ldots p}$.

Therefore

$$U_n = \frac{n(n+1)\ldots\ldots(n+p)}{(p+1)\,.\,1\,.\,2\ldots\ldots p} = \frac{n(n+1)\ldots\ldots(n+\overline{p+1}-1)}{1\,.\,2\ldots\ldots\overline{p+1}}.$$

and $U_0 = 0$.

Therefore $U_n - U_0$ or $S_n = \dfrac{n(n+1)\ldots\ldots(n+\overline{p+1}-1)}{1\,.\,2\ldots\ldots\overline{p+1}}$.

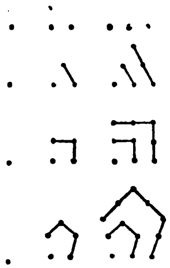
Hence, by assuming S_n to be formed according to a certain law for the pth order, we have proved it to be formed according to the *same law* for the $(p+1)$th order. Now, we know it is true for the 2nd and 3rd orders. It is therefore true for the 4th, and therefore for the 5th, and so generally true.

Hence for the rth order,

$$S_n = \frac{n(n+1)\ldots\ldots(n+r-1)}{1\,.\,2\ldots\ldots r}.$$

Polygonal Numbers.

97. These numbers are called **linear, triangular, square, pentagonal,** and generally **polygonal,** from the fact that if a dot be *taken to represent* unity, they may all be represented *in the form of the* corresponding polygons. Thus—



Linear Numbers, &c., or 1, 2, 3, 4, &c.

Triangular „ &c., or 1, 3, 6, 10, &c.

Square „ &c., or 1, 4, 9, 16, &c.

Pentagonal „ &c., or 1, 5, 12, 22, &c.

The *first order* of polygonal numbers is the series in which each term is *unity*.

And hence, by observing the above diagrams, it is easy to see that the several orders may be arranged as follows :—

1st order,	1,	,	1 ,	1	, &c.
2nd „	1,	1 + 1,	1 + 1 + 1,	1 + 1 + 1 + 1,	&c.
3rd „	1,	1 + 2,	1 + 2 + 3,	1 + 2 + 3 + 4,	&c.
4th „	1,	1 + 3,	1 + 3 + 5,	1 + 3 + 5 + 7,	&c.
5th „	1,	1 + 4,	1 + 4 + 7,	1 + 4 + 7 + 10,	&c.
&c.					

The law is evident. The nth number of each order after the first is the sum of n terms of an A. P., whose first term is unity; and the common difference in the rth order is $r - 2$.

Hence the nth term of the rth order

$$= 1 + (1 + \overline{r-2}) + (1 + 2 \cdot \overline{r-2}) + \text{\&c., to } n \text{ terms}$$

$$= \left\{2 + \overline{n-1} \cdot (r-2)\right\} \frac{n}{2}$$

$$= n + \tfrac{1}{2} n (n-1)(r-2).$$

Hence, for the rth order, we have u_n

$$= n + \tfrac{1}{2}(r-2) \cdot (n-1) n.$$

Hence, $U_n = \dfrac{n(n+1)}{2} + \tfrac{1}{2}(r-2) \cdot \dfrac{(n-1)\,n(n+1)}{3}$,

and $\therefore\ U_0 = 0.$

Hence (Art. 93) for the rth order,

$$S_n = \frac{n}{6}(n+1)\left\{(r-2)(n-1) + 3\right\}.$$

COR. Let $r = 3, 4, 5$, successively, then,

For triangular numbers, $S_n = \frac{1}{6} n(n + 1)(n + 2)$.

„ square „ $S_n = \frac{1}{6} n(n + 1)(2n + 1)$.

„ pentagonal „ $S_n = \frac{1}{6} n(n + 1)(3n - 3 + 3)$

$= \frac{1}{2} n^2(n + 1)$.

98. *To find the number of cannon balls in a pyramidal heap.*

(1.) When the base is an *equilateral triangle.*

Let n be the number of balls in a side of the base.

Then, number of balls in lowest layer

$$= 1 + 2 + 3 + \ldots + n = \tfrac{1}{2} n(n + 1),$$

the nth triangular number.

Hence, by Art. 93, $S_n = \frac{1}{6} n(n + 1)(n + 2)$.

(2.) When the base is a *square.*

We have, similarly, $S_n = \frac{1}{6} n(n + 1)(2n + 1)$.

(3.) When the pile is *deficient*, the first m courses being removed.

Here, if S_n = the number in the *complete* pile,
and S_m = the number in the removed courses,
$S_n - S_m$ = the number in the incomplete pile.

99. *To find the number of balls when the base is rectangular, and not square.*

The pile will now terminate in a single row at the top.

Let l be the length of the top row.

Then, the nth row, reckoned from the top, will contain $l + n - 1$ balls, and its breadth will be n.

Then, number of balls in the lowest or nth layer

$$= (l + n - 1)\, n.$$

Hence $u_n = (l + n - 1)n = l \,.\, n + (n - 1)n$.

Therefore (Art. 94)

$$U_n = l \,.\, \frac{n(n + 1)}{2} + \frac{(n - 1)n(n + 1)}{3}, \text{ or, since } U_0 = 0,$$

$$S_n = l \cdot \frac{n(n+1)}{2} + \frac{(n-1)\,n(n+1)}{3}$$
$$= \tfrac{1}{6}\,n(n+1)\,\{3\,l + 2(n-1)\}$$
$$= \tfrac{1}{6}\,n(n+1)\,\{3\,(l+n-1) - n + 1\}$$
$$= \tfrac{1}{6}\,n(n+1)\,(3\,l_1 - n + 1),$$

when l_1 is the length, and n the breadth, of the lowest row.

Method of Differences.

100. Suppose we have to find the sum of the series

3, 6, 11, 18, 27, 38, &c.

Taking the differences of each two consecutive terms, and then the differences of each two of these differences, and so on, until the differences become a series of identical terms, we have:

Series, ...	3, 6, 11, 18, 27, 38, &c.
1st order of differences, .	3, 5, 7, 9, 11, &c.
2nd order of differences, .	2, 2, 2, 2, &c.

It will be seen that *the* n*th term of any of these series is the sum of its* 1st term, *and the first* (n − 1) *terms of the succeeding series*.. (A).

Thus, in the 1st order,

$$u_n = 3 + 2(n-1).$$

Therefore, by Art. 94, $U_n = 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)\,n}{2 \cdot 1}$

$$= 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)n}{1 \cdot 2}; \text{ or, since } U_0 = 0$$

$$S_n = 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)\,n}{1 \cdot 2}.$$

$$\therefore\ S_{n-1} = 3 \cdot \frac{n-1}{1} + 2 \cdot \frac{(n-2)\,(n-1)}{1 \cdot 2}.$$

Hence, by (A) above, in the given series,

$$u_n = 3 + 3 \cdot \frac{n-1}{1} + 2 \cdot \frac{(n-2)\,(n-1)}{1 \cdot 2}.$$

$$\therefore U_n = 3 \cdot \frac{n}{1} + 3 \cdot \frac{(n-1)n}{1 \cdot 2} + 2 \cdot \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3};$$

or, since $U_0 = 0$,

$$S_n = 3n + 3 \cdot \frac{n(n-1)}{1 \cdot 2} + 2 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

$$= \frac{n}{6}(n^2 + 3n + 13).$$

We shall in the next article give the general theorem.

101. *If* u_1, u_2, u_3, *&c. be a series, and* d_1, d_2, d_3, *&c. the first terms of the successive order of differences, the differences in the* r*th order being identical, then we have*

$$S_n = nu_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \ldots .$$
$$+ \frac{n(n-1) \ldots . (n-r)}{\lfloor r+1} d_r.$$

In the $(r-1)$th order, we have $u_n = d_{r-1} + (n-1)d_r.$
Therefore $U_n = \frac{n}{1} d_{r-1} + \frac{(n-1)n}{1 \cdot 2} d_r$; and therefore, since

$$U_0 = 0, \; S_n = \frac{n}{1} d_{r-1} + \frac{(n-1)n}{1 \cdot 2} d_r.$$

Therefore $S_{n-1} = \frac{n-1}{1} d_{r-1} + \frac{(n-2)(n-1)}{1 \cdot 2} d_r.$

In the $(r-2)$th order, we therefore have

$$u_n = d_{r-2} + \frac{n-1}{1} d_{r-1} + \frac{(n-2)(n-1)}{1 \cdot 2} d_r.$$

Therefore $U_n = \frac{n}{1} d_{r-2} + \frac{(n-1)n}{1 \cdot 2} d_{r-1} + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_r;$

and therefore, since $U_0 = 0$,

$$S_n = \frac{n}{1} d_{r-2} + \frac{(n-1)n}{1 \cdot 2} d_{r-1} + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_r.$$

We may easily show therefore by induction that—

For the $(r - p)$th order

$$S_n = \frac{n}{1} d_{r-p} + \frac{(n-1)n}{1 \cdot 2} d_{r-p+1} + \dots\dots$$
$$+ \frac{(n-p)(n-p+1)\dots\dots(n-1)n}{1 \cdot 2 \dots\dots (p+1)} d_r.$$

Put $p = r$, then for the series *preceding* the 1st order of differences, that is, for the given series, since $d_0 = u_1$, we have

$$S_n = \frac{n}{1} u_1 + \frac{(n-1)n}{1 \cdot 2} d_1 + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_2 + \dots\dots$$
$$+ \frac{(n-r)\dots\dots(n-1)n}{1 \cdot 2 \dots\dots (r+1)} d_r,$$
$$= nu_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \dots\dots$$
$$+ \frac{n(-1)\dots\dots(n-r)}{\underline{|r+1}} d_r.$$

Ex. XV.

Find the generating functions of the following six series:

1. $1 + 8x + 22x^2 + 50x^3 + 106x^4 + \dots\dots$
2. $1 + 3x + x^2 - 5x^3 - 7x^4 - \dots\dots$
3. $2 + 13x + 38x^2 + 61x^3 - 22x^4 - \dots\dots$
4. $3 + 11x + 57x^2 + 309x^3 + 1683x^4 + \dots\dots$
5. $9 - 11x - 52x^2 + 129x^3 + 235x^4 - \dots\dots$
6. $6 + 16x + 25x^2 + 17x^3 - 33x^4 - 142x^5 - 277x^6 - \dots\dots$

Sum to n terms, and to infinity, the following eight series:

7. $3 - 13x + 69x^2 - 337x^3 + 170x^4 - \dots\dots$
8. $6 - 33x + 249x^2 - 1707x^3 + 12021x^4 - \dots\dots$
9. $1 - 2x + 7x^2 - 8x^3 + 37x^4 - 14x^5 + 211x^6 + \dots\dots$
10. $1 - 7x + 22x^2 - 70x^3 + 211x^4 - 637x^5 + 1912x^6 - \dots\dots$
11. $2 - 11x + 54x^2 - 271x^3 + 1354x^4 - \dots\dots$

12. $1 + 4x + 10x^2 + 34x^3 - 26x^4 + \ldots.$

13. $3 + 14x + 36x^2 + 69x^3 + 113x^4 + 168x^5 + 234x^6 + \ldots.$

14. $4 + 29x + 131x^2 + 491x^3 + 1679x^4 + 5459x^5 + 17231x^6 + \ldots.$

15. Find the scale of relation of the recurring series

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots.$$

16. Find the generating function of the series

$$a + (a + b)x + (a + 2b)x^2 + (a + 3b)x^3 + \ldots.$$

17. If $a_0, a_1, a_2, \ldots$ form as A. P., and $b_0, b_1, b_2, \ldots$ form a G. P., show that the series

$$\frac{a_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots \text{ is recurring.}$$

18. If $1 + p_1x + p_2x^2$ is the scale of relation of the series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, show that

$$a_n(a^2_n - a_{n-1}a_{n+1}) + a_{n-1}(a_{n-1}a_{n+2} - a_na_{n+1}) + a_{n-2}(a^2_{n+1} - a_na_{n+2}) = 0.$$

Sum to n terms the following ten series:

19. $1 . 3 . 5 + 2 . 4 . 6 + 3 . 5 . 7 + \ldots.$

20. $(x+1)(x+2)(x+3) + (x+2)(x+3)(x+4) + \ldots.$

21. $\frac{1}{2 . 4 . 6} + \frac{1}{3 . 5 . 7} + \frac{1}{4 . 6 . 8} + \ldots.$

22. $\frac{4}{2 . 3 . 4} + \frac{7}{3 . 4 . 5} + \frac{10}{4 . 5 . 6} + \ldots.$

23. $\frac{1^3 + 1}{2} + \frac{2^3 + 1}{3} + \frac{3^3 + 1}{4} + \ldots.$

24. $\frac{1}{(a+b)(a+2b)(a+3b)} + \frac{1}{(a+2b)(a+3b)(a+4b)} + \ldots.$

25. $1^2 + 3^2 + 5^2 + \ldots.$

26. $1^3 + 3^3 + 5^3 + \ldots.$

27. $\dfrac{1}{1\,.\,2} + \dfrac{2}{1\,.\,2\,.\,3} + \dfrac{3}{1\,.\,2\,.\,3\,.\,4} + \ldots .$

28. $\dfrac{1}{\lfloor 3} + \dfrac{12}{\lfloor 4} + \ldots . + \dfrac{3n^2 + 2n - 4}{\lfloor n + 2}.$

29. $\dfrac{1}{m} + \dfrac{m-n}{m(m-1)} + \dfrac{(m-n)(m-n-1)}{m(m-1)(m-2)} + \ldots$ to $m - n + 1$ terms.

30. $\dfrac{1}{2\,.\,4} + \dfrac{1\,.\,3}{2\,.\,4\,.\,6} \cdot \dfrac{1}{4} + \dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8} \cdot \dfrac{1}{4^2} + \ldots$ to infinity.

31. Find the number of shot in a triangular pile, each side of whose base contains 10 balls.

32. There are 12 balls in the length, and 8 in the breadth of the base of a rectangular pile, and it terminates in a single row. Find the number of balls.

33. Find the number of balls in a rectangular pile when m, n represent respectively the number in the length and breadth of the base, and there are p layers.

34. If $x \div \left\{(1 - x)^2 - cx\right\}$ be expanded in a series of ascending powers of x, show that the coefficient of x^r is

$$r\left\{1 + \frac{r^2 - 1}{2\,.\,3}c + \frac{(r^2 - 1)(r^2 - 4)}{2\,.\,3\,.\,4\,.\,5}c^2 + \frac{(r^2 - 1)(r^2 - 4)(r^2 - 9)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}c^3 + \text{\&c.}\right\}.$$

35. Show that

$$\frac{1}{m} - n\,.\,\frac{1}{m+p} + \frac{n(n-1)}{1\,.\,2}\,.\,\frac{1}{m+2p} - \text{\&c.} + (-1)^n\,.\,\frac{1}{m+np} = \frac{\lfloor n\, p^n}{m(m+p)\ldots .(m+np)}.$$

36. If p_r be the coefficient of the $(r + 1)$th term of $(1 + x)^n$, where n is a positive integer, then

$$\frac{p_1}{p_0} + \frac{2p_2}{p_1} + \frac{3p_3}{p_2} + \ldots . + \frac{np_n}{p_{n-1}} = \frac{1}{2}n(n + 1),$$

and $(p_0 + p_1)(p_1 + p_2)(p_2 + p_3) + \ldots . (p_{n-1} + p_n)$

$$= \frac{p_1 p_2 p_3 \ldots . p_n}{1\,.\,2\,.\,3 \ldots . n}(n + 1)^n.$$

CHAPTER XII.

Scales of Notation.

102. An ordinary number as 6437 may be thus written:

$$6 . 10^3 + 4 . 10^2 + 3 . 10 + 7.$$

We call 10 the radix of the decimal system of notation.

And generally, if r be the radix of any other scale, and $p_0, p_1, p_2, \ldots . p_{n-1}$, the n digits commencing with the one at the right hand, we may express a number N thus:

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots . + p_2r^2 + p_1r + p_0.$$

It is easy to see that in order to express all numbers in any given scale, it is necessary and sufficient to have, besides zero, one digit less than the radix of the scale.

Thus, in ordinary notation, we have 9 digits.

Hence also, for the duodenary scale we shall require characters to represent *ten* and *eleven*. It is usual to represent them by t and e respectively.

Ex. Express 1420 (senary scale), and $1te3$ (duodenary scale), as common numbers.

1420 (senary scale) $= 1 . 6^3 + 4 . 6^2 + 2 . 6 + 0 = 372.$

$1te3$ (duodenary scale) $= 1 . 12^3 + 10 . 12^2 + 11 . 12 + 3 = 3303.$

103. *To transform an integer from any scale of notation to any other scale.*

Let N be the number, and r the radix of the scale in which we are to express it.

We shall therefore have

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots . + p_2r^2 + p_1r + p_0 \ldots (1),$$

where p_0, p_1, p_2, &c., are the digits whose values are required.

From (1), dividing each side by r, we have

$$\text{Quotient} = p_{n-1}r^{n-2} + p_{n-2}r^{n-3} + \ldots . + p_2r + p_1 \ldots (2),$$

and remainder $= p_0$.

We obtain then the first or right hand digit by dividing the given number by the proposed radix, and taking the remainder.

Again, dividing each side of (2) by r, we get—

$$\text{Quotient} = p_{n-1}r^{n-3} + p_{n-2}r^{n-4} + \dots + p_2 \dots (3),$$

and Remainder $= p_1$.

Hence, *we obtain the second digit by dividing the first quotient by the radix, and taking the remainder.*

And so we may proceed until we have found all the digits. We have therefore the following rule:—

RULE.—*Divide the given number by the proposed radix, then the quotient so obtained again by the radix, and so on. Then the successive remainders are the respective digits required, commencing with the right hand one.*

Ex. 1. Transform the common number 3526 into the *ternary* and *septenary* scales.

3	3526	
3	1175	1
3	391	2
3	130	1
3	43	1
3	14	1
3	4	2
3	1	1
		1

7	3526	
7	503	5
7	71	6
7	10	1
7	1	3
		1

Hence 3526, in the common scale, is expressed by *11211121* in the ternary scale, and by 13165 in the septenary scale.

Ex. 2. Transpose 35*et* from the duodenary scale to the septenary scale.

First method.

Expressing 35*et* as a common number, we have

$$35et = 3 \,.\, 12^3 + 5 \,.\, 12^2 + 11 \,.\, 12 + 10 = 6046.$$

Then, as in the last example,

7	6046	
7	863	 5
7	123	 2
7	17	 4
7	2	 3
		 2

Therefore, the number required in the septenary scale is 23425.

Second method.

We may proceed to divide the given number, 35*et*; at once by 7. Thus—

7	35*et*	
7	5*e*	 5
7	*t*3	 2
7	15	 4
7	2	 3
		 2

The operation requires a little explanation. Dividing 35 by 7, we get 5, and 6 over; for 35 in the duodenary scale is $3 \times 12 + 5$ or 41 in the common scale.

Then dividing $6e$ by 7 we get e, and 6 over, for

$$6e = 6 \times 12 + 11 = 83. \quad \text{And so on.}$$

In the next examples, the student is recommended to work out, as above, the process for himself.

Ex. 3. The numbers 357 and 76 are in the duodenary scale; find their product.

$$\begin{array}{r} 357 \\ 76 \\ \hline 1896 \\ 2031 \\ \hline 21et6 \\ \hline \end{array}$$

Ex. 4. Find the quotient of 1420422 divided by 241, both numbers being in the senary scale.

$$\begin{array}{l} 241)1420422(3502 \\ 1203 \\ \hline 2134 \\ 2125 \\ \hline 522 \\ 522 \\ \hline \end{array}$$

Ex. 5. Find the area of a room which measures 27 ft. 9 in. by 10 ft. 10 in.

Expressing these in the duodenary scale, we have

$$\begin{array}{r} 23{\cdot}9 \\ 16{\cdot}t \\ \hline 1e16 \\ 11t6 \\ 239 \\ \hline \end{array}$$

$376{\cdot}76$ = 376·76 sq ft. in the duodenary scale.

This expressed in the common scale

$= (3 \cdot 12^2 + 7 \cdot 12 + 6)$ sq. ft. $(7 \times 12 + 6)$ sq. in.

$= 522$ sq. ft. 90 sq. in.

104. *To transpose a given proper fraction from any scale of notation to any other scale.*

If we transform the numerator and denominator each into the new scale, we transpose the fraction into the new scale.

It may, however, be required to transform it into a fraction having the form of a decimal in ordinary notation. Such fractions are called **radix fractions.**

We proceed to investigate the method of doing this.

Let F be the given proper fraction,

And let q_1, q_2, q_3, &c., be the digits of the radix fraction, commencing from the left.

We then have

$$F = \frac{q_1}{r} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \&c. \qquad (1).$$

where q_1, q_2, q_3, &c., are integers whose values are required.

Multiplying each side of (1) by r, we have

$$Fr = q_1 + \frac{q_2}{r} + \frac{q_3}{r_2} + \&c. \qquad (2).$$

Hence, the first digit, q_1, of the radix fraction is found by multiplying the given fraction by the proposed radix, and taking the integral portion.

The remaining fractional part is $\frac{q_2}{r} + \frac{q_3}{r_2} + \&c.$, and it is therefore evident that the second digit q_2 of the radix fraction is found by multiplying this by the radix. And so on.

Hence we have the following rule for bringing a given fraction to a radix fraction.

RULE.—*Multiply the given fraction by the radix, and the integral portion of the product is the first digit. Multiply the fractional part by the radix, and the integral portion of the product is the second digit. And so on.*

Ex. 1. Transform ·8125 from the common scale to the scale whose radix is 8.

$$\begin{array}{rl} \cdot 8125 & \\ 8 & \\ \hline 6\cdot 5000, & \therefore q_1 = 6. \\ 8 & \\ \hline 4\cdot 0 \quad , & \therefore q_2 = 4. \end{array}$$

Hence the required radix fraction is ·64.

Proof.—·64, in the scale whose radix is 8, $= \frac{6}{8} + \frac{4}{8^2}$ $= \cdot 75 + \cdot 0625 = \cdot 8125.$

Ex. 2. Transform 713·44 in the scale eight to the scale six.

$$\begin{array}{r|l} 6 & 713 \\ \hline 6 & 114 \ldots\ldots 3 = p_0, \\ \hline 6 & 14 \ldots\ldots 4 = p_1, \\ \hline 6 & 2 \ldots\ldots 0 = p_2, \\ \hline & \ldots\ldots 2 = p_3. \end{array} \qquad \begin{array}{rl} \cdot 44 & \\ 6 & \\ \hline 3\cdot 30, & \therefore q_1 = 3. \\ 6 & \\ \hline 2\cdot 2\ , & \therefore q_2 = 2. \\ 6 & \\ \hline 1\cdot 4\ , & \therefore q_3 = 1. \\ 6 & \\ \hline 3\cdot 0\ , & \therefore q_4 = 3. \end{array}$$

Hence the number required is 2043·3213.

105. *A number whose radix is* r *is divisible by* r – 1, *when the sum of its digits is so divisible.*

Let the number N be such that

$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots + p_2r^2 + p_1r + p_0 \ldots$ (1),
where $p_0, p_1, p_2, \ldots, p_{n-1}$, are the digits commencing at the right hand.

We have

$$N = p_{n-1}(r^{n-1} - 1) + p_{n-2}(r^{n-2} - 1) + \dots p_2(r^2 - 1) + p_1(r - 1) + (p_{n-1} + p_{n-2} + \dots + p_2 + p_1 + p_0).$$

Now each of the quantities $r^{n-1} - 1, r^{n-2} - 1, \dots, r^2 - 1, r - 1$, is divisible by $r - 1$. Hence the whole of the right side of this equation is divisible by $r - 1$, if $(p_{n-1} + p_{n-2} + \dots + p_2 + p_1 + p_0)$ is so divisible; *i.e.*,

N is divisible by $r - 1$, when $(p_{n-1} + p_{n-2} + \dots + p_2 + p_1 + p_0)$, the sum of its digits, is so divisible.

COR. *Hence also, a number whose radix is* r, *will leave the same remainder when divided by* r − 1, *as the sum of its digits leaves when divided by* r − 1.

106. *A number whose radix is* r *is divisible by* r + 1, *when the difference between the sum of the digits in the odd places and the sum of the digits in the even places is so divisible.*

We have

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots + p_3r^3 + p_2r^2 + p_1r + p_0 \dots (1).$$

$$= p_{n-1}\{r^{n-1} - (-1)^{n-1}\} + p_{n-2}\{r^{n-2} - (-1)^{n-2}\} + \dots + p_3(r^3 + 1) + p_2(r^2 - 1) + p_1(r + 1) + \{p_0 - p_1 + p_2 - p_3 + \dots + (-1)^{n-1}p_{n-1}\}.$$

Now (Vol. I., Art. 30) each of the quantities $r + 1, r^2 - 1, r^3 + 1, \dots, r^{n-1} - (-1)^{n-1}$, is divisible by $r + 1$.

Hence the whole of the right side of this equation is divisible by $r + 1$, if $\{p_0 - p_1 + p_2 - p_3 + \dots (-1)^{n-1}p_{n-1}\}$ is so divisible.

But $p_0 - p_1 - p_2 - p_3 + \dots + (-1)^{n-1}p_{n-1}$ is the difference between the sums of the digits in the odd and even places.

Hence N is divisible by $r + 1$, when the difference between the sums of the digits in the odd and even places is so divisible.

COR. Hence, also, *when a number, whose radix is* r, *is divided by* r + 1, *it will leave the same remainder as the difference between the sums of the digits in the odd and even places will leave when so divided.*

Ex. XVI.

1. Express the common numbers 654 and 387 in the senary scale.

2. Transform 372 and 516 from the denary to the septenary scale, multiply the results together, and divide the product by 516 (septenary scale). Express the quotient in the denary scale.

3. Transform 4163 from the septenary to the scale whose radix is eight.

4. The number 86 expressed in another scale is 95. Find the radix.

5. The number 172 expressed in a different scale is 124. What is the radix.

6. If the number 516 be divided by 2, the quotient when expressed in another scale has the same digits as the original number. Find the scale.

7. Transform 79·875 from the denary scale to the scale whose radix is 8.

8. Express 4521 (senary scale) in the duodenary scale.

9. Find the area of a room measuring 52 ft. 9 in. by 30 ft. 7 in., expressing your result in the duodenary scale.

10. Simplify $\frac{313 \times 1034\frac{2}{3}}{45\frac{5}{10} \times 210}$, the number being in scale six. Give your answer in scale seven.

11. The number 52614 is in scale seven. Find its square root in that scale.

12. *Reduce* $\cdot\dot{3}\dot{2}$ (scale six) to a vulgar fraction in the *same scale.*

13. Show that 1331 is a cube number in any scale above three.

14. The difference between any number and the number composed of the same digits in reverse order is divisible by $r - 1$, where r is the radix of the scale.

15. Show that the difference between two numbers having the same digits is divisible by $r - 1$, whatever be the order of the digits in the second number.

16. Given a system of weights 1 lb., 3 lbs., 3^2 lbs., &c., (one of each kind); show how to weigh 1135 lbs.

17. If $\phi(n)$ represent the remainder after dividing a number n by $r - 1$, show that

$$\phi\{\phi(p) \cdot \phi(q)\} = \phi(pq).$$

18. If the digits of a number n be added together, the digits of the result added together, and so on, until the sum is represented by one digit; and we represent the final result by $F(n)$, show that

$$F(n) \cdot F(p) \cdot F(q) \ldots \text{ and } F(npq \ldots)$$

are either equal or differ by a multiple of $(r - 1)$.

19. If a number whose digits are p_0, p_1, p_2, is a perfect square in any scale, find the relation between p_0, p_1, p_2.

20. Show that if the sum of two numbers is a multiple of the radix, the sum of the last digits of their cubes is equal to the radix.

21. Every number having its digits repeated in threes, and being in any scale of the series—*four*, *seven*, *ten*, *thirteen*, &c.—is divisible by 3, and the quotient contains a factor, the sum of whose digits is equal to the radix.

22. The sum of the numbers (scale s) represented by the permutations of any three digits of the scale amongst each other, is always divisible by the number (s 1) of the scale whose radix is $(s + 1)$.

23. Every number less than 2^{n+1} can be expressed by the sum of certain terms of the series

$$1, 2, 2^2, \ldots, 2^n.$$

24. Every number less than 3^{n+1} can be expressed by terms of the series

$$1,\ 3,\ 3^2,\ \ldots\ldots,\ 3^n,$$

the coefficients of the terms being either $\pm$ 1.

CHAPTER XIII.

Properties of Numbers.

107. By the term **number** in the present chapter we mean a positive integer.

Owing to the extensive nature of the Theory of Numbers we shall be able to give in this work only the more elementary propositions.

A **Prime** number is one which has no divisor except itself and unity.

Two numbers are said to be **prime to each other** when they contain no other common divisor than *unity*.

108. *If a number* a *be prime to each of the numbers* b *and* c, *it is prime to their product* bc.

For suppose a, b, c to be each broken up into their elementary factors.

Then a contains no factor except unity, which b contains; and neither does it contain any factor except unity, which c contains.

Now, bc when broken up into elementary prime factors cannot contain any other such factors than what can be found in b and c.

Therefore a contains no factor common to bc other than unity.

Hence a is prime to bc.

Cor. 1. Hence if a be prime to any number of quantities b, c, d, &c., it is prime to their product bcd, &c.; and if a be prime to any quantity b, it is prime to each of the factors of b.

COR. 2. If a be prime to b, it is also prime to $bbb \ldots$, that is, *to any power* of b.

COR. 3. If a be a divisor of bc, and be prime to b, it must be a divisor of c.

COR. 4. If $\frac{a}{b}$ is a fraction in its lowest terms, then $\frac{a^n}{b^n}$ is in its lowest terms.

109. *If* $\frac{a}{b}$ *be a fraction in its lowest terms, and* $\frac{a}{b} = \frac{c}{d}$; *then will* c *and* d *be equimultiples of* a *and* b.

For $\frac{a}{b}$ is the fraction $\frac{c}{d}$ reduced to its *lowest terms.*

Now in order to express $\frac{c}{d}$ in its lowest terms we have to divide numerator and denominator by the greatest common measure x, suppose.

We must therefore have $\frac{c}{x} = a$, or $c = ax$;

$$\text{and } \frac{d}{x} = b, \text{ or } d = bx.$$

110. *If* N *be an integer, and* $\sqrt[n]{N}$ *a surd, we cannot express the latter by a fraction.*

For suppose $\sqrt[n]{N} = \frac{a}{b}$, we then have $N = \frac{a^n}{b^n}$, ... (1).

Now if $\frac{a}{b}$ be not in its lowest terms, we may so reduce it.

Then we shall have $\frac{a^n}{b^n}$ in its lowest terms, and also a fraction.

Hence in (1) we have, *an integer = a fraction,* which is absurd.

111. *Any number is prime if it cannot be divided by some number equal to, or less than, its square root.*

For every number N which is not a *prime* is composed of two factors, as a and b, so that we have

$$N = ab.$$

Now if $a = b$, we have $a = b = \sqrt{N}$, so that N is divisible by $\sqrt{N}$.

Again, if $a > b$, we have $b < \sqrt{N}$;

And, if $a < b$, we have $a < \sqrt{N}$.

Hence, for every number not prime, there always exists a divisor, either equal to, or less than, its square root; and therefore, every number not having such a divisor is prime.

Ex. To determine whether 101 is a prime number.

We have $\sqrt{101} < 11$.

Now, as 101 is odd, we need only try the prime divisors 3, 5, 7; and we find it is not divisible by either of them.

Hence 101 is a prime number.

112. *No rational algebraical formula can contain prime numbers only.*

For, if possible, suppose that the expression

$$p + qx + rx^2 + sx^3 + \&c. \ldots.$$

expresses prime numbers only.

Suppose that when $x = m$, the expression equals P; and when $x = m + nP$, the value of the expression is P_1.

Then we have

$$P = p + qm + rm^2 + sm^3 + \&c. \ldots\ldots\ldots\ldots \quad (1),$$

$$\text{and } P_1 = p + q(m + nP) + r(m + nP)^2 + s(m + nP)^3 + \&c.$$

$$= (p + qm + rm^2 + sm^3 + \&c.)$$

$$+ (qn + 2rmn + 3sm^2n + \&c.)P$$

$$+ (rn^2 + 3smn^2 + \&c.)P^2 + \&c.$$

$$= P + \text{terms containing P as a factor.}$$

Hence P_1 is divisible by P, and therefore not prime.

And hence the algebraical expression

$$p + qx + rx^2 + sx^3 + \&c.$$

cannot represent prime numbers only.

Remark. There are several remarkable algebraical formulæ *which represent* a large number of primes, Thus $x^2 + x + 41$, by making $x = 0, 1, 2, 3$, &c. 39, will give a series of forty numbers, all primes.

$x^2 + x + 17$ gives similarly seventeen primes.

$2x^2 + 29$ gives twenty-nine primes.

$2^n + 1$ is prime for $n = 1, 2, 4, 8$, or 16.

113. *The number of primes is infinite.*

For suppose p to be the greatest prime number. Then the product of all the prime numbers up to p is

$$2 . 3 . 5 . 7 \dots . p.$$

And this product is divisible by each of the prime numbers. Hence

$$(2 . 3 . 5 . 7 \dots . p) + 1$$

is divisible by none of the prime numbers $2, 3, 5, 7, \dots . p$.

Now, if this quantity is a prime, it is a greater prime than p; and if it be a composite number, it must have a prime factor greater than p, since none of the prime numbers 2, 3, 5, 7, ... p will divide it.

Hence, in any case, there exists a prime number greater than any supposed prime p.

114. *If* a *be prime to* b, *and each of the terms of the series*

$$b, 2b, 3b, \dots . (a - 1) b$$

be divided by a, *all the remainders will be different.*

For suppose any two terms of this series, mb and m_1b, to have the same remainder, where m and m_1 are of course each less than a.

Then their difference $(mb - m_1b)$ will be divisible by a.

But since $mb - m_1b = (m - m_1)b$, it is composed of two factors, one of which is prime to a, and the other of which is less than a. It therefore cannot be divisible by a.

Hence no terms of the given series can leave the same remainder when divided by a.

Cor. 1. Since all these remainders are less than a, and they are all different, and since there are $(a - 1)$ terms in the given series, it follows that the series

$$1, 2, 3, \ldots . a - 1$$

represents the remainders, not regarding their order.

Cor. 2. If a be prime to b, the terms of the A. P.

$$b + c, 2b + c, 3b + c, \ldots . (a - 1) b + c,$$

when divided by a, will give for remainders the series

$$1, 2, 3, \ldots . (a - 1),$$

no regard being had to their order.

Fermat's Theorem.

115. *If* n *be a prime number, and* N *be a number prime to* n, *then* $N^{n-1} - 1$ *is divisible by* n.

We shall give two demonstrations of this important theorem.

First proof.

By Art. 114, Cor. 1, the terms of the series

$$N, 2N, 3N, \ldots , (n - 1) N,$$

when divided by n, give for remainders (disregarding their order) the series

$$1, 2, 3, \ldots , (n - 1).$$

Hence, assuming $q_1, q_2, q_3, \ldots , q_{n-1}$ as the quotients corresponding to these remainders, we must have

$$N . 2N . 3N \ldots . (n - 1) N$$

$$= (1 + q_1 n)(2 + q_2 n)(3 + q_3 n) \ldots . (\overline{n-1} + q_{n-1} n) \text{; or,}$$

$N^{n-1} . 1 . 2 . 3 \ldots . (n - 1) = 1 . 2 . 3 \ldots . (n - 1) +$ terms containing n as a factor

$$= 1 . 2 . 3 \ldots . (n - 1) + Mn, \text{ suppose.}$$

$$\therefore (N^{n-1} - 1) . 1 . 2 . 3 \ldots . (n - 1) = Mn.$$

Now n, being a prime number, is prime to each of the factors $1, 2, 3, \ldots, (n-1)$. And hence

$$\frac{M}{1 \cdot 2 \cdot 3 \ldots (n-1)} \text{ is an integer,}$$

since $N^{n-1} - 1$ and n are integers.

Hence we have $N^{n-1} - 1 = Pn$, when P is integral.

$\therefore$ $N^{n-1} - 1$ is divisible by n.

Second proof.

We have $N^n = \{(N-1) + 1\}^n =$

$$(N-1)^n + n(N-1)^{n-1} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \ldots r}(N-1)^{n-r}$$
$$+ \ldots + n(N-1) + 1.$$

Now every term of the second side of this identity, excepting the first and last, is divisible by n.

For the general coefficient $\frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \ldots r}$ is an integer, and is so divisible, since r is $< n$, and n is prime. Hence we have

$$N^n = (N-1)^n + 1 + P_0 n, \text{ where } P_0 \text{ is an integer.}$$
$$\therefore \; N^n - N = (N-1)^n - (N-1) + P_0 n;$$

or, putting successively, $N-1, N-2, \ldots, 2$, for N, we have

$$(N-1)^n - (N-1) = (N-2)^n - (N-2) + P_1 n,$$
$$(N-2)^n - (N-2) = (N-3)^n - (N-3) + P_2 n,$$
$$\&c. = \&c.$$
$$2^n - 2 = 1^n - 1 + P_{n-2}$$

where P_1, P_2, &c., are integers.

Hence adding these equations, we have—

$$N^n - N = 1^n - 1 + (P_0 + P_1 + P_2 + \ldots P_{n-2})n$$
$$= Pn, \text{ suppose, where P is an integer.}$$

Or, $N(N^{n-1} - 1) = Pn$.

Now n is prime to N, and therefore $\frac{P}{N}$ is integral.

Hence $N^{n-1} - 1$ is divisible by n.

Extension of Fermat's Theorem.

116. *If* n *be prime to* N, *and* ϕ(n) *represent how many numbers there are less than* n, *and prime to it, then*

$$N^{\phi(n)} - 1 \textit{ is divisible by } n.$$

Let $a, b, c, d, \ldots . m,$ ....(1).
be the $\phi(n)$ numbers which are less than n and prime to it. Then if the series be multiplied by N we have the second series

$aN, bN, cN, \ldots, mN,$ ....(2).

Now it may be shown, as in Art. 114, that if the terms of the series be severally divided by n, we shall have for remainders, neglecting their order, the series (1).

Hence, as in Art. 115, assuming q_a, q_b, q_c, &c., q_n as the quotients corresponding to these remainders, we must have

$$aN \,.\, bN \,.\, cN \ldots . mN$$
$$= (a + q_1n)\,(b + q_2n)\,(c + q_3n) \ldots . (m + q_mn)\text{; or,}$$
$$abc \ldots . m \,.\, N^{\phi(n)}$$
$$= abc \ldots . m + \text{terms containing } n \text{ as a factor}$$
$$= abc \ldots . m + Mn, \text{ suppose.}$$
$$\therefore \left\{N^{\phi(n)} - 1\right\} abc \ldots . m = Mn.$$

Now each of the quantities $a, b, c, \ldots, m$ is prime to n and less than n. Hence n is prime to the product $abc \ldots . m$.

Therefore $N^{\phi(n)} - 1$ is divisible by n.

Cor. Fermat's theorem follows at once from this.

For suppose n a prime number, then the numbers less than n and prime to it are

$$1, 2, 3, \ldots, (n-1).$$

Hence, here $\phi(n) = n - 1$.

Hence, if n be a prime number, and n be prime to N,

$$N^{n-1} - 1 \text{ is divisible by } n.$$

Wilson's Theorem.

117. *If* n *be a prime number, then*

$$\underline{|n-1} + 1$$

is divisible by n.

Let a represent one of the numbers of the series

$$1, 2, 3, \ldots, (n-1) \ldots\ldots (1).$$

a is therefore prime to n.

Multiplying each term of this series by a, we have the second series

$$a, 2a, 3a, \ldots (n-1)a \ldots\ldots (2).$$

Now (Art. 114, Cor. 1) if the terms of this series be divided by n, we have for remainders, not regarding their order, the series in (1).

Hence some one of the numbers in series (2) give *unity* for a remainder when divided by n.

If we represent this number by ma, we have

$$ma - 1 \text{ divisible by } n = b_0 n, \text{ suppose} \ldots\ldots (3).$$

Now m cannot be equal to a, except

$$a = 1, \text{ or } a = n - 1 \ldots\ldots (4).$$

For suppose we have $m = a$, we then have from (3) $a^2 - 1 = b_0 n$ or $(a + 1)(a - 1) = b_0 n$.

Now n is a prime number;

Hence either $a + 1$ or $a - 1$ is divisible by n.

But a is less than n, and hence when n divides $a + 1$, we must have $a = n - 1$.

And so, when n divides $a - 1$, we must have $a = 1$.

Hence, leaving out the numbers 1 and $n - 1$, we may say that it is always possible that *any one* of the numbers

$$2, 3, 4, \ldots, (n-2),$$

may be multiplied by *some other* number of the same series, *so as to form a product* of the form $b_0 n + 1$.

Suppose then we multiply these numbers so as to form such products as $b_1n + 1$, $b_2n + 1$, &c. By multiplying together these products we have

$$2 . 3 . 4 \dots . (n - 2) = (b_1n + 1)(b_2n + 1)\dots\dots\dots$$
$$= Mn + 1,$$

where Mn represents the terms having n as a factor.

Hence, multiplying each side by $n - 1$, we have

$$1 . 2 . 3 \dots . (n - 1) = (Mn + 1)(n - 1)$$
$$= M(n - 1)n + n - 1; \text{ or}$$
$$1 . 2 . 3 \dots . (n - 1) + 1 = \{M(n - 1) + 1\}n.$$

Hence $\lfloor n - 1 + 1$ is divisible by n.

Cor. Hence $\left(1 . 2 . 3 \dots . \frac{n-1}{2}\right)^2 \pm 1$ *is divisible by* n, *when* n *is prime, and* > 2, *the upper or lower sign being taken as* n *is of the form* 4 m + 1 *or* 4 m − 1.

Since $\frac{n-1}{2}$ is integral, we have

$$1 . 2 . 3 \dots . (n - 1)$$
$$= \left(1 . 2 . 3 \dots . \frac{n-1}{2}\right) . \left(\frac{n+1}{2} . \frac{n+3}{6} \dots . \overline{n-3} . \overline{n-2} . \overline{n-1}\right)$$
$$= 1 . 2 . 3 \dots . \frac{n-1}{2} . \left\{\left(n - \frac{n-1}{2}\right) \dots . \overline{n-3} . \overline{n-2} . \overline{n-1}\right\}$$
$$= 1 . 2 . 3 \dots . \frac{n-1}{2} . \left\{Mn + (-1)^{\frac{n-1}{2}} 1 . 2 . 3 \dots \frac{n-1}{2}\right\}$$

where Mn represents terms containing n as a factor;

$$= \left(1 . 2 . 3 \dots . \frac{n-1}{2}\right) Mn + (-1)^{\frac{n-1}{2}} \left(1 . 2 . 3 \dots . \frac{n-1}{2}\right)^2;$$

or, adding *unity* to each side, $\lfloor n - 1 + 1$

$$= \left(1 . 2 . 3 \dots . \frac{n-1}{2}\right) Mn + (-1)^{\frac{n-1}{2}} . \left(1 . 2 . 3 \dots . \frac{n-1}{2}\right)^2 + 1.$$

Now we have just shown that $\lfloor n - 1 + 1$ is divisible *by n.*

Hence $(-1)^{\frac{n-1}{2}}\left(1 \cdot 2 \cdot 3 \ldots \frac{n-1}{2}\right)^2 + 1$ is divisible by n.

Now $(-1)^{\frac{n-1}{2}} = \pm 1$, according as n is of the form of $4n + 1$ or $4n - 1$.

Hence the truth of the proposition.

118. *If* a, b, c, ... , l *are a series of numbers prime to each other, and* ϕ (n) *represents how many numbers there are which are prime to* n *and less than it, then*

$$\phi(\text{abc} \ldots \text{l}) = \phi(\text{a}) \cdot \phi(\text{b}) \cdot \phi(\text{c}) \ldots \phi(\text{l}).$$

First, suppose N = ab.

We may arrange the numbers from 1 to ab in b columns thus:

$$\begin{array}{ccccc} 1, & 2, & 3,\ldots & & k,\ldots a, \\ a+1, & a+2, & a+3,\ldots & & a+k,\ldots 2a, \\ 2a+1, & 2a+2, & 2a+3,\ldots & & 2a+k,\ldots 3a, \\ \vdots & \vdots & \vdots & & \vdots \quad\quad \vdots \\ (b-1)a+1, & (b-1)a+2, & (b-1)a+3 \ldots & & (b-1)a+k,\ldots ab. \end{array}$$

If we consider any column as the one commencing with k, we see that all the numbers in it are prime to a, or all the numbers contain some factor common with a, according as k is prime to a or not.

Hence the number of columns which contain all the numbers of the column prime to a is equal to the number of quantities in the series 1, 2, 3, . . . k, . . . a, which are prime to a. And no other column contains any number prime to a.

Hence $\phi(a)$ is the number of columns containing numbers prime to a out of all the numbers from 1 to ab............(1).

Again, by Art. 114, Cor. 1, if the numbers in any column, as that commencing with k, be divided by b, the remainders, when k is prime to b, will form the series

$$1, 2, 3, \ldots, (b-1).$$

Hence in every column there are as many numbers prime to b as there are of such numbers in the series $1, 2, 3, \ldots . (b-1)$.

Therefore $\phi(b)$ is the number of quantities in each column which are prime to b..(2).

Hence from (1) and (2) we conclude that—

The number of quantities in all the columns which are prime to a and prime to b, and therefore prime to ab, is $\phi(a) \,.\, \phi(b)$.

$$\text{Therefore } \phi(ab) = \phi(a) \,.\, \phi(b).$$

If $N = abc \ldots . l$, we have

$$\begin{aligned} \phi(abc \ldots . l) &= \phi(a) \,.\, \phi(bc \ldots . l) \\ &= \phi(a) \,.\, \phi(b) \,.\, \phi(c \ldots l) = \&c. \\ &= \phi(a) \,.\, \phi(b) \,.\, \phi(c) \ldots . \phi(l). \quad Q.E.D. \end{aligned}$$

NOTE. It will be seen that we consider here *unity* as prime to a, b, c, &c.

119. *To find the number of positive integers which are less than* N *and prime to it.*

Suppose $N = a^p b^q c^r \ldots . l^s$, where $a, b, c, \ldots, l$ are prime factors.

Now the number of positive integers which are less than a^p and prime to it is found by subtracting from a^p the number of quantities in the series

$$a, 2a, 3a, \ldots, a^{p-1}a \text{ or } a^p.$$

The number of these is evidently a^{p-1}.

Hence we must have $\phi(a^p) = a^p - a^{p-1}$,

$$\text{or } \phi(a^p) = a^p\left(1 - \frac{1}{a}\right); \text{ and so}$$

$$\begin{aligned} \phi(b^q) &= b^q\left(1 - \frac{1}{b}\right), \\ \phi(c^r) &= c^r\left(1 - \frac{1}{c}\right), \\ \&c., &= \&c. \\ \phi(l^s) &= l^s\left(1 - \frac{1}{l}\right). \end{aligned}$$

Hence, multiplying, we have

$$\phi(a^p) \,.\, \phi(b^q) \,.\, \phi(c^r) \,\ldots\ldots\, \phi(l^s)$$

$$= a^p b^q c^r \ldots\ldots l^s \,.\, \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\ldots\left(1-\frac{1}{l}\right).$$

$$= \mathrm{N} \,.\, \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\ldots\left(1-\frac{1}{l}\right).$$

But by last Art.,

$$\phi(\mathrm{N} =) = \phi\, a^p) \,.\, \phi(b^q) \,.\, \phi(c^r) \,\ldots\ldots\, \phi(l^s).$$

Hence $\phi(\mathrm{N}) = \mathrm{N}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\left(1-\frac{1}{l}\right).$

120. *To find the number of divisors in a given integer.*

Suppose $\mathrm{N} = a^p b^q c^r \ldots\ldots$, where $a, b, c \ldots\ldots$ are primes.

Now N is divisible by every power of a not greater than a^p.

Hence, including *unity* as one of its divisors, N is divisible by every term of the series

$$1, a, a^2, \ldots, a^p.$$

And so it is divisible by every term of each of the series

$$1, b, b^2, \ldots, b^q,$$
$$1, c, c^2, \ldots, c^r,$$
&c.

N is therefore divisible by every combination of the terms of these series.

Hence N is divisible by every term of the product

$$(1 + a + a^2 + \ldots\ldots a^p)(1 + b + b^2 + \ldots\ldots + b^q)$$
$$(1 + c + c^2 + \ldots\ldots + c^r)\ldots\ldots,$$

and by no other quantity.

Now the *number of terms* of this product is found by putting $a = b = c =$ &c. $= 1$.

Therefore, number of divisors of N, including itself and unity

$$= (p + 1)(q + 1)(r + 1)\ldots\ldots$$

Cor. 1. Since every divisor has a corresponding divisor for a quotient, it follows that the number of ways in which N may be broken into two factors is half the number of divisors.

(i.) When N is *not a perfect square*,

One at least of the indices is odd, and therefore

$$(p + 1)(q + 1)(r + 1) \dots .$$

is even.

Hence, number of ways in which N may be broken up into two factors

$$= \tfrac{1}{2}(p + 1)(q + 1)(r + 1) \dots .$$

(ii.) When N *is a perfect square*,

All the indices, $p, q, r, \dots .$ are even, and therefore

$$(p + 1)(q + 1)(r + 1) \dots .$$

is odd.

