QUATERNIONS

AS THE

RESULT OF ALGEBRAIC OPERATIONS

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PREFACE

Beginners in the subject of Quaternions are generally bewildered by the arbitrary manner in which the subject is developed. They are forcibly introduced into a new domain where the familiar rules of combination of symbols are not valid. New magnitudes are arbitrarily assumed, subject to arbitrary laws. The reader finds the logic consistent and the results concordant with those of his previous courses, but he hardly knows why. He finds himself in a new country, but thoroughly and bewilderingly uncertain as to how he got there.

It is in the attempt to avoid this uncertain journey, to lead the student from the known to the unknown by familiar steps, by steps which require no arbitrary limitations of former laws, but merely their adaptation to new circumstances, that these class notes have grown into their present shape.

The backbone of the method of presentation is the use of a one-to-one correspondence between the mathematical concept and what I have ventured to call its idiographic symbol, that is, a symbol whose spatial properties are the same as the mathematical properties of the concept it symbolizes. From this similarity of properties there exists a one-to-one correspondence between the results of spatial operations upon the symbols and the corresponding mathematical operations upon the concept.

These idiographic symbols are strokes, spherical shells, and vectors, corresponding respectively to magnitudes having size, and correlated sense of opposition, scalars, and magnitudes having size, sense, and direction.

Spatial operations upon these symbols are used as suggestions for a one-to-one corresponding interpretation for for the mathematical concept.

These spatial operations are rational and logical and require no " standing loose for a time to logical accuracy." * As they are rational and logical, so their interpretations are rational and logical, and the reader does not lose his sense of logical sequence. There is no " removal of barriers, of limitations, of conditions." * Multiplication is the same from beginning to end, whether applied to scalars, vectors, or quaternions. Commutivity of factors may be permissible in some cases and not in others, but this is a mere incident and not an essential element of the operation.

The reader is not mystified by arbitrarily defining multiplication of one vector into another as the turning through a right angle, etc., and left to wonder how one line can do anything to another. In fact, the operations are not defined *a priori* at all, but *taking the properties of discrete quantities as symbols of operations which the reader is to perform,* we find six possible operations, addition, subtraction, multiplication, division, reversion, and mean reversion. These operations are defined accordingly *a posteriori* as results of causes, not arbitrarily as assumptions.

The performance of these operations upon scalars leads to or *evolves* successively vectors and quaternions. Thus quaternions are evolved from discrete magnitudes, not arbitrarily, but of necessity, and along certain fixed and preordained

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* Kelland and Tait, and others.

lines, by rules which the properties of discrete magnitudes necessitate, and which cannot be altered or varied, and with which the reader is already familiar. The reader is not disturbed by the thought, Suppose we had made some other assumption, what then? No assumptions are made. He simply follows the road suggested by the properties of discrete magnitudes, and can arrive at but one result.

We make no laws, lay down no rules, make no modifications or limitations. The only way in which we exercise any choice is in the rational application of the laws we discover to the proper operands and in a proper and logical manner.

The "interpretation of our results" is not made to "depend upon the definition" as a foundation. The foundation is the *properties of discrete magnitudes*, and the definitions are merely rational statements of the results of these properties being used as *suggestions for operations* to be performed by the reader.

Considerable stress has been laid upon the avoidance of the sole use of mere typographical symbols and upon the auxiliary use of idiographic symbols upon which spatial operations can be performed; as in the use of strokes for merely reversible magnitudes, a spherical shell for scalar magnitudes, arc strokes for quaternion multiplication, two vector factors for the corresponding quaternion, etc., thus making the treatment concrete and avoiding the difficulties of abstractness.

The original features of the book are those specified above, coupled with the general heuristic method by which the student hews out his own concepts as he goes along. The results, the examples, the applications, and the terms used are those found in every treatise on the subject, of which I have made free use and to whom should be accredited these features: particularly Hamilton, Kelland and Tait, Tait, Laisant, Molenbrock, Hathaway.

As these notes are only intended as an introduction, not an overabundance of examples or formulæ has been provided, nor have any applications been made to problems in Geometry and Physics. These will be found in the works cited. Nor has the subject of differentiation been touched upon.

The author's own experience with this method of presentation of the subject to beginners has been encouraging. It is hoped others will have the same experience.

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QUATERNIONS

CHAPTER I

MATHEMATICAL OPERATIONS UPON DISCRETE MAGNITUDES

1. Mathematics is for school purposes the science of magnitude.

Magnitude has been defined as that which can be increased or diminished, and of two kinds, continuous magnitude (*continuum*), which is usually called quantity and answers to the question, How much? and discrete magnitude (*discreta*), which is usually called number and answers to the question, How many?

Quantity is differentness from nothingness, or quantitatively expressed, difference from nothing.

2. When we attempt to express quantity in symbols or terms, or to conceptualize it, we find ourselves compelled to express it in terms of some arbitrary unit adopted as a standard. The symbol or concept for the measure of quantity is called *number*, expressing its differentness from oneness, the oneness of the standard.

In the case of natural units, soul, personality, inhabitant, etc., the differentness from oneness which distinguishes a group from an individual is number, giving rise to the so-called primary numbers, 1, 2, 3, . . .

Number is differentness from oneness. The whole essence of number is to differ from another number. It belongs to the primary categories of time, space, matter, etc. No one of these can exist without the presence of number, that is without a oneness and a difference from that oneness. The very existence of time, space, matter implies multiplicity, three-foldness.

The limitations of the mind require symbols to represent the different concepts so that the mind can posit its judgments and conclusions to await its return to make use of them. Otherwise the mind soon becomes clogged with the impedimenta of its own creation.

Upon the fortunate choice of these symbols depends very greatly the progress in their use. A striking illustration of this is seen in the enormous advantage of the Hindu position system over the clumsy notation of the Greeks and Romans.

3. In the infancy of the subject pebbles (*calculi*) and other concrete objects were the symbols used. And as a negative pebble was impossible of imagination, negative numbers were imaginary. Even as late as the sixteenth century they were called by Cardan in his Ars Magna, *numeri ficti*, imaginary numbers. After they were dignified with the title of real numbers, expressions containing $\sqrt{-1}$ would intrude themselves, under the operations of mathematical analysis, and as there was no way of writing a numerical symbol whose square would be a negative number, these were in their turn imaginaries, even to the present day. So late a writer as De Morgan (1831) speaks of them as void of meaning, self-contradictory, and absurd, though of great utility in the formal mechanism of Algebra.

But it is not the number that is imaginary, it is the application of it to certain data or symbols that produces the imaginary features. Fractions are imaginary when counting souls, inhabitants, events, phenomena, etc. Negative numbers are imaginary when applied to length, bulk, intensity, etc. The imaginaries of to-day become real when applied to the proper symbol.

Before searching for a symbol we must inquire about the properties of primary number and the operations performable upon it. What are these performable operations, guiding ourselves by the properties of number itself? This guidance is necessary, the properties of a thing, if we thus speak of number, being the determining factors as to what purposes it can be put.

4. Number is differentness from unity. We can increase this differentness quantitatively by combining the differentness of one number with the differentness of another number. This operation we call **addition,** and by means of it we can evolve all the primary numbers from unity. In this two numbers are taken accumulatively.

5. We can imagine the differentness from oneness to have a correlated sense of opposition, like debit and credit, affirmation and negation, an antagonistic differentness. One is the reversal of the other, the negative of the other; and the process of converting one into the other is called **reversion.**

Numbers having this correlated sense of opposition are said to have size and sense. An example could be the accumulative collection of material into the form of a mound or tumulus. The workmen could proceed decumulatively (this word is not in the lexicons, but its usefulness is apparent) by undoing their former work, and might proceed below the bed rock. We might not call this a negative tumulus, but it would be the negative of the tumulus concept and the number concept would be a negative number.

6. We can diminish the differentness from oneness which is characteristic of a number by the characteristic differentness of some other number, giving rise to the operation called **subtraction.** The continued application of this process to numbers having size and sense leads to negative numbers. Here the numbers are taken decumulatively. As addition is accumulative combination so subtraction is decumulative combination of numbers without regard to the size of the subtrahend.

We can consider the characteristic differentness of one number as a mandate for an operation to be performed upon another number. Now this characteristic differentness from oneness or unity may be evolutory or involutory, differentness *from* unity, or differentness *toward* unity, a differentness which evolves the number from unity, or a differentness which converts a number into unity.

7. If we take the evolutory differentness as the mandate, then the operation is called **multiplication,** the doing to the operand (multiplicand) what was done to unity to produce the operator (multiplier). If the first operand and all the operators in a series of operations are the same, the operation is called **involution.** Involution works *from* unity with the first operand and all the operators alike.

A curious error is liable to creep in here unless care be exercised. For example, in $\sqrt{2}\cdot 3$, one is liable to say that $\sqrt{2}$ is derived from 1 by doubling 1 and taking the $\sqrt{}$ of the result. This applied to 3 gives the result $\sqrt{2\cdot 3}$, which is, of course, wrong. The error consists in taking the square root of the *result,* whereas multiplication is the doing what was done to *unity,* and not what was done to the *results* of operations upon unity. To get $\sqrt{2}$ from unity, we really operate upon unity with the tensor or stretcher $1.4142\ldots = \sqrt{2}$.

8. If we take the differentness *toward* unity as the mandate, then the operation is called **division,** the doing to the operand (dividend) what was done to the operator (divisor) to produce unity.

If all the operators and the last operand in a series of operations are the same, the operation is called **evolution,** and gives rise to irrational and so-called imaginary numbers, e.g., $\sqrt{2}$, $\sqrt{-1}$, etc.

Evolution works *towards* unity with all the operators and the last operand alike.

