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PURE MATHEMATICS,

INCLUDING

ARITHMETIC, ALGEBRA, GEOMETRY,

AND

PLANE TRIGONOMETRY.

BY

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TO THE REVEREND

WILLIAM FRY, M.A.,

HONORARY CANON OF PETERBOROUGH,

AND SECRETARY OF THE

LEICESTER ARCHIDIACONAL BOARD OF EDUCATION,

This Work is Offered,

AS A TRIBUTE OF RESPECT FROM ONE OF HIS

OLD PUPILS.

PREFACE.

THE special object of the present work is to meet the requirements of the Science and Art Department's Examinations in the first three stages of Pure Mathematics as set down in the Syllabus of the Science Directory. This will account for the arrangement of the subject matter. I hope, however, that it will be found not unsuitable as a general class-book in Elementary Mathematics.

In the Arithmetical Section my object has been to deduce the rules from first principles, avoiding as much as possible algebraical considerations.

The Geometry consists simply of the first three books of Euclid's Elements with exercises, the only point calling for remark being the marginal notes. I have found it useful in class teaching to put down upon a blackboard the chief steps of the proposition—just those points, in fact, which it is necessary to retain in the memory; and to encourage the pupil to depend upon himself for supplying the connecting links. This skeleton, as it were, of the proposition is placed in the margin, with the hope that it will be specially appreciated by many students of the industrial classes to whom the language of Euclid is ordinarily an insuperable barrier. Particular reference is here made to such of those classes as have grown up to almost manhood without any mathematical training.

In the Algebraical Section of Stage I., the more difficult examples are set particularly for the exercise of those students who take up the work of Stage II. without having previously used this book for Stage I.

The amount of Plane Trigonometry included in this volume is small, extending only to the solution of triangles and the simpler cases of heights and distances. There is, however, sufficient to fully cover the requirements of Stage II. of the Government Syllabus. The treatment of the higher parts of Algebra and Plane Trigonometry, as well as that of Spherical Trigonometry, is reserved for a second volume.

E. A.

LEICESTER, *November, 1873.*

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MATHEMATICS.

FIRST STAGE.

SECTION I.

ARITHMETIC.

CHAPTER I.

THE FUNDAMENTAL PRINCIPLES AND RULES APPLIED TO WHOLE NUMBERS AND DECIMAL FRACTIONS.

Notation and Numeration.

1. We learn from elementary books on Arithmetic, that figures have a local as well as an intrinsic value, and that the local value of a figure increases tenfold, or diminishes tenfold, according as its position is changed from right to left, or from left to right. Thus, commencing with the right hand figure of an ordinary number, the respective figures of the number stand for units, tens, hundreds, thousands, &c.; or, beginning with the left hand figure, which, we will suppose, stands for thousands, the respective figures represent thousands, hundreds, tens, units. Let us carry this principle a little further. Take the figures 68754, and suppose that 7 represents 7 units; the question then arises as to the number represented by 68754. Now, as 7 is the units' figure, we have evidently, by the above principle, 6 hundreds, 8 tens, 7 units; and further, remembering that the local value of a figure decreases tenfold for every remove to the right, the 5, on our supposition, must represent 5 tenths, and the 4 must represent 4 hundredths. Let us, as is usual in numbers thus represented, mark the units' figure by placing a dot to the right of it. Thus, 357·2605 will then represent 3 hundreds, 5 tens, 7 units, 2 tenths, 6 hundredths, 5 ten-thousandths; and to take one other example, ·3065, where the units' figure, though not expressed, is actually 0, will represent 3 tenths, 6 thousandths, 5 ten-thousandths. The dot is called the decimal point, and the digits to the right are called decimals, because they represent portions of the unit obtained by cutting it up into a number of equal parts, which is always some *power* of 10. It may be remarked, that 10 is called the first power of 10; 100, or 10 × 10, the second power, sometimes written 10^2; 1000, or 10 × 10 × 10, the third power, written 10^3, and so on.

To make the subject clear, let us see what the decimals, ·237, ·2370, ·0237 respectively represent. Now, the digits 2, 3, 7, in the first two decimals, are in exactly the same position with regard to the decimal point, and the respective digits in each have the same absolute value; moreover, the cipher affixed to the right of the decimal ·2370, has no intrinsic value, and hence the two decimals, ·237, and ·2370, have the same absolute value. And since the reasoning is the same, no matter how many ciphers are *affixed* to the right, we get the following important principle:—

The value of a decimal is not altered by affixing ciphers to the right.

We will now compare the first and third of our examples, namely, the decimals ·237 and ·0237. The cipher which is here *prefixed* to the left, has again no intrinsic value; but it has removed the digits 2, 3, 7, one stage to the right, and has, therefore, diminished their local value tenfold. The effect of prefixing the cipher, is therefore to diminish the absolute value of the decimal tenfold, and as every additional cipher so prefixed has a similar effect, we get another fundamental principle, as follows:—

The value of a decimal is diminished tenfold for every cipher prefixed.

After the above it is easy to see that we have only to remove the decimal point one place to the right or left, in order to increase or diminish respectively the value of a decimal tenfold. And if we allow the term decimal to include numbers which are greater than unity, as 35·721, we may extend the principle thus:—

A decimal may be divided by 10, by removing the dot *one* place to the left; by 100 or 10^2 by removing the dot *two* places to the left; by 1000 or 10^3 by removing the dot *three* places to the left, and so on; and further, a decimal may be multiplied by 10, 100, 1000, &c., that is, by 10, 10^2, 10^3, &c., by removing the dot 1, 2, 3, &c., places respectively to the right. Thus, 6872·3476 divided by 10, 100, 1000 respectively, becomes 687·23476, 68·723476, 6·8723476; and multiplied by the same becomes 68723·476, 687234·76, 6872347·6 respectively.

Addition and Subtraction.

2. If the student has understood the preceding article, he will at once perceive that, provided we keep the units' figure under the units' figure in every case, there is no difference between the addition and subtraction of decimals, and the addition and subtraction of ordinary integers. All he has to take care of is that the decimal points are kept under each other.

Ex. 1.—Add together 325·02, ·647, 5·6073, ·00214, 290, and 4·7001. Proceeding as in ordinary addition:

```
325·02
   ·647
  5·6073
   ·00214
290·
  4·7001
---------
625·97654
```

Ex. 2.—Take 6·291 from 18·3064. Proceeding as in ordinary subtraction:

```
18·3064
 6·291
-------
12·0154
```

Ex. 3.—Find the difference between 15·02 and ·6732.

```
15·02
  ·6732
―――――――
14·3468
```

Note.—In an example of this kind, where the number of decimal figures in the lower line exceeds the number in the upper, it is advisable to mentally supply ciphers to make up the deficiency in the upper line. This may be done, as we have seen, without altering the value of the upper line.

Multiplication.

3. Suppose we have to multiply 2·935 by 6·34, and let us suppose the dot in each case removed to the extreme right. Then (Art. 1), we have multiplied the number 2·935 by 1000. and the number 6·34 by 100, and we have obtained the numbers 2935· and 634· respectively. As these numbers are integers, we may omit the dot, and write them 2935 and 634. Now $2935 \times 634 = 1860790$, but as we increased our original numbers one thousand and one hundredfold respectively, it is evident that our product is increased 1000×100, or one hundred thousandfold. Dividing, therefore, the above result, 1860790 by 100000, or what is the same thing (Art. 1), writing it 1860790· and removing the dot 5 places to the left, we get for our product of the numbers 2·935 and 6·34 the result, 18·60790. We may remark that the number of decimal figures in the product, namely, 5, is the sum of the numbers of decimal figures in the two given numbers.

We have, therefore, the following rule for multiplication:—Multiply the given numbers exactly as integers, regardless of the decimal points, and after the operation is finished, point off as many decimal figures in the product as there are together in the multiplier and multiplicand.

Ex. 1.—Multiply 6·35 by ·1703.

```
  ·1703
   6·35
―――――――
   8515
  5109
10218
―――――――
1081405
```

Now, the number of decimal figures in the multiplier and multiplicand together, is (4 + 2), or 6, and therefore we mark off 6 decimal figures in our product. This gives us 1·081405.

Ex. 2.—Multiply ·0063, by ·017.

$$\begin{array}{r} \cdot0063 \\ \cdot017 \\ \hline 441 \\ 63 \\ \hline 1071 \end{array}$$

And pointing off (4 + 3), or 7 decimal figures, we obtain for our product ·0001071.

Division.

4. Suppose we have to divide ·76875 by 6·25. We will proceed as in the case of multiplication, by imagining the decimal points in each number removed to the extreme right. The numbers will then be 76875·, and 625·, or, omitting the dot, as they are now integers, they will be 76875, and 625.

Proceed now as in ordinary division (which operation it is unnecessary to explain), and we get for our quotient 123.

Now we must remember that we have increased our dividend 100,000 or 10^5-fold, and that, consequently, our quotient will require to be diminished 10^5-fold. This is done (Art. 1) by removing the dot of the number 123· five places to the *left*. But, before doing that, we know that the divisor has been increased 100 or 10^2-fold, and on this account our quotient must be increased 10^2-fold. This is done (Art. 1) by removing the dot two places to the *right*. Hence, to get the true quotient of ·76875 by 6·25, we must remove the dot from the extreme right of the number 123·, five minus two, or three places to the left. Now three is the excess of the number of decimal figures in the dividend over the number in the divisor. We hence arrive at the following rule :—

Proceed as in ordinary division, and when all the figures of the dividend have been brought down, and the remainder, if any, obtained, cut off as many decimal figures in the quotient, as the number of decimal figures in the dividend exceeds the number in the divisor.

NOTE.—When the number of decimal figures in the dividend is less than the number in the divisor, affix a sufficient number of ciphers to make the number of decimals in the dividend *equal* to the number in the divisor. After finishing the operation of ordinary division, there will be *no* decimal figures to cut off in the quotient. If there be a remainder, and the division carried on further, by affixing ciphers to the successive remainders, all the quotient figures thus obtained will be decimals.

Ex. 1.—Divide 117·85088 by 6·272.

```
6·272)117·85088(1879
      6272
      ----
      55130
      50176
      -----
       49548
       43904
       -----
        56448
        56448
        -----
```

We see that there are five decimal figures in the dividend, and three in the divisor, and so we cut off $(5 - 3)$ or 2 in the quotient. The answer is therefore 18·79.

Ex. 2.—Divide 527·2 by ·0008.

Here it will be necessary to affix three ciphers to the dividend, and the operation will stand thus—

```
·0008)527·2000
      --------
       659000
```

As there is no remainder, and the number of decimal figures in the dividend is equal to that in the divisor, we have none to cut off. The answer is therefore 659000.

Ex. 3.—Divide 463·7 by 2·769 to four places of decimals.

Here we must affix two ciphers to the dividend, and the operation, as far as the ordinary remainder of long division, stands thus :—

```
2·769)463·700(167
      2769
       18680
       16614
        ·20660
         19383
          1277
```

The quotient up to this point is integral; but, as we have a remainder, we must continue the operation of division, first placing a dot at the right of the figures in the quotient, and affixing a cipher to the present and each successive remainder, until we have the requisite number of decimals in the quotient. By thus proceeding, it is easy to see we arrive at an answer—167·4611.

Ex. I.

1. Increase the numbers 4·523, 29, ·02367, ·07 respectively 10, 100, 1,000, 10,000-fold.

2. Divide by inspection the numbers 0·05, 1111, 4·0020, 45 respectively by 100, 10,000, 1,000, 10.

3. Express in words ·3467, 34·67, ·0003467, 3·467; and compare the values of the last three with the first.

4. Add together—
   (1.) 6·732, 14·9, ·0064, 14·27006.
   (2.) ·00291, ·291, 29, 29100·9.
   (3.) ·821, 29·60, 29·6, ·0029.

5. By how much does 5 exceed 4·2763, and 16·021 exceed 12·70009?

6. Find the value of—
   (1.) 74·25 + ·0067 − 3·0298 + 1·032 − 2·73.
   (2.) 3·276 − 8·2409 + 10·0326 − ·00091.
   (3.) 2·5 − ·00029 − 7·364 + 5·2791.

7. What number added to four thousandths will give three hundredths, and what number subtracted from 8,000 units will give 291 units 29 hundredths?

8. Find value of—
(1.) ·3 × ·3. (2.) 9·001 × 27·06. (3.) 0·403 × ·009. (4.) ·17 × ·017 × 100. (5.) ·3 × ·005 × 6·4. (6.) $(·4)^2 \times (·032)^2$.

9. Find the quotient of—
(1.) 79·4 by ·397. (2.) 5·928 by 4742·4. (3.) 28 by ·007. (4.) ·6426 by 2·8. (5.) $(·24)^2$ by 9·6. (6.) 1·806 by $(1·9)^2$.

10. Given the quotient ·00073, the dividend 124·1, find the divisor when there is no remainder.

11. What is the value of—
(1.) $\{2 - ·815\} \div \{·201 + ·039 - ·002\}$?
(2.) $\{(·693)^2 - (·307)^2\} \div \{·693 - ·307\}$?

12. If I add ·061 to a certain number, and then divide the result by 290, I get ·0009 for a quotient; what is the number?

CHAPTER II.

THE TREATMENT OF FRACTIONS CONSIDERED AS RATIOS.

5. A **Fraction** is a part or parts of a whole. It is generally expressed by two numbers, the one placed above the other and separated by a line. The lower number expresses the number of equal parts into which the whole quantity has been divided, and the upper number, how many of those parts are taken. Thus $\frac{3}{5}$ is a fraction, and tells us that unity has been divided into 5 equal parts, and that we have taken 3 of those parts. The fraction $\frac{3}{5}$ is read three fifths; each of the equal parts into which unity has been divided being called a fifth.

The denominator of a fraction is the lower number, and therefore shows the number of equal parts into which we have divided the unit.

The numerator is the upper number, and tells us how many of these equal parts are taken.

When the numerator is less than the denominator, the quantity expressed is actually less than a whole. The quantity is therefore a real or proper fraction. Again, when the numerator and denominator are both integral numbers, the fraction is termed a simple fraction.

Thus $\frac{3}{7}$, $\frac{7}{9}$, $\frac{10}{11}$, are both proper and simple fractions.

It is however usual to include in the term fraction every expression which contains one or more simple fractions, with or without integral numbers.

Thus $\frac{2}{3}$, $\frac{4}{1}$, $\frac{12}{5}$, $6\frac{1}{2}$, $\dfrac{3}{2\frac{1}{9}}$, $\dfrac{4\frac{1}{5}}{7\frac{1}{3}}$, $4\frac{1}{7}$ of $\frac{5}{6}$ of $\dfrac{2\frac{1}{8}}{\frac{1}{16}}$, are all included in the term fraction.

They are, moreover, called **vulgar fractions** to distinguish them from decimals, which, as will be shown further on, may be looked upon as fractions, according to the above definition, whose denominators are powers of 10, and not expressed but understood.

It is convenient to classify fractions as follows :—

(1.) A **proper fraction** is one whose numerator is less than its denominator, as $\frac{2}{7}$, $\frac{3}{5}$, $\frac{12}{13}$, $\dfrac{2\frac{1}{4}}{9}$.

(2.) An **improper fraction** is one whose numerator is not less than its denominator, as $\frac{9}{4}$, $\frac{8}{8}$, $\frac{12}{11}$, $\dfrac{5\frac{1}{4}}{2\frac{1}{6}}$.

(3.) A **simple fraction** is one whose numerator and denominator are both integral numbers, as $\frac{5}{3}$, $\frac{2}{7}$, $\frac{6}{19}$.

(4.) A **mixed number** is a fraction expressed by an integer and a simple fraction, as $2\frac{1}{3}$, $4\frac{2}{5}$, $3\frac{5}{7}$.

(5.) A **complex fraction** has its numerator, or denominator, or both, in a fractional form, as $\dfrac{3}{5\frac{1}{7}}$, $\dfrac{2\frac{1}{3}}{6\frac{1}{9}}$, $\dfrac{\frac{5}{7}}{11}$.

(6.) A **compound fraction** is a fraction of a quantity which is itself fractional, as $\frac{5}{7}$ of $2\frac{1}{3}$, $2\frac{1}{9}$ of $6\frac{1}{15}$ of $\dfrac{3}{7\frac{1}{6}}$.

6. In the preceding article we have spoken of fractions in the ordinary way. We will now approach them from a different point of view.

By the term ratio we understand the result of the comparison of two quantities with regard to magnitude. There are two kinds of ratios—ratio by difference or subtraction, and ratio by quotient or division. Thus we may consider how much one quantity exceeds another, or we may consider how many times one quantity contains another. The former kind is called the **arithmetical ratio**, and the latter the **geometrical ratio**. We shall speak only of the latter.

DEFINITION.—The ratio between two quantities is that multiple, part, or parts which the former is of the latter.

It is evident that a ratio can exist only between quantities of the same kind; thus, we may compare 12 horses and 6 horses, but not 9 men and 4 miles. And if the quantities are reduced to the same denomination, we may treat the quantities as abstract, just as we find the quotient of one concrete quantity by another, by reducing them both to the same denomination, and dividing as if they were abstract quantities.

Now, according to what has been stated above, the ratio of 12 to 7, or, as it is usually written, 12 : 7, is obtained by dividing 12 by 7; and this is the same thing as dividing unity or 1 into 7 equal parts, and computing how much 12 of such parts amount to. It hence follows that the fraction $\frac{12}{7}$ is properly expressed by the ratio 12 : 7.

The first term of a ratio is called the **antecedent**, and the second is called the **consequent**; and hence we may consider a fraction as a ratio, the numerator being the antecedent of the ratio, and the denominator the consequent.

When the antecedent is *equal* to the consequent, the ratio is said to be a ratio of equality; and it is said to be a ratio of *less or greater inequality* according as the antecedent is less or greater than the consequent.

Thus, 6 : 6 is a ratio of equality.
3 : 4 is a ratio of less inequality.
11 : 9 is a ratio of greater inequality.

The student will therefore have no difficulty in assenting to the following definitions:—

(1.) A proper fraction is a ratio of less inequality.

(2.) An improper fraction is a ratio of equality or of greater inequality.

(3.) A simple fraction is a ratio whose terms are integers.

Thus, $\frac{3}{5} = 3 : 5$ is a simple fraction.

(4.) A mixed number is a ratio of greater inequality, whose antecedent has been actually divided by its consequent, and the result expressed as an integer and simple fraction.

Thus, $2\frac{3}{7} = \frac{17}{7} = 17 : 7$.

(5.) A complex fraction is a ratio, whose antecedent, or consequent, or both, are not integers.

Thus, $\frac{3\frac{1}{2}}{1\frac{3}{4}}, \frac{2}{7\frac{1}{6}}, \frac{9\frac{1}{8}}{5}$ = respectively to $3\frac{1}{2} : 1\frac{3}{4},\ 2 : 7\frac{1}{6},\ 9\frac{1}{8} : 5$ are complex fractions.

(6.) A compound fraction is an expression containing two or more ratios to be compounded together.

Thus $\frac{3}{4}$ of $\frac{7}{9}$ contains the ratios $3 : 4$ and $7 : 9$ to be compounded; $\frac{2\frac{1}{2}}{7}$ of $\frac{3}{6\frac{1}{4}}$ of $\frac{7\frac{1}{2}}{9}$ contains the ratios $2\frac{1}{2} : 7$, $3 : 6\frac{1}{4}$, $7\frac{1}{2} : 9$ to be compounded.

7. *A fraction whose numerator and denominator are multiplied or divided by the same quantity is not altered in value.*

Suppose, for example, we multiply the numerator and denominator of the fraction $\frac{3}{7}$ each by 4, we get $\frac{3}{7} = \frac{12}{28}$. Now the ratio of $3 : 7$ is, from the definition of a ratio, four times as small as the ratio $(3 \times 4) : 7$ or $12 : 7$; and the ratio $12 : 7$ is, for the same reason, four times as great as the ratio $12 : (7 \times 4)$ or $12 : 28$. It therefore follows that the ratio $3 : 7$ is exactly equal to the ratio $12 : 28$, and consequently $\frac{3}{7} = \frac{12}{28}$.

Again, suppose we divided each term of the fraction $\frac{15}{27}$ by 3, we get $\frac{15}{27} = \frac{5}{9}$. Now the ratio of $15 : 27$ is, from the definition of a ratio, three times as great as the ratio $(15 \div 3) : 27$ or $5 : 27$; and, again, the ratio $5 : 27$ is three times as small as the ratio $5 : (27 \div 3)$ or $5 : 9$. It therefore follows that the ratio $15 : 27$ = the ratio $5 : 9$, and consequently $\frac{15}{27} = \frac{5}{9}$.

Cor.—An integer may be expressed as a fraction with any given denominator.

For we may consider an integer as a ratio whose consequent is 1, and we may multiply each term of this ratio by any given number without altering its value.

Thus—$6 = \frac{6}{1} = \frac{6 \times 7}{1 \times 7} = \frac{42}{7}$.

Or, $6 = \frac{6}{1} = \frac{6 \times 9}{1 \times 9} = \frac{54}{9}$.

8. To multiply a fraction by a whole number, we may either multiply the numerator by the number, or divide the denominator by it.

Ex. $\frac{8}{9} \times 3 = \frac{8 \times 3}{9} = \frac{24}{9}$; or we may proceed thus—

$\frac{8}{9} \times 3 = \frac{8}{9 \div 3} = \frac{8}{3}$.

As to the first method—

The ratio 8 : 9 will be evidently increased three times if we multiply its antecedent by 3; this follows from the definition of a ratio. We thus get the ratio (8 × 3) : 9 or 24 : 9.

It therefore follows that $\frac{8}{9} \times 3 = \frac{24}{9}$.

As to the second method—

The ratio 8 : 9 will be evidently increased three times if we make the consequent three times as small; and we thus get the ratio 8 : (9 ÷ 3) or 8 : 3; and hence it follows that $\frac{8}{9} \times 3 = \frac{8}{3}$.

It may be remarked that the two results, $\frac{24}{9}$ and $\frac{8}{3}$, are of exactly the same value (Art. 7), since the latter may be obtained from the former by dividing each of its terms by 3.

In actual practice we sometimes pursue the first method, and sometimes the second. If the denominator of the given fraction contains the multiplier as a factor, it is more convenient to use the second method, thus:—

$$\frac{11}{12} \times 3 = \frac{11}{12 \div 3} = \frac{11}{4}.$$

On the other hand, when the denominator does not contain the multiplier as a factor, we use the first method, thus:—

(1.) $\frac{7}{13} \times 5 = \frac{7 \times 5}{13} = \frac{35}{13}$.

(2.) $\frac{9}{55} \times 10 = \frac{9 \times 10}{55}$.

Here we see that the numerator and denominator have a common factor 5, and therefore, by Art. 7, if we divide them both by it, we have:—

$$\frac{9}{55} \times 10 = \frac{9 \times 2}{11} = \frac{18}{11}.$$

9. To divide a fraction by a whole number, we may either multiply the denominator by the number, or we may divide the numerator by it.

Ex.—Divide $\frac{12}{17}$ by 4.

We may proceed thus, (1.) $\frac{12}{17} \div 4 = \frac{12 \div 4}{17} = \frac{3}{17}$.

(2.) $\frac{12}{17} \div 4 = \frac{12}{17 \times 4} = \frac{12}{68}$.

As to the first method—

The ratio of 12 : 17 will evidently be diminished 4 times if we divide its antecedent 12 by 4. We thus get the ratio (12 ÷ 4) : 17 or 3 : 17 ; and it therefore follows that $\frac{12}{17} \div 4 = \frac{3}{17}$.

As to the second method—

The ratio of 12 : 17 can also be diminished 4 times by increasing its consequent or divisor 4 times, so that we thus get the ratio 12 : (17 × 4) or 12 : 68. It therefore follows that $\frac{12}{17} \div 4 = \frac{12}{68}$.

It may be remarked, as in Art. 8, that the two results, $\frac{3}{17}$ and $\frac{12}{68}$, have exactly the same value, for the latter can be obtained from the former by multiplying each of its terms by 4 (see Art. 7).

And again, in actual practice, we usually take the first method when the numerator contains the divisor as a factor, but not otherwise. Thus—

(1.) $\frac{18}{19} \div 6 = \frac{18 \div 6}{19} = \frac{3}{19}$.

(2.) $\frac{11}{13} \div 5 = \frac{11}{13 \times 5} = \frac{11}{65}$.

(3.) $\frac{12}{17} \div 8 = \frac{12}{17 \times 8}$.

Here it is convenient to divide the numerator and denominator by the common factor 4 (Art. 7).

We then have $\frac{12}{17} \div 8 = \frac{12 \div 4}{17 \times (8 \div 4)} = \frac{3}{17 \times 2} = \frac{3}{34}$.

10. To reduce a mixed number to an improper fraction. Looking at our definition of a mixed number (Art. 6), the following rule is evident:

Multiply the integral part by the denominator of the fractional part, and add in the numerator ; this gives the required numerator, and the denominator of the fractional part is the required denominator.

Ex.—Reduce $5\frac{2}{9}$, $7\frac{3}{5}$ to improper fractions.

$$(1.)\ 5\tfrac{2}{9} = \frac{5 \times 9 + 2}{9} = \tfrac{47}{9}.$$
$$(2.)\ 7\tfrac{3}{5} = \frac{7 \times 5 + 3}{5} = \tfrac{38}{5}.$$

11. To reduce a complex fraction to its equivalent simple fraction. Before stating a rule, let us take an example. Suppose we have to reduce $\dfrac{3\frac{1}{5}}{5\frac{2}{9}}$ to an equivalent simple fraction.

Now, by the last Art., $\dfrac{3\frac{1}{5}}{5\frac{2}{9}} = \dfrac{\frac{3 \times 5 + 1}{5}}{\frac{5 \times 9 + 2}{9}} = \dfrac{\frac{16}{5}}{\frac{47}{9}}$.

Again, the ratio $\frac{16}{5} : \frac{47}{9}$ will not be altered in value if we multiply both its terms by the same quantity. Let us multiply them by 9 and it becomes $\frac{16}{5} \times 9 : \frac{47}{9} \times 9$. Now, by Art. 8, $\frac{16}{5} \times 9 = \frac{16 \times 9}{5}$ and $\frac{47}{9} \times 9 = \frac{47}{9 \div 9} = \frac{47}{1} = 47$. The ratio then becomes $\frac{16 \times 9}{5} : 47$. We will again multiply the terms of this ratio by the same quantity, viz. by 5, and we get the ratio $\frac{16 \times 9}{5} \times 5 : 47 \times 5$. Now, by Art. 8, $\frac{16 \times 9}{5} \times 5 = \frac{16 \times 9}{5 \div 5} = \frac{16 \times 9}{1} = 16 \times 9$. Hence the ratio $\frac{16}{5} : \frac{47}{9}$ is equivalent to the ratio $16 \times 9 : 47 \times 5$, and hence the fraction $\dfrac{\frac{16}{5}}{\frac{47}{9}} = \dfrac{16 \times 9}{47 \times 5}$.

Now 16 and 9 are called the *extreme* terms of the complex fraction $\dfrac{\frac{16}{5}}{\frac{47}{9}}$, and 5 and 47 are called its *mean* terms.

We arrive then at the following rule :—

Rule.—Bring the numerator and denominator to the form of simple fractions, then multiply together the extreme terms for a new numerator, and the mean terms for a new denominator.

$$\text{Thus, } \frac{7\frac{1}{5}}{2\frac{1}{9}} = \frac{\frac{7 \times 5 + 1}{5}}{\frac{2 \times 9 + 1}{9}} = \frac{\frac{36}{5}}{\frac{19}{9}} = \frac{36 \times 9}{5 \times 19} = \tfrac{324}{95} = 3\tfrac{39}{95}.$$

12. To reduce a compound fraction to its equivalent simple fraction.

Let it be required to find the simple fraction equivalent to the compound fraction $\frac{3}{4}$ of $\frac{5}{7}$.

Now $\frac{3}{4}$ of $\frac{5}{7}$ is the ratio $3:4$, where the unit of this ratio is $\frac{5}{7}$. It is therefore, from the definition of ratio, equal to 3 times this unit divided by 4.

Now 3 times $\frac{5}{7} = \frac{5}{7} \times 3 = \frac{5 \times 3}{7}$ (Art. 8),

And $\therefore$ 3 times $\frac{5}{7} \div 4 = \frac{5 \times 3}{7} \div 4 = \frac{5 \times 3}{7 \times 4} = \frac{3 \times 5}{4 \times 7}$.

Hence we arrive at the result—

$$\frac{3}{4} \text{ of } \frac{5}{7} = \frac{3 \times 5}{4 \times 7}.$$

And in the same way we might show that

$$\frac{7}{8} \text{ of } \frac{4}{9} \text{ of } \frac{3}{5} \text{ of } 11 = \frac{7 \times 4 \times 3 \times 11}{8 \times 9 \times 5 \times 1}.$$

Hence the rule :—

RULE.—Multiply together the several numerators for a new numerator, and the several denominators for a new denominator.

Ex. $3\frac{1}{8}$ of $2\frac{1}{2}$ of $6 = \frac{3 \times 8 + 1}{8}$ of $\frac{2 \times 2 + 1}{2}$ of $\frac{6}{1} =$
$\frac{25}{8}$ of $\frac{5}{2}$ of $\frac{6}{1} = \frac{25 \times 5 \times 6}{8 \times 2 \times 1}$.

Ex. II.

1. Reduce the following to improper fractions—
$3\frac{1}{7}$, $4\frac{7}{9}$, $100\frac{7}{10}$, $35\frac{12}{17}$, $1\frac{11}{999}$, $11\frac{25}{64}$.

2. Reduce the integer 19 to sixths, tenths, thirteenths, eighteenths, nineteenths, and twentieths.

3. Bring the following fractions to integers, and reduce them respectively to fourths, sixths, eighths, tenths, twelfths, and fourteenths—
$\frac{36}{6}$, $\frac{91}{13}$, $\frac{84}{7}$, $\frac{1331}{121}$, $\frac{27}{3}$, $\frac{70}{5}$.

4. Multiply the following fractions each by 10, 11, 12—
$\frac{5}{7}$, $\frac{3}{16}$, $\frac{9}{12}$, $1\frac{2}{3}$, $7\frac{1}{9}$, $3\frac{1}{8}$.

5. By how much does 8 times the fraction $\frac{29}{4}$ exceed the quotient of $\frac{666}{37}$ by 3?

6. Divide the following fractions each by 6, 7, 18—
$\frac{37}{7}$, $\frac{14}{17}$, $1\frac{5}{6}$, $3\frac{7}{13}$, $103\frac{1}{7}$, $11\frac{5}{11}$.

7. Diminish the following ratios respectively 6, 7, 8-fold—
$12:5$, $3\frac{3}{13}:4$, $9\frac{7}{9}:2\frac{1}{5}$.

8. Simplify the expressions—

(1.) $\frac{1}{2}$ of $\frac{3}{7}$ of $\frac{5}{8}$ of $1\frac{5}{9}$. (2.) $\frac{7}{3}$ of 12 of $\frac{11}{14}$ of $\frac{3}{11}$.
(3.) $\frac{4}{11}$ of $3\frac{1}{7}$ of $\frac{7}{9}$ of $1\frac{1}{8}$. (4.) $\frac{11}{10}$ of 2 of $\frac{14}{9}$ of $1\frac{1}{11}$ of $\frac{5}{8}$ of $\frac{3}{7}$. (5.) $2\frac{1}{5}$ of $\frac{4}{9}$ of $1\frac{1}{2}$ of $4\frac{1}{11}$.

9. Reduce to simple fractions—

$$\frac{1\frac{1}{4}}{2\frac{1}{2}}, \frac{2\frac{2}{7}}{3\frac{3}{14}}, \frac{5\frac{1}{3} \text{ of } 6\frac{1}{2}}{3\frac{5}{7}}, \frac{2\frac{1}{4}}{7\frac{1}{9} \text{ of } 3\frac{15}{16}}, \frac{2\frac{1}{9} \text{ of } 3}{9\frac{1}{2} \text{ of } 3\frac{1}{3}}, 7\frac{1}{9} \text{ of } \frac{3\frac{3}{5}}{4\frac{4}{7}} \text{ of } \frac{1}{14}.$$

10. What is the difference between $2\frac{3}{7}$ of $3\frac{1}{9}$ of $2\frac{13}{34}$ and $\frac{9\frac{6}{7}}{1\frac{1}{2}}$.

11. Find the integral value of $\frac{8\frac{1}{7} \text{ of } 2\frac{1}{3} - 2\frac{4}{5} \text{ of } 1\frac{3}{7}}{3\frac{1}{2} \text{ of } \frac{6}{7} + \frac{8}{19} \text{ of } 4\frac{3}{4}}$.

12. Add the sum of the fractions $\frac{4\frac{1}{5}}{\frac{3}{10}}, \frac{8\frac{1}{3}}{2\frac{1}{12}}$ to their difference.

13. To reduce a fraction to its lowest terms.

Def. A fraction is in its lowest terms when its numerator and denominator have no common factor, or are prime to each other. (Among such common factors, we include either the numerator or denominator itself, when one of them happens to be a divisor of the other.)

When the numerator and denominator have a common factor, we may divide them both by it (Art. 7) without altering the value of the fraction. Now, the highest common factor of two or more numbers is called their greatest common measure, usually written G.C.M.

Hence we have the following rule :—

Rule.—To reduce a fraction to its lowest terms, divide the numerator and denominator by their G.C.M.

Ex.—Reduce $\frac{24}{84}$ to its lowest terms.

We can easily see that 12 is the G.C.M. of 24 and 84 ; hence, dividing each by 12, we get—

$$\frac{24}{84} = \frac{24 \div 12}{84 \div 12} = \frac{2}{7}, \text{ the fraction required.}$$

14. It is not, however, always easy to tell by inspection the G.C.M. ; but, before giving a general method for determining it, it will be useful to make a few remarks as to the divisibility of numbers in certain cases.

A number is divisible as follows :—

By 2, when it is *even*.

By 3, when the sum of its digits is divisible by 3.

By 4, when the number formed by the last two figures is divisible by 4.

By 5, when it ends in 5 or 0.

By 6, when it is even, and is also divisible by 3.

By 8, when the number formed by the last three figures is divisible by 8.

By 9, when the sum of its digits is divisible by 9.

By 10, when it ends in 0.

By 11, when the sum of the digits in the odd places (that is, the sum of the 1st, 3rd, 5th, &c.) is equal to the sum of the digits in the even places, or the one exceeds the other by a multiple of 11.

By 12, when the number formed by its last two figures is divisible by 4, and the sum of its digits is a multiple of 3.

We may add also :—

(1.) A number is divisible by 37, when it is composed of digits which are repeated three times, or any multiple of three times, as 111, 333, 444444, &c.

(2.) A number which has three figures repeated in the same order is divisible by 7, 11, 13.

Thus 271271, 165165, 23023 are divisible by 7, 11, and 13; for the last may be written 023023.

(3.) A number which has four figures repeated in the same order is divisible by 73 and 137.

Thus 53245324, 2760276 are both divisible by 73 and 137; for the last may be written 02760276.

Hence a fraction may often then be reduced to its lowest terms by gradually striking out factors determined by inspection.

Ex.—Reduce $\frac{792}{2244}$ to its lowest terms.

Now 792 and 2244 are each divisible by 4, for the numbers 92 and 44, which are formed by the last two figures of each, are evidently so. Hence, dividing numerator and denominator by 4, we have $\frac{792}{2244} = \frac{792 \div 4}{2244 \div 4} = \frac{198}{561}$; each is evidently divisible by 3. Hence $\frac{792}{2224} = \frac{198 \div 3}{561 \div 3} = \frac{66}{187}$;

and each is now evidently divisible by 11. Hence $\frac{792}{2244} = \frac{66 \div 11}{187 \div 11} = \frac{6}{17}$.

REMARK.—It may sometimes happen that, although we are able to tell by inspection some of the factors of numerator and denominator, none of them are common to both numerator and denominator. We cannot then strike them out, but we may use them to determine what would be left supposing they are struck out, and we may thus often come upon the G.C.M. of both numerator and denominator.

Ex.—Reduce $\frac{474}{2133}$ to its lowest terms.

Now 474 is even, and therefore divisible by 2; thus $474 \div 2 = 237$. Again, 2133 is divisible by 9, for the sum of its digits, viz., $(2 + 1 + 3 + 3)$ or 9 is so divisible; thus, $2133 \div 9 = 237$.

We have thus learned that, although 2 and 9 are not common factors, 237 is a common factor, and, in fact, the G.C.M. Dividing the numerator and denominator of the given fraction by 237, we get $\frac{474}{2133} = \frac{474 \div 237}{2133 \div 237} = \frac{2}{9}$.

We proceed now to give the general method of determining the G.C.M.

15. To find the G.C.M. of two numbers.

RULE.—Divide the greater number by the less, and if there be no remainder, the less is the G.C.M.; but if there be a remainder, make a divisor of this remainder, and a dividend of the first divisor; if there be a remainder again, make a divisor of it, and a dividend of the preceding divisor, and so on until there be no remainder. The last divisor will be the G.C.M.

Ex.—Find the G.C.M. of 282 and 799.

The operation will stand thus—

```
282)799(2
    564
    ---
    235)282(1
        235
        ---
         47)235(5
            235
            ---
```

or thus—

282	799	2
	564	
235	282	1
	235	
47	235	5
	235	

The G.C.M. is therefore 47.

The reason of this rule is easy to see. 47 divides 235, and it therefore divides 235 + 47 or 282. Hence it is a common divisor of 282 and 235, and it is therefore a common divisor of 282 and 282 × 2 + 235 or of 282 and 799.

It is, moreover, the highest common divisor; for every number which divides 799 and 282 must also divide 799 − 282 × 2 or 235; and hence every number which is a measure of 799 and 282 is also a measure of 235 and 282. Similarly, every number which divides 235 and 282 will also divide 282 − 235 or 47, and hence every measure of 799 and 282 is a measure of 47. But no higher number than 47 can divide 47, and therefore 47 is the G.C.M of 799 and 282.

16. It is now easy to see how any fraction may be reduced to its lowest terms.

Ex.—Reduce $\frac{148}{703}$ to its lowest terms.

148	703	4
	592	
111	148	1
	111	
37	111	3
	111	

Hence 37 is the G.C.M, and dividing numerator and denominator by it, we get

$$\frac{148}{703} = \frac{4}{19}.$$

17. To find the G.C.M. of more than two numbers.

The following rule needs no explanation:—

Rule.—Find the G.C.M. of any two of the numbers, then the G.C.M. of this result and the third number, and so on. The last result will be the G.C.M. required.

Ex.—Find the G.C.M. of 282, 987, 658, 1128.
The operation will stand thus—

282	987	
	846	
141	282	2
	282	

141	658	4
	564	
94	141	1
	94	
47	94	2
	94	

47	1128	24
	94	
	188	
	188	

Hence the G.C.M. is 47.

Ex. III.

1. Resolve into prime factors 44, 64, 150, 252, 1269, 462.

2. Find the prime factor of 1386, 1720, 4608, 21175, 15972, 14256.

3. A number is said to be a perfect number when it is equal to the sum of its aliquot parts. Show that the following are each perfect numbers—6, 28, 496, 8128.

4. Show that 284 and 220 are a pair of amicable numbers; that is, such a pair that each is equal to the sum of the divisors of the other, unity being here counted as a divisor.

5. Reduce, by inspection, to their lowest terms, the following fractions:

(1.) $\frac{6}{15}$, $\frac{18}{24}$, $\frac{35}{40}$, $\frac{121}{990}$, $\frac{632}{3950}$, $\frac{548}{1233}$.
(2.) $\frac{95}{160}$, $\frac{75}{108}$, $\frac{165}{176}$, $\frac{1539}{2052}$, $\frac{185}{555}$, $\frac{1836}{2052}$.
(3.) $\frac{168}{216}$, $\frac{540}{900}$, $\frac{726}{1331}$, $\frac{204}{432}$, $\frac{16016}{35035}$, $\frac{1573}{713713}$.

6. Find the G.C.M. of

(1.) 304, 323. (2.) 413, 448. (3.) 377, 533. (4.) 1866, 2832. (5.) 1189, 1517. (6.) 4374, 5103. (7.) 168, 378, 602. (8.) 539, 616, 792. (9.) 780, 1092, 2145.

7. Reduce to their lowest terms the following fractions—

(1.) $\frac{975}{1625}$, $\frac{952}{1224}$, $\frac{702}{2574}$, $\frac{2068}{6721}$, $\frac{1392}{1972}$, $\frac{1555}{2799}$.
(2.) $\frac{1226}{1839}$, $\frac{474}{553}$, $\frac{1053}{1215}$, $\frac{194}{1746}$, $\frac{1302}{1344}$, $\frac{1599}{2106}$.

8. Show, without applying the rule for the G.C.M., that

$$\frac{7344}{8208} = \frac{17}{19}, \text{ and that } \frac{225423}{362637} = \frac{23}{37}.$$

9. It rains in a certain district 634 days out of every 2,219; express this fact in the simplest way possible.

10. Out of 1,659 men engaged in a battle, only 1,185 answered the roll-call in the evening; express by a ratio in its simplest terms the number missing in relation to the whole.

11. There are 40 numbers less than 100, and prime to it; what are they?

12. A, B, C can do a piece of work respectively in 318, 477, 795 hours; express the relative value of A, B, C as workmen by the simplest integral numbers possible.

The Least Common Multiple.

18. It is often necessary to express fractions as equivalent fractions, having a common denominator; and it is, moreover, convenient to have this denominator as small as possible. Now, there are always an infinite number of numbers which will contain each of the given denominators as a factor, and our problem is therefore to obtain the least of such numbers.

Def.—The least common multiple (L.C.M.) of two or more numbers is the *least* number which contains each of the given numbers exactly.

Rule.—Arrange the numbers in a line, putting one of them as a divisor. Strike out the greatest factor common to this divisor, and each of the numbers separately, and place the several quotients in the line below; at the same time bring down every number prime to the divisor. Repeat the operation upon the second line, and so on until we have a line of numbers prime to each other. Multiply the several divisors and the numbers in the lowest line together, and their continued product will be the least common multiple.

Ex. 1.—Find the L.C.M. of 12, 16, 36, 45, 60.

36	12,	16,	36,	45,	60
5	1,	4,	1,	5,	5
		4,		1,	1

Hence the L.C.M. is $36 \times 5 \times 4 = 720$.

Ex. 2.—Find the L.C.M. of 15, 21, 60, 84, 140.

60	15,	21,	60,	84,	140
7	1,	7,	1,	7,	7
		1,		1,	1

Hence the L.C.M. is $60 \times 7 = 420$.

19. Reduction of fractions to equivalent fractions having the least common denominator.

It is evident that the least common denominator cannot be a less number than the L.C.M., and therefore the following rule needs no explanation :—

Rule.—Divide the L.C.M. of the given denominators by each denominator in turn, and multiply the corresponding numerator by the quotient. The product thus obtained is the new numerator, and the L.C.M. is the least common denominator.

Ex. 1.—Reduce $\frac{2}{3}$, $\frac{5}{12}$, $\frac{7}{8}$, $\frac{11}{10}$ to equivalent fractions, having the least common denominator.

$$\begin{array}{r|l} 12 & 3,\ 12,\ 8,\ 10 \\ \hline & 1,\ \ 1,\ 2,\ \ 5 \end{array}$$

$\therefore$ The L.C.M. is $12 \times 2 \times 5 = 120$.

Dividing 120 by the respective denominators, we get as quotients 40, 10, 15, 12; and hence the given fractions become

$$\frac{2\times40}{120}, \frac{5\times10}{120}, \frac{7\times15}{120}, \frac{11\times12}{120};$$

$$\text{Or, } \frac{80}{120}, \frac{50}{120}, \frac{105}{120}, \frac{132}{120}.$$

Ex. 2. Reduce to their least common denominator the following:—$\frac{11}{30}$, $\frac{13}{18}$, $\frac{52}{45}$, $\frac{29}{63}$.

$$\begin{array}{r|l} 45 & 30,\ 18,\ 45,\ 63 \\ \hline 2 & \ \ 2,\ \ \ 2,\ \ \ 1,\ \ \ 7 \\ \hline & \ \ 1,\ \ \ 1,\ \ \ \ \ \ \ \ \ 7 \end{array}$$

Hence the L.C.M. is $45 \times 2 \times 7 = 630$.

Dividing 630 by the respective denominators, we get for quotients 21, 35, 14, 10. Hence the required fractions are

$$\frac{11\times21}{630}, \frac{13\times35}{630}, \frac{52\times14}{630}, \frac{29\times10}{630}; \text{ or, } \frac{231}{630}, \frac{455}{630}, \frac{728}{630}, \frac{290}{630}.$$

Note 1.—The operation of dividing the L.C.M. of the denominators is often simplified by using the L.C.M. in its factorial form. Thus, in our present example, the L.C.M. is $45 \times 2 \times 7$. Now it is easy to see that the quotient of this by 30 or $3 \times 2 \times 5$ is 3×7 or 21, and that the quotient by 18 or 9×2 is 5×7 or 35, and so on. We thus avoid the process of long division.

Note 2.—It is sometimes necessary, and generally advisable, especially for beginners, to reduce the given fractions to their lowest terms before applying the rule for the least common denominator. Thus, the least common denominator of the fractions $\frac{2}{5}$, $\frac{9}{12}$, $\frac{11}{20}$, taken as they are, is 60; whereas, if we reduce the second fraction to its lowest terms by striking out the factor 3, common to both numerator and denominator, the fractions become $\frac{2}{5}$, $\frac{3}{4}$, $\frac{11}{20}$, and the least common denominator is 20. If, however, the denominator of any such fraction not in its lowest terms is contained in the L.C.M. of the denominators, when the fractions are all in their lowest terms, it is unnecessary to reduce the fraction to its lowest terms.

We strongly recommend the beginner, however, to always commence by reducing the given fractions to their lowest terms.

Ex. IV.

1. Find the L.C.M. of—
   (1.) 2, 6, 8, 12. (2.) 4, 9, 10, 14. (3.) 15, 21, 40, 45. (4.) 12, 20, 35, 126. (5.) 39, 65, 52, 140. (6.) 37, 60, 222, 225.

2. Reduce to their least common denominator—
   (1.) $\frac{3}{4}$, $\frac{7}{5}$, $\frac{2}{15}$, $\frac{11}{20}$. (2.) $\frac{2}{7}$, $\frac{5}{12}$, $\frac{13}{28}$, $\frac{11}{21}$. (3.) $\frac{4}{3}$, $\frac{2}{9}$, $\frac{11}{12}$, $\frac{13}{14}$. (4.) $\frac{1}{3}$, $\frac{4}{20}$, $\frac{1}{7}$. (5.) $\frac{3}{4}$, $\frac{7}{16}$, $\frac{38}{162}$, $\frac{175}{360}$. (6.) $\frac{77}{126}$, $\frac{19}{153}$, $\frac{51}{289}$.

3. A can run round a ring in three minutes, B in four minutes, and C in six minutes, and they start together. In how many minutes will they all be again at the starting point?

4. Arrange the fractions $\frac{7}{13}$, $\frac{11}{28}$, $\frac{15}{39}$, $\frac{17}{21}$, in order of magnitude.

5. Multiply the greatest of the fractions $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$ by 339.

6. Divide the least of the fractions $\frac{8}{43}$, $\frac{3}{14}$, $\frac{10}{49}$, by 6.

7. Reduce to a simple fraction the complex fraction having the greater of the fractions $\frac{3}{5}$, $\frac{2}{3}$ in the numerator, and the less in the denominator.

8. Which of the fractions $\frac{1}{4}$, $\frac{4}{19}$ is nearer to $\frac{1}{5}$?

9. Show that $\dfrac{3\frac{1}{9}}{9\frac{1}{3}}$ is less than $\dfrac{2\frac{1}{5}}{5\frac{1}{2}}$.

10. Arrange in order of magnitude the following:—

$$\frac{4\frac{3}{4} \text{ of } \frac{4}{9}}{6\frac{1}{3}}, \quad \frac{1\frac{1}{6} \text{ of } \frac{3}{5}}{1\frac{3}{4} \text{ of } 2\frac{4}{7}}, \quad \frac{1\frac{1}{3}}{13\frac{2}{3} \text{ of } \frac{1}{4}}.$$

11. Show that nine times the less of the fractions $\dfrac{\frac{14}{15}}{1\frac{1}{5}}$, $\dfrac{3\frac{1}{2}}{4}$ is eight times the greater.

12. Show that the ratio 18 : 7 is a ratio of greater inequality than the ratio 41 : 16.

Addition of Fractions.

20. We have shown (Art. 6) that the numerator and denominator respectively represent the antecedent and consequent of a ratio; and it is evident from the definition of a ratio (Art. 6) that the sum of two ratios having the same consequent is equal to a ratio whose antecedent is the sum of the given antecedents, and whose consequent is unaltered. Hence we have the following rule for addition of fractions—

RULE.—Bring the given fractions to their least common denominator, add together the numerators thus obtained, and place under the sum the least common denominator.

Ex. 1.—Add together $\frac{2}{3}$, $\frac{10}{21}$, $\frac{9}{14}$, $\frac{5}{6}$.

The least common denominator is easily found to be 42. Dividing this by each of the given denominators, we get as quotients, 14, 2, 3, 7.

Hence, $\frac{2}{3} + \frac{10}{21} + \frac{9}{14} + \frac{5}{6} = \frac{2\times14}{42} + \frac{10\times2}{42} + \frac{9\times3}{42} + \frac{5\times7}{42}$

$= \frac{28}{42} + \frac{20}{42} + \frac{27}{42} + \frac{35}{42} = \frac{28+20+27+35}{42}$

$= \frac{110}{42} = 2\frac{26}{42} = 2\frac{13}{21}$.

Ex. 2.—Add together $3\frac{1}{5}$, $2\frac{7}{9}$, $1\frac{11}{30}$, $4\frac{7}{12}$.

The sum of the integral parts of the given fractions $= 3 + 2 + 1 + 4 = 10$. Hence, $3\frac{1}{5} + 2\frac{7}{9} + 1\frac{11}{30} + 4\frac{7}{12} = 10 + \frac{1}{5} + \frac{7}{9} + \frac{11}{30} + \frac{7}{12}$.

The least common denominator is easily found to be 180. Dividing this by each of the given denominators we get as quotients 36, 20, 6, 15.

Hence the required sum

$$= 10 + \frac{1\times36}{180} + \frac{7\times20}{180} + \frac{11\times6}{180} + \frac{7\times15}{180}$$
$$= 10 + \frac{36}{180} + \frac{140}{180} + \frac{66}{180} + \frac{105}{180}$$
$$= 10 + \frac{36+140+66+105}{180} = 10 + \frac{347}{180}$$
$$= 10 + 1\frac{167}{180} = 11\frac{167}{180}.$$

Subtraction of Fractions.

21. After the preceding article there will be no difficulty in comprehending the following rule—

RULE.—Bring the given fractions to their least common denominator, subtract the numerators thus obtained, and under the difference place the least common denominator.

Ex. 1.—Subtract $\frac{3}{5}$ from $\frac{7}{8}$.

$$\frac{7}{8} - \frac{3}{5} = \frac{7\times5}{40} - \frac{3\times8}{40} = \frac{35}{40} - \frac{24}{40} = \frac{35-24}{40} = \frac{11}{40}.$$

Ex. 2.—Take $6\frac{7}{9}$ from $9\frac{1}{6}$.

$$9\tfrac{1}{6} - 6\tfrac{7}{9} = 3\tfrac{1}{6} - \tfrac{7}{9} = 3 + \tfrac{1\times3}{18} - \tfrac{7\times2}{18}$$
$$= 3 + \tfrac{3}{18} - \tfrac{14}{18} = 3 + \tfrac{3-14}{18}.$$

Here the number to be subtracted is the greater. We shall, however, take the *less* from the *greater*, and put the *negative* sign to the remainder, meaning by this that the remainder has yet to be subtracted.

Hence, $9\frac{1}{6} - 6\frac{7}{9} = 3 - \frac{11}{18} = 2 + (1 - \frac{11}{18})$; now $\frac{11}{18}$ taken from unity or $\frac{18}{18}$ evidently leaves $\frac{7}{18}$.

$$\therefore \text{required result} = 2 + \tfrac{7}{18} = 2\tfrac{7}{18}.$$

We will now give an example involving both addition and subtraction.

Ex. 3.—Find the value of $6\frac{1}{3} - 7\frac{2}{9} + 4\frac{5}{12} - 1\frac{5}{18}$.

Whenever we have an expression involving both + and − signs, the simplest method is to add together all the quantities affected with the *plus* sign, and likewise those affected with the *minus* sign; then taking the difference between these two sums, we place the sign of the greater sum before the result.

Thus, taking first the integers, we have $6 - 7 + 4 - 1 = 10 - 8 = 2$ (it must be remembered that the sign + is understood before a number which appears without a sign when it stands alone or at the head of an expression).

Hence the given expression $= 2 + \frac{1}{3} - \frac{2}{9} + \frac{5}{12} - \frac{5}{18}$ $= 2 + \frac{1\times12}{36} - \frac{2\times4}{36} + \frac{5\times3}{36} - \frac{5\times2}{36} = 2 + \frac{12-8+15-10}{36}$ $= 2\frac{27-18}{36} = 2\frac{9}{36} = 2\frac{1}{4}$.

We shall give one more example in order to show how brackets are to be treated.

Ex. 4.—Find the value of $7\frac{1}{8} - (2\frac{1}{5} - 3\frac{1}{2}) + (4\frac{1}{6} - 1\frac{1}{3}) - (2\frac{3}{10} + 3\frac{1}{12})$.

The general rule, which the student will better understand when we come to Algebra, is this—

When a *minus* sign stands before a bracket, it changes all the signs within on removing the bracket; but when a *plus* sign stands before the bracket, the latter may be removed *without* changing any of the signs within.

Thus, taking the expression $-(2\frac{1}{5} - 3\frac{1}{2})$:—

We must remember that the sign of $2\frac{1}{5}$ when within the bracket is, according to a remark made in Ex. 3, *understood* to be *plus*. In fact, though it is unusual, we might write the expression thus: $-(+2\frac{1}{5} - 3\frac{1}{2})$.

Now, remove the bracket, and it becomes $-2\frac{1}{5} + 3\frac{1}{2}$.

$$\text{Again } +(4\tfrac{1}{6} - 1\tfrac{1}{3}) = +4\tfrac{1}{6} - 1\tfrac{1}{3}.$$
$$\text{And } -(2\tfrac{3}{10} + 3\tfrac{1}{12}) = -2\tfrac{3}{10} - 3\tfrac{1}{12}.$$

Hence our given expression—

$$= 7\tfrac{1}{8} - 2\tfrac{1}{5} + 3\tfrac{1}{2} + 4\tfrac{1}{6} - 1\tfrac{1}{3} - 2\tfrac{3}{10} - 3\tfrac{1}{12}$$
$$= 14 - 8 + \tfrac{1}{8} - \tfrac{1}{5} + \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{3} - \tfrac{3}{10} - \tfrac{1}{12}$$
$$= 6 + \tfrac{1\times15}{120} - \tfrac{1\times24}{120} + \tfrac{1\times60}{120} + \tfrac{1\times20}{120} - \tfrac{1\times40}{120} - \tfrac{3\times12}{120} - \tfrac{1\times10}{120}$$
$$= 6 + \tfrac{15-24+60+20-40-36-10}{120} = 6 + \tfrac{95-110}{120}$$
$$= 6 - \tfrac{15}{120} = 6 - \tfrac{1}{8} = 5 + (1 - \tfrac{1}{8}) = 5\tfrac{7}{8}.$$

Ex. V.

1. Add together (1.) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$; (2.) $\frac{3}{5}, \frac{7}{10}, \frac{11}{15}$; (3.) $\frac{4}{9}, \frac{5}{12}, \frac{7}{18}$.
2. Find the sum of $3\frac{4}{5}, 17\frac{11}{60}, 2\frac{3}{10}, 1\frac{7}{15}, \frac{9}{8}, 3\frac{2}{3}$.
3. Add together $2\frac{7}{9}, \frac{3\frac{1}{2}}{4}, \frac{5}{2\frac{1}{7}}, 2\frac{2}{3}$ of $1\frac{1}{8}$, $5\frac{1}{5}$ of $1\frac{1}{13}$.
4. Find the difference between (1.) $\frac{7}{8}$ and $\frac{3}{4}$; (2.) $\frac{5}{6}$ and $\frac{4}{5}$; (3.) $\frac{6}{7}$ and $\frac{7}{8}$.
5. Subtract (1.) $6\frac{1}{3}$ from $8\frac{7}{12}$; (2.) $3\frac{7}{9}$ from $4\frac{1}{15}$; (3.) $2\frac{7}{10}$ from $6\frac{1}{3}$.
6. Take $\frac{1 - \frac{13}{16}}{1\frac{2}{5}}$ from $\frac{1\frac{9}{16}}{2 - 1\frac{1}{4}}$
7. Find the value of $1\frac{3}{28} - 2\frac{6}{7} + 3\frac{1}{8} - \frac{17}{14}$.
8. Simplify the expression $(2\frac{1}{2} - 1\frac{1}{3}) - (3\frac{5}{6} - 7\frac{3}{4})$.
9. By how much does $3\frac{4}{13} - 1\frac{3}{5}$ exceed $2\frac{19}{26} - \frac{7}{10}$?
10. Take the difference of $6\frac{7}{17}$ and $1\frac{11}{351}$ from their sum.
11. Add the difference of the same two fractions to their sum.
12. Find the value of the expression $\frac{3\frac{1}{3} \text{ of } 1\frac{2}{5}}{1\frac{3}{4}} - \left(\frac{6}{2\frac{1}{2}} - \frac{3\frac{3}{4}}{5}\right)$

Multiplication of Fractions.

22. RULE.—Multiply together the numerators of the fractions for a new numerator, and the denominators for a new denominator.

The reason of this rule is easily seen. Let it be required to find the product of $\frac{3}{5}$ and $\frac{7}{8}$, or the value of $\frac{3}{5} \times \frac{7}{8}$.

Now, what is the meaning of multiplying the ratio 3 : 5 by the ratio 7 : 8? It means evidently that the ratio 3 : 5 is to be multiplied by 7, and the result divided by 8.

Now (Art. 8) the ratio 3 : 5, when multiplied by 7, becomes $3 \times 7 : 5$; and (Art. 9) the ratio $3 \times 7 : 5$, when divided by 8, becomes $3 \times 7 : 5 \times 8$; and we have

$$\frac{3}{5} \times \frac{7}{8} = \frac{3 \times 7}{5 \times 8}.$$

And so on for any number of fractions. Hence the above rule.

Ex 1.—Multiply together the fractions $\frac{5}{6}$, $\frac{3}{8}$, $\frac{12}{25}$.

$$\frac{5}{6} \times \frac{3}{8} \times \frac{12}{25} = \frac{5 \times 3 \times 12}{6 \times 8 \times 25}$$

Before actually performing the operation of multiplication, it is advisable to strike out any factor common to both numerator and denominator. We see that 5 is common to 5 and 25, 3 common to 3 and 6, 4 common to 12 and 8, and we then have

$$\frac{5}{6} \times \frac{3}{8} \times \frac{12}{25} = \frac{1 \times 1 \times 3}{2 \times 2 \times 5} = \frac{3}{20}.$$

The whole operation is sometimes written thus—

$$\frac{5}{6} \times \frac{3}{8} \times \frac{12}{25} = \frac{\cancel{5} \times \cancel{3} \times \overset{3}{\cancel{12}}}{\underset{2}{\cancel{6}} \times \underset{2}{\cancel{8}} \times \underset{5}{\cancel{25}}} = \frac{3}{20}.$$

Ex. 2.—Multiply together $2\frac{3}{5}$, $3\frac{13}{29}$, $1\frac{1}{20}$, $\frac{87}{91}$.

$$2\frac{3}{5} \times 3\frac{13}{29} \times 1\frac{1}{20} \times \frac{87}{91} = \frac{13}{5} \times \frac{100}{29} \times \frac{21}{20} \times \frac{87}{91}$$

$$= \frac{\cancel{13} \times \overset{20}{\cancel{100}} \times \overset{3}{\cancel{21}} \times \overset{3}{\cancel{87}}}{\cancel{5} \times \cancel{29} \times \cancel{20} \times \underset{7}{\cancel{91}}} = \frac{9}{1} = 9.$$

Division of Fractions.

23. RULE.—Invert the divisor, and proceed as in multiplication.

To explain this rule, let us endeavour to divide $\frac{7}{9}$ by $\frac{5}{8}$. We may evidently consider the required quotient as nothing else than the ratio $\frac{7}{9} : \frac{5}{8}$, and this, by the reasoning of Art. 11, is equivalent to the ratio $7 \times 8 : 9 \times 5$, and we hence get

$$\frac{7}{9} \div \frac{5}{8} = \frac{7 \times 8}{9 \times 5}.$$

Now, $\frac{8}{5}$ is the divisor $\frac{5}{8}$ when inverted, and hence the above rule.

Ex. 1.—Divide $\frac{9}{10}$ by $\frac{3}{8}$.

$$\frac{9}{10} \div \frac{3}{8} = \frac{9}{10} \times \frac{8}{3} = \frac{\overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{\underset{5}{\cancel{10}} \times \cancel{3}} = \frac{12}{5} = 2\tfrac{2}{5}.$$

Ex. 2.—Divide $1\frac{2}{3}$ of $7\frac{1}{8}$ by $3\frac{1}{3}$ of $3\frac{2}{9}$.

$$1\tfrac{2}{3} \text{ of } 7\tfrac{1}{8} \div 3\tfrac{1}{3} \text{ of } 3\tfrac{2}{9} = \frac{5}{3} \times \frac{57}{8} \div \left(\frac{10}{3} \times \frac{29}{9}\right)$$
$$= \frac{\cancel{5}}{\cancel{3}} \times \frac{57}{8} \times \frac{\cancel{3}}{\underset{2}{\cancel{10}}} \times \frac{9}{29} = \frac{513}{464} = 1\tfrac{49}{464}.$$

We have introduced a bracket on the right side of the first equality, for otherwise the sign $\div$ affects only the first fraction $\frac{10}{3}$. On the other side a bracket is unnecessary, for the sign $\div$ standing before a *compound* fraction (not *two* fractions) affects the whole.

Ex. 3.—Simplify the expression

$$1\tfrac{2}{15} \div 6\tfrac{2}{5} \times 4\tfrac{1}{4} \div 2\tfrac{2}{3} \text{ of } 24\tfrac{1}{12}.$$

The given expression $= \frac{17}{15} \div \frac{32}{5} \times \frac{17}{4} \div \left(\frac{8}{3} \times \frac{289}{12}\right)$

$$= \frac{\cancel{17}}{\cancel{15}} \times \frac{\cancel{5}}{32} \times \frac{\cancel{17}}{\cancel{4}} \times \frac{3}{8} \times \frac{\overset{3}{\cancel{12}}}{\cancel{289}} = \frac{3}{256}.$$

Ex. VI.

1. Find the sum, difference, and product of $2\frac{1}{8}$ and $1\frac{3}{4}$.

2. Multiply the sum of the fractions $3\frac{3}{5}$, $2\frac{7}{10}$ by their difference.

3. Simplify the expression $\left\{(3\frac{2}{17})^2 - (1\frac{1}{17}\frac{5}{7})^2\right\} \div \left\{3\frac{2}{17} - 1\frac{1}{17}\frac{5}{7}\right\}$.

4. Reduce to a simple fraction each of the following expressions—

(1.) $1\frac{5}{13} \div 7\frac{1}{5} \times 8\frac{2}{3} \div 2\frac{2}{3}$ of $20\frac{1}{4}$.

(2.) $1\frac{5}{13} \div 7\frac{1}{5}$ of $8\frac{2}{3} \div 2\frac{2}{3} \div 20\frac{1}{4}$.

5. What is the difference between $(8\frac{1}{4} - 3\frac{7}{8})$ and $(5\frac{1}{6} - 4\frac{1}{12})$?

6. Divide $\dfrac{4}{3 + \dfrac{2\frac{8}{21}}{7\frac{1}{7}}}$ by $\dfrac{\frac{6}{7} \text{ of } 1\frac{2}{5}}{4}$.

7. Reduce to a simple fraction $3 + \dfrac{1}{7 + \frac{1}{15}}$.

8. Simplify the expressions (1.) $\dfrac{\frac{3}{4} - \frac{5}{12}}{\frac{5}{6} - \frac{5}{8}} \div \dfrac{2\frac{1}{2} - 1\frac{5}{6}}{1\frac{3}{4} + \frac{7}{6}}$.

(2.) $\dfrac{6\frac{1}{8}}{\frac{1}{3} \text{ of } 1\frac{2}{7} \text{ of } 2\frac{1}{3}} \times \left\{3\frac{1}{2} - \left(2\frac{1}{6} - 1\frac{1}{8}\right)\right\}$.

9. Find the quotient of $103\frac{16}{29}$ by $30\frac{1}{3}$ of $\frac{9}{203}$.

10. The cost of $7\frac{3}{5}$ articles is £$65\frac{2}{5}$, what is the cost of each article?

11. Find the cost of $89\frac{1}{7}$ articles, when one cost £$4\frac{1}{12}$.

12. The sum of two quantities is $34\frac{7}{12}$, and their difference is $6\frac{11}{12}$; required the greater.

Reduction of Fractions to Decimals.

24. If we place a decimal point to the right of an integer, and add as many ciphers as we please, it is clear, from Art. 1, that we do not alter its value. And hence a given ratio, as 3 : 8, is not altered in value by writing it 3·000 : 8; and further, dividing each of its terms by 8, according to the rule for division of decimals, it becomes ·375 : 1. It therefore follows, putting each of these ratios in a fractional form, that

$$\frac{3}{8} = \frac{3{\cdot}000}{8} = \frac{{\cdot}375}{1} = {\cdot}375.$$

We get, therefore, the following rule:—

Rule.—Place a decimal point to the right of the numerator, and add as many ciphers as may be thought necessary. Divide the new numerator by the given denominator, according to the rule for division of decimals, and, if necessary, add ciphers to the successive remainders until the division terminates, or until we have obtained as many decimal figures as required.

Ex. 1.—Reduce $\frac{5}{32}$ to a decimal.

$$
\begin{array}{l}
32)5\cdot0(\cdot15625 \\
\underline{32} \\
\ \ 180 \\
\ \ \underline{160} \\
\ \ \ \ 200 \\
\ \ \ \ \underline{192} \\
\ \ \ \ \ \ \ \ 80 \\
\ \ \ \ \ \ \ \ \underline{64} \\
\ \ \ \ \ \ \ \ 160 \\
\ \ \ \ \ \ \ \ \underline{160}
\end{array}
$$

Hence $\frac{5}{32} = \cdot15625$.

Ex. 2.—Reduce $\frac{5}{296}$ to a decimal.

$$
\begin{array}{l}
296)5\cdot00(\cdot016\dot{8}9\dot{1} \\
\ \ \ \underline{296} \\
\ \ \ 2040 \\
\ \ \ \underline{1776} \\
\ \ \ \ \ 2640 \\
\ \ \ \ \ \underline{2368} \\
\ \ \ \ \ \ \ 2720 \\
\ \ \ \ \ \ \ \underline{2664} \\
\ \ \ \ \ \ \ \ \ \ 560 \\
\ \ \ \ \ \ \ \ \ \ \underline{296} \\
\ \ \ \ \ \ \ \ \ \ 264
\end{array}
$$

It will be seen that we have arrived at a remainder, 264, exactly the same as the second remainder; and that, therefore, the quotient figures 891 will continually repeat, and that the division will never terminate. We call 891 the recurring period of the decimal, and it is usual to indicate the fact of its recurrence by placing dots over its first and last figures, as above.

We have, therefore, as a result, $\frac{5}{296} = \cdot016\dot{8}9\dot{1}$.

NOTE.—It is easy to see that no fraction, *reduced to its lowest terms*, whose denominator contains any prime factor, other than 2 or 5, can be expressed as a terminating decimal. For every terminating decimal is an exact number of tenths, hundredths, &c., and may, therefore, be transformed into a fraction, having some power of 10 as its denominator. Now, if we wish to bring a fraction already in its lowest terms to an equivalent fraction having some power of 10 for its denominator, it can only be done by *multiplying* its numerator and denominator by some integer; and it is impossible to obtain any power of 10 by multiplication only, from a number which contains any prime factor, other than 2 or 5.

Reduction of Terminating Decimals to Fractions.

25. Remembering (Art. 1) that any given terminating decimal may be considered as derived from an integer by diminishing it 10, or 10^2, 10^3, &c. fold, we have

$$\cdot 345 = 345 \div 10^3.$$

Hence, ·345 is the value of the ratio $345 : 10^3$; and we have, also, $\cdot 345 = \frac{345}{10^3} = \frac{345}{1000}$.

The following rule is, therefore, clear:—

Rule.—Make a numerator of the integer, formed by taking away the decimal point; and for a denominator put 1, followed by as many ciphers as there are given decimal figures.

Ex. 1.— $\cdot 625 = \frac{625}{1000} = \frac{5 \times 125}{8 \times 125} = \frac{5}{8}$.

$3\cdot 95 = 3\frac{95}{100} = 3\frac{19 \times 5}{20 \times 5} = 3\frac{19}{20}$.

$\cdot 0025 = \frac{25}{10000} = \frac{25}{400 \times 25} = \frac{1}{400}$.

Reduction of Recurring or Circulating Decimals to Fractions.

26. There are two kinds, and it is convenient to treat them separately.

(1.) *Pure circulating decimals*, where the whole of the decimal figures recur.

Rule.—Take away the decimal point and the dots, make a numerator of the integer thus obtained, and place under this as denominator as many *nines* as there are recurring figures.

The following example will make this rule clear.

Ex.—Reduce $\cdot\dot{2}0\dot{7}$ to a fraction.

The value of the decimals is evidently not altered by writing it $\cdot 207\dot{2}0\dot{7}$. Let us remove the decimal point three places to the right, or, what is the same thing (Art. 1), let us multiply the given decimal by 1000.

We then get $207\cdot\dot{2}0\dot{7}$ as the value of 1000 times the given decimal. Now the number $207\cdot\dot{2}0\dot{7}$ includes the integer 207, and the given decimal $\cdot\dot{2}0\dot{7}$; and it therefore follows that the integral part 207 is $(1000 - 1)$, or 999 times the value of the given decimal.

Hence, dividing it by 999, we get

$$\cdot\dot{2}0\dot{7} = 207 \div 999 = \tfrac{207}{999}.$$

(2.) *Mixed circulating decimals.*—Where part only of the figures recurs.

Rule.—Take away the decimal point and the dots, subtract from the integer thus obtained the integer formed by the figures which do not recur, and make a numerator of the result. Then, for a denominator, place underneath as many *nines* as there are recurring figures, followed by as many ciphers as there are figures which do not recur.

We shall make this rule clear by the following example:—

Ex.—Reduce $\cdot 24\dot{5}7\dot{3}$ to a fraction.

Let us remove the decimal point *two* places to the right, thus, by Art. 1, multiplying the given decimal by 100; we then get

$$100 \text{ times the value of } \cdot 24\dot{5}7\dot{3} = 24\cdot\dot{5}7\dot{3}.$$

Now, by case (1) above, $24\cdot\dot{5}7\dot{3} = 24\tfrac{573}{999}$; or reducing to an improper fraction, and noticing that $24 \times 999 = 24000 - 24$, we have

$$24\cdot\dot{5}7\dot{3} = \frac{24000 - 24 + 573}{999} = \frac{24573 - 24}{999}.$$

Hence, dividing this result by 100, we get

$$\cdot 24\dot{5}7\dot{3} = \frac{24573 - 24}{999} \div 100 = \frac{24573 - 24}{99900} \text{ (Art. 24.)}$$

Ex. 1.—$\cdot\dot{4}2857\dot{1} = \frac{428571}{999999} = \frac{3 \times 142857}{7 \times 142857} = \frac{3}{7}.$

Ex. 2.—$\cdot 27\dot{0}\dot{9} = \frac{2709 - 27}{9900} = \frac{2682}{9900} = \frac{149 \times 2 \times 9}{550 \times 2 \times 9} = \frac{149}{550}.$

27. In arithmetical operations involving circulating decimals, and, indeed, any decimals having a large number of decimal figures, it is generally sufficient to obtain a result correct to a given number of decimals.

1. In *addition* and *subtraction* we obtain this result most easily by using in our operation *one* or *two* more figures of the given decimals than are required in the result.

Ex. 1.—Add together (correct to five places) the following:—

$$\cdot\dot{3}02\dot{6},\ 6\cdot729\dot{4},\ \cdot01\dot{6},\ \cdot4163729.$$

$$\begin{array}{r} \cdot3026026 \\ 6\cdot7294444 \\ \cdot0166666 \\ \cdot4163729 \\ \hline 7\cdot4650865. \end{array} \quad \text{Ans. } 7\cdot46508.$$

NOTE.—If our object is to obtain a decimal of five places which shall give the *approximate* sum of the given decimals we must write 7·46509 as the approximate sum; for 7·46509 is nearer to the true value of 7·465086 &c. than 7·46508. The general rule is to increase by 1 the decimal figure at which we stop, *when the next figure is 5 or above 5.*

Ex. 2.—Find the difference (correct to six places) of $3\cdot07\dot{4}\dot{5}$ and $4\cdot\dot{2}6\dot{3}$, and express the approximate difference by a decimal of five places.

$$\begin{array}{r} 4\cdot26326326 \\ 3\cdot07454545 \\ \hline 1\cdot18871781 \end{array}$$

Hence the difference correct to six places is 1·188717, and the required approximate difference 1·18872.

2. In *multiplication* and *division* of circulating decimals it is generally preferable to reduce the given decimals to fractions, bring out the result in a fractional form, and afterwards reduce this to a decimal.

Ex. VII.

1. Express as a decimal the sum of the following fractions:—

$$\tfrac{3}{8},\ \tfrac{15}{96},\ \tfrac{7}{112},\ \tfrac{35}{28},\ \tfrac{7}{64},\ \tfrac{1}{4}.$$

2. Reduce to fractions the following decimals :—

$$\cdot35,\ \cdot026,\ \cdot\dot{1}\dot{6},\ \cdot\dot{1}4285\dot{7},\ \cdot1\dot{6},\ \cdot4\dot{2}8571\dot{4}.$$

3. Find the value (correct to six places) of $\cdot2\dot{3}\dot{7} + 3\cdot\dot{8}1\dot{6} - 6\cdot023\dot{5} + 4\cdot2\dot{9} - \cdot0\dot{0}\dot{2} + 1\cdot37\dot{4}$.

4. Add together $\cdot 6\dot{2}$, $\cdot\dot{0}3\dot{7}$, $2\cdot 47\dot{6}\dot{1}$, $\cdot 8\dot{1}0\dot{6}$, $\cdot\dot{7}$, $\cdot 375$.

5. Find the product and quotient of $3\cdot\dot{5}\dot{4}$ by $4\cdot\dot{3}$.

6. Simplify the expression—

$$(4\cdot\dot{6} \times \cdot\dot{4}2857\dot{1} - 2\cdot 2 \times \cdot\dot{3}\dot{6})\,(1 - \cdot 1\dot{6}).$$

In the next six examples the dots are signs of multiplication, and you are required to give the values of the expressions correct to six places.

7. $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \&c.$

8. $\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \&c.$

9. $\frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \&c.$

10. $1 + \frac{1}{8^2} + \frac{1}{8^4} + \frac{1}{8^6} + \&c.$

11. $16\left\{\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \&c.\right\} - 4\left\{\frac{1}{239} - \frac{1}{3\cdot 239^3} + \&c.\right\}$

12. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7}.$

CHAPTER III.

APPLICATION OF THE PRECEDING ARTICLES TO CONCRETE QUANTITIES.

To find the Value of a Fraction of a Concrete Quantity.

28. Rule.—*Multiply the concrete quantity by the numerator of the fraction, and divide the product by the denominator.*

Ex. 1.—Find the value of $\frac{3}{7}$ of 2 tons, 3 cwt. 21 lbs.

$$\tfrac{3}{7} \text{ of 2 tons, 3 cwt. 21 lbs.} = \frac{(\text{2 tons, 3 cwt. 21 lbs.}) \times 3}{7} = \frac{\text{6 tons, 9 cwt. 2 qrs. 7 lbs.}}{7}$$
$$= \text{18 cwt. 2 qrs. 1 lb.} \quad \textit{Ans.}$$

Ex. 2.—Find the value of $3\frac{2}{9}$ of £12. 6s. $2\frac{1}{4}$d.

		£.	s.	d.
3 times £12. 6s. $2\frac{1}{4}$d.	$= (\text{£12. 6s. } 2\frac{1}{4}\text{d.}) \times 3 =$	36	18	$6\frac{3}{4}$
$\frac{2}{9}$ of £12. 6s. $2\frac{1}{4}$d.	$= \dfrac{(\text{£12. 6s. } 2\frac{1}{4}\text{d.}) \times 2}{9}$			
	$= \dfrac{\text{£24. 12s. } 4\frac{1}{2}\text{d.}}{9} =$	2	14	$8\frac{1}{2}$
Hence, adding, the value required	=	£39	13	$3\frac{1}{4}$

Ex. 3.—Find the value of $\frac{3}{7}$ of 4 miles + $\frac{7}{9}$ of 3 fur. + $\frac{5}{21}$ of 8 poles.

			Mile.	Fur.	Poles.	Yards.
$\frac{3}{7}$ of 4 miles	$= \frac{(4 \times 3)\text{ miles}}{7}$	$= \frac{12\text{ miles}}{7} =$	1	5	28	$3\frac{1}{7}$
$\frac{7}{9}$ of 3 fur.	$= \frac{(3 \times 7)\text{ fur.}}{9}$	$= \frac{21\text{ fur.}}{9} =$	0	2	13	$1\frac{5}{6}$
$\frac{5}{21}$ of 8 poles	$= \frac{(8 \times 5)\text{ poles}}{21}$	$= \frac{40\text{ poles}}{21} =$	0	0	1	$4\frac{41}{42}$
Hence, adding, the value required		=	2	0	3	$4\frac{19}{42}$

NOTE.—The addition of the yards is thus effected :—

$$(3\tfrac{1}{7} + 1\tfrac{5}{6} + 4\tfrac{41}{42}) \text{ yds.} = (8 + \tfrac{6+35+41}{42}) \text{ yds.} = (8 + 1\tfrac{20}{21}) \text{ yds.}$$
$$= 9\tfrac{20}{21} \text{ yds.} = 1 \text{ pole}, (3\tfrac{1}{2} + \tfrac{20}{21}) \text{ yds.}$$
$$= 1 \text{ pole}, (3 + \tfrac{21+40}{42}) \text{ yds.} = 1 \text{ pole}, 4\tfrac{19}{42} \text{ yds.}$$

Ex. VIII.

Find the values of—

1. $\frac{2}{3}$ of £1; $\frac{3}{5}$ of 1s.; $\frac{2}{9}$ of 12s.; $\frac{3}{7}$ of £3.
2. $\frac{7}{8}$ of £5; $\frac{2}{7}$ of a guinea; $\frac{5}{11}$ of 2s. 6d.; $\frac{3}{4}$ of a crown.
3. $3\frac{2}{3}$ of £1. 12s.; $2\frac{1}{2}$ of £3. 10s.; $7\frac{9}{11}$ of £3. 4s. $5\frac{1}{2}$.
4. $\frac{6}{7}$ of 1 ton; $\frac{3}{5}$ of 1 qr.; $\frac{4}{9}$ of 1 stone; $\frac{3}{11}$ of 9 lbs.
5. $3\frac{2}{7}$ mile; $\frac{4}{9}$ of 3 fur.; $\frac{2}{7}$ of 15 poles; $3\frac{1}{7}$ of $2\frac{1}{5}$ of 12 yds.
6. $\dfrac{8}{20\frac{1}{3}}$ leap year; $\dfrac{2}{1\frac{3}{4}}$ lunar month; $\dfrac{5\frac{1}{9}}{10\frac{2}{3}}$ of 10 h. 15′ 12″.
7. 12 lbs. 3 oz. 7 dwt. 5 grs. $\times\ 3\frac{1}{7}$; 10 oz. 4 gr. $\div\ 16\frac{1}{2}$.

8. $8\frac{3}{8}$ of 4 ac. 3 po. $-\frac{7}{16}$ of 1 sq. mile $+ 1\frac{2}{9}$ of 3 r. 20 sq. yds.

9. ($5\frac{3}{4}$ of $\frac{9}{38} \div 1\frac{4}{19}$) of 35° 36′ 25″; ($2\frac{3}{16} \div 1\frac{1}{8}$ of $1\frac{2}{3}$) of 30°.

10. $\frac{3}{10}$ of 1 lb. Troy + $\frac{5}{6}$ of 1 lb. Troy $-\frac{3}{56}$ of 1 lb. Avoirdupois (= 7000 grains).

11. $\frac{2}{9}$ of $\dfrac{5\frac{1}{2} \text{ of 3 guineas}}{\frac{3}{7} \text{ of £1. 3s. 10d.}}$; $\frac{2}{9}$ of 1 ton + $\frac{3}{7}$ of 3 qrs. $-\frac{1}{18}$ of 7 cwt.

12. $2\frac{3}{13}$ of 15 h. 10′ $13\frac{1}{5}$″ $-\frac{5}{27}$ of 1 day, 12 h. 11′ $12\frac{1}{9}$″.

To Reduce a given Quantity to the Fraction of any other given Quantity of the same kind.

29. Rule.—Reduce both the given quantities to the same denomination, and the fraction required will have the number of units in the first quantity as numerator, and that of the second quantity as denominator.

Ex. 1.—Reduce 3s. 8d. to the fraction of £1.

3s. 8d. = 44d,
and £1 = 240d.

Hence, the fraction required $= \frac{44}{240} = \frac{11}{60}$.

Or, better, thus:

3s. 8d. = 11 fourpences,
and £1 = 60 fourpences.

∴ Fraction required $= \frac{11}{60}$.

Note.—It is always best to keep the denominations to which the given quantities are reduced as high as possible.

Ex. 2.—Reduce $\frac{1}{7}$ of a moidore to the fraction of $2\frac{1}{2}$ guineas.

$\frac{1}{7}$ of a moidore = ($\frac{1}{7} \times 27$)s., and $2\frac{1}{2}$ guineas = ($2\frac{1}{2} \times 21$)s.

Hence, the fraction required $= \dfrac{\frac{1}{7} \times 27}{2\frac{1}{2} \times 21} = \dfrac{\frac{1}{7} \times 27}{\frac{5}{2} \times 21} = \dfrac{1 \times 27 \times 2}{5 \times 21 \times 7}$

$= \dfrac{1 \times 9 \times 2}{5 \times 7 \times 7} = \dfrac{18}{245}$

Ex. 3.—Reduce 3 cwt. $2\frac{1}{9}$ qrs. to the fraction of 4 cwt. 2 qrs. 4 lbs.

3 cwt. $2\frac{1}{9}$ qrs. = $14\frac{1}{9}$ qrs., and 4 cwt. 2 qrs. 4 lbs. = 4 cwt. $2\frac{1}{7}$ qrs. = $18\frac{1}{7}$ qrs.

∴ Fraction required $= \dfrac{14\frac{1}{9}}{18\frac{1}{7}} = \dfrac{\frac{127}{9}}{\frac{127}{7}} = \dfrac{127 \times 7}{9 \times 127} = \frac{7}{9}$.

Ex. IX.

Reduce—

1. 1s. 8d. to the fraction of £1; $7\frac{1}{2}$d. to the fraction of 10s.

2. 2s. 4d. to the fraction of 10s. 8d.; 1s. $7\frac{1}{2}$d. to the fraction of 3s. $4\frac{1}{2}$d.

3. 3 qrs. 15 lbs. to the fraction of 1 ton; 2 stones 10 oz. to the fraction of 3 cwt.

4. 3 lbs. avoirdupois to troy weight; 10 lbs. 3 oz. 4 dwt. troy to avoirdupois.

5. 3 quires, 10 sheets to the fraction of 2 reams, 3 quires; 3 ft. $8\frac{1}{5}$ in. to the fraction of 3 yards.

6. $30^\circ\ 3'\ 12''$ to the fraction of a right angle $(=90^\circ)$; $57^\circ\ 16'\ 21\frac{9}{11}''$ to the fraction of two right angles.

What fraction is—

7. $\frac{3}{4}$ ac. of $3\frac{1}{2}$ ac.; $2\frac{1}{3}$ days of 17 weeks?

8. $\dfrac{\cdot125+1\cdot875}{\cdot0140625}$ acres of $19\frac{1}{5}$ poles; $\dfrac{3\cdot1\dot{6}\times1\cdot4}{2\cdot375\times\cdot\dot{1}}$ yds. of $3\frac{1}{7}$ m?

9. $\left(\dfrac{3}{4\frac{1}{5}}\text{ rood}+\dfrac{7\frac{1}{6}}{4\frac{7}{9}}\text{ poles}+\dfrac{1\frac{1}{2}\times1\frac{4}{11}}{2\frac{1}{2}}\text{ yd.}\right)$ of 3 acres?

10. $\dfrac{\frac{5}{7}}{2\frac{8}{21}}$ gal. of $\dfrac{3\frac{7}{26}}{2\frac{6}{13}}$ pipes?

11. What fraction of his original income has a person left after paying a tax of 4d. in the £?

12. A garden roller is 2 ft. 6 in. wide, and it is rolled at the rate of 1 mile in 20 minutes: find in what fraction of a day a man will roll $\frac{1}{6}$ of an acre.

To Find the Value of a Decimal of a Concrete Quantity.

30. Rule.—Multiply the given decimal by the number of units in the concrete quantity when expressed in terms of one denomination, and the integral part of the result will be the number of units of this denomination. Then multiply the decimal part of this denomination by the number of units connecting it with the next lower and the integral part will be units of this latter denomination, and so on, Ex. 1.—Find the value of ·325 of £3. 10s.

£3. 10s. = 70s.; proceeding then according to rule, we have :

```
 ·325
   70
22·750s.
   12
 9·00d.
```

Ans.: 22s. 9d. or £1. 2s. 9d.

Ex. 2.—Find the value of ·546875 of 3 tons.

```
 ·546875
       3
1·640625 tons.
      20
12·812500 cwt.
      4
3·2500 qrs.
  28
 200
 50
7·00 lbs.
```

Ans.: 1 ton, 12 cwt. 3 qrs. 7 lbs.

Ex. 3.—Find the value of 6·66875 acres.

```
6·66875 acres.
      4
2·67500 roods.
     40
27·000 poles.
```

Ans.: 6 acres, 2 roods, 27 poles.

Ex. 4.—Find the value of $\cdot 31\dot{6}$ of £1.

First Method.

```
     £
·3166667 nearly.
      20
6·3333340s.
      12
4·000008d.
```

Ans.: 6s. 4d.

Second Method.

$$\cdot 31\dot{6} = \frac{316 - 31}{900} = \frac{285}{900}$$

$$= \frac{15 \times 19}{15 \times 60} = \frac{19}{60}$$

Then by rule for reduction of fractions,

£$\frac{19}{60}$ = 6s. 4d. Ans.

The latter method is preferable when perfect accuracy is required.

Ex. X.

Find the value of—

1. £·375; £·98125; £·815625.

2. $\cdot 41\dot{6}$ of £3; $\cdot\dot{4}2857\dot{1}$ of 6s. 5d.; $8\cdot\dot{5}7142\dot{8}$ of 3s. $0\frac{3}{4}$d.

3. ·625 of 1 ton; ·046875 of 3 tons; $4\cdot\dot{3}\dot{9}$ of 1 cwt. 53 lbs.

4. 1·6671875 acres; ·3475 rood; $\cdot\dot{0}7692\dot{3}$ of 5 acres, 6 poles.

5. $\cdot 208\dot{3}$ of 1 ream; $\cdot 458\dot{3}$ quire; $\cdot 38\dot{3}$ of 3 reams, 12 sheets.

6. ·3078125 pipe; ·490625 tun; ·37125 bushel.

7. Express in grains $\cdot\dot{1}4285\dot{7}$ of 4 lbs. avoirdupois, and express the result in troy weight.

8. Express 10 oz. 3 dwt. 14 grs. as the decimal of 1 lb. avoirdupois.

9. What is the sum of $\cdot\dot{6}$ of 1 guinea, $\cdot 08\dot{3}$ of 1 crown, and $\cdot\dot{0}3\dot{7}$ of £1. 0s. 3d.

10. Find the value of

$$\left(4 \div \frac{\cdot 2041\dot{6} \times 7\cdot 5}{\cdot 21875}\right) \text{cwt.} + (\cdot 95 \text{ ton} - \dot{7}6923\dot{0} \text{ qrs.}) \times 26.$$

11. What is the value of

$$\left(3 + \cfrac{\cdot\dot{3}}{1 + \cfrac{\cdot\dot{1}}{1 + \cfrac{\cdot\dot{3}}{1 + \cfrac{\cdot\dot{1}}{\cdot\dot{3}}}}}\right) \text{ of 1 cwt. 35 lbs.?}$$

12. Simplify $\left(\frac{1 + \cdot 1\dot{6}}{1 - \cdot 1\dot{6}} + \frac{1 - \cdot 1\dot{6}}{1 + \cdot 1\dot{6}}\right)$ of 1 oz. 15 dwts.

To Reduce from one Denomination to the Decimal of another Denomination of the same kind.

31. Rule.—Bring the given quantity to the fraction of the proposed denomination, and reduce this fraction to a decimal.

Ex. 1.—Reduce 3s. 3d. to the decimal of 8s. $1\frac{1}{2}$d.

$$3\text{s. }3\text{d.} = 26 \text{ three-halfpence},$$
$$8\text{s. }1\tfrac{1}{2}\text{d.} = 65 \text{ three-halfpence},$$
$$\text{and the fraction } \tfrac{26}{65} = \frac{2 \times 13}{5 \times 13} = \tfrac{2}{5} = \cdot 4.$$

Hence ·4 is the decimal required.

Ex. 2.—Reduce 3 qrs. 21 lbs. to the decimal of 2 cwt. 3 qrs.

$$3 \text{ qrs. } 21 \text{ lbs.} = 105 \text{ lbs.}$$
$$2 \text{ cwt. } 3 \text{ qrs.} = 308 \text{ lbs.}$$
$$\text{and the fraction } \tfrac{105}{308} = \frac{15 \times 7}{44 \times 7} = \tfrac{15}{44} = \frac{3\cdot 75}{11} = \cdot 34\dot{0}\dot{9}.$$

Hence $\cdot 34\dot{0}\dot{9}$ is the decimal required.

Ex. 3.—Reduce 18s. $8\frac{1}{4}$d. to the decimal of £1.

The rule may be applied most concisely as follows:—

4	1·
12	8·25
20	18·6875
	·934375

∴ ·934375 is the decimal required.

The farthing is first reduced to the decimal of a penny, and the 8d. prefixed; then 8·25d. are reduced to the decimal of a shilling, and the 18s. prefixed; lastly, 18·6875s. are reduced to the decimal of £1.

Ex. 4.—Reduce 3 qrs. 21 lbs. to the decimal of 1 ton.

28 {	7	21
	4	3·
	4	3·75
	20	·9375
		·046875

Hence ·046875 is the decimal required.

Ex. XI.

Reduce—

1. 12s. 6d., 10s. $7\frac{1}{2}$d., 11s. $1\frac{1}{2}$d., 18s. $6\frac{3}{4}$d., each to the decimal of £1.

2. 13s. $0\frac{3}{4}$d., 10s. $8\frac{1}{4}$d., 9s. 6d., each to the decimal of 15s. $5\frac{1}{4}$d.

3. 7s. $8\frac{3}{4}$d. to the decimal of a guinea; and 3s. $2\frac{1}{4}$d. to the decimal of a moidore.

4. 1 qr. 7 lbs. to the decimal of 1 ton; 3 cwt. 3 qrs. 20 lbs. to the decimal of 3 tons.

5. $3\frac{1}{2}$ lbs. to the decimal of 10 cwt.; 14 oz. to the decimal of 3 cwt. 2 qrs.

6. 20 grs. to the decimal of 1 lb. Troy; 3 dwts. 16 grs. to the decimal of 4 oz. 11 dwts.

7. 1 rood, 10 poles to the decimal of 1 acre; 3 roods, $15\frac{1}{8}$ square yards to the decimal of 5 acres.

8. $\frac{3}{8}$ hours to the decimal of 10 weeks; 7 h. 18′ to the decimal of 1 year (365 days).

9. Bring the sum of $\frac{4}{15}$ of 9 hours, $\frac{5}{8}$ of $12\frac{1}{2}$ days, $\frac{2}{3}$ of $7\frac{1}{2}$ minutes, to the decimal of a week.

10. Express a pound troy as the decimal of a pound avoirdupois.

11. Reduce the sum of 6 lbs. $6\frac{6}{7}$ oz. avoirdupois, and 8 oz. 6 dwts. 16 grs. to the decimal of 1 ton.

12. Express $2{\cdot}3\dot{6}$ of 4s. $-$ $\cdot5\dot{1}\dot{8}$ of 9s. 2d. + $1{\cdot}458\dot{3}$ of 6d. as the decimal of £5.

CHAPTER IV.

THE METRIC SYSTEM.

