Quadratic Involutions on the Plane Rational Quartic

DISSERTATION

Submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

BY
THOMAS BRYCE ASHCRAFT

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§1. *The General Theory of Involution Curves of a Plane Rational Curve of Order n.*

Let R^n denote a plane rational curve of order n, and let it be given by the equation

$$(1)\qquad \begin{aligned} x_0 &= a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_n \\ x_1 &= b_0t^n + b_1t^{n-1} + b_2t^{n-2} + \cdots + b_n \\ x_2 &= c_0t^n + c_1t^{n-1} + c_2t^{n-2} + \cdots + c_n \end{aligned}$$

If we join the parameters t_1 and t_2 by a line, where t_1 and t_2 are in an involution of the form

$$(2)\qquad A\,t_1t_2 + B(t_1 + t_2) + C = 0,$$

we shall show that *the locus of this line is a rational curve of class n-1 which touches R^n 3(n-2) times and meets it in 2(n-2)(n-3) other points.* This class curve will be called an involution curve, and will be denoted by r^{n-1}.

Cut the curve R^n by any line

$$(3)\qquad (\xi x) \equiv \xi_0x_0 + \xi_1x_1 + \xi_2x_2 = 0$$

and we have

$$(4)\qquad (\alpha_0\xi)\,t^n + (\alpha_1\xi)\,t^{n-1} + \cdots + (\alpha_n\xi) = 0.$$

For convenience suppose we choose the involution with 0 and ∞ as double points. Then t and $-t$ must satisfy the last equation, and we have for n even, say $n = 2m$,

(5) $$(\alpha_0\xi)\, t^{2m} + (\alpha_1\xi)\, t^{2m-1} + \text{-----} + (\alpha_n\xi) = 0,$$

(6) $$(\alpha_0\xi)\, t^{2m} - (\alpha_1\xi)\, t^{2m-1} + \text{-----} + (\alpha_n\xi) = 0.$$

Whence by addition and then subtraction we get

(7) $$(\alpha_0\xi)\, t^{2m} + (\alpha_2\xi)\, t^{2m-2} + \text{-----} + (\alpha_n\xi) = 0,$$

(8) $$(\alpha_1\xi)\, t^{2m-2} + .(\alpha_3\xi)\, t^{2m-4} + \text{----} + (\alpha_{n-1}\xi) = 0.$$

Since only even powers occur we can divide the exponent by 2 and write

(9) $$(\alpha_0\xi)\, t^{m} + (\alpha_2\xi)\, t^{m-1} + \text{-----} + (\alpha_n\xi) = 0,$$

(10) $$(\alpha_1\xi)\, t^{m-1} + (\alpha_3\xi)\, t^{m-2} + \text{-----} + (\alpha_{n-1}\xi) = 0.$$

Eliminating ξ_0, ξ_1, ξ_2 from equations (3), (9) and (10), we have the locus required in determinant form

(11) $$\begin{vmatrix} x_0 & , & x_1 & , & x_2 \\ f_0(t^m) & , & f_1(t^m) & , & f_2(t^m) \\ f_0'(t^{m-1}) & , & f_1'(t^{m-1}) & , & f_2'(t^{m-1}) \end{vmatrix} = 0.$$

Or written parametrically its equation is

$$\xi_i = F_i(t^{2m-1}) \quad ,$$

and since $n = 2m$ we have as the representation of the involution curve

(12) $$\xi_i = F_i(t^{n-1}),$$

which is a rational class curve of order n-1. Similar argument holds for n odd.

The point to be emphasized is that the parameter may be replaced by a new one which reduces the degree by one half; that is a T replaces a quadratic in t. This new parameter may be chosen in a triple infinity of ways depending on the ratios of α, β, ν, δ in a transformation of the form

$$\lambda = \frac{\alpha t + \beta}{\nu t + \delta}\ .$$

If the double points of the involution are given by $(a\, t)^2$, then we choose any two quadratics apolar to $(a\, t)^2$, say $(\alpha t)^2$ and $(\beta\, t)^2$; then any convenient member of the pencil $(\alpha\, t)^2 + \lambda\, (\beta\, t)^2$ will serve as a new parameter T.

To find the number of contacts of the r^{n-1} with the R^n, we shall consider first the R^4 and its involution cubic r^3. The R^4 is of class six so there are 18 common lines. There are three ways in which we may have common lines. A line meets the curve in four points, t_1, t_2, t_3, t_4. Let t_1 and t_2 come together so that line is a tangent. We have common lines when

1) Points t_1 and t_2 are a pair of the involution,
2) Points t_3 and t_4 are a pair of the involution,
3) Points t_1 and t_3 are a pair of the involution.

Case 1) can happen twice, viz. when the line cuts out either of the double points of the involution. This accounts for two common lines.

In case 2) a tangent at t meets the curve again, say at t_1. For a given t there are two t_1's, and for a given t_1 there are 4 t's, since there are four tangents from a point on the curve, the curve being of class six. The relation connecting t and t_1 is

$$f_1(t^4)\, t_1^2 + f_2(t^4)\, t_1 + f_3(t^4) = 0.$$

The condition that the roots t_1 be in an involution is of the fourth degree in t, which means four common lines for case 2). Case 3) must contain all the other common lines, that is twelve. This case happens when t_1 and t_3 are a pair of the involution, but t_2 and t_3 are as well a pair of the involution; therefore the twelve common lines are six repeated. In other words the R^4 has six contacts with the r^3.

This is easily extended to the general case of R^n. The R^n is of class $2(n-1)$, so the R^n and the r^{n-1} have $2(n-1)^2$ common lines. There will always be two of these accounted for in case 1), corresponding to the double points of the involution.

For case 2) the equation connecting a point of tangency t and a point of intersection of the tangent at t_1 is of degree $2n$-4 in t and n-2 in t_1, and is of the form

$$f\,(t^{2n-4},\ t_1^{n-2}) = 0.$$

For two values of t_1 to be in a given involution is a condition of degree n-3 in the coefficients of t_1 and hence of degree $2(n-2)(n-3)$ in t; this then is the number of common lines for case 2). Subtracting the common lines for case 1) and case 2) from the total number we have for case 3) $2(n-1)^2 - 2(n-2)(n-3) - 2 = 6n-12$. But since these pair off we have in general $3n-6$ contacts of R^n and r^{n-1}.

The order of r^{n-1} is $2n - 4$, so the R^n and its involution curve intersect in $2n(n-2)$ points. The contacts count for $6n - 12$ intersections, so there are $2(n-2)(n-3)$ remaining intersections.