But $\sqrt{N}$ is a divisor, and hence, since $N = \sqrt{N} . \sqrt{N}$, there is *one way* of resolving N into two factors, by means of *one divisor*.

Hence, number of ways in which N may be resolved into two factors

$$= \tfrac{1}{2}\left\{(p + 1)(q + 1)(r +) \dots . + 1\right\}.$$

It must be remembered that (N, 1) is one of the pairs of factors.

Cor. 2. *The sum of the divisors of any number* N, including among them itself and unity,

$$= \text{the product } (1 + a + a^2 + \dots . + a^p)(1 + b + b^2 + \dots . + b^q)$$
$$(1 + c + c^2 + \dots . + c^r) \dots .$$
$$= \frac{a^{p+1} - 1}{a - 1} \cdot \frac{b^{q+1} - 1}{b - 1} \cdot \frac{c^{r+1} - 1}{c - 1} \dots .$$

Cor. 3. If n be the number of prime factors, $a, b, c,$ &c. (not including N or unity), in N,

The number of ways in which N may be resolved into *two factors prime to each other* $= 2^{n-1}$.

For we must have a^p in one factor of any pair; and so we must have b^q, c^r, &c.; and we cannot have any lower powers of a, b, c, &c., in any of the factors which are prime to each other.

Hence the *number of ways* in which we may resolve N into two factors prime to each other is the same as that of the number abc . . . . ;

We have therefore, by Cor. 1, making there $p = q = r = \&c. = 1$,

Number of ways in which N may be resolved into two factors prime to each other,

$$= \tfrac{1}{2}(1 + 1)(1 + 1)(1 + 1) \dots \text{ to } n \text{ factors} = 2^{n-1}.$$

Ex. XVII.

1. If any number divide each of two other numbers, it will divide their sum and difference.

2. If two numbers be prime to each other, then will their sum and difference be prime to each of them.

3. If a square number be multiplied by a square, the product will be a square.

4. All numbers belong to one of the forms $3n$, $3n \pm 1$.

5. Every square number is of the form $5n$ or $5n \pm 1$.

6. Every cube number is of the form $7n$ or $7n \pm 1$.

7. A number which is a perfect square will divide by 5 without a remainder, or will leave 1 or 4 for a remainder.

8. Show that $n(n^2 + 5)$ is divisible by 6, when n is an integer.

9. Show that $n(n^2 - 1)(n^2 - 4)$ is divisible by 120, when n is an integer.

10. All even square numbers are of the form $16n$ or $4(4n + 1)$, and all odd squares of the form $8n + 1$.

11. Every number of the form $4n + 1$ is the sum of two squares.

12. If the difference between a and b is divisible by p, a is said to be congruous to b, for the modulus p. Hence show that if $a \equiv b$ (mod. p), and $c \equiv d$ (mod. p), then $a + c \equiv b + d$ (mod. p).

13. If $a \equiv b$ (mod. p), then $a^n \equiv b^n$ (mod. p), and $ma \equiv mb$ (mod. p).

14. If $ma \equiv mb$ (mod. p), what is the condition that $a \equiv b$ (mod. p)?

15. If the congruence $\phi(x) \equiv 0$ (mod. p) is satisfied by $x = a$, show that every number of the form $a + mp$ is also a solution, and that among these numbers there always exists one, and only one, between the limits 0 and $p - 1$ inclusive.

16. If $ax + b \equiv 0$ (mod. p), how many solutions are possible in the three cases: (i.) when a is prime to p; (ii.) when a and p have a common measure, which will not divide b; (iii.) when a, b, p have a common measure?

17. If p be a prime number, show that

$$(x + 1)(x + 2)(x + 3) \dots (x + p - 1)$$
$$\equiv x^{p-1} - 1 \text{ (mod. } p),$$

and hence deduce Wilson's theorem.

18. If $2^n - 1$ be a prime number, then will $2^{n-1}(2^n - 1)$ be a perfect number—that is, a number which is equal to the sum of all its aliquot parts, or its divisors.

19. If $3 \cdot 2^n - 1$, $6 \cdot 2^n - 1$, $18 \cdot 2^{2n} - 1$ are prime numbers, show that

$$2^{n+1}(18 \cdot 2^{2n} - 1), \text{ and } 2^{n+1}(3 \cdot 2^n - 1)(6 \cdot 2^n - 1),$$

are amicable numbers—that is, numbers each of which is equal to the sum of the divisors of the other.

20. Any power of the sum of two squares may be separated into the sum of two squares.

21. If a, b, and n are positive integers, and b less than $2a - 1$, show that the integral part of $(a + \sqrt{a^2 - b})^n$ is an odd number.

22. If n be a prime number, and m be odd, show that $m^n + 4$ cannot be a perfect square.

23. If ϕ (N) is the number of integers not greater than N and prime to it, and 1, d_1, d_2, &c., are all the divisors of N, then

$$\phi(1) + \phi(d_1) + \phi(d_2) + \dots = N.$$

24. If p be the G. C. M. of m and n, then $x^p - 1$ is the G. C. M. of $x^m - 1$ and $x^n - 1$.

25. If p be a prime number, and a one of the numbers $1, 2, 3, \dots \overline{p-1}$, then $a + kp$ contains square numbers or not according as $a^{\frac{p-1}{2}} - 1$, or $a^{\frac{p-1}{2}} + 1$, is divisible by p.

26. With the notation of Ex. 23, show that, when N is an odd number, $\phi(2N) = \phi(N)$.

27. Find the sum of the positive integers which are less than a given number (N) and prime to it.

28. Show that the sum of the squares of these integers

$$= \frac{N^3}{3}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\dots + \frac{N}{6}(1-a)(1-b)(1-c)\dots;$$

and that the sum of their cubes

$$= \frac{N^2}{4}\left\{N^2\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\dots + (1-a)(1-b)(1-c)\dots\right\},$$

where $a, b, c, \dots$, are the prime factors of N.

29. If p be a prime number, then the congruence

$$a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \dots + a_m \equiv 0 \pmod{p}$$

cannot admit of more than m solutions less than p.

30. The congruence $x^p \equiv 1 \pmod{p}$ has always $\phi(p-1)$ roots, where $\phi(n)$ represents the number of integers less than n and prime to it, and where p is a prime number.

CHAPTER XIV.

THEORY OF INDICES. IMAGINARY QUANTITIES.

Indices.

121. In the first volume of this work we have not altogether omitted this subject; we shall here complete the theory as far as space will permit.

I. *Positive integral indices.*

It is convenient to conceive of a^m, where m is a positive integer, as the result of *multiplying* unity by m factors each equal to a.

Thus $a^m = 1\,(a\,.\,a\,.\,a\,.\,.\,.\,.$ to m factors).

II. *Negative integral indices.*

Here, remembering that *negative* is the *reverse* of *positive*, we shall be quite consistent if we allow a^{-m} to represent the result of *dividing* unity by a, and the quotients successively by a for m operations.

Thus, $a^{-m} = 1 \div (a\,.\,a\,.\,a\,.\,.\,.\,.$ to m factors $= \dfrac{1}{a^m}$.

We get from this $a^m \times a^{-m} = 1$.

Now, by *assuming* that the formula $a^m \times a^n = a^{m+n}$, which we know holds for positive integral indices, to also hold when

$$n = -m, \text{ we have } a^m \times a^{-m} = a^{m-m} = a^0;$$

and we conclude that $a^0 = 1$.

We have no need, however, to *assume* the principle at all.

For, referring to our definitions of a^m and a^{-m}, it at once follows, since a^m represents the result of *multiplying* unity by a, and the successive products by a for m operations, and that a^{-m} represents the result of *dividing* unity by a, and the successive quotients by a for m operations, that

a^0 represents unity neither multiplied nor divided at all by a; *that is, a^0 is consistently* interpreted to mean unity.

Hence, whether m be a positive or negative integer, or zero, a^m admits of the same interpretation, provided we remember the algebraical meaning of *positive* and *negative*.

Again, the law that $a^m \times a^n = a^{m+n}$ holds good if either m or n, or both, be negative integers.

For, suppose $n = -p$, when p is a positive integer, we have

$$a^m \times a^n = a^m \times a^{-p} = a^m \times \frac{1}{a^p} = \frac{a^m}{a^p}.$$

If $m > p$, then $\frac{a^m}{a^p} = a^{m-p} = a^{m+n}$.

If $m < p$, then $\frac{a^m}{a^p} = \frac{1}{a^{p-m}} = a^{-(p-m)} = a^{m-p} = a^{m+n}$.

Hence $a^m \times a^n = a^{m+n}$, when n is negative.

And it may be similarly shown when both are negative.

Fractional Indices.

122. If m and n are any integers, it follows at once from the above that

$$(a^m)^n = a^{mn} = a^{m \times n}.$$

Hence, a^{mn} or $a^{m \times n}$ is the correct symbol for the nth *power* of a^m. And therefore $a^{m \div n}$ or $a^{\frac{m}{n}}$ will consistently represent the nth *root* of a^m. And this is true whether m and n be either one or both positive or negative.

(i.) We shall now show that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$, where p, q, r, s are integers, and thus show that the law $a^m \times a^n = a^{m+n}$ holds when m and n are fractional.

We have $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \sqrt[qs]{(a^p)^s} = \sqrt[qs]{a^{ps}}$ ....(1).

and $a^{\frac{r}{s}} = \sqrt[s]{a^r} = \sqrt[qs]{(a^r)^q} = \sqrt[qs]{a^{qr}}$....................(2).

$$\therefore \quad p^{\frac{p}{q}} \times a^{\frac{r}{s}} = \sqrt[qs]{a^{ps} \times a^{qr}} = \sqrt[qs]{a^{ps+qr}}$$

$$= a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

(ii.) We shall now show that $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}}$, where p, q, r, s are integers, and thus that the law $(a^m)^n = a^{mn}$ holds when m and n are fractional.

We have $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \sqrt[s]{\left(a^{\frac{p}{q}}\right)^r}$,

or by what has just been proved,

$$= \sqrt[s]{a^{\frac{pr}{q}}}$$

$$= \sqrt[s]{\sqrt[q]{a^{pr}}} = \sqrt[qs]{a^{pr}} = a^{\frac{pr}{qs}}.$$

Hence it has been universally established that the laws

(i.) $a^m a^n = a^{m+n}$, and

(ii.) $(a^m)^n = a^{mn}$,

are universally true, whether m, n be integral or fractional, positive or negative.

Imaginary Quantities.

123. An expression of the form $a + b\sqrt{-1}$ is called an **imaginary quantity,** a and b being real quantities. If we allow that $b\sqrt{-1}$ vanishes when $b = 0$, then the whole expression vanishes when $a = 0$, and $b = 0$.

Conversely, when $a + b\sqrt{-1} = 0$, *we must have* a = 0, b = 0.

For, transposing, $a = -b\sqrt{-1}$; or, squaring,

$$a^2 = b^2(-1) = -b^2;\ \text{or,}$$

$$a^2 + b^2 = 0.$$

And the only values of a and b which satisfy this equality are

$$a = 0, b = 0.$$

124. *If* a + b $\sqrt{-1}$ = c + d $\sqrt{-1}$, *then* a = c, *and* b = d.

For, transposing,

$$(a - c) + (b - d)\sqrt{-1} = 0.$$

Hence, by the last Art., $a - c = 0$, or $a = c$.
and $b - d = 0$, or $b = d$.

125. Def.—Two imaginary expressions are **conjugate** when they differ only in the sign of the coefficient of $\sqrt{-1}$.

Thus $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are conjugate expressions.

It follows, therefore, that

(i.) *The sum of two conjugate imaginary expressions is* REAL.

For $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$.

(ii.) *The difference of two conjugate imaginary expressions is* IMAGINARY.

The square root of the product of an imaginary expression by its conjugate is called its **modulus**.

Thus, since $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$, we call $\sqrt{a^2 + b^2}$ the modulus of $a + b\sqrt{-1}$, or of $a - b\sqrt{-1}$.

126. *If* M (p) *and* M (q) *represent respectively the moduli of* p *and* q, *then*

$$M(p) \cdot M(q) = M(pq).$$

Suppose $p = a + b\sqrt{-1}$,
and $q = c + d\sqrt{-1}$.

Then $M(p) = \sqrt{a^2 + b^2}$, and $M(q) = \sqrt{c^2 + d^2}$.

Therefore $M(p) \cdot M(q) = \sqrt{(a^2 + b^2)(c^2 + d^2)}$........ (1).

But $pq = (a + b\sqrt{-1})(c + d\sqrt{-1})$
$= (ac - bd) + (ad + bc)\sqrt{-1}$.

$\therefore M(pq) = \sqrt{(ac - bd)^2 + (ad + bc)^2}$
$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$........................ (2).

Hence from (1),

$$M(p) \cdot M(q) = M(pq).$$

127. *The addition, subtraction, multiplication, or division of quantities of the form* $A + B\sqrt{-1}$ *gives a result of the same form.*

Let $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$ be the quantities.

(i.) *Addition* and *subtraction.*

$$(a + b\sqrt{-1}) \pm (c + d\sqrt{-1}) = (a \pm c) + (b \pm d)\sqrt{-1}.$$

(ii.) *Multiplication.*

$$(a + b\sqrt{-1})(c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}.$$

(iii.) *Division.*

$$\frac{a + b\sqrt{-1}}{c + d\sqrt{-1}} = \frac{(a + b\sqrt{-1})(c - d\sqrt{-1})}{(c + d\sqrt{-1})(c - d\sqrt{-1})}$$

$$= \frac{(ac + bd) + (bc - ad)\sqrt{-1}}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\sqrt{-1}.$$

128. *To find the powers of* $\sqrt{-1}$.

Every number must be of the form $4m$, $4m + 1$, $4m + 2$, or $4m + 3$.

$$\text{Now} \quad (\sqrt{-1})^{4m} = \{(\sqrt{-1}^2)\}^{2m} = (-1)^{2m} = 1.$$

$$(\sqrt{-1})^{4m+1} = (\sqrt{-1})^{4m}.(\sqrt{-1}) = 1.\sqrt{-1} = \sqrt{-1}.$$

$$(\sqrt{-1})^{4m+2} = (\sqrt{-1})^{4m}.(\sqrt{-1})^2 = 1(-1) = -1.$$

$$(\sqrt{-1})^{4m+3} = (\sqrt{-1})^{4m}.(\sqrt{-1})^2(\sqrt{-1})$$

$$= 1(-1)\sqrt{-1} = -\sqrt{-1}.$$

129. *To find the cube roots of unity.*

These will evidently be the three values of x which satisfy the equation $x^3 = 1$, or $x^3 - 1 = 0$.

We have

$$(x - 1)(x^2 + x + 1) = 0.$$

Therefore $x - 1 = 0$, and $x^2 + x + 1 = 0$, will both furnish solutions.

From the former, $x = 1$;

And from the latter, $x = \frac{1}{2}(-1 \pm \sqrt{-3})$.

Hence the cube roots of unity are $1, \frac{1}{2}(-1 \pm \sqrt{-3})$.
We may verify this result by *cubing* these quantities.
In each case the cube is *unity*.

COR. Similarly, the cube roots of -1 are

$$-1, \tfrac{1}{2}(1 \pm \sqrt{-3}).$$

And so, the fourth roots of 1 are $\pm 1, \pm\sqrt{-1}$;

and the fourth roots of -1 are $\pm \dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}$.

Miscellaneous Examples.

1. Show that abc is $> (a + b - c)(a + c - b)(b + c - a)$, except when $a = b = c$.

2. If the number n be divided into any two parts, the difference of the squares of the parts is n times the difference of the parts.

3. Find to five places of decimals the value of x from the equation

$$\frac{2}{3 + x} + \frac{1}{5 - x} = 4.$$

4. Show that

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)$$
$$= (adf + bcf + bed - ace)^2 + (bce + aed + acf - bdf)^2.$$

5. Prove that

$$25\left\{(y - z)^7 + (z - x)^7 + (x - y)^7\right\}$$
$$\times \left\{(y - z)^3 + (z - x)^3 + (x - y)^3\right\}$$
$$= 21\left\{(y - z)^5 + (z - x)^5 + (x \mp y)^5\right\}^2.$$

6. Clear of surds the equations

$$\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 0, \text{ and}$$
$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{u} = 0.$$

7. Find x and y from the equations

$$x + \sqrt{\frac{x}{y}} = \frac{42}{y}, \text{ and}$$
$$2x^2y + 3x\sqrt{y} = 324.$$

8. Prove that

$$1 - n^2 + \frac{n^2(n^2 - 1^2)}{1^2 . 2^2} + \frac{n^2(n^2 - 1^2)(n^2 - 2^2)}{1^2 . 2^2 . 3^2} + \ldots\ldots = 0,$$

when n is a positive integer.

9. Show that the sum of the squares of the coefficients of

$$(1 - x + x^2)^n$$

$$= \frac{2n}{(\lfloor n)^2}\left\{1 + \frac{n^2}{\lfloor 2} + \frac{n^2(n-1)^2}{\lfloor 4} + \ldots\ldots\right\}.$$

10. For what values of x is $x + \frac{3}{x} >$ or < 4; and find the least value of $\frac{(1+x)(2+x)}{3+x}$.

11. If α, β be the roots of the quadratic $ax^2 + bx + c = 0$, show that the equation whose roots are $\frac{\alpha^4}{\beta}$ and $\frac{\beta^4}{\alpha}$ is

$$a^4cx^2 - b(b^4 - 5ab^2c + 5a^2c^2)x + ac^4 = 0.$$

12. What is the interpretation of the expression $x^{\sqrt{3}}$?

13. Show that unless $a = b = c = \ldots\ldots$, then

$$\frac{bcd\ldots\ldots}{(b-a)(c-a)(d-a)\ \ldots} + \frac{acd\ldots\ldots}{(a-b)(c-b)(d-b)\ldots\ldots} + \&c. = 1.$$

14. Sum the series $1^p + 2^p + 3^p + \ldots\ldots + n^p$.

15. Show that

$$\tfrac{1}{5}\{(x-y)^5 + (y-z)^5 + (z-x)^5\}$$

$$= \tfrac{1}{6}\{(x-y)^3 + (y-z)^3 + (z-x)^3\}\{(x-y)^2 + (y-z)^2 + (z-x)^2\}.$$

16. Show that

$$\frac{a^{n+2}}{(a-b)(c-a)} + \frac{b^{n+2}}{(a-b)(b-c)} + \frac{c^{n+2}}{(b-c)(c-a)}$$

is always equal to the sum of the homogeneous products of *n dimensions of a, b, c.*

17. The integral part of $(1 + \sqrt{3})^{2m+1}$ is divisible by 2^{m+1}.

18. If $E^n = y + \sqrt{1 + y^2}$, then $y = \dfrac{E^n - E^{-n}}{2}$.

19. Resolve into factors the expression

$$1 + a_0 + (1 + a_0) a_1 + (1 + a_0)(1 + a_1) a_2 \ldots . + \{(1 + a_0)(1 + a_1) \ldots . (1 + a_{n-1}) a_n\}.$$

20. Show that

$$\sqrt{\frac{b}{2} + \sqrt{ab - a^2}} + \sqrt{\frac{b}{2} - \sqrt{ab - a^2}} = \sqrt{2a}.$$

21. Show that $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^n + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^n = 2$, if n be a multiple of 3, and is equal to -1 if n be any other integer.

22. Show how to sum the series $a_m x^m + a_{m+n} x^{m+n} + a_{m+2n} x^{m+2n} +$ &c., when the sum of $a_0 + a_1 x + a_2 x^2 + \ldots$ is known.

23. If $(a + b\sqrt{-1})^{\frac{1}{n}} = c + d\sqrt{-1}$, then

$$(a - b\sqrt{-1})^{\frac{1}{n}} = c - d\sqrt{-1}.$$

24. If a_1, a_2, a_3, &c., a_n be unequal, and $m < n - 1$, then

$$\frac{a_1^m}{(a_1 - a_2)(a_1 - a_3) \ldots . (a_1 - a_n)} + \frac{a_2^m}{(a_2 - a_1)(a_2 - a_3) \ldots . (a_2 - a_n)} + \ldots . + \frac{a_n^m}{(a_n - a_1)(a_n - a_2) \ldots . (a_n - a_{n-1})} = 0.$$

25. If $\dfrac{ad - bc}{a - b - c + d} = \dfrac{ac - bd}{a - b - d + c}$, then each of them

$$= \tfrac{1}{4}(a + b + c + d).$$

26. Resolve into partial fractions

$$\frac{x^2 + mx + n}{(x + a)(x + b)(x + c)}.$$

27. Find the number of homogeneous products of n quantities a, b, c, &c., of r dimensions. Hence find the number of terms in the expansion of $(a + b + c + \&c.)^n$, n being integral.

28. Find a factor which will rationalize $a^{\frac{1}{2}} - b^{\frac{2}{3}}$, and extract the square root of $37 - 20\sqrt{3}$.

29. If $ax^3 = by^3 = cz^3$, and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$, show that $ax^2 + by^2 + cz^2 = (a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^3 d^2$.

30. If a debt a at compound interest is discharged in n years by annual payments of $\frac{a}{m}$, show that

$$(1 + r)^n (1 - mr) = 1.$$

31. If $x = h\sqrt{s^2 + \cdot 0075\, a^2}$, find the value of x when $h = 28\cdot 4$ metres, $s = 3\cdot 2$ metres, $a = 5\cdot 2$ kilometres. Of what *dimension* is x?

32. Show that $\frac{p + q + r + s}{a + b + c + d}$ lies between the greatest and least of the fractions $\frac{p}{a}, \frac{q}{b}, \frac{r}{c}, \frac{s}{d}$, but differs from the average value of those quantities.

33. If $u_r = \frac{x \,.\, \overline{x + 1} \ldots \overline{x + r - 1}}{\lfloor r}$, show that

$$u_1 + u_2 + \ldots + u_n = \frac{\overline{x + 1} \,.\, \overline{x + 2} \ldots \overline{x + n}}{\lfloor n} - 1.$$

34. Show that the minimum value of

$$\frac{(a + x)(b + x)}{x} \text{ is } (\sqrt{a} + \sqrt{b})^2.$$

35. If $10^x = 2$, find x as a continued fraction.

36. Calculate T from the following formula:

$$T = \frac{2A}{mS\sqrt{2g}}\sqrt{H},$$

when $A = 3503\cdot 6$ sq. ft., $S = 13\cdot 532$ sq. ft., $m = \cdot 548$, $g = 32\cdot 1998$ ft., $H = 6\cdot 3945$ ft.

37. Reduce to its simplest form

$$\frac{(yz - ax)^2 - (ac - y^2)(ab - z^2)}{(bc - x^2)(yz - ax) - (xz - by)(xy - cz)}$$

38. If p and q are each less than 1, show that

$$\frac{\log(1 - p)}{\log(1 - q)} \text{ is } > \frac{p - pq}{q}, \text{ and } < \frac{p}{q - pq}.$$

39. If $x = \dfrac{2ab + b^2}{a^2 + ab + b^2}$ and $y = \dfrac{a^2 - b^2}{a^2 + ab + b^2}$, show that $x^3 + y = y^3 + x$.

40. Prove that

$$\log_e x = \frac{1}{n} \cdot {}^{\infty}S_m \left(\frac{x^{mn} - 1}{m(x^n + 1)^m}\right),$$

where ${}^rS_m f(m)$ denotes the sum of r terms of the series whose mth term is $f(m)$.

41. Show that the integral part of $(5 + \sqrt{24})^n$ is odd if n be a positive integer.

42. If $\dfrac{A}{x} = \dfrac{B}{y} = \dfrac{C}{z}$, and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, then

$$\frac{A^2}{a^2} + \frac{B^2}{b^2} + \frac{C^2}{c^2} = \frac{A^2 + B^2 + C^2}{x^2 + y^2 + z^2}.$$

43. Write down the coefficient of t^{2n} in the expression

$$1 - \tfrac{1}{2}(e^t - 1) + \tfrac{1}{3}(e^t - 1)^2 - \&c.$$
$$+ \frac{1}{2n + 1}(e^t - 1)^{2n} - \&c.$$

44. On a sum of money borrowed, interest is paid at the rate of 5 per cent. After a time £600 of the loan is paid off, and the interest on the remainder reduced to 4 per cent., and the yearly interest is now lessened one-third. What was the sum borrowed?

45. If $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$, find x in terms of y by the method of indeterminate coefficients.

46. Show how the solution of the equation $ax + by = c$ in *positive integers* may be obtained by the properties of

continued and converging fractions. Solve $17x + 19y = 250$, in positive integers.

47. In the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$, find the value of $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma +$ &c., where α, β, γ are the roots of the equation.

48. If $\dfrac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)} + \dfrac{\beta^3}{(\beta-\alpha)(\beta-\gamma)} + \dfrac{\gamma^3}{(\gamma-\alpha)(\gamma-\beta)} = \mathrm{L}$,

$$\frac{\alpha^4}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^4}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^4}{(\gamma-\alpha)(\gamma-\beta)} = \mathrm{M},$$

$$\frac{\alpha^5}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^5}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^5}{(\gamma-\alpha)(\gamma-\beta)} = \mathrm{N},$$

show that $\mathrm{LN} - \mathrm{M}^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 - \alpha\beta\gamma(\alpha + \beta + \gamma)$.

49. Sum to n terms, and to infinity

$$\frac{x}{(1+x)(1+ax)} + \frac{ax}{(1+ax)(1+a^2x)} + \frac{a^2x}{(1+a^2x)(1+a^3x)} + \text{\&c.}$$

50. Show that the result of eliminating x, y, z from the equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0,$$

is the same as the result of elimination from

$$a_1x + a_2y + a_3z = 0,$$
$$b_1x + b_2y + b_3z = 0,$$
$$c_1x + c_2y + c_3z = 0.$$

SECTION II.

PLANE TRIGONOMETRY.

CHAPTER I.

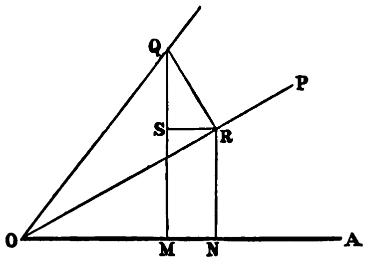
FUNCTIONS OF THE SUM AND DIFFERENCE OF ANGLES. MULTIPLES AND SUB-MULTIPLES OF ANGLES.

1. *To express the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles.*

We shall first take the case where the sum of the angles in question is less than a right angle.

Let $\angle$ AOP = A, $\angle$ POQ = B.

Then $\angle$ AOQ = A + B; and considering OA as the initial line, OQ is *the bounding line of the compound angle* (A + B).



In the *bounding line* OQ take any point Q, and from Q draw QR perpendicular to OP, and meeting it in R; also draw QM, RN perpendiculars to OA, and RS perpendicular to QM, and therefore also parallel to OA.

Then $\angle SQR = 90° - \angle SRQ = \angle SRO = \angle AOP = A$.

Now, sin (A + B)

$$= \sin AOQ = \frac{QM}{OQ} = \frac{SM + QS}{OQ} = \frac{RN + QS}{OQ}$$

$$= \frac{RN}{OQ} + \frac{QS}{OQ}$$

$$= \frac{RN}{OR} \cdot \frac{OR}{OQ} + \frac{QS}{QR} \cdot \frac{QR}{OQ}$$

$$= \sin AOP \cdot \cos POQ + \cos SQR \cdot \sin POQ$$

$$= \sin A \cos B + \cos A \sin B.$$

Again, cos (A + B)

$$= \cos AOQ = \frac{OM}{OQ} = \frac{ON - MN}{OQ}$$

$$= \frac{ON - SR}{OQ} = \frac{ON}{OQ} - \frac{SR}{OQ}$$

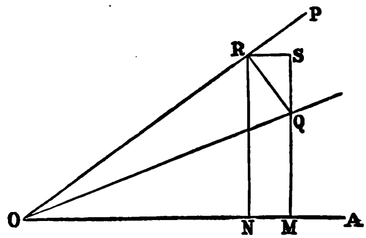
$$= \frac{ON}{OR} \cdot \frac{OR}{OQ} - \frac{SR}{QR} \cdot \frac{QR}{OQ}$$

$$= \cos AOP \cdot \cos POQ - \sin SQR \cdot \sin POQ$$

$$= \cos A \cos B - \sin A \sin B.$$

2. *To express the sine and cosine of the difference of two angles in terms of the sines and cosines of the angles.*

Let ∠ AOP = A, ∠ POQ = B.



Then ∠ AOQ = A − B, and OQ is *the bounding line of the compound angle* (A − B).

In the *bounding line* OQ take any point Q, and from Q draw QR perpendicular to OP, meeting it in R; also draw QM, RN perpendiculars to OA, and RS perpendicular to MQ produced, and therefore also parallel to OA.

Then $\angle RQS = 90° - \angle QRS = \angle PRS = \angle AOP = A.$

$$\text{Now, } \sin(A - B) = \sin AOQ = \frac{QM}{OQ} = \frac{SM - QS}{OQ} = \frac{RN - QS}{OQ}$$

$$= \frac{RN}{OQ} - \frac{QS}{OQ}$$

$$= \frac{RN}{OR} \cdot \frac{OR}{OQ} - \frac{QS}{QR} \cdot \frac{QR}{OQ}$$

$$= \sin AOP \, . \cos POQ - \cos RQS \, . \sin POQ$$

$$= \sin A \cos B - \cos A \sin B.$$

And $\cos(A - B)$

$$= \cos AOQ = \frac{OM}{OQ} = \frac{ON + NM}{OQ}$$

$$= \frac{ON + RS}{OQ} = \frac{ON}{OQ} + \frac{RS}{OQ}$$

$$= \frac{ON}{OR} \cdot \frac{OR}{OQ} + \frac{RS}{QR} \cdot \frac{QR}{OQ}$$

$$= \cos AOP \, . \cos POQ + \sin RQS \, . \sin POQ$$

$$= \cos A \cos B + \sin A \sin B.$$

It is easy to see that the two formulæ just obtained may be obtained from those in the last article by putting $-B$ for B.

We will collect for reference the results of this and the last article:

Sin $(A + B) = \sin A \cos B + \cos A \sin B$.........(1).
Sin $(A - B) = \sin A \cos B - \cos A \sin B$.........(2).
Cos $(A + B) = \cos A \cos B - \sin A \sin B$.........(3).
Cos $(A - B) = \cos A \cos B + \sin A \sin B$.........(4).

3. By means of these formulæ we may easily obtain other useful formulæ.

We have,

(1) + (2), $\sin(A+B)+\sin(A-B)=2\sin A\cos B$......(5).

(1) − (2), $\sin(A+B)-\sin(A-B)=2\cos A\sin B$......(6).

(3) + (4), $\cos(A+B)+\cos(A-B)=2\cos A\cos B$......(7).

(4) − (3), $\cos(A-B)-\cos(A+B)=2\sin A\sin B$......(8).

(The student will be careful to remember that first side of (8) has its terms in an order the reverse of the other three.)

Put $A+B=C$, and $A-B=D$;

and $\therefore\ A=\dfrac{C+D}{2}$, $B=\dfrac{C-D}{2}$,

then we have, from (5), (6), (7), (8),

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\text{..........(9)},$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\text{........(10)},$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\text{........(11)},$$

$$\cos D-\cos C=2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\text{........(12)}.$$

Again, we have,

$\text{Sin}(A+B).\sin(A-B)$

$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$

$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$

$=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$

$=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$......................(13).

And $\cos(A+B).\cos(A-B)$

$=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$

$=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$

$=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$.......................(14).

The formulæ obtained in this article are extremely useful, and are often referred to in trigonometrical transformations.

Thus (i.),

$$\sin 7\theta + \sin 5\theta = \sin(6\theta + \theta) + \sin(6\theta - \theta); \text{ or, by (5),}$$
$$= 2\sin 6\theta \,.\, \cos\theta.$$

Or, at once by (9),

$$\text{Sin } 7\theta + \sin 5\theta = 2\sin\frac{7\theta + 5\theta}{2}\cos\frac{7\theta - 5\theta}{2}$$
$$= 2\sin 6\theta \,.\, \cos\theta.$$

Again (ii.),

$$\cos\theta - \cos 3\theta = \cos(2\theta - \theta) - \cos(2\theta + \theta); \text{ or, by (8),}$$
$$= 2\sin 2\theta \,.\, \sin\theta.$$

Or, at once by (12),

$$\cos\theta - \cos 3\theta = 2\sin\frac{3\theta + \theta}{2}\sin\frac{3\theta - \theta}{2} = 2\sin 2\theta \,.\, \sin\theta.$$

And so (iii.)

$$\sin(\alpha - \beta) + \sin(\beta - \gamma) + (\sin\gamma - \delta) + \sin(\delta - \alpha)$$
$$= 2\sin\frac{\overline{\alpha - \beta} + \overline{\beta - \gamma}}{2}\cos\frac{\overline{\alpha - \beta} - \overline{\beta - \gamma}}{2}$$
$$+ 2\sin\frac{\overline{\gamma - \delta} + \overline{\delta - \alpha}}{2}\cos\frac{\overline{\gamma - \delta} - \overline{\delta - \alpha}}{2}$$
$$= 2\sin\frac{\alpha - \gamma}{2}\cos\frac{\alpha - 2\beta + \gamma}{2} + 2\sin\frac{\gamma - \alpha}{2}\cos\frac{\alpha + \gamma - 2\delta}{2}$$
$$= 2\sin\frac{\alpha - \gamma}{2}\left\{\cos\frac{\alpha - 2\beta + \gamma}{2} - \cos\frac{\alpha + \gamma - 2\delta}{2}\right\}$$
$$= 2\sin\frac{\alpha - \gamma}{2} \times$$
$$\left\{\cos\left(\frac{\alpha - \beta + \gamma - \delta}{2} - \frac{\beta - \delta}{2}\right) - \cos\left(\frac{\alpha - \beta + \gamma - \delta}{2} + \frac{\beta - \delta}{2}\right)\right\};$$

or, by (8),

$$= 2\sin\frac{\alpha - \gamma}{2} \,.\, 2\sin\frac{\alpha - \beta + \gamma - \delta}{2}\sin\frac{\beta - \delta}{2}$$
$$= 4\sin\frac{\alpha - \gamma}{2} \,.\, \sin\frac{\beta - \delta}{2} \,.\, \sin\frac{\alpha - \beta + \gamma - \delta}{2}.$$

4. It will be remembered that formula (1) of Art. 2 was obtained in Vol. I. of this work by a less direct method, and we there found formulæ for $\cos\frac{A}{2}$, $\tan\frac{A}{2}$, &c.

We shall now obtain these formulæ, with others, from the results of the last Art.

We have $\sin(A+B) = \sin A\cos B + \cos A\sin B$,

and $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Put $B = A$, then since $A + B = 2A$, we at once have

$$\sin 2A = \sin A\cos A + \cos A\sin A$$
$$= 2\sin A\cos A \quad\ldots\ldots(1),$$

$$\text{and } \cos 2A = \cos A\cos A - \sin A.\sin A$$
$$\left.\begin{aligned} &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}\right\}\quad\ldots\ldots(2).$$

And so, if we put $\frac{A}{2}$ for A, and therefore A for 2 A, we get

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\quad\ldots\ldots(3),$$

$$\text{and } \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$

$$= 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}\quad\ldots\ldots(4).$$

5. *To find* $\tan(A \pm B)$, $\cot(A \pm B)$.

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A\cos B \pm \cos A\sin B}{\cos A\cos B \mp \sin A\sin B}$$

$$= \frac{\frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B}}{1 \mp \frac{\sin A}{\cos A}.\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}\quad\ldots\ldots(1).$$

And so—

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A} \qquad \text{(2).}$$

COR. 1. Put B = A, and take the upper sign, then,

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \qquad \text{(3),}$$

$$\text{and } \cot 2A = \frac{\cot^2 A - 1}{2\cot A} \qquad \text{(4).}$$

Again, put $\frac{A}{2}$ for A, and therefore A for 2 A, then,

$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}} \qquad \text{(5),}$$

$$\text{and } \cot A = \frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}} \qquad \text{(6).}$$

COR. 2. Put A = 45°, then since tan 45° = 1, we have,

$$\tan(45° \pm B) = \frac{1 \pm \tan B}{1 \mp \tan B}.$$

6. *To expand* sin 3 A, cos 3 A, tan 3 A, cot 3 A.

$$\begin{aligned}
\sin 3A &= \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A \\
&= 2\sin A \cos A \cdot \cos A + (1 - 2\sin^2 A)\sin A \\
&= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A \\
&= 3\sin A - 4\sin^3 A \qquad \text{(1).}
\end{aligned}$$

$$\begin{aligned}
\cos 3A &= \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A \\
&= (2\cos^2 A - 1)\cos A - 2\sin A \cos A \cdot \sin A \\
&= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A \\
&= 4\cos^3 A - 3\cos A \qquad \text{(2).}
\end{aligned}$$

$$\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A};$$

or, dividing numerator and denominator by $\cos^3 A$,

$$\tan 3A = \frac{3 \cdot \frac{\sin A}{\cos A} \cdot \frac{1}{\cos^2 A} - 4 \cdot \frac{\sin^3 A}{\cos^3 A}}{4 - 3 \cdot \frac{1}{\cos^2 A}}$$

$$= \frac{3 \tan A (1 + \tan^2 A) - 4 \tan^3 A}{4 - 3(1 + \tan^2 A)}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \text{ ....(3).}$$

$$\cot 3A = \frac{1}{\tan 3A}; \text{ or, by (3),}$$

$$= \frac{1 - 3\tan^2 A}{3 \tan A - \tan^3 A};$$

or, dividing numerator and denominator by $\tan^3 A$,

$$= \frac{\frac{1}{\tan^3 A} - 3 \cdot \frac{1}{\tan A}}{3 \cdot \frac{1}{\tan^2 A} - 1}$$

$$= \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1} \text{ ....(4).}$$

COR. Put $\frac{A}{3}$ for A, and therefore A for 3 A, we have—

$$\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3},$$

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3},$$

$$\tan A = \frac{3 \tan \frac{A}{3} - \tan^3 \frac{A}{3}}{1 - 3 \tan^2 \frac{A}{3}},$$

$$\cot A = \frac{\cot^3 \frac{A}{3} - 3 \cot \frac{A}{3}}{3 \cot^2 \frac{A}{3} - 1}.$$

7. ***To express the sine and cosine of a multiple angle in terms of sines and cosines of inferior multiples.***

5—II.

We have, by Art. 2,

$$\sin (n+1) A + \sin (n-1) A = 2 \sin n A \cos A$$

$$\therefore \sin (n+1) A = 2 \sin n A \cos A - \sin (n-1) A \quad \ldots\ldots (1).$$

And so

$$\cos (n+1) A = 2 \cos n A \cos A - \cos (n-1) A \quad \ldots\ldots (2).$$

8. *To find the sine, cosine, and tangent, of* (A + B + C).

N (i.) $\sin (A+B+C) = \sin (A+B) \cos C + \cos (A+B) \sin C$

$$= (\sin A \cos B + \cos A \sin B) \cos C + (\cos A \cos B - \sin A \sin B) \sin C$$

$$= \sin A . \cos B \cos C + \sin B . \cos A \cos C + \sin C . \cos A \cos B - \sin A \sin B \sin C.$$

(ii.) $\cos(A+B+C) = \cos (A+B) \cos C - \sin (A+B) \sin C$

$$= (\cos A \cos B - \sin A \sin B) \cos C - (\sin A \cos B + \cos A \sin B) \sin C$$

$$= \cos A \cos B \cos C - \cos A . \sin B \sin C - \cos B . \sin A \sin C - \cos C . \sin A \sin B.$$

(iii.) $\tan(A+B+C) = \dfrac{\sin (A+B+C)}{\cos (A+B+C)}$

$$= \frac{\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C}{\cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin A \sin C - \cos C \sin A \sin B}.$$

Dividing numerator and denominator by cos A cos B cos C, we have,

$$\tan(A+B+C) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C}}{1 - \frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}\cdot\frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \,.\, \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}.$$

Cor. 1. If $A + B + C = 90°$, then, since $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90° = \infty$, we have, from (i.),

$$1 = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C, \text{ or}$$

$$\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B = 1 + \sin A \sin B \sin C \ldots\ldots (1).$$

Also, from (ii.), we similarly have,

$$\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B = \cos A \cos B \cos C \ldots\ldots (2).$$

And from this, dividing each side by cos A cos B cos C, we have,

$$\tan B \tan C + \tan A \tan C + \tan A \tan B = 1 \ldots\ldots (3).$$

Cor. 2. So when $A + B + C = 180°$, the student will easily get the following formulæ:

$$\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B = \sin A \sin B \sin C \ldots\ldots (4).$$

$$\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B = \cos A \cos B \cos C + 1 \ldots\ldots (5).$$

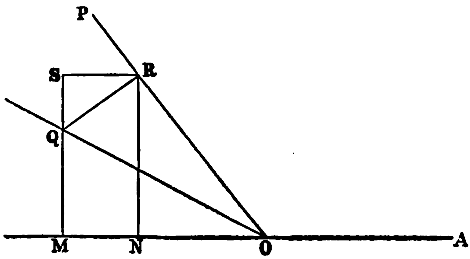
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \ldots\ldots (6).$$

9. *To extend the formulæ of Arts. 1 and 2 to angles greater than a right angle.*

We shall take the case only when A is greater than a right angle, (A + B) being less than two right angles. Other cases will form good exercises for the student, and will not offer much difficulty.

Let $\angle AOP = A$, $\angle POQ = B$;

$\therefore \angle AOQ = A + B$.



Take any point Q in OQ the bounding line of the angle (A + B), and draw QR perpendicular to OP. From R and Q draw RN and QM perpendiculars to AO *produced*, and from R draw RS perpendicular to MQ *produced*.

Then $\angle SQR = 90^\circ - \angle SRQ = \angle PRS$

$= \angle POM = 180^\circ - \angle AOP = 180^\circ - A.$

Now, $\sin(A + B) = \sin AOQ$

$$= \frac{QM}{OQ} = \frac{SM - SQ}{OQ} = \frac{RN - SQ}{OQ}$$

$$= \frac{RN}{OQ} - \frac{SQ}{OQ} = \frac{RN}{OR} \cdot \frac{OR}{OQ} - \frac{SQ}{QR} \cdot \frac{QR}{OQ}$$

$$= \sin AOP . \cos POQ - \cos SQR . \sin POQ$$

$$= \sin A \cos B - \cos(180^\circ - A) \sin B.$$

Hence, since $\cos(180^\circ - A) = -\cos A$, we easily get—

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

And in the same way we may obtain expressions for *cos (A + B), and the* sine and cosine of (A − B).

10. *To find the trigonometrical ratios of* 18°.

Put $\theta = 18°$, then $5\,\theta = 90°$,

and $\therefore\ 2\,\theta = 90° - 3\,\theta$

$\therefore\ \sin 2\,\theta = \sin\,(90° - 3\,\theta) = \cos 3\,\theta$, or

$2 \sin \theta \,.\, \cos \theta = 4 \cos^3 \theta - 3 \cos \theta$;

or, dividing each side by $\cos \theta$,

$$2 \sin \theta = 4 \cos^2 \theta - 3$$
$$= 4\,(1 - \sin^2 \theta) - 3\text{; or}$$
$$4 \sin^2 \theta + 2 \sin \theta = 1\text{, or}$$
$$\sin^2 \theta + \tfrac{1}{2} \sin \theta = \tfrac{1}{4}.$$

Solving this quadratic for $\sin \theta$, we have—

$$\sin \theta = \frac{\pm \sqrt{5} - 1}{4}.$$

Now, as $\theta = 18°$, the *sine* must be positive, and we must therefore take the upper sign;

Hence we have $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$.

From this result, remembering that cos 18° is positive, we get—

$$\cos 18° = \sqrt{1 - \left(\frac{\sqrt{5} - 1}{4}\right)} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

And hence we may easily obtain the values of tan 18°, cot 18°, &c.

11. *To find the trigonometrical ratios of* 36°.

We have,

$$\cos 36° = \cos 2\,(18°) = 1 - 2 \sin^2 18°$$
$$= 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2 = 1 - \frac{3 - \sqrt{5}}{4} = \frac{\sqrt{5} + 1}{4}.$$

And hence,

$$36° = \sqrt{1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

COR. Since sin 18° = cos 72°, sin 36° = cos 54°, &c., we may arrange the results of the last two articles, thus:

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4} = \cos 72^\circ,$$

$$\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \sin 72^\circ,$$

$$\sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \cos 54^\circ,$$

$$\cos 36^\circ = \frac{\sqrt{5} + 1}{4} = \sin 54^\circ.$$

Inverse Functions.

12. When the *sine, cosine, tangent,* &c., of an unknown angle are given, it is sometimes convenient to express the angle by an inverse function.

Thus, by $\sin^{-1} s$ is meant the angle whose *sine* is s;
by $\cos^{-1} c$ " " *cosine* is c; and
by $\tan^{-1} t$ " " *tangent* is t.

13. *To show that*

$$\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \frac{t_1 + t_2}{1 - t_1 t_2}.$$

Let $\tan^{-1} t_1 = A$, and $\therefore t_1 = \tan A$.
Also, let $\tan^{-1} t_2 = B$, and $\therefore t_2 = \tan B$. } ....(1).

Now $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$.

Hence, by Art. 12,

$$A + B = \tan^{-1} \frac{\tan A + \tan B}{1 - \tan A . \tan B}.$$

Or, substituting from (1),

$$\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \frac{t_1 + t_2}{1 - t_1 t_2}.$$

COR. Hence also,

$$\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2},$$

$$\text{and } 2\tan^{-1} t_1 = \tan^{-1} \frac{2t}{1 - t^2}.$$

14. *To show that*

$$\sin^{-1} s_1 + \sin^{-1} s_2 = \sin^{-1}\left\{s_1\sqrt{1 - s_2^2} + s_2\sqrt{1 - s_1^2}\right\}.$$

$$\left.\begin{array}{l}\text{Let } \sin^{-1} s_1 = A, \text{ and } \therefore s_1 = \sin A. \\ \text{Also let } \sin^{-1} s_2 = B, \text{ and } \therefore s_2 = \sin B.\end{array}\right\} \ldots\ldots\ldots\ldots(1).$$

Then we have,

$$\cos A = \sqrt{1 - s_1^2}, \text{ and } \cos B = \sqrt{1 - s_2^2} \ldots\ldots\ldots\ldots(2).$$

Now,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B;$$

hence, by Art. 12,

$$A + B = \sin^{-1}\{\sin A \cos B + \cos A \sin B\}.$$

Or, substituting from (1) and (2),

$$\sin^{-1} s_1 + \sin^{-1} s_2 = \sin^{-1}\left\{s_1\sqrt{1 - s_2^2} + s_2\sqrt{1 - s_1^2}\right\}.$$

COR. So we get,

$$\cos^{-1} s_1 + \cos^{-1} s_2 = \cos^{-1}\left\{s_1 s_2 - \sqrt{1 - s_1^2}.\sqrt{1 - s_2^2}\right\}.$$

15. We shall now give a few examples of questions solved by the aid of the preceding formulæ.

Ex. 1. Show that

$$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}.$$

[As the required expression is to contain sin 3 A in the numerator, and sin 5 A in the denominator, we shall not alter the terms containing these quantities in the given expression. We observe that the first and last terms of the numerator may be thrown into the form—

$$\sin (3A - 2A) + \sin (3A + 2A),$$

where we have the sum of the *sines* of the difference and sum of two *angles.* We therefore apply the formulæ of Art. 2. And a similar plan may be pursued in the denominator. Thus :\

$$\frac{\sin A + 2 \sin 3 A + \sin 5 A}{\sin 3 A + 2 \sin 5 A + \sin 7 A}$$

$$= \frac{\sin (3 A - 2 A) + \sin (3 A + 2 A) + 2 \sin 3 A}{\sin (5 A - 2 A) + \sin (5 A + 2 A) + 2 \sin 5 A};$$

or, by Art. 3,

$$= \frac{2 \sin 3 A \cos 2 A + 2 \sin 3 A}{2 \sin 5 A \cos 2 A + 2 \sin 5 A}$$

$$= \frac{2 \sin 3 A (\cos 2 A + 1)}{2 \sin 5 A (\cos 2 A + 1)} = \frac{\sin 3 A}{\sin 5 A}.$$

Ex. 2. Prove that

$$\frac{\cos A - \cos 5 A}{\sin 5 A - \sin A} = \tan 3 A.$$

[Our result is to be tan 3 A or $\frac{\sin 3 A}{\cos 3 A}$; we must then try to transform the denominator so as to contain the factor cos 3 A. Throwing it into the form of the difference of the *sines* of the sum and difference of two angles, we shall easily get the required result. And a similar method will apply to the numerator. Thus :]

$$\frac{\cos A - \cos 5 A}{\sin 5 A - \sin A} = \frac{\cos (3 A - 2 A) - \cos (3 A + 2 A)}{\sin (3 A + 2 A) - \sin (3 A - 2 A)}$$

$$= \frac{2 \sin 3 A \sin 2 A}{2 \cos 3 A \sin 2 A} = \tan 3 A.$$

Ex. 3. Show that

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = 45°.$$

By Art. 13, Cor.,

$$\tan^{-1} 1 - \tan^{-1} \tfrac{1}{5} = \tan^{-1} \frac{1 - \frac{1}{5}}{1 + 1 \left(\frac{1}{5}\right)} = \tan^{-1} \tfrac{2}{3},$$

$$\tan^{-1} \tfrac{2}{3} - \tan^{-1} \tfrac{1}{5} = \tan^{-1} \frac{\frac{2}{3} - \frac{1}{5}}{1 + \frac{2}{3} \left(\frac{1}{5}\right)} = \tan^{-1} \tfrac{7}{17},$$

$$\tan^{-1} \tfrac{7}{17} - \tan^{-1} \tfrac{1}{5} = \tan^{-1} \frac{\frac{7}{17} - \frac{1}{5}}{1 + \frac{7}{17} \left(\frac{1}{5}\right)} = \tan^{-1} \tfrac{9}{46},$$

$$\tan^{-1} \tfrac{9}{46} - \tan^{-1} \tfrac{1}{5} = \tan^{-1} \frac{\frac{9}{46} - \frac{1}{5}}{1 + \frac{9}{46} \left(\frac{1}{5}\right)} = \tan^{-1} \left(- \tfrac{1}{239}\right).$$

Adding together these equations, and reducing, we have,

$$\tan^{-1} 1 - 4 \tan^{-1} \tfrac{1}{5} = \tan^{-1}(-\tfrac{1}{239}) = -\tan^{-1} \tfrac{1}{239}.$$

Or, transposing,

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = \tan^{-1} 1.$$

Now $\tan^{-1} 1$ is the angle whose tangent is unity, and therefore $\tan^{-1} 1 = 45°$.