32.—The fundamental unit of the metric system is the *metre*. A metre is the ten-millionth, or $\frac{1}{10^7}$ part of 90° of the earth's meridian, and measures 39·3708 English inches. In order to express multiples and sub-multiples of this unit, and, indeed, of *any* unit in the metric system, we make use of one or more of the following prefixes:—

Deka, . .	10 times.	**Deci**, . .	10th.
Hecto, .	100 ,,	**Centi**, . .	100th.
Kilo, . .	1,000 ,,	**Milli**, . .	1,000th.
Myria, .	10,000 ,,		

We will arrange these prefixes and the word *unit* in order according to their signification, thus—

Myria, Kilo, Hecto, Deka, Unit, Deci, Centi, Milli.

Now, as we read this line from left to right, it is evident that the words have a signification decreasing tenfold in value; and as we read it from right to left, they have a signification increasing tenfold in value.

It therefore follows that figures placed under the above words have a local as well as an intrinsic value; and further, if when a figure is wanting to complete the series, its place be filled up by a cipher, it will be seen that the local value corresponds exactly with the ordinary decimal notation.

Moreover, we have only to place a mark (in fact, a decimal point) at the right of the figure standing under any of the words of the above memorial line, and the given quantity is at once expressed in the denomination corresponding to that figure.

Thus, taking the metre as our unit:

	Myriam.	Kilom.	Dekam.	Metres.	Decim.	Millim.	
	3	2	5	4	7	8	
Myriam.	Kilom.	Hectom.	Dekam.	Metre.	Decim.	Centim.	Millim.
= 3	2	0	5	4	7	0	8

= 3·2054708 myriametres.
= 32·054708 kilometres.
= 320·54708 hectometres.
= 3205·4708 dekametres.
= 32054·708 metres.
= 320547·08 decimetres.
= 3205470·8 centimetres.
= 32054708· or 32054708 millimetres.

The following rule for expressing any quantity in terms of any one multiple or sub-multiple of the unit, or of the unit itself, is therefore evident:—

Rule.—Put ciphers in the place of any multiple, unit, or sub-multiple absent in the series, and write the figures in close order, as in the ordinary decimal notation. Then place a decimal point at the *right* of the figure corresponding to the denomination in which we wish to express the given quantity.

Ex. 1.—Express 5 myriam. 3 hectom. 6 decim. as metres.

Filling up with ciphers the vacant spaces, we have—

Myriam.	Kilom.	Hectom.	Dekam.	Metres.	Decim.
5	0	3	0	0	6

= 50300·6 metres.

Ex. 2.—Express 3 dekam. 4 decim. as myriametres.

Filling up with ciphers, we have—

Myriam.	Kilom.	Hectom.	Dekam.	Metres.	Decim.	Centim.	Millim.
0	0	0	3	0	4	0	0

= 0·0030400 myriametres = 0·00304 myriametres.

(The student will see that it was unnecessary here to extend the series beyond decimetres.)

Ex. 3.—Express 13 metres 502 millimetres as kilometres. We may write the given quantity thus—

Kilom.	Hectom.	Deckam.	Metres.	Decim.	Centim.	Millim.
0	0	1	3	5	0	2

= 0·013502 kilometres.

We have hitherto spoken only of the fundamental unit, and its multiples and sub-multiples. We shall hereafter (Art. 35) explain the principal derived units, viz., the Gram, the Are, the Stere, the Litre, and the Franc; but as the multiples and sub-multiples of these derived units bear the same relation respectively to the corresponding derived unit, as in the case of the fundamental unit, all the preceding remarks relative to the multiples and sub-multiples of the fundamental unit apply equally to those of the Gram, the Are, the Stere, the Litre, and the Franc.

With regard to the units, multiples, and sub-multiples of square and cubic measure, properly so called, it is necessary to make a few remarks.

33. Square Measure.—The unit of square measure is the *square metre;* and since the series *myriam., kilom., hectom., dekam., metre.,* &c., decrease in value tenfold when read from left to right, and increases similarly when read from right to left, it follows that the series square myriam., square kilom., square hectom., square dekam., square metre, &c., will decrease or increase 10^2 or 100-fold. Hence we see that in square measure the multiples and sub-multiples increase or decrease successively 100-fold, and, therefore, when quantities in square measure are expressed by the ordinary decimal notation, each multiple or sub-multiple must occupy the place of two figures, a cipher being supplied when we have less than ten of any multiple or sub-multiple, and two ciphers when there is any blank in the series.

	Sq. kilom.	Sq. hectom.	Sq. dekam.	Sq. metre.	Sq. centim.
Ex.	10	3	15	3	5

	Sq. kilom.	Sq. hectom.	Sq. dekam.	Sq. metre.	Sq. decim.	Sq. centim.
=	10	03	15	03	00	05

= 10·0315030005 square kilometres.
= 1003·15030005 ,, hectometres.
= 100315·030005 ,, dekametres.
= 10031503·0005 ,, metres.
= 1003150300·05 ,, decimetres.
= 100315030005 ,, centimetres.

34. Cubic Measure.—The unit of cubic measure is the *cubic metre*, and hence after the remarks in the last article, since $10^3 = 1000$, when quantities in cubic measure are expressed by the ordinary decimal notation, the units, multiples, and sub-multiples must respectively occupy the place of three figures, ciphers being supplied to fill up blank spaces when necessary.

Ex. 1. 325 cubic metres 51 cubic decimetres.
= 325 ,, ,, 051 ,, ,,
= 325·051 cubic metres.
= 325051 cubic decimetres.

Ex. 2.—25 cubic metres 3 cubic decim. 40 cubic centim.
= 25 cubic metres 003 cubic decim. 040 cubic centim.
= 25·003040 cubic metres.
= 25003·040 ,, decimetres.
= 25003040 ,, centimetres.

35. Derived Units.—The principal derived units of the metric system are—

1. The **Gram**, for measures of weight.

The *gram* is the weight of a cubic centimetre of distilled water at the temperature of 4° C.

1 gram = 15·4323 grains, 1 grain = ·0648 gram.

2. The **Are**, for land measure.

The *are* is a square whose side measures 10 metres; it is therefore equal to a square dekametre, or 100 square metres.

1 are = 119·6033 square yards, 1 hectare = 2·471 acres, 1 acre = ·405 hectare.

3. The **Stere,** for fire-wood.

The *stere* is equivalent to a cubic metre. It is therefore the solidity of a cube whose edge measures 1 metre.

4. The **Litre,** for measures of capacity.

The *litre* is a capacity equal to the volume of a cube whose edge measures a decimetre or 10 centimetres. It is therefore equal to a cubic decimetre or 1,000 cubic centimetres, and 1,000 litres are equivalent to a cubic metre.

1 litre = ·2201 gallon, 1 gallon = 1·543 litres, 11 gallons = 50 litres nearly.

5. The **Franc,** for money.

The *franc* is a coin weighing 5 grams, and composed of an alloy, nine-tenths of which are silver and one-tenth copper.

The following table exhibits at a glance the fundamental unit and the above derived units, together with the multiples and sub-multiples at present in use :—

TABLE OF THE METRIC SYSTEM OF WEIGHTS AND MEASURES.

MULTIPLES.				UNITS.	SUB-MULTIPLES.		
10,000	1,000	100	10		10th.	100th.	1,000th.
Myria.	Kilo.	Hecto.	Deka.	METRE, Long Measure.	Deci.	Centi.	Milli.
Myria.	Kilo.	Hecto.	Deka.	GRAM, Weight.	Deci.	Centi.	Milli.
		Hecto.	..	ARE, Land Measure.		Centi.	
			Deka.	STERE, Solid Measure.	Deci.		
		Hecto.	Deka.	LITRE, Capacity.	Deci.	Centi.	Milli.
				FRANC, Money.	Decime.	Centime.	

A quintal = 100 kilog. = 2 cwt. nearly; a millier, or tonneau de mer, = 10 quintals = 20 cwt. nearly.

Ex. XII.

Examples upon the Multiples and Sub-Multiples of the Units.

1. Express each of the following as metres—
15 myriam.; 20 kilom.; 1 hectom.; 27 dekam.; 25 decim.; 100 centim.; 345 centim.; 5294 millim.

2. How many centimetres in the following—
46 myriam.; 30 kilom.; 295 hectom.; 1·5 dekam.; 3·95 metres, 295 millim.?

3. Express according to the metric table—
20 kilog. 29 dekag.; 18 kilog. 85 decig.; 123 hectog. 13 centig.; 12 dekag. 296 millig.; 153 centig. 3 millig; 3427 millig.

4. How many decigrams in the following—
16 kilog. 12 centig; 25 hectog. 10 grams; 39,645 millig.; 20 kilog. 35 dekag. 5 grams?

5. Express in ares—
10 hectar.; 296 centiar.; 29 hectar. 3 centiar.; 3 hectar. 12 centiar.; 376,543 centiar.

6. How many square decimetres are there in the following—
100 sq. kilom.; 10 sq. hectom.; 5 sq. dekam.; 3498 sq. met.; 46 sq. met.?

7. How many steres in the following quantities—
15 dekas.; 394 decis.; 9 dekas. 2 decis.; 186 dekas. 3 decis.; 3764 decis.; 4 decis.?

8. Express as cubic metres—
10,000 cubic decim.; 1,234,567 cubic centim.; 372,456,126 cubic millim.; 1,000,000 cubic centim.; 639 cubic centim.; 293 cubic decim.

9. Express as litres—
3 kilol. 2 hectol. 3 litres; 4 decil.; 2 kilol. 3 millil.; 76,384 millil.; 2934 centil.; 830 dekal.; 34,576 decil.

10. Express as litres, as dekalitres, and as centilitres—
18 kilol. 3 hectol. 4 decil. 5 centil. 3 millil.

11. How many francs in the following sums of money—
100 cent.; 736 dec.; 24,645 cent.; 5 cent. 25 dec.; 1695 cent.?

12. How many centimes and how many decimes are expressed by the following—

13 francs; 7 fr. 13 c.; 12 fr. 3 dec. 5 c.; 29 c.; 3 fr. 2 dec.; 18 fr. 4 c.?

Addition, Subtraction, Multiplication, and Division in the Metric System.

36. Since (Art. 33) all quantities in the metrical system may be expressed as one denomination by figures whose local as well as intrinsic values follow the decimal system of notation, it is evident that when they are so expressed we may add, subtract, multiply, or divide them exactly as ordinary integers and decimals.

Ex. 1.—Add together 49 metres 36 centim.; 3 kilom. 2 dekam. 3 decim.; 2 hectom. 3 metres 25 centim.; and 13 dekam. 327 millim.

Metres.		Millimetres.
49·36		49360
3020·3		3020300
203·25	or thus, by integers—	203250
130·327		130327
3403·237		3403237

Ans.: 3 kilom. 4 hectom. 3 metres, 2 decim. 3 centim. 7 millim.

Ex. 2.—Subtract 3 kilog. 2 dekag. 37 millig. from 10 myriag. 25 grams, 369 millig.

Kilograms.		Milligrams.
100·025369		100025369
3·020037	or thus, by integers—	3020037
97·005332		97005332

Ans.: 9 myriag. 7 kilog. 5 grams, 3 decig. 3 centig. 2 millig.

Ex. 3.—Multiply 12 dekasteres, 3 steres, 5 decisteres by 23.

Steres.		Decisteres.
123·5		1235
23		23
3705	or thus, by integers—	3705
2470		2470
2840·5 steres.		28405 decisteres.

Ans.: 284 dekasteres, 5 decisteres.

Ex. 4.—Multiply 455,602 cubic centimetres by 36.

Cubic Metres.		Cubic Centimetres.
0·455602		455602
36		36
2733612	or thus, by integers—	2733612
1366806		1366806
16·401672		16401672

Ans.: 16 cub. met. 401 cub. decim. 672 cub. centim.

Ex. 5.—Divide 1369 kilol. 35 lit. 36 centil. by 72.

	Kilolitres.			Centilitres.
72 { 9	1369·03536		72 { 9	136903536
8	152·11504	or thus, by integers—	8	15211504
	19·01438			1901438

Ans.: 19 kilol. 1 dekal. 4 lit. 3 decil. 8 centil., or 19 kilol. 12 lit. 38 centil.

Ex. 6.—How many times is 12 sq. dekam. 3 sq. met. 15 sq. decim. contained in 216 sq. dekam. 56 sq. met. 70 sq. decim.?

Sq. Dekam. Sq. Dekam.		Sq. Decim. Sq. Decim.
12·0315)216·5670(18		120315)2165670(18
120315		120315
962520	or thus, by integers—	962520
962520		962520

Ans.: 18.

Ex. XIII.

1. Add together—

(1.) 3 metres, 2 decim. 4 centim.; 18 metres, 219 millim.; 4 kilom. 2 hectom. 3 dekam. 14 centim.; 12 kilom. 36 metres.

(2.) 74006 hectom., 3216 kilom.; 12 myriam. 2167 metres.

(3.) 4 sq. met. 42 sq. decim.; 12 sq. dekam. 18 sq. decim.; 82 sq. met. 3250 sq. decim.; 3·271 sq. met.

(4.) 18 cub. met. 186 cub. decim.; 39·207365 cub. met. 30761 cub. centim.; 12 cub. met. 124·27 cub. decim.

(5.) 25 kilog. 235 grams; 3072 centig.; 13 kilog. 51 grams, 63 millig. 8132·07 decig.

(6.) 319 hectar. 4 ares, 51 centiar.; 93·712 hectar. 23756·27 ares; 6 hectar. 4 centiar.

(7.) 3 steres, 5 decis.; 209 steres, 4 decis.; 25·76 steres, 13·027 dekas.

(8.) 51 kilol. 126 lit. 32 centil.; 123 lit. 3 centil. 15·02703 kilol.; 12 kilol. 3·27602 hectol.

(9.) 161 fr. 35 c.; 32 fr. 4 c.; 8276 c.; 10·26 fr.; 16 decimes, 5 c.

2. Subtract—

(1.) 4 metres, 372 millim. from 16 hectom. 5·06 metres.

(2.) 30765 centim. from 12 kilom. 4 metres, 9 millim.

(3.) 3 sq. met. 89 sq. decim. from 1 sq. dekam. 7 sq. decim.

(4.) 12·0324 cub. met. from 18 cub. met. 29 cub. millim.

(5.) 39 grams, 65 millig. from 6 kilog. 12 grams.

(6.) 8 hectar. 19·08 ar. from 32 hectar. 70 ar. 2 centiar.

(7.) 9 dekas. 6 decis. from 50 dekas. 2 decis.

(8.) 6 kilol. 6 millil. from 700 kilol. 3 lit. 3 centil.

(9.) 65 c. from 3 fr.; and 2 fr. 4 c. from 100 fr. 60 c.

3. Multiply—

(1.) 10 metres, 35 millim. by 7, 11, 13.

(2.) 18 kilom. 3·07 metres by 27, 48, 64.

(3.) 3·0625 sq. met. by 16, 18, 35.

(4.) 4 cub. met. 10 cub. decim. 5 cub. millim. by 19, 23, 26.

(5.) 7364 hectog. 9·31 decig. by 15, 25, 20.

(6.) 12 hectar. 3 centiar. by 30, 50, 40.

(7.) 416 steres, 2·9 decis. by 100, 150, 60.

(8.) 612305·06 litres by 12, 14, 16.

(9.) 39 fr. 10 c. by 75, 105, 135.

4. A merchant owed 1500 fr., and he gave in payment 69 metres of cloth at 3 fr. 4 c. per metre, 48 metres of silk at 8 fr. 65 c., 13·5 metres of calico at 75 c. How much does he still owe?

5. Make out the following bill—

		fr.	c.		
44 hectol. of oil, . .	at	0	75	the	litre.
66 kilog. 125 gr. of sugar,	„	1	25	„	kilog.
375 gr. of pepper, . .	„	3	5	„	„
128·75 hectog. of soap, .	„	1	75	„	„
562 gr. 5 decig. of coffee, .	„	0	30	„	hectog.

6. Divide—

(1.) 17 metres, 16 centim. by 11, 12, 13.
(2.) 41 kilom. 82 dekam. by 15, 16, 17.
(3.) 29 sq. met., 2740 sq. centim. by 14, 21, 35.
(4.) 376·38 cub. met. by 9, 27, 45.
(5.) 4 kilog. 14 dekag. 18 decig. by 22, 33, 55.
(6.) 8 hectar. 58 ares by 65, 60, 55.
(7.) 12 dekas. 1·2 decis. by 12, 13, 91.
(8.) 36 myrial. 4 kilol. 16 lit. 7 decil. by 9, 18, 27.
(9.) 7339 fr. 50 c. by 25, 30, 75.

7. Find the price of—

(1.) A metre, when 2 met. 80 centim. cost 70 fr.
(2.) A square decim., when 30 sq. met. cost 450 fr. 30 c.
(3.) A cubic metre, when 15 cub. decim. cost 361 fr. 80 c.
(4.) A hectometre, when 3 kilom. 125 metres cost 10 fr. 25 c.
(5.) A kilog. of coffee, when 7 hectog. 50 grams cost 1 fr. 35 c.
(6.) A hectare, when 4265 fr. 2·50 c. is the price of 149 ares, 25 centiar.
(7.) A stere, when 125 dekas. 4 decis. cost 20631 fr. 60 c.
(8.) A decilitre, when 47 dekal. 5 litres cost 570 francs.
(9.) A cub. centim., when 1 cubic metre cost 10,000 fr.

8. How many times is—

(1.) 1 kilom. 470 met. 38 centim. contained in 36759·50 metres?
(2.) 12 sq. decim. 75 sq. centim. contained in 10 sq. met. 20 sq. decim.?
(3.) 13 sq. met. 25 sq. decim. contained in 318 sq. dekam.?

(4.) 31 cub. met. 725 cub. decim. contained in 45684 cub. met.?

(5.) 345 millig. contained in 165 kilog. 6 hectog.?

(6.) 275 centiar. contained in 396 hectar.?

(7.) 7 steres, 2·5 decis. contained in 29 dekas.?

(8.) 4 kilog. 5 grams, contained in 38 myriag. 4 kilog. 480 grams?

(9.) 8 centimes contained in 10 francs?

9. A merchant bought 95 litres of wine for 118 fr. 75 c., and sold it at a loss of 10 c. per litre. What was the selling price per kilolitre?

10. To make 12 suits of clothes, it required 40 metres of stuff 90 centim. wide. How much stuff will it take if the width is 80 centim.?

11. How many cubic decimetres of iron are there in a bar weighing 280 kilog. 368 grams, when one cubic centim. weighs 7 grams 788 millig.?

12. An iron wire, 126 metres long, is cut into pieces 3 centim. 2·5 millim. long. How many pieces are there?

Relation between the Metric Units and the English System of Weights and Measures.

37. We shall work a few examples to show how quantities expressed in the metric system may be expressed in the English system, and *vice versa*.

Ex. 1.—Reduce 10 kilom. 321 metres to English measure.

$$\begin{aligned} 10 \text{ kilom. } 321 \text{ metres} &= 10321 \text{ metres.} \\ &= (10321 \times 1{\cdot}094) \text{ yards.} \\ &= \tfrac{10321 \times 1{\cdot}094}{1760} \text{ miles.} \\ &= 6 \text{ miles } 731{\cdot}174 \text{ yards.} \end{aligned}$$

Ex. 2.—Express 2 miles, 309 yards in the metric system.

$$\begin{aligned} 2 \text{ miles, } 309 \text{ yards} &= (2 \times 1760 + 309) \text{ yards.} \\ &= 3829 \text{ yards} = \tfrac{3829}{1{\cdot}094} \text{ metres.} \\ &= 3500 \text{ metres.} \\ &= 3 \text{ kilom. } 500 \text{ metres.} \end{aligned}$$

Ex. 3.—Reduce 1 ton to kilograms, having given 1 gram = 15·4323 grains.

$$1 \text{ ton} = 20 \times 112 \times 7000 \text{ grains.}$$
$$= \frac{20 \times 112 \times 7000}{15 \cdot 4323} \text{ grams.}$$
$$= 1016050 \cdot 7507 \text{ grams nearly.}$$
$$= 1016 \text{ kilog. } 50 \text{ grams, } 750 \cdot 7 \text{ millig. nearly.}$$

Ex. 4.—Express £13. 17s. 4½d. in the pound and mil system.

(£1 = 10 florins, 1 florin = 10 cents, 1 cent = 10 mils.)

Reducing the given sum to the decimal of a pound, we have—

£13. 17s. 4½d. = £13·86875.
= £13. 8 fl. 6 cent. 8¾ mil.

Ex. XIV.

1. Express a mile in the metric system, having given that a metre = 39·3708 inches.

2. An are contains 1076·43 square feet. Reduce 53 ares 25 centiares to English measure.

3. The area of a room is 22 sq. met. 26 square decim. Express this in English measure (1 metre = 39·3708 inches).

4. A block of marble measures 3 feet, 3 inches in length, 2 feet, 6 inches in depth, and 3 feet, 9 inches in width. What is the solid content expressed in cub. centim.?

5. In 1235 litres how many gallons, when 50 litres = 11 gallons nearly?

6. Supposing a franc to be equivalent to 9½d., reduce £44. 13s. to francs.

7. Taking £1 sterling as equal to 25·22 francs, reduce £2. 13s. 7½d. to francs.

8. In 1852 France reaped about 47850000 hectol. of wheat. Express this in gallons, assuming 1 gallon = 4·543 litres.

9. The ceiling of a room contains 83 sq. met. 53·96 sq. decim. What will be the expense of painting it at 10d. a square yard (1 metre = 1·094 yard)?

10. Find the cost of 2000 kilog. of sugar at 0 fr. 50 c. per lb.

11. In England the unit of work is the foot-pound, and in the metric system it is the kilogram-metre. Reduce 62 metric units of work to English units, taking 1 gram = 15·4323 grains, and 1 metre = 39·3708 inches.

12. The pressure of the atmosphere is $14\frac{3}{4}$ lbs. upon the square inch. Find the pressure in kilograms upon the square centimetre.

CHAPTER V.

PROPORTION.

38. Proportion is the equality of ratios.

Thus, since the ratio $6 : 8 = \frac{6}{8} = \frac{3}{4} = \frac{15}{20}$,
we have ratio $6 : 8 =$ ratio $15 : 20$;

and we say that the numbers 6, 8, 15, 20 form a proportion. We generally express the fact thus—

$$6 : 8 :: 15 : 20.$$

It is easy to find *by trial* that *the product of the extreme terms is equal to the product of the means.*

Thus, we have $6 \times 20 = 8 \times 15$.

We may prove this property of the terms of a proportion to hold generally as follows:—

Suppose we have given the proportion $12 : 21 :: 20 : 35$.

It follows, from our definition above, that $\frac{12}{21} = \frac{20}{35}$, and multiplying each of these fractions by the product of their denominators, viz., by 21×35, we have

$$\frac{12}{21} \times 21 \times 35 = \frac{20}{35} \times 35 \times 21.$$

Now (Art. 8), $\frac{12}{21} \times 21 = \frac{12}{1} = 12$, and $\frac{20}{35} \times 35 = \frac{20}{1} = 20$, and we hence have $12 \times 35 = 20 \times 21$.

Now, we have not in our reasoning taken into account the actual value of the terms of the given proportion; and it is therefore evident that a similar result will follow from every proportion, and we may hence conclude generally:—

In every proportion *the product of the extremes is equal to the product of the means.*

39. Having given any three terms of a proportion, to find the remaining one.

Since the product of the extremes is equal to the product of the means, the following rule is evident:—

Rule.—If the required term be a mean, divide the product of the extremes by the other mean; but if the required term be an extreme, divide the product of the means by the other extreme.

Ex. 1.—28, 24, 30 are respectively the 1st, 3rd, and 4th terms of a proportion, required the 2nd term.

We have— 28 : required term : : 24 : 30

$\therefore$ required term $= \frac{28 \times 30}{24} = \frac{7 \times 5}{1} = 35.$

Ex. 2.—10, 45, 16 are respectively the 1st, 2nd, and 3rd terms, required the 4th term.

We have— 10 : 45 : : 16 : required term

$\therefore$ required term $= \frac{45 \times 16}{10} = \frac{9 \times 8}{1} = 72.$

Ex. 3.—2 hours, 45 minutes, 8 men are respectively the 1st, 2nd, and 3rd terms, required the 4th term.

We must express (Art. 6) the 1st and 2nd terms in the same denomination, and the proportion will stand thus—

Min. Min. Men.
120 : 45 : : 8 : required term.

Now, the ratio of the first two terms is the same as the ratio of the abstract numbers 120 and 45; and the 4th term must be of the same denomination as the 3rd term, otherwise the 3rd and 4th terms could not form a ratio.

We have therefore—

Required term $= \frac{45 \times 8}{120}$ men $= \frac{3 \times 1}{1}$ men $= 3$ men.

Simple Proportion.

40. In Arithmetic we divide Proportion into Simple and Compound. Simple Proportion is the equality of two simple ratios, and therefore contains four simple terms; and the usual problem is to find the fourth term, having given the *first three* terms.

When we know the exact order of the given terms, the fourth term is, of course (Art. 38), found thus—

Rule.—Multiply the 2nd and 3rd terms together, and divide by the 1st.

The formal arrangement of the three given terms in their proper order is called *the statement;* and the only difficulty, therefore, in working a sum in Simple Proportion, or Single Rule of Three, as it is called, consists in stating it.

We shall work a few examples to illustrate the mode of doing this.

Ex. 1.—If 12 men earn £18, what will 15 men earn under the same circumstances?

We have here *two kinds* of terms, *men* and *earnings*, and whatever ratio any given number of men bears to any second given number of men, it is evident that it must be equal to the ratio of the earnings of the first lot of men to the earnings of the second lot, and we may therefore write—

$$\overset{\text{Men.}}{12} : \overset{\text{Men.}}{15} = £18 : \text{2nd earnings.}$$

$$\text{or} \quad \overset{\text{Men.}}{12} : \overset{\text{Men.}}{15} :: £18 : \text{2nd earnings.}$$

As the first two terms are of the same denomination, their ratio is not altered by treating them as abstract quantities, and the denomination of the 4th term must be the same as that of the 3rd.

Hence we have—

$$\text{Ans.} : \quad = £\frac{18 \times 15}{12} = £\frac{3 \times 15}{2} = £22.\ 10s.$$

Ex. 2.—If 18 men do a piece of work in 25 days, in what time will 20 men do it?

The two kinds of terms we have here to consider are *men* and *time*. In doing work we know that the *time* will *diminish* exactly as the number of *men increases*, and hence the ratio of the *second* lot of men to the *first* lot will be equal to the ratio of the *given* time to the *time required*. We therefore have—

$$\overset{\text{Men.}}{20} : \overset{\text{Men.}}{18} :: \overset{\text{Days.}}{25} : \text{required time.}$$

$$\therefore \text{Ans.} : \quad = \frac{18 \times 25}{20} \text{ days} = \frac{9 \times 5}{2} \text{ days} = 22{\cdot}5 \text{ days.}$$

We have reasoned out the above examples thus to show that the working of problems in Rule of Three depends upon the principle of the equality of ratios. Practically, however, we proceed as follows :—

Ex. 1.—If 12 men earn £18, what will 15 men earn under the same circumstances?

We are required to find *earnings*, and we therefore put down for the 3rd term the given *earnings*, thus—

$$: \quad :: \quad \overset{\pounds}{18}$$

The question is with regard to 15 men instead of 12 men, and we know their earnings must be *greater*. We therefore place the *greater* of these terms in the 2nd place and the other in the 1st, and the statement becomes—

Men.	Men.	£.	
12 :	15 :	18 :	required earnings.

∴ as before—

$$\text{Ans.} = \pounds\frac{15 \times 18}{12} = \pounds\frac{15 \times 3}{2} = \pounds 22.\ 10s.$$

Ex. 2.—If 18 men do a piece of work in 25 days, in what time will 20 men do it?

We are required to find *time*, and we place therefore the given time, viz., 25 days, in the 3rd place.

Again, the question is with regard to 20 men instead of 18 men. Now, we know that 20 men require *less time* than 18 men to do a piece of work, and we hence place the less of these terms in the 2nd place. The statement then becomes—

Men.	Men.	Days.	
20 :	18 : :	25 :	required time.

∴ as before—

$$\text{Ans.:} = \frac{18 \times 25}{20} \text{ days} = \frac{9 \times 5}{2} \text{ days} = 22{\cdot}5 \text{ days.}$$

Ex. XV.

1. If 12 articles cost £15, what will 624 cost?
2. What is the price of 35 loaves, when 29 loaves cost 15s. $8\frac{1}{2}$d.?
3. If I get 140 metres of cloth for 541 fr. 70 c., what must I pay for 89 metres, 3 decim.?

4. If 4 cubic metres of water run into a cistern in 18 minutes, in what time will it be full, supposing it to be 4 metres long, 6 metres, 25 centim. deep, and 35 decim. wide?

5. If the carriage of a parcel for the first 50 miles be 1s. 3d., and if the rate be reduced by one-third for distances beyond, how far can the parcel be carried for 1s. 7d.?

6. If a half-kilogram of sugar cost 1 fr. 10 c., what will be the cost of 3 kilog. 625 grams.?

7. There are two pieces of the same kind of cloth, measuring 43 yards and 57 yards respectively, and the second costs £1. 9s. 2d. more than the first. What is the cost of the first?

8. A garrison of 720 men have provisions for 35 days, and after 7 days 120 more men arrive. How long will the provisions last?

9. After paying 4d. in the pound income-tax a person has £299. 18s. 4d. left. What was the amount of his original income?

10. Two clocks, one of which gains 3 minutes and the other loses 5 minutes per day, are put right at noon on Monday. What is the time by the second clock when the first indicates 4 P.M. on the following Thursday?

11. When will the hands of a clock be exactly 30 minute divisions apart between 2 and 3 o'clock?

12. If I lend a friend £120 for 9 months, how long ought he to lend me £270?

Compound Proportion.

41. Compound Proportion is an equality between ratios, one of which at least is a ratio compounded of two or more simple ratios.

Arithmetical questions depending on Compound Proportion are generally said to belong to the Double Rule of Three; and the proportion consists of an equality between a ratio, on the one hand, compounded of two or more simple ratios; and, on the other hand, a simple ratio, whose consequent is required.

The following examples will illustrate the method of working questions in this rule:—

Ex. 1.—If 12 horses eat 20 bushels of corn in 8 days, in what time will 24 horses eat 16 bushels?

$$\left.\begin{array}{l}24 \text{ horses} : 12 \text{ horses} \\ 20 \text{ bushels} : 16 \text{ bushels}\end{array}\right\} :: 8 \text{ days} : \text{required time.}$$

Explanation.—We are required to find *time*; and so, as in simple proportion, we put in the 3rd place the given time, viz., 8 days.

Leaving, for the present, the quantity eaten out of consideration, we know that 24 horses require less time to consume a given quantity of food than 12 horses do; we therefore place the *less* of these two terms in the 2nd place, and the other in the 1st place.

(The statement up to this point is 24 horses : 12 horses : : 8 days : required time, and we might obtain 4 days as an answer, irrespective of the quantity eaten. We might now place this answer in the 3rd term of another simple proportion, and take the *quantity eaten* into consideration, irrespective of the number of horses, thus getting an answer depending both upon the *number of horses* and the *quantity eaten*. It is more convenient, however, to proceed thus :)

Again, taking into consideration the *quantity eaten*, and *leaving out* of consideration the other given pair of terms, we see that *less time* is required to eat 16 bushels than to eat 20 bushels. We, therefore, put the *less term* in the 2nd place, and the other in the 1st.

Now, treating the terms of the ratios which occupy the 1st and 2nd places as abstract quantities, and compounding them, we have:

$$24 \times 20 : 12 \times 16 :: 8 \text{ days} : \text{required time.}$$

$$\therefore \text{ required time} = \frac{12 \times 16 \times 8}{24 \times 20} \text{ days} = 3{\cdot}2 \text{ days.}$$

Ex. 2.—How much bread can I get for 9d. when wheat is at 18s. a bushel, if the fourpenny loaf weigh 3 lbs. when wheat is at 20s. a bushel?

Proceeding as in Example 1, we have

$$\left.\begin{array}{l}4\text{d.} : 9\text{d.} \\ 18\text{s.} : 20\text{s.}\end{array}\right\} :: 3 \text{ lbs.} : \text{weight required.}$$

Or, $4 \times 18 : 9 \times 20 :: 3$ lbs. : weight required.

$$\therefore \text{ weight required} = \frac{9 \times 20 \times 3}{4 \times 18} \text{ lbs.} = 7{\cdot}5 \text{ lbs.}$$

Ex. XVI.

1. If 15 men can build a wall 81 feet long in 18 days, how many men can build 135 feet of the same kind of wall in 30 days?

2. In 4 days, 18 workmen can dig a ditch 162 yards long, 7 feet wide, and 12 feet deep. What must be the depth of a ditch which 45 workmen can dig in 7 days, supposing it to be 387 yards long and 5 feet wide?

3. A traveller, going 15 hours a day, walks 1500 kilometres in 20 days. How far will he go in 30 days, walking 12 hours a day with the same velocity? Express your answer in English miles.

4. Two men are partners; one puts in a capital of £800, and receives as 6 months' profit £120. What is the capital of the other, who receives £3375 as 9 months' profit?

5. Two tourists having spent £1. 16s. 8d. in $2\frac{1}{2}$ days, meet three others with whom they continue their tour, and they spend while together £21. 1s. 8d., at the same rate per day. Required how long they were in company.

6. If 16 men and 10 boys do a piece of work in 10 days, in how many days would 8 men and 18 boys do a piece 7 times as great, supposing the work of 5 boys equal that of 2 men?

7. Supposing the rate of carriage to be diminished one-third after the first 50 miles, find the cost of carrying 16 cwt. for 40 miles, when 12 cwt. can be carried 100 miles for 4s. 2d.

8. A cistern is 8 metres, 4 decim. long, 1 metre, 8 centim. wide, and 275 centim. deep. Find the depth of another cistern of equal capacity whose length is 7 metres, 2 decim., and width 11 decim.

9. Persons whose incomes are less than £300 per annum are taxed upon £80 less than their income. Supposing 3 persons, having equal incomes, to pay £7 in the aggregate, at 4d. in the pound, find the total tax upon 14 persons, each having incomes 3 times as great.

10. If 12 horses eat 10 acres of grass in 16 weeks, and 18 horses eat 10 acres in 8 weeks, how many horses would eat 40 acres in 6 weeks, the grass being supposed to grow uniformly?

11. A boat, propelled by 8 oars, which take 28 strokes per minute, goes at the rate of $9\frac{1}{3}$ miles per hour. Find the rate of a boat propelled by 6 oars, which take 36 strokes per minute, the work done by each stroke of the latter being one-sixth less than that by each stroke of the former.

12. If 4 men and 10 women can do a piece of work in 8 days, which 12 women and 20 children can do in 4 days, in what time will 6 men, 18 women and 5 children do a work three times as great?

CHAPTER VI.

APPLICATION TO ORDINARY QUESTIONS OF COMMERCE AND TRADE.

Interest.

42. Interest is the money paid for the use of money.

The **Principal** is the money lent, and the **Amount** is the sum of the interest and principal.

The **Rate** of interest is the money paid for a given sum for a given time. £100 is in practice the given sum, and one year the given time. Thus, if £4 be paid for the use of £100 for one year, the rate of interest is £4 per cent. per annum, or as we generally say, 4 per cent.

Simple Interest is interest calculated on the original principal only.

Compound Interest is the interest which arises from adding the interest for each year to the principal of that year, and calculating interest for the next year upon the amount so obtained.

Simple Interest.

43. Rule.—Multiply the principal by the rate per cent. and by the number of years; then divide the product by 100, and the quotient will be the simple interest.

Ex. 1.—Find the simple interest of £420 for 3 years at 5 per cent.

```
   £
  420
    5
 ----
 2100
    3
 ----
£63·00   Ans.: £63.
```

Reason for this process—

£100 gains £5 in 1 year, and the question is to find the gain upon £420 in 3 years.

Hence, proceeding as in Double Rule of Three, we have—

$100 \times 1 : 420 \times 3 :: £5$: interest required.

$\therefore$ interest required $= £\frac{420 \times 5 \times 3}{100}$, which is exactly as stated in the rule.

Ex. 2.—Find the simple interest of £352. 1s. 8d. from March 16 to August 21, 1873, at 4 per cent.

The following process will be easily understood:—

```
     £    s.  d.
    352   1   8
              4
 ----------------
 £14·08   6   8
     20
  -----
  1·66s.
     12
  -----
  8·00d.
```

$\therefore$ £14. 1s. 8d. is the interest for 1 year.

Now, from March 16 to August 21 are 158 days; hence we have—

365 days : 158 days :: £14. 1s. 8d. : Interest required.

$\therefore$ interest required = £6. 1s. $11\frac{9}{73}$d.

Ex. XVII.

Find the simple interest of—

1. £350 for 4 years at 5 per cent.
2. £295. 2s. 1d. for $3\frac{1}{2}$ years at 4 per cent.

3. £375. 8s. 4d. for $2\frac{1}{3}$ years at $4\frac{1}{2}$ per cent.
4. £160 from Feb. 1 to June 12, 1872, at $7\frac{1}{2}$ per cent.
5. £48 for 7 months at $1\frac{1}{4}$ per cent. per month.
6. £219. 4s. 2d. for 6 years at $1\frac{2}{3}$ per cent.

In the six following examples, understand simple interest.

7. At what rate per cent. will £129. 8s. 4d. gain £6. 3s. $5\frac{1}{2}$d. in $2\frac{1}{2}$ years?

8. A certain sum amounts in 3 years at $7\frac{1}{2}$ per cent. to £289. 16s. $3\frac{1}{2}$d.; find the original sum.

9. In what time will £175. 6s. 3d. amount to £192. 7s. $10\frac{1}{2}$d. at $2\frac{1}{2}$ per cent?

10. What sum will amount in 2 years 9 months at 4 per cent. to £427. 7s?

11. If £320 gain £9 in 13 months, in what time will £480 gain £6 at the same rate?

12. Find the interest of £29. 7s. 5d. for 6 months at $5\frac{5}{19}$ per cent.

Compound Interest.

44. Rule 1.—Find the interest for one year as in Simple Interest, and add it to the principal; then find the interest for one year upon this amount reckoned as *principal* for the second year, and add it to the second year's principal, and so on. Subtract the original principal from the amount so obtained for the given number of years, and the result will be the compound interest required.

Rule. 2.—Divide the given rate per cent. by 100, putting the result in a decimal form, and place *unity* before the decimal point. Raise the number thus obtained to a power corresponding to the given number of years, multiply the principal by the result, and we get the *amount* for the given number of years.

Thus, supposing 5 to be the rate per cent., we have, dividing by 100 and placing *unity* before the result, the number 1·05. Then, if the given number of years be 4, and £162 the principal, we have, according to rule, amount $= £162 \times (1{\cdot}05)^4$.

The first rule requires no explanation; the second rule may be explained thus:—

Ex.—Find the compound interest of £360 for 3 years at 4 per cent.

Now, interest for £100 for 1 year = £4,
∴ " £1 " = £·04;
hence, amount of £1 " = £1 + £·04 = £1·04.

We thus see that the amount of £1 for 1 year at 4 per cent. is 1·04 times the original sum. It therefore follows that—

Amount of the £1·04 for 1 year = 1·04 times £1·04,
∴ amount of £1 " 2 years = $£(1·04 \times 1·04)$
$= £(1·04)^2$;
and so, amount of £1 " 3 years = 1·04 times $£(1·04)^2$
$= £(1·04)^3$,
hence, amount of £360 " " = 360 times $£(1·04)^3$
$= £360 \times (1·04)^3$.

The compound interest is then found by subtracting from this the original principal.

Ex. 1.—Find the compound interest of £570 for 3 years at 5 per cent.

(We shall work this by Rule 1, and for convenience shall keep our quantities in a decimal form.)

£	
570	
5	
£28·50	= interest for first year.
570	
£598·5	= principal for second year.
5	
£29·925	= interest " "
598·5	
£628·425	= principal for third year.
5	
£31·42125	= interest " "
628·425	
£659·84625	= amount at end of third year, } therefore
570	= original principal; } subtracting,
£89·84625	= compound interest for 3 years.

Ex. 2.—Find the compound interest of £327. 12s. 6d. for 4 years at 3 per cent.

Now, £327. 12s. 6d. = £327·625.

Hence, by Rule 2—

Amount = £327·625 × $(1·03)^4$ = £368. 14s. $10\frac{3}{4}$d. nearly.

Ex. 3.—What sum of money, if put out for 2 years at 4 per cent., will amount to £324. 9s. $7\frac{1}{5}$d., compound interest being reckoned?

By Rule 2, the principal may be found by dividing the amount by $(1·04)^2$.

Now, given amount = £324. 9s. $7\frac{1}{5}$d. = £324·48.

Hence principal required = £324·48 ÷ $(1·04)^2$ = £300.

Ex. XVIII.

Find the compound interest of

1. £284 for 2 years at 4 per cent.
2. £312. 12s. $7\frac{1}{2}$d. for 3 years at 5 per cent.
3. £283. 10s. for 2 years at $3\frac{1}{2}$ per cent.
4. £605. 12s. 6d. for 4 years at 4 per cent.
5. What is the difference between the simple and compound interest of £150 for 2 years at 6 per cent.?
6. Find the amount of £381. 1 florin 3 cents 5 mils for 3 years at 5 per cent.

(£1 = 10 florins, 1 florin = 10 cents, 1 cent = 10 mils.)

7. Find the amount of £250 for 2 years at 4 per cent. per annum, interest being payable half-yearly.
8. What sum will amount in 3 years at $4\frac{1}{2}$ per cent. compound interest to £200?
9. A town has 200,000 inhabitants, and it increases at the rate of 5 per cent. per annum; find the number of inhabitants at the end of 3 years.
10. Find the difference in amount of £350 for 3 years at 4 per cent. simple interest, and £420 for 2 years at 5 per cent. compound interest.
11. How much would a person who lays by £50 a year at 5 per cent. compound interest, draw out at the end of 4 years?
12. A person expects to receive £450 in 3 years; what present sum is equivalent to this, reckoning compound interest at 4 per cent.?

Discount.

45. When money is paid before it is due, the payee may, of course, put out the money at interest for the rest of the term, and thereby increase it. It therefore follows that the amount which ought to be paid for the discharge of an account before its proper time should be such a sum that, if put out at interest for the remainder of the term, will just amount to the original sum in question.

Thus £102. 10s. (interest being reckoned at 5 per cent. per annum) payable 6 months hence, would be fully discharged by paying £100 at once. For £100 in 6 months at 5 per cent. per annum would amount to £102. 10s. Hence, the payee ought to remit £2. 10s. from the full account. The amount remitted is called **discount.**

It will be seen, therefore, that the discount on £102. 10s. due 6 months hence at 5 per cent. is £2. 10s.

Bankers, however, are in the habit of charging *interest* instead of *discount.* The banker's discount, therefore, on £100 due 6 months hence at 5 per cent. is £2. 10s.

Hence, the *true* discount on £102. 10s. due 6 months at 5 per cent. is the same as the *banker's* discount on £100 under the same circumstances; and bankers' discount on any given sum is in excess of the true discount.

Tradesmen's bills are legally due *three days* after the term for which they are drawn is completed. This extension of time is called *three days of grace.* When a bill falls due on a Sunday, it is usual in England to meet it on the previous Saturday.

Ex. 1.—Find the difference between the true discount and the banker's discount on £306 due 4 months hence at 6 per cent.

Now, £100 would in 4 months gain $\frac{1}{3}$ of £6, or £2. Hence, the true discount on £102 due four months hence at 6 per cent. is £2, and therefore we have—

£102 : £306 :: £2 : true discount required.

$$\text{Hence, true discount} = £\frac{306 \times 2}{102} = £6.$$

Again, proceeding according to the rule for simple interest—

$$\text{Banker's discount} = \frac{£306 \times 6 \times \frac{1}{3}}{100} = £\frac{612}{100} = £6.\ 2s.\ 4\tfrac{4}{5}d.$$

Hence, the excess of the banker's discount over the true discount is 2s. 4$\frac{4}{5}$d.

Ex. 2.—A bill of £350 drawn on March 15, at 6 months, is cashed on May 20, 1872; what is the banker's discount at 6 per cent.?

The bill is legally due on Sept 18, and from May 20 to Sept. 18 are 121 days.

Now, the *interest* on £350 for 1 year at 6 per cent. is easily found to be £21. Hence,

365 days : 121 days :: £21 . banker's discount required;

and ∴ banker's discount = £$\frac{21 \times 121}{365}$ = £6. 19s. 2$\frac{58}{73}$d.

Percentages.

46. There are many questions which relate to ordinary commercial transactions which may be worked exactly as if we had to find the simple interest for one year—*e.g.*, questions in commission, brokerage, insurance, &c.

Commission is a sum of money charged, by an agent for buying or selling goods, at a certain rate per cent. upon the value of the goods.

Brokerage is similar to commission, but it is charged upon money transactions instead of upon the sale of goods.

Insurance is a sum charged per cent. upon the value of property, the said value being paid to the insured in case of loss from causes as per agreement.

Ex. 1.—Find the brokerage on £625. 5s. at 4$\frac{1}{2}$ per cent.

£	*s.*	*d.*
625	5	0
		4$\frac{1}{2}$
2501	0	0
312	12	6
28·13	12	6
20		
2·72s.		
12		
8·70		

Ans.: £28. 2s. 8$\frac{7}{10}$d.

Ex. 2.—The rate of insurance is $4\frac{1}{4}$ per cent., and the value of some property insured is worth £766. What will be the annual payment, so that, in case of fire, the owner may receive back his premium, as well as the value of his property?

If, instead of paying £4. 5s. as insurance on every £100, he pays £4. 5s. upon every (£100 – £4. 5s.) or upon every £95. 15s.; then, in case of fire, he will receive £100 for a damage of £95. 15s., and thus have the value of his property and the amount of his premium.

The problem is, therefore—

If £4. 5s. is the premium on £95. 15s., what is the premium on £766?

Hence—

£95. 15s. : £766 : : £4. 5s. : premium required.

And, therefore, premium = £34.

Ex. XIX.

1. Find the banker's discount on £412, due 6 months hence, at 6 per cent.

2. By how much does the banker's discount on £100, due 3 months hence, at 5 per cent., exceed the true?

3. What discount would be charged upon a bill drawn for £320, on April 15, at 4 months, and presented for payment on June 3 (discount at 7 per cent.)?

4. Find the discount on a bill drawn on Aug. 3 for £200, at 6 months, and cashed on Sept. 10, discount being reckoned at 6 per cent.

5. A man buys goods for £250, being allowed 6 months credit, and he immediately sells them for the same amount, allowing 3 months credit. What does he gain by the transaction, interest being reckoned at 5 per cent.?

6. Find the brokerage on £352. 17s. 6d. at 3 per cent.

7. What is the brokerage on £4500 at $\frac{1}{8}$ per cent.?

8. What is the commission on the sale of goods to the amount of £850 at 5 per cent.?

9. A person insures for £1050 at $3\frac{3}{4}$ per cent. What is his annual premium?

10. What will be the annual premium on property worth £965, so that the insured may obtain his premium back again with the value of his property, in case of loss—insurance being at $3\frac{1}{2}$ per cent.?

11. The brokerage on a certain sum at $\frac{1}{4}$ per cent. is £5. 10s. $7\frac{1}{2}$d. Find the sum.

12. Together with a commission of 4 per cent., goods cost a person £339. 8s. 5d. Find the cost price to the agent.

Stocks and Shares.

47. When a large amount of capital is to be raised, a company is generally formed, which raises the money by the issue of shares. We will suppose a person to hold a £100 share; he will then be entitled to such a part of the profits of the company as £100 is of the whole capital. If there be a great demand for these shares, persons holding them may dispose of them for more than the nominal value, say for £106; whereas, if they are very little in demand, the seller may be glad to sell at, perhaps, £70. In the first instance, we should say that the shares were at 106, or at 6 *premium;* and in the second instance, that the shares were at 70, or at 30 *discount.* If the selling price of the shares is £100, they are said to be at *par.*

So, when we read that the Three per Cent. Consols are quoted at $96\frac{3}{8}$, it means that an acknowledgment of indebtedness on the part of the Government to the amount of £100, bearing interest at 3 per cent. per annum, may be bought for £$96\frac{3}{8}$.

The buying and selling of stocks and shares is carried on by brokers, who charge a percentage from $\frac{1}{8}$ to $\frac{1}{2}$, sometimes upon the *nominal* value of the stock, but mostly upon the *actual cash value.* When brokerage is to be taken into account in any example it will be mentioned, and it will be estimated by the first method, unless specified.

Ex. 1.—What is the value of £1000 stock at $89\frac{5}{16}$ per cent.?

$$\text{No. of cents. stock} = \frac{1000}{100} = 10$$

$$\therefore \text{ value required} = £89\tfrac{5}{16} \times 10 = £893.\ 2s.\ 6d.$$

Ex. 2.—What would be the cost of £1180 stock at $153\frac{1}{4}$ per cent., including brokerage at $\frac{1}{4}$ per cent.?

$$\text{No. of cents. stock} = \frac{1180}{100} = \frac{118}{10},$$

$$\text{and total cost of each cent.} = £(153\tfrac{1}{4} + \tfrac{1}{4}) = £153\tfrac{1}{2}.$$

$$\text{Hence, required cost} = £153\tfrac{1}{2} \times \frac{118}{10} = £1811.\ 6s.$$

Ex. 3.—What is the annual income arising from investing £6510 in the Four per Cents. at 93?

$$\text{Here, price of £100 stock} = £93,$$

$$\text{and } \therefore \text{ No. of cents. stock for £6510} = \frac{6510}{93} = 70.$$

$$\text{Hence, annual income} = £4 \times 70 = £280.$$

Ex. XX.

1. Find the cost of £750 stock at $92\frac{1}{8}$ per cent.

2. I sell out £325 Three per Cent. Consols at 94. What do I get after allowing the broker $\frac{1}{4}$ per cent. upon the cash he receives for the sale?

3. Invest £5065. 9s. 9d. in the Three per Cent. Consols at $91\frac{5}{8}$.

4. What would be the cost of £413. 1s. 9d. reduced Three per Cents. at $92\frac{3}{8}$, including a brokerage of $\frac{1}{4}$ per cent. upon the cost to the broker?

5. What would be the proceeds of the sale of £6228 India Five per Cent. stock at $111\frac{1}{4}$, deducting $\frac{1}{4}$ per cent. upon the selling price for brokerage?

6. If I sell £8160 Spanish Three per Cent. Bonds at 31 per cent, and invest the proceeds, less $\frac{1}{8}$ per cent. brokerage, in Indian Railway Five per Cent. Stock at 107, what will be the difference in my annual income?

7. What is the value sterling of \$4000 American Bonds at $93\frac{5}{8}$ per cent. (\$ = 4s. 6d.)?

8. What is the value of \$37,000 United States Bonds at $93\frac{5}{16}$ per cent.?

9. Invest £954. 0s. 7d. in India Five per Cent. Stock at $110\frac{3}{8}$, allowing $\frac{1}{8}$ per cent. brokerage.

10. What is the sterling value of 5000 francs Italian Bonds at 67 per cent. (Exchange 25 fr.)?

11. Find the sterling value of 6000 guilders Dutch $2\frac{1}{2}$ per Cent. Bonds at $56\frac{1}{4}$ (Exchange 12 guilders).

12. Invest £1025 in French Rentes at $51\frac{1}{8}$, allowing $\frac{1}{8}$ per cent. brokerage (Exchange 25 fr.).

Annuities.

48. Annuities are annual payments, the first payment being due at the end of a year. When an annuity is left untouched for a number of years, its *amount* is properly obtained by allowing compound interest.

49. To find the amount of an annuity of a given sum for a given time, at a given rate per cent.

Let us suppose the annuity to be £1, the time 4 years, and 5 the rate per cent. The first payment will not be due till the end of a year; so that at the end of 4 years it will have been accumulating for 3 years at compound interest; so the next payment, not being due till the end of the second year, will have been accumulating for 2 years at compound interest; and so on, the last payment being made when due.

Hence, the amount of an annuity of £1, for 4 years at 5 per cent., will be (reversing the above order) as follows :—

£1 + amount of £1 for 1 year + amount of £1 for 2 years + amount of £1 for 3 years (compound interest being reckoned).

And it is evident that for any other annuity we may follow the same method, and at the end multiply the sum by the number of £ in the annuity.

Ex.—Find the amount of an annuity of £300 for 4 years at 5 per cent.

We shall find the respective *amounts* by Rule 2, Art. 44. Thus—

£1·05	= amount for 1 year of £1 at 5 per cent.			
1·05				
525				
105				
£1·1025 =	„ „	2 years	„	„
1·05				
55125				
11025				
1·157625 =	„ „	3 years	„	„

Hence, amount of annuity of £1 for 4 years—

= £1 + £1·05 + £1·1025 + £1·157625 = £4·310125.

∴ amount of given annuity = £4·310125 × 300 = £1293. 0s. 9d.

50. To find the present value of an annuity to continue for a given time.

By the present value of an annuity to continue a given number of years is meant such a lump sum which, paid down at once, would, by accumulating at compound interest for the same time, amount to just the same sum as the annuity itself if it were allowed to accumulate. In the absence of algebraical symbols, we shall best illustrate the method of finding such a lump sum by an example.

Ex.—Find the present value of an annuity of £50 for 3 years at 4 per cent.

As in Art. 49, we find the amount of the annuity—

$$= \{£1 + £1{\cdot}04 + £(1{\cdot}04)^2\} \times 50 = £156{\cdot}08.$$

Now, whatever be the present value required, we know that its amount in 3 years at 4 per cent. compound interest is found (Art. 44) by multiplying it by $(1{\cdot}04)^3$ or 1·124864.

It, therefore, follows that if we know this amount beforehand we can find the present value by dividing it by 1·124864.

Hence, present value = £156·08 ÷ 1·124864 = £138·755 nearly.

51. To find the present value of an annuity to continue for ever.

It is evident that the sum we require is one which, put out at interest, will annually produce a sum equal to that of the annuity itself. The problem then is simply this—

Having given the interest for 1 year of a certain sum, and the rate per cent., to find the principal.

We have therefore the following rule :—

RULE.—Divide the given annuity by the rate per cent., and multiply by 100, and the result is the present value.

Ex.—How much must a gentleman invest at 5 per cent. in order to endow a charity with £60 a year.

Present value of the annuity of £60 to continue for ever—

$$= £\frac{60 \times 100}{5} = £1200.$$

There are many other questions connected with annuities which are, however, best left till the student has a knowledge of Logarithms.

Ex. XXI.

Find the amount of an annuity of—

1. £120 for 3 years at 4 per cent.
2. £250 for 4 years at $4\frac{1}{2}$ per cent.
3. £321 for 5 years at 5 per cent.
4. What is the present value of an annuity of £80, to continue for six years, at 6 per cent.?
5. A person who, according to the tables of mortality, is likely to live 10 years, wishes to insure an annual payment of £40 during life. What sum must he pay down, reckoning interest at 5 per cent.? (Give the result to four places of decimals.)
6. A house produces a clear rental of £30. How many years' purchase is it worth, interest being reckoned at 5 per cent.?
7. A gentleman invested a sum of money in the Three per Cent. Consols, in order that an annual payment of 7s. 6d. a year might be made in bread for ever. What sum did he invest?
8. Find the present value of a pension of £120 a-year, *payable half-yearly* for 5 years, interest being at the rate of 5 per cent. per annum.

9. A house, which ordinarily lets for £80 a-year, is leased for a term of four years, at a rent of £20, a certain sum being paid in addition at the time of letting. Find this latter amount.

10. What is the present value of a freehold which produces a clear rental of £50, but which cannot be entered upon for two years, reckoning interest at 5 per cent.?

11. Find the annuity which in four years, at 4 per cent., will amount to £100.

12. A corporation borrows a sum of £3000 at 4 per cent. What annual payment will clear off the debt in ten years? (Give the result correct to four places of decimals.)

Profit and Loss.

52. All questions involving the loss or gain per cent. by any transaction belong to this rule, and may be generally worked by Proportion.

Ex. 1.—A man buys goods at 5s. and sells them at 5s. 8d. Find his gain per cent.

The actual gain upon 5s. is 8d., and we are required to find the gain upon £100.

Now 5s. : £100 : : 8d. : gain upon £100,

$\therefore$ gain upon £100 = £$\frac{100 \times 8}{5 \times 12}$ = £$13\frac{1}{3}$,

or, required gain per cent = $13\frac{1}{3}$.

Ex. 2.—By selling goods at 6s. 3d. there is a gain of 25 per cent. What will be the selling price to gain 10 per cent.?

Now, selling price of goods which cost £100, so as to gain 25 per cent, is £125, and that to gain 10 per cent. is £110.

Hence £125 : £110 : : 6s. 3d. : selling price required;

from which, selling price required = 5s. 6d.

Ex. 3.—Find the cost price when articles sold at 1s. 9d. entail a loss of $12\frac{1}{2}$ per cent.

Now, articles which cost £100 when sold at a loss of $12\frac{1}{2}$ per cent. must sell for £$87\frac{1}{2}$.

Hence £$87\frac{1}{2}$: 1s. 9d. : : £100 : cost price required;

from which, cost price = 2s.

Ex. XXII.

1. Find the cost price of goods which are sold at a loss of 10 per cent. for 4s. $10\frac{1}{2}$d.

2. Goods which are sold for 7s. 11d. entail a loss of 5 per cent. What should be the price to gain 30 per cent.?

3. A tradesman reduces his goods $7\frac{1}{2}$ per cent. What was the original price of an article which now fetches £1. 7s. 9d.?

4. In what proportion must tea at 4s. 2d. be mixed with tea at 6s. a pound, so that a grocer may sell the mixture at 5s. 6d. and gain by the sale 10 per cent.?

5. A quantity of silk, after paying a duty of $12\frac{1}{2}$ per cent., cost £54. Find the original cost price.

6. An innkeeper buys $37\frac{1}{2}$ gallons of brandy at 14s. a gallon, and adds to it sufficient water to enable him to sell it at the same price and gain 12 per cent. How much water does he add?

7. By selling goods at 8s. 2d. a tradesman gains $16\frac{2}{3}$ per cent. What will be the gain or loss per cent. by selling at 6s. $1\frac{1}{2}$d.

8. A company has a capital of £750,000, and the working expenses for the year have been £42,123. 12s. 6d. What must have been the gross receipts in order that the shareholders may receive a dividend of 4 per cent.?

9. If stock which is bought at $91\frac{1}{2}$ is immediately sold at $91\frac{5}{8}$, what is the gain per cent.?

10. A person buys goods at 6 months' credit and sells them for cash at the nominal cost price immediately. What is his gain per cent.? (Interest 5 per cent.)

11. Goods are marked at a ready-money price and a credit price allowing 12 months. The credit price is £4. 9s. 3d., what is the ready-money price?

12. Goods are now being sold at 10 per cent. loss. How much per cent. must be put upon the selling price in order that they may be sold at 20 per cent. gain?

Square Root and Cube Root.

53. To avoid unnecessary repetition, the student is referred to the articles on Involution, Algebra, stage I., where the arithmetical principles and methods are explained.

Estimates.

54. The following specimens will give the student an idea of what he may expect to meet with under the head of Estimates. It is usual, in ordinary transactions, to use certain abbreviations; as *cub.* for cubic measure, *sup.* for superficial measure, *run.* for running or lineal measure. Builders, too, are in the habit of calling twelfths of a foot—whether it be cubical, superficial, or lineal measure—by the name of inches. The names yards, feet, inches, are often written thus: yds., ′, ″.

Ex. 1.—Digger, Bricklayer, and Mason.

Yds.	Ft.	In.				£	s.	d.
25	0	0	Cub.	Digging in trenches, filling, wheeling, and carting away,	1/			
232	0	0	Sup.	9″ reduced common brickwork in mortar, pointed on both sides,	4/2			
5	0	0	,,	4½″ trimmer arches to hearths,	2/3			
16	4	0	,,	Best blue brick on edge, paving in cement,	4/6			
6	0	0	,,	6″ blue and red quarries do.,	3/10			
14	0	0	Run.	12″ round blue coping bricks in cement,	2/4			
9	0	0	,,	Best red brick flat steps do.,	1/2			
33	2	0	,,	Extra only to splayed brick angles to doors and windows,	0/2½			
12	1	0	,,	Do. to plinth in cement,	0/3			
5	0	0	,,	Do. to sailing course to chimney,	0/2			
12	0	0	,,	Do. to cornice to eaves of best red bricks, three courses,	0/5½			
			No.	2 core chimney flues,	1/6			
			,,	1 set sink,	4/			
			,,	4 do. stoves,	5/3			
			,,	1 do. copper,	5/			
			,,	12 do. ornamental air gratings,	0/6			
	19	6	Run.	Gable coping of hard stone 13″ × 4½″, twice splayed,	1/9			
	15	0	,,	2½ rubbed hearth and back hearth,	1/2			
	7	9	,,	Solid York step 14″ × 6″, tooled,	2/6			
			No.	2 knee stones to gables 18″ × 14″ × 10″, ea.	7/6			
			,,	1 apex stone do., 16″ × 14″ × 12″,	8/			
			,,	1 York stone sink 4′ 0″ × 1′ 9″, with rounded corners and hole cut for waste-pipe,	10/			
			,,	2 stone chimney-pieces with 7″ × 1½″, chamfered jambs, mantel, and shelf, ... ea.	24/			
				Total,				

Ex. 2.—Carpenter and Joiner. (A square = 100 sq. ft.)

Sq.	Ft.	In.				£	s.	d.
	178	6	Cub.	Fir framed-in roof timbers,................	2/9			
	97	0	,,	Do. do., floors,.................	2/8			
	45	0	Sup.	Centreing to trimmer hearths, ...	0/4			
				Do. to 3 openings 5 0″ wide, with segment heads in $1\frac{1}{2}$ brick wall,...............ea.	2/			
				Do. to 4 do. 3′ 0″ wide do.,.............ea.	1/6			
	326	0	,,	Labour in planing roof and floor-timbers,..	$0/0\frac{3}{4}$			
	139	0	Run.	Do. in stop chamfering edges of do.,.......	$0/0\frac{1}{2}$			
	36	0	,,	7″ × $1\frac{1}{2}$″ ridge board,....................	0/3			
	21	0	,,	3″ × $1\frac{1}{2}$″ chamfered fillet to eaves,.........	$0/1\frac{1}{2}$			
3	45	0	Sup.	Inch clean red deal batten for boarded floors, wrought,............................	26/			
	61	0	Run.	7″ × 1″ torus skirting, plugged,...........	0/3			
			No.	2 labour to mitred margins to hearths, ...	$0/9\frac{1}{2}$			
			,,	16 mitres to skirting, ...ea.	0/2			
	114	3	Sup.	Inch deal treads and risers, ploughed, tongued, and screwed on 3 – 7″ × $2\frac{1}{2}$″ carriages,..............................	0/9			
	21	9	Run.	$1\frac{1}{4}$″ wall-string housed, for treads and risers,..............................	0/8			
	14	6	,,	$1\frac{1}{2}$″ close-string do., and sunk and beaded,..	0/9			
	14	6	,,	3″ × $2\frac{1}{2}$″ rounded oak hand-rail, French-polished,..............................	2/			
	12	8		9″ × 1″ beaded fascia,....................	0/4			
	36	6	Sup.	2″ – 6 panel doors, bead, flush, and square,	1/3			
	23	7	,,	$1\frac{1}{2}$″ do. do.,	1/1			
	30	0	Run.	$4\frac{1}{2}$″ × 3″ rebated and beaded frame,.......	0/6			
	26	0	,,	$4\frac{1}{2}$″ × 2″ do. do., ...	$0/4\frac{1}{2}$			
	60	0	,,	3″ × $1\frac{1}{2}$″ moulding, mitred,..............	0/3			
	57	0	Sup.	2″ ovalo-moulded sashes, double hung, in deal-cased frames, with oak sunk and weathered sills, ...	1/6			
	37	0	,,	$1\frac{3}{4}$″ do. do.,	1/4			
	22	6	Run.	$\frac{7}{8}$″ mitred bead,.........................	0/1			
	35	0	,,	$5\frac{1}{2}$″ × $\frac{3}{4}$″ bead lining,....................	$0/3\frac{1}{2}$			
			No.	11 inch-beaded centre boards,............	0/6			
				Total,..................				

MISCELLANEOUS EXAMPLES.

(*Selected from University and other Examination Papers.*)

1. Show by an easy example that the division of one whole number by another is equivalent to a series of subtractions.
Divide 1·02 by $\frac{17}{24}$ of ·144.

2. If the Three per Cents. are at $91\frac{1}{8}$, what interest does this give on £100? (Omit brokerage and fractions of a penny.)

3. How many lbs. in ·321875 of a ton weight? Convert it into kilograms (omitting fractions), assuming that a cubic decimetre of distilled water weighs 15432·35 grains.

4. Reduce to their simplest forms $\dfrac{\sqrt{1{\cdot}75} - \sqrt{{\cdot}63}}{\sqrt{3{\cdot}5} - \sqrt{2{\cdot}1}}$ and $\dfrac{\sqrt[3]{5{\cdot}12} + \sqrt[3]{{\cdot}03375}}{\sqrt[3]{80} - \sqrt[3]{{\cdot}01}}$.

5. Convert $\frac{1}{128}$ into a decimal fraction, and find the vulgar fraction corresponding to the recurring decimal $\cdot 22\dot{2}9\dot{7}$.

6. Show, by proper attention to the value of the figures, in multiplying one number by another, that the order in which the figures of the multiplier are taken is of no importance. Multiply 61·143 by 47·982 correctly to three places of decimals, beginning with the left hand figure of the multiplier, and use as few figures as possible.

7. Extract the square root of 1095·61, and find to three places of decimals the value of $\dfrac{4}{\sqrt{5} - 1}$.

8. Find the compound interest of £55 for one year, payable quarterly, at 5 per cent. per annum.
A person bought into the Three per Cents. at 98, and after receiving three years' interest he sold at 90. How much per cent. on the sum invested did he gain or lose?

9. Three gardeners working all day can plant a field in 10 days; but one of them having other employment can only work half time. How long will it take them to complete the work?

10. What fraction of a crown is $\frac{3}{5}$ of 6s. 8d.? What is the value of $\frac{2}{7}$ of a guinea? Reduce $11\frac{3}{4}$d. to a decimal of a pound, correct to five places of decimals.

11. Reduce the expressions—

$$\tfrac{3}{20} + \tfrac{5}{12} + \tfrac{4}{15} - \tfrac{5}{6}, \text{ and } \sqrt{\left\{\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}} + \frac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{3}-\frac{1}{4}}\right\}}.$$

Multiply $49\frac{15}{16}$ by $50\frac{1}{16}$, and add $\frac{1}{256}$ to the result.
Divide $(2\frac{1}{16})^3 - 1$ by $(2\frac{1}{16})^2 + 3\frac{1}{16}$.

12. A bankrupt's estate amounts to £910. 3s. $1\frac{1}{2}$d., and his debts to £1875. What can he pay in the pound? and what will a creditor lose on a debt of £57?

13. A person having invested a sum of money in the Three per Cent. Consols received annually therefrom £233, after deducting the income-tax of 7d. in the pound. What is the sum of money? What can the stock be sold for when Consols are at $94\frac{1}{8}$.

14. Find the value of $\frac{\cdot 003 \times \cdot 004}{\cdot 006}$, and of $\frac{\cdot 10724}{\cdot 003125}$.

15. Prove the rule for finding the value of a circulating decimal, and divide 4·367 by the circulating decimal $\cdot 0\dot{5}\dot{2}$.

Reduce to its simplest form the quantity $\frac{3+\sqrt{5}}{3-\sqrt{5}} \times \frac{8-\sqrt{20}}{8+\sqrt{20}}$

16. Three persons, A, B, C, hold a pasture in common, for which they are to pay £30 per annum. A put in 7 oxen for 3 months; B, 9 oxen for 5 months; and C, 4 oxen for 12 months. How much rent ought each to pay?

17. Calculate to four places of decimals the value of the expression $\frac{\frac{5}{8} \text{ of } \cdot 31416}{\sqrt{\cdot 93}}$

18. Find the least common multiple of 16, 24, and 30, and explain the method.

19. What should be the price of English standard silver, 37-40ths fine, in order that the par of exchange between England and France should be 25 fr. 22 c.—200 francs being coined from 1 kilogram of silver, 9-10ths fine? (1 kilog. = 15·434 grains).

20. A person buys 100 shares in a company for £3,500; after receiving four half-yearly dividends of 15s. 4d., 20s. 10d., 30s. 4d., and 38s. 9d. per share, he sells at a profit of 43 per cent.; reckoning the simple interest of money at 4 per cent., how much above that interest has he gained?

21. The price of Three per Cent. Consols is $90\frac{3}{8}$; what sum must be invested in order to purchase £24 per annum; and what is the rate of interest on the money invested?

22. Three partners in trade contribute respectively the sums of £438, £292, £730, with the agreement that each was to receive 5 per cent. on their respective investments, and that the remainder of the gains of the firm, if any, was to be divided between them in the proportion of the sums originally advanced. The whole gain of the firm was £200. What was each man's share?

23. If 25 tons of goods are purchased for £37. 10s. and sold at 35s. a ton, what is the gain per ton?

At what rate per ton should the goods have been sold in order to obtain a profit of £9. 7s. 6d.?

24. Find the value of $\frac{5}{17}$ of £3. 12s. $11\frac{1}{2}$d.; and find the fraction that 3 miles, 2 fur. 100 yards is of 12 leagues, 2 fur. 20 yards.

25. The sum of £9040. 16s. is placed in the Three and a Half per Cents. at 94; find the income obtained, allowing on the stock purchased $\frac{1}{8}$th per cent. to the broker, and $\frac{1}{20}$ per cent. for other expenses.

26. Express as a fraction $\cdot 200\dot{1}2\dot{3}$, and express as a recurring decimal $\cdot 0\dot{1}\dot{2} \div \cdot 00\dot{1}3\dot{2}$.

27. By the reduction of the income-tax from 7d. in the pound to 5d. a person saves £28. 2s. 6d. a year; what is his income?

28. If 81 bushels of wheat are consumed by 56 men in 5 days, how long will 16 men take to consume 28 bushels?

29. Find the square root of $\cdot\dot{1}$, and prove that $\sqrt{\cdot 69\dot{4}} = 8\dot{3}$.

30. The periods of three planets which move uniformly in circular orbits round the sun are respectively 200, 250, and 300 days. Supposing that their positions relative to each other and to the sun to be given at any moment, determine how many days must elapse before they again have exactly the same relative positions.

SECTION II.

GEOMETRY.

EUCLID'S ELEMENTS, BOOK I.

Definitions.

1. A **point** is that which has position, but not magnitude.

2. A **line** is length without breadth.

3. The **extremities of a line** are points.

4. A **straight line** is that which lies evenly between its extreme points.

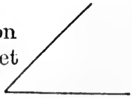
5. A **superficies** (or **surface**) is that which has only length and breadth.

6. The **extremities of a superficies** are lines.

7. A **plane superficies** is that in which any two points being taken, the straight line between them lies wholly in that superficies.

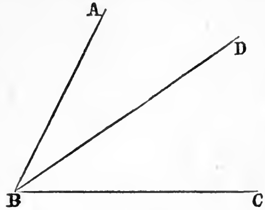
8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

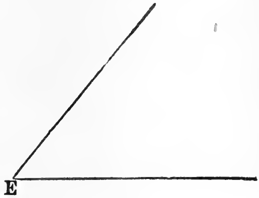
9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



NOTE.—When several angles are at one point B, any one of them is expressed by three letters, of which the middle letter is B, and the first letter is on one of the straight lines which contain the angle, and the last letter on the other line.

Thus, the angle contained by the straight lines AB and BC is expressed either by ABC or CBA, and the angle contained by AB and

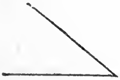




BD is expressed either by ABD or DBA. When there is only one angle at any given point, it may be expressed by the letter at that point, as the angle E.

10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle;** and the straight line which stands on the other is called a **perpendicular** to it.

11. An **obtuse angle** is that which is greater than a right angle.



12. An **acute angle** is that which is less than a right angle.

13. A **term** or **boundary** is the extremity of anything.

14. A **figure** is that which is enclosed by one or more boundaries.

15. A **circle** is a plane figure contained by one line, which is called the **circumference,** and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the **centre** of the circle, [and any straight line drawn from the centre to the circumference is called a **radius** of the circle].

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

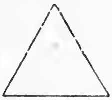
20. **Rectilineal figures** are those which are contained by straight lines.

21. **Trilateral figures,** or **triangles,** by three straight lines.

22. **Quadrilateral figures,** by four straight lines.

23. **Multilateral figures,** or **polygons,** by more than four straight lines.

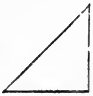
24. Of three-sided figures an **equilateral triangle** is that which has three equal sides.



25. An **isosceles triangle** is that which has only two sides equal.

26. A **scalene triangle** is that which has three unequal sides.

27. A **right-angled triangle** is that which has a right angle.

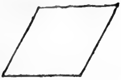


28. An **obtuse-angled triangle** is that which has an obtuse angle.

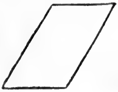
29. An **acute-angled triangle** is that which has three acute angles.

30. Of four-sided figures, a **square** is that which has all its sides equal, and all its angles right angles.

31. An **oblong** is that which has all its angles right angles, but not all its sides equal.



32. A **rhombus** is that which has all its sides equal, but its angles are not right angles.



33. A **rhomboid** is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

34. **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.

35. A **parallelogram** is a four-sided figure of which the opposite sides are parallel; and the **diagonal** is the straight line joining two of its opposite angles. All other four-sided figures are called **trapeziums.**

Postulates.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, at any distance from that centre.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same are equal to one another.

7. Things which are halves of the same are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot inclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced shall at length meet on that side on which are the angles which are less than two right angles.

Explanation of Terms and Abbreviations.

An **Axiom** is a truth admitted without demonstration.

A **Theorem** is a truth which is capable of being demonstrated from previously demonstrated or admitted truths.

A **Postulate** states a geometrical process, the power of effecting which is required to be admitted.

A **Problem** proposes to effect something by means of admitted processes, or by means of processes or constructions, the power of effecting which has been previously demonstrated.

A **Corollary** to a proposition is an inference which may be easily deduced from that proposition.

The sign $=$ is used to express *equality*.

$\angle$ means *angle*, and $\triangle$ signifies *triangle*, The sign > signifies "is greater than," and < "is less than."

+ expresses addition; thus AB + BC is the line whose length is the *sum* of the lengths of AB and BC.

− expresses subtraction; thus AB − BC is the excess of the length of the line AB above that of BC.

AB^2 means the square described upon the straight line AB.

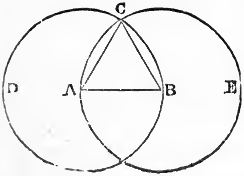
Proposition 1.—Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

From centres A and B, and radius=AB, describe circles.



CONSTRUCTION.—From the centre A, at the distance AB, describe the circle BCD (Post. 3).

From the centre B, at the distance BA, describe the circle ACE (Post. 3).

From the point C, in which the circles cut one another, draw the straight lines CA, CB to the points A and B (Post. 1).

Then ABC *shall be an equilateral triangle.*

AC=AB. PROOF.—Because the point A is the centre of the circle BCD, AC is equal to AB (Def. 15).

BC=AB. Because the point B is the centre of the circle ACE, BC is equal to BA (Def. 15).

AC and BC each=AB. Therefore AC and BC are each of them equal to AB.

AC=BC. But things which are equal to the same thing are equal to one another. Therefore AC is equal to BC (Ax. 1).

∴AB=BC=CA. Therefore AB, BC, and CA are equal to one another.

Therefore the triangle ABC is equilateral, and it is described on the given straight line AB. *Which was to be done.*

Proposition 2.—Problem.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

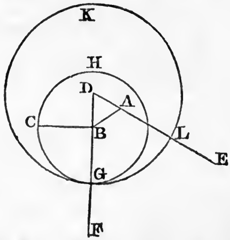
It is required to draw from the point A a straight line equal to BC.

Construction.—From the point A to B draw the straight line AB (Post. 1). Draw AB.

Upon AB describe the equilateral triangle DAB (Book I., Prop. 1). Δ DAB equilateral.

Produce the straight lines DA, DB, to E and F (Post. 2).

From the centre B, at the distance BC, describe the circle CGH, meeting DF in G (Post. 3). B as centre.



From the centre D, at the distance DG, describe the circle GKL, meeting DE in L (Post. 3). D as centre.

Then AL *shall be equal to* BC.

Proof.—Because the point B is the centre of the circle CGH, BC is equal to BG (Def. 15). BC=BG.

Because the point D is the centre of the circle GKL, DL is equal to DG (Def. 15). DL=DG.

But DA, DB, parts of them, are equal (Construction). DA=DB.

Therefore the remainder AL is equal to the remainder BG (Ax. 3). ∴ AL = BG.

But it has been shown that BC is equal to BG.

Therefore AL and BC are each of them equal to BG. ∴ AL and BC each = BG.

But things which are equal to the same thing are equal to one another, therefore AL is equal to BC (Ax. 1).

Therefore from the given point A a straight line AL has been drawn equal to the given straight line BC. *Which was to be done.* ∴ AL = BC.

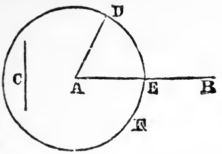
Proposition 3.—Problem.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB, the greater, a part equal to C, the less.

Make AD = C.



A as centre and radius AD.

CONSTRUCTION.—From the point A draw the straight line AD equal to C (I. 2).

From the centre A, at the distance AD, describe the circle DEF, cutting AB in E (Post. 3).

Then AE *shall be equal to* C.

PROOF.—Because the point A is the centre of the circle DEF, AE is equal to AD (Def. 15). — AE=AD.

But C is also equal to AD (Construction). — AD=C.

Therefore AE and C are each of them equal to AD. — AE and C each=AD.

Therefore AE is equal to C (Ax. 1). — ∴ AE=C.

Therefore, from AB, the greater of two given straight lines, a part AE has been cut off, equal to C, the less. *Q. E. F.**

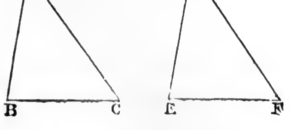
Proposition 4.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another: they shall have their bases, or third sides, equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite. Or,

If two sides and the contained angle of one triangle be respectively equal to those of another, the triangles are equal in every respect.

Let ABC, DEF be two triangles which have

The two sides AB, AC, equal to the two sides DE, DF, each to each, viz., AB equal to DE, and AC equal to DF. — AB=DE. AC=DF.



And the angle BAC equal to the angle EDF :—then— — ∠ BAC = ∠ EDF.

The base BC shall be equal to the base EF;

The triangle ABC shall be equal to the triangle DEF;

* Q. E. F. is an abbreviation for *quod erat faciendum*, that is "*which was to be done.*"

And the other angles to which the equal sides are opposite, shall be equal, each to each, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

PROOF.—For if the triangle ABC be applied to (*or placed upon*) the triangle DEF, Suppose Δ ABC put upon Δ DEF.

So that the point A may be on the point D, and the straight line AB on the straight line DE,

The point B shall coincide with the point E, because AB is equal to DE (Hypothesis).

And AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF (Hyp.).

Therefore also the point C shall coincide with the point F, because the straight line AC is equal to DF (Hyp.).

But the point B was proved to coincide with the point E.

Therefore the base BC shall coincide with the base EF.

Because the point B coinciding with E, and C with F, if the base BC do not coincide with the base EF, two straight lines would enclose a space, which is impossible (Ax. 10).

Therefore the base BC coincides with the base EF, and is therefore equal to it (Ax. 8). BC=EF.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it (Ax. 8). $\therefore \Delta$ ABC $= \Delta$ DEF.

And the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz., the angle ABC to DEF, and the angle ACB to DFE. $\angle$ ABC $=$ $\angle$ DEF. $\angle$ ACB $=$ $\angle$ DFE.

Therefore, if two triangles have, &c. (see Enunciation). *Which was to be shown.*

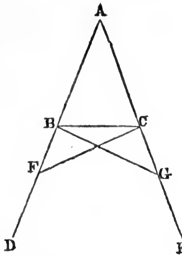
Proposition 5.—Theorem.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to the side AC. AB $=$ AC.

Let the straight lines AB, AC (*the equal sides of the triangle*), be produced to D and E.

The angle ABC shall be equal to the angle ACB (*angles at the base*), And the angle CBD shall be equal to the angle BCE (*angles upon the other side of the base*).



CONSTRUCTION.—In BD take any point F.

AG = AF. From AE, the greater, cut off AG, equal to AF, the less (I. 3).

Join FC, GB.

PROOF.—Because AF is equal to AG (Construction), and AB is equal to AC (Hyp.),

FA, AC respectively =GA, AB. Therefore the two sides FA, AC are equal to the two sides GA, AB, each to each;

And they contain the angle FAG, common to the two triangles AFC, AGB.

∴ FC=GB and Δ AFC = Δ AGB. Therefore the base FC is equal to the base GB (I. 4);

And the triangle AFC to the triangle AGB (I. 4);

∠ ACF = ∠ ABG. ∠ AFC = ∠ AGB. And the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite, viz., the angle ACF to the angle ABG, and the angle AFC to the angle AGB (I. 4).

And because the whole AF is equal to the whole AG, of which the parts AB, AC, are equal (Hyp.),

BF = CG. The remainder BF is equal to the remainder CG (Ax. 3).

And FC was proved to be equal to GB;

Therefore the two sides BF, FC are equal to the two sides CG, GB, each to each.

And the angle BFC was proved equal to the angle CGB;

Therefore the triangles BFC, CGB are equal; and their other angles are equal, each to each, to which the equal sides are opposite (I. 4).

∴ ∠ FBC = ∠ GCB. ∠ BCF = ∠ CBG. Therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.

And since it has been demonstrated that the whole angle ABG is equal to the whole angle ACF, and that the parts of these, the angles CBG, BCF, are also equal,

∴ ∠ ABC = ∠ ACB. Therefore the remaining angle ABC is equal to the remaining angle ACB (Ax. 3),

Which are the angles at the base of the triangle ABC.

And it has been proved that the angle FBC is equal to the angle GCB (Dem. 11),

Which are the angles upon the other side of the base,

Therefore the angles at the base, &c. (see Enunciation). *Which was to be shown.*

Corollary.—Hence every equilateral triangle is also equiangular.

Proposition 6.—Theorem.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB.

The side AB shall be equal to the side AC.

For if AB be not equal to AC, one of them is greater than the other. Let AB be the greater. Suppose AB > AC.

Construction.—From AB, the greater, cut off a part DB, equal to AC, the less (I. 3). Make DB = AC.

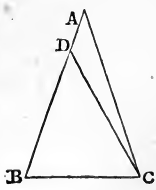
Join DC.

Proof.—Because in the triangles DBC, ACB, DB is equal to AC, and BC is common to both,

Therefore the two sides DB, BC are equal to the two sides AC, CB, each to each;

And the angle DBC is equal to the angle ACB (Hyp.)

Therefore the base DC is equal to the base AB (I. 4).



And the triangle DBC is equal to the triangle ACB (I. 4), the less to the greater, which is absurd. Δ DBC = Δ ACB.

Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, if two angles, &c. *Q. E. D.* *

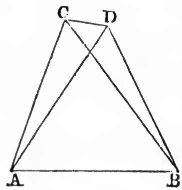
Corollary.—Hence every equiangular triangle is also equilateral.

* Q. E. D. is an abbreviation for *quod erat demonstrandum*, that is, "*which was to be shown or proved.*"

Proposition 7.—Theorem.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

Let the triangles ACB, ADB, upon the same base AB, and on the same side of it, have, if possible,



Suppose CA = DA. Their sides CA, DA, terminated in the extremity A of the base, equal to one another;

CB = DB. And their sides CB, DB, terminated in the extremity B of the base, likewise equal to one another.

Case I.—Let the vertex of each triangle be without the other triangle.

Construction.—Join CD.

Proof.—Because AC is equal to AD (Hyp.),

The triangle ADC is an isosceles triangle, and the angle
∠ ACD = ∠ ADC. ACD is therefore equal to the angle ADC (I. 5).

But the angle ACD is greater than the angle BCD (Ax. 9).

Therefore the angle ADC is also greater than BCD.

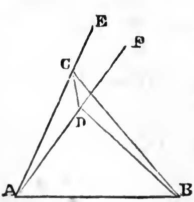
BDC > BCD. Much more then is the angle BDC greater than BCD.

Again, because BC is equal to BD (Hyp.),

The triangle BCD is an isosceles triangle, and the angle
∠ BDC = ∠ BCD. BDC is equal to the angle BCD (I. 5).

But the angle BDC has been shown to be greater than the angle BCD (Dem. 5).

∠ BDC = and > ∠ BCD. Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible.



Case II.—Let the vertex of one of the triangles fall within the other.

Construction.—Produce AC, AD to E and F, and join CD.

Proof.—Because AC is equal to AD (Hyp.),

Again ∠ ECD = ∠ FDC. The triangle ADC is an isosceles triangle, and the angles ECD, FDC, upon the other side of its base CD, are equal to one another (I. 5).

But the angle ECD is greater than the angle BCD (Ax. 9).

Therefore the angle FDC is likewise greater than BCD.

Much more then is the angle BDC greater than BCD. ∠ BDC > ∠ BCD.

Again, because BC is equal to BD (Hyp.),

The triangle BDC is an isosceles triangle, and the angle BDC is equal to the angle BCD (I. 5). ∠ BDC = ∠ BCD.

But the angle BDC has been shown to be greater than the angle BCD.

Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible. ∴ ∠ BDC = and > ∠ BCD.

Therefore, upon the same base, &c. *Q. E. D.*

Proposition 8.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other. Or,

If two triangles have three sides of the one respectively equal to the three sides of the other, they are equal in every respect, those angles being equal which are opposite to the equal sides.

Let ABC, DEF be two triangles which have

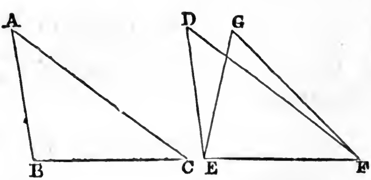
The two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF, Given AB = DE, AC = DF,

And the base BC equal to the base EF. and BC = EF.

The angle BAC shall be equal to the angle EDF.

Proof.—For if the triangle ABC be applied to the triangle DEF,

So that the point B may be on E, and the straight line BC on EF,



The point C shall coincide with the point F, because BC is equal to EF (Hyp.). Make BC coincide with EF.

Therefore, BC coinciding with EF, BA and AC shall coincide with ED and DF.

For if the base BC coincides with the base EF, But the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation, as EG, GF,

Then upon the same base, and on the same side of it, there will be two triangles, which have their sides terminated in one extremity of the base equal to one another, and likewise their sides, which are terminated in the other extremity. But this is impossible (I. 7).

∴ BA, AC respectively coincide with ED, DF.

Therefore, if the base BC coincides with the base EF, the sides BA, AC must coincide with the sides ED, DF.

Therefore the angle BAC coincides with the angle EDF, and is equal to it (Ax. 8).

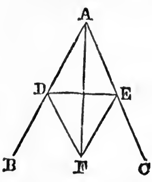
Also the triangle ABC coincides with the triangle DEF and is therefore equal to it in every respect (Ax. 8).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 9.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let BAC be the given rectilineal angle.



It is required to bisect it.

Make AE = AD.

CONSTRUCTION.—Take any point D in AB. From AC cut off AE equal to AD (I. 3). Join DE.

Δ DEF equilateral.

Upon DE, on the side remote from A, describe an equilateral triangle DEF (I. 1). Join AF.

Then the straight line AF *shall bisect the angle* BAC.

PROOF.—Because AD is equal to AE (Const.), and AF is common to the two triangles DAF, EAF;

The two sides DA, AF are equal to the two sides EA, AF, each to each;

And the base DF is equal to the base EF (Const.);

∴ ∠ DAF = ∠ EAF.

Therefore the angle DAF is equal to the angle EAF (I. 8).

Therefore the given rectilineal angle BAC is bisected by the straight line AF. *Q. E. F.*

Proposition 10.—Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.

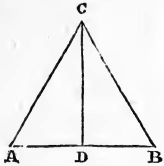
It is required to divide it into two equal parts.

Construction.—Upon AB describe the equilateral triangle ABC (I. 1).

Make Δ ABC equilateral and ∠ ACD = ∠ BCD.

Bisect the angle ACB by the straight line CD (I. 9).

Then AB *shall be cut into two equal parts in the point* D.



Proof.—Because AC is equal to CB (Const.), and CD common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD (Const.);

Therefore the base AD is equal to the base DB (I. 4). ∴ AD = DB.

Therefore the straight line AB is divided into two equal parts in the point D. *Q. E. F.*

Proposition 11.—Problem.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C a given point in it.

It is required to draw a straight line from the point C at right angles to AB.

Construction.—Take any point D in AC.

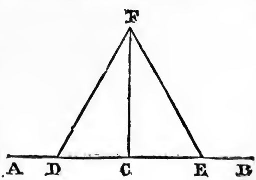
Make CE equal to CD (I. 3).

Upon DE describe the equilateral triangle DFE (I. 1).

Make CE = CD and Δ DEF equilateral.

Join FC.

Then FC *shall be at right angles to* AB.



Proof.—Because DC is equal to CE (Const.), and FC common to the two triangles DCF, ECF;

The two sides DC, CF, are equal to the two sides EC, CF, each to each;

And the base DF is equal to the base EF (Const.);

Therefore the angle DCF is equal to the angle ECF (I. 8); ∠ DCF = ∠ ECF.

And they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (Def. 10);

∴ ∠ DCF, ∠ ECF are right angles.

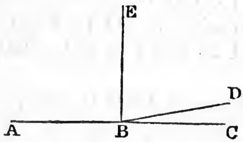
Therefore each of the angles DCF, ECF is a right angle.

Therefore from the given point C in the given straight line AB, a straight line FC has been drawn at right angles to AB. *Q. E. F.*

Corollary.—By help of this problem, it may be demonstrated that

Two straight lines cannot have a common segment.

If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.



Make ∠ ABE a right ∠

Construction.—From the point B, draw BE at right angles to AB (I. 11).

∠ CBE = ∠ EBA.

Proof. — Because ABC is a straight line, the angle CBE is equal to the angle EBA (Def. 10).

and ∠ DBE = ∠ EBA.

Also, because ABD is a straight line, the angle DBE is equal to the angle EBA (Def. 10).

∴ ∠ DBE = ∠ CBE.

Therefore the angle DBE is equal to the angle CBE. The less to the greater; which is impossible.

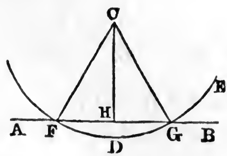
Therefore two straight lines cannot have a common segment.

Proposition 12.—Problem.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it.

It is required to draw from the point C, a straight line perpendicular to AB.



Construction.—Take any point D upon the other side of AB.

CD as radius.

From the centre C, at the distance CD, describe the circle EGF, meeting AB in F and G (Post. 3).

Bisect FG in H.

Bisect FG in H (I. 10).

Join CF, CH, CG.

Then CH *shall be perpendicular to* AB.

Proof.—Because FH is equal to HG (Const.), and HC common to the two triangles FHC, GHC;

The two sides FH, HC are equal to the two sides GH, HC, each to each;

And the base CF is equal to the base CG (Def. 15);

Therefore the angle CHF is equal to the angle CHG (I. 8), and they are adjacent angles.

∴ adjacent angles CHF, CHG are equal.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it (Def. 10).

Therefore, from the given point C, a perpendicular has been drawn to the given straight line AB. *Q. E. F.*

Proposition 13.—Theorem.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

These angles shall either be two right angles, or shall together be equal to two right angles.

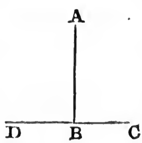
Proof.—If the angle CBA be equal to the angle ABD, each of them is a right angle (Def. 10).

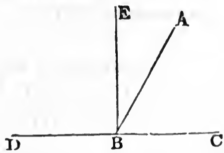
But if the angle CBA be not equal to the angle ABD, from the point B, draw BE at right angles to CD (I. 11).

Therefore the angles CBE, EBD, are two right angles.

Make ∠ CBE = ∠ EBD = a right ∠.

Now the angle CBE is equal to the two angles CBA, ABE; to each of these equals add the angle EBD.





Therefore the angles CBE, EBD, are equal to the three angles CBA, ABE, EBD (Ax. 2).

Again, the angle DBA is equal to the two angles DBE, EBA; to each of these equals add the angle ABC.

Therefore the angles DBA, ABC, are equal to the three angles DBE, EBA, ABC (Ax. 2).

∴ ∠ CBE + ∠ EBD = ∠ CBA + ∠ ABE + ∠ EBD, also ∠ DBA + ∠ ABC = ∠ DBE + ∠ EBA + ∠ ABC.

But the angles CBE, EBD have been shown to be equal to the same three angles;

And things which are equal to the same thing are equal to one another;

∴ ∠ CBE + ∠ EBD = ∠ DBA + ∠ ABC.

Therefore the angles CBE, EBD, are equal to the angles DBA, ABC (Ax. 1).

But the angles CBE, EBD are two right angles.

Therefore the angles DBA, ABC, are together equal to two right angles (Ax. 1).

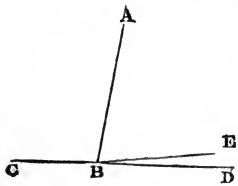
Therefore, the angles which one straight line, &c. *Q. E. D.*

Proposition 14.—Theorem.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

Given ∠ ABC + ∠ ABD = two right angles.