If the parameters of a node of R^n are in the involution, then the node is a factor of the involution curve, and the remaining factor is an r^{n-2}. This part of the locus being a double point of the R^n will count for two contacts;

hence the remaining part, r^{n-2}, will have only $3n - 8$ contacts. If n-3 nodes are made a part of the locus then we always get for the remaining part of the locus an r^2 with n contacts or full contact, that is a conic all of whose intersections are contacts. Since two sets of the involution determine the involution it is a condition on the R^n for n-3 nodes to be in the involution for n greater than 5.*

We shall now consider the R^3 and find the involution conic. We found that there are in general $3n$-6 contacts, so in this case we get three contacts and no extra intersections. Let the double points of the involution be 0 and ∞, and let the reference triangle be the tangents at the double points and their join. Then the equation of R^3 may be

$$\begin{aligned} x_0 &= a_0t^3 + b_0t^2 \\ x_1 &= c_1t + d_1 \\ x_2 &= b_2t^2 + c_2t. \end{aligned} \tag{1}$$

If 0 and ∞ are the double points of the involution it is of the form

$$t_1 + t_2 = 0, \tag{2}$$

and the line whose locus is the involution conic will join t and $-t$ of the R^3. Such a line is given by

$$\begin{vmatrix} x_0 & , & x_1 & , & x_2 \\ a_0t^3 + b_0t^2 & , & c_1t + d_1 & , & b_2t^2 + c_2t \\ -a_0t^3 + b_0t^2 & , & -c_1t + d_1 & , & b_2t^2 - c_2t \end{vmatrix} = 0.$$

This determinant is readily seen to reduce to the following:

$$\begin{vmatrix} x_0 & , & x_1 & , & x_2 \\ b_0t^2 & , & d_1 & , & b_2t^2 \\ a_0t^2 & , & c_1 & , & c_2 \end{vmatrix} = 0 \tag{4}$$

which is parametrically

$$\begin{aligned} \xi_0 &= -b_2c_1t^2 + c_2d_1 \\ \xi_1 &= a_0b_2t^4 - b_0c_2t^2 \\ \xi_2 &= (b_0c_1 - a_0d_1)\, t^2. \end{aligned} \tag{5}$$

---

*In a Desargues Configuration B there is a sextic with nodes at the ten points of the Configuration. Any three nodes on a line would be in an involution, hence we could get ten conics having full contact with the sextic.

It is seen that only even powers of t occur so we replace the parameter t^2 by a new one, t' say. For convenience we drop the primes and write the equation in the form

$$
(6) \qquad \begin{aligned}
\xi_0 &= -b_2c_1t + c_2d_1 \\
\xi_1 &= a_0b_2t^2 - b_0c_2t \\
\xi_2 &= (b_0c_1 - a_0d_1)\, t
\end{aligned}
$$

This is the involution conic in line form but we want the point form. We have

$$
(7) \qquad \begin{aligned}
x_0 &= a_0b_2\,(a_0d_1 - b_0c_1)\, t^2 \\
x_1 &= c_2d_1\,(a_0d_1 - b_0c_1) \\
x_2 &= -a_0b_2c_1t^2 + 2\,a_0b_2c_2d_1t - b_0c_2^2d_1.
\end{aligned}
$$

Now in order to get the intersections of this involution conic with the R^3 we must eliminate the parameter from the equation of the conic and thus get an equation of the second degree in x. If we then substitute for the x_i's their values in the equation of the R^3 we obtain a sextic in t which will give the intersections of the two curves.

Eliminating t from (7) we get

$$
(8) \qquad \begin{aligned}
& b_2^2c_1^2x_0^2 + b_0^2c_2^2x_1^2 + (a_0^2d_1^2 - 2\,a_0b_0c_1d_1 + b_0^2c_1^2)\, x_2^2 \\
& + 2\,(a_0b_0c_2d_1 - b_0^2c_1c_2)\, x_1x_2 + 2\,(a_0b_2c_1d_1 - b_0b_2c_1^2)\, x_0x_2 \\
& \qquad + 2\,(b_0b_2c_1c_2 - 2\,a_0b_2c_2d_1)\, x_0x_1 = 0.
\end{aligned}
$$

If we now substitute for the x_i's their values in equation (1) we get

$$
(9) \qquad \begin{aligned}
& a_0^2b_2^2c_1^2t^6 + 2\,a_0^2b_2^2c_1d_1t^5 + (a_0^2b_2^2d_1^2 - 2a_0^2b_2c_1c_2d_1)\, t^4 \\
& - (2\,a_0^2b_2c_2d_1^2 - 2\,a_0b_0b_2c_1c_2d_1)\, t^3 + (a_0^2c_2^2d_1^2 - 2\,a_0b_0b_2c_2d_1^2)\, t^2 \\
& \qquad + 2\,a_0b_0c_2^2d_1^2t + b_0^2c_2^2d_1^2 = 0.
\end{aligned}
$$

This sextic is seen to be the square of the cubic

$$
(10) \qquad a_0b_2c_1t^3 + a_0b_2d_1t^2 - a_0c_2d_1t - b_0c_2d_1 = 0,
$$

which gives the parameters of the three points of contact of the R^3 and its involution conic.

We shall now prove that *the points of contact of an R^3 and its involution conic are given by the Jacobian of the cubic giving the parameters of the three flexes of R^3 and the quadratic which gives the double points of the involution.*

We shall consider the R^3 given by equation (1), and the involution whose double points are 0 and ∞. The cubic giving the flexes is the fundamental cubic, that is, the unique cubic apolar to each of the three binary cubics in (1).

Calculating that cubic in the usual way we have

(11) $$a_0 b_2 c_1 t^3 + 3\, a_0 b_2 d_1 t^2 + 3\, a_0 c_2 d_1 t + b_0 c_2 d_1 = 0.$$

The quadratic giving the double points under consideration is

(12) $$t = 0.$$

The Jacobian of (11) and (12) is

(13) $$a_0 b_2 c_1 t^3 + a_0 b_2 d_1 t^2 - a_0 c_2 d_1 t - b_0 c_2 d_1 = 0,$$

and is just the same cubic as (10), and the theorem is proved.

We shall now consider R^4 and its involution cubic r^3. The number of contacts we found to be in general $3n$-6 which is just the number of flexes of an R^n. Since the Jacobian of the flex cubic and the quadratic of the double points of the involution gave the contacts of R^3 and r^2, it seems natural to look for some such relation in the case of R^4. The Jacobian of the flex equation in general and the quadratic giving the roots of the involution will always be of the right degree in t, $3n$-6, to give the points of contact of R^n and r^{n-1}. But in the case of R^4 we find the degree in the coefficients not the same as those of the contact equation. We shall find by a symbolic method the degree of the contact equation in the coefficients of the fundamental involution, as well as in the coefficients of the quadratic of the involution.

Suppose the fundamental involution of R^4 is given by

(1) $$(\alpha t)^4 + \lambda\, (\beta t)^4 = 0.$$

Let the double points of the involution be $(Qt)^2 = 0$, and let one set of the involution t_1 and t_2 be given by $(a\, t)^2$. Let the line on t_1 and t_2 meet the R^4 again at T_1 and T_2.

Since every line section is apolar to the fundamental involution we have

(2) $$|\alpha a|\ (\alpha\, T_1)\ (\alpha\, T_2) = 0 \quad \text{and}$$

(3) $$|\beta a|\ (\beta T_1)\ (\beta T_2) = 0.$$

Eliminating T_2 from (2) and (3) we get

(4) $\quad |\alpha\, \beta|\, |\alpha\, a|^2\, |\beta\, a_1{}^2|\, (\alpha\, T_1)\, (\beta T_1) = 0.$

Since every set of the involution is apolar to $(Qt)^2$, we have,

(5) $\quad |aQ|^2 = 0.$

Again, since when t_1 or t_2 is T_1, there is a contact, we have

(6) $\quad (a\, T_1)^2 = 0.$

Solving for the a's in (5) and (6) we find them to be of the first degree in the Q's and of the second degree in the T's. If these values of the a's are put in (4) we get an equation of the first degree in the determinants of the fundamental involution, of the first degree in the Q's, and of the sixth degree in T. This is the contact equation.

The flex sextic is the first transvectant of the fundamental involution and is of the first degree in the determinants of the fundamental involution. The Jacobian of the flex sextic F and the quadratic Q giving the roots of the involution is a sextic J_1 which is of the first degree in the determinants of the fundamental involution, but only of the first degree in the Q's. Taking the Jacobian of J_1 and Q we get a sextic J_2 which is of the second degree in the Q's and the first degree in the determinants of the fundamental involution.