9. One of the most important phases of evolution is $\sqrt{a}$, the breaking up of the operation of passing from unity to a into two equal steps, so that the repetition of the first step, $\sqrt{a}$, shall produce a. When $a=-1$, we get $\sqrt{-1}$.

Since -1 is reversion, $\sqrt{-1}$ is called **mean reversion.**

Unlike addition and subtraction, which merely adjoin the elements without change, multiplication and division are transformational operations which change the operand into an entirely new and different number. They are purely algebraic, being limited, unlike addition and subtraction, to operations upon discrete number. A not entirely satisfactory illustration of the difference between the two operations of addition and multiplication is the putting of two canes together for addition; for multiplication we would be compelled because the first was triangular, black, smooth, and uniform in size to transform the second cane which was round, white, knotted, tapering, and with a knob on the end, into a triangular, black, smooth, prismatic stick. It is transformed into an entirely new and distinct form, and its original features have disappeared. So in numbers, addition and subtraction are merely adjunction without change of the original elements. Multiplication and division are destructive transformations, the destruction of one element and the production of an entirely new and different one, like the growth of a plant from a seed.

CHAPTER II

IDIOGRAPHS

10. Mathematical operations are conducted by means of symbols, and upon our choice of symbols depends largely the success of our operations.

In the dawn of mathematics, the fingers or other concrete objects were the symbols used, and a negative quantity was purely imaginary. With the introduction of written symbols of quantity and the concept of debit and credit, the negative quantity lost its imaginary quality and became real. In these symbols, however, $\sqrt{-a}$ remained imaginary, because the spatial and typographical properties of the symbol and the mathematical properties of the thing it represented did not agree.

11. With the introduction of the symbol called a **stroke,** *a straight line upon the surface of the paper,* to represent a magnitude of a given size, the space properties of the symbol and the mathematical properties of the thing symbolized became the same, and spatial operations upon the symbols corresponded to mathematical operations upon the thing symbolized, and could be used to interpret and control the mathematical operations.

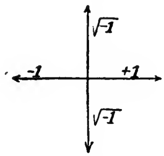
Symbols whose spatial properties are the same as the mathematical properties of the things represented might be called *idiographs* (ἴδιος, proper, peculiar).

12. Representing a magnitude having sense of generation, sense of correlated opposition, by the idiograph $\longrightarrow$, and of course its reversal by ←—, how can we convert —→ into ←—?

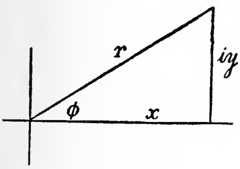
Obviously, only in one way, so long as we retain its *stroke* characteristics, namely, by swinging it through an angle of 180° in the surface of the paper. To attempt to swing it out of the surface of the paper is to lose its *stroke* characteristics, to give it absolute direction in space and render it no longer an idiograph.

The revolution of 180° can be broken into two equal steps of 90° each. Hence ↑ or ↓ must be the idiographic equivalent of $\sqrt{-}$.

Combining these into one diagram and assuming the normal revolution as counter-clockwise, we get the idiographic diagram with the corresponding typographical symbols.

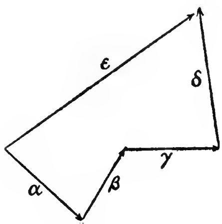


13. This is the well-known Argand diagram, affording a simple method of representing relatively directed quantities, or as they are generally called, *complex quantities*, the general type of which is $x+\sqrt{-1}y$, where x represents the normal or the reversed portion and $\sqrt{-1}y$ the mean reversed portion. It is generally written $x+iy$, i standing for $\sqrt{-1}$. The rectangular co-ordinates x and iy determine a point, whose distance from the origin, r, is called the **modulus,** and whose angular distance from the axis of x, the angle ϕ, is called the **amplitude** of the point or complex quantity $x+iy$.



14. The idiographic symbol for a magnitude, —→, showing its size and sense, we have already designated as a stroke, a stroke forward or a stroke backward. Two forward strokes need not be represented by the same symbol on the paper. The forwardness is in reference to its own backwardness, and has no reference to the forwardness or backwardness of other strokes. A stroke is a straight line in a plane, symbolizing a given magnitude in size and in relation to its sense of normalcy or of reversion, or a condition between these. Two strokes are equal when they have the same lengths and the same direction in a plane as regards a standard normal direction in the plane.

15. If several strokes be taken in succession the sum (result) of them is the same as the stroke from the beginning of the first, to the end of the last, when they are arranged end to end so as to be successive. Thus



$$\alpha+\beta+\gamma+\delta=\varepsilon.$$

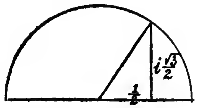
Strokes will for the present be represented typographically by lowercase Greek letters, as above.

The reader must notice carefully that it is not the lengths of the stroke which are added, but the results of the strokes, including both length and direction on the paper.

It is easily seen that *the order of the strokes is immaterial*, and that *any number of consecutive strokes can be replaced by their sum. The addition of strokes is a commutative and associative operation*, that is, the order and mode of grouping has no effect on the result.

A stroke is subtracted by reversing its direction and adding.

16. If we attempt to break up the operation of reversal into three or more equal and similar operations, for example, three, as shown in the diagram, we find that $\sqrt[3]{-1}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$, that is, the first of the three equal operations is expressible in terms of the operation of mean reversion, and of course the second operation likewise.



Similarly for a greater number of steps. Hence mean reversion is evidently the unique operation, a multiple of which is reversion, all the other partial equal operations whose continued application results in reversion being expressible in this one.

17. Hence there are six unique and fundamental operations which can be performed upon a discrete magnitude: addition, subtraction, reversal, multiplication, division, and mean reversion, and no others.

By fundamental operations is meant operations based upon the properties of discrete magnitudes, size, and sense of correlated opposition.

18. Mere discrete magnitudes, considering size only, are **scalars,** that is, they can be scaled off on a scale either in a normal or in a reversed direction.

CHAPTER III

SPACE IDIOGRAPHS

19. Space idiographs. In space we cannot idiographically represent a scalar by a line, for that would be assigning to the symbol a characteristic direction, which the magnitude it represents does not possess. If we are to represent a scalar magnitude in space by any idiographic symbol, the only one which seems available as possessing perfect symmetry and therefore devoid of direction is a spherical shell.

Just as we can assume our + unit of heat, pressure, etc., anywhere on the scale, so we can posit our +spherical shell anywhere in space.

Likewise as the − unit of heat, etc., would naturally adjoin the + unit, along the scale, so naturally we should expect the − unit shell to adjoin the + unit shell in some position determined by previous assignment of direction. According to this previously determined direction we shall have units of α direction, of β direction, etc., where α and β denote *direction,* not magnitudes, unless we say unit magnitude, just as previously we had units of heat sense, credit sense, etc.

20. How can we break up the operation of transforming the + shell into the − shell into two similar and equal operations. The + shell can be changed into the − shell by the repetition of two different operations:

A. By revolving the $-$ shell about its point of contact with the $+$ shell through an angle of $90°$ twice;

B. By moving the elements of the $-$ shell perpendicularly to the common line of centers in the proportion $\sin\theta$ (where $\theta = \cos^{-1}$ harmonic displacement * of the shell element) and moving the resulting configuration, a directed line, one-half its dimensions toward the correlatively reversed position of the original operand, i.e., toward the position of the $+$ shell. A repetition of this operation would produce the $+$ shell.

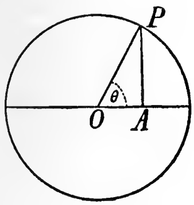
21. Operation A is excluded by reason of its lack of definiteness, leaving operation B as the operation producing the mean reversed state.

Since we are dealing with iodiographs these spatial operations must have a one-to-one correspondence with mathematical operations performed upon the things they symbolize.

22. Hence a mean reversed scalar is represented in all its properties by a directed magnitude in space, a *vector*, as it is called, which is definitely directed as soon as the $+$ and $-$ shells are posited.

23. A **vector** is any magnitude having direction of extension in space, a directed line, plane, etc., such as velocity, impulse, force, etc.

Vectors are *equal* when they possess the same quantitative



* If P represents an element of the shell, OA, the projection of the radius on a given diameter is its harmonic displacement,

$$\theta = \cos^{-1} OA.$$

and qualitative properties, viz., magnitude, sense, and direction of extension. Direction of extension is that property which prevents the vectors from coinciding (in whole or in part) when brought together.

Parallel vectors of the same length are equal. Vectors can be made coinitial without altering their properties.

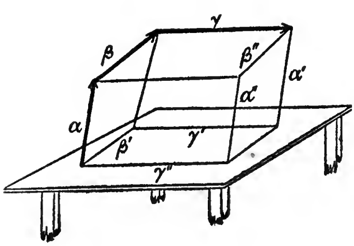
24. It is customary to indicate the unit vector in a given direction by Greek letters α, β, γ, . . . Generally three unit vectors at right angles to each other are assumed as reference units. These are designated by i, j, k.

25. So far we have recognized two kinds of magnitudes, scalars and vectors, and six operations, addition, subtraction, reversion, multiplication, division, and mean reversion.

Applying these operations to scalars we find that they all produce scalars again, except in the case of mean reversion, and that produces a vector. This gave us the second kind of magnitude, to which we will now proceed to apply the six fundamental operations.

26. Reversal is merely the turning of the vector into the opposite direction, as the word implies. The result is sometimes called a **revector**.

27. Addition and subtraction of vectors. Subtraction of vectors is merely addition with the minuend reverted. Vectors are of the nature of strokes, with the property of absolute direction added. The laws governing the addition of strokes evidently hold here also. Thus, *vector addition is commutative and associative, and this whether the vectors are coplanar or not.* Thus



$$\alpha+\beta+\gamma=\beta'+\gamma'+\alpha'$$
$$=\gamma''+\alpha''+\beta'',\text{etc.,}$$

where α, β, γ are the three edges of a parallelopiped. The same reasoning would apply to additional vectors.