At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.



BD shall be in the same straight line with BC.

If possible, let CBE be a straight line.

For if BD be not in the same straight line with BC, let BE be in the same straight line with it.

Proof.—Because CBE is a straight line, and AB meets it in B.

Therefore the adjacent angles ABC, ABE are together equal to two right angles (I. 13).

But the angles ABC, ABD, are also together equal to two right angles (Hyp.);

Therefore the angles ABC, ABE, are equal to the angles ABC, ABD (Ax. 1).

Take away the common angle ABC.

∴ ∠ ABE = ∠ ABD.

The remaining angle ABE is equal to the remaining angle ABD (Ax. 3), the less to the greater, which is impossible;

Therefore BE is not in the same straight line with BC.

And, in like manner, it may be demonstrated that no other can be in the same straight line with it but BD.

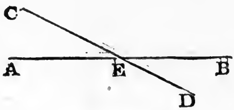
Therefore BD is in the same straight line with BC.

Therefore, if at a point, &c. *Q. E. D.*

Proposition 15.—Theorem.

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E.



The angle AEC shall be equal to angle DEB, and the angle CEB to the angle AED.

Proof.—Because the straight line AE makes with CD, the angles CEA, AED, these angles are together equal to two right angles (I. 13). ∠ CEA + ∠ AED = 2 right angles.

Again, because the straight line DE makes with AB the angles AED, DEB, these also are together equal to two right angles (I. 13). ∠ AED + ∠ DEB = 2 right angles.

But the angles CEA, AED have been shown to be together equal to two right angles,

Therefore the angles CEA, AED are equal to the angles AED, DEB (Ax. 1).

Take away the common angle AED.

The remaining angle CEA is equal to the remaining angle DEB (Ax. 3). ∴ ∠ CEA = ∠ DEB.

In the same manner it can be shown that the angles CEB, AED are equal.

Therefore, if two straight lines, &c. *Q. E. D.*

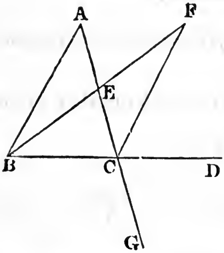
Corollary 1.—From this it is manifest that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

Corollary 2.—And, consequently, that all the angles made by any number of lines meeting in one point are together equal to four right angles, provided that no one of the angles be included in any other angle.

Proposition 16.—Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D. The exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.



Make AE = EC. and EF = BE.

CONSTRUCTION.—Bisect AC in E (I. 10).

Join BE, and produce it to F, making EF equal to BE (I. 3), and join FC.

PROOF.—Because AE is equal to EC, and BE equal to EF (Const.),

AE, EB are equal to CE, EF, each to each;

And the angle AEB is equal to the angle CEF, because they are opposite vertical angles (I. 15).

Therefore the base AB is equal to the base CF (I. 4);

And the triangle AEB to the triangle CEF (I. 4);

And the remaining angles to the remaining angles, each to each, to which the equal sides are opposite.

$\therefore \angle$ BAE $= \angle$ ECF.

Therefore the angle BAE is equal to the angle ECF (I. 4).

But the angle ECD is greater than the angle ECF (Ax. 9);

Therefore the angle ACD is greater than the angle BAE.

$\therefore \angle$ ACD $> \angle$ BAE.

In the same manner, if BC be bisected, and the side AC be produced to G, it may be proved that the angle BCG (or its equal ACD), is greater than the angle ABC.

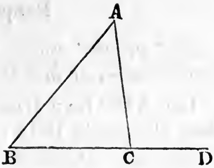
Therefore, if one side, &c. *Q. E. D.*

Proposition 17.—Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.

Any two of its angles together shall be less than two right angles.



CONSTRUCTION.—Produce BC to D.

PROOF.—Because ACD is the exterior angle of the triangle ABC, it is greater than the interior and opposite angle ABC (I. 16). ∠ ACD > ∠ ABC.

To each of these add the angle ACB. Add to each ∠ ACB.

Therefore the angles ACD, ACB are greater than the angles ABC, ACB (Ax. 4). ∠ ABC + ∠ ACB < 2 right angles.

But the angles ACD, ACB are together equal to two right angles (I. 13);

Therefore the angles ABC, ACB are together less than two right angles.

In like manner, it may be proved that the angles BAC, ACB, as also the angles CAB, ABC are together less than two right angles.

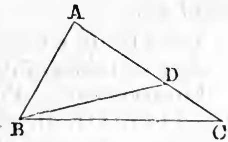
Therefore, any two angles, &c. *Q. E. D.*

Proposition 18.—Theorem.

The greater side of every triangle is opposite the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB. AC > AB.

The angle ABC shall be greater than the angle BCA.



CONSTRUCTION.—Because AC is greater than AB, make AD equal to AB (I. 3), and join BD. Make AD = AB.

PROOF.—Because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle BCD (I. 16). ∠ ADB > ∠ BCD, and

But the angle ADB is equal to the angle ABD; the triangle BAD being isosceles (I. 5), ∠ ADB = ∠ ABD and

Therefore the angle ABD is greater than the angle BCD (*or* ACB). ∴ ∠ ABD > ∠ BCD.

Much more then is the angle ABC greater than the angle ACB.

Therefore, the greater side, &c. *Q. E. D.*

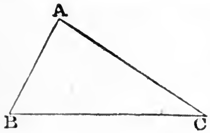
Proposition 19.—Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Given ∠ ABC > ∠ BCA.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA;

The side AC shall be greater than the side AB.



PROOF.—If AC be not greater than AB, it must either be equal to or less than AB.

It is not equal, for then the angle ABC would be equal to the angle BCA (I. 5); but it is not (Hyp.);

AC not = AB.

Therefore AC is not equal to AB.

Neither is AC less than AB, for then the angle ABC would be less than the angle BCA (I. 18); but it is not (Hyp.);

AC not < AB.

Therefore AC is not less than AB.

And it has been proved that AC is not equal to AB;

Therefore AC is greater than AB.

Therefore, the greater angle, &c. *Q. E. D.*

Proposition 20.—Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle;

Any two sides of it are together greater than the third side.

Make AD = AC.

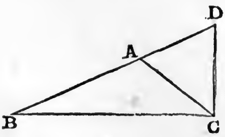
CONSTRUCTION.—Produce BA to the point D, making AD equal to AC (I. 3), and join DC.

PROOF.—Because DA is equal to AC, the angle ADC is equal to the angle ACD (I. 5).

But the angle BCD is greater than the angle ACD (Ax. 9);

∠ BCD > ∠ BDC.

Therefore the angle BCD is greater than the angle ADC (*or* BDC).



And because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater angle is subtended by the greater side;

∴ DB > BC.

Therefore the side DB is greater than the side BC (I. 19).

But BD is equal to BA and AC;
Therefore BA, AC are greater than BC. ∴ BA + AC > BC.

In the same manner it may be proved that AB, BC are greater than AC; and BC, CA greater than AB.

Therefore any two sides, &c. *Q. E. D.*

Proposition 21.—Theorem.

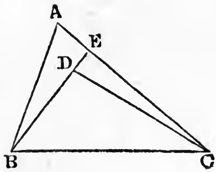
If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle;

BD, DC shall be less than the sides BA, AC;

But BD, DC shall contain an angle BDC greater than the angle BAC.

Construction.—Produce BD to E.



Proof.—1. Because two sides of a triangle are greater than the third side (I. 20), the two sides BA, AE, of the triangle BAE are greater than BE.

To each of these add EC.

Therefore the sides BA, AC, are greater than BE, EC (Ax. 4). BA + AC > BE + EC.

Again, because the two sides CE, ED, of the triangle CED are greater than CD (I. 20),

To each of these add DB.

Therefore CE, EB are greater than CD, DB (Ax. 4).

But it has been shown that BA, AC are greater than BE, EC; ∴ EC + EB > CD + DB.

Much more then are BA, AC greater than BD, DC.

Proof.—2. Again, because the exterior angle of a triangle is greater than the interior and opposite angle (I. 16), therefore BDC, the exterior angle of the triangle CDE, is greater than CED or CEB. Again ∠ BDC > ∠ CEB,

For the same reason, CEB, the exterior angle of the triangle ABE, is greater than the angle BAE or BAC. and ∠ CEB > ∠ BAE.

And it has been shown that the angle BDC is greater than CEB;

∴ ∠ BDC > ∠ BAC. Much more then is the angle BDC greater than the angle BAC.

Therefore, if from the ends, &c. *Q. E. D.*

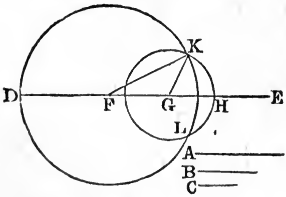
Proposition 22.—Problem.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these lines must be greater than the third (I. 20).

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third—namely, A and B greater than C, A and C greater than B, and B and C greater than A;

It is required to make a triangle of which the sides shall be equal to A, B, and C, each to each.

Construction.—Take a straight line DE terminated at the point D, but unlimited towards E.



DF, FG, GH respectively = A, B, C. Make DF equal to A, FG equal to B, and GH equal to C (I. 3).

FD as radius. From the centre F, at the distance FD, describe the circle DKL (Post. 3).

and GH as radius. From the centre G, at the distance GH, describe the circle HLK (Post. 3).

Join KF, KG.

Then the triangle KFG *shall have its sides equal to the three straight lines* A, B, C.

Proof.—Because the point F is the centre of the circle DKL, FD is equal to FK (Def. 15).

But FD is equal to A (Const.);

FK = A. Therefore FK is equal to A (Ax. 1).

Again, because the point G is the centre of the circle HLK, GH is equal to GK (Def. 15).

But GH is equal to C (Const.);

GK = C. Therefore GK is equal to C (Ax. 1), And FG is equal to B (Const.); FG = B

Therefore the three straight lines KF, FG, GK are equal to the three A, B, C, each to each.

Therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C. *Q. E. F.*

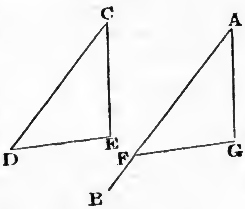
Proposition 23.—Problem.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to take an angle at the point A, in the straight line AB, equal to the rectilineal angle DCE.

CONSTRUCTION. — In CD, CE, take any points D, E, and join DE.



On AB construct a triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC — namely, AF equal to CD, FG to DE, and AG to EC (I. 22); Make △ AFG so that AF = CD FG = DE AG = CE.

Then the angle FAG *shall be equal to the angle* DCE.

PROOF.—Because DC, CE are equal to FA, AG, each to each, and the base DE equal to the base FG (Const.),

The angle DCE is equal to the angle FAG (I. 8). Then ∠ DCE = ∠ FAG

Therefore, at the given point A, in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE. *Q. E. F.*

Proposition 24.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF, be two triangles which have

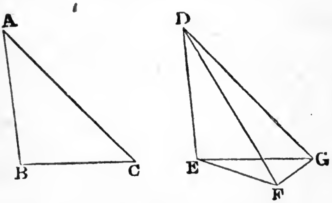
The two sides AB, AC equal to the two DE, DF, each to each—namely, AB to DE, and AC to DF,

But the angle BAC greater than the angle EDF;

The base BC shall be greater than the base EF.

Suppose DF > DE.

CONSTRUCTION.—Let the side DF of the triangle DEF be greater than its side DE.



Make ∠ EDG = ∠ BAC.

Then at the point D, in the straight line ED, make the angle EDG equal to the angle BAC (I. 23).

Make DG = AC, and = DF.

Make DG equal to AC or DF (I. 3).

Join EG, GF.

PROOF.—Because AB is equal to DE (Hyp.), and AC to DG (Const.), the two sides BA, AC are equal to the two ED, DG, each to each;

And the angle BAC is equal to the angle EDG (Const.);

∴ BC = EG.

Therefore the base BC is equal to the base EG (I. 4).

And because DG is equal to DF (Const.), the angle DFG is equal to the angle DGF (I. 5).

But the angle DGF is greater than the angle EGF (Ax. 9);

and ∠ EFG > ∠ EGF.

Therefore the angle DFG is greater than the angle EGF;

Much more then is the angle EFG greater than the angle EGF.

And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is subtended by the greater side,

∴ EG > EF.

Therefore the side EG is greater than the side EF (I. 19).

But EG was proved equal to BC;

Therefore BC is greater than EF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 25.—Theorem.

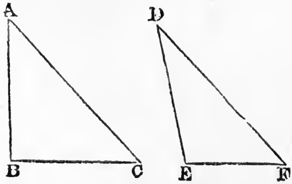
If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

Let ABC, DEF, be two triangles, which have

The two sides AB, AC equal to the two sides DE, DF, each to each—namely, AB to DE, and AC to DF,

But the base BC greater than the base EF;

The angle BAC shall be greater than the angle EDF.



PROOF.—For if the angle BAC be not greater than the angle EDF, it must either be equal to it or less.

But the angle BAC is not equal to the angle EDF, for then the base BC would be equal to the base EF (I. 4), but it is not (Hyp.);

Therefore the angle BAC is not equal to the angle EDF; $\angle$ BAC not $= \angle$ EDF.

Neither is the angle BAC less than the angle EDF, for then the base BC would be less than the base EF (I. 24), but it is not (Hyp.),

Therefore the angle BAC is not less than the angle EDF. $\angle$ BAC not $< \angle$ EDF.

And it has been proved that the angle BAC is not equal to the angle EDF;

Therefore the angle BAC is greater than the angle EDF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 26.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side—namely, either the side adjacent to the equal angles in each, or the side opposite to them; then shall the other sides be equal, each to each; and also the third angle of the one equal to the third angle of the other. Or,

If two angles and a side in one triangle be respectively equal to two angles and a corresponding side in another triangle, the triangles shall be equal in every respect.

Let ABC, DEF be two triangles, which have

The angles ABC, BCA equal to the angles DEF, EFD, each to each—namely, ABC to DEF, and BCA to EFD;

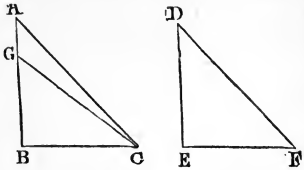
Also one side equal to one side.

CASE 1.—First, let the sides adjacent to the equal angles in each be equal—namely, BC to EF; Given BC = EF.

Then shall the side AB be equal to DE, the side AC to DF, and the angle BAC to the angle EDF.

Suppose AB > DE.

For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater of the two.



Make BG = DE.

Construction.—Make BG equal to DE (I. 3), and join GC.

Proof.—Because BG is equal to DE (Const.), and BC is equal to EF (Hyp.), the two sides GB, BC are equal to the two sides DE, EF, each to each.

And the angle GBC is equal to the angle DEF (Hyp.);
Therefore the base GC is equal to the base DF (I. 4),
And the triangle GBC to the triangle DEF (I. 4),
And the other angles to the other angles, each to each, to which the equal sides are opposite;

∴ ∠ GCB = ∠ DFE.

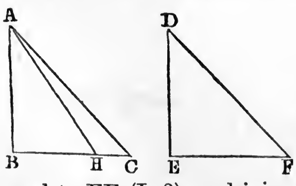
Therefore the angle GCB is equal to the angle DFE (I. 4).
But the angle DFE is equal to the angle BCA (Hyp.);
Therefore the angle GCB is equal to the angle BCA (Ax. 1), the less to the greater, which is impossible;

AB not unequal to DE.

Therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF (Hyp.);
Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,
And the angle ABC is equal to the angle DEF (Hyp.);
Therefore the base AC is equal to the base DF (I. 4),
And the third angle BAC to the third angle EDF (I. 4).

Case 2.—Next, let the sides which are opposite to the equal angles in each triangle be equal to one another—namely, AB equal to DE.

Likewise in this case the other sides shall be equal, AC to DF, and BC to EF; and also the angle BAC to the angle EDF.



Suppose BC > EF

For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater of the two.

Make BH = EF.

Construction.—Make BH equal to EF (I. 3), and join AH.

PROOF.—Because BH is equal to EF (Const.), and AB is equal to DE (Hyp.), the two sides AB, BH are equal to the two sides DE, EF, each to each, BH = EF, AB = DE.

And the angle ABH is equal to the angle DEF (Hyp.); ∠ ABH = ∠ DEF.

Therefore the base AH is equal to the base DF (I. 4),

And the triangle ABH to the triangle DEF (I. 4),

And the other angles to the other angles, each to each, to which the sides are opposite;

Therefore the angle BHA is equal to the angle EFD (I. 4). ∴ ∠ BHA = ∠ EFD = ∠ BCA.

But the angle EFD is equal to the angle BCA (Hyp.);

Therefore the angle BHA is also equal to the angle BCA (Ax. 1);

That is, the exterior angle BHA of the triangle AHC, is equal to its interior and opposite angle BCA, which is impossible (I. 16);

Therefore BC is not unequal to EF—that is, it is equal to it; and AB is equal to DE (Hyp.); BC not unequal to EF.

Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,

And the angle ABC is equal to the angle DEF (Hyp.);

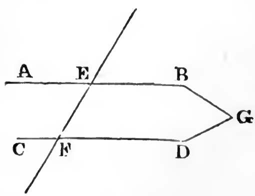
Therefore the base AC is equal to the base DF (I. 4),

And the third angle BAC is equal to the third angle EDF (I. 4).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 27.—Theorem.

If a straight line falling upon two other straight lines make the alternate angles equal to one another, these two straight lines shall be parallel.



Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another. Given ∠ AEF = ∠ EFD.

AB shall be parallel to CD.

For if AB and CD be not parallel, they will meet if produced, either towards B, D, or towards A, C. Let them be produced, and meet towards B, D, in the point G.

∠ AEF > ∠ EFG, and also = ∠ EFG.

PROOF.—Then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG (I. 16).

But the angle AEF is also equal to EFG (Hyp.), which is impossible;

Therefore AB and CD, being produced, do not meet towards B, D.

In like manner it may be shown that they do not meet towards A, C.

But those straight lines in the same plane which being produced ever so far both ways do not meet are parallel (Def. 34);

Therefore AB is parallel to CB.

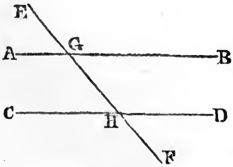
Therefore, if a straight line, &c. *Q. E. D.*

Proposition 28.—Theorem.

If a straight line falling upon two other straight lines make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make—

The exterior angle EGB equal to the interior and opposite angle GHD, upon the same side;



Or make the interior angles on the same side, the angles BGH, GHD, together equal to two right angles;

AB shall be parallel to CD.

PROOF 1.—Because the angle EGB is equal to the angle GHD (Hyp.),

And the angle EGB is equal to the angle AGH (I. 15);

∠ AGH = ∠ GHD.

Therefore the angle AGH is equal to the angle GHD (Ax. 1), and these angles are alternate;

Therefore AB is parallel to CD (I. 27).

PROOF 2.—Again, because the angles BGH, GHD are equal to two right angles (Hyp.),

And the angles BGH, AGH are also equal to two right angles (I. 13).

Therefore the angles BGH, AGH are equal to the angles BGH, GHD (Ax. 1). ∴ ∠ BGH + ∠ AGH = ∠ BGH + ∠ GHD.

Take away the common angle BGH.

Therefore the remaining angle AGH is equal to the remaining angle GHD (Ax. 3), and they are alternate angles. ∴ ∠ AGH = ∠ GHD

Therefore AB is parallel to CD (I. 27).

Therefore, if a straight line, &c. *Q. E. D.*

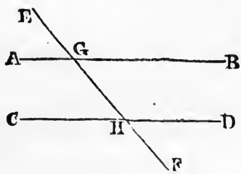
Proposition 29.—Theorem.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side; and also the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD;

The alternate angles AGH, GHD shall be equal to one another.

The exterior angle EGB shall be equal to GHD, the interior and opposite angle upon the same side;



And the two interior angles on the same side BGH, GHD shall be together equal to two right angles.

For if AGH be not equal to GHD, one of them must be greater than the other. Let AGH be the greater. ∠ AGH > GHD. (suppose.)

Proof.—Then the angle AGH is greater than the angle GHD; to each of them add the angle BGH.

Therefore the angles BGH, AGH are greater than the angles BGH, GHD (Ax. 4).

But the angles BGH, AGH are together equal to two right angles (I. 3).

Therefore the angles BGH, GHD are less than two right angles. ∴ ∠ BGH + ∠ GHD <two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles (Ax. 12);

Hence AB and CD meet, and are parallel.

Therefore the straight lines AB, CD will meet if produced far enough.

But they cannot meet, because they are parallel straight lines (Hyp.);

∴ ∠AGH not unequal to ∠ GHD.

Therefore the angle AGH is not unequal to the angle GHD—that is, it is equal to it.

But the angle AGH is equal to the angle EGB (I. 15);

and ∠EGB = ∠GHD,

Therefore the angle EGB is equal to the angle GHD (Ax. 1).

Add to each of these the angle BGH.

Therefore the angles EGB, BGH, are equal to the angles BGH, GHD (Ax. 2).

But the angles EGB, BGH, are equal to two right angles (I. 13).

also ∠BGH + ∠GHD = two right angles.

Therefore also BGH, GHD, are equal to two right angles (Ax. 1).

Therefore, if a straight line, &c. *Q. E. D.*

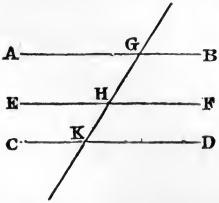
Proposition 30.—Theorem.

Straight lines which are parallel to the same straight lines are parallel to one another.

Let AB, CD be each of them parallel to EF;

AB shall be parallel to CD.

Construction.—Let the straight line GHK cut AB, EF, CD.



∠ AGH or ∠ AGK = ∠ GHF,

Proof.—Because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (I. 29).

and ∠ GHF = ∠ GKD.

Again, because GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD (I. 29).

And it was shown that the angle AGK is equal to the angle GHF;

∴ ∠ AGK = ∠ GKD.

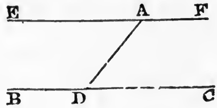
Therefore the angle AGK is equal to the angle GKD (Ax. 1), and they are alternate angles;

Therefore AB is parallel to CD (I. 27).
Therefore, straight lines, &c. *Q. E. D.*

Proposition 31.—Problem.

To draw a straight line through a given point, parallel to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw a straight line through the point A, parallel to BC.



CONSTRUCTION.—In BC take any point D, and join AD.

At the point A, in the straight line AD, make the angle DAE equal to the angle ADC (I. 23). Make ∠ DAE = ∠ ADC.

Produce the straight line EA to F.
Then EF *shall be parallel to* BC.

PROOF.—Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another; They are alternate angles.

Therefore EF is parallel to BC (I. 27).

Therefore, the straight line EAF is drawn through the given point A, parallel to the given straight line BC. *Q. E. F.*

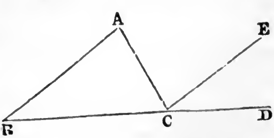
Proposition 32.—Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D;

The exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC;

And the three interior angles of the triangle—namely, ABC, BCA, CAB, shall be equal to two right angles.



CONSTRUCTION.—Through the point C, draw CE parallel to AB (I. 31). Make CE parallel to AB.

Then ∠ BAC = ∠ ACE,

PROOF.—Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (I. 29).

and ∠ ECD = ∠ ABC.

Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC (I. 29).

But the angle ACE was shown to be equal to the angle BAC;

∴ ∠ ACD = ∠ BAC + ∠ ABC.

Therefore the whole exterior angle ACD is equal to the two interior and opposite angles BAC, ABC (Ax. 2).

Add ∠ ACB.

To each of these equals add the angle ACB.

Therefore the angles ACD, ACB are equal to the three angles CBA, BAC, ACB (Ax. 2).

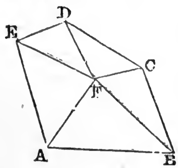
But the angles ACD, ACB are equal to two right angles (I. 13);

Therefore also the angles CBA, BAC, ACB are equal to two right angles (Ax. 1).

Therefore, if a side of any triangle, &c. *Q. E. D.*

COROLLARY 1.—*All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure ABCDE can, by drawing straight lines from a point F within the figure to each angle, be divided into as many triangles as the figure has sides.



And, by the preceding proposition, the angles of each triangle are equal to two right angles.

Therefore all the angles of the triangles are equal to twice as many right angles as there are triangles; that is, as there are sides of the figure.

But the same angles are equal to the angles of the figure, together with the angles at the point F;

And the angles at the point F, which is the common vertex of all the triangles, are equal to four right angles (I. 15, Cor. 2);

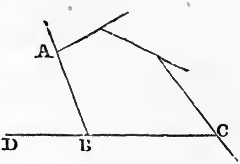
Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Corollary 2.—*All the exterior angles of any rectilineal figure are together equal to four right angles.*

The interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles (I. 13);

Therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides.

But all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides (I. 32, Cor. 1);



Therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles (Ax. 1).

Take away the interior angles which are common;

Therefore all the exterior angles are equal to four right angles (Ax. 3).

Proposition 33.—Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

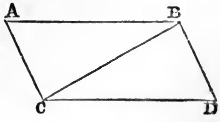
Let AB and CD be equal and parallel straight lines joined towards the same parts by the straight lines AC and BD;

AC and BD shall be equal and parallel.

Construction.—Join BC.

Proof.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29). ∠ ABC = ∠ BCD.

Because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two sides DC, CB, each to each;



And the angle ABC was proved to be equal to the angle BCD;

Therefore the base AC is equal to the base BD (I. 4), ∴ AC = BD,

And the triangle ABC is equal to the triangle BCD (I. 4),

And the other angles are equal to the other angles, each to each, to which the equal sides are opposite;

and ∠ ACB = ∠ CBD.

Therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;

Therefore AC is parallel to BD (I. 27); and it was shown to be equal to it.

Therefore, the straight lines, &c. *Q. E. D.*

Proposition 34.—Theorem.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram—that is, divides it into two equal parts.

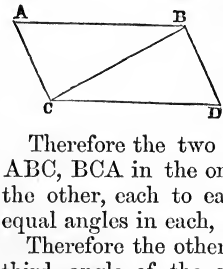
Let ACDB be a parallelogram, of which BC is a diagonal;

The opposite sides and angles of the figure shall be equal to one another,

And the diagonal BC shall bisect it.

∠ ABC = ∠ BCD,

Proof.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another (I. 29);



and ∠ ACB = ∠ CBD.

Because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another (I. 29);

Therefore the two triangles ABC, BCD have two angles, ABC, BCA in the one, equal to two angles, BCD, CBD in the other, each to each; and the side BC, adjacent to the equal angles in each, is common to both triangles.

∴ AB = CD, AC = BD, ∠ BAC = ∠ CDB,

Therefore the other sides are equal, each to each, and the third angle of the one to the third angle of the other—namely, AB equal to CD, AC to BD, and the angle BAC to the angle CDB (I. 26).

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB,

and ∴ ∠ ABD = ∠ ACD,

Therefore the whole angle ABD is equal to the whole angle ACD (Ax. 2).

And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diagonal bisects it.

For AB being equal to CD, and BC common,

The two sides AB, BC are equal to the two sides CD and CB, each to each.

And the angle ABC has been shown to be equal to the angle BCD;

Therefore the triangle ABC is equal to the triangle BCD (I. 4), also △ ABC = △ BCD.

And the diagonal BC divides the parallelogram ABCD into two equal parts.

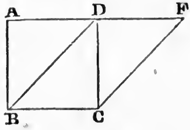
Therefore, the opposite sides, &c. *Q. E. D.*

Proposition 35.—Theorem.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

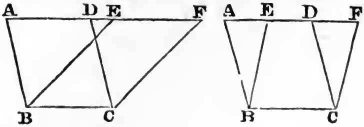
Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC;

The parallelogram ABCD shall be equal to the parallelogram EBCF.



Case 1.—If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain that each of the parallelograms is double of the triangle DBC (I. 34), and that they are therefore equal to one another (Ax. 6).

Case 2.—But if the sides AD, EF, opposite to the base BC, of the parallelograms ABCD, EBCF, be not terminated in the same point, then—



Proof.—Because ABCD is a parallelogram, AD is equal to BC (I. 34). AD = BC.

For the same reason EF is equal to BC; EF = BC.

Therefore AD is equal to EF (Ax. 1), and DE is common;

Therefore the whole, or the remainder, AE, is equal to the whole, or the remainder, DF (Ax. 2, or 3), ∴AE=DF.

And AB is equal to DC (I. 34).

Therefore the two EA, AB are equal to the two FD, DC, each to each;

And the exterior angle FDC is equal to the interior EAB (I. 29);

Therefore the base EB is equal to the base FC (I. 4),

Hence △ EAB = △ FDC.

And the triangle EAB equal to the triangle FDC (I. 4).

Take the triangle FDC from the trapezium ABCF, and from the same trapezium ABCF, take the triangle EAB, and the remainders are equal (Ax. 3),

That is, the parallelogram ABCD is equal to the parallelogram EBCF.

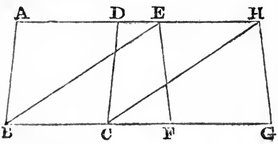
Therefore, parallelograms, &c. *Q. E. D.*

Proposition 36.—Theorem.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG;

The parallelogram ABCD shall be equal to the parallelogram EFGH.



Construction.—Join BE, CH.

Proof.—Because BC is equal to FG (Hyp.), and FG to EH (I. 34),

BC = EH,

Therefore BC is equal to EH (Ax. 1); and they are parallels, and joined towards the same parts by the straight lines BE, CH.

But straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel (I. 33);

and BE = CH.

Therefore BE, CH are both equal and parallel;

EBCH a parallelogram, equal each of the given ones.

Therefore EBCH is a parallelogram (Def. 35),

And it is equal to the parallelogram ABCD, because they are on the same base BC, and between the same parallels BC, AH (I. 35).

For the like reason, the parallelogram EFGH is equal to the same parallelogram EBCH;

Therefore the parallelogram ABCD is equal to the parallelogram EFGH (Ax. 1).

Therefore, parallelograms, &c. *Q. E. D.*

Proposition 37.—Theorem.

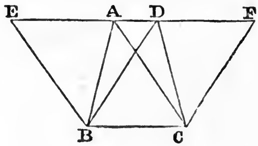
Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC;

The triangle ABC shall be equal to the triangle DBC.

Construction.—Produce AD both ways, to the points E, F.

Through B draw BE parallel to CA, and through C draw CF parallel to BD (I. 31).



Proof. — Then each of the figures EBCA, DBCF, is a parallelogram (Def. 35), and they are equal to one another, because they are on the same base BC, and between the same parallels BC, EF (I. 35.); Figures EBCA and DBCF are equal;

And the triangle ABC is half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34); and the triangles are respectively half of these.

And the triangle DBC is half of the parallelogram DBCF, because the diagonal DC bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DBC.

Therefore, triangles, &c. *Q. E. D.*

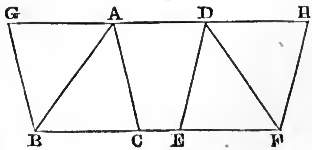
Proposition 38.—Theorem.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF, be on equal bases BC, EF, and between the same parallels BF, AD.

The triangle ABC shall be equal to the triangle DEF.

Construction.—Produce AD both ways to the points G, H.



Through B draw BG parallel to CA, and through F draw FH parallel to ED (I. 31).

Proof.—Then each of the figures GBCA. DEFH, is a Figures GBCA and DEFH are equal;

parallelogram (Def. 35), and they are equal to one another, because they are on equal bases BC, EF, and between the same parallels BF, GH (I. 36);

and the triangles are half of these respectively. And the triangle ABC is half of the parallelogram GBCA, because the diagonal AB bisects it (I. 34);

And the triangle DEF is half of the parallelogram DEFH, because the diagonal DF bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DEF.

Therefore, triangles, &c. *Q. E. D.*

Proposition 39.—Theorem.

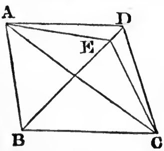
Equal triangles upon the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and on the same side of it;

They shall be between the same parallels.

CONSTRUCTION.—Join AD; AD shall be parallel to BC.

AE parallel to BC suppose. For if it is not, through A draw AE parallel to BC (I. 31), and join EC.



PROOF.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37).

But the triangle ABC is equal to the triangle DBC (Hyp.);

Then △ DBC = △ EBC, an absurdity. Therefore the triangle DBC is equal to the triangle EBC (Ax. 1), the greater equal to the less, which is impossible;

Therefore AE is not parallel to BC.

In the same manner, it can be demonstrated that no line passing through A can be parallel to BC, except AD;

Therefore AD is parallel to BC.

Therefore, equal triangles, &c. *Q. E. D.*

Proposition 40.—Theorem.

Equal triangles upon the same side of equal bases, that are in the same straight line, are between the same parallels.

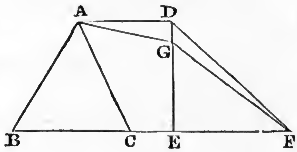
Let the equal triangles ABC, DEF, be upon the same side of equal bases BC, EF, in the same straight line BF, The triangles ABC, DEF shall be between the same parallels.

Construction.—Join AD; AD shall be parallel to BF.

For if it is not, through A draw AG parallel to BF (I. 31), and join GF.

AG parallel to BF suppose.

Proof.—The triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and are between the same parallels BF, AG (I. 38).



But the triangle ABC is equal to the triangle DEF;

Therefore the triangle DEF is equal to the triangle GEF (Ax. 1), the greater equal to the less, which is impossible;

△ DEF = △ GEF, an absurdity.

Therefore AG is not parallel to BF.

In the same manner, it can be demonstrated that no line, passing through A, can be parallel to BF, except AD;

Therefore AD is parallel to BF.

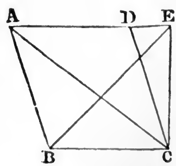
Therefore, equal triangles, &c.

Proposition 41.—Theorem.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE;

The parallelogram ABCD shall be double of the triangle EBC.



Construction.—Join AC.

Proof.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37).

△ ABC = △ EBC.

But the parallelogram ABCD is double of the triangle ABC, because the diagonal AC bisects the parallelogram (I. 34).

And parallelogram = 2 △ ABC.

Therefore the parallelogram ABCD is also double of the triangle EBC (Ax. 1).

Therefore, if a parallelogram, &c. *Q. E. D.*

Proposition 42.—Problem.

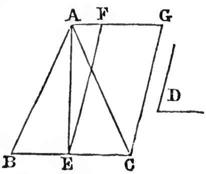
To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle;

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

Make BE = EC

and ∠ CEF = D



Construction.—Bisect BC in E (I. 10), and join AE.

At the point E, in the straight line CE, make the angle CEF equal to D (I. 23).

Through A draw AFG parallel to EC (I. 31).

Through C draw CG parallel to EF (I. 31).

Then FECG *is the parallelogram required.*

Proof.—Because BE is equal to EC (Const.), the triangle ABE is equal to the triangle AEC, since they are upon equal bases and between the same parallels (I. 38);

△ ABC = 2 △ AEC, and also figure FECG = 2 △ AEC.

Therefore the triangle ABC is double of the triangle AEC.

But the parallelogram FECG is also double of the triangle AEC, because they are upon the same base, and between the same parallels (I. 41);

Therefore the parallelogram FECG is equal to the triangle ABC (Ax. 6),

And it has one of its angles CEF equal to the given angle D (Const.).

Therefore a parallelogram FECG has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. *Q. E. F.*

Proposition 43.—Theorem.

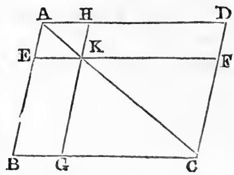
The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements.

The complement BK shall be equal to the complement KD.

PROOF.—Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (I. 34). △ ABC = △ ADC.

Again, because AEKH is a parallelogram, and AK its diagonal, the triangle AEK is equal to the triangle AHK (I. 34). Also △ AEK = △ AHK.



For the like reason the triangle KGC is equal to the triangle KFC. And △ KGC = △ KFC.

Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC,

The triangles AEK, KGC are equal to the triangles AHK, KFC (Ax. 2).

But the whole triangle ABC was proved equal to the whole triangle ADC;

Therefore the remaining complement BK is equal to the remaining complement KD (Ax. 3). ∴ BK = KD.

Therefore, the complements, &c. *Q. E. D.*

Proposition 44.—Problem.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make parallelogram BEFG = △ C, and ∠ at B = ∠ D, and EBA a straight line.

CONSTRUCTION 1.—Make the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D (I. 42);

And let the parallelogram BEFG be made so that BE may be in the same straight line with AB.

Produce FG to H.

Through A draw AH parallel to BG or EF (I. 31).

Join HB.

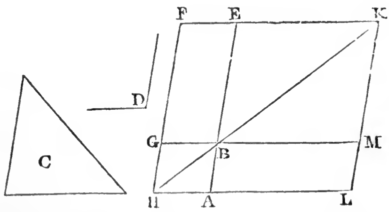
PROOF 1.—Because the straight line HF falls on the parallels AH, EF, the angles AHF, HFE are together equal to two right angles (I. 29).

Therefore the angles BHF, HFE are together less than two right angles (Ax. 9). But straight lines which with another straight line make the interior angles on the same side together less than two right angles, will meet on that side, if produced far enough (Ax. 12);

HB and FE meet.

Therefore HB and FE shall meet if produced.

CONSTRUCTION 2.—Produce HB and FE towards BE, and let them meet in K.



Through K draw KL parallel to EA or FH (I. 31).

Produce HA, GB to the points L, M.

Then LB *shall be the parallelogram required.*

PROOF 2.—Because HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are the parallelograms about HK; and LB, BF are the complements;

Figures LB = BF.

Therefore the complement LB is equal to the complement BF (I. 43).

But BF is equal to the triangle C (Const.);
Therefore LB is equal to the triangle C (Ax. 1).

But BF = △ C, ∴ LB = △ C;

And because the angle GBE is equal to the angle ABM (I. 15), and likewise to the angle D (Const.);
Therefore the angle ABM is equal to the angle D (Ax. 1).
Therefore, the parallelogram LB is applied to the straight line AB, and is equal to the triangle C, and has the angle ABM equal to the angle D. *Q. E. F.*

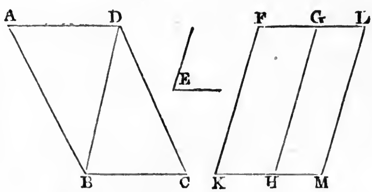
Also ∠ ABM = ∠ GBE = ∠ D.

Proposition 45.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.



Construction.—Join DB.

Describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E (I. 42).

To the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E (I. 44).

Make FH = △ ADB. Apply to GH, GM = △ DBC, with ∠ GHM = ∠ E.

Then the figure FKML *shall be the parallelogram required.*

Proof.—Because the angle E is equal to each of the angles FKH, GHM (Const.),

Therefore the angle FKH is equal to the angle GHM (Ax. 1).

Add to each of these equals the angle KHG;

Therefore the angles FKH, KHG are equal to the angles KHG, GHM (Ax. 2).

But FKH, KHG are equal to two right angles (I. 29);

Therefore also KHG, GHM are equal to two right angles (Ax. 1).

And because at the point H, in the straight line GH, the two straight lines KH, HM, on the opposite sides of it, make the adjacent angles together equal to two right angles,

Then KHM is a straight line;

Therefore KH is in the same straight line with HM (I. 14).

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal (I. 29).

Add to each of these equals the angle HGL;

Therefore the angles MHG, HGL are equal to the angles HGF, HGL (Ax. 2).

But the angles MHG, HGL are equal to two right angles (I. 29);

Therefore also the angles HGF, HGL are equal to two right angles,

and FGL is a straight line.

And therefore FG is in the same straight line with GL (I. 14).

And because KF is parallel to HG, and HG parallel to ML (Const.);

Therefore KF is parallel to ML (I. 30).

And KM, FL are parallels (Const);

∴ KFLM a parallelogram.

Therefore KFLM is a parallelogram (Def. 35).

And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC equal to the parallelogram GM (Const.),

And the figure ABCD is equal to it.

Therefore the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM (Ax. 2).

Therefore, the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E. *Q. E. F.*

Corollary.—From this it is manifest how to apply to a given straight line a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure—namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on (I. 44).

Proposition 46.—Problem.

To describe a square upon a given straight line.

Let AB be the given straight line;

It is required to describe a square upon AB.

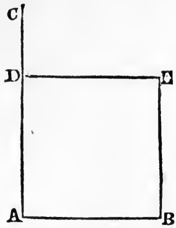
CONSTRUCTION.—From the point A draw AC at right angles to AB (I. 11),

And make AD equal to AB (I. 3).

Through the point D draw DE parallel to AB (I. 31).

Through the point B draw BE parallel to AD (I. 31).

Then ADEB *shall be the square required.*



PROOF.—Because DE is parallel to AB, and BE parallel to AD (Const.), therefore ADEB is a parallelogram; ADEB a parallelogram.

Therefore AB is equal to DE, and AD to BE (I. 34).

But AB is equal to AD (Const.);

Therefore the four straight lines BA, AD, DE, EB are equal to one another (Ax. 1),

And the parallelogram ADEB is therefore equilateral. It is equilateral.

Likewise all its angles are right angles.

For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles (I. 29).

But BAD is a right angle (Const.);

Therefore also ADE is a right angle (Ax. 3).

But the opposite angles of parallelograms are equal (I. 34);

Therefore each of the opposite angles ABE, BED is a right angle (Ax. 1);

Therefore the figure ADEB is rectangular; and it has been proved to be equilateral; therefore it is a square (Def. 30). It is rectangular.

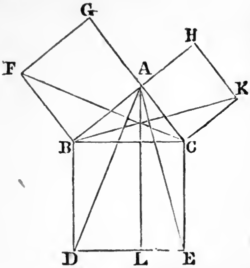
Therefore, the figure ADEB is a square, and it is described upon the given straight line AB. *Q. E. F.* ∴ a square.

COROLLARY.—Hence every parallelogram that has one right angle has all its angles right angles.

Proposition 47.—Theorem.

In any right-angled triangle, the square which is described upon the side opposite to the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC;



The square described upon the side BC shall be equal to the squares described upon BA, AC.

Construction. — On BC describe the square BDEC (I. 46).

On BA, AC describe the squares GB, HC (I. 46).

Through A draw AL parallel to BD or CE (I. 31).

Join AD, FC.

Proof.—Because the angle BAC is a right angle (Hyp.), and that the angle BAG is also a right angle (Def. 30),

The two straight lines AC, AG, upon opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;

CAG is a straight line.

Therefore CA is in the same straight line with AG (I. 14).

BAH is a straight line.

For the same reason, AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle (Ax. 11); add to each the angle ABC.

Therefore the whole angle DBA is equal to the whole angle FBC (Ax. 2).

And because the two sides AB, BD are equal to the two sides FB, BC, each to each (Def. 30), and the angle DBA equal to the angle FBC;

△ ABD = △ FBC.

Therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC (I. 4).

Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD, and between the same parallels BD, AL (I. 41).

And the square GB is double of the triangle FBC, because they are on the same base FB, and between the same parallels FB, GC (I. 41).

Hence parallelogram BL = square GB, and parallelogram CL = square HC.

But the doubles of equals are equal (Ax. 6), therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shown that the parallelogram CL is equal to the square HC.

Therefore the whole square BDEC is equal to the two squares GB, HC (Ax. 2);

And the square BDEC is described on the straight line BC, and the squares GB, HC upon BA, AC. ∴ $BC^2 = BA^2 + AC^2$

Therefore the square described upon the side BC is equal to the squares described upon the sides BA, AC.

Therefore, in any right-angled triangle, &c. *Q. E. D.*

Proposition 48.—Theorem.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares described upon the other sides BA, AC;

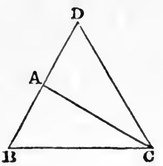
The angle BAC shall be a right angle.

Construction.—From the point A draw AD at right angles to AC (I. 11). Draw AD at right angles to AC.

Make AD equal to BA (I. 3), and join DC. *(Do not produce BA.)*

Proof.—Because DA is equal to AB, the square on DA is equal to the square on BA.

To each of these add the square on AC.



Therefore the squares on DA, AC are equal to the squares on BA, AC (Ax. 2).

But because the angle DAC is a right angle (Const.), the square on DC is equal to the squares on DA, AC (I. 47),

And the square on BC is equal to the squares on BA, AC (Hyp.);

Therefore the square on DC is equal to the square on BC (Ax. 1);

And therefore the side DC is equal to the side BC. Then $DC^2 = BC^2$, and $DC = BC$.

And because the side DA is equal to AB (Const.), and AC common to the two triangles DAC, BAC, the two sides DA, AC are equal to the two sides BA, AC, each to each.

And the base DC has been proved equal to the base BC;

Hence ∠DAC = ∠BAC.

Therefore the angle DAC is equal to BAC (I. 8).
But DAC is a right angle (Const.);
Therefore also BAC is a right angle (Ax. 1).
Therefore, if the square, &c. *Q. E. D.*

EXERCISES.

Prop. 1—15.

1. From the greater of two given straight lines to cut off a portion which is three times as long as the less.
2. The line bisecting the vertical angle of an isosceles triangle also bisects the base.
3. Prove Euc. I. 5, by the method of *super-position*.
4. In the figure to Euc. I. 5, show that the line joining A with the point of intersection of BG and FC, makes equal angles with AB and AC.
5. ABC is an isosceles triangle, whose base is BC, and AD is perpendicular to BC; every point in AD is equally distant from B and C.
6. Show that the sum of the sum and difference of two given straight lines is twice the greater, and that the difference of the sum and difference is twice the less.
7. Prove the same property with regard to angles.
8. Make an angle which shall be three-fourths of a right angle.
9. If, with the extremities of a given line as centres, circles be drawn intersecting in two points, the line joining the points of intersection will be perpendicular to the given line, and will also bisect it.
10. Find a point which is at a given distance from a given point and from a given line.
11. Show that the sum of the angles round a given point are together equal to four right angles.
12. If the exterior angle of a triangle and its adjacent interior angle be bisected, the bisecting lines will be at right angles.
13. If three points, A, B, C, be taken not in the same straight line, and AB and AC be joined and bisected by perpendiculars which meet in D, show that DA, DB, DC are equal to each other.

Prop. 16—32.

14. The perpendiculars from the angular points upon the opposite sides of a triangle meet in a point.
15. To construct an isosceles triangle on a given base, the sides being each of them double the given base.

16. Describe an isosceles triangle having a given base, and whose vertical angle is half a right angle.

17. AB is a straight line, C and D are points on the same side of it; find a point E in AB such that the sum of CE and ED shall be a minimum.

18. Having given two sides of a triangle and an angle, construct the triangle. Examine the cases when there will be (1.) one solution; (2.) two solutions; (3.) none.

19. Given an angle of a triangle and the sum and difference of the two sides including the angle, to construct the triangle.

20. Show that each of the angles of an equilateral triangle is two-thirds of a right angle, and hence show how to trisect a right angle.

21. If two angles of a triangle be bisected by lines drawn from the angular points to a given point within, then the line bisecting the third angle will pass through the same point.

22. The difference of any two sides of a triangle is less than the third side.

23. If the angles at the base of a right-angled isosceles triangle be bisected, the bisecting line includes an angle which is three halves of a right angle.

24. The sum of the lines drawn from any point within a polygon to the angular points is greater than half the sum of the sides of the polygon.

Prop. 33—48.

25. Show that the diagonals of a square bisect each other at right angles, and that the square described upon a semi-diagonal is half the given square.

26. Divide a given line into any number of equal parts, and hence show how to divide a line similarly to a given line.

27. If D and E be respectively the middle points of the sides BC and AC of the triangle ABC, and AD and BE be joined, and intersect in G, show that GD and GE are respectively one-third of AD and BE.

28. The lines drawn to the bisections of the sides of a triangle from the opposite angles meet in a point.

29. Describe a square which is five times a given square.

30. Show that a square, hexagon, and dodecagon will fill up the space round a point.

31. Divide a square into three equal areas, by lines drawn parallel to one of the diagonals.

32. Upon a given straight line construct a regular octagon.

33. Divide a given triangle into equal triangles by lines drawn from one of the angles.

34. If any two angles of a quadrilateral are together equal to two right angles, show that the sum of the other two is two right angles.

35. The area of a trapezium having two parallel sides is equal to half the rectangle contained by the perpendicular distance between the parallel sides of the trapezium, and the sum of the parallel sides.

36. The area of any trapezium is half the rectangle contained by one of the diagonals of the trapezium, and the sum of the perpendiculars let fall upon it from the opposite angles.

37. If the middle points of the sides of a triangle be joined, the lines form a triangle whose area is one-fourth that of the given triangle.

38. If the sides of a triangle be such that they are respectively the sum of two given lines, the difference of the same two lines, and twice the side of a square equal to the rectangle contained by these lines, the triangle shall be right-angled, having the right angle opposite to the first-named side.

39. If a point be taken within a triangle such that the lengths of the perpendiculars upon the sides are equal, show that the area of the rectangle contained by one of the perpendiculars and the perimeter of the triangle is double the area of the triangle.

40. In the last problem, if O be the given point, and OD, OE, OF the respective perpendiculars upon the sides BC, AC, and AB, show that the sum of the squares upon AD, OB, and DC exceeds the sum of the squares upon AF, BD, and CD by three times the square upon either of the perpendiculars.

41. Having given the lengths of the segments AF, BD, CE, in Problem 40, construct the triangle.

42. Draw a line, the square upon which shall be seven times the square upon a given line.

43. Draw a line, the square upon which shall be equal to the sum or difference of two given squares.

44. Reduce a given polygon to an equivalent triangle.

45. Divide a triangle into equal areas by drawing a line from a given point in a side.

46. Do the same with a given parallelogram.

47. If in the fig., Euc. I. 47, the square on the hypothenuse be on the other side, show how the other two squares may be made to cover exactly the square on the hypothenuse.

48. The area of a quadrilateral whose diagonals are at right angles is half the rectangle contained by the diagonals.

SECTION III.

ALGEBRA.

CHAPTER I.

ELEMENTARY PRINCIPLES.

1. Algebra treats of numbers, the numbers being represented by letters (symbols of *quantity*), affected with certain symbols of *quality*, and connected by symbols of *operation*.

It is easy to see that these symbols of quantity may be dealt with very much as we deal with concrete quantities in arithmetic. Thus, allowing the letter a to stand for the number of units in any quantity, and allowing also $2a$, $3a$, $4a$, &c., to stand respectively for *twice, thrice, four times,* &c., as large a quantity as the letter a, it at once follows that we may perform the operations of addition, subtraction, multiplication, and division upon these symbols exactly as we do in ordinary arithmetic upon concrete quantities. For instance, $4a$ and $6a$ make $10a$, $9a$ exceeds $5a$ by $4a$, $15a$ is 5 times $3a$, and $7a$ is contained 8 times in $56a$.

Neither is it necessary in these operations to *state*, or even to *know* the exact number of units for which any symbol of quantity stands, nor indeed the nature of these units; it is simply sufficient that it is a symbol of *quantity*. Thus, in the science of chemistry, we use a weight called a *crith;* and a person unacquainted with chemistry might not know whether a crith were a measure of length, weight, or capacity, or indeed whether it were a measure at all, yet he would at once allow that 6 criths and 5 criths are 11 criths, that twice 4 criths are 8 criths, &c.

The Signs + and − as Symbols of Operation.

2. In purely arithmetical operations, the signs + and − are respectively the signs of addition and subtraction. In this sense, too, they are used in algebra.

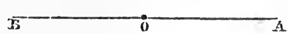
Thus, $a + b$ means that b is to be *added* to a, and $a - b$ means that b is to be *subtracted* from a.

Hence, as long as a and b represent ordinary arithmetical numbers, $a + b$ admits of easy interpretation, as also does $a - b$, when b is not greater than a. But when b is greater than a, the expression $a - b$ has no arithmetical meaning. By an extension, however, of the use of the signs + and −, we are able to give such expressions an intelligible signification, whatever may be the quantities represented by a and b.

Positive and Negative Quantities.—The Signs + and − as Symbols of Affection or Quality.

3. Def.—A **positive** quantity is one which is affected with a + sign, and a **negative** quantity is one which is affected with a − sign.

Let BA be a straight line, and O a point in the line; and suppose a person, starting from O, to walk a miles in the direction OA. Suppose also another person, starting from the same or any other point in BA to walk a miles in the direction OB. These persons will thus walk a miles each in *exactly opposite* directions. Now, we call one of these directions *positive* (it matters not which) and the other *negative*. Let us take the direction OA as *positive*. We then have the first person walking a miles in a *positive* direction, and the second walking a miles in a *negative* direction. We represent these distances algebraically by $+ a$ and $- a$ respectively.

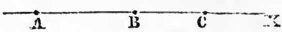


It will therefore be seen that the signs + and − have no effect upon the magnitudes of quantities, but that they express the *quality* or *affection* of the quantities before which they stand.

Again, suppose a person in business to get a profit of £6, while another suffers a loss of £6. We may express these facts algebraically in two ways. We may consider *gain* as positive, and *loss* as negative gain, and say that the former has gained + 6 pounds, while the latter has gained − 6 pounds. Or we may consider *loss* as positive, and *gain* as negative loss, and say that the former has lost − 6 pounds, while the latter has lost + 6 pounds. We hence see that the gain of + 6 is equal to a loss of − 6, and that a gain of − 6 is equal to a loss of + 6.

The Sum of Algebraical Quantities.

4. Let a distance AB be measured to the *right* along the line AX. And let a further distance BC be measured from B in the *same* direction. By the *sum* of these lines we mean the resulting distance of the point C from the original point A, that is to say, the distance AC.



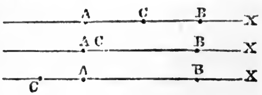
(It may be remarked that we *add* the line BC to the line AB by measuring BC in its *own proper direction* from the extremity B of AB. It is hardly necessary to remind the student that both lines are in the same straight line AX.)

Let us represent the distances AB and BC by $+ a$ and $+ b$ respectively; then the *algebraical sum* of the lines will be represented writing these quantities side by side, each with its *own proper sign* of affection.

Thus the sum of the distances AB and BC is expressed by $+ a + b$, or, as it is usual to omit the + sign of a positive quantity when the quantity stands alone or at the head of an algebraical expression, the sum of AB and BC is expressed by $a + b$.

Hence, the interpretation of $a + b$ is that it represents the distance AC.

Again, taking as above $+ a$ to represent the distance AB along the straight line AX, and measured to the *right*, let a distance BC be measured from B in the same straight line AX, but this time to the *left*.



Let the latter distance be represented by $-b$.

Then, on the principle above, AC is the *sum* of these distances, and this *sum* is represented algebraically by $+a-b$ or $a-b$.

(It will be seen that the distance BC is again measured from B in its *own proper direction*, and that the resultant distance AC of the point C is again the *sum* of the line AB and BC.)

There will evidently be three cases, viz. :—

1. When the distance BC is less in magnitude than AB, in which case the point C is on the right of A, and the distance AC is *positive.*

2. When the distance BC is *equal* in magnitude to AB, in which case the point C coincides with A, and the distance AC is *zero.*

3. When the distance BC is *greater* in magnitude than AB, the point C being then on the left of A, and the distance is *negative.*

Now, $a-b$ in all these cases represents the distance AC. It therefore admits of intelligible interpretation whether b be less than, equal to, or greater than a.

And, since the distance AC is obtained in the first two cases by subtracting the distance BC from that of AB, and in the second case by subtracting as far as AB will allow of subtraction, and measuring the remainder to be subtracted in an *opposite* direction, it follows that—

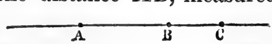
The sign $-$, which, standing before a letter, is a symbol of quality, becomes at once a symbol of subtraction in all cases when the quantity in question is placed immediately after any other given quantity with its proper sign of affection.

Hence also we may conclude that the addition of a *negative* quantity is equivalent to the subtraction of the corresponding *positive* quantity.

5. We may prove in a similar way that—

The subtraction of a *negative* quantity is equivalent to the addition of the corresponding *positive* quantity.

Let, as before, $+a$ represent the distance AB, measured from the point A to the *right*, and let it be required to *subtract* from $+a$ the distance represented by $-b$.



Now, in the last article, we *added* a distance to a given distance AB, by measuring the second distance in its *own direction* from the extremity B. We shall therefore be consistent if we *subtract* a given distance $-b$ by measuring this distance in a direction *exactly opposite* to its own direction, from the same extremity B.

Now, the direction of $-b$ is to the *left*. If, therefore, we measure a distance BC to the *right*, equal in magnitude to the distance to be subtracted, we obtain a distance AC which is correctly represented by $a-(-b)$. But AC is also correctly represented by $a+b$, and hence it follows that $a-(-b)=a+b$.

We may apply the above principle to all magnitudes which admit of continuous and indefinite extension; as, for instance, to forces which pull and push, attract and repel; to time past and time to come, to temperatures above zero and below zero, to money due and money owed, to distance up and distance down, &c., in all which cases, having represented one by a quantity affected with a $+$ sign, we may represent the other by a quantity affected with a $-$ sign.

6. *In expressing the sum of a number of quantities, the order of the terms is immaterial.*

We will take, as our illustration, a body subject to various alterations of temperature, and we will suppose the temperature of the body, before the changes in question, to be *zero* or 0°. Let the temperature now undergo the following changes—viz., a rise of $a°$, a fall of $b°$, a fall of $c°$, and, lastly, a rise of $d°$. Let us consider a *rise* as positive, and therefore a *fall* as negative. We may then represent these changes respectively by $+a$, $-b$, $-c$, $+d$.

And it is further evident that the resulting temperature will be represented by the *sum* of these quantities, which, as previously written, will be $a-b-c+d$.

But again, it is plain that the resulting temperature of the body will not be affected if these changes of temperature take place in the reverse order, or in any other order. Thus, suppose the temperature first falls $c°$, then rises $a°$, then rises $d°$, and, lastly, falls $b°$, it is evident that the final temperature will be the same as before. And the *sum* of the quantities $-c$, $+a$, $+d$, $-b$, represents this final temperature.

Now, expressing the sum of these quantities by writing them (Art. 4) side by side with their proper signs of affection, and in the order in which they stand, we have $-c+a+d-b$ for the *sum.*

It therefore follows, since we might have chosen any other order of these terms with a similar result, that the sum of any number of quantities, $+a$, $-b$, $-c$, $+d$, may be expressed by writing the terms side by side with their proper signs of affection *in any order whatever.*

Nevertheless, for convenience, and for other reasons, we write the terms *generally* in *alphabetical order,* or we arrange them according to the *power* (see Art. 16) of some particular letter.

7. We may sum up the results and remarks of the last four articles as follows :—

1. *Positive* and *negative* are used in exactly opposite senses.

2. The sign $+$ before an algebraical quantity *affirms* the quality of the quantity as represented without the sign in question.

Thus, $+(+a) = +a$, and $+(-b) = -b$.

3. The sign $-$ before an algebraical quantity *reverses* the quality of the quantity as represented without the sign in question.

Thus, $-(+a) = -a$, and $-(-b) = +b$.

4. The algebraical subtraction of a quantity is the same as the addition of the quantity with the sign of affection *reversed.*

5. The algebraical addition of quantities is expressed by writing the quantities down side by side with their proper signs of affection. And they *may* be written down in any order, though we generally write them in *alphabetical* order, or arranged according to the power (Art. 16) of some letter.

Brackets.

8. **Brackets**—( ), { }, [ ]—are used, for the most part, whenever we wish to consider an algebraical expression containing more than one term *as a whole.*

Thus, if we wish to express that the quantity $3a + 7b$ is to be added as a whole to $4a$, we write—

$$4a + (3a + 7b),$$

and, while inclosed within brackets, we think and speak of $3a + 7b$ as *one quantity.*

Again, if we wish to express that $b - c$ is to be subtracted from a, we write—

$$a - (b - c).$$

Let us consider what is the result of subtracting $(b - c)$ from a. We may evidently, if we please, subtract b first, then afterwards $-c$ from the quantity so obtained, without affecting this result.

Now, we know by Art. 7 (4.) that this is equivalent to *adding* the quantities $-b$ and $+c$ successively.

Now, the sum of a, $-b$, $+c$, is $a - b + c$.

We have therefore $a - (b - c) = a - b + c$.

We observe that the sign of b *within* the brackets is $+$, and that of c is $-$, whereas, in our final result, these signs are both reversed. And we hence arrive at the following important principle :—

When a *minus* sign stands before a bracket, its effect on removing the brackets is to *reverse* the sign of affection of every term *within.*

And it is evident that we may, by a similar course of reasoning, arrive at a principle equally important, viz. :—

When a *plus* sign stands before a bracket, its effect on removing the bracket is to *affirm* the sign of affection of every term within.

We shall show, in Art. 9, the use of brackets in expressing the product or quotient of quantities.

Though, as stated above, brackets are, for the most part, used to group together as a whole a number of quantities, they are sometimes used to inclose single terms. Thus, in Art. 5, we have the expression $a - (-b)$. Now, the brackets are used here to express that the *negative quantity* is to be subtracted as a *negative quantity.* And, in the same way, the expression $a + (-b)$ indicates that the *negative quantity* $(-b)$ is to be *added* to the quantity a. When one pair only is required we generally use the brackets ( ) ; if, however, a quantity already in brackets is to be inclosed in a second pair, we use { }, as in the expression—

$$3a - \{6b + (4c - d)\}.$$

If a third pair be required we use the brackets [ ], and finally, we sometimes find it convenient to group a number of terms by means of a *vinculum*, thus—

$$4x - [6x - \{5y + (3z - \overline{7x - y})\} + 2y].$$

It must be remembered that the *vinculum* has in an expression exactly the same force as brackets.

9. We shall, in this Article, show how to find the value of a few algebraical expressions, as illustrations of the foregoing principles :—

Ex. 1.—If $a = 1$, $b = 2$, $c = 4$, find the value of $3a + 5b + 7c$.

We have only to substitute the value of the letters in the given expression, putting a sign of multiplication to avoid ambiguity.

Thus we have—

$$3a + 5b + 7c = 3 \times 1 + 5 \times 2 + 7 \times 4.$$
$$= 3 + 10 + 28 = 41.$$

Ex. 2.—If $x = 5$, $y = 2$, $z = 6$, find the value of $4x + 3y - 9z$.

Here, $4x + 3y - 9z = 4 \times 5 + 3 \times 2 - 9 \times 6$
$= 20 + 6 - 54$
$= 26 - 54.$

The negative quantity is here the larger, and exceeds the positive quantity by 28, and hence (Art. 4 (3),) the result will be *negative.*

We therefore have—

$$4x + 3y - 9z = -28.$$

Ex. 3.—If $x = 1$, $y = 3$, $z = 0$, find the value of—

$$3x - \{2y + (6z - \overline{5x - y})\}.$$

The given expression—

$$= 3 \times 1 - \{2 \times 3 + (6 \times 0 - \overline{5 \times 1 - 3})\}$$
$$= 3 - \{6 + (0 - \overline{5 - 3})\}$$

$$= 3 - \left\{ 6 + (0 - 2) \right\}.$$
$$= 3 - (6 - 2) = 3 - 4 = -1.$$

It may be advisable to simplify expressions of this kind before substituting the given value of the letters. The method of doing this, however, is deferred till we come to Chapter II.

Ex. I.

If $a = 1$, $b = 2$, $c = 3$, $d = 0$, $e = 4$, find the value of the following :—

1. $4a + 2b$, $3b + 7c$, $6a + 4d$, $4c - 7e$.
2. $a + b + c$, $a - b + c$, $b + c - a$, $a + b - c$.
3. $3a + 7b - 4c$, $2a + 7d + 3c$, $7a - 10b + 2c$.
4. $6a - (2b - 3c)$, $7b + (2a - 4d)$.
5. $3c + (6a - \overline{7b + c})$, $2b - \left\{ 7c + (4d - b) \right\}$.
6. $\left\{ 7b - (2d - c) \right\} - (4b - c + e)$.

If $x = 2$, $y = 3$, $z = 4$, find the sum and difference of the following expressions :—

7. $3x - 4y$ and $3y - 4x$.
8. $7(x - y)$ and $4(y - z)$.
9. $3x - (7y + 4z)$ and $8y + 5z - 3x$.
10. $12y - \overline{6y + 4z}$ and $12y + \overline{6y + 4z}$.
11. $x - y - z$ and $x - (y - z)$.
12. $x - (-3y)$ and $y - \left\{ 3x - (-3z) \right\}$.

Product of Two or more Quantities.

10. The product of two or more quantities may be expressed in several ways.

Thus, the product of a, b, c may be written as follows :—

1. abc, by placing the letters side by side without *any sign* between them.
2. $a \times b \times c$, by placing between them the sign $\times$.
3. $a \,.\, b \,.\, c$, by placing a dot between them.

4. $(a)(b)(c)$, by inclosing the quantities in brackets, writing them without any sign between the brackets.

When, however, the quantities are negative, or either of them consists of more than one term, it is *best* to inclose such quantities in brackets, and, in most cases, it is *necessary* to do so.

Thus, the product of a, $-b$, c would not be correctly expressed by $a - bc$ (for this means, that the product of b and c is to be subtracted from a), but must be written $a(-b)c$. Again, the product of $2a + 3b$ and $a + 5b$ cannot be written $2a + 3ba + 5b$, as this expression means that three times the product of $b \times a$ is to be added to $2a$, and then $5b$ to the result. The product is correctly expressed thus—

$$(2a + 3b)(a + 5b).$$

(The student cannot be too strongly cautioned against leaving out brackets in cases of this kind.)

The Order of the Letters.

11. It is evident that a times $b = b$ times a; for, if we arrange a rows of b things so as to have a horizontal rows and b vertical columns, we may either consider the number of *rows* or the number of columns. In the former case we have a times b things, and, in the latter, b times a things.

We may therefore write the letters whose product we wish to express in any order.

Thus, the product of a, b, c may be written in either of the following ways:—

$$abc, acb, bac, cba, bca, cab.$$

For convenience, however, and for reasons which the student will see as he proceeds with the subject, we write them in the order of the alphabet, unless there happen to be special reasons to the contrary.

Rule of Signs in Multiplication.

12. It was shown (Art. 3) that a *minus* sign does not affect the magnitude of a quantity, but simply its *affection* or *quality*. It therefore follows that the product of $+ a$ and $- b$ will be the same in magnitude as that of $+ a$ and $+ b$, or of $a \times b$—that is, will be equal in magnitude to ab. But it is evident that the quality of the product will be the same as that of $- b$, for it is a times *a negative* quantity.

We therefore have $+ a (- b) = - ab$.

So also, the product of $- a$ and $+ b$, will have the same magnitude as that of $+ a$ and $+ b$—that is, its magnitude will be ab. But since, to take a geometrical illustration, a line of length, b, drawn *negatively once, negatively twice*, &c., must give a negative result, the quality of the product in question must be negative.

We therefore have $(- a) (+ b) = - ab$.

Again, the product of $- a$ and $- b$ will be equal in magnitude to ab, and its quality will be evidently the *reverse* of the quality of the product of $- a$ and $+ b$, and the quality of the latter product is above shown to be *negative*. The product of $- a \times - b$ will be therefore positive.

Hence we have $(- a) (- b) = + ab$.

Collecting these results, and remembering that $(+ a) (+ b) = + ab$, we have—

$$(+ a) (+ b) = + ab.$$
$$(+ a) (- b) = - ab.$$
$$(- a) (+ b) = - ab.$$
$$(- a) (- b) = + ab.$$

We have then the following rule of signs in multiplication :—

Rule.—Like signs give +, and unlike signs give −.

Quotient of Two Quantities.

13. The quotient of two quantities is expressed in either of the two following ways :—

1. By placing the divisor under the dividend, separated by a line.

Thus, the quotient of a by $b = \frac{a}{b}$.

2. By placing the divisor after the dividend with a sign $\div$ between them; thus, $a \div b$.

When either of the quantities is negative, it is better, if the second be used, to inclose the negative quantity, if the divisor, in brackets.

Thus, the quotient of a by $-b$ may be written $a \div -b$, but it is better written $a \div (-b)$.

And when either of the quantities contains more than one term, it must always be inclosed in brackets when expressed by the second method.

Thus, the quotient of $a + 2b$ by $2c = (a + 2b) \div 2c$, not $a + 2b \div 2c$, for this would mean that the quotient of $2b$ by $2c$ is to be added to a.

Rule of Signs in Division.

It is evident from the last article that—

$$(+ab) \div (+a) = +b.$$
$$(-ab) \div (+a) = -b.$$
$$(-ab) \div (-a) = +b.$$
$$(+ab) \div (-a) = -b.$$

We have therefore the following rule:—

RULE.—In division, as in multiplication, like signs give $+$, and unlike signs give $-$.

Coefficients.

14. When a quantity can be broken up into two factors, each of those factors is called a **coefficient** of the other.

Thus, taking the quantity $4abc$, we see that 4 is the coefficient of abc, $4b$ that of ac, $4c$ that of ab, $4bc$ that of a, &c. We call 4 a *numerical* coefficient; but when the coefficient contains a letter or letters we call it a *literal* coefficient. We often speak of the numerical coefficient of a quantity as *the coefficient* of the quantity. It is, moreover, unusual to write down unity as a numerical coefficient, and so, when an algebraical quantity has no expressed numerical coefficient, we may, conversely, consider *unity* as such.

15. *Like quantities* are generally defined as those which differ only in their *numerical* coefficients. Thus, $9x$, $3y$, $4xyz$, $12ab$ are respectively *like* to the quantities $10x$, $5y$, $6xyz$, $4ab$, while $3x^2$, $6xy$, $7b^2$ are called *unlike*.

It is sometimes convenient, however, to *consider as like quantities* those which may contain perhaps one *common* letter only, though containing other letters which are not common. Thus, for the purposes of addition and subtraction, we may consider $3ax$ and $4bx$ as *like quantities*, having $3a$ and $4b$ respectively as their coefficients. Now, their sum, since they are respectively equivalent to $3a$ and $4b$ times x, will be equivalent to $(3a + 4b)$ times x, and may then be written $(3a + 4b)x$.

Hence we learn that addition and subtraction of quantities having any common factor may, considering the uncommon factor as coefficient, be expressed according to the following rule :—

Add together the coefficients of the common factor, and take the sum as a new coefficient of the common factor.

And a similar rule will apply to the subtraction of such quantities.

Thus, we have the sum of $4axy$, $-2bcy$, $3dey$,

$$= (4ax - 2bc + 3de)y.$$

Powers.

16. We have seen (Art. 10) that abc means that a is to be multiplied by b, and the product by c. And so, if we wish to express that a is to be multiplied by a, and this product again by a, we might write the expression thus: aaa. Instead of this, however, we usually place a small figure at the top and on the right hand of the letter a, to indicate how many times the letter appears as a factor. In this case, therefore, we write a^3. We call quantities of this form **powers** of the letter in question; and so, remembering that a may be considered as a^1 on this principle, we have :—

a, the first power of a; a^2, the *second* power of a; a^3, the *third* power of a, &c. The figure written at the right hand at the top of the letter is called the **index** or **exponent**, and it is usual to call a^2, a^3 respectively the *square* and *cube* of a, from the fact they express respectively the *area* of a square whose side is a, and the *volume* of cube whose edge is a.

17. The **square root** of a quantity is that quantity which, when raised to the *second power* or *squared*, will give the original quantity.

It is generally written $\sqrt{\ }$. Thus, $\sqrt{16} = 4$, $\sqrt{144} = 12$.

The **cube root** of a quantity is that quantity which, when raised to the *third power* or *cubed*, will give the original quantity.

It is generally written $\sqrt[3]{\ }$. Thus, $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$, $\sqrt[3]{1728} = 12$. And so the fourth, fifth, &c., roots are indicated by the symbols $\sqrt[4]{\ }$, $\sqrt[5]{\ }$, &c., respectively.

18. The **dimensions** of an algebraical quantity are the sum of the indices or exponents of the *literal* factors.

Thus, the dimensions of $3a^2b^3c^4 = 2 + 3 + 4 = 9$, and the dimensions of $abx^2 = 1 + 1 + 2 = 4$.

19. A **homogeneous** expression is one in which the dimensions of every term are the same.

Thus, $a^3 + 3a^2b + 3ab^2 + b^3$ is *homogeneous*,
Whereas, $5a + 3ab + 7a^2bc$ is *not homogeneous*.

Ex. II.

If $a = 2$, $b = 3$, $c = 0$, $d = 1$,

Find the values of—

1. $6a^2 + 3b^2 - 5c^2$; $ab + ac + bc$, $bc + bd + cd$.
2. $a^3 + 3a^2b + 3ab^2 + b^3$; $a^3 + b^3 + c^3 - 3abc$.
3. $(3a + 7b)(4a - 9b)$; $(a^2 + b^2)(a + b)(a - b)$.
4. $6\{2a^3 - 4(2b^3 - \overline{2c^3 - d^3})\}$; $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$.

If $a = 1$, $b = -2$, $c = -3$, $d = 0$, $e = 4$,

Find the values of—

5. $(a + b + c + d + e)^2$; $(a^2 + 2ab + b^2 - c^2) \div (a + b + c)$.
6. $\dfrac{a^4 - d^4}{a^3 + a^2d + ad^2 + d^3}$; $\dfrac{c^3 + 3c^2d + 3cd^2 + d^3}{b^3 - 3b^2c + 3bc^2 - c^3}$.
7. $\dfrac{\sqrt{c^2 + e^2} - \sqrt{5a^2 + b^2}}{ab + ac + bc}$; $\sqrt[3]{c^3 + 3c^2 + 3c + 1}$.

8. $3(4b + 5c)^2 + 4(c + e)^2$; *abcde*.

If $x = 3$, $y = 4$, $z = 0$,

Find the value of—

9. $(3x - \sqrt{x^2 + y^2})^2 (2x + \sqrt[3]{x^2 + y^2 + 2})$.

10. $\{5x^2 + 2(y + 2)^2\}\{5x^2 - 2(y + z)^2\}$.

11. $x^y + y^x + z^x$.

12. $(x^3 - y^3) \div \sqrt[3]{\{3x^3 + 3(3x^2 + 3xy + y^2)y\}}$.

CHAPTER II.

ADDITION SUBTRACTION, MULTIPLICATION, AND DIVISION.

Addition.

20. Rule.—Arrange the terms of the given quantities so that *like* quantities may be under each other; add separately the positive and negative coefficients of each column; take the difference and prefix the sign of the *greater*, and annex the common letter.

(When the coefficients are all positive or all negative, we, of course, simply add them together and prefix the common sign for the coefficient of the sum.)

Ex. 1.

$$\begin{array}{rrrr}
 & 3a & +5b & -3c \\
 & 2a & -7b & +4c \\
 & 5a & +b & -2c \\
 & 4a & -3b & +8c \\
\hline
\text{Ans.} & 14a & -4b & +7c
\end{array}$$

Ex. 2.

$$\begin{array}{rrrrr}
 & -3a & +7b & +c & -4d \\
 & 2a & -2b & +5c & +3d \\
 & -7a & -3b & +2c & -6d \\
 & 2a & +5b & -8c & +6d \\
\hline
\text{Ans.} & -6a & +7b & & -d
\end{array}$$

Ex. 3. Add together $5x^2 - 3y^2 + 3y$, $6y^2 + 7xy - 4x$, $4xy + 6x - 5$, $-2x^2 - 3xy + 2$.

Arranging like quantities in each expression under each other, we have :—

$$\begin{array}{rrrrrrr}
 & 5x^2 & & -3y^2 & & +3y & \\
 & & 7xy & +6y^2 & -4x & & \\
 & & 4xy & & +6x & & -5 \\
 & -2x^2 & -3xy & & & & +2 \\
\hline
\text{Ans.} & 3x^2 & +8xy & +3y^2 & +2x & +3y & -3
\end{array}$$

Ex. III.

Add together-

1. $3a - 2b$, $4a + 7b$, $2a + 3b$, $a - 5b$.

2. $9a^2 + 7b^2$, $-3a^2 + 4b^2$, $a^2 + b^2$, $4a^2 - 12b^2$.

3. $a + b + c$, $3a + 2b + 3c$, $-4a + 7b - c$, $2b + 5c$.

4. $x - y - z$, $y - x - z$, $z - x - y$, $x + y + z$.

5. $3a^2 - 4ab + 6b^2$, $7ab - a^2 - b^2$, $2a^2 - 3ab - 4b^2$, $4a^2 + ab - b^2$.

6. $2x^4 - 7x^2 + 3$, $-4x^3 + 6x^2 - 2x + 7$, $x^4 - 2x^3 - 4x$, $6x^3 - 9x - 12$.

7. $2a^2 + 7ab + 3b^2 - 6a - 5b - 2$, $a^2 + 3a - 2b + 9$, $9ab - 2a - 3b + 4$, $-3a^2 - 12ab - 3b^2 + 5a + 10b - 15$.

8. $x^3 - x^2y - x^2z + xy^2 - xyz + xz^2$, $x^2y - xy^2 - xyz + y^3 - y^2z + yz^2$, $x^2z - xyz - xz^2 + y^2z - yz^2 + z^3$.

9. $x^4 + x^2y^2 + x^3y$, $-x^3y - x^2y^2 - xy^3$, $y^4 + xy^3 + x^2y^2$.

10. $a^3 + ab^2 + ac^2 + 2a^2b - 2a^2c - 2abc$, $a^2b + b^3 + bc^2 + 2ab^2 - 2abc - 2b^2c$, $a^2c + b^2c + c^3 + 2abc - 2ac^2 - 2bc^2$.

11. $x^4 - xy^3 + xz^3 - 3x^3y + 3x^3z$, $3x^2y^2 + 3x^2z^2 + 3xy^2z - 3xyz^2 - 6x^2yz$, $y^4 - x^3y - yz^3 + 3x^2y^2 - 3x^2yz$, $-3xy^3 - 3xyz^2 - 3y^3z + 3y^2z^2 + 6xy^2z$, $z^4 + x^3z - y^3z - 3x^2yz + 3x^2z^2$, $3xy^2z + 3xz^3 + 3y^2z^2 - 3yz^3 - 6xyz^2$.

12. $a^4 - a^3b + 3a^2c^2 + ab^2c - 3abc^2 - b^3c$, $a^3b - a^3c + 3abc^2 - 3ac^3 - b^2c^2 + b^3c$, $a^3c - a^3d + 3ac^3 - 3ac^2d + b^2c^2 - b^2cd$, $-a^4 + a^3d - 3a^2c^2 + 3ac^2d - ab^2c + b^2cd$.

Subtraction.

21. We have seen, Art. 7 (4.), that the subtraction of a quantity is equivalent to the addition of the same quantity with its sign of affection reversed. We therefore have the following rule :—

RULE.—Change the sign of each term of the subtrahend, and proceed as in addition.

Ex. 1.

$$\begin{array}{r} 5x + 6y - 3z \\ 2x - 2y + 2z \\ \hline \text{Ans. } 3x + 8y - 5z. \end{array}$$

Ex. 2.

$$\begin{array}{r} 6a^2 - 3ab + 4b^2 \\ -2a^2 - 3ab - 2b^2 \\ \hline \text{Ans. } 8a^2 \qquad\quad + 6b^2. \end{array}$$

Ex. 3.

$$\begin{array}{rrrr} 2x^3 & & +\ 6xy^2 & +\ 3y^3 \\ x^3 & +\ 3x^2y & +\ 3xy^2 & +\ 2y^3 \\ \hline x^3 & -\ 3x^2y & +\ 3xy^2 & +\ y^3. \end{array}$$

Ex. IV.

1. From $6a + 7b + 3c$ take $2a + 5b - 2c$.

2. From $2x - 3y - 8z$ take $6x - 5y - 2z$.

3. Take $5a^2 + 3ab + 4b^2 + 3a + 7b + 8$ from $6a^2 + 3b^2 - 2a$.

4. Take $6a^4 + 8a^2x^2 + x^4$ from $8a^4 + 6a^2x^2 + 2x^4$.

5. Subtract the sum of the quantities $a^4 + 2a^2b^2 + b^4$, $a^4 - 2a^2b^2 + b^4$ from $6a^4 + 8a^2b^2 + 6b^4$.

6. From $x^3 + y^3 + z^3 - 3xyz$ take $4x^3 + y^3 + 4z^3 + 3x^2z + 3xz^2 - 3xyz$.

7. From $3x^4 + 3ax^3 - 9a^2x^2 + a^3x - a^4$ take $2x^4 + 4ax^3 + 4a^3x + a^4$.

8. Take $a^3 - 5a^2b + 7ab^2 - 2b^3$ from the sum of $2a^3 - 9a^2b + 11ab^2 - 3b^3$ and $b^3 - 4ab^2 + 4a^2b - a^3$.

9. Subtract $a + b + c + d$ from $e + f + g + h$.

10. Take $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ from $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, and subtract the result from their sum.

11. Add together the given quantities in the last example, and subtract the result from $3x^4 + 10x^2y^2 + 3y^4$.

12. Take $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ from $2a^2 + 2b^2 + 4ab - c^2$.

Brackets—*continued.*

22. It was shown, in Art. 8, that, when a quantity inclosed in brackets is to be added, we may remove the sign (+) of addition and the brackets without changing the sign of the terms within the brackets. On the contrary, when the quantity in brackets is to be subtracted, or has the sign *minus* before it, we must change the sign of every term within the brackets on removing the brackets and the sign of subtraction.

We shall now see how to simplify expressions involving brackets connected by the signs of addition and subtraction;—

Ex. 1. Simplify $(3a + 5b) - (6b - 2c) + (-2a + b - 3c)$.

The given expression $= 3a + 5b - 6b + 2c - 2a + b - 3c$, or adding together the like quantities,

$$= a - c.$$

Ex. 2. Reduce to its simplest form—

$$a - (b - c) + \left\{ b + (a - c) \right\} - \left\{ (a - b) - c \right\}.$$

(When a pair of brackets is inclosed within another pair, it is convenient to remove the inner one first.)

Hence the given expression—

$$= a - b + c + \left\{ b + a - c \right\} - \left\{ a - b - c \right\}$$

$$= a - b + c + b + a - c - a + b + c = a + b + c.$$

Ex. 3. Simplify the expression $\frac{3x}{4} - \frac{x - 7}{2} + \frac{6x - 9}{8}$.

The line separating the numerator and denominator of a fraction is a species of vinculum, since it serves to show that the *whole* numerator is to be divided by the *whole* denominator. Hence, on breaking up the two latter fractions into fractions having one term only in the numerator, we have—

$$\frac{3x}{4} - \frac{x-7}{2} + \frac{6x-9}{8} = \frac{3}{4}x - \frac{1}{2}x + \frac{7}{2} + \frac{6}{8}x - \frac{9}{8} = x + \frac{19}{8}.$$

23. As it is often necessary to inclose quantities within brackets, we shall now show how this is done.

The following rule needs no explanation :—

Rule.—When a number of terms is inclosed within brackets, if the sign placed before the brackets be +, the terms must be written down with their signs of affection *unchanged;* but, if the sign placed before the brackets be −, the sign of affection of every term placed within the brackets must be *changed.*

Thus we may express $a + b - c - d$ in any of the following ways :—

$$a + b - c - d = a + (b - c - d) = (a + b) - (c + d)$$
$$= a - (-b + c + d) = (a + b - d) - c, \text{ \&c.}$$

(When the word *sign* is used in future, the student is to understand *sign of affection*, unless otherwise expressed.)

Ex. V.

Simplify the expressions—

1. $3x - (2x - y) + (5x + 3y - 2z) - (7x - 4y - 3z)$.

2. $(2a^3 + 2a^2b + 2ab^2) - (2a^3 + a^2b + ab^2 - b^3) + (a^3 - a^2b - ab^2 - 2b^3)$.

3. $1 - (1 - 2x) - \{3 - (4 - 3x)\} + \{5 - (4x - \overline{7 - x})\}$.

4. $6x^3 - (2 - 3x + x^2) + \{-7 + (5x - \overline{8x - 2})\} - (3 - \overline{3 - 3x})$.

Group together the terms of the four following expressions, so that—

(i.) The first two and last two are inclosed in brackets.
(ii.) The last three are inclosed in brackets.
(iii.) The first three are so inclosed, and an inner pair of brackets used to inclose the second and third.

5. $a - b + c - d$.

6. $-6a + 7b - 3c + 5d$.

7. $-4x^3 + 12x^2y - 12xy^2 + 4y^3$.

8. $a^3 - b^3 - c^3 + 3abc$.

Add together—

9. $ax^2 + bxy + cy^2$, $-dx^2 - axy + ey^2$, $bx^2 + 3cxy + fy^2$, $-2ax^2 - 2by^2$.

10. $ax - cy - ez$, $-bx + dy + fz$, $cx - ey - gz$, $-dx + fy + hz$.

Subtract—

11. $(2a + b)x - (3b + c)y + (4c + d)z$ from $(3a - d)x + (4b - a)y + (5c - b)z$.

12. $(y - z)a^2 + (z - x)ab + (x - y)b^2$ from $(y - x)a^2 - (y - z)ab - (z - x)b^2$.

Multiplication.

24. Remembering the definition (Art. 14) of a coefficient, it follows that the product of two terms having coefficients is found by multiplying the product of the coefficients by the product of the remaining factors.

Thus, the product of $4a$ and $3b$, or $4a \times 3b = 12ab$.

Again, the terms to be multiplied may contain like letters.

Now, $a^3 = aaa$, $a^5 = aaaaa$, and hence it follows that

$$a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8.$$

And, generally, we have

$$a^m \times a^n = aaa\ldots\text{to } m \text{ factors} \times aaa\ldots\text{to } n \text{ factors},$$
$$= aaa\ldots\text{to } (m + n) \text{ factors} = a^{m+n}.$$

It therefore follows that the product of the *powers* of like letters is found by adding together the *exponents* of the like letters.

Thus, $a^4 \times a^2 = a^6$, and $a^3 \times a = a^4$, &c.

And, further, as regards the sign of the product, we have seen, in Art. 12, that *like signs* give $+$, and *unlike signs* give $-$.

There are three things, then, which must be attended to in the multiplication of algebraical terms, viz.:—

1. The *signs.*—Like signs give $+$, and unlike signs give $-$.

2. The *coefficients.*—These are to be multiplied like ordinary numbers.

3. The *letters.*—The exponents of like letters are to be added together, and the powers so obtained written side by side with the unlike letters.

	Ex. 1.		Ex. 2.		Ex. 3.
	$6\,a^2b$		$-\,3\,xyz$		$-\,5\,abd$
	$2\,a^3c$		x^2yz^3		$-\,2\,bcd^2$
Ans.	$12\,a^5bc$	Ans.	$-\,3\,x^3y^2z^4$	Ans.	$10\,ab^2cd^3$

25. Whenever the multiplicand or multiplier, or both, contains more than one term, it is evident the product is found by multiplying each term of the multiplicand separately by each term of the multiplier, and adding together the separate products.

	Ex. 4.	Ex. 5.
	$3\,a^2 - 7\,ab + 5\,b^2$	$x^3 - 3\,x^2y + 3\,xy^2 - y^3$
	$6\,ab$	$-\,2\,xy$
Ans.	$18\,a^3b - 42\,a^2b^2 + 30\,ab^3$	$-\,2\,x^4y + 6\,x^3y^2 - 6\,x^2y^3 + 2\,xy^4$

Ex. 6.

$$\begin{array}{r}x^2 + xy + y^2 \\ x^2 - xy + y^2 \\ \hline x^4 + x^3y + x^2y^2 \qquad\qquad\quad \\ - x^3y - x^2y^2 - xy^3 \quad \\ x^2y^2 + xy^3 + y^4 \\ \hline \text{Ans. } x^4 \qquad + x^2y^2 \qquad + y^4.\end{array}$$

Ex. 7.

$$\begin{array}{l}x^2 + (a + b)\,x + ab \\ x + c \\ \hline x^3 + (a + b)\,x^2 + abx \\ \qquad\qquad cx^2 + (ac + bc)\,x + abc \\ \hline x^3 + (a + b + c)\,x^2 + (ab + ac + bc)\,x + abc.\end{array}$$

It will be observed that the terms in the above examples are all arranged according to the power of some letter. Thus, in Ex. 4, they are arranged according to the descending powers of a; and in Exs. 5, 6, 7, they are arranged according to the descending powers of x. It matters not whether they are arranged according to ascending or descending powers, and the result would be the same if they are not so arranged. It is then, however, much easier to collect like terms, as they generally fall under each other. When we come to division we shall find it *necessary* to arrange the terms according to the power of some letter.

Ex. VI.

1. Multiply $3a + 2b$ by $4a - 3b$, and $6x + 7y$ by $3x - 5y$.

2. Multiply $x^2 + 2xy - 2y^2$ by $x - 2y$, and $15x^2 + 17xy - 4y^2$ by $2x + y$.

3. Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$, and $a^2 + b^2$ by $a^2 - b^2$.

4. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$, and the product by $x^4 + y^4$.

5. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

6. Find the continued product of $x^4 + y^4$, $x^2 + y^2$, $x + y$, $x - y$.

7. Develop the expressions $(a + b)^3$, and $(x - y)^4$.

8. Multiply $5a^3 + 15a^2 + 45a + 135$ by $a^2 - 2a - 3$.

9. Multiply $1 + 3x - 7x^2$ by $x - 2$, and $a - x$ by $x^2 + ax + a^2$.

10. Multiply $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$ by $a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4$.

11. Multiply $a + bx + cx^2 + dx^3$ by $cx + f$.

12. Find the product of $x^2 + px + q$ and $x - a$.

13. Find the continued product of $x + a$, $x + b$, $x + c$.

26. We have explained in the last Article the general method to be pursued in Multiplication. There are, however, many algebraical expressions which may be multiplied by inspection.

I. Expressions of the form $(a + b)^2$.

It is easily found by long multiplication that

$$(a + b)^2 = a^2 + 2ab + b^2,$$

and this, expressed in ordinary language, may be read thus :—

The square of the sum of two quantities is the square of the first, plus twice their product, plus the square of the second.

(This rule is evidently true whether the quantities are *positive* or *negative.*)

Ex. 1. $(3a + 7b)^2 = (3a)^2 + 2(3a)(7b) + (7b)^2 = 9a^2 + 42ab + 49b^2$.

Ex. 2. $(6x - 5)^2 = (6x)^2 + 2(6x)(-5) + (-5)^2 = 36x^2 - 60x + 25$.

Ex. 3.
$$\begin{aligned}(a + b + c)^2 &= \{(a + b) + c\}^2 \\ &= (a + b)^2 + 2(a + b)c + c^2 \\ &= (a^2 + 2ab + b^2) + 2(ac + bc) + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\end{aligned}$$

Remark.—It is a very common mistake with beginners to write down $a^2 + b^2$ as the square of $(a + b)$, thereby leaving out *twice the product of the quantities a and b.* They should impress this formula, viz. :—

$$(a + b)^2 = a^2 + 2ab + b^2$$

thoroughly upon the mind.

II. The form $(a + b)(a - b)$.

It may be easily found by long multiplication that

$$(a + b)(a - b) = a^2 - b^2,$$

which, in ordinary language, may be thus expressed :—

The product of the sum and difference of two quantities is the difference of the squares of the quantities.

This formula may be applied in all cases where the terms of the quantities to be multiplied are of the same magnitude, but when *some* of them differ in sign.

Ex. 1. $(a + b + c + d)(a + b - c - d)$

$= \{(a+b)+(c+d)\}\{(a + b) - (c + d)\}$

$= (a + b)^2 - (c + d)^2 = (a^2 + 2\,ab + b^2) - (c^2 + 2\,cd + d^2)$

$= a^2 + b^2 - c^2 - d^2 + 2\,ab - 2\,cd.$

Ex. 2. $(a^3 - 3\,a^2b + 3\,ab^2 - b^3)(a^3 + 3\,a^2b + 3\,ab^2 + b^3)$

$= \{(a^3 + 3\,ab^2) - (3\,a^2b + b^3)\}\{(a^3 + 3\,ab^2) + (3\,a^2b + b^3)\}$

$= (a^3 + 3\,ab^2)^2 - (3\,a^2b + b^3)^2$

$= (a^6 + 6\,a^4b^2 + 9\,a^2b^4) - (9\,a^4b^2 + 6\,a^2b^4 + b^6)$

$= a^6 - 3\,a^4b^2 + 3\,a^2b^4 - b^6.$

The principle to be adopted in all such cases is to find what terms in the given quantities are *exactly alike*, and put them *first* in brackets, when the remaining terms will fall into the proper form. Thus—

$(3\,a + 7\,b + 5\,c - d)(3\,a - 7\,b + 5\,c + d) = \{(3\,a + 5\,c) + (7\,b - d)\}\{(3\,a + 5\,c) - (7\,b - d)\}$.

And $(3\,a - 7\,b - 5\,c + d)(3\,a + 7\,b + 5\,c + d) = \{(3\,a + d) - (7\,b + 5\,c)\}\{(3\,a + d) + (7\,b + 5\,c)\}$.

(III.) The form $(x + a)(x + b)$.

By multiplication we find that—

$$(x + a)(x + b) = x^2 + (a + b)\,x + ab,$$

which in ordinary language may be thus expressed :—

The product of two binomials containing a common term and an uncommon term, is the square of the common term, plus the sum of the uncommon terms multiplied into the common term, plus the product of the uncommon term.