Now we propose to show that *K, the sextic giving the points of contact of R^4 and r^3, can be built from F, Q, J_2, Δ, and q, where F, Q, J_2 have the meaning just given, and where Δ is the discriminant of Q, and q is the third transvectant of the two members of the fundamental involution.* The possible combinations that are of the same degree as K are easily seen. We shall show that

$$K = \lambda\, \Delta\, F + \mu\, J_2 + v\, q\, Q^2,$$

where λ, μ, v are constants to be determined. Let the R^4 be referred to two flex tangents and the line joining these flexes whose parameters are 0 and ∞. Its parametric equation will be

$$\text{(1)}\qquad \begin{aligned} x_0 &= a\,t^4 + b\,t^3 + c\,t^2 \\ x_1 &= b\,t^3 + c\,t^2 + d\,t \\ x_2 &= c\,t^2 + d\,t + e \end{aligned}$$

We shall choose the involution whose sets are t and $-t$, hence whose double points are given by

(2) $\qquad Q \equiv 2t = 0.$

Since we are not interested in the equation of r^3, we proceed to find the equation giving the points of contact with the R^4. Calculating the fundamental involution of R^4, that is two quartics which are apolar to the three binary quartics in (1), we get

(3) $\qquad bct^4 - 6bet^2 - 4cet = 0,$ and

(4) $\qquad 4act^3 + 6adt^2 - cd = 0.$

The polarized form of (3) and (4), that is where S refers to t_1, t_2, t_3, t_4, is

(5) $\qquad bcS_4 - beS_2 - ceS_1 = 0$ and

(6) $\qquad acS_3 + adS_2 - cd = 0,$

which is known to be the condition that four points lie on a line. Now let two of the t's, say t_1 and t_2 be equal to t, and let σ refer to t_3 and t_4. Then (5) and (6) become

(7) $\qquad (bct^2 - be)\,\sigma_2 - (2bet + ce)\,\sigma_1 - bet^2 - 2cet = 0,$

(8) $\qquad (2act + ab)\,\sigma_2 + (act^2 + 2adt)\,\sigma_1 + adt^2 - cd = 0.$

Taking also the equation

(9) $\qquad \sigma_2 - \tau\sigma_1 + \tau^2 = 0,$

and eliminating the σ's from (7), (8), (9) we have

(10) $$\begin{vmatrix} bct^2 - be, & -2bet - ce, & -bet^2 - 2cet \\ 2act + ad, & act^2 + 2adt, & adt^2 - cd \\ 1, & -\tau, & \tau^2 \end{vmatrix} = 0.$$

This is an equation which is obviously the equation giving the parameters of the six flexes when τ is t, and giving the parameters of the six points of contact of R^4 and r^3 when τ is $-t$. Putting $\tau = t$ and developing (10) we get

(11) $$F \equiv abc^2t^6 + 3abcdt^5 + 6abcet^4 + (8ac^2e - bc^2d)t^3 + 6acdet^2 + 3bcdet + c^2de = 0.$$

If $\tau = -t$, (10) becomes

(12) $$K \equiv abc^2t^6 + abcdt^5 + 2abcet^4 + bc^2dt^3 + 2acdet^2 + bcdet + c^2de = 0.$$

The Jacobian of F and Q is

(13) $$J_1 \equiv abc^2t^6 + 2abcdt^5 + 2abcet^4 - 2acdet^2 - 2bcdet - c^2de = 0.$$

The Jacobian of J_1 and Q is

(14) $$J_2 \equiv 3abc^2t^6 + 4abcdt^5 + 2abcet^4 + 2acdet^2 + 4bcdet + 3c^2de = 0.$$

The quadratic q is the third transvectant of the members of the fundamental involution.

Taking the third transvectant of (3) and (4) we have

(15) $$q \equiv 3abcet^2 + (2ac^2e + bc^2d)t + 3acde = 0.$$

Forming the product of q and Q^2, we get

(16) $$qQ^2 \equiv 12abcet^4 + (8ac^2e + 4bc^2d)t^3 + 12acdet^2 = 0.$$

The discriminant of Q is

(17) $$\Delta = 1.$$

Writing down now

$$K = \lambda\Delta F + \mu J_2 + \nu q Q^2,$$

we find that K is given for

$$\lambda = -1/5, \mu = 2/5, \nu = 1/5.$$

Or to avoid fractions we have finally

(18) $$5K = qQ^2 + 2J_2 - \Delta F.$$

If two nodes are in the involution, these two nodes are a factor of the r^3 and the remaining factor is some other point. We shall show *that the remaining factor of r^3 is the third node.*

Let the parameters of one node be given by $t^2 + a$, a second by $t^2 + b$, and the third by a general quadratic $c_0 t^2 + c_1 t + c_2$. The R^4 referred to its nodes has the equation

$$
\begin{aligned}
x_0 &= (t^2 + a)\,(c_0 t^2 + c_1 t + c_2) \\
x_1 &= (t^2 + b)\,(c_0 t^2 + c_1 t + c_2) \qquad (19) \\
x_2 &= (t^2 + a)\,(t^2 + b)
\end{aligned}
$$

The sets of the involution are, if two nodes are in it, t and $-t$. The equation of a line joining t and $-t$ is given by the determinant

$$
\begin{vmatrix}
x_0, & x_1, & x_2 \\
(t^2+a)\,(c_0t^2+c_1t+c_2), & (t^2+b)\,(c_0t^2+c_1t+c_2), & (t^2+a)\,(t^2+b) \\
(t^2+a)\,(c_0t^2-c_1t+c_2), & (t^2+b)\,(c_0t^2-c_1t+c_2), & (t^2+a)\,(t^2+b)
\end{vmatrix} = 0.
$$

If now we take the sum of the second and third rows for a new second row, and their difference for a new third row, and remove the factor c_1t from the third row, we have

$$
(20) \qquad
\begin{vmatrix}
x_0 & , & x_1 & , & x_2 \\
(t^2 + a)\,(c_0 t^2 + c_2) & , & (t^2 + b)\,(c_0 t^2 + c_2) & , & (t^2 + a)\,(t^2 + b) \\
t^2 + a & , & t^2 + b & , & 0
\end{vmatrix} = 0.
$$

Replacing t^2 by T and expressing this equation in terms of ξ_i's we have after removing common factors

$$
\begin{aligned}
\xi_0 &= -(T + b) \\
\xi_1 &= T + a \qquad (21) \\
\xi_2 &= 0
\end{aligned}
$$

which shows on the face of it the other node to be the rest of r^3.

§ 2. *Involutions Determined by Two Double Lines of the Plane Rational Quartic.*

If lines are drawn on the meet of any two double lines of the rational quartic, we obtain a quadratic involution. That is to say, while such a line meets the curve in four points the parameters pair off, t_1 and t_2 say.

By choosing 0 and ∞ as the points of contact of one double tangent we may write the curve

$$(1) \qquad \begin{aligned} x_0 &= 4t^2 \\ x_1 &= (a_1 t^2 + 2 b_1 t + c_1)^2 \\ x_2 &= (a_2 t^2 + 2 b_2 t + c_2)^2 . \end{aligned}$$

Any line on the meet of x_0 and x_1 is of the form $x_0 - \lambda^2 x_1 = 0$, or

$$(2) \qquad 4t^2 - \lambda^2 (a_1 t^2 + 2 b_1 t + c_1)^2 = 0$$

which breaks into factors

$$(3) \qquad [2t - \lambda (a_1 t^2 + 2 b_1 t + c_1)] [2t + \lambda (a_1 t^2 + 2 b_1 t + c_1)] = 0.$$

If t_1 is a root of the first factor, then

$$(4) \qquad \lambda = \frac{2t_1}{a_1 t^2 + 2 b_1 t + c_1}$$

Substituting this value we have, after removing the factor $t - t_1$,

$$(5) \qquad I_{(0,1)} \equiv a_1 t_1 t_2 - c_1 = 0,$$

and this is the quadratic involution from the meet of x_0 and x_1.