28. The following equations are self evident:

$$\alpha+\alpha+\alpha \ldots \text{ to } m \text{ terms} = m\alpha,$$

$$-\alpha+(-\alpha)+ \ldots \text{ to } m \text{ terms} = m(-\alpha) = -m\alpha,$$

$$m\alpha+m\beta=m(\alpha+\beta), \qquad m\alpha-m\beta=m(\alpha-\beta).$$

29. If α, β, γ be three coinitial vectors, then any fourth coinitial vector, ξ, can be expressed as

$$\xi = x\alpha+y\beta+z\gamma,$$

ξ being the diagonal of the parallelopiped whose edges are $x\alpha$, $y\beta$, and $z\gamma$.

30. If α is a unit vector, and $m\alpha=A$, then m indicated generally by the symbol TA, which expresses the length of the vector A, is called the **tensor** (*tendere*, to stretch) of the vector A. α, denoted by UA is called the **unit vector** of A. Therefore

$$A = TA \cdot UA.$$

Vectors will be denoted by capital Greek letters when the tensor and unit part are to be emphasized; by lower-case Greek letters when the question of length is not important; and by the corresponding lower-case English and Greek letters when speaking of the tensor and unit part separately.

Thus the same vector may be indicated by A, $a\alpha$, α.

The tensor is signless, just as any length, the height of a steeple, for instance, is signless; or the height of a man. A man cannot be -5 feet tall.

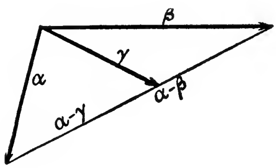
Exercises in Vector Combinations

31. If $l\alpha+m\beta=0$, then $l=0$, $m=0$, for in no other way can two strokes in different directions cancel each other so as to leave the pen at the point of beginning, unless $\alpha=x\beta$, i.e., unless α is parallel to β.

32. If
$$l\alpha+m\beta=l_1\alpha+m_1\beta,$$
that is,
$$(l-l_1)\alpha+(m-m_1)\beta=0,$$
then,
$$l=l_1, \quad m=m_1.$$

33. If $l\alpha+m\beta+n\gamma=0$, and l, m, n are not zero, then α, β, and γ are coplanar, for $l\alpha$ and $m\beta$ determine a plane which contains the *ends* of $n\gamma$, and therefore $n\gamma$ itself.

34. If $l\alpha+m\beta+n\gamma=0$, and l, m, n are not zero but $l+m+n=0$, then $(l+m+n)\alpha=0$, and subtracting the first equation, we get



$$m(\alpha-\beta)+n(\alpha-\gamma)=0,$$

whence $\alpha-\beta$ and $\alpha-\gamma$ are parallel (§ 31). But $\alpha-\gamma$ connects the ends of α and γ, and $\alpha-\beta$ the ends of α and β, hence *if* $l+m+n=0$, α, β, *and* γ *terminate in the same line.*

35. Conversely, *if* α, β, *and* γ *terminate collinearly and* $l\alpha+m\beta+n\gamma=0$, *then* $l+m+n=0$.

For by condition, $\quad \alpha-\beta=x(\alpha-\gamma),$

or
$$(1-x)\alpha-\beta+x\gamma=0,$$
in which
$$1-x-1+x=0. \qquad \text{Q.E.D.}$$

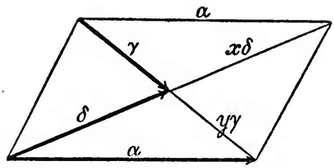
36. If $a\alpha + b\beta + c\gamma + \ldots = d\delta$, then evidently $a+b+c + \ldots > d$, or $TA+TB+TC+\ldots > TD$,

or, Sum of the tensors > tensor of the sum, $\Sigma T > T\Sigma$,

or the distance a man travels > his distance from home.

If the vectors are *parallel*, $\Sigma T = T\Sigma$.

37. The *diagonals of a parallelogram mutually bisect each other.*



$\alpha = \delta + y\gamma = \gamma + x\delta.$

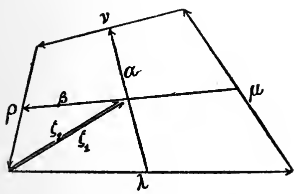
$\therefore$ § 31,

$\delta = x\delta, \qquad \gamma = y\gamma,$

Q.E.D.

γ and δ being parts of the diagonals to the point of intersection, and $y\gamma$ and $x\delta$ the remaining portions respectively.

38. *The lines joining the middle points of the opposite sides of any quadrilateral, whether plane or gauche, mutually bisect each other.*



One bisector is $\alpha = \frac{1}{2}\lambda + \mu + \frac{1}{2}\nu.$

The other is $\beta = \frac{\mu}{2} + \nu + \frac{\rho}{2}.$

Find the vectors from any assumed point to the middle points of these bisectors and compare the results.

Thus the vector from the beginning of λ to the middle point of α is

$$\zeta_1 = \frac{\lambda}{2} + \frac{\alpha}{2} = \frac{\lambda}{2} + \frac{1}{2}\left(\frac{1}{2}\lambda + \mu + \frac{\nu}{2}\right) = \frac{3}{4}\lambda + \frac{1}{2}\mu + \frac{1}{4}\nu.$$

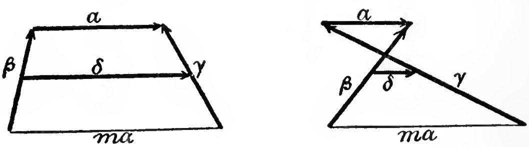
The vector from the same point to the middle point of β is

$$\zeta_2 = \lambda + \frac{\mu}{2} + \frac{\beta}{2} = \lambda + \frac{\mu}{2} + \frac{1}{2}\left(\frac{\mu}{2} + \nu + \frac{\rho}{2}\right)$$

$$= \lambda + \frac{3}{4}\mu + \frac{1}{2}\nu - \frac{1}{2}\ \frac{\lambda + \mu + \nu}{2} = \frac{3}{4}\lambda + \frac{1}{2}\mu + \frac{1}{4}\nu.$$

$\therefore\ \zeta_1 = \zeta_2$, and the middle points coincide. Q.E.D.

39. If the ends of two parallel vectors be connected by straight lines, the join (connecting line) of the middle points of the straight lines is half the sum or difference of



the tensors of the parallel vectors: i.e., *the median line of a trapezoid is half the algebraic sum of the bases.*

$$\delta = \frac{\beta}{2} \pm \alpha - \frac{\gamma}{2}, \qquad \text{taken along } \alpha;$$

$$\delta = -\frac{\beta}{2} + m\alpha + \frac{\gamma}{2}, \qquad \text{taken along } m\alpha.$$

Whence by addition, $\delta = \dfrac{m\alpha \pm \alpha}{2}.$

$\therefore$ § 36, $d = \dfrac{ma \pm a}{2}.$ Q.E.D.

CHAPTER IV

MULTIPLICATION OF UNIT VECTORS

40. Parallel vectors. Remembering that multiplication is the performing by the reader on the multiplicand of an operation which is symbolized by the multiplier, viz., the operation which produced the multiplier *from* unity, we must in the product ii * ask what operation is the first i the symbol of. The answer is, of course, of one of two equal operations whose successive applications shall produce reversal. Now i can be reversed by the repetition of each of two methods. The one we have designated as operation A (§ 20). The other we have designated as operation B. Since the multiplier is *exactly* the same as the multiplicand in all its properties, we must if possible use exactly the same operation not only in kind, but also in detail, that produced the multiplier, that is, operation B (§ 20).

This amounts to the repetition upon the multiplicand i of the same operation which produced it from unity, and of course results in -1, see § 20. That is,

$$ii = -1.$$

Hence, *Multiplication (performance of an operation symbolized by the multiplier) of one unit vector into another parallel to it produces reversion.*

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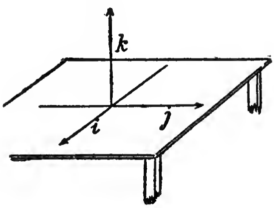
* i = some directed $\sqrt{-1}$ = some directed mean reversed scalar.

41. Perpendicular vectors. Since the multiplier is now perpendicular to the multiplicand, we must take it as the symbol of an operation to be performed on the multiplicand, the same in kind but as far removed in detail from that which would have been used had the multiplier been parallel to the multiplicand as perpendicularity is removed from parallelism.

This we must do in order to take into account the perpendicularity of direction as opposed to parallelism. The operation of mean reversion which produced the multiplier was operation B (§ 20). Therefore we must use operation A. Let i and j be operator and operand respectively. The only position into which j can be revolved such that the reversal of the signs of the two factors will give the same result is k, one of perpendicularity to both factors. Thus



$$ij = k \quad \text{and} \quad -i \cdot -j = k,$$

since $-i$ bears exactly the same relation to $-j$ that i does to k, and must therefore have the same effect. Any other position than k for the product of ij would not do this.

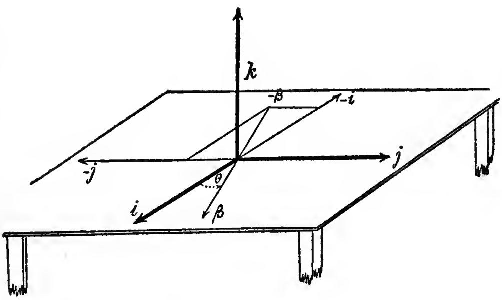
Hence, *The multiplication (performance of an operation symbolized by the multiplier) of one unit vector into another perpendicular to it results in the turning of the multiplicand through a right angle in (to) a plane perpendicular to the operator.*

42. Inclined vectors. Naturally the result will be a combination of those of §§ 40, 41, that is partly scalar and partly vector, or

$$\alpha\beta = -\cos\theta + \varepsilon \sin\theta,$$

where α and β are the two unit vectors inclined at an angle θ, and ε a unit vector perpendicular to α and β, since this formula satisfies both the limiting cases (§§ 40, 41).