Thus—

$(x + 2)(x + 3) = x^2 + (2 + 3)x + 2 \times 3 = x^2 + 5x + 6.$

$(x + 6)(x - 1) = x^2 + (6 - 1)x + 6(-1) = x^2 + 5x - 6.$

$(x - 5)(x - 7) = x^2 - (5 + 7)x + (-5)(-7) = x^2 - 12x + 35.$

$(x + 3a)(x - 5a) = x^2 + (3a - 5a)x + (3a)(-5a) = x^2 - 2ax - 15a^2.$

And so,

$$(x + a + b)(x + c + d) = (x + \overline{a + b})(x + \overline{c + d})$$
$$= x^2 + (a + b + c + d)x + (a + b)(c + d).$$

We may extend this formula to any number of factors.

Thus, by multiplication, we find that—

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$

$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + acd + bcd)x + abcd.$

Law of Formation of the Terms.

1. It will be seen that the coefficient of the first term is in each case *unity;* that of the second term, the sum of the uncommon letters taken *singly;* that of the third term, the sum of the uncommon letters taken *two together;* that of the fourth term, the sum of the uncommon letters taken *three together*, &c.

2. The power of the common letter is in the first term that of the number of the binomials, and it sinks *one* every term.

Ex. $(x + 1)(x + 2)(x + 3) = x^3 + (1 + 2 + 3)x^2 + (1 + 2 + 1 + 3 + 2 + 3)x + 1 + 2 + 3 = x^3 + 6x^2 + 11x + 6.$

(IV.) The form $(a + b + c + d + \&c.)^2$.

By ordinary multiplication or otherwise we have $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$; and a similar result follows if we take a larger number of terms.

We may hence deduce the following rule :—

The square of the sum of any number of quantities is the sum of the squares of the quantities together with twice the sum of the products formed by multiplying the first quantity into all that follow separately, then the second into all that follow, the third into all that follow, &c.

Ex. 1. $(1 + 2x + 3x^2)^2 = 1 \quad + 4x^2 \quad + 9x^4$
$$\begin{array}{l} \qquad\qquad 4x + 6x^2 + 12x^3 \\ = 1 + 4x + 10x^2 + 12x^3 + 9x^4. \end{array}$$

Ex. 2. $(a^3 + 3a^2b + 3ab^2 + b^3)^2$
$$\begin{array}{l} = a^6 \quad + 9a^4b^2 \quad + 9a^2b^4 \\ \quad 6a^5b + 6a^4b^2 + 2a^3b^3 \quad + b^6 \\ \quad 18a^3b^3 + 6a^2b^4 + 6ab^5 \\ = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6. \end{array}$$

Ex. VII.

1. Find by inspection the squares of $x - y$, $3a - 5b$, $4c^2 + d^2$, $3x^2 - 2y^2$.

2. Find the continued product of $a + b$, $a^2 + b^2$, $a^4 + b^4$, and $a - b$.

3. Multiply $mx + ny$ by $mx - ny$, and $5a^2 - 3b^2$ by $10a^2 + 6b^2$.

4. Find the value of $(a + b + c + d)(a - b + c - d)$, and of $(a + b - c - d)(a + b + c + d)$.

5. Show that $(x^2 + 2xy + y^2)(x^2 - 2xy + y^2) = x^4 - 2x^2y^2 + y^4$.

6. Multiply $x + 5$ separately by $x + 1$, $x + 2$, $x - 3$, $x - 5$.

7. What is the continued product of $x - 2$, $x + 3$, $x - 6$, $x + 5$?

8. If $2s = a + b + c$, find the value of $s(s - a)(s - b)(s - c)$.

9. If $2s = a + b + c + d$, what is the value of—
$$(2s - 2a)^2 + (2s - 2b)^2 + (2s - 2c)^2 + (2s - 2d)^2?$$

Prove that—

10. $(ax + by)^2 + (cx + dy)^2 + (ay - bx)^2 + (cy - dx)^2 = (a^2 + b^2 + c^2 + d^2)(x^2 + y^2)$.

11. $\left\{(ac - bd)x + (ad + bc)y\right\}^2 + \left\{(ac - bd)y - (ad + bc)x\right\}^2 = (a^2 + b^2)(c^2 + d^2)(x^2 + y^2)$.

12. $(ax + by + cz)^2 = (a + b + c)(ax^2 + by^2 + cz^2) - ab(x - y)^2 - ac(x - z)^2 - bc(y - z)^2$.

13. $(x + a)(x + b)(x + c) = (x - a)(x - b)(x - c) + 2\{(a + b + c)x^2 + abc\}$.

14. $(m + n + p + q)^2 = (m + n)^2 + (m + p)^2 + (m + q)^2 + (n + p)^2 + (n + q)^2 + (p + q)^2 - 2(m^2 + n^2 + p^2 + q^2)$.

15. $(a^2 + b^2 + c^2)(m^2 + n^2 + p^2 + q^2) = (am + bn + cp)^2 + (an - bm + cq)^2 + (ap - bq - cm)^2 + (aq + bp - cn)^2$.

16. $(a - b)(b - c) + (b - c)(c - a) + (c - a)(a - b) = 3(ab + ac + bc) - (a + b + c)^2$.

17. $(x + y + z + a)^2 - (x - y - z + a)^2 = 4(x + a)(y + z)$.

18. $\{(a - b)^2 + 4ab\}\{(a + b)^2 - 4ab\}\{a^4 - b^4 + 2ab(a^2 - b^2)\} = (a + b)^5(a - b)^3$.

19. $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$.

20. $4(a^2 + b^2)(c^2 + d^2) = (a + b)^2(c + d)^2 + (a - b)^2(c - d)^2 + (a + b)^2(c - d)^2 + (a - b)^2(c + d)^2$.

21. $8(a^4 + b^4) = (a + b)^4 + 6(a^2 - b^2)^2 + (a - b)^4$.

22. $(x - y)^2 + (y - z)^2 + (z - x)^2 + 2(x - y)(y - z) + 2(x - y)(z - x) + 2(y - z)(z - x) = 0$.

23. $(a + b + c)(b + c - a)(a + c - b)(a + b - c) = 4a^2b^2$, when $a^2 + b^2 = c^2$.

24. $\{2(ax^m + by^n)^2 + (ay^n - bx^m)^2\} = (a^2 + b^2)\{(x^m + y^n)^2 + (x^m - y^n)^2\}$.

Division.

27. As in multiplication, we have especially three things to attend to, viz. :—

1. The *signs.*

We have learnt (Art. 13) that like signs give +, and unlike signs give −.

2. The *coefficients.*

Understanding here the *numerical* coefficients, it is plain that they may be divided as ordinary arithmetical quantities.

3. The *letters.*

As the product of the quantities a and b is expressed by ab, it follows that the quotient of ab by either of the factors a or b will give the remaining factor b or a respectively.

Thus, $ab \div a$ or $\frac{ab}{a} = b$, and $ab \div b$ or $\frac{ab}{b} = a$.

And so, $xyz \div xz = y$, and $pqrs \div qs = pr$.

We may then conclude that *when the divisor is contained as a factor in the dividend, the quotient is found by omitting from the dividend those of its factors which constitute the divisor.*

If the divisor be *not* contained as an exact factor in the dividend, we may then express the quotient symbolically.

Thus, $xy \div ab = \frac{xy}{ab}$.

When, however, the dividend and divisor have a common factor, it is plain that we may, as in arithmetic, strike out of the numerator and denominator of the symbolical quotient this common factor.

Thus, $5\,abc \div bd = \frac{5\,abc}{bd} = \frac{5\,ac}{d}$.

And $16\,xyz \div 10\,axz = \frac{16\,xyz}{10\,axz} = \frac{8\,y}{5\,a}$.

28. A power of a quantity is divided by any other power of the same quantity by subtracting the index of the divisor from that of the dividend, the quotient being that power of the quantity whose index is the remainder so obtained.

1. Let the power of the quantity in the dividend be the *higher*.

We have $a^5 = aaaaa$, and $a^3 = aaa$.

$$\therefore a^5 \div a^3 = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}.$$

Or, generally, m being greater than n, since

$a^m = aaa \ldots$ to m factors, and $a^n = aaa \ldots$ to n factors,

we have—

$$a^m \div a^n = \frac{aaa \ldots \text{to } m \text{ factors}}{aaa \ldots \text{to } n \text{ factors}} = aaa \ldots (m - n) \text{ factors} = a^{m-n}.$$

2. Let the power of the quantity in the dividend be the *lower*.

Suppose we have to divide a^4 by a^7.

We have $a^4 \div a^7 = \dfrac{aaaa}{aaaaaaa} = \dfrac{1}{aaa} = \dfrac{1}{a^3}$.

We may, however, so express the result that it shall agree exactly with the proposition at the head of this article.

For we may conceive of a^3 as representing the *product* of *unity* and the quantity a^3. We shall therefore be perfectly consistent if we allow a^3 to represent the *quotient* of *unity* by the quantity a^3.

We shall then have $a^{-3} = \dfrac{1}{a^3}$, and hence we get from the above result—

$$a^4 \div a^7 = a^{-3} = a^{4-7}.$$

Or, generally, m being less than n, we have—

$$\frac{a^m}{a^n} = \frac{aaa\ldots\ldots\text{to } m \text{ factors}}{aaa\ldots\ldots\text{to } n \text{ factors}} = \frac{1}{aaa\ldots\ldots(n-m)\text{ factors}}$$

$= \dfrac{1}{a^{n-m}}$; or, using the notation just explained,

$$= a^{-(n-m)} = a^{-n+m} = a^{m-n}.$$

3. Let the powers of the quantities in the dividend and divisor be *equal.*

It is evident that their quotient is *unity.*

Thus, $\dfrac{a^4}{a^4} = \dfrac{aaaa}{aaaa} = \dfrac{1}{1} = 1.$

And so, $\dfrac{a^m}{a^m} = 1.$

If, however, we *assume* the principle proved in the two cases above to hold here, we have—

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

It follows therefore that $a^0 = 1$.

Cor.—From the above interpretation of negative indices it follows that the same rules for multiplication and division of quantities involving them may be applied as in the case of positive indices.

Thus, $a^5 \times a^{-2} = a^5 \times \dfrac{1}{a^2} = \dfrac{a^5}{a^2} = a^{5-2}.$

And so, $a^4 \div a^{-3} = a^{4-(-3)} = a^7.$

Ex. 1. $\dfrac{20\,x^2yz}{4\,xz} = 5\,xy.$

Ex. 2. $\dfrac{-\,45\,a^3b^2c^4}{15\,ab^2c^2} = -\,3\,a^2c^2.$

Ex. 3. $\dfrac{-\,7\,a^3 + 21\,a^2b - 14\,ab^2}{-\,7\,a} = a^2 - 3\,ab + 2\,b^2.$

Ex. 4. $\dfrac{4\,x^5yz^3 - 6\,x^2y^3z^4 + 42\,xy^4z^4}{3\,x^2y^4z^5}$

$$= \frac{4\,x^3}{3\,y^3z^2} - \frac{2}{yz} + \frac{14}{xz}, \text{ or}$$

$$= \frac{4}{3}x^3y^{-3}z^{-2} - 2\,y^{-1}z^{-1} + 14\,x^{-1}z^{-1}.$$

Ex. 5. Divide $3\,a^2 + 13\,ab - 10\,b^2$ by $a + 5\,b$.

When the divisor, as in this example, contains more than one term, it is generally convenient to follow the method of arithmetical long division. Thus—

$$\begin{array}{l}
a + 5\,b)\; 3\,a^2 + 13\,ab - 10\,b^2\,(3\,a - 2\,b \\
\qquad\quad\;\; \underline{3\,a^2 + 15\,ab} \\
\qquad\qquad\quad - 2\,ab - 10\,b^2 \\
\qquad\qquad\quad \underline{- 2\,ab - 10\,b^2}
\end{array}$$

Ex. 6. Divide $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.

$$\begin{array}{l}
a^2 + ab + b^2)\; a^4 + a^2b^2 + b^4\,(a^2 - ab + b^2 \\
\qquad\qquad\quad\;\; \underline{a^4 + a^3b + a^2b^2} \\
\qquad\qquad\qquad - a^3b + b^4 \\
\qquad\qquad\qquad \underline{- a^3b - a^2b^2 - ab^3} \\
\qquad\qquad\qquad\qquad\quad a^2b^2 + ab^3 + b^4 \\
\qquad\qquad\qquad\qquad\quad \underline{a^2b^2 + ab^3 + b^4}
\end{array}$$

It will be seen that in the last two examples care has been taken to keep the terms of the divisor, dividend, and successive remainders arranged according to the ascending or descending powers of some letter. In these cases we have arranged the terms according to the descending powers of a, and, as therefore follows, according to the ascending powers of b. Want of care in this respect will often render the operation of finding the true quotient *tiresome*, if not impossible. The next two examples will illustrate this point. They may be attempted first by the student, keeping the terms in the order as given.

Ex. 7. Divide $2 - 7x - 15x^2$ by $5x - 1$.

$$
\begin{array}{l}
-1 + 5x)\ 2 - 7x - 15x^2\ (-2 - 3x \\
\qquad\quad 2 - 10x \\
\qquad\qquad\quad 3x - 15x^2 \\
\qquad\qquad\quad 3x - 15x^2
\end{array}
$$

Or thus,
$$
\begin{array}{l}
5x - 1)\ -15x^2 - 7x + 2\ (-3x - 2 \\
\qquad\quad -15x^2 + 3x \\
\qquad\qquad\quad -10x + 2 \\
\qquad\qquad\quad -10x + 2
\end{array}
$$

Ex. 8. Divide $x^3 + y^3$ by $x^{-1} + y^{-1}$.

Here we have the powers of x in the dividend descending, while in the divisor they are ascending. Arranging them in the divisor as in the dividend, the operation is easy. Thus—

$$
\begin{array}{l}
y^{-1} + x^{-1})x^3 + y^3\ (x^3y - x^2y^2 + xy^3 \\
\qquad\quad x^3 + x^2y \\
\qquad\qquad -x^2y + y^3 \\
\qquad\qquad -x^2y - xy^2 \\
\qquad\qquad\qquad xy^2 + y^3 \\
\qquad\qquad\qquad xy^2 + y^3
\end{array}
$$

We shall work the next example in two ways to illustrate, firstly, the above point again; and, secondly, to show how the operation may be sometimes abbreviated by the use of brackets.

Ex. 9. Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

$$
\begin{array}{l}
x + y + z)x^3 - 3xyz + y^3 + z^3(x^2 - xy - xz + y^2 - yz + z^2 \\
\qquad x^3 + x^2y + x^2z \\
\qquad\quad -x^2y - x^2z - 3xyz \\
\qquad\quad -x^2y - xy^2 - xyz \\
\qquad\qquad -x^2z + xy^2 - 2xyz \\
\qquad\qquad -x^2z \qquad\ - xyz - xz^2 \\
\qquad\qquad\qquad xy^2 - xyz + xz^2 + y^3 \\
\qquad\qquad\qquad xy^2 \qquad\qquad\quad + y^3 + y^2z \\
\qquad\qquad\qquad\quad -xyz + xz^2 - y^2z \\
\qquad\qquad\qquad\quad -xyz \qquad\quad - y^2z - yz^2 \\
\qquad\qquad\qquad\qquad\qquad xz^2 + yz^2 + z^3 \\
\qquad\qquad\qquad\qquad\qquad xz^2 + yz^2 + z^3
\end{array}
$$

Or thus, inclosing the last two terms of the divisor in brackets—

$$
\begin{array}{l}
x+(y+z)\,)\,x^3-3\,xyz+y^3+z^3\,(\,x^2-(y+z)\,x+(y^2-yz+z^2) \\
\qquad\qquad x^3+(y+z)\,x^2 \\
\qquad\qquad\quad -(y+z)\,x^2-3\,xyz \\
\qquad\qquad\quad -(y+z)\,x^2-(y^2+2\,yz+z^2)\,x \\
\qquad\qquad\qquad\qquad (y^2-yz+z^2)\,x+y^3+z^3 \\
\qquad\qquad\qquad\qquad (y^2-yz+z^2)\,x+y^3+z^3
\end{array}
$$

It will be seen in both the above operations that we have brought down the terms of the dividend only when the subtrahends indicated they were required. This often prevents much useless repetition.

Ex. VIII.

Find the quotient of—

1. $28\,a^2b - 7\,ab^2 + 14\,b^3$ by $7\,b$; $3\,x^2y^2 - 12\,xy^3$ by $3\,xy$.

2. $-6\,a^5b + 15\,a^4b^2 - 20\,a^3b^3$ by $-3\,ab$; $4\,x^3y^2 + 6\,x^2y^3 + 4\,xy^4$ by $2\,xy^2$.

3. $ax^{2m} + bx^my^n + cy^{2n}$ by x^{m+n}; $ax^{2m-2n} + bx^{m-n}y^n + cy^{2n}$ by x^{m-n}.

4. $30\,x^2 + 2\,xy - 12\,y^2$ by $5\,x - 3\,y$; and by $6\,x + 4\,y$.

5. $1 + 2\,x + 3\,x^2 + 2\,x^3 + x^4$ by $1 + x + x^2$.

6. $12 - 19\,x - 21\,x^2$ by $7\,x - 3$; and by $3\,x + 4$.

7. $x^4 - 4\,x^2y^2 + 12\,xy^3 - 9\,y^4$ by $x - 3\,y$; and by $x + y$.

8. $x^5 - y^5$ by $x - y$; and $x^5 + y^5$ by $x + y$.

9. $acx^{m+n} + adx^my^m + bcx^ny^n + bdy^{m+n}$ by $cx^n + dy^m$.

10. $a^6 + a^4b^2 + a^2b^4 + b^6$ by $a^3 - a^2b - ab^2 + b^3$.

11. $b^2 + ab + bc + ac$ by $a + b$; and $a^2 + ab + bc + ac$ by $a + c$.

12. $a + (a + b)\,x + (a + b + c)\,x^2 + (a + b + c)\,x^3 + (b + c)\,x^4 + cx^5$ by $1 + x + x^2 + x^3$.

13. $a^5 - pa^4 + qa^3 - qa^2 + pa - 1$ by $a - 1$.

14. $a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd$ by $a - b + c + d$; and that of this quotient by $a - b - c - d$.

15. Show that the remainder, after the division of $x^4 - px^3 + qx^2 - rx + s$ by $x - a$, is $a^4 - pa^3 + qa^2 - ra + s$.

16. Divide $x^4 - y^4$ by $x^{-2} - y^{-2}$, and $x^2 + x^{-2} + 2$ by $x + x^{-1}$.

17. Show that the quotient of 1 by $1 + x$, is $1 - x + x^2 - x^3 +$ &c. *ad infinitum.*

18. Show that the quotient of 1 by $1 - 3x + 3x^2 - x^3$ is $1 + 3x + 6x^2 + 10x^3 + 15x^4 +$ &c. *ad infinitum.*

19. Divide $(x + y)^4 - 2(x + y)^2 z^2 + z^4$ by $x + y - z$.

20. Divide $(a - c)^3 - 3(a - c)^2(b - d) + 3(a - c)(b - d)^2 - (b - d)^3$ by $a - b - c + d$.

Factors.

29. The *ordinary* method of finding the quotient of two algebraical quantities having been explained in the last article, we shall now proceed to show how, in certain cases, this method may be avoided, and the quotient written down at sight. It may be remarked at the outset, that the resolution of algebraical expressions into their elementary factors is a subject of very great importance, and one which the student will do well to thoroughly master.

(I.) The form $x^2 + 2ax + a^2$.

We have seen (Art. 26) that $x^2 + 2ax + a^2 = (x + a)^2$. Hence the *sum of the squares* of two quantities, together with twice the product of the quantities, is equal to the *square of the sum* of the quantities.

And hence, any algebraical expression, which can be thrown into the form $(x^2 + 2ax + a^2)$, is of necessity a perfect square.

Thus—

$x^2 + 6x + 9 = x^2 + 2(3)x + 3^2 = (x + 3)^2$.

$a^2 - 10ab + 25b^2 = a^2 + 2a(-5b) + (-5b)^2 = (a - 5b)^2$.

$16a^2x^2 - 56abxy + 49b^2y^2 = (4ax)^2 + 2(4ax)(-7by) + (-7by)^2 = (4ax - 7by)^2$.

$3\,ab^2x^2 - 6\,abcxy + 3\,ac^2y^2 = 3\,a\,(b^2x^2 - 2\,bcxy + c^2y^2) = 3\,a\left\{(bx)^2 - 2\,(bx)\,(cy) + (cy)^2\right\} = 3\,a\,(bx - cy)^2.$

(II.) The form $a^2 - b^2$.

We have seen (Art. 26, II.) that $a^2 - b^2 = (a + b)\,(a - b)$.

Hence the *difference of the squares* of two quantities is equal to the product of the sum and difference of the quantities. Thus—

$$a^2x^2 - b^2y^2 = (ax)^2 - (by)^2 = (ax + by)\,(ax - by).$$

$$\begin{aligned} x^4 - y^4 &= (x^2)^2 - (y^2)^2 = (x^2 + y^2)\,(x^2 - y^2) \\ &= (x^2 + y^2)\,(x + y)\,(x - y). \end{aligned}$$

$$\begin{aligned} &a^2 + b^2 - c^2 - d^2 + 2\,ab - 2\,cd \\ &\qquad = (a^2 + 2\,ab + b^2) - (c^2 + 2\,cd + d^2) \\ &\qquad = (a + b)^2 - (c + d)^2 \\ &\qquad = \left\{(a + b) + (c + d)\right\}\left\{(a + b) - (c + d)\right\} \\ &\qquad = (a + b + c + d)\,(a + b - c - d). \end{aligned}$$

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2\,x^2y^2 + y^4) - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= \left\{(x^2 + y^2) + xy\right\}\left\{(x^2 + y^2) - xy\right\} \\ &= (x^2 + xy + y^2)\,(x^2 - xy + y^2). \end{aligned}$$

(III.) The form $x^2 + px + q$.

This form evidently includes both the preceding, for the first form—viz., $x^2 + 2\,ax + a^2$ is included, since q may be the square of half p; and the second is included—viz., $x^2 - a^2$, since we may have $p = o$, and q a negative square quantity.

Now, the resolution into elementary rational factors of the quantity $x^2 + px + q$ is not always possible; but, since (Art. 26, III.),

$$x^2 + (a + b)\,x + ab = (x + a)\,(x + b),$$

we have the following rule, when the quantity admits of resolution.

RULE.—If the third term q of the quantity $x^2 + px + q$ can be broken up into two factors, a and b, such that the sum of these will give the coefficient of x, then the elementary factors of $x^2 + px + q$ are $x + a$ and $x + b$.

Thus, $x^2 + 7\,x + 12 = (x + 3)\,(x + 4)$; for the *product* of 3 and 4 is 12, and their *sum* is 7, the coefficient of x.

$x^2 - x - 30 = (x - 6)(x + 5)$; for the *product* of -6 and 5 is -30, and their *sum* is -1.

And so, $x^2 - 18x + 32 = (x - 2)(x - 16)$,

And $a^2 + 3ab - 108b^2 = a^2 + (12b - 9b)a + (12b)(-9b) = (a + 12b)(a - 9b)$.

(IV.) The form $ax^2 + bx + c$.

This is the *general* form of a trinomial. The following remarks, though equally applying to each of the three preceding forms, are especially intended to be *practically* applied to trinomials not included by them.

The above form will include such expressions as the following:—$20x^2 + 11x - 42$, $6x^2 - 37x + 55$.

It is evident that the product of the first terms of the factors will be the first term of the given trinomial, and that the product of the last terms of the factors will be the third term of the given trinomial.

And, further, when the third term is *negative*, the last term of one factor must have the sign $+$, and the last term of the other the sign $-$; but, when the third term is *positive*, the last terms of the factors must have the same sign as the middle term.

Thus, $12x^2 - 31x - 30 = (4x + 3)(3x - 10)$.

Here the factors of $12x^2$ are either $3x$ and $4x$, $6x$ and $2x$, $12x$ and x, and the factors of 30 either 5 and 6, 3 and 10, 2 and 15, 1 and 30; and we must give a $+$ sign to one of each of these latter pairs, and a $-$ sign to the other. It is easily found on trial that, in order to obtain $-31x$ as the middle term, the factors of the trinomial must be $4x + 3$ and $3x - 10$.

So we have $10a^2 - 41ab + 21b^2 = (5a - 3b)(2a - 7b)$, and $acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$.

(V.) The forms $x^n + y^n$ and $x^n - y^n$.

We shall show in the next article that a rational integral algebraical expression, involving x, contains $x - a$ as a factor when it vanishes on substituting a for x.

Hence, $x^n + y^n$ and $x^n - y^n$ must each vanish on putting y for x, if they contain $x - y$ as a factor, n being an integer.

The former becomes $y^n + y^n$ or $2y^n$, and the latter $y^n - y^n$ or 0. We therefore conclude that—

$x^n + y^n$ does not contain $x - y$ as a factor, and that—

$x^n - y^n$ does contain $x - y$ as a factor, whether n be *even* or *odd*.

Again, on the same principle, they must each vanish if they contain $x + y$ as a factor, on putting $-y$ for x.

The former becomes $(-y)^n + y^n$, which vanishes when n is *odd*, and the latter becomes $(-y)^n - y^n$, which vanishes when n is *even*.

Hence we conclude that—

$x^n + y^n$ contains $x + y$ as a factor when n is *odd*, and
$x^n - y^n$ contains $x + y$ as a factor when n is *even*.

Now, the quotient of either of these quantities by $x + y$ or $x - y$ can in any particular case be found by long division. We thus find that—

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4),$$
$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3),$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

The law of formation of the co-factor in each case is easy to see; and if we may *assume* this apparent law as generally true, we may conclude that, when an algebraical quantity is of the form $x^n + y^n$ or $x^n - y^n$, and it contains $x + y$ or $x - y$ as a factor, the law of formation of the co-factor is as follows:—

Law of Formation of Co-Factor.

1. The terms are homogeneous, and of dimensions one degree lower than the given expression, the power of x in the first term being $n - 1$, and diminishing each successive term by *unity;* and the power of y increasing each successive term by *unity,* and first appearing in the second term.

2. The coefficient of every term is *unity.*

3. The *signs* are alternately $+$ and $-$, when $x + y$ is the corresponding elementary factor; and are all $+$, when $x - y$ is the corresponding elementary factor.

Ex. 1. $a^5 + 32 = a^5 + 2^5 = (a + 2)(a^4 + a^3 \cdot 2 + a^2 \cdot 2^2 + a \cdot 2^3 + 2^4) = (a + 2)(a^4 + 2a^3 + 4a^2 + 8a + 16)$.

Ex. 2. $a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3) = (a + b)(a^2 - ab + b^2) \cdot (a - b)(a^2 + ab + b^2 = (a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$.

30. *The remainder of the division of a rational integral function of* x *by* x − a *may be found by putting* a *for* x *in the given function.*

Def.—A **function** of x is an algebraical expression involving x; and a rational integral function of x is an expression of the form $ax^n + bx^{n-1} + \&c. + sx + t$, where all the powers of x are integral and positive.

Let $f(x)$* be a rational integral function of x, and suppose Q to be the quotient, and R the remainder on dividing the function by $x - a$.

Then, evidently—

$$Q(x - a) + R = f(x) \text{ identically.}$$

And this identity must hold for all values of x, and therefore holds when $x = a$.

In this case we have $Q(a - a) + R = f(a)$

$$0 + R = f(a)$$

$$\text{or } R = f(a).$$

Now $f(a)$ is the result of putting a for x in the given function, and is, as we have just shown, the remainder on dividing the given function by $x - a$.

Cor. 1. When there is no remainder, we must, of course, have $f(a) = 0$. Hence, a given rational integral function of x vanishes when a is put for x, if it be divisible by $x - a$.

Ex. 1. The remainder, after the division of $2x^3 - 5x^2 + 6x + 7$ by $x - 2$ is 15.

For, putting $x = 2$, we have—

$$2x^3 - 5x^2 + 6x + 7 = 2 \cdot 2^3 - 5 \cdot 2^2 + 6 \cdot 2 + 7 = 15.$$

Ex. 2. The function—

$$x^3 - 2x^2 + 5x - 52 \text{ is divisible by } x - 4.$$

For, putting $x = 4$, we have—

$$x^3 - 2x^2 + 5x - 52 = 4^3 - 2 \cdot 4^2 + 5 \cdot 4 - 52 = 0.$$

Cor. 2. Any rational integral function of x is divisible by $x - 1$, when the sum of the coefficients of the terms is *zero*.

For, putting $x = 1$ in the given function, it is evident that it is reduced to the sum of its coefficients, which sum must be zero if the function be divisible by $x - 1$.

Ex. Each of the following functions is divisible by $x - 1$, viz.:—

$3x^4 + 7x^3 - x^2 + 12x - 21$, $5x^7 - 2x - 3$,

$(a - b)x^2 + (b - c)x + (c - a)$, $(a + b)^2x^2 - 4abx - (a - b)^2$.

* The expression $f(x)$ must not be considered to mean the product of f and x, but as a *symbol* used for convenience.

Cor. 3. Any rational integral function of x is divisible by $x + 1$, when the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers.

(The term independent of x is always to be considered as the coefficient of an *even* power).

Let $ax^n + ax^{n-1} + \&c. + rx^2 + sx + t$ be a rational integral function of x.

Put $x = -1$, then we have, if the function be divisible by $x + 1$—

$$a(-1)^n + b(-1)^{n-1} + \&c. + r(-1)^2 + s(-1) + t = 0.$$

Suppose n to be even, then evidently $(-1)^n = (-1)(-1)(-1)$...to an *even* number of factors $= +1$.

And so $(-1)^{n-1} = (-1)(-1)(-1)$...to an *odd* number of factors $= -1$; and so on.

Hence we get $a - b + \&c. + r - s + t = 0$, and this must evidently require the condition that the sum of the positive quantities is equal to the sum of the negative, and, therefore, that the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers. And a similar result will follow if we suppose n to be *odd*.

Ex. Each of the following functions is divisible by $x + 1$, viz.:—

$x^3 + 5x^2 + 7x + 3$, $5x^5 - 4x^4 + 8x^2 - 2x - 1$, $a^2x^4 - (a + 1)(a + 2)x^2 + 2x + 3a + 4$, $px^3 + (q + r)x^2 + (q + r)x + p$.

Ex. IX.

Resolve into elementary factors—

1. $x^2 - 9a^2$, $16y^4 - 25z^4$, $24a^2 - 54b^2$, $8x^3 - 27y^3$.
2. $x^4 - xy^3$, $a^8 - b^8$, $xy^4 + x^4y$, $2x^3y^2z - 8xy^2z^3$.
3. $a^4 - 4b^4$, $x^4 + x^2y^2 + y^4$, $a^4 - 2a^2b^2 + b^4$, $a^2 + b^2 - c^2 + 2ab$.
4. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$, $a^2 - b^2 - c^2 + d^2 + 2bc + 2ad$, $a^2 - (b - c)^2$.
5. $(x + 7)^2 - (x + 2)^2$, $(x + 5)^2 - (x + 2)^2$, $(2a + b)^2 - (a - c)^2$.
6. $(x^3 - y^3)^2 + 4(x^4 + x^2y^2 + y^4)x^2y^2$, $(x^2 + y^2)^4 - 5(x^2 + y^2)^2x^2y^2 + 4x^4y^4$.

7. $x^2 - 3x - 70$, $x^2 + 11x + 10$, $a^2 - 15ab + 56b^2$, $x^2 - 4x - 192$.

8. $a^2x^2 + abxy - 42b^2y^2$, $3ax^2 - 24ax - 60a$, $24ac - 5ac^2 + ac^3$.

9. $6x^2 - 11x - 35$, $8x^2 + 6x - 135$, $18x^2 - 21x - 72$, $20x^2 - 11x - 42$.

10. $3x^3y + 10x^2y^2 + 3xy^3$, $20x^3 + 12ax^2 + 25bx^2 + 15abx$, $m^2x^2 + (mq + mp)x + pq$.

Write down the quotient of—

11. $x^4 - 16$ by $x - 2$, $3x^5 + 96$ by $x + 2$, $x^6 - 27$ by $x^2 - 3$.

12. $(a + b)^3 - (c + d)^3$ by $a + b + c + d$, $a^2 - b^2 + c^2 - d^2 + 2ac + 2bd$ by $a + b + c - d$.

Find the remainder after the division of—

13. $x^4 + px^3 + qx^2 + rx + s$ by $x - a$, $x^4 + a^4$ by $x^2 - a^2$.

14. $x^3 - 5x^2 + 7x - 9$ by $x + 3$, $x^4 - 3x + 7$ by $x - 2$.

Show that—

15. $5x^5 - 3x^3 + 7x^2 - 8x - 1$ is divisible by $x - 1$.

16. $2x^4 - 3x^3 + x^2 - 7x - 13$ is divisible by $x + 1$.

CHAPTER III.

Involution and Evolution.

Involution.

31. Involution is the operation by which we obtain the *powers* of quantities. This can of course be done by multiplication, but the results obtained by the actual multiplication of simple forms enable us to develop without multiplication more complex forms. As the subject requires the aid of the Binomial Theorem, we shall here show how to develop a few only of the more simple expressions.

32. The power of a single term is obtained by raising the

coefficient of the term to the power in question, and *multiplying* the exponents of the letters of the term by the exponent of the power in question.

Thus, $(a^3)^2 = a^3 \times a^3 = a^{3+3} = a^{3 \times 2}$,

$(a^m)^n = a^m \times a^m \times a^m \ldots\ldots \text{to } n \text{ factors} = a^{m+m+m+\ldots \text{ to } n \text{ terms}}$
$= a^{mn}$.

And so, $(4\,a^2b^3c^4)^3 = 4^3\,a^{2\times 3}b^{3\times 3}c^{4\times 3} = 64\,a^6b^9c^{12}$.

33. Development of the *third*, *fourth*, and *fifth* powers of $a + b$.

We know (Art. 26, I.) that $(a+b)^2 = a^2 + 2\,ab + b^2$.

$\therefore\ (a+b)^3 = (a^2 + 2\,ab + b^2)(a+b) = a^3 + 3\,a^2b + 3\,ab^2 + b^3$;

also, $(a+b)^4 = (a^3 + 3\,ab^2 + ab^2 + b^3)(a+b) = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$;

and $(a+b)^5 = (a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4)(a+b)$
$= a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5$.

The following law of the formation of the terms is evident:—

Law of Formation of Terms.

1. The first term contains a raised to the given power, and the power of a decreases by *unity* in each successive term, while the power of b (which first appears in the second term) increases by unity in each successive term, till it reaches the power of the given quantity.

2. The first coefficient is unity, and the coefficient of any term is found by multiplying the previous coefficient by the exponent of a in the previous term, and dividing the product by the number of terms hitherto developed.

Ex. 1. $(2\,x + 3\,y)^3 = (2\,x)^3 + 3\,(2\,x)^2(3\,y) + 3\,(2\,x)(3\,y)^2 + (3\,y)^3 = 8\,x^3 + 36\,x^2y + 54\,xy^2 + 27\,y^3$.

Ex. 2. $(a + b + c)^3 = (\overline{a + b} + c)^3 = (a+b)^3 + 3\,(a+b)^2c + 3\,(a+b)\,c^2 + c^3 = (a^3 + 3\,a^2b + 3\,ab^2 + b^3) + 3\,(a^2 + 2\,ab + b^2)\,c + 3\,(a+b)\,c^2 + c^3 = a^3 + b^3 + c^3 + 3\,a^2b + 3\,a^2c + 3\,ab^2 + 3\,b^2c + 3\,ac^2 + 3\,bc^2 + 6\,abc$.

(In the following examples the above law may be *assumed* as *generally* true.)

Ex. X.

1. Find the values of $(a^2b^3)^2$, $(-3\,ab^3)^3$, $(2\,a^2bc)^4$, $(-x^3yz^2)^5$.

Expand—

2. $(a + 3b)^4$, $(2a + b)^4$, $(a - b)^5$, $(3a - 4b)^3$.

3. $(2m + 1)^4$, $(5x + 2)^3$, $(3a - 4c)^4$, $(-a - b)^3$.

4. $(x^2 + x + 1)^2$, $(3a - b + 4c - d)^2$, $(a + 2b - c)^2$ $(3a + 3b + 3c)^2$.

5. $(1 + x - x^2)^3$, $(ax + by + cz)^3$, $\left\{(a+b)x - (c+d)y\right\}^3$.

6. $(1 + x)^7$, $(1 + x + x^2)^4$, $(a + bx + cx^2)^4$.

7. $(ax - by)^8$, $(3x + y)^3 (3x - y)^3$, $(x^2 + xy + y^2)^3 (x - y)^3$.

8. $(x^4 + x^2y^2 + y^4)^2 (x + y)^2 (x - y)^2$,

$$\left\{(a + b)^2 - 4ab\right\}^3 \left\{(a - b)^2 + 4ab\right\}.$$

Simplify—

9. $(a + b + c)^3 - 3(a + b + c)^2 c + 3(a + b + c)c^2 - c^3$.

10. $(a - b)^3 + 3(a - b)^2 (b - c) + 3(a - b)(b - c)^2 + (b - c)^3$.

11. $(1 + x + 3x^2 + 3x^3)^3 + (1 - x + 3x^2 - 3x^3)^3$.

12. $\left\{(x + y)^3 - (x^3 + y^3)\right\}^3 - 27x^3y^3(x^3 + y^3)$.

Evolution.

34. Evolution is the operation by which we obtain the *roots* of quantities.

Since the *square* or *second power* of a^3 is $(a^3)^2$ or a^6, we call (Art. 17) a^3 the *square root* or the *second root* of a^6.

And so, since the *cube* or *third power* of a^2 is $(a^2)^3$ or a^6, we call a^2 the *cube root* or the *third root* of a^6.

So, generally, since the nth power of $a^m = (a^m)^n = a^{mn}$, we call a^m the nth root of a^{mn}.

Thus, we have $\sqrt{a^6} = a^3$, $\sqrt[3]{a^6} = a^2$, $\sqrt[n]{a^{mn}} = a^m$.

Hence, in the case of quantities consisting of a single letter with a given exponent, when the given exponent contains as a factor the *number* indicating the root, we must divide the *given exponent* by this number, the quotient being the exponent of the root.

(We shall see farther on that this rule holds when the given exponent is not so divisible, the root in this case being called a surd).

35. Since the product of an *even* number of *negative factors* must give a *positive* result, and the product of an *odd* number of *negative* factors a *negative* result, it follows that—

I. When the root is indicated by an *even* number—

1. The root of a *positive* quantity may be written either with a + or − sign.

Thus, $\sqrt[2]{9a^2} = \pm 3a$, $\sqrt[4]{16b^4} = \pm 2b$.

2. The root of a *negative* quantity is impossible.

Thus, $\sqrt[2]{-a^2}$, $\sqrt[4]{-a^4b^{12}}$, &c., are impossible quantities.

II. When the root is indicated by an *odd* number, the root has always the sign of the given quantity.

Thus, $\sqrt[3]{-27b^6} = -3b^2$, $\sqrt[5]{32x^{10}y^5} = 2x^2y$.

(It may be remarked that the theory of *impossible quantities* forms an important branch of Algebra, which the student cannot yet enter upon. According to that theory, all quantities have as many roots as the number indicating the root.)

Square Root.

36. We shall now develop the method of finding the square root of a given quantity.

Ex. 1. Find the square root of $a^2 + 2ab + b^2$.

We know that $a^2 + 2ab + b^2 = (a + b)^2$.

Hence, $a + b$ is the square root of $a^2 + 2ab + b^2$ or $a^2 + (2a + b)b$.

Now, it is evident that the first term a of the root is the square root of the first term a^2 of the given quantity; and if this term be subtracted, there remains $2ab + b^2$, from which to determine b the second term of the root. Now, b is contained in $2ab + b^2$ or $(2a + b)b$ exactly $(2a + b)$ times. Hence it follows that the second term of the root is found by dividing the remainder by twice the first term of the root, and, if we wish to arrange our work in a way similar to long division, it is evident that we first take for our divisor $2a + b$, that is, twice the first term of the root added to the second term, which multiplied by b the second term, and subtracted, leaves no remainder.

Thus the whole process may be arranged as follows :—

$$
\begin{array}{l|l}
 & a^2 + 2ab + b^2 (a + b \\
 & a^2 \\
\hline
2a + b & \quad 2ab + b^2 \\
 & \quad 2ab + b^2 \\
\end{array}
$$

Ex. 2. Find the square root of $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

Now we know (Art. 26) that—

$$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \text{ or } (a + b)^2 + 2(a + b)c + c^2 = (a + b + c)^2.$$

And if we compare the form $(a + b)^2 + 2(a + b)c + c^2$ with the form $a^2 + 2ab + b^2$, it is evident that, having obtained as in the last example the first two terms $a + b$, we shall by continuing the process obtain the third term. Thus—

$$
\begin{array}{l|l}
 & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 (a + b + c \\
 & a^2 \\
\hline
2a + b & \quad 2ab + b^2 \\
 & \quad 2ab + b^2 \\
\hline
2a + 2b + c & \qquad 2ac + 2bc + c^2 \\
 & \qquad 2ac + 2bc + c^2 \\
\end{array}
$$

We may deduce from the above examples the following general rule :—

Rule.

1. Arrange the terms of the given quantity according to the ascending or descending powers of some letter, and take the square root of the first term for the first term of the quotient.

2. Subtract the square of the quotient, and bring down the next two terms of the given quantity.

3. Double the quotient, and place the result as a *trial divisor;* then, dividing the first of the terms brought down by this trial divisor to obtain the second term of the root, add the quotient so obtained to the first term of the root, and also to the trial divisor, to obtain a *complete* divisor.

4. Multiply the complete divisor by the second term of the root, and subtract the product, as in long division, from the terms brought down.

5. If there be any remainder or more terms to bring down, double the whole quotient for a trial divisor, and divide the

remainder by the new trial divisor, to obtain the third term of the root; and so on.

Ex. 3. Find the square root of $1 - 4x + 10x^2 - 12x^3 + 9x^4$. The terms are here arranged according to the ascending powers of x. Then proceeding according to rule, we have—

$$
\begin{array}{l|l}
 & 1 - 4x + 10x^2 - 12x^3 + 9x^4 \ (1 - 2x + 3x^2 \\
 & 1 \\
\hline
2 - 2x & \quad - 4x + 10x^2 \\
 & \quad - 4x + \ \ 4x^2 \\
\hline
2 - 4x + 3x^2 & \qquad\qquad 6x^2 - 12x^3 + 9x^4 \\
 & \qquad\qquad 6x^2 - 12x^3 + 9x^4 \\
\hline
\end{array}
$$

The student will observe that twice the quotient is most easily obtained by bringing down the previous complete divisor with its last term doubled.

Square Root of Numerical Quantities.

37. It is easy to apply the above method to numerical quantities.

Since $1^2 = 1, 10^2 = 100, 100^2 = 10{,}000, 1{,}000^2 = 1{,}000{,}000$, &c., it is evident that the square roots of numbers having less than three figures must contain one figure only ;

That those having not less than three and less than five must contain two figures and two only ;

Those having not less than five and less than seven must contain three figures and three only ; and so on.

Hence it follows that, if a dot be placed over the units' figure, and over every alternate figure to the left, the number of dots will give the number of figures in the square root.

Thus, the square roots of the numbers $1\dot{4}1\dot{3}7\dot{6}$ and $\dot{1}5\dot{2}2\dot{7}5\dot{6}$ have three and four figures respectively.

In the number $1\dot{4}1\dot{3}7\dot{6}$ we call 14, 13, 76 respectively the first, second, and third periods. So in the number $\dot{1}5\dot{2}2\dot{7}5\dot{6}$, the first, second, third, and fourth periods are respectively 1, 52, 27, 56.

It is evident that the number of periods correspond to the number of figures in the square root, and it will be seen that the figures of each period are used in the operation for the corresponding figure of the root.

Ex. Find the square root of 565504.

Pointing off the number, we find the first period to be 56. Now, the greatest square in 56 is 49, and the square next greater is 64. Hence the number lies between the square of 700 and 800; and, following the algebraical method, 700 will be the first term of the root.

The operation will stand thus—

	$\dot{5}6\dot{5}5\dot{0}4\ (\overset{a}{700} + \overset{b}{50} + \overset{c}{2} = 752$
	$490000 = a^2$
$2a + b = 1400 + 50 = 1450$	75504
	$72500 = 2ab + b^2$
$2a + 2b + c = 1400 + 100 + 2 = 1502$	3004
	$3004 = 2ac + 2bc + c^2$

Or, omitting the useless ciphers, and bringing down one period of figures at a time, the operation will stand thus—

	565504(752
	49
145	755
	725
1502	3004
	3004

Ex. 1. Find the square root of 6091024.

	6091024(2468
	4
44	209
	176
486	3310
	2916
4928	39424
	39421

Ex. 2. Find the square root of 83521.

	83521(289
	4
48	435
	384
569	5121
	5121

It will be observed that the second remainder, 51, is greater than the previous complete divisor, 48, and it might be supposed, therefore, that the second figure in the root should be 9 instead of 8.

Now, the square of $(a + 1)$ exceeds the square of a by $2a + 1$.

Thus, $(a + 1)^2 - a^2 = (a^2 + 2a + 1) - a^2 = 2a + 1.$

Hence it follows that, so long as any remainder is less than twice the corresponding number in the root + 1, we may be certain that we have taken the figure of the root sufficiently large.

Thus, since the remainder is less than $28 \times 2 + 1$ or 57, we may be certain that 8 is the correct figure and not 9.

Square Root of a Decimal.

38. It is evident that the *square* of any number containing one, two, three, &c., decimal figures, will contain *two, four, six*, &c., decimal figures respectively; and, hence, conversely, every decimal considered as a square must contain an even number of decimal figures, and its square root must contain half this even number of figures. It will then be necessary to add a cipher when the given number of decimal figures is odd.

Further, since decimals and integers follow the same system of notation, it is evident that if a dot be placed over the *units'* figure of the given number, the pointing off may be performed with regard to the integral part exactly as in integers, there being no necessity to point off the decimals, only taking care to bring them down in pairs, and putting a decimal point in the quotient when the first pair is brought down.

And again, if an integer be given which is not a perfect square, we may, by affixing to the right of it a decimal point and an even number of ciphers, gradually approximate to the square root as nearly as we please.

Ex. 3. Find the square root of 1·8225.

```
        1·8225(1·35
        1
 23      82
         69
 265     1325
         1325
```

Ex. 4. Find the square root of 247 to four places of decimals.

Instead of adding a decimal point and eight ciphers to the right of the given number, we will proceed in the ordinary way till we arrive at a remainder. Then putting a decimal point in the quotient, we shall add two ciphers to this and each successive remainder.

```
          247(15·7162
          1
 25       147
          125
 307       2200
           2149
 3141        5100
             3141
 31426       195900
             188556
 314322        734400
               628644
               105756
```

39. It will be shown hereafter that *when* n + 1 *figures of a square root have been obtained by the ordinary method,* n *figures more may be obtained by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is* 2 n + 1.

Ex. Find the square root of 29 to six places of decimals.

The square root required will evidently contain *seven* figures. We shall therefore find the first *four* figures by the ordinary method, and the other *three* by the above method. Thus—

	29(5·385164	
	25	
103	400	
	309	
1068	9100	
	8544	
10765	55600	
	53825	
10770	17750	then by division,
	10770	
	69800	
	64620	
	51800	
	43080	
Ans. 5·385164.	8720	

Cube Root.

40. We will next develop the method of finding the cube root of a quantity.

Ex. 1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We know (Art. 33) that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Hence $a + b$ is the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We see then that, the quantity being arranged according to the powers of a, the first term a of the cube root is the cube root of the first term of the given quantity; and if this term a^3 be subtracted, there remains $3a^2b + 3ab^2 + b^3$.

We see again that if this remainder be divided by $3a^2$, its first term gives b the second term of the root, and, further, if it be divided by b, we get $3a^2 + 3ab + b^2$ as a quotient. If we wish therefore to arrange the whole process in a way similar to ordinary division, it is evident that we must write as a divisor $3a^2 + 3ab + b^2$, in order that after multiplication by the quotient figure b we may obtain a quantity which when subtracted shall leave no remainder. The operation will then stand thus—

$$\begin{array}{rl} & a^3 + 3a^2b + 3ab^2 + b^3(a + b \\ & a^3 \\ \hline 3a^2 + 3ab + b^2 & \quad 3a^2b + 3ab^2 + b^3 \\ & \quad 3a^2b + 3ab^2 + b^3 \\ \hline \end{array}$$

We call $3a^2$ the *trial divisor*, because by means of it we search for the second term of the cube root. Having obtained this second term, we then form *the complete divisor* $3a^2 + 3ab + b^2$.

Ex. 2. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

$$\begin{array}{rl} & 8x^3 - 36x^2y + 54xy^2 - 27y^3(2x - 3y \\ & 8x^3 \\ \hline 12x^2 - 18xy + 9y^2 & \quad - 36x^2y + 54xy^2 - 27y^3 \\ & \quad - 36x^2y + 54xy^2 - 27y^3 \\ \hline \end{array}$$

EXPLANATION.—We find the cube root of $8x^3$ to be $2x$. This is then the first term of the quotient, and corresponds to a in the previous example. We now require $3a^2$ for a trial divisor. This, of course, $= 3(2x)^2 = 12x^2$. Subtracting the first term of the given quantity and dividing the first term of the remainder by this trial divisor, we obtain $-3y$ for the quotient. This forms the second term of the root, and corresponds to b in the last example. We now easily obtain $3ab$ and b^2.

Thus, $3ab = 3(2x)(-3y) = -18xy$, and $b^2 = (-3y)^2 = 9y^2$.

Hence, the complete trial divisor, corresponding to $3a^2 + 3ab + b^2$ in the last example, $= 12x^2 - 18xy + 9y^2$.

Multiplying now by $-3y$ the quotient, we obtain $-36x^2y + 54xy^2 - 27y^3$, which subtracted leaves no remainder.

Hence, $2x - 3y$ is the cube root.

Ex. 3. Find the cube root of—

$$x^6 + 6x^5y - 40x^3y^3 + 96xy^5 - 64y^6.$$

$$
\begin{array}{l}
x^6 + 6x^5y - 40x^3y^3 + 96xy^5 - 64y^6 \left\{ \begin{array}{l} x^2 + 2xy \\ -4y^2 \end{array} \right. \\
x^6 \\
\hline
\quad 6x^5y \qquad\qquad - 40x^3y^3 \\
\quad 6x^5y + 12x^4y^2 + 8x^3y^3 \\
\hline
\quad -12x^4y^2 - 48x^3y^3 + 96xy^5 - 64y^6 \\
\quad -12x^4y^2 - 48x^3y^3 + 96xy^5 - 64y^6 \\
\hline
\end{array}
$$

$$3x^4 + 6x^3y + 4x^2y^2$$

$$
\begin{array}{rl}
3(x^2 + 2xy)^2 = & 3x^4 + 12x^3y + 12x^2y^2 \\
3(x^2 + 2xy)(-4y^2) = & \qquad\qquad -12x^2y^2 - 24xy^3 \\
(-4y^2)^2 = & \qquad\qquad\qquad\qquad 16y^4 \\
\hline
& 3x^4 + 12x^3y \qquad - 24xy^3 + 16y^4
\end{array}
$$

It will be evident that the first two terms are obtained exactly as in Exs. 1 and 2. To obtain the next term we treat the terms already found as corresponding to a in Ex. 1. Then we obtain the trial divisor $3(x^2 + 2xy)^2$ corresponding to $3a^2$, and afterwards having divided the first term of the remainder by the first term of the trial divisor in order to get the term corresponding to b, we obtain in order $3(x^2 + 2xy)(-4y^2)$ corresponding to $3ab$, then $(-4y^2)^2$ corresponding to b^2, and lastly, by addition, $3x^4 + 12x^3y - 24xy^3 + 16y^4$, being the complete divisor, corresponding to $3a^2 + 3ab + b^2$ of Ex. 1. We then conclude the operation as before.

Cube Root of Numerical Quantities.

41. We shall now apply the above method to numerical quantities. It may be shown, as in Art. 34, that, if we place a dot over the *units' figure*, and over every *third* figure to the left, the number of periods so formed will be the number of figures in the root.

Ex. 1. Find the cube root of 262144.

$$
\begin{array}{rrl}
 & & 26\dot{2}14\dot{4}(\overset{a}{60} + \overset{b}{4} = 64 \\
 & & 216000 \\
3a^2 = & 10800 & \overline{46144} \\
3ab = & 720 & \\
b^2 = & 16 & \\
3a^2 + 3ab + b^2 = & \overline{11536} & \underline{46144}
\end{array}
$$

Explanation.—Pointing off the given number, we find the first period to be 262, and that the cube root consists of two figures. Now, the greatest perfect cube in 262 is 216, which is the cube of 6. Hence, the given number lies between the cubes of 60 and 70; and following the algebraical method, 60 will be the first term of the cube. This, we see, corresponds to a in the algebraical method.

We first then subtract the cube of a—viz., 216000, which leaves as a remainder 46144.

We now write down $3a^2$ or $3(60)^2 = 10800$, which is the trial divisor for determining b. Dividing then by this value of $3a^2$, we find $b = 4$, which is the second term of the cube root. We next obtain $3ab = 3(60)(4) = 720$, and $b^2 = 4^2 = 16$, and so by addition we get $3a^2 + 3ab + b^2 = 11536$, which is the complete divisor. Multiplying this then by the quotient figure, we subtract the product, and, there being no remainder, we find the cube root to be 64.

We may omit the useless ciphers in the above operation, if, remembering the local value of figures when numbers are expressed in ordinary notation, we take care to place the right-hand figure of the value of $3ab$ one place to the right of the corresponding figure of the value of $3a^2$; and also to place the right hand figure of b^2 one figure further still to the right.

The operation will then stand thus—

$$
\begin{array}{lrl}
 & & 26\dot{2}14\dot{4}(64 \\
 & & 216 \\
3 \times 6^2 = & 108 & \overline{46144} \\
3 \times 6 \times 4 = & 72 & \\
4^2 = & 16 & \\
 & \overline{11536} & \underline{46144}
\end{array}
$$

Ex. 2. Find the cube root of 102503232.

$$
\begin{array}{lrl}
 & & 102503232(468 \\
 & & 64 \\
\hline
3 \times 4^2 = & 48 & 38503 \\
3 \times 4 \times 6 = & 72 & \\
6^2 = & 36 & \\
\hline
 & 5556 & 33336 \\
 & 36 & \overline{5167232} \\
\hline
3 \times 46^2 = & 6348 & \\
3 \times 46 \times 8 = & 1104 & \\
8^2 = & 64 & \\
\hline
 & 645904 & 5167232 \\
\end{array}
$$

EXPLANATION.—The first two figures of the root are obtained as in Ex. 1. We then treat the number they form, viz., 46, as corresponding to a in the algebraical model, omitting useless ciphers. Obtaining then $3a^2$ or $3 \times 46^2 = 6348$, we find 8 to be the next figure of the root. Then writing under this, $3ab$ or $3 \times 46 \times 8 = 1104$, and afterwards b^2 or $8^2 = 64$, taking care as to the positions of the right-hand figures, and adding, we get 645904 as the complete divisor. Then as before.

REMARK.—It is unnecessary to be at the trouble to find the value of 3×46^2 by ordinary multiplication. For referring to the algebraical model, and writing here the successive terms of the complete divisor, and adding, we have—

$$
\begin{array}{rr}
 & 3a^2 \\
 & 3ab \\
 & b^2 \\
\hline
\text{Sum} = & 3a^2 + 3ab + b^2 \\
 & b^2 \\
\hline
\end{array}
$$

If we now again write down b^2 under this sum, and then add up *the last four lines*, we get—

$$3a^2 + 6ab + 3b^2, \text{ or } 3(a^2 + 2ab + b^2) = 3(a + b)^2.$$

This is three times the square of the first two terms of the root.

It therefore follows that, if, as in the above example, after completing the operation for finding the first two figures of the cube root, we write under the complete divisor just obtained the value of the square of the second figure, and then add together the last four lines thus obtained, we get three times the square of the quotient for a partial divisor by which to determine the next figure of the root.

The four lines to be added are in the above example bracketed. This method will be found to materially shorten the work, for it may be similarly applied to find the trial divisor when the cube root consists of any number of figures

Cube Root of a Decimal.

42. We know that the *cube* of any number containing one, two, three, &c., decimal figures will contain three, six, nine, &c., decimal figures respectively, and hence, conversely, every decimal considered as a cube must contain a number of decimal figures which is a multiple of *three*, and the number of decimal figures in the *cube root* must be *one-third* of the number contained in the given *cube*. It will then be necessary to add ciphers when the given number of decimal figures is not a multiple of 3.

And by continuing the reasoning of Art. 37, if a dot be placed over the units' figure and over every third figure to the left, it will be sufficient to bring down the decimal figures *three* at a time, putting a decimal point in the quotient when the first three are brought down.

And further, if an integer be given which is not a perfect cube, we may proceed in the ordinary way till we arrive at a remainder, and then, putting a decimal point in the quotient, by affixing *three* ciphers to this and each successive remainder, approximate to the cube root as nearly as we please.

Ex. 3. Find the cube root of 395·446904.

		$39\dot{5}{\cdot}446904(7{\cdot}34$
		343
$3 \times 7^2 =$	147	52446
$3 \times 7 \times 3 =$	63	
$3^2 =$	9	
	15339	46017
	9	6429904
$3 \times 73^2 =$	15987	
$3 \times 73 \times 4 =$	876	
$4^2 =$	16	
	1607476	6429904

43. We shall show farther on that *when* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *figures more may be obtained by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is* 2 n + 2.

We may apply this principle with advantage when we require the cube root of number to a given number of decimals.

Ex. Find the cube root of 87 to five places of decimals.

The required cube root will evidently contain 6 figures, and since 6 here corresponds to $2n + 2$ above, it is evident that $n = 2$. Hence, we shall find 4 (that is, $n + 2$) figures by the ordinary method, and then 2 more by division.

The operation will stand thus—

		$\dot{8}7$(4·43104
		64
$3 \times 4^2 =$	48	23000
$3 \times 4 \times 4 =$	48	
$4^2 =$	16	
	5296	21184
	16	1816000
$3 \times 44^2 =$	5808	
$3 \times 44 \times 3 =$	396	
$3^2 =$	9	
	584769	1754307
	9	61693000
$3 \times 443^2 =$	588747	
$3 \times 443 \times 1 =$	1329	
$1^2 =$	1	
	58887991	58887991
	1	280500900
	58901283	235605132
		44895768

Ans. 4·43104.

Ex. XI.

Find the square roots of—

1. $4x^2y^4z^6$, $16a^4y^4$, $x^4 + 2a^2x^2 + a^4$.
2. $4x^6 - 12x^5y + 29x^4y^2 - 30x^3y^3 + 25x^2y^4$.

3. $25a^4 - 30a^3b + 19a^2b^2 - 6ab^3 + b^4$.

4. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.

5. $a^2 + 2abx + (2ac + b^2)x^2 + 2(ad + bc)x^3 + (2bd + c^2)x^4 + 2cdx^5 + d^2x^6$.

6. $a^4x^{2n} - 6a^3x^{2n-1} + 17a^2x^{2n-2} - 24ax^{2n-3} + 16x^{2n-4}$.

7. $x^2 + 2 + x^{-2}$, $a^2x^{-2} - 2 + a^{-2}x^2$.

8. $9x^{2m} - 3a^2x^m + 25a^2 - 30ax^m + \frac{a^4}{4} + 5a^3$.

Find the square roots of—

9. 1296, 6241, 42849, 83521.

10. 10650·24, ·000576, $\dot{\cdot 1}$, $\frac{48}{243}$.

11. $\sqrt{17}$, $\sqrt{1\cdot 5}$, $\frac{2}{\sqrt{5}+1}$, $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

Give the values correct to four places of decimals of—

12. $\frac{\frac{5}{8} \text{ of } \cdot 31416}{\sqrt{93}}$, $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} + \frac{\sqrt{10}-2}{\sqrt{10}+2}$, 3·1416 of $\sqrt{\frac{40}{32\cdot 16}}$.

Find the cube roots of—

13. $8a^3b^6y^{12}$, $125x^{12}y^3$, $a^3 + 6a^2b + 12ab^2 + 8b^3$.

14. $x^{12} + 9x^{10} + 6x^8 - 99x^6 - 42x^4 + 441x^2 - 343$.

15. $x^3 + 3x^2y + 3xy^2 + y^3 - 6cxy - 3cx^2 - 3cy^2 + 3c^2x + 3c^2y - c^3$.

16. $x^3 + x^{-3} + 3(x + x^{-1})$, $x^3y^{-3} + 3x^2y^{-2} + 3xy^{-1} + 1$.

Find the cube roots of—

17. 5849513501832, 1371·330631.

18. 20·346417; $\cdot 0\dot{3}\dot{7}$, $\frac{108}{1372}$.

Give the value of the following correct to four places of decimals:—

19. $\frac{\sqrt[3]{5\cdot 12} + \sqrt[3]{\cdot 03375}}{\sqrt[3]{80} - \sqrt[3]{\cdot 01}}$, $\frac{1}{\sqrt[3]{4} + \sqrt[3]{2} + 1}$.

20. $\frac{\sqrt[3]{\cdot 625} + \sqrt[3]{\cdot 04}}{\sqrt{\cdot 05} + \sqrt{\cdot 04}}$ of $\frac{\sqrt{5}+2}{7}$.

21. $(\sqrt{7} + 2)(\sqrt{7} - 1)$, $(5 + \sqrt{3})(4 + \sqrt{12})$.

22. $\sqrt{11 + 6\sqrt{2}}$, $\sqrt[3]{6 + 15\sqrt{3}}$.

23. $\sqrt{\frac{3}{4} - \frac{6 + 2\sqrt{5}}{16}} \cdot \frac{\sqrt{5} + 1}{4}$.

24. $a^3(b - c) - b^3(a - c) + c^3(a - b)$, where $a = \sqrt{1{\cdot}2}$, $b = -\sqrt{{\cdot}3}$, and $c = -\sqrt[6]{{\cdot}027}$.

CHAPTER IV.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

Greatest Common Measure.

44. In Arithmetic (page 24) we defined the G.C.M. of two or more numbers as their highest common factor. In Algebra the same definition will suffice, provided we understand by the term *highest common factor, the factor of highest dimensions* (Art. 18). This, it need hardly be remarked, does not necessarily correspond to the factor of highest numerical value.

45. To find the G.C.M. of two quantities.

RULE.—Let A and B be the quantities, of which A is not of lower dimensions than B. Divide A by B, until a remainder is obtained of lower dimensions than B. Take this remainder as a new divisor, and the preceding divisor A as a new dividend, and divide till a remainder is again obtained of lower dimensions than the divisor; and so on. The last divisor is the G.C.M.

Before giving the general theory of the G.C.M. we shall work out a few examples.

Ex. 1. Find the G.C.M. of $x^2 - 6x - 27$ and $2x^2 - 11x - 63$.

According to the above rule, the operation is as follows:—

$$\begin{array}{llll}
x^2 - 6x - 27) & 2x^2 - 11x - 63\,(2 & & \\
 & \underline{2x^2 - 12x - 54} & & \\
 & \qquad\quad x - 9) & x^2 - 6x - 27\,(x + 3 & \\
 & & \underline{x^2 - 9x} & \\
 & & \qquad 3x - 27 & \\
 & & \qquad \underline{3x - 27} &
\end{array}$$

$\therefore$ The G.C.M. is $x - 9$.

Ex. 2. Find the G.C.M. of $10x^3 + 31x^2 - 63x$ and $14x^3 + 51x^2 - 54x$.

We may tell by inspection that x is a common factor, which we therefore strike out of both, only *taking care to reserve* it. The quantities then become—

$$10x^2 + 31x - 63, \text{ and } 14x^2 + 51x - 54.$$

We may now proceed according to rule, taking the former as divisor. We see, however, that the coefficient of the first term of the dividend is not exactly divisible by the coefficient of the first term of the divisor. Multiply therefore (to avoid fractions) the dividend by such a number as will make it so divisible, viz., by 5. This will not affect the G.C.M., as 5 is not *a factor* of the first expression, viz., $10x^2 + 31x - 63$.

It may as well be here mentioned that the G.C.M. of two quantities cannot be affected by the multiplication or division of *one* of the quantities by any quantity which is not a measure of the other. We shall, for a similar reason, reject certain factors or introduce them into any of the remainders or dividends during the operation. (See Art. 47).

$$\begin{array}{r|l}
 & 14x^2 + 51x - 54 \\
 & \quad 5 \\
\hline
10x^2 + 31x - 63) & 70x^2 + 255x - 270(7 \\
 & 70x^2 + 217x - 441 \\
\hline
 & \qquad\quad 38x + 171
\end{array}$$

Rejecting the factor 19 of this remainder, we have—

$$\begin{array}{l}
2x + 9)10x^2 + 31x - 63(5x - 7 \\
\qquad\quad 10x^2 + 45x \\
\hline
\qquad\qquad\quad - 14x - 63 \\
\qquad\qquad\quad - 14x - 63 \\
\hline
\end{array}$$

Hence, $2x + 9$ is the last divisor, and multiplying this by x, the common measure struck out at the commencement, we find the G.C.M to be $x(2x + 9)$ or $2x^2 + 9x$.

Ex. 3. Find the G.C.M. of $x^6-7x^5-3x^4-5x^3+42x^2-34x-21$, and $x^5-11x^4+25x^3+19x^2-49x-21$.

$$x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21)\ x^6 - 7x^5 - 3x^4 - 5x^3 + 42x^2 - 34x - 21\ (x+4$$
$$x^6 - 11x^5 + 25x^4 + 19x^3 - 49x^2 - 21x$$
$$\overline{4x^5 - 28x^4 - 24x^3 + 91x^2 - 13x - 21}$$
$$4x^5 - 44x^4 + 100x^3 + 76x^2 - 196x - 84$$
$$\overline{16x^4 - 124x^3 + 15x^2 + 183x + 63}$$

Multiplying the preceding divisor by 16, and taking the result for a dividend, we have—

$$16x^4 - 124x^3 + 15x^2 + 183x + 63)\ 16x^5 - 176x^4 + 400x^3 + 304x^2 - 784x - 336\ (x$$
$$16x^5 - 124x^4 + 15x^3 + 183x^2 + 63x$$
$$\overline{-\ 52x^4 + 385x^3 + 121x^2 - 847x - 336}$$

(Multiplying this remainder by 4)

$$4$$
$$\overline{-208x^4 + 1540x^3 + 484x^2 - 3388x - 1344\ (-13}$$
$$-208x^4 + 1612x^3 - 195x^2 - 2379x - 819$$
$$\overline{-\ 72x^3 + 679x^2 - 1009x - 525\ -}$$

Multiplying the preceding divisor by 9, and taking the result for a dividend, we have—

$$-72x^3 + 679x^2 - 1009x - 525)\ 144x^4 - 1116x^3 + 135x^2 + 1647x + 567\ (-2x$$
$$144x^4 - 1358x^3 + 2018x^2 + 1050x$$
$$\overline{242x^3 - 1883x^2 + 597x + 567}$$

(Multiplying this remainder by 36)

$$36$$
$$\overline{8712x^3 - 67788x^2 + 21492x + 20412\ (121}$$
$$8712x^3 - 82159x^2 + 122089x + 63525$$
$$\overline{14371x^2 - 100597x - 43113}$$

Dividing this remainder by 14371, and taking the quotient for a new divisor, we have—

$$
\begin{array}{r}
x^2 - 7x - 3)\ -72x^3 + 679x^2 - 1009x - 525(-72x + 175 \\
\underline{-72x^3 + 504x^2 + 216x } \\
175x^2 - 1225x - 525 \\
\underline{175x^2 - 1225x - 525}
\end{array}
$$

$\therefore\ x^2 - 7x - 3$ is the G.C.M.

It will be seen that we have introduced and rejected factors during the operation in order to avoid fractional coefficients. This, as will be seen from the general theory, will not affect the result, provided that no factor thus introduced or rejected is a measure of the corresponding divisor or dividend, as the case may be.

Theory of the Greatest Common Measure.

46. Let A and B be the two algebraical quantities, and the operation as indicated by the rule (Art. 45) be performed. Thus, let A be divided by B, with quotient p and remainder C. Then let B be divided by C, with quotient q, and remainder D. Lastly, let C be divided by D, with quotient r, and remainder zero.

$$
\begin{array}{l}
B)A(p \\
\quad \underline{pB} \\
\quad C)B(q \\
\qquad \underline{qC} \\
\qquad D)C(r \\
\qquad\quad \underline{rD} \\
\qquad\quad\ 0
\end{array}
$$

Then we are required to show that D is the G.C.M. of A and B.

(1.) D is a common measure of A and B.

Now, we have $C = rD$, $B = qC + D$, $A = pB + C$. Hence, D is a measure of C, and therefore of qC. It is therefore a measure of $qC + D$ or B. Hence, also, D is a measure of pB, and since it is also a measure of C, it must be a measure of $pB + C$ or A. But we have shown it to be a measure of B. Hence, D is a common measure of A and B.

(2.) D is the G.C.M. of A and B.

For every measure of A and B will divide $A - pB$ or C; and hence every measure of A and B will divide $B - qC$ or D. Now, D cannot be divided by any quantity higher than D, and, therefore, there cannot exist a measure of A and B higher than D. Hence, D is the G.C.M. of A and B.

47. *A factor which does not contain any factor common to both A and B may be rejected at any stage of the process.*

Let the operation stand thus:—

$$B = mB' \text{ suppose,}$$

$$\begin{array}{ll} B')A(p & \qquad C')B'(q \\ \underline{pB'} & \qquad \underline{qC'} \\ C = nC' \text{ suppose,} & \qquad D)C'(r \\ & \qquad \underline{rD} \\ & \qquad 0 \end{array}$$

where neither m nor n contains any factor common to A and B.

It will be an exercise for the student to show that D is the G.C.M. of A and B.

48. *A factor, which has no factor that the divisor has, may be introduced into the dividend at any stage of the process.*

The operation may stand thus—

$B)mA(p$, where m has no factor that B has;

$$\begin{array}{l} \underline{pB} \\ C)nB(q \end{array}$$

where n has no factor that C has;

$$\begin{array}{l} \underline{qC} \\ D)C(r \\ \underline{rD} \\ 0 \end{array}$$

As in Arts. 46, 47, it may be easily shown that D is the G.C.M.

Both the above principles are made use of in working out Ex. 3 Art. 45.

49. When a common factor can be found by inspection, it is advisable to strike it out of the given expressions. Then, having found by the ordinary process the G.C.M. of the resulting quantities, we must multiply the G.C.M. so found by the rejected factor.

Thus, $4x$ is common to the quantities $4x^3 - 20x^2 + 24x$, and $4x^3 + 16x^2 - 84x$.

Rejecting it, we get $x^2 - 5x + 6$, and $x^2 + 4x - 21$; whose G.C.M. is easily found to be $x - 2$.

Multiplying by $4x$, we find the required G.C.M. to be $4x^2 - 8x$.

50. By a little ingenuity on the part of the student in breaking up the given expressions into factors, the ordinary and often tedious process of finding the G.C.M. may be avoided. The limits of our space will allow us only one example.

Ex. Find the G.C.M. of $3x^3 + 4x^2 - 10x + 3$, and $15x^3 + 47x^2 + 13x - 12$.

The first expression contains $x - 1$ as a factor (Art. 30), for the sum of its coefficients is *zero*. The other factor may be obtained thus—

$$\begin{aligned} 3x^3 + 4x^2 - 10x + 3 &= 3x^3 - 3x^2 + 7x^2 - 7x - 3x + 3 \\ &= 3x^2(x-1) + 7x(x-1) - 3(x-1) \\ &= (3x^2 + 7x - 3)(x - 1). \end{aligned}$$

Now, $3x^2 + 7x - 3$ is not further resolvable, and $x - 1$ is evidently (Art. 30) not a factor of $15x^3 + 34x^2 + 13x - 12$. It is, therefore, very probable that $3x^2 + 7x - 3$ is the G.C.M. required.

We may test it thus—

$$\begin{aligned} 15x^3 + 47x^2 + 13x - 12 &= 15x^3 + 35x^2 - 15x + 12x^2 + 28x - 12 \\ &= 5x(3x^2 + 7x - 3) + 4(3x^2 + 7x - 3) \\ &= (5x + 4)(3x^2 + 7x - 3). \end{aligned}$$

Hence, $3x^2 + 7x - 3$ is the G.C.M. required.

G.C.M. of Three or More Quantities.

51. The G.C.M. of three or more quantities may be found thus—

Rule.—Find the G.C.M. of any two of the quantities, then the G.C.M. of the G.C.M. so found and a third quantity, and so on. The last found G.C.M. will be the G.C.M. required.

Ex. XII.

Find the G.C.M. of the following—

1. $x^2 - 5x + 6$ and $x^2 + 3x - 18$.
2. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 5x^2 + 7x + 3$.
3. $2x^3 + 10x^2 - 18x - 90$ and $3x^3 + 16x^2 - 26x - 141$.
4. $x^2 + (a + b\ x + ab$ and $x^2 + (a + c)x + ac$.
5. $a^3 - b^3$ and $a^3 + a^2b + ab^2$.

6. $x^3 - 4x + 3$ and $x^3 + 4x^2 - 5$.

7. $4x^3 - 32x^2 + 85x - 75$ and $3x^3 - 15x^2 + 15x + 9$.

8. $9x^2 - 3xy + 2y - 4$ and $6x^4 - 4x^3 - 9xy^2 + 6y^2$.

9. $48x^4 + 8x^3 + 31x^2 + 15x$ and $24x^4 + 22x^3 + 17x^2 + 5x$.

10. $15a^3 + a^2b - 3ab^2 + 2b^3$ and $54a^2b^2 - 24b^4$.

11. $3x^3 - (3c + d + 1)x^2 - (2a + b - 3c - d + 2)x + 2a + b$ and $2x^2 - (a + b + 2)x + a + b$.

12. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.

13. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ and $9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2$.

14. $e^xx^2 + e^x + x^2 + 1$ and $e^{2x}x^4 - e^{2x} + x^4 - 1$.

15. $ax^4 + (b + c)x^2 - ax - b - c$ and $ex^3 - (f - g)x^2 + (f - e)x - g$.

16. $4x^4 + 2x^3 + 4x^2 + 39x - 9$, $8x^4 + 20x^2 + 51x + 9$, and $2x^4 + x^3 + 3x^2 + 18x$.

17. $ax^3 - (c + 1)x^2 + (c + 1)x - a$, $bx^4 - (b + d)x^3 + (c + d)x^2 - (c + e)x + e$, and $(c + 1)x^5 + (d + 2)x^4 - (d + 1)x^3 - (c + 2)x^2$.

18. $a^3 - b^3 + c^3 + 3abc$ and $a^2 - b^2 + c^2 + 2ac$.

Least Common Multiple.

52. When two or more algebraical expressions are arranged according to the powers of some letter, the expression of lowest dimensions which is divisible by each of the given expressions is called the L.C.M.

53. The L.C.M. of monomials and of expressions whose factors are apparent may be found by inspection.

Ex. 1. Find the L.C.M. of ab, ac, ad, bc, bd, cd.

If we form an expression, whose elementary factors contain each of the elementary factors of the given quantities, we shall evidently have a *common multiple;* and if no elementary factor of this expression is of a higher power than the highest power of the same factor in the given quantities, we shall get the L.C.M.

Hence, the required L.C.M. $= abcd$.

Ex. 2. Find the L.C.M. of—

$(a - b)(b - c), (a - b)(c - a), (b - c)(c - a)$.

Ans. $(a - b)(b - c)(c - a)$.

Ex. 3. Find the L.C.M. of $a(x + 1)$, $b(x^2 - 1)$, $c(x^2 + 2x - 3)$, $d(x^2 + 4x - 3)$. We may write the given expressions thus—

$a(x + 1), b(x + 1)(x - 1)$,

$c(x - 1)(x + 3), d(x + 1)(x + 3)$.

Hence, the required L.C.M. $= abcd(x - 1)(x + 1)(x + 3)$.

Ex. 4. Find the L.C.M. of $a^2 - ax + x^2, a^2 + ax + x^2, a^3 + x^3, a^3 - x^3$.

Now (Art. 29) $a^3 + x^3 = (a + x)(a^2 - ax + x^2)$,
and $a^3 - x^3 = (a - x)(a^2 + ax + x^2)$.

Hence the required L.C.M.—

$= (a + x)(a - x)(a^2 + ax + x^2)(a^2 - ax + x^2) = a^6 - x^6$.

54. *The L.C.M. of two quantities is found by dividing their product by their G.C.M.*

Let a and b be the two quantities, and d the G.C.M.;

And suppose $a = pd$ and $b = qd$.

It is evident that p and q contain no common factor. Hence pq is the L.C.M. of p and q; and, therefore, no expression of lower dimensions than pqd can possibly be divisible by pd and qd.

Hence pqd is the L.C.M. of pd and qd, or of a and b.

Now $pqd = pd \times qd \div d = a \times b \div d$, and hence the rule.

55. To find the L.C.M. of *three* or more quantities.

Rule.—Find the L.C.M. of two of the quantities, then the L.C.M. of the expression thus obtained and a third quantity, and so on. The last expression so found is the L.C.M. required.

We shall prove this rule in the case of *three* quantities.

Let a, b, c be the quantities, and m be the L.C.M. of a and b.

Then the L.C.M. of m and c is the L.C.M. required.

For every common multiple of m and c is a common multiple of a, b, c. And every common multiple of a and b

must contain the m, their least common multiple. Hence, every common multiple of a, b, c must be a common multiple of m and c, and the converse is also true. Hence, the L.C.M. of m and c is the L.C.M. of a, b, c.

Ex. XIII.

Find the L.C.M. of—

1. axy^2, $3a^2x^3y$, $4a^3y^3$, $6x^2y^2$.
2. $5a^2b^2$, $6a^2c^2$, $4b^2c^2$.
3. $(a - b)(b - c)$, $(b - a)(a - c)$, $(c - a)(c - b)$.
4. $ax(x + a)$, $a^2(x - a)$, $x^2 - a^2$.
5. $x^2 + 3x + 2$, $x^2 + 4x + 3$, $x^2 + 5x + 6$.
6. $x^2 - x - 30$, $x^2 - 11x + 30$, $x^2 - 25$.
7. $6x^2 + 37x + 56$, $8x^2 + 38x + 35$, $12x^2 + 47x + 40$.
8. $5(x^2 - x + 1)$, $6(x^2 + 1)$, $7(x^3 + 1)$.
9. $x^4 + a^2x^2 + a^4$, $a^2x^2 + a^3x + a^4$, $ax^2 - a^2x + a^3$.
10. $x^2 + (a + b)x + ab$, $x^2 + (a + c)x + ac$, $x^2 + (b + c)x + bc$.
11. $1 - x$, $1 + x$, $1 + x^2$, $1 + x^4$, $1 + x^8$.
12. $x^3 + 6x^2 + 11x + 6$, $x^3 - 6x^2 - 25x + 150$.
13. $a^3 - 3ab(a - b) - b^3$, $a^3 - b^3$, $a^3 + a^2b + ab^2$.
14. $x^4 - 1$, $6x^5 + 5x^4 + 8x^3 + 4x^2 + 2x - 1$.
15. $a^4 - 2a^2b^2 + b^4$, $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
16. $3x^3 - 4x + 1$, $2x^3 - 7x + 5$, $4x^4 + 6x^3 + 10x$.
17. $3x^2 + 6x - 24$, $x^3 - 12x + 16$, $5x^4 - 22x - 36$.
18. $a^3 - ab^2$, $b^3 - a^2b$, $ab^2 - b^3$, $a^2b - a^3$.
19. $3x^4 - 48$, $5x^2 - 20$, $3x^2 - 16x + 20$.
20. $x^8 - y^8$, $x^4 - 2x^2y^2 + y^4$, $x^3 + x^2y + xy^2 + y^3$.
21. $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$ and $x^5 - ax^4 + a^2x^3 - a^3x^2 + a^4x - a^5$.
22. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$ and $a^2 - b^2 - c^2 + d^2 + 2ad - 2bc$.

23. $a^3 + b^3 + c^3 - 3abc$ and $(a + b)^2 + 2(a + b)c + c^2$.

24. $(a + b)^2 - (c + d)^2$, $(a + c)^2 - (b - d)^2$, $(a + d)^2 - (b - c)^2$, $(c + d)^2 - (a - b)^2$, $(b + d)^2 - (a - c)^2$, and $(b + c)^2 - (a - d)^2$.

CHAPTER V.

Fractions.

56. It is unnecessary to repeat here the propositions relating to fractions which were proved in Arithmetic, Chap. II. of this work. The student will see that, by substituting general symbols for the particular figures there used, the reasoning will equally hold. We shall work out a few examples to show the method of dealing with them in algebra.

Ex. 1. Simplify the fraction $\dfrac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15}$.

By inspection (Art. 30) we see that $x - 3$ is a factor of numerator and denominator. We have then—

$$\frac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15} = \frac{x^2(x - 3) + x(x - 3) + 4(x - 3)}{(x - 3)(x + 5)}$$

$$= \frac{(x^2 + x + 4)(x - 3)}{(x - 3)(x + 5)} = \frac{x^2 + x + 4}{x + 5}. \quad \text{Ans.}$$

Ex. 2. Find the value of $\dfrac{1}{a + b} + \dfrac{1}{a - b} - \dfrac{2a}{a^2 + b^2}$.

$$\frac{1}{a + b} + \frac{1}{a - b} - \frac{2a}{a^2 + b^2} = \frac{(a - b) + (a + b)}{(a + b)(a - b)} - \frac{2a}{a^2 + b^2}$$

$$= \frac{2a}{a^2 - b^2} - \frac{2a}{a^2 + b^2} = 2a\left(\frac{1}{a^2 - b^2} - \frac{1}{a^2 + b^2}\right)$$

$$= 2a \cdot \frac{(a^2 + b^2) - (a^2 - b^2)}{(a^2 - b^2)(a^2 + b^2)} = 2a \cdot \frac{2b^2}{a^4 - b^4} = \frac{4ab^2}{a^4 - b^4}.$$

Ex. 3. Find the value of $\frac{bc}{(a-b)(a-c)} + \frac{ac}{(b-a)(b-c)}$ $+ \frac{ab}{(c-a)(c-b)}$.

The second denominator has a factor, $(b - a)$, which differs from a factor, $(a - b)$, of the first denominator in sign only. We shall therefore change the sign of the second fraction, and also of its first factor. This will not alter its value.

And, similarly, we find that by changing the signs of each of the factors of the third denominator we shall have them in a form corresponding to factors of the first and second denominators. The sign of the third fraction will not be changed, as the sign of the denominator will, on the whole, be unchanged.

The given expression then will stand thus—

$$= \frac{bc}{(a-b)(a-c)} - \frac{ac}{(a-b)(b-c)} + \frac{ab}{(a-c)(b-c)}$$

$$= \frac{bc(b-c) - ac(a-c) + ab(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{bc(b-c) - a^2c + ac^2 + a^2b - ab^2}{(a-b)(a-c)(b-c)}\text{, or, re-arranging,}$$

$$= \frac{a^2(b-c) - a(b^2-c^2) + bc(b-c)}{(a-b)(a-c)(b-c)}\text{, then, dividing nume-}$$

rator and denominator by $b - c$,

$$= \frac{a^2 - a(b+c) + bc}{(a-b)(a-c)} = \frac{(a-b)(a-c)}{(a-b)(a-c)} = 1.$$

Ex. 4. Simplify—

$$\left\{a^2 - \frac{4a^2x - 3ax^2 + x^3}{a+x}\right\} \times \left\{a^2 + \frac{4a^2x + 3ax^2 + x^3}{a-x}\right\}.$$

The given expression—

$$= \frac{a^2(a+x) - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^2(a-x) + 4a^2x + 3ax^2 + x^3}{a-x}$$

$$= \frac{a^3 + a^2x - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^3 - a^2x + 4a^2x + 3ax^2 + x^3}{a-x}$$

$$= \frac{a^3 - 3a^2x + 3ax^2 - x^3}{a + x} \times \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a - x}$$

$$= \frac{(a - x)^3}{a + x} \times \frac{(a + x)^3}{a - x} = \frac{(a - x)^2}{1} \times \frac{(a + x)^2}{1} = (a^2 - x^2)^2.$$

Ex. 5. Divide—

$$(a + b)\left\{\frac{1}{x - a} + \frac{1}{x + b}\right\} \text{ by } (a - b)\left\{\frac{a}{(x-a)^2} + \frac{b}{(x + b)^2}\right\}.$$

Now—

$$\frac{(a + b)\left\{\dfrac{1}{x - a} - \dfrac{1}{x + b}\right\}}{(a - b)\left\{\dfrac{a}{(x - a)^2} + \dfrac{b}{(x + b)^2}\right\}} = \frac{(a + b)\,.\,\dfrac{(x + b) - (x - a)}{(x - a)(x + b)}}{(a - b)\,.\,\dfrac{a(x + b)^2 + b(x - a)^2}{(x - a)^2(x + b)^2}},$$

or, reducing—

$$= \frac{(a + b)\,.\,\dfrac{a + b}{(x - a)(x + b)}}{(a - b)\,.\,\dfrac{(a + b)x^2 + ab(a + b)}{(x - a)^2(x + b)^2}} = \frac{(a + b)\,.\,\dfrac{1}{1}}{(a - b)\,.\,\dfrac{x^2 + ab}{(x - a)(x + b)}}$$

$$= \frac{(a + b)(x - a)(x + b)}{(a - b)(x^2 + ab)}.$$

Ex. XIV.

Simplify the following expressions:—

1. $\dfrac{x^2 - 5x + 4}{x^2 + 2x - 24}, \dfrac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}.$

2. $\dfrac{6x^2 + 29x + 35}{14x^2 + 39x + 10}, \dfrac{2x^3 + 7x - 9}{5x^3 - 3x^2 - 4x + 2}.$

3. $\dfrac{a^4 - 2a^2b^2 + b^4}{a^3 - 4a^2b + 4ab^2 - b^3}, \dfrac{24a^3 - 28a^2b + 6ab^2 - 7b^3}{6a^2 + 11ab - 21b^2}.$

4. $\dfrac{x^2(y^2 - z^2) - xy(2y^2 + yz - z^2) + y^3(y + z)}{x^2(y + z)^2 - xy(2y^2 + 3yz + z^2) + y^3(y + z)}, \dfrac{x^4 + x^2y^2 + y^4}{x^6 - y^6}.$

5. $\dfrac{1}{a + b} + \dfrac{1}{a - b}, \dfrac{1}{a - b} - \dfrac{1}{a + b}.$

6. $\frac{a}{a+b} + \frac{b}{a-b} - \frac{ab}{ab-b^2} + \frac{ab}{a^2+ab}.$

7. $\frac{1}{4(x-1)} - \frac{x-1}{4(x^2+1)} - \frac{x-1}{2(x^2+1)^2}.$

8. $\frac{5}{x+1} - \frac{2}{(x+2)^2} - \frac{18}{5(x+2)} - \frac{7x+1}{5(x^2+1)}.$

9. $\frac{11}{x+1} - \frac{11}{x+3} - \frac{14}{(x+1)^2}.$

10. $\frac{8}{(x-1)^4} - \frac{4}{(x-1)^3} - \frac{2}{(x-1)^2} + \frac{16x+14}{3(x^2-1)} - \frac{16x-8}{3(x^2-x+1)}.$

11. $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}.$

12. $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)}.$

13. $\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)}.$

14. $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}.$

15. $\frac{1}{\left(\frac{a}{b}-1\right)\left(\frac{a}{c}-1\right)} + \frac{1}{\left(\frac{b}{a}-1\right)\left(\frac{b}{c}-1\right)} + \frac{1}{\left(\frac{c}{a}-1\right)\left(\frac{c}{b}-1\right)}.$

16. $\frac{(a+b)^2+(b-c)^2+(a+c)^2}{(a+b)(b-c)(a+c)} - \frac{2}{a+c} - \frac{2}{b-c} + \frac{2}{a+b}.$

17. $\left\{\frac{1}{a+x} + \frac{1}{a-x} + \frac{2a}{a^2+x^2}\right\} \cdot \left\{\frac{1}{a+x} - \frac{1}{a-x} - \frac{2x}{a^2+x^2}\right\}.$

18. $\left\{\frac{a^2+b^2}{a^2-b^2} + \frac{a-b}{a+b}\right\} \div \left\{2 + \frac{a^2+b^2}{a^2-b^2} - \frac{a-b}{a+b}\right\}.$

19. $\left\{1 - \frac{2x^2}{a^2} + \frac{2x^4}{a^2(a^2+x^2)}\right\} \times \left\{\frac{2a^4}{x^2(a^2-x^2)} - \frac{2a^2}{x^2} - 1\right\}.$

20. $\frac{a}{c} + \frac{b}{a} + \frac{c}{b} - \frac{a^2\left(\frac{b}{c}+1\right) + b^2\left(\frac{c}{a}+1\right) + c^2\left(\frac{a}{b}+1\right)}{ab+bc+ca}.$

21. $\left\{\left(\frac{x+y}{x-y}\right)^2+1\right\}\cdot\left\{\left(\frac{x+z}{x-z}\right)^2+1\right\}\cdot\left\{\left(\frac{y+z}{y-z}\right)^2+1\right\}\times$

$$\frac{x^2(y-z)+y^2(z-x)+z^2(x-y)}{x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4+2x^2y^2z^2}.$$

22. $\left\{\frac{1-x^2}{1-x^3}+\frac{1-x}{1-x+x^2}\right\}\div\left\{\frac{1+x}{1+x+x^2}-\frac{1-x^2}{1-x^3}\right\}.$

23. $\frac{a^2}{(a+b)(a+c)(x-a)}+\frac{b^2}{(a+b)(b-c)(x+b)}-$

$$\frac{c^2}{(a+c)(b-c)(x+c)}.$$

24. $\left\{\frac{9x^2}{a^2}-\frac{24x}{a}+16-\frac{25a^2}{x^2}\right\}\div\left\{\frac{3x}{a^2}-\frac{4}{a}-\frac{5}{x}\right\}.$

25. $\left\{x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)\right\}\times\left\{\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^3(x^2-1)}\right\}^3.$

26. $\dfrac{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2-2}{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2+2}.$

27. $\frac{b^2}{(a+b)^2}+\frac{a^2}{(a-b)^2}-\frac{2b}{a+b}-\frac{2a}{a-b}+2.$

28. $\frac{x^2}{(x-y)(a+x)}+\frac{y^2-2xy}{(x-y)(a+y)}+\frac{y^2}{(a+y)^2}.$

29. $\dfrac{\frac{x}{y}+\frac{y}{x}-1}{\frac{x^3}{y^3}-1}\times\dfrac{\frac{x}{y}+\frac{y}{x}+1}{\frac{x^3}{y^3}+1}\times\left(\frac{x^4}{y^4}-\frac{x^2}{y^2}\right).$

30. $\left\{\frac{1}{x}-\dfrac{1}{x+\dfrac{1}{y+\frac{1}{z}}}\right\}\left(y+\frac{1}{x}+\frac{1}{z}\right).$

CHAPTER VI.

Simple Equations.

57. When two algebraical expressions are connected by the sign $(=)$, they are said to form an equation.

When the equality is such that it is true for *all* values of the letters in the given expressions, it is called *an identity.*

Thus, $(x+a)(x+b) = x^2 + (a+b)x + ab$ and $(a+b)^2 - (a-b)^2 = 4ab$ are identities.

58. When the condition of equality is such that some one or other of the letters must have particular values or a limited number of values, the statement of equality is termed an *equation of condition*, or, more briefly, *an equation.*

Thus, it may be found on trial that the equality

$$4x + 2 = 3x + 5$$

is true only when $x = 3$. Such an expression is therefore an equation.

59. The letters of an equation to which particular or a limited number of values must be given are termed *unknown* quantities.

Equations may contain *one, two, three,* or more unknown quantities.

The determination of the particular value or values of the unknown quantities is called the *solution* of the equation, and each of the values which *satisfies* the equation is said to be a *root* of the equation.

60. The expressions on the left and right sides of the sign $(=)$ are termed the first and second sides respectively. It follows, therefore, that—

1. *If both sides of an equation be multiplied by the same quantity, the equation still subsists.*

2. *If both sides be divided by the same quantity, the equation still holds.*

3. *Any term may be transposed from one side to the other if the sign of the term be changed.*

Thus, if $3x + a = b$, we must have also

$$3x = b - a,$$

for this results from subtracting a from each side of the equation.

4. *The equation holds if every term on both sides has its sign changed.*

Thus, if $ax + b = cx - d$, we may reason as follows:—

The quantity $(ax + b)$ looked upon as a whole is given equal to the quantity $(cx - d)$ looked upon as a whole. If we change the *qualities* of these quantities, they will evidently be still equal.

$$\text{Hence, } -(ax + b) = -(cx - d)$$
$$\text{or, } -ax - b = -cx + d.$$

Now, this is the result of changing the sign of every term on both sides of the given equation.

5. *The sides of an equation may be reversed without destroying the equality.*

Thus, if $mx + n = px + q$, it must also follow that

$$px + q = mx + n.$$

6. *The sides of an equation may be raised to the* SAME POWER, *or we may extract the same root of both sides, and the equation still subsists.*

61. Simple equations are those in which the unknown quantities are not higher than the first degree, when the equations are reduced to a rational integral form.