We shall denote the double lines by 0, 1, 2, 3, and $I_{(i,j)}$ will denote the quadratic involution obtained by drawing lines on the meet of any two double lines i and j.

We shall show *that the double points of* $I_{(0,1)}$ *are given by the points of contact of the two remaining tangents from the meet of the double lines* 0 *and* 1.

The double points of $I_{(0,1)}$ are given by

$$(6) \qquad a_1 t^2 - c_1 = 0.$$

It is easily seen that the Jacobian of the quadratics which give the points of contact of the double lines give the points of contact of the two remaining tangents from their meet; for the Jacobian of two squared quadratics is the product of the quadratics and their Jacobian. Forming the Jacobian of the double lines 0 and 1 we have

$$(7) \qquad a_1 t^2 - c_1 = 0 ,$$

which gives precisely the roots of $I_{(0,1)}$.

We shall next prove *that the points of contact of the two tangents that may be drawn to the quartic from the meet of any two double lines are on a line through the meet of the other two double lines.* Having proved that the roots of the involution $I_{(0,1)}$ are the points of contact of the other two tangents from the meet of 0 and 1, we have only to show that the points of contact of tangents from the meet of the double lines 2 and 3 are in the involution $I_{(0,1)}$, that is to say, that the roots of $I_{(2,3)}$ are in $I_{(0,1)}$.

Not having the equation of the double line 3, we make use of the well-known fact that the three catalectic sets of the fundamental involution give the three sets of two tangents from the meets of double lines, such as from 0 and 1, and 2 and 3. The fundamental involution of the quartic given by (1) takes the form

$$(8) \qquad |a_1 b_1 b_2 c_2| t^4 + |a_1 b_1 c_2^2| t^3 + |b_2 c_2 c_1^2| t + \lambda (|a_1 b_1 a_2^2| t^3 + |b_2 c_2 a_1^2| t + |a_1 b_1 b_2 c_2|) = 0,$$

where $|a_1 b_1 b_2 c_2|$ denotes the determinant

$$\begin{vmatrix} a_1 b_1 , & a_2 b_2 \\ b_1 c_1 , & b_2 c_2 \end{vmatrix}, \qquad \text{and } |a_1 b_1 c_2^2| \text{ is } \qquad \begin{vmatrix} a_1 b_1 , & a_2 b_2 \\ c_1^2 , & c_2^2 \end{vmatrix}$$

and so on.

Writing down the g_3 of (8) we have

$$(9) \quad \begin{vmatrix} |a_1 b_1 b_2 c_2| & , |a_1 b_1 c_2^2| + \lambda |a_1 b_1 a_2^2|, & 0 \\ |a_1 b_1 c_2^2| + \lambda\, |a_1 b_1 a_2^2|, & 0 & , |b_2 c_2 c_1^2| + \lambda |b_2 c_2 a_1^2| \\ 0 & , |b_2 c_2 c_1^2| + \lambda |b_2 c_2 a_1^2|, & \lambda\, |a_1 b_1 b_2 c_2| \end{vmatrix} = 0.$$

This is a cubic in λ whose roots are at once found to be

$$-\frac{c_1^2}{a_1^2}, \quad -\frac{c_2^2}{a_2^2}, \quad -\frac{(b_1 c_2 - b_2 c_1)^2}{(a_1 b_2 - a_2 b_1)^2}.$$

Substituting $\lambda = \dfrac{c_1^2}{a_1^2}$ in (8) we get, after removing the factor $b_1(a_1 c_2 - a_2 c_1)$,

$$(10) \qquad a_1^2 b_2 t^4 + a_1 (a_1 c_2 + a_2 c_1) t^3 - c_1 (a_1 c_2 + a_2 c_1) t - b_2 c_1^2 = 0.$$

This factors into

$$(11) \qquad [a_1 t^2 - c_1] [a_1 b_2 t^2 + (a_1 c_2 + a_2 c_1) t + b_2 c_1] = 0,$$

the first factor giving the points of contact of tangents from (0, 1) and the second factor giving the points of contact of the two tangents from (2, 3), where (0, 1) means the meet of 0 and 1, and so on. We have now to show that the roots of

$$(12) \qquad a_1 b_2 t^2 + (a_1 c_2 + a_2 c_1) t + b_2 c_1 = 0$$

are in $I_{(0,1)}$. The only condition necessary is that the product of the roots shall be c^1/a_1, and this is obviously so in the case of (12).

We get $I_{(2,3)}$ by polarizing (12). Thus

$$(13) \qquad I_{(2,3)} \equiv 2 a_1 b_2 t_1 t_2 + (a_1 c_2 + a_2 c_1)(t_1 + t_2) + 2 b_2 c_1 = 0.$$

It is then also readily seen that the roots of $I_{(0,1)}$ are a set of $I_{(2,3)}$. The fact that the double points of $I_{(0,1)}$ are in $I_{(2,3)}$ and also that the double points of $I_{(2,3)}$ are a set of $I_{(0,1)}$ says the two involutions are commutative, that is $I_{(0,1)} I_{(2,3)} = I_{(2,3)} I_{(0,1)}$. Thus there are a single infinity of four-points on the curve for which (0, 1) and (2,3) are diagonal points. If we allow this four-point to run around the curve, (0, 1) and (2, 3) will be fixed diagonal points and the third diagonal point will have a locus. There will be three such loci corresponding to the three ways in which we may pair off the double lines. This question will be considered in a subsequent paragraph of this paper.

We have proved that *the points of contact of the two remaining tangents from the meet of any two double lines lie on a line through the meet of the other two.* There are then six such lines and we shall prove *they are on four points.* We shall denote by $L_{(0,1)}$ the line on the points of contact of tangents from (0, 1), that is from the meet of the double lines 0 and 1. The other five lines are similarly named. It is obvious that if three lines are on a point they must be such as $L_{(0,1)}$, $L_{(0,2)}$ $L_{(0,3)}$. To get these three lines we need the double points of $I_{(0,1)}$, $I_{(0,2)}$ and $I_{(0,3)}$. We have found the roots of $I_{(0,1)}$ to be

(14) $$a_1 t^2 - c_1 = 0.$$

From symmetry the roots of $I_{(0,2)}$ are

(15) $$a_2 t^2 - c_2 = 0.$$

Now finding the roots of $I_{(1,2)}$ and again making use of the catalectic sets we find the roots of $I_{(0,3)}$ to be

(16) $$(a_1 b_2 - a_2 b_1) t^2 - (b_1 c_2 - b_2 c_1) = 0.$$

The roots in these three involutions are given by equations with no middle term; hence the three lines wanted are in each case on parameters t and $-t$. From equation (1) we get the line joining t and $-t$ as the determinant

(17) $$\begin{vmatrix} x_0 , & x_1 , & x_2 \\ 4t^2 , & (a_1 t^2 + 2 b_1 t + c_1)^2 , & (a_2 t^2 + 2 b_2 t + c_2)^2 \\ 4t^2 , & (a_1 t^2 - 2 b_1 t + c_1)^2 , & (a_2 t^2 - 2 b_2 t + c_2)^2 \end{vmatrix} = 0.$$