Hence we have

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = 45°.$$

Ex. 4. Eliminate θ from the equations,

$$(a + b) \tan (\theta - \phi) = (a - b) \tan (\theta + \phi) \ldots\ldots\ldots (1),$$

$$a \cos 2\phi + b \cos 2\theta = c \ldots\ldots\ldots (2).$$

From (1),

$$\frac{a + b}{a - b} = \frac{\tan (\theta + \phi)}{\tan (\theta - \phi)} = \frac{\sin (\theta + \phi) \cos (\theta - \phi)}{\cos (\theta + \phi) \sin (\theta - \phi)}.$$

Hence,

$$\frac{a}{b} = \frac{\sin (\theta + \phi) \cos (\theta - \phi) + \cos (\theta + \phi) \sin (\theta - \phi)}{\sin (\theta + \phi) \cos (\theta - \phi) - \cos (\theta + \phi) \sin (\theta - \phi)}$$

$$= \frac{\sin\{(\theta + \phi) + (\theta - \phi)\}}{\sin\{(\theta + \phi) - (\theta - \phi)\}} = \frac{\sin 2\theta}{\sin 2\phi}$$

$$\therefore b \sin 2\theta = a \sin 2\phi \ldots \ldots\ldots (3).$$

We have also from (2),

$$b \cos 2\theta = c - a \cos 2\phi \ldots \ldots\ldots (4).$$

Squaring (3) and (4), and adding, we have,

$$b^2 (\sin^2 2\theta + \cos^2 2\theta) = a^2 \sin^2 2\phi + (c - a \cos 2\phi)^2, \text{ or}$$

$$b^2 = a^2 (\sin^2 2\phi + \cos^2 2\phi) + c^2 - 2ac \cos 2\phi.$$

$$\therefore b^2 = a^2 + c^2 - 2ac \cos 2\phi, \text{ or}$$

$2ac \cos 2\phi = a^2 + c^2 - b^2$, the relation required.

Ex. I.

1. *If* $\sin a = \dfrac{1}{\sqrt{2}}$, and $\sin \beta = \tfrac{1}{2}$, then $\sin (a - \beta) = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$.

2. If $\sin \alpha = \frac{1}{3}$, and $\sin \beta = \frac{1}{5}$, then $\cos(\alpha + \beta)$

$$= \frac{8\sqrt{3} - 1}{15}.$$

3. Show that $\cos 2x = \frac{\sqrt{3}}{2}$, and $\sin 2x = \frac{1}{2}$, when

$$\sin x = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

4. When $\cos x = \frac{1}{2}$, and $\cos y = \frac{\sqrt{3} + 1}{2\sqrt{2}}$, then $\cos(x - y)$

$$= \frac{1}{\sqrt{2}}.$$

5. If $\sin \alpha = \frac{1}{4}(\sqrt{6} - \sqrt{2})$, show that

$$\tan(45° + \alpha) + \tan(45° - \alpha) = \sqrt{3} - \frac{1}{\sqrt{3}}.$$

6. Given $\sin 72° = \frac{\sqrt{5 + \sqrt{5}}}{2\sqrt{2}}$, show that

$$\tan 36° = \sqrt{5 - 2\sqrt{5}}.$$

7. Given $\sin 18° = \frac{\sqrt{5} - 1}{4}$, show that

$$\sin 9° = \sqrt{\frac{4 - \sqrt{10 + 2\sqrt{5}}}{8}}, \quad \cos 9° = \sqrt{\frac{4 + \sqrt{10 + 2\sqrt{5}}}{8}}.$$

8. Find the trigonometrical ratios of $7°\frac{1}{2}$, $22°\frac{1}{2}$, $37°\frac{1}{2}$.

9. If $\alpha = 7°\frac{1}{2}$, show that $\frac{\cos 3\alpha}{\cos 7\alpha} = \frac{\sin 9\alpha}{\sin 5\alpha}$.

10. If $\sin \alpha = \frac{4}{5}$, $\sin \beta = \frac{5}{13}$, $\sin \gamma = \frac{16}{65}$, show that

$$\sin(\alpha + \beta + \gamma) = 1.$$

11. If $\tan A = \frac{1}{x - 1}$, and $\cot B = x$, then

$$\tan(A - B) = \frac{1}{x^2 - x + 1}.$$

Adding together these equations, and reducing, we have,

$$\tan^{-1} 1 - 4 \tan^{-1} \tfrac{1}{5} = \tan^{-1}(-\tfrac{1}{239}) = -\tan^{-1} \tfrac{1}{239}.$$

Or, transposing,

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = \tan^{-1} 1.$$

Now $\tan^{-1} 1$ is the angle whose tangent is unity, and therefore $\tan^{-1} 1 = 45°$.

Hence we have

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = 45°.$$

Ex. 4. Eliminate θ from the equations,

$$(a + b) \tan(\theta - \phi) = (a - b) \tan(\theta + \phi) \quad \ldots\ldots\ldots(1),$$

$$a \cos 2\phi + b \cos 2\theta = c \quad \ldots\ldots\ldots(2).$$

From (1),

$$\frac{a + b}{a - b} = \frac{\tan(\theta + \phi)}{\tan(\theta - \phi)} = \frac{\sin(\theta + \phi)\cos(\theta - \phi)}{\cos(\theta + \phi)\sin(\theta - \phi)}.$$

Hence,

$$\frac{a}{b} = \frac{\sin(\theta + \phi)\cos(\theta - \phi) + \cos(\theta + \phi)\sin(\theta - \phi)}{\sin(\theta + \phi)\cos(\theta - \phi) - \cos(\theta + \phi)\sin(\theta - \phi)}$$

$$= \frac{\sin\{(\theta + \phi) + (\theta - \phi)\}}{\sin\{(\theta + \phi) - (\theta - \phi)\}} = \frac{\sin 2\theta}{\sin 2\phi}.$$

$$\therefore b \sin 2\theta = a \sin 2\phi \ldots \quad \ldots\ldots\ldots(3).$$

We have also from (2),

$$b \cos 2\theta = c - a \cos 2\phi \ldots \quad \ldots\ldots\ldots(4).$$

Squaring (3) and (4), and adding, we have,

$$b^2 (\sin^2 2\theta + \cos^2 2\theta) = a^2 \sin^2 2\phi + (c - a \cos 2\phi)^2, \text{ or}$$

$$b^2 = a^2 (\sin^2 2\phi + \cos^2 2\phi) + c^2 - 2ac \cos 2\phi.$$

$\therefore b^2 = a^2 + c^2 - 2ac \cos 2\phi$, or

$2ac \cos 2\phi = a^2 + c^2 - b^2$, the relation required.

Ex. I.

1. If $\sin a = \dfrac{1}{\sqrt{2}}$, and $\sin \beta = \tfrac{1}{2}$, then $\sin(a - \beta) = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$.

2. If $\sin \alpha = \frac{1}{3}$, and $\sin \beta = \frac{1}{5}$, then $\cos (\alpha + \beta)$

$$= \frac{8\sqrt{3} - 1}{15}.$$

3. Show that $\cos 2x = \frac{\sqrt{3}}{2}$, and $\sin 2x = \frac{1}{2}$, when

$$\sin x = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

4. When $\cos x = \frac{1}{2}$, and $\cos y = \frac{\sqrt{3} + 1}{2\sqrt{2}}$, then $\cos (x - y)$

$$= \frac{1}{\sqrt{2}}.$$

5. If $\sin \alpha = \frac{1}{4}(\sqrt{6} - \sqrt{2})$, show that

$$\tan (45^\circ + \alpha) + \tan (45^\circ - \alpha) = \sqrt{3} - \frac{1}{\sqrt{3}}.$$

6. Given $\sin 72^\circ = \frac{\sqrt{5 + \sqrt{5}}}{2\sqrt{2}}$, show that

$$\tan 36^\circ = \sqrt{5 - 2\sqrt{5}}.$$

7. Given $\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$, show that

$$\sin 9^\circ = \sqrt{\frac{4 - \sqrt{10 + 2\sqrt{5}}}{8}}, \quad \cos 9^\circ = \sqrt{\frac{4 + \sqrt{10 + 2\sqrt{5}}}{8}}.$$

8. Find the trigonometrical ratios of $7^\circ\frac{1}{2}$, $22^\circ\frac{1}{2}$, $37^\circ\frac{1}{2}$.

9. If $\alpha = 7^\circ\frac{1}{2}$, show that $\frac{\cos 3\alpha}{\cos 7\alpha} = \frac{\sin 9\alpha}{\sin 5\alpha}$.

10. If $\sin \alpha = \frac{4}{5}$, $\sin \beta = \frac{5}{13}$, $\sin \gamma = \frac{16}{65}$, show that

$$\sin (\alpha + \beta + \gamma) = 1.$$

11. If $\tan A = \frac{1}{x - 1}$, and $\cot B = x$, then

$$\tan (A - B) = \frac{1}{x^2 - x + 1}.$$

42. $\sin^2 A \cdot \sin\frac{B+C}{2}\sin\frac{B-C}{2}$

$$+\sin^2 B\sin\frac{C+A}{2}\sin\frac{C-A}{2}$$

$$+\sin^2 C\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$= (\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A).$

43. $\sin A \cdot \sin\frac{B+C}{2}\sin\frac{B-C}{2}$

$$+\sin B \cdot \sin\frac{C+A}{2}\sin\frac{C-A}{2}$$

$$+\sin C \cdot \sin\frac{A+B}{2}\sin\frac{A-B}{2} = 0.$$

44. $\cos(2A + B)$
$= \cos(A + 2B)\sec(A - B) - \sin(2A + B)\tan(A - B).$

45. $\sin 2(\alpha - \beta) + \sin 2(\beta - \gamma) + \sin 2(\gamma - \alpha)$
$= 4\sin(\alpha - \beta)\sin(\beta - \gamma)\sin(\alpha - \gamma).$

46. $\cos 2(\alpha - \beta) + \cos 2(\beta - \gamma) + \cos 2(\gamma - \alpha)$
$= 4\cos(\alpha - \beta)\cos(\beta - \gamma)\cos(\gamma - \alpha) - 1.$

47. $\tan(\alpha + \beta + \gamma)$

$$= \frac{\sin 3\alpha\sin(\beta-\gamma)+\sin 3\beta\sin(\gamma-\alpha)+\sin 3\gamma\sin(\alpha-\beta)}{\cos 3\alpha\sin(\beta-\gamma)+\cos 3\beta\sin(\gamma-\alpha)+\cos 3\gamma\sin(\alpha-\beta)}.$$

48. $\frac{1}{2\sin\alpha -}\;\frac{1}{2\sin\alpha -}\;\frac{1}{2\sin\alpha -}\cdots\cdots\frac{1}{2\sin\alpha + x}$

$$= \frac{\sin n\alpha + x\cos\overline{n-1}\cdot\alpha}{x\sin n\alpha - \cos\overline{n+1}\cdot\alpha},\ n \text{ even};$$

$$\text{and } = \frac{\cos n\alpha - x\sin\overline{n-1}\cdot\alpha}{x\cos n\alpha + \sin\overline{n+1}\cdot\alpha},\ n \text{ odd},$$

where n is the number of quotients of the continued fraction.

If A, B, C are the angles of a triangle, show that:

49. $\cos A + \cos B + \cos C$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1.$$

50. $\text{Sin A} + \sin \text{B} + \sin \text{C}$

$$= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

51. $\text{Cot} \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$

$$= \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

52. $\text{Sin}^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}$

$$= 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

53. $\text{Cos}^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}$

$$= 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

54. $\text{Sin } 2\text{A} + \sin 2\text{B} + \sin 2\text{C}$

$$= 4 \sin \text{A} \,.\, \sin \text{B} \,.\, \sin \text{C}.$$

55. $\text{Cos } 2\text{A} + \cos 2\text{B} + \cos 2\text{C}$

$$= -4 \cos \text{A} \cos \text{B} \cos \text{C} - 1.$$

56. $\text{Sin} \frac{A + B}{2} + \sin \frac{A + C}{2} + \sin \frac{B + C}{2}$

$$= 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$$

57. $\text{Sin} \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}$

$$+ 4 \sin \frac{A + B}{4} \sin \frac{A + C}{4} \sin \frac{B + C}{4} = 1.$$

58. $\text{Cot A} \cot \text{B} + \cot \text{A} \cot \text{C} + \cot \text{B} \cot \text{C} = 1.$

59. $\text{Tan} \frac{A}{2} \tan \frac{B}{2} + \tan \frac{A}{2} \tan \frac{C}{2}$

$$+ \tan \frac{B}{2} \tan \frac{C}{2} = 1.$$

60. $\operatorname{Cos}^2 \frac{C}{2}$

$$= \frac{(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C)}{4 \sin B \sin C}.$$

61. $\operatorname{Tan}^2 \frac{A}{2}$

$$= \frac{(\sin A + \sin C - \sin B)(\sin A + \sin B - \sin C)}{(\sin A + \sin B + \sin C)(\sin B + \sin C - \sin A)}$$

62. $\operatorname{Cot} mA \cot mB + \cot mA \cot mB$
$+ \cot mB \cot mC = 1.$

If $A + B + C = 90°$, show that:

63. $\operatorname{Cos} 2A + \cos 2B + \cos 2C$
$= 1 + 4 \sin A \sin B \sin C.$

64. $\operatorname{Tan} A + \tan B + \tan C$
$= \tan A \tan B \tan C + \sec A \sec B \sec C.$

65. $\frac{\operatorname{Cos} B + \sin C - \sin A}{\operatorname{Cos} A + \sin C - \sin B} = \frac{1 + \tan \frac{1}{2} B}{1 + \tan \frac{1}{2} A}.$

66. $(\operatorname{Tan} A + \tan B + \tan C)(\cot A + \cot B + \cot C)$
$= 1 + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$

If α, β, γ be in A. P., show that:

67. $\operatorname{Sin} \alpha - \sin \gamma = 2 \sin (\alpha - \beta) \cos B.$

68. $\frac{\operatorname{Tan} \beta}{\operatorname{Tan} (\beta - \gamma)} = \frac{\sin \alpha + \sin \gamma}{\sin \alpha - \sin \gamma} = \frac{\tan \frac{1}{2}(\alpha + \gamma)}{\tan \frac{1}{2}(\alpha - \gamma)}.$

69. $\frac{1}{\operatorname{Tan} \alpha + \tan \gamma} + \frac{1}{2} \tan \beta$

$$= \frac{1}{\cot \alpha + \cot \gamma} + \frac{1}{2} \cot \beta.$$

70. If $\alpha + \beta + \gamma = 180°$, and the sines and cosines of α, β, γ be in A. P., prove that:

(1) $\operatorname{Sin} \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2} \gamma = \sin \frac{1}{2} \alpha \sin \frac{1}{2} (\beta - \gamma).$

(2) $\operatorname{Cos} \beta = 4 \cos \frac{1}{2} \alpha \sin \frac{1}{2} \beta \cos \frac{1}{2} \gamma + 1.$

Prove that

71. $\text{Tan}^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45°.$

72. $\text{Tan}^{-1}\frac{4}{3} + \tan^{-1}7 = 135°.$

73. $\text{Sin}^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}\frac{3}{\sqrt{10}} = 45°.$

74. $\text{Tan}^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{3}{41} + \tan^{-1}\frac{1}{5} = 45°.$

75. $2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right) = \cos^{-1}\frac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}.$

76. $\text{Cos}^{-1}\frac{\tan\theta + \tan\alpha}{1 + \tan\theta\tan\alpha}$

$$= 2\tan^{-1}\left\{\sqrt{\tan(45° - \theta)\tan(45° - \alpha)}\right\}.$$

77. If $1 + \cos A + \cos B + \cos C = 0$, where $A + B + C = 0$, show that either A, B, or C is an odd multiple of 180°.

78. If $A + B + C = (3 \pm 1)\frac{\pi}{2}$,

$$1 \pm 2\cos A\cos B\cos C = \cos^2 A + \cos^2 B + \cos^2 C;$$

but, if $A + B + C = (2 \pm 1)\frac{\pi}{2}$,

$$1 \pm 2\sin A\sin B\sin C = \sin^2 A + \sin^2 B + \sin^2 C.$$

79. Show that, unless $A = \frac{2x + 1}{r}\pi$,

$$\frac{\sin(n - r)A + 2\sin nA + \sin(n + r)A}{\sin(m - r) + 2\sin mA + \sin(m + r)A} = \frac{\sin nA}{\sin mA},$$

and explain the reason for the exception.

80. Given $x = a\cos(\phi + \alpha)$, $y = b\cos(\phi + \beta)$, $z = c\cos(\phi + \gamma)$, show that

$$\frac{x}{a}\sin(\beta - \gamma) + \frac{y}{b}\sin(\gamma - \alpha) + \frac{z}{c}\sin(\alpha - \beta) = 0.$$

81. If the tangents of the angles of a triangle are in geometric progression, so that $\tan A = r \tan B = r^2 \tan C$, show that

$$\tan C = \left(\frac{1 + r + r^2}{r^3}\right)^{\frac{1}{2}}, \text{ and that } \sin 2C = r \sin 2A.$$

82. Eliminate θ from the equations,

$$a \cos(\theta + \alpha) = x, \text{ and } b \cos(\theta + \beta) = y.$$

83. Prove that in any triangle,

$$(\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C)(\cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C - \cos \tfrac{1}{2}A)$$
$$\times (\cos \tfrac{1}{2}C + \cos \tfrac{1}{2}A - \cos \tfrac{1}{2}B)(\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B - \cos \tfrac{1}{2}C)$$
$$= 4 \cos^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C.$$

84. Transform the equation,

$$\alpha^2 \cos^4 \tfrac{1}{2}A + \beta^2 \cos^4 \tfrac{1}{2}B + \gamma^2 \cos^4 \tfrac{1}{2}C$$
$$- 2\alpha\beta \cos^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}B - 2\beta\gamma \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C$$
$$- 2\gamma\alpha \cos^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}A = 0,$$

into the form

$$(l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C)$$
$$= p(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C),$$

determining the values of l, m, n, p.

85. If $1, \frac{p_1}{q_1}, \frac{p_2}{q_2}$, &c., be successive convergents to $\sqrt{2}$, then will

$$2 \tan^{-1} \frac{q_{n-1}}{q_n} + (-1)^n \tan^{-1} \frac{1}{q_{2n}} = \frac{\pi}{4}.$$

86. Given $\cos \theta = \frac{\sin \beta}{\sin \alpha}$, $\cos \phi = \frac{\sin \gamma}{\sin \alpha}$, $\cos(\theta + \phi) = \sin \beta \sin \gamma$, then

$$\tan^2 \alpha = \tan^2 \beta + \tan^2 \gamma.$$

87. Given that $\tan 3\theta = (2 \pm \sqrt{3}) \tan \theta$, show that $\tan^2 3\theta = 1$.

88. If $\tan \frac{1}{2}\alpha = \tan^3 \frac{1}{2}\beta$, and $\tan \beta = 2 \tan \phi$: then will $2\phi = \alpha + \beta$.

89. If $a\cos(x+\alpha) = b\cos(x+\beta) = c\cos(x+\gamma)$; then

$$\cos(\alpha-\beta) = \frac{c^2-a^2-b^2}{2ab}, \text{ and } \tan\gamma = \frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}.$$

90. If $\sin(\beta + x) + \sin(2\alpha - \beta + x)$
$= (\cos^2\beta - \cos^2\alpha)\sin(\alpha + 2x) - \sin\alpha(\sin^2\beta - \sin^2\alpha)$;
then will

$$\sec x = \sin(\alpha+\beta)\tan(\alpha-\beta).$$

91. If $\sin\frac{\theta}{2} = m\sin\frac{\phi}{2}$, and $\theta + \phi = \alpha$, then

$$\tan\tfrac{1}{4}(\theta-\phi) = \frac{m-1}{m+1}\tan\tfrac{1}{4}\alpha.$$

92. If $\tan\theta = m\tan\phi$, and $\theta + \phi = \alpha$, then

$$\sin\alpha = \frac{m+1}{m-1}\sin(\theta-\phi).$$

93. If $\{1 + m\cos(x+z)\}\sin y = (1 + m\cos z)\sin(x+y)$, then $\tan\frac{1}{2}x = \cot y + m\cos(y-z)\operatorname{cosec} y$.

94. If $2\tan(x-y-z) + \tan z = \tan(x-z)$, then

$$\tan z = \frac{\sin y - \sin(x-y)\cos x}{\sin(x-y)\sin x}.$$

95. If $\sin\theta = \cos^2\alpha - \sin^2\alpha\sqrt{1-c^2\cos^2\theta}$, then

$$1 - \tan\frac{\theta}{2} = \pm\left(1 + \tan\frac{\theta}{2}\right)\tan\alpha\sqrt{1-c^2\sin^2\alpha}.$$

96. If $\tan\alpha = \tan^3\frac{1}{2}\theta$, and $\cos 2\theta = \frac{1}{3}(2m^2 - 5)$, then will $\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha = \left(\frac{2}{m}\right)^{\frac{2}{3}}$.

97. If $\qquad x\cos\phi = (1 - x\sin\phi)\tan\alpha$,
and $\qquad x - \sin\phi = \cos\phi\,.\tan\beta$,
then will $\qquad \alpha - \beta = \phi$.

98. If $m - \operatorname{cosec}\theta = \sin\theta$, $n = \sec\theta - \cos\theta$, then will

$$m^{\frac{2}{3}} n^{\frac{2}{3}} \left(m^{\frac{2}{3}} + n^{\frac{2}{3}}\right) = 1.$$

99. If $\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$,
where $\alpha + \beta + \gamma = 180°$,
then will $\cot\theta = \cot\alpha + \cot\beta + \cot\gamma$,
and $\cot^2\theta = \cot^2\alpha + \cot^2\beta + \cot^2\gamma + 2$.

100. If $\cos(\phi - \theta + \alpha)\cos(\theta - \alpha) = \cos(\phi - \theta - \alpha)\cos(\theta + \alpha)$, show that each of these quantities

$$= \tfrac{1}{2}\left\{\cos\phi + \cos 2\alpha \, . \cos(2\theta - \phi)\right\}.$$

CHAPTER II.

CIRCULAR MEASURE. GENERAL VALUES OF ANGLES.

Circular Measure.

16. We have explained how angles are measured by degrees and by grades. There is another method of measuring angles, and one much used in the higher branches of Mathematical Analysis. The fundamental proposition we have to establish is as follows:—*If the vertex of an angle be taken as centre, and an arc be drawn cutting the lines forming the angle, then the angle is measured by the ratio which the length of the arc intercepted between the lines bears to the radius.*

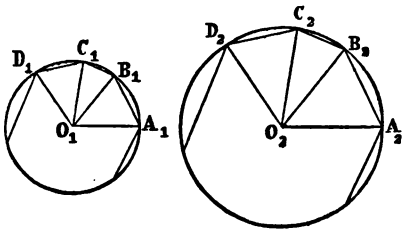
We shall first prove the following preliminary propositions.

17. *The circumference of a circle bears to its diameter a constant ratio.*

Let two circles be drawn with centres O_1 and O_2.

And inscribe in the former a polygon having any number of sides A_1B_1, B_1C_1, C_1D_1, &c.

Also, inscribe in the latter a similar polygon $A_2B_2C_2D_2$, &c. Join O_1A_1, O_1B_1, &c.



Then evidently, $$\frac{A_1B_1}{A_1O_1} = \frac{A_2B_2}{A_2O_2},$$

And so, $$\frac{B_1C_1}{A_1O_1} = \frac{B_2C_2}{A_2O_2}, \text{ \&c.}$$

Hence, by addition,

$$\frac{A_1B_1 + B_1C_1 + \text{\&c.}}{A_1O_1} = \frac{A_2B_2 + B_2C_2 + \text{\&c}}{A_2O_2}.$$

Now, suppose the sides of the polygon to diminish indefinitely, and their number to increase indefinitely.

Then, the sum of the sides of the polygon in each circle will be the circumference of the circles. Hence, we have,

$$\frac{\text{circumference of } A_1B_1C_1D_1\text{\&c.}}{\text{rad. } A_1O_1} = \frac{\text{circumference of } A_2B_2C_2D_2 \text{ \&c.}}{\text{rad. } A_2O_2}$$

Hence the circumference of a circle bears to its diameter a constant ratio.

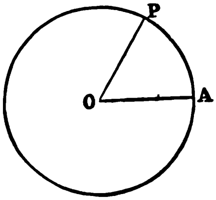
It is usual to represent this ratio by the symbol π.

Hence the circumference of a circle $= 2\, r\pi$.

The value of π will be computed farther on. Its value is 3·14159 . . . . , converging fractions to which are

$$\frac{1}{3}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \text{ \&c.}$$

18. *To find the angle whose arc is equal in length to the radius of the circle.*



Let a circle be drawn with centre O and radius OA, and let the angle AOP be such that its arc AP is equal in length to the radius.

Then we have

$$\text{Arc AP} = r \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now (Euc. VI. 33) angles at the centre are as the arcs on which they stand; and hence

$$\frac{\angle \text{AOP}}{4 \text{ right angles}} = \frac{\text{arc AP}}{\text{circumference of circle}}, \text{ or}$$

$$\frac{\angle \text{AOP}}{360^\circ} = \frac{r}{2r\pi} = \frac{1}{2\pi}$$

$$\therefore \angle \text{AOP} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}.$$

Now, by the last article, π is a constant quantity; and hence the angle corresponding to the arc which is equal in length to the radius has the **constant value** $\frac{180^\circ}{\pi}$, whatever be the radius of the circle.

If we take this angle as a unit or standard of reference, then any other angle may be expressed in terms of this unit.

Suppose a given angle A to be θ times this standard angle.

Then θ is called the **circular measure** of the angle A.

Cor. 1. Hence if θ be the circular measure of an angle A, we have

$$\text{A}^\circ = \frac{180^\circ}{\pi}\theta = 57^\circ{\cdot}29577 \times \theta.$$

Sometimes it is convenient to express the angle at once in seconds; we then have, since $57^\circ{\cdot}29577 = 206265''$ nearly,

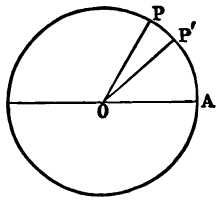
$$\therefore \text{angle} = 206265'' \times \theta.$$

COR. 2. Number of grades in angle A

$$= \frac{200}{\pi}\theta = 63 \cdot 66197 \ldots \times \theta.$$

19. *The circular measure of an angle is the ratio of the length of the arc to the radius; or,*

$$\textit{circular measure} = \frac{arc}{radius}.$$



Let AP be an arc which is equal in length to the radius;

And let AOP′ be an angle whose circular measure is required.

Then, by Art. 18,

$$\text{Circular measure of } \angle\ AOP' = \frac{\angle AOP'}{\angle AOP};\ \text{or (Euc. VI. 33)},$$

$$= \frac{\text{arc AP}'}{\text{arc AP}};$$

But arc AP = radius, and hence

$$\text{Circular measure of } \angle\ AOP' = \frac{\text{arc}}{\text{radius}}.$$

COR. If a represent the length of the arc, we have

$$\theta = \frac{a}{r}, \text{ or } a = r\theta.$$

It is sometimes convenient to represent an angle by its circular measure; thus, since

$$\text{circular measure of } 180^\circ = \frac{r\pi}{r} = \pi,$$

it is convenient to use π instead of 180°.

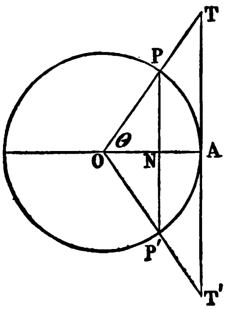
We may then write

$$\sin A = \sin(\pi - A)$$
$$\cos A = -\cos(\pi - A), \text{ \&c.}$$

In the same way we allow the Greek letters θ, ϕ, ψ, &c., to represent the circular measure, and the angles themselves.

The student will have no difficulty in this respect, when he considers that the *unit* of circular measure is $\frac{180°}{\pi}$.

20. *When* $\theta = 0$, *the limit of each of the ratios* $\frac{\sin\theta}{\theta}$, *and* $\frac{\tan\theta}{\theta}$ *is unity.*



Let OA be an initial line, and AOP any angle, the arc corresponding to which is AP.

Make an angle AOP′ equal to AOP.

Join PNP′; it will cut OA at right angles in N.

Draw the line TT′ touching the circle at A, and meeting OP and OP′ produced in T and T′.

Then it is easily seen that PN = NP′, and TA = AT′ in magnitude.

And further,

PNP′, arc PAP′, TAT′ are in ascending order of magnitude.

But $\quad \text{PNP}' = 2\,\text{PN} = 2r\sin\theta,$

$\quad \text{arc PAP}' = 2\,\text{arc PA} = 2r\theta,$

and $\quad \text{TAT}' = 2\,\text{TA} = 2r\tan\theta.$

Hence,

$2r\sin\theta$, $2r\theta$, $2r\tan\theta$ are in ascending order of magnitude.

$\therefore \sin\theta, \theta, \tan\theta$,, ,, ... (1).

Hence also,

$$\frac{\sin\theta}{\sin\theta}, \frac{\sin\theta}{\theta}, \frac{\sin\theta}{\tan\theta}, \text{ are in descending order,}$$

or, $\quad 1, \frac{\sin\theta}{\theta}, \cos\theta$, in descending order.

But (Vol. I., page 345) the limit of $\cos\theta$ is 1, when $\theta = 0$.

Hence the limit of $\dfrac{\sin\theta}{\theta} = 1$, when $\theta = 0$,

and $\dfrac{\tan\theta}{\theta} = \dfrac{1}{\cos\theta} \cdot \dfrac{\sin\theta}{\theta} = 1 \times 1 = 1$, when $\theta = 0$.

COR. Hence, when θ is small, we have,

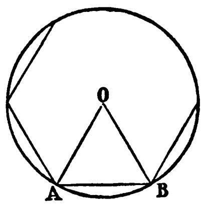
$$\left.\begin{array}{l}\sin\theta = \theta \\ \tan\theta = \theta\end{array}\right\} \text{very nearly.}$$

And further, when θ is very small,

$$\frac{\sin n\theta}{\sin\theta} = \frac{\sin n\theta}{n\theta} \cdot \frac{n\theta}{\sin\theta} = n \cdot \frac{\sin n\theta}{n\theta} \div \frac{\sin\theta}{\theta} = n.$$

21. *To find the areas of a regular polygon and a circle.*

Let AB be the side of a regular polygon of n sides;



And let O be the centre of the circumscribing circle, whose radius is r, suppose.

Then, since the angles about a point are together equal to 2π, we have,

$$\angle \text{AOB} = \frac{2\pi}{n}.$$

$$\therefore \text{ area of } \Delta \text{ AOB} = \tfrac{1}{2} \text{AO} \cdot \text{OB} \sin\frac{2\pi}{n} = \frac{r^2}{2} \sin\frac{2\pi}{n}.$$

And since the polygon consists of n such triangles, we therefore have

$$\textit{area of regular polygon of } n \textit{ sides} = \frac{nr^2}{2} \sin\frac{2\pi}{n} \;\ldots\ldots\; (1).$$

To find the area of a circle whose radius is r, we may conceive the circle to be *the limit of the regular* polygon of n sides, when n is indefinitely increased.

We then have, *when* n *is infinite,*
area of circle whose radius is r

$$= \frac{nr^2}{2} \cdot \frac{2\pi}{n} \cdot \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}; \text{ or, since } \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1, \text{ when } n = \infty,$$

$$= \frac{nr^2}{2} \cdot \frac{2\pi}{n} \cdot 1,$$

$$= r^2\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

General Value of Angles.

22. *To find a general expression for all angles having the same sine; that is, to show that, when* n *is integral,*

$$\sin \theta = \sin \left\{ n\pi + (-1)^n \theta \right\}.$$

We have, $\sin \theta = \sin (\pi - \theta)$, and $\sin \theta = \sin (2\pi + \theta)$; and we may increase or diminish the angle on the right side of these equations by any multiple of 2π.

We may therefore write

$$\sin \theta = \sin \left\{ (2n + 1)\pi - \theta \right\},$$

$$\text{and } \sin \theta = \sin \left\{ 2n\pi + \theta \right\}.$$

We observe that when θ has a *negative* sign, the multiple of π is *odd*, and when θ has a *positive* sign, the multiple of π is *even.*

Moreover, $(-1)^n$ is $-$ or $+$ according as n is odd or even.

The expressions $(2n + 1)\pi - \theta$, and $2n\pi + \theta$, are then both included in the single form

$$n\pi + (-1)^n \theta,$$

where n is any integer.

Hence, for all integral values of n, we have

$$\sin \theta = \sin \left\{ n\pi + (-1)^n \theta \right\}.$$

23. *To find a general expression for all angles having the same cosine; that is, to show that, when* n *is integral,*

$$\cos\theta = \cos(2\,n\pi \pm \theta).$$

We have $\cos\theta = \cos(\pm\theta)$;

And we may increase or diminish the angle $(\pm\theta)$ by any multiple of 2π, without altering the value of the cosine.

Hence, adding $2\,n\pi$, where n is any integer, we have

$$\cos\theta = \cos(2\,n\pi \pm \theta).$$

24. *To show that* $\tan\theta = \tan(n\pi + \theta)$, *where* n *is any integer.*

We have $\tan\theta = \tan(\pi + \theta)$;

Hence it follows that the tangent of an angle is not altered by *adding* π to the angle;

And since we may write the above equation thus:

$$\tan(\pi + \theta) = \tan\theta,$$

it follows that the tangent is unaltered, if we *subtract* π from the angle.

Hence we conclude that *any multiple of* π may be added or subtracted without changing the value of the tangent.

We therefore have

$$\tan\theta = \tan(n\pi + \theta),$$

where n is any integer.

Cor. We may easily gather from the results of the last three articles, that

$$\cot\theta = \cot(n\pi + \theta),$$
$$\sec\theta = \sec(2\,n\pi \pm \theta),$$
$$\operatorname{cosec}\theta = \operatorname{cosec}\left\{n\pi + (-1)^n\theta\right\}.$$

It would be a useful exercise for the student to illustrate these results *geometrically*.

Ex. II.

1. Find the circular measure of 1°, 1^g, $1'$, $1''$.

2. The circular measure of the sum of two angles is $\frac{5}{12}\pi$, and the greater angle exceeds the less by 17°. Find the angles.

3. Find the length of any arc of 45° in a circle whose radius is 100 feet.

4. Show that the radius of a circle is 35 feet, when the length of any arc of 60° is $36\frac{2}{3}$.

5. The semi-diameter of a disc whose distance is d, is found to be $10'$; show that the area of the disc is $\frac{d^2\pi^3}{(1080)^2}$ nearly.

6. Show that $\sin\theta < \theta$, and $> \theta - \frac{\theta^3}{4}$; and hence, prove that

$$\sin 10'' = \cdot 000048481368 \ldots.$$

7. Prove that at a distance of 3 miles, a length of 5 feet will subtend an angle of $1''$ nearly.

8. If h be the height of a mountain, r the radius of the earth, and d the distance which a person can see from the mountain top; prove that $d = \sqrt{2rh}$ nearly.

Solve the following equations:

9. $\text{Sin}\,\theta + \cos\theta = \sqrt{2}$.

10. $\text{Sin}\,\theta + \sin 3\theta = \sin 2\theta$.

11. $\text{Sin}\,5\theta - \sin 3\theta = \sin\theta$.

12. $\text{Sin}\,m\theta = \cos n\theta$.

13. $\text{Cos}\,\theta - \cos 2\theta = 1$.

14. $\text{Tan}\,5\theta \pm \tan 2\theta = 0$.

15. When the tangent of half an angle is found from the tangent of the whole angle, it has two values. Why is this?

16. Show, if α be one value of θ which satisfies the equation $\sin\theta = \sin\alpha$, that the three values of $\sin\frac{\theta}{3}$ will be

$$\sin\frac{\alpha}{3},\ \sin\left(\frac{\pi}{3}-\frac{\alpha}{3}\right),\ -\sin\left(\frac{\pi}{3}+\frac{\alpha}{3}\right).$$

17. Having given that $2\sin\frac{\theta}{2} = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$, where either sign of the radicals may be taken, explain the meaning of the four values of $\sin\frac{\theta}{2}$ thus obtained, by means of the formula

$$\sin\theta = \sin\left\{n\pi + (-1)^n\theta\right\}.$$

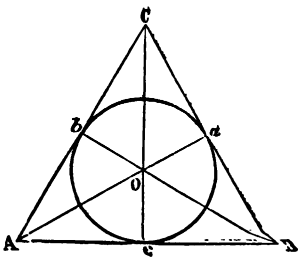
18. A sector, whose angle is θ, is cut from a circle, and formed into a cone. Show that the vertical angle of the cone is $\sin^{-1}\frac{\theta}{2\pi}$.

CHAPTER III.

INSCRIBED, ESCRIBED, AND CIRCUMSCRIBED CIRCLES OF A TRIANGLE. QUADRILATERALS.

The Inscribed Circle.

25. Let O be the centre of the inscribed circle of the triangle ABC, and let the circle touch the sides in a, b, c.



Then Oa, Ob, Oc are perpendicular to the sides, and

$$Oa = Ob = Oc = r.$$

Let S represent the area of the Δ ABC.

Then,

$$S = \Delta BOC + \Delta AOC + \Delta AOB$$

$$= \tfrac{1}{2} ar + \tfrac{1}{2} br + \tfrac{1}{2} cr = \tfrac{1}{2}(a + b + c) r = sr.$$

$$\therefore r = \frac{S}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \ldots\ldots\ldots\ldots (1).$$

Since $S = \tfrac{1}{2} bc \sin A$, and $s = \tfrac{1}{2}(a + b + c)$.

We may evidently also write,

$$r = \frac{bc}{a+b+c} \sin A = \frac{ac}{a+b+c} \sin B$$

$$= \frac{ab}{a+b+c} \sin C \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Cor. 1. Since the angles A, B, C are bisected by the lines AO, BO, CO respectively, we have

$$\frac{r}{AO} = \sin \tfrac{1}{2} A \text{ or } AO = \frac{r}{\sin \tfrac{1}{2} A} = \frac{2r}{\sin A} \cos \tfrac{1}{2} A;$$

or, by (2),

$$AO = \frac{2bc}{a+b+c} \cos \tfrac{1}{2} A; \text{ and so}$$

$$BO = \frac{2ac}{a+b+c} \cos \tfrac{1}{2} B$$

$$CO = \frac{2ab}{a+b+c} \cos \tfrac{1}{2} C.$$

Cor. 2. Since $Ab = Ac$, $Bc = Ba$, $Ca = Cb$, we have

$$2\,Ab + 2\,Bc + 2\,Ca = a + b + c = 2s, \text{ or}$$

$$Ab + Bc + Ca = s, \text{ or } Ab + Bc + Cb = s;$$

$$\text{or } b + Bc = s; \therefore Bc = Ba = s - b.$$

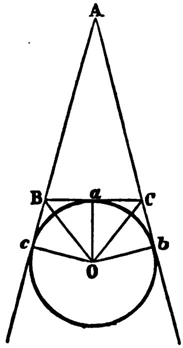
And so,

$$Ab = Ac = s - a, \text{ and } Ca = Cb = s - c.$$

The Escribed Circles.

26. An **escribed circle** is one which touches one side of a triangle, and two others produced. Every triangle has therefore the escribed circles.

We shall denote those which touch the sides a, b, c respectively by r_a, r_b, r_c.



Let the sides AB, AC of the Δ ABC be produced. Bisect the exterior angles at B and C by lines meeting in O.

Then it may be easily shown that O is the centre of a circle touching the side BC and the sides AB, AC produced.

Let it touch these lines respectively in a, b, c.

Then we have Oa, Ob, Oc perpendiculars to BC, AC, AB respectively, and

$$Oa = Ob = Oc = r_a.$$

$$\begin{aligned} \text{Now } S &= \Delta\,ABC \\ &= \Delta\,AOC + \Delta\,AOB - \Delta\,BOC \\ &= \tfrac{1}{2}r_a b + \tfrac{1}{2}r_a c - \tfrac{1}{2}r_a a = \tfrac{1}{2}(b + c - a)r_a \\ &= (s - a)r_a. \end{aligned}$$

$$\therefore r_a = \frac{S}{s - a} \quad\text{..........(1)},$$

$$\text{and so } r_b = \frac{S}{s - b} \quad\text{..........(2)},$$

$$r_c = \frac{S}{s - c} \quad\text{..........(3)}.$$

Cor. Hence,

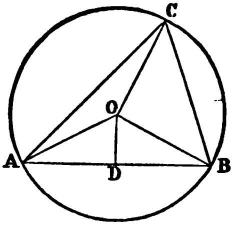
$$\text{(i.)}\ \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{(s - a) + (s - b) + (s - c)}{S} = \frac{s}{S} = \frac{1}{r}.$$

$$\text{(ii.)}\ rr_ar_br_c = \frac{S}{s} \cdot \frac{S}{s - a} \cdot \frac{S}{s - b} \cdot \frac{S}{s - c} = \frac{S^4}{S^2} = S^2.$$

$$\therefore S = \sqrt{rr_ar_br_c}.$$

The Circumscribed Circle.

27. Let O be the centre of the circumscribed circle.



Then AO = BO = CO = R, suppose.

Now (Euc. III. 20),

$$\angle AOB = 2\angle ACB = 2C.$$

Draw OD perpendicular to AB, then evidently

$$\angle AOD = \tfrac{1}{2} AOB = C.$$

And $AD = \frac{1}{2} AB = \frac{1}{2} c$.

Now $AO = \dfrac{AD}{\sin AOD} = \dfrac{\frac{1}{2}c}{\sin C}$, or

$$R = \frac{C}{2\sin B}.\quad \text{Hence we may write}$$

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

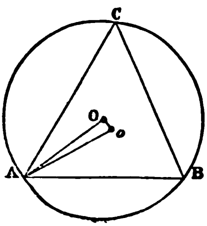
And since $\dfrac{a}{2\sin A} = \dfrac{abc}{4\,.\,\frac{1}{2}\,bc\sin A} = \dfrac{abc}{4S}$, we also have

$$R = \frac{abc}{4S} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}\ldots\ldots\ldots(2).$$

Cor. Since (Art. 24) $r = \dfrac{S}{s}$, we have

$$Rr = \frac{abc}{4S}\cdot\frac{S}{s} = \frac{abc}{4s} = \frac{abc}{2(a+b+c)}.$$

28. *The distance between the centres of the circumscribed and inscribed circles is* $\sqrt{R^2 - 2Rr}$.



Let O be the centre of the circumscribed circle, and *o* the centre of the inscribed circle.

Join AO, A*o*, O*o*, then

$$Oo^2 = AO^2 + Ao^2 - 2\,AO\,.\,Ao\,.\cos OAo\ldots(1).$$

Now AO = R; and, by Art. 25, Cor.,

$$Ao = \frac{r}{\sin \frac{1}{2} A}.$$

Also, $\angle$ OAo = $\angle$ OAB − $\angle$ oAB = (90° − C) − ½ A
= ½ (B − C).

Hence, from (1),

$$Oo^2 = R^2 + \frac{r^2}{\sin^2 \frac{1}{2} A} - \frac{2 Rr}{\sin \frac{1}{2} A} \cos \tfrac{1}{2} (B - C) \quad \text{.........} (2).$$

$$\text{But } \frac{r}{\sin \frac{1}{2} A} = \frac{S}{s} \cdot \sqrt{\frac{bc}{(s - b)(s - c)}} = \sqrt{bc \cdot \frac{s - a}{s}},$$

and cos ½ (B − C)

$$= \cos \tfrac{1}{2} B \cdot \cos \tfrac{1}{2} C + \sin \tfrac{1}{2} B \cdot \sin \tfrac{1}{2} C$$

$$= \sqrt{\frac{s(s-b)}{ac}} \cdot \sqrt{\frac{s(s-c)}{ab}} + \sqrt{\frac{(s-a)(s-c)}{ac}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \left(\frac{s}{a} + \frac{s-a}{a} \right) \sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{2s-a}{a} \sin \tfrac{1}{2} A;$$

$$\therefore \frac{\cos \frac{1}{2} (B - C)}{\sin \frac{1}{2} A} = \frac{2s - a}{a}.$$

Also (Art. 27, Cor.) $Rr = \dfrac{abc}{4s}$.

Hence

$$Oo^2 = R^2 + bc \cdot \frac{s - a}{s} - 2 \cdot \frac{abc}{4s} \cdot \frac{2s - a}{a}$$

$$= R^2 - \frac{abc}{2s}, \text{ or, by Art. 27, Cor.,}$$

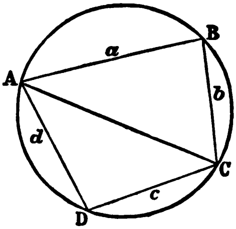
$$Oo^2 = R^2 - 2 Rr.$$

Cor. Similarly may we show that the distances of the centres of the escribed circles from the centre of the circumscribed circle are respectively

$$\sqrt{R^2 + 2 Rr_a}, \ \sqrt{R^2 + 2 Rr_b}, \ \sqrt{R^2 + 2 Rr_c}.$$

Quadrilaterals.

29. *To find the area of a quadrilateral which can be inscribed in a circle.*



Let ABCD be the quadrilateral;

Then (Euc. III., 22),

$$A + C = 180°,$$

$$\text{and } B + D = 180°.$$

Let us represent the sides AB, BC, CD, DA respectively by a, b, c, d.

Join AC.

Then area ABCD $= \Delta\,ABC + \Delta\,ADC$

$$= \tfrac{1}{2}\,ab \sin B + \tfrac{1}{2}\,cd \sin D.$$

But $\sin D = \sin(180° - B) = \sin B.$

Hence area ABCD $= \frac{1}{2}(ab + cd)\sin B$ .... (1).

Now, from the triangle ABC,

$$a^2 + b^2 - AC^2 = 2\,ab \cos B \text{.............. (2).}$$

And from the triangle ADC,

$$c^2 + d^2 - AC^2 = 2\,cd \cos D$$

$$= -2\,cd \cos B \text{.......... (3).}$$

(2) − (3) then

$$a^2 + b^2 - c^2 - d^2 = 2(ab + cd)\cos B;$$

$$\therefore \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

Hence $\sin^2 B$

$$= 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}, \text{ or, arranging,}$$

$$= \frac{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}{4(ab + cd)^2}.$$

Let $2s = a + b + c + d$, we then easily get

$$\sin^2 B = \frac{4(s-a)(s-b)(s-c)(s-d)}{(ab+cd)^2}, \text{ or}$$

$$\sin B = \frac{2}{ab+cd}\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Hence, substituting in (1), we have

Area ABCD $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Cor. 1. Let $d = 0$, that is, let the quadrilateral become the Δ ABC.

Then $S = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$.

Cor. 2. We may find the length of the diagonal AC thus:

From (2)

$$AC^2 = a^2 + b^2 - 2ab\cos B$$

$$= a^2 + b^2 - 2ab \cdot \frac{a^2+b^2-c^2-d^2}{2(ab+cd)};$$

or, simplifying,

$$= \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

$$\therefore AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

And so, $$BD = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

Hence, $AC \cdot BD = ac + bd$.

Cor. 3. We also easily find

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-a)(s-d)}{ad+bc}} = \cos \tfrac{1}{2} C,$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{ad+bc}} = \sin \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}} = \cot \tfrac{1}{2} C.$$

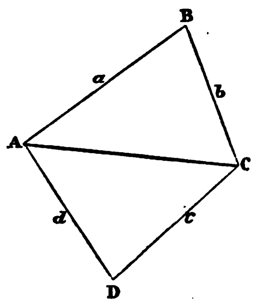
COR. 4. If R be the radius of the circumscribing circle, then, since the circle circumscribes the Δ ABC,

$$R = \frac{AC}{2 \sin B}; \text{ or, by substitution and reduction,}$$

$$R = \frac{1}{4}\sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}.$$

30. *To express the area of any quadrilateral in terms of its sides, and two of its opposite angles.*

We have,



$$S = \Delta ABC + \Delta ADC$$

$$= \tfrac{1}{2}(ab \sin B + cd \sin D) \ldots\ldots\ldots\ldots (1).$$

Now $\cos B = \dfrac{a^2 + b^2 - AC^2}{2ab}$, and $\cos D = \dfrac{c^2 + d^2 - AC^2}{2cd}$.