The following is the general method adopted in solving a simple equation involving only one unknown quantity—

1. *Clear of fractions if necessary.*

2. *Transpose all the terms involving the unknown quantity to the first side of the equation, and all the remaining terms to the second side.*

3. *Simplify both sides if necessary, and divide both sides by the coefficient of the unknown quantity.*

Ex. 1. Solve the equation $5x + 6 = 3x + 12$.

Transposing the terms, we have—

$$5x - 3x = 12 - 6.$$

Now, simplifying, we get—

$$2x = 6;$$

and dividing each side by the coefficient of the unknown quantity, viz., by 2, we have—

$$x = 6 \div 2 = 3.$$

VERIFICATION.—Putting the value 3 for x in each side of the given equation, the first side becomes $5 \times 3 + 6$ or 21; and the second side becomes $3 \times 3 + 12$ or 21. The value of x found therefore *satisfies* the given equation.

Ex. 2. Given $\dfrac{x-2}{2} + \dfrac{x}{3} = 20 - \dfrac{x-6}{2}$, find x.

Clearing of fractions, by multiplying *every term* on each side by the L.C.M. of the denominators, viz., by 6, we get—

$$3(x-2) + 2x = 20 \times 6 - 3(x-6).$$

(Beginners often neglect to multiply integral terms such as 20 by the L.C.M.)

or $3x - 6 + 2x = 120 - 3x + 18$, or, transposing,

$$3x + 2x + 3x = 120 + 18 + 6, \text{ or, simplifying,}$$

$$8x = 144,$$

or dividing each side by 8, the coefficient of x, we have—

$$x = 18, \text{ the value required.}$$

(It will be good practice for the student to verify this result as in the last example).

Ex. 3. $\dfrac{4x-21}{7} + 7\tfrac{5}{6} + \dfrac{7x-28}{3} = x + 3\tfrac{3}{4} - \dfrac{9-7x}{8} + \tfrac{1}{12}.$

It is sometimes convenient to first *partially* clear off fractions. Thus, multiplying each side by 72, we have—

$$\frac{72(4x-21)}{7} + 47 \times 12 + 24(7x-28)$$
$$= 72x + 15 \times 18 - 9(9-7x) + 6;$$

$$\text{or } \frac{288x}{7} - 72 \times 3 + 564 + 168x - 672,$$
$$= 72x + 270 - 81 + 63x + 6;$$

or, transposing,

$$41\tfrac{1}{7}x + 168x - 72x - 63x$$
$$= 270 - 81 + 6 + 216 - 564 + 672;$$

or, simplifying,

$$74\tfrac{1}{7}x = 519; \text{ or, multiplying each side by 7,}$$
$$519x = 519 \times 7;$$
$$\therefore x = 7.$$

Ex. XIV.

1. $5x + 2 = 2x + 11.$

2. $\frac{x}{4} + \frac{x}{6} + \frac{x}{8} = 2\frac{1}{6}.$

3. $2x + a = 3x - b.$

4. $3(x-7) + 4x = 2(2x - 4) + 2.$

5. $\frac{x-1}{2} + \frac{x+3}{4} = \frac{2x-7}{6} + \frac{8x-1}{12}.$

6. $\frac{7x-8}{13} + \frac{3x-2}{7} = \frac{10x+3}{11} - \frac{7x-3}{18}.$

7. $\frac{4x-15}{9} - \frac{2x+3}{6} = \frac{5x-1}{12} - 3\frac{1}{18}.$

8. $\frac{8(5x+2)}{3} - \frac{2x-1}{8} = \frac{17x-2}{4} + \frac{5\frac{3}{8} + 80x}{7}.$

9. $ax + bc = bx + ac.$

10. $\frac{x}{a} + \frac{a}{b} = \frac{x}{b} + \frac{b}{a}.$

11. $\frac{x-a}{b} + \frac{x-b}{a} = 2 + \frac{cx - c^2}{ab}.$

12. $abx + b^3 = b^2x + a^3.$

13. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = ab + ac + bc.$

14. $\frac{x+b}{a} = \frac{a+x}{b}.$

15. $\frac{ax + bx + cx}{abc} = a + b + c.$

16. $x - \frac{a^2 - 3bx}{a^2} - b^2 = \frac{bx}{a} + \frac{6bx - 5a^2}{2a^2} - \frac{bx + 4a}{4a}.$

17. $\cdot 15x + \cdot 025 = \cdot 075x + \cdot 175.$

18. $\frac{3 - \cdot 125x}{8} + \cdot 1\dot{6} = \frac{2 + \cdot 1875x}{7} - \cdot 08\dot{3}.$

19. $\dfrac{1}{2x} + \dfrac{1}{12x} = \dfrac{1}{6x} + 1\tfrac{2}{3}.$

20. $(x + a)(x + b) = (x + c)(x + d).$

21. $(x - a)(x - b) = (x - \overline{a + b})^2.$

22. $\dfrac{1}{ax} + \dfrac{1}{bx} + \dfrac{1}{cx} = \tfrac{1}{2}(a + b + c)^2 - \tfrac{1}{2}\left(\dfrac{a}{bcx} + \dfrac{b}{acx} + \dfrac{c}{abx}\right).$

23. $\dfrac{x - a}{bc} + \dfrac{x - b}{ac} + \dfrac{x - c}{ab} = 2\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right).$

24. $\dfrac{1 - ax}{bc} + \dfrac{1 - bx}{ac} + \dfrac{1 - cx}{ab} = \left(\dfrac{2}{a} + \dfrac{2}{b} + \dfrac{2}{c}\right)x.$

Abbreviated Methods for Particular Cases.

62. When the unknown quantity is involved in both numerator and denominator, it is often convenient to reduce such fractions to *mixed numbers*.

Ex. Solve the equation $\dfrac{6x - 7}{x + 2} - \dfrac{12x + 18}{3x - 5} = 2.$

By division we get $\dfrac{6x - 7}{x + 2} = 6 - \dfrac{19}{x + 2};$

and $\dfrac{12x + 18}{3x - 5} = 4 + \dfrac{38}{3x - 5}.$

Hence the given equation becomes—

$$\left(6 - \frac{19}{x + 2}\right) - \left(4 + \frac{38}{3x - 5}\right) = 2;$$

$$\text{or } 6 - \frac{19}{x + 2} - 4 - \frac{38}{3x - 5} = 2;$$

$$\text{or, transposing, } -\frac{19}{x + 2} = \frac{38}{3x - 5} + 2 - 6 + 4;$$

$$\text{or } -\frac{19}{x + 2} = \frac{38}{3x - 5};$$

or, dividing each side by -19, we have—

$$\frac{1}{x + 2} = -\frac{2}{3x - 5}.$$

Hence, multiplying each side by the L.C.M. of the denominators—

$$3x - 5 = -2(x + 2) = -2x - 4,$$
$$\text{or } 3x + 2x = -4 + 5,$$
$$\text{or } 5x = 1,$$
$$\therefore x = \tfrac{1}{5}.$$

63. *When each side of an equation consists solely of a single fraction, the numerator of either fraction may change places with the denominator of the other.*

Let $\frac{a}{b} = \frac{p}{q}$ be the equation.

Multiply each side by b, then, by Art. 60 (1.)—

$$\frac{a}{b} \times b = \frac{p}{q} \times b, \text{ or } a = \frac{pb}{q}.$$

Divide each side by p, then, by Art. 60 (2.)—

$$\frac{a}{p} = \frac{pb}{q} \div p, \text{ or } \frac{a}{p} = \frac{b}{q}.$$

Here the denominator b of the first side of the given equation has changed places with the numerator p of the second side.

And similarly we may show that $\frac{q}{b} = \frac{p}{a}$, where the other numerator and denominator have changed places.

Cor. *The two sides of an equation of the form* $\frac{a}{b} = \frac{p}{q}$ *may be inverted.*

For interchanging p and b in the last result, viz., $\frac{q}{b} = \frac{p}{a}$, we get $\frac{q}{p} = \frac{b}{a}$, and therefore also, by Art. 60 (5.), we have $\frac{b}{a} = \frac{q}{p}$.

(The student is *cautioned* against inverting the *separate terms* of the two sides of an equation when there are more than one term on each side.)

64. When each side of an equation consists solely of a single fraction, we may perform the following operations :—

1. *We may add or subtract the numerator and denominator of* EACH *fraction for a new numerator or denominator, and retain either the original numerator or denominator for the other term of the fraction, both sides being always similarly treated.*

Thus, if $\frac{a}{b} = \frac{p}{q}$, we have—

$$\text{(i.)}\ \frac{a+b}{b} = \frac{p+q}{q}, \quad \text{(ii.)}\ \frac{a-b}{b} = \frac{p-q}{q},$$

$$\text{(iii.)}\ \frac{a+b}{a} = \frac{p+q}{p}, \quad \text{(iv.)}\ \frac{a-b}{a} = \frac{p-q}{p},$$

or, (v.) we may have equations formed by inverting each of these.

These results are easily obtained—

For, since $\frac{a}{b} = \frac{p}{q}$, we have, adding unity to each side—

$$\frac{a}{b} + 1 = \frac{p}{q} + 1 \text{ or } \frac{a+b}{b} = \frac{p+q}{q}.$$

And so, by subtracting unity from each side, we get—

$$\frac{a-b}{b} = \frac{p-q}{q}, \text{ and so on.}$$

2. *We may take the* SUMS *of the numerator and denominator of each for new numerators or denominators, and the* DIFFERENCES *for the other terms of the fraction; and* VICE VERSA, *both sides being always similarly treated.*

Thus, if $\frac{a}{b} = \frac{c}{d}$, we have also $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

and $\frac{a-b}{a+b} = \frac{c-d}{c+d}$,

for we have just shown that $\frac{a+b}{b} = \frac{p+q}{q}$,

and $\frac{a-b}{b} = \frac{p-q}{q}$.

Hence, dividing equals by equals, we get—

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{p+q}{q} \div \frac{p-q}{q},$$

or $\dfrac{a+b}{a-b} = \dfrac{p+q}{p-q}$, which is the first result.

And inverting each side, we have, by Art. 63 (Cor.)—

$$\frac{a-b}{a+b} = \frac{p-q}{p+q},$$

Ex. 1. Solve the equation $\dfrac{mx+a+b}{nx-c-d} = \dfrac{mx+a+c}{nx-b-d}$.

By Art. 63, we have $\dfrac{mx+a+b}{mx+a+c} = \dfrac{nx-c-d}{nx-b-d}$.

Then, by Art. 64 (1.), retaining the numerators and taking the *differences* for new denominators, we have—

$$\frac{mx+a+b}{b-c} = \frac{nx-c-d}{b-c};$$

or, multiplying each side by $(b-c)$—

$mx + a + b = nx - c - d$; or, transposing—

$$(m-n)x = -(a+b+c+d); \therefore x = -\frac{a+b+c+d}{m-n}.$$

Ex. 2. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = a.$

We may consider the quantity a as a fraction whose denominator is *unity*, or as $\dfrac{a}{1}$.

Then, Art. 64 (2.), taking the *sum* and *difference*, we have—

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}} = \frac{a+1}{a-1}; \text{ or—}$$

$$\frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{a+1}{a-1}; \text{ or, squaring—}$$

$$\frac{a+x}{a-x} = \frac{a^2+2a+1}{a^2-2a+1}.$$

Again, taking the *difference* and *sum*, we have—

$$\frac{2x}{2a} = \frac{4a}{2a^2 + 2}; \text{ or—}$$

$$\frac{x}{a} = \frac{2a}{a^2 + 1}.$$

$$\therefore x = \frac{2a^2}{a^2 + 1}.$$

65. We now give an example to show that sometimes the easy solution depends on an advantageous arrangement of the terms on the two sides of the equation.

Ex. Solve $\sqrt{x + 4} + \sqrt{x - 3} = 7$.

Transposing, we have—

$\sqrt{x + 4} = 7 - \sqrt{x - 3}$; squaring, then—

$$x + 4 = 49 - 14\sqrt{x - 3} + (x - 3);$$

subtracting x from each side and transposing, then—

$14\sqrt{x - 3} = 49 - 3 - 4 = 42.$

$\therefore \sqrt{x - 3} = 3$; or squaring,

$$x - 3 = 9; \quad \therefore x = 3 + 9 = 12.$$

Should the student commence by squaring at once, he will render the equation more complicated.

Ex. XVI.

1. $\dfrac{3x + 7}{x + 4} = \dfrac{3x - 13}{x - 4}.$

2. $(x - a)(x - b) = (x - c)(x - d).$

3. $\dfrac{3x + 13}{15} - \dfrac{3x + 10}{5x - 50} = \dfrac{x}{5}.$

4. $\dfrac{1 - 25x}{15} - \dfrac{3 - 2\frac{1}{2}x}{14(x-1)} = \dfrac{28 - 5x}{3} - \dfrac{10x - 11}{30} + \dfrac{x}{3}.$

5. $\dfrac{x - 4}{6x + 5} + \dfrac{3x - 13}{18x - 6} = \dfrac{1}{3}.$

6. $\dfrac{3}{1 - 2x} - \dfrac{5 - 2x}{7 - 2x} = 1 - \dfrac{4x^2 - 2}{7 - 16x + 4x^2}.$

7. $\dfrac{18x - 22}{13 - 2x} + 6x + \dfrac{1 + 16x}{8} = 13\tfrac{1}{4} - \dfrac{101 - 64x}{8}.$

8. $\dfrac{6x + 5}{3x + 1} + \dfrac{58\frac{1}{2} + 14x}{9 + 2x} = 9.$

9. $\dfrac{3}{4x - 9} + \dfrac{1\frac{1}{2}}{6x - 21} = \dfrac{1}{x - 2\frac{1}{2}}.$

10. $\dfrac{4x - 7}{2x - 9} + \dfrac{2 - 14x}{7} + \dfrac{3\frac{1}{3} + x}{14} = \dfrac{10 - 3\frac{6}{7}x}{2} - \dfrac{19}{21}.$

11. $\dfrac{6x^2 - 7x - 6\frac{2}{3}}{2x - 3} - \dfrac{9x^2 - 12x - 19}{3x - 5} = \dfrac{17}{6x - 7}.$

12. $\dfrac{ax + m + 1}{ax + m - 1} + \dfrac{ax + n}{ax + n - 2} = \dfrac{ax + m}{ax + m - 2} + \dfrac{ax + n + 1}{ax + n - 1}.$

13. $\dfrac{\sqrt{x - a + b} - \sqrt{x + a - b}}{\sqrt{x - a + b} + \sqrt{x + a - b}} = \dfrac{a - b}{a + b}.$

14. $\dfrac{1 - \sqrt{1 - \sqrt{1 - x}}}{1 + \sqrt{1 - \sqrt{1 - x}}} = b.$

15. $\sqrt{2x + 10} + \sqrt{2x - 2} = 6.$

16. $\sqrt{8 - x} - \dfrac{3}{\sqrt{1 - x}} = \sqrt{1 - x}.$

17. $\sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}} = 2.$

18. $\dfrac{ax + 1 + \sqrt{a^2x^2 - 1}}{ax + 1 - \sqrt{a^2x^2 - 1}} = b.$

19. $\dfrac{a}{\sqrt{x} - \sqrt{b}} - \dfrac{b}{\sqrt{x} - \sqrt{a}} = \dfrac{a - b}{\sqrt{x}}.$

20. $\dfrac{a}{\sqrt{x} + \sqrt{a}} = \dfrac{a}{\sqrt{x} - \sqrt{a}} + \sqrt{a}.$

21. $\sqrt{\dfrac{\sqrt{x} + a}{\sqrt{x} - a}} - \sqrt{\dfrac{\sqrt{x} - a}{\sqrt{x} + a}} = \sqrt{x - a^2}.$

22. $\frac{1}{x-2} - \frac{2 + \frac{5}{2}x^2 - \frac{1}{2}x^3}{6 - 5x + x^2} - \frac{1}{2}x = \frac{5}{x-3}$.

23. $\frac{4x+5}{x+1} + \frac{x+5}{x+4} = \frac{2x+5}{x+2} - \frac{x^2-10}{x+3} + x$.

24. $\frac{ax+b}{cx+d} + \frac{cx+e}{ax+f} = \frac{a^2+c^2}{ac}$.

Problems producing Simple Equations involving One Unknown Quantity.

66. To solve an algebraical problem we represent the required or unknown quantity by a letter, as x, and then express the given conditions in algebraical language. Thus we form an equation, the solution of which gives the required value of the unknown quantity.

Ex. 1. My purse and money are together worth 24 shillings, and the money is worth seven times the purse. Find the value of each.

Let x = the value in shillings of the purse,
Then $7x$ = " " money.

Now, by problem the value of both together is 24 shillings. Hence we have—

$$x + 7x = 24$$
$$\text{or } 8x = 24$$
$$\therefore x = 3, \text{ the value in shillings of the purse,}$$
$$\text{and } \therefore \text{ also } 7x = 7 \times 3 = 21, \quad " \quad \text{money.}$$

Ex. 2. What number is that to which, if 36 be added, the sum shall be equal to 3 times the number?

Let x = the number;
$\therefore x + 36$ = the sum when 36 is added,
and $3x$ = 3 times the number.

Hence, by problem—

$$x + 36 = 3x,$$
$$\text{or } x - 3x = -36,$$
$$\text{or } -2x = -36;$$
$$\therefore x = \frac{-36}{-2} = 18, \text{ the number required.}$$

Ex. 3. The distance between two towns is such that a train, whose speed is 30 miles an hour, takes 1 hour more in going 10 miles over 5 times the distance than a train whose speed is 20 miles an hour takes in going within 4 miles of 3 times the distance. Find the distance between the towns.

Let x = the distance required in miles.

Then $5x + 10 = 5$ times the distance together with 10 miles,

and $\frac{5x + 10}{30}$ = time in hours to travel this distance at 30 miles an hour.

And so, $\frac{3x - 4}{20}$ = time in hours to travel 4 miles less than 3 times the required distance, at 20 miles an hour.

But by the problem the former of these times exceed the latter by 1 hour.

Hence $\frac{5x + 10}{30} - \frac{3x - 4}{20} = 1.$

From this we easily find $x = 28$.

Hence 28 miles is the distance required.

Ex. 4. Find the price of an article, when as many can be bought for 1s. 4d. as can be bought for 2s. after the price has been raised 1d.

Let x = the price required in *pence;*

Then $\frac{16}{x}$ = number of articles bought for 1s. 4d.

And $x + 1$ = the raised price in *pence.*

$\therefore \frac{24}{x + 1}$ = number of articles bought for 2s. at the raised price.

But, by the problem, the number of articles in each case is the same.

Hence $\frac{16}{x} = \frac{24}{x + 1}$, from which $x = 2$.

Hence 2d. is the price required.

Ex. 5. A man sells geese at as many shillings each as the number he has, and having returned 5s., finds that if he had

had 2 more to sell on the same condition, and had returned 3s., he would have had 38s. more. How many had he?

Let x = the number required;

Then $x^2 - 5$ = number of shillings received.

Also, on the second supposition,

$(x + 2)^2 - 3$ = number of shillings he would have received.

Now, by the problem, this latter number is 38 more than the former.

Hence $(x + 2)^2 - 3 = x^2 - 5 + 38$, from which we find $x = 8$.

Ex. 6. A waterman finds that he can row 5 miles in $\frac{3}{4}$ hour with the tide, and that it takes him $1\frac{1}{2}$ hours to row the same distance against the tide when it is but half as strong. What is the velocity of the tide?

Let x = the velocity of the tide in miles per hour.

Now the velocity of the boat when going *with* the tide $= 5 \div \frac{3}{4} = \frac{20}{3}$.

$\therefore \frac{20}{3} - x$ = velocity of the boat when there is no tide.

Again, velocity of boat *against* the tide when it is half as strong $= 5 \div 1\frac{1}{2} = \frac{10}{3}$.

$\therefore \dfrac{10}{3} + \dfrac{x}{2}$ = velocity of the boat, when there is no tide.

Hence we have—

$$\frac{10}{3} + \frac{x}{2} = \frac{20}{3} - x; \text{ from which}$$
$$x = 2\tfrac{2}{9}.$$

$\therefore$ The velocity required is $2\frac{2}{9}$ miles per hour.

Ex. XVII.

1. If I add 25 to 3 times a certain number, I obtain the same result as if I subtract 25 from 8 times the number. Find the number.

2. Divide 70 into 2 such parts that the one shall be as much above half the number as the other is above 15.

3. Divide £720 among A, B, and C, so that B may have twice as much as C, and A as much as B and C together.

4. There are two trains, one of which goes 5 miles an hour faster than the other, and the former performs a journey of 100 miles, while the latter goes 75 miles. Find their respective rates.

5. A horse when let out for hire brings in *a clear gain* of 10s. per day, but costs 1s. 6d. daily for food. At the end of 30 days his master had gained £11. 11s. Required the number of days for which he was hired.

6. A and B have 4 guineas between them, and play at hazard. B loses $\frac{1}{6}$ of his money, and afterwards gains $\frac{1}{10}$ of what he then had. It is then seen that B has as much money as A had at the end of the first game. How much had each at first?

7. A and B have respectively an equal number of florins and crowns. B pays a debt of 4s. to A, and then A's money is just half B's. Find what each had.

8. A workman, instead of adopting the 9 hours' system, worked 10 hours daily, and had a corresponding rise of wages. By this means his wages were increased 4s. weekly. Find his original wages.

9. A person who has regular wages of 26s. weekly, thinking to better himself, takes a job at higher wages. He is, however, put on half-time during 20 weeks of the year, and finds himself at the end of the year £4. 12s. worse off. Required his increased wages.

10. A company of men, arranged in a hollow square 4 deep, numbered 144. What was the number in a side of the square?

11. In an examination paper there were two series of questions, and the questions of the second series carried each 3 marks more than those of the first series. A candidate who attempted 3 of the first series, obtaining half marks for them, and 5 of the second series, obtaining for these full marks, got altogether 80 marks. Find the number of marks attached to each question of the first series.

12. A grocer has tea at 3s. 4d. and at 4s. He sells altogether 64 pounds, thereby realizing £12. How much did he sell of each?

13. If 11 be subtracted from 5 times a certain number, and the remainder divided by 6, the quotient will exceed by 2 the quotient obtained by subtracting 3 from 4 times the number and dividing the remainder by 7. Find the number.

14. A garrison of 1,250 men were provisioned for 64 days; but after 22 days a certain number were called away, and it was found that the remaining provisions lasted the number left for 70 days. Find the number told off.

15. At a railway station £15 was taken for single fares, and £33. 15s. for returns. The number of return tickets exceeded the single tickets by 10, and the price of a return ticket was half as much again as a single ticket. Find the fare for a single journey.

16. In a tour lately made round the world, the distance travelled by water was 20,000 miles, and by land 8,000 miles; and the whole time taken was 220 days. Supposing the rate by water to be two-thirds of that by land, find the number of days travelled by land.

17. The distance between A and B is 32 miles. A person starting from A, at the rate of 4 miles an hour, meets another who started from B half an hour later, at a rate of $3\frac{1}{2}$ miles an hour. At what point will they meet?

18. There are two clocks, one of which gains twice as much per day as the second loses, and they are set right at noon on Monday. When it is noon on Thursday by the first clock, it is 11·50 A.M. by the second. What is the gain per day of the first clock?

19. A draper raises his goods a certain rate per cent., and afterwards reduces them to the original price by lowering them $13\frac{1}{23}$ per cent. Find the original rise per cent.

20. Required the distance between two towns such that a person can perform the journey one hour sooner when he walks 4 miles an hour than when he walks $3\frac{1}{2}$ miles an hour?

21. The sum of £12. 15s. is paid away with an equal number of sovereigns, crowns, and sixpences. Required the number of each.

22. A walks along an inclined plane at a certain rate, and B walks along the base of the plane at a rate of one-third of

a mile per hour less than A. The inclination of the plane is such that A is always vertically over B, and that at the end of half an hour they are exactly five-sixths of a mile apart. Find the respective rates of A and B.

23. There is a direct road over a hill between two stations at the foot of each side. The distance on the one side from the foot to the top is 5 miles, and the road down the other side forms a right angle with the road up. It is also known to be 1 mile less down the hill than the direct distance by tunnel between the two stations. Find the distance down the hill.

24. Two trains, whose respective lengths are 122 and 98 yards, and the former of which is going at the rate of 35 miles an hour, pass each other in 30 seconds. Find the rate and relative direction of the second train.

25. A man bought a number of sheep for £132, and having lost 10, and sold 20 that were diseased at 6s. per head below cost price, disposed of the remainder for £116, thereby realizing his outlay. How many did he buy?

26. A boy spends 10s. in oranges and apples. The oranges were bought at 5 for 6d., and the apples at 3 for 2d.; and their number together amounted to 132. What did he spend on each?

27. If B is allowed 2 hours more time than A takes to do a piece of work, he will do 4 times as much, and if C is allowed 1 hour more than A, he also can do 4 times as much. Moreover, D requires 4 hours more than A to do the piece of work. Also, the work done by A and B together is the same as that done by C and D together in the same time. Required the respective times for A, B, C, D to do a piece of work.

28. A person sells out £1,200 Three and a Half per Cent. stock, and invests the money in Two and a Half per Cents., whose price is 14 lower than the first-named stock. The loss in annual income is £7. Find the price of the first-named stock.

29. The banker's discount on a certain sum of money at 5 per cent. per annum is equal to the true discount on a sum £50 larger. Find the sum.

30. An express train, which ought to perform its journey in $2\frac{1}{2}$ hours, after having gone uniformly 80 miles, finds itself 6 minutes behind. However, by increasing the speed to as many miles per hour as there were miles in half the journey, it just arrived at its destination in time. Find the original speed of the train, and the length of the journey.

31. A vessel contains a quantity of spirit (sp. gr. ·9) and water, and a cylinder of wood (sp. gr. ·92), whose length is 10 inches, floats upright, so as to be just covered by the spirit. Find how much of the cylinder floats in the water.

32. A mixture of hydrogen and oxygen is found to condense when fired, to 16 vols. of steam. Now, every 3 vols. of such a mixture is known to condense to 2 vols., when the original gases are in the proportion of 2 : 1. Find the quantity of each.

33. A mixture of 100 grams of sodic and potassic sulphates yielded a gram of baric sulphate. Now, each gram of sodic sulphate yields b grams of baric sulphate, and each gram of potassic sulphate yields c grams of baric sulphate. Find the amount of sodic and potassic sulphates in the mixture.

34. If a oxen consume b acres of grass in c weeks, and a' oxen consume b' acres of grass in c' weeks, the grass growing uniformly, find the week's growth of an acre.

35. The freezing and boiling points of a common thermometer are marked 32° and 212° respectively; those on the centigrade thermometer are marked 0° and 100°. At what temperature do the graduations agree?

36. A person going at the rate of a miles per hour finds himself b hours too late when he has c miles farther to go. How much must he increase his speed to reach home in time?

Simultaneous Equations of the First Degree with two Unknown Quantities.

67. Suppose we have given the equation $3x - 4y = 5$, then by ascribing to y a series of values we get a corresponding series of values for x.

Thus we may have $\left.\begin{matrix} x = 3 \\ y = 1 \end{matrix}\right\}, \left.\begin{matrix} x = 7 \\ y = 4 \end{matrix}\right\}, \left.\begin{matrix} x = 11 \\ y = 7 \end{matrix}\right\}$, &c.

Again, if another equation, as $4x + y = 32$, be given, we may in the same way obtain a series of pairs of values which satisfy it. And, further, if the two equations are *distinct* and *compatible*, there is always *a pair of values common to the two equations*. This pair of values then satisfies both equations, and the equations are called **simultaneous** equations.

The methods of solving simultaneous equations will now be explained.

First Method.—*Equalize the coefficients of one of the unknown quantities in both equations, and add or subtract the equations so obtained, so as to obtain an equation with one unknown quantity.*

Ex.
$$\left.\begin{aligned} 4x + 3y &= 17 \quad \ldots\ldots\ldots (1) \\ 9x - 5y &= 3 \quad \ldots\ldots\ldots (2) \end{aligned}\right\}.$$

From (1), multiplying each side by 9, we get—
$$36x + 27y = 153 \quad \ldots\ldots\ldots (3).$$

And from (2), multiplying each side by 4, we get—
$$36x - 20y = 12 \quad \ldots\ldots\ldots (4).$$

(3) – (4), then $47y = 141$
$$\therefore y = 3.$$

Hence, substituting in (1), we have—
$$4x + 3 \times 3 = 17, \text{ from which we get—}$$
$$x = 2.$$

Hence, the solution required is $x = 2$, $y = 3$.

Second Method.—*Express one of the unknown quantities in terms of the other by means of either equation, and substitute its value in the other.*

Taking the same example, we have—
$$\left.\begin{aligned} 4x + 3y &= 17 \quad \ldots\ldots\ldots (1) \\ 9x - 5y &= 3 \quad \ldots\ldots\ldots (2) \end{aligned}\right\}.$$

From (1) we have $4x = 17 - 3y$, or $x = \dfrac{17 - 3y}{4} \ldots (3)$.

Substituting this value of x in (2) we have—
$$9\left(\frac{17 - 3y}{4}\right) - 5y = 3,$$
$$\text{or, } 153 - 27y - 20y = 12, \text{ from which—}$$
$$y = 3.$$

Then from (3) we get $x = \frac{17 - 3 \times 3}{4} = 2.$

THIRD METHOD.—*Express one of the unknown quantities in terms of the other by means of each equation, and equate the expression.*

We will this time express in each case y in terms of x.

$$\left.\begin{aligned} \text{We have } 4x + 3y &= 17 \ldots\ldots\ldots\ldots(1) \\ 9x - 5y &= 3 \ldots\ldots\ldots\ldots(2) \end{aligned}\right\}.$$

From (1) $3y = 17 - 4x$, or $y = \frac{17 - 4x}{3} \ldots(3).$

And from (2) $5y = 9x - 3$, or $y = \frac{9x - 3}{5} \ldots\ldots(4).$

Equating (3) and (4), then $\frac{17 - 4x}{3} = \frac{9x - 3}{5}$, from which

$$x = 2.$$

Then from (3) by substitution, $y = \frac{17 - 4 \times 2}{3} = 3.$

Simultaneous Equations of the First Degree of Three or more Unknown Quantities.

68. In the case of *three* unknown quantities we may obtain, from the three given equations, two equations with two unknown quantities, and then, by a similar method, from the two obtain an equation with one unknown quantity; and a like method may be pursued for more than three unknown quantities.

Ex.

$$\left.\begin{aligned} 3x + y + 4z &= 25 \ldots\ldots (1) \\ 4x + 3y - 5z &= -3 \ldots\ldots (2) \\ 6x + 7y - 8z &= 1 \ldots\ldots (3) \end{aligned}\right\}$$

From (1) $12x + 4y + 16z = 100 \ldots\ldots (4)$

And from (2) $12x + 9y - 15z = -9 \ldots\ldots (5)$

(5) − (4), then $5y - 31z = -109 \ldots\ldots (6)$

Again, from (1) $6x + 2y + 8z = 50 \ldots\ldots (7)$

(3) − (7), then $5y - 16z = -49 \ldots\ldots (8)$

(8) − (6), then $15z = 60$

$\therefore z = 4$

Hence from (8), by substitution—

$$5y - 16 \times 4 = -49,$$
$$\text{from which } y = 3.$$

And hence from (1), by substitution of the known values of y and z—

$$3x + 3 + 4 \times 4 = 25,$$
$$\text{from which } x = 2.$$

Hence the respective values of x, y, z, are 2, 3, 4.

Ex. XVIII.

1. $6x + y = 22$, $5x + 3y = 27$.
2. $4x - 3y = 14$, $6x + 5y = 40$.
3. $3x + 5y = 44$, $y - x = 4$.
4. $\frac{x}{4} + \frac{y}{3} = 2\frac{3}{3}$, $\frac{3x}{5} + \frac{y}{10} = 2\frac{9}{10}$.
5. $\frac{x}{6} + \frac{y}{4} = 4$, $\frac{x}{4} + \frac{y}{6} = 4\frac{1}{3}$.
6. $\frac{x}{8} + \frac{y}{6} = 1\frac{1}{4}$, $x + \frac{y}{3} = 4$.
7. $3x + 4y + z = 11$, $2x + y + 5z = 19$, $5x + 2y + 3z = 18$.
8. $7x + 2y + 3z = 20$, $3x - 4y + 2z = 1$, $5y - 2x + 7z = 29$.
9. $2x - 5y = 3$, $3y - 2z = -1$, $4x + 2z = 20$.
10. $ax + by = c$, $a_1x + b_1y = c_1$.
11. $x + y = a$, $y + x = b$, $x + z = c$.
12. $ax + by = d$, $by + cz = e$, $ax + cz = f$.
13. $\frac{3x - 2y}{4} + \frac{5x + 2y}{6} = 5\frac{1}{2}$, $\frac{x + 2y}{7} - \frac{3x - 2y}{2} = 1$.
14. $\frac{x + y}{10} + \frac{x - y}{15} = \frac{23}{30}$, $\frac{x + y}{15} + \frac{x - y}{10} = \frac{7}{30}$.
15. $\frac{3}{x} + \frac{9}{y} = 1\frac{7}{8}$, $\frac{15}{x} - \frac{11}{y} = 2\frac{3}{8}$.

16. $\dfrac{4}{x-2} + \dfrac{6}{y} = 5\tfrac{1}{2}$, $\dfrac{4}{y} - \dfrac{1}{x-2} = 0$.

17. $4(x + 3y) - 3(x + y) = \frac{1}{3}(x + y)(x + 3y)$,
$\dfrac{20}{x + y} + \dfrac{7}{x + 3y} = 3$.

18. $73y - 5x = (x - 5y)(x + 3y)$,
$\dfrac{2}{x - 5y} - \dfrac{5}{x + 3y} = \dfrac{7}{33}$.

19. $\dfrac{1}{x} + \dfrac{1}{y} = a$, $\dfrac{1}{y} + \dfrac{1}{z} = b$, $\dfrac{1}{x} + \dfrac{1}{z} = c$.

20. $3x - 2y = 0$, $4y - 3z = 0$, $5z - 6u = 14$, $7u - 2x = 3$.

21. $3x - 2y = 5z - 6y = 7x - 4z = 1$.

22. $x + y + z = a$, $y + z + u = b$, $z + u + x = c$, $u + x + y = d$.

23. $\dfrac{x}{m} + \dfrac{y}{n} = a$, $\dfrac{y}{n} + \dfrac{z}{p} = b$, $\dfrac{x}{m} + \dfrac{z}{p} = c$.

24. $x + ay + bz = \alpha$, $y + az + bx = \beta$, $z + ax + by = \gamma$.

25.
$$x + ay + a^2z + a^3 = 0.$$
$$x + by + b^2z + b^3 = 0.$$
$$x + cy + c^2z + c^3 = 0.$$

26.
$$x + ay + a^2z + a^3u + a^4 = 0.$$
$$x + by + b^2z + b^3u + b^4 = 0.$$
$$x + cy + c^2z + c^3u + c^4 = 0.$$
$$x + dy + d^2z + d^3u + d^4 = 0.$$

27.
$$x + ay = \alpha.$$
$$y + bz = \beta.$$
$$z + ct = \gamma.$$
$$t + dx = \delta.$$
$$u + ex = \epsilon.$$

28.
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = d$$
$$\frac{a'}{x} + \frac{b'}{y} + \frac{c'}{z} = d'$$
$$\frac{a''}{x} + \frac{b''}{y} + \frac{c''}{z} = d''$$

29.
$$x_1 + 2x_2 + 3x_3 + \&c. + nx_n = a_1.$$
$$x_2 + 2x_3 + 3x_4 + \&c. + nx_1 = a_2.$$
$$\&c. = \&c.$$
$$x_n + 2x_1 + 3x_2 + \&c. + nx_{n-1} = a_n.$$

30. $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=\dfrac{x}{b}+\dfrac{y}{c}+\dfrac{z}{a}=\dfrac{x}{c}+\dfrac{y}{a}+\dfrac{z}{b}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}.$

Problems producing Simultaneous Equations of the First Degree.

69. There are certain problems which may be solved with much greater facility by the introduction of more than one unknown quantity.

Ex. 1. Nine men and seven women receive together £3. 2s. 8d., and three men receive 10d. more than four women. Find the receipts of each.

Let x and y respectively represent in pence the receipts of a man and woman.

Then, since £3. 2s. 8d. = 752d., expressing the conditions of the problem in algebraical language—

$$\left.\begin{array}{l} 9x + 7y = 752 \\ 3x - 4y = 10 \end{array}\right\}\ \text{Solving these equations, we find—}$$

$$x = 54,\ y = 38.$$

Hence, the receipts of a man and woman are respectively 54d. and 38d., or 4s. 6d. and 3s. 2d.

Ex. 2. Find a fraction such that, if we diminish its numerator by 1, it becomes equal to $\frac{1}{7}$; and if we increase its denominator by 1, it becomes equal to $\frac{1}{6}$.

Let $\dfrac{x}{y}$ be the required fraction.

Then, by problem, $\left.\begin{array}{l} \dfrac{x-1}{y} = \dfrac{1}{7} \\ \text{and } \dfrac{x}{y+1} = \dfrac{1}{6} \end{array}\right\}$, which equations, when solved, give—

$$x = 6,\ y = 35.$$

$\therefore\ \dfrac{6}{35}$ is the fraction required.

Ex. 3. A's money, together with twice B's and thrice C's, amounts to £38; B's money, together with twice C's and thrice A's, to £35; and C's money, together with twice A's and thrice B's, also to £35. Find the money of each.

Let x, y, z be respectively the number of pounds each has. Then we have—

$$\left.\begin{aligned} x + 2y + 3z &= 38 \\ y + 2z + 3x &= 35 \\ z + 2x + 3y &= 35 \end{aligned}\right\} \text{; from which we get—}$$

$$x = 5,\ y = 6,\ z = 7.$$

Hence, £5, £6, £7 are the respective moneys of A, B, and C.

Ex. XIX.

1. A and B engage in play. A puts down half-a-crown to B's florin. They play twenty games, and then it is found that A has won 2s. How many games did each win?

2. There is a number, the sum of whose two digits is 10, and, if 36 be added to the number, the digits change places. Find the number.

3. A grocer has tea at 3s. 4d. a lb., and sugar at 4s. a stone. He sells £2 worth of the two. If he had raised the tea 10 per cent., and lowered the sugar 12½ per cent., and sold the same quantities of each, his profits would have been 1s. 6d. more. Find the quantity sold of each.

4. Eight times the numerator of a certain fraction exceeds three times the denominator by 3, and five times the numerator added to twice the denominator make 29. Find the fraction.

5. If the number of cows in a field were doubled, there would be 65 cows and horses together; but, if the number of horses be doubled, and that of the cows halved, there would be 46. How many are there of each?

6. Thirty shillings are spent in brandy, and 42s. in gin; 19 bottles being purchased in all. Had the 42s. been spent in brandy, and the 30s. in gin, 17 bottles only would have been bought. Find the cost per bottle of each.

7. A fishmonger receives 240 mackerel. He sells a certain number at 4 for a shilling, but the rest being seized as bad fish, and he being fined 10s., finds himself a loser by 9s. Had he sold them at 3 for a shilling, he would have been a gainer by 5s., if 13 more fish had been seized. How many did he sell, and what did he pay for the lot?

8. Three persons invest their money at 3, 4, 5 per cent. interest respectively. The total amount of interest is £38, and the interest of the first and third together is $2\frac{1}{6}$ that of the second; while the total interest would have been £34 had the rates been 5, 4, 3 per cent. respectively. Find the capital of each.

9. A toll-gate keeper receives 8s. 8d. for the toll of a number of horses, oxen, and sheep, the tolls for each being respectively $1\frac{1}{2}$d. 1d., $\frac{1}{2}$d. Had there been twice as many sheep and the number of horses diminished accordingly, he would have received 7s. 2d. Had the oxen passed through free, and the tolls for a horse and sheep respectively been 2d. and $\frac{3}{4}$d., he would have received 9s. $1\frac{1}{2}$d. Find the number of each.

10. A, B, C start from the same place. B after a quarter of an hour doubles his rate, while C, who falls, after ten minutes diminishes his rate $\frac{1}{6}$th. At the end of half an hour A is $\frac{1}{4}$ mile before B, and $\frac{1}{2}$ mile before C, and it is observed that the total distance which would have been walked by the three, had they each continued to walk uniformly from the first, is $6\frac{1}{4}$ miles. Find the original rate of each.

11. A, B, C, working 3, 4, 5 hours respectively, can do $2\frac{1}{24}$ pieces of work; if they each work an hour more, they can finish an extra $\frac{13}{24}$ of a piece; and, if C does not work, the other two, working for 1 and 6 hours respectively, can together finish 1 piece. Find the time required for A, B, C to finish separately a piece of work.

12. There are three numbers such that, if the first be increased by 6, and the second diminished by 5, the product of the results is the product of the first two numbers; if the second be increased by 2 and the third diminished by 3, the product of the results is the product of the second and third; and, if the first be increased by 3 and the third diminished by 6, the product is that of the first and third. Find the numbers.

13. A person performed a journey of $22\frac{1}{2}$ miles, partly by carriage at 10 miles an hour, partly by train at 36 miles an hour, and the remainder by walking at 4 miles an hour. He did the whole in 1 hour 50 minutes. Had he walked the

first portion, and performed the last by carriage, it would have taken him 2 hours $30\frac{1}{2}$ minutes. Find the respective distances by carriage, train, and walking.

14. A and B start from two places C and D, distant 28 miles, and it is found that A reaches D 3 hours after they meet. Had the distance between C and D been 35 miles, A would have reached a point 28 miles from C 2 hours after he met B. Find the respective rates of A and B.

15. Three trains—a luggage, ordinary, and express—move along three parallel pairs of rails, the distance between the stations being 120 miles. The first two start from the same station, and the express from the opposite. The luggage train, starting 2 hours first, is overtaken by the ordinary in 2 hours; and the express train, starting 1 hour after the ordinary, meets the luggage in 1 hour $7\frac{1}{2}$ minutes. Had all three started from the same station, the ordinary would have been overtaken in 2 hours. Find the respective rates of the trains.

16. If (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) be the respective compositions by weight of three mixtures of three substances, and d_1, d_2, d_3 be the respective prices of the mixtures, find the price per unit of weight of each substance.

17. By alloying two ingots of gold in two given proportions, we form two new ingots of which the fineness of each is known. What is the fineness of each of the given ingots?

18. A group of n persons play as follows:—The first plays with the second and loses $\frac{1}{a}$ of what he has, the second then plays with the third and loses $\frac{1}{a}$ of what he has, the third then with the fourth, losing $\frac{1}{a}$ of what he has, &c., the nth with the first losing $\frac{1}{a}$ of what he has. At the end they each have b. What had each at first?

MATHEMATICS.

SECOND STAGE.

SECTION I.

GEOMETRY.

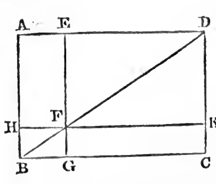
EUCLID'S ELEMENTS, BOOK II.

Definitions.

1. A **rectangle,** or **right-angled parallelogram,** is said to be contained by any two of the straight lines which contain one of the right angles.

2. **In any parallelogram,** the **figure** which is composed of either of the parallelograms about a diameter, together with the two complements, is called a *gnomon.*

Thus the parallelogram HG, together with the complements AF, FC, is a gnomon, which is briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



The rectangle under, or contained by two lines, as AB and BC, is concisely expressed thus:—AB, BC.

Proposition 1.—Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E;

The rectangle contained by the straight lines A and BC shall be equal to the rectangle contained by A and BD, together with that contained by A and DE, and that contained by A and EC. A · BC = A · BD + A · DE + A · EC.

Construction.—From the point B draw BF at right angles to BC (I. 11),

And make BG equal to A (I. 3).

Through G draw GH parallel to BC (I. 31).

And through the points D, E, C draw DK, EL, CH parallel to BG (I. 31).

Proof.—Then the rectangle BH is equal to the rectangles BK, DL, EH. Since BH = BK + DL + EH.

But BH is contained by A and BC, for it is contained by GB and BC, and GB is equal to A (Const.);

And BK is contained by A and BD, for it is contained by GB and BD, and GB is equal to A;

And DL is contained by A and DE, because DK is equal to BG, which is equal to A (I. 34);

And in like manner EH is contained by A and EC;

Therefore the rectangle contained by A and BC is equal to the several rectangles contained by A and BD, by A and DE, and by A and EC.

Therefore, if there be two straight lines, &c. *Q. E. D.*

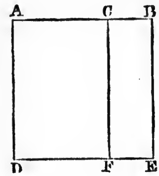
Proposition 2.—Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole line and each of its parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts in the point C;

AB · BC + AB · AC = AB².

The rectangle contained by AB and BC, together with the rectangle contained by AB and AC, shall be equal to the square on AB.



CONSTRUCTION.—Upon AB describe the square ADEB (I. 46).

Through C draw CF parallel to AD or BE (I. 31).

PROOF.—Then AE is equal to the rectangles AF and CE.

But AE is the square on AB;

For AB² is the sum of its parts AF + CE.

Therefore the square on AB is equal to the rectangles AF and CE.

And AF is the rectangle contained by BA and AC, for it is contained by DA and AC, of which DA is equal to BA;

And CE is contained by AB and BC, for BE is equal to AB.

And ∴ = AB · AC + AB · BC.

Therefore the rectangle AB, AC, together with the rectangle AB, BC, is equal to the square on AB.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 3.—Theorem.

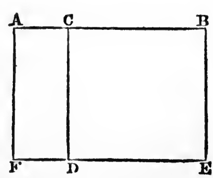
If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts in the point C;

AB · BC = BC² + AC · BC.

The rectangle AB · BC shall be equal to the square on BC, together with the rectangle AC · CB.

CONSTRUCTION.—Upon BC describe the square CDEB (I. 46).



Produce ED to F; and through A draw AF parallel to CD or BE (I. 31).

For AE = AD + CE.

PROOF.—Then the rectangle AE is equal to the rectangles AD and CE.

But AE is the rectangle contained by AB and BC, for it is contained by AB and BE, of which BE is equal to BC;

And AD is contained by AC and CB, for CD is equal to CB;

And CE is the square on BC.

Therefore the rectangle AB, BC is equal to the square on BC, together with the rectangle AC, CB.

Therefore, if a straight line, &c. *Q. E. D.*

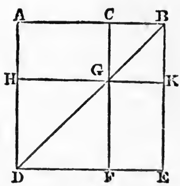
Proposition 4.—Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C;

The square on AB shall be equal to the squares on AC and CB, together with twice the rectangle contained by AC and CB. $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$.

Construction.—Upon AB describe the square ADEB (I. 46), and join BD.



Through C draw CGF parallel to AD or BE (I. 31).

Through G draw HGK parallel to AB or DE (I. 31).

Proof.—Because CF is parallel to AD, and BD falls upon them,

Therefore the exterior angle BGC is equal to the interior and opposite angle ADB (I. 29).

Because AB is equal to AD, being sides of a square, the angle ADB is equal to the angle ABD (I. 5);

Therefore the angle CGB is equal to the angle CBG (Ax. 1); Show first that CK is a square $= CB^2$.

Therefore the side BC is equal to the side CG (I. 6).

But CB is also equal to GK, and CG to BK (I. 34);

Therefore the figure CGKB is equilateral.

It is likewise rectangular.

For since CG is parallel to BK, and CB meets them, the angles KBC and GCB are together equal to two right angles (I. 29).

But KBC is a right angle (Const.), therefore GCB is a right angle (Ax. 3).

Therefore also the angles CGK, GKB, opposite to these, are right angles (I. 34).

Therefore CGKB is rectangular; and it has been proved equilateral; therefore it is a square; and it is upon the side CB.

So also $HF = AC^2$

For the same reason HF is also a square, and it is on the side HG, which is equal to AC (I. 34).

Therefore HF and CK are the squares on AC and CB.

And because the complement AG is equal to the complement GE (I. 43),

And $AG + GE = 2AC \cdot CB$.

And that AG is the rectangle contained by AC and CG, that is, by AC and CB,

Therefore GE is also equal to the rectangle AC, CB;

Therefore AG, GE are together equal to twice the rectangle AC, CB;

And HF, CK are the squares on AC and CB.

Therefore the four figures HF, CK, AG, GE are equal to the squares on AC and CB, together with twice the rectangle AC, CB.

But HF, CK, AG, GE, make up the whole figure ADEB, which is the square on AB;

∴ whole figure or $AB^2 = AC^2 + BC^2, + 2AC \cdot CB$.

Therefore the square on AB is equal to the squares on AC and CB and twice the rectangle $AC \cdot CB$.

Therefore, if a straight line, &c. *Q. E. D.*

Corollary.—From this demonstration it follows that the parallelograms about the diameter of a square are likewise squares.

Proposition 5.—Theorem.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be bisected in C, and divided unequally in D;

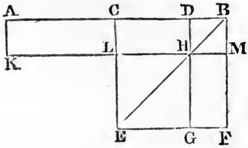
$AD \cdot DB + CD^2 = CB^2$.

The rectangle AD, BD, together with the square on CD, shall be equal to the square on CB.

Construction.—Upon CB describe the square CEFB (I. 46), and join BE, Through D draw DHG parallel to CE or BF (I. 31).

Through H draw KLM parallel to CB or EF.

And through A draw AK parallel to CL or BM.



Proof.—Then the complement CH is equal to the complement HF (I. 43).

To each of these add DM; therefore the whole CM is equal to the whole DF (Ax. 2).

But CM is equal to AL (I. 36), because AC is equal to CB (Hyp.); For AL=CM =DF.

Therefore also AL is equal to DF (Ax. 1).

To each of these add CH; therefore the whole AH is equal to DF and CH (Ax. 2). ∴ AH= DF+CH.

But AH is contained by AD and BD, since DH is equal to DB (II. 4, cor.),

And DF, together with CH, is the gnomon CMG;

Therefore the gnomon CMG is equal to the rectangle AD, DB. ∴ CMG. =AD · DB.

To each of these equals add LG, which is equal to the square on CD (II. 4, cor., and I · 34); Add to each LG or CD^2.

Therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square on CD.

But the gnomon CMG and LG make up the whole figure CEFB, which is the square on CB;

Therefore the rectangle AD, DB, together with the square on CD, is equal to the square on CB. ∴ CB^2 =AD · DB $+CD^2$.

Therefore, if a straight line, &c. *Q. E. D.*

Corollary.—From this proposition it is manifest that the difference of the squares on two unequal lines AC, CD is equal to the rectangle contained by their sum and difference.

Proposition 6.—Theorem.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D;

AD · DB + CB² = CD².

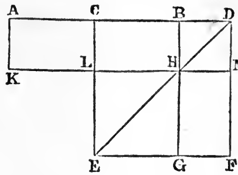
The rectangle AD, DB, together with the square on CB, shall be equal to the square on CD.

Construction.—Upon CD describe the square CEFD (I. 46), and join DE.

Through B draw BHG parallel to CE or DF (I. 31).

Through H draw KLM parallel to AD or EF.

And through A draw AK parallel to CL or DM.



For AL = CH = HF.

Proof.—Because AC is equal to CB (Hyp.), the rectangle AL is equal to CH (I. 36).

But CH is equal to HF (I. 43), therefore AL is equal to HF (Ax. 14).

∴ AM or AD · DB = CMG.

To each of these add CM; therefore the whole AM is equal to the gnomon CMG (Ax. 2).

But AM is the rectangle contained by AD and DB, since DM is equal to DB (II. 4, cor.);

Therefore the gnomon CMG is equal to the rectangle AD, DB (Ax. 1).

Add to each LG or CB².

Add to each of these LG, which is equal to the square on CB (II. 4, cor., and I. 34);

Therefore the rectangle AD, DB, together with the square on CB, is equal to the gnomon CMG and the figure LG.

But the gnomon CMG and LG make up the whole figure CEFD, which is the square on CD;

∴ AD · DB + CB² = CD².

Therefore the rectangle AD, DB, together with the square on CB, is equal to the square on CD.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 7.—Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts in the point C;

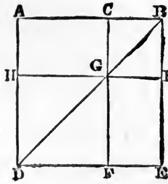
AB² + BC² = 2 AB · BC + AC².

The squares on AB and BC shall be equal to twice the rectangle AB, BC, together with the square on AC.

CONSTRUCTION.—Upon AB describe the square ADEB (I. 46), and join BD.

Through C draw CGF parallel to AD or BE (I. 31).

Through G draw HGK parallel to AB or DE (I. 31).



PROOF.—Then AG is equal to GE (I. 43).

To each of these add CK; therefore the whole AK is equal to the whole CE; For AK=CE.

Therefore AK and CE are double of AK.

But AK and CE are the gnomon AKF, together with the square CK;

Therefore the gnomon AKF, together with the square CK, is double of AK. ∴ AKF + CK = 2AB·BC.

But twice the rectangle AB, BC is also double of AK, for BK is equal to BC (II. 4, cor.);

Therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC.

To each of these equals add HF, which is equal to the square on AC (II. 4, cor., and I. 34); Add HF or AC² to each equal.

Therefore the gnomon AKF, together with the squares CK and HF, is equal to twice the rectangle AB, BC, together with the square on AC.

But the gnomon AKF, together with the squares CK and HF, make up the whole figure ADEB and CK, which are the squares on AB and BC; ∴ $AB^2 + BC^2 = 2\,AB \cdot BC + AC^2$.

Therefore the squares on AB and BC are equal to twice the rectangle AB, BC, together with the square on AC.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 8.—Theorem.

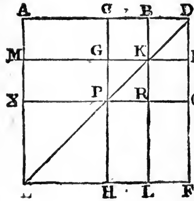
If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole line and the first mentioned part.

Let the straight line AB be divided into any two parts in the point C;

$4\,AB \cdot BC + AC^2 = (AB+BC)^2$. Four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.

CONSTRUCTION.—Produce AB to D, so that BD may be equal to CB (Post. 2, and I. 3).

Upon AD describe the square AEFD (I. 46),



And construct two figures such as in the preceding propositions.

PROOF.—Because CB is equal to BD (Const.), CB to GK, and BD to KN (Ax. 1),

For the same reason PR is equal to RO.

For BN=CK =GR=RN. And because CB is equal to BD, and GK to KN,

Therefore the rectangle CK is equal to BN, and GR to RN (I. 36).

But CK is equal to RN, because they are the complements of the parallelogram CO (I. 43);

Therefore also BN is equal to GR (Ax. 1).

∴ the four together = 4 CK, Therefore the four rectangles BN, CK, GR, RN are equal to one another, and so the four are quadruple of one of them, CK.

And the rectangles AG, MP, PL, RF, are equal to each other, and are together = 4 AG. Again, because CB is equal to BD (Const.);

And that BD is equal to BK, that is CG (II. 4, Cor., and I. 34);

And that CB is equal to GK, that is GP (I. 34, and II. 4, cor.);

Therefore CG is equal to GP (Ax. 1).

And because CG is equal to GP, and PR to RO,

The rectangle AG is equal to MP, and PL to RF (I. 36).

But MP is equal to PL, because they are complements of the parallelogram ML (I. 43), and AG is equal to RF (Ax. 1);

Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so the four are quadruple of one of them, AG.

And it was demonstrated that the four CK, BN, GR, and RN are quadruple of CK;

Therefore the eight rectangles which make up the gnomon AOH are quadruple of AK.

And because AK is the rectangle contained by AB and BC, for BK is equal to BC; ∴ Gnomon AOH = 4(CK+AG) = 4 AK = 4 AB·BC

Therefore four times the rectangle AB, BC is quadruple of AK.

But the gnomon AOH was demonstrated to be quadruple of AK;

Therefore four times the rectangle AB, BC is equal to the gnomon AOH (Ax. 1).

To each of these add XH, which is equal to the square on AC (II. 4, cor., and I. 34); Hence adding XH or AC^2,

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the gnomon AOH and the square XH.

But the gnomon AOH and the square XH make up the figure AEFD, which is the square on AD;

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD, that is, on the line made up of AB and BC together. $4\,AB\cdot BC + AC^2 = AF = (AB+BC)^2$.

Therefore, if a straight line, &c. *Q. E. D.*

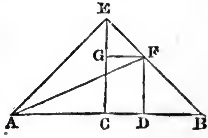
Proposition 9.—Theorem.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D;

The squares on AD and DB shall be together double of the squares on AC and CD. $AD^2+DB^2 = 2\,(AC^2+CD^2)$.

Construction.—From the point C draw CE at right angles to AB (I. 11), and make it equal to AC or CB (I. 3), and join EA, EB.



Through D draw DF parallel to CE (I. 31).

Through F draw FG parallel to BA (I. 31), and join AF.

Proof.—Because AC is equal to CE (Const.), the angle EAC is equal to the angle AEC (I. 5).

And because the angle ACE is a right angle (Const.), the angles AEC and EAC together make one right angle (I. 32), and they are equal to one another;

For AEC=½ a right ∠ = CEB.

Therefore each of the angles AEC and EAC is half a right angle.

For the same reason, each of the angles CEB and EBC is half a right angle;

∴ AEB is a right ∠.

Therefore the whole angle AEB is a right angle.

And because the angle GEF is half a right angle, and the angle EGF a right angle, for it is equal to the interior and opposite angle ECB (I. 29),

Therefore the remaining angle EFG is half a right angle;

Therefore the angle GEF is equal to the angle EFG, and the side EG is equal to the side GF (I. 6).

EG=GF.

Again, because the angle at B is half a right angle, and the angle FDB a right angle, for it is equal to the interior and opposite angle ECB (I. 29),

Therefore the remaining angle BFD is half a right angle;

Therefore the angle at B is equal to the angle BFD, and the side DF is equal to the side DB (I. 6).

And DF=DB.

And because AC is equal to CE (Const.), the square on AC is equal to the square on CE;

Therefore the squares on AC and CE are double of the square on AC.

But $AE^2 = 2AC^2$.

But the square on AE is equal to the squares on AC and CE, because the angle ACE is a right angle (I. 47);

Therefore the square on AE is double of the square on AC.

Again, because EG is equal to GF (Const.), the square on EG is equal to the square on GF;

Therefore the squares on EG and GF are double of the square on GF.

So also $EF^2 = 2GF^2 = 2\,CD^2$.

But the square on EF is equal to the squares on EG and GF, because the angle EGF is a right angle (I. 47);

Therefore the square on EF is double of the square on GF.

And GF is equal to CD (I. 34);

Therefore the square on EF is double of the square on CD.

But it has been demonstrated that the square on AE is also double of the square on AC;

∴ $AE^2 + EF^2 = 2(AC^2 + CD^2)$,

Therefore the squares on AE and EF are double of the squares on AC and CD.

But the square on AF is equal to the squares on AE and EF, because the angle AEF is a right angle (I. 47); But $AE^2+EF^2 = AF^2 = AD^2+DF^2 = AD^2+DB^2$.

Therefore the square on AF is double of the squares on AC and CD.

But the squares on AD and DF are equal to the square on AF, because the angle ADF is a right angle (I. 47);

Therefore the squares on AD and DF are double of the squares on AC and CD.

And DF is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD. $\therefore AD^2+DB = 2(AC^2+CD^2)$

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 10.—Theorem.

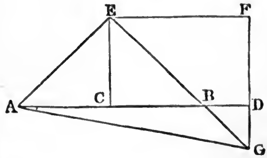
If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D; $AD^2+DB^2 = 2(AC^2+CD^2)$.

The squares on AD and DB shall be together double of the squares on AC and CD.

Construction.—From the point C draw CE at right angles to AB, and make it equal to AC or CB (I. 11, I. 3), and join AE, EB.

Through E draw EF parallel to AB, and through D draw DF parallel to CE (I. 31). Then because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal to two right angles (I. 29); therefore the angles BEF, EFD are less than two right angles; therefore EB, FD will meet, if produced towards B and D (Ax. 12).



Let them meet in G, and join AG.

Proof.—Because AC is equal to CE (Const.), the angle CEA is equal to the angle EAC (I. 5).

And the angle ACE is a right angle; therefore each of the angles CEA and EAC is half a right angle (I. 32).

As in Prop. 9.

For the same reason each of the angles CEB and EBC is half a right angle;

AEB is a right ∠.

Therefore the whole angle AEB is a right angle.

And because the angle EBC is half a right angle, the angle DBG, which is vertically opposite, is also half a right angle (I. 15);

But the angle BDG is a right angle, because it is equal to the alternate angle DCE (I. 29);

Therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG;

And BD=DG.

Therefore also the side BD is equal to the side DG (I. 6).

Again, because the angle EGF is half a right angle, and the angle at F a right angle, for it is equal to the opposite angle ECD (I. 34);

Therefore the remaining angle FEG is half a right angle (I. 32), and therefore equal to the angle EGF;

Also GF=FE.

Therefore also the side GF is equal to the side FE (I. 6).

And because EC is equal to CA, the square on EC is equal to the square on CA;

Therefore the squares on EC and CA are double of the square on CA.

Again as in Prop. 9.

But the square on AE is equal to the squares on EC and CA (I. 47);

$AE^2 = 2\,AC^2$.

Therefore the square on AE is double of the square on AC.

Again, because GF is equal to FE, the square on GF is equal to the square on FE;

Therefore the squares on GF and FE are double of the square on FE.

But the square on EG is equal to the squares on GF and FE (I. 47);

Therefore the square on EG is double of the square on FE.

And FE is equal to CD (I. 34),

And $EG^2 = 2\,CD^2$.

Therefore the square on EG is double of the square on CD.

But it has been demonstrated that the square on AE is double of the square on AC;

$\therefore AE^2 + E\ ^2 = 2(AC^2 + CD^2)$.

Therefore the squares on AE and EG are double of the squares on AC and CD.

But $AE^2 + EG^2 = AG^2$.

But the square on AG is equal to the squares on AE and EG (I. 47);

Therefore the square on AG is double of the squares on AC and CD. $\therefore AG^2 = AD^2 + DB^2$.

But the squares on AD and DG are equal to the square on AG (I. 47);

Therefore the squares on AD and DG are double of the squares on AC and CD. $\therefore AD^2 + DB^2 = 2(AC^2 + CD^2)$.

And DG is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

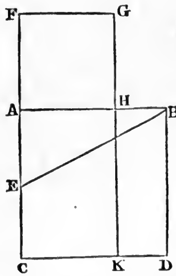
Therefore, if a straight line, &c. *Q.E.D.*

Proposition 11.—Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.

Let AB be the given straight line.

It is required to divide AB into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.



Construction.—Upon AB describe the square ABDC (I. 46).

Bisect AC in E (I. 10), and join BE.

Produce CA to F, and make EF equal to EB (I. 3).

Upon AF describe the square AFGH (I. 46).

Produce GH to K.

Then AB shall be divided in H, so that the rectangle AB, BH is equal to the square on AH.

Proof.—Because the straight line AC is bisected in E, and produced to F,

The rectangle CF, FA, together with the square on AE, is equal to the square on EF (II. 6). $CF \cdot FA + AE^2 = EF^2 = EB^2$

But EF is equal to EB (Const.);

Therefore the rectangle CF, FA, together with the square on AE, is equal to the square on EB.

But the square on EB is equal to the squares on AE and AB, because the angle EAB is a right angle (I. 47); $= AB^2 + AE^2$.

Therefore the rectangle CF, FA, together with the square on AE, is equal to the squares on AE and AB.

Take away the square on AE, which is common to both;

$CF \cdot FA = AB^2$.

Therefore the remaining rectangle CF, FA is equal to the square on AB (Ax. 3).

But the figure FK is the rectangle contained by CF and FA, for FA is equal to FG;

And AD is the square on AB;

$\therefore$ FK=AD

Therefore the figure FK is equal to AD.

Take away AK, then FH=HD or $AB \cdot BH = AH^2$.

Take away the common part AK, and the remainder FH is equal to the remainder HD (Ax. 3).

But HD is the rectangle contained by AB and BH, for AB is equal to BD;

And FH is the square on AH;

Therefore the rectangle AB,BH is equal to the square on AH.

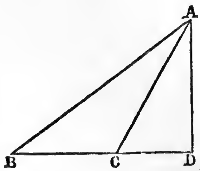
Therefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square on AH. *Q.E.F.*

Proposition 12.—Theorem.

In obtuse-angled triangles if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB; and from the point A let AD be drawn perpendicular to BC produced.

For $AB^2 = AC^2 + CB^2 + 2\,BC \cdot CD$.



The square on AB shall be greater than the squares on AC and CB by twice the rectangle BC, CD.

Proof.—Because the straight line BD is divided into two parts in the point C,

$BD^2 = BC^2 + CD^2 + 2\,BC \cdot CD$.

The square on BD is equal to the squares on BC and CD, and twice the rectangle BC, CD (II. 4).

To each of these equals add the square on DA;

Therefore the squares on BD and DA are equal to the squares on BC, CD, DA, and twice the rectangle BC, CD.

But the square on BA is equal to the squares on BD and DA, because the angle at D is a right angle (I. 47);

And the square on CA is equal to the squares on CD and DA (I. 47);

$\therefore BD^2+DA^2$ or $AB^2 = BC^2 + (CD^2 + DA^2) + 2BC \cdot CD = BC^2 + AC^2 + 2BC \cdot CD$.

Therefore the square on BA is equal to the squares on BC and CA, and twice the rectangle BC, CD; that is, the square on BA is greater than the squares on BC and CA by twice the rectangle BC, CD.

Therefore, in obtuse-angled triangles, &c. *Q.E.D.*

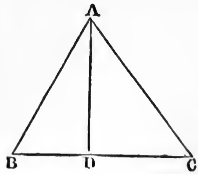
Proposition 13.—Theorem.

In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

Let ABC be any triangle, and the angle at B an acute angle; and on BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle (I. 12).

The square on AC, opposite to the angle B, shall be less than the squares on CB and BA, by twice the rectangle CB, BD.

$AC^2 = CB^2 + AB^2 - 2CB \cdot BD$.



Case I.—First, let AD fall within the triangle ABC.

Proof.—Because the straight line CB is divided into two parts in the point D,

The squares on CB and BD are equal to twice the rectangle contained by CB, BD, and the square on DC (II. 7).

For $CB^2 + BD^2 = 2CB \cdot BD + DC^2$.

To each of these equals add the square on DA.

Therefore the squares on CB, BD, DA are equal to twice the rectangle CB, BD, and the squares on AD and DC.

But the square on AB is equal to the squares on BD and DA, because the angle BDA is a right angle (I. 47);

And the square on AC is equal to the squares on AD and DC (I. 47);

$\therefore CB^2 + (BD^2 + DA^2) = 2CB \cdot BD + (AD^2 + DC^2)$, or $CB^2 + BA^2 = 2CB \cdot DB + AC^2$.

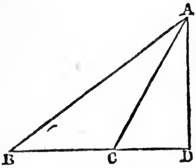
Therefore the squares on CB and BA are equal to the

$\therefore AC^2 = CB^2 + BA^2 - 2CB \cdot BD.$

square on AC, and twice the rectangle CB, BD; that is, the square on AC alone is less than the squares on CB and BA by twice the rectangle CB, BD.

Case II.—Secondly, let AD fall without the triangle ABC.

Proof.—Because the angle at D is a right angle (Const.), the angle ACB is greater than a right angle (I. 16);



$AB^2 = AC^2 + CB^2 + BC \cdot CD.$

Therefore the square on AB is equal to the squares on AC and CB, and twice the rectangle BC, CD (II. 12).

To each of these equals add the square on BC.

$\therefore AB^2 + BC^2 = AC^2 + 2(BC^2 + BC \cdot CD).$

Therefore the squares on AB and BC are equal to the square on AC, and twice the square on BC, and twice the rectangle BC, CD (Ax. 2).

But because BD is divided into two parts at C,

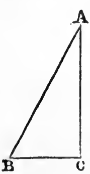
The rectangle DB, BC is equal to the rectangle BC, CD and the square on BC (II. 3);

Now $DB \cdot BC = BC \cdot CD + BC^2.$

And the doubles of these are equal, that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square on BC;

$\therefore AC^2 = AB^2 + BC^2 - 2BC \cdot DB.$

Therefore the squares on AB and BC are equal to the square on AC, and twice the rectangle DB, BC; that is, the square on AC alone is less than the squares on AB and BC by twice the rectangle DB, BC.



Case III.—Lastly, let the side AC be perpendicular to BC.

Proof.—Then BC is the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares on AB and BC are equal to the square on AC, and twice the square on BC (I. 47, and Ax. 2).

Therefore, in every triangle, &c. *Q.E.D.*

Proposition 14.—Problem.

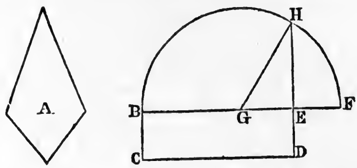
To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure.

It is required to describe a square that shall be equal to A.

Construction.—Describe the rectangular parallelogram BCDE equal to the rectilineal figure A (I. 45).

If then the sides of it, BE, ED, are equal to one another it is a square, and what was required is now done.



But if they are not equal, produce one of them, BE, to F, and make EF equal to ED (I. 3).

Bisect BF in G (I. 10), and from the centre G, at the distance GB, or GF, describe the semicircle BHF;

Produce DE to H, and join GH;

Then the square described upon EH shall be equal to the rectilineal figure A.

Proof.—Because the straight line BF is divided into two equal parts in the point G, and into two unequal parts in the point E,

The rectangle BE,EF, together with the square on GE, is equal to the square on GF (II. 5). **$BE \cdot EF + GE^2 = GF^2 = GH^2 = GE^2 + EH^2$.**

But GF is equal to GH;

Therefore the rectangle BE,EF, together with the square on GE, is equal to the square on GH.

But the square on GH is equal to the squares on GE and EH (I. 47);

Therefore the rectangle BE,EF, together with the square on GE, is equal to the squares on GE and EH. **$\therefore BE \cdot EF$ or $BD = EH^2$.**

Take away the square on GE, which is common to both;

Therefore the rectangle BE,EF is equal to the square on EH (Ax. 3).

But the rectangle contained by BE and EF is the parallelogram BD, because EF is equal to ED (Const.);

Therefore BD is equal to the square on EH.

But BD is equal to the rectilineal figure A (Const.);

Therefore the square on EH is equal to the rectilineal figure A. **Hence $EH^2 = A$.**

Therefore, a square has been made equal to the given rectilineal figure A, viz., the square described on EH. *Q.E.F.*

EXERCISES ON BOOK II.

Prop. 1—11.

1. Divide a given straight line into two such parts that the rectangle contained by them may be the greatest possible.

2. The sum of the squares of two straight lines is never less than twice the rectangle contained by the straight lines.

3. Divide a given straight line into two parts such that the squares of the whole line and one of the parts shall be equal to twice the square of the other part.

4. Given the sum of two straight lines and the difference of their squares, to find the lines.

5. In any triangle the difference of the squares of the sides is equal to the rectangle contained by the sum and difference of the parts into which the base is divided by a perpendicular from the vertical angle.

6. Divide a given straight line into such parts that the sum of their squares may be equal to a given square.

7. If ABCD be any rectangle, A and C being opposite angles, and O any point either within or without the rectangle—$OA^2 + OC^2 = OB^2 + OD^2$.

8. Let the straight line AB be divided into any two parts in the point C. Bisect CB in D, and take a point E in AC such that EC = CD. Then shall $AD^2 = AE^2 + AC \cdot CB$.

9. If a point C be taken in AB, and AB be produced to D so that BD and AC are equal, show that the squares described upon AD and AC together exceed the square upon AB by twice the rectangle contained by AE and AC.

10. From the hypothenuse of a right-angled triangle portions are cut off equal to the adjacent sides. Show that the square on the middle segment is equal to twice the rectangle under the extreme segments.

11. If a straight line be divided into any number of parts, the square of the whole line is equal to the sum of the squares of the parts, together with twice the rectangles of the parts taken two and two together.

12. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC, cutting in D and E either the side or sides produced, and EB be joined; prove that $BE^2 = BC \cdot DE + CE^2$.

Prop. 12—14.

13. In any triangle show that the sum of the squares on the sides is equal to twice the square on half the base, and twice the square on the line drawn from the vertex to the middle of the base.

14. If squares are described on the sides of any triangle, find the difference between the sum of two of the squares and the third square, and show from your result what this becomes when the angle opposite the third square is a right angle.

15. Show also what the difference becomes when the vertex of the triangle is depressed until it coincide with the base.

16. The square on any straight line drawn from the vertex of an isosceles triangle, together with the rectangle contained by the segments of the base, is equal to the square upon a side of the triangle.

17. If a side of a triangle be bisected, and a perpendicular drawn from the middle point of the base to meet the side, then the square of the altitude of the triangle exceeds the square upon half the base by twice the rectangle contained by the side and the straight line between the points of section of the side.

18. In any triangle ABC, if perpendiculars be drawn from each of the angles upon the opposite sides, or opposite sides produced, meeting them respectively in D, E, F, show that—

$$BA^2 + AC^2 + CB^2 = 2\,AE \cdot AC + 2\,CD \cdot CB + 2\,BF \cdot BA;$$

all lines being measured in the *same direction* round the triangle.

19. Construct a square equal to the sum of the areas of two given rectilinear figures.

20. The base of a triangle is 63 ft., and the sides 25 ft. and 52 ft. respectively. Show that the segments of the base, made by a perpendicular from the vertex, are 15 ft. and 48 ft. respectively, and that the area of the triangle is 630 sq. ft.

21. In the same triangle, show that the length of the line joining the vertex with the middle of the base is 22·9 ft.

22. A ladder, 45 ft. long, reaches to a certain height against a wall, but, when turned over without moving the foot, must be shortened 6 ft. in order to reach the same height on the opposite side. Supposing the width of the street to be 42 ft., show that the height to which the ladder reaches is 36 ft.

23. The base and altitude of a triangle are 8 in. and 9 in. respectively; show that its area is equal to a square whose side is 6 in. Prove your result by construction.

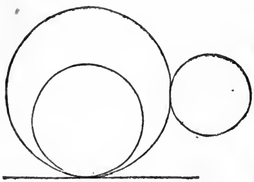
24. On the supposition that lines can be always expressed *exactly* in terms of some unit of length, what geometrical propositions may be deduced from the following algebraical identities?—

(1.) $(a + b)^2 = a^2 + 2\,ab + b^2$
(2.) $(a + b)\,(a - b) + b^2 = a^2$
(3.) $(2\,a + b)\,b + a^2 = (a + b)^2$
(4.) $(a + b)^2 + b^2 = 2\,(a + b)\,b + a^2$
(5.) $4\,(a + b)\,b + b^2 = (a + 2\,b)^2$
(6.) $(a + b)^2 + (a - b)^2 = 2\,a^2 + 2\,b^2$
(7.) $(2\,a + b)^2 + b^2 = 2\,a^2 + 2\,(a + b)^2$

EUCLID'S ELEMENTS, BOOK III.

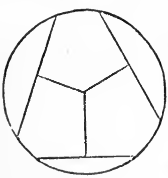
Definitions.

1. **Equal circles** are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.



2. **A straight line is said to touch a circle,** or to be a tangent to it, when it meets the circle, and being produced does not cut it.

3. **Circles** are said to **touch one another,** which meet, but do not cut one another.

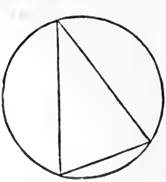


4. Straight lines are said to be **equally distant** from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

5. And the straight line on which the greater perpendicular falls, is said to be **farther** from the centre.

6. **A segment of a circle** is the figure contained by a straight line and the circumference which it cuts off.

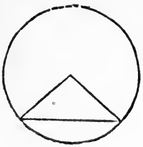
7. The **angle of a segment** is that which is contained by the straight line and the circumference.



8. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line, which is the base of the segment.

9. An angle is said to **insist** or **stand upon** the circumference intercepted between the straight lines that contain the angle.

10. **A sector** of a circle is the figure contained by two straight lines drawn from the centre and the circumference between them.



11. **Similar segments** of circles are those which contain equal angles.

[Any portion of the circumference is called an *arc*, and the *chord* of an arc is the straight line joining its extremities.]

Proposition 1.—Problem.

To find the centre of a given circle.

Let ABC be the given circle.

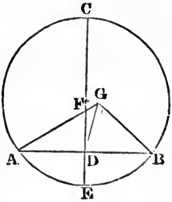
It is required to find its centre.

CONSTRUCTION.—Draw within the circle any chord AB, and bisect it in D (I. 10).

From the point D draw DC at right angles to AB (I. 11).

Produce CD to meet the circumference in E, and bisect CE in F (I. 10).

Then the point F shall be the centre of the circle ABC.



PROOF.—For if F be not the centre, if possible let G be the centre; and join GA, GD, GB. Suppose G the centre.

Then, because DA is equal to DB (Const.), and DG common to the two triangles ADG, BDG;

The two sides AD, DG are equal to the two sides BD, DG, each to each;

And the base GA is equal to the base GB, being radii of the same circle;

Therefore the angle ADG is equal to the angle BDG (I. 8). $\therefore \angle ADG = \angle BDG$.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (I. Def. 10).

Therefore the angle GDB is a right angle.

But the angle FDB is also a right angle (Const.);

∴ ∠GDB = ∠FDB. Therefore the angle GDB is equal to the angle FDB (Ax. 11), the less to the greater; which is impossible.

Therefore G is not the centre of the circle ABC.

In the same manner it may be shown that no point which is not in CE can be the centre.

And since the centre is in CE, it must be in F, its point of bisection.

Therefore F is the centre of the circle ABC: which was to be found.

Corollary.—From this it is manifest that, if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

Proposition 2.—Theorem.

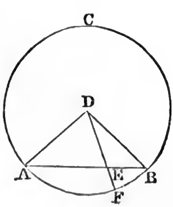
If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A and B any two points in the circumference.

The straight line drawn from A to B shall fall within the circle.

Construction.—Find D the centre of the circle ABC (III. 1), and join DA, DB.

In AB take any point E; join DE, and produce it to the circumference in F.



Proof.—Because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5).

∠DBE = ∠DAB, and ∴ ∠DEB > DBE. And because AE, a side of the triangle DAE, is produced to B, the exterior angle DEB is greater than the interior and opposite angle DAE (I. 16).

But the angle DAE was proved to be equal to the angle DBE;

Therefore the angle DEB is also greater than DBE.

But the greater angle is subtended by the greater side (I. 19);

∴ DB > DE. Therefore DB is greater than DE.

But DB is equal to DF; therefore DF is greater than DE, and the point E is therefore within the circle.

In the same manner it may be proved that every point in AB lies within the circle.

Therefore the straight line AB lies within the circle.

Therefore, if any two points, &c. *Q.E.D.*

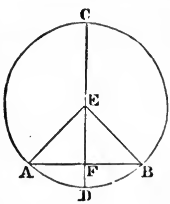
Proposition 3.—Theorem.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cut it at right angles, it shall bisect it.

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre.

CD shall cut AB at right angles.

Construction.—Take E, the centre of the circle (III. 1), and join EA, EB.



Proof.—Because AF is equal to FB (Hyp.), and FE common to the two triangles AFE, BFE, and the base EA equal to the base EB (I. Def. 15),

Therefore the angle AFE is equal to the angle BFE (I. 8);

Therefore each of the angles AFE, BFE is a right angle (I. def. 10);

Triangles AFE and BEF are equal in every respect.

Therefore the straight line CD, drawn through the centre, bisecting another, AB, that does not pass through the centre, also cuts it at right angles.

Conversely, let CD cut AB at right angles.

CD shall bisect AB; that is, AF shall be equal to FB—the same construction being made.

Proof.—Because the radii EA, EB are equal, the angle EAF is equal to the angle EBF (I. 5),

And the angle AFE is equal to the angle BFE (Hyp.),

Therefore, in the two triangles EAF, EBF, there are two angles in the one equal to two angles in the other, each to each, and the side EF, which is opposite to one of the equal angles in each, is common to both;

Therefore their other sides are equal (I. 26);

Therefore AF is equal to FB.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 4.—Theorem.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another at the point E, and do not both pass through the centre.

AC, BD shall not bisect one another.

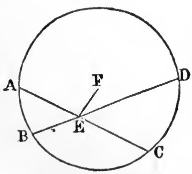
If one of the lines pass through the centre, it is plain that it cannot be bisected by the other, which does not pass through the centre.

If they bisect each other,

But if neither of them pass through the centre, if possible, let AE be equal to EC, and BE to ED.

CONSTRUCTION.—Take F, the centre of the circle (III. 1), and join EF.

EF bisects AC at right angles.



Because FE, a straight line drawn through the centre, bisects another line AC, which does not pass through the centre (Hyp.), therefore it cuts it at right angles (III. 3);

Therefore the angle FEA is a right angle.

And EF bisects BD at right angles.

Again, because the straight line FE bisects the straight line BD, which does not pass through the centre (Hyp.), therefore it cuts it at right angles (III. 3);

Therefore the angle FEB is a right angle.

But the angle FEA was shown to be a right angle;

Therefore the angle FEA is equal to the angle FEB, the less to the greater, which is impossible;

∴ ∠FEA = ∠FEB.

Therefore AC, BD do not bisect each other.

Therefore, if in a circle, &c. *Q.E.D.*

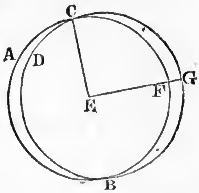
Proposition 5.—Theorem.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C.

They shall not have the same centre.

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG, meeting the circles in F and G.



PROOF.—Because E is the centre of the circle ABC, EC is equal to EF (I. Def. 15). EC=EF

And because E is the centre of the circle CDG, EC is equal to EG. and =EG.

But EC was shown to be equal to EF;

Therefore EF is equal to EG (Ax. 1), the less to the greater, ∴ EF=EG. which is impossible;

Therefore E is not the centre of the circles ABC, CDG.

Therefore, if two circles, &c. *Q.E.D.*

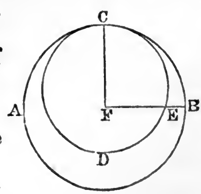
Proposition 6.—Theorem.

If one circle touch another internally, they shall not have the same centre.

Let the circle CDE touch the circle ABC internally in the point C.

They shall not have the same centre.

CONSTRUCTION.—For, if it be possible, let F be their centre; If they have the same centre F, join FC, and draw any straight line FEB, meeting the circles in E and B.



PROOF.—Because F is the centre of the circle ABC, FC is equal to FB (I Def. 15).

And because F is the centre of the circle CDE, FC is equal to FE. FC=FE and=FB.

But FC was shown to be equal to FB;

Therefore FE is equal to FB (Ax. 1), the less to the greater, ∴ FE=FB which is impossible;

Therefore F is not the centre of the circles ABC, CDE.

Therefore, if one circle, &c. *Q.E.D.*

Proposition 7.—Theorem.

If any point be taken in the diameter of a circle, which is not the centre of all the straight lines which can be drawn from this point to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.

Let ABCD be a circle, AD its diameter, and E its centre, in which let any point F be taken which is not the centre.

FA > FB > FC > FG &c., > FD the least.

Of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA, which passes through the centre, shall be the greatest;

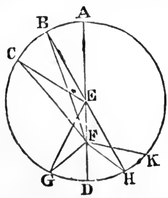
FD, the other part of the diameter AD, shall be the least;

And of the others, FB, the nearer to FA, shall be greater than FC, the more remote; and FC greater than FG.

CONSTRUCTION.—Join BE, CE, GE.

For BE+EF or AF > BF

PROOF.—Because any two sides of a triangle are greater than the third side, BE, EF are greater than BF (I. 20).



But AE is equal to BE; therefore AE, EF, that is, AF is greater than BF.

Again, because BE is equal to CE, and EF common to the two triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF, each to each.

But the angle BEF is greater than the angle CEF;

And base BF > base FC

Therefore the base FB is greater than the base FC (I. 24).

In the same manner it may be shown that FC is greater than FG.

Again GF+FE > EG and ∴ > ED.

Again, because GF, FE are greater than EG (I. 20), and that EG is equal to ED;

Therefore GF, FE are greater than ED.

∴ GF > FD

Take away the common part FE, and the remainder GF is greater than the remainder FD (Ax. 5).

Therefore FA is the greatest, and FD the least, of all the straight lines from F to the circumference; and FB is greater than FC, and FC than FG.

Also, there cannot be drawn more than two equal straight lines from the point F to the circumference, one on each side of the shortest line.

Construction.—At the point E, in the straight line EF, make the angle FEH equal to the angle FEG (I. 23), and join FH.

Proof.—Because EG is equal to EH, and EF common to the two triangles GEF, HEF, the two sides EG, EF are equal to the two sides EH, EF, each to each; **Triangles GEF and HEF are equal in every respect.**

And the angle GEF is equal to the angle HEF (Const.);

Therefore the base FG is equal to the base FH (I. 4).

But, besides FH, no other straight line can be drawn from F to the circumference equal to FG.

For, if it be possible, let FK be equal to FG;

Then, because FK is equal to FG, and FH is also equal to FG, therefore FH is equal to FK; **And if FK =FG, it also = FH, which is impossible.**

That is, a line nearer to that which passes through the centre is equal to a line which is more remote; which is impossible by what has been already shown.

Therefore, if any point, &c. *Q.E.D.*

Proposition 8.—Theorem.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre; of those which fall on the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the least line.

Let ABC be a circle, and D any point without it, and from D let the straight lines DA, DE, DF, DC be drawn to the circumference, of which DA passes through the centre.

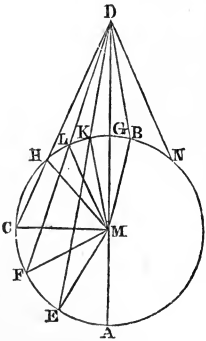
DA>DE >DF >DC,

Of those which fall on the concave part of the circumference AEFC, the greatest shall be DA, which passes through the centre, and the nearer to it shall be greater than the more remote, viz., DE greater than DF, and DF greater than DC.

and DG<DK <DL <DH

But of those which fall on the convex circumference GKLH, the least shall be DG between the point D and the diameter AG, and the nearer to it shall be less than the more remote, viz., DK less than DL, and DL less than DH.

Construction.—Take M, the centre of the circle ABC (III. 1), and join ME, MF, MC, MH, ML, MK.



Proof.—Because any two sides of a triangle are greater than the third side, EM, MD are greater than ED (I. 20).

But EM is equal to AM; therefore AM, MD are greater than ED—that is, AD is greater than ED.

For EM+MD or AD> ED.

Again, because EM is equal to FM, and MD common to the two triangles EMD, FMD; the two sides EM, MD are equal to the two sides FM, MD, each to each;

But the angle EMD is greater than the angle FMD;

Therefore the base ED is greater than the base FD (I. 24).

Also base ED > base FD, &c.

In like manner it may be shown that FD is greater than CD;

Therefore DA is the greatest, and DE greater than DF and DF greater than DC.

Again MK+KD >MD, or >MK+ DG. ∴KD> DG, and ∴DG <KD, &c.

Again, because MK, KD are greater than MD (I. 20), and MK is equal to MG;

The remainder KD is greater than the remainder GD—that is, GD is less than KD.

And because MK, DK are drawn to the point K within the triangle MLD from M and D, the extremities of its side MD;

Therefore MK, DK are less than ML, LD (I. 21).

But MK is equal to ML; therefore the remainder KD is less than the remainder LD.

In like manner it may be shown that LD is less than HD.

Therefore DG is the least, and KD less than DL, and DL less than DH.

Also, there can be drawn only two equal straight lines from the point D to the circumference, one on each side of the least line.

CONSTRUCTION.—At the point M, in the straight line MD, make the angle DMB equal to the angle DMK (I. 23), and join DB.

PROOF.—Because MK is equal to MB, and MD common to the two triangles KMD, BMD; the two sides KM, MD are equal to the two sides BM, MD, each to each; **Triangles KMD and BMD are equal in every respect.**

And the angle DMK is equal to the angle DMB (Const);

Therefore the base DK is equal to the base DB (I. 4).

But, besides DB, no other straight line can be drawn from D to the circumference equal to DK.

For, if it be possible, let DN be equal to DK.

Then, because DN is equal to DK, and DB is also equal to DK, therefore DB is equal to DN (Ax. 1); **DN=DK and ∴ =DB which is impossible.**

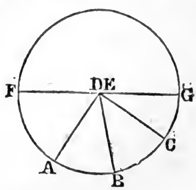
That is, a line nearer to the least is equal to one which is more remote; which is impossible by what has been already shown.

Therefore, if any point, &c. *Q.E.D.*

Proposition 9.—Theorem.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there can be drawn more than two equal straight lines, viz., DA, DB, DC.



The point D shall be the centre of the circle.

CONSTRUCTION.—For if not, let E be the centre; join DE, and produce it to the circumference in F and G.

PROOF.—Then FG is a diameter of the circle ABC (I. Def. 17).

If D be not the centre.

And because in FG, the diameter of the circle ABC, there is taken the point D, not the centre;

DG > DC > DB > DA.

Therefore DG is the greatest straight line from D to the circumference, and DC is greater than DB, and DB greater than DA (III. 7);

But they are also equal.

But these lines are likewise equal, by hypothesis; which is impossible.

Therefore E is not the centre of the circle ABC.

In like manner it may be demonstrated that any other point than D is not the centre;

Therefore D is the centre of the circle ABC.

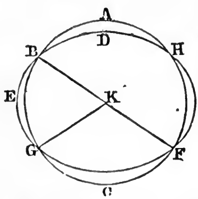
Therefore, if a point, &c. *Q.E.D.*

Proposition 10.—Theorem.

One circumference of a circle cannot cut another in more than two points.

If possible,

CONSTRUCTION.—If it be possible, let the circumference ABC cut the circumference DEF in more than two points, viz., in the points B, G, F.



Take the centre K of the circle ABC (III. 1), and join KB, KG, KF.

PROOF.—Because K is the centre of the circle ABC, the radii KB, KG, KF are all equal.

the two circles have the same centre.

And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, therefore K is the centre of the circle DEF (III. 9).

But K is also the centre of the circle ABC (Const.);

Therefore the same point is the centre of two circles which cut one another; which is impossible (III. 5).

Therefore, one circumference, &c. *Q.E.D.*

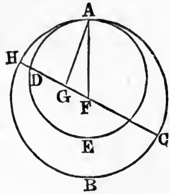
Proposition 11.—Theorem.

If one circle touch another internally in any point, the straight line which joins their centres, being produced, shall pass through that point.

Let the circle ADE touch the circle ABC internally in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE.

The straight line which joins their centres, being produced, shall pass through the point of contact A.

CONSTRUCTION.—For, if not, let it pass otherwise, if possible, as FGDH. Join AF and AG. If not,



PROOF.—Because AG, GF are greater than AF (I. 20), and AF is equal to HF (I. def. 15);

Therefore AG, GF are greater than HF.

Take away the common part GF, and the remainder AG is greater than the remainder HG. AG > HG.

But AG is equal to DG (I. Def. 15); But AG = DG.

Therefore DG is greater than HG, the less than the greater; which is impossible. ∴ DG > HG.

Therefore the straight line which joins the centres, being produced, cannot fall otherwise than upon the point A, that is, it must pass through it.

Therefore, if one circle, &c. *Q.E.D.*

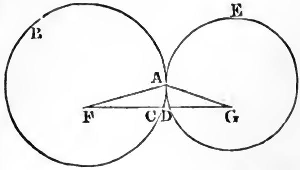
Proposition 12.—Theorem.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE;

The straight line which joins their centres shall pass through the point of contact A.

CONSTRUCTION.—For, if not, let it pass otherwise, if possible, as FCDG. Join FA and AG. If not,



PROOF.—Because F is the centre of the circle ABC, FA is equal to FC (I. Def. 15).

And because G is the centre of the circle ADE, GA is equal to GD;

Therefore FA, AG are equal to FC, DG (Ax. 2).

FG is > FA+AG, but it is also *less*. Therefore the whole FG is greater than FA, AG.

But FG is also less than FA, AG (I. 20); which is impossible.

Therefore the straight line which joins the centres of the circles shall not pass otherwise than through the point A, that is, it must pass through it.

Therefore, if two circles, &c. *Q.E.D.*

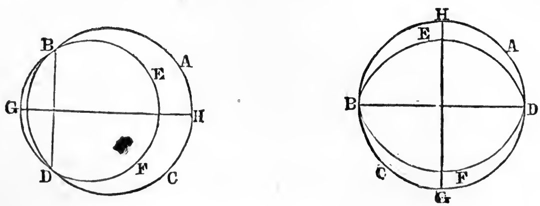
Proposition 13.—Theorem.

One circle cannot touch another in more points than one, whether it touch it internally or externally.

I. First, let the circle EBF touch the circle ABC internally in the point B.

Then EBF cannot touch ABC in any other point.

If possible, let it touch in D also; CONSTRUCTION.—If it be possible, let EBF touch ABC in another point D; join BD, and draw GH bisecting BD at right angles (I. 10, 11).



PROOF.—Because the two points B, D are in the circumference of each of the circles, the straight line BD falls within each of them (III. 2):

Therefore the centre of each circle is in the straight line GH, which bisects BD at right angles (III. 1 cor.)

then GH passes through the point of contact, which it does not. Therefore GH passes through the point of contact (III. 11).

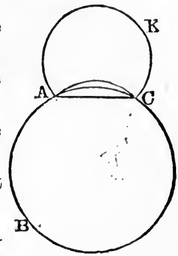
But GH does not pass through the point of contact, because the points B, D are out of the line of GH; which is absurd.

Therefore one circle cannot touch another internally in more points than one.

II. Next, let the circle ACK touch the circle ABC externally in the point A.

Then ACK cannot touch ABC in any other point.