Hence, *The multiplication (performance of an operation symbolized by the operator) of one unit vector into another inclined to it at an angle θ, thus producing the mean reversed*



state induced by the operator symbol, turns the operand through a right angle into a plane perpendicular to the multiplier, makes its length sin θ and adds a scalar, $-\cos\theta$.

43. We can get the same result as follows:

$i\beta$, the mean reversed state of β, must be as to direction some vector perpendicular to the plane of i and β, since $-i\cdot-\beta$ must produce the same result as $i\beta$. Hence, tentatively,

$$i\beta = sk,$$

where s is some scalar. Operating again with i to see if the second application produces reversal, we get

$$i\cdot i\beta = i\cdot sk = s\cdot ik = s\cdot -j,$$

which is not reversal, but which would be, except as to length, perhaps, by the addition of $-ci$, where c is some scalar. But this would require

$$i\beta = -c + sk, \qquad \text{since } i \cdot i\beta = -ic + i \cdot sk$$
$$= -ic - sj.$$

Now if $-ic - sj = -\beta$, then $c = \cos\theta$, $s = \sin\theta$,

and we have as before $i\beta = -\cos\theta + k\sin\theta$.

44. Exercises in unit reference vectors.

$ij = k$, but $ji = -k$; $jk = i$, but $kj = -i$.

Hence the factors are not *commutative*.

$i \cdot jk = i \cdot i = -1$, $ij \cdot k = k \cdot k = -1$.

Hence $i \cdot jk = ij \cdot k$, or the factors are *associative*.

$ki = j$, $i \cdot jk = ii = i^2 = -1$,

$ji = -k$, $j \cdot ki = jj = -1$,

$i \cdot -j = -k$, $ijk = jki = kij = -1$,

$jjiii = j^2 i^3 = i$, $k \cdot ji = k \cdot -k = -k^2 = 1$.

$ij \cdot k = kk = k^2 = -1$.

CHAPTER V

QUATERNIONS

45. Having ascertained that the product of two vectors α, β is

$$\alpha\beta = -\cos\theta + \varepsilon\sin\theta,$$

we can, § 29, express $\varepsilon\sin\theta$ in terms of i, j, k, viz.:

$$\varepsilon\sin\theta = xi + yj + zk, \quad \text{or} \quad \alpha\beta = -\cos\theta + xi + yj + zk,$$

which, being composed of four terms, a scalar and three vectors, is called a **quaternion,** and will be symbolized by q.

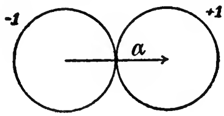
A quaternion is evidently composed of a scalar plus a vector. Later (§ 61) we shall find that, conversely, a scalar plus a vector is a quaternion.

46. The plane of the factors (multiplier and multiplicand) of a quaternion is called the **plane of the quaternion.**

The **plane of a vector** is the plane perpendicular to it.

ε is called the **axis** of $\alpha\beta$. The most convenient method of defining it seems to be: The unit vector toward the north pole when the multiplicand is to the east of the multiplier, the equator being the plane of the quaternion; toward the south pole when the multiplicand is to the west.

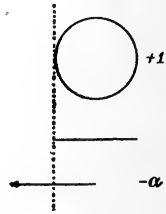
47. Meaning of $\frac{1}{\alpha}$. By the rule for division we must first ascertain what must be done to α to produce **1.** Returning to our idoigraphic shell; to convert α into **+1,** we must, repeating the operation which produced **it,** apply operation B (§ 20), and then direct it one-half its dimensions towards its correlatively reversed position. Performing these operations on the



numerator

we get by application of B,

and then by directing it,



whence $$\frac{1}{\alpha}=-\alpha.$$

This is verified by the fact that $\alpha\cdot-\alpha=-\alpha^2=-\cdot-1=1$, hence $\frac{1}{\alpha}=-\alpha$, since $\alpha\frac{1}{\alpha}=1$, being a functional * operation followed by the inverse operation and therefore resulting in the original operand.

Hence, *the reciprocal of a unit vector is the unit vector reversed.*

48. Meaning of† $\frac{\mathbf{j}}{\mathbf{i}}$, $\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$, etc. In a similar manner $\frac{j}{i}=k$,

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* See Appendix.

† $\frac{j}{i}$ means $j\cdot\frac{1}{i}$ and not $\frac{1}{i}\cdot j$. Thus we can write $\frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma}=\alpha\frac{1}{\beta}\beta\frac{1}{\gamma}=\alpha\frac{1}{\gamma}=\frac{\alpha}{\gamma}$. But we cannot write $\frac{\beta}{\gamma}\cdot\frac{\alpha}{\beta}=\frac{\alpha}{\gamma}$, for $\beta\frac{1}{\gamma}\alpha\frac{1}{\beta}=\beta(-\gamma)\alpha(-\beta)$ does not allow the β's to cancel each other, the vector factors not being commutative, § 44.

and $\frac{\alpha}{\beta}=\cos\theta-\varepsilon\sin\theta$, since this satisfies both the limiting cases $\frac{i}{i}=1$ and $\frac{j}{i}=+k$, θ being the angle from α to β.

Examples for Practice

$\frac{-k}{i}=j.$ $\frac{ik}{j}=-1.$ $\frac{j}{k}=-i.$ $\frac{i}{j}\frac{j}{k}=\frac{i}{k}.$

$\frac{-i}{j}=k.$ $-i\cdot -j=k$ $\frac{1}{i}=-i.$

$\frac{-j}{k}=i.$ $i^2j^2k^2=-(ijk)^2.$ $\frac{j}{k}\cdot\frac{i}{j}=-i\cdot -k=-\cdot j=\frac{k}{i}\neq\frac{i}{k}$

$\frac{j}{-k}=i.$ $\frac{i}{j}j=i.$ $\frac{-j}{k}\cdot\frac{k}{j}=i\cdot -i=i^2.$

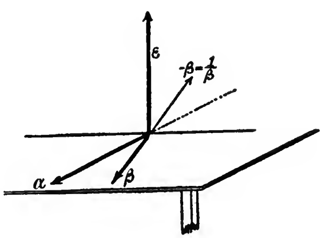
$\frac{k}{j}=i.$ $j\frac{i}{j}\neq i.$ $ikj=kji=jik=-i^2=-j^2=1.$

49. Since $\quad \alpha\alpha\frac{1}{\alpha\alpha}=\alpha\cdot -\alpha=1=\frac{1}{\alpha\alpha}\alpha\alpha=-\alpha\cdot\alpha,$

therefore a *vector is commutative with its reciprocal.*

50. Since $\frac{1}{\beta}=-\beta$, we can write

$$\frac{1}{\beta}\beta=-\beta\cdot\beta=1.$$



Hence, § 42,

$$\frac{\alpha}{\beta}=\alpha\cdot\frac{1}{\beta}=\alpha(-\beta)$$

$$=-\cos(\pi+\theta)+\varepsilon\sin(\pi+\theta)$$

$$=\cos\theta-\varepsilon\sin\theta,$$

where, as before, θ is the angle from α to β.

Similarly.

$$\beta\alpha = -\cos(-\theta) + \varepsilon \sin(-\theta) = -\cos\theta - \varepsilon \sin\theta,$$

$$\frac{1}{\beta}\alpha = -\cos(\pi - \theta) + \varepsilon \sin(\pi - \theta) = \cos\theta + \varepsilon \sin\theta.$$

Hence $$\frac{1}{\beta}\alpha \neq \left(\frac{\alpha}{\beta} = \alpha\frac{1}{\beta}\right),$$

$$\frac{1}{\alpha}\beta = -\cos(\pi + \theta) + \varepsilon \sin(\pi + \theta) = \cos\theta - \varepsilon \sin\theta,$$

$$\frac{\beta}{\alpha} = -\cos(\pi - \theta) + \varepsilon \sin(\pi - \theta) = \cos\theta + \varepsilon \sin\theta.$$

51. Introducing the tensors of α and β, and collecting the results, we have,

$$\alpha\beta = ab(-\cos\theta + \varepsilon \sin\theta), \qquad \frac{\alpha}{\beta} = \frac{a}{b}(\cos\theta - \varepsilon \sin\theta),$$

$$\beta\alpha = ab(-\cos\theta - \varepsilon \sin\theta), \qquad \frac{\beta}{\alpha} = \frac{b}{a}(\cos\theta + \varepsilon \sin\theta).$$

52. *Distributivity of the vector multiplier.* Let α, β, γ be three unit vectors, making with each other the angles ϕ, θ, $2a$, as shown; δ is not a unit vector.