From these we easily get,

$$2ab \cos B - 2cd \cos D = a^2 + b^2 - c^2 - d^2; \text{ or,}$$

$$2ab\left(2\cos^2\frac{B}{2} - 1\right) - 2cd\left(1 - 2\sin^2\frac{D}{2}\right) = a^2 + b^2 - c^2 - d^2;$$

or, arranging, &c.,

$$(a + b)^2 - (c - d)^2 = 4ab\cos^2\frac{B}{2} + 4cd\sin^2\frac{D}{2} \ldots\ldots\ldots\ldots (2);$$

and so

$$(c + d)^2 - (a - b)^2 = 4\,ab \sin^2 \frac{B}{2} + 4\,cd \cos^2 \frac{D}{2} \text{.........................(3).}$$

(2) × (3), then, putting $2s = a + b + c + d$, we get

$$(s-a)(s-b)(s-c)(s-d) = a^2b^2 \sin^2 \frac{B}{2} \cos^2 \frac{B}{2} + c^2d^2 \sin^2 \frac{D}{2} \cos^2 \frac{D}{2} + abcd\left(\sin^2 \frac{B}{2} \sin^2 \frac{D}{2} + \cos^2 \frac{B}{2} \cos^2 \frac{D}{2}\right)$$

$$= \left(ab \sin \frac{B}{2} \cos \frac{B}{2} + cd \sin \frac{D}{2} \cos \frac{D}{2}\right)^2 + abcd\left(\cos \frac{B}{2} \cos \frac{D}{2} - \sin \frac{B}{2} \sin \frac{D}{2}\right)^2$$

$$= \tfrac{1}{4}\,(ab \sin B + cd \sin D)^2 + abcd \cos^2 \tfrac{1}{2}(B + D).$$

Hence, substituting from (1),

$$(s - a)(s - b)(s - c)(s - d) = S^2 + abcd \cos^2 \tfrac{1}{2}(B + D).$$

$$\therefore S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \tfrac{1}{2}(B + D). \text{ ....(4).}$$

And since $A + B + C + D = 2\pi$, we have $\frac{1}{2}(A + C) = \pi - \frac{1}{2}(B + D)$.

$$\therefore \cos \tfrac{1}{2}(A + C) = -\cos \tfrac{1}{2}(B + D), \text{ or}$$

$$\cos^2 \tfrac{1}{2}(A + C) = \cos^2 \tfrac{1}{2}(B + D).$$

Hence from (4), if 2ω = the sum of either two opposite angles, we have

$$S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \omega.$$

COR. When the quadrilateral is inscribed in a circle we have

$$2\omega = 180°, \text{ or } \omega = 90°.$$

Hence,

$$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 90°,$$

or $S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, as in Art. 29.

31. *To find the area of a quadrilateral described about a circle.*

It is easy to show that $a + c = b + d$.

$$\therefore s = \tfrac{1}{2}(a + c + a + c) = a + c \text{ or } b + d.$$

$$\therefore s - a = c, \quad s - b = d, \quad s - c = a, \quad s - d = b.$$

Hence $(s-a)(s-b)(s-c)(s-d) = abcd$.

Then, by Art. 30,

$$S^2 = abcd - abcd \cos^2 \omega = abcd \sin^2 \omega.$$

$$\therefore S = \sqrt{abcd} \,.\, \sin \omega,$$

where 2ω is the sum of two opposite angles of the quadrilateral.

CHAPTER IV.

SUBSIDIARY ANGLES.

32. When a trigonometrical formula is not adapted to logarithmic computation, it is usual to introduce what is called a **subsidiary angle** into the expression in order to put it into the form of a product of factors.

Since the value of the *sine* and *cosine* of an angle ranges from zero to unity, it follows that if a be not greater than b, it is perfectly legitimate to assume either

$$\sin \phi = \frac{a}{b}, \text{ or } \cos \phi = \frac{a}{b}.$$

Again, since the value of the *tangent* may range from *zero* to *infinity*, it is always legitimate to assume, for any real values of a and b, that

$$\tan \phi = \frac{a}{b}.$$

Still, as will be seen farther on, certain assumptions, although legitimate, are objectionable in particular cases. We shall not, however, enter upon this subject in the present chapter.

33. *To adapt to logarithmic computation the formula*

$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

(i.) We have

$$c^2 = a^2 + b^2 - 2\,ab\left(2\cos^2\frac{C}{2} - 1\right)$$

$$= (a+b)^2 - 4\,ab\cos^2\frac{A}{2}$$

$$= (a+b)^2\left\{1 - \frac{4\,ab}{(a+b)^2}\cos^2\frac{A}{2}\right\} \quad \ldots\ldots (1).$$

Now *assume*

$$\sin^2\theta = \frac{4\,ab}{(a+b)^2}\cos^2\frac{A}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

(This assumption is allowable, for $(a+b)^2$ is never less than $4\,ab$.)

We then have

$$c^2 = (a+b)^2(1 - \sin^2\theta) = (a+b)^2\cos^2\theta.$$

$\therefore\ c = (a+b)\cos\theta$, the form required .... (3).

It will be seen that θ may be obtained from (2) by means of logarithms, and then c from the equation just found.

Thus, from (2), $2\,\mathrm{L}\sin\theta$

$$= 2\log 2 + \log a + \log b - 2\log(a+b) + 2\,\mathrm{L}\cos\frac{A}{2},$$

And from (3),

$$\log c = \log (a + b) + \mathrm{L} \cos \theta - 10.$$

(ii.) *Or thus:*

$$c^2 = a^2 + b^2 - 2\,ab\left(1 - 2 \sin^2 \frac{C}{2}\right)$$

$$= (a - b)^2 + 4\,ab \sin^2 \frac{C}{2}$$

$$= (a - b)^2 \left\{ 1 + \frac{4\,ab}{(a - b)^2} \sin^2 \frac{C}{2} \right\} \ldots\ldots\ldots (4).$$

Now $\frac{4\,ab}{(a - b)^2} \sin^2 \frac{C}{2}$ may have *any positive* value, and hence there is some angle θ, such that

$$\tan^2 \theta = \frac{4\,ab}{(a - b)^2} \sin^2 \frac{C}{2} \ldots\ldots\ldots\ldots (5).$$

Then, from (4),

$$c^2 = (a - b)^2 (1 + \tan^2 \theta) = (a - b)^2 \sec^2 \theta;$$

$$\therefore c = (a - b) \sec \theta \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

We then have, from (5),

$2\,\mathrm{L} \tan \theta$

$$= 2 \log 2 + \log a + \log b - 2 \log (a - b) + 2\,\mathrm{L} \sin \frac{C}{2};$$

And from (6),

$$\log c = \log (a - b) + \mathrm{L} \sec \theta - 10.$$

(iii.)

$$c^2 = (a^2 + b^2)\left(\cos \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2\,ab\left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= (a + b)^2 \sin^2 \frac{C}{2} + (a - b)^2 \cos^2 \frac{C}{2}$$

$$= (a + b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left(\frac{a - b}{a + b}\right)^2 \cot^2 \frac{C}{2} \right\} \ldots\ldots\ldots\ldots (7),$$

$$\text{or} = (a - b)^2 \cos^2 \frac{C}{2} \left\{ 1 + \left(\frac{a + b}{a - b}\right)^2 \tan^2 \frac{C}{2} \right\} \dots\dots\dots\dots (8).$$

Either of these expressions may now be easily adapted to logarithmic computation.

34. *Adapt* $\sqrt{a + b} + \sqrt{a - b}$ *to logarithmic computation.*

Since b is not greater than a, we may assume $\cos \phi = \frac{b}{a}$.

Then

$$\sqrt{a + b} + \sqrt{a - b} = \sqrt{a}\{\sqrt{1 + \cos \phi} + \sqrt{1 - \cos \phi}\}$$

$$= \sqrt{a}\left\{\sqrt{2} \cos \frac{\phi}{2} + \sqrt{2} \sin \frac{\phi}{2}\right\} = 2\sqrt{a} \sin \left(45^\circ + \frac{\phi}{2}\right).$$

35. *Given the formula*

$$(\cos \delta' \cos a - \cos \delta) \cos h - \cos \delta' . \sin a . \sin h$$
$$= \tan l (\sin \delta - \sin \delta'),$$

to find h *in a form adapted to logarithmic computation.*

We have

$$\cos \delta' \sin a \left\{\frac{\cos \delta' \cos a - \cos \delta}{\cos \delta' \sin a} \cos h - \sin h\right\}$$
$$= \tan l (\sin \delta - \sin \delta').$$

$$\text{Put } \cot \phi = \frac{\cos \delta' \cos a - \cos \delta}{\cos \delta' \sin a} \dots\dots\dots\dots (1)$$

then

$$\cos \delta' \sin a (\cot \phi \cos h - \sin h) = \tan l (\sin \delta - \sin \delta'); \text{ or}$$

$$\cos \delta' \sin a . \frac{\cos (\phi + h)}{\sin \phi}$$
$$= \tan l . 2 \cos \tfrac{1}{2} (\delta + \delta') \sin \tfrac{1}{2} (\delta - \delta').$$

$$\therefore \cos (\phi + h) = \frac{2 \sin \phi . \tan l}{\cos \delta' \sin a} \cos \tfrac{1}{2} (\delta + \delta') \sin \tfrac{1}{2} (\delta - \delta'),$$

the form required.

Ex. III.

Miscellaneous Exercises.

1. Adapt $a + b + c + d +$ &c., to logarithmic computation.

2. Solve the equation $\sin 5\theta = \cos 2\theta$.

3. If t_r represent the sum of the products of $\tan \alpha$, $\tan \beta$, $\tan \gamma$, &c., taken r together, show that, when there are n angles,

$$\tan(\alpha + \beta + \gamma + \ldots \kappa)$$

$$= \frac{t_1 - t_3 + t_5 - \ldots . (-1)^{\frac{n}{2}-1} t_{n-1}}{1 - t_2 + t_4 - \ldots . (-1)^{\frac{n}{2}} t_n}, \; n \text{ even};$$

$$= \frac{t_1 - t_3 + t_5 - \ldots . (-1)^{\frac{n-1}{2}} t_n}{1 - t_2 + t_4 - \ldots . (-1)^{\frac{n-1}{2}} t_{n-1}}, \; n \text{ odd}.$$

4. In any triangle

$$(b + c)\sin \tfrac{1}{2}\mathrm{A} \sin \tfrac{1}{2}(\mathrm{B} - \mathrm{C}) = (b - c)\cos \tfrac{1}{2}\mathrm{A} \cos \tfrac{1}{2}(\mathrm{B} - \mathrm{C}).$$

5. Show that

$$\frac{\sin 3\mathrm{A} - \cos 3\mathrm{A}}{\sin 3\mathrm{A} + \cos 3\mathrm{A}} = \frac{2\sin 2\mathrm{A} - 1}{2\sin 2\mathrm{A} + 1}\tan(45^\circ - \mathrm{A}).$$

6. In any triangle the perpendiculars from the angles upon the opposite side are inversely proportional to the sides.

7. If $\tan 2\alpha = \dfrac{pr + qs}{pq + rs}$, and $\tan 2\beta = \dfrac{pr - qs}{pq + rs}$;

then $\qquad \tan(\alpha - \beta) = \dfrac{s}{p}$ or $-\dfrac{p}{s}$,

and $\qquad \tan(\alpha + \beta) = \dfrac{r}{q}$ or $-\dfrac{q}{r}$.

8. If $\tan u = \tan a \cos b$, and $\tan v = \tan b \cos a$,

then $\qquad \sec(u \pm v) = \sec a \sec b \pm \tan a \tan b$,

and $\qquad \tan(u \pm v) = \tan a \sec b \pm \sec a \tan b$.

9. If D be the middle point of BC in the triangle ABC, DE be drawn perpendicular to AC, and EF perpendicular to BC, then $C = \frac{1}{2} \sin^{-1} \frac{4\,EF}{a}$.

10. The lines joining the angles of a triangle to the middle points of the opposite sides, meet in a point.

11. Show that

(i.) $\log \sec 2\theta = 2\left\{\tan^2\theta + \frac{1}{3}\tan^6\theta + \frac{1}{5}\tan^{10}\theta + \&c\right\}$.

(ii.) $2 \log \sec \theta = \tan^2\theta - \frac{1}{2}\tan^4\theta + \frac{1}{3}\tan^6\theta - \&c.$

12. Show that

(i.) $\tan\theta + \frac{1}{2\tan\theta +}\ \frac{1}{2\tan\theta +} \ldots\ldots = \sec\theta.$

(ii.) $2\cot\theta + \frac{1}{2\cot\theta +}\ \frac{1}{2\cot\theta +} \ldots\ldots = \cot\frac{\theta}{2}.$

13. If the sides a, b, c of a triangle be in A. P., show that the area is to that of an equilateral triangle having the same perimeter as $\sqrt{1 - \frac{4(a-b)^2}{b^2}}$ to unity.

14. The area of a sector of a circle is half the product of the radius into the length of the arc.

15. Given that

$$a^2 \cos\alpha \cos\beta + a(\sin\alpha + \sin\beta) + 1 = 0,$$
$$a^2 \cos\alpha \cos\gamma + a(\sin\alpha + \sin\gamma) + 1 = 0,$$

show that

$$a^2 \cos\beta \cos\gamma + a(\sin\beta + \sin\gamma) + 1 = 0,$$

and $\cos\alpha + \cos\beta + \cos\gamma = \cos(\alpha + \beta + \gamma)$,

where β and γ are unequal and less than π.

16. Show that

$$\frac{1}{a + b\cos x} = \frac{\sec^2\frac{1}{2}x}{(a+b) + (a-b)\tan^2\frac{1}{2}x}.$$

17. If a point O be taken either within or without a triangle ABC then $\frac{\sin ABO \cdot \sin BCO \cdot \sin CAO}{\sin CBO \cdot \sin ACO \cdot \sin BAO} = 1.$

18. In any triangle ABC, show that

$$\tan \tfrac{1}{2} A \,.\, \tan \tfrac{1}{2} B \,.\, \tan \tfrac{1}{2} C = \frac{S}{s^2}.$$

19. If l be the length of the line drawn from the angle A to the opposite side so as to bisect the angle, then

$$\cos \tfrac{1}{2} A = \frac{l}{2}\left(\frac{1}{b} + \frac{1}{c}\right).$$

20. If p, q, r be the perpendiculars upon the sides of a triangle from the opposite angles, show that

$$\frac{a^2b^2c^2}{b^3c^3 + a^3c^3 + a^3b^3} = \frac{pqr}{p^3 + q^3 + r^3}.$$

21. If $\cos(a - b)\cos(b - c)\cos(c - a) = \frac{1}{4}$, then

$$\cos 3(a - b) \,.\, \cos 3(b - c) \,.\, \cos 3(c - a)$$
$$= 3 \cos 2(a - b) \,.\, \cos 2(b - c) \,.\, \cos 2(c - a) + \tfrac{1}{4}.$$

22. If a point O be taken within or without an equilateral triangle ABC, and upon BO an equilateral triangle OBD be described so that the angles ABO and CBD are equal, then will

$$\cos DOC = \frac{BO^2 + CO^2 - AO^2}{2\, BO \,.\, CO}.$$

23. In the ambiguous case where a, b, A are given, and $b > a$, show that if C_1, C_2 are the values of the third angle, and c_1, c_2 the values of the third side, then

(i.) $C_1 + C_2 = 2 \tan^{-1}(\cot A)$.

(ii.) The distance between the centres of the circumscribing circles of the triangles $= \dfrac{c_1 + c_2}{2 \sin A}$.

24. If the centre of gravity of a triangle be joined to the angular points, and R_a, R_b, R_c be respectively the radii of the circles circumscribing the triangles thus formed, then

$$R_a^2 R_b^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) + R_b^2 R_c^2 \left(\frac{1}{b^2} - \frac{1}{c^2}\right) + R_c^2 R_a^2 \left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 0.$$

25. The area of the triangle formed by joining the centres of three circles of radii r_1, r_2, r_3 which touch each other *externally is* $\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}$.

26. If the middle points of the sides of a triangle be joined to the opposite angles, and r_1, r_2, r_3, &c., be the radii of the circles inscribed in the triangles thus formed, and R_1, R_2, R_3, &c., the radii of the circumscribed circles,

then $\quad R_1\, R_3\, R_5 = R_2\, R_4\, R_6,$

and $\quad \dfrac{1}{r_1} + \dfrac{1}{r_3} + \dfrac{1}{r_5} = \dfrac{1}{r_2} + \dfrac{1}{r_4} + \dfrac{1}{r_6}.$

27. If r be the radius of the inscribed, and r_a, r_b, r_c those of the escribed circles, then $\dfrac{1}{r} = \dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c}$.

28. If R be the radius of the circumscribed circle, show that

$$r_a + r_b + r_c - r = 4\,R = \frac{(r_a + r_b)\,(r_b + r_c)\,(r_c + r_a)}{r_a r_b + r_b r_c + r_c r_a}.$$

29. If p_a, p_b, p_c be the perpendiculars upon the sides a, b, c, then $\dfrac{1}{r_a} = \dfrac{1}{p_b} + \dfrac{1}{p_c} - \dfrac{1}{p_a}$.

30. If a point P be taken either within or without a triangle, and perpendiculars Pa, Pb, Pc be drawn upon the sides, show that the area of the triangle formed by joining the centres of the circles circumscribing the triangle Pab, Pac, Pbc = $\frac{1}{4}$ S.

31. Two circles have a common radius r, and a circle is described touching this radius, and the two circles; show that the radius of the circle touching the three circles is $\frac{3}{16}\,r$.

32. In any triangle, if $\dfrac{\tan C}{\tan B} = \dfrac{\tan B}{\tan A}$, show that each of these quantities $= \dfrac{\sin 2\,A}{\sin 2\,C}$.

33. If $\tan \frac{1}{2}\,\alpha = \tan^3 \frac{1}{2}\,\gamma$, and $\tan \gamma = 2 \tan \beta$, show that α, β, γ are in A. P.

34. Show, by the binomial theorem, that

$$(1 + \tfrac{1}{2} \sin \theta + \tfrac{3}{8} \sin^2 \theta + \&c.)\,(1 - \tfrac{1}{2} \sin \theta + \tfrac{3}{8} \sin^2 \theta - \&c.) = \sec \theta.$$

35. If α, β, γ be the distances between the centres of the escribed circles respectively, and α_1, β_1, γ_1, be the distances between the centre of the inscribed circle and the centres of the escribed, then $\frac{\alpha}{\alpha_1} = \cot\frac{A}{2}$, $\frac{\beta}{\beta_1} = \cot\frac{B}{2}$, $\frac{\gamma}{\gamma_1} = \cot\frac{C}{2}$.

36. Show that the area of the inscribed circle is to the area of the triangle as $\pi : \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

37. The area of a triangle $= \sqrt{rr_ar_br_c}$.

38. If $\text{vers}^{-1}\frac{x}{a} - \text{vers}^{-1}\frac{bx}{a} = \text{vers}^{-1}(1 - b)$;

$$\text{then } \frac{x}{a} = \pm\sqrt{\frac{2b}{1+b}}.$$

39. Show that

$$4\,R = \frac{(r_a + r_b)(r_a + r_c)(r_b + r_c)}{r_ar_b + r_ar_c + r_br_c} = \frac{\alpha\beta\gamma}{\sqrt{\sigma(\sigma - \alpha)(\sigma - \beta)(\sigma - \gamma)}},$$

when $2\,\sigma = \alpha + \beta + \gamma$.

40. Three circles touch each other externally whose radii are a, b, c. Show that the tangents at the points of contact meet in a point, and that its distance from each of them is

$$\left(\frac{abc}{a + b + c}\right)^{\frac{1}{2}}.$$

CHAPTER V.

HEIGHTS AND DISTANCES.

36. In the first volume of this work we have explained at some length the methods to be followed in the simpler cases of heights and distances. We shall work out an example or two involving more complicated considerations as specimens of the general course to be pursued. First, however, we shall explain the Vernier Scale, the Sextant, and the Theodolite.

The Vernier Scale.

37. The vernier scale consists of a small scale CD which slides along the edge of any ordinary graduated scale AB.

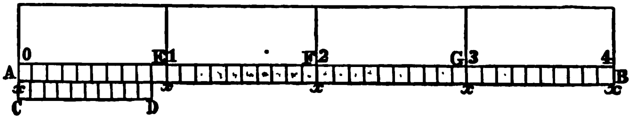


Fig 1.

With a straight scale, the vernier is straight; and with a graduated portion of the arc of a circle, the vernier is a small circular arc.

Suppose the scale AB to be divided into equal divisions (say inches), AE, EF, &c., and each of these divisions again into tenths of an inch. Then it would be possible to express any length in inches and tenths of an inch by the scale AB alone.

Let the length of CD be *nine* of these small divisions, and suppose we wish to measure the exact length of a line *ac* in the diagram below.

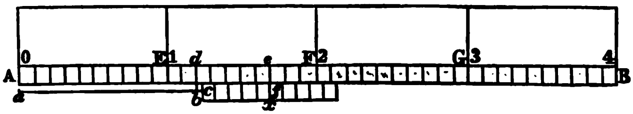


Fig 2.

We see, by the scale AB, that its length is something over *one* inch and *two tenths.* By means of the vernier CD we can express this excess in hundredths.

Let the xth *division* of the vernier correspond with a division of the scale AB, as at *f*.

$$\text{Now each division of the vernier} = \frac{1}{10} \text{ of } \frac{9}{10} \text{ in.} = \frac{9}{100} \text{ in.}$$

$$\text{Also } bc + cf = de.$$

$$\text{Hence } bc + \frac{9}{100}x = \frac{1}{10}x.$$

$$\therefore bc = \left(\frac{1}{10} - \frac{9}{100}\right)x = \frac{x}{100}.$$

Hence we learn that the excess over the length given by the scale AB is as many hundredths of an inch as corresponds to *the division* of the vernier which concides with a division of the scale.

We have supposed here the divisions of the vernier to be reckoned *from the left.* If, as in some verniers, they are reckoned from the right, we must take 11 divisions of the scale instead of 9, and divide it into 10 equal parts.

In that case, if the xth division correspond with the scale, we have, remembering that the divisions of the vernier are reckoned from the right, and that each division of the vernier

$$= \frac{1}{10} \text{ of } \frac{11}{10} \text{ in.} = \frac{11}{100} \text{ in.},$$

$$bc + (10 - x)\frac{11}{100} = (11 - x)\frac{1}{10}.$$

$$\therefore bc = \frac{x}{100}.$$

In the case of a graduated arc of a circle, where the arc is marked for degrees, then, if 59 divisions of the scale correspond to 60 divisions of the vernier, and the vernier is read from left to right, and if the xth division of the vernier corresponds with a division of the scale, we should learn that the arc ought to be increased to the amount of x'' above the scale reading.

The Sextant.

38. The sextant consists of a graduated arc AB, whose centre is C, formed upon a plate ABC. An arm CA carries a mirror fixed at its centre C in a plane at right angles to the plate. The arm is moveable about an axis through C, carrying the mirror with it, and terminates in an index which moves along the arc AB. When the index points to *zero, the* mirror C and the zero point are in the same *vertical plane.*

D is also a mirror, the lower half only of which is silvered; it is immoveable, and fixed parallel to the position of C, when the index points to zero.

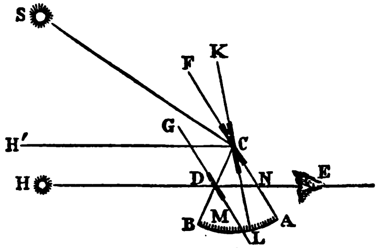


Fig. 3.

Let A be the zero point, and suppose the index to be at A. The instrument may be used either to observe the altitude of a star, or to obtain the angle which two stars subtend at the observer's eye.

Let H and S be two stars. If we require simply the altitude of S, we may suppose H to be a point on the horizon in the same meridian as S.

We shall now explain however the general principles involved in taking the angle subtended by any two stars.

H and S are both so distant that they may be supposed to emit parallel rays.

Let H be observed, by an eye placed at E, by direct vision through the centre D of the lower mirror.

Now hold the instrument so that a pencil of rays H'C parallel to HD may, after impinging upon the centre C of the upper mirror, be reflected so as to fall upon the lower mirror at D.

Then FCH' and ACD are respectively *the complements of the angles of incidence and reflection* with respect to the mirror C.

$$\therefore \angle FCH' = \angle ACD,$$

according to the law of reflection.

Again, CDG is the complement of the angle of incidence upon D.

Now, as the mirrors are parallel, $\angle ACD = \angle CDG$.

Hence $\angle FCH' = \angle CDG$.

But since CH′ is parallel to NH, we have also

$$\angle FCH' = \angle CNH = \angle NDM.$$

Hence, $\angle CDG = \angle NDM$.

But CDG is the complement of the angle of incidence of the pencil CD upon D. Hence, by the law of reflection, NDM is the complement of the angle of reflection.

Therefore, the reflected ray will proceed along the line DNE.

Hence the eye will see the object H both by direct vision, and by reflection, when the mirrors are parallel, or when the index points to zero.

Let the arm be now moved until the star S appears by reflection to coincide with H.

Suppose the index to point to L.

Then SCH′ is the angle subtended by the two stars at the eye.

Hence a pencil of rays SC will, after reflection, proceed along CD, and thence along DE.

$$\therefore \angle SCK = \angle DCL,$$

since they are respectively the complements of the angles of incidence and reflection.

$$\text{Now, } \angle SCK = \angle FCH' + \angle FCK - \angle SCH'$$
$$= \angle FCH' + \angle ACL - \angle SCH'.$$

$$\text{And } \angle DCL = \angle DCA - \angle ACL$$
$$= \angle FCH' - \angle ACL.$$

Then, equating these, we have

$$\angle FCH' - \angle ACL = \angle FCH' + \angle ACL - \angle SCH'.$$
$$\therefore \angle SCH' = 2\angle ACL.$$

Hence the angle subtended by the two stars is *twice* the angle through which the index has been turned.

It is usual to graduate the arc AB by dividing it into *twice the number of degrees* it actually contains, so that the reading from the instrument gives the *true value* of the angle required.

A vernier is attached to the graduated arc to enable the observer to take the reading with greater exactness.

The Theodolite.

39. This instrument is used for taking the angles of elevation of objects, and the horizontal angle between the objects.

It consists of a vertical circle carrying a telescope, whose axis is in the plane of the vertical circle, and moving about a horizontal axis, so that the line of vision can be directed to any point in the plane of the vertical circle; and also, of a horizontal circle carrying the vertical circle and telescope, and moveable in a horizontal plane, so that the vertical circle and telescope can be brought into any azimuth.

If the line of vision be successively directed to any two objects A and B, then an index upon the vertical circle gives the angles of elevation of the objects; and the difference of the indications of the index on the horizontal circle gives the horizontal angle between the objects.

The horizontal circle carries another, called a vernier circle, which has a horizontal motion independent of the horizontal circle. There are generally two verniers upon the horizontal circle, to enable the observer to correct for centreing and graduation. Two readings are made, and the mean of the two will give the corrected angle. The vertical circle also carries a vernier for a similar purpose.

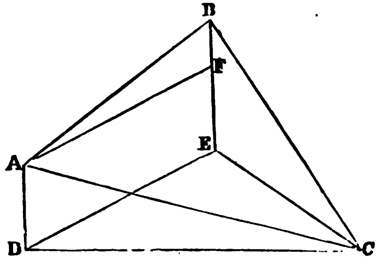
Before proceeding to use the instrument care must be taken that it stands firm, that the horizontal circle is truly horizontal, that the line of collimation is properly adjusted, and that the vertical circle moves in a truly vertical plane.

The theodolite often carries a compass, so that if the zero of the horizontal circle corresponds to the magnetic meridian, the horizontal angle between the object and the zero point gives the *bearing* of the object.

Ex. 1. Two places B, C, inaccessible from each other, are distant a from another place A on the side of a hill. From the lower place C there are observed the horizontal angle (α) between A and B, and also the elevations (λ, μ) of A and B. Find the distance BC.

Draw CD, CE horizontal, meeting verticals through A, B respectively in D, E.

Join DE, which is also horizontal; and draw AF parallel to DE.



Then we have:

$$AB = AC = a,$$
$$\angle DCE = \alpha,$$
$$\angle ACD = \lambda,$$
$$\angle BCE = \mu.$$

Now AC^2

$$= AB^2 = AF^2 + BF^2$$
$$= DE^2 + (BE - AD)^2$$
$$= (EC^2 + DC^2 - 2\,EC\,.\,DC \cos ECD) + (BE - AD)^2; \text{ or,}$$
$$a^2 = BC^2 \cos^2 \mu + AC^2 \cos^2 \lambda - 2\,BC\,.\,AC \cos \lambda \cos \mu \cos \alpha$$
$$+ (BC \sin \mu - AC \sin \lambda)^2; \text{ or, simplifying,}$$
$$= BC^2 + AC^2 - 2\,AC\,.\,BC \cos \lambda \cos \mu \cos \alpha - 2\,AC\,.\,BC \sin \lambda \sin \mu$$
$$= BC^2 + a^2 - 2\,a BC\ (\cos \lambda \cos \mu \cos \alpha + \sin \lambda \sin \mu).$$

Hence

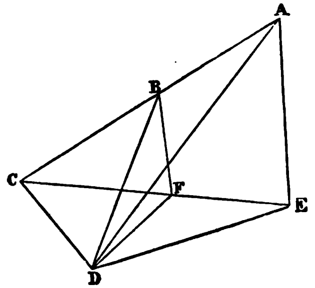
$$BC = 2\,a\,(\cos \lambda \cos \mu \cos \alpha + \sin \lambda \sin \mu); \text{ or, arranging,}$$
$$= 2\,a \left\{ \cos (\lambda - \mu) \cos^2 \frac{\alpha}{2} - \cos (\lambda + \mu) \sin^2 \frac{\alpha}{2} \right\}.$$

Ex. 2. An object A on the top of a hill is just seen from a point C behind the summit of hill B. A distance CD is measured along the road, and the following observations are made. At D the elevations of A, B are observed respectively to be α, β; and the angles ADC, ADB, ACD are found to be γ, δ, ϵ. Find the heights of A and B.

Draw CE, DE horizontal, and AE, BF vertical. Join DF.

We have:

$$CD = a,$$
$$\angle ADE = \alpha,$$
$$\angle BDF = \beta,$$
$$\angle ADC = \gamma,$$
$$\angle ADB = \delta,$$
$$\angle ACD = \epsilon.$$



Now

$$AE = AD \sin ADE$$

$$= CD \cdot \frac{\sin ACD}{\sin CAD} \sin ADE$$

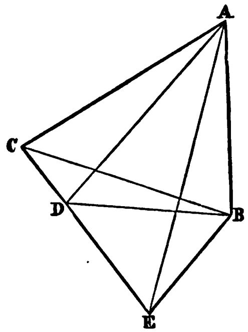
$$= a \cdot \frac{\sin \epsilon \cdot \sin \alpha}{\sin \{180^\circ - (\gamma + \epsilon)\}} = a \cdot \frac{\sin \alpha \cdot \sin \epsilon}{\sin (\gamma + \epsilon)}.$$

Again

$$BF = BD \sin BDF = CD \cdot \frac{\sin ACD}{\sin CBD} \sin BDF.$$

$$= a \cdot \frac{\sin \epsilon}{\sin \{180^\circ - (\gamma - \delta + \epsilon)\}} \sin \beta = a \cdot \frac{\sin \beta \sin \epsilon}{\sin (\gamma + \epsilon - \delta)}.$$

Ex. 3. *Given the elevations of an object from three* points in a straight line in the horizontal plane, to find the height of the object.



Let A be the object; C, D, E the three points in the straight line CE.

Draw AB vertical, meeting the horizon in B.

Let CD = a, DE = b.........(1).

$\angle$ ACB = α, $\angle$ ADB = β, $\angle$ AEB = γ; and let h be the height required.

We have

$$\left.\begin{array}{l} BC = h \cot \alpha \\ BD = h \cot \beta \\ BE = h \cot \gamma \end{array}\right\} \text{...............(2).}$$

Now $\cos BDC = \dfrac{CD^2 + BD^2 - CB^2}{2\,CD\,.\,BD}$,

And $\cos BDE = \dfrac{BD^2 + DE^2 - BE^2}{2\,BD\,.\,DE}$.

But $\cos BDC = -\cos(180° - BDC) = -\cos BDE$.

Hence, $\dfrac{CD^2 + BD^2 - CB^2}{2\,CD\,.\,BD} = -\dfrac{BD^2 + DE^2 - BE^2}{2\,BD\,.\,DE}$; or

$$DE\,(CD^2 + BD^2 - CB^2) = -CD\,(BD^2 + DE^2 - BE^2).$$

Or, substituting from (1) and (2), we have

$$b\,(a^2 + h^2 \cot^2 \beta - h^2 \cot^2 \alpha) = -a\,(h^2 \cot^2 \beta + b^2 - h^2 \cot^2 \gamma);$$

or, $h^2 \{a \cot^2 \gamma - (a + b) \cot^2 \beta + b \cot^2 \alpha\} = ab\,(a + b)$,

$$\therefore h = \sqrt{\frac{ab\,(a + b)}{a \cot^2 \gamma - (a + b) \cot^2 \beta + b \cot^2 \alpha}}.$$

Ex. IV.

1. An object was observed from the deck of a ship to have a bearing of $\alpha°$ from the N.; and after a and b hours' sailing in the same direction at a uniform rate, its bearing was found to be $\beta°$ and $\gamma°$ respectively. Find the direction of sailing.

2. A staff at the top of a tower subtends to the eye of a spectator on the ground an angle θ, and on going a feet nearer to the foot of the tower, it was found to subtend the same angle θ. What is the distance of the tower?

3. Three observers in a straight line, the middle one of which is at a distance a, b respectively from the other two, observe the elevation of a bird to be α, β, γ respectively at the same instant. Find the distance of each observer from the vertical through the bird.

4. There are two hills, the height of the second being h. From the top of the former the angles of depression of two objects in a straight line with both the hills and in the horizontal plane are α_1 and β_1. From the top of the second hill they are α_2 and β_2. Find the height of the former hill.

5. A gun is fired from a fort A, and the intervals between seeing the flash and hearing the report at two stations B, C are t_1, t_2 respectively. D is a point in the straight line BC whose distance from A is required, having given v the velocity of sound, BD $= a$, DC $= b$.

6. From a certain point on the slope of a hill the elevations α, β of two objects on the hill are observed, and angle γ they subtend at the eye. If l be the length of the hill in a direct road from the foot, find its height.

7. Two ships are sailing uniformly with velocities u, v. At a certain instant their distances from the point of intersection of their courses was a, b, respectively. Show that if d be their minimum distance from each other, and θ the inclination of their courses, then

$$(av - bu)^2 \sin^2 \theta = d^2 (u^2 + v^2 - 2uv \cos \theta).$$

8. The shadow of a column whose height is h is found to be a in length. When the sun has attained double his altitude, it is found to be b. Find the relation between a, b, h.

9. From a ship a point of land was observed to bear $\alpha°$ from the N., and after sailing a miles in a N.W. direction the bearing was $\beta°$ from the N. Find the distance of the land from each point of observation.

10. From the top of a mountain whose height is h miles the depression of the horizon was $a°$. Show that the radius of the earth $= h \cot a \cot \frac{a}{2}$.

11. Wishing to know the breadth of a river at an inaccessible point A, I observed from a point D at a distance a from the bank the angle θ subtended at my eye by the point A and the opposite point; and also, on walking direct to the bank, I observed the subtended angle to be ϕ. Find the breadth.

12. From a balloon the depressions of two towns whose distance apart is a, are found to be α and β respectively, and the angle they subtend at the balloon is γ. Find the height of the balloon.

CHAPTER VI.

DEMOIVRE'S THEOREM. CALCULATION OF THE VALUE OF π.

40. *To show that*

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)\ldots.$$
$$= \cos(\alpha + \beta + \gamma + \ldots.) + \sqrt{-1}\sin(\alpha + \beta + \gamma + \ldots.).$$

We have, by multiplication,

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)$$
$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + \sqrt{-1}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= \cos(\alpha + \beta) + \sqrt{-1}\sin(\alpha + \beta).$$

And hence also by multiplication,

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)$$
$$= \cos(\alpha+\beta+\gamma) + \sqrt{-1}\sin(\alpha+\beta+\gamma).$$

Assume the law to hold for r factors corresponding to the angles $\alpha, \beta, \gamma \ldots\ldots \kappa$.

Then we have

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)\ldots\ldots(\cos\kappa + \sqrt{-1}\sin\kappa)$$
$$= \cos(\alpha+\beta+\gamma+\ldots\ldots\kappa) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots\ldots\kappa).$$

Multiply each side of this equality by another factor $\cos\lambda + \sqrt{-1}\sin\lambda$.

Then

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)\ldots\ldots(\cos\lambda + \sqrt{-1}\sin\lambda)$$
$$= \{\cos(\alpha+\beta+\gamma+\ldots\ldots\kappa) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots\ldots\kappa)\}\{(\cos\lambda + \sqrt{-1}\sin\lambda)\};$$

or, by multiplication,

$$= \cos(\alpha+\beta+\gamma+\ldots\ldots\lambda) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots\ldots\lambda).$$

Hence, by assuming the law to hold for r factors, we have been able to prove it to hold for $(r+1)$ factors.

Now we know by actual multiplication that it is true for 2 and 3 factors. It is therefore true for 4, and hence for 5, 6, &c.

We may thus conclude it to be true for any number of factors.

Hence generally,

$$(\cos \alpha + \sqrt{-1} \sin \alpha) (\cos \beta + \sqrt{-1} \sin \beta) (\cos \gamma + \sqrt{-1} \sin \gamma) \dots$$
$$= \cos (\alpha + \beta + \gamma + \dots) + \sqrt{-1} \sin (\alpha + \beta + \gamma + \dots).$$

COR. Since $(\cos \alpha + \sqrt{-1} \sin \alpha) (\cos \alpha - \sqrt{-1} \sin \alpha) = \cos^2 \alpha + \sin^2 \alpha = 1$, it at once follows that we may write the above thus :

$$(\cos \alpha \pm \sqrt{-1} \sin \alpha) (\cos \beta \pm \sqrt{-1} \sin \beta) (\cos \gamma \pm \sqrt{-1} \sin \gamma) \dots$$
$$= \cos (\alpha + \beta + \gamma + \dots) \pm \sqrt{-1} \sin (\alpha + \beta + \gamma + \dots),$$

the upper or lower sign being taken throughout.

Demoivre's Theorem.

41. *To show that for all values of* n, *positive or negative, integral or fractional,*

$$(\cos \theta \pm \sqrt{-1} \sin \theta)^n = \cos n\theta \pm \sqrt{-1} \sin \theta.$$

As in the last Art., we need only trouble ourselves with the *upper sign.*

We have (Art. 40),

$$(\cos \alpha + \sqrt{-1} \sin \alpha) (\cos \beta + \sqrt{-1} \sin \beta) (\cos \gamma + \sqrt{-1} \sin \gamma) \dots$$
$$= \cos (\alpha + \beta + \gamma + \dots) + \sqrt{-1} \sin (\alpha + \beta + \gamma \dots).$$

Let $\alpha = \beta = \gamma = \ldots\ldots = \theta$, and let there be n factors, then, since

$$\alpha + \beta + \gamma + \ldots\ldots = n\theta,$$

we have,

(i.) *When* n *is a positive integer,*

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta \quad \ldots\ldots\ldots (1).$$

Now let $n = -m$, where m is a positive integer.

We have

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \frac{1}{(\cos\theta + \sqrt{-1}\sin\theta)^m}; \text{ or by (1),}$$

$$= \frac{1}{\cos m\theta + \sqrt{-1}\sin m\theta}$$

$$= \frac{\cos m\theta - \sqrt{-1}\sin m\theta}{(\cos m\theta + \sqrt{-1}\sin m\theta)(\cos m\theta - \sqrt{-1}\sin m\theta)}$$

$$= \frac{\cos m\theta - \sqrt{-1}\sin m\theta}{1}$$

$$= \cos(-m)\theta + \sqrt{-1}\sin(-m)\theta$$

$$= \cos n\theta + \sqrt{-1}\sin n\theta.$$

Hence

(ii.) *The formula is true when* n *is a negative integer.*

Again, we have, when m and n are any integers,

$$\left(\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta\right)^n = \cos n\left(\frac{m}{n}\theta\right) + \sqrt{-1}\sin n\left(\frac{m}{n}\theta\right)$$

$$= \cos m\theta + \sqrt{-1}\sin m\theta$$

$$= (\cos\theta + \sqrt{-1}\sin\theta)^m.$$

Or, taking the nth root of each side,

$$\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta = (\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}.$$

Hence putting n for $\frac{m}{n}$, we have, where n is a fraction,

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta.$$

Therefore, *for all values of* n,

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta.$$

In the next Art. we shall show that $\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta$ is only *one* of n values of the expression $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$, where $\frac{m}{n}$ is a fraction in its lowest terms.

42. *To find the* n *values of the expression* $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$, *where* $\frac{m}{n}$ *is a fraction in its lowest terms.*

Since $\cos\theta = \cos(2r\pi + \theta)$, and $\sin\theta = \sin(2r\pi + \theta)$, where r is an integer, we have

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \{\cos(2r\pi + \theta) + \sqrt{-1}\sin(2r\pi + \theta)\}^{\frac{m}{n}}$$

$$= \cos\frac{m}{n}(2r\pi + \theta) + \sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta)\ldots\ldots\ldots(1).$$

By giving r the values $0, 1, 2, 3 \ldots\ldots (n-1)$ successively, we obtain n *different* values of the expression $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$.

For, suppose $r = p$, and $r = q$, where p and q are each less than n, gave the *same values*,

Then evidently $\frac{m}{n}(2p\pi + \theta)$, and $\frac{m}{n}(2q\pi + \theta)$, must differ by a multiple of 2π.

Hence $\frac{m}{n}\{(2q\pi + \theta) - (2p\pi + \theta)\}$, or $\frac{m}{n}(p - q)\,2\pi$ is a multiple of 2π.

Therefore $\frac{m}{n}(p - q)$ is an integer, which is absurd, since $\frac{m}{n}$ is in its lowest terms, and p and q are each less than n.

Moreover, no value of r beyond the limits of the series $0, 1, 2, 3, \ldots\ldots (n-1)$ will give any value of $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$ which is not given by the values in this series.

For suppose $r = an + r'$, when a is an integer either positive or negative, and r' one of the values 0, 1, 2, 3 . . . . $(n - 1)$.

Then

$$\cos\frac{m}{n}(2r\pi + \theta) = \cos\frac{m}{n}\{2(an + r')\pi + \theta\} = \cos\{2am\pi + \frac{m}{n}(2r'\pi + \theta)\}$$

$$= \cos\frac{m}{n}(2r'\pi + \theta).$$

And so, $\sin\frac{m}{n}(2r\pi + \theta) = \sin\frac{m}{n}(2r'\pi + \theta)$.

Hence, when $r = an + r'$,

$(\cos\theta + \sqrt{-1}\,.\,\sin\theta)^{\frac{m}{n}} = \cos\frac{m}{n}(2r'\pi + \theta) + \sqrt{-1}\sin\frac{m}{n}(2r'\pi + \theta)$, which is one of the values obtained from giving to r the series of values 0, 1, 2, 3 . . . . $(n - 1)$.

Hence,

$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \cos\frac{m}{n}(2r\pi + \theta) + \sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta)$, where r may have each of the values 0, 1, 2, 3, ... $(n - 1)$.

43. *To expand* $\sin n\theta$, $\cos n\theta$, $\tan n\theta$ *in terms of the ratios of* θ.

We have $$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n;$$

or, expanding by the Binomial Theorem and arranging,

$$\cos n\theta + \sqrt{-1}\sin n\theta$$

$$= \left\{\cos^n\theta - \frac{n(n-1)}{1\,.\,2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\cos^{n-4}\theta\sin^4\theta - \&c.\right\}$$

$$+ \sqrt{-1}\left\{n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cos^{n-3}\theta\sin^3\theta + \&c.\right\}.$$

Equating the possible and impossible parts of this equality, we have

$$\cos n\theta = \cos^n\theta\left\{1 - \frac{n(n-1)}{1\,.\,2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\tan^4\theta - \&c.\right\}\ldots\ldots(1),$$

$$\text{and }\sin n\theta = \cos^n\theta\left\{n\tan\theta - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\tan^3\theta + \&c.\right\}\ldots\ldots\ldots(2).$$

Hence (2) ÷ (1), we have

$$\tan n\theta = \frac{n\tan\theta - \dfrac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\tan^3\theta + \&c.}{1 - \dfrac{n(n-1)}{1\,.\,2}\tan^2\theta + \dfrac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\tan^4\theta - \&c.}\ldots\ldots(3).$$

44. *To express* $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ *as series involving powers of* α.

We have (Art. 43),

$$\cos n\theta = \cos^n\theta\left\{1 - \frac{n(n-1)}{1\,.\,2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\tan^4\theta - \&c.\right\}.$$

Put $n\theta = a$, and therefore $n = \frac{a}{\theta}$; then, arranging,

$$\cos a = \cos^n \theta \left\{1 - \frac{a(a-\theta)}{1 \cdot 2}\left(\frac{\tan \theta}{\theta}\right)^2 + \frac{a(a-\theta)(a-2\theta)(a-3\theta)}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{\tan \theta}{\theta}\right)^4 - \&c.\right\}.$$

Let n be infinite, and therefore the limit of θ be zero.

Then, limit of $\cos \theta = 1$, and of $\frac{\tan \theta}{\theta} = 1$.

Hence, we have

$$\cos a = \left\{1 - \frac{a \cdot a}{1 \cdot 2}(1)^2 + \frac{a \cdot a \cdot a \cdot a}{1 \cdot 2 \cdot 3 \cdot 4}(1)^4 - \&c.\right\};$$

$$\text{or, } \cos a = 1 - \frac{a^2}{1 \cdot 2} + \frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c. \ldots\ldots (1);$$

and so

$$\sin a = a - \frac{a^3}{1 \cdot 2 \cdot 3} + \frac{a^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c. \ldots\ldots (2).$$

Dividing (2) by (1), then

$$\tan a = a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c. \ldots\ldots (3).$$

Exponential Expressions.

45. *To find exponential expressions for* $\sin\theta$, $\cos\theta$, $\tan\theta$.

We have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.}$$

Put successively $\theta\sqrt{-1}$ and $-\theta\sqrt{-1}$ for x, then

$$e^{\theta\sqrt{-1}} = 1 + \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1\,.\,2} - \frac{\theta^3\sqrt{-1}}{1\,.\,2\,.\,3} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.}$$

and $$e^{-\theta\sqrt{-1}} = 1 - \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1\,.\,2} + \frac{\theta^3\sqrt{-1}}{1\,.\,2\,.\,3} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}$$

Hence, adding,

$$e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}} = 2\left\{1 - \frac{\theta^2}{1\,.\,2} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}\right\} = 2\cos\theta.$$

$$\therefore \cos\theta = \tfrac{1}{2}\left(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right) \ldots\ldots\ldots\ldots (1).$$

And so, subtracting,

$$\sin\theta = \frac{1}{2\sqrt{-1}}\left(e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}\right) \ldots\ldots\ldots\ldots (2).$$

(2) ÷ (1), then

$$\tan\theta = \frac{1}{\sqrt{-1}} \cdot \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}} = \frac{1}{\sqrt{-1}} \cdot \frac{e^{2\theta\sqrt{-1}} - 1}{e^{2\theta\sqrt{-1}} + 1} \cdot \text{ .... } (3).$$

Cor. From (1) and (2) we get

$$e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta,$$

$$e^{-\theta\sqrt{-1}} = \cos\theta - \sqrt{-1}\sin\theta.$$

Determination of the Value of π.

46. *Gregory's Series.* To show that

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \text{\&c.}$$

We have (Art. 45, Cor.),

$$\frac{e^{\theta\sqrt{-1}}}{e^{-\theta\sqrt{-1}}} = \frac{\cos\theta + \sqrt{-1}\sin\theta}{\cos\theta - \sqrt{-1}\sin\theta}, \text{ or}$$

$$e^{2\theta\sqrt{-1}} = \frac{1 + \sqrt{-1}\tan\theta}{1 - \sqrt{-1}\tan\theta}.$$

Taking $\log_e$ of each side of this equality, then

$$2\,\theta\sqrt{-1} = \log_e \frac{1 + \sqrt{-1}\tan\theta}{1 - \sqrt{-1}\tan\theta}$$

$$= 2\left\{\sqrt{-1}\tan\theta - \frac{\sqrt{-1}\tan^3\theta}{3} + \frac{\sqrt{-1}\tan^5\theta}{5} - \&c.\right\}.$$

$$\therefore\ \theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \&c.$$

COR. Put $\theta = \tan^{-1} a$, or $\tan\theta = a$, then

$$\tan^{-1} a = a - \tfrac{1}{3}a^3 + \tfrac{1}{5}a^5 - \&c.$$

5—II.