CONSTRUCTION.—If it be possible, let ACK touch ABC in another point C. Join AC. If possible let it touch in C also;



PROOF.—Because the points A, C are in the circumference of the circle ACK, the straight line AC must fall within the circle ACK (III. 2).

But the circle ACK is without the circle ABC (Hyp.);

Therefore the straight line AC is without the circle ABC. then AC falls without the circle ABC, which is absurd.

But because the two points A, C are in the circumference of the circle ABC, the straight line AC falls within the circle ABC (III. 2); which is absurd.

Therefore one circle cannot touch another externally in more points than one.

And it has been shown that one circle cannot touch another internally in more points than one.

Therefore, one circle, &c. *Q.E.D.*

Proposition 14.—Theorem.

Equal straight lines in a circle are equally distant from the centre; and, conversely, those which are equally distant from the centre are equal to one another.

Let the straight lines AB, CD, in the circle ABDC, be equal to one another.

Then they shall be equally distant from the centre.

CONSTRUCTION.—Take E, the centre of the circle ABDC (III. 1).

From E draw EF, EG, perpendiculars to AB, CD (I. 12).

Join EA, EC.

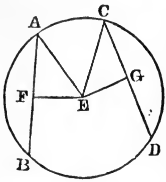
PROOF.—Because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects it (III. 3).

Therefore AF is equal to FB, and AB is double of AF.

For the like reason CD is double of CG.

AF = CG, But AB is equal to CD (Hyp.); therefore AF is equal to CG (Ax. 7).

And because AE is equal to CE, the square on AE is equal to the square on CE.



But the squares on AF, FE are equal to the square on AE, because the angle AFE is a right angle (I. 47).

For the like reason the squares on CG, GE are equal to the square on CE;

and $AF^2+FE^2 = CG^2 + EG^2$. Therefore the squares on AF, FE are equal to the squares on CG, GE (Ax. 1).

But the square on AF is equal to the square on CG, because AF is equal to CG;

Therefore the remaining square on FE is equal to the remaining square on GE (Ax. 3);

∴ EF=EG. And therefore the straight line EF is equal to the straight line EG.

But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (III. Def. 4);

Therefore AB, CD are equally distant from the centre.

Conversely, let the straight lines AB, CD be equally distant from the centre, that is, let EF be equal to EG;

Then AB shall be equal to CD.

Here EF = EG, and $AF^2+EF^2 = CG^2 + EG^2$. PROOF.—The same construction being made, it may be demonstrated, as before, that AB is double AF, and CD double of CG, and that the squares on EF, FA are equal to the squares on EG, GC.

But the square on EF is equal to the square on EG, because EF is equal to EG (Hyp.);

Therefore the remaining square on FA is equal to the remaining square on GC (Ax. 3),

∴ AF = CG, &c. And therefore the straight line AF is equal to the straight line CG.

But AB was shown to be double of AF, and CD double of CG;

Therefore AB is equal to CD (Ax. 6);

Therefore, equal straight lines, &c. *Q.E.D.*

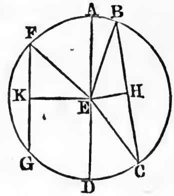
Proposition 15.—Theorem.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG.

Then AD shall be greater than any straight line CB, which is not a diameter; and BC shall be greater than FG.

Construction.—From the centre E draw EH EK perpendiculars to BC, FG (I. 12), and join EB, EC, EF.



Proof.—Because AE is equal to BE, and ED to EC,

Therefore AD is equal to BE, EC. BE + EC or AD > BC,

But BE, EC are greater than BC (I. 20);

Therefore also AD is greater than BC.

And because BC is nearer to the centre than FG (Hyp.), EH is less than EK (III. Def. 5). and EH < EK.

But, as was demonstrated in the preceding proposition, BC is double of BH, and FG double of FK, and the squares on EH, HB are equal to the squares on EK, KF. ∴ since $EH^2 + HB^2 = EK^2 + KF^2$,

But the square on EH is less than the square on EK, because EH is less than EK;

Therefore the square on HB is greater than the square on KF, and the straight line BH greater than FK; HB > FK.

And therefore BC is greater than FG.

Conversely, let BC be greater than FG.

Then BC shall be nearer to the centre than FG, that is, the same construction being made, EH shall be less than EK.

Proof.—Because BC is greater than FG, BH is greater than FK.

But the squares on BH, HE are equal to the squares on FK, KE;

And the square on BH is greater than the square on FK, because BH is greater than FK;

Therefore the square on HE is less than the square on KE, and the straight line EH less than EK;

And therefore BC is nearer to the centre than FG (III. def. 5).

Therefore, the diameter, &c. *Q.E.D.*

Proposition 16.—Theorem.

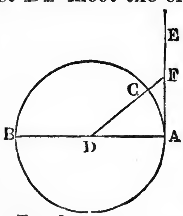
The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and a straight line, making an acute angle with the diameter at its extremity, cuts the circle.

Let ABC be a circle, of which D is the centre, and AB a diameter, and AE a line drawn from A perpendicular to AB.

The straight line AE shall fall without the circle.

Take any point F in AE,

CONSTRUCTION.—In AE take any point F; join DF, and let DF meet the circle in C.



PROOF.—Because DAF is a right angle, it is greater than the angle AFD (I. 17);

Therefore DF is greater than DA (I. 19).

then DF > DA and ∴ > DC.

But DA is equal to DC; therefore DF is greater than DC.

Therefore the point F is without the circle.

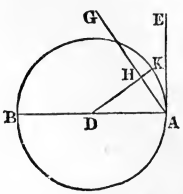
In the same manner it may be shown that any other point in AE, except the point A, is without the circle.

Therefore AE falls without the circle.

Again, let AG make with the diameter the angle DAG less than a right angle.

The line AG shall fall within the circle, and hence cut it.

Draw DH at right angles to HG, then DH < DA, and ∴ < DK.



CONSTRUCTION.—From D draw DH at right angles to AG, and meeting the circumference in K (I. 12).

PROOF.—Because DHA is a right angle, and DAH less than a right angle;

Therefore the side DH is less than the side DA (I. 19).

But DK is equal to DA; therefore DH is less than DK.

Therefore the point H is within the circle.

Therefore the straight line AG cuts the circle.
Therefore, the straight line, &c. *Q.E.D.*

Corollary.—From this it is manifest that the straight line which is drawn at right angles to the diameter of a circle, from the extremity of it, touches the circle (III. Def. 2); and that it touches it only in one point, because if it did meet the circle in two points it would fall within it (III. 2). Also it is evident that there can be but one straight line which touches the circle in the same point.

Proposition 17.—Problem.

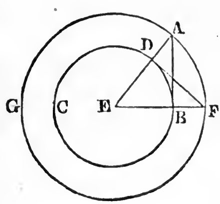
To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let the given point A be without the given circle BCD.

It is required to draw from A a straight line which shall touch the given circle.

Construction.—Find the centre E of the circle (III. 1), and join AE.

From the centre E, at the distance EA, describe the circle AFG.



From the point D draw DF at right angles to EA (I. 11), and join EBF and AB.

Then AB shall touch the circle BCD.

Proof.—Because E is the centre of the circles AFG, BCD, EA is equal to EF, and ED to EB;

Therefore the two sides AE, EB are equal to the two sides FE, ED, each to each; EA, ED respectively = EF, EB and ∠ E common;

And the angle at E is common to the two triangles AEB, FED;

Therefore the base AB is equal to the base FD, and the triangle AEB to the triangle FED, and the other angles to the other angles, each to each, to which the equal sides are opposite (I. 4);

Therefore the angle ABE is equal to the angle FDE. ∴ ∠ ABE = ∠ FDE a right angle.

But the angle FDE is a right angle (Const.);

Therefore the angle ABE is a right angle (Ax. 1).

And EB is drawn from the centre (Const.)

But the straight line drawn at right angles to a diameter of a circle, from the extremity of it, touches the circle (III. 16, cor.);

∴ AB touches the circle.

Therefore AB touches the circle, and it is drawn from the given point A.

Next, let the given point be in the circumference of the circle, at the point D.

Draw DE to the centre E, and DF at right angles to DE;

Then DF touches the circle (III. 16, cor.)

Therefore, from the given points A and D, straight lines, AB and DF, have been drawn, touching the given circle BCD. *Q.E.F.*

Proposition 18.—Theorem.

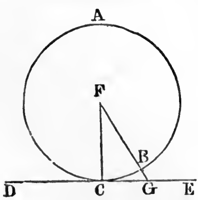
If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C; take the centre F (III. 1), and draw the straight line FC.

FC shall be perpendicular to DE.

If not, suppose FG perpendicular.

CONSTRUCTION.—For, if not, let FG be drawn from the point F perpendicular to DE, meeting the circumference in B.



PROOF.—Because FGC is a right angle (Hyp.), FCG is an acute angle (I. 17), and to the greater angle the greater side is opposite (I. 19);

Therefore FC is greater than FG.

Then must FB > FG.

But FC is equal to FB; therefore FB is greater than FG; the part greater than the whole, which is impossible.

Therefore FG is not perpendicular to DE.

In the same manner it may be shown that no other straight line from F is perpendicular to DE, but FC; therefore FC is perpendicular to DE.

Therefore, if a straight line, &c. *Q.E.D.*

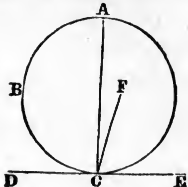
Proposition 19.—Theorem.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE.

The centre of the circle shall be in CA.

CONSTRUCTION.—For, if not, if possible, let F be the centre, and join CF. If not, take F the centre, out of the line.



PROOF.—Because DE touches the circle ABC, and FC is drawn from the assumed centre to the point of contact,

Therefore FC is perpendicular to DE (III. 18);

Therefore FCE is a right angle.

But the angle ACE is also a right angle (Const.);

Therefore the angle FCE is equal to the angle ACE; the less to the greater, which is impossible. Then ∠ ACE = ∠ FCE, being right angles.

Therefore F is not the centre of the circle ABC.

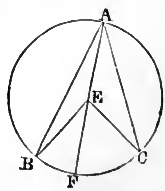
In the same manner it may be shown that no other point which is not in CA is the centre; therefore the centre is in CA.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 20.—Theorem.

The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same arc.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base.



The angle BEC shall be double of the angle BAC.

CASE I.—First, let the centre E of the circle be within the angle BAC.

CONSTRUCTION.—Join AE, and produce it to the circumference in F, PROOF.—Because EA is equal to EB, the angle EAB is equal to the angle EBA (I. 5);

Therefore the angles EAB, EBA are double of the angle EAB.

∠ BEF = ∠ EAB + ∠ EBA =2 ∠ EAB, and so ∠ FEC = 2 ∠ EAC.

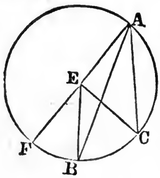
But the angle BEF is equal to the angles EAB, EBA (I. 32);

Therefore the angle BEF is double of the angle EAB.

For the same reason the angle FEC is double of the angle EAC.

∴ ∠ BEC =2 ∠ BAC.

Therefore the whole angle BEC is double of the whole angle BAC.



CASE II.—Next, let the centre E of the circle be without the angle BAC.

CONSTRUCTION.—Join AE, and produce it to meet the circumference in F.

∠ FEC = 2 ∠ EAC, and ∠ FEB =2 ∠ CAB.

PROOF.—It may be demonstrated, as in the first case, that the angle FEC is double of the angle FAC, and that FEB, a part of FEC, is double of FAB, a part of FAC;

∴ taking the difference ∠ BEC = 2 ∠ BAC.

Therefore the remaining angle BEC is double of the remaining angle BAC.

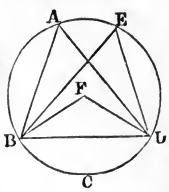
Therefore, the angle at the centre, &c. *Q.E.D.*

Proposition 21.—Theorem.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED.

The angles BAD, BED shall be equal to one another.



CASE I.—First, let the segment BAED be greater than a semicircle.

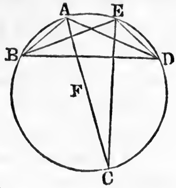
CONSTRUCTION.—Take F, the centre of the circle ABCD (III. 1), and join BF, DF.

PROOF.—Because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same arc for their base, namely, BCD;

∠ BFD = 2 ∠ BAD,

Therefore the angle BFD is double of the angle BAD (III. 20).

For the same reason, the angle BFD is double of the angle BED; — and also = 2 ∠ BED.



Therefore the angle BAD is equal to the angle BED (Ax. 7). — ∴ ∠ BAD = ∠ BED.

Case II.—Next, let the segment BAED be not greater than a semicircle.

Construction.—Draw AF to the centre, and produce it to C, and join CE.

Proof.—Then the segment BADC is greater than a semicircle, and therefore the angles BAC, BEC in it are equal by the first case. — ∠ BAC = ∠ BEC,

For the same reason, because the segment CBED is greater than a semicircle, the angles CAD, CED are equal. — and ∠ CAD = ∠ CED.

Therefore the whole angle BAD is equal to the whole angle BED (Ax. 2). — ∴ ∠ BAD = ∠ BED.

Therefore, the angles in the same segment, &c. *Q.E.D.*

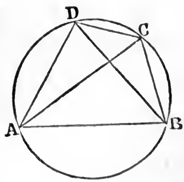
Proposition 22.—Theorem.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABCD.

Any two of its opposite angles shall be together equal to two right angles.

Construction.—Join AC, BD.



Proof.—Because the three angles of every triangle are together equal to two right angles (I. 32),

The three angles of the triangle CAB, namely, CAB, ACB, ABC, are together equal to two right angles. — ∠ CAB + ∠ ACB + ∠ ABC = 2 right angles.

But the angle CAB is equal to the angle CDB, because they are in the same segment CDAB (III. 21); — The first two together = ∠ CDB + ∠ ADB = ∠ ADC.

And the angle ACB is equal to the angle ADB, because they are in the same segment ADCB;

Therefore the two angles CAB, ACB are together equal to the whole angle ADC (Ax. 2).

To each of these equals add the angle ABC;

Therefore the three angles CAB, ACB, ABC are equal to the two angles ABC, ADC.

But the angles CAB, ACB, ABC are together equal to two right angles (I. 32);

∴ ∠ ADC + ∠ ABC = 2 right angles.

Therefore also the angles ABC, ADC are together equal to two right angles.

In like manner it may be shown that the angles BAD, BCD are together equal to two right angles.

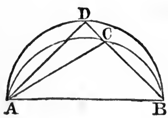
Therefore, the opposite angles, &c. *Q.E.D.*

Proposition 23.—Theorem.

Upon the same straight line, and on the same side of it, there cannot be two similar segments of circles not coinciding with one another.

If possible,

If it be possible, on the same straight line AB, and on the same side of it, let there be two similar segments of circles ACB, ADB not coinciding with one another.



Construction.—Then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (III. 10);

Therefore one of the segments must fall within the other.

Let ACB fall within ADB; draw the straight line BCD, and join AC, AD.

Proof.—Because the segment ACB is similar to the segment ADB (Hyp.), and that similar segments of circles contain equal angles (III. Def. 11);

exterior ∠ ACB = interior and opposite ∠ ADC.

Therefore the angle ACB is equal to the angle ADB; that is, the exterior angle of the triangle ACD, equal to the interior and opposite angle; which is impossible (I. 16).

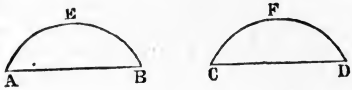
Therefore, there cannot be two similar segments of circles on the same straight line, and on the same side of it, which do not coincide. *Q.E.D.*

Proposition 24.—Theorem.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD.

The segment AEB shall be equal to the segment CFD.



They are equal, because they must coincide by Prop. 23.

Proof.—For if the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and the straight line AB on the straight line CD,

Then the point B shall coincide with the point D, because AB is equal to CD.

And the straight line AB coinciding with CD, the segment AEB must coincide with the segment CFD (III. 23); and is therefore equal to it.

Therefore, similar segments, &c. *Q.E.D.*

Proposition 25.—Problem.

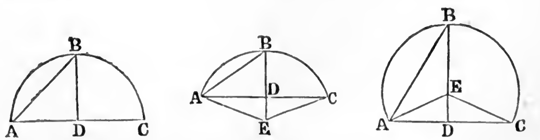
A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle.

It is required to describe the circle of which ABC is a segment.

Construction.—Bisect AC in D (I. 10).

From the point D draw DB at right angles to AC (I. 11), and join AB.



Case I.—First, let the angles ABD, BAD be equal to one another.

Then D shall be the centre of the circle required.

When $\angle ABD = \angle BAD$, $DA = DB = DC$.

PROOF.—Because the angle ABD is equal to the angle BAD (Hyp.);

Therefore DB is equal to DA (I. 6).

But DA is equal to DC (Const.);

Therefore DB is equal to DC (Ax. 1).

Therefore the three straight lines DA, DB, DC are all equal;

and $\therefore$ D the centre.

And therefore D is the centre of the circle (III. 9).

Hence, if from the centre D, at the distance of any of the three lines, DA, DB, DC a circle be described, it will pass through the other two points, and be the circle required.

CASE II.—Next, let the angles ABD, BAD be not equal to one another.

Make $\angle BAE = \angle ABD$.

CONSTRUCTION.—At the point A, in the straight line AB, make the angle BAE equal to the angle ABD (I. 23);

Produce BD, if necessary, to E, and join EC.

Then E shall be the centre of the circle required.

$\therefore EA = EB$

PROOF.—Because the angle BAE is equal to the angle ABE (Const.), EA is equal to EB (I. 6).

And because AD is equal to CD (Const.), and DE is common to the two triangles ADE, CDE,

The two sides AD, DE are equal to the two sides CD, DE, each to each;

And the angle ADE is equal to the angle CDE, for each of them is a right angle (Const.);

and $EA = EC$.

Therefore the base EA is equal to the base EC (I. 4).

But EA was shown to be equal to EB;

Therefore EB is equal to EC (Ax. 1).

$\therefore EA = EB = EC$,

Therefore the three straight lines EA, EB, EC are all equal;

and therefore E is the centre.

And therefore E is the centre of the circle (III. 9).

Hence, if from the centre E, at the distance of any of the three lines EA, EB, EC, a circle be described, it will pass through the other two points, and be the circle required.

In the *first* case, it is evident that, because the centre D is in AC, the segment ABC is a semicircle.

In the *second* case, if the angle ABD be greater than BAD, the centre E falls without the segment ABC, which is therefore less than a semicircle;

But if the angle ABD be less than the angle BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle.

Therefore, a segment of a circle being given, the circle has been described of which it is a segment. *Q.E.F.*

Proposition 26.—Theorem.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or at the circumferences.

Let ABC, DEF be equal circles, having the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences.

The arc BKC shall be equal to the arc ELF.

Construction.—Join BC, EF.

Proof.—Because the circles ABC, DEF are equal (Hyp.), the straight lines from their centres are equal (III. def. 1);

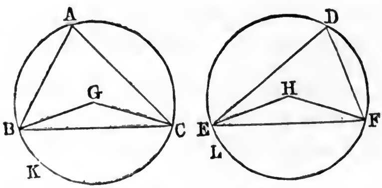
Therefore the two sides BG, GC are equal to the two sides EH, HF, each to each;

And the angle at G is equal to the angle at H (Hyp.);

Therefore the base BC is equal to the base EF (I. 4).

Triangles BGC and EHF are equal in every respect.

And because the angle at A is equal to the angle at D (Hyp.),



The segment BAC is similar to the segment EDF (III. def. 11),

And they are on equal straight lines BC, EF.

But similar segments of circles on equal straight lines are equal to one another (III. 24);

∴ segments BAC and EDF are similar and on equal straight lines.

Therefore the segment BAC is equal to the segment EDF.

∴ are equal.

But the whole circle ABC is equal to the whole circle DEF (Hyp.);

Therefore the remaining segment BKC is equal to the remaining segment ELF (Ax. 3).

∴ arc BKC = arc ELF.

Therefore the arc BKC is equal to the arc ELF.

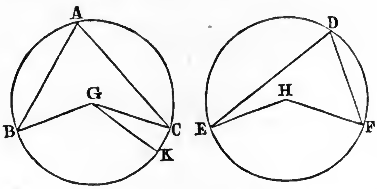
Therefore, in equal circles, &c. *Q.E.D.*

Proposition 27.—Theorem.

In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or at the circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF, at their centres, and the angles BAC, EDF, at their circumferences, stand on equal arcs BC, EF.

The angle BGC shall be equal to the angle EHF, and the angle BAC equal to the angle EDF.



Construction.—If the angle BGC be equal to the angle EHF, it is manifest that the angle BAC is also equal to the angle EDF (III. 20, ax. 7).

If one ∠ is greater than the other, the corresponding arc is greater.

But, if not, one of them must be the greater. Let BGC be the greater, and at the point G, in the straight line BG, make the angle BGK equal to the angle EHF (I. 23).

Proof.—Because the angle BGK is equal to the angle EHF, and that in equal circles equal angles stand on equal arcs, when they are at the centres (III. 26);

Therefore the arc BK is equal to the arc EF.

But the arc EF is equal to the arc BC (Hyp.);

Therefore the arc BK is equal to the arc BC (Ax. 1); the less to the greater, which is impossible.

Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it.

And the angle at A is half of the angle BGC, and the angle at D is half of the angle EHF (III. 20);

Therefore the angle at A is equal to the angle at D (Ax. 7).

Therefore, in equal circles, &c. *Q.E.D.*

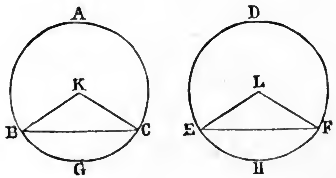
Proposition 28.—Theorem.

In equal circles, equal chords cut off equal arcs, the greater equal to the greater, and the less equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF.

The greater arc BAC shall be equal to the greater arc EDF, and the less arc BGC equal to the less arc EHF.

CONSTRUCTION.—Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF. **Take K and L the centres.**



PROOF.—Because the circles ABC, DEF are equal, their radii are equal (III. def. 1).

Therefore the two sides BK, KC are equal to the two sides EL, LF, each to each;

And the base BC is equal to the base EF (Hyp.);

Therefore the angle BKC is equal to the angle ELF (I. 8). **Triangles BKC and ELF are equal in every respect.**

But in equal circles equal angles stand on equal arcs, when they are at the centres (III. 26);

Therefore the arc BGC is equal to the arc EHF.

But the whole circle ABC is equal to the whole circle DEF (Hyp.);

Therefore the remaining arc BAC is equal to the remaining arc EDF (Ax. 3).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 29.—Theorem.

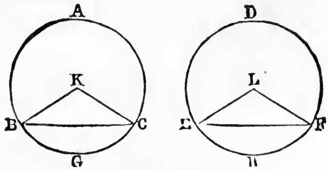
In equal circles equal arcs are subtended by equal chords.

Let ABC, DEF be equal circles, and let BGC, EHF be equal arcs in them, and join BC, EF.

The chord BC shall be equal to the chord EF.

Take K and L the centres.

CONSTRUCTION.—Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF.



Then ∠ BKC = ∠ ELF,

PROOF.—Because the arc BGC is equal to the arc EHF (Hyp.), the angle BKC is equal to the angle ELF (III. 27).

And because the circles ABC, DEF are equal (Hyp.), their radii are equal (III. def. 1).

Therefore the two sides BK, KC are equal to the two sides EL, LF, each to each; and they contain equal angles;

and so base BC = base EF.

Therefore the base BC is equal to the base EF (I. 4).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 30.—Problem.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc.

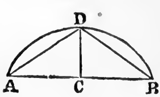
It is required to bisect it.

CONSTRUCTION.—Join AB, and bisect it in C (I. 10).

From the point C draw CD at right angles to AB (I. 11), and join AD and DB.

Then the arc ABD shall be bisected in the point D.

In the triangles ADC and CDB.



PROOF.—Because AC is equal to CB (Const.), and CD is common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD, because each of them is a right angle (Const.);

Therefore the base AD is equal to the base BD (I. 4). base AD = base BD.

But equal chords cut off equal arcs, the greater equal to the greater, and the less equal to the less (III. 28);

And each of the arcs AD, DB is less than a semicircle, because DC, if produced, is a diameter (III. 1, cor.);

Therefore the arc AD is equal to the arc DB.

Therefore, the given arc is bisected in D. *Q.E.F.*

Proposition 31.—Theorem.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABC be a circle, of which BC is a diameter, and E the centre; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC.

The angle in the semicircle BAC shall be a right angle;

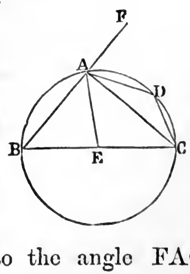
The angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle;

The angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.

Construction.—Join AE, and produce BA to F.

Proof.—Because EA is equal to EB (I. Def. 15),

The angle EAB is equal to the angle EBA (I. 5);



And, because EA is equal to EC,

The angle EAC is equal to the angle ECA;

Therefore the whole angle BAC is equal to the two angles ABC, ACB (Ax. 2).

But FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB (I. 32).

Therefore the angle BAC is equal to the angle FAC (Ax. 1),

∠ BAE + ∠ EAC, or ∠ BAC = ∠ ABC + ∠ ACB = ∠ FAC and ∴ a right angle.

And therefore each of them is a right angle (I. Def. 10);

Therefore the angle in a semicircle BAC is a right angle.

And because the two angles ABC, BAC, of the triangle ABC, are together less than two right angles (I. 17), and that BAC has been shown to be a right angle;

∴ ∠ ABC < a right angle.

Therefore the angle ABC is less than a right angle.

Therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.

And, because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are together equal to two right angles (III. 22);

Therefore the angles ABC, ADC are together equal to two right angles.

But the angle ABC has been shown to be less than a right angle;

Hence ∠ ADC > a right angle, by Prop. 32.

Therefore the angle ADC is greater than a right angle;

Therefore the angle in a segment ADC, less than a semicircle, is greater than a right angle.

Therefore, the angle, &c. *Q.E.D.*

Corollary.—From this demonstration it is manifest that, if one angle of a triangle be equal to the other two, it is a right angle.

For the angle adjacent to it is equal to the same two angles (I. 32).

And, when the adjacent angles are equal, they are right angles (I. def. 10).

Proposition 32.—Theorem.

The angles contained by a tangent to a circle and a chord drawn from the point of contact are equal to the angles in the alternate segments of the circle.

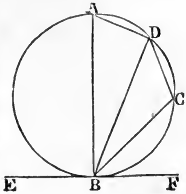
Let EF be a tangent to the circle ABCD, and BD a chord drawn from the point of contact B, cutting the circle.

The angles which BD makes with the tangent EF shall be equal to the angles in the alternate segments of the circle;

That is, the angle DBF shall be equal to the angle in the segment BAD, and the angle DBE shall be equal to the angle in the segment BCD.

Construction.—From the point B draw BA at right angles to EF (I. 11).

Take any point C in the circumference BD, and join AD, DC, CB.



Proof.—Because the straight line EF touches the circle ABCD at the point B (Hyp.), and BA is drawn at right angles to the tangent from the point of contact B (Const.),

The centre of the circle is in BA (III. 19). The centre is in BA.

Therefore the angle ADB, being in a semicircle, is a right angle (III. 31). ∴ ADB is a right angle,

Therefore the other two angles BAD, ABD are equal to a right angle (I. 32). and ∠ BAD + ∠ ABD = a right angle

But ABF is also a right angle (Const.);

Therefore the angle ABF is equal to the angles BAD, ABD. = ABF.

From each of these equals take away the common angle ABD;

Therefore the remaining angle DBF is equal to the remaining angle BAD, which is in the alternate segment of the circle (Ax. 3). ∴ ∠ BAD = ∠ DBF.

And because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are together equal to two right angles (III. 22). Also, ∠ BCD + ∠ BAD = 2 right angles

But the angles DBF, DBE are together equal to two right angles (I. 13);

Therefore the angles DBF, DBE are together equal to the angles BAD, BCD.

And the angle DBF has been shown equal to the angle BAD; = ∠ DBF + ∠ DBE.

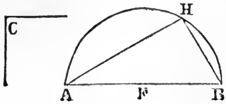
Therefore the remaining angle DBE is equal to the angle BCD, which is in the alternate segment of the circle (Ax. 3). ∴ ∠ DBE = ∠ BCD.

Therefore, the angles, &c. *Q.E.D.*

Proposition 33.—Problem.

Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle, It is required to describe, on the given straight line AB, a segment of a circle, containing an angle equal to the angle C.



CASE I.—Let the angle C be a right angle.

CONSTRUCTION.—Bisect AB in F (I. 10).

From the centre F, at the distance FB, describe the semicircle AHB.

Then AHB shall be the segment required.

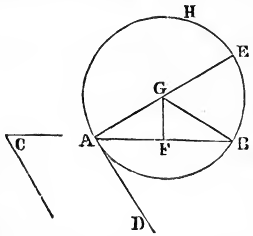
Angle in a semicircle is a right angle.

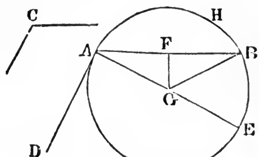
PROOF.—Because AHB is a semicircle, the angle AHB in it is a right angle, and therefore equal to the angle C (III. 31).

CASE II.—Let C be not a right angle.

At point A make ∠ BAD = C;

CONSTRUCTION.—At the point A, in the straight line AB, make the angle BAD equal to the angle C (I. 23).





and draw AE at right angles to AD. From F, middle of AB, draw perpendicular, meeting AE in G.

From the point A draw AE at right angles to AD (I. 11).

Bisect AB in F (I. 10).

From the point F draw FG at right angles to AB (I. 11), and join GB.

Because AF is equal to BF (Const.), and FG is common to the two triangles AFG, BFG;

The two sides AF, FG are equal to the two sides BF, FG, each to each;

And the angle AFG is equal to the angle BFG (Const.);

Therefore the base AG is equal to the base BG (I. 4).

Then G is the centre of a circle passing through A and B,

And the circle described from the centre G, at the distance GA, will therefore pass through the point B.

Let this circle be described; and let it be AHB.

The segment AHB shall contain an angle equal to the given rectilineal angle C.

Proof.—Because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE (Const.);

And AD touches the circle,

Therefore AD touches the circle (III. 16, cor.)

Because AB is drawn from the point of contact A, the angle DAB is equal to the angle in the alternate segment AHB (III. 32).

But the angle DAB is equal to the angle C (Const.);

Therefore the angle in the segment AHB is equal to the angle C (Ax. 1).

and ∴ ∠ in AHB = ∠ DAB or C.

Therefore, on the given straight line AB, the segment AHB of a circle has been described, containing an angle equal to the given angle C. *Q.E.F.*

Proposition 34.—Problem.

From a given circle to cut off a segment which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle.

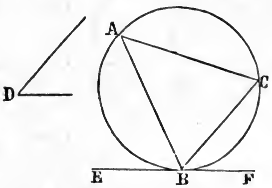
It is required to cut off from the circle ABC a segment that shall contain an angle equal to the angle D.

Construction.—Draw the straight line EF touching the circle ABC in the point B (III. 17);

Draw tangent EBF,

And at the point B, in the straight line BF, make the angle FBC equal to the angle D (I. 23).

and make ∠ FBC = given ∠.



Then the segment BAC shall contain an angle equal to the given angle D.

Proof.—Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B (Const.);

Therefore the angle FBC is equal to the angle in the alternate segment BAC of the circle (III. 32).

But the angle FBC is equal to the angle D (Const.);

Therefore the angle in the segment BAC is equal to the angle D (Ax. 1).

∴ ∠ BAC = ∠ FBC = ∠ D.

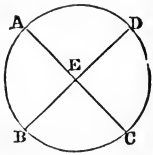
Therefore, from the given circle ABC, the segment BAC has been cut off, containing an angle equal to the given angle D. *Q.E.F.*

Proposition 35.—Theorem.

If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD cut one another in the point E, within the circle ABCD.

The rectangle contained by AE and EC shall be equal to the rectangle contained by BE and ED.

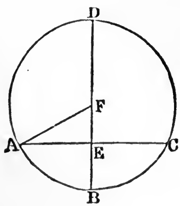


CASE I.—Let AC, BD pass each of them through the centre.

PROOF.—Because E is the centre, EA, EB, EC, ED are all equal (I. def. 15);

Therefore the rectangle AE, EC is equal to the rectangle BE, ED.

CASE II.—Let one of them, BD, pass through the centre, and cut the other, AC, which does not pass through the centre, at right angles, in the point E.



CONSTRUCTION.—Bisect BD in F, then F is the centre of the circle; join AF.

PROOF.—Because BD, which passes through the centre, cuts AC, which does not pass through the centre, at right angles in E (Hyp.);

AE = EC. Therefore AE is equal to EC (III. 3).

And because BD is cut into two equal parts in the point F, and into two unequal parts in the point E,

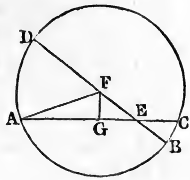
$BE \cdot ED + EF^2 = FB^2 = AF^2 = AE^2 + EF^2$. The rectangle BE, ED, together with the square on EF, is equal to the square on FB (II. 5); that is, the square on AF.

But the square on AF is equal to the squares on AE, EF (I. 47);

Therefore the rectangle BE, ED, together with the square on EF, is equal to the squares on AE, EF (Ax. 1).

Take away the common square on EF;

$\therefore BE \cdot ED = AE^2 = AE \cdot EC$. Then the remaining rectangle BE, ED is equal to the remaining square on AE; that is, to the rectangle AE, EC, since AE is equal to EC, CASE III.—Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in the point E, but not at right angles.



CONSTRUCTION.—Bisect BD in F, then F is the centre of the circle.

Bisect BD in F the centre.

Join AF, and from F draw FG perpendicular to AC (I. 12).

Draw FG at right angles to AC.

PROOF.—Then AG is equal to GC (III. 3).

∴ AG = GC.

Therefore the rectangle AE, EC, together with the square on EG, is equal to the square on AG (II. 5).

Now, $AE \cdot EC + EG^2 = AG^2$.

To each of these equals add the square on GF;

Then the rectangle AE, EC, together with the squares on EG, GF, is equal to the squares on AG, GF (Ax. 2).

But the squares on EG, GF are equal to the square on EF;

And the squares on AG, GF are equal to the square on AF (I. 47).

Therefore the rectangle AE, EC, together with the square on EF, is equal to the square on AF; that is, the square on FB.

$\therefore AE \cdot EC + EF^2 = AF^2 = FB^2 = BE \cdot ED + EF^2$.

But the square on FB is equal to the rectangle BE, ED, together with the square on EF (II. 5);

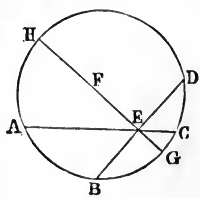
Therefore the rectangle AE, EC, together with the square on EF, is equal to the rectangle BE, ED, together with the square on EF.

Take away the common square on EF;

And the remaining rectangle AE, EC is equal to the remaining rectangle BE, ED (Ax. 3).

$\therefore AE \cdot EC = BE \cdot ED$.

CASE IV.—Let neither of the straight lines AC, BD pass through the centre.



CONSTRUCTION.—Take the centre F (III. 1), and through E, the intersection of the lines AC, BD, draw the diameter GEFH.

PROOF.—Because the rectangle GE, EH is equal, as has been shown, to the rectangle AE, EC, and also to the rectangle BE, ED;

Again, $GE \cdot EH = AE \cdot EC$ and $= BE \cdot ED$, as just shown.

Therefore the rectangle AE, EC is equal to the rectangle BE, ED (Ax. 1).

$\therefore AE \cdot EC = BE \cdot ED$.

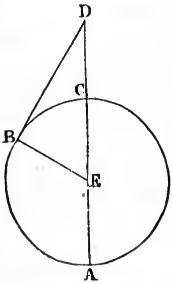
Therefore, if two straight lines, &c. *Q.E.D.*

Proposition 36.—Theorem.

If from a point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

Let D be any point without the circle ABC, and let DCA, DB be two straight lines drawn from it, of which DCA cuts the circle, and DB touches it.

The rectangle AD, DC shall be equal to the square on DB.



CASE I.—Let DCA pass through the centre E, and join EB.

PROOF.—Then EBD is a right angle (III. 18).

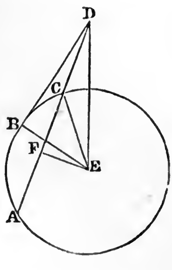
AD·DC + EC² = ED². And because the straight line AC is bisected in E, and produced to D, the rectangle AD, DC, together with the square on EC, is equal to the square on ED (II. 6).

But EC is equal to EB;

Therefore the rectangle AD, DC, together with the square on EB, is equal to the square on ED.

But the square on ED is equal to the squares on EB, BD, because EBD is a right angle (I. 47);

∴ AD·DC + EB² = BD²+EB². Therefore the rectangle AD, DC, together with the square on EB, is equal to the squares on EB, BD.



Take away the common square on EB;

∴ AD·DC = BD². Then the remaining rectangle AD, DC is equal to the square on DB (Ax. 3).

CASE II.—Let DCA not pass through the centre of the circle ABC.

Draw EF perpendicular to AC. CONSTRUCTION.—Take the centre E (III. 1), and draw EF perpendicular to AC (I. 12), and join EB, EC, ED.

PROOF.—Because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it also bisects it (III. 3);

Therefore AF is equal to FC. $\therefore$ AF = FC.

And because the straight line AC is bisected in F and produced to D, the rectangle AD, DC, together with the square on FC, is equal to the square on FD (II. 6). $\therefore AD \cdot DC + FC^2 = FD^2$.

To each of these equals add the square on FE;

Therefore the rectangle AD, DC, together with the squares on CF, FE, is equal to the squares on DF, FE (Ax. 2).

But the squares on CF, FE are equal to the square on CE, because CFE is a right angle (I. 47);

And the squares on DF, FE are equal to the square on DE;

Therefore the rectangle AD, DC, together with the square on CE, is equal to the square on DE. $\therefore AD \cdot DC + EC^2 = DE^2$.

But CE is equal to BE;

Therefore the rectangle AD, DC, together with the square on BE, is equal to the square on DE.

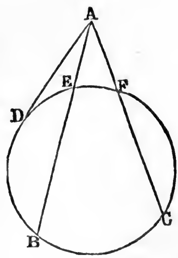
But the square on DE is equal to the squares on DB, BE, because EBD is a right angle (I. 47);

Therefore the rectangle AD, DC, together with the square on BE, is equal to the squares on DB, BE. $\therefore AD \cdot DC + BE^2 = DB^2 + BE^2$.

Take away the common square on BE;

Then the remaining rectangle AD, DC is equal to the square on DB (Ax. 3). $\therefore AD \cdot DC = DB^2$.

Therefore, if from any point, &c. *Q.E.D.*



Corollary.—If from any point without a circle there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another; namely, the rectangle BA, AE is equal to the rectangle CA, AF; for each of them is equal to the square on the straight line AD, which touches the circle.

Proposition 37.—Theorem.

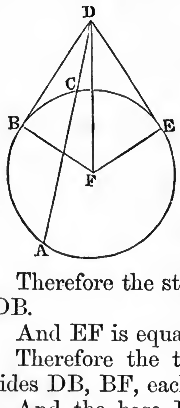
If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on *the line which meets the circle, the line which meets the circle shall touch it.*

Let any point D be taken without the circle ABC, and from it let two straight lines, DCA, DB, be drawn, of which DCA cuts the circle, and DB meets it; and let the rectangle AD, DC be equal to the square on DB.

Then DB shall touch the circle.

Draw DE touching the circle.

CONSTRUCTION.—Draw the straight line DE, touching the circle ABC (III. 17);



Find F the centre (III. 1) and join FB, FD, FE.

PROOF.—Then FED is a right angle (III. 18).

And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE (III. 36).

But the rectangle AD, DC is equal to the square on DB (Hyp.);

Therefore the square on DE is equal to the square on DB (Ax. 1);

Then DE = DB.

Therefore the straight line DE is equal to the straight line DB.

And EF is equal to BF (I. Def. 15);

Therefore the two sides DE, EF are equal to the two sides DB, BF, each to each;

And triangles DBF and DEF are equal in every respect.

And the base DF is common to the two triangles DEF, DBF;

Therefore the angle DEF is equal to the angle DBF (I. 8).

But DEF is a right angle (Const.);

Therefore also DBF is a right angle (Ax. 1).

∴ DBF is a right angle;

And BF, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches the circle (III. 16, Cor.);

and therefore DB touches the circle.

Therefore DB touches the circle ABC.

Therefore, if from a point, &c. *Q.E.D.*

EXERCISES ON BOOK III.

Prop. 1—15.

1. Two straight lines intersect. Describe a circle passing through the point of intersection and two other points, one in each straight line.

2. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circumferences are equal.

3. Show that the centre of a circle may be found by drawing perpendiculars from the middle points of any two chords.

4. Through a given point, which is not the centre, draw the least line to meet the circumference of a given circle, whether the given point be within or without the circle.

5. The sum of the squares of any two chords in a circle, together with four times the sum of the squares of the perpendiculars on them from the centre, is equal to twice the square of the diameter.

6. With a given radius, describe a circle passing through the centre of a given circle and a point in its circumference.

7. If two chords of a circle are given in magnitude and position, describe the circle.

8. Describe a circle which shall touch a given circle in a given point, and shall also touch another given circle.

9. If, from any point in the diameter of a circle, straight lines be drawn to the extremities of a parallel chord, the squares of these lines are together equal to the squares of the segments into which the diameter is divided.

10. If two circles touch each other externally, and parallel diameters be drawn, the straight line joining extremities of these diameters will pass through the point of contact.

11. Draw three circles of given radii touching each other.

12. If a circle of constant radius touch a given circle, it will always touch the same concentric circle.

13. If a chord of constant length be inscribed in a circle, it will always touch the same concentric circle.

14. The locus of the middle points of chords parallel to a given straight line is a line drawn through the centre perpendicular to the parallel chords.

Prop. 16—30.

15. Show that the two tangents from an external point are equal in length.

16. Draw a tangent to a given circle, making a given angle with a given straight line.

17. If a polygon having an even number of sides be inscribed in a circle, the sums of the alternate angles are equal.

18. If such a polygon be described about a circle, the sums of the alternate sides are each equal to half the perimeter of the polygon.

19. If a polygon be inscribed in a circle, the sum of the angles in the segments exterior to the polygon, together with two right angles, is equal to twice as many right angles as the polygon has sides.

20. Draw the common tangents to two given circles.

21. From a given point draw a straight line cutting a given circle, so that the intercepted segment of the line may have a given length.

22. The straight line which joins the extremities of equal arcs towards the same parts are parallel.

23. Any parallelogram described about a circle is equilateral, and any parallelogram inscribed in a circle is rectangular.

24. Two opposite sides of a quadrilateral circumscribing a circle touch the circle at extremities of a diameter. Show that the area of the quadrilateral is equal to one-half the rectangle contained by the diameter, and the sum of the other sides.

Prop. 31—37.

25. A tangent is drawn to a circle of 21 inches diameter from a point 17·5 inches from the centre. Find the length of the tangent.

26. Show that a man 6 feet high, standing at the sea level, has a view of 3 miles (approximately) in every direction, along a horizontal plane passing through his eye.

27. The angle between a tangent to a circle and the chord through the point of contact is equal to half the angle which the chord subtends at the centre.

28. From a given point P, within or without a circle, draw a straight line cutting the circle in A and B such that PA shall be three-fourths of PB.

Ex. Let the circle be of 1·5 inches radius, and point P 3·5 inches from its centre. Prove your construction by scale.

29. The greatest rectangle which can be inscribed in a circle is a square whose area is equal to half that of the square described upon the diameter as side.

30. If the base and vertical angle of a triangle remain constant in magnitude while the sides vary, show that the locus of the middle point of the base is a circle.

31. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex, to construct the triangle.

32. Show that the locus of the middle point of a straight line, which moves with its extremities upon two straight lines at right angles to each other, is a circle.

33. Show how to produce a straight line, that the rectangle contained by the given line, and the whole line thus produced, may be equal to the square of the part produced.

Ex. If the length of the given line be 2 inches, show geometrically that the length of the part produced is $(\sqrt{5} + 1)$ inches.

34. Given the height and chord of a segment of a circle to find the radius of the circle.

Ex. If the chord be 24 inches, and the height of the segment be 4 inches, show that the radius of the circle is 20 inches.

35. Show that the locus of the middle points of chords which pass through a fixed point is the circle described as diameter upon the line joining the fixed point and the centre of the given circle.

36. Let ACDB be a semicircle whose diameter is AB, and AD, BC any two chords intersecting in P; prove that

$$AB^2 = DA \cdot AP + CB \cdot BP.$$

MATHEMATICS.

SECOND STAGE.

SECTION II.

ALGEBRA.

CHAPTER I.

QUADRATIC EQUATIONS.

1. Equations of this class, when reduced to a rational integral form, contain the *square* of the unknown quantity, but no higher powers.

When the equation contains the *square only* of the unknown quantity, and *not the first power*, it is called a *pure quadratic.*

Thus, $x^2 - 25 = 0$, $4x^2 + 10 = 19$, $5x^2 = 180$, are *pure quadratics.* When the equation contains the *square* of the unknown quantity, *as well as the first power,* it is called an *adfected quadratic.*

Thus, $x^2 - 5x = 6$, $x^2 - x - 30 = 0$, $2x^2 + x + 3 = 6$, are *adfected quadratics.*

Pure Quadratics.

2. To solve these, we treat them exactly as we do simple equations, until we obtain the value of the *square* of the unknown quantity; then, taking the square root of each side, we obtain the *value* of the unknown quantity. It will be seen (Stage I., Alg.; Art. 35) that the unknown quantity in a pure quadratic has always two values, *equal* in magnitude, but of *opposite* sign.

Ex. 1. Given $3x^2 + 12 = 687$, find x.

We have $3x^2 = 687 - 12 = 675$, or $x^2 = 225$.

$$\therefore x = \pm 15.$$

Ex. 2. Given $\dfrac{2x+7}{2x^2-7x} - \dfrac{2x-7}{2x^2+7x} = \dfrac{56}{6x^2-99}$, find x.

Bringing the fractions on the first side to a common denominator, we have—

$$\frac{(2x+7)^2 - (2x-7)^2}{x(4x^2-49)} = \frac{56}{6x^2-99},$$

$$\text{or, } \frac{56x}{x(4x^2-49)} = \frac{56}{6x^2-99},$$

$$\text{or, } \frac{1}{4x^2-49} = \frac{1}{6x^2-99}, \text{ clearing of fractions;}$$

then, $6x^2 - 99 = 4x^2 - 49$, from which

$$x^2 = 25$$

$$x = \pm 5.$$

Adfected Quadratics.

3. *Solution by completing the square.*

Suppose we have given the equation $x^2 + 2ax = 3a^2$, to find x.

It will be remembered that $(x^2 + 2ax + a^2)$ is a *perfect square*, viz., the square of $(x + a)$. It is evident, then, that the first side of our equation will become a perfect square by the addition of a^2 as a third term.

Adding, then, a^2 to each side of the given equation, we have—

$$x^2 + 2ax + a^2 = 4a^2, \text{ or}$$

$$(x+a)^2 = 4a^2.$$

Taking now the square root of each side, we have—

$$x + a = \pm 2a.$$

$$\therefore x = \pm 2a - a.$$

$$= a \text{ or } -3a.$$

We may remark that the quantity a^2, added to the expression $x^2 + 2ax$ in order to make it a *perfect square*, is the square of half the coefficient of x. The operation itself is called *completing the square.*

An adfected quadratic may therefore be solved as follows:—

1. *Reduce and arrange it until all the terms involving* x *are on the first side, and the coefficient of* x^2 *is unity.*

2. COMPLETE THE SQUARE *by adding to each side the* SQUARE OF HALF *the coefficient of* x.

3. *Take the square root of each side, put a double sign to the second side, and transpose the term of the first side not involving* x.

Ex. 1. Solve the equation $3x^2 + 18x + 4 = 52$.

We have $3x^2 + 18x = 52 - 4 = 48$; or, dividing each side by 3, $\quad x^2 + 6x = 16$.

Here, the coefficient of x is 6, the half of which is 3. Adding then the square of 3 to each side, to *complete the square*, we have—

$$x^2 + 6x + 3^2 = 16 + 9 = 25.$$

Taking the square root of each side, we have—

$$x + 3 = \pm 5.$$

$$\therefore x = \pm 5 - 3 = 2 \text{ or } -8.$$

Ex. 2. Given $\dfrac{x+3}{x+4} - \dfrac{x+1}{x+2} = \dfrac{4x+9}{2x+7} - \dfrac{12x+17}{6x+16}$, find x.

We have $\left(1 - \dfrac{1}{x+4}\right) - \left(1 - \dfrac{1}{x+2}\right) = \left(2 - \dfrac{5}{2x+7}\right)$
$- \left(2 - \dfrac{15}{6x+16}\right)$;

or, simplifying, $\dfrac{1}{x+2} - \dfrac{1}{x+4} = \dfrac{15}{6x+16} - \dfrac{5}{2x+7}$;

or, $\dfrac{(x+4)-(x+2)}{(x+2)(x+4)} = \dfrac{15(2x+7) - 5(6x+16)}{(6x+16)(2x+7)}$;

or, simplifying, $\dfrac{2}{x^2+6x+8} = \dfrac{25}{12x^2+74x+112}$;

or, clearing of fractions—

$$24x^2 + 148x + 224 = 25x^2 + 150x + 200;$$

or, transposing and reducing, $-x^2 - 2x = -24$;
or, changing the sign of each side, then—

$$x^2 + 2x = 24;$$

or, completing the square, $x^2 + 2x + 1^2 = 24 + 1 = 25$.

Taking the square root of each side, we have—

$$x + 1 = \pm 5$$
$$\therefore x = \pm 5 - 1 = 4 \text{ or } -6.$$

Ex. 3. Solve the equation $x^2 + 6x + 25 = 0$.

We have $x^2 + 6x = -25$;

or, completing the square—

$$x^2 + 6x + 3^2 = -25 + 9 = -16;$$

or, extracting the square root of each side—

$$x + 3 = \pm\sqrt{-16}$$
$$\therefore x = -3 \pm \sqrt{-16}.$$

As the quantity $\sqrt{-16}$ has no exact or approximate value, the given equation has no real roots. The roots are therefore said to be imaginary.

4. *Solution by breaking into factors.*

We have seen (Stage I., Alg., Art. 30) that it is often easy to find by inspection the factors of quadratic expressions. We may make use of this knowledge to solve quadratic equations.

Ex. 1. Solve the equation $x^2 + 5x = 66$.

Transposing all to the first side, we have—

$$x^2 + 5x - 66 = 0.$$

And, resolving the first side into its elementary factors, we get—

$$(x - 6)(x + 11) = 0.$$

Now, if either of these factors is put equal to 0, the equation is satisfied.

Hence, we have, $x - 6 = 0$, and $x + 11 = 0$;
or, $x = 6$, and $x = -11$.

$\therefore$ 6 and -11 are the roots required.

Ex. 2. Given $x^2 - (a + b)x + ab = 0$, find x.

We have $(x - a)(x - b) = 0$.

Now this equation is satisfied by making either of the factors $= 0$.

$$\left.\begin{array}{r} \text{Hence, } x - a = 0, \text{ or } x = a; \\ \text{and, } x - b = 0, \text{ or } x = b; \end{array}\right\}$$

$\therefore a, b$ are the roots required.

Ex. I.

1. $4x^2 - 7 = 29$.
2. $5x^2 + 6 = 86$.
3. $\dfrac{3x^2 - 7}{2} = 34$.
4. $\frac{5}{2}x^2 + \frac{3}{8} = 6$.
5. $3x - \dfrac{12}{x} = 0$.
6. $2(a^2 + b^2) - x^2 = (a - b)^2$.
7. * $x\sqrt{24 + x^2} = 4 + x^2$.
8. * $\sqrt{x - a} = \sqrt{x - \sqrt{b^2 + x^2}}$.
9. $\dfrac{ab}{x} + \sqrt{\dfrac{a^2b^2}{x^2} - b^2} = x$.
10. $\dfrac{1}{ax - \sqrt{a^2x^2 - 1}} - \dfrac{1}{ax + \sqrt{a^2x^2 - 1}} = ax$.
11. $\dfrac{\sqrt[3]{a^2 + x^2} + \sqrt[3]{a^2 - x^2}}{\sqrt[3]{a^2 + x^2} - \sqrt[3]{a^2 - x^2}} = b$.
12. $\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = a$.
13. $x^2 - 5x + 6 = 0$.
14. $x^2 - x = 72$.
15. $3x^2 - x = 2$.
16. $4x^2 - 9x = 28$.
17. $6x^2 + x - 35 = 0$.
18. $x^2 + 6\frac{3}{5}x + 8 = 0$.
19. $ax^2 + bx + c = 0$.
20. $3x - \dfrac{14}{2x - 3} = 4x - 7$.
21. $x - \dfrac{42}{x} = 1$.
22. $2x + 1 - \dfrac{26}{2x + 1} = 11$.

* See Remark, page 304.

23. $(a + bx)(cx + d) = (a + b)(c + dx).$

24. $(a - b)(x^2 - b^2) = (a + b)(x - b)^2.$

25. $(a + b)(x - a)(x - b) = abx.$

26. $\frac{a}{x} + \frac{x}{b} = \frac{b}{x} + \frac{x}{a} + \frac{a^2 - b^2}{ab}.$

27. $ax^2 - (c + d)x = bx^2 - \frac{cd}{a - b}.$

28. $a^3x^2 - (a - b)^2 x + a^3 = b^3x^2 + (a^2 + ab + b^2)^2x + b^3.$

29. $\frac{23}{x + 2} - \frac{2}{x} = 3\frac{1}{3}.$

30. $\frac{x + 1}{2x + 3} + \frac{7}{13} = \frac{2x + 5}{4x - 5}.$

31. $\frac{2x + 9}{x} + \frac{x}{2x + 9} = 5\frac{1}{5}.$

32. $\frac{x + 2}{x + 3} + \frac{x - 10}{x + 8} = 0.$

33. $\frac{3x - 7}{4x - 2} + \frac{5x + 3}{7x + 4} = \frac{5}{9}.$

34. $\frac{6x + 5}{2x - 7} + \frac{4x - 1}{x - 2} = \frac{7x + 1}{x - 3}.$

35. $\frac{2x + 1}{x + 2} - \frac{x + 2}{4x + 4} = \frac{7x + 8}{4x + 13}.$

36. $\frac{3x}{x - 2} - \frac{2x + 9}{x + 2} = \frac{2x + 24}{2x - 1}.$

37. $\frac{3x^2 - 2x + 7}{6x^2 - 4x + 11} = \frac{x^2 + 7x - 3}{2x^2 + 14x - 9}.$

38. $\frac{3x^2 + 10}{x} - \frac{x^2 + 2x + 3}{x + 2} = \frac{2x^2 + 2x + 10}{x + 1}.$

39. * $\sqrt{5x + 6} + \sqrt{x + 10} = 4.$

40. $\sqrt{3x - 4} - \sqrt{2x - 4} = \sqrt{x}.$

41. * $\sqrt{a^2 + b^2 + x} + \sqrt{a^2 + b^2 - x} = \sqrt{\frac{2ax}{b}}.$

* See Remark, page 304.

42. * $\sqrt{2a + x} + \sqrt{4a - x} = 2\sqrt{x - a}$.

43. * $\sqrt{ax + b} + \sqrt{bx + a} = \sqrt{(a - b)x + a + b + 2\sqrt{ab}}$.

44. * $\sqrt{x + \frac{b}{a}} + \sqrt{x + \frac{a}{b}} = \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{b}}\right)\sqrt{a + b}$.

45. $\left(\frac{a}{b} - 1\right) \cdot \frac{x - c}{b} = \left(\frac{a^2}{b^2} + 1\right) \cdot \frac{a + b + c}{x} - 2$.

46. $(a - b)(x + b - c)(x + c - a) + a^2(x + b - c) + b^2(x + c - a) + c^2(x + a - b = 0)$.

47. Show that, if x be real, the value of $x + \frac{1}{x}$ cannot lie between 2 and $-$ 2.

48. If, in the equation $x - \frac{x^2}{20} = a$, the quantity x be real, show that a cannot be greater than 5.

Equations which may be Solved like Quadratics.

5. Certain equations of a degree higher than the second may be solved like quadratics. It will be seen that, although it is impossible to lay down definite rules for the treatment of every such equation, the object to be attained is either (1.) To throw the equation into the form—

$$X^2 + pX = q,$$

when X is an expression containing the unknown quantity, and solving when possible this equation for X; or (2.), To strike out from each side a factor containing the unknown quantity, thus reducing the dimensions of the equation, and obtaining a value or values of the unknown quantity by equating this factor to zero.

Ex. 1. Solve the equation $x^4 + 144 = 25x^2$.

We have, transposing, $x^4 - 25x^2 = -144$;

or, $(x^2)^2 - 25(x^2) = -144$;

* See Remark, page 304.

or, completing the square—

$$(x^2)^2 - 25\,(x^2) + (\tfrac{25}{2})^2 = \tfrac{625}{4} - 144 = \tfrac{49}{4}.$$

$$\therefore x^2 - \tfrac{25}{2} = \pm \tfrac{7}{2}.$$

$$\therefore x^2 = \tfrac{25}{2} \pm \tfrac{7}{2} = 16 \text{ or } 9.$$

$$\text{Hence, } x = \pm 4 \text{ or } \pm 3.$$

The given equation has therefore *four* roots, being as many roots as the *degree* of the equation.

Ex. 2. Solve $x^6 + 3\,x^3 = 88$.

We have, $(x^3)^2 + 3\,(x^3) = 88$; or, transposing,

$$(x^3)^2 + 3\,(x^3) - 88 = 0;$$

or, $(x^3 - 8)\,(x^3 + 11) = 0$.

Hence, the given equation is satisfied by—

$$x^3 - 8 = 0, \text{ and also by } x^3 + 11 = 0.$$

We have then $x^3 - 2^3 = 0$, and $x^3 + (\sqrt[3]{11})^3 = 0$;

or, breaking into factors, we have—

$$(x - 2)\,(x^2 + x \cdot 2 + 2^2) = 0, \text{ and}$$

$$(x + \sqrt[3]{11})\left\{x^2 - x \cdot \sqrt[3]{11} + (\sqrt[3]{11})^2\right\} = 0.$$

From the *first* of these equations, we get—

$$x - 2 = 0 \text{ or } x = 2,$$

and $x^2 + 2\,x + 4 = 0$, from which two other roots may be found.

And from the *second* equation, we get—

$$x + \sqrt[3]{11} = 0, \text{ or } x = -\sqrt[3]{11},$$

and $x^2 - x\sqrt[3]{11} + \sqrt[3]{121} = 0$, which gives two other roots.

We have therefore shown how to obtain the six roots of the given equation.

Ex. 3. Solve—

$$23\,x^2 - 75\,x - 6\,x\sqrt{4\,x^2 - 9\,x + 9} + 40 = 0.$$

Changing the sign of each side, and transposing, we have—

$$6\,x\sqrt{4\,x^2 - 9\,x + 9} = 23\,x^2 - 75\,x + 40;$$

adding to each side $13\,x^2 - 9\,x + 9$, then—

$$13\,x^2 - 9\,x + 9 + 6\,x\sqrt{4\,x^2 - 9\,x + 9} = 36\,x^2 - 84\,x + 49;$$

or, $(4\,x^2 - 9\,x + 9) + 2\,(3\,x)\sqrt{4\,x^2 - 9\,x + 9} + (3\,x)^2$
$= (6\,x)^2 - 2\,(6\,x) \cdot 7 + 7^2;$

or, $(\sqrt{4x^2 - 9x + 9} + 3x)^2 = (6x - 7)^2$.

$\therefore \sqrt{4x^2 - 9x + 9} + 3x = \pm (6x - 7)$,............(1.);

or, taking the upper sign—

$$\sqrt{4x^2 - 9x + 9} = (6x - 7) - 3x = 3 - 7.$$

Hence, squaring each side—

$$4x^2 - 9x + 9 = 9x^2 - 42x + 49;$$

$$\text{or, } 5x^2 - 33x + 40 = 0;$$

$$\text{or, } (x - 5)(5x - 8) = 0.$$

$$\therefore x = 5 \text{ or } \tfrac{8}{5}.$$

Again, taking the lower sign of (1), we have—

$\sqrt{4x^2 - 9x + 9} = -(6x - 7) - 3x = -9x + 7$.........(2.);

or, squaring—

$$4x^2 - 9x + 9 = 81x^2 - 126x + 49;$$

$$\text{from which } 77x^2 - 117x + 40 = 0;$$

$$\text{or, } (x - 1)(77x - 40) = 0.$$

$$\therefore x = 1 \text{ or } \tfrac{40}{77}.$$

Hence, the roots of the given equation are 5, $\tfrac{8}{5}$, 1, $\tfrac{40}{77}$.

Remark.—If we proceed to verify these values of x, we shall find that the last two values—viz., 1 and $\tfrac{40}{77}$—will not satisfy the given equation unless we obtain the value of $\sqrt{4x^2 - 9x + 9}$ by means of the equation from which these last roots were found.

Thus from (2) we find, on putting 1 and $\tfrac{40}{77}$ successively for x—

$$\sqrt{4x^2 - 9x + 9} = -9(1) + 7 = -2,$$

$$\text{and } \sqrt{4x^2 - 9x + 9} = -9(\tfrac{40}{77}) + 7 = 2\tfrac{25}{77};$$

and on substituting either of these values of $\sqrt{4x^2 - 9x + 9}$ along with the corresponding value of x, the given equation is satisfied.

Ex. 4. Solve—

$$(x + b + c)(x + b - c)(b + c - x)$$
$$= (a + b + c)(a + b - c)(b + c - a).$$

By inspection we see that a is one of the roots. We shall therefore so arrange the equation as to be able to strike out $x - a$ as a factor of each side (Art. 5).

We have—

$$\frac{(x + b + c)(x + b - c)}{(a + b + c)(a + b - c)} = \frac{b + c - a}{b + c - x};$$

$$\text{or, } \frac{(x + b)^2 - c^2}{(a + b)^2 - c^2} = \frac{b + c - a}{b + c - x};$$

or, taking the *difference* of numerator and denominator—

$$\frac{(x + b)^2 - (a + b)^2}{(a + b)^2 - c^2} = \frac{x - a}{b + c - x};$$

$$\text{or, } \frac{(x - a)(x + 2b - a)}{(a + b)^2 - c^2} = \frac{x - a}{b + c - x}.$$

Dividing each side by $x - a$, we have—

$$\frac{x + 2b - a}{(a + b)^2 - c^2} = \frac{1}{b + c - x}, \text{ and } x - a = 0, \text{ or } x = a;$$

Hence also $(x + 2b - a)(b + c - x) = (a + b)^2 - c^2$; an ordinary quadratic from which two other roots may be determined.

Ex. 5. * Solve the equation—

$$a^2b^2x^{\frac{1}{q}} - 4a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{p+q}{2pq}} = (a - b)^2x^{\frac{1}{p}}.$$

Dividing each side by $x^{\frac{1}{p}}$, we have—

$$a^2b^2x^{\frac{p-q}{pq}} - 4a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{p-q}{2pq}} = (a - b)^2;$$

$$\text{or, } \left(abx^{\frac{p-q}{2pq}}\right)^2 - 4a^{\frac{1}{2}}b^{\frac{1}{2}}\left(abx^{\frac{p-q}{2pq}}\right) = (a - b)^2;$$

or, completing the square—

$$\left(abx^{\frac{p-q}{2pq}}\right)^2 - 4a^{\frac{1}{2}}b^{\frac{1}{2}}\left(abx^{\frac{p-q}{2pq}}\right) + \left(2a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^2$$

$$= (a - b)^2 + 4ab = (a + b)^2.$$

* We shall *assume* in this example that the laws of multiplication and division proved in Algebra, Stage I., Arts. 25, 27, hold for fractional indices.

Hence, extracting the square root—

$$abx^{\frac{p-q}{2pq}} - 2a^{\frac{1}{2}}b^{\frac{1}{2}} = \pm (a + b).$$

$$\therefore abx^{\frac{p-q}{2pq}} = \pm (a + b) + 2a^{\frac{1}{2}}b^{\frac{1}{2}} = (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 \text{ or } -(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2;$$

$$\text{or, } x^{\frac{p-q}{2pq}} = \left(\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)^2 \text{ or } -\left(\frac{a^{\frac{1}{2}} - b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)^2;$$

$$= \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}\right)^2 \text{ or } -\left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}}\right)^2.$$

Then, raising each side to the $(2pq)$th power, we have—

$$x^{p-q} = \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}\right)^{4pq} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}}\right)^{4pq}.$$

Hence, taking the $(p - q)$th root, we get—

$$x = \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}}$$

$$= \left(\frac{1}{b^{\frac{1}{2}}} \pm \frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}}$$

We have also, since the factor $x^{\frac{1}{p}}$ has been struck out—

$$x^{\frac{1}{p}} = 0, \text{ and } \therefore x = 0 \text{ as another solution.}$$

Ex. II.

1. $4x^4 - 11x^2 = 225.$
2. $5x^6 - 17x^3 = 184.$
3. $x^2 + 5x^{-2} = 25\frac{1}{5}.$
4. $(x^2 + 3x + 3)^2 + 2x^2 = 189 - 6x.$
5. $3x^{-4} - 17x^{-2} = 1450.$
6. $x^n + \frac{6}{x^n} = 5.$
7. $\sqrt{x + 12} + \sqrt[4]{x + 12} = 6.$
8. $\sqrt{x^2 + 9} + 20 = x^2 + 9.$
9. $\left(\frac{12}{x} - x\right)^2 + \frac{60}{x} = 6 + 5x.$

10. $(3x + 4) + \sqrt{3x + 4} - 12 = 0.$

11. $x^2 + 2x = \dfrac{2}{x + 4} - 1.$

12. $\dfrac{\sqrt[n]{a - x}}{x} - \dfrac{\sqrt[n]{a - x}}{a} = \dfrac{\sqrt[n]{x}}{a}.$

13. $9x + 24\sqrt{x} - \dfrac{4}{\sqrt{x}}\left(\dfrac{1}{\sqrt{x}} + 9\right) = 65.$

14. $x = \sqrt{a} \cdot \sqrt{b - x} - \sqrt{b} \cdot \sqrt{x + a}.$

15. $\dfrac{1}{(a - \sqrt{a^2 - x^2})} + \dfrac{1}{(a + \sqrt{a^2 - x^2})^2} = \dfrac{a^2 + 2x^2}{x^4}.$

16. $\dfrac{x^2}{b} - 2\sqrt{\dfrac{a^2 - ax + x^2}{b}} = \dfrac{ax}{b} - 1.$

17. $x^{\frac{p+q}{2pq}} - \dfrac{1}{2} \cdot \dfrac{a^2 - b^2}{a^2 + b^2}\left(x^{\frac{1}{p}} + x^{\frac{1}{q}}\right) = 0.$

18. $\sqrt{\dfrac{a^2 + x^2}{x^2}} - \sqrt{\dfrac{x^2}{a^2 + x^2}} = 1.$

19. $(a^{4m} + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1).$

20. $\dfrac{x^2}{x^2 - 9} - \dfrac{975}{256} \cdot \dfrac{x^2 - 9}{x^2} + \dfrac{7}{8} = 0.$

21. $\dfrac{12}{x + 1}\left\{\dfrac{3}{x + 1} + \dfrac{2}{x}\right\} + \dfrac{4}{x^2} = 9.$

22. $6x^2 - 5x - 8\sqrt{3x^2 + 5x - 4} = 12.$

23. $2x^2 + 5x - 2x\sqrt{2x^2 - 7x + 1} = 35.$

24. $20x^2 - 9ax - 8x\sqrt{5x^2 - 3ax + 2a^2} = 7a^2.$

25. $x^2(x - 2)^2 + 6x^2(x - 2) = 24x + 36 - 5x^2.$

26. $2x^2 + 20x - \sqrt{3x^2 - x + 5} = 105.$

27. $4x^2 + 12x - 2x\sqrt{4x^2 - 2x + 19} = 30.$

28. $x^4 + x^2 + 1 = a(x^2 + x + 1).$

29. $x^6 - 1 = 0.$

30. $(x^2 - a)^3 + (x^2 - b)^3 + (x^2 - c)^3$
$= 3(x^2 - a)(x^2 - b)(x^2 - c) + 9ax^4 - a(a + b + c)^2.$

31. $(x - b)(x^2 - c^2) = (a - b)(a^2 - c^2).$

32. $(x - a)(x - b)(x - c) = (m - a)(m - b)(m - c).$

33. $(x - a)(x - b)(x - c) = (a + b)(a + c)(b + c).$

34. $(n - 1)x(x^2 + ax + a^2) = a^3 - x^3.$

35. $(ax - b)^3 + (cx - d)^3 = (a + c)^3x^3 - (b + d)^3.$

36. $x^{2^n} + x^{2^{n-1}} + 1 = a(x^2 + x + 1).$

Simultaneous Quadratic Equations.

6. The following worked examples are given as specimens of the methods to be employed, but it must be understood that practice alone will give the student complete mastery over equations of this class.

Ex. 1. Solve
$$\left.\begin{aligned} x^2 + y^2 &= 20 \ldots\ldots(1.) \\ x + y &= 6 \ldots\ldots(2.) \end{aligned}\right\}$$

As we have given the *sum* of the unknown quantities, we shall work for the *difference.*

From (2), multiplying each side by 2, we have—

$$2x^2 \qquad + 2y^2 = 40$$

and from (1), squaring, $\quad x^2 + 2xy + y^2 = 36$

Then, subtracting, $\quad x^2 - 2xy + y^2 = 4;$

and, taking the square root, we have $x - y = \pm 2 \ldots\ldots(3).$

(2) + (3), then $2x = 8$ or 4, and $\therefore x = 4$ or 2.
(2) − (3), then $2y = 4$ or 8, and $\therefore y = 2$ or 4.

NOTE.—Having found that $x = 4$, $y = 2$, we might have told by inspection that the values $x = 2$, $y = 4$, would also satisfy the given equations, for x and y are similarly involved in both equations.

Ex. 2. Solve
$$\left.\begin{aligned} x^2 - y^2 &= 5 \ldots\ldots(1.) \\ xy &= 6 \ldots\ldots(2.) \end{aligned}\right\}$$

As we have given here the *difference of the squares* of the unknown quantities, it will be convenient to work for the *sum of the squares.*

From (1.), squaring, $x^4 - 2x^2y^2 + y^4 = 25$,
and from (2.), squaring, &c., $4x^2y^2 = 144$
Then, adding, $x^4 + 2x^2y^2 + y^4 = 169$
and taking the square root, $x^2 + y^2 = \pm = 13 \ldots\ldots (3)$.

(3.) + (1.), then, $2x^2 = 18$ or -8, or $x^2 = 9$ or -4,
and $\therefore x = \pm 3$ or $\pm 2\sqrt{-1}$.

(3.) − (1.), then, $2y^2 = 8$ or -18, or $y^2 = 4$ or -9,
and $\therefore y = \pm 2$ or $\pm 3\sqrt{-1}$.

NOTE.—The student will see that the pairs of values which satisfy the given equations are, $x = 3, y = 2$; $x = -3, y = -2$; $x = 2\sqrt{-1}$, $y = 3\sqrt{-1}$; $x = -2\sqrt{-1}$, $y = -3\sqrt{-1}$.

Ex. 3. Solve
$$\left.\begin{aligned} x^2 + y &= 11x \ldots\ldots\ldots\ldots\ldots\ldots (1.) \\ y^2 + x &= 11y \ldots\ldots\ldots\ldots\ldots\ldots (2.) \end{aligned}\right\}$$

Subtracting, then, $x^2 - y^2 - x + y = 11x - 11y$;
or, $x^2 - y^2 = 12(x - y)$.

Now $(x - y)$ is a factor of each side, and hence, striking it out, we have—

$$x + y = 12 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.),$$
and also $$x - y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.).$$

Equations (3.) and (4.) may not be used as simultaneous equations, but *each of them* may be used in turn with either of the given equations.

Thus, taking equations (3.) and (1.), we have—

(1.) − (3.), $x^2 - x = 11x - 12$,
from which $x = 6 \pm 2\sqrt{6}$;
and hence from (3.), by substitution, we easily get-
$$y = 6 \mp 2\sqrt{6}.$$

Again, taking equations (4.) and (1.)—

we have, from (4.) $x = y$,
and $\therefore$ from (1.), $x^2 + x = 11x$ or $x^2 = 10x$,
from which $x = 10$ or 0;
and so, from (4.), $y = 10$ or 0.

Hence, the pairs of values satisfying the given equations are—

$$x = 10, y = 10; x = 0, y = 0;$$
$$x = 6 + 2\sqrt{6}, y = 6 - 2\sqrt{6};$$
$$x = 6 - 2\sqrt{6}, y = 6 + 2\sqrt{6}.$$

NOTE.—It is worth while remarking that when *each of the terms* of the given equations contain at least one of the unknown quantities, the values $x = 0$, $y = 0$ will always satisfy.

Ex. 4. Solve $3x^2 - 2xy = 55$ .... (1.)
$x^2 - 5xy + 8y^2 = 7$ .... (2.)

Multiplying the equations together *crosswise*, we get—

$$55x^2 - 275xy + 440y^2 = 21x^2 - 14xy;$$

or, transposing, $34x^2 - 261xy + 440y^2 = 0$;
or, $(2x - 5y)(17x - 88y) = 0$,
from which $2x = 5y$, and $17x = 88y$.

Each of these equations taken *in turn* with either (1.) or (2) will easily give the required values of x and y.

Ex. 5. $x^4 + y^4 = 337$ ....(1.),
$x + y = 7$ ....(2.)

From (2), raising each side to the *fourth power*, we have—

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401$;.........(3.)
(3) − (1), then $4x^3y + 6x^2y^2 + 4xy^3 = 2064$;
or, $2x^3y + 3x^2y^2 + 2xy^3 = 1032$;
or, arranging, $2xy(x + y)^2 - x^2y^2 = 1032$;
but from (2), $(x + y)^2 = 49$,
and hence, $2xy(49) - x^2y^2 = 1032$;
or, $x^2y^2 - 98xy + 1032 = 0$,
from which $xy = 12$ or 86,......(4.)

From (2) and (4), $x - y$ may now be easily obtained, and hence also the required values of x and y.

Ex. III.

1. $x + y = 5, xy = 6.$
2. $x - y = 2, xy = 15.$
3. $x^2 + y^2 = 25, xy = 12.$
4. $x^2 + y^2 = 20, x + y = 6.$

5. $x^2 + y^2 = 29,\ x - y = 3.$
6. $x^2 - y^2 = 13,\ (x - y)^2 = 1.$
7. $x^2 - y^2 = 27,\ xy = 18.$
8. $x^2 - y^2 = 12,\ x + y = 6.$
9. $x^2 + y^2 = 53,\ x^2 - y^2 = -45.$
10. $x^2 + xy = 28,\ y^2 + xy = 21.$
11. $x^2 + xy + y^2 = 19,\ xy - x^2 = -3.$
12. $x + y = 13,\ \sqrt{x} + \sqrt{y} = 5.$
13. $x^2 + xy + y^2 = 84,\ x + \sqrt{xy} + y = 14.$
14. $x^3 + y^3 = 35,\ x^2y + xy^2 = 30.$
15. $\frac{1}{x} + \frac{1}{y} = a,\ \frac{1}{x^2} + \frac{1}{y^2} = b.$
16. $x + y = a(x - y),\ x^2 + y^2 = b^2.$
17. $x^4 - y^4 = a,\ x^2 - y^2 = b.$
18. $x + y = 5,\ x^3 + y^3 = 35.$
19. $x + y = 5,\ x^5 + y^5 = 275.$
20. $x^2 + y^2 = \frac{13}{5}(x + y),\ xy = 6.$
21. $x - y = 2,\ x^3 - y^3 = 98.$
22. $xy(x + y) = a,\ x^2y^2(x^2 + y^2) = b.$
23. $xy(x + y) = 30,\ x^2y^2(x^5 + y^5) = 9900.$
24. $4x^2 - 3xy = 18,\ 5y^2 - 2xy = 8.$
25. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 35,\ x^{\frac{1}{2}} + y^{\frac{1}{2}} = \frac{30}{x^{\frac{1}{2}}y^{\frac{1}{2}}}.$
26. $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x,\ x^{\frac{1}{2}} + y^{\frac{1}{3}} = x.$
27. $xy + 6 = \frac{216}{xy},\ x + y + 4 = \frac{77}{x + y}.$
28. $(x + y)^2 + 2(x - y)^2 = 3(x + y)(x - y),\ x^2 + y^2 = 10.$
29. $x^2 + 10xy + y^2 = \frac{73}{5}(x^2 - y^2),\ x^2 + 5y^2 = x + 13y.$
30. $x^4 - 2x^2y^2 + y^4 = 1 + 4xy,\ x^2(x + 1) + y^2(y + 1) = xy.$
31. $3x + y - 9 = \frac{9x^2 - y^2 - 117}{3x - y - 6} = \frac{8x^2 - y^2 + 1}{4x + y + 1}.$

32. $\dfrac{x+2}{y-2}+\dfrac{y+2}{x-2}=5\tfrac{1}{2},\ \dfrac{x+2}{y+2}+\dfrac{y-2}{x-2}=1\tfrac{5}{6}.$

33. $\dfrac{\sqrt{x^2-y^2}+(x-y)}{\sqrt{x^2-y^2}-(x-y)}=3+2\sqrt{2},\ \dfrac{1}{y}-\dfrac{1}{x}=\dfrac{2}{xy}.$

34. $y-2\sqrt{5x^2+y+3}=32-5x^2,$

$\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^2+\left(\dfrac{x}{y}+\dfrac{y}{x}\right)=14\tfrac{4}{9}.$

35. $\dfrac{x}{y}-\dfrac{y}{x}-3=-\tfrac{9}{4}\cdot\dfrac{xy}{x^2-y^2},\ 3x+y=7.$

36. $(x+y)\,xy=c\,(bx+ay),$
$xy\,(bx+ay)=x^2y^2+abc\,(x+y-c).$

37. $y^4=x^2(ay-bx),\ x^2=ax-by.$

38. $\dfrac{\sqrt{y^2+1}+1}{y}=\dfrac{\sqrt{x+9}+3}{\sqrt{x}},\ x(y+1)^2=36\,(y^3+\tfrac{16}{9}).$

39. $x=\dfrac{a}{y+z},\ y=\dfrac{b}{x+z},\ z=\dfrac{c}{x+y}.$

40. $x+y+z=6,\ xy+xz+yz=11,\ xyz=6.$

41. $x^2-yz=0,\ y^2-xz=0,\ z^2-xy=0.$

42. $xyz=a^2(x+y)=b^2(y+z)=c^2(x+z).$

43. $x^2+y^2+z^2=\dfrac{a^2}{x^2}=\dfrac{b^2}{y^2}=\dfrac{c^2}{z^2}.$

44. $x+y+z+u=4a+4b,$
$xy+xz+xu+yz+yu+zu=6a^2+12ab+6b^2,$
$xyz+xyu+xzu+yzu=4a^3+12a^2b+12ab^2+4b^3,$
$xyzu=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

45. $x^2y^2+xy^2z+x^2yz=a,$
$y^2z^2+xy^2z+xyz^2=b,$
$x^2z^2+x^2yz+xyz^2=c.$

46. $(x+y)^3+z^3=1125,$
$x+y+z=15,$
$xy=24.$

47. If $ax^3=by^3=cz^3$, and $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=a$, show that $ax^2+by^2+cz^2=(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3\,a^2.$

48. Given $R = 1 + r$, $P = \frac{A}{r}(1 - R^{-n})$, $M = PR^n$, show that $R = A\left(\frac{1}{P} - \frac{1}{M}\right) + 1$.

CHAPTER II.

Problems Producing Quadratic Equations.

7. We shall now discuss one or two problems whose solutions depend upon quadratic equations.

Ex. 1. A person raised his goods a certain rate per cent., and found that to bring them back to the original price he must lower them $3\frac{1}{3}$ less per cent. than he had raised them. Find the original rise per cent.

Let x = the original rise per cent.,

then $\frac{x}{100 + x} \cdot 100$ = the *fall per cent.* to bring them to the original price.

Hence, by problem—

$$x - \frac{100\,x}{100 + x} = 3\tfrac{1}{3}, \text{ which solved, gives}$$
$$x = 20 \text{ or } -16\tfrac{2}{3}.$$

The value $x = 20$ is alone applicable to the problem. Remembering, however, the algebraical meaning of the *negative* sign, it is easy to see that the second value, $x = -16\frac{2}{3}$, gives us the solution of the following problem:—

A person *lowered* his goods a certain rate per cent., and found that to bring them back to the original price he must *raise* them $3\frac{1}{3}$ more per cent. than he had *lowered* them. Find the original fall per cent.

The above solution tells us that the fall required is $16\frac{2}{3}$ per cent.

Had we worked the latter problem first, we should have obtained $x = 16\frac{2}{3}$ or -20, the value $x = -20$ indicating the solution of the former problem.

Ex. 2. Find a number such that when multiplied by its deficiency from 100 the product is 196.

Let x = the number,
then $100 - x$ = its deficiency from 100.
Hence, by problem—

$$x(100 - x) = 196, \text{ or } x^2 - 100x + 196 = 0;$$

from which, $x = 2$ or 98.

Both these values will be found to be consistent with the conditions of the problem.

Ex. 3. The number of men required to build a house is such that, when four times the number is subtracted from three times the square of the number, the result is 160. Find the number of men.

Let x = the number of men,
then, by problem—

$$3x^2 - 4x = 160, \text{ from which}$$
$$x = 8 \text{ or } -6\tfrac{2}{3}.$$

The value $x = 8$ is alone applicable to the problem as it stands. If, however, we may conceive of a *fractional* number of men—and this we may easily do here by supposing a *boy's* work to be equal to $\frac{2}{3}$ of a man's—we find that the second result gives us the solution of the following problem:—

The number of men required to build a house is such that when four times the number is *added to* three times the square of the number, the result is 160. Find the number.

The answer, as above indicated, is 6 men and 1 boy, where a boy is worth $\frac{2}{3}$ of a man.

The student will find, however, that in some cases there is no *obvious* interpretation to the second result, owing occasionally to the fact that certain *terms* are used in the problem to which the results will not apply, and indeed that the algebraical expression of the conditions of the problem is more general than the language of the problem itself.

Ex. IV.

1. Find a number whose square is equal to the product of two other numbers, one of which is less by 6 than the required number, and the other greater by 9 than twice that number.

2. When the numerator and denominator of a certain fraction are each increased by unity the fraction is increased by $\frac{1}{190}$, and when they are each diminished by unity the fraction is diminished by $\frac{1}{95}$. Find the fraction.

3. The mean proportional between the excess of a certain number above 21, and its defect from 37, is 28. Find the number.

4. A number of articles, which were bought for £4, cost each 3 shillings more than half as many shillings as there were articles. Find the number of articles.

5. There is a square court-yard, such that if its length be increased by 10 feet, and its breadth diminished by 20 feet, its area would be 5,104 feet. Required the side of the square.

6. If the number of shillings given for an article be added to the number of articles which can be bought at the same price for 18 shillings, the result is 11. Find the price.

7. Two travellers set out to meet each other from two places 180 miles distant; the first goes 3 miles an hour, and the second goes 1 mile more per hour than one-fourth of the number of hours before they meet. When will they meet?

8. A farmer bought a number of calves, sheep, and pigs, the number of calves being equal to that of the sheep and pigs together. For the calves he gave 64s. a head, and for the sheep twice as many shillings as there were sheep. He paid £153. 12s. for the calves and sheep together, and £36. 12s. for the pigs—a pig costing as much as a sheep and calf together. Find the cost of the sheep per head.

9. There are two squares, and an oblong whose sides are equal to those of the squares, and it is noted that three times the area of the first square exceeds four times the area of the oblong by 3 square feet, while twice the area of the square, together with three times the area of the rectangle, is 36 square feet. Required the sides of the squares.

10. The sum of two quantities is equal to 6 times the square of their product, and the sum of their cubes is equal to 36 times the product of their fifth powers. Find the quantities.

11. The solid content of a rectangular parallelopiped is 60 cubic feet, and the total area of the side is 98 square feet, while the sum of the edges is 48 feet. Required the dimensions.

12. The products of the number of units of length in the sides of a polygon of n sides, when taken $n - 1$, together are respectively a_1, a_2, a_3, &c., a_n. Required the lengths of the sides.

13. A, B, and C can together do a piece of work in a day, and C's rates of work is the product of the rates of A and B. Moreover, C is one-fifth as good a workman as A and B together. Find the respective times required for A, B, C to do a piece of work.

14. The compound interest of a certain sum of money for 3 years is a, and the third year's interest is b. Find the principal and the rate per cent.

15. A owes B £a due m months hence, and also £b due n months hence. Find the equated time, reckoning interest at 5 per cent. per annum.

16. Find three quantities such that the sum of any two is equal to the reciprocal of the third.

17. Find three magnitudes, when the quotients arising from dividing the products of every two by the other are respectively a, b, c.

18. The sum of three quantities is 9, the sum of their products, taken two and two together, is 23, and their continued product is 15. Show that the three quantities are the roots of the equation $a^3 - 9a^2 + 23a - 15 = 0$.

CHAPTER III.

Indices.

8. We shall reserve the discussion of the complete theory of Indices for Vol. II., confining ourselves here to a few simple cases, and giving a few examples involving fractional and negative exponents.

9. *Fractional exponents.*

Def.—The numerator of a fractional exponent indicates the power to which the quantity must be raised, and the denominator the root which must be taken of the power so obtained.

Thus, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$, $a^{\frac{5}{7}} = \sqrt[7]{a^5}$, $a^{\frac{6}{3}} = \sqrt[3]{a^6} = a^2$;

and generally $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

The above definition is that which follows at once if we *assume* the law proved in Art. 24, page 159, viz., $a^m \times a^n = a^{m+n}$ to be true, whatever be the value of m and n. Thus we have—

$$\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots\ldots\ldots\ldots\text{to } n \text{ factors}$$

$$= a^{\frac{m}{n} + \frac{m}{n} + \frac{m}{n} \text{ to } n \text{ terms}} = a^{\frac{m}{n}\cdot n} = a^m.$$

Hence, taking the nth root,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

10. *To show that* $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

Let $x = \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \sqrt[s]{\left(a^{\frac{p}{q}}\right)^r}$, by Def., Art. 9.

$$\therefore\ x^s = \left(a^{\frac{p}{q}}\right)^r = \left(\sqrt[q]{a^p}\right)^r,$$

or, raising each side to the qth power, we have—

$$x^{qs} = (a^p)^r = a^{pr}.$$

Hence, taking the (qs)th root, we have $x = a^{\frac{pr}{qs}}$.

11. *To show that* $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$.

Now, $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = \sqrt[n]{a^m} \times \sqrt[n]{b^m} = \sqrt[n]{a^m b^m} = \sqrt[n]{(ab)^m}$

$= (ab)^{\frac{m}{n}}$, by Def., Art. 9.

And so, $a^{\frac{m}{n}} \div b^{\frac{m}{n}} = \left(\frac{a}{b}\right)^{\frac{m}{n}}$.

Ex. 1. Multiply together $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$ by $a^{\frac{1}{3}}b^{\frac{1}{5}}c^{\frac{1}{7}}$.

Adding together the indices of like letters, we have—

$$\tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6},\ \tfrac{1}{3} + \tfrac{1}{5} = \tfrac{8}{15},\ \tfrac{1}{4} + \tfrac{1}{7} = \tfrac{11}{28}.$$

Hence, the required product is $a^{\frac{5}{6}}b^{\frac{8}{15}}c^{\frac{11}{28}}$.

Ex. 2. Multiply $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

$$
\begin{array}{lllll}
x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & & & & \\
x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y & & & & \\
\hline
x^2 + x^{\frac{3}{2}}y^{\frac{1}{2}} & + xy & & & \\
\quad - x^{\frac{3}{2}}y^{\frac{1}{2}} & - xy & - x^{\frac{1}{2}}y^{\frac{3}{2}} & & \\
 & xy & + x^{\frac{1}{2}}y^{\frac{3}{2}} & + y^2 & \\
\hline
x^2 & + xy & & + y^2 &
\end{array}
$$

Ex. 3. Divide $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

$$
\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}})\,x - y\,(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\qquad\quad \underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}} \\
\qquad\quad x^{\frac{2}{3}}y^{\frac{1}{3}} \\
\qquad\quad \underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{2}} \\
\qquad\qquad\quad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\
\qquad\qquad\quad \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}
$$

Ex. V.

Find the value of—

1. $(a^6)^{\frac{1}{3}}$, $a^{\frac{8}{4}}$, $(a^{-2})^{-\frac{1}{4}}$, $(a^{\frac{3}{5}})^{\frac{5}{3}}$.

2. $(a + x)^{\frac{1}{3}}(a^2 + 2\,ax + x^2)^{\frac{1}{3}}$, $(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}}\,(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{1}{2}}$.

Multiply together—

3. $a^{\frac{1}{2}} - a^{\frac{1}{4}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$ and $a^{\frac{1}{2}} + a^{\frac{1}{4}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$.

4. $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ and $x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$.

5. $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ and $x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

Divide—

6. $a^3 - b$ by $a - b^{\frac{1}{3}}$, $a^{\frac{3}{4}} - b^{\frac{3}{4}}$ by $a^{\frac{3}{8}} - b^{\frac{3}{8}}$.

7. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} + x^{-\frac{1}{2}}y$.

8. $a + b + c - 3\,a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.

9. $a^{\frac{7}{3}} - a^2b^{-\frac{2}{3}} - a^{\frac{1}{3}}b + b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{3}} - b^{-\frac{1}{3}}$.

Show that—

10. $(x^{2m} + x^{2n})^{\frac{1}{mn}} = x^{\frac{1}{m} + \frac{1}{n}} \left\{ x^{m-n} + x^{n-m} \right\}^{\frac{1}{mn}}$.

11. $\frac{a^2 - b^2}{a^2 + b^2} \left\{ x^{\frac{1}{p}} + x^{\frac{1}{q}} \right\} - 2x^{\frac{p+q}{2pq}}$

$$= \frac{a^2 - b^2}{a^2 + b^2} \left\{ x^{\frac{1}{2p}} - x^{\frac{1}{2q}} \right\} - \frac{4b^2}{a^2 + b^2} x^{\frac{p+q}{2pq}}.$$

12. $(x^m - a^m) \div (x^{\frac{m}{2^n}} - a^{\frac{m}{2^n}})$

$$= (x^{\frac{m}{2}} + a^{\frac{m}{2}})(x^{\frac{m}{4}} + a^{\frac{m}{4}})(x^{\frac{m}{8}} + a^{\frac{m}{8}}) \ldots\ldots (x^{\frac{m}{2^n}} + a^{\frac{m}{2^n}}).$$

Find the square roots of—

13. $a + b + c + 2(a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}c^{\frac{1}{2}} + b^{\frac{1}{2}}c^{\frac{1}{2}})$.

14. $4xy^{\frac{4}{3}} - 12x^{\frac{3}{2}}y + 17x^2y^{\frac{2}{3}} - 12x^{\frac{5}{2}}y^{\frac{1}{3}} + 4x^3$.

15. $a^2b^{-1} - 4ab^{-\frac{1}{2}} - 8a^{-1}b^{\frac{1}{2}} + 4a^{-2}b + 8$.

Find the cube roots of—

16. $x^4 + 9x^{\frac{10}{3}} + 6x^{\frac{8}{3}} - 99x^2 - 42x^{\frac{4}{3}} + 441x^{\frac{2}{3}} - 343$.

17. $x^6y^{-1} + 3x^4y^{-\frac{2}{3}} + 3x^2y^{-\frac{1}{3}} + 1$.

18. $ab(1 + 3a^{-\frac{1}{3}}b^{\frac{1}{3}} + 3a^{-\frac{2}{3}}b^{\frac{2}{3}} + a^{-1}b)(ab^{-1} - 3a^{\frac{2}{3}}b^{-\frac{2}{3}} + 3a^{\frac{1}{3}}b^{-\frac{1}{3}} + 1)$.

CHAPTER IV.

SURDS.

12. A **surd** quantity is one in which the root indicated cannot be denoted without the use of a fractional index.

Thus, the following quantities are surds :—

$$\sqrt{a},\ \sqrt[4]{a^2 + x^2},\ \sqrt[n]{a^2 + b^2 + c^2},\ \sqrt{\frac{x + y}{a}},\ \frac{\sqrt[4]{a + x}}{\sqrt[3]{x + y}}.$$

Since, from what has been explained in the last chapter, these quantities may be written thus—

$$a^{\frac{1}{2}},\ (a^2 + x^2)^{\frac{1}{4}},\ (a^2 + b^2 + c^2)^{\frac{1}{n}},\ \left(\frac{x + y}{a}\right)^{\frac{1}{2}},\ \frac{(a + x)^{\frac{1}{4}}}{(x + y)^{\frac{1}{3}}},$$

it follows that surds may be dealt with exactly as we deal with their equivalent expressions with fractional indices.

It is evident that rational quantities may be put in the *form* of surds, and conversely, expressions which have the *form* of surds may sometimes be rational quantities.

Thus, $a^2 = \sqrt[3]{(a^2)^3} = \sqrt[3]{a^6}$;

and $\sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3} = \sqrt[3]{(a + b)^3} = a + b$.