ε is the axis of $\alpha\beta$, η of $\alpha\gamma$, ζ of $\alpha\delta$; coplanar, since the the planes of the three quaternions have the common edge α.

$$\beta + \gamma = \delta, \qquad |\delta| = 2\cos a.$$

[$|\delta|$ means length of δ. $\cos a = \frac{1}{2}$ diag. of the parallelogram on β, γ.]

$$\alpha\beta = -\cos\theta + \varepsilon \sin\theta, \qquad \alpha\gamma = -\cos\phi + \eta \sin\phi.$$

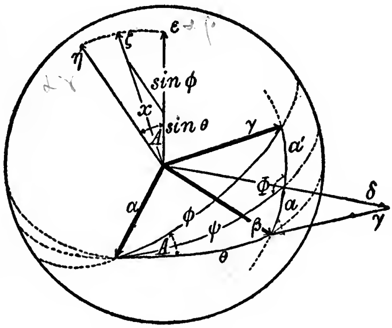
$$\therefore \quad \alpha\beta + \alpha\gamma = -\cos\theta - \cos\phi + \varepsilon \sin\theta + \eta \sin\phi.$$

But by trigonometry, since the angle between the axes is the same as the angle between the planes of the quaternions, and since $a=a'$ makes the sines of these angles proportional to the sines of the adjacent sides, that is, $\sin\theta$, $\sin\phi$, therefore $\varepsilon\sin\theta+\eta\sin\phi$ will lie along the ζ axis and

$$(1)\qquad \alpha\beta+\alpha\gamma=-\cos\theta-\cos\phi+x\zeta,$$

where x is some unknown tensor.

(2) But $\alpha(\beta+\gamma)=\alpha\delta=2\cos a(-\cos\psi+\zeta\sin\psi)$, and we now have to show that this agrees with $\alpha\beta+\alpha\gamma$.



By trigonometry,

$$\cos\phi=\cos\psi\cos a+\sin\psi\sin a\cos\Phi$$

$$\cos\theta=\cos\psi\cos a-\sin\psi\sin a\cos\Phi$$

$$\therefore\quad \cos\Phi=\frac{\cos\phi-\cos\psi\cos a}{\sin\psi\sin a}=\frac{\cos\psi\cos a-\cos\theta}{\sin\psi\sin a},$$

$$\therefore\quad \frac{\cos\phi+\cos\theta}{2\cos a}=\cos\psi.$$

(3) $\therefore \quad -2\cos a\cos\psi = -\cos\phi - \cos\theta.$

By trigonometry again,

$$x^2 = \sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi\cos A,$$

$$\cos 2a = \cos\phi\cos\theta + \sin\phi\sin\theta\cos A.$$

$$\begin{aligned}\therefore \quad x^2 &= \sin^2\theta + \sin^2\phi + 2(\cos 2a - \cos\phi\cos\theta)\\ &= \sin^2\theta + \sin^2\phi + 2(\cos^2 a - \sin^2 a - \cos\phi\cos\theta)\\ &= 1 - \cos^2\theta + 1 - \cos^2\phi + 2\cos^2 a - 2 + 2\cos^2 a\\ &\qquad - 2\cos\phi\cos\theta\\ &= 4\cos^2 a - \cos^2\theta - 2\cos\theta\cos\phi - \cos^2\phi\\ &= 4\cos^2 a - (\cos\phi + \cos\theta)^2.\end{aligned}$$

But

$$\sin\psi = \sqrt{1 - \left(\frac{\cos\phi + \cos\theta}{2\cos a}\right)^2} = \frac{\sqrt{4\cos^2 a - (\cos\phi + \cos^2\theta)^2}}{2\cos a}.$$

(4) $\therefore \quad x = 2\cos a\sin\psi.$

$$\begin{aligned}\therefore \quad \alpha\beta + \alpha\gamma &= -2\cos a\cos\psi + 2\cos a\sin\psi\cdot\zeta \qquad (1), (3), (4)\\ &= 2\cos a(-\cos\psi + \sin\psi\cdot\zeta)\\ &= \alpha(\beta+\gamma). \qquad \text{Q.E.D.} \quad (2)\end{aligned}$$

Hence in vector multiplication, the multiplier is distributive over the operand.

53. If $\alpha\beta = ab(-\cos\theta + \varepsilon\sin\theta) = q$.

ab	is called the	**tensor** of q and is symbolized by $ab = Tq$.
$-ab\cos\theta$	"	**scalar part** of q and is symbolized by Sq.
$ab\sin\theta\cdot\varepsilon$	"	**vector part** of q and is symbolized by Vq.
$ab\sin\theta$	"	**tensor of the vector part** and is symbolized by TVq.
$-\cos\theta + \varepsilon\sin\theta$	"	**unit part** and is symbolized by Uq.
ε	"	**unit vector of the vector part** and is symbolized by UVq.
$-\cos\theta$	"	**scalar of the unit part** and is symbolized by SVq.
$\sin\theta\cdot\varepsilon$	"	**vector part of the unit part** and is symbolized by VUq.
$\sin\theta$	"	**tensor of the vector part of the unit part** and is symbolized by $TVUq$.

Exercises

54. Show that

$$(TVq)^2 = -(Vq)^2, \qquad UVq = \frac{Vq}{TVq},$$

$$(Tq)^2 = (Sq)^2 - (Vq)^2, \qquad TVUq = \frac{TVq}{Tq}.$$

If $q = w + xi + yj + zk$, § 45, show that $Sq = w$.

$$Vq = xi + yj + zk, \qquad (1) \quad Tq = \sqrt{w^2 + x^2 + y^2 + z^2},$$

$$TVq = \sqrt{x^2 + y^2 + z^2}, \qquad Uq = \frac{w + xi + yj + zk}{\sqrt{w^2 + x^2 + y^2 + z^2}}.$$

$$UVq = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}},$$

If q and r are quaternions

$$(2) \qquad T \cdot qr = TqTr, \qquad U \cdot qr = UqUr, \qquad U\frac{q}{r} = \frac{Uq}{Ur}.$$

If q and r degenerate to α, $\quad T \cdot \alpha^2 = -a^2. \qquad [\alpha^2 = -1.$

55. By § 54, Eqs. (1), (2),

$$Tqr = \sqrt{W^2 + X^2 + Y^2 + Z^2},$$

where

$$Tq = \sqrt{w_1^2 + x_1^2 + y_1^2 + z_1^2}, \qquad Tr = \sqrt{w_2^2 + x_2^2 + y_2^2 + z_2^2}.$$

$$\therefore \; W^2 + X^2 + Y^2 + Z^2$$
$$= (w_1^2 + x_1^2 + y_1^2 + z_1^2)(w_2^2 + x_2^2 + y_2^2 + z_2^2),$$

or (Euler's Theorem) the sum of four squares may be resolved into two factors, each of which is the sum of four squares.

56. Show that

$$S\alpha\beta = S\beta\alpha, \qquad \alpha\beta - \beta\alpha = 2V\alpha\beta,$$

$$V\alpha\beta = -V\beta\alpha, \qquad \alpha\beta \cdot \beta\alpha = (S\alpha\beta)^2 - (V\alpha\beta)^2$$

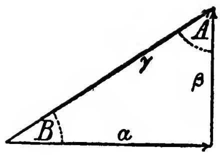
$$\alpha\beta + \beta\alpha = 2S\alpha\beta, \quad . \quad (1) \qquad = (T\alpha\beta)^2.$$

$$(2) \qquad Sx = x, \qquad Sij = 0, \qquad Vij = k.$$

57. Example. $\alpha+\beta=\gamma$.

$$(\alpha+\beta)^2=\gamma^2=\alpha^2+\alpha\beta+\beta\alpha+\beta^2,$$

or § 56, eq. (1), $-a^2+2S\alpha\beta-b^2=-c^2$.



If this is a right triangle, at C, then § 56, eq. (2), $S\alpha\beta=0$ and $a^2+b^2=c^2$.

If not a right triangle, this becomes

$$c^2=a^2+b^2-2ab\cos c. \qquad \text{(Law of Cosines. Trig.)}$$

58. Geometric meaning of TVαβ and Sαβ.

$$V\alpha\beta=ab\sin\theta\cdot\varepsilon,$$

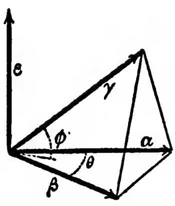
$$TV\alpha\beta=ab\sin\theta$$

$$=\text{parallelogram on } \alpha\beta.$$

$$S\alpha\beta=-ab\cos\theta$$

$=-$(one tensor · projection of the other upon it).

59. Meaning of Sαβγ. Suppose α, β, γ unit vectors, θ = angle between α, β, and ϕ = angle between γ and plane of $\alpha\beta$.



$$\alpha\beta=-\cos\theta+\varepsilon\sin\theta,$$

$$S\alpha\beta\gamma=S\cdot(-\cos\theta+\varepsilon\sin\theta)\gamma$$

$$=S\varepsilon\gamma\sin\theta.$$

But $S\varepsilon\gamma=-\sin\phi$,

$$\therefore\quad S\alpha\beta\gamma=-\sin\phi\sin\theta.$$

If α, β, γ are not unit vectors, but have the lengths a b, c, respectively, then

$$S\alpha\beta\gamma=-abc\sin\theta\sin\phi$$

$=-$volume of the parallelopiped on α, β, γ, as edges.

60. If $S\alpha\beta\gamma=0$, α, β, γ are coplanar, and vice versa.

61. *The sum of a scalar and a vector is a quaternion.* Let

$$w=\text{some scalar}; \qquad A=a\alpha=\text{some vector}.$$

Then
$$w+a\alpha=\sqrt{w^2+a^2}\left(\frac{w}{\sqrt{w^2+a^2}}+\alpha\frac{a}{\sqrt{w^2+a^2}}\right)$$

$$=\text{a tensor }(\cos\phi+\alpha\sin\phi)$$

$$=\text{a quaternion.} \quad (\S\ 51.)$$ Q.E.D.

62. Idiographic proof. We can always construct in a plane perpendicular to the vector a right-angle triangle, one of whose legs shall equal the diameter of the sphere which is the idiograph of the scalar, and the other leg the tensor of the given vector. If we denote the hypothenuse of this triangle by ab, then the leg corresponding to the scalar length will be $ab\cos\theta$, and the other leg will be $ab\sin\theta$, θ being the angle adjacent to the first leg constructed.