R **47.** *To prove Euler's series for determining the value of* π, viz. :

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \&c.\right) + \left(\frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \ldots\right).$$

We have
$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = \tan^{-1}\frac{\tfrac{1}{2} + \tfrac{1}{3}}{1 - \tfrac{1}{2}\,.\,\tfrac{1}{3}} = \tan^{-1} 1.$$

But
$$\tan^{-1} 1 = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

Hence $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$; or, by the last Art.,

$$= \left(\frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \ldots\right) + \left(\frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \ldots\right).$$

48. *To prove Machin's series,* viz. :

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \ldots\right) - \left\{\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \ldots\right\}.$$

We have (Art. 15),

$$\frac{\pi}{4} = 4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239},$$ which gives the series required.

Miscellaneous Propositions.

49. *To prove Euler's series for* sin θ *and* cos θ, *without the aid of Demoivre's Theorem.*

Since $\sin(-\theta) = -\sin\theta$, it is evident that the value of $\sin\theta$ changes sign with θ. Hence the series for $\sin\theta$ in powers of θ can contain only odd powers of θ.

And since $\cos(-\theta) = \cos\theta$, it may be similarly reasoned that the series for $\cos\theta$ in powers of θ contains only even powers.

Assume then $$\sin\theta = a_1\theta + a_3\theta^3 + a_5\theta^5 + \ldots \quad (1),$$

and $$\cos\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots \quad (2).$$

From (1), $$\frac{\sin\theta}{\theta} = a_1 + a_3\theta^2 + a_5\theta^4 + \ldots ;$$

Now, as this is true for all values of θ, it is true for the value $\theta = 0$; we then have

$$\frac{\sin\theta}{\theta} = 1 \text{; and hence } 1 = a_1.$$

And so from (2), when $\theta = 0$, we have $1 = a_0$.

We may therefore write

$$\sin\theta = \theta + a_3\theta^3 + a_5\theta^5 + \dots \qquad \dots\dots\dots\dots(3),$$

and

$$\cos\theta = 1 + a_2\theta^2 + a_4\theta^4 + \dots \qquad \dots\dots\dots\dots(4).$$

(3) + (4), then

$$\cos\theta + \sin\theta = 1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + \dots (5).$$

Put $\theta + \phi$ for θ; then, arranging,

$$(\cos\theta + \sin\theta)\cos\phi + (\cos\theta - \sin\theta)\sin\phi$$
$$= 1 + (\theta + \phi) + a_2(\theta + \phi)^2 + a_3(\theta + \phi)^3 + a_4(\theta + \phi)^4$$
$$+ a_5(\theta + \phi)^5 + \dots \qquad \dots\dots\dots\dots\dots(6).$$

But from (3), (4), and (5), we get

$$(\cos\theta + \sin\theta)\cos\phi + (\cos\theta - \sin\theta)\sin\phi$$
$$= (1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + \dots)$$
$$(1 + a_2\phi^2 + a_4\phi^4 + \dots)$$
$$+ (1 - \theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - a_5\theta^5 + \dots)$$
$$(\theta + a_3\theta^3 + a_5\theta^5 + \dots)\dots\dots\dots\dots\dots(7).$$

Equating co-efficients of like powers of $\theta\phi$, $\theta^2\phi$, &c., in (6) and (7), then

$$2a_2 = -1, \therefore a_2 = -\frac{1}{1.2}.$$

$$3a_3 = a_2, \quad \therefore a_3 = -\frac{1}{1.2.3}.$$

$$4a_4 = -a_3, \therefore a_4 = \frac{1}{1.2.3.4}.$$

$$5a_5 = a_4, \quad \therefore a_5 = \frac{1}{1.2.3.4.5}.$$

&c. = &c.

Hence from (3) and (4) we get by substitution,

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \text{\&c.},$$

and

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \text{\&c.}$$

50. *To show that if* $2\cos\theta = x + \frac{1}{x}$, *then*

$$2\sqrt{-1}\sin\theta = x - \frac{1}{x};$$

and also that

$$2\cos n\theta = x^n + \frac{1}{x^n},\ \textit{and}\ 2\sqrt{-1}\sin n\theta = x^n - \frac{1}{x^n}.$$

We have $2\cos\theta = x + \frac{1}{x}$; or, squaring,

$$4\cos^2\theta = \left(x + \frac{1}{x}\right)^2; \text{ or}$$

$$-4(1-\cos^2\theta) = \left(x + \frac{1}{x}\right)^2 - 4; \text{ or}$$

$$-4\sin^2\theta = \left(x - \frac{1}{x}\right)^2.$$

$$\therefore 2\sqrt{-1}\sin\theta = x - \frac{1}{x}.$$

Hence also,

$$2(\cos\theta + \sqrt{-1}\sin\theta) = \left(x + \frac{1}{x}\right) + \left(x - \frac{1}{x}\right) = 2x.$$

$$\therefore \cos\theta + \sqrt{-1}\sin\theta = x \quad\text{.................(1).}$$

And so, $$\cos\theta - \sqrt{-1}\sin\theta = \frac{1}{x} \quad\text{.................(2).}$$

From (1), by Demoivre's Theorem,

$$\cos n\theta + \sqrt{-1}\sin n\theta = x^n;$$

and from (2), $$\cos n\theta - \sqrt{-1}\sin n\theta = \frac{1}{x^n}.$$

From these two equations, the required results at once follow.

51. *To expand* $\sin^n \theta$ *in terms of sines or cosines of multiples of* θ, *when* n *is a positive integer.*

Put $2 \cos \theta = x + \frac{1}{x}$, and therefore $2\sqrt{-1} \sin \theta = x - \frac{1}{x}$.

Then $\quad (2\sqrt{-1} \sin \theta)^n = \left(x - \frac{1}{x}\right)^n$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} - \dots + \frac{n(n-1)}{1.2}x^2\left(-\frac{1}{x}\right)^{n-2} + nx\left(-\frac{1}{x}\right)^{n-1} + \left(-\frac{1}{x}\right)^n \dots (1).$$

(i.) *Let* n *be even.*

Then, as there are $n + 1$ terms in (1),

the middle term is the $\left(\frac{n}{2} + 1\right)$th term

$$= \frac{n(n-1)\dots\left(n - \frac{n}{2} + 1\right)}{1.2\dots\dots\frac{n}{2}} x^{\frac{n}{2}}\left(-\frac{1}{x}\right)^{\frac{n}{2}}$$

$$= (-1)^{\frac{n}{2}} \cdot \frac{n(n-1)\dots\left(\frac{n}{2} + 1\right)}{1.2\dots\frac{n}{2}}.$$

We have therefore from (1), collecting terms equidistant from the beginning and end of the series,

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1 \cdot 2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right)$$

$$+ \cdots\cdots + (-1)^{\frac{n}{2}} \cdot \frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1 \cdot 2 \ldots\ldots \frac{n}{2}}$$

$$= 2\cos n\theta - n \cdot 2\cos(n-2)\theta + \frac{n(n-1)}{1 \cdot 2}\, 2\cos(n-4)\theta$$

$$+ \cdots\cdots + (-1)^{\frac{n}{2}} \frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1 \cdot 2 \ldots\ldots \frac{n}{2}}.$$

$$\therefore 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1 \cdot 2}\cos(n-4)\theta$$

$$- \cdots\cdots + \tfrac{1}{2}(-1)^{\frac{n}{2}} \cdot \frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1 \cdot 2 \ldots\ldots \frac{n}{2}}.$$

(ii.) *Let* n *be odd.*

We may similarly show that

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\sin(n-4)\theta$$

$$-\ldots\ldots + (-1)^{\frac{n-1}{2}}\frac{n(n-1)\ldots\ldots\frac{1}{2}(n+3)}{1\,.\,2\ldots\ldots\frac{1}{2}(n-1)}\,.\,\sin\theta.$$

52. *To expand* $\cos^n\theta$ *in terms of the cosines of multiples of* θ.

We may show exactly as in the last Art. that:

(i.) *When* n *is even,*

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta$$

$$+\ldots\ldots + \tfrac{1}{2}\,.\,\frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\ldots\ldots\frac{n}{2}}.$$

(ii.) *When* n *is odd,*

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta$$

$$+\ldots\ldots + \frac{n(n-1)\ldots\ldots\frac{n+3}{2}}{1\,.\,2\ldots\ldots\frac{n-1}{2}}\cos\theta.$$

53. *If* $\tan\alpha = \cos\omega \,.\, \tan l$, *to show that*

$$l - \alpha = \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2l - \tfrac{1}{2}\tan^4 \tfrac{1}{2}\omega \,.\, \sin 4l + \text{\&c.};$$ *and that*

$$l - \alpha = \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2\alpha + \tfrac{1}{2}\tan^4 \tfrac{1}{2}\omega \,.\, \sin 4\alpha + \text{\&c.}$$

We have (Art 33),

$$\frac{1}{\sqrt{-1}} \cdot \frac{e^{2\alpha\sqrt{-1}} - 1}{e^{2\alpha\sqrt{-1}} + 1} = \cos\omega \,.\, \frac{1}{\sqrt{-1}} \frac{e^{2l\sqrt{-1}} - 1}{e^{2l\sqrt{-1}} + 1}; \text{ or,}$$

$$\frac{e^{2\alpha\sqrt{-1}} - 1}{e^{2\alpha\sqrt{-1}} + 1} = \frac{(e^{2l\sqrt{-1}} - 1)\cos\omega}{e^{2l\sqrt{-1}} + 1}.$$

Taking the sum and difference of numerator and denominator, then

$$e^{2\alpha\sqrt{-1}} = \frac{e^{2l\sqrt{-1}}(1 + \cos\omega) + (1 - \cos\omega)}{e^{2l\sqrt{-1}}(1 - \cos\omega) + (1 + \cos\omega)}; \text{ or, arranging, \&c.,}$$

$$= \frac{e^{2l\sqrt{-1}} + \tan^2 \frac{1}{2}\omega}{e^{2l\sqrt{-1}} \tan^2 \frac{1}{2}\omega + 1}$$

$$= e^{2l\sqrt{-1}} \frac{1 + \tan^2 \frac{1}{2}\omega \,.\, e^{-2l\sqrt{-1}}}{1 + \tan^2 \frac{1}{2}\omega \,.\, e^{2l\sqrt{-1}}}.$$

Taking logarithms of each side, and expanding, then

$$2\,a\sqrt{-1} = 2\,l\sqrt{-1} + \left\{\tan^2\tfrac{1}{2}\omega\;\; e^{-2l\sqrt{-1}} - \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,.\,e^{-4l\sqrt{-1}} + \&c.\right\}$$

$$-\left\{\tan^2\tfrac{1}{2}\omega\,.\,e^{2l\sqrt{-1}} - \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,.\,e^{4l\sqrt{-1}} + \&c.\right\}$$

$$= 2\,l\sqrt{-1} - \tan^2\tfrac{1}{2}\omega\,(e^{2l\sqrt{-1}} - e^{-2l\sqrt{-1}}) + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,(e^{4l\sqrt{-1}} - e^{-4l\sqrt{-1}}) - \&c.$$

$$= 2\,l\sqrt{-1} - \tan^2\tfrac{1}{2}\omega\,.\,2\sqrt{-1}\sin 2\,l + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,.\,2\sqrt{-1}\sin 4\,l - \&c.$$

Or, dividing each side by $2\sqrt{-1}$, then

$$a = l - \tan^2\tfrac{1}{2}\omega\,.\,\sin 2\,l + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,.\,\sin 4\,l - \&c.$$

Hence, transposing,

$$l - a = \tan^2\tfrac{1}{2}\omega\,.\,\sin 2\,l - \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,.\,\sin 4\,l + \&c.$$

And similarly, by taking the given equation in the form $\tan l = \dfrac{1}{\cos\omega}\,.\,\tan a$, we get the second required result.

54. *To resolve* $\sin\theta$ *and* $\cos\theta$ *into the product of quadratic factors in terms of* θ.

The equation $\sin\theta = 0$ is satisfied by $\theta = n\pi$, when n may be zero, or any integer, positive or negative.

Hence the values $\theta = 0, \pm \pi, \pm 2\pi, \pm 3\pi$, &c., represent the only possible solutions of the equation.

Therefore, by the Theory of Equations,

$$\sin \theta = a\theta (\theta + \pi)(\theta - \pi)(\theta + 2\pi)(\theta - 2\pi)(\theta + 3\pi)(\theta - 3\pi) \ldots .$$

$$= a\theta (\theta^2 - \pi^2)(\theta^2 - 2^2\pi^2)(\theta^2 - 3^2\pi^2) \ldots . \quad \ldots\ldots (1),$$

where a is some constant quantity to be determined.

We may determine a thus:

Since the equation (1) may be written

$$\frac{\sin \theta}{\theta} = a(\theta^2 - \pi^2)(\theta^2 - 2^2\pi^2)(\theta^2 - 3^2\pi^2) \ldots .,$$

and the equation is true when $\theta = 0$, we have, since $\frac{\sin \theta}{\theta} = 1$, when $\theta = 0$,

$$1 = a(-\pi^2)(-2^2\pi^2)(-3^2\pi^2) \ldots .$$

$$= \pm a\pi^2 . 2^2\pi^2 . 3^2\pi^2 \ldots . \quad \ldots\ldots (2).$$

But from (1) we get

$$\sin \theta = \pm a\pi^2 . 2^2\pi^2 . 3^2\pi^2 \ldots . \theta \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \ldots .$$

Hence, substituting from (2),

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots \quad \ldots\ldots\ldots\ldots(A).$$

To find the expression for $\cos\theta$.

The equation $\cos\theta = 0$ is satisfied by $\theta = (2n+1)\frac{\pi}{2}$, when n may be zero, or any integer, positive or negative.

Hence the values $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, &c. represent the only possible solutions of the equation.

Therefore, by the Theory of Equations,

$$\cos\theta = a\left(\theta + \frac{\pi}{2}\right)\left(\theta - \frac{\pi}{2}\right)\left(\theta + \frac{3\pi}{2}\right)\left(\theta - \frac{3\pi}{2}\right)\left(\theta + \frac{5\pi}{2}\right)\left(\theta - \frac{5\pi}{2}\right)\ldots$$

$$= a\left(\theta^2 - \frac{\pi^2}{2^2}\right)\left(\theta^2 - \frac{3^2\pi^2}{2^2}\right)\left(\theta^2 - \frac{5^2\pi^2}{2^2}\right)\ldots \quad \ldots\ldots\ldots\ldots(3),$$

where a is some constant to be determined.

Put $\theta = 0$, then, since $\cos\theta = 1$, when $\theta = 0$, we have

$$1 = a\left(-\frac{\pi^2}{2^2}\right)\left(-\frac{3^2\pi^2}{2^2}\right)\left(-\frac{5^2\pi^2}{2^2}\right)\ldots$$

$$= \pm a\,\frac{\pi^2}{2^2}\cdot\frac{3^2\pi^2}{2^2}\cdot\frac{5^2\pi^2}{2^2}\ldots \quad \ldots\ldots\ldots\ldots(4).$$

But from (3)

$$\cos\theta = \pm a \cdot \frac{\pi^2}{2^2} \cdot \frac{3\,\pi^2}{2^2} \cdots\cdots \left(1 - \frac{4\,\theta^2}{\pi^2}\right)\left(1 - \frac{4\,\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\,\theta^2}{5^2\pi^2}\right)\cdots\cdots$$

Hence, substituting from (4),

$$\cos\theta = \left(1 - \frac{4\,\theta^2}{\pi^2}\right)\left(1 - \frac{4\,\theta^2}{3^2\,\pi^2}\right)\left(1 - \frac{4\,\theta^2}{5^2\,\pi^2}\right)\cdots\cdots \qquad \ldots\text{(B)}.$$

COR. From the result for sin θ, we may obtain *Wallis's expression for* π.

In (A), put $\theta = \frac{\pi}{2}$, then since $\sin\frac{\pi}{2} = 1$, and $\frac{\pi}{\theta} = 2$, we have

$$1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^2 \cdot 2^2}\right)\left(1 - \frac{1}{3^2 \cdot 2^2}\right)\cdots\cdots$$

$$= \frac{\pi}{2} \cdot \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{5 \cdot 7}{6^2} \cdots\cdots$$

$$\therefore \frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots\cdots$$

55. *To resolve* $x^n - 1$ *into factors.*

We have, from the equation $x^n - 1 = 0$,

$$x^n = 1 = \cos 2\,m\pi \pm \sqrt{-1}\,\sin 2\,m\pi,$$

where m is any integer, including zero.

$$\therefore x = (\cos 2\,m\pi \pm \sqrt{-1}\,\sin 2\,m\pi)^{\frac{1}{n}}$$

$$= \cos\frac{2\,m\pi}{n} \pm \sqrt{-1}\,\sin\frac{2\,m\pi}{n}.$$

(i.) *Let* n *be even.*

Then by giving m all integral positive values from 0 to $\frac{n}{2}$, we get + 1 for each of the first pair of values, and − 1 for each of the last pair. For all other values of m, the values of x are different.

For suppose p and q to be two values of m between the limits 0 and $\frac{n}{2}$, such that

$$\cos\frac{2p\pi}{n} + \sqrt{-1}\sin\frac{2p\pi}{n} = \cos\frac{2q\pi}{n} \pm \sqrt{-1}\sin\frac{2q\pi}{n}.$$

Then we have $$\cos\frac{2p\pi}{n} = \cos\frac{2q\pi}{n},$$

and $$\sin\frac{2p\pi}{n} = \pm\sin\frac{2q\pi}{n}.$$

Hence (Art. 21), we must therefore have

$$\frac{2p\pi}{n} = 2r\pi \pm \frac{2q\pi}{n},$$

where r is any integer.

$$\text{or, } \frac{p}{n} = r \pm \frac{q}{n}; \quad \text{or, } \frac{p \mp q}{n} = r, \text{ an integer.}$$

But p and q are by supposition each $< \frac{n}{2}$, and are different.

Hence $\frac{p \mp q}{n}$ cannot be integral.

Hence the n values of x obtained from the equation

$$x = \frac{\cos 2m\pi}{n} \pm \sqrt{-1}\sin\frac{2m\pi}{n},$$

by giving m all integral values from 0 to $\frac{n}{2}$, are all different.

It must be remembered that we get really $(n + 2)$ values, the first pair of which are equal, viz. + 1, and the last pair equal, viz. − 1.

Further, no value of m beyond the limits 0 to $\frac{n}{2}$ will give any other value for x.

For any integer greater than $\frac{n}{2}$ may be put in the form $sn + t$, where t is not greater *in magnitude* than $\frac{n}{2}$.

Put then $m = sn + t$, where s and t are integral, and t not $> \frac{n}{2}$, but may be negative.

$$\text{Then} \quad \cos\frac{2m\pi}{n} \pm \sqrt{-1}\sin\frac{2m\pi}{n} = \cos\left(2s + \frac{2t}{n}\right)\pi \pm \sqrt{-1}\sin\left(2s + \frac{2t}{n}\right)\pi$$
$$= \cos\frac{2t\pi}{n} \pm \sqrt{-1}\sin\frac{2t\pi}{n},$$

which is *equal to* one of the values before obtained.

The only possible values of x which satisfy the equation $x^n - 1 = 0$, when n is even, are hence the following series:

$$\pm 1,\ \cos\frac{2\pi}{n} \pm \sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} \pm \sqrt{-1}\sin\frac{4\pi}{n},\ \&c.,\ \cos\frac{(n-2)\pi}{n} \pm \sqrt{-1}\sin\frac{(n-2)\pi}{n}.$$

Hence, by the Theory of Equations, since the coefficient of x^n in the equation $x^n - 1 = 0$, is unity, we must have

$$x^n - 1 = (x+1)(x-1) \cdot \left\{x - \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)\right\}\left\{x - \left(\cos\frac{2\pi}{n} - \sqrt{-1}\sin\frac{2\pi}{n}\right)\right\}$$
$$\ldots \left\{x - \left(\cos\frac{\overline{n-2}\,.\,\pi}{n} + \sqrt{-1}\sin\frac{\overline{n-2}\,.\,\pi}{n}\right)\right\}\left\{x - \left(\cos\frac{\overline{n-2}\,.\,\pi}{n} - \sqrt{-1}\sin\frac{\overline{n-2}\,.\,\pi}{n}\right)\right\}.$$

Or, simplifying, we have, when n is *even*, $x^n - 1$

$$= (x^2 - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots\ldots\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)\ldots(1).$$

(ii.) And so, *when* n *is odd*, we similarly get: $x^n - 1$

$$= (x - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots\ldots\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right)\ldots(2).$$

Cor. 1. By an exactly similar process, we get,

When n *is even*,

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots\ldots\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right)\ldots\ldots(3).$$

And, *when* n *is odd*,

$$x^n + 1 = (x + 1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots\ldots\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)\ldots\ldots(4).$$

Cor. 2. Many factorials may be obtained from the above expressions.

(i.) In (2), dividing each side by $x - 1$, and puttting $x = 1$, then

since $\dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \&c. + x + 1 = n$, when $x = 1$, we have

Further, no value of m beyond the limits 0 to $\frac{n}{2}$ will give any other value for x.

For any integer greater than $\frac{n}{2}$ may be put in the form $sn + t$, where t is not greater *in magnitude* than $\frac{n}{2}$.

Put then $m = sn + t$, where s and t are integral, and t not $> \frac{n}{2}$, but may be negative.

$$\text{Then} \quad \cos\frac{2m\pi}{n} \pm \sqrt{-1}\sin\frac{2m\pi}{n} = \cos\left(2s + \frac{2t}{n}\right)\pi \pm \sqrt{-1}\sin\left(2s + \frac{2t}{n}\right)\pi$$
$$= \cos\frac{2t\pi}{n} \pm \sqrt{-1}\sin\frac{2t\pi}{n},$$

which is *equal to* one of the values before obtained.

The only possible values of x which satisfy the equation $x^n - 1 = 0$, when n is even, are hence the following series:

$$\pm 1,\ \cos\frac{2\pi}{n} \pm \sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} \pm \sqrt{-1}\sin\frac{4\pi}{n},\ \&c.,\ \cos\frac{(n-2)\pi}{n} \pm \sqrt{-1}\sin\frac{(n-2)\pi}{n}.$$

Hence, by the Theory of Equations, since the coefficient of x^n in the equation $x^n - 1 = 0$, is unity, we must have

$$x^n - 1 = (x+1)(x-1) \cdot \left\{x - \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)\right\}\left\{x - \left(\cos\frac{2\pi}{n} - \sqrt{-1}\sin\frac{2\pi}{n}\right)\right\}$$
$$\ldots\left\{x - \left(\cos\frac{\overline{n-2}\,.\,\pi}{n} + \sqrt{-1}\sin\frac{\overline{n-2}\,.\,\pi}{n}\right)\right\}\left\{x - \left(\cos\frac{\overline{n-2}\,.\,\pi}{n} - \sqrt{-1}\sin\frac{\overline{n-2}\,.\,\pi}{n}\right)\right\}.$$

Or, simplifying, we have, when n is *even*, $x^n - 1$

$$= (x^2 - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right) \ldots (1).$$

(ii.) And so, *when* n *is odd*, we similarly get: $x^n - 1$

$$= (x - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right) \ldots (2).$$

Cor. 1. By an exactly similar process, we get,

When n *is even*,

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right) \ldots\ldots (3).$$

And, *when* n *is odd*,

$$x^n + 1 = (x + 1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right) \ldots\ldots (4).$$

Cor. 2. Many factorials may be obtained from the above expressions.

(i.) In (2), dividing each side by $x - 1$, and puttting $x = 1$, then

since $\dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \&c. + x + 1 = n$, when $x = 1$, we have

$$n = \left(2 - 2\cos\frac{2\pi}{n}\right)\left(2 - 2\cos\frac{4\pi}{n}\right)\ldots\ldots\left(2 - 2\cos\frac{n-1}{n}\pi\right), \textit{ when n is odd};$$

$$= 2^2\sin^2\frac{\pi}{n}\cdot 2^2\sin^2\frac{2\pi}{n}\ldots\ldots 2^2\sin^2\frac{n-1}{2n}\pi;$$

$$\text{or, } n = 2^{n-1}\cdot\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\ldots\ldots\sin^2\frac{n-1}{2n}\pi.$$

$$\therefore \sqrt{n} = 2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\sin\frac{n-1}{2n}\pi, \textit{ when n is odd.}$$

We take the positive sign of the square root, because every sine is positive, every angle being between 0 and $\frac{\pi}{2}$.

(ii.) So from (1), by putting $2n$ for n, and similarly reasoning, we get for all integral values of n:

$$\sqrt{n} = 2^{n-1}\cdot\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\ldots\ldots\sin\frac{n-1}{2n}\pi.$$

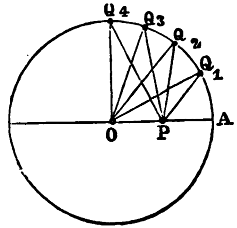
with many similar results.

56. *Demoivre's Property of the Circle.*

Let P be any point within or without a circle whose centre is O.

And let the whole circumference be divided into n equal parts Q_1Q_2, Q_2Q_3, &c., so that

$$Q_1Q_2 = Q_2Q_3 = \&c. = \frac{2\pi}{n}.$$



Join O and P with the points Q_1, Q_2, &c.;

And let $\angle POQ_1 = \theta$.

Then will

$$OP^{2n} - 2\,OP^n . OQ_1^n \cos n\theta + OQ_1^n = PQ_1^2 . PQ_2^2 \ldots . PQ_n^2.$$

For we have

$$\angle POQ_1 = \theta,\ \angle POQ_2 = \theta + \frac{2\pi}{n},\ \angle POQ_3 = \theta + \frac{4\pi}{n}, \&c.$$

Hence
$$PQ_1^2 = OP^2 - 2\,OP . r\cos\theta + r^2,$$

$$PQ_2^2 = OP^2 - 2\,OP . r\cos\left(\theta + \frac{2\pi}{n}\right) + r^2,$$

$$PQ_3^2 = OP^2 - 2\,OP . r\cos\left(\theta + \frac{4\pi}{n}\right) + r^2,$$

. . . .

$$PQ_n^2 = OP^2 - 2\,OP . r\cos\left(\theta + \frac{n-1 . 2\pi}{n}\right) + r^2.$$

5—II.

Therefore, multiplying these equations together,

$$PQ_1^2 \,.\, PQ_2^2 \,.\, PQ_3^2 \ldots PQ_n^2$$

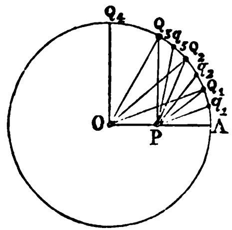
$$= (OP^2 - 2\,OP\,.\,r\cos\theta + r^2)\left\{OP^2 - 2\,OP\,.\,r\cos\left(\theta + \frac{2\pi}{n}\right) + r^2\right\}$$

$$\times\left\{OP^2 - 2\,OP\,.\,r\cos\left(\theta + \frac{4\pi}{n}\right) + r^2\right\} \ldots \left\{OP^2 - 2\,OP\,.\,r\cos\left(\theta + \frac{\overline{n-1}\,.\,2\pi}{n}\right) + r^2\right\}.$$

Now it may be easily shown, as in Art. 54, that the resolution of $OP^{2n} - 2\,OP^n\,.\,r^n\cos n\theta + r^{2n}$ gives the expression on the right side of this equation.

Hence $$OP^{2n} - 2\,OP^n\,.\,r^n\cos n\theta + r^{2n} = PQ_1^2\,.\,PQ_2^2\,.\,PQ_3^2 \ldots PQ_n^2,$$
which proves the proposition.

57. *Cotes's Property of the Circle.*



Let OP meet the circle in A so that A corresponds with Q_n. Also let the arcs AQ_1, Q_1Q_2, &c., be bisected in q_1, q_2, q_3, &c.

Then shall (i.) $OP^n \sim OQ_n^n = PQ_1\ \ PQ_2 \ldots PQ_n$; and

(ii.) $OP^n + OQ_n^n = Pq_1\ \ Pq_2 \ldots Pq_n$.

For we have $$PQ_1^2 = OP^2 - 2\,OP\,.\,r\cos\frac{2\pi}{n} + r^2,$$

$$PQ_2^2 = OP^2 - 2\,OP\,.\,r\cos\frac{4\pi}{n} + r^2, \text{ \&c.}$$

Hence $$PQ_1^2 . PQ_2^2 \dots . PQ_n^2$$

$$= (OP^2 - 2\,OP . r \cos\frac{2\pi}{n} + r^2)(OP^2 - 2\,OP . r\cos\frac{4\pi}{n} + r^2)$$

$$\dots . (OP^2 - 2\,OP . r\cos\frac{n . 2\pi}{n} + r^2).$$

Or, as in last Art.,

$$PQ_1^2 . PQ_2^2 \dots . PQ_n^2 = OP^{2n} - 2\,OP^n . r\cos n . \frac{2\pi}{n} + r^{2n} = (OP^n \sim r^n)^2.$$

Taking the square roots, and putting OQ_n for r, we have

$$\therefore OP^n \sim OQ_n^n = PQ_1 . PQ_2 \dots . PQ_n, \text{ which proves (i).}$$

Put $2n$ for n, then, since q_1, q_2, &c., are the intermediate divisions of the arcs, and the last two points of division are q_n, Q_n, we have

$$OP^{2n} \sim OQ_n^{2n} = PQ_1 . Pq_1 . PQ_2 . Pq_2 \dots . Pq_n . PQ_n.$$

Dividing this result by the former, then

$$OP^n + OQ_n^n = Pq_1 . Pq_2 \dots . Pq_n, \text{ which proves (ii.)}$$

Ex. VI.

1. Find a formula which gives all the values of $\sqrt[n]{1}$, where n is a positive integer.
2. Find the nth roots of -1.

3. Prove that

$$\cos n\theta = (-1)^{\frac{n}{2}} \left\{ 1 - \frac{n^2}{1 \,.\, 2} \cos^2\theta + \frac{n^2(n^2 - 2^2)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \cos^4\theta - \&c. \right\}, \qquad (n \text{ even});$$

$$= (-1)^{\frac{n-1}{2}} \left\{ n \cos\theta - \frac{n(n^2 - 1^2)}{1 \,.\, 2 \,.\, 3} \cos^3\theta + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \cos^5\theta - \&c. \right\}, (n \text{ odd}).$$

4. Show that

$$\sin n\theta = (-1)^{\frac{n}{2}-1} \sin\theta \left\{ n \cos\theta - \frac{n(n^2 - 2^2)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \cos^3\theta + \&c. \right\}, \qquad (n \text{ even});$$

$$= (-1)^{\frac{n-1}{2}} \sin\theta \left\{ 1 - \frac{n^2 - 1^2}{1 \,.\, 2} \cos^2\theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \cos^4\theta - \&c. \right\}, (n \text{ odd}).$$

5. Having given the series for $\sin\theta$ and $\cos\theta$ in powers of θ, and the factorial expressions of Art. 53, show that

$$(1),\quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c. = \frac{\pi^2}{6},$$

$$(2),\quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{\pi^2}{8}.$$

6. Show that, when $a = \dfrac{\pi}{2n}$,

$$2^{n-1} \sin a \sin 3a \sin 5a \ldots \sin(2n - 1)a = 1,$$

and $2^{n-1} \sin 2a \sin 4a \sin 6a \ldots \sin(2n - 2)a = n.$

Prove that

7. $\text{Sin}\,\theta = 2^{n-1} \sin\frac{\theta}{n} \sin\frac{\pi + \theta}{n} \cdot \sin\frac{2\pi + \theta}{n} \cdots \sin\frac{(n-1)\pi + \theta}{n}.$

8. $2^{n-1} \cos\frac{\theta}{n} \cos\frac{\pi + \theta}{n} \cdot \cos\frac{2\pi + \theta}{n} \cdots \cos\frac{(n-1)\pi + \theta}{n} = \pm \sin\theta,\ n \text{ even,}$

$$= \pm \cos\theta,\ n \text{ odd.}$$

9. $\text{Tan}\,\frac{\pi}{4n} \tan\frac{5\pi}{4n} \tan\frac{9\pi}{4n} \cdots \tan\frac{4n-3}{4n}\pi = \pm 1.$

10. Show, by substituting the values of $\cos 2\theta$, and $\cos\theta$, when expressed as series in powers of θ, that

$$\cos 2\theta = 2\cos^2\theta - 1.$$

11. Show that

$$e^x + 2\cos\theta + e^{-x} = 4\cos\frac{\theta + x\sqrt{-1}}{2}\cos\frac{\theta - x\sqrt{-1}}{2},$$

And hence show how to resolve $e^x + 2\cos\theta + e^{-x}$ into real quadratic factors.

12. Prove that

$$\log\sec\theta = \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \cdots$$

13. Show that, when x is very large,

$$\sqrt{\pi x} = \frac{2 \cdot 4 \cdot 6 \cdots 2x}{1 \cdot 3 \cdot 5 \cdots (2x-1)}.$$

14. Prove that

$$e^x + e^{-x} = 2\left(1 + \frac{2^2 x^2}{\pi^2}\right)\left(1 + \frac{2^2 x^2}{3^2 \pi^2}\right) \ldots .$$

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2 \pi^2}\right) \ldots .$$

15. If $\sin \theta = \sin m \,.\, \sin (n + \theta)$, then

$$\theta = \sin m \sin n + \tfrac{1}{2} \sin^2 m \,.\, \sin 2n + \tfrac{1}{3} \sin^3 m \,.\, \sin 3n + \ldots .$$

16. If $\tan m = a \,.\, \tan n$, show that

$$m = n + \frac{a-1}{a+1} \sin 2n + \tfrac{1}{2}\left(\frac{a-1}{a+1}\right)^2 \sin 4n + \text{\&c.},$$

$$n = m - \frac{a-1}{a+1} \sin 2m + \tfrac{1}{2}\left(\frac{a-1}{a+1}\right)^2 \sin 4m - \text{\&c.}$$

17. If $\sin (\alpha + \beta \sqrt{-1}) = a + b\sqrt{-1}$, then

$$a = (e^{\beta} + e^{-\beta}) \sin \alpha,$$

$$\text{and} \quad b = (e^{\beta} - e^{-\beta}) \cos \alpha.$$

18. Find the roots of the equation $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

CHAPTER VII.

SUMMATION OF TRIGONOMETRICAL SERIES.

58. *To find the sums of a series of sines and cosines of angles in A. P.*

Let α, $\alpha + \beta$, $\alpha + 2\beta$, ..., $\alpha + \overline{n-1}\,.\,\beta$ be the angles.

We have

$$2\sin\frac{\beta}{2}\sin\alpha = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right),$$

$$2\sin\frac{\beta}{2}\sin(\alpha + \beta) = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right),$$

$$2\sin\frac{\beta}{2}\sin(\alpha + 2\beta) = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\sin\frac{\beta}{2}\sin(\alpha + \overline{n-1}\,.\,\beta) = \cos\left(\alpha + \frac{2n-3}{2}\beta\right) - \cos\left(\alpha - \frac{2n-1}{2}\beta\right).$$

Hence, adding, we get

$$2\sin\frac{\beta}{2}\left\{\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin(\alpha + \overline{n-1}\,.\,\beta)\right\}$$

$$= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{2n-1}{2}\beta\right) = 2\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}.$$

$$\therefore \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots + \sin(\alpha + \overline{n - 1}\,.\,\beta)$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

And similarly we find that

$$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots\ldots + \cos(\alpha + \overline{n - 1}\,.\,\beta)$$

$$= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

Cor. 1. If we put $\beta = \alpha$, we have

(i.) $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots\ldots + \sin n\alpha = \dfrac{\sin\frac{1}{2}(n+1)\alpha \sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}$,

(ii.) $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots\ldots + \cos n\alpha = \dfrac{\cos\frac{1}{2}(n+1)\alpha \sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}$.

Cor. 2. So also we may find the sum of a series of *squares, cubes*, &c., of sines and cosines of angles in A. P.

Thus by Art. 55, $\quad 16\sin^5\alpha = \sin 5\alpha - 5\sin 3\alpha + 10\sin\alpha.$

$$\therefore\ 16\left\{\sin^5 \alpha + \sin^5 (\alpha + \beta) + \sin^5 (\alpha + 2\,\beta) + \ldots\ldots + \sin^5 (\alpha + \overline{n-1}\,.\,\beta)\right\}$$

$$= \left\{\sin 5\,\alpha + \sin (5\,\alpha + 5\,\beta) + \ldots\ldots + \sin (5\,\alpha + \overline{n-1}\,.\,5\,\beta)\right\}$$

$$- 5\left\{\sin 3\,\alpha + \sin (3\,\alpha + 3\,\beta) + \ldots\ldots + \sin (\alpha + \overline{n-1}\,.\,\beta)\right\}$$

$$+ 10\left\{\sin \alpha + \sin (\alpha + \beta) + \ldots\ldots + \sin (\alpha + \overline{n-1}\,.\,\beta)\right\}.$$

From which the required sum is easily found by (A) above.

59. *To sum* $\sin \alpha - \sin (\alpha + \beta) + \sin (\alpha + 2\,\beta) - \&c.$ *to* n *terms,*

and $\cos \alpha - \cos (\alpha + \beta) + \cos (\alpha + 2\,\beta) - \&c.$ *to* n *terms.*

$$2 \cos \frac{\beta}{2} \sin \alpha = \sin \left(\alpha + \frac{\beta}{2}\right) + \sin \left(\alpha - \frac{\beta}{2}\right),$$

$$-2 \cos \frac{\beta}{2} \sin (\alpha + \beta) = - \sin \left(\alpha + \frac{3\,\beta}{2}\right) - \sin \left(\alpha + \frac{\beta}{2}\right),$$

$$2 \cos \frac{\beta}{2} \sin (\alpha + 2\,\beta = \sin \left(\alpha + \frac{5\,\beta}{2}\right) + \sin \left(\alpha + \frac{3\,\beta}{2}\right),$$

. . . .

$$(-1)^{n-1}\, 2 \cos \frac{\beta}{2} \sin (\alpha + \overline{n-1}\,.\,\beta) = (-1)^{n-1} \left\{\sin \left(\alpha + \frac{2\,n-1}{2}\,\beta\right) + \left(\alpha + \frac{2\,n-3}{2}\,\beta\right)\right\}.$$

Hence, adding, we get

$$2\cos\frac{\beta}{2}\left\{\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha + 2\beta) - \ldots\ldots + (-1)^{n-1}\sin(\alpha + \overline{n-1}\,.\,\beta)\right\}$$

$$= \sin\left(\alpha - \frac{\beta}{2}\right) + (-1)^{n-1}\sin\left(\alpha + \frac{2n-1}{2}\beta\right).$$

$$\therefore \sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots\ldots + (-1)^{n-1}\sin(\alpha + \overline{n-1}\,.\,\beta)$$

$$= \frac{\sin\left(\alpha - \frac{\beta}{2}\right) + (-1)^{n-1}\sin\left(\alpha + \frac{2n-1}{2}\beta\right)}{2\cos\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\cos\frac{n\beta}{2}}{\cos\frac{\beta}{2}},\ n \text{ odd; and}$$

$$= -\frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\cos\frac{\beta}{2}},\ n \text{ even.}$$

60. *To sum the series*

$$\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots\ldots + x^{n-1}\sin(\alpha + \overline{n-1}\,\beta),$$

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \ldots\ldots + x^{n-1}\cos(\alpha + \overline{n-1}\,\beta).$$

We have

$$2\sqrt{-1}\left\{\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \left(e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\right) + x\left\{e^{(\alpha+\beta)\sqrt{-1}} - e^{(\alpha+\beta)\sqrt{-1}}\right\} + x^2\left\{e^{(\alpha+2\beta)\sqrt{-1}} - e^{-(\alpha+2\beta)\sqrt{-1}}\right\}$$

$$+ \ldots \text{ to } n \text{ terms.}$$

$$= e^{\alpha\sqrt{-1}}\left\{1 + xe^{\beta\sqrt{-1}} + x^2e^{2\beta\sqrt{-1}} + \ldots x^{n-1}\,.\,e^{(n-1)\beta\sqrt{-1}}\right\}$$

$$- e^{-\alpha\sqrt{-1}}\left\{1 + xe^{-\beta\sqrt{-1}} + x^2e^{-2\beta\sqrt{-1}} + \ldots + x^{n-1}e^{-(n-1)\beta\sqrt{-1}}\right\},$$

$$= e^{\alpha\sqrt{-1}}\,.\,\frac{x^ne^{n\beta\sqrt{-1}} - 1}{xe^{\beta\sqrt{-1}} - 1} - e^{-\alpha\sqrt{-1}}\,.\,\frac{x^ne^{-n\beta\sqrt{-1}} - 1}{xe^{-\beta\sqrt{-1}} \quad 1}.$$

Or, bringing to a common denominator, and arranging, &c., we easily get

$$\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots + x^{n-1}\sin(\alpha + \overline{n-1}.\beta)$$

$$= \frac{\sin\alpha - x\sin(\alpha-\beta) - x^n\sin(\alpha+n\beta) + x^{n+1}\sin(\alpha+\overline{n-1}.\beta)}{1 - 2x\cos\beta + 2x^2} \ldots\ldots\ldots\ldots (1).$$

And similarly we have

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \dots\, x^{n-1}\cos(\alpha+\overline{n-1}\,\beta)$$

$$= \frac{\cos\alpha - x\cos(\alpha-\beta) - x^n\cos(\alpha+n\beta) + x^{n+1}\cos(\alpha+\overline{n-1}\,.\,\beta)}{1 - 2x\cos\beta + x^2} \dots\dots\dots (2).$$

Cor. By making n infinite, we get, when $x < 1$, the limits of the corresponding infinite series.

And similarly a variety of series of this kind may be summed.

The student who is acquainted with the Differential Calculus will at once see that, by differentiating the result in (1) with respect to α, we get the result in (2).

And similarly, and with the aid of the Integral Calculus, the sums of very complicated series may be easily obtained.

61. *To sum the series*

$$\operatorname{cosec}\alpha + \operatorname{cosec}2\alpha + \operatorname{cosec}2^2\alpha + \dots + \operatorname{cosec}2^{n-1}\alpha.$$

We have

$$\operatorname{cosec}\alpha = \cot\frac{\alpha}{2} - \cot\alpha,$$

$$\operatorname{cosec}2\alpha = \cot\alpha - \cot 2\alpha,$$

$$\operatorname{cosec}2^2\alpha = \cot 2\alpha - \cot 2^2\alpha,$$

$$\dots\dots\dots$$

$$\operatorname{cosec}2^{n-1} = \cot 2^{n-2}\alpha - \cot 2^{n-1}\alpha.$$

Therefore adding,

$$\operatorname{cosec} \alpha + \operatorname{cosec} 2\alpha + \operatorname{cosec} 2^2 \alpha + \ldots + \operatorname{cosec} 2^{n-1} \alpha$$

$$= \cot \frac{\alpha}{2} - \cot 2^{n-1} \alpha.$$

62. *To sum the series*

$$\tan \alpha + \frac{1}{2} \tan \frac{\alpha}{2} + \frac{1}{2^2} \tan \frac{\alpha}{2^2} + \ldots + \frac{1}{2^{n-1}} \tan \frac{\alpha}{2^{n-1}}.$$

We have

$$\tan \alpha = \cot \alpha - 2 \cot 2\alpha$$

$$\frac{1}{2} \tan \frac{\alpha}{2} = \frac{1}{2} \cot \frac{\alpha}{2} - \cot \alpha$$

$$\frac{1}{2^2} \tan \frac{\alpha}{2^2} = \frac{1}{2^2} \cot \frac{\alpha}{2^2} - \frac{1}{2} \cot \frac{\alpha}{2}$$

. . . .

$$\frac{1}{2^{n-1}} \tan \frac{\alpha}{2^{n-1}} = \frac{1}{2^{n-1}} \cot \frac{\alpha}{2^{n-1}} - \frac{1}{2^{n-2}} \cot \frac{\alpha}{2^{n-2}}.$$

Hence, adding,

$$\text{Sum to } n \text{ terms} = \frac{1}{2^{n-1}} \cot \frac{\alpha}{2^{n-1}} - 2 \cot 2\alpha.$$

COR. Let n be infinite,

Then,
$$\text{limit of } \frac{1}{2^{n-1}} \cot \frac{\alpha}{2^{n-1}} = \text{limit of } \frac{\frac{\alpha}{2^{n-1}}}{\tan \frac{\alpha}{2^{n-1}}} \cdot \frac{1}{\alpha} = \frac{1}{\alpha}.$$

$$\therefore \Sigma = \frac{1}{\alpha} - 2 \cot 2 \alpha.$$

Ex. VII.

Show that

1. $x \sin \theta + x^2 \sin 2 \theta + x^3 \sin 3 \theta + \ldots$ to n terms
$$= \frac{x \sin \theta + x^{n+2} \sin n\theta - x^{n+1} \sin (n+1) \theta}{x^2 - 2 x \cos \theta + 1}.$$

2. $(\operatorname{Tan} \alpha + \cot \alpha) + (\tan 2 \alpha + \cot 2 \alpha) + (\tan 2^2 \alpha + \cot 2^2 \alpha) + \ldots$ to n terms
$$= 2 \cot \alpha - 2 \cot 2^n \alpha.$$

3. $\operatorname{Tan} \alpha + 2 \tan 2 \alpha + 2^2 \tan 2^2 \alpha + \ldots$ to n terms $= \cot \alpha - 2^n \cot 2^n \alpha$.

4. $\operatorname{Sin} \alpha + \sin 3 \alpha + \sin 5 \alpha + \ldots + \sin (2n - 1) \alpha = \frac{\sin^2 n\alpha}{\sin \alpha}$.

5. $\operatorname{Cos} \alpha + \cos 3 \alpha + \cos 5 \alpha + \ldots + \cos (2n - 1) \alpha = \frac{\sin 2 n\alpha}{2 \sin \alpha}$.

6. $\cos^3 \theta + \cos^3 (\theta + \alpha) + \cos^3 (\theta + 2\alpha) + \ldots$ to n terms

$$= \frac{3}{4} \cdot \frac{\cos\left\{\theta + \frac{n-1}{2}\alpha\right\} \sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}} + \frac{1}{4} \cdot \frac{\cos\left\{3\theta + \frac{n-1}{2} 3\alpha\right\} \sin\frac{3n\alpha}{2}}{\sin\frac{3\alpha}{2}}.$$

7. $\cos\alpha \cos(\alpha+\beta) + \cos(\alpha+\beta)\cos(\alpha+2\beta) + \cos(\alpha+3\beta)\cos(\alpha+4\beta) + \ldots$ to n terms

$$= \frac{n}{2}\cos\beta + \frac{\cos(2\alpha + n\beta)\sin n\beta}{2\sin\beta}.$$

8. $\cot^{-1}(1 + 1 + 1^2) + \cot^{-1}(1 + 2 + 2^2) + \ldots$ to n terms $= \cot^{-1}\frac{n+2}{n}$.

9. $\sec\alpha \sec 2\alpha + \sec 2\alpha \sec 3\alpha + \sec 3\alpha \sec 4\alpha + \ldots$ to n terms

$$= \operatorname{cosec}\alpha \left\{\tan(n+1)\alpha - \tan\alpha\right\}.$$

10. $\left(\frac{2}{\cos 2\theta}\right)^2 + \left(\frac{2^2}{\cos 2^2\theta}\right)^2 + \left(\frac{2^3}{\cos 2^3\theta}\right)^3 + \ldots$ to n terms

$$= \left(\frac{2^{n+1}}{\cos 2^{n+1}\theta}\right)^2 - \left(\frac{2}{\sin 2\theta}\right)^2.$$

11. $\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)^2 + \left(\frac{1}{2^2}\operatorname{cosec}\frac{\theta}{2^2}\right)^3 + \left(\frac{1}{2^3}\operatorname{cosec}\frac{\theta}{2^3}\right)^3 + \ldots$ to n terms

$$= (\operatorname{cosec}\theta)^2 - \left(\frac{1}{2^n}\operatorname{cosec}\frac{\theta}{2^n}\right)^2.$$

12. $\operatorname{Sin}\frac{\theta}{2}\left(\frac{1}{2}\sec\frac{\theta}{2}\right)^3 + \sin\frac{\theta}{2^2}\left(\frac{1}{2}\sec\frac{\theta}{2^2}\right)^3 +$ to n terms

$$= \cos\frac{\theta}{2^n}\left(\frac{1}{2^n}\operatorname{cosec}\frac{\theta}{2^n}\right)^3 - \cos\alpha \,.\, \operatorname{cosec}^3\theta.$$

The sum to infinity of

13. $\operatorname{Sin}\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots\ldots = \frac{\sin\alpha - x\sin(\alpha-\beta)}{1 - 2x\cos\beta + x^2}.$

14. $x\cos(\alpha+\beta) + \frac{x^2}{1.2}\cos(\alpha+2\beta) + \frac{x^3}{1.2.3}\cos(\alpha+3\beta) + \ldots\ldots$

$$= e^{x\cos\beta}\cos(\alpha + x\sin\beta) - \cos\alpha.$$

15. $x\sin(\alpha+\beta) + \frac{x^2}{1.2}\sin(\alpha+2\beta) + \frac{x^3}{1.2.3}\cos(\alpha+3\beta) + \ldots\ldots$

$$= e^{x\cos\beta}\sin(\alpha + x\sin\beta) - \sin\alpha.$$

16. $x\cos(\theta+\alpha) + \frac{x(x-1)}{1.2}\cos(\theta+2\alpha) + \frac{x(x-1)(x-2)}{1.2.3}\cos(\theta+3\alpha) + \ldots$

$$= 2^x\cos(\theta + \tfrac{1}{2}x\alpha)\cos^x\tfrac{1}{2}\alpha - \cos\theta.$$

17. $x\sin(\theta+\alpha) + \frac{x(x-1)}{1.2}\sin(\theta+2\alpha) + \frac{x(x-1)(x-2)}{1.2.3}\sin(\theta+3\alpha) + \ldots\ldots$

$$= 2^x\sin(\theta + \tfrac{1}{2}x\alpha)\cos^x\tfrac{1}{2}\alpha - \sin\theta.$$

18. Show that $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$.

19. If $a = \frac{\pi}{17}$, show that

$$\cos a + \cos 9a + \cos 13a + \cos 15a = \tfrac{1}{4}(1 + \sqrt{17}),$$
$$\cos 3a + \cos 5a + \cos 7a + \cos 11a = \tfrac{1}{4}(1 - \sqrt{17}).$$

20. If $a = \frac{\pi}{17}$, show that

$$(\cos a + \cos 13a)(\cos 9a + \cos 15a)$$
$$= (\cos 3a + \cos 5a)(\cos 7a + \cos 11a) = -\tfrac{1}{4}.$$

21. If $\phi = \frac{\pi}{13}$, show that

$$\cos 5\phi + \cos 7\phi + \cos 11\phi = \frac{1 - \sqrt{13}}{4},$$

and $$\cos\phi + \cos 3\phi + \cos 9\phi = \frac{1 + \sqrt{13}}{4}.$$

22. If a circle be inscribed in an equilateral triangle, and in the angular points other circles described, and so on, *ad infinitum*; then the sum of the areas of this series of circles $= \frac{11}{96} a^2\pi$, where a is the side of the triangle.