13. *A mixed quantity may be expressed as a surd.*

Thus, $3\sqrt[3]{5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5} = \sqrt[3]{3^3 \times 5} = \sqrt[3]{135}$,

and so, $x\sqrt[n]{y} = \sqrt[n]{x^n} \cdot \sqrt[n]{y} = \sqrt[n]{x^n y}$.

14. Conversely, *a surd may be expressed as a mixed quantity,* when the root of any factor can be obtained.

Thus, $\sqrt{18a^3b^2} = \sqrt{9a^2b^2 \times 2a}$

$= \sqrt{9a^2b^2} \cdot \sqrt{2a} = 3ab\sqrt{2a}$.

And $\sqrt[3]{(a^2 + b^2)^6 x^4 y^5} = \sqrt[3]{(a^2 + b^2)^6 x^3 y^3 \times xy^2}$

$= \sqrt[3]{(a^2 + b^2)^6 x^3 y^3} \cdot \sqrt[3]{xy^2} = (a^2 + b^2)^2 xy\sqrt[3]{xy^2}$.

15. *Fractional surd expressions may be so expressed that the surd portion may be integral.*

The process is called *rationalizing the denominator*, and is worth special notice.

Ex. 1. $\sqrt{\frac{3}{7}} = \sqrt{\frac{3 \times 7}{7^2}} = \frac{\sqrt{21}}{\sqrt{7^2}} = \frac{\sqrt{21}}{7}$.

It is much easier to find approximately the value of $\sqrt{21}$, and divide the result by 7, than to find the values of $\sqrt{3}$ and $\sqrt{7}$, and divide the former by the latter.

Ex. 2. Reduce to its simplest form $\sqrt{\frac{xy}{b - c}}$.

$$\sqrt{\frac{xy}{b - c}} = \sqrt{\frac{xy(b - c)}{(b - c)^2}} = \frac{\sqrt{xy(b - c)}}{b - c}.$$

Ex. 3. Find the arithmetical value of $\frac{4}{2 - \sqrt{3}}$.

The denominator is the *difference* of two quantities, one of hich is a *quadratic* surd.

Now, we know that $(2 - \sqrt{3})(2 + \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$, and hence we see that by multiplying numerator and denominator by the *sum* of the quantities in the denominator we can obtain the denominator in a *rational form.*

$$\text{Thus, } \frac{4}{2 - \sqrt{3}} = \frac{4(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{4(2 + \sqrt{3})}{2^2 - (\sqrt{3})^2}$$

$$\frac{4(2 + \sqrt{3})}{4 - 3} = \frac{4(2 + \sqrt{3})}{1} = 4(2 + \sqrt{3})$$

$$= 4(2 + 1{\cdot}73205) = 4(3{\cdot}73205.)$$

$$= 14{\cdot}92820.$$

$$\text{na so, } \frac{4}{4\sqrt{2} + 3\sqrt{3}} = \frac{4(4\sqrt{2} - 3\sqrt{3})}{(4\sqrt{2} + 3\sqrt{3})(4\sqrt{2} - 3\sqrt{3})}$$

$$= \frac{4(4\sqrt{2} - 3\sqrt{3})}{(4\sqrt{2})^2 - (3\sqrt{3})^2} = \frac{4(4\sqrt{2} - 3\sqrt{3})}{32 - 27} = \tfrac{4}{5}(4\sqrt{2} - 3\sqrt{3}).$$

We shall now give an example when the surds are not quadratic.

Ex 4. Rationalize the denominator of $\frac{a}{x^{\frac{1}{4}} - y^{\frac{1}{3}}}$.

Since $(x^{\frac{1}{4}})^{12} - (y^{\frac{1}{3}})^{12}$ is (Art. 29, page 175) divisible by $x^{\frac{1}{4}} - y^{\frac{1}{3}}$, it follows that the rationalizing factor is their quotient, which is easily found.

16. *Surds may be reduced to a common index.*

Ex. 1. Express $\sqrt[m]{a}$ and $\sqrt[n]{b}$ as surds having a common index.

Since $\sqrt[m]{a} = a^{\frac{1}{m}}$, and $\sqrt[n]{b} = b^{\frac{1}{n}}$, it follows that, by reducing the fractional indices to a common denominator, the given surds become respectively $a^{\frac{n}{mn}}$, $b^{\frac{m}{mn}}$, or $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.

Ex. 2. Reduce $\sqrt[4]{a^3b}$ and $\sqrt[3]{x^5y^2}$ to a common index.

The least common denominator of the fractional indices of the given surds is 4×3 or 12. Hence we proceed as follows:—

$$\sqrt[4]{a^3b} = (a^3b)^{\frac{1}{4}} = (a^3b)^{\frac{3}{12}} = \sqrt[12]{(a^3b)^3} = \sqrt[12]{a^9b^3},$$
$$\sqrt[3]{x^5y^2} = (x^5y^2)^{\frac{1}{3}} = (x^5y^2)^{\frac{4}{12}} = \sqrt[12]{(x^5y^2)^4} = \sqrt[12]{x^{20}y^8}.$$

When the student has had a little practice, the first two steps of each of the operations may be omitted.

17. *Addition and subtraction of similar surds.*

DEF. Similar surds are those which have the same irrational factors.

Ex. 1. Find the sum of $\sqrt{12}$, $5\sqrt{27}$, $-2\sqrt{75}$.

We have—

$$\begin{aligned} &\sqrt{12} + 5\sqrt{27} - 2\sqrt{75} \\ &= \sqrt{2^2 \times 3} + 5\sqrt{3^2 \times 3} - 2\sqrt{5^2 \times 3} \\ &= 2\sqrt{3} + 5 \times 3\sqrt{3} - 2 \times 5\sqrt{3} \\ &= 2\sqrt{3} + 15\sqrt{3} - 10\sqrt{3} \\ &= (2 + 15 - 10)\sqrt{3} = 7\sqrt{3}. \end{aligned}$$

Ex. 2. Simplify—

$$\sqrt{\frac{a^2b + 2ab^2 + b^3}{a^2 - 2ab + b^2}} - \sqrt{\frac{a^2b - 2ab^2 + b^3}{a^2 + 2ab + b^2}}.$$

The given expression—

$$= \sqrt{\frac{(a + b)^2b}{(a - b)^2}} - \sqrt{\frac{(a - b)^2b}{(a + b)^2}}$$
$$= \frac{a + b}{a - b}\sqrt{b} - \frac{a - b}{a + b}\sqrt{b} = \left(\frac{a + b}{a - b} - \frac{a - b}{a + b}\right)\sqrt{b}$$
$$= \frac{(a + b)^2 - (a - b)^2}{(a - b)(a + b)}\sqrt{b} = \frac{4ab}{a^2 - b^2}\sqrt{b}.$$

18. *Multiplication and division of surds.*

The following examples will best illustrate these operations:—

Ex. 1. Multiply $a\sqrt{x^3yz}$ by $b\sqrt{xy^3u}$.

We have, $a\sqrt{x^3yz} \times b\sqrt{xy^3u} = ab\sqrt{x^3yz \times xy^3u}$

$$= ab\sqrt{x^4y^4uz} = abx^2y^2\sqrt{uz}.$$

Ex. 2. Multiply $a\sqrt{b} + c\sqrt{d}$ by $a - \sqrt{bd}$.

Arranging as in the case of rational quantities, we have—

$$\begin{array}{r} a\sqrt{b} + c\sqrt{d} \\ a - \sqrt{bd} \\ \hline a^2\sqrt{b} + ac\sqrt{d} \\ - ab\sqrt{d} - cd\sqrt{b} \\ \hline (a^2 - cd)\sqrt{b} + a(c - b)\sqrt{d} \end{array}$$

Ex. 3. Divide $a\sqrt{b}$ by $b\sqrt{a}$.

We have, $\dfrac{a\sqrt{b}}{b\sqrt{a}} = \dfrac{a\sqrt{b}.\sqrt{a}}{b\sqrt{a}.\sqrt{a}} = \dfrac{a\sqrt{ab}}{b.a} = \dfrac{1}{b}\sqrt{ab}$.

When the divisor is a compound quantity it will *generally* be the best to express the surds as quantities with fractional indices, and proceed as in ordinary division.

19. *The square root of a rational quantity cannot be partly rational and partly irrational.*

If possible, let $\sqrt{a} = m + \sqrt{b}$;

then, squaring, $a = m^2 + 2m\sqrt{b} + b$;

or, $2m\sqrt{b} = a - (m^2 + b)$;

or, $\sqrt{b} = \dfrac{a - (m^2 + b)}{2m}$;

that is, an irrational quantity is equal to a rational quantity, which is absurd.

20. *To find the square root of a binomial, one of whose terms is a quadratic surd.*

Let $a + \sqrt{b}$ be the binomial.

Assume $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,........................(1);

then, squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$,................(2).

Equating the rational and irrational parts (Art. 19), we have—

$$x + y = a \text{..........................(3.)},$$
$$\text{and } 2\sqrt{xy} = \sqrt{b} \text{ or } 4xy = b \text{...............(4.)}$$

From (3) and (4) we easily find $x = \frac{1}{2}(a + \sqrt{a^2 - b})$,

and $y = \frac{1}{2}(a - \sqrt{a^2 - b})$.

Hence, from (1), the square root required is—

$$\sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} + \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})}.$$

NOTE.—It is evident that, unless $(a^2 - b)$ is a perfect square, our result is more complicated than the original expression, and therefore the above method fails in that case.

Ex. 1. Find the square root of $14 + 6\sqrt{5}$.

Let $\sqrt{14 + 6\sqrt{5}} = \sqrt{x} + \sqrt{y}$............................(1.)

Squaring, then, $14 + 6\sqrt{5} = x + y + 2\sqrt{xy}$.

Hence, equating the rational and irrational parts—

$$x + y = 14 \text{........................(2.)},$$
$$2\sqrt{xy} = 6\sqrt{5} \text{ or } 4xy = 180 \text{................(3).}$$

From (2) and (3) we easily find $x = 9$, $y = 5$.

Hence the square root required is $\sqrt{9} + \sqrt{5}$ or $3 + \sqrt{5}$.

Ex. 2. Find the square root of $39 + \sqrt{1496}$.

Let $\sqrt{39 + \sqrt{1496}} = \sqrt{x} + \sqrt{y}$.

Squaring, &c., we have, $x + y = 39$;

and $4xy = 1496$.

From these equations we easily find $x = 22$, $y = 17$.

Hence, the square root required is $\sqrt{22} + \sqrt{17}$.

21. *The square roots of quantities of this kind may often be found by inspection.*

Ex. 1. Find the square root of $19 + 8\sqrt{3}$.

We shall throw this expression into the form $a^2 + 2ab + b^2$, which we know is a perfect square.

Dividing the irrational term by 2, we have $4\sqrt{3}$. Now all we have to do is to break this up into two such factors that the sum of their squares shall be 19. The factors are evidently 4 and $\sqrt{3}$.

Thus, we have $19 + 8\sqrt{3} = (4)^2 + 2(4)\sqrt{3} + (\sqrt{3})^2$
$= (4 + \sqrt{3})^2.$

The square root is therefore $4 + \sqrt{3}$.

Ex. 2. Find the square root of $29 + 12\sqrt{5}$.

We have $29 + 12\sqrt{5} = (3)^2 + 2(3)2\sqrt{5} + (2\sqrt{5})^2$
$= (3 + 2\sqrt{5})^2.$

The square root is therefore $3 + 2\sqrt{5}$.

Ex. VI.

Express with fractional indices—

1. $\sqrt{x^5}$, $\sqrt[3]{a^5b^2}$, $\sqrt[4]{x^2y^3}$, $\sqrt[8]{a^2b^4}$.

2. $\dfrac{xy}{\sqrt{ab}}$, $\dfrac{\sqrt{c^3d}}{\sqrt{xy}}$, $\dfrac{\sqrt[n]{x^m}}{\sqrt[m]{a^n}}$.

Reduce to entire surds—

3. $3\sqrt{3}$, $4\sqrt{\frac{3}{2}}$, $\frac{3}{5}\sqrt{15}$, $3\sqrt[3]{\frac{1}{3}}$.

4. $4 \cdot 2^{\frac{1}{2}}$, $9 \cdot 3^{-\frac{1}{2}}$, $4 \cdot 2^{-\frac{3}{4}}$, $\frac{3}{2}\left(\frac{2}{3}\right)^{-\frac{1}{2}}$.

5. $3\sqrt{ab}$, $a\sqrt{\dfrac{bc}{a}}$, $(a + x)\sqrt[3]{\dfrac{a - x}{(a + x)^2}}$.

Reduce to a common index—

6. $\sqrt[3]{2}$, $\sqrt{3}$.
7. $\sqrt[4]{2}$, $\sqrt[3]{3}$.
8. $2\sqrt{2}$, $3\sqrt[3]{5}$.
9. $\sqrt[m]{a}$, $\sqrt[n]{b}$.
10. $(a + x)^{\frac{1}{2}}$, $\sqrt[3]{a - x}$.
11. $a^{\frac{1}{p} + \frac{1}{q}}$, $b^{\frac{1}{p} - \frac{1}{q}}$.

Simplify—

12. $\sqrt{12}$, $\sqrt[3]{48}$, $3\sqrt{28}$, $\frac{1}{2}\sqrt[3]{648}$.

13. $\sqrt{4a^3 + 4a^2b}$, $\sqrt[3]{a^3b^3 + b^6}$, $\sqrt[3]{\dfrac{a + b}{64\,a}}$.

14. $\sqrt{\dfrac{1}{x^2} - \dfrac{2a}{x} + a^2}$, $\sqrt[3]{\dfrac{a^2}{x^2} - \dfrac{3\,a^2}{x} + 3\,a^2 - a^2x}$.

15. $\sqrt[3]{\dfrac{x^3 + 1}{9\,x^2} + \dfrac{x + 1}{3\,x}}$, $\sqrt[3]{\dfrac{(x + a)^2\,(x^2 - a^2)}{(x - a)^2\,(x + a)}}$.

Find the value of—

16. $\sqrt{12} + \sqrt{48} - 2\sqrt{3}$, $\sqrt{56} + \sqrt[3]{189}$.

17. $\sqrt[3]{\dfrac{a^4x^2y^3}{b}} - \sqrt[3]{a^4b^2x^2} + \sqrt[3]{\dfrac{ax^2c^3}{b}}$.

18. $\sqrt[3]{27\,a^{m+6}\,b^3} - \sqrt[3]{8a^{4m+9}} + 3\sqrt[3]{64\,a^m}$.

Multiply—

19. $a + \sqrt{ab} + b$ by $\sqrt{a} - \sqrt{b}$, $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $\sqrt{a} - \sqrt{b}$.

20. $(x + y)^{\frac{1}{2}}$ by $(x + y)^{\frac{1}{3}}$, $a + b\sqrt{d}$ by $a^2 - ab\sqrt{d} + b^2d$.

21. $c\sqrt{\dfrac{a}{a + b}} + \sqrt{\dfrac{1}{b}}$ by $\dfrac{c^2}{d}\sqrt{a\,(a + b)} - \sqrt{b^3}$.

22. $a^{\frac{1}{2}} + b^{\frac{1}{3}} + c^{\frac{1}{4}} + d^{\frac{1}{5}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{3}} + c^{\frac{1}{4}} - d^{\frac{1}{5}}$.

Divide—

23. $x^2 + xy + y^2$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

24. $x^3 - y^2$ by $x^{\frac{1}{2}} + y^{\frac{1}{3}}$.

Rationalize the denominators of—

25. $\sqrt{\dfrac{5}{3}}$, $\dfrac{4}{\sqrt{7}}$, $\sqrt[3]{\dfrac{7}{5}}$.

26. $\dfrac{3}{2 + \sqrt{3}}$, $\dfrac{4}{3\sqrt{2} - 2\sqrt{3}}$, $\dfrac{1}{\sqrt{5} - \sqrt{3}}$.

27. $\dfrac{3}{1 + \sqrt{2} + \sqrt{3}}$, $\dfrac{2}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$, $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$.

28. $\dfrac{a}{x^{\frac{1}{2}} - y^{\frac{1}{3}}}$, $\dfrac{1}{\sqrt[3]{2} + \sqrt{3}}$, $\dfrac{b}{x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y}$.

Find the square roots of—

29. $11 + 4\sqrt{7}$, $8 + 2\sqrt{15}$, $30 - 10\sqrt{5}$.

30. $8 + 2\sqrt{12}$, $9 - 6\sqrt{2}$, $20 - 10\sqrt{3}$.

CHAPTER V.

RATIO AND PROPORTION.

Ratio.

22. The student is referred to Chapter II. of the Arithmetic section of this work for definitions and observations which need not be repeated here.

23. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by increasing the terms of the ratio by the same quantity.*

Let $a : b$ or $\frac{a}{b}$ be the ratio, and let each of its terms be increased by m. It will then become $\frac{a + m}{b + m}$.

Now, $\frac{a + m}{b + m} \gtrless \frac{a}{b}$, as $(a + m)b \gtrless (b + m)a$, or, as $ab + bm \gtrless ab + am$; or, as $bm \gtrless am$, or as $b \gtrless a$.

Hence the ratio $\frac{a}{b}$ is *increased* when $b > a$, that is, when it is a ratio of less inequality; and is diminished when $b < a$, that is, when it is a ratio of greater inequality.

Cor. It may be shown in the same way that—

A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by diminishing the terms of the ratio by the same quantity.

24. When the difference between the antecedent and consequent is small compared with either, the ratio of the higher powers of the terms is found by doubling, trebling, &c., their difference.

Let $a + x : a$ or $\frac{a + x}{a}$ be the ratio, where x is small compared with a.

Then $\frac{(a + x)^2}{a^2} = \frac{a^2 + 2ax + x^2}{a^2} = 1 + \frac{2x}{a}$ nearly $= \frac{a + 2x}{a}$ nearly.

$\frac{(a+x)^3}{a^3} = \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a^3} = 1 + \frac{3x}{a}$ nearly $=$ $\frac{a + 3x}{a}$ nearly; and so on.

Ex. $(1002)^2 : (1000)^2 = 1004 : 1000$ nearly.
$(1002)^3 : (1000)^3 = 1006 : 1000$ nearly.

Proportion.

25. Proportion, as has been already said, is the relation of equality expressed between ratios.

Thus, the expression $a : b = c : d$,

or $a : b :: c : d$,

or $\frac{a}{b} = \frac{c}{d}$,

is called a proportion.

26. The following results are easily obtained :—

(1.) Since $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$ or $\frac{a}{c} = \frac{b}{d}$,

$\therefore a : c :: b : d$ (*alternando*).

(2.) $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$ or $\frac{b}{a} = \frac{d}{c}$,

$\therefore b : a :: d : c$ (*invertendo*).

Also, by Art. 64, page 214, we have—

(3.) $a + b : b :: c + d : d$ (*componendo*).

(4.) $a - b : b :: c - d : d$ (*dividendo*).

(5.) $a - b : a :: c - d : c$ (*convertendo*).

(6.) $a + b : a - b :: c + d : c - d$ (*componendo* and *dividendo*).

27. If $a : b :: c : d$ and $e : f :: g : h$, we may compound the proportions.

Thus we have $\frac{a}{b} = \frac{c}{d}$ ......(1), and $\frac{e}{f} = \frac{g}{h}$ ......(2).

(1) $\times$ (2), then, $\frac{ae}{bf} = \frac{cg}{dh}$,

or $ae : bf :: cg : dh$.

And in the same way we may show that, if the corresponding terms of any number of proportions be multiplied together, the products will be proportional.

28. *If three quantities are in continued proportion, the first has to the third the duplicate ratio of what it has to the second.*

Let a, b, c be the given quantities in continued proportion; then—

$$\frac{a}{b} = \frac{b}{c}.$$

$$\text{Hence, } \frac{a}{c} \text{ or } \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2};$$

$$\therefore a : c :: a^2 : b^2.$$

And, similarly, if a, b, c, d are four quantities *in continued proportion*, $a : d :: a^3 : b^3$, that is—

The first has to the fourth the triplicate ratio of what it has to the second; and so on, for any number of quantities.

29. We shall now give one or two examples of problems in Proportion.

Ex. 1. If $a : b :: c : d$, prove that $\dfrac{a^3 + c^3}{b^3 + d^3} = \left(\dfrac{a + c}{b + d}\right)^3.$

Let $\dfrac{a}{b} = \dfrac{c}{d} = x$; $\therefore a = bx$, and $c = dx$.

Hence, $\dfrac{a^3 + c^3}{(a + c)^3} = \dfrac{(bx)^3 + (dx)^3}{(bx + dx)^3} = \dfrac{b^3 + d^3}{(b + d)^3}$, then, *alternando*, $\dfrac{a^3 + c^3}{b^3 + d^3} = \left(\dfrac{a + c}{b + d}\right)^3.$

Ex. 2. If $a : b :: c : d$, prove that $\dfrac{a + b}{a - b} = \dfrac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}.$

Let $\dfrac{a}{b} = \dfrac{c}{d} = x$; $\therefore a = bx$, and $c = dx$.

$$\text{Hence, } \frac{a + b}{a - b} = \frac{bx + b}{bx - b} = \frac{x + 1}{x - 1} = \frac{\sqrt{bd}\,.\,x + \sqrt{bd}}{\sqrt{bd}\,.\,x - \sqrt{bd}}$$

$$= \frac{\sqrt{bx\,.\,dx} + \sqrt{bd}}{\sqrt{bx\,.\,dx} - \sqrt{bd}} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}.$$

Or, it may be worked thus—

Since $\frac{a}{b} = \frac{c}{d}$, we have $\sqrt{\frac{a}{b}} = \sqrt{\frac{c}{d}}$,

Or, $\sqrt{\frac{a}{b}} \times \sqrt{\frac{a}{b}} = \sqrt{\frac{c}{d}} \times \sqrt{\frac{a}{b}}$, or $\frac{a}{b} = \frac{\sqrt{ac}}{\sqrt{bd}}$,

Hence, by Art. 26—

$$\frac{a+b}{a-b} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}.$$

Ex. 3. If $a : b :: c : d :: e : f$, show that

$$\frac{a}{b} = \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r}\right)^{\frac{1}{r}}$$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = x$..................................(1).

$$\therefore \frac{a^r}{b^r} = \frac{c^r}{d^r} = \frac{e^r}{f^r} = x^r.$$

Hence, $\left.\begin{aligned} a^r &= b^r x^r, \therefore ma^r = mb^r x^r \\ c^r &= d^r x^r, \therefore nc^r = nd^r x^r \\ e^r &= f^r x^r, \therefore pe^r = pf^r x^r \end{aligned}\right\}$, and $\therefore$ by addition,

$$ma^r + nc^r + pe^r = (mb^r + nd^r + pf^r)x^r.$$

$$\therefore \frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r} = x^r, \text{ or} \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r}\right)^{\frac{1}{r}} = x \ldots (2).$$

$\therefore$ Equating (1) and (2), we have—

$$\frac{a}{b} = \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r}\right)^{\frac{1}{r}}. \quad Q.E.D.$$

Ex. VII.

1. Compare the ratios $a + b : a - b$, and $a^2 + b^2 : a^2 - b^2$.

2. Which is the greater of the ratios $a + b : 2a$, and $2b : a + b$?

3. What quantity must be subtracted from the consequent of the ratio $a : b$ in order to make it equal to the ratio $c : d$?

4. Compound the ratios $1 - x^2 : 1 + y$, $x - xy^2 : 1 + x^2$, and $1 : x - x^2$.

5. There are two numbers in the ratio of 6 : 7, but if 10 be added to each they are in the ratio of 8 : 9. Find the numbers.

6. In what cases is $x + \frac{6}{x} >$ or < 5?

7. If $\frac{a}{x - y} = \frac{b}{y - z} = \frac{c}{z - x}$, show that $a + b + c = 0$.

8. Find the value of x when the ratio $x + 2a : x + 2b$ is the duplicate ratio of $2x + a + c : 2x + b + c$.

9. Find x when the ratio $x - b : x + 2a - b$ is the triplicate ratio of $x - a : x + a - b$.

10. If $\frac{x + y}{a + b} = \frac{y + z}{b + c} = \frac{x + z}{c + a}$, show that each of the fractions is equal to $\frac{x + y + z}{a + b + c}$, and that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

11. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each is equivalent to $\frac{la + mc + nc}{lb + md + nf}$; hence, show that—

$$\frac{a}{2z + 2x - y} = \frac{b}{2x + 2y - z} = \frac{c}{2y + 2z - x},$$

when $\frac{x}{2a + 2b - c} = \frac{y}{2b + 2c - a} = \frac{z}{2c + 2a - b}$.

12. If $a : b :: c : d$, then

$$a + b : c + d :: \sqrt{a^2 + ab + b^2} : \sqrt{c^2 + cd + d^2}.$$

13. Find a fourth proportional to the quantities—

$$\frac{x + 1}{x - 1}, \frac{x^2 + x + 1}{x^2 - x + 1}, \frac{x^3 + 1}{x^3 - 1}.$$

14. Find c in terms of a and b when—

(1.) $a : a :: a - b : b - c$.
(2.) $a : b :: a - b : b - c$.
(3.) $a : c :: a - b : b - c$.

15. If a, b, c are in continued proportion, show that $\frac{a+b}{b+c}$, b, bc are also in continued proportion.

16. If $a : b :: c : d$, then—

$$\sqrt{a^{2n} + b^{2n}} : \sqrt{c^{2n} + d^{2n}} :: (a - b)^n : (c - d)^n.$$

17. From a vessel containing a cubic inches of hydrogen gas, b cubic inches are withdrawn, the vessel being filled up with oxygen at the same pressure. Show that if this operation be repeated n times successively, the quantity of hydrogen remaining in the vessel is $\frac{(a-b)^n}{a^{n-1}}$ cubic inches, when reduced to the original pressure.

18. If, in Ex. 34, page 225, (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) are corresponding terms respectively, show that $a_1b_2b_3(c_2 - c_3) + a_2b_3b_1(c_3 - c_1) + a_3b_1b_2(c_1 - c_2) = 0$.

SECTION III.

PLANE TRIGONOMETRY.

CHAPTER I.

MODES OF MEASURING ANGLES BY DEGREES AND GRADES.

1. We are able to determine geometrically a right angle, and it might therefore be taken as the unit of angular measurement. Practically, however, it is too large, and so we take a determinate part of a right angle as a standard.

In England we divide a right angle into 90 equal parts, called *degrees*, and we further subdivide a *degree* into 60 equal parts, called *minutes*, and again a *minute* into 60 equal parts, called *seconds*. This is the English or sexagesimal method.

In France the right angle is divided into 100 equal parts, called *grades*, a *grade* into a hundred equal parts, called *minutes*, and a *minute* into 100 equal parts, called *seconds*. This is the French or centesimal method, and its advantages are those of the metric system generally.

The symbols $^\circ$, $'$, $''$, are used to express English *degrees, minutes, seconds* respectively, and the symbols ^{g}, $^`$, $^{``}$, to express French *grades, minutes, seconds* respectively.

Conversion of English and French Units.

2. Let D = the number of degrees in an angle,
and G = the number of grades in the same angle;

then $\frac{D}{90}$ expresses the angle in terms of a right angle;

and so also does $\frac{G}{100}$.

Hence, $\frac{D}{90} = \frac{G}{100}$ or $\frac{D}{9} = \frac{G}{10}$.

$$\therefore\ D = \frac{9}{10}G = G - \frac{1}{10}G \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\text{and } G = \frac{10}{9}D = D + \frac{1}{9}D \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence the following rules:—

1. *To convert grades into degrees.*

From the number of grades SUBTRACT $\frac{1}{10}$, and the remainder is the number of degrees.

2. *To convert degrees into grades.*

To the number of degrees ADD $\frac{1}{9}$, and the sum is the number of grades.

Ex. 1. Convert $13^g\ 18'\ 75''$ into English measure.

$$\begin{aligned} \text{No. of grades} &= 13\cdot1875 \\ \text{Subtract } \tfrac{1}{10} \text{ of this} &= \ \ 1\cdot31875 \\ \therefore \text{No of degrees} &= 11\cdot86875 \\ & \qquad\quad 60 \\ & \quad 52'\cdot12500 \\ & \qquad\quad 60 \\ & \qquad 7''\cdot500 \end{aligned}$$

Ans. $11^\circ\ 52'\ 7''\cdot5$.

Ex. 2. Convert $18^\circ\ 7'\ 30''$ into French measure.

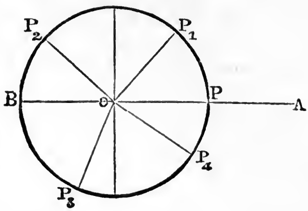
$$\begin{aligned} \text{No. of degrees} &= 18\cdot125 \\ \text{Add } \tfrac{1}{9} \text{ of this} &= \ \ 2\cdot013\dot{8} \\ \therefore \text{No. of grades} &= 20\cdot138\dot{8} \end{aligned}$$

Ans. $20^g\ 13'\ 88''\cdot\dot{8}$.

3. An angle may be conceived to be generated by the revolution of a line about a fixed point. Thus—

Let OA be an initial line, and let a line, OP, starting from OA, revolve with O as centre, and take up successively the positions OP_1, OP_2, OP_3, OP_4.

Now the magnitude of an angle may be measured by *the amount of turning required to generate it.* When, therefore, the revolving line reaches the position OB, we may conceive an angle to have been generated whose magnitude is *two right angles.* And, further, when the revolving line assumes the positions OP_3, OP_4, the angles AOP_3, AOP_4 (the letters being read in the direction of revolution) are angles whose magnitudes are each *greater than two right angles.* Indeed, when the revolving line again reaches the position OP, we may conceive an angle to have been generated whose magnitude is four right angles. Lastly, if the revolution of the line OP be continued, we may conceive of angles being generated to whose magnitude there is no limit.



Ex. I.

1. Express $39° \ 22' \ 30''$ in French measure, and $13^g \ 15^` \ 75^{``}$ in English measure.

2. One of the angles at the base of an isosceles triangle is 50°. Express the vertical angle in grades.

3. Divide an angle of n degrees into two such parts that the number of degrees in one part may be twice the number of grades in the other.

4. Two angles of a triangle are respectively $a°$, b^g, express the other angle in degrees and grades.

5. If $\frac{3}{4}$ of a right angle be the unit of measurement, express an angle which contains 22·5 degrees.

6. Show how to reduce English seconds to French seconds.

7. If the unit of measurement be 8°, what is the value of 10^g.

8. If two of the angles of a triangle be expressed in grades, and the third in degrees, they are respectively as the numbers 5, 15, 18. Find the angles.

9. What is the value in degrees and grades of an angle

which is the result of the revolution of a line $3\frac{1}{5}$ times round.

10. In what quadrants are the following angles found :—
$145°$, 96^g, $327°$, 272^g, $272°$.

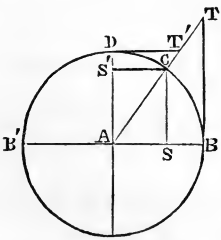
11. If $a°$ be taken as the unit of angular measurement, express an angle containing b^g.

12. What is the unit of measurement when a expresses $\frac{p}{q}$ of a right angle?

CHAPTER II.

THE GONIOMETRIC FUNCTIONS.

4. It was formerly usual in works on Trigonometry to give the following definitions :—



Let a circle be described from centre A, with radius AB supposed to be unity, then—

(1.) The *sine* of an arc BC is the perpendicular from one extremity, C, of the arc upon the diameter passing through the other extremity B.

Thus CS *is the* SINE *of the arc BC.*

(2.) The *cosine* of an arc is the *sine of the complement* of the arc.

Thus, since DC is the complement of BC, S'C *is the* COSINE *of the arc BC.*

(3.) The *tangent* of an arc BC is a line drawn from one extremity, B, of the arc touching the circle, and terminated in the diameter which passes through the other extremity, C, of the arc.

Thus, BT *is the* TANGENT *of the arc BC.*

(4.) The *cotangent* of an arc is the *tangent of the complement* of the arc.

Thus, DT' *is the* COTANGENT *of the arc BC.*

(5.) The *secant* of an arc BC is a line drawn from the centre through one extremity, C, of the arc, and terminated in the tangent at the other extremity.

Thus, AT *is the* SECANT *of the arc BC.*

(6.) The *cosecant* of an arc is the *secant of the complement* of the arc.

Thus, AT' *is the* COSECANT *of the arc BC.*

(7.) The *versed sine* is that portion of the radius upon which the *sine* falls, which is included between the sine and the extremity of the arc.

Thus, SB *is the* VERSED SINE *of the arc BC.*

(8.) The *coversed sine* is the *versed sine of the complement* of the arc.

Thus, S′D *is the* COVERSED SINE *of the arc BC.*

(9.) The *suversed sine* is the *versed sine of the supplement* of the arc.

Thus, B′S *is the* SUVERSED SINE *of the arc BC.*

Representing the arc BC by A, it is usual to write the above functions thus :—Sin A, cos A, tan A, cot A, sec A, cosec A, vers A, covers A, suvers A.

By mere inspection, the student will see that the following relations hold :—

(1.) $\text{Sin A} = \text{CS} = \frac{\text{CS}}{1} = \frac{\text{CS}}{\text{AS}} = \frac{\text{AD}}{\text{AT}'} = \frac{1}{\text{cosec A}}.$

(2.) $\text{Cos A} = \text{S}'\text{C} = \frac{\text{AS}}{1} = \frac{\text{AS}}{\text{AC}} = \frac{\text{AB}}{\text{AT}} = \frac{1}{\text{sec A}}.$

(3.) $\text{Tan A} = \text{BT} = \frac{\text{BT}}{1} = \frac{\text{BT}}{\text{AB}} = \frac{\text{AD}}{\text{DT}'} = \frac{1}{\text{cot A}}.$

(4.) $\text{Sin}^2\text{ A} + \cos^2\text{ A} = \text{CS}^2 + \text{S}'\text{C}^2 = \text{CS}^2 + \text{AS}^2 = \text{AC}^2 = 1.$

(5.) $\text{Sec}^2\text{ A} = \text{AT}^2 = \text{AB}^2 + \text{BT}^2 = 1 + \tan^2\text{ A}.$

(6.) $\text{Cosec}^2\text{ A} = \text{AT}'^2 = \text{AD}^2 + \text{DT}'^2 = 1 + \cot^2\text{ A}.$

(7.) $\text{Tan A} = \text{BT} = \frac{\text{BT}}{\text{AB}} = \frac{\text{CS}}{\text{AS}} = \frac{\text{CS}}{\text{S}'\text{C}} = \frac{\sin\text{ A}}{\cos\text{ A}}.$

(8.) $\text{Vers A} = \text{SB} = \text{AB} - \text{AS} = \text{AB} - \text{S}'\text{C} = 1 - \cos\text{ A}.$

(9.) $\text{Covers A} = \text{S}'\text{D} = \text{AD} - \text{AS}' = \text{AD} - \text{CS} = 1 - \sin\text{ A}.$

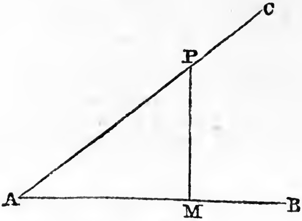
(10.) $\text{Suvers A} = \text{B}'\text{S} = \text{B}'\text{A} + \text{AS} = 1 + \cos\text{ A}.$

It is more convenient, however, to define the *sine*, *cosine*, &c., as in the next article, according to which definitions they are commonly called TRIGONOMETRICAL RATIOS. The student will see that if the above definitions be so far modified that, instead of the *lines* themselves, the goniometric functions be taken as *the ratios* which the lines respectively bear to the radius, they are included in the definitions of the next article.

Trigonometrical Ratios.

5. Let BAC be any angle, which we may denote by A, and P any point in the line AC. Draw PM perpendicular to AB.

Then (1.) $\text{Sin A} = \frac{\text{perpendicular}}{\text{hyp.}} = \frac{PM}{AP}.$



(2.) $\text{Cos A} = \frac{\text{base}}{\text{hyp.}} = \frac{AM}{AP}.$

(3.) $\text{Tan A} = \frac{\text{perpendicular}}{\text{base}} = \frac{PM}{AM}.$

(4.) $\text{Cot A} = \frac{\text{base}}{\text{perpendicular}} = \frac{AM}{PM}.$

(5.) $\text{Sec A} = \frac{\text{hyp.}}{\text{base}} = \frac{AP}{AM}.$

(6.) $\text{Cosec A} = \frac{\text{hyp.}}{\text{perpendicular}} = \frac{AP}{PM}.$

(7.) The *versed sine* is the remainder after subtracting the *cosine* from unity, or—

$$\text{vers A} = 1 - \cos A.$$

(8.) The *coversed sine* is the remainder after subtracting the *sine* from unity, or—

$$\text{covers A} = 1 - \sin A.$$

(9.) The *suversed sine* is the sum of the *cosine* and unity, or—

$$\text{suvers A} = 1 + \cos A.$$

The last three are not much used in practice.

6. Comparing (1.) and (6.), (2.) and (5.), (3.) and (4.), of the last article, we see at once that the *sine* and *cosecant*, the *cosine* and *secant*, and the *tangent* and *cotangent*, are respectively each the reciprocal of the other.

We therefore have—

(1.) $\text{Sin A} = \frac{1}{\text{cosec A}}$, $\text{cosec A} = \frac{1}{\text{sin A}}$.

(2.) $\text{Cos A} = \frac{1}{\text{sec A}}$, $\text{sec A} = \frac{1}{\text{cos A}}$.

(3.) $\text{Tan A} = \frac{1}{\text{cot A}}$, $\text{cot A} = \frac{1}{\text{tan A}}$.

Further—

(4.) $\text{Sin}^2 \text{ A} + \cos^2 \text{A} = \frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} = \frac{PM^2 + AM^2}{AP^2}$

$= \frac{AP^2}{AP^2} = 1.$

Hence also, transposing and taking the square root—

(5.) $\text{Sin A} = \sqrt{1 - \cos^2 \text{A}}.$

(6.) $\text{Cos A} = \sqrt{1 - \sin^2 \text{A}}.$

And again—

(7.) $\text{Sec}^2 \text{A} = \frac{AP^2}{AM^2} = \frac{AM^2 + PM^2}{AM^2} = 1 + \frac{PM^2}{AM^2} = 1 + \tan^2 \text{A}.$

(8.) $\text{Cosec}^2 \text{ A} = \frac{AP^2}{PM^2} = \frac{PM^2 + AM^2}{PM^2} = 1 + \frac{AM^2}{PM^2}$

$= 1 + \cot^2 \text{A}.$

(9.) $\text{Tan A} = \frac{PM}{AM} = \frac{PM}{AP} \div \frac{AM}{AP} = \sin \text{A} \div \cos \text{A}$

$= \frac{\sin \text{A}}{\cos \text{A}}.$

(10.) $\text{Cot A} = \frac{AM}{PM} = \frac{AM}{AP} \div \frac{PM}{AP} = \cos \text{A} \div \sin \text{A}$

$= \frac{\cos \text{A}}{\sin \text{A}}.$

The student must make himself thoroughly master of the results in this article.

7. To express the trigonometrical ratios in terms of the *sine.*

(1.) $\text{Cos A} = \sqrt{1 - \sin^2 A}$, by Art. 6 (6.)

(2.) $\text{Tan A} = \dfrac{\sin A}{\cos A}$, by Art. 6 (9.),

$$= \frac{\sin A}{\sqrt{1 - \sin^2 A}}.$$

(3.) $\text{Cot A} = \dfrac{\cos A}{\sin A}$, by Art. 6 (10.),

$$= \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

(4.) $\text{Sec A} = \dfrac{1}{\cos A}$, by Art. 6 (2.),

$$= \frac{1}{\sqrt{1 - \sin^2 A}}.$$

(5.) $\text{Cosec A} = \dfrac{1}{\sin A}$, by Art. 6 (1.)

Ex. If $\sin A = \frac{3}{5}$, find the other trigonometrical ratios.

We have, $\cos A = \sqrt{1 - (\frac{3}{5})^2} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.

$$\tan A = \frac{\sin A}{\cos A} = \tfrac{3}{5} \div \tfrac{4}{5} = \tfrac{3}{4}.$$

8. To express the trigonometrical ratios in terms of the *cotangent.*

(1.) $\text{Sin A} = \dfrac{1}{\text{cosec A}} = \dfrac{1}{\sqrt{1 + \cot^2 A}}$, by Art. 6 (8.)

(2.) $\text{Cos A} = \dfrac{1}{\sec A} = \dfrac{1}{\sqrt{1 + \tan^2 A}}$, by Art. 6 (7.)

$$= \frac{1}{\sqrt{1 + \dfrac{1}{\cot^2 A}}} = \frac{\cot A}{\sqrt{\cot^2 A + 1}}.$$

(3.) $\text{Tan A} = \dfrac{1}{\cot A}$.

(4.) $\text{Sec A} = \dfrac{1}{\cos A} = 1 \div \dfrac{\cot A}{\sqrt{\cot^2 A + 1}}$, by (2.) above,

$$= \frac{\sqrt{\cot^2 A + 1}}{\cot A}.$$

(5.) $\text{Cosec A} = \sqrt{1 + \cot^2 A}$, by Art. 6 (8.)

And in the same way the trigonometrical ratios may be expressed in terms of any one of them.

Ex. II.

1. Given $\sin A = \frac{12}{13}$, find the other trigonometrical ratios.

2. Given $\tan A = \frac{24}{25}$, find the remaining trigonometrical ratios.

3. If $\cot A = a$, show that $\sin A = \dfrac{1}{\sqrt{1 + a^2}}$.

4. If $\text{vers } A = b$, then $\tan A = \dfrac{\sqrt{2b - b^2}}{1 - b}$.

5. Construct by scale and compass an angle (1.) whose cosine is $\frac{3}{7}$; (2.) whose tangent is $\frac{5}{6}$; (3.) whose secant is $\sqrt{2}$; (4.) whose cotangent is $2 + \sqrt{3}$.

Prove the following identities :—

6. $(\text{Sin A} + \cos A)^2 + (\sin A - \cos A)^2 = 2.$

7. $\text{Sec}^2 A + \cos^2 B \,.\, \text{cosec}^2 B = \text{cosec}^2 B + \sin^2 A \,.\, \sec^2 A.$

8. $\text{Sec}^2 A \,.\, \text{cosec}^2 A = \sec^2 A + \text{cosec}^2 A.$

9. $\text{Sec A} \,.\, \text{cosec A} = \tan A + \cot A.$

10. $\text{Sin}^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(\sin^4 A + \cos^4 A).$

11. $\dfrac{\text{Sec A} + \tan A}{\text{Cosec B} + \cot B} = \dfrac{\text{cosec B} - \cot B}{\sec A - \tan A}.$

12. $1 + \sin A \cos A = \dfrac{\sin^4 A + \sin^2 A \cos^2 A + \cos^4 A}{1 - \sin A \cos A}.$

13. $(x \cos \theta + y \sin \theta)(x \sin \theta + y \cos \theta) - (x \cos \theta - y \sin \theta)(x \sin \theta - y \cos \theta) = 2xy.$

14. $(a \sin\theta \cos\phi + r\cos\theta\cos\phi)(b\sin\theta\sin\phi + r\sin\theta\cos\phi)$
$-(b\sin\theta\cos\phi - r\sin\theta\sin\phi)(a\sin\theta\sin\phi + r\cos\theta\sin\phi)$
$= r\sin\theta(r\cos\theta + a\sin\theta)$.

15. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that $x^2 + y^2 + z^2 = r^2$.

16. If $a = b\cos C + c\cos B$, $b = a\cos C + c\cos A$, $c = a\cos B + b\cos A$, show that $a^2 + b^2 - c^2 = 2ab\cos C$.

17. Given $\sin^2 A + 3\sin A = \frac{7}{4}$, find $\sin A$.

18. Given $\cos^2 A - \sin A = \frac{9}{25}$, find $\cos A$.

19. Solve $\sin A - \cos A = \frac{1}{\sqrt{2}}$, for $\sin A$.

20. Find $\tan A$, when $\tan A + 1 = \sqrt{\frac{3}{2}}.\sec A$.

21. Given $a\cos A = b\sin A + a$, find $\cot A$.

22. Given $\tan^3 A - 7\tan A + 6 = 0$, find $\tan A$.

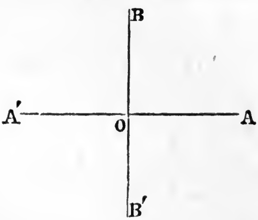
23. Show that $\sqrt{1 + 2\sin A\cos A} + \sqrt{1 - 2\sin A\cos A}$ $= 2\cos A$ or $2\sin A$, according as A is between 0° and 45°, or between 45° and 90°.

24. Given $m\sin^2 A + n\sin^2 B = a\cos^2 A$,
$m\cos^2 A + n\cos^2 B = b\sin^2 A$, find $\sin A$ and $\sin B$.

CHAPTER III.

CONTRARIETY OF SIGNS.—CHANGES OF MAGNITUDE AND SIGN OF THE TRIGONOMETRICAL RATIOS THROUGH THE FOUR QUADRANTS.

9. We have explained at some length the meaning and use of the signs + and − in algebra. They have a similar interpretation in trigonometry.

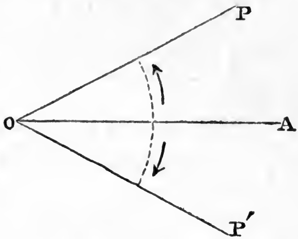


1. *Lines.*—Draw the horizontal line A′A, and draw BB′ at right angles, meeting it in O. Then considering O as origin,

(1.) All lines drawn to the *right* parallel to A′A are called *positive*, and all lines drawn to the *left* parallel to A′A are called *negative.*

(2.) All lines drawn *upwards* parallel to B′B are called *positive,* and all lines drawn *downwards* parallel to B′B are called *negative.*

(3.) Lines drawn in every other direction are considered positive, as is therefore the revolving line by which angles may be conceived to be generated.



2. *Angles.*—A similar convention is made for angles. Let OA be an initial line, and let a revolving line about the centre O take up the positions OP and OP′. Then—

(1.) That direction of revolution is considered *positive* which is *contrary* to that of the hands of a watch, and the angle generated is a *positive* angle.

(The positive direction is then *upwards.*)

Thus, AOP is a *positive* angle.

(2.) The *negative* direction of revolution is *the same* as that of the hands of a watch, and the angle thus generated is a *negative* angle.

(The negative direction is then *downwards.*)

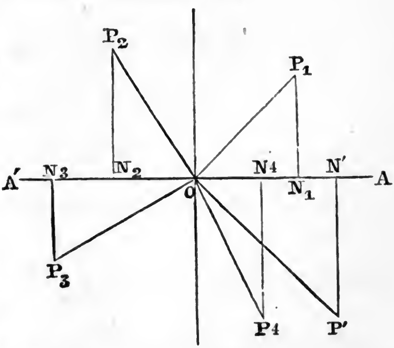
Thus, AOP′ is a negative angle.

Hence, if the angles AOP and AOP′ be of the same magnitude, and

We have—

$$\angle AOP = A,$$

then $\angle AOP' = -A$.



10. We will now examine the trigonometrical ratios for angles greater than a right angle, and for negative angles.

Let OP_1, OP_2, OP_3, OP_4 represent the position of the revolving line at any period of revolution in the several quadrants respectively,

And let P_1N_1, P_2N_2, P_3N_3, P_4N_4 be the respective perpendiculars from the end of the revolving line upon the initial line.

Then P_1N_1, P_2N_2, P_3N_3, P_4N_4 are respectively the perpendiculars corresponding to the angles generated.

Also, ON_1, ON_2, ON_3, ON_4 are respectively the *bases* of the right-angled triangles with respect to the angles in question.

We have then in the *second* quadrant—

$$\text{Sin } AOP_2 = \frac{P_2N_2}{OP_2}, \cos AOP_2 = \frac{ON_2}{OP_2},$$

$$\tan AOP_2, = \frac{P_2N_2}{ON_2}, \text{ \&c.}$$

It is therefore evident that the relations between the trigonometrical ratios, which were proved to exist in Art. 7, also hold for angles in the second quadrant—that is, angles between 90° and 180°.

And in the same way we may show that they hold for angles in the third, fourth, or any quadrant.

And again, if we suppose the line to revolve in a *negative* direction, and take the position OP', we shall have $P'N'$ the perpendicular corresponding to the negative angle AOP', and ON' the base.

$$\text{Hence, } \sin AOP' = \frac{P'N'}{OP'}, \cos AOP' = \frac{ON'}{OP'},$$

$$\tan AOP' = \frac{P'N'}{ON'}, \text{ \&c.}$$

And the relations proved in Art. 7 may be also similarly proved to exist here.

Hence the relations proved in Art. 7 hold for any angles whatever.

Changes of Magnitude and Sign of the Trigonometrical Ratios.

11. Let OP_1, OP_2, OP_3, OP_4 be positions of the revolving line in the several quadrants respectively; P_1N_1, P_2N_2,

P_3N_3, P_4N_4, the respective perpendiculars; and ON_1, ON_2, ON_3, ON_4, the *bases* of the corresponding right-angled triangles.

Then—

(1.) *In the first quadrant*—

$$\text{Sin AOP}_1 = \frac{P_1N_1}{OP_1}, \ \cos \text{AOP}_1 = \frac{ON_1}{OP_1},$$
$$\tan \text{AOP}_1 = \frac{P_1N_1}{OP_1}, \ \&c.$$

At the *commencement* of the motion of the revolving line, the angle $AOP_1 = 0°$;

Also, the perpendicular $P_1N_1 = 0$,

And the base $ON = OP_1$.

Hence, we have—

$$* \ \text{Sin } 0° = \frac{0}{OP_1} = 0, \ \cos\ 0° = \frac{OP_1}{OP_1} = 1,$$
$$\tan\ 0° = \frac{0}{OP_1}, = 0.$$

As the revolving line moves from OA towards OB, P_1N_1 increases and ON_1 diminishes; and when it arrives at OB, we have $P_1N_1 = OP_1$, and $ON_1 = 0$. But the angle generated is now a right angle. Hence we have—

$$\text{Sin } 90° = \frac{OP_1}{OP_1} = 1, \ \cos 90° = \frac{0}{OP_1} = 1,$$
$$\tan 90° = \frac{OP_1}{0} = \infty.$$

Hence, as the angle increases from 0° to 90°—
The *sine* changes in magnitude from 0 to 1 and is +.
The *cosine* changes in magnitude from 1 to 0 and is +.
The *tangent* changes in magnitude from 0 to ∞ and is +.

(2.) *In the second quadrant*—

Here the *perpendicular* P_2N_2 is +,
and the *base* ON_2 is −.

* The student ought properly to look upon the values 0, 1, 0 here obtained as the limiting values of the *sine*, *cosine*, and *tangent* respectively, when the angle is indefinitely diminished.

Hence the *sine* during the second quadrant is +, the *cosine* is −, and the *tangent* is −.

Again, as the revolving line moves from OB to OA′, the perpendicular P_2N_2 diminishes until it becomes zero. Also, the base ON_2 increases in magnitude, until it finally coincides with OA′, and ∴ = − OP_2. But the angle now described is 180°.

Hence we have—

$$\text{Sin } 180° = \frac{0}{OP_2} = 0, \cos 180° = -\frac{OP_2}{OP_2} = -1,$$

$$\tan 180° = -\frac{0}{OP_2} = 0, \&c.$$

Hence in the second quadrant—
The *sine* changes in magnitude from 1 to 0, and is *positive.*
The *cosine* changes in magnitude from 0 to 1, and is *negative.*
The *tangent* changes in magnitude from ∞ to 0, and is *negative.*

And in the same way may we trace the changes of magnitude and sign in the third and fourth quadrants.

Thus we shall find—

(3.) *In the third quadrant—*
The *sine* changes in magnitude from 0 to 1, and is *negative.*
The *cosine* changes in magnitude from 1 to 0, and is *negative.*
The *tangent* changes in magnitude from 0 to ∞, and is *positive.*

(4.) *In the fourth quadrant—*
The *sine* changes in magnitude from 1 to 0, and is *negative.*
The *cosine* changes in magnitude from 0 to 1, and is *positive.*
The *tangent* changes in magnitude from ∞ to 0, and is *negative.*

Moreover, as the cosecant, secant, and cotangent are respectively the reciprocals of the *sine, cosine,* and *tangent,* it follows that their *signs* will follow respectively the latter, and that their magnitudes will be their reciprocals.

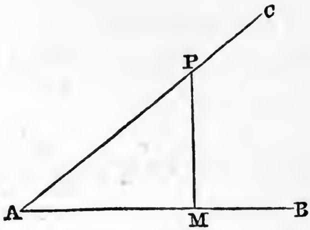
CHAPTER IV.

TRIGONOMETRICAL RATIOS CONTINUED. ARITHMETICAL VALUES OF THE TRIGONOMETRICAL RATIOS OF 30°, 45°, 60°, &c.

12. *To prove that* sin A = cos (90° − A), and that
cos A = sin (90° − A).

Using the same figure as in Art. 5, we have—

$$\text{Sin A} = \frac{PM}{AP} = \cos APM.$$



But $\angle APM = 90° - A$,

$\therefore \text{Sin A} = \cos (90° - A)$,

and similarly—

$\cos A = \sin (90° - A)$,

$\tan A = \cot (90° - A)$,

$\cot A = \tan (90° - A)$,

$\sec A = \text{cosec} (90° - A)$,

$\text{cosec} A = \sec (90° - A)$.

13. *Ratios of* 45°.

In the last figure, suppose $\angle PAM = 45°$, then also $\angle APM = 90° - 45° = 45°$. And hence $\angle PAM = \angle APM$, and $\therefore PM = AM$ (Euc. I., 6).

Hence, also—

$$AP \text{ or } \sqrt{AM^2 + PM^2} = \sqrt{2\,AM^2} \text{ or } \sqrt{2\,PM^2}.$$

$$\therefore AP = AM\sqrt{2} \text{ or } PM\sqrt{2}.$$

Hence we have—

$$\text{Sin } 45° = \sin PAM = \frac{PM}{AP} = \frac{PM}{PM\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45°, \text{ by}$$

Art. 8.

$$\text{Tan } 45° = \tan PAM = \frac{PM}{AM} = \frac{PM}{PM} = 1 = \cot 45°, \text{ by Art. 8.}$$

$$\text{Sec } 45° = \sec PAM = \frac{AP}{AM} = \frac{AM\sqrt{2}}{AM} = \sqrt{2} = \text{cosec } 45°,$$

by Art. 8.

14. *Ratios of* 30° and 60°.

In the same diagram, suppose $\angle PAM = 30°$, then $\angle APM = 90° - 30° = 60°$.

Hence, if we conceive another triangle equal in every respect to APM to be described on the other side of AM, the whole would form an equilateral triangle whose side is AP.

Hence, $PM = \frac{1}{2} AP$.

Now $AM = \sqrt{AP^2 - PM^2}$, $\therefore AM = \sqrt{AP^2 - (\frac{1}{2} AP)^2}$ $= \frac{\sqrt{3}}{2} AP$.

We hence have—

$$\text{Sin } 30^\circ = \sin PAM = \frac{PM}{AP} = \frac{\frac{1}{2} AP}{AP} = \tfrac{1}{2} = \cos 60^\circ, \text{ by Art. 8.}$$

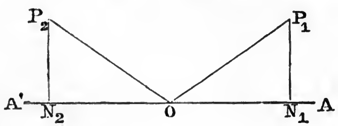
$$\text{Cos } 30^\circ = \cos PAM = \frac{AM}{AP} = \frac{\frac{\sqrt{3}}{2} AP}{AP} = \frac{\sqrt{3}}{2} = \sin 60^\circ, \text{ by}$$

Art. 8.

$$\text{Tan } 30^\circ = \frac{PM}{AM} = \frac{\frac{1}{2} AP}{\frac{\sqrt{3}}{2} AP} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \cot 60^\circ, \text{ by}$$

Art. 8.

15. *To show that* $\sin(180^\circ - A) = \sin A$,
$\cos(180^\circ - A) = -\cos A$,



Let $\angle AOP_1 = A$,

And let the revolving line describe an angle $AOP_2 = 180^\circ - A$;

Then $\angle A'OP_2 = 180^\circ - (180^\circ - A) = A$;

Hence, $\angle AOP_1 = A'OP_2$.

Hence, also (Euc. I., 26), if P_1N_1, P_2N_2 be drawn perpendicular to AA', $P_1N_1 = P_2N_2$, $ON_2 = -ON_1$,

We have therefore—

$$\text{Sin } (180^\circ - A) = \sin AOP_2 = \frac{P_2N_2}{OP_2} = \frac{P_1N_1}{OP_1} = \sin A.$$

$$\text{Cos } (180^\circ - A) = \cos AOP_2 = \frac{ON_2}{OP_2} = -\frac{ON_1}{OP_1} = -\cos A.$$

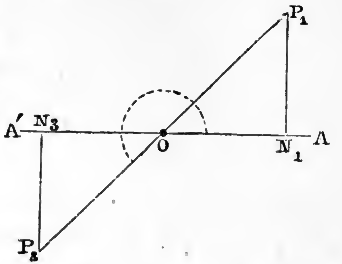
And similarly—

Tan $(180^\circ - A) = -\tan A$, cot $(180^\circ - A) = -\cot A$.
Sec $(180^\circ - A) = -\sec A$, cosec $(180^\circ - A) = \text{cosec } A$.

16. *To show that* $\sin(180° + A) = -\sin A$,
$\cos(180° + A) = -\cos A$,

Let $AOP_1 = A$,

And let the revolving line take a position such that P_1P_3 is a straight line.



Then, evidently, $\angle AOP_3 = 180° + A$.

Then, as in last Article—

$P_3N_3 = -P_1N_1$,

$ON_3 = -ON_1$.

Hence—

$$\text{Sin}(180° + A) = \sin AOP_3 = \frac{P_3N_3}{OP_3} = -\frac{P_1N_1}{OP_1} = -\sin A.$$

$$\text{Cos}(180° + A) = \cos AOP_3 = \frac{ON_3}{OP_3} = -\frac{ON_1}{OP_1} = -\cos A.$$

And similarly—

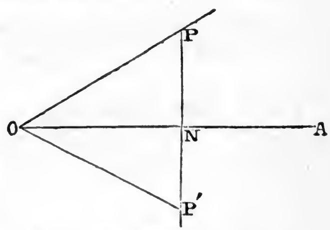
$\text{Tan}(180° + A) = \tan A$, $\cot(180° + A) = \cot A$.

$\text{Sec}(180° + A) = -\sec A$, $\text{cosec}(180° + A) = -\text{cosec} A$.

17. *To show that* $\sin(-A) = -\sin A$, and
$\cos(-A) = \cos A$,

Let $\angle AOP = A$,

And let the revolving line describe an angle $AOP' = -A$.



Then evidently, if PNP' be drawn perpendicular to OA, we have (Euc. I., 26) $P'N = -PN$.

Hence—

$$\text{Sin}(-A) = \sin AOP' = \frac{P'N}{OP'} = -\frac{PN}{OP} = -\sin A,$$

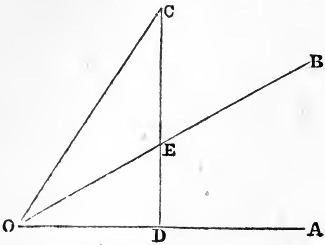
$$\text{Cos}(-A) = \cos AOP' = \frac{ON}{OP'} = \frac{ON}{OP} = \cos A.$$

And similarly—

$\text{Tan}(-A) = -\tan A$, $\cot(-A) = -\cot A$,

$\text{Sec}(-A) = \sec A$, $\text{cosec}(-A) = -\text{cosec} A$.

Although the results of Arts. 12, 15, 16, 17 have been proved from diagrams where A is less than a right angle, the student will have no difficulty, if he has understood the proofs, in deducing the same results for angles of any magnitude whatever.



18. *To show that* $\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A}$,

Let $\angle AOC = A$;

Bisect it by the straight line OB, so that $\angle AOB = \frac{A}{2}$, and draw CD perpendicular to OA, meeting OB in E.

Then $\tan\frac{A}{2} = \tan EOD = \frac{ED}{OD}$........................(1).

Now (Euc. VI., 2), $\frac{OD}{OC} = \frac{ED}{EC}$, and $\therefore \frac{OD}{OC+OD} = \frac{ED}{CD}$, or

$$\frac{ED}{OD} = \frac{CD}{OC + OD} = \frac{CD(OC - OD)}{OC^2 - OD^2} = \frac{CD(OC - OD)}{CD^2}$$

$$= \frac{OC - OD}{CD} = \frac{OC - OC\cos A}{OC\sin A} = \frac{1 - \cos A}{\sin A}. \quad Q.E.D.$$

Cor 1. Hence, squaring—

$$\text{Tan}^2\frac{A}{2} = \frac{(1 - \cos A)^2}{\sin^2 A} = \frac{(1 - \cos A)^2}{1 - \cos^2 A} = \frac{1 - \cos A}{1 + \cos A}.$$

$$\therefore \text{Art. 64, page 215, } \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}} = \frac{\cos A}{1}.$$

$$\therefore \text{Cos } A = \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}} \text{.............................(1).}$$

$$= \frac{1 - \dfrac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}}{1 + \dfrac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}} = \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}}$$

$$= \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

$$= \cos^2 \frac{A}{2} - \left(1 - \cos^2 \frac{A}{2}\right)$$

$$= 2 \cos^2 \frac{A}{2} - 1 \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

$$= 2\left(1 - \sin^2 \frac{A}{2}\right) - 1 = 1 - 2 \sin^2 \frac{A}{2} \quad (4.)$$

19. *To find the trigonometrical ratios of* 15°, 75°, 120°, 135°, 150°.

(1.) *Ratios of* 120°.

$$\text{Sin } 120° = \sin (180° - 120°) = \sin 60° = \frac{\sqrt{3}}{2},$$

$$\text{Cos } 120° = -\cos (180° - 120°) = -\cos 60° = -\frac{1}{2},$$

$$\text{Tan } 120° = -\tan (180° - 120°) = -\tan 60° = -\sqrt{3}, \text{ \&c.}$$

(2.) *Ratios of* 150°.

$$\text{Sin } 150° = \sin (180° - 150°) = \sin 30° = \frac{1}{2},$$

$$\text{Cos } 150° = -\cos (180° - 150°) = -\cos 30° = -\frac{\sqrt{3}}{2},$$

$$\text{Tan } 150° = -\tan (180° - 150°) = -\tan 30 = -\frac{1}{\sqrt{3}},$$

&c.

(3.) *Ratios of* 15°.

By last Art., $\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A}$; put A = 30°, or $\frac{A}{2} = 15°$,

$$\text{then } \tan 15° = \frac{1 - \cos 30°}{\sin 30°} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}.$$

From this result we easily get, Art. 8,

$$\text{Sin } 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}},\ \cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}},\ \&c.$$

(4.) *Ratios of* 75°.

We have, $\sin 75° = \sin(90° - 15°) = \cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}}$,

$$\cos 75° = \sin(90° - 15°) = \sin 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

$$\tan 75° = \tan(90° - 15°) = \cot 15° = \frac{1}{2 - \sqrt{3}}$$

$$= 2 + \sqrt{3},\ \&c.$$

(5.) *Ratios of* 135°.

We have, $\sin 135° = \sin(180° - 135°) = \sin 45° = \frac{1}{\sqrt{2}}$,

$$\cos 135° = -\cos(180° - 135°) = -\cos 45°$$

$$= -\frac{1}{\sqrt{2}},$$

$$\tan 135° = -\tan(180° - 135°) = -\tan 45°$$
$$= -1,\ \&c.$$

Ex. III.

1. Define a negative angle, and show that tan (– A) = – tan A, when A lies between – 90° and – 180°.

2. Trace the changes of sign of sin A . cos A through the four quadrants.

3. Trace the changes of sign of cos A + sin A, and of cos A – sin A, as A changes from – 45° to 315°.

4. Assuming generally that $\cos 2A = \cos^2 A - \sin^2 A$, trace the changes of sign of $\cos 2A$ as A changes from $-45°$ to $315°$.

5. Write down the sines of $210°$, $165°$, $240°$, $-120°$.

6. Show that $\sin(90° + A) = \cos A$, and $\cos(90° + A) = -\sin A$, for any value of A from $0°$ to $180°$.

7. Assuming generally that $2\cos^2\frac{A}{2} = 1 + \cos A$, and $2\sin^2\frac{A}{2} = 1 - \cos A$, show that $\sqrt{2}\cos\frac{A}{2} = -\sqrt{1 + \cos A}$ and $\sqrt{2}\sin\frac{A}{2} = -\sqrt{1 - \cos A}$, when A lies between $360°$ and $540°$.

8. Given $\cos A = 1 - 2\sin^2\frac{A}{2}$, show that $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$.

9. Hence show that $2\cos\frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$, when $\frac{A}{2}$ lies between $135°$ and $225°$.

Solve the following equations:—

10. $\text{Cos}^2 A + \frac{3}{8}\cos A = \frac{7}{16}$.

11. $\text{Tan}\,\theta + 5\cot\theta = 6$.

12. $\text{Sin}\,A + \sec A = \frac{2}{\sqrt{3}} + \frac{1}{2}$.

13. $2\cos^2 A = 3\sin A$.

14. $\text{Sin}(A + B) = \cos(A - B) = \frac{\sqrt{3}}{2}$.

15. $\text{Tan}^2 A = 2\sin^2 A$.

16. $\text{Sin}(3A + 75°) = \cos(2A - 15°)$.

17. $\text{Sec}\,\theta + \cos\theta = \frac{5}{2\sqrt{3}}\tan\theta$.

18. $\text{Tan}\,\theta + \cot\theta = 4$.

CHAPTER V.

LOGARITHMS.

20. Def.—The **logarithm** of a number to a given base is the index of the power to which the base must be raised to obtain the number.

Thus, we may obtain the numbers 1, 10, 100, 1,000, 10,000, &c., by raising the base 10 to the powers 0, 1, 2, 3, 4, &c., respectively; and hence, by the above definition, we have—

$$\text{Log}_{10} 1 = 0, \log_{10} 10 = 1, \log_{10} 100 = 2, \log_{10} 1{,}000 = 3, \text{\&c.},$$

the suffix 10 being added to the word *log* to indicate that the base is 10. It is usual, however, in common logarithms to omit this suffix; and hence, when there is no base expressed, the student will understand 10.

Again, the numbers 1, 2, 4, 8, 16, &c., may be obtained by finding the values of 2^0, 2^1, 2^2, 2^3, 2^4, &c., respectively, and hence we have by definition—

$$\text{Log}_2 1 = 0, \log_2 2 = 1, \log_2 4 = 2, \log_2 8 = 3, \text{\&c.}$$

So also we find $\log_4 16 = 2, \log_5 125 = 3, \log_3 81 = 4$, &c.

Ex. Find $\log_4 256$, $\log_{36} 216$, and the logarithm of 9 to base $\sqrt{3}$.

$\text{Log}_4 256 = \log_4 4^4 = 4$, by definition.

$\text{Log}_{36} 216 = \log_{36} 6^3 = \log_{36} (6^2)^{\frac{3}{2}} = \log_{36} 36^{\frac{3}{2}} = \frac{3}{2}$, by definition.

$\text{Log}_{\sqrt{3}} 9 = \log_{\sqrt{3}} 3^2 = \log_{\sqrt{3}} (\sqrt{3})^4 = 4$, by definition.

Characteristics of Ordinary Logarithms.

21. Def.—The **characteristic** of a logarithm is the integral part of the logarithm, and the fractional part (generally expressed as a decimal) is called the **mantissa**.

1. *Numbers containing integer digits.*

Every number containing n digits in its integral part must lie between 10^{n-1} and 10^n.

Thus, 6 lies between 10^0 and 10^1, 29 lies between 10^1 and 10^2, 839 lies between 10^2 and 10^3, &c.

Hence the ordinary logarithms of all numbers having n integer digits lies between $(n - 1)$ and n.

The integral portion or characteristic of the logarithm of a number having n integer digits is therefore $(n - 1)$.

Hence we have the following rule :—

RULE 1.—*The characteristic of the logarithm of a number having integer digits is* ONE *less than the number of integer digits.*

Thus, the characteristics of the logarithms of 32, 713·54, 8·7168, 56452, 73607·9 are respectively 1, 2, 0, 4, 4.

2. *Numbers less than unity expressed as decimals.*

All such numbers having n zeros immediately after the decimal point lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, or between 10^{-n} and $10^{-(n+1)}$.

Thus, ·3 lies between 1 and ·1, or between 1 and $\frac{1}{10}$, or 10^0 and 10^{-1};

·027 lies between ·1 and ·01, or between $\frac{1}{10}$ and $\frac{1}{10^2}$, or 10^{-1} and 10^{-2};

·000354 lies between ·001 and ·0001, or between $\frac{1}{10^3}$ and $\frac{1}{10^4}$, or 10^{-3} and 10^{-4}, and so on.

Hence, by Def., Art. 20, the logarithm of any number having n zeros immediately after the decimal point lies between $-n$ and $-(n + 1)$. Hence, the logarithm is *negative*, and the integral part of this negative quantity is n. It is however usual to write all the mantissæ of logarithms as *positive* quantities, and the *negative integral* part of the logarithm will be the next higher negative integer, viz., $-(n + 1)$.

We have therefore the following rule :—

RULE 2.—*The characteristic of the logarithm of a number less than unity, and expressed as a decimal, is the negative* *integer next greater than the number of zeros immediately after the decimal point.*

Thus, the characteristics of the logarithms of ·3, ·0076, ·02535, ·7687, are respectively − 1, − 3, − 2, − 1.

22. *The logarithm of the* PRODUCT *of two numbers is the* SUM *of the logarithms of the numbers.*

Let m and n be the numbers, and let a be the base. Since m and n must be each some power of a, integral or fractional, positive or negative, assume—

$$\left.\begin{aligned} m &= a^x, \\ n &= a^y. \end{aligned}\right\} \text{ Then, by definition of a logarithm,}$$

$$x = \log_a m, \text{ and } y = \log_a n.$$

Now we have $mn = a^x . a^y = a^{x+y}$, and hence, by definition,

$$\log_a mn = x + y; \text{ we therefore have—}$$

$$\log_a (mn) = \log_a m + \log_a n. \quad Q.E.D.$$

COR. This proposition may be extended to any number of factors.

Thus, $\log_a (mnpq) = \log_a m + \log_a n + \log_a p + \log_a q$.

23. *The logarithm of the* QUOTIENT *of two numbers is found by* SUBTRACTING *the logarithm of the denominator from the logarithm of the numerator.*

Assuming, as in the last Art., we have—

$$x = \log_a m, \; y = \log_a n.$$

$$\text{Also, } \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}, \text{ and hence, by definition,}$$

$$\log_a \frac{m}{n} = x - y; \text{ we therefore have—}$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n. \quad Q.E.D.$$

24. *The logarithm of the* POWER *of a number is found by* MULTIPLYING *the logarithm of the number by the* INDEX *of the power.*

Let it be required to find $\log_a N^p$.

Assume $N = a^x$, and therefore $x = \log_a N$.

We have $N^p = (a^x)^p = a^{px}$, and hence, by definition,

$$\log_a N^p = px = p \log_a N. \quad Q.E.D.$$

Ex. 1.—Given log 2 = ·3010300, and log 3 = ·4771213, find the logarithms of 18, 15, ·125, 6·75.

Log 18 = $\log (2 \times 3^2) = \log 2 + 2 \log 3$ = ·3010300 + 2(·4771213) = 1·2552726.

Log 15 = $\log (3 \times \frac{10}{2}) = \log 3 + \log 10 - \log 2$ = ·4771213 + 1 − ·3010300 = 1·1760913.

Log ·125 = $\log\left(\frac{1}{2^3}\right) = \log 1 - 3 \log 2 = 0 - 3 \times \cdot3010300$

= − ·9030900 = − 1 + (1 − ·9030900) = − 1 + ·0969100, or, as usually written, = $\overline{1}\cdot0969100$.

$\text{Log}\, 6\cdot75 = \log \frac{27}{4} = \log \frac{3^3}{2^2} = 3 \log 3 - 2 \log 2 = 3(\cdot4771213)$

− 2(·3010300) = ·8293039.

Ex. 2. Find the logarithm of $\frac{(2\cdot4)^{\frac{1}{2}} \times (\cdot375)^4}{(2\cdot43)^5 \times (\cdot032)^{-\frac{1}{3}}}$, having given log 2 and log 3.

We have—

$$\text{Log N} = \tfrac{1}{2} \log 2\cdot4 + 4 \log \cdot375 - 5 \log 2\cdot43 - (-\tfrac{1}{3}) \log \cdot032.$$

$$= \frac{1}{2} \log \frac{2^3 \times 3}{10} + 4 \log \frac{3}{2^3} - 5 \log \frac{3^5}{10^2} + \frac{1}{3} \log \frac{2^5}{10^3}$$

$$= \tfrac{1}{2} (3 \log 2 + \log 3 - \log 10) + 4 (\log 3 - 3 \log 2) - 5 (5 \log 3 - 2 \log 10) + \tfrac{1}{3} (5 \log 2 - 3 \log 10)$$

$$= (\tfrac{3}{2} - 12 + \tfrac{5}{3}) \log 2 + (\tfrac{1}{2} + 4 - 25) \log 3 + (-\tfrac{1}{2} + 10 - 1) \log 10$$

$$= -\tfrac{53}{6} \times \cdot3010300 - \tfrac{41}{2} \times \cdot4771213 + \tfrac{17}{2} \times 1$$

$$= -2\cdot6590983 - 9\cdot7809867 + 8\cdot5 = -3\cdot9400850$$

$$= \overline{4} + (4 - 3\cdot9400850) = \overline{4}\cdot0599150.$$

Ex. IV.

1. Find the logarithm to base 4 of the following numbers: 16, 64, 2, ·25, ·0625, 8.

2. Find the value of $\log_8 32$, $\log_{\sqrt[3]{5}} 25$, $\log_{\cdot 81} \cdot 729$.

3. Given log 2 = ·3010300, and log 3 = ·4771213, find the logarithms of 12, 36, 45, 75, ·04, 3·75, $\cdot\dot{6}$, $\cdot\dot{0}7\dot{4}$.

4. Given log 20763 = 4·3172901, what is the logarithm of 2·0763, 2076·3, ·020763, ·0020763?

5. Write down the characteristics of the common logarithms of 29·6, ·25402, ·0034, 6176·003.

6. Given log 20·912 = 1·3203956, what numbers correspond to the following logarithms:—$\bar{2}$·3203956, 6·3203956, $\bar{1}$·3203956, 4·3203956?

7. Given log 20·713 = 1·3162430, and log 20714 = 3·3162640, find log ·2071457.

8. Given log 3·4937 = ·5432856, and log 3·4938 = ·5432980, find the number whose logarithm is $\bar{3}$·5432930.

9. Given log 1·05 = ·0211893, log 2·7 = 1·4313638, log 135 = 2·1303338, find the value of $\log \dfrac{(2{\cdot}7)^{\frac{1}{3}} \times \sqrt[5]{13{\cdot}5}}{(1{\cdot}05)^6}$.

10. Given log 18 = 1·2552725, and log 2·4 = ·3802112, find the value of log ·00135.

11. What are the characteristics of $\log_3 1167$, and $\log_4 1965$?

12. Having log 2 = ·3010300, and log 3 = ·4771213, find x when $18^x = 125$.

CHAPTER VI.

THE USE OF TABLES.

25. Tables have been formed of the logarithms of all numbers from 1 to 100,000, and we shall now show how they are practically used. We shall not enter here upon the method of forming the tables themselves.

The following is a specimen of the way in which the logarithms of numbers are usually tabulated:—

No.	0	1	2	3	4	5	6	7	8	9	D.
7990	9025468	5522	5577	5631	5685	5740	5794	5848	5903	5957	
91	6011	6066	6120	6174	6229	6283	6337	6392	6446	6500	
92	6555	6609	6663	6718	6772	6826	6881	6935	6989	7044	
93	7098	7152	7207	7261	7315	7370	7424	7478	7533	7587	
94	7641	7696	7750	7804	7859	7913	7967	8022	8076	8130	54
95	8185	8239	8293	8348	8402	8456	8511	8565	8619	8674	
96	8728	8782	8836	8891	8945	8999	9054	9108	9162	9217	
97	9271	9325	9380	9434	9488	9542	9597	9651	9705	9760	
98	9814	9868	9923	9977	$\bar{0}$031	0085	0140	0194	0248	0303	
99	9030357	0411	0466	0520	0574	0628	0683	0737	0791	0846	
8000	0900	0954	1008	1063	1117	1171	1226	1280	1334	1388	
D. { 54	P. {	5	11	16	22	27	32	38	43	49	

Thus, if the number consist of four figures only, we have simply to copy out the figures in the column headed 0, prefix a decimal point, and the proper characteristic.

Ex. Log 7991 = 3·9026011, log 7·995 = ·9028185.

When we speak of a number consisting of four figures only, we include such numbers as ·003654, ·07682, &c., the number of zeros immediately following the decimal points not being counted.

$$\text{Thus, } \log \cdot 07997 = \bar{2}\cdot 9029271$$
$$\log \cdot 007992 = \bar{3}\cdot 9026555.$$

When the number contains five figures, as, for instance, 79936, we look along the line containing the first four figures —viz., 7993—of the number until the eye rests upon the column headed 6, the fifth figure. We then take the first three figures of the column headed 0, and affix the four figures of the column headed 6 in the horizontal line of the first four figures of the number.

$$\text{Thus, } \log 79936 = 4\cdot 9027424$$
$$\log \cdot 079927 = \bar{2}\cdot 9026935.$$

It will be seen from the portion of the logarithmic table above extracted, that when the first three figures of the logarithm—viz., 902—have been once printed, they are not repeated, but must be understood to belong to every four figures in each column, until they are superseded by higher figures, as 903. When, however, this change is intended to be made at any place not at the commencement of a horizontal row, the first of the four figures corresponding to the change is usually printed either in different type, or, as above, with a bar over it. Thus we have above $\bar{0}031$, indicating that from this point we must prefix 903 instead of 902.

$$\text{Thus, } \log 79{\cdot}986 = 1{\cdot}9030140,$$
$$\log {\cdot}0079987 = \bar{3}{\cdot}9030194.$$

26. *To find the logarithm of a number not contained in the tables.*

Ex. Find the logarithm of 799·1635.

Since* the *mantissa* of the number 79916·35 is the same as the *mantissa* of the given number, and that the first five figures are contained in the tables, we may proceed as follows—

(1.) Take out from the tables the mantissa corresponding to the number 79916. This is ·9026337.

(2.) Take out the mantissa of the *next higher* number in the tables—viz., 79917. This is ·9026392.

(3.) Find the difference between these mantissæ. This is called the tabular difference, being the difference of the mantissæ for a difference of *unity* in the numbers. We find tab. diff. = ·0000055, which we call D.

(4.) Then *assuming* that small differences in numbers are proportional to the differences of the corresponding logarithms, we find the difference for ·35 = ·35 × ·0000055 = ·0000019, retaining only 7 figures. This is often called *d*.

(5.) Now adding this value of *d* to the mantissa for the number 79916, we get the mantissa corresponding to the number 79916·35.

(6.) Lastly, prefix to this mantissa the proper characteristic.

The whole operation may stand thus—

$$\text{M. of log } 79916 = {\cdot}9026337 \ldots\ldots\ldots\ldots\ldots (1).$$
$$\text{M. of log } 79917 = {\cdot}9026392$$
$$\text{Tabular difference or D} = {\cdot}0000055$$

* Thus log 79916·35 = log (100 × 799·1635) = 2 + log 799·1635.

Hence, difference for ·35 or d

$$= \cdot 35 \times \cdot 0000055 = \cdot 0000019 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, adding (1.) and (2.)—

$$\text{M. of log } 79916\cdot 35 = \cdot 9026356.$$
$$\therefore \log 799\cdot 1635 = 2\cdot 9026356.$$

or better thus, omitting the useless ciphers—

$$\text{M. of log } 79916 = \cdot 9026337$$
$$\text{M. of log } 79917 = \cdot 9026392$$
$$\therefore D = 55$$
$$\text{Hence, } d = \cdot 35 \times 55 = 19$$
$$\therefore \text{M. of log } 79916\cdot 35 = \cdot 9026356, \text{ as before.}$$

In the next article we shall show how the required difference may be obtained by inspection from the tables.

27. *Proportional parts.*

We saw in the example just worked that the tab. diff. (omitting the useless ciphers) is 55, and if we examine the table in Art. 25, we shall find the difference between the mantissæ of any two consecutive numbers there to be 54 or 55—generally 54. The number 54 is therefore placed in a separate column at the right of the table, and headed D.

The student will understand that the tab. diff. changes from time to time, and is not *always* 54 or 55.

Now assuming as in (4.) of the last article, we have—

Diff. for	·1 = 54 × ·1 = 5	Diff. for	·6 = 54 × ·6 = 32
,,	·2 = 54 × ·2 = 11	,,	·7 = 54 × ·7 = 38
,,	·3 = 54 × ·3 = 16	,,	·8 = 54 × ·8 = 43
,,	·4 = 54 × ·4 = 22	,,	·9 = 54 × ·9 = 49
,,	·5 = 54 × ·5 = 27		

We find therefore the numbers 5, 11, 16, 22, 27, 32, 38, 43, 49 placed in a horizontal row at the bottom marked P, in the columns respectively headed 1, 2, 3, 4, &c.

Hence, if we require the difference for (say) ·7, we take out the number 38 from the horizontal row marked P, instead of being at the trouble to find it by actual computation.

The following example will illustrate how we proceed when we require the difference for a decimal containing more than one decimal figure. No explanation is needed.

Ex. Find log 7994·3726—

$$\begin{array}{rll}
\text{M. of log } 79943 & = & \cdot 9027804 \\
\text{Diff. for } 7 & = & 38 \\
\text{,, } \quad 2 & = & 1\,1 \\
\text{,, } \quad 6 & = & \quad 32 \\
\hline
\therefore \text{ M. of log } 79943726 & = & \cdot 9027843 \\
\text{Hence, log } 7994\cdot 3726 & = & 3\cdot 9027843.
\end{array}$$

28. *Having given the logarithm of a number to find the number.*

After the explanations of Art. 26, the method of working the following examples will be easily understood :—

Ex. 1. Find the number whose logarithm is $\bar{1}\cdot 9030173$. Taking from the tables the mantissæ next above and below, we have—

$$\begin{array}{rll}
\cdot 9030194 & = & \text{M. of log } 79987 \\
\cdot 9030140 & = & \text{M. of log } 79986 \ldots\ldots\ldots\ldots\ldots(1). \\
\therefore \qquad 54 & = & \text{D.} \\
\text{Again, } \cdot 9030173 & = & \text{M. of log N} \ldots\ldots\ldots\ldots\ldots\ldots(2).
\end{array}$$

Hence, subtracting (1) from (2)—

33 = d, the difference between the logarithms of the required number and the next lower.

Now $\frac{33}{54}$ = ·61, the difference between the next lower number and the required number.

Hence ·9030173 = M. of log 79986·61 ;

$\therefore$ $\bar{1}\cdot 9030173$ = log ·7998661 ;

$\therefore$ ·7998661 is the number required.

Ex. 2.* Find the value of $\dfrac{(1\cdot 023)^3 \times (\cdot 00123)^{\frac{1}{4}}}{(1\cdot 32756)^4}$.

We have—

$$\log N = 3 \log 1\cdot 023 + \tfrac{1}{4} \log \cdot 00123 - 4 \log 1\cdot 32756.$$

$$\begin{array}{rlll}
\text{Now, } 3 \log 1\cdot 023 & = 3 \times \cdot 0098756 & = & \cdot 0276268 \\
\tfrac{1}{4} \log \cdot 00123 & = \tfrac{1}{4}\,(\bar{3}\cdot 0899051) & & \\
 & = \tfrac{1}{4}\,(\bar{4} + 1\cdot 0899051) & = & \bar{1}\cdot 2724763 \\
\hline
\end{array}$$

* The logarithms used in this example are taken from the tables.

$\therefore$ adding, 3 log. 1·023 + $\frac{1}{4}$ log ·00123 $= \overline{1}$·3001031
Again, M. of log 13275 = ·1230345
and diff. for ·6 = ·196
$\therefore$ M. of log 13275·6 = ·1230541
$\therefore$ 4 log 1·32756 = 4 × ·1230541 = ·4922164
Then, subtracting, log N $= \overline{2}$·5078867
Hence we have, ·5078867 = M. of log N,
and ·5078828 = M. of log 32202;
$\therefore$ 39 = d,
also 135 = D,
and $\frac{39}{135}$ = ·29.
$\therefore$ ·5078867 = M. of log 32202·29;
$\therefore$ $\overline{2}$·5078867 = log ·03220229.
Hence ·03220229 is the number required.

Trigonometrical Tables.

29. We use trigonometrical tables much in the same way as we do tables of ordinary logarithms of numbers.

Tables have been formed of natural sines, cosines, &c., and also of logarithmic sines, cosines, &c. It is with the latter only we shall now deal, though many of our remarks apply equally to the former.

As the values of the natural sines and cosines of all angles between 0° and 90° are (Art. 11) less than *unity*, it follows (Art. 21) that their logarithms are negative. To avoid, however, printing them in a negative form, and for other reasons, it is usual to add 10 to their real value, and hence in using them we must allow for this. The same thing is also done in the case of logarithmic tangents, cotangents, secants, and cosecants.

We generally express the *true* logarithmic sine by log sin, and the *tabular* logarithmic sine by L sin.

Hence, we have, log sin A = L sin A − 10,
log cos A = L cos A − 10, &c.

It must be remembered in using the tables that, although (Art. 11) the sine, secant, and tangent of an angle *increase* as the angle *increases* from 0° to 90°, yet the cosine, cosecant, and cotangent *diminish* as the angle *increases*.

Hence, when any angle is not exactly contained in the tables, we must *add* the difference in the case of a *sine*, *secant*, or *tangent;* but *subtract* it in the case of a *cosine*, *cosecant*, or *cotangent*.

And, conversely, when the given logarithm is not contained exactly in the tables, we must in the case of the *sine*, *secant*, or *tangent* take out the next *lower* tabular logarithm as corresponding to the angle next lower; but in the case of a *cosine*, *cosecant*, or *cotangent*, we must take out the next *higher* tabular logarithm as corresponding to the angle next lower in the tables.

We shall *assume* that small differences in the angles are proportional to the corresponding differences of the logarithmic trigonometrical ratios

Ex. 1. Find L sin 56° 28′ 24″.

Referring to tables, we have—

$$\text{L}\sin 56^\circ\, 28' = 9{\cdot}9209393$$

$$\text{Tab. diff. for } 60'' \text{ or D} = 836$$

$$\therefore \text{ diff. for } 24'' \text{ or } d = \frac{24}{60} \times 836 = 334$$

$$\therefore \text{L}\sin 56^\circ\, 28'\, 24'' = 9{\cdot}9209727$$

Ex. 2. Find L cos 29° 31′ 28″.

$$\text{Now L}\cos 29^\circ\, 31' = 9{\cdot}9396253$$

$$\text{Tab. diff. for } 60'' \text{ or D} = -716$$

$$\therefore \text{ diff. for } 28'' = -\frac{28}{60} \times 716 = -334$$

$$\therefore \text{L}\cos 29^\circ\, 31'\, 28'' = 9{\cdot}9395919$$

Ex. 3. Find the angle A, when L tan A $= 9{\cdot}8658585$

$$\text{We have } 9{\cdot}8658585 = \text{L}\tan \text{A}.$$

$$\text{Next } lower,\ 9{\cdot}8657702 = \text{L}\tan 36^\circ\, 17',$$

$$883 = \text{difference or } d,$$

$$\text{Also, } 2648 = \text{tab. diff. for } 60'' = \text{D},$$

$$\text{And } \frac{d}{\text{D}} \times 60'' = \frac{883}{2648} \times 60'' = 20''.$$

Hence, $9{\cdot}8658585 = \text{L}\tan 36^\circ\, 17'\, 20''$.

Ex. 4. Find the angle A, when L cot A = 10·0397936.

$$\begin{aligned} \text{We have, } 10{\cdot}0397936 &= \text{L cot A},\\ \text{Next } \textit{higher}, \ \underline{10{\cdot}0399770} &= \text{L cot } 42^\circ\ 22',\\ 1834 &= \text{difference or } d,\\ \text{Also, } \quad 2537 &= \text{tab. diff. for } 60'' \text{ or D}, \end{aligned}$$

$$\text{And } \frac{d}{\text{D}} \times 60'' = \frac{1834}{2537} \times 60'' = 43''.$$

Hence, 10·0397936 = L cot 42° 22′ 43″.

Ex. V.

1. Given log 47582 = 4·6774427, and log 47583 = 4·6774518, find log 47·58275.

2. Given log 5·2404 = ·7193644, and log 524·05 = 2·7193727, find log ·5240463.

3. Given log ·56145 = $\bar{1}$·7493111, and log 56·146 = 1·7493188, find log $\sqrt[7]{\cdot 05614581}$.

4. Given log 61683 = 4·7901655, and log 616·84 = 2·7901725, find the number whose logarithm is $\bar{2}$·7901693.

5. Find the value of $(1{\cdot}05)^{15}$, having given log 1·05 = ·0211893, log 20789 = 4·3178336, and log 20790 = 4·3178545.

6. Find the compound interest of £120 for 10 years at 4 per cent. per annum, having given log 1·04 = ·0170333, log 14802 = 4·1703204, and log 14803 = 4·1703497.

7. A corporation borrows £8,630 at 4½ per cent. compound interest, what annual payment will clear off the debt in 20 years?

Log 1·045 = ·0191163, log 4·1464 = ·6176712, and log 4·1465 = ·6176817.

8. Find the value of $\dfrac{(1{\cdot}032)^{10} \times \sqrt[3]{37{\cdot}62}}{(\cdot 347215)^6}$, having given

Log 1·032 = ·0136797, log 34722 = 4·5406047.
Log 3762 = 3·5754188, log 26202 = ·4183344.
Log 34721 = 4·5405922, log 26203 = ·4183510.

9. Find L sin 32° 28′ 31″, having given L sin 32° 28′ = 9·7298197, and L sin 32° 29′ = 9·7300182.

10. Find L cosec 43° 48′ 16″, having given L sin 43° 48′ = 9·8401959, and L sin 43° 49′ = 9·8403276.

11. Required the angle whose logarithmic cotangent is 10·1322449, having given L cot 36° 25′ = 10·1321127, L cot 36° 26′ = 10·1318483.

12. Construct a table of proportional parts, having given 163 as the tabular difference.

13. In what time will a sum of money double itself at 5 per cent. per annum, compound interest?

14. Find x when $1{\cdot}03^x = 1{\cdot}2143$, having given that log 1·03 = ·0128372, and log 12143 = 4·0843260.

15. Solve the equation $2^{2x-1} - 40 = 9 \cdot 2^x$, having given log 2 = ·3010300.

16. Given L cos 32° 45′ = 9·9341986, D = 752, find L cos 32° 45′ 12″, and L sec 32° 45′ 20″.

17. Given L tan 28° 38′ = 10·2628291, D = 3003, find L tan 28° 37′ 15″, and L cot 28° 38′ 42″.

18. Find the angle whose logarithmic cosine is 9·9590635, having given—

L cos 24° 29′ = 9·9590805,
L cos 24° 31′ = 9·9589653.

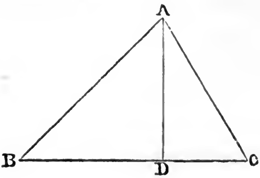
CHAPTER VII.

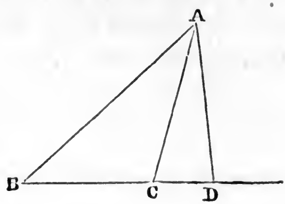
PROPERTIES OF TRIANGLES.

30. *The sines of the angles of a triangle are proportional to the opposite sides.*

We shall designate the sides opposite to the angles A, B, C, by the small letters a, b, c, respectively.

Draw AD perpendicular to BC, or to BC produced.





Then $\sin B = \dfrac{AD}{AB} = \dfrac{AD}{c}$ ....(1).

And $\sin C = \dfrac{AD}{AC} = \dfrac{AD}{b}$ ....(2).

Hence, (1) ÷ (2), $\dfrac{\sin B}{\sin C} = \dfrac{AD}{c} \div \dfrac{AD}{b} = \dfrac{b}{c}$,

$$\text{or } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

It follows, therefore, from the symmetrical nature of this equation, that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. *Q.E.D.*

31. *In any triangle,* $\cos C = \dfrac{a^2 + b^2 - c^2}{2\,ab}$.

Taking the figures of the last article, we have—

(1.) *When C is an acute angle—*
By Euc. II., 13, $AB^2 = BC^2 + AC^2 - 2\,BC.CD$.

$$\text{Now } \frac{CD}{AC} = \cos C, \text{ or } CD = AC \cos C.$$

Hence we have, $AB^2 = BC^2 + AC^2 - 2\,BC.AC \cos C$, or $c^2 = a^2 + b^2 - 2\,ab \cos C$.

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

(2.) *When C is an obtuse angle*, as in the second figure—

By Euc. II., 12, $AB^2 = BC^2 + AC^2 + 2\,BC.CD$;

and $CD = AC\cos ACD = AC\cos(180° - C) = -AC\cos C$.

Hence, $AB^2 = BC^2 + AC^2 + 2\,BC(-AC\cos C)$;

or, $c^2 = a^2 + b^2 - 2\,ab\cos C$.