On the sides including θ, lay off respectively the distances a and b as vectors. Then the product of these two vectors will be a quaternion,

$$-ab\cos\theta+ab\sin\theta\cdot\varepsilon,$$

which will be the given sum, and at the same time the product of two vectors.

63. We are now able to distinguish between the different branches of mathematics of discrete magnitudes.

First, the mathematics of numbers having size only. This is **Arithmetic,** the algebra of tensors. The operations are four only, *addition, subtraction, multiplication,* and *division.* The operands are tensors.

Secondly, the mathematics of numbers having size and sense, using in addition to the previous operations, *reversion.* Its operands are *scalars,* generally symbolized by letters and numerals. This is **Algebra,** in its broad sense, including Calculus and allied subjects.

The usual line of demarcation between arithmetic and algebra, the use or non-use of literal characters, is unphilosophical and erroneous in its significance. If the letters represent tensors (e.g., number of girls, boys, children), $g+b=c$ is arithmetic purely, and $4-5=-1$ is pure algebra.

Thirdly, the mathematics of numbers having size and sense, and which adds to the previous operations that of *mean reversion.* This is the algebra of **Complex Functions.** Its operands are scalars, symbolized by *strokes* as well as by letters and numerals.

Fourthly, that branch of mathematics which deals with numbers having size, sense, and direction. Its operands are *scalars* and *vectors* and combinations of these. This is the subject of **Quaternions.**

CHAPTER VI

KINDS OF QUATERNIONS

64. The reciprocal of a quaternion, denoted by Rq, is the factor into which the quaternion must be multiplied in order to produce unity.

If $$q = a\alpha b\beta,$$

then $$a\alpha b\beta \frac{1}{b\beta} \cdot \frac{1}{a\alpha} = \alpha\beta\frac{1}{\beta} \cdot \frac{1^*}{\alpha} = \alpha\frac{1}{\alpha} = 1.$$

Hence the reciprocal of a quaternion is found by taking the product, in reverse order, of the reciprocals of the factors.

$$R\frac{a\alpha}{b\beta} = b\beta\frac{1}{a\alpha}. \qquad \frac{1}{\alpha\beta} = \frac{1}{q} = \frac{1}{-\cos\theta + \varepsilon\sin\theta} = -\cos\theta - \varepsilon\sin\theta.$$

$$\frac{1}{\alpha} \cdot \frac{1}{\beta} = -\cos\theta + \varepsilon\sin\theta. \qquad \therefore \quad \frac{1}{\alpha\beta} \neq \frac{1}{\alpha} \cdot \frac{1}{\beta}.$$

$$\frac{1}{\beta} \cdot \frac{1}{\alpha} = \frac{1}{\alpha\beta} \neq \frac{1}{\alpha} \cdot \frac{1}{\beta}.$$

If $\alpha\beta = q$, $$Rq = \frac{1}{\alpha\beta} = q^{-1}.$$

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* The assumption made here that $\beta\frac{1}{\beta} \cdot \alpha = \beta \cdot \frac{1}{\beta}\alpha$ is justified later, in § 90, since all vectors are (§ 97) quaternions.

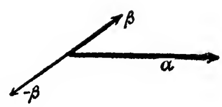
$$R\alpha\beta=\frac{1}{ab}(-\cos\theta-\varepsilon\sin\theta),\qquad \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1,\qquad \alpha\beta\frac{1}{\beta\alpha}\neq 1.$$

$$\frac{\alpha}{\beta}\cdot\frac{1}{\frac{\alpha}{\beta}}=\frac{\alpha}{\beta}\cdot\frac{1}{\alpha\frac{1}{\beta}}\neq\frac{\alpha}{\beta}\cdot\frac{1}{\alpha}\beta,\qquad \alpha\beta\cdot\frac{1}{\alpha\beta}=q\frac{1}{q}=1=\frac{1}{\alpha\beta}\cdot\alpha\beta=\frac{1}{q}\cdot q.$$

Hence, a quaternion and its reciprocal are commutative: $qq^{-1}=1=q^{-1}q$.

65. Opposite quaternions are those in which one factor is reversed.

$$\alpha\beta=ab(-\cos\theta+\varepsilon\sin\theta)$$
$$=ab(\cos(\pi-\theta)+\varepsilon\sin(\pi-\theta)),$$
$$\alpha(-\beta)=ab(\cos\theta-\varepsilon\sin\theta).$$



Hence if $\alpha\beta=q$, $\qquad \alpha(-\beta)=-q$

and $\qquad q+(-q)=0,$

or the sum of two opposite quaternions is zero.

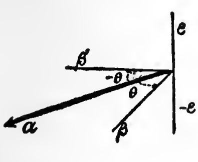
$$\frac{-q}{q}=-1,$$

or the quotient of two opposite quaternions is -1.

$$Tq=T(-q)=ab.$$

Opposite quaternions have a common plane, equal tensors, supplementary angles and opposite axes.

66. The Conjugate of a quaternion, denoted by Kq, has one of its factors turned an equal angle across the other factor, the order of the factors not being changed: it reverses the angle of the original quaternion.



$$Kq=K\alpha\beta=\alpha\beta'$$
$$=ab(-\cos(-\theta)+\varepsilon\sin(-\theta))$$
$$=ab(-\cos\theta-\varepsilon\sin\theta).$$

$$Kq = Sq - Vq. \qquad\qquad Kx = x.$$

$$q + Kq = 2Sq. \qquad\qquad \alpha\beta = K\beta\alpha.$$

$$q - Kq = 2Vq. \qquad \S\ 55. \quad Kq = w - xi - yj - kz.$$

Conjugate quaternions have a common plane, equal angles between the factors, equal angles (see § 76), equal tensors, and opposite axes. (§ 46.)

67. Since $\quad TKq = Tq \quad$ and $\quad UKq = \dfrac{1}{Uq}.$

$$\therefore \quad Kq = TKq \cdot UKq = \frac{Tq}{Uq}.$$

Whence by multiplication and division,

$$qKq = (Tq)^2, \qquad\qquad \frac{q}{Kq} = (Uq)^2.$$

$$Kq = K\alpha\beta = Tq\frac{1}{Uq} = ab(-\cos\theta - \varepsilon\sin\theta) = \beta\alpha.$$

$$\therefore \quad K\alpha\beta = \beta\alpha \quad \text{and} \quad K\frac{\alpha}{\beta} = \frac{1}{\beta}\alpha.$$

Evidently, $Kxq = xab(-\cos\theta - \varepsilon\sin\theta) = xKq.$
Making $x = -1$, we have

$$K(-q) = -Kq.$$

68.
$$K\frac{1}{\dfrac{\alpha}{\beta}} = K\frac{\beta}{\alpha} = T\frac{\beta}{\alpha}\frac{1}{U\dfrac{\beta}{\alpha}} = T\frac{\beta}{\alpha}\cdot U\frac{\alpha}{\beta};$$

$$\frac{1}{K\dfrac{\alpha}{\beta}} = \frac{1}{T\dfrac{\alpha}{\beta}\,\dfrac{1}{U\dfrac{\alpha}{\beta}}} = T\frac{\beta}{\alpha}U\frac{\alpha}{\beta}; \qquad\qquad \therefore \quad K\frac{1}{q} = \frac{1}{Kq}.$$

Confirm this by diagram, using $\alpha\beta$ instead of $\dfrac{\alpha}{\beta}$.

69. Collecting the q, q^{-1}, $-q$, and Kq into a table, we have:

			T	θ_1	U
$\frac{\alpha}{\beta}$	q	$\frac{a}{b}(-\cos\phi+\varepsilon\sin\phi)$	Tq	ϕ	Uq
$\frac{\beta}{\alpha}$	q^{-1}	$\frac{b}{a}(-\cos\phi-\varepsilon\sin\phi)$	$\frac{1}{Tq}$	$-\phi$	$\frac{1}{Uq}$
$\frac{-\alpha}{\beta}$	$-q$	$\frac{a}{b}(\cos\phi-\varepsilon\sin\phi)$	Tq	$\pi+\phi$	$\frac{1}{Uq}$
$\frac{1}{\beta}\alpha$	Kq	$\frac{a}{b}(-\cos\phi-\varepsilon\sin\phi)$	Tq	$-\phi$	$\frac{1}{Uq}$

θ_1 indicates the angle from the multiplier to the multiplicand, and not the angle between α and β.

From this table it is evident that if $\phi=0$, $KS=S$, i.e., *The conjugate of a scalar is the scalar itself.*

If $\phi=\frac{\pi}{2}$, $KV=-V$, *The conjugate of a vector is its opposite.*

70. $RRq=\left(\frac{1}{q}\right)^{-1}=q$, The reciprocal of the reciprocal

$-(-q)=q$, The opposite of the opposite

$KKq=q$, The conjugate of the conjugate

} is the quaternion itself.

71. Equality of quaternions. If α, β, γ, δ are four coplanar unit vectors with the same angle θ between α and β, and γ and δ, then

$$\alpha\beta=-\cos\theta+\varepsilon\sin\theta, \qquad \gamma\delta=-\cos\theta+\varepsilon\sin\theta,$$

or

$$\alpha\beta=\gamma\delta.$$

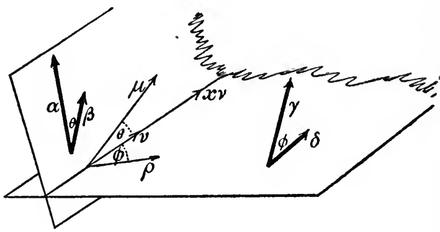
Similarly, $$\frac{\alpha}{\beta}=\frac{\gamma}{\delta}=\cos\theta-\varepsilon\sin\theta.$$

Hence, *Revolving the factors of a quaternion in the plane of the quaternion does not alter the quaternion.*

Quaternions having the same S and V parts must be the same. Diplanar quaternions, that is, quaternions not having their factors in the same plane, cannot have the same V part, and therefore cannot be equal. To be equal, two quaternions must be coplanar.