23. If a circle is inscribed in a square, and between this circle and the angular points circles described, and so on *ad infinitum*: show that the sum of the areas of this series of circles $\frac{1}{4}(44\sqrt{2} - 61) a^2\pi$, where a is the side of the square.

24. Circles are described in triangles whose bases are the sides of a regular polygon of n sides, and whose vertices are the angular points of the polygon; show that:

(i.) The sum of the radii of the circles $= 2r\left(1 - n\sin^2\frac{\pi}{2n}\right)$,

(ii.) The sum of the areas of the circles $= 4\sin^2\frac{\pi}{2n}\left\{n\sin^2\frac{\pi}{2n} + \frac{n-4}{2}\right\}r^2\pi$,

where r is the radius of the circle circumscribing the polygon.

CHAPTER VIII.

CONSTRUCTION OF TRIGONOMETRICAL TABLES.

63. By means of the formula $\sin\theta < \theta$, and $> \theta - \frac{\theta^3}{4}$, it may be easily shown that

$$\sin 10'' = \cdot 000048481368 \ldots;$$ and from this, that

$$\cos 10'' = \cdot 9999999988248 \ldots.$$

Knowing these, we may find the sines and cosines of all angles which are multiples of 10″.

Thus we have:

$$\sin(n+1)A + \sin(n-1)A = 2\sin nA\cos A.$$

$$\therefore \sin(n+1)A = 2\cos A \,.\, \sin nA - \sin(n-1)A.$$

Put A = 10″, and n = 1, 2, 3, &c., successively, then

$$\sin 20'' = 2\cos 10'' \,.\, \sin 10'',$$
$$\sin 30'' = 2\cos 10'' \,.\, \sin 20'' - \sin 10''$$
$$\sin 40'' = 2\cos 10'' \,.\, \sin 30'' - \sin 20''$$
$$\&c. = \&c.$$

In this manner we may find the sines of the series of angles whose common difference is 10″.

64. When the sines of all such angles up to 45° have been thus found, we may calculate the sines of the angles from 45° to 90°, by means of the following formula, which is easily proved:

$$\sin(45° + A) = \sin(45° - A) + \sqrt{2} \,.\, \sin A.$$

Or, having found the sines of all such angles up to 60°, we may find the remainder up to 90° more easily by the formula—

$$\sin(60° + A) = \sin(60° - A) + \sin A.$$

Having found the *sines* it is unnecessary to find the *cosines* independently, since we have

$$\cos A = \sin(90° - A).$$

The *tangents*, *secants*, *cotangents*, *cosecants* are, of course, easily obtained from the sines and cosines.

65. *Formulæ of Verification.*

As the trigonometrical ratios are found from each other, any error made is continued. It is therefore necessary to test the accuracy of the results. This may be done by what are called *formulæ of verification.* Amongst others we may mention the following :

(i.) $\text{Sin A} = \frac{1}{2}\left\{\sqrt{1 + \sin 2\text{A}} - \sqrt{1 - \sin 2\text{A}}\right\}$,

(ii.) $\text{Cos A} = \frac{1}{2}\left\{\sqrt{1 + \sin 2\text{A}} + \sqrt{1 - \sin 2\text{A}}\right\}$, where A in each case is less than 45°.

(iii.) *Euler's formula.*

$$\text{Sin A} + \sin(36° - \text{A}) + \sin(72° + \text{A}) = \sin(36° + \text{A}) + \sin(72° - \text{A}).$$

We may prove this formula thus :

$$\sin(72° + \text{A}) - \sin(72° - \text{A}) = 2\cos 72° . \sin\text{A} = 2\sin 18° . \sin\text{A}$$

$$= \frac{\sqrt{5} - 1}{2}\sin\text{A} \ldots\ldots\ldots\ldots (1).$$

And so

$$\sin(36° + \text{A}) - \sin(36° - \text{A}) = 2\cos 36° . \sin\text{A} = \frac{\sqrt{5} + 1}{2}\sin\text{A} \ldots\ldots\ldots (2).$$

(1) – (2), then,

$$\sin(72° + \text{A}) - \sin(72° - \text{A}) - \sin(36° + \text{A}) + \sin(36° - \text{A}) = \sin\text{A}; \text{ or}$$

$$\sin\text{A} + \sin(36° - \text{A}) + \sin(72° + \text{A}) = \sin(36° + \text{A}) + \sin(72° - \text{A}).$$

And similarly we may prove the following : viz.,

(iv.) *Legendre's formula.*

$$\text{Cos A or } \sin(90° - \text{A}) = \sin(54° + \text{A}) + \sin(54° - \text{A}) - \sin(18° + \text{A}) - \sin(18° - \text{A}).$$

As an example, suppose we make in (iii) and (iv), A = 11°, we have

$$\sin 11° + \sin 25° + \sin 83° = \sin 47° + \sin 61°; \text{ and}$$
$$\sin 79° = \sin 65° + \sin 43° - \sin 29° - \sin 7°.$$

By taking from the tables each of these *sines*, it is an easy matter to test the accuracy of the results there tabulated.

Logarithmic Tables of Trigonometrical Ratios.

66. Having determined the natural sines, cosines, &c., of angles, we may obtain from them the corresponding logarithmic trigonometrical ratios.

It is usual, however, to determine these independently.

67. *To find the logarithmic sine and cosine of an angle.*

We have (Art. 42),

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Put $\theta = \frac{m}{n}\cdot\frac{\pi}{2}$, or $\frac{\theta}{\pi} = \frac{m}{2n}$; then, taking the logarithms of each side, we have

$$\log_{10}\sin\left(\frac{m}{n}\,\frac{\pi}{2}\right) = \log_{10}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right) + \log_{10}\left(1 - \frac{m^2}{2^2 n^2}\right) + \log_{10}\left(1 - \frac{m^2}{4^2 n^2}\right) + \log_{10}\left(1 - \frac{m^2}{6^2 n^2}\right) + \ldots.$$

Or expanding, arranging, &c., we have,

$$\log_{10}\sin\left(\frac{m}{2}\cdot\frac{\pi}{2}\right) = \log_{10} m + \log_{10}(2n + m) + \log_{10}(2n - m) + \log_{10}\pi - 3(\log_{10} n + \log_{10} 2)$$

$$- \frac{1}{\log_e 10}\left\{\begin{array}{l} \left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots\right)\frac{m^2}{n^2} \\ + \frac{1}{2}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \ldots\right)\frac{m^4}{n^4} \\ + \frac{1}{3}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \ldots\right)\frac{m^6}{n^6} \\ + \ldots \end{array}\right\}.$$

From which, by giving m and n different values, so that $m < n$, we obtain logarithmic sines of all angles $< 90°$.

So from the formula

$$\cos\theta = \left(1 - \frac{2^2\theta^2}{\pi^2}\right)\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right)\left(1 - \frac{2^2\theta}{5^2\pi^2}\right)\ldots$$

we may directly obtain the logarithmic cosines.

The values of the other logarithmic trigonometrical ratios are then easily found. As explained in Vol. I., the tabular values are found by adding 10 to the values thus obtained.

Theory of Proportional Parts.

68. *The increment of the sine of an angle is in general approximately proportional to the small increment of the angle.*

Let the angle A receive an increment h, then the increment of sin A will be

$$\sin(A + h) - \sin A.$$

Denoting this by $\Delta \sin A$, we have

$$\begin{aligned}\Delta \sin A &= \sin(A + h) - \sin A \\ &= \sin A \,.\, \cos h + \cos A \,.\, \sin h - \sin A \\ &= \sin h \,.\, \cos A - (1 - \cos h) \sin A \\ &= \sin h \cos A \left\{ 1 - \frac{1 - \cos h}{\sin h} \,.\, \tan A \right\} \\ &= \sin h \cos A \left\{ 1 - \tan \frac{h}{2} \,.\, \tan A \right\}.\end{aligned}$$

If h be a small increment, we may put $\sin h = h$.

And unless tan A be very large, that is, unless A be very nearly 90°, the value of $\tan \frac{h}{2} \,.\, \tan A$ will be also very small, and may be neglected.

Hence, *unless* A *be very nearly a right angle,* we have

$$\Delta \sin A = h \cos A, \text{ very nearly.}$$

Hence, in determining any angle from its *sine*, we may adopt the principle of proportional parts, unless the angle be very nearly a right angle.

69. *The increment of the cosine of an angle is in general approximately proportional to the small increment of the angle.*

We have

$$\begin{aligned}\Delta \cos A &= \cos(A + h) - \cos A \\ &= \cos A \cos h - \sin A \sin h - \cos A \\ &= - \sin h \sin A - (1 - \cos h) \cos A\end{aligned}$$

$$= -\sin h \sin A \left\{ 1 + \frac{1 - \cos h}{\sin h} \cot A \right\}$$

$$= -\sin h \sin A \left(1 + \tan \frac{h}{2} \cot A\right).$$

Now, when A is very small, cot A is very large.

Hence, when the increment h is small, we find that,

Unless A be a very small angle,

$$\Delta \cos h = -h \sin A, \text{ very nearly.}$$

It will be noticed that the cosine diminishes as the angle increases.

70. *The increment of the tangent of an angle is in general proportional to the small increment of the angle.*

We have

$$\Delta \tan A = \tan (A + h) - \tan A$$

$$= \frac{\sin (A + h)}{\cos (A + h)} - \frac{\sin A}{\cos A} = \frac{\sin (\overline{A + h} - A)}{\cos (A + h) \cos A}$$

$$= \frac{\sin h}{\cos (A + h) \cos A}$$

$$= \frac{\tan h}{\cos^2 A\,(1 - \tan h \,.\, \tan A)}.$$

Hence, as before, if h be small, and A be not nearly 90°, we have approximately

$$\Delta \tan A = \frac{h}{\cos^2 A} = h \sec^2 A \text{............(1).}$$

And so we may show that,

Unless A be a very small angle,

$$\Delta \cot A = -h \operatorname{cosec}^2 A \text{..................(2).}$$

Also, that,

When A is *neither very small, nor nearly equal to* 90°,

$$\Delta \sec A = h \sin A \sec^2 A \text{................(3),}$$

and $$\Delta \operatorname{cosec} A = -h \cos A \operatorname{cosec}^2 A \text{..........(4).}$$

71. *The increment of the values of the logarithmic trigonometrical ratios is in general proportional to the small increment of the angle.*

(i.) Δ L sin A.

We have $\sin(A + h) = \sin A \cos h + \cos A \sin h$
$= \sin A + h \cos A$, approximately.

$$\therefore \frac{\sin(A+h)}{\sin A} = 1 + h \cot A.$$

Taking the logarithms, we have

$$\log \sin(A+h) - \log \sin A = \log(1 + h \cot A).$$

But $\log \sin(A+h) - \log \sin A = \text{L} \sin(A+h) - \text{L} \sin A$
$= \Delta \text{L} \sin A.$

And

$$\log(1 + h \cot A) = \mu \left\{ h \cot A - \tfrac{1}{2} h^2 \cot^2 A + \&c. \right\}$$
$$= \mu h \cot A, \text{ approximately.}$$

Hence $\Delta \text{L} \sin A = \mu h \cot A$,

except in cases to be discussed in the next article.

So we may easily show that

(ii.) $\Delta \text{L} \cos A = -\mu h \tan A$.

And also that

(iii.) $\Delta \text{L} \tan A = 2 \mu h \operatorname{cosec} 2A$.

(iv.) $\Delta \text{L} \cot A = -2 \mu h \operatorname{cosec} 2A$.

(v.) $\Delta \text{L} \operatorname{cosec} A = -\mu h \cot A$.

(vi.) $\Delta \text{L} \sec A = \mu h \tan A$.

72. The limits of our space compel us to omit the purely trigonometrical investigations as to the amount of error, and the cases of failure in the principle of proportional parts.

To the student who is acquainted with the Differential Calculus we will indicate how this may be done by Taylor's Theorem.

We have, by Taylor's Theorem.

$$\sin(A + h) = \sin A + h \cos A - \frac{h^2}{1 \cdot 2} \sin(A + \theta h),$$

where θ *is some* proper fraction.

$$\text{Hence } \Delta \sin A = h \cos A - \frac{h^2}{1 . 2} \sin (A + \theta h) \ldots\ldots\ldots (1).$$

Hence, by taking $\Delta \sin A = h \cos A$, *the error is less than* $\frac{h^2}{1 . 2}$.

Again, when A is a small angle, $\frac{h^2}{1 . 2} \sin (A + \theta h)$ is extremely small.

But when A is nearly equal to 90°,

$\sin (A + \theta h)$ *is very nearly unity.*

Hence the value of $\frac{h^2}{1 . 2} \sin (A + \theta h)$ is sensibly comparable with $h \cos A$, as $\cos A$ is then very small.

Hence the differences between the sines of angles which are nearly 90° are very small, or, as it is called, *insensible;* and the second term of the series in (1) cannot be neglected, so that the increments of the sines are not proportional to the increments of the angle. They are in this case said to be *irregular.*

We hence conclude that:

The differences of consecutive sines, when the angle is nearly 90°, are *insensible* and *irregular.*

And so we may show that for cosines they are *insensible* when the angles are very small; for tangents they are *irregular* when the angles are nearly 90°; for cotangents *irregular* when the angles are very small; for secants *insensible* when the angles are small, and *irregular* when nearly 90°; and for cosecants *irregular* when the angles are small, and *insensible* when nearly 90°. *Irregularity* accompanies *insensibility.*

Again, by Taylor's Theorem,

$$\log \sin (A + h)$$
$$= \log \sin A + \mu h \cot A - \frac{\mu h^2}{2} \operatorname{cosec}^2 (A + \theta h),$$

where θ *is* some proper fraction.

$$\therefore \Delta L \sin A = \mu h \cot A - \frac{\mu h^2}{2} \operatorname{cosec}^2 (A + \theta h).$$

Hence, if we may neglect the second term, we have approximately,

$$\Delta L \sin A = \mu h \cot A.$$

When A is very small, $\operatorname{cosec}^2 (A + \theta h)$ is very large, and the second term may not be neglected.

Hence the differences of consecutive logarithmic sines are irregular *when the angle is very small.*

Again, when A is very nearly equal to 90°,

$$\operatorname{cosec}^2 (A + \theta h) \text{ is very nearly equal to unity.}$$

Hence the value of $\frac{\mu h^2}{2} \operatorname{cosec}^2 (A + \theta h)$ is comparable to $\mu h \cot A$, for cot A is then very small.

Hence the differences are *insensible* when the angles are nearly 90°.

And in a similar way it may be shown that the principle of proportional parts fails for every logarithmic trigonometrical ratio, when the angle is either very small or very nearly equal to a right angle. We may express these results as follows:

	Angles Small.	Angles nearly 90°.
Logarithmic:		
(1.) Sines and cosecants, ...	irregular, ...	insensible.
(2.) Tangents and cotangents,	irregular, ...	irregular.
(3.) Cosines and secants, ...	insensible,	irregular.

73. Hence, when an angle is very small or nearly 90°, it is

impracticable to use the principle of proportional parts in the case of logarithmic trigonometrical functions taken from ordinary tables.

We sometimes however require to know the *number of seconds* in a very small angle. The methods which may be followed will be now explained.

First Method.

Tables are constructed for a few degrees to intervals of a second, instead of to intervals of 10″, or of a minute.

Here h must be less than one second, and the greatest value of $\frac{h^2}{2}\operatorname{cosec}^2(A+\theta h)$ is $\frac{h^2}{2}$, where h is the circular measure of *one second.* It is therefore small enough to be neglected.

Second Method. Maskelyne's Method.

We have $\sin\alpha = \theta - \frac{\theta^3}{1.2.3} + \ldots = \alpha - \frac{\alpha^3}{1.2.3}$ nearly.

And $\cos\alpha = 1 - \frac{\alpha^2}{1.2} + \ldots = 1 - \frac{\alpha^2}{2}$ nearly.

$$\therefore \frac{\sin\alpha}{\alpha} = 1 - \frac{\alpha^2}{6} = \left(1 - \frac{\alpha^2}{2}\right)^{\frac{1}{3}}, \text{ nearly.}$$

$$= (\cos\alpha)^{\frac{1}{3}}, \text{ nearly; or } \sin\alpha = \alpha(\cos\alpha)^{\frac{1}{3}}.$$

Let α be an angle containing n'', then

Circular measure of the angle $= n \sin 1''$

or $\alpha = n\sin 1''$.

$\therefore \sin\alpha = n\sin 1''(\cos\alpha)^{\frac{1}{3}}$, or

$\sin n'' = n\sin 1''(\cos n'')^{\frac{1}{3}}$; or, taking logarithms.

$$L\sin n'' = \log n + L\sin 1'' + \tfrac{1}{3}(L\cos n'' - 10)$$
$$= \log n + L\sin 1'' - \tfrac{1}{3}(10 - L\cos n'')\ldots\ldots(1).$$

And therefore

$$\log n = L\sin n'' + \tfrac{1}{3}(10 - L\cos n'') - L\sin 1''\ldots(2).$$

Since the angle n'' is small, we may, in finding n from $L\sin n''$ or in finding $L\sin n''$ from n, make use of an *approximate value* of L cos n'' from the tables; for its variation is very small when the variation of n is small.

We may find a similar value for the *tangent* of a small angle.

$$\tan a^2 = \frac{\sin n}{\cos a} = \left(a - \frac{a^3}{6}\right)\left(1 - \frac{a^2}{2}\right)^{-1}$$

$$= \left(a - \frac{a^3}{6}\right)\left(1 + \frac{a^2}{2}\right) = a + \frac{a^3}{3}.$$

$$\therefore \frac{\tan a}{a} = 1 + \frac{a^2}{3} = \left(1 - \frac{a^2}{2}\right)^{-\frac{2}{3}} = (\cos a)^{-\frac{2}{3}}$$

Hence, as before

$$\text{L} \tan n'' = \log n + \text{L} \tan 1'' - \tfrac{2}{3} (\text{L} \cos n'' - 10).$$

Third Method—Delambre's Method.

Delambre constructed a table for *every second* to a degree for $\log \frac{\sin a}{a} + \text{L} \sin 1''$.

We have, if a be the circular measure of an angle of n''.

$$\log \frac{\sin a}{a} = \log \frac{\sin n''}{n \sin 1''}$$

$$= \text{L} \sin n'' - \log n - \text{L} \sin 1''.$$

$$\therefore \text{L} \sin n'' = \log n + (\log \frac{\sin a}{a} + \text{L} \sin 1'') \ldots\ldots\ldots\ldots (3),$$

and $\log n = \text{L} \sin n'' - (\log \frac{\sin a}{a} + \text{L} \sin 1'') \ldots\ldots\ldots\ldots (4).$

Suppose the L sin of the angle is known, and we require to find n.

We can find *approximately*, to the nearest integer, the value of n, and then making use of this to obtain from the tables the value of $(\log \frac{\sin a}{a} + \text{L} \sin 1'')$, we can at once find $\log n$, and then n from a table of logarithms of numbers.

By making use of the approximate value of n we are of course liable to error. That the error is not sensible will be seen as follows:

We have

$$\frac{\sin \alpha}{\alpha} = 1 - \frac{\alpha^2}{1.2.3} + \frac{\alpha^4}{1.2.3.4.5} - \ldots = 1 - \frac{\alpha^2}{6}, \text{ nearly.}$$

$$\therefore \log_{10} \frac{\sin \alpha}{\alpha} = \mu \log \left(1 - \frac{\alpha^2}{6}\right)$$

$$= \mu \left\{ - \frac{\alpha^2}{6} - \tfrac{1}{2}\left(\frac{\alpha^2}{6}\right) - \&c. \right\}.$$

$$\therefore \Delta \log_{10} \frac{\sin \alpha}{\alpha}$$

$$= - \mu \left\{ \tfrac{1}{6}\{(\alpha + h)^2 - \alpha^2\} + \tfrac{1}{72}\{(\alpha + h)^4 - \alpha^4\} + \&c. \right\}$$

$$= - \mu \left(\frac{\alpha}{3} + \frac{\alpha^3}{18} + \ldots\right) h = - \frac{\mu h}{3} \alpha \text{ very nearly.}$$

$$\therefore \Delta \log_{10} \frac{\sin \alpha}{\alpha} \div h = - \frac{\mu}{3} \alpha.$$

Hence, the variation of $\log \dfrac{\sin \alpha}{\alpha}$ is much less rapid than that of α, and the error will therefore be insensible.

74. Since, Art. 70, $\left.\begin{array}{l}\Delta \sin A = h \cos A, \\ \text{and } \Delta \cos A = - h \sin A,\end{array}\right\}$ approximately,

we see that the differences for consecutive sines vary approximately as the cosine, and the differences for consecutive cosines vary approximately as the sine.

Now, when the angle is $< 45°$, *sine* is $<$ *cosine.*
and „ „ $> 45°$, *sine* is $>$ *cosine.*

Hence, for angles less than 45°, the differences for consecutive sines will be greater than the differences for consecutive cosines, and *vice versâ.*

Hence, for angles less than 45°, it is better to determine them from their *sines* than from their cosines; and for angles greater than 45° it is better to determine them from their *cosines.* And the same rule should be observed in the use of *logarithmic* sines and cosines.

SECTION III.

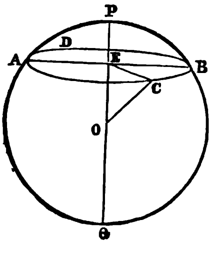
SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS, AND INTRODUCTORY PROPOSITIONS.

1. A sphere is a body, every point of the surface of which is equidistant from a certain point within it, called the centre. We may also define a sphere as the solid generated by the revolution of a semicircle about its diameter or axis.

2. *Every section of a sphere made by a plane is a circle.*



Let ABCD be the section made by a plane passing through the points A and B, of the sphere whose centre is O.

From O draw OE at right angles to the plane, meeting it in E.

Take any point C in the circumference of the section, and draw the line EC.

Then (Euc. XI., Def. 3), OE is perpendicular to EC.

Hence, $EC^2 = OC^2 - OE^2$.

$\therefore EC = \sqrt{OC^2 - OE^2}$, a constant quantity, since

OC is the radius of the sphere, and OE the distance of the plane from the centre of the sphere.

Hence, it follows that all straight lines drawn from the point E to the circumference of the section, are equal to one another.

The section is, therefore, a circle, whose centre is E.

COR. 1. When the plane passes through the centre of the sphere, the section is a circle, whose radius is that of the sphere.

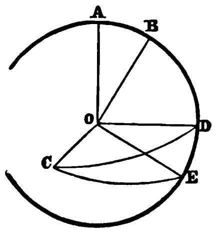
DEF. A **great circle** is a section of a sphere made by a plane passing through the centre of the sphere, and has its radius equal therefore to the radius of the sphere.

All other sections are termed **small circles.**

If OE be produced both ways, to meet the sphere in P and Q, the points P and Q, where the perpendicular to the plane of the section meets the surface of the sphere, are called **the poles of the circle** ACDB.

COR. 2. It follows from this definition that all arcs of great circles, and all chords, drawn from a pole to the circumference of the corresponding circle, are equal to each other.

3. *The angle subtended at the centre of the sphere by the arc of a great circle, which joins the poles of two great circles, is equal to the angle of inclination of the planes of the circles.*



Let CD, CE be the two given great circles;

A, B, their poles respectively, O the centre of the sphere, and ABDE the great circle passing through the poles A, B, and meeting CD, CE, respectively in DE.

Then AO is perpendicular to CO, and OB is also perpendicular to CO.

Hence CO is perpendicular to both AO and OB. It is therefore perpendicular to every line drawn from O in the plane AOB (Euc. XI., 4).

Hence CO is perpendicular to OD and OE.

Now OC is the intersection of the planes OCD and OCE, and OD and OE are lines drawn respectively in those planes.

Hence (Euc. XI., Def. 6) DOE is the angle of inclination of the planes OCD and OCE.

Now $\angle$ AOB = $\angle$ AOD − $\angle$ BOD

= $\angle$ BOE − $\angle$ BOD = $\angle$ DOE.

Def. The angle between the arcs of two great circles is the **angle of inclination** of the planes of the circles.

Hence $\angle$ DCE = $\angle$ DOE = $\angle$ AOB.

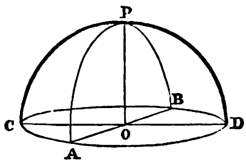
Cor. Hence by Art. 2, any circle of a sphere is at right angles to the great circles which pass through its poles.

4. *Two great circles bisect each other.*

As the planes of all great circles pass through the centre of the sphere, it follows that a diameter of the sphere is the line of intersection of any two circles. Each segment of the great circles must therefore be a semicircle.

5. *Arcs of great circles drawn from a pole of a great circle to points in its circumference are quadrants.*

Let P be the pole of the great circle ACBD, and let O be the centre of the sphere, which therefore lies in the plane of ACBD.



Then PO is at right angles to the plane ACBD.

Draw PA, PB, arcs of great circles, from P to the points A, B.

Then, since PO is perpendicular to the plane ACBD,

$\angle$ AOP is a right angle.

Hence, the arc AP, which subtends the angle AOP, must be a quadrant. And similarly, PB is a quadrant.

Def. Great circles drawn through the pole of a great circle are called **secondaries** to that great circle.

Thus, in the figure, Art. 3, where C is a pole of ABDE, the arcs CD, CE are arcs of secondaries to the great circle ABDE.

COR. Since the arc DE measures the angle of inclination DOE, it follows that *the angle between any two great circles is measured by the arc intercepted on the great circle to which they are secondaries.*

6. *If from a point on the surface of a sphere there can be drawn two arcs of great circles, at right angles to a given circle, which two arcs are not parts of the same circle, that point is a pole of the given circle.*

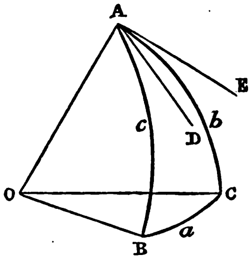
Since the arcs are at right angles to the given circle, their planes must intersect in a line which is perpendicular to the plane of the given circle, and this line must pass through the centre of the given circle and sphere.

Hence, the line is the axis of the given circle, and the point from which the arcs spring is a pole of the given circle.

CHAPTER II.

SPHERICAL TRIANGLES.

7. **Spherical Trigonometry** investigates the relations which subsist between the angles of the plane faces of a solid angle, and the inclinations of the plane faces to each other.



Let O be the centre of a sphere, and let a solid angle be formed at O, by the three planes, BOC, AOC, AOB.

These planes will cut the surface of the sphere in the arcs of great circles, BC, AC, AB.

Then ABC is called a spherical triangle.

In Spherical Trigonometry the arcs of great circles only are concerned, and it must be understood, that when arcs are in future mentioned, they are arcs of great circles unless it be otherwise expressed.

BC, AC, AB are called the sides of the spherical triangle ABC, and are usually expressed respectively by the letters a, b, c.

If AD, AE be drawn respectively tangents to the arcs AB, AC, then OA is perpendicular to both AD, AE.

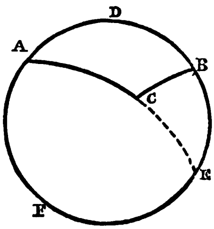
Hence (Euc. XI., Def. 6),

$\angle$ DAE is the angle of inclination of the planes AOB, AOC.

Hence, *the angle between two sides*, AB, AC, *of a spherical triangle, is the angle between the tangents drawn at the point of intersection of the arcs.*

It is convenient in Spherical Trigonometry to limit the sides of a triangle to less than a semicircle.

8. *Every angle of a spherical triangle is less than two right angles.*



For suppose, if possible, that a triangle is formed of the arcs AC, BC, AEB, where the angle ACB is greater than two right angles.

Since the angle ACB is greater than two right angles, if we produce AC it will meet the circle ACBD in some point between A and B on the side remote from D.

Let it be produced and meet ACB in E.

Then (Art. 4), arc AFE is a semicircle, and therefore AEB is greater than a semicircle, and consequently the triangle we have supposed is not one considered in Spherical Trigonometry.

9. *Any two sides of a spherical triangle are together greater than the third side, and the sum of the three sides is less than the circumference of a great circle.*

Using the figure of Art. 7, we have, by Euc. XI., 20,

Any two of the angles AOB, AOC, BOC, are together greater than the third.

Now these angles are respectively subtended by the arcs AB, AC, BC.

Hence any two of the sides AB, AC, BC of the triangle ABC, are together greater than the third.

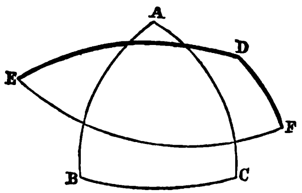
Again, by Euc. XI., 21,

The three angles AOB, AOC, BOC, forming the solid angle at O, are together less than four right angles.

And four right angles is the angle subtended at the centre by the circumference of a great circle.

Hence, AB, AC, BC, are together less than the circumference of a great circle.

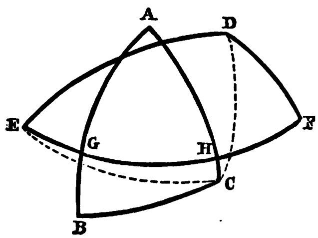
The Polar Triangle.



10. Def. If the points D, E, F be respectively the poles of the sides BC, AC, AB of the triangle ABC, then the triangle DEF is the **polar triangle** of the *primitive triangle* ABC.

As the sides of the triangle ABC have each two poles, it is evident we may form *eight* triangles having for their angular points poles of the triangle ABC. Thus, if D′, E′, F′ be the other poles corresponding to D, E, F, we may form the triangle DEF, D′E′F′, DE′F′, D′EF, ED′F′, E′DF, FD′E′, F′DE. The triangle DEF is, however, the only one when the poles D, E, F lie *towards the same parts* with the corresponding angles A, B, C. This is the triangle meant in the above definition.

11. *If one triangle is the polar triangle of another, the latter is the polar triangle of the former.*



Let DEF be the polar triangle of ABC, then shall ABC be the polar triangle of DEF.

Since D is the pole of BC, then DC is a quadrant, and since E is the pole AC, then EC is a quadrant.

Hence the arcs CE, CD, drawn from the point C to the great circle ED, are quadrants; and hence C is the pole of ED.

And, similarly, we may show that A, B are respectively the poles of EF, DF.

Hence ABC is the polar triangle of DEF.

The polar triangle is often called the **supplemental triangle**, from the property proved in the next article.

12. *The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.*

For suppose the arc EF to meet the arcs AB, AC in the points G, H, producing the arc EF, if necessary.

Then (Art. 5),

$$\begin{aligned} \angle A &= GH = EH - EG \\ &= EH - (EF - GF) \\ &= EH + GF - EF. \end{aligned}$$

But EH and GF are quadrants, and subtend, therefore, angles each equal to $\frac{\pi}{2}$. And EF is the side of the polar triangle corresponding to the angle A.

Hence, if a', b', c', represent respectively the sides of the polar triangle corresponding to the sides a, b, c of the primitive triangle, we have

$$A = \pi - a',$$

And so $\qquad B = \pi - b', \ C = \pi - c'.$

And again, similarly, if A', B', C', be the angles of the polar triangle,

$$a = \pi - A', \ b = \pi - B', \ c = \pi - C'.$$

COR. Hence, *if a general relation be established between the sides and angles of a spherical triangle, we shall still have a true relation, if, for the angles, we substitute the supplements of the respective sides; and for the sides, the supplements of the respective angles.*

Thus, it will be shown farther on that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad \text{.............. (1).}$$

Hence, in the polar triangle,

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'} \quad \text{..............(2).}$$

But

$$A' = \pi - a, \; a' = \pi - A, \; b' = \pi - B, \; c' = \pi - C.$$

Hence, substituting in (2), we have

$$\cos(\pi - a) = \frac{\cos(\pi - A) - \cos(\pi - B)\cos(\pi - C)}{\sin(\pi - B)\sin(\pi - C)}.$$

This is just what we at once obtain by replacing in (1) the angle and sides by the supplements of the corresponding side and angles respectively.

The result, when simplified, gives

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

The proposition here established is one of fundamental importance.

13. *The sum of the angles of a spherical triangle is greater than two right angles, and less than six right angles.*

By Art. 9, $\quad a + b + c < 2\pi.$

Put $\pi - A$, $\pi - B$, $\pi - C$ for a, b, c respectively, then

$$(\pi - A) + (\pi - B) + (\pi - C) < 2\pi.$$

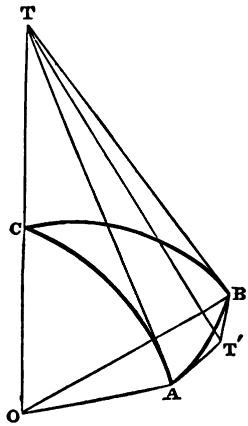
Hence $\quad A + B + C > \pi.$

Again, Art. 8, each of the angles A, B, C is $< \pi$.

Hence $\quad A + B + C < 3\pi.$

$$\therefore A + B + C > \pi, \text{ and } < 3\pi. \quad \text{Q.E.D.}$$

14. *The angles at the base of an isosceles spherical triangle are equal to one another.*



Let ABC be a spherical triangle, having the side BC equal to the side AC.

Then shall ∠ A = ∠ B.

First, let the two given sides be less than quadrants.

At A, B draw tangents AT, BT. These will meet OC produced in the same point T, since CA = CB.

Again, draw AT′, BT′, tangents to the arc AB, then AT′ = BT′.

Also, we have AT = BT.

Hence, in the two triangles ATT′, BTT′ the sides AT, AT′ are respectively equal to the sides BT, BT′, and TT′ is common.

∴ ∠ TAT′ = ∠ TBT′.

But (Art. 7) these are respectively the angles A and B.

∴ ∠ A = ∠ B.

If the sides AC, BC are greater than quadrants, the tangents AT, BT will meet OC produced in the opposite direction, and the proof will be similar.

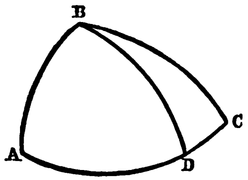
If the sides AC, BC are quadrants, then (Art. 3) the angles A, B are right angles.

Cor. Conversely, from the same figure it may be shown that if the angles A and B are equal, then the side BC is equal to the side AC.

Hence, also, *every equilateral triangle is equiangular, and every equiangular triangle is equilateral.*

15. *If two angles of a triangle be unequal, the side opposite to the greater angle is greater than the side opposite to the less.*

Let the angle B be greater than the angle A.



Then shall AC be greater than BC.

At B make the angle ABD equal to the angle at A.

Then (Art. 14,) $BD = AD$.

Now (Art. 9,) $BD + DC > BC$.

Hence, $AD + DC > BC$;
that is, $AC > BC$.

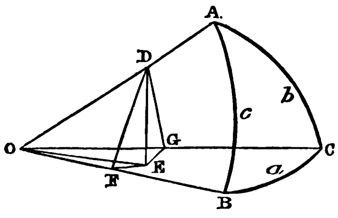
And similarly, the converse proposition may be proved, viz.:

If two sides of a triangle be unequal, the angle opposite to the greater side is greater than the angle opposite to the less.

CHAPTER III.

RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS OF THE SIDES AND ANGLES OF SPHERICAL TRIANGLES.

16. *The sines of the angles are proportional to the sines of the opposite sides.*



Let ABC be the spherical triangle, O the centre of the sphere.

Join OA, OB, OC, and from any point D in OA, draw DE perpendicular to the plane BOC; also, from D draw DF, DG perpendicular respectively to OB, OC. Join OE, EF, EG.

Then, DEG, DEF are right angles since DE is perpendicular to the plane BOC.

$$\therefore EF^2 = DF^2 - DE^2$$
$$= (OD^2 - OF^2) - (OD^2 - OE^2) = OE^2 - OF^2.$$

Hence (Euc. I., 48), ∠ OFE is a right angle. And so ∠ OGE is a right angle.

Hence DF, EF are each at right angles to OF.

Therefore, ∠ DFE is the inclination of the planes AOB, BOC (Euc. XI., Def. 6).

$$\therefore \angle \text{DFE} = \text{B}.$$

And so, $\angle \text{DGE} = \text{C}$.

Now $\text{DE} = \text{DF} \sin \text{DFE} = \text{OD} . \sin \text{AOB} . \sin \text{DFE}$

$$= \text{OD} \sin c \sin \text{B}.$$

And so $\text{DE} = \text{OD} \sin b \sin \text{C}$.

Hence $\sin c \sin \text{B} = \sin b \sin \text{C}$.

$$\therefore \frac{\sin \text{B}}{\sin b} = \frac{\sin \text{C}}{\sin c}.$$

It at once follows from the form of this result, that

$$\frac{\sin \text{A}}{\sin a} = \frac{\sin \text{B}}{\sin b} = \frac{\sin \text{C}}{\sin c}.$$

In the figure the sides are all less than quadrants, and the perpendicular DE falls within the triangle BOC. Any other case will be found to admit of a similar proof.

17. *To express the cosine of an angle in terms of the sines and cosines of the sides.*

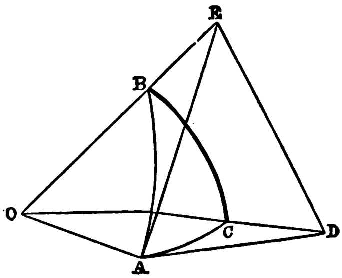
Let ABC be the triangle, O the centre of the sphere.

Join OA, OB, OC.

At A draw the lines AD, AE tangents to AC, AB respectively, and meeting OC, OB in D, E.

Join DE.

Then (Art. 7), ∠ DAE = ∠ BAC = A.



From the triangle EOD, we have

$$\text{DE}^2 = \text{OE}^2 + \text{OD}^2 - 2\,\text{OE} . \text{OD} \cos \text{DOE}$$

$$= \text{OE}^2 + \text{OD}^2 - 2\,\text{OE} . \text{OD} \cos a.$$

And from the triangle EAD, we have

$$DE^2 = AE^2 + AD^2 - 2\,AE\,.\,AD \cos DAE$$
$$= AE^2 + AD^2 - 2\,AE\,.\,AD \cos A.$$

Hence, equating these results,

$$AE^2 + AD^2 - 2\,AE\,.\,AD \cos A$$
$$= OE^2 + OD^2 - 2\,OE\,.\,OD \cos a.$$

$$\therefore\ 2\,OE\,.\,OD \cos a$$
$$= (OE^2 - AE^2) + (OD^2 - AD^2) + 2\,AE\,.\,AD \cos A$$
$$= OA^2 + OA^2 + 2\,AE\,.\,AD \cos A.$$

$$\therefore\ \cos a = \frac{OA}{OD}\cdot\frac{OA}{OE} + \frac{AE}{OD}\cdot\frac{AD}{OE}\cos A,$$

or $\cos a = \cos b \cos c + \sin b \sin c \cos A.$

And so

$$\cos b = \cos a \cos c + \sin b \sin c \cos B,$$

and $\cos c = \cos a \cos b + \sin a \sin c \cos C.$

From these formulæ, the cosines of the angles may be expressed in terms of the sines and cosines of the sides (Art. 12).

18. *To express the cosine of a side in terms of the sines and cosines of the angles.*

We have, by the last Art.,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Hence (Art. 12), replacing A by $\pi - a$, and a, b, c respectively by $\pi - A$, $\pi - B$, $\pi - C$, we get

$$\cos(\pi - A)$$
$$= \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a);$$

or, simplifying

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

And so $\cos B = -\cos A \cos C + \sin A \sin C \cos b,$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

From these formulæ we have at once, $\cos a$, $\cos b$, $\cos c$ expressed in terms of the angles A, B, C; thus

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

19. *To express the sine of an angle in terms of the functions of the sides.*

We have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$.

$$\therefore \sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c)}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Hence, taking the square root of each side, and taking the positive sign of the radical, as sin A is always positive, we have

$$\sin A = \frac{1}{\sin b \sin c} \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}.$$

Cor. This result, by taking the expression in (1) and breaking into factors, may be easily expressed in the following form:

$$\sin A = \frac{2}{\sin b \sin c} \sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)},$$

where $2s = a + b + c$.

And so

$$\sin B = \frac{2}{\sin a \sin c}\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-}$$

$$\sin C = \frac{2}{\sin a \sin b}\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-}$$

20. *To express the sines, cosines, and tangents of* $\frac{A}{2}, \frac{B}{2}$, *in terms of the functions of the sides.*

We have, Art. 17,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence

$$2 \sin^2 \frac{A}{2} = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos (b-c) - \cos a}{\sin b \sin c}.$$

From which

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}.$$

And so

$$\cos \frac{A}{2} = \sqrt{\frac{\cos s \sin (s-a)}{\sin b \sin c}}.$$

Hence also, since

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}, \text{ we have}$$

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}.$$

We have taken the positive sign of the radicals becaus $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, $\tan \frac{A}{2}$ are always positive.

From the form of these results we at once obtain similar formulæ for $\frac{B}{2}, \frac{C}{2}$.

The student cannot fail to see the similarity between the expressions obtained in the last two articles, and the expressions for sin A, sin $\frac{A}{2}$, &c., in Plane Trigonometry.

21. *To express the functions of the sides and half sides in terms of the functions of the angles.*

We have (Art. 18),

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

And by proceeding as in Arts. 19, 20, we can obtain sin a, sin $\frac{a}{2}$, cos $\frac{a}{2}$, &c., as wethe n obtained sin A, sin $\frac{A}{2}$, cos $\frac{A}{2}$, &c.

We may get these results however by means of the properties of the supplemental triangle (Art. 12).

The student may easily see that we then get

$$\sin a = \frac{2}{\sin B \sin C} \sqrt{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)} \quad (1),$$

$$\sin \frac{a}{2} = \sqrt{-\frac{\cos S \cos (S-A)}{\sin B \sin C}} \ldots\ldots\ldots\ldots (2),$$

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}} \ldots\ldots\ldots (3),$$

$$\tan \frac{a}{2} = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}} \ldots\ldots (4),$$

where $2S = A + B + C$.

We have taken here again the positive signs of the radicals, because a is less than two right angles, and therefore $\sin a$, $\sin \frac{a}{2}$, &c., are all positive.

We can show that the expression in (1), (2), (4) are all real, thus:

By Art. 13, $2\,S > \pi$, and $< 3\,\pi$.

$$\therefore S > \frac{\pi}{2}, \text{ and } < \frac{3\,\pi}{2}.$$

Hence $\cos S$ is negative.

Again (Art. 9), $a < b + c$.

$\therefore$ (Art. 12), $\pi - A < \pi - B + \pi - C$, or $B + C - A < \pi$.

$$\therefore S - A < \frac{\pi}{2}.$$

Again $B + C - A$ is always greater than $-\pi$, for B and C must have some positive value, and A cannot be greater than π.

Therefore $S - A$ is always algebraically $> -\frac{\pi}{2}$.

Hence, $\cos (S - A)$ is always positive, and similarly $\cos (S - B)$, $\cos (S - C)$ are positive.

Hence the expressions for $\sin a$, $\sin \frac{a}{2}$, &c., are real quantities.

22. *To show that*

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.$$

We have $\cos a = \cos b \cos c + \sin b \sin c \cos A$;

also $\cos c = \cos a \cos b + \sin a \sin b \cos C$,

and $\sin c = \sin a \cdot \dfrac{\sin C}{\sin A}$.

Substituting for these values of sin c and cos c, then

$$\cos a = \cos b\,(\cos a \cos b + \sin a \sin b \cos C) + \sin b \,.\, \sin a \frac{\sin C}{\sin A} \,.\, \cos A\,;$$

or, $$\cos a\,(1 - \cos^2 b) = \sin a \sin b \cot A \sin C + \sin a \sin b \cos b \cos C\,;$$

or, putting $\sin^2 b$ for $1 - \cos^2 b$, and dividing each side by $\sin a \sin b$, then

$$\cot a \sin b = \cot A \sin C + \cos b \cos C \quad \ldots\ldots (1).$$

From the form of this result, we also have

$$\cot b \sin a = \cot B \sin C + \cos a \cos C \quad \ldots\ldots (2).$$

$$\cot b \sin c = \cot B \sin A + \cos c \cos A \quad \ldots\ldots (3).$$

$$\cot c \sin b = \cot C \sin A + \cos b \cos A \quad \ldots\ldots (4).$$

$$\cot c \sin a = \cot C \sin B + \cos a \cos B \quad \ldots\ldots (5).$$

$$\cot a \sin c = \cot A \sin B + \cos c \cos B \quad \ldots\ldots (6).$$

In these results we may of course replace the angles by the supplements of the sides, if we also replace sides by the supplements of the angles. But it will be found that by so doing we obtain no new formulæ.

We thus obtain (6) from (1), (3) from (2), and (5) from (4).

23. *Napier's Analogies—To show that*

$$\text{(i.)}\ \tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2},$$

$$\text{(ii.)}\ \tan \tfrac{1}{2}(\mathrm{A} - \mathrm{B}) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{\mathrm{C}}{2},$$

$$\text{(iii.)}\ \tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(\mathrm{A} - \mathrm{B})}{\cos \frac{1}{2}(\mathrm{A} + \mathrm{B})} \tan \frac{c}{2},$$

$$\text{(iv.)}\ \tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(\mathrm{A} - \mathrm{B})}{\sin \frac{1}{2}(\mathrm{A} + \mathrm{B})} \tan \frac{c}{2}.$$

We have $$\frac{\sin \mathrm{A}}{\sin a} = \frac{\sin \mathrm{B}}{\sin b}.$$

Hence it is easily shown (see Vol. I., page 215), that each of the quantities

$$\frac{\sin \mathrm{A}}{\sin a} \text{ or } \frac{\sin \mathrm{B}}{\sin b} = \frac{\sin \mathrm{A} \pm \sin \mathrm{B}}{\sin a \pm \sin b} \quad \text{.........(1)},$$

both the upper or both the lower signs being taken.

Now $$\cos \mathrm{A} + \cos \mathrm{B} \cos \mathrm{C} = \sin \mathrm{B} \sin \mathrm{C} \cos a = \frac{\sin \mathrm{B}}{\sin b} . \cos a \sin b \sin \mathrm{C} \quad \text{.........(2)},$$

and $$\cos \mathrm{B} + \cos \mathrm{A} \cos \mathrm{C} = \sin \mathrm{A} \sin \mathrm{C} \cos b = \frac{\sin \mathrm{A}}{\sin a} . \sin a \cos b . \sin \mathrm{C} \quad \text{.........(3)}.$$

Adding then, by (1) above, taking the upper sign,

$$(\cos \mathrm{A} + \cos \mathrm{B})(1 + \cos \mathrm{C}) = \frac{\sin \mathrm{A} + \sin \mathrm{B}}{\sin a + \sin b} . \sin (a + b) \sin \mathrm{C}; \text{ or,}$$

$$\frac{\sin \mathrm{A} + \sin \mathrm{B}}{\cos \mathrm{A} + \cos \mathrm{B}} = \frac{\sin a + \sin b}{\sin (a + b)} \cdot \frac{1 + \cos \mathrm{C}}{\sin \mathrm{C}}.$$

Hence, simplifying, we have

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

5—II.

And so, by taking the lower sign in (1), we have

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

Now writing $\pi - A$ for a, $\pi - B$ for b, &c., we have, on simplifying,

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2}.$$

x and

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2}.$$

The last two may be of course obtained directly by commencing with the identity

$$\cos a - \cos b \cos c = \sin b \sin c \cos A,$$

instead of as in (2).