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}, \text{ as before.}$$

From the *form* of this result we have also—

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2\,bc},$$

$$\text{Cos } B = \frac{a^2 + c^2 - b^2}{2\,ac}.$$

32. *To express the sine of any angle of a triangle in terms of the sides.*

We have—

$$\text{Sin}^2 A = 1 - \cos^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2\,bc}\right)^2$$

$$= \frac{(2\,bc)^2 - (b^2 + c^2 - a^2)^2}{(2\,bc)^2}$$

$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{(2\,bc)^2}$$

$$= \frac{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}{(2\,bc)^2}.$$

Hence, taking the square root, and taking the positive sign, because (Art. 11) the sin A is always positive when A is the angle of a triangle, we have—

$$\text{Sin } A = \frac{1}{2\,bc}\sqrt{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}.$$

Let $a + b + c = 2\,s$, then $b + c - a = 2(s - a)$,

$a + c - b = 2(s - b)$, $a + b - c = 2(s - c)$.

$$\text{Hence, } \sin A = \frac{1}{2\,bc}\sqrt{2\,s \, . \, 2(s - a) \, . \, 2(s - b) \, . \, 2(s - c)}$$

$$= \frac{2}{bc}\sqrt{s(s - a)(s - b)(s - c)}.$$

From the *form* of this result we have also—

$$\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)},$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

33. *To find the area of a triangle.*

Using the figures of Art. 30, we have—

Δ ABC $= \frac{1}{2}$ BC . AD, or, since AD $=$ AC sin C,

$= \frac{1}{2}$ BC . AC sin C

$= \frac{1}{2} ab \sin C$ ....(1).

From the *form* of this result we have also—

Δ ABC $= \frac{1}{2} ac \sin B$ ....(2).

and Δ ABC $= \frac{1}{2} bc \sin A$.......................................(3).

The results in (1), (2), (3) express the area of a triangle in terms of two sides, and the included angle.

We will now express the area in terms of the three sides. We have—

$$\Delta\, ABC = \tfrac{1}{2} ab \sin C, \text{ or, by last Art.,}$$

$$= \tfrac{1}{2} ab \,.\, \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \text{ ....(4).}$$

34. *To express the sine, cosine, and tangent of half an angle of a triangle in terms of the sides.*

We have, Art. 18,

$$1 - 2\sin^2\frac{A}{2} = \cos A = \frac{b^2+c^2-a^2}{2bc}.$$

$$\therefore 2\sin^2\frac{A}{2} = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+c-b)(a+b-c)}{2bc} = \frac{2(s-b)\,.\,2(s-c)}{2bc};$$

$$\text{or, } \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}.$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \text{ ....(1).}$$

The *form* of this result gives us also—

$$\text{Sin}\,\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Again, Art. 18, we have—

$$2\cos^2\frac{A}{2} - 1 = \cos A = \frac{b^2+c^2-a^2}{2\,bc};$$

$$\text{or, } 2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2\,bc} = \frac{(b+c)^2-a^2}{2\,bc}$$

$$= \frac{(a+b+c)(b+c-a)}{2\,bc} = \frac{2\,s\,.\,2\,(s-a)}{2\,bc};$$

$$\text{or, } \cos^2\frac{A}{2} = \frac{s\,(s-a)}{bc}.$$

$$\therefore\ \cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The *form* of this result gives us also—

$$\cos\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ac}},\ \cos\frac{C}{2} = \sqrt{\frac{s\,(s-c)}{ab}}.$$

Now, (1) ÷ (2), we have—

$$\frac{\text{Sin}\,\frac{A}{2}}{\text{Cos}\,\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}},\ \text{or}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The *form* of this result gives us also—

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}},\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

35. *In any triangle* ABC, $a = b\cos C + c\cos B$.
Using the figures of Art. 30, we have—

(1.) When C is acute—

$$BD = AB\cos B,\ \text{and } DC = AC\cos C.$$
$$\therefore\ BD + DC = AC\cos C + AB\cos B,\ \text{or}$$
$$a = b\cos C + c\cos B.$$

(2.) When C is obtuse—

$$BD = AB \cos B,$$

and $DC = AC \cos ACD = AC \cos (180° - C) = -AC \cos C.$

$$\therefore BD - DC = AC \cos C + AB \cos B,$$

$$\text{or } a = b \cos C + c \cos B, \text{ as before.}$$

The *form* of this result gives us also—

$$b = a \cos C + c \cos A, \; c = a \cos B + b \cos A.$$

Cor. 1. Hence $\sin (B + C) = \sin B \cos C + \cos B \sin C.$

For we have, $1 = \frac{b}{a} \cos C + \frac{c}{a} \cos B$, or, by Art. 30,

$$= \frac{\sin B}{\sin A} \cos C + \frac{\sin C}{\sin A} \cos B,$$

$$\text{or, } \sin A = \sin B \cos C + \sin C \cos B.$$

But $\sin A = \sin (180° - A) = \sin (B + C).$

$\therefore$ Sin $(B + C) = \sin B \cos C + \cos B \sin C.$

Cor 2. Hence also—

$$\sin (B - C) = \sin B \cos C - \cos B \sin C.$$

For, since this result has been proved for any angles of a triangle, it will be also true for a triangle which contains an angle *supplementary* to B; that is, it will be true, if we put $180° - B$ for B. We then have—

$$\text{Sin } (180° - B + C) = \sin (180° - B) \cos C + \cos (180° - B) \sin C.$$

But $\sin (180° - B + C) = \sin (180° - \overline{B - C}) = \sin (B - C),$

$$\sin (180° - B) = \sin B, \; \cos (180° - B) = -\cos B.$$

Hence, $\sin (B - C) = \sin B \cos C - \cos B \cos C.$

Note.—The results of Cor. 1 and Cor. 2 have been proved only for angles less than two right angles. We shall see in Vol. II. they are generally true.

36. *In any triangle* ABC—

$$\text{Tan } \tfrac{1}{2} (B - C) = \frac{b - c}{b + c} \cot \tfrac{1}{2} A.$$

We have, Art. 30,

$$\frac{\text{Sin } B}{\text{Sin } C} = \frac{b}{c}, \text{ or } \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b - c}{b + c} \ldots\ldots\ldots\ldots (1).$$

Now, $\sin B = \sin\left\{\frac{B+C}{2} + \frac{B-C}{2}\right\}$, or, by Cor. 1, Art. 35,

$$= \sin\frac{B+C}{2}\cos\frac{B-C}{2} + \cos\frac{B+C}{2}\sin\frac{B-C}{2} \ldots\ldots(2).$$

and $\sin C = \sin\left\{\frac{B+C}{2} - \frac{B-C}{2}\right\}$, or, by Cor. 2, Art. 35,

$$= \sin\frac{B+C}{2}\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\sin\frac{B-C}{2} \ldots\ldots(3).$$

(2) − (3), then $\sin B - \sin C = 2\cos\frac{B+C}{2}\sin\frac{B-C}{2}$,

and (2) + (3), then $\sin B + \sin C = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$.

$$\therefore \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \cot\frac{B+C}{2}\tan\frac{B-C}{2} = \cot\left(90^\circ - \frac{A}{2}\right)\tan\frac{B-C}{2}$$

$$= \tan\frac{A}{2}\tan\frac{B-C}{2} \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

(4) = (1), then $\tan\frac{A}{2}\tan\frac{B-C}{2} = \frac{b-c}{b+c}$.

$$\therefore \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\tfrac{1}{2}A. \quad \textit{Q.E.D.}$$

Ex. VI.

1. If $a = 5$, $C = 30^\circ$, $\sin A = \frac{1}{5}$, find c.
2. Find a, having given $b = 12$, $c = 15$, $A = 60^\circ$.
3. Find $\tan\frac{A}{2}$, when $a = 6$, $b = 7$, $c = 8$.
4. What is the area of a triangle whose sides are 48, 52, 20?
5. Given two sides of a triangle to be 18 and 24, and the included angle 45°, find the area.

6. Two of the sides of a triangle are as $2 : 1$, and the included angle is $60°$, find the other angles.

7. In any triangle show that—

$$b \cos C - c \cos B = \sqrt{a^2 - 4\,bc \cos B \cos C}.$$

8. Show that the area of a triangle $= \frac{1}{2}\,a^2.\frac{\sin B \sin C}{\sin A}$.

9. An object is observed from two stations 100 yards apart, and the angles subtended by the distance between the object and either station are $45°$ and $60°$ respectively. Find the distance of the object from each station.

10. An observation is made from a point known to be distant 120 and 230 yards respectively from two trees, and the angle which the trees subtend is found to be $120°$. Find the distance between the trees.

11. If $\left.\begin{aligned} a \sin^2\theta + b\cos^2\theta &= m, \\ b\sin^2\phi + a\cos^2\phi &= n, \\ a\tan\theta &= b\tan\phi, \end{aligned}\right\}$ then $\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}$.

12. If a', b', c' be the sides of the triangle formed by joining the feet of the perpendiculars from the angles A, B, C of the triangle ABC upon the opposite sides, then—

$$\frac{a'}{a^2} + \frac{b'}{b^2} + \frac{c'}{c^2} = \frac{a^2 + b^2 + c^2}{2\,abc}.$$

13. A perpendicular AD is drawn from the angle A of a triangle, meeting the opposite side BC and D; and from D a perpendicular is drawn to AC, meeting it in E. Show that $DE = b \sin C \cos C$.

14. Show that the length of AD in the last example—

$$= \frac{bc \sin A + ac \sin B + ab \sin C}{3\,a}.$$

15. Show that $\sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \frac{1}{2}\,ab$, when the triangle is right-angled at C.

16. Show that $(a + b + c)^2 \sin A \sin B$

$$= (\sin A + \sin B + \sin C)^2\,ab.$$

17. Show that in any triangle—

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

18. The sides of a triangle ABC are in arithmetical progression; show that its area $= \frac{b\sqrt{3}}{2}\sqrt{(2a - b)(3b - 2a)}$.

CHAPTER VIII.

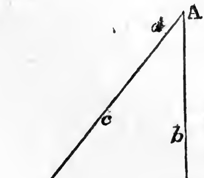
SOLUTION OF RIGHT-ANGLED TRIANGLES.

37. A triangle can always be determined when any three elements, with the exception of the three angles, are given. In the latter case we have only the same data as when two angles are given, for the third can always be found by subtracting the sum of the other two from two right angles (Euc. I., 32).

Hence a right-angled triangle can always be determined when any two elements, other than the two acute angles, are given besides the right angle. And when one of the acute angles is given, the other may be obtained by subtracting it from a right angle.

We have the following cases:—

CASE 1. *When the two sides containing the right angle are given.*



We shall take C as the right angle in every case.

Now $\tan A = \frac{a}{b}$;

or, $L\tan A - 10 = \log a - \log b$;

or, $L\tan A = 10 + \log a - \log b$...(1).

This determines A, and we then have

$B = 90° - A$..........................(2).

Also, $\frac{a}{c} = \sin A$, or $\log a - \log c = L\sin A - 10$.

$\therefore \log c = 10 + \log a - L\sin A$..........................(3).

Hence the three elements, A, B, c, are determined.

CASE 2. *When the hypothenuse and a side are given.*

Let a be the given side.

We have $\sin A = \frac{a}{c}$, or $L \sin A - 10 = \log a - \log c$.

$$\therefore L \sin A = 10 + \log a - \log c \ldots\ldots\ldots\ldots(1),$$
$$\text{and } B = 90^\circ - A \ldots\ldots\ldots\ldots(2),$$
$$\text{also } b^2 = c^2 - a^2 = (c + a)(c - a);$$
$$\therefore \log b = \tfrac{1}{2}\left\{\log(c + a) + \log(c - a)\right\} \ldots\ldots(3).$$

CASE 3. *When an acute angle and a side are given.*

Let A, a be the given angle and side.

Then $B = 90^\circ - A \ldots\ldots\ldots\ldots(1)$,

also $\frac{b}{a} = \tan B$, or $\log b = L \tan B - 10 + \log a \ldots\ldots(2)$,

and $\frac{a}{c} = \sin A$, or $\log a - \log c = L \sin A - 10$,

or $\log c = 10 + \log a - L \sin A$.

CASE 4. *When the hypothenuse and an acute angle are given.*

Let A be the given acute angle.

We have $B = 90^\circ - A \ldots\ldots\ldots\ldots(1)$.

Also $\frac{a}{c} = \sin A$, or $\log a = \log c + L \sin A - 10 \ldots(2)$.

And $\frac{b}{c} = \cos A$, or $\log b = \log c + L \cos A - 10 \ldots.(3)$.

It is evident, from Art. 30, that when the angles only of a triangle are known, we can determine the *ratio* only of the three sides of the triangle to each other.

Ex. 1. Given $A = 23^\circ\ 41'$, $a = 35$, solve the triangle.

This is an example of Case 3.

We have $B = 90^\circ - 23^\circ\ 41' = 66^\circ\ 19'$.

$$\begin{aligned}
\text{Again, } \log b &= L \tan B - 10 - \log a \\
&= L \tan 66^\circ\ 19' - 10 - \log 35 \\
&= 10{\cdot}3579092 - 10 + 1{\cdot}5440680 \\
&= 1{\cdot}9019772 = \log 79{\cdot}795 \\
\therefore b &= 79{\cdot}795.
\end{aligned}$$

Also $\log c = 10 + \log a - \text{L} \sin \text{A}$
$= 10 + \log 35 - \text{L} \sin 23° 41'$
$= 10 + 1{\cdot}5440680 - 9{\cdot}6038817$
$= 1{\cdot}9401863 = \log 87{\cdot}134$
$\therefore c = 87{\cdot}134.$

Ex. 2. Given $a = 214$, $b = 317$, solve the triangle.
This falls under Case 1.
We have $\text{L} \tan \text{A} = 10 + \log a - \log b$
$= 10 + \log 214 - \log 317$
$= 10 + 2{\cdot}3304138 - \log 2{\cdot}5010593$
$= 9{\cdot}8293545,$
Next lower in tables is $9{\cdot}8292599 = \text{L} \tan 34° 1'$;
$\therefore d = 946.$
Also, by tables, $\text{D} = 2724,$

$$\text{And } \frac{d}{\text{D}} \times 60'' = \frac{946 \times 60''}{2724} = 21'' \text{ nearly.}$$

$\therefore \text{L} \tan \text{A} = \text{L} \tan 34° 1' 21''$,
or $\text{A} = 34° 1' 21''$.
Hence $\text{B} = 90 - 34° 1' 21'' = 45° 58' 39''$.

And similarly may c be determined.

Ex. VII.

1. Given $a = 32$, $\text{A} = 63° 45'$, find b.
   Log 32 = 1·5051500, L cot 63° 45′ = 9·6929750,
   Log 15780 = 4·1981070, log 15781 = 4·1901345.
2. Given $c = 151$, $\text{A} = 37° 42'$, find a.
   Log 151 = 2·1789769, L sin 37° 42′ = 9·7864157,
   Log 92340 = 4·9653899, tab. diff. = 47.
3. Given $a = 60$, $c = 65$, find b, A.
   Log 2 = ·3010300, log 3 = ·4771213,
   Log 65 = 1·8129134, L sin 67° 22′ = 9·9651953,
   L sin 67° 23′ = 9·9652480.
4. Given $a = 73$, $b = 84$, find A, c.
   Log 73 = 1·8633229, L tan 40° 59′ = 9·9389079,
   Log 84 = 1·9242793, L tan 41° = 9·9391631,
   L sin 40° 59′ = 9·8167975, L sin 41° = 9·8169429,
   Log 111·288 = 2·0464479.

5. Given B = $71° 41' 10''$, $c = 24$, find b.
Log 24 = 1·3802112, L cos $18° 18'$ = 9·9774609,
L cos $18° 19'$ = 9·9774191, log 2278·4 = 3·3576300,
Log 2278·5 = 3·3576490.

6. Given $a = 293$, $c = 751$, find b.
Log 1044 = 3·0187005, log 458 = 2·6608655,
Log 691·49 = 2·8397830.

7. Given $a = 12$, A = $30°$, find b, c.

8. Given $c = 10$, B = $75°$, find a, b.

9. Given $a = 17$, $c = 34$, find A, b.

10. Given $a = 5$, $b = 5\sqrt{3}$, find A.

11. Given $a = 28$, B = $15°$, find the length of the perpendicular from C on AB.

12. CD is the perpendicular from C on AB, and DE the perpendicular from D on BC. Given B = $60°$, $a = 20$, find DE.

CHAPTER IX.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

38. *Given the three sides of a triangle, to find the remaining parts.*

We have, Art. 31, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, from which A may be determined; and from similar formulæ we may find B and C. These formulæ are not however adapted to logarithmic computation. We shall therefore find it generally advisable to use the formulæ of Art. 34.

We have, $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

From either of these formulæ we can determine A, and from similar formulæ determine the other angles.

39. *Given one side and two angles, to find the remaining parts.*

Of course the third angle is at once known. Let a be the given side.

We have, Art. 30, $b = \dfrac{a \sin B}{\sin A}$,

$$c = \frac{a \sin C}{\sin A},$$

both of which formulæ are adapted to logarithmic computation.

40. *Given two sides and the included angle, to find the remaining parts.*

Let b, c be the given sides, and A the included angle.

We have, Art. 36, $\tan \frac{1}{2}(B - C) = \dfrac{b - c}{b + c} \cot \frac{1}{2} A.$

This formula is adapted to logarithmic computation, and determines $\frac{1}{2}(B - C)$.

We know also $\frac{1}{2}(B + C)$, for it is the complement of $\frac{1}{2}$ A. Hence, B and C are easily determined.

Then, Art. 30, $a = \dfrac{b \sin A}{\sin B}$, which determines a.

NOTE.—When the two given sides are equal, the solution may be effected more easily by drawing a perpendicular from the given angle upon the opposite side, and so bisecting it. By drawing a figure, it is easily seen that, in this case, $B = C = 90° - \frac{1}{2} A$, and $a = 2b \cos B$.

41. *Given two sides and an angle opposite to one of them, to find the remaining parts.*

Let a, b, B be the given elements.

Then we have, Art. 30, $\sin A = \dfrac{a}{b} \sin B.$

(1.) Let the value of $\dfrac{a}{b} \sin B$ be *unity*.

We then have $\sin A = 1 = \sin 90°$, and $\therefore A = 90°$. Hence the triangle is right-angled at A.

(2.) Let the value of $\frac{a}{b}\sin B$ be > 1.

We then have $\sin A > 1$, which is impossible.

Hence, in this case, it is impossible to form a triangle with the given elements.

(3.) Let the value of $\frac{a}{b}\sin B$ be < 1.

Then, since, Art. 15, the sine of an angle is the same as the sine of its supplement, there are two values of A which satisfy the equality, $\sin A = \frac{a}{b}\sin B$, and these values are supplementary.

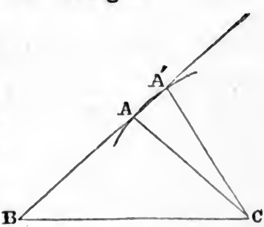
Let A, A′ be the two values, then the relation between them is $A + A' = 180°$.

If a is not greater than b, then A is not greater than B, and there is no doubt as to which value of A is to be taken. If, however, a is greater than b—that is, if the given angle is opposite to the less of the given sides, we must have A greater than B, and both values of A may satisfy this condition. This particular case, when *the given angle is opposite to the less of the given sides*, is called *the ambiguous case.* We will illustrate this geometrically.

42. *The ambiguous case.*

Let a, b, B be given to construct the triangle.

Draw the line BC equal to the given side a, and draw BA making an angle B with BC.



Then with centre C and radius CA equal to b describe an arc meeting BA or BA produced in A and A′, and join CA and CA′.

Each of the triangles ABC, A′BC satisfies the given conditions.

For, in the triangle ABC, we have $BC = a$, $AC = b$, and $\angle ABC = B$; and in the triangle A′BC, we have $BC = a$, $A'C = b$, and $\angle A'BC = B$.

Again, the sides BA and BA′ correspond to the two values of c which are obtained from the two values of A in the equality $\sin A = \frac{a}{b} \sin B$ (see last Art.).

And the angles ACB and A′CB correspond to the two values of C which would also be found.

COR. If a perpendicular CD be drawn from C upon AA′, and if c' and c be the lengths of BA′ and BA respectively, it may be easily shown that $c' + c = 2a \cos B$, and $c' \sim c = 2b \cos A$.

Ex. VIII.

Solve the following triangles, having given—

1. $b = 12, c = 6, A = 60°$.

2. $a = 18, b = 18\sqrt{2}, A = 30°$.

3. $a = 5\sqrt{3}, b = 5\sqrt{2}, c = \frac{5}{2}(\sqrt{6} + \sqrt{2})$.

4. $a = 12, B = 60°, C = 15°$.

5. $a = 3\sqrt{2} + \sqrt{6}, b = 6, C = 45°$.

6. $a = 10\sqrt{3}, b = 15\sqrt{2}, A = 45°$.

Given—

7. $b = 251, c = 372, A = 40° 32'$, find B and C.
Log 121 = 2·0827854, L cot 20° 16′ = 10·4326795,
Log 623 = 2·7944880, L tan 27° 44′ = 9·7207827,
L tan 27° 45′ = 9·7210893.

8. $a = 237, b = 341, B = 28° 24'$, find A.
Log 237 = 2·3747483, L sin 28° 24′ = 9·6772640,
Log 341 = 2·5327544, L sin 19° 18′ = 9·5191904,
L sin 19° 19′ = 9·5195510.

9. $C = 26° 32'$, and $a : b :: 3 : 5$, find A, B.
Log 2 = ·3010300, L cot 13° 16′ = 10·6275008,
L tan 46° 40′ = 10·0252805, L tan 46° 41′ = 10·0255336.

10. $a = 14, b = 16, c = 18$, find A, B.
log 2 = ·3010300, log 3 = ·4771213,
L tan 24° 5′ = 9·6502809, L tan 24° 6′ = 9·6506199,
L tan 29° 12′ = 9·7473194, L tan 29° 13′ = 9·7476160.

11. $a = 3, b = 2$ $A = 60°$, find B, C, and c.
Log 2 = ·3010300, log 3 = ·4771213,
L sin 35° 15′ = 9·7612851, L sin 35° 16′ = 9·7614638,
Log 1·3797 = ·1397847, log 1·3798 = ·1398161.

12. $a = 5, b = 6, c = 7$, find A.
Log 2 = ·3010300, L tan 22° 12′ = 9·6107586,
Log 3 = ·4771213, L tan 22° 13′ = 9·6111196.

13. If c, c' be the two values of the third side in the ambiguous case when a, b, A are given, show that—

$$(c - c')^2 + (c + c')^2 \tan^2 A = 4a^2.$$

14. If a, b, A are given, show from the equation—

$$b^2 + c^2 - 2bc \cos A = a^2;$$

that if c and c' be the two values of the third side—

$$cc' = b^2 - a^2, \text{ and } c + c' = 2b \cos A.$$

15. Show also from the same equation that there is no ambiguous case when $a = b \sin A$, and that c is impossible when $a < b \sin A$.

16. Having $a - b$, A, B, solve the triangle.

17. Given the ratios of the sides, and the angle A, solve the triangle.

18. If in a triangle $\tan \frac{1}{2}(B - C) = \tan^2 \frac{\phi}{2} \cot \frac{A}{2}$, show that $b \cos \phi = c$.

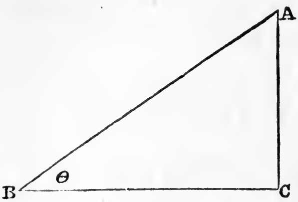
CHAPTER X.

HEIGHTS AND DISTANCES.

43. We shall now show how the principles of the previous chapters are practically applied in determining heights and distances.

We have not space here to describe the instruments by which angles are practically measured, but we shall assume that they can be measured to almost any degree of accuracy.

44. *To find the height of an accessible object.*



Let AC be the object, and let any distance BC from its foot be measured.

At B let the *angle of elevation* ABC be observed.

Suppose BC = a, $\angle$ ABC = θ.

Then, we have—

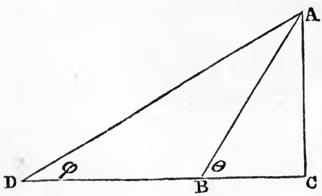
$$\frac{AC}{BC} = \tan ABC \text{ or } \frac{AC}{a} = \tan\theta.$$

$\therefore$ AC = $a\tan\theta$, the height required.

Ex. Let $a = 200$, and $\theta = 30°$.

$$\text{Then AC} = 200\tan 30° = 200 \cdot \frac{1}{\sqrt{3}} = \frac{200\sqrt{3}}{3}.$$

45. *To find the height of an inaccessible object.*



At any point B in the horizontal plane of the base let the *angle of elevation* ABC be observed.

Measure a convenient distance BD in the straight line CB produced, and observe the *angle of elevation* ADC.

Let BD = a, $\angle$ ABC = θ, $\angle$ ADB = ϕ.

Then, Euc. I. 32, $\angle$ BAD = $\theta - \phi$.

$$\text{Now, } \frac{AB}{BD} = \frac{\sin ADB}{\sin BAD} \text{ or } \frac{AB}{a} = \frac{\sin\phi}{\sin(\theta-\phi)},$$

$$\therefore AB = a \cdot \frac{\sin\phi}{\sin(\theta-\phi)} \quad \text{......} \quad (1).$$

$$\text{Again, } \frac{AC}{AB} = \sin ABC = \sin\theta, \ \therefore AC = AB\sin\theta,$$

$$\text{or, from (1) } AC = a\frac{\sin\theta\sin\phi}{\sin(\theta-\phi)}.$$

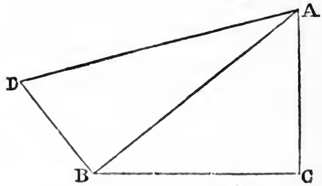
Ex. Let BD = 120, $\theta = 60°$, $\phi = 45°$.

$$\text{Then, AC} = 120 \cdot \frac{\sin 60° \sin 45°}{\sin(60° - 45°)}.$$

$$= 120 \frac{\sin 60° \sin 45°}{\sin 15°} = 120 \,.\, \frac{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}}$$

$$= 120 \,.\, \frac{\sqrt{3}}{\sqrt{3} - 1} = 60\,(3 + \sqrt{3}).$$

46. *To find the height of an inaccessible object when it is not convenient to measure any distance in a line with the base of the object.*



Let a distance BD be measured in any direction in the same horizontal plane as BC, and let the angles ABC, ABD, ADB be observed.

Let $BD = a$, $\angle ABC = \alpha$

$\angle ABD = \beta$, $\angle ADB = \gamma$.

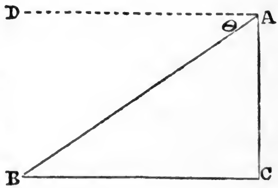
Then, Euc. I., 32, $\angle BAD = 108° - (\beta + \gamma)$.

$$\text{Now, } \frac{AB}{BD} = \frac{\sin ADB}{\sin BAD} = \frac{\sin \gamma}{\sin \{180° - (\beta + \gamma)\}}, \text{ or}$$

$$\frac{AB}{a} = \frac{\sin \gamma}{\sin (\beta + \gamma)}, \;\because\; AB = a \,.\, \frac{\sin \gamma}{\sin (\beta + \gamma)} \;\ldots\ldots (1).$$

$$\text{Again, } \frac{AC}{AB} = \sin ABC = \sin \alpha, \;\therefore\; AC = AB \,.\, \sin \alpha,$$

or from (1), $AC = a . \dfrac{\sin \alpha \,.\, \sin \gamma}{\sin (\beta + \gamma)}$.



47. *To find the distance of an object by observation from the top of a tower whose height is known.*

Let B be the object in the same horizontal plane with c the foot of the tower, and let *the angle of depression* DAB be observed.

Let $AC = h$, $\angle DAB = \theta$,

Then we have, $\frac{BC}{AC} = \cot ABC = \cot DAB$, or $\frac{BC}{h} = \cot \theta$,

$\therefore BC = h \cot \theta$, the distance required.

COR. Suppose B to be not in the same level with C. Let m be the height of B above C, then B is on the same level with a point which is $h - m$ from A. $\therefore BC = (h - m) \cot \theta$.

Ex. IX.

1. Find the height of a tower 200 yards distant when it subtends an angle of 15°.

2. From the top of a tower, the angle of depression of a point in the horizontal plane at the foot of the tower was 30°. Given the height to be 60 ft., find the distance of the point.

3. The angle of depression of two consecutive milestones in a direct line with the summit of a hill were observed to be 60° and 30°. Find the height of the hill.

4. An object is observed from a ship to be due E. After sailing due S. for six miles it is observed to be N.E. Find the distance from the last position of the ship.

5. There is an object A, and two stations B and C are taken in the same plane. Given that BC = 50, ∠ ABC = 60°, ∠ ACB = 30°, find AB, AC.

6. The elevation of a tower is found to be 45°, and on approaching 60 feet nearer the elevation is 75°. Find the height of the tower.

7. Wishing to know the breadth of a river I observed an object on the opposite bank, and, having walked along the side of the river a distance of 100 yards, found the angle subtended by the object and my first station to be 30°. Find the breadth of the river.

8. A person, standing exactly opposite to the centre of an oblong which measures 16 ft. by 12 ft., and such that the line drawn from the centre to his eye is at right angles to the oblong, observes that the diagonal subtends an angle of 60°. Find his distance.

9. The angles of elevation of the summit of one tower, whose height is h, are observed from the base and summit of

another, and found to be θ and ϕ respectively. Show that the height of the second tower—

$$= h \cdot \frac{\tan \theta}{\tan\theta - \tan\phi}.$$

10. From the top of a tower the angles of depression of two objects in a direct line, and whose distance from each other is a, are α, β respectively. Show that the height of the tower—

$$= \frac{a}{\cot \beta - \cot \alpha}.$$

11. A person, having walked a distance a from one corner along a side of an oblong, observes that the side immediately behind him subtends an angle α, and the side in front an angle β. Show that the dimensions of the oblong are—

$$a \tan \alpha, \ a\ (1 + \tan \alpha \cot \beta).$$

12. Three points A, B, C form a triangle whose sides are a, b, c respectively, and a person standing at a point S, such that SA is at right angles to BC, observes that the side AC subtends an angle θ. Show that the distance of S from B—

$$= \frac{1}{2a} \sqrt{(a^2 + c^2 - b^2)^2 + (a^2 + b^2 - c^2)^2 \cot^2 \theta}.$$

13. A person walks a yards from A to E along AB the side of a triangle ABC, and observes that the angle AEC $= \alpha$; he also walks b yards from B to F along BA, and observes $\angle$ CFB $= \beta$. Having given AB $= c$, find BC and AC.

14. A tower is observed from three stations A, B, C, in a straight line not meeting the tower, to subtend angles α, β, γ respectively. Show that if AB $= a$, BC $= b$, the height of the tower—

$$= \sqrt{\frac{ab(a + b)}{a \cot^2 \gamma - (a + b) \cot^2 \beta + b \cot^2 \alpha}}.$$

When are the conditions impossible?

15. From two stations whose distance apart is a, and which are due W. and due S. respectively of one end of a wall, the angles subtended by the wall are each α. Show that the length of the wall is $a \sin \alpha$.

16. The angles of elevation of the top of a tower, whose height is h and standing on a hill, are α, β, when observed from two stations a miles distant, and in a direct line up the hill. Show that if θ be the slope of the hill—

$$\cos\theta = \frac{a}{h} \cdot \frac{\sin\alpha\sin\beta}{\sin(\beta - \alpha)}.$$

17. The elevation of a tower was observed to be α, but on walking in the horizontal plane a distance a at right angles to the line joining the first position and the foot of the tower, the elevation was β. Show that the height of the tower was—

$$\frac{a}{\sqrt{\cot^2\beta - \cot^2\alpha}}.$$

18. The angles of depression of two objects in the same horizontal plane, as seen from the top of a tower, are θ and ϕ respectively, and the angle they subtend is α. Show that if h be the height of the tower, the distance between the objects—

$$= h\sqrt{\operatorname{cosec}^2\theta + \operatorname{cosec}^2\phi - 2\operatorname{cosec}\theta\operatorname{cosec}\phi\cos\alpha}.$$

ANSWERS.

I.—Page 15.

1. 45·23, 290, ·2367, ·7, &c.

2. ·0005, 11·11, ·040020, ·45, &c.

3. Three thousand four hundred and sixty-seven thousandths; thirty-four, and sixty-seven hundredths; three thousand four hundred and sixty-seven millionths; three, and four hundred and sixty-seven thousandths.

4. 35·90846, 29130·19391, 60·0239.

5. ·7237, 3·32091.

6. 69·5289, 5·06679, ·41481.

7. ·026, 7708·71.

8. ·09, 24·356706, ·003627, ·289, ·0096, ·00016384.

9. 200, ·00125, 4000, ·2295, ·006, ·5002, &c.

10. 170000. **11.** 4·97 &c., 1. **12.** ·2.

II.—Page 23.

1. $\frac{22}{7}$, $\frac{43}{9}$, $\frac{1007}{10}$, $\frac{607}{17}$, $\frac{1010}{999}$, $\frac{729}{64}$.

2. $\frac{114}{6}$, $\frac{190}{10}$, $\frac{247}{13}$, $\frac{342}{18}$, $\frac{361}{19}$, $\frac{380}{20}$.

3. 6, 7, 12, 11, 9, 14; $\frac{24}{4}$, $\frac{42}{6}$, $\frac{96}{8}$, $\frac{110}{10}$, $\frac{108}{12}$, $\frac{196}{14}$.

4. $7\frac{1}{7}$, $1\frac{7}{8}$, $7\frac{3}{5}$, $16\frac{2}{3}$, $71\frac{1}{9}$, $31\frac{1}{4}$, &c. **5.** 52.

6. $\frac{37}{42}$, $\frac{7}{51}$, $\frac{11}{36}$, $\frac{23}{39}$, $17\frac{4}{21}$, $1\frac{10}{11}$; $\frac{37}{49}$, &c.

7. $\frac{2}{5}$, $\frac{7}{52}$, $\frac{20}{27}$; $\frac{12}{35}$, $\frac{3}{26}$, $\frac{40}{63}$; $\frac{3}{10}$, $\frac{21}{208}$, $\frac{5}{9}$.

8. $\frac{5}{24}$, 6, 1, 1, 6. **9.** $\frac{1}{2}$, $\frac{32}{45}$, $9\frac{1}{3}$, $\frac{9}{112}$, $\frac{1}{5}$, $\frac{2}{5}$.

10. $11\frac{3}{7}$. **11.** 3. **12.** 28.

III.—Page 28.

1. 4×11, 2^6, $2 \times 3 \times 5^2$, $2^2 \times 3^2 \times 7$, $3^3 \times 47$, $2 \times 3 \times 7 \times 11$.

2. $2 \times 7 \times 3^2 \times 11$, $2^3 \times 5 \times 43$, $2^9 \times 3^2$, $5^3 \times 7 \times 11^2$, $2^2 \times 3 \times 11^3$, $2^4 \times 3^4 \times 11$.

5. $\frac{2}{5}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{11}{90}$, $\frac{4}{25}$, $\frac{4}{9}$; $\frac{19}{32}$, $\frac{25}{36}$, $\frac{15}{16}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{17}{19}$; $\frac{7}{9}$, $\frac{3}{5}$, $\frac{6}{11}$, $\frac{17}{36}$, $\frac{16}{35}$, $\frac{11}{4991}$.

6. 19, 7, 13, 6, 41, 729, 14, 11, 39.

7. $\frac{3}{5}$, $\frac{7}{9}$, $\frac{3}{11}$, $\frac{4}{13}$, $\frac{12}{17}$, $\frac{5}{9}$; $\frac{2}{3}$, $\frac{6}{7}$, $\frac{13}{15}$, $\frac{1}{9}$, $\frac{31}{32}$, $\frac{41}{54}$.

9. 2 days out of every 7.

10. 2 : 7. 12. 15, 10, 6.

IV.—Page 31.

1. 24, 1260, 2520, 1260, 5460, 33300.

2. $\frac{45}{60}$, $\frac{84}{60}$, $\frac{8}{60}$, $\frac{33}{60}$; $\frac{24}{84}$, $\frac{35}{84}$, $\frac{39}{84}$, $\frac{44}{84}$; $\frac{336}{252}$, $\frac{56}{252}$, $\frac{231}{252}$, $\frac{234}{252}$; $\frac{35}{105}$, $\frac{21}{105}$, $\frac{15}{105}$; $\frac{972}{1296}$, $\frac{567}{1296}$, $\frac{304}{1296}$, $\frac{630}{1296}$; $\frac{187}{306}$, $\frac{38}{306}$, $\frac{54}{306}$.

3. 12. 4. $\frac{17}{21}$, $\frac{7}{13}$, $\frac{11}{28}$, $\frac{15}{39}$. 5. $1065\frac{3}{7}$.

6. $\frac{4}{129}$. 7. $1\frac{1}{9}$. 8. $\frac{4}{19}$. 10. $\frac{16}{41}$, $\frac{1}{3}$, $\frac{7}{45}$.

V.—Page 34.

1. $\frac{7}{8}$, $2\frac{1}{30}$, $1\frac{1}{4}$. 2. $29\frac{13}{24}$. 3. $14\frac{211}{360}$.

4. $\frac{1}{8}$, $\frac{1}{30}$, $\frac{1}{56}$. 5. $2\frac{1}{4}$, $\frac{13}{45}$, $3\frac{19}{30}$. 6. $2\frac{25}{84}$.

7. $\frac{9}{56}$. 8. $5\frac{1}{12}$. 9. $\frac{3}{130}$.

10. $2\frac{22}{351}$. 11. $12\frac{14}{17}$. 12. $1\frac{1}{60}$.

VI.—Page 36.

1. $3\frac{7}{8}$, $1\frac{3}{8}$, $3\frac{23}{32}$. 2. $5\frac{67}{100}$. 3. 5.

4. $\frac{5}{162}$, $\frac{5}{12168}$. 5. $3\frac{7}{24}$. 6. 4.

7. $\frac{333}{106}$. 8. $2\frac{662}{1125}$, $15\frac{11}{192}$.

9. 77. 10. £8. $12\frac{2}{19}$s.

11. £352. 6s. 8d. 12. $20\frac{3}{4}$.

VII.—Page 41.

1. 2·203125.
2. $\frac{7}{20}$, $\frac{13}{500}$, $\frac{16}{99}$, $\frac{1}{7}$, $\frac{1}{6}$, $\frac{3}{7}$.
3. 3·703059239.
4. $5\cdot098\dot{8}0926\dot{3}$.
5. $15\frac{4}{11}$, $\frac{9}{11}$.
6. 1.
7. 2·718281.
8. ·321750.
9. ·367879.
10. 1·015873.
11. 3·141592.
12. $\cdot\dot{8}5714\dot{2}$.

VIII.—Page 43.

1. 13s. 4d., $7\frac{1}{5}$d., 2s. 8d., £1. 5s. $8\frac{4}{7}$d.
2. £4. 7s. 6d., 6s., 1s. $1\frac{7}{11}$d., 3s. 9d.
3. £5. 17s. 4d., £8. 15s., £25. 3s. $11\frac{4}{11}$d.
4. 17 cwt. 16 lbs., $16\frac{4}{5}$ lbs., $6\frac{2}{9}$ lbs., $2\frac{5}{11}$ lbs.
5. 3 m. 2 f. $62\frac{6}{7}$ yds., $293\frac{1}{3}$ yds., 4 po. $1\frac{4}{7}$ yd., $82\frac{34}{35}$ yds.
6. 144 days, 32 days, 4 hrs. 54′ 47″.
7. 38 lbs. 7 oz. 2 dwt. $15\frac{5}{7}$ grs., 12 dwt. $6\frac{6}{49}$ grs.
8. – 245 ac. 1 r. 27 po. $12\frac{31}{288}$ yds.
9. 40° 3′ $28\frac{1}{8}$″, 35°.
10. 6153 grains.
11. $7\frac{7}{13}$; 4 cwt. 1 qr. $14\frac{2}{9}$ lbs.
12. 1 day 3 hrs. 8′ $24\frac{15487}{15795}$″.

IX.—Page 45.

1. $\frac{1}{12}$, $\frac{1}{16}$.
2. $\frac{7}{32}$, $\frac{13}{27}$.
3. $\frac{99}{2240}$, $\frac{229}{2688}$.
4. 3 lbs. 7 oz. 15 dwts., $8\frac{76}{175}$ lbs.
5. $\frac{41}{516}$, $\frac{221}{540}$.
6. $\frac{1127}{3375}$, $\frac{7}{22}$.
7. $\frac{3}{14}$, $\frac{1}{51}$.
8. $1185\frac{5}{27}$, $\frac{147}{48400}$.
9. $\frac{1121175}{1788864}$.
10. $\frac{16}{8925}$.
11. $\frac{59}{60}$.
12. $\frac{11}{1440}$ day of 24 hrs.

X.—Page 47.

1. 7s. 6d., 19s. $7\frac{1}{2}$d., 16s. $3\frac{3}{4}$d.
2. £1. 5s., 2s. 9d., £1. 6s. 3d.

3. 12 cwt. 2 qrs., 2 cwt. 3 qrs. 7 lbs., 6 cwt. 1 qr. 25 lbs.

4. 1 ac. 2 r. $26\frac{3}{4}$ po., $13\frac{9}{10}$ po., 1 r. 22 po.

5. 4 quires 4 sheets, 11 sheets, 23 quires $4\frac{3}{5}$ sheets.

6. 38 galls. 3 qts. $0\frac{1}{40}$ pts., 1 hhd. 60 galls. 2 qts. $1\frac{1}{5}$ pt. nearly, 1 pk. 0 gall. 3 qts. $1\frac{3}{4}$ pt. nearly.

7. 4000 grains, 8 oz. 6 dwts. 16 grs. **8.** ·698 lbs.

9. 15s. 2d. **10.** 24 tons 9 cwt. 2 qrs. 8 lbs.

11. 4 cwt. 1 qr. 10 lbs. **12.** 3 oz. 14 dwts.

XI.—Page 48.

1. ·625, ·53125, ·55625, ·928125.

2. $\cdot\dot{8}4615\dot{3}$, $\cdot\dot{6}9230\dot{7}$, $\cdot\dot{6}1538\dot{4}$. **3.** $\cdot 3680\dot{3}$, $\cdot 1180\dot{5}$.

4. ·015625, $\cdot 065\dot{4}7619\dot{0}$. **5.** ·003125, $\cdot 002232\dot{1}4285\dot{7}$.

6. $\cdot 00347\dot{2}$, $\cdot\dot{0}4029\dot{3}$. **7.** ·3125, ·150625.

8. $\cdot 0002232\dot{1}4285\dot{7}$, $\cdot 0008\dot{3}$. **9.** $1\cdot 13085\dot{3}1746\dot{0}$.

10. $\cdot 82\dot{2}8571\dot{4}$. **11.**

12. ·0544575

XII.—Page 54.

1. 150000, 20000, 100, 270, 2·5, 1, 3·45, 5·294.

2. 46000000, 3000000, 2950000, 1500, 395, 29·5.

3. 2 myriag. 0 kilog. 2 hectog. 9 dekag., 1 myriag. 8 kilog. 0 hectog. 0 dekag. 8 grm. 5 decig., 1 myriag. 2 kilog. 3 hectog. 0 dekag. 0 grm. 1 decig. 3 centig., 1 hectog. 2 dekag. 0 grm. 2 decig. 9 centig. 6 millig., 1 grm. 5 decig. 3 centig. 3 millig., 3 grm. 4 decig. 2 centig. 7 millig.

4. 160001·2, 25100, 396·45, 203550.

5. 1000, 2·96, 2900·03, 300·12, 3765·43.

6. 10000000000, 10000000, 50000, 349800, 4600.

7. 150, 39·4, 90·2, 1860·3, 3·764, ·4.

8. 10, 1·234567, ·372456126, 1, ·000639, ·293.

9. 3203, ·4, 2000·003, 76·384, 29·34, 8300, 3457·6.

10. 18300·453, 1830·0453, 1830045·3.

11. 1, 73·6, 246·45, 2·55, 16·95.

12. 1300, 130; 713, 71·3; 1235, 123·5; 2·9, 320, 32, 1804, 180·4.

XIII.—PAGE 56.

1. 16287·599 m., 10738767 m., 1322·371 sq. m., 69·548396 cub. m., 39129·99 grm., 65632·02 ares, 368·93 st., 78603·982 lit., 288·06 fr.

2. 1600·688 m., 11696·359 m., 96·18 sq. m., 5·967600029 cub. m., 5972·935 grm., 2450·94 ares, 409·6 st., 694003·024 lit., 2·35 fr., 98·56 fr.

3. (1) 70·245 m., 110·385 m., 130·455 m.; (2) 486082·89 m., &c.; (3) 49 sq. m., &c.; (4) 76·190000095 cub. m., &c.; (5) 11046013·965 gr., &c.; (6) 36000·9 ar., &c.; (7) 41629 st., &c.; (8) 7347660·72 lit., &c.; (9) 2932·50 fr., &c.

4. 864 fr. 91·5 c. 5. 3408 fr. $1\frac{7}{8}$ c.

6. (1) 1·56 m., 1·43 m., 1·32 m.; (2) 27788 m., 2613·75 m., 2460 m.; (3) 2·0910 sq. m., &c.; (4) 41·82 cub. m., &c.; (5) 188·263 grm., &c.; (6) 13·2 ar., &c.; (7) 10·01 st., &c.; (8) 40446·3 lit., &c.; (9) 293·58 fr., &c.

7. 25 fr., ·15010 fr., 24120 fr., ·328 fr., 1·80 fr., 2857 fr. 64 c. nearly, 16·5 fr., 12 c., ·01 fr.

8. 25, 80, 2400, 1440, 480000, 14400, 4, 96, 125.

9. 1150 fr. 10. 45. 11. 36. 12. 3877 nearly.

XIV.—PAGE 60.

1. 1609·314 m. 2. 57319·8975 ft. 3. 239·613 sq. ft.
4. 862784. 5. 271·7. 6. 1128 fr.
7. 67 fr. 62 c. nearly. 8. 1053268765 galls.
9. £4. 3s. 4d. 10. 2204 fr. 61 c.
11. 447·39. 12. 1·0392.

XV.—PAGE 64.

1. £480. 2. 8s. $11\frac{1}{2}$d. 3. 344 fr. $48\frac{1}{4}$ c.
4. $39' 22\frac{1}{2}''$. 5. 70. 6. 7 fr. $97\frac{1}{2}$ c.

7. £4. 9s. 7d. 8. 31 days. 9. £305.

10. 3 h. $34\frac{1038}{1443}'$ P.M. 11. $32'\ 43\frac{7}{11}''$ past 2.

12. 4 months.

XVI.—Page 67.

1. 15. 2. $30\frac{33}{43}$ ft.

3. 1125 miles (take 8 kilom. = 5 miles).

4. £15,000. 5. $11\frac{1}{2}$ days. 6. $92\frac{2}{19}$ days.

7. 2s. 8d. 8. 52·5 m. 9. £154.

10. 88 horses. 11. $7\frac{1}{2}$ miles. 12. 12 days.

XVII.—Page 69.

1. £70. 2. £41. 6s. $2\frac{1}{2}$d. 3. £39. 8s. $4\frac{1}{2}$d.

4. £4. 6s. $9\frac{39}{73}$d. 5. £4. 4s. 6. £21. 18s. 5d.

7. $1\frac{1410}{1553}$. 8. £236. 11s. 8d. 9. $3\frac{839}{935}$.

10. £385. 11. $5\frac{7}{9}$ months. 12. 15s. $5\frac{1}{2}$d.

XVIII.—Page 72.

1. £23. 3s. 5·856d. 2. £49. 5s. 6·84d.

3. £20. 3s. 10·149d. 4. £102. 17s. 4·941d.

5. 10s. 9·6d. 6. £441. 2 fl. 1 c. 1·40 m.

7. £270. 12s. 1·929d. 8. $£200 \div (1{\cdot}045)^3$.

9. £231525. 10. £71. 1s.

11. $£50\{(1{\cdot}05)^3 + (1{\cdot}05)^2 + (1{\cdot}05)\} = £165.\ 10s.\ 1\frac{1}{2}d.$

12. $£450 \div (1{\cdot}04)^3$.

XIX.—Page 75.

1. £12. 7s. 2·4d. 2. $3\frac{19}{27}$d.

3. £4. 13s. 3·38d. 4. £4. 17s. 11·66d.

5. £3. 0s. $2\frac{746}{1107}$d. 6. £10. 11s. 8·7d.

7. £5. 12s. 6d. 8. £42. 10s.

9. £39. 7s. 6d. 10. £35.

11. £2212. 10s. 12. £326. 7s. $3\frac{23}{36}$d.

XX.—Page 77.

1. £690. 18s. 9d.
2. £304. 14s. $8\frac{7}{10}$d.
3. £5528. 10s. nearly.
4. £382. 10s. 10d.
5. £6911. 3s. $6\frac{81}{100}$d.
6. £126. 14s. 10d. nearly.
7. £842. 12s. 6d.
8. £7768. 5s. $3\frac{3}{4}$d.
9. £863. 7s. 6d. nearly.
10. £134.
11. £281. 5s.
12. 50,000 francs.

XXI.—Page 80.

1. £374. 11s. $10\frac{2}{25}$d.
2. £1069. 10s. $11\frac{1}{2}$d. nearly.
3. £1773. 14s. $6\frac{1}{2}$d.
4. £393. 7s. $8\frac{1}{2}$d. nearly.
5. £308·8693.
6. 20.
7. 12s. 10d.
8. £522·0411.
9. £217·7937.
10. £1000 $\div$ $(1{\cdot}05)^2$.
11. £23. 11s. nearly.
12. £369·8728.

XXII.—Page 82.

1. 5s. 5d.
2. 10s. 10d.
3. £1. 10s.
4. 6 to 5.
5. £48.
6. $4\frac{1}{2}$ gallons.
7. He loses $12\frac{1}{2}$ per cent.
8. £72,123. 12s. 6d.
9. £$\frac{25}{183}$.
10. $2\frac{1}{2}$.
11. £4. 5s.
12. $33\frac{1}{3}$.

Miscellaneous Examples.—Page 85.

1. 10.
2. £3. 5s. 10d.
3. 721 lbs., 321 kilog.
4. $\sqrt{\cdot 5} + \sqrt{\cdot 3}$, $\frac{1}{2}$.
5. ·0078125, $\frac{33}{148}$.
6. 2933·73.
7. 33·1, 3·236.
8. £2. 16s. $0\frac{1}{4}$d., he lost $12\frac{1}{3}$ per cent.
9. 12.
10. $\frac{4}{5}$, 6s., ·04895.
11. 0, 4, 2500, $1\frac{1}{16}$.
12. £29. 6s. $7\frac{1}{2}$d., £0. 9s. $8\frac{1}{2}$d.
13. £8000 stock, £7530.
14. £·002, 34·3168.
15. 83·8967, 1·9387.
16. £$5\frac{1}{9}$, £$11\frac{1}{9}$, £$12\frac{1}{9}$.
17. ·2036.
18. 240.
19. 4s. 7·4d. per ounce.
20. £1763. 1s.

21. £723, £3. 6s. 4d. **22.** £60, £40, £100.
23. 5s., £1. 17s. 6d. **24.** £1. 1s. $5\frac{1}{2}$d, $11\frac{20}{291}$.
25. £336, $\frac{291}{3191}$. **26.** $\frac{66641}{333000}$, $9 \cdot \dot{0}\dot{9}$.
27. £3375. **28.** $6\frac{4}{81}$. **29.** $\cdot\dot{3}$. **30.** 3000 days.

ALGEBRA—STAGE I.

I.—Page 149.

1. 8, 27, 6, – 16. **2.** 6, 2, 4, 0. **3.** 5, 11, – 7.
4. 11, 16. **5.** – 2, – 15. **6.** 8. **7.** – 5, – 7.
8. – 11, 8. **9.** 7, – 75. **10.** 72, – 68.
11. – 2, – 3. **12.** – 10, 32.

II.—Page 154.

1. 51, 6, 3. **2.** 35. **3.** – 513, – 65.
4. – 1224, 30. **5.** 0, 2. **6.** 1, – 27. **7.** 1, – 2.
8. 1591, 0. **9.** 144. **10.** 1521.
11. 145. **12.** $-\frac{37}{\sqrt[3]{1029}}$.

III.—Page 156.

1. $10a + 3b$. **2.** $11a^2$. **3.** $12b + 8c$. **4.** 0.
5. $8a^2 + ab$. **6.** $3x^4 - x^2 - 15x - 2$. **7.** $4ab - 4$.
8. $x^3 + y^3 + z^3 - 3xyz$. **9.** $x^4 + x^2y^2 + y^4$.
10. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2 - a^2c - ac^2 - b^2c - bc^2 - 2abc$.
11. $x^4 + y^4 + z^4 - 4x^3y + 4x^3z - 4xy^3 - 4y^3z + 4xz^3 - 4yz^3 + 6x^2y^2 + 6xz^2 + 6y^2z^2 - 12x^2yz + 12xy^2z - 12xyz^2$. **12.** 0.

IV.—Page 157.

1. $4a + 2b + 5c$. **2.** $-4x + 2y - 6z$.
3. $a^2 - 3ab - b^2 - 5a - 7b - 8$.

4. $2a^4 - 2a^2x^2 + x^4$. 5. $4a^4 + 8a^2b^2 + 4b^4$.
6. $-3x^3 - 3z^3 - 3x^2z - 3xz^2$.
7. $x^4 - ax^3 - 9a^2x^2 - 3a^3x - 2a^4$. 8. 0.
9. $-a - b - c - d + e + f + g + h$.
10. $2x^4 - 8x^3y + 12x^2y^2 - 8xy^3 + 2y^4$.
11. $x^4 + 2x^2y^2 + y^4$.
12. $a^2 + b^2 - 2c^2 + 2ab - 2ac - 2bc$.

V.—Page 159.

1. $-x + 8y + z$. 2. $a^3 - b^3$. 3. $13 - 6x$.
4. $5x^2 - 3x - 7$.
5. $(a + b) + (c - d)$, $a - (b - c + d)$,
$\{a - (b - c)\} - d$.
6. $-(6a - 7b) - (3c - 5d)$, $-6a + (7b - 3c + 5d)$, $-\{6a - (7b - 3c)\} + 5d$.
7. $-(4x^3 - 12x^2y) - (12xy^2 - xy^3)$,
$-4x^3 + (12x^2y - 12xy^2 + 4y^3)$,
$-\{4x^3 - (12x^2y - 12xy^2)\} + 4y^3$.
8. $(a^3 - b^3) - (c^3 - 3abc)$, $a^3 - (b^3 + c^3 - 3abc)$, $\{a^3 - (b^3 + c^3)\} + 3abc$.
9. $-(a - b + d)x^2 - (a - b - 3c)xy$
$- (2b - c - e - f)y^2$.
10. $(a - b + c - d)x - (c - d + e - f)y$
$- (e - f + g - h)z$.
11. $(a - b - d)x - (a - 7b - c)y - (b - c + d)z$.
12. $(2 - x)a^2 + (x - y)ab + (y - z)b^2$.

VI.—Page 161.

1. $12a^2 - ab - 6b^2$, $18x^2 - 9xy - 35y^2$.
2. $x^3 - 2xy^2 + 4y^3$, $30x^3 + 49x^2y + 9xy^2 - y^3$.
3. $a^4 + 2a^2b^2 + b^4$, $a^4 - b^4$. 4. $x^4 - y^4$, $x^8 - y^8$.
5. $a^3 + b^3 + c^3 - 3abc$. 6. $x^8 - y^8$.
7. $a^3 + 3a^2b + 3ab^2 + b^3$.
8. $5a^5 + 5a^4 - 405a - 405$.

9. $-7x^3 + 17x^2 - 5x - 2$, $a^3 - x^3$.

10. $a^8 + 2a^6b^2 + 3a^4b^4 + 2a^2b^6 + b^8$.

11. $af + (ac + bf)x + (b + f)cx^2 + (c^2 + df)x^3 + cdx^4$.

12. $x^3 + (p - a)x^2 + (q - ap)x - aq$.

13. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

VII.—PAGE 165.

1. $x^2 - 2xy + y^2$, $9a^2 - 30ab + 25b^2$, $16c^4 + 8c^2d^2 + d^4$, $9x^4 - 12x^2y^2 + 4y^4$.

2. $a^8 - b^8$. 3. $m^2x^2 - n^2y^2$, $5a^4 - 18b^4$.

4. $(a + c)^2 - (b + d)^2$, $(a + b)^2 - (c + d)^2$.

6. $x^2 + 4x - 5$, $x^2 + 7x + 10$, $x^2 + 2x - 15$, $x^2 - 25$.

7. $x^4 - 37x^2 - 24x + 180$.

8. $\frac{1}{16}\{4a^2b^2 - (a^2 - b^2 - c^2)^2\}$.

9. $4(a^2 + b^2 + c^2 + d^2)$.

VIII.—PAGE 171.

1. $4a^2 - ab + 2b^2$, $xy - 4y^2$.

2. $2a^4 - 5a^3b + \frac{20}{3}a^2b^2$, $2x^2 + 3xy + 2y^2$.

3. $ax^{m-n} + bx^{-n}y^n + cx^{-(m+n)}y^{2n}$,
$ax^{m-n} + by^n + cx^{n-m}y^{2n}$.

4. $6x + 4y$, $5x - 3y$. 5. $1 + x + x^2$.

6. $-3x - 4$, $-7x + 3$.

7. $x^3 - 3x^2y + 5xy^2 + 27y^3 + \dfrac{72y^4}{x - 3y}$,

$$x^3 - x^2y - 3xy^2 + 15y^3 - \frac{24y^4}{x + y}.$$

8. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$, $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

9. $ax^m + by^n$. 10. $a^3 + a^2b + ab^2 + b^3$.

11. $b + c$, $a + b$. 12. $a + bx + cx^2$.

13. $a^4 - (p - 1)a^3 - (p - q - 1)a^2 - (p - 1)a + 1$.

14. The given expression is—
$(a + b + c - d)(a + b - c + d)(a - b + c + d)(a - b - c - d)$:

16. $-x^4y^2 - x^2y^4$, $x + x^{-1}$.
17. $(x+y)^3 + (x+y)^2z - (x+y)z^2 - z^3$.
20. $(a-c)^2 - 2(a-c)(b-d) + (b-d)^2$.

IX.—Page 177.

1. $(x+3a)(x-3a)$, $(4y^2+5z^2)(4y^2-5z^2)$, $6(2a+3b)(2a-3b)$, $(2x-3y)(4x^2+6xy+9y^2)$.
2. $x(x-y)(x^2+xy+y^2)$, $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$, $xy(x+y)(x^2-xy+y^2)$, $2xy^2z(x+2z)(x-2z)$.
3. $(a^2-2b^2)(a^2+2b^2)$, $(x^2+xy+y^2)(x^2-xy+y^2)$, $(a+b)^2(a-b)^3(a+b+c)(a+b-c)$.
4. $(a+b+c+d)(a+b-c-d)$, $(a+b-c+d)(a-b+c+d)$, $(a+b-c)(a-b+c)$.
5. $5(2x+9)$, $3(2x+7)$, $(3a+b-c)(a+b-c)$.
6. $(x^2+xy+y^2)\{(x-y)(x^3-y^3)+4x^2y^2(x^2-xy-y^2)\}$, $(x^2+xy+y^2)(x^2-xy+y^2)(x+y)^2(x-y)^2$.
7. $(x-10)(x+7)$, $(x+1)(x+10)$, $(a-7b)(a-8b)$, $(x-16)(x+12)$.
8. $(ax+7by)(ax-6by)$, $3a(x-10)(x+2)$, $ac(c-8)(c+3)$.
9. $(3x+5)(2x-7)$, $(2x+9)(4x-15)$, $3(2x+3)(3x-8)$, $(4x-7)(5x+6)$.
10. $xy(3x+y)(x+3y)$, $x(5x+3a)(4x+5b)$, $(mx+p)(nx+q)$.
11. x^3+2x^2+4x+8, $3x^4-6x^3+12x^2-24x+48$, x^4+3x^2+9.
12. $(a+b)^2-(a+b)(c+d)+(c+d)^2$, $a-b+c+d$.
13. $a^4+pa^3+qa^2+ra+s$, $2a^4$. 14. -102, 17.

X.—Page 179.

1. a^4b^6, $-27a^3b^9$, $16a^8b^2c^2$, $-x^9y^2z^6$.

2. $a^4 + 12a^3b + 54a^2b^2 + 108ab^3 + 81b^4$,
$16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$,
$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$,
$27a^3 - 108a^2b + 144ab^2 - 64b^3$.

3. $16m^4 + 32m^3 + 24m^2 + 8m + 1$,
$125x^3 + 150x^2 + 60x + 8$,
$81a^4 - 432a^3c + 216a^2c^2 - 1152ac^3 + 256c^4$,
$-a^3 - 3a^2b - 3ab^2 - b^3$.

4. $x^4 + 2x^3 + 3x^2 + 2x + 1$,
$9a^2 + b^2 + 16c^2 + d^2 - 6ab + 24ac - 6ad - 8bc + 2bd - 8cd$,
$a^2 + 4b^2 + c^2 + 4ab - 2ac - 4bc$,
$9(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)$.

5. $1 + 3x - 5x^3 + 3x^5 - x^6$, $(ax + by)^3 + 3(ax + by)\ cz +$ &c.

6. $1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$,
$(1 + x)^4 + 4(1 + x)^3x^2 + 6(1 + x)^2x^4 + 4(1 + x)x^6 + x^8$,
$(a + bx)^4 + 4(a + bx)^3\,cx^2 +$ &c.

7. $a^8x^8 - 8a^7bx^7y + 28a^6b^2x^6y^2 - 56a^5b^3x^5y^3 + 70a^4b^4x^4y^4 - 56a^3b^5x^3y^5 + 28a^2b^2x^2y^6 - 8ab^2xy^2 + b^8y^8$,
$729x^6 - 243x^4y^2 + 27x^2y^4 - y^6$,
$x^9 - 3x^6y^3 + 3x^3y^6 - y^9$.

8. $x^{12} - 2x^6y^6 + y^{12}$,
$a^{12} - 6a^{10}b^2 + 15a^8b^4 - 20a^6b^6 + 15a^4b^8 - 6a^2b^{10} + b^{12}$.

9. $a^3 + 3a^2b + 3ab^2 + b^3$. 10. $(a - c)^3$.

11 $2(1 + 3x^2)^4$. 12. $81x^4y^4(x + y)$.

XI.—Page 193.

1. $2xy^2z^3$, $4a^2y^2$, $x^2 + a^2$. 2. $2x^3 - 3x^2y + 5x^2y^2$.

3. $5a^2 - 3ab + b^2$. 4. $1 - 2x + 3x^2 - 4x^3$.

5. $a + bx + cx^2 + dx^3$. 6. $a^2x^n - 3ax^{n-1} + 4x^{n-2}$.

7. $x + x^{-1}$, $ax^{-1} - a^{-1}x$ 8. $3x^{\frac{m}{2}} - \frac{a^2}{2}\,5a$.

9. 36, 79, 207, 289. 10. $\dot{1}03{\cdot}2$, $\cdot 024$, $\cdot\dot{3}$, $\frac{4}{9}$.

11. 4·123, 1·224, ·618, 3·732.

12. ·0203, $4{\cdot}\dot{6}$, 3·5036. 13. $2ab^2y^4$, $5x^4y$, $a + 2b$.

14. $x^4 + 3x^2 - 7$.
15. $x + y - c$.
16. $x + x^{-1}$, $xy^{-1} + 1$.
17. 18018, 11·11.
18. 2·73, $\cdot\dot{3}$, $\frac{3}{7}$.
19. ·5, ·2599.
20. $\sqrt[3]{5} = 1{\cdot}709$.
21. 7·6457, 50·2487.
22. 4·4142, 3·7320.
23. ·25.
24. 0.

XII.—Page 200.

1. $x - 3$.
2. $x^2 + 4x + 3$.
3. $x + 3$.
4. $x + a$.
5. $a^2 + ab + b^2$.
6. $x - 1$.
7. $x - 3$.
8. $3x - 2$.
9. $12x^2 + 5x$.
10. $3a + 2b$.
11. $x - 1$.
12. $2x^3 - 4x^2 + x - 1$.
13. $2a + 3b + c$.
14. $x^2 + 1$.
15. $x - 1$.
16. $2x^2 - 3x + 9$.
17. 1.
18. $a + b + c$.

XIII.—Page 203.

1. $12a^3x^3y^3$.
2. $60a^2b^2c^2$.
3. $(a - b)(b - c)(c - a)$.
4. $a^2x(x^2 - a^2)$.
5. $(x + 1)(x + 2)(x + 3)$.
6. $(x - 6)(x + 5)(x - 5)$.
7. $(2x + 7)(3x + 8)(4x + 5)$.
8. $210(x^2 + 1)(x^3 + 1)$.
9. $a^2(x^4 + a^2x^2 + a^4)$.
10. $(x + a)(x + b)(x + c)$.
11. $1 - x^{16}$.
12. $(x + 1)(x + 2)(x + 3)(x - 5)(x - 6)(x + 5)$.
13. $a(a - b)^3(a^2 - ab + b^2)$.
14. $(x - 1)(x + 1)(x^2 + 1)(6x^3 + 5x^2 + 2x - 1)$.
15. $(a^2 - b^2)^4$.
16. $2x(x^2 - 1)(3x^2 + 3x - 1)(2x^2 + 2x - 5)(2x^2 - 2x + 5)$.
17. $3(x - 2)^2(x + 4)(5x^3 + 10x^2 + 20x + 18)$.
18. $a^2b^2(a + b)(a - b)$.
19. $15(3x - 10)(x^4 - 16)$.
20. $(x^2 - y^2)^2(x^2 + y^2)(x^4 + y^4)$.
21. $x^6 - a^6$.
22. $(a + b + c + d)(a + b - c - d)(a - b - c + d)$.
23. $(a + b + c)(a^3 + b^3 + c^3 - 3abc)$.
24. $(a + b + c + d)(b + c + d - a)(a + c + d - b)$
$(a + b + d - c)(a + b + c - d)(a + b - c - d)$.

XIV.—Page 206.

1. $\frac{x-6}{x+6}$, $\frac{x^2-2x+2}{x^2+5x+5}$.

2. $\frac{3x+7}{7x+2}$, $\frac{2x^2+9}{5x^2+2x-2}$.

3. $\frac{(a-b)(a+b)^4}{a^2-3ab+b^2}$, $\frac{4a^2+b^2}{a+3b}$.

4. $\frac{x(y-z)-y^2}{x(y+z)-y^2}$, $\frac{1}{x^2-y^2}$.

5. $\frac{2a}{a^2-b^2}$, $\frac{2b}{a^2-b^2}$.

6. $\frac{a-b}{a+b}$.

7. $\frac{x^3-x^2+3x-1}{2(x-1)(x^2+1)^2}$.

8. $\frac{10}{(x+1)(x+2)^2(x^2+1)}$.

9. $\frac{8x-20}{(x+1)^2(x+3)}$.

10. $\frac{8x^2+8}{(x-1)^4(x^3+1)}$.

11. 0. 12. 0. 13. 1. 14. $a+b+c$. 15. 1. 16. 0.

17. $-\frac{16a^5x}{(a^2-x^2)^2}$.

18. $\frac{a^2-ab+b^2}{a^2+ab-b^2}$.

19. 1. 20. 1.

21. $\frac{8}{(x-y)(y-z)(x-z)}$.

22. $\frac{1}{x^3}$.

23. $\frac{1}{(x-a)(x+b)(x+c)}$.

24. $\frac{3x^2}{a}-4x+5a$.

25. 1.

26. $\frac{4a^2b^2}{(a^2-b^2)^2}$.

27. $\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{a-b}\right)^2$.

28. $\frac{a^2(x-y)}{(a+x)(a+y)^3}$.

29. 1.

30. $\frac{1}{x^2}$.

XV.—Page 212.

1. 3. 2. 4. 3. $a+b$. 4. 5.

5. 6. 6. 3. 7. 5. 8. 2.

9. c. 10. $a + b$. 11. $a + b + c$. 12. $\frac{a^2 + ab + b^2}{b}$.

13. abc. 14. $-(a + b)$. 15. abc. 16. $\frac{2a(b^2 - 5)}{3b - 4a}$.

17. 2. 18. 8. 19. $\frac{1}{4}$. 20. $\frac{cd - ab}{a + b - c - d}$.

21. $\frac{a^2 + ab + b^2}{a + b}$. 22. $\frac{1}{abc}$. 23. $a + b + c$.

24. $\frac{1}{a + b + c}$.

XVI.—Page 217.

1. 6. 2. $\frac{ab - cd}{a + b - c - d}$. 3. 40.

4. $\frac{3956}{3971}$. 5. $-\frac{31}{159}$. 6. $-\frac{7}{8}$.

7. $1\frac{1}{2}$. 8. 3. 9. $-1\frac{1}{2}$.

10. 8. 11. $1\frac{41}{93}$. 12. $\frac{3 - m - n}{2a}$.

13. $-\frac{a^2 + b^2}{a + b}$. 14. $2\left(\frac{1 - b}{1 + b}\right)^2 - \left(\frac{1 - b}{1 + b}\right)^4$.

15. 3. 16. -8. 17. 1. 18. $\frac{b^2 + 1}{2ab}$.

19. $(\sqrt{a} + \sqrt{b})^2$. 20. $-a$. 21. $a^2 + 2a$.

22. $\frac{5}{7}$. 23. $-2\frac{1}{2}$. 24. $\frac{af(bc - ad) + (cf - ae)cd}{a^2(bc - ad) - (cf - ae)c^2}$.

XVII.—Page 221.

1. 10. 2. 45, 25. 3. £360, £240, £120.
4. 20, 15 miles per hour. 5. 24.
6. 36s., 48s. 7. 12. 8. 36s. 9. 30s.
10. 13. 11. 10. 12. 24 lbs., 40 lbs.

13. 13. 14. 750. 15. 15s. 16. $46\frac{6}{19}$.
17. 18 miles from A. 18. $\frac{24}{647}$ hr.
19. 15 per cent. 20. 28 miles. 21. 10.
22. $4\frac{1}{3}$, 4 miles per hour. 23. 12 miles.
24. 20 miles per hour in same direction.
25. 120. 26. 6s., 4s. 27. 2, 1, $\frac{4}{3}$, 6 hours.
28. 98 per cent. 29. £1,000. 30. $38\frac{2}{21}$ miles per hour.
31. 2 inches. 32. 16 vols. of hydrogen, 8 of oxygen.

33. $\frac{1 - 100c}{b - c}, \frac{100b - 1}{b - c}$. 34. $\frac{ab'c - a'bc'}{cc'(a'b - ab')}$.

35. $-40°$. 36. $\frac{a^2b}{c - ab}$.

XVIII.—Page 228.

1. 3, 4. 2. 5, 2. 3. 3, 7. 4. 4, 5.
5. 12, 8. 6. 2, 6. 7. 1, 2, 3. 8. 1, 2, 3.

9. 4, 1, 2. 10. $\frac{b_1c - bc_1}{ab_1 - a_1b}, \frac{ac_1 - a_1c}{ab_1 - a_1b}$.

11. $x = \frac{a + c - b}{2}$, &c. 12. $x = \frac{d + f - e}{2a}$.

13. 4, 5. 14. 3, 8. 15. 4, 8. 16. 3, 4.

17. 3, 5. 18. 8, 1. 19. $x = \frac{2}{a + c - b}$, &c.

20. 2, 3, 4, 1. 21. 3, 4, 5. 22. $x = \frac{a + c + d - b}{3}$, &c.

23. $x = \frac{m(a + c - b)}{2}$, &c. 30. $x = y = z = 1$.

XIX.—Page 231.

1. 12, 8. 2. 37. 3. 6 lbs., 5 stones.
4. $\frac{3}{7}$. 5. 17 horses, 24 cows.

6. 6s., 3s. 7. 220 and £2. 14s.

8. 200, 300, 400. 9. 48, 23, 18. 10. 5, 3, $4\frac{1}{2}$.

11. 4, 6, 8 hours. 12. 6, 10, 18. 13. $7\frac{1}{2}$, 12, 3.

14. 4, 3 miles per hour. 15. 16, 32, 48 miles per hour.

16. $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$, from which x, y, z.

ALGEBRA—STAGE II.

I.—Page 300.

1. ± 3. 2. ± 4. 3. ± 5.

4. $\pm \frac{3}{2}$. 5. ± 2. 6. $\pm (a + b)$.

7. ± 1. 8. $\pm \sqrt{a^2 - b^2}$. 9. $\pm \sqrt{2ab - b^2}$.

10. $\pm \frac{2}{3a}\sqrt{3}$. 11. $\pm a \sqrt{\frac{2b^2 + 1}{b^3 + 3b}}$.

12. $\pm \frac{a + 1}{2\sqrt{a}}$. 13. 2, 3. 14. 9, $-$ 8.

15. 1, $-\frac{2}{3}$. 16. 4, $-\frac{7}{4}$. 17. $-\frac{7}{3}$, $-\frac{5}{2}$.

18. $-$ 5, $-\frac{8}{5}$. 19. $\frac{1}{2}(b \quad \sqrt{b^2 - 4ac})$.

20. 5, $3\frac{1}{2}$. 21. 7, $-$ 6. 22. 6, $-\frac{3}{2}$.

23. 1, $\frac{ad - (a + b)c}{bc}$. 24. b, a.

25. $a + b$, $\frac{ab}{a + b}$. 26. a, b. 27. $\frac{c}{a - b}$, $\frac{d}{a - b}$.

28. $\frac{a^2 + ab + b^2}{a - b}$, $\frac{a - b}{a^2 - ab + b^2}$. 29. 4, $\frac{3}{10}$.

30. 5, $-\frac{73}{56}$. 31. 3, $-$ 5. 32. 2, $-3\frac{1}{2}$.

33. 2, $-\frac{133}{229}$. 34. 5, $\frac{1}{4}$. 35. 8, $-\frac{8}{7}$.

36. 3, $-\frac{26}{23}$. 37. 2, $2\frac{1}{2}$. 38. 4, $-\frac{5}{3}$.

39. 15, $-$ 1. 40. 2, 0. 41. 0, $2ab$.

42. $a, 3\frac{2}{5}a$. 43. $0, \dfrac{a^2 + b^2 + 2b\sqrt{ab}}{(b-a)b}$.

44. $1, \dfrac{a^2b^2(a^2 - b^2)^2}{\{(a^2+b^2)^2 + 2ab(a+b)^2\}^2 - 4a^4b^4}$.

45. $a + b + c, -\dfrac{a^2 + b^2}{a - b}$.

II.—Page 306.

1. ± 3. 2. $2, -\sqrt[3]{2\frac{3}{5}}$. 3. $\pm 5, \pm \dfrac{1}{\sqrt{5}}$.

4. $2, -5$. 5. $\pm \frac{1}{5}$. 6. $\sqrt[n]{2}, \sqrt[n]{3}$. 7. $4, 69$.

8. $\pm 4, \pm \sqrt{7}$. 9. $3, -4, 3 \pm \sqrt{21}$. 10. $\frac{5}{3}, 4$.

11. $-2, -2 \pm \sqrt{3}$. 12. $\dfrac{a}{2}$.

13. $4, \frac{1}{9}, -\frac{1}{6}(-13 \pm \sqrt{145})$. 14. $0, b - a$.

15. $\pm \frac{1}{2}a\sqrt{3}$. 16. $\dfrac{a}{2} \pm \sqrt{b - 2a\sqrt{b} + \dfrac{a^2}{2}}$.

17. $0, \left(\dfrac{a \pm b}{a \mp b}\right)^{\frac{2pq}{p-q}}$. 18. $\pm \dfrac{a}{\sqrt{2}}\sqrt{\sqrt{5} - 1}$.

19. $\left(\dfrac{a^{2m} \pm 1}{a^{2m} \mp 1}\right)^2$. 20. $\pm 5, \pm \sqrt{3\tfrac{5}{2}\tfrac{1}{3}}$.

21. $2, -\frac{1}{3}, \frac{1}{6}(-11 \pm \sqrt{97})$. 22. $4, \frac{5}{6}$.

23. $5, 3\frac{1}{2}, \frac{1}{4}(-7 \pm \sqrt{53})$. 24. $3a$.

25. $3, -2$. 26. $4, -14\frac{1}{2}, \frac{1}{2}(-19 \pm \sqrt{741})$.

27. $3, -2\frac{1}{2}, 1\frac{5}{3}$. 28. $\frac{1}{2}(1 \pm \sqrt{4a - 3})$.

29. ± 1, and $x^4 + x^2 + 1 = 0$. 30. $\pm \sqrt{\dfrac{a + b + c}{3}}$.

31. $a, \frac{1}{2}\{\sqrt{4c^2 - 3a^2 + 2ab + b^2} - (a - b)\}$.

32. $x = m$ is one solution.

33. $x = a + b + c$ is one solution.

34. $x = \frac{a}{m}$, and $a^2 + ax + x^2 = 0$.

35. $x = \frac{b + d}{a + c}$ is one solution.

36. $x^2 - x + 1 = 0$ gives two solutions.

III.—PAGE 310.

1. $x = 2, 3, y = 3, 2$. 2. $x = 5, -3, y = 3, -5$.

3. $x = 3, 4, y = 4, 3$. 4. $x = 2, 4, y = 4, 3$.

5. $x = 5, -2, y = 2, -5$. 6. $\pm 7, \pm 6$.

7. $\pm 6, \pm 3$. 8. $4, 2$. 9. $\pm 2, \pm 7$.

10. $\pm 4, \pm 3$. 11. $x = \pm 3, \frac{1}{\sqrt{3}}, y = \pm 2, -\frac{8}{\sqrt{3}}$.

12. $x = 9, 4, y = 4, 9$. 13. $x = 2, 8, y = 8, 2$.

14. $x = 3, 2, y = 2, 3$.

15. $x = \frac{2}{a \pm \sqrt{2b^2 - a^2}}, y = \frac{2}{a \mp \sqrt{2b^2 - a^2}}$.

16. $\pm \frac{(a + 1)b}{\sqrt{2(a^2 + 1)}}, \pm \frac{(a - 1)b}{2\sqrt{a^2 + 1}}$.

17. $\pm \sqrt{\frac{a + b^2}{2b}}, \pm \sqrt{\frac{a - b^2}{2b}}$.

18. $x = 2, 3, y = 3, 2$. 19. $x = 2, 3, y = 3, 2$.

20. $x = 2, 3, y = 3, 2$ 21. $x = 5, -3, y = 3, -5$.

22. $x = \frac{a + \sqrt{2b - a^2}}{\sqrt[3]{4a^2 - 4b}}$. 23. $x = 2, 3, y = 3, 2$.

24. $x = \pm 3, \mp \frac{15}{8}\sqrt{\frac{32}{35}}, y = \pm 2, \pm \sqrt{\frac{32}{35}}$

25. $x = 4, 9, y = 9, 4$. 26. $x = 4, 1, y = 8, 0$.

27. $x = 4, 3, y = 3, 4$. 28. $\pm 3, \pm 1$.

29. $x = 0, 3, -\frac{1352}{1145}, y = 0, 2, \frac{1768}{1145}$.

30. $0, -1$. 31. $x = 4, \frac{1708}{559}, y = 3, -\frac{6243}{559}$.

32. $x = 5, 6, y = 4$. 33. $x = 3, 1, y = 1, 0$.

34. $x = 3, -\frac{138}{45}, y = 1, -\frac{46}{45}$.

35. $x = 2, 7, y = 4, -14$.

36. $x + y = c$, and $xy = bx + ay$.

37. $x = 0, a - b, y = 0, a - b$.

38. $\frac{51}{2} \pm 9\sqrt{\frac{35}{12}}, \frac{1}{2} \pm \sqrt{\frac{35}{12}}$.

39. $x = \sqrt{\frac{(a + c - b)(a + b - c)}{2(b + c - a)}}$.

40. $1, 2, 3$. 41. $\pm 1, 0, 0$.

42. $x = y = z = 0$, also $\frac{1}{xy} = \frac{1}{2}\left(\frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a^2}\right)$,

$$\frac{1}{yz} = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{c^2} - \frac{1}{b^2}\right), \text{ \&c.}$$

43. $\frac{a}{\sqrt[4]{a^2 + b^2 + c^2}}, \frac{b}{\sqrt[4]{a^2 + b^2 + c^2}}, \frac{c}{\sqrt[4]{a^2 + b^2 + c^2}}$.

44. $x = y = z = u = a + b$.

45. $\frac{a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}}{(a + b + c)^{\frac{1}{4}}}, \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}}}{(a + b + c)^{\frac{1}{4}}}$, &c.

46. $x = 6, 4, y = 4, 6, z = 5$.

IV.—Page 315.

1. 9, − 6. 2. Numerator $2\frac{89}{95}$, denominator 3.

3. 35 or 23. 4. 10, − 16. 5. 78, − 68 ft.

6. 2s., 9s. 7. 20 hours. 8. 48 shillings.

V.—Page 318.

1. $a^2, a^2, a^{\frac{1}{2}}, a$. 2. $a + x, (a - x)^{\frac{1}{2}}$.

3. $a + a^{\frac{1}{2}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$. 4. $x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}$.

5. $xy^{-1} + x^{-1}y + 1$. 6. $a^2 + ab^{\frac{1}{3}} + b^{\frac{2}{3}}$, $a^{\frac{3}{8}} + b^{\frac{3}{8}}$.

7. $x - x^{\frac{1}{2}}y$. 8. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}}$.

9. $a^2b^{-\frac{1}{3}} - b^{\frac{2}{3}}$. 13. $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}$.

14. $2x^{\frac{1}{2}}y^{\frac{2}{3}} - 3xy^{\frac{1}{3}} + 2x^{\frac{3}{2}}$. 15. $ab^{-\frac{1}{2}} - 2 + 2a^{-1}b^{\frac{1}{2}}$.

16. $x^{\frac{4}{3}} + 3x^{\frac{2}{3}} - 7$. 17. $x^2y^{-\frac{1}{3}} + 1$. 18. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^2$.

VI.—Page 325.

1. $x^{\frac{5}{2}}$, $a^{\frac{5}{3}}b^{\frac{2}{3}}$, $x^{\frac{1}{2}}y^{\frac{3}{4}}$, $a^{\frac{1}{4}}b^{\frac{1}{2}}$.

2. $xya^{-\frac{1}{2}}b^{-\frac{1}{2}}$, $c^{\frac{3}{2}}d^{\frac{1}{2}}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$, $a^{-\frac{n}{m}}x^{\frac{m}{n}}$.

3. $\sqrt{27}$, $\sqrt{24}$, $\sqrt{\frac{27}{5}}$, $\sqrt[3]{9}$. 4. $\sqrt{32}$, $\sqrt{27}$, $\sqrt[4]{32}$, $\sqrt{\frac{27}{8}}$.

5. $\sqrt{9ab}$, $\sqrt{abc}$, $\sqrt[3]{a^2 - x^2}$. 6. $\sqrt[6]{4}$, $\sqrt[6]{27}$.

7. $\sqrt[12]{8}$, $\sqrt[12]{81}$. 8. $\sqrt{8}$, $\sqrt[3]{135}$. 9. $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.

10. $\sqrt[6]{(a + x)^3}$, $\sqrt[6]{(a - x)^2}$. 11. $a^{\frac{p+q}{pq}}$, $b^{\frac{q-p}{pq}}$.

12. $2\sqrt{3}$, $2\sqrt[3]{6}$, $6\sqrt{7}$, $3\sqrt[3]{3}$.

13. $2a\sqrt{a + b}$, $b\sqrt[3]{a^3 + b^3}$, $\frac{1}{4a}\sqrt[3]{a^3 + a^2b}$.

14. $\frac{1}{x} - a$, $\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}} - a^{\frac{2}{3}}x^{\frac{1}{3}}$.

15. $\frac{x + 1}{3x}\sqrt{3x}$, $\frac{1}{x - a}\sqrt[3]{(x^2 - a^2)^2}$.

16. $4\sqrt{3}$, $5\sqrt[3]{7}$. 17. $\frac{ay - ab + c}{b}\sqrt{ab^2x^2}$.

18. $(3a^2b - 2a^{m+3} - 12)\sqrt[3]{a^m}$.

19. $a^{\frac{3}{2}} - b^{\frac{3}{2}}$, $a - b$. 20. $(x + y)^{\frac{5}{6}}$, $a^3 - b^3d^{\frac{3}{2}}$.

21. $\frac{ac^3}{d} - c\sqrt{\frac{ab^3}{a + b}} + \frac{c^2}{d}\sqrt{\frac{a}{b}(a + b)} - b$.

22. $a - b^{\frac{2}{3}} + c^{\frac{1}{2}} - d^{\frac{2}{5}} + 2\,a^{\frac{1}{2}}c^{\frac{1}{4}} - 2\,b^{\frac{1}{3}}c^{\frac{1}{6}}$.

23. $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$:

24. $x^{\frac{5}{2}} - x^2y^{\frac{1}{3}} + x^{\frac{3}{2}}y^{\frac{2}{3}} - xy + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{3}}$.

25. $\frac{1}{3}\sqrt{15}$, $\frac{4}{7}\sqrt{7}$, $\frac{1}{5}\sqrt[3]{175}$.

26. $3\,(2 - \sqrt{3})$, $\frac{2}{3}\,(3\sqrt{2} - 2\sqrt{3})$, $\frac{1}{2}\,(\sqrt{5} + \sqrt{3})$.

27. $\frac{3}{4}\,(2 + \sqrt{2} - \sqrt{6})$, $\frac{1}{6}\,(2\sqrt{3} + 3\sqrt{2} - \sqrt{30})$, $5 + 2\sqrt{6}$.

28. $a\,(x^{\frac{5}{2}} - x^2y^{\frac{1}{3}} + x^{\frac{3}{2}}y^{\frac{2}{3}} - xy + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{3}})$,

$\frac{1}{23}\,(3^{\frac{5}{2}} - 3^2 . 2^{\frac{1}{3}} + 3^{\frac{3}{2}} . 2^{\frac{2}{3}} - 3 . 2 + 3^{\frac{1}{2}} . 2^{\frac{4}{3}} - 2^{\frac{5}{3}})$,

$\frac{b}{x^2 + xy + y^2}\,(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$.

29. $2 + \sqrt{7}$, $\sqrt{5} + \sqrt{3}$, $5 - \sqrt{5}$.

30. $\sqrt{6} + \sqrt{2}$, $\sqrt{6} - \sqrt{3}$, $\sqrt{15} - \sqrt{5}$.

VII.—Page 330.

1. $a^2 + b^2 : a^2 - b^2$ is the greater, if $a > b$.

2. The former. 3. $b - \frac{ad}{c}$.

4. $(1 - y)\,(1 + x) : 1 + x^2$. 5. 30, 35.

6. Less than 5 when x lies between 2 and 3 ; greater than 5 for all values beyond these limits.

8. x satisfies the equation—
$4\,x^2 - (a + b - 6\,c)x + 2\,(c^2 - ab) = 0$.

9. $\frac{b}{2}$. 10. 1. 14. $2\,b - a$, $\frac{b^2}{a^2}$, $\frac{ab}{2\,a - b}$.

PLANE TRIGONOMETRY.

I.—Page 335.

1. $43 \cdot 750^g$, $11° \; 50' \; 3 \cdot 03''$. 2. $88^g \; 88' \; 88 \cdot \dot{8}''$.

3. $\frac{10}{29} n$ the number of grades.

4. $180 - (a + \frac{9}{10} b)$ degrees, $200 - (\frac{10}{9} a + b)$ grades.

5. $\frac{1}{3}$. 7. $\frac{9}{8}$. 8. 25^g, 75^g, $90°$. 9. $576°$, 648^g.

10. 2nd, 1st, 4th, 3rd, 4th. 11. $\frac{9}{10}$, $\frac{b}{a}$. 12. $\frac{90°.p}{qa}$.

II.—Page 341.

1. Cos A $= \frac{5}{13}$, tan A $= \frac{12}{5}$ &c.

2. Sin A $= \frac{24}{\sqrt{1201}}$, cos A $= \frac{25}{\sqrt{1201}}$, &c.

17. $\frac{1}{2}$. 18. $\frac{1}{10} (2 \sqrt{89} - 5)$. 19. $\frac{\sqrt{3} + 1}{2 \sqrt{2}}$.

20. $2 + \sqrt{3}$. 21. $\infty, - \frac{a^2 - b^2}{2 ab}$. 22. 1.

24. Sin A $= \sqrt{\frac{a - (m + n)}{a + b}}$.

III.—Page 352.

2. From 0° to 90°, +; from 90° to 180°, −; from 180° to 270°, +; from 270° to 360°, −.

3. Cos A + sin A is +, +, −, −, as A lies between − 45° and 45°, 45° and 135°, 135° and 225°, 225° and 315° respectively. The corresponding signs of cos A − sin A are +, −, −, +.

4. Cos 2 A is +, −, +, − respectively, as in Question 3.

6. $- \frac{1}{2}, \frac{\sqrt{3} - 1}{2 \sqrt{2}}, - \frac{\sqrt{3}}{2}, - \frac{\sqrt{3}}{2}$.

10. A = 60°. 11. A = 45°, and tan A = − 6.

12. A = 30°. 13. A = 30°.

14. A = 45°, B = 15°. 15. A = 0°, 45°, 135°.

16. A = 6°. 17. A = 60°. 18. $\theta = 2 \pm \sqrt{3}$.

IV.—Page 358.

1. 2, 3, $\frac{1}{2}$, − 1, − 2, 1·5. 2. $\frac{5}{3}$, 6, 1·5.

3. 1·0791812, 1·5563025, 1·6532125, 1·8750613, $\bar{2}$·6020600, ·5740313, $\bar{1}$·8239087, $\bar{2}$·8696761.

4. 1·3172901, 3·3172901, $\bar{2}$·3172901, $\bar{3}$·3172901.

5. 1, $\bar{1}$, $\bar{3}$, $\bar{3}$.

6. ·020912, 2091200, ·20912, 20912. 7. $\bar{1}$·3162760.

8. 3493·768. 9. ·2427189. 10. $\bar{3}$·1303338.

11. 6, 5. 12. 1·67.

V.—Page 365.

1. 1·6774495. 2. 6·7193696. 3. $\bar{1}$·8213310.

4. ·0616835. 5. 2·07892. 6. £177·63 nearly.

7. £663·449. 8. 2620·248. 9. 9·7299222.

10. 9·8401608. 11. 36° 26′ 30″.

VI.—Page 372.

1. $c = 12·5$. 2. $\sqrt{189}$. 3. $\text{Tan}\frac{A}{2} = \frac{5}{27}$.

4. 480. 5. $108\sqrt{2}$. 6. B = 90°, C = 30°.

9. $C = 50\sqrt{6}(\sqrt{3}-1)$, $b = 100(\sqrt{3}-1)$.

10. $a = 10\sqrt{949}$.

VII.—Page 376.

1. $b = 15 \cdot 78065$. 2. $a = 92 \cdot 34057$.
3. $b = 25$, $A = 67° 22' 49''$.
4. $A = 40° 59' 32''$, $c = 111 \cdot 288$.
5. $b = 22 \cdot 78438$. 6. $b = 691 \cdot 49$.
7. $b = 12\sqrt{3}$, $c = 24$.
8. $a = \frac{5}{2}(\sqrt{6} - \sqrt{2})$, $b = \frac{5}{2}(\sqrt{6} + \sqrt{2})$.
9. $A = 30°$, $b = 17\sqrt{3}$. 10. $A = 30°$.
11. $CD = 7(\sqrt{6} - \sqrt{2})$. 12. $DE = 5\sqrt{3}$.

VIII.—Page 380.

1. $a = 6\sqrt{3}$, $B = 90°$, $C = 30°$.
2. $B = 45°$ or $135°$, $C = 105°$ or $15°$, $c = 9(\sqrt{6} \pm \sqrt{2})$.
3. $A = 60°$, $B = 45°$, $C = 75°$.
4. $A = 105°$, $b = 6(3\sqrt{2} - \sqrt{6})$, $c = 12(2 - \sqrt{3})$.
5. $A = 75°$, $B = 60°$, $c = 2\sqrt{6}$.
6. $B = 60°$ or $120°$, $C = 75°$ or $15°$, $c = 5(3 \pm \sqrt{3})$.
7. $B = 41° 59' 23''$, $C = 97° 28' 37''$.
8. $A = 19° 18' 11''$.
9. $A = 30° 3' 26''$, $B = 123° 24' 34''$.
10. $A = 48° 11' 22''$, $B = 58° 24' 42''$.
11. $B = 35° 15' 52''$, $C = 84° 44' 8''$, $c = 137 \cdot 9796$.
12. $A = 44° 24' 56''$.

IX.—Page 384.

1. $200(2 - \sqrt{3})$. 2. $60\sqrt{3}$.
3. $\frac{1}{2}\sqrt{3}$. 4. $6\sqrt{2}$.

5. $6(\sqrt{3}-1), 3\sqrt{6}(\sqrt{3}-1)$.

6. $30(\sqrt{3}+1)$. 7. $\frac{100}{3}\sqrt{3}$.

8. $10\sqrt{3}$.

13. $\sqrt{\left\{\left(\frac{x+y-c}{2}\sec\theta\right)^2+(c-y)x\right\}}$,

$\sqrt{\left\{\left(\frac{x+y-c}{2}\sec\theta\right)^2+(c-x)y\right\}}$.

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