72. Two quaternions can always be transformed so as to have the same vector for the numerator of one and the denominator of the other, or what is the same thing, the multiplicand of one the multiplier of the other.

For moving the quaternions in their planes until the denominator of one and the numerator of the other lie



along the line of intersection of the planes, a vector along this line can be taken as the numerator of one quaternion quotient and the denominator of the other. Thus

$$\frac{\alpha}{\beta}=\frac{\mu}{\nu}. \qquad \frac{\gamma}{\delta}=\frac{x\nu}{\rho}=\frac{\nu}{\frac{\rho}{x}}. \qquad \text{Q.E.D.}$$

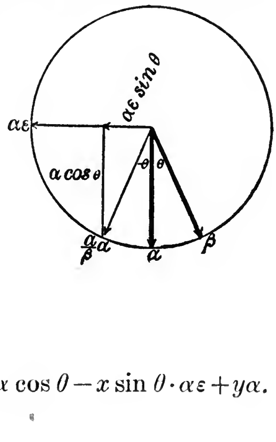
CHAPTER VII

THE QUATERNION AS A MULTIPLIER

73. *Into the reciprocal of its multiplicand factor.*

$$\frac{\alpha}{\beta}\beta = q\beta = \alpha.$$

Result: *The quaternion multiplier turns the multiplicand through the angle* $-\theta$, in the plane of the quaternion.



74. *Into its mulitplier factor.*

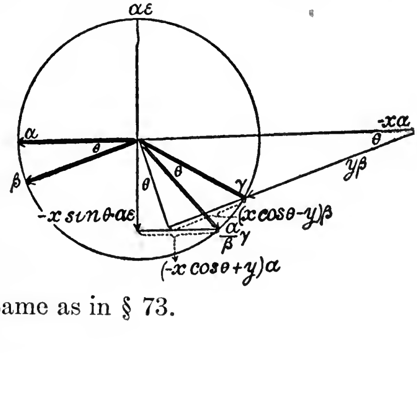
$$q\alpha = \frac{\alpha}{\beta}\alpha = \alpha \cos\theta - \varepsilon\alpha \sin\theta$$
$$= \alpha\frac{1}{\beta}\alpha = \alpha\cos\theta + \alpha\varepsilon \sin\theta.$$

[§ 51.

Result: Same as in § 73.

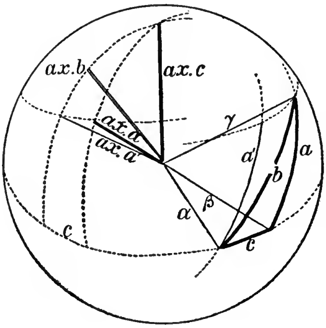
75. *Into any coplanar factor,* γ.

$$\frac{\alpha}{\beta}\gamma = \frac{\alpha}{\beta}(-x\alpha + y\beta) = -x\alpha\cdot\frac{1}{\beta}\alpha + y\frac{\alpha}{\beta}\beta$$
$$= -x\alpha(\cos\theta + \varepsilon\sin\theta) + y\alpha = -x\alpha\cos\theta - x\sin\theta\cdot\alpha\varepsilon + y\alpha.$$



Result: Same as in § 73.

76. *Into any quaternion.* Let the quaternions be reduced, § 72, to a common numerator and denominator respectively, viz.:



$$\frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma}=\frac{\alpha}{\gamma}.$$

Denote the arcs which determine the planes of the quaternions by c, a, b respectively, as shown in the figure. The effect of the quaternion whose axis is $ax\cdot c$ operating upon the quaternion whose axis is $ax\cdot a$ is to give a new quaternion whose axis is $ax\cdot b$.

This can be analyzed as follows: Suppose we move the arc a horizontally along c, being careful not to change its inclination to c, until it takes the position a'. The $ax\cdot a$ will move horizontally around $ax\cdot c$ (conical revolution) until it takes the position $ax\cdot a'$. Then as we turn a' down to coincidence with b around α as an axis, $ax\cdot a'$ will move up the arc passing through $ax\cdot c$ (the equator of α) until it takes the position $ax\cdot b$.

That is, the multiplication of $\frac{\alpha}{\beta}$ into $\frac{\beta}{\gamma}$ *turns the axis of* $\frac{\beta}{\gamma}$ horizontally, i.e., parallel to the plane of $\frac{\alpha}{\beta}$ (conical revolution) through an angle equal to that between α and β, and raises or lowers it through an angle which depends entirely upon $\frac{\beta}{\gamma}$. Whatever the position of γ with reference to β, the horizontal revolution is always the same, dependent upon α and β.

Hence, the *effect of* $\frac{\alpha}{\beta}$ *as a multiplier is generally to revolve the axis of the multiplicand conically through a definite angle around the axis of the multiplier.* This angle ($-\theta$ here), the angle through which the axis of the operand is moved, is called the **angle of the quaternion,** and is designated by $\angle q$, or in these notes by D.

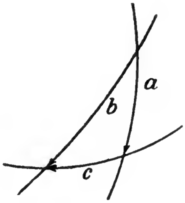
77. Since the effect of a quaternion, q, as a multiplier is to revolve the multiplicand (axis) through the angle $\angle q$ around the $ax \cdot q$, the arc of a great circle which measures $\angle q$ on the equator of $ax \cdot q$, can be taken as an indirect measure of Uq.

Just as a quaternion can be revolved in its plane without alteration of value, so the arc representing it can be moved in its *great circle* without alteration of value.

Arcs in different great circles cannot be equal. If they are semicircles, any semicircle will have the same effect.

78. Representing the quaternions $\frac{\alpha}{\beta}, \frac{\beta}{\gamma}, \frac{\alpha}{\gamma}$, by their arcs (arc strokes) c, a, b respectively, we have, considering the arcs as arc strokes on the sphere surface similar to the plane strokes of § 12,

$$\frac{\alpha}{\beta} \cdot \frac{\beta}{\gamma} = \frac{\alpha}{\gamma} \quad \text{or} \quad a + c = b,$$



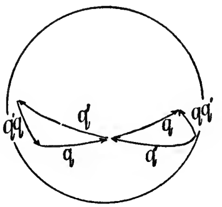
and we have reduced the multiplication of quaternions to the addition of arc strokes.

In the same way

$$\frac{\beta}{\alpha} \cdot \frac{\alpha}{\gamma} = \frac{\beta}{\gamma}, \quad b + (-c) = a.$$

79. Notice carefully that the arc strokes corresponding to the quaternion factors read from the left, are *added up from the right.*

80. Evidently from the figure qq' is quite a different arc from $q'q$, and hence, *The addition of arc strokes is not commutative,* unless they are coplanar; and correspondingly, *the multiplication of quaternions is not commutative,* unless the quaternions are coplanar.



Rule for multiplication by arc strokes. To the arc stroke of the multiplicand add the arc stroke of the multiplier.

Rule for division by arc strokes. Reverse the arc stroke of the divisor and add the arc stroke of the dividend.

Show that

$$qKq = Kq \cdot q. \qquad q(-q) = -q \cdot q.$$

$$q\frac{1}{q} = \frac{1}{q}q = qq^{-1} = q^{-1}q.$$

81. From this we see that the *product of a quaternion, is a quaternion,* the sum of two arc strokes being an arc stroke.

Take two quaternions in the form of a scalar plus a vector, and show that their product is a scalar plus a vector, and therefore, § 61, a quaternion.

82. By multiplying the various forms of a quaternion, $\alpha\beta$, $\beta\alpha$, $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$, etc., into some vector, whatever will give the result most readily, we find the following angles for the quaternions, θ being the angle from α to β, θ_1 the angle from the multiplier to the multiplicand, and D the angle of the quaternion.

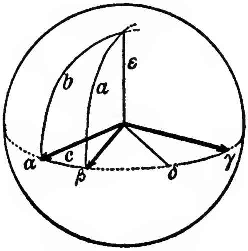
q	D	θ_1	q	D	θ_1
$\alpha\beta$	$\pi-\theta$	θ	$\frac{1}{\beta}\alpha$	θ	$\pi-\theta$
$\beta\alpha$	$\pi+\theta$	$-\theta$	$\frac{1}{\alpha}\beta$	$-\theta$	$\pi+\theta$
$\frac{\alpha}{\beta}$	$-\theta$	$\pi+\theta$	$-\alpha\beta$	$-\theta$	$-\pi+\theta$
$\frac{\beta}{\alpha}$	θ	$\pi-\theta$			

Inspection of the table shows that: The angle, D, of a quaternion is the supplement of the angle between the factors measured from the multiplier to the multiplicand.