COR. Hence $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ *are of the same affection, that is, are either both less or both greater than a right angle.*

For we learn from (i.) that $\tan \frac{1}{2}(A + B)$ and $\cos \frac{1}{2}(a + b)$ *are of the same sign*, since $\cos \frac{1}{2}(a - b)$ and $\cot \frac{C}{2}$ are necessarily positive, $\frac{1}{2}(a - b)$ and $\frac{C}{2}$ being each less than $\frac{\pi}{2}$.

Hence $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ are both less or both greater than a right angle together.

24. *Formulæ of Gauss.* To show that:

$$\text{(i.)}\quad \cos \tfrac{1}{2} c \cos \tfrac{1}{2} (A + B) = \sin \tfrac{1}{2} C \cos \tfrac{1}{2} (a + b),$$

$$\text{(ii.)}\quad \sin \tfrac{1}{2} c \cos \tfrac{1}{2} (A - B) = \sin \tfrac{1}{2} C \sin \tfrac{1}{2} (a + b),$$

$$\text{(iii.)}\quad \cos \tfrac{1}{2} c \sin \tfrac{1}{2} (A + B) = \cos \tfrac{1}{2} C \cos \tfrac{1}{2} (a - b),$$

$$\text{(iv.)}\quad \sin \tfrac{1}{2} c \sin \tfrac{1}{2} (A - B) = \cos \tfrac{1}{2} C \sin \tfrac{1}{2} (a - b).$$

Now $\cos c = \cos a \cos b + \sin a \sin b \cos C$.

Hence $1 + \cos c = 1 + \cos a \cos b + \sin a \sin b \cos C$; or

$$2 \cos^2 \tfrac{1}{2} c = (1 + \cos a \cos b)\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) + \sin a \sin b \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= \left\{1 + \cos (a - b)\right\} \cos^2 \frac{C}{2} + \left\{1 + \cos (a + b)\right\} \sin^2 \frac{C}{2}$$

$$= 2 \cos^2 \tfrac{1}{2} (a - b) \cos^2 \frac{C}{2} + 2 \cos^2 \tfrac{1}{2} (a + b) \sin^2 \frac{C}{2}.$$

$$\therefore \cos^2 \tfrac{1}{2} c = \cos^2 \tfrac{1}{2} (a - b) \cos^2 \frac{C}{2} + \cos^2 \tfrac{1}{2} (a + b) \sin^2 \frac{C}{2} \quad \text{.......... (1).}$$

And so

$$\sin^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2} (a - b) \cos^2 \tfrac{1}{2} C + \sin^2 \tfrac{1}{2} (a + b) \sin^2 \frac{C}{2} \quad \text{.......... (2).}$$

But (Art. 23), by Napier's Analogies,

$$\tan^2 \tfrac{1}{2}(A + B) = \frac{\cos^2 \frac{1}{2}(a - b) \cos^2 \frac{C}{2}}{\cos^2 \frac{1}{2}(a + b) \sin^2 \frac{C}{2}};$$ adding unity to each side, then

$$\sec^2 \tfrac{1}{2}(A + B) = \frac{\cos^2 \frac{1}{2}(a - b) \cos^2 \frac{C}{2} + \cos^2 \frac{1}{2}(a + b) \sin^2 \frac{C}{2}}{\cos^2 \frac{1}{2}(a + b) \sin^2 \frac{C}{2}};$$ or by (1),

$$\frac{1}{\cos^2 \frac{1}{2}(A + B)} = \frac{\cos^2 \frac{1}{2} c}{\cos^2 \frac{1}{2}(a + b) \sin^2 \frac{C}{2}};$$ or

$$\cos^2 \tfrac{1}{2} c \cos^2 \tfrac{1}{2}(A + B) = \sin^2 \tfrac{1}{2} C \cos^2 \tfrac{1}{2}(a + b).$$

Taking the square root of each side, and remembering (Art. 23) that $\frac{1}{2}(A + B)$, $\frac{1}{2}(a + b)$ are of the same affection, and that $\frac{1}{2} c$, $\frac{1}{2} C$ are each less than $\frac{\pi}{2}$, we have

$$\cos \tfrac{1}{2} c \cos \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2} C \cos \tfrac{1}{2}(a + b) \quad \text{.........(3).}$$

And similarly

$$\sin \tfrac{1}{2} c \, . \cos \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2} C \, . \sin \tfrac{1}{2}(a + b) \quad \text{.........(4).}$$

But (Art. 23),

$$\text{(i.)} \quad \tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2} \quad \text{.........(5),}$$

and

$$\text{(ii.)} \quad \tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2} \quad \text{.........(6).}$$

(3) × (5), then

$$\cos \tfrac{1}{2} c \,.\, \sin \tfrac{1}{2} (A + B) = \cos \tfrac{1}{2} C \cos \tfrac{1}{2} (a - b).$$

And (4) × (6), then

$$\sin \tfrac{1}{2} c \,.\, \sin \tfrac{1}{2} (A - B) = \cos \tfrac{1}{2} C \,.\, \sin \tfrac{1}{2} (a - b).$$

Ex. I.

In any right-angled triangle ABC, right angled at C, show that

1. $2 \cos \frac{c}{2} = \sqrt{2 + 2 \cos a \cos b}.$

2. $\tan^2 \frac{a}{2} = \tan \frac{c + b}{2} \tan \frac{c - b}{2}.$

3. $\sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} - \sin^2 \frac{c}{2} = 2 \sin^2 \frac{a}{2} \sin^2 \frac{b}{2}.$

4. $\sin (c - b) = \tan^2 \frac{A}{2} \sin (c + b).$

5. $\sin (a - b) \cot \frac{1}{2} (A - B) = \sin (a + b) \tan \frac{1}{2} (A + B).$

6. $\tan^2 \left(\frac{\pi}{4} + \frac{c}{2}\right) = \tan \frac{1}{2} (A + a) \cot \frac{1}{2} (A - a).$

$$\tan^2 \left(\frac{\pi}{4} + \frac{A}{2}\right) = \cot \tfrac{1}{2} (B - b) \cot \tfrac{1}{2} (B + b).$$

7. In any right-angled triangle, if α, β be the arcs respectively drawn perpendicular to the hypothenuse and bisecting it, then

$$\cot \alpha = \sqrt{\cot^2 a + \cot^2 b}, \quad \cot \beta = \frac{\cos a + \cos b}{\sqrt{\sin^2 a + \sin^2 b}}.$$

8. The perpendicular drawn from the pole of any circle upon the great circle chord, bisects the chord.

9. If perpendiculars be drawn from the angles of a triangle ABC, meeting the opposite sides in D, E, F, show that

$$\frac{\tan BD}{\tan CD} \cdot \frac{\tan EC}{\tan EA} \cdot \frac{\tan FA}{\tan FB} = 1.$$

10. If the sides BC, CA, AB be cut by the arc of a great circle respectively in a, b, c, show that

$$\sin \mathrm{A}c \sin \mathrm{B}a \sin \mathrm{C}b = \sin c\mathrm{B} \sin a\mathrm{C} \sin b\mathrm{A}.$$

11. Let P be any point on the sphere, and draw through P arcs of great circles to the angular points, meeting the sides of a triangle in D, E, F respectively, then

$$\sin \mathrm{AF} \sin \mathrm{BD} \sin \mathrm{CE} = \sin \mathrm{CD} \sin \mathrm{BF} \sin \mathrm{AE}.$$

12. If great circles be drawn from any point on a sphere to the angles of a polygon, the products of the sines of the alternate angles are equal.

13. In any equilateral triangle whose side is A,

$$\cot^2 \frac{\mathrm{A}}{2} = 2 \cos a + 1,$$

$$\text{and } \tan^2 \frac{a}{2} = 1 - 2 \cos \mathrm{A}.$$

14. In any equilateral triangle whose sides are quadrants, if α, β, γ be the lengths of arcs of great circles from a point in the triangle to the angular points, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

15. If δ, z be the declination and zenith distance of a heavenly body, and l the latitude, show that the cosine of the azimuth $= \dfrac{\sin \delta - \sin l \cos z}{\cos l \sin z}$.

16. If l be the latitude of a place on the earth's surface, δ the sun's declination; then if $2\,t$ be the length of a day in solar hours, show that

$$\cos 15^\circ . t = \tan l \tan \delta.$$

17. If l_1, l_2 be the latitudes of two stars, and λ their difference in longitude, show that their distance apart can be found from the formula

$$\cos \theta = \frac{\sin l_1 \cos (l_2 - \omega)}{\sin \omega},$$

where θ is the distance, and $\cot \omega = \cot l_1 \cos \lambda$.

18. If a, l, δ be respectfully the sun's right ascension,

longitude, and declination; and ω be the obliquity of the ecliptic, prove the following relations:

$$\tan a = \cos \omega \tan l,$$
$$\tan \delta = \sin a \tan \omega,$$
$$\sin \delta = \sin \omega \sin l,$$
$$\cos l = \cos a \cos \delta.$$

CHAPTER IV.

SOLUTION OF SPHERICAL TRIANGLES.

25. In general when any three elements of a spherical triangle are given, the other three can be determined. For the angles and sides are connected together by six equations of the form

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \text{ or } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

When the trangle is *right-angled*, then the solution is generally possible if we know two other elements; and so also, when the triangle is *quadrantal*, or has one of its sides a quadrant, two other elements are generally sufficient for a complete solution.

I.—Right-Angled Triangles.

26. We have:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \ldots\ldots\ldots\ldots (1).$$

$$\cos c \sin A \sin B = \cos C + \cos A \cos B \ldots\ldots\ldots (2).$$

$$\cos A + \cos B \cos C = \cos a \sin B . \sin C \ldots\ldots (3).$$

$$\sin a = \frac{\sin c \sin A}{\sin C} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

$$\cot a \sin b = \cot A \sin C + \cos b \cos C \ldots\ldots\ldots (5).$$

$$\cot C \sin B + \cos a \cos B = \cot c \sin a \ldots\ldots\ldots (6).$$

Let C be the right angle of the spherical triangle ABC.

Then C = 90°, sin C = 1, cos C = 0, cot C = 0.

Hence from (1),

$$\cos c = \cos a \cos b;$$

from (2), $$\cos c = \cot A \cot B;$$

from (3), $$\cos A = \cos a \sin B;$$

and so, $$\cos B = \cos b \sin A.$$

Also from (4),

$$\sin a = \sin c \sin A;$$

and so, $$\sin b = \sin c \sin B.$$

And from (5),

$$\sin b = \tan a \cot A$$

and so, $$\sin a = \tan b \cot B.$$

And again, from (6),

$$\cos B = \cot c \tan a;$$

and so, $$\cos A = \cot c \tan b.$$

We may write these results as follows:

(i.) $$\sin\left(\frac{\pi}{2} - c\right) = \cos a \cos b = \tan\left(\frac{\pi}{2} - A\right) \tan\left(\frac{\pi}{2} - B\right).$$

(ii.) $$\sin\left(\frac{\pi}{2} - A\right) = \cos a \cos\left(\frac{\pi}{2} - B\right) = \tan\left(\frac{\pi}{2} - c\right) \tan b.$$

(iii.) $$\sin\left(\frac{\pi}{2} - B\right) = \cos b \cos\left(\frac{\pi}{2} - A\right) = \tan\left(\frac{\pi}{2} - c\right) \tan a.$$

(iv.) $$\sin a = \cos\left(\frac{\pi}{2} - c\right) \cos\left(\frac{\pi}{2} - A\right)$$

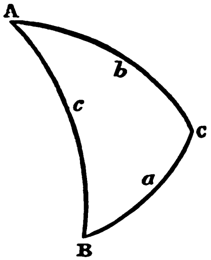
$$= \tan b \tan\left(\frac{\pi}{2} - B\right).$$

(v.) $$\sin b = \cos\left(\frac{\pi}{2} - c\right) \cos\left(\frac{\pi}{2} - B\right)$$

$$= \tan a \tan\left(\frac{\pi}{2} - A\right).$$

We can now explain *Napier's Rules.*

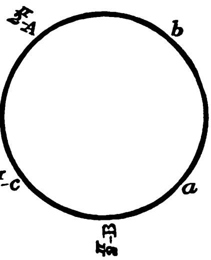
Leaving the right angle C out of consideration, we have *five* remaining elements.



Again, if we read the remaining elements in order *round the triangle* commencing with any one of them, there will always be two adjacent parts, and two opposite parts.

Instead however of reading A, c, B, we shall read the *complements of the angles* and *the complement of the hypothenuse.*

We shall then have what Napier called *the circular parts,* viz., a, b, $\frac{\pi}{2} - A$, $\frac{\pi}{2} - c$, $\frac{\pi}{2} - B$, any one of which may be the starting point, or what he called *the middle part.*



The formulæ proved above may then be expressed by the following rules, which the student can easily verify for himself.

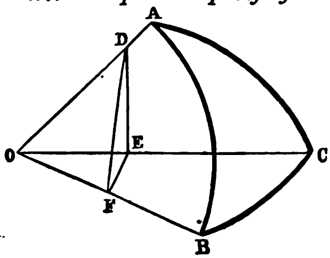
Napier's Rules:

(1.) *Sine of the middle part*
= *product of cosines of opposite parts.*

(2.) *Sine of the middle part*
= *product of tangent of adjacent parts.*

It will be seen that all the possible cases of these rules are exemplified in the formula just proved.

27. *Independent proof of the relations in the last Article.*



Let ABC be a spherical triangle having the right angle at C, and let O be the centre of the sphere.

In OA take any point D; draw DE perpendicular to OC, and from E draw EF perpendicular to OB.

Join DF.

Then, because C is a right angle, and DE is perpendicular to OC, DE is perpendicular to EF.

We shall now show that DFO is a right angle.

We have

$$DF^2 = DE^2 + EF^2 = (OD^2 - OE^2) + (OE^2 - OF^2)$$
$$= OD^2 - OF^2.$$

Hence (Euc. I., 48), DF is perpendicular to OF.

Hence also,

$\angle$ DFE is the inclination of the planes AOB and BOC.

$$\therefore \angle DFE = B.$$

Now $$\frac{OF}{OD} = \frac{OF}{OE} \cdot \frac{OE}{OD}; \therefore \cos c = \cos a \cos b \dots\dots (1).$$

$$\left.\begin{array}{l} \dfrac{DE}{OD} = \dfrac{DF}{OD} \cdot \dfrac{DE}{DF}; \therefore \sin b = \sin c \sin B; \\ \text{and so,} \quad \sin a = \sin c \sin A \end{array}\right\} \dots\dots (2).$$

$$\left.\begin{array}{l} \dfrac{EF}{OE} = \dfrac{DE}{OE} \cdot \dfrac{EF}{DE}; \therefore \sin a = \tan b . \cot B; \\ \text{and so,} \quad \sin b = \tan a \cot A \end{array}\right\} \dots\dots (3).$$

$$\left.\begin{array}{l} \dfrac{EF}{DF} = \dfrac{EF}{OF} \cdot \dfrac{OF}{DF}; \therefore \cos B = \tan a . \cot c; \\ \text{and so,} \quad \cos A = \tan b \cot c \end{array}\right\} \dots\dots (4).$$

From (3), $\sin a \sin b = \tan a \tan b . \cot A \cot B$; or

$$1 = \frac{1}{\cos a \cos b} \cot A \cot B; \text{ or, by (1),}$$
$$= \frac{1}{\cos c} \cot A \cot B.$$

$$\therefore \cos c = \cot A \cot B \dots\dots (5).$$

Again (2) $\times$ (4), we have

$$\sin a \cos B = \sin c \sin A . \tan a \cot c = \sin A \tan a \cos c; \text{ or,}$$

$$\cos B = \frac{\sin A \tan a \cos c}{\sin a} = \frac{\cos c}{\cos a} \sin A; \text{ or by (1),}$$

$$\left.\begin{array}{l} \cos B = \cos b \sin A; \\ \text{and so,} \\ \cos A = \cos a \sin B \end{array}\right\} \dots\dots (6).$$

COR. (i.) From (1) we learn that $\cos c$ has the same sign as the product of the cosines of a and b. Hence either all the cosines of the sides are positive, or only one is positive.

Hence, in any right-angled spherical triangle either *all the sides are quadrants*, or *all the sides are less than quadrants, or else one side is less than a quadrant, and the other two greater than quadrants.*

(ii.) *Either angle and the opposite side are of the same affection;* that is, A and a are both less, equal to, or greater than $\frac{\pi}{2}$, and so are B and b.

For we have from (3), $\sin b = \tan a \cot A = \frac{\tan a}{\tan A}$.

Now $\sin b$ must be positive, and hence $\tan a$ and $\tan A$ *must have the same sign.*

(iii.) *From a point of a sphere the shortest great circle that can be drawn to a given great circle is the perpendicular.*

For we have $\sin a = \sin c \sin A$.

Hence $\sin a$ is less than $\sin c$, unless $\sin A = 1$, or $A = 90°$.

Hence $\sin c$ is greater than $\sin a$, or c greater than a, unless $A = 90°$, when $c = a$. In this case both a and c are perpendiculars.

(iv.) *The hypothenuse is acute if the sides,* a, b *are of the same affection, or the angles* A, B *opposite to them are of the same affection, but otherwise obtuse.*

For we have $\cos c = \cos a \cos b$.

Hence if a is $< \frac{\pi}{2}$ and $b > \frac{\pi}{2}$, $\cos c$ is negative and $c > \frac{\pi}{2}$.

Also if a is $> \frac{\pi}{2}$ and $b > \frac{\pi}{2}$, or $a < \frac{\pi}{2}$, and $b < \frac{\pi}{2}$, $\cos c$ is positive, and $c < \frac{\pi}{2}$.

And hence also, by (ii.), c is $< \frac{\pi}{2}$ when A, B are of the same affection, but otherwise $< \frac{\pi}{2}$.

28. *When in a right-angled triangle we have given an angle and the side opposite to it, the solution is generally ambiguous.*

Let A, a be given.

We have

$$\sin a = \sin c \sin A, \text{ or } \sin c = \frac{\sin a}{\sin A} \ldots\ldots\ldots\ldots (1).$$

$$\sin b = \tan a \cot A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\cot A = \cos a \sin B, \text{ or } \sin B = \frac{\cos A}{\cos a} \ldots\ldots\ldots\ldots (3).$$

Hence c, b, B are determined from their sines, and generally therefore we may expect two values of each.

(i.) If sin a *be greater than* sin A, then C is impossible; and hence, since (Art. 27) a and A are of the same affection, we must have $a < A$ *when both are acute*, and $a > A$ *when both are obtuse*.

Otherwise $\sin a < \sin A$, and the solution impossible.

(ii.) *If* sin a *be less than* sin A, then c has two values.

Each of these values of c however gives *only one* value for b, and *only one* for B, since

$$\cos c = \cos a \cos b, \text{ or } \cos b = \frac{\cos c}{\cos a};$$

and

$$\cos c = \cot A \cot B, \text{ or } \cot B = \frac{\cos c}{\cos A}.$$

Hence there are in this case *two triangles* which answer the conditions, but only two.

(iii.) If $\sin a = \sin A$, and therefore $a = A$ (since a and A are of the same affection), we have

$$\sin c = 1 = \sin \frac{\pi}{2}, \therefore c = \frac{\pi}{2}.$$

Hence also, from (2) and (3),

$$b = \frac{\pi}{2}, \text{ and } B = \frac{\pi}{2}.$$

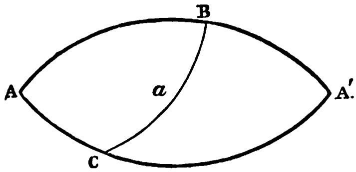
Therefore c, b, B are each right angles. And hence, since C is a right angle, A is the pole of a, and the two triangles ABC, A′BC (see below) will be *symmetrically* equal.

In the case where $a = A = \frac{\pi}{2}$, we have from (2) $\sin b = \frac{0}{0}$, and from (3) $\sin B = \frac{0}{0}$.

Hence b, B are indeterminate.

29. We may illustrate the above remarks geometrically thus:

We have given A, a; and the triangle ABC is right angled at C.



Produce AB, AC to meet at A′.

Then evidently A′ = A.

Hence we have two triangles ABC, A′BC answering the given conditions, for they each have an angle equal to A, and they have a common side a, and a right angle at C.

If A and a are both right angles, then evidently B is the pole of AC.

Hence (Art. 5), $B = b$, but the values are indeterminate.

There is no other ambiguous case in the solution of right-angled triangles, as we now proceed to show.

30. *Given a side* a *and an adjacent angle* B.

We have, by Napier's Rules,

$$\cos B = \tan a \cot c, \therefore \tan c = \frac{\tan a}{\cos B}.$$

Also, $\quad \sin a = \tan b \cot B.$

$$\therefore \tan b = \sin a \tan B.$$

And $\quad \cos A = \cos a \sin B.$

Hence b, c, A are determined without any ambiguity, and *there is no* impossible case.

31. *Given the two sides* a, b.

By Napier's Rules, we have

$$\cos c = \cos a \cos b,$$

$$\sin a = \tan b \cot B, \therefore \cot B = \frac{\sin a}{\tan b}.$$

and $$\sin b = \tan a \cot A, \therefore \cot A = \frac{\sin b}{\tan a}.$$

Hence there is no ambiguity in determining the values of c, A, B, and the triangle is always possible.

32. *Given the hypothenuse* c *and an angle* A.

By Napier's Rules,

$$\cos A = \tan b \cot c, \therefore \tan b = \tan c \cos A \text{.........(1)}.$$

Also $\cos c = \cot A \cot B, \therefore \cot B = \cos c \tan A$.......(2).

And $\sin a = \sin c \sin A$ ....(3).

From (1) and (2) we obtain b, B without any ambiguity, and their values are always possible.

In (3) there is an *apparent ambiguity* in the determination of a. We must remember however that a and A *are of the same affection*. Hence, if A be given greater than a right angle, we must take the value of a which is such; and so when A is less than a right angle, the value of a must be taken less than a right angle.

Hence there is no ambiguity, and the triangle is always possible.

From (1) and (2), when A, c are both right angles, we find B, b are indeterminate, for we have $\tan b$ and $\cot B$ each taking the form $\infty \times 0$.

33. *Given the hypothenuse* c *and a side* a.

By Napier's Rules, we have

$$\sin a = \sin c \sin A, \text{ or } \sin A = \frac{\sin a}{\sin c} \text{...........(1)}.$$

Also, $\cos B = \tan a \cot c$....................................(2).

And $\cos c = \cos a \cos b$, or $\cos b = \dfrac{\cos c}{\cos a}$ ....(3).

From (2) and (3) we obtain B, b without ambiguity; and from (1) we obtain A without ambiguity, since A and a are of the same affection.

Again, we see that the solution is impossible if we obtain from (1), (2), (3) values of sin A, cos B, or cos b greater than unity.

Hence, for a possible solution, we cannot have

$$\sin c < \sin a,\quad \cos c > \cos a,\quad \tan c < \tan a.$$

And hence for a possible solution we must have c between the limits a and $\pi - a$.

34. *Given the two angles* A, B.

By Napier's Rules, we have

$$\cos A = \cos a \sin B, \quad \therefore \cos a = \frac{\cos A}{\sin B} \quad\text{.........(1)},$$

$$\cos B = \cos b \sin A, \quad \therefore \cos b = \frac{\cos B}{\sin A} \quad\text{..........(2)},$$

$$\cos c = \cot A \cot B \quad\text{..............................(3)}.$$

Hence the values of a, b, c can be determined without ambiguity.

We learn however from (1), (2), and (3) that the solution is not always possible.

For we cannot have cos B > sin A where sin A may be positive or negative, and the inequality is with regard to magnitude only.

(i.) *Let* A *be less than* $\frac{\pi}{2}$.

Then since cos B *cannot be* > $\cos\left(\frac{\pi}{2} - A\right)$,

and cannot be > $\cos\left(\frac{\pi}{2} + A\right)$ in magnitude,

B must lie between $\frac{\pi}{2} - A$ and $\frac{\pi}{2} + A$.

(ii.) *Let* A *be greater than* $\frac{\pi}{2}$.

Then $\pi - A$ is less than $\frac{\pi}{2}$, and we have, as in the last case,

$$B \text{ must lie between } \frac{\pi}{2} - (\pi - A) \text{ and } \frac{\pi}{2} + (\pi - A).$$

$$\therefore \ B \text{ must lie between } A - \frac{\pi}{2} \text{ and } \frac{3\pi}{2} - A.$$

When the side or angle required is small, or nearly equal to one right angle, or to two right angles, we must be careful as to what formula we use, and proceed conformably to the principles laid down in Plane Trigonometry, Art. 70. If a convenient formula cannot be obtained by Napier's Rules, we may transform the formula.

Thus, knowing that $\cos c = \cot A \cot B$, it follows easily

that
$$\sin\frac{c}{2} = \sqrt{\frac{-\cos(A+B)}{2\sin A \sin B}},$$

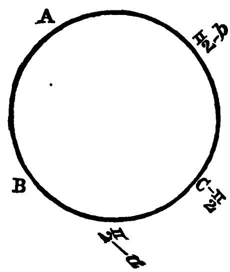
and
$$\cos\frac{c}{2} = \sqrt{\frac{\cos(A-B)}{2\sin A \sin B}}.$$

And so on.

II.—Quadrantal Triangles.

35. *A quadrantal triangle* ABC, *of which* c *is the quadrantal side, may be solved by Napier's Rules, if we take as the circular parts (the quadrantal side being neglected),*

$$\frac{\pi}{2} - a, \ \frac{\pi}{2} - b, \ A, \ B, \ C - \frac{\pi}{2}.$$



Let A′B′C′ be the polar triangle; then

$$A' = \pi - a, \ B' = \pi - b, \ C' = \pi - c = \frac{\pi}{2},$$

$$a' = \pi - A, \ b' = \pi - B, \ c' = \pi - C.$$

Applying Napier's Rules to the polar right-angled triangle A′B′C′, we have

$$\cos c' = \cos a' \cos b' = \cot A' \cot B',$$
$$\sin a' = \sin c' \sin A' = \tan b' \cot B',$$
$$\sin b' = \sin c' \sin B' = \tan a' \cot A',$$
$$\cos A' = \cos a' \sin B' = \tan b' \cot c',$$
$$\cos B' = \cos b' \sin A' = \tan a' \cot c'.$$

Substituting the above values of A′, B′, &c., in these equations, and arranging, we have

$$\sin\left(C - \frac{\pi}{2}\right) = \cos A \cos B$$
$$= \tan\left(\frac{\pi}{2} - a\right) \tan\left(\frac{\pi}{2} - b\right),$$
$$\sin A = \cos\left(\frac{\pi}{2} - a\right) \cos\left(C - \frac{\pi}{2}\right)$$
$$= \tan\left(\frac{\pi}{2} - b\right) \tan B,$$
$$\sin B = \cos\left(\frac{\pi}{2} - b\right) \cos\left(C - \frac{\pi}{2}\right)$$
$$= \tan\left(\frac{\pi}{2} - a\right) \tan A,$$
$$\sin\left(\frac{\pi}{2} - a\right) = \cos\left(\frac{\pi}{2} - b\right) \cos A$$
$$= \tan B \,.\, \tan\left(C - \frac{\pi}{2}\right),$$
$$\sin\left(\frac{\pi}{2} - b\right) = \cos\left(\frac{\pi}{2} - a\right) \cos B$$
$$= \tan A \,.\, \tan\left(C - \frac{\pi}{2}\right).$$

These results prove the proposition.

III.—Oblique-Angled Triangles.

36. CASE I. *Given the three sides* (a, b, c).

We may determine the angles from any of the formulæ in Art. 17. In certain cases the particular one chosen must depend upon the principles laid down in Plane Trigonometry, Art. 70.

37. CASE II. *Given the three angles* (A, B, C).

We may determine the sides from the formulæ in Art. 18, with the precautions mentioned in last Art.

38. CASE III. *Given two sides and the included angle* (a, C, b).

By Napier's Analogies

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2},$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

Hence A, B are determined.

Then, since $\sin c = \sin a \,.\, \dfrac{\sin C}{\sin A}$, c is also determined.

As c is determined from its *sine*, there may be some uncertainty as to which of the two values obtainable we are to take. We may generally determine this point by remembering that the greater side of a spherical triangle is opposite to the greater angle.

We may, however, determine c from formulæ which give no ambiguous values. We do this in the next article.

39. *To find* c *without previously finding the values of* A *and* B.

We have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
$$= \cos b (\cos a + \sin a \tan b \cos C) \ldots\ldots\ldots (1).$$

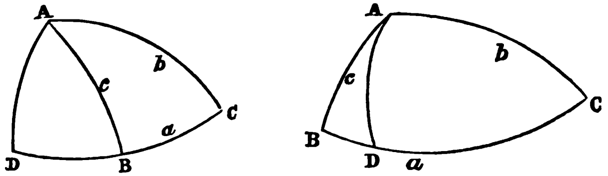
Let $\tan \theta = \tan b \cos C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

Then, from (1),

$$\cos c = \cos b\,(\cos a + \sin a \tan \theta) = \frac{\cos b}{\cos \theta} \cos (a - \theta) \ldots\ldots (3).$$

Hence c can be obtained independently by formulæ adapted to logarithmic computation, and there is no ambiguity. Moreover the triangle is always possible.

We may also solve the triangle by resolving it into two right-angled triangles by drawing from A an arc perpendicular to BC, or BC produced.



We have, by Napier's Rules, from the right-angled triangle ACD,

$\cos C = \cot b \tan CD$; $\therefore \tan CD = \tan b \cos C \ldots\ldots\ldots (4)$.

Hence CD is known; and then, since $BD = BC \backsim CD \ldots (5)$,

BD is also determined.

Again, from the right-angled triangle ADB,

$$\cos c = \cos AD \cos DB \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Also we have

$$\cos b = \cos AD \cos CD, \text{ or } \cos AD = \frac{\cos b}{\cos CD};$$

$$\therefore \cos c = \frac{\cos b}{\cos CD} . \cos DB \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

If we compare (4) with (2), we see that CD is nothing more than θ in the previous assumption. We then have

$$CD = \theta, \text{ and also } BD = a \backsim \theta.$$

B may be also found independently of A, as follows:

We have $\quad \sin CD = \tan AD \cot C \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$,

and $\quad \sin BD = \tan AD . \cot ABD \ldots\ldots\ldots\ldots (9)$.

$(8) \div (9)$, then $\dfrac{\sin CD}{\sin BD} = \dfrac{\cot C}{\cot ABD}$.

$$\therefore \tan ABD = \frac{\sin CD}{\sin BD} . \tan C = \frac{\sin \theta}{\sin (a \sim \theta)} \tan C.$$

Here ABD is the angle B or its supplement, according as BD is $a - \theta$ or $\theta - a$.

40. Case IV. *Given two angles and the included side* (A, c, B).

By Napier's Analogies

$$\tan \tfrac{1}{2} (a + b) = \frac{\cos \frac{1}{2} (A - B)}{\cos \frac{1}{2} (A + B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2} (a - b) = \frac{\sin \frac{1}{2} (A - B)}{\sin \frac{1}{2} (A + B)} \tan \tfrac{1}{2} c.$$

Hence a, b are determined.

Then, since $\sin C = \sin A \dfrac{\sin c}{\sin a}$,

C is also determined.

As in Case III., there may be an uncertainty as to which of the two obtainable values of C is admissible. We may however generally settle the point on the principle that the greater angle of a triangle is opposite to the greater side.

We may also find C from formulæ where there is no ambiguity, as in the next article.

41. *To find* C *without previously finding the values of* a *and* b

$$\begin{aligned} \text{We have } \cos C &= -\cos A \cos B + \sin A \sin B \cos c \\ &= \cos B\,(-\cos A + \sin A \tan B \cos c) \ldots (1). \end{aligned}$$

Let $\qquad \cot \phi = \tan B \cos c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

Then, from (1), we have $\cos C = \dfrac{\cos B}{\sin \phi} \sin (A - \phi) \ldots (3)$,

from which C can be determined without ambiguity.

Moreover the solution is always possible.

We may also solve the triangle, as in the last case, by drawing a perpendicular from A upon BC or BC produced.

The *process* corresponds exactly to that in Art. 39.

42. CASE V. *Given two sides and the angle opposite to one of them* (a, b, A).

We have $$\sin B = \frac{\sin b}{\sin a} \sin A \qquad (1).$$

Also we have $$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B) \qquad (2).$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b) \qquad (3).$$

From (1) B is determined. (We shall reserve the discussion of the cases of ambiguity till Art. 47).

And then from (2), (3) we determine C and *c*.

43. *To determine* C *and* c *independently of* B.

We have (Art. 22),

$$\cot a \sin b = \cos b \cos C + \sin C \cot A$$
$$= \cos b \left(\cos C + \frac{\cot A}{\cos b} \sin C\right) \qquad (1.)$$

Put $$\tan \phi = \frac{\cot A}{\cos b} \qquad (2).$$

Then $$\cot a \sin b = \cos b (\cos C + \tan \phi \sin C)$$
$$= \cos b \frac{\cos (C - \phi)}{\cos \phi}.$$

$$\therefore \cos (C - \phi) = \cos \phi \cot a \tan b \qquad (3).$$

From (2) we obtain ϕ, and then from (3), having found $C - \phi$, we obtain C.

Again, $$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$= \cos b (\cos c + \sin c \tan b \cos A) \qquad (4).$$

Put $$\tan \theta = \tan b \cos A \qquad (5).$$

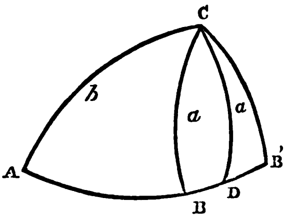
Then, $$\cos a = \cos b (\cos c + \sin c \tan \theta)$$
$$= \cos b . \frac{\cos (c - \theta)}{\cos \theta}.$$

$$\therefore \cos (c - \theta) = \frac{\cos a \cos \theta}{\cos b} \qquad (6).$$

From this equation we obtain $c - \theta$, and then having obtained θ from (5), we get c.

Since from (3) the value of $C - \phi$ may be positive or negative, there may be an ambiguity as in the last Article. And the same remark applies to equation (5).

44. *Geometrical illustration of* CASE V.



Draw CA = b, and make the angle CAB = $\angle$ A.

Also draw CB and CB′ each equal to a.

Then we have the two triangles CAB, CAB′ answering the given conditions.

Draw CD perpendicular to AB′. Then evidently

$$BD = DB', \text{ and } \angle BCD = \angle B'CD.$$

From the right-angled triangle ACD, we have

$$\cos b = \cot A \cot ACD, \text{ or } \cot ACD = \frac{\cos b}{\cot A},$$

or $\tan ACD = \dfrac{\cot A}{\cos b}$, which determines $\angle$ ACD.......... (1).

Comparing this equation with (2) of the last Art., we find $\angle ACD = \phi$.

Again we have $\cos ACD = \tan CD \cot b$,

and $\cos BCD = \tan CD \cot a$.

Hence $$\frac{\cos BCD}{\cos ACD} = \frac{\cot a}{\cot b} = \frac{\tan b}{\tan a};$$

$$\therefore \cos BCD = \cos ACD \cdot \frac{\tan b}{\tan a},$$

which determines $\angle$ BCD or BCD′.

Hence the two values of C are determined.

Again, we have

$$\cos A = \tan AD \cot b, \text{ or}$$

$$\tan AD = \tan b \cos A, \text{ which determines AD} \ldots\ldots (2).$$

Comparing this equation with (5) of last Art. we find $AD = \theta$.

Also we have $\cos b = \cos CD \cos AD,$
and $\cos a = \cos CD \cos BD;$ from which

$$\cos BD = \frac{\cos a}{\cos b} \cos AD, \text{ which determines}$$

BD or B′D.

Hence the two values of c are determined.

45. Case VI. *Given two angles and the side opposite to one of them* (A, B, a).

We have $$\sin b = \frac{\sin B}{\sin A} \sin a \quad \text{.......................(1)}.$$

Also we have

$$\tan \tfrac{1}{2} C = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B) \quad \text{......(2)},$$

and $$\tan \tfrac{1}{2} c = \frac{\cos \tfrac{1}{2}(A+B)}{\cos \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b) \quad \text{......(3)}.$$

From (1) we obtain b, and then C and c from (2) and (3). The ambiguities are analogous to those of Case V.

46. *To determine* C *and* c *independently of* b.

We have (Art. 22),

$$\cot A \sin B = \cot a \sin c - \cos c \cos B$$
$$= \cos B \left(\frac{\cot a \sin c}{\cos B} - \cos c\right) \quad \text{..............(1)}.$$

Put $$\cot \theta = \frac{\cot a}{\cos B} \quad \text{..(2)}.$$

Then, from (1),

$$\cot A \sin B = \cos B (\cot \theta \sin c - \cos c)$$
$$= \frac{\cos B}{\sin \theta} \sin (c - \theta);$$

$$\therefore \sin (c - \theta) = \sin \theta \cot A \tan B \quad \text{.....................(3)}.$$

Hence we obtain $c - \theta$, and therefore knowing θ from (2), we get c.

Again $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

$= \cos B\,(-\cos C + \tan B \cos a \,.\, \sin C) \ldots (4).$

Put $\cot \phi = \tan B \cos a$ ....(5).

Then

$$\cos A = \cos B\,(-\cos C + \cot \phi \sin C) = \frac{\cos B}{\sin \phi} \sin (C - \phi).$$

$$\therefore \sin (C - \phi) = \frac{\sin \phi \cos A}{\cos B} \ldots\ldots (6).$$

Hence, knowing ϕ from (5), after obtaining $C - \phi$ from (6), we get C.

Since in (3) we obtain $c - \theta$, and in (6) we get $C - \phi$ from their sines, there may be ambiguities.

It may be shown, as in Art. 44, if a perpendicular CD be drawn, that θ corresponds to B′D or BD, and ϕ to ∠ BCD or B′CD.

47. *Ambiguities of* CASE V.

We will first take *the particular case* when $a = b$.

Then $A = B$, and there is no ambiguity as regards B.

The formulæ (Art. 23) now give $\cot \frac{1}{2} C = \cos a \tan A$,

and $\tan \frac{1}{2} c = \tan a \cos A$.

Now $\frac{1}{2} C$, $\frac{1}{2} c$ are $< \frac{\pi}{2}$, and hence their tangents are positive. But $\cos a$ and $\tan A$, and also $\tan a$ and $\cos A$, have the *same sign* only when A, a are of the same affection.

Hence the solution *is impossible* when A, a *are not of the same affection.*

If $A = a = \frac{\pi}{2}$, then $\cos a = 0$, $\cos A = 0$, $\tan a = \infty$, $\tan A = \infty$.

Hence the values of $\cot \frac{1}{2} C$, $\tan \frac{1}{2} c$ assume the indeterminate form $0 \times \infty$.

C and c are therefore indeterminate, and there are an *infinite number of solutions.*

In all other cases where A, a *are of the same affection*, $\cot \frac{1}{2} C$ and $\tan \frac{1}{2} c$ will be positive. There will therefore be a possible solution, and *only one*.

We shall now consider the more general case.

We have $\sin B = \frac{\sin b}{\sin a} \sin A$.

When $\sin b \sin A > \sin a$, there can be no possible value of B.

When $\sin b \sin A$ *is not greater* than $\sin a$, there will generally be two values of B, which we may call B and B′. They are supplementary to each other. Let B be that which is not greater than B′. Now as A may be $<$, $=$, or $> \frac{\pi}{2}$, it is convenient to take these three cases in turn.

We may remark

since $\qquad \tan \frac{1}{2} C = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{1}{2}(A - B)$,

and $\qquad \tan \frac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \frac{1}{2}(a - b)$,

that C, c will be always possible, and not otherwise, when $\frac{1}{2}(a - b)$, $\frac{1}{2}(A - B)$, and therefore $a - b$, $A - B$, are of the same sign.

Hence we have to see that $A - B$, $a - b$, and also $A - B'$, $a - b$ have the same sign.

Let A *be less than a right angle.*

1. Let $b < \frac{\pi}{2}$.

(i.) When $a < b$, we must have $A < B$, or $B > A$.

Now the equation $\sin B = \frac{\sin b}{\sin a} \sin A$..................(1),

gives us $B > A$, since $\sin a < \sin b$ when $b < \frac{\pi}{2}$.

Hence we have both B, $B' > A$, and there are *two solutions.*

(ii.) When $a = b$, we have shown, since here a, A are each $< \frac{\pi}{2}$, that there is *one solution.*

(iii.) When $a > b$, we have $A > B$;

Now if $a + b = \pi$, or $\sin a = \sin b$,
equation (1) above gives $\sin B = \sin A$; that is $B = A$, and $B' > A$.

Neither of them is possible.

Again, if $a + b > \pi$, we have, since $b < \frac{\pi}{2}$, $\sin a < \sin b$.

Hence $\sin B > \sin A$, or $B > A$, and therefore $B' > A$.

Neither of these is possible.

Also, if $a + b < \pi$, we have, since $b < \frac{\pi}{2}$, $\sin a > \sin b$;

$\therefore \sin B < \sin A$, $\therefore B < A$.

Hence B' or $\pi - B$ is $> A$, being greater than a right angle.

The *first* value only of B is therefore admissible. Hence there is *one possible solution.*

2. Let $b = \frac{\pi}{2}$.

(i.) When $a < b$, we have $\sin B > \sin A$.

Hence B, B′ are both $> A$, and there are *two solutions.*

(ii.) When $a = b$, as above shown, since A, a are not *now of the same affection*, there will be *no solution.*

(iii.) When $a > b$, we have $\sin a < \sin b$, since $b = \frac{\pi}{2}$.

Therefore $\sin B > \sin A$, and both values of B are $> A$, which is impossible.

There is therefore *no solution.*

3. Let $b > \frac{\pi}{2}$.

(i.) When $a < b$, we must have $A < B$.

If $a + b = \pi$, $\sin a = \sin b$, and therefore $\sin B = \sin A$.

Hence, the *only possible* solution is $B' = A$.

If $a + b > \pi$, then (Art. 23), $A + B > \pi$, and there can be only *one solution*.

If $a + b < \pi$, then (Art. 23), $A + B < \pi$, and there may, therefore, be *two solutions*.

(ii.) When $a = b$, as above shown, there is *no solution*, since A, a, are not of the same affection.

(iii.) When $a > b$, we have, since $b > \frac{\pi}{2}$,

$$\sin a < \sin b.$$

Hence, $\sin B > \sin A$,

And both B, B', are $> A$, which values are impossible, since $a > b$.

Hence there is *no solution*.

Collecting the above results, we have:

I. *When* A *is less than a right angle.*

The number of solutions is seen from the table:

	$b < \frac{\pi}{2}$	$b = \frac{\pi}{2}$	$b > \frac{\pi}{2}$
$a < b$	*two*	*two*	$a + b < \pi$, *two* $a + b \geqq \pi$, *one*
$a = b$	*one*	*none*	*none*
$a > b$	$a + b < \pi$, *one* $a + b \geqq \pi$, *none*	*none*	*none*

And by similar reasoning we obtain the following tables for the cases where A is equal to, or greater than a right angle.

II. *When* A *is equal to a right angle.*

	$b < \frac{\pi}{2}$	$b = \frac{\pi}{2}$	$b > \frac{\pi}{2}$
$a < b$	*none*	*none*	$a + b > \pi$, *one* $a + b \overline{\underline{<}} \pi$, *none*
$a = b$	*none*	*infinite*	*none*
$a > b$	$a + b < \pi$, *one* $a + b \overline{\underline{>}} \pi$, *none*	*none*	*none*

III. *When* A *is greater than a right angle.*

	$b < \frac{\pi}{2}$	$b = \frac{\pi}{2}$	$b > \frac{\pi}{2}$
$a < b$	*none*	*none*	$a + b > \pi$, *one* $a + b \overline{\underline{<}} \pi$, *none*
$a = b$	*none*	*none*	*one*
$a > b$	$a + b > \pi$, *two* $a + b \overline{\underline{<}} \pi$, *one*	*two*	*two*

The ambiguities which arise in Case VI. may be discussed in a similar manner.

Ex. II.

1. Show that

$$\cos \tfrac{1}{2}(a+b)\cos \tfrac{1}{2}(a-b)\sin C = \cos^2 \tfrac{1}{2}c \sin (A+B),$$

and

$$\sin \tfrac{1}{2}(a+b)\sin \tfrac{1}{2}(a-b)\sin C = \sin^2 \tfrac{1}{2}c \sin (A-B).$$

2. Prove that, when a triangle ABC is equal and similar to its polar triangle,

$$\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1.$$

3. If $a + b + c = \pi$, show that

$$\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2,$$

$$\tan a = \frac{\sin \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}.$$

4. If $C = A + B$, and the side AB be bisected in D, then CD, AD, BD are equal to each other.

Prove that:

5. $\sin a \sin b \sin c \sin A \sin B \sin C = 4\,Nn$ (see Art. 49),

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{N}{n}.$$

6. $\sin^2 a \sin^2 b \sin^2 c \sin A \sin B \sin C = 8\,n^3,$
$\sin^2 A \sin^2 B \sin^2 C \sin a \sin b \sin c = 8\,N^3.$

7. $\sin^2 s = n \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$

$$\cos^2 S = N \tan \frac{a}{2} \tan \frac{b}{2} \tan \frac{c}{2}.$$

8. $\sin^2 \tfrac{1}{2}(A-B) \sin^2 \tfrac{1}{2}c + \sin^2 \tfrac{1}{2}(A+B)\cos^2 \tfrac{1}{2}c = \cos^2 \tfrac{1}{2}C,$
$\sin^2 \tfrac{1}{2}(a-b)\cos^2 \tfrac{1}{2}C + \sin^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C = \sin^2 \tfrac{1}{2}c.$

9. $\cos^2 \tfrac{1}{2}(A-B)\sin^2 \tfrac{1}{2}c + \cos^2 \tfrac{1}{2}(A+B)\cos^2 \tfrac{1}{2}c = \sin^2 \tfrac{1}{2}C,$
$\cos^2 \tfrac{1}{2}(a-b)\cos^2 \tfrac{1}{2}C + \cos^2 \tfrac{1}{2}(a+b)\cos^2 \tfrac{1}{2}C = \cos^2 \tfrac{1}{2}c.$

Solve the six following right-angled triangles, C being the right angle :

10. $a = 62° 45'$, $B = 135° 21'$.

11. $a = 127° 21' 30''$, $b = 80° 32' 27''$.

12. $c = 132° 15' 17''$, $B = 63° 51' 24''$.

13. $c = 56° 21' 35''$, $b = 140° 11' 38''$.

14. $A = 41° 31' 48''$, $B = 118° 56' 10''$.

15. $A = 156° 20' 30''$, $a = 65° 15' 45''$.

16. A ship sails from the Cape of Good Hope in lat. 33° 56′ S., lon. 18° 23′ E., and crosses the Equator in lon. 28° 18′ W. What is the least distance which she can have run in geographical miles?

17. The elevation of the North Pole is 45°, and I observe a bright star on the horizon at an angular distance of 45° from the north part of the horizon. What is its angular distance from the North Pole?

18. Given δ the sun's declination, and a his altitude when due E, what is the latitude of the place of observation?

Solve the following triangles :—

19. $a = 35° 21' 40''$, $b = 126° 25' 32''$, $c = 29° 24' 26''$.

20. $A = 118° 6' 12''$, $B = 56° 29' 14''$, $C = 87° 15' 18''$.

21. $a = 19° 54' 24''$, $b = 56° 29' 16''$, $C = 85° 31' 20''$.

22. $A = 89° 56'$, $B = 61° 28'$, $c = 45° 29'$.

23. $a = 48° 10' 18''$, $b = 64° 15' 20''$, $A = 56° 29' 10''$.

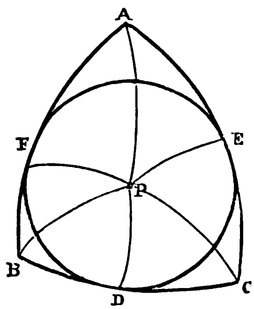
24. $A = 84° 15' 17''$, $B = 29° 13' 11''$, $a = 59° 16' 18''$.

CHAPTER V.

INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES.

Inscribed Circle.

48. *To find the radius of the small circle inscribed in a given spherical triangle.*



Let ABC be the spherical triangle.

Bisect the angles A, B, by arcs meeting in the point P. From P draw the arcs PD, PE, PF perpendicular to the sides. Then it is easily seen that

$$PD = PE = PF,$$

and that P is the pole of the small circle DEF.

Also AE = AF, CD = CE, BD = BF.