Expanding and removing extraneous factors and placing $t^2 = c_1/a_1$ we have

(18) $$L_{(01)} \equiv [\, a_1^2 b_1 c_2^2 + a_2^2 b_1 c_1^2 - 2a_1^2 b_2 c_1 c_2 - 2a_1 a_2 b_2 c_1^2 + 2\, a_1 a_2 b_1 c_1 c_2 + 4\, a_1 b_1 b_2^2 c_1 - 2\, a_1 b_1^2 b_2 c_2 - 2\, a_2 b_1^2 b_2 c_1]\, x_0 + [\, 2\, b_2 (a_1 c_2 + a_2 c_1)]\, x_1 - 4\, a_2 b_1 c_1 x_2 = 0.$$

If $t^2 = c_2/a_2$ we have

(19) $$L_{(0,2)} \equiv [\, -a_1^2 b_2 c_2^2 - a_2^2 b_2 c_1^2 + 2a_2^2 b_1 c_1 c_2 + 2a_1 a_2 b_1 c_2^2 - 2a_1 a_2 b_2 c_1 c_2 - 4\, a_2 b_1^2 b_2 c_2 + 2\, a_1 b_1 b_2^2 c_2 + 2\, a_2 b_1 b_2^2 c_1]\, x_0 + 4\, a_2 b_2 c_2 x_1 - [\, 2\, b_1 (a_1 c_2 + a_2 c_1)\,]\, x_2 = 0.$$

If $t^2 = (b_1 c_2 - b_2 c_1)/(a_1 b_2 - a_2 b_1)$ we have

$$\text{(20)} \qquad L_{(0,3)} \equiv b_2^2 x_1 - b_1^2 x_2 = 0.$$

If $L_{(0,1)}$, $L_{(0,2)}$, $L_{(0,3)}$ are on a point the determinant of their coefficients must vanish. The determinant may be written

$$\begin{vmatrix} b_1(a_1^2c_2^2+a_2^2c_1^2)-2a_1b_2c_1(a_1c_2+a_2c_1)+2a_1b_1c_2(a_2c_1-b_1b_2)+2b_1b_2c_1(2a_1b_2-a_2b_1), & a_1c_2+a_2c_1, & 2a_1c_1 \\ -b_2(a_1^2c_2^2+a_2^2c_1^2)+2a_2b_1c_2(a_1c_2+a_2c_1)-2a_2b_2c_1(a_1c_2-b_1b_2)-2b_1b_2c_2(2a_2b_1-a_1b_2), & 2a_2c_2, & a_1c_2+a_2c_1 \\ 0, & b_2, & b_1 \end{vmatrix}$$

This expanded is readily seen to vanish which proves the theorem that *the six lines on the points of contact of tangents from the meets of any two double lines form a complete four-point.*

These six lines together with the four double lines form a Desargues Configuration B. That is we have two triangles perspective from a point and having homologous sides meeting in three collinear points.

We shall now study the four-point more in detail. The one point obtained was that determined by the lines $L_{(0,1)}$, $L_{(0,2)}$ and $L_{(0,3)}$. Thus the point is paired off with the double line 0. In the same way each of the four points is paired with a double line. Now there is reason to believe from other considerations, that these points are in some way related to the Stahl conic N, which is the locus of the flex lines of cubic osculants of the rational quartic. We shall show that *the four points are the polar points of the four double lines as to the conic N.*

If the quartic is written

$$\text{(21)} \qquad x_i = a_i t^4 + 4 b_i t^3 + 6 c_i t^2 + 4 d_i t + e_i, \; [i = 1, 2, 3],$$

it is known that N takes the form

$$\text{(22)} \qquad -36(b\,c\,x)(c\,d\,x) + 12(a\,d\,x)(c\,d\,x) + 12(b\,e\,x)(b\,c\,x) + 4(b\,d\,x)^2 + (a\,e\,x)^2 + 4(a\,b\,x)(d\,e\,x) - 8(a\,d\,x)(b\,e\,x) = 0,$$

where $(b\,c\,x) = \begin{vmatrix} b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \\ x_0 & x_1 & x_2 \end{vmatrix}$, and so on.

Taking the quartic as given by (1) N is

$$(23)\quad x_0^2[-4\,|a_1b_1(a_2c_2+2b_2^2)|\,|b_2c_2(a_1c_1+2b_1^2)|+4\,|b_2c_2(a_1c_1+2b_1^2)|\,|a_1^2b_2c_2|$$
$$+4\,|a_1b_1(a_2c_2+2b_2^2)|\,|a_1b_1c_2^2|+4\,|a_1b_1b_2c_2|^2+|a_1^2c_2^2|^2$$
$$+4\,|a_1^2a_2b_2|\,|b_1c_1c_2^2|-8\,|a_1^2b_2c_2|\,|a_1b_1c_2^2|\,]$$
$$+x_1^2[16a_2b_2^2c_2]+x_2^2[16a_1b_1^2c_1]+x_1x_2[-16b_1b_2(a_1c_2+a_2c_1)]$$
$$+x_0x_2[-8\,b_1c_1|a_1b_1(a_2c_2+2b_2^2)|+8\,a_1b_1\,|b_2c_2(a_1c_1+2b_1^2)|$$
$$+8\,b_1c_1\,|a_1^2b_2c_2|-8\,a_1b_1\,|a_1b_1c_2^2|\,]$$
$$+x_0x_1[8\,b_2c_2\,|a_1b_1(a_2c_2+2b_2^2)|-8\,a_2b_2\,|b_2c_2(a_1c_1+2b_1^2)|$$
$$-8\,b_2c_2\,|a_1^2b_2c_2|+8\,a_2b_2\,|a_1b_1c_2^2|\,]=0,$$

where $|a_1b_1(a_2c_2+2b_2^2)|=a_1b_1(a_2c_2+2b_2^2)-a_2b_2(a_1c_1+2b_1^2)$, and so on.

The coordinates of the point in question are found by getting the intersection of any two of the three lines, say $L_{(0,1)}$ and $L_{(0,3)}$. We have at once

$$(24)\quad \begin{aligned} x_0 &= 2\,b_1b_2(c_1\,|a_1b_2|-a_1\,|b_1c_2|) \\ x_1 &= b_1^2[2\,c_1|a_1b_1(a_2c_2+2\,b_2^2)|+2\,b_1^2b_2|a_2c_1|+a_1^2c_2|b_1c_2|+c_1|a_1^2b_1c_1|] \\ x_2 &= b_2^2[2\,c_1\,|a_1b_1(a_2c_2+2\,b_2^2)|+2\,b_1^2b_2|a_2c_1|+a_1^2c_2|b_1c_2|+c_1\,|a_1^2b_1c_1|] \end{aligned}$$

where the expressions within the bars have the same meaning as in (23). The question now is whether this point and the double line o are pole and polar with regard to N. To prove this we only have to find the derivative of N as to x_1, and then as to x_2, and see if the two resulting lines pass through the point represented by (24).

$$(25)\quad \begin{aligned} Dx_1N \equiv\; & x_0[b_2c_2|a_1b_1(a_2c_2+2\,b_2^2)|-a_2b_2\,|b_2c_2(a_1c_1+2\,b_1^2)| \\ & -b_2c_2\,|a_1^2b_2c_2|+a_2b_2\,|a_1b_1c_2^2|\,] \\ & +x_1[4\,a_2b_2^2c_2]+x_2[-2\,a_1b_1b_2c_2-2\,a_2b_1b_2c_1]=0. \end{aligned}$$

$$(26)\quad \begin{aligned} Dx_2N \equiv\; & x_0[-b_1c_1\,|a_1b_1(a_2c_2+2\,b_2^2)|+a_1b_1\,|b_2c_2(a_1c_1+2b_1^2)| \\ & +b_1c_1\,|a_1^2b_2c_2|-a_1b_1\,|a_1b_1c_2^2|\,] \\ & +x_1[-2\,a_1b_1b_2c_2-2\,a_2b_1b_2c_1]+x_2[4\,a_1b_1^2c_1]=0. \end{aligned}$$

Substituting the coordinates of the point given by (24) in (25) and (26) we find that each of them is satisfied, and the theorem is proved.