Hence $AE + CD + BF = \frac{1}{2}(a + b + c) = s$,

or $AE + CD + BD = s, \therefore AE = s - BC = s - a.$

Now, from the right-angled triangle APE, we have

$$\tan PE = \sin AE . \tan PAE = \sin(s - a)\tan\frac{A}{2};$$

$$\text{or} \quad \tan r = \sin(s - a)\tan\frac{A}{2} \quad \text{..........(1)}.$$

Hence, by Art. 20,

$$\tan r = \sin(s - a) . \sqrt{\frac{\sin(s - b)\sin(s - c)}{\sin s \sin(s - a)}},$$

$$\text{or} \quad \tan r = \sqrt{\frac{\sin(s - a)\sin(s - b)\sin(s - c)}{\sin s}} \quad \text{..........(2)}.$$

We may express the radius in terms of the angles thus:

We have again

$$\tan r = \sin(s - a)\tan\tfrac{1}{2}A = \sin\left\{\tfrac{1}{2}(b + c) - \tfrac{1}{2}a\right\}\tan\tfrac{1}{2}A \text{; or expanding,}$$

$$= \left\{\sin\tfrac{1}{2}(b + c)\sin\tfrac{1}{2}A \,.\, \cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(b + c)\sin\tfrac{1}{2}A \,.\, \sin\tfrac{1}{2}a\right\}\frac{1}{\cos\frac{1}{2}A};$$

or, by Gauss's Theorem,

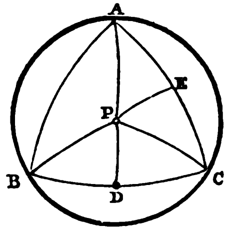
$$= \left\{\cos\tfrac{1}{2}(B - C)\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(B + C)\cos\tfrac{1}{2}a\sin\tfrac{1}{2}a\right\}\cdot\frac{1}{\cos\frac{1}{2}A}$$

$$= \frac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A}\sin a.$$

Hence (Art. 20), by substitution and reduction,

$$\tan r = \frac{\sqrt{-\cos S\cos(S - A)\cos(S - B)\cos(S - C)}}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} \quad\text{......................(3).}$$

Circumscribed Circle.



49. *To find the radius of the small circle which may be circumscribed about a given triangle.*

Let P be the pole of the circumscribed circle.

Then it may be easily shown that the arcs of great circles drawn perpendicularly from P upon the sides BC, AC also bisect those sides in D, E.

We have also ∠ PAB = ∠ PBA, ∠ PBD = ∠ PCD, and ∠ PAC = ∠ PCA.

Hence it follows that $\angle$ PCB = S − A.

We have from the right-angled triangle PCD,

$$\cos \text{PCD} = \tan \text{CD} \cot \text{PC}.$$

$$\therefore \tan \text{PC} = \frac{\tan \text{CD}}{\cos \text{PCD}}; \text{ or}$$

$$\tan \text{R} = \frac{\tan \frac{1}{2} a}{\cos (\text{S} - \text{A})} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence (Art. 21), by substitution and reduction,

$$\tan \text{R} = \sqrt{\frac{-\cos \text{S}}{\cos (\text{S} - \text{A}) \cos (\text{S} - \text{B}) \cos (\text{S} - \text{C})}} \ldots (2).$$

Again, proceeding as in the last Art., we get

$$\tan \text{R} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}} \ldots (3).$$

Cor. 1. Multiplying this last result by (2) of the last Art., we have

$$\tan \text{R} \tan r = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sin s}$$

$$= \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sin \frac{1}{2} (a + b + c)}.$$

Cor. 2. It is convenient to represent the expressions

$$\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)},$$

and $\sqrt{-\cos \text{S} \cos (\text{S} - \text{A}) \cos (\text{S} - \text{B}) \cos (\text{S} - \text{C})}$,

respectively by n and N.

Then, $$\tan r = \frac{n}{\sin s} = \frac{\text{N}}{2 \cos \frac{1}{2} \text{A} \cos \frac{1}{2} \text{B} \cos \frac{1}{2} \text{C}},$$

and $$\tan \text{R} = \frac{\cos \text{S}}{\text{N}} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{n}.$$

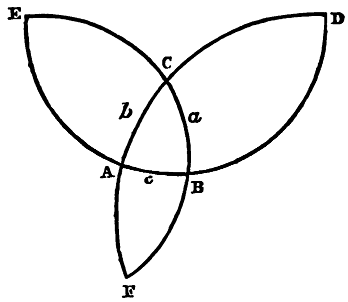
Escribed Circles.

50. *To find the radius of the small circle which touches one side of a spherical triangle and the other sides produced.*

Let ABC be the triangle, and let its sides be produced to meet in D, E, F.

Then evidently the angles of the triangle BCD are equal to A, $\pi - B$, $\pi - C$;

And the sides are a, $\pi - b$, $\pi - c$.



Now the radius of the small circle touching BC, and AB, AC produced, is the inscribed circle of the triangle BCD.

5—II.

Let r_a represent the radius of the escribed circle touching the side a; then if we suppose a, $b_{,}$, $c_{,}$, the sides of BCD, we have

$$\tan r_a = \sqrt{\frac{\sin(s_{,} - a)\sin(s_{,} - b_{,})\sin(s_{,} - c_{,})}{\sin s_{,}}},$$

But $s_{,} = \frac{1}{2}(a + \overline{\pi - b} + \overline{\pi - c})$, $\therefore \sin s_{,} = \sin(s - a)$.

And so
$$\sin(s_{,} - a) = \sin s$$
$$\sin(s_{,} - b_{,}) = \sin(s - c)$$
$$\sin(s_{,} - c_{,}) = \sin(s - b).$$

Hence, by substitution,

$$\tan r_a = \sqrt{\frac{\sin s \sin(s - b)\sin(s - c)}{\sin(s - a)}} = \frac{n}{\sin(s - a)} \quad \text{.......... (1).}$$

So
$$\tan r_b = \sqrt{\frac{\sin s \sin(s - a)\sin(s - c)}{\sin(s - b)}} = \frac{n}{\sin(s - b)} \quad \text{.......... (2),}$$

and
$$\tan r_c = \sqrt{\frac{\sin s \sin(s - a)\sin(s - b)}{\sin(s - c)}} = \frac{n}{\sin(s - c)} \quad \text{.......... (3).}$$

And similarly we express these radii in terms of the angles in the following form:

$$\tan r_a = \frac{\cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A} \sin a = \frac{N}{2 \cos \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} \quad (4).$$

Cor. If R_a, R_b, R_c represent respectively the small circles described about the triangles BCD, CAE, ABF, then we have similarly

$$\left.\begin{aligned} \tan R_a &= \frac{\tan \frac{1}{2} a}{- \cos S} = \frac{\cos (S - A)}{N} \\ &= \frac{\sin \frac{1}{2} a}{\sin \frac{1}{2} b \sin \frac{1}{2} c} \cdot \frac{1}{\sin A} \\ &= \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} \end{aligned}\right\}.$$

And similar expressions for $\tan R_b$, $\tan R_c$.

The four triangles, ABC and the three formed by producing the sides to meet in D, E, F, are called **associated triangles**, the triangle ABC being the **fundamental triangle.**

Ex. III.

Prove the following relations:

1. $\tan r \tan r_a \tan r_b \tan r_c = n^2$.
2. $\cot R \cot R_a \cot R_b \cot R_c = N^2$.
3. $\cot r \tan r_a \tan r_b \tan r_c = \sin^2 s$.
4. $\tan r \tan r_b \tan r_c \cot r_a = \sin^2 (s - a)$.
5. $\cot R \cot R_b \cot R_c \tan R_a = \cos^2 (S - A)$.
6. $\tan R \cot R_a \cot R_b \cot R_c = \cos^2 S$.
7. $\cot R \cot R_a + \cot R_b \cot R_c = \sin B \sin C$.
8. $\tan r \tan r_a + \tan r_b \tan r_c = \sin b \cdot \sin c$.
9. The sum of the products of the tangents of the radii of the circles inscribed in the associated triangles, when taken two and two together, is equal to the sum of the products of the sines of the sides of the fundamental triangle when taken *two together.*

10. A similar property belongs to the radii of the circumscribed circles of the associated triangles.

11. In any spherical triangle, if the vertical angle and the difference between the base and the other two sides be given, the radius of the inscribed circle will be *constant*, and the centre *a fixed point.*

12. Prove also the corresponding theorem for the circumscribing circle.

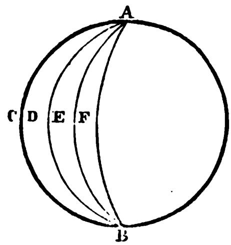
CHAPTER VI.

AREA OF A SPHERICAL TRIANGLE. SPHERICAL EXCESS.

51. *To find the area of a lune.*

Def. The portion of the surface of a sphere which is included between two great semicircles is called a **Lune.**

Let the lunes ACBD, ADBE, AEBF, have each the same angle A.



Then, since each may by superposition be made to exactly coincide with another, they are equal to each other.

Hence it easily follows that any lune, whatever the area of ACBD, will bear to the area of *any* lune ACBF, the ratio of the angle A to the angle CAB.

Instead of the lune ACBF let us take the whole sphere whose angle is 2π.

Then $$\frac{\text{area of lune whose angle is A}}{\text{area of sphere}} = \frac{A}{2\pi}.$$

If r be the radius of the sphere, then—

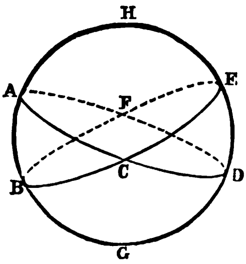
$$\textit{Surface of a sphere} = 4\pi r^2.$$

Hence, $$\textit{Area of lune} = \frac{A}{2\pi} \cdot 4\pi r^2 = 2Ar^2.$$

(The student will remember that A here is the circular measure of the angle.)

52. *Area of spherical triangle.*

Let ABC be the spherical triangle; produce the arcs forming the sides to meet again two and two. This will be the case when each has become a semicircle.



Now in the triangles AFB and DCE we have

$$\angle \text{AFB} = \angle \text{DCE}.$$

Also $\text{DC} = \text{AD} - \text{AC}$

$= \text{CAF} - \text{AC} = \text{AF},$

and $\text{EC} = \text{BE} - \text{BC}$

$= \text{CBF} - \text{BC} = \text{BF}.$

Hence the triangles AFB and DCE are equal.

Let Σ represent the area of the Δ ABC, then

$$\Sigma = \text{ABGDEH} - (\text{BGDC} + \text{AHEC} + \text{DCE})$$

$$= \text{hemisphere} - (\text{lune AGD} - \Delta\ \text{ABC}) - (\text{lune BAE} - \Delta\ \text{ABC}) - \Delta\ \text{AFB};$$

And Δ AFB = lune CAFB $-$ Δ ABC. Hence

$$\Sigma = 2\pi r^2 - (2\,\text{A}r^2 - \Sigma) - (2\,\text{B}r^2 - \Sigma) - (2\,\text{C}r^2 - \Sigma)$$

$$= 2r^2(\pi - \text{A} - \text{B} - \text{C}) + 3\Sigma.$$

$\therefore\ \Sigma = (\text{A} + \text{B} + \text{C} - \pi)\,r^2$; where A, B, C represent the circular measures of the angles of the triangle.

If A, B, C represent the numbers of degrees, we have

$$\Sigma = \frac{\text{A} + \text{B} + \text{C} - 180}{180} \cdot \pi r^2$$

Def. The excess of the sum of the angles of a triangle above two right angles is called **the spherical excess.**

Hence, if E be the number of degrees in the spherical excess, we have

$$\text{A} + \text{B} + \text{C} - 180 = \text{E},$$

And $$\Sigma = \frac{\text{E}}{180} \cdot \pi r^2.$$

COR. Since a spherical polygon may be divided into as many triangles as the polygon has sides, by drawing arcs of great circles from a point within it to the angles, we have, if n be the number of sides,

Area of polygon

$$= \text{area of } n \text{ triangles}$$

$$= (\text{sum of angles of the triangles} - n \,.\, 180°)\,\frac{\pi r^2}{180°}$$

$$= (\text{sum of angles of the polygon} - \overline{n-2} \,.\, 180°)\,\frac{\pi r^2}{180°}.$$

53. *Cagnoli's Theorem. To show that*

$$\sin \tfrac{1}{2} E = \frac{\sqrt{\sin s \sin (s-a) \sin (s-b) (s-c)}}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}$$

We have, $\sin \frac{1}{2} E$

$$= \sin \tfrac{1}{2} (A + B + C - \pi) = - \cos \tfrac{1}{2} (A + B + C)$$

$$= - \left\{\cos \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} C - \sin \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} C\right\}$$

$$= \sin \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} C.$$

Or, by Gauss's Theorem,

$$= \frac{\cos \frac{1}{2} (a-b)}{\cos \frac{1}{2} c} \cos \tfrac{1}{2} C \sin \tfrac{1}{2} C - \frac{\cos \frac{1}{2} (a+b)}{\cos \frac{1}{2} c} \sin \tfrac{1}{2} C \cos \tfrac{1}{2} C.$$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin C \text{ ; or, by Art. 20,}$$

$\sin \frac{1}{2} E$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \cdot \frac{2}{\sin a \sin b} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

$$= \frac{\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

54. *Llhuillier's Theorem. To show that*

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

$\tan \tfrac{1}{4} E$

$$= \frac{2 \sin \frac{1}{4}(A+B+C-\pi) \, . \cos \frac{1}{4}(A+B-C+\pi)}{2 \cos \frac{1}{4}(A+B+C-\pi) \, . \cos \frac{1}{4}(A+B-C+\pi)};$$

or,

$$= \frac{\sin \frac{1}{2}(A+B) - \sin \frac{1}{2}(\pi - C)}{\cos \frac{1}{2}(A+B) + \cos \frac{1}{2}(\pi - C)} = \frac{\sin \frac{1}{2}(A+B) - \cos \frac{1}{2} C}{\cos \frac{1}{2}(A+B) + \sin \frac{1}{2} C}.$$

Or, by Gauss's Theorem, after reduction,

$\tan \tfrac{1}{4} E$

$$= \frac{\cos \frac{1}{2}(a-b) - \cos \frac{1}{2} c}{\cos \frac{1}{2}(a+b) + \cos \frac{1}{2} c} \cdot \frac{\cos \frac{1}{2} C}{\sin \frac{1}{2} C},$$

$$= \frac{\sin \frac{1}{4}(b+c-a) \sin \frac{1}{4}(a+c-b)}{\cos \frac{1}{4}(a+b+c) \cos \frac{1}{4}(a+b-c)} \cot \tfrac{1}{2} C,$$

$$= \frac{\sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b)}{\cos \frac{1}{2} s \cos \frac{1}{2}(s-c)} \cdot \sqrt{\frac{\sin s \sin (s-c)}{\sin (s-a) \sin (s-b)}},$$

$$= \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

Other formulæ, involving the spherical excess, follow in the exercises upon this chapter.

Ex. IV.

Prove the following relations :

1. $\cot \tfrac{1}{2} E = \dfrac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}.$

2. $\cos \tfrac{1}{2} E = \dfrac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$

3. $\sin \tfrac{1}{2} E = \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sec \tfrac{1}{2} c \sin C.$

4. $\sin \tfrac{1}{4} E = \dfrac{\sin \frac{1}{2} s \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-c)}{\cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$

5. $\cos \frac{1}{4} E = \dfrac{\cos \frac{1}{2} s \cos \frac{1}{2} (s - a) \cos \frac{1}{2} (s - b) \cos \frac{1}{2} (s - c)}{\cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}$.

6. $\tan \frac{1}{4} E \tan \frac{1}{4} E_a \tan \frac{1}{4} E_b \tan \frac{1}{4} E_c = \tan^2 \frac{1}{2} s$, where E, E_a, E_b, E_c are respectively the spherical excesses of the associated triangles.

7. $\cot \frac{1}{4} E \cot \frac{1}{4} E_b \cot \frac{1}{4} E_c \tan \frac{1}{4} E_a = \cot^2 \frac{1}{2} (s - a)$.

8. If E', E'_a, E'_b, E'_c be the spherical excesses of the polar triangle of the associated triangles, then

$$\cot^2 \tfrac{1}{4} E' = \cot \tfrac{1}{4} E_a \cot \tfrac{1}{4} E_b \cot \tfrac{1}{4} E_c \tan \tfrac{1}{4} E,$$
$$\cot^2 \tfrac{1}{4} E'_a = \cot \tfrac{1}{4} E \cot \tfrac{1}{4} E_b \cot \tfrac{1}{4} E_c \tan \tfrac{1}{4} E_a.$$
$$\&c. = \&c.$$

9. $\tan \frac{1}{4} E \tan \frac{1}{4} E' = \tan \frac{1}{4} E_a \tan \frac{1}{4} E'_a = \tan \frac{1}{4} E_b \tan E'_b$
$= \tan \frac{1}{4} E_c \tan \frac{1}{4} E'_c$.

10. $\cot \frac{1}{2} E$
$$= \frac{\cot r_a + \cot r_b + \cot r_c - \cot r}{\tan r_a + \tan r_b + \tan r_c - \tan r} \sqrt{\tan r \tan r_a \tan r_b \tan r_c}.$$

11. Prove similar formulæ for E_a, E_b, E_c.

12. If E be the excess of the triangle each of whose angles is 120°, and E′ that of the polar triangle, then

$$\tan \tfrac{1}{3} E : \tan \tfrac{1}{3} E' :: 27 + 8\sqrt{6} : 5.$$

CHAPTER VII.

POLYHEDRONS.

55. By a **Polyhedron** is meant a solid bounded by plane rectilineal figures; and it is called **regular** when the bounding surfaces are similar and equal. We have space for only a few of the more elementary propositions.

56. *In any polyhedron, if* F *be the number of faces,* S *the number of solid angles,* E *the number of edges, then*

$$S + F = E + 2.$$

Take any point within the polyhedron, and draw to this point lines from each of the solid angles. Describe a sphere with this point as centre, and let its surface be cut by these lines. Draw arcs of great circles connecting the points where the lines meet the surface.

Then the surface of the sphere will be divided into as many spherical polygons as there are faces in the solid.

Now, Art. 52, Cor.,

Area of one of these polygons

$$= \left\{\text{sum of the angles of the polygon} - \overline{n-2}\,.\,180^\circ\right\}\frac{\pi r^2}{180^\circ}.$$

Hence, total area of all the polygons

$$= \left\{\text{sum of angles of all the polygons} - (2\,\text{E} - 2\,\text{F})\,180^\circ\right\}\frac{\pi r^2}{180^\circ}.$$

$$= \left\{\text{S}\,.\,360^\circ - (2\,\text{E} - 2\,\text{F})\,180^\circ\right\}\frac{\pi r^2}{180^\circ} = 2\,(\text{S} - \text{E} + \text{F})\,\pi r^2.$$

But the total area of all the polygons

$$= \text{surface of the sphere} = 4\,\pi r^2.$$

Hence $\qquad 2\,(\text{S} - \text{E} + \text{F})\,\pi r^2 = 4\,\pi r^2,$

or $\qquad \text{S} - \text{E} + \text{F} = 2.$

$$\therefore \text{S} + \text{F} = \text{E} + 2.$$

Cor. If m be the number of sides in each face, and n the number of plane angles in every solid angle, then in any regular polyhedron

$$2\,\text{E} = m\text{F} = n\text{S}.$$

57. *The sum of all the plane angles which form the solid angles of any polyhedron* $= (\text{S} - 2)\,360^\circ$.

Now, Sum of all the plane angles

= sum of all the interior angles of the faces...... (1).

Now, if m be the number of sides in any one face, then (*Euc.* I., 32) Sum of the interior angles of that face $= (m - 2)\,180^\circ$, And the whole number of sides in the faces is 2 E.

Hence, Total sum of the interior angles of all the faces = (2 E − 2 F) 180°.

Hence from (1),

$$\text{Sum of all the plane angles} = (2\,E - 2\,F)\,180° = (E - F)\,360°, \text{ or (Art. 56)}, = (S - 2)\,360°.$$

58. *In any regular polyhedron,*

$$S = \frac{4\,m}{2\,(m + n) - mn},\ E = \frac{2\,mn}{2\,(m + n) - mn},$$

$$F = \frac{4\,n}{2\,(m + n) - mn}.$$

We have (Art. 56), $S + F = E + 2$,

and $$2\,E = mF = nS.$$

The above values of S, E, F at once follow from these equations.

59. *There are only five regular polyhedrons.*

The expressions for S, E, F just found must always be positive integers.

Hence $2\,(m + n) - mn > 0$, or $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$.

Now n, the number of plane angles in a solid angle, cannot be less than 3.

Hence $\frac{1}{m} > \frac{1}{2} - \frac{1}{3}$, or $> \frac{1}{6}$.

And hence neither m nor n can be less than 3 or greater than 6.

And on trial, the only admissible values of m and n are *3, 4, 5.*

If we substitute these values in the above values of S, E, F, we find the only pairs of values are as follows:

m	n	S	E	F	Name of regular Polyhedron.
3	3	4	6	4	Tetrahedron or Regular Pyramid.
4	3	8	12	6	Hexahedron or Cube.
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

COR. Since it is not necessary for the relations

$$2\,E = mF = nS,$$

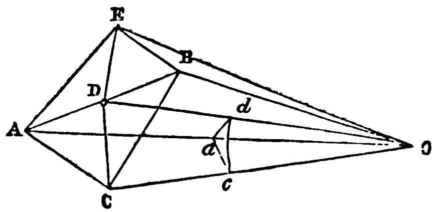
and

$$S + F = E + 2,$$

that the faces should be equilateral and equiangular, and all equal, it follows that we may conclude the following more general proposition.

There can only be five solid bodies each of which has all its faces of the same number of sides, and all its solid angles formed of the same number of plane angles.

60. *To find the inclination of two contiguous faces of a regular polyhedron.*



Let AB be the common edge of two faces; C, E, the centres of the faces. Bisect AB in D, and join DC, DE. Then evidently CD, CE are perpendicular to AB.

$\therefore \angle$ CDE = I, the inclination of the faces.

Draw CO, EO at right angles to CD, DE respectively, and both in the plane CDE. Join OA, OB, OD.

Let a sphere be described with O as centre, so as to meet OA, OC, OD in a, c, d respectively, and form the spherical triangle acd.

Now AB is perpendicular to CD, CE, and it is therefore perpendicular to the plane CDE. Hence the plane AOB in which AB lies is also perpendicular to the plane CDE or the plane COE.

But ad lies in the plane AOB, and cd lies in the plane COE.

Hence, the angle adc of the spherical triangle acd is a right angle.

Now, if m be the number of sides in each face of the polyhedron, and n the number of plane angles forming each solid angle,

$$\angle acd = \angle \text{ACD} = \tfrac{1}{2}\,\text{ACB} = \tfrac{1}{2}\cdot\frac{360^\circ}{m} = \frac{180^\circ}{m},$$

And $\angle\, cad = \frac{1}{2}$ one of the equal angles formed upon the sphere round the point a.

$$= \tfrac{1}{2}\cdot\frac{360^\circ}{n} = \frac{180^\circ}{n}.$$

By Napier's Rules, we have from the right-angled triangle acd

$$\cos cad = \cos cd \sin acd.$$

$$\therefore \cos\frac{180^\circ}{n} = \cos cd \sin\frac{180^\circ}{m}\ldots\ldots\ldots\ldots\ldots(1).$$

But $cd = \angle \text{DOC} = \frac{1}{2}\,\text{EOC} = \frac{1}{2}(180^\circ - \text{CDE}) = 90^\circ - \frac{1}{2}\text{I}$

$$\therefore \cos cd = \sin\tfrac{1}{2}\text{I}.$$

Hence, from (1),

$$\cos\frac{180°}{n} = \sin\tfrac{1}{2}\text{I}\sin\frac{180°}{m}$$

$$\therefore \sin\tfrac{1}{2}\text{I} = \frac{\cos\frac{180°}{n}}{\sin\frac{180°}{m}}.$$

Cor. The following results are easily obtained:

(1.) For the *Tetrahedron*, $\sin\frac{1}{2}\text{I} = \dfrac{1}{\sqrt{3}}$; $\therefore \cos\text{I} = \frac{1}{3}$.

(2.) *Hexahedron or Cube*, $\sin\frac{1}{2}\text{I} = \dfrac{1}{\sqrt{2}}$;

$$\therefore \cos\text{I} = 1, \text{ or I} = 90°.$$

(3.) *Octahedron*, $\sin\frac{1}{2}\text{I} = \sqrt{\frac{2}{3}}$, $\therefore \cos\text{I} = -\frac{1}{3}$.

(4.) *Dodecahedron*, $\sin\frac{1}{2}\text{I} = \dfrac{2}{\sqrt{10 - 2\sqrt{5}}}$;

$$\therefore \cos\text{I} = -\tfrac{1}{5}\sqrt{5}.$$

(5.) *Icosahedron*, $\sin\frac{1}{2}\text{I} = \dfrac{\sqrt{5}+1}{2\sqrt{3}}$;

$$\therefore \cos\text{I} = -\tfrac{1}{3}\sqrt{5}.$$

61. *To find the radii of the spheres inscribed in and described about a regular polyhedron.*

In the last diagram, since OC, OE are at right angles to the planes ABC, ABE, they are equal to each other.

Hence O is the centre of the inscribed and circumscribed spheres.

Let $\quad$ AB $= a$, OA $=$ R, OC $= r$.

From the right-angled triangle *acd*, we have, by Napier's *Rules*,

$$\cos ac = \cot acd \cot cad.$$

But $\cos ac = \cos \text{AOC} = \dfrac{\text{OC}}{\text{OA}} = \dfrac{r}{\text{R}}$.

Hence (Art. 60), $\dfrac{r}{\text{R}} = \cot \dfrac{180^\circ}{m} \cot \dfrac{180^\circ}{n}$ .... (1).

Again,

$$\text{OC} = \text{CD} \tan \text{CDO}; \text{ or } r = \text{CD} \tan \frac{\text{I}}{2},$$

and

$$\text{CD} = \text{AD} \cot \text{ACD} = \frac{a}{2} \cot \frac{180^\circ}{m}.$$

Hence

$$r = \frac{a}{2} \cot \frac{180^\circ}{m} \tan \frac{\text{I}}{2}.$$

And therefore from (1),

$$\text{R} = \frac{a}{2} \tan \frac{180^\circ}{n} \tan \frac{\text{I}}{2}.$$

62. *To find the surface and volume of a regular polyhedron.*

Now as m is the number of sides in each face, we have:

$$\text{Area of each face} = \frac{ma^2}{4} \cot \frac{180^\circ}{m};$$

$$\text{Hence, whole surface} = \frac{\text{F}ma^2}{4} \cot \frac{180^\circ}{m}.$$

Again, Volume of each pyramid, having O for vertex and a face for base,

$$= \tfrac{1}{3} \text{ corresponding prism}$$

$$= \tfrac{1}{3} r \,.\, \frac{ma^2}{4} \cot \frac{180^\circ}{m}$$

$$= \frac{mra^2}{12} \cot \frac{180^\circ}{m}.$$

$$\text{Hence whole volume} = \frac{\text{F}mra^2}{12} \cot \frac{180^\circ}{m}.$$

Ex. V.

1. Show how to fit together a number of equal regular tetrahedrons, and equal regular octahedrons to form a solid mass, without interstices or gaps.

2. A pyramid stands on a square base, and its altitude is half the side of the base. What are the dihedral angles at the vertex?

3. Similar polyhedrons are to one another in the triplicate ratio of their homologous edges, and their convex surfaces are in the duplicate ratio of their edges.

4. Show by a plane construction how to find the angle contained by any two of the planes AOB, BOC, COA, which form a solid angle at O.

5. When two solid angles are contained each by three plane angles which are respectively equal, the dihedral angles of the one will be equal to the dihedral angles of the other, each to each.

6. In a given dodecahedron inscribe a cube.

7. If R, r be the radii of the circumscribed and inscribed spheres, show that:

(1.) For the tetrahedron, $R = 3r$.

(2.) For the cube or octahedron, $R = \sqrt{3} \cdot r$.

8. In the regular icosahedron, the distance of the regular pentagon, which passes through five of the solid angles, from the centre is one-half the radius of the sphere circumscribing the pentagon.

9. In a pyramid on a square base each edge meeting the vertex is twice the length of the base. Find the inclination of two contiguous faces.

10. If α, β, γ be the angles which a line passing through an angle makes with each of the edges of a cube, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

11. If A, B, C, D, be four points in a plane, and $A_{,}$, $B_{,}$, $C_{,}$, $D_{,}$ their projections on any other plane, then the volumes of $ABCD_{,}$ and $A_{,}B_{,}C_{,}D$ are equal to each other.

12. If a, b, c be three edges of a tetrahedron, and d, e, f the three opposite edges, then, if I be the angle between a and d,

$$\cos I = \frac{(b^2 + e^2) - (c^2 + f^2)}{2\,ad}.$$

ANSWERS.

ALGEBRA.

I.—Page 13.

1. $\frac{8}{3}\sqrt{165}$. 2. $cx^2 + bx + a = 0$.

3. $(x + \gamma)^2 - p(x + \gamma) + q = 0$. 4. 4.

5. $a(ac' - a'c)^2 + b(a'b - ab')(ac' - a'c) + c(a'b - ab')^2 = 0$.

7. $m \pm n \pm p = 0$.

8. $x^2 - (p + \sqrt{p^2 - 4q})\,x + p\sqrt{p^2 - 4q} = 0$.

12. $(3x + 4y + 5)(2x + 3y - 2)$.

II.—Page 17.

1. 27. 2. 1. 3. 3, 5.

4. $\frac{e_{\prime}d_{\prime}^2}{ed^2}$. 5. 156 . 8. 6. $s = 16t^2$.

7. 6¾ tons 8. $m^2p : 20n^2q$.

9. $\frac{mc}{m+1} \cdot \frac{n^2}{a^2}$, $\frac{c}{m+1} \cdot \left(\frac{n}{b}\right)^{\frac{2}{3}}$. 11. ·067.

12. 7·231 ; 23·717.

III.—Page 26.

1. 10. 2. $\frac{|\underline{10}}{|\underline{2}\;|\underline{2}\;|\underline{3}}$.

3. $\frac{|\underline{12}}{|\underline{2}\;|\underline{2}}$, $\frac{|\underline{7}}{|\underline{3}\;|\underline{2}}$. 5. $\frac{|\underline{9}}{(|\underline{2})^3\;|\underline{3}}$.

6. $\frac{|\underline{9}}{2}$, if no person is to have the same neighbours twice.

7. 12. 8. 7. 12. m^n. 16. 10.

17. $\frac{3\,|\underline{100}}{|\underline{12}\;|\underline{88}}$.

IV.—Page 31.

1. 38, 288. 2. $17\frac{1}{4}$. 3. -18.

4. 150. 5. 50^2. 6. $13a$.

7. $10x$. 8. $84\frac{1}{3}$. 9. 5, 8, 11, ...

10. $229\frac{1}{5}$. 11. 2.

13. $a = \frac{P(1-q) - Q(1-p)}{p-q}$, $d = \frac{P-Q}{p-q}$.

20. $(3n-7)\frac{n}{2}$. 21. 3, 5, 7. 22. 1, 2, 3, 4.

23. An A. P. where p, q, r are the P^{th}, Q^{th}, R^{th} terms respectively. The common differences are the reciprocals of each other.

24. 9 or $-10\frac{2}{3}$.

V.—Page 36.

1. 96. 2. $\frac{7}{125}$. 3. $60\frac{2}{3}$. 4. $\frac{28}{15}\left\{\left(\frac{5}{2}\right)^{12} - 1\right\}$

5. $\frac{9}{2}\left\{1 - \left(-\frac{1}{3}\right)^n\right\}$. 6. $1 - \frac{1}{2^{10}}$ 7. 2.

8. $\frac{3}{8}$. 9. $5\frac{5}{19}$. 10. $7\frac{1}{9}$. 11 $\frac{6}{11}$

12. $\frac{a^3b^2}{ab+1}$. 13. $\frac{(a-b)(a+b)^2}{a+b-1}$. 14. $\frac{1}{4}(11 + 5\sqrt{5})$.

15. $2\frac{1}{4}$. 16. $2\frac{65}{72}$. 17. $\frac{1}{4}(\sqrt{3} + 1)^3$.

18. $\frac{5}{16}$. 19. $a = -\frac{513}{341}$. 20. $a^{\frac{1}{3}}b^{\frac{2}{3}}$.

23. $a = \frac{2\,s_1\,s_2}{s_1^{\,2} + s_2}, r = \frac{s_1^{\,2} - s_2}{s_1^{\,2} + s_2}$. 35. $2\,(2^n - 1) + \frac{1}{2}\,n\,(n+1)$.

VI.—Page 41.

1. $\frac{8}{61}$, $\frac{4}{57}$, $\frac{8}{167}$, &c. 2. $\frac{21}{11}$, $\frac{43}{23}$, $\frac{7}{4}$, &c.

3. − 6, ∞, 6, 3 and 1, $\frac{6}{7}$, $\frac{3}{4}$, $\frac{2}{3}$.

5. The common difference of the A. P. is $\frac{Q - P}{PQ\,(p - q)}$.

6. $\frac{2\,ac - bc - ab}{2\,b - a - c}$.

10. $\frac{20}{123}\,(4 - \sqrt{7})$, $\frac{15}{41}$, $\frac{20}{123}\,(4 + \sqrt{7})$.

VII.—Page 63.

7. $126\,a^5b^4$, $126\,a^4b^5$. 8. $252\,(15\,x)^5$. 9. $-20\,a^6b^6$.

10. $2^n \cdot \frac{1 \cdot 3 \cdot 5 \ldots \ldots (2\,n - 1)}{\lfloor n}$. 11. $\frac{135}{8}\,x^2$, $\frac{45}{32}\,x^3$.

12. $-35 \cdot 4^4 \cdot 3^3\,a^2b$, $35 \cdot 4^3 \cdot 3^4 \cdot a^{\frac{3}{2}}b^{\frac{4}{3}}$.

13. $\frac{1}{3^{\,r+\frac{1}{5}}} \cdot \frac{1 \cdot 6 \cdot 11 \ldots \ldots (5\,r - 4)}{\lfloor r} \cdot (5\,x)^r$.

14. $\frac{\lfloor n}{\lfloor r \;\lfloor n - r}\,a^{n-r}b^rx^{n-2r}$. 15. $a^{-\frac{1}{p}(r+1)}b^{\frac{r}{p}}$.

21. 1st term. 22. 1st term.

VIII.—Page 72.

1. $\frac{1}{2}\,y + \frac{5}{8}\,y^2 + \frac{25}{16}\,y^3 +$ &c.

2. $1 + x - 2\,x^3 - 5\,x^4 -$ &c.

3. $1 + x + \frac{1}{3}\,x^2 - x^3 +$ &c.

4. $5\,x + 5\,x^2 + 35\,x^3 +$ &c. 5. $\frac{1}{6}\,n\,(n+1)\,(2\,n+1)$.

7. $\dfrac{3}{x+3} - \dfrac{2}{x+5}$.   8. $\frac{1}{14}\left\{\dfrac{1}{x^2-6} - \dfrac{1}{x^2+8}\right\}$.

9. $\dfrac{1}{2(x+1)} - \dfrac{1}{x+2} + \dfrac{1}{2(x+3)}$.

10. $\dfrac{1}{x} - \dfrac{1}{(x+1)^2} - \dfrac{2}{x+1} + \dfrac{x}{1+x+x^2}$.

11. $\dfrac{1}{3x-1} + \dfrac{2}{x-1} - \dfrac{1}{x}$.

12. $\dfrac{1}{1+x} - \dfrac{x-2}{1-x+x^2}$.

14. $\dfrac{1}{(a-b)(a-c)(x+a)} + \dfrac{1}{(b-a)(b-c)(x+b)}$
$+ \dfrac{1}{(c-a)(c-b)(x+c)}$.

15. $\dfrac{a}{(a-b)(c-a)(x+a)} + \dfrac{b}{(b-c)(a-b)(x+b)}$
$+ \dfrac{c}{(c-a)(b-c)(x+c)}$.

16. $\dfrac{a^2}{(a-b)(a-c)(x+a)} + \dfrac{b^2}{(b-c)(b-a)(x+b)}$
$+ \dfrac{c^2}{(c-a)(c-b)(x+c)}$.

17. $\frac{1}{7}\left\{\dfrac{1}{x+3} - \dfrac{x}{x^2+3x+7}\right\}$.

18. $\dfrac{a^2-ma+n}{(a-b)(a-c)(x+a)} + \dfrac{b^2-mb+n}{(b-a)(b-c)(x+b)}$
$+ \dfrac{c^2-mc+n}{(c-a)(c-b)(x+c)}$.

X.—Page 90.

1. 16.   2. 111.   3. $\frac{5715}{131075}$.

4. -84600. 5. $12\,a_0^2a_1a_3 + 6\,a_0^2a_2^2 + 12\,a_0a_1^2a_2 + a_1^4$.

6. $\frac{19}{27}$. 7. 420. 8. 10.

9. 840. 10. $6a_0^2a_3^2 + 24\,a_0a_1a_2a_3 + 4\,a_1^3a_3$.

11. 0. 12. $\frac{16 \,.\, 15 \,.\, 14 \,.\, 13}{1 \,.\, 2 \,.\, 3 \,.\, 4}$.

13. $1 + x + \frac{2}{3}x^2 - \frac{2}{3}x^3 - \frac{31}{9}x^4$.

14. $1 + \frac{7}{2}x^2 + \frac{1}{2}x^3 - \frac{49}{8}x^4$.

XI.—Page 99.

1. £350 $(1{\cdot}035)^{12}$. 2. $n = \frac{\log 2}{\log 1{\cdot}05}$.

3. £1250 $\left\{(1{\cdot}04)^{10} - 1\right\}$. 4. $\frac{150{\cdot}76}{1 - (1{\cdot}04)^{-35}}$.

5. $6\,\frac{69}{116}$ months. 6. $n = \frac{\log 5}{\log 1{\cdot}06}$.

7. $6\frac{1}{4}$. 8. See Ex. 2, page 99.

9. $n = 20 \,.\, \frac{\log 6}{\log e}$, reckoning 5 per cent. per annum.

10. $96\,(1{\cdot}06)^5$. 11. $\frac{drR^a}{R^c - 1}$.

12. $\frac{\log a - \log (a - br)}{\log (1 + r)}$.

XII.—Page 111.

1. $2 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{19}$. 5. See Art. 72.

12. $x = 7, 12, 17$, &c.
$y = 2, 5, 8$, &c.

XIII.—Page 121.

1. $3 + \frac{1}{6+}\,\frac{1}{6+\ldots\ldots}$.   2. $4 + \frac{1}{8+}\,\frac{1}{8+\ldots\ldots}$.

3. $4 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+\ldots\ldots}$.   4. $5 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+\ldots\ldots}$.

5. $1 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+\ldots\ldots}$.   6. $2 + \frac{1}{1+}\,\frac{1}{8+\ldots\ldots}$.

7. $3 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{10+\ldots\ldots}$.   8. $3 + \frac{1}{1+}\,\frac{1}{2+\ldots\ldots}$.

9. $2 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+\ldots\ldots}$. $3 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+\ldots\ldots}$.

10. $1 + \frac{1}{1+}\,\frac{1}{12+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+\ldots\ldots}$.

11. $2 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+\ldots\ldots}$, $\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+\ldots\ldots}$.

12. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{1+\ldots\ldots}$, $-\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+\ldots\ldots}$.

13. $5 + \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+\ldots\ldots}$,

$1 + \frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+\ldots\ldots}$.

14. $2a \pm \left(a + \frac{1}{2a+\ldots\ldots}\right)$.   15. $\sqrt{26}$.

19. $\frac{9+\sqrt{15}}{7}$.

20. $2, \frac{5}{2}, \frac{7}{3}, \frac{12}{5}$, &c.; $1, 2, \frac{25}{13}, \frac{27}{14}$; $2, \frac{7}{3}, \frac{9}{4}, \frac{16}{7}$.

23. No.

XV. Page 146.

1. $\frac{1+5x}{1-3x+2x^2}$.   2. $\frac{1+2x}{1-x+2x^2}$.

3. $\dfrac{2+5x}{1-4x+7x^2}$. 4. $\dfrac{3-7x}{1-6x+3x^2}$.

5. $\dfrac{9-2x}{1+x+7x^2}$. 6. $\dfrac{6-2x+x^2}{1-3x+4x^2-x^3}$.

7. $\Sigma = \dfrac{3-4x}{(1-2x)(1+5x)}$. 8. $\Sigma = \dfrac{6-3x}{(1+7x)(1-2x)}$.

9. $\Sigma = \dfrac{1-2x}{(1+x)(1+2x)(1-3x)}$.

10. $\Sigma = \dfrac{1-4x}{(1-3x)(1-x)(1+x)}$.

11. $\Sigma = \dfrac{2-3x}{(1-x)(1+5x)}$. 12. $\Sigma = \dfrac{5+3x}{(1-3x)(1+2x)}$.

13. $\Sigma = \dfrac{3+5x+3x^2}{(1-x)^3}$.

14. $\Sigma = \dfrac{4+5x+x^2}{(1-x)(1-2x)(1-3x)}$.

15. $1-3x+3x^2-x^3$. 16. $\dfrac{a-(a-b)x}{(1-x)^2}$.

19. $\frac{1}{4}n(n+1)(n+2)(n+3)+n(n+1)(n+2)+\frac{3}{2}n(n+1)$.

20. $\frac{1}{4}\{(x+n)(x+n+1)(x+n+2)(x+n+3)-x(x+1)(x+2)(x+3)\}$

21. $\frac{1}{4}\left\{\dfrac{23}{120}-\dfrac{1}{(n+2)(n+4)}-\dfrac{1}{(n+3)(n+5)}\right\}$.

22. $\frac{5}{6}-\dfrac{3n+5}{(n+2)(n+3)}$. 23. $\dfrac{n}{3}(n^2+2)$.

24. $\dfrac{1}{2b}\left\{\dfrac{1}{(a+b)(a+2b)}-\dfrac{1}{(a+\overline{n+1}\,.\,b)(a+\overline{n+2}\,.\,b)}\right\}$.

25. $\dfrac{n}{3}(4n^2-1)$. 26. $n^2(2n^2-1)$.

27. $1-\dfrac{1}{\lfloor n+1}$.

28. $1 + \dfrac{4}{\lfloor n+2} - \dfrac{3}{\lfloor n+1}$. **29.** $\dfrac{1}{n}$.

XVI.—Page 156.

1. 3010, 1443. **2.** 744. **3.** 2672.
4. 9. **5.** 12. **6.** 7.
7. 117·7. **8.** *e* 41. **9.** *e* 25·33.
10. 15. **11.** 222. **12.** $\frac{32}{55}$.
16. 3^7, 1 in one scale, and 3^6, 3^5, 3^4 in the other.
19. $p_1^2 = 4 p_0 p_2$.

XVII.—Page 171.

16. (i.) one; (ii.) equal to G.C.M.; (iii.) may be reduced to (i.) and (ii.).

Miscellaneous Examples.—Page 179.

3. $\frac{1}{8}(9 \pm \sqrt{833})$. **6.** Transpose, cube, &c.
7. 4, 9. **10.** According as it is beyond or between 3 and 1; $2\sqrt{2} - 3$.
19. $(1 + a_0)(1 + a_1) \ldots . (1 + a_n)$.
26. $\dfrac{a^2 - ma + n}{(b-a)(c-a)} \cdot \dfrac{1}{x - a} + \&c.$
28. The rationalized expression is $a^5 - b^6$; $5 - 2\sqrt{3}$.
31. 12789·8 sq. met. **35.** $\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{9+}\,\dfrac{1}{2+\ldots\ldots}$.
36. 301·9. **37.** $\dfrac{a}{x}$.
43. $\dfrac{B_{2n}}{\lfloor 2n}$, where B_1, B_2, &c., have the following relation:

$$rB_1 + \frac{r(r-1)}{1 \,.\, 2} B_2 + \ldots . + \frac{r(r-1)}{1 \,.\, 2} B_{r-2} + rB_{r-2} = 0.$$

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45. $x = y + \frac{1}{3} y^3 + \frac{1}{5} y^5 + \frac{7}{15} y^7 + \&c.$

46. $x = 8, y = 6.$ 49. $\frac{1}{(a - 1)(1 + x)}.$

PLANE TRIGONOMETRY.

I.—Page 200.

8. $\sin 7°\frac{1}{2} = \sqrt{\frac{2\sqrt{2} - \sqrt{3} - 1}{4\sqrt{2}}}, \sin 22°\frac{1}{2} = \sqrt{\frac{2 - \sqrt{2}}{2}},$

$\sin 37°\frac{1}{2} = \sqrt{\frac{2\sqrt{2} - \sqrt{3} + 1}{4\sqrt{2}}}.$

12. $\cos^{-1}\frac{x}{a} - a = \cos^{-1}\frac{y}{b} - \beta.$

II.—Page 218.

1. ·01745, ·015705, ·000291, ·00000485.

2. 27° 15′; 10° 15′ 3. 78·54.

9. $2\theta = n\pi + (-1)^n \frac{\pi}{4}.$ 10. $\theta = 2n\pi \pm \frac{\pi}{3}, \theta = \frac{1}{2} n\pi.$

11. $4\theta = 2n\pi \pm \frac{\pi}{3}, \theta = n\pi.$

12. $(m \pm n)\theta = (4r \pm 1)\frac{\pi}{2}.$

13. $\theta = (6n \pm 1)\frac{\pi}{3}$, or $(4n \pm 1)\frac{\pi}{2}.$

14. $5\theta = n\pi \mp 2\theta.$

15. $\tan\frac{1}{2}(n\pi + \theta)$ has different values according as n is even or odd.

III.—Page 232.

1. Put $b = a \tan^2 \theta$, then $a + b = a \sec^2 \theta$, &c.

2. $5\theta = 2n\pi \pm 2\theta$.

IV.—Page 244.

1. $\tan \theta = \dfrac{a \sin \alpha \sin (\beta - \gamma) - b \sin \gamma \sin (\alpha - \beta)}{a \cos \alpha \sin (\beta - \gamma) - b \cos \gamma \sin (\alpha - \beta)}$.

2. $\frac{1}{2}(a + l \cot \theta)$.

3. See Ex. 3, page 243. 4. $\dfrac{\cot \alpha_2 - \cot \beta_2}{\cot \alpha_1 - \cot \beta_1} . h$.

5. $\left\{ \dfrac{v^2 (at_2^2 - bt_1^2)}{a - b} + ab \right\}^{\frac{1}{2}}$.

6. $h = \dfrac{l}{\sin \gamma} \sqrt{\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \cos \gamma}$.

8. $h^2 = a^2 - 2ab$.

9. $\dfrac{a}{\sqrt{2}} \cdot \dfrac{\sin \beta + \cos \beta}{\sin (\beta - \alpha)}$, $\dfrac{a}{\sqrt{2}} \cdot \dfrac{\sin \alpha + \cos \alpha}{\sin (\beta - \alpha)}$.

11. x is found from the equation

$$\frac{a^2}{x^2} + \frac{a}{x} = \cot \phi (\cot \theta - \cot \phi).$$

12. $\dfrac{a}{\sqrt{\operatorname{cosec}^2 \alpha + \operatorname{cosec}^2 \beta - 2 \operatorname{cosec} \alpha \operatorname{cosec} \beta \cos \gamma}}$.

V.—Page 275.

1. $\cos \dfrac{2m\pi}{n} + \sqrt{-1} . \sin \dfrac{2m\pi}{n}$.

2. $\cos \dfrac{(2m + 1)\pi}{n} + \sqrt{-1} \sin \dfrac{(2m + 1)\pi}{n}$, where m may have any value from 0 to $n - 1$.

18. -1, $\cos \dfrac{\pi}{3} \pm \sqrt{-1} \sin \dfrac{\pi}{3}$, $\cos \dfrac{2\pi}{3} \pm \sqrt{-1} \sin \dfrac{2\pi}{3}$.

SPHERICAL TRIGONOMETRY.

II.—Page 350.

10. $A = 71° 47' 45''$, $b = 138° 42' 35''$, $c = 110° 7' 23''$.
11. $A = 126° 58' 51''$, $B = 82° 27' 22''$, $c = 95° 43' 24''$.
12. $a = 158° 10' 3''$, $b = 41° 38' 22''$, $A = 108° 16' 32''$.
13. $A = 123° 40' 40''$, $B = 50° 15' 41''$, $a = 136° 8' 50''$.
14. $a = 31° 11' 47''$, $b = 136° 51' 56''$, $c = 128° 37' 32''$.
15. Impossible.
16. $41° 20' 44''$.
17. $60°$.
18. $\sin \lambda = \frac{\sin \delta}{\sin a}$.
19. Impossible.
20. $a = 122° 16' 6''$.
21. $A = 25° 39' 37''$, $B = 79° 28' 39''$.
22. $a = 64° 46' 27''$, $b = 52° 37' 57''$.
23. $\sin b \cdot \sin A > \sin a$.
24. $b = 24° 56' 40''$.

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2 Physical Map of Europe—shewing Mountains and Rivers, High and Low Lands.
3 Physical Map of Asia—shewing as above.
4 Physical Map of Africa—shewing as above.
5 Physical Map of North America—shewing as above.
6 Physical Map of South America—shewing as above.
7 Physical Map of the British Isles—shewing as above.
8 Various Sections across the Continents.
9 Various Sections in the Oceans, Diagram of Suez Canal, &c.
10 Maps of the World—shewing the Mean Temperature of the Air—March to August.
11 Maps of the World—shewing the Mean Temperature of the Air—September to February.
12 Map of the World—shewing the Distribution of Constant, Periodical, and Variable Winds; Limits of Trade Winds, &c.
13 Map of the World—shewing the proportionate distribution of *Rain*, &c.
14 Map of the World—illustrating Earthquakes and the distribution of Volcanoes.
15 Map of the World—shewing the Ocean Currents and Basins, and principal River Systems.
16 Map of the World—shewing the Geographical Distribution of Birds and Fishes.
17 Map of the World—shewing Geographical Distribution of Quadrupeds and Reptiles.
18 Map of the World—shewing the Geographical Distribution of Plants.
19 Map of the World—shewing the Distribution of Mankind according to Races.
20 Geological Map of the British Isles.