We have seen that there is a single infinity of four-points on the curve for which (0,1) and (2,3) are fixed diagonal points, and we now want to find *the locus of the third diagonal point.* Dr. J. R. Conner has suggested and kindly given me the proof that the projection of the intersection of two circular cylinders touching and intersecting at right angles is a general rational quartic. The general rational plane quartic may be considered as a projection of a space quartic with a node.

A quartic in space is the intersection of two quadric cones. Let A_1 and B_1 be the vertices of two quadric cones on the curve. Choose the plane at infinity as a plane on A_1 B_1. This plane meets each cone in a pair of lines. Take the absolute as a conic touching the four lines and apolar to the pair of points A_1 and B_1. Then the curve is the intersection of two circular cylinders touching and intersecting at right angles*

We shall denote this space quartic with one node by the symbol S^4, and the plane rational quartic into which S^4 projects by R^4. Now consider S^4. Take the line normal to the two cylinders at the node. The osculating planes of the two branches through the node contain this line. There is a single infinity of planes on this line, each of which meets S^4 in two points other than the node. We have thus a quadratic involution. Its double points will be given when the plane osculates one of the branches at the node; that is to say the nodal parameters give the double points of the involution. That tells us that all these planes cut out from S^4 quadratics apolar to the quadratic giving the node. Projecting from a point M of space, the two tangent planes to each cylinder from M go into the four double lines of R^4, thus giving two pairs of double lines. The involutions, which we have previously discussed, on the meets of pairs of double lines of R^4 are cut out of S^4 by the generators of the cylinders. Hence the double points of such an involution are given by the generators that touch S^4. It is then easy to see that the parameters of the isolated node are apolar to the double points of each involution; that is the Jacobian of these pairs of points gives the nodal parameters. Also it is here obvious that the double points of one involution are a set of the other, and that the node is in both involutions. This shows again the paring off of the double lines by choosing a node.

The involutions on the meets of double lines of R^4 are the projections of points cut out on S^4 by the generators of the cylinder. We have on S^4 a

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*Cf. Marletta: *Studio geometrico della quartica golba razionale. Annali di Ma. Series 3, Vol. 8, pp. 111 ff.*

system of corners of rectangles cut out by planes on A_1 and B_1; these groups of four points on R^4 are obtained from any pair of involutions, such as $I_{(0,1)}$ and $I_{(2,3)}$. We may remark in passing that it is obvious that the parameters of the groups of four points of which we are speaking are a pencil of quartics,—precisely a syzygetic pencil, being built on a quartic and its Hessian—,as it is easily seen from the space figure that the pencil contains three perfect squares.

The diagonals of the rectangles intersect in a point whose locus is just that of the third diagonal point which we started out to find. It is seen that *this locus is a line, and it is the normal to the cylinders at the node.* This line projects into a line in the plane which passes through a node and cuts out a pair of parameters from R^4 harmonic to the nodal parameters. It is evident there is only one such line for each node. We thus get three such lines, and Professor Morley has proved that these three lines are on a point. We have then the theorem: *With each pairing of the double lines of R^4 we obtain groups of four-points on R^4, which have the meets of the pairs of double lines as fixed diagonal points, and whose third diagonal point has for a locus the line on the isolated node and meeting R^4 in harmonic pairs.*

§ 3. *The Case with Three Flex Tangents on a Point.*

It is well known that the locus of lines which meet a rational quartic in sets of four self apolar points is a conic, g_2. In particular the flex tangents are such lines; hence the six flex tangents touch the g_2 conic. Now if g_2 breaks up, it breaks up into two points. Then three flex tangents are on one point and three are on the other necessarily.

First take a quartic curve referred to two flex tangents and the line joining the two flexes:

$$(1)\qquad \begin{aligned} x_0 &= a\,t^4 + 4\,b\,t^3 \\ x_1 &= 4\,b\,t^3 + 6\,c\,t^2 + 4\,d\,t \\ x_2 &= 4\,d\,t + e \end{aligned}$$

We now have two flex tangents on the meet of x_0 and x_2. If $a = 2\,b$, and $e = 2\,d$ we shall have a third on that point, and the curve takes the form:

$$(2)\qquad \begin{aligned} x_0 &= 2\,b\,t^4 + 4\,b\,t^3 \\ x_1 &= 4\,b\,t^3 + 6\,c\,t^2 + 4\,d\,t \\ x_2 &= 4\,d\,t + 2\,d \end{aligned}$$

The three flexes have the parameters $0, \infty, -1$.

The cubic giving them is then

$$(3)\qquad t^2 + t = 0.$$

The flex tangents meet the curve again, and the cubic giving the parameters of these three points is

$$(4)\qquad 2\,t^3 + 3\,t^2 - 3\,t - 2 = 0,$$

which is seen to be the cubicovariant of the flexes given by (3).

We wish to find the cubic giving the parameters of the other three flexes. The Jacobian of the two members of the fundamental involution gives the sextic of flexes. Calculating the fundamental involution of (2) we have

(5) $\qquad 2bt^2 + 2ct + c = 0,$ and

(6) $\qquad ct^4 + 2ct^3 + 2dt^2 = 0.$

The Jacobian of (5) and (6) is

(7) $\qquad 2bt^5 + (3c + 2b)t^4 + 6ct^3 + (3c + 2d)t^2 + 2dt = 0.$

The factors of (7) are

(8) $\qquad t^2 + t = 0,$ and

(9) $\qquad 2bt^3 + 3ct^2 + 3ct + 2d = 0.$

We shall speak of the two points on each of which are three flex tangents as the g_2 points, and the line on them as the g_2 line. Now the six tangents from one of these points is a cubic squared, say $[(\alpha t)^3]^2$, and the six on the other is say $[(\beta t)^3]^2$. The six tangents from any point on the g_2 line will be in the pencil

$$[(\alpha t)^3]^2 - \lambda^2[(\beta t)^3]^2 = 0, \text{ or}$$

(10) $\qquad [(\alpha t)^3 + \lambda(\beta t)^3][(\alpha t)^3 - \lambda(\beta t)^3] = 0.$

This shows that the tangents all along the g_2 line pair off into two sets of three. In our notation the pencil of cubics is

(11) $\qquad 2bt^3 + 3ct^2 + 3ct + 2d + \lambda(t^2 + t) = 0,$

which is readily seen to be a set of apolar cubics.

The equation of the g_2 line is $x_0 - x_2 = 0$. Hence the parameters of the four points in which this line meets the curve are given by

(12) $\qquad bt^4 + 2bt^3 - 2dt - d = 0.$

At these points two of the six tangents to the curve come together. The pairing of the tangents must be this tangent at the point with two of the remaining four, and this same tangent with the other two; or, the tangent at the point taken twice with one of the four, and then the remaining three paired. The latter is the case, as is proved by taking the Jacobian of the two members of (11) and obtaining (12). We note too that (12) is a self-apolar quartic which tells us again that the pencil of cubics are apolar, since they have a self-apolar quartic for Jacobian.

The pencil of cubics has a unique apolar quartic and its Hessian is the Jacobian of the cubics. Finding the quartic we have

(13) $$Q \equiv b^2 t^4 + 4 b d t^3 + 6 b d t^2 + 4 b d t + d^2 = 0.$$

The Hessian of this quartic is at once verified to be (12).

At the four points in which the g_2 line meets the curve two tangents come together and we have just seen that one other tangent is paired with this one taken twice. There are then four such tangents. They are given by the Steinerian of Q to which the system is apolar, for if the Hessian has a root that is a double root of the cubic, the other root of the cubic is a root of the Steinerian. The Steinerian of a quartic Q is known to be $g_3 Q + \lambda g_2 H = 0$, when g_2 and g_3 are the invariants of Q, and H is its Hessian. But g_2 of our Q is zero, so the Steinerian is the quartic over again.

We assert further that *Q is the quartic of which the system of cubics are the first polars.* Salmon tells us how to find the quartic when the cubics are given. It is $12 H (J) + g_2 J$, where J is the Jacobian of the cubics, H the Hessian of J, and g_2 the self apolarity condition of J. But

(14) $$J \equiv b t^4 + 2 b t^3 - 2 d t - d = 0,$$

and $g_2 = 0$, and

(15) $$H (J) \equiv b^2 t^4 + 4 b d t^3 + 6 b d t^3 + 4 b d t + d^2 = 0.$$

This is just Q over again.

Q is then the quartic to which the system of cubics is apolar, as well as the quartic of which they are the first polars, and besides it is its own Steinerian, and gives the points of contact of the four tangents which are paired with the tangents counted twice at the points where the g_2 line meets the curve.

Consider now the pencil of cubics. To any one of the pencil we have a definite corresponding point t_1 on the curve, viz. the point with regard to which the cubic is the polar of Q. But paired with that cubic there is a second cubic, and we get a point t_2 on the curve from it in a similar way. We have then a *quadratic involution* set up on the curve. Since the two cubics come together at the g_2 points, these points correspond to the double points of the involution. We can find the quadratics giving the double points in the following manner. The coefficients of the polarized form of Q must be proportional to the coefficients of the pencil of cubics. We can thus determine λ in terms of t_1.

Polarizing Q we have

$$(16)\qquad (b^2 t_1 + b\,d)\,t^3 + 3\,(b\,d\,t_1 + b\,d\,)\,t^2 + 3\,(b\,d\,t_1 + b\,d)\,t + b\,d\,t_1 + d^2 = 0.$$

The pencil of cubics is

$$(17)\qquad 2\,b\,t^3 + (\,3\,c + \lambda\,)\,t^2 + (\,3\,c + \lambda\,)\,t + 2\,d = 0,$$

hence

$$(18)\qquad \lambda = \frac{(6\,b\,d - 3\,b\,c\,)\,t_1 + 6\,b\,d - 3\,c\,d}{b\,t_1 + d}\;.$$

Since our base cubics are those at the g_2 points, the double points of the involution will be given for $\lambda = 0$, and $\lambda = \infty$; that is one is given by the numerator of (18) and the other by the denominator of (18).

We now look for this pair of points on the curve. The conic on the flexes meets the curve again in two other points q. If the fundamental involution of the curve is $(\alpha\, t)^4 + \lambda\,(\beta\, t)^4$ then

$$(19)\qquad q \equiv (\alpha\,\beta\,)^3\,(\,\alpha\,t\,)\,(\,\beta\,t\,) = 0.$$

Using the fundamental involution as given by (5) and (6) we get

$$(20)\qquad q \equiv (\,b - c\,)\,t^2 - c\,t + (\,d - c\,) = 0.$$

Now operating with q on the pencil of cubics we obtain

$$(21)\qquad \lambda = \frac{(\,3\,b\,c - 6\,b\,d\,)\,t_1 + 3\,c\,d - 6\,b\,d}{b\,t_1 + d}\;.$$

It is to be noticed that λ in (21) is just the negative of λ in (18). This shows a sort of cross working of the quartic of which the cubics are the first polars, and the quadratic q; that is if C_1 is the polar cubic as to t_1, and C_2 is the polar cubic as to t_2, then q operating on C_1 gives t_2 and q operating on C_2 gives t_1.

The double points of the quadratic involution are the points with regard to which the polars of the quartic are taken to give the two base cubics of the pencil, that is the flex cubics. Now at these double points we have a tangent from some point on the q_2 line; that is each of these belong to one of the cubics of the pencil. We seek the *relation of this cubic to the flex cubic.*

The polar of (13) as to the double point $b\,t + d$ gives the flex cubic

(22) $$t^2 + t = 0.$$

We can find the cubic to which $b\,t + d$ belongs by equating coefficients in the identity

(23) $$2\,b\,t^3 + (3\,c + \lambda)\,t^2 + (3\,c + \lambda)\,t + 2\,d \equiv (b\,t + d)\,(a_1 t^2 + a_2 t + a_3),$$

and we get

(24) $$(b\,t + d)(t^2 + t + 1) = 0.$$

The last factor gives the pair of tangents to which $b\,t + d$ belongs, and it is seen *to be the Hessian of* (22).

Now the cubic $t^2 + t = 0$ which we have considered is special only geometrically; it is one of the pencil and behaves as any other member of the pencil. So *any two of the three tangents given by* (24) *are the Hessian of some cubic of the pencil.* We look for the three cubics thus obtained from (24). The factored form of (24) is

(25) $$(b\,t + d)\,(t - w)\,(t - w^2) = 0.$$

We assert that *the three cubics, of which any two of the three tangents given by* (25) *are the Hessian, are just the cubics which are the polars of the quartic Q as to the three roots of* (25). The polar of Q as to $t = w$ is

(26) $$(b^2 w + b\,d)\,t^3 - 3\,b\,d\,w^2\,t^2 - 3\,b\,d\,w^2\,t + b\,d\,w + d^2 = 0.$$

The Hessian of (26) is

(27) $(bt + d)(t - w^2) = 0,$

and our assertion is proved. That is *we have a system of cubics which are the first polars of a quartic, such that if the roots of any one be,* t_1, t_2, t_3, *then the cubic which is the polar of the quartic as to any one of the* t's, *and is therefore one of the pencil, has the other two* t'*s for its Hessian.*

§ 4. *The Case with Three Double Lines on a Point.*

First of all we shall find the parametric equation of a quartic curve with a *syzygetic* point, that is with three double lines on a point. We may choose three points of the curve, say 0, ∞, 1. Let 0 and ∞ be the points of contact of one double tangent, say $x_1 = t^2$.
Let x_0 be a double tangent with points of contact 1 and α; then

$$x_0 = (t - 1)^2 (t - \alpha)^2.$$

If there is another double tangent on the meet of x_0 and x_1, it is of the form $x_0 + \lambda x_1 = 0$, and we find that $\alpha = 1$, and $\lambda = 4$.
The fourth double tangent will be written generally, and the equation of the curve is

$$(1) \qquad \begin{aligned} x_0 &= (t - 1)^2 (t + 1)^2 \\ x_1 &= t^2 \\ x_2 &= (t^2 - s_1 t + s_2)^2. \end{aligned}$$

By transformation we can write the curve in the better form

$$(2) \qquad \begin{aligned} x_0 &= t^4 + 1 \\ x_1 &= 6 t^2 \\ x_2 &= 4 t^3 + 4 s_2 t + m. \end{aligned} \qquad \left[m = -\frac{2(s_2{}^2 - 1)}{s_1} \right]$$

The three double lines on a point are

$$x_1 = 0, \quad 3 x_0 + x_1 = 0, \; 3 x_0 - x_1 = 0.$$

The Hessian of the three double lines is

(3) $$3x_0^2 + x_1^2 = 0.$$

The cubicovariant of the three double lines is

(4) $$x_0(x_1^2 - x_0^2) = 0.$$

If we cut (2) by any line

(5) $$(\xi x) \equiv \xi_0 x + \xi_1 x_1 + \xi_2 x_2 = 0,$$

we have a quartic in t

(6) $$\xi_0 t^4 + 4\xi_2 t^3 + 6\xi_1 t^2 + 4\xi_2 s_2 t + m\xi_2 + \xi_0 = 0.$$

Now if we form the two invariants, g_2 and g_3, of (6) we get a line conic and a line cubic.

We shall prove that *the lines to the g_2 conic from the syzygetic point are the Hessian pair of the three double lines on that point*; *and that the three lines to the g_3 cubic from the syzygetic point are the cubicovariant of the three double lines.*

Writing the g_2 of (6) we get

(7) $$\xi_0^2 + 3\xi_1^2 - 4s_2\xi_2^2 + m\xi_0\xi_2 = 0.$$

Put $\xi_2 = 0$ and we get the tangents to g_2 from the syzygetic point. Changing the result into points we have

(8) $$3x_0^2 + x_1^2 = 0.$$

Next writing the g_3 of (6) we have

(9) $$\xi_1^3 + m\xi_2^3 - \xi_0^2\xi_1 + (s_2^2+1)\xi_0^3\xi_2^2 + 2s_2\xi_1\xi_2^2 + m\xi_0\xi_1\xi_2 = 0.$$

Setting $\xi_2 = 0$, and changing the result into points we have as the tangents from the syzygetic point

(10) $$x_0(x_1^2 - x_0^2) = 0,$$

and our theorem is proved.

The Stahl conic N was defined in section 2 of this paper. The cubic osculant at a point τ of the quartic (2) is

$$(11)\qquad \begin{aligned} x_0 &= \tau t^3 + 1 \\ x_1 &= 3 t^2 + 3 \tau t \\ x_2 &= t^3 + 3 \tau t^2 + 3 s_2 t + s_2 \tau + m. \end{aligned}$$

Cut this cubic by two lines $(u x)$ and $(v x)$. Making them cut in sets of apolar points we get the equation

$$(12)\qquad x_1 (3 s_2 - 3 \tau^2) + x_2 (s_2 \tau^2 + m \tau - 1) = 0.$$

For a given τ this a line, the line of flexes of the cubic osculant at the point τ of the quartic. For a varying τ we get the locus of this line, which locus is the Stahl conic N. Its parametric equation is

$$(13)\qquad \begin{aligned} \xi_0 &= 3 s_2 - 3 \tau^2 \\ \xi_1 &= s_2 \tau^2 + m \tau - 1. \\ \xi_2 &= 0 \end{aligned}$$

Since $\xi_2 = 0$ all the flex lines of the cubic osculants pass through that point, which is the syzygetic point. That is *N is the syzygetic point counted twice.*

The cubic osculants at the points of contact of double tangents have interesting properties. Consider the double tangent with points of contact 1 and -1. The cubic osculant at $\tau = 1$ is

$$(14)\qquad \begin{aligned} x_0 &= t^3 + 1 \\ x_1 &= 3 t^2 + 3 t \\ x_2 &= t^3 + 3 t^2 + 3 s_2 t + s_2 + m. \end{aligned}$$

This curve passes through N and of course has a flex there. The flex tangent is

$$(15)\qquad x_0 + x_1 = 0.$$

The cubic osculant at $\tau = -1$ is

$$(16)\qquad \begin{aligned} x_0 &= - t^3 + 1 \\ x_1 &= 3 t^2 + 3 t \\ x_2 &= t^3 - 3 t^2 + 3 s_2 t - s_2 + m \end{aligned}$$

It also passes through N, and has then the same flex tangent as (14). Hence the theorem: *When three double lines are on a point the cubic osculants at the two points of contact of any one of these three double lines pass through the syzygetic point, each having a flex there, and both have the same flex tangent.*

The three such tangents are also the three tangents to g_3 from the syzygetic point.

We have seen that a line on the points of contact of the two tangents from the meet of any two double tangents passes through the meet of the other two double tangents. In the syzygetic case the six such lines are on the syzygetic point, three of them being the double lines themselves. The equations of the other three may be found by use of the catalectic sets of the fundamental involution.

Again the Jacobians of the factors of the fundamental involution give the nodes, and thus we can get the equation of the lines to the nodes from the syzygetic point.

We have then four sets of three lines on the syzygetic point, viz.,
The three double lines

$$\text{I}\qquad \begin{array}{ll} x_1 = 0 , & \text{(A)} \\ 3x_0 + x_1 = 0 , & \text{(B)} \\ 3x_0 - x_1 = 0 . & \text{(C)} \end{array}$$

The lines to g_3

$$\text{II}\qquad \begin{array}{ll} x_0 = 0 , & \text{(A)} \\ x_1 - x_0 = 0 , & \text{(B)} \\ x_1 + x_0 = 0 . & \text{(C)} \end{array}$$

The lines to the nodes

$$\text{III}\qquad \begin{array}{ll} 6s_2x_0 + (s_2^2 + 1)x_1 = 0 , & \text{(A)} \\ 3(s_2 - 1)^2x_0 - [2s_1^2 + (s_2 - 1)^2]x_1 = 0, & \text{(B)} \\ 3(s_2 + 1)^2x_0 - [2s_1^2 - (s_2 + 1)^2]x_1 = 0. & \text{(C)} \end{array}$$

Lines on the points of contact of tangents from the meets of the three double lines on the syzygetic point with the fourth double line

$$\text{IV}\qquad \begin{array}{ll} 6s_2x_0 - (s_2^2 + 1)x_1 = 0 , & \text{(A)} \\ 3s_1^2x_0 - [2(s_2 - 1_2)^2 + s_1^2]x_1 = 0, & \text{(B)} \\ 3s_1^2x_0 - [2(s_2 + 1)^2 - s_1^2]x_1 = 0. & \text{(C)} \end{array}$$

We have already seen that the *second set is the cubicovariant of the first set.* We say furthermore that *the lines marked (A) are harmonic*, because it is readily seen that they are of the form

$$(\alpha x) = 0, \ (\beta x) = 0, (\alpha x) + \lambda (\beta x) = 0, \ (\alpha x) - \lambda (\beta x) = 0.$$

The same is true of those marked (B), and of those marked (C).

In conclusion we may add that the study of the syzygetic case of the rational quartic is the same as that of a conic and four points such that conics on them cut out a syzygetic pencil from the given conic. For if we call the nodes d_0,d_1, d_2, and the syzygetic point N, and then invert the quartic into a conic α by the transformation $y_i = 1/x_i$, the four points all behave alike. The double lines become conics which touch the conic α twice. Three of these meet at N. The conic on d_0, d_1, d_2, and bitangent to α would correspond to the fourth double line. We have four points d_0, d_1, d_2, N, such that conics on them cut out a syzygetic pencil from α, that is such that three conics can be drawn on them bitangent to α.

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VITA

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