83. Representing the angle measured from α to β by θ, and the angle of the quaternion by D, the angle of $\alpha\beta$ by ψ, the angle of $\frac{\alpha}{\beta}$ by ϕ, and abbreviating $\cos D+\varepsilon \sin D$ by $\operatorname{ces} D$, we get the following table:

$\alpha\beta$	$\operatorname{ces} \psi$	$\operatorname{ces} (\pi+\phi)$	$\operatorname{ces} (\pi-\theta)$	$\operatorname{ces} D$
$\beta\alpha$	$\operatorname{ces} (2\pi-\psi)$	$\operatorname{ces} (\pi-\phi)$	$\operatorname{ces} (\pi+\theta)$	$\operatorname{ces} D$
$\frac{\alpha}{\beta}$	$\operatorname{ces} (\pi+\psi)$	$\operatorname{ces} \phi$	$\operatorname{ces} (-\theta)$	$\operatorname{ces} D$
$\frac{\beta}{\alpha}$	$\operatorname{ces} (\pi-\psi)$	$\operatorname{ces} (2\pi-\phi)$	$\operatorname{ces} \theta$	$\operatorname{ces} D$
$\frac{1}{\beta}\alpha$	$\operatorname{ces} (\pi-\psi)$	$\operatorname{ces} (-\phi)$	$\operatorname{ces} \theta$	$\operatorname{ces} D$
$\frac{1}{\alpha}\beta$	$\operatorname{ces} (\pi+\psi)$	$\operatorname{ces} \phi$	$\operatorname{ces} (-\theta)$	$\operatorname{ces} D$

84. Since a vector is the special case of a quaternion, one whose angle is $\frac{\pi}{2}$, we can represent a vector by an arc stroke of $\frac{\pi}{2}$ on its equator. As an example take the case $\frac{\alpha}{\beta}\gamma$.



$$\text{arc } \frac{\alpha}{\beta} = c, \qquad \text{arc } \gamma = a,$$

$$\frac{\alpha}{\beta}\gamma = a + c = b = \text{arc } \delta,$$

and γ has been revolved through $\angle q$.

85. Collecting the results of these articles we have the table on the following page, where

$$\theta_1 = \pi - D, \qquad D = \angle q, \qquad \phi = \angle \alpha\beta, \qquad \phi = \angle\frac{\alpha}{\beta} = \angle\frac{1}{\alpha}\beta,$$

$$\theta = \angle\frac{\beta}{\alpha} = \angle\frac{1}{\beta}\alpha. \qquad \theta_1 = \text{angle from first factor to the second.}$$

θ = angle from α to β.

86. Since quaternions are multiplied by adding their arc strokes, the square, q^2, of a quaternion will have for its representative arc stroke twice the stroke of q. Similarly, one-half the arc stroke will be representative of $q^{\frac{1}{2}}$. But since there are two arcs D and $-(2\pi - D)$, $q^{\frac{1}{2}}$ will have two representative strokes, $\frac{1}{2}D$ and $-\pi + \frac{D}{2}$, either of which if doubled will give the stroke of q. Hence, *Rule for extraction of square roots of unit quaternions: Halve the angles of the quaternion.*

q	θ_1	Uq	D of cεs D	Tq	$\angle q$			Rq	Sq	Vq
$\alpha\beta$	θ	$-\cos\theta+\sin\theta$	$\pi-\theta$	ab	$\pi-\theta$	ψ	$\pi+\phi$	$\frac{1}{\beta\alpha}$	$-ab\cos\theta$	$ab\sin\theta\cdot\varepsilon$
$\beta\alpha$	$-\theta$	$-\cos\theta-\varepsilon\sin\theta$	$\pi+\theta$	ab	$\pi+\theta$	$-\psi$	$\pi-\phi$	$\frac{1}{\alpha\beta}$	$-ab\cos\theta$	$-ab\sin\theta\cdot\varepsilon$
$\frac{\alpha}{\beta}$	$\pi+\theta$	$\cos(-\theta)+\varepsilon\sin(-\theta)$	$-\theta$	$\frac{a}{b}$	$-\theta$	$\pi+\psi$	ϕ	$\frac{\beta}{\alpha}$	$\frac{a}{b}\cos\theta$	$-\frac{a}{b}\sin\theta\cdot\varepsilon$
$\frac{\beta}{\alpha}$	$\pi-\theta$	$\cos\theta+\varepsilon\sin\theta$	θ	$\frac{b}{a}$	θ	$\pi-\psi$	$-\phi$	$\frac{\alpha}{\beta}$	$\frac{b}{a}\cos\theta$	$\frac{b}{a}\sin\theta\cdot\varepsilon$
$\frac{1}{\alpha}\beta$	$\pi+\theta$	$\cos\theta-\varepsilon\sin\theta$	$-\theta$	$\frac{b}{a}$	$-\theta$	$\pi+\psi$	ϕ	$\frac{1}{\beta}\alpha$	$\frac{a}{b}\cos\theta$	$-\frac{b}{a}\sin\theta\cdot\varepsilon$
$\frac{1}{\beta}\alpha$	$\pi-\theta$	$\cos\theta+\varepsilon\sin\theta$	θ	$\frac{a}{b}$	θ	$\pi-\psi$	$-\phi$	$\frac{1}{\alpha}\beta$	$\frac{b}{a}\cos\theta$	$\frac{a}{b}\sin\theta\cdot\varepsilon$
$-\alpha\beta$	$\pi+\theta$	$\cos\theta-\varepsilon\sin\theta$	$-\theta$	ab	$-\theta$	$\pi+\psi$	ϕ	$\frac{-1}{\beta\alpha}$	$ab\cos\theta$	$-ab\sin\theta\cdot\varepsilon$
ε	$\frac{\pi}{2}$	ε	$\frac{\pi}{2}$	1	$\frac{\pi}{2}$					

87. Since the angle of the quaternion is the supplement of the angles measured from the multiplier to the multiplicand, that is, $\pi - D = \theta_1$, a better working rule would be:

For square root. Revert the multiplicand and halve the angles between this and the multiplier, for the factors of $q^{\frac{1}{2}}$.

Similarly, a working *rule for squaring quaternions: Double the angle between the factors and revert the multiplicand in its new position* for the factors of q^2.

In $\delta^2 = \beta\gamma$, or $\frac{\delta}{\beta} = \sqrt{\frac{\gamma}{\beta}}$, δ bisects the angle between β and γ.

88. Using the arcs of quaternions, show that

(1) $\text{arc } q^2 = \text{arc } q + \text{arc } q = 2 \text{ arc } q \pm 2n\pi.$

(2) $\text{arc } q^{\frac{1}{2}} = \frac{1}{2}(\text{arc } q \pm 2n\pi).$

(3) $\text{arc } q^n = \text{arc } q + \text{arc } q + \ldots = n \text{ arc } q \pm 2n\pi.$

(4) $\text{arc } q^{\frac{1}{n}} = \frac{1}{n}(\text{arc } q \pm 2n\pi).$ (5) $(pq)^2 \neq p^2q^2.$

89. When the quaternion is a scalar, its arc is some multiple of π. All coinitial arcs of the same length, π, for instance, will represent will represent the same scalar, -1, however the arcs are situated; that is, whatever the direction of their axes. In halving these arcs to extract square roots we will get an infinite number of answers, one for each arc, each corresponding to its own axis. Or in other words, there are an infinite number of vectors the square of which is a scalar.

Similar reasoning shows that there are an infinite number of quaternion nth roots of a scalar. On this account the roots of a scalar are limited to scalars.

From Eq. (2) of § 88 we get $(pq)^{\frac{1}{2}} = \cos \frac{D}{2} + \varepsilon \sin \frac{D}{2}$;

repetition of which gives $(pq)^\circ = \cos 0 = 1$,

thus illustrating for the quaternion number the familiar fact of algebra.

CHAPTER VIII

PRODUCTS OF QUATERNIONS

90. Having three quaternions, let the first two be reduced, § 72, to a common denominator and numerator respectively, viz., $q=\frac{\alpha}{\beta}$, $r=\frac{\beta}{\gamma}$; and the latter two also, viz., $r=\frac{\mu}{\nu}$, $s=\frac{\nu}{\rho}$, where, evidently, since they represent the same quaternion,

$$\frac{\beta}{\gamma}=\frac{\mu}{\nu}.$$

The product of the three quaternions is

$$\frac{\alpha}{\beta}\left(\frac{\beta}{\gamma}=\frac{\mu}{\nu}\right)\frac{\nu}{\rho}=qrs.$$

Associating the first two factors, we have,

$$qr\cdot s=\frac{\alpha}{\beta}\frac{\beta}{\gamma}\cdot\frac{\nu}{\rho}=\frac{\alpha}{\gamma}\cdot\frac{\nu}{\rho}=\alpha\cdot\frac{1}{\gamma}\nu\cdot\frac{1}{\rho}.$$

Associating the second two, we have

$$q\cdot rs=\frac{\alpha}{\beta}\left(\frac{\mu}{\nu}\cdot\frac{\nu}{\rho}\right)=\frac{\alpha}{\beta}\cdot\frac{\mu}{\rho}=\alpha\cdot\frac{1}{\beta}\mu\cdot\frac{1}{\rho}.$$

But if $\frac{\beta}{\gamma}=\frac{\mu}{\nu}$ then $\frac{\gamma}{\beta}=\frac{\nu}{\mu}$.

Multiplying this by $\frac{1}{\gamma}$ and into μ, we have

$$\frac{1}{\beta}\mu=\frac{1}{\gamma}\nu.$$

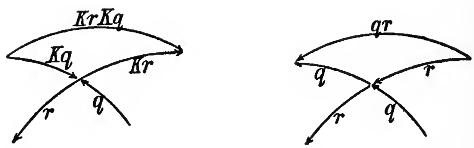
Whence $$\frac{\alpha}{\gamma}\cdot\frac{\nu}{\rho}=\frac{\alpha}{\beta}\cdot\frac{\mu}{\rho} \quad \text{or} \quad qr\cdot s=q\cdot rs,$$

or *quaternions are associative in multiplication.*

91. Since, § 67,

$$Kq=\frac{Tq}{Uq}=Tq(-\cos\theta-\varepsilon\sin\theta)=Tq(-\cos(-\theta)+\varepsilon\sin(-\theta)),$$

the arc representing Kq will be the reverse of that representing q, or $\quad \text{arc } Kq=-\text{arc } q.$



Inspection of the figures shows that

$$-\text{arc } qr=+\text{arc } KrKq.$$

Hence

$$K(qr)=KrKq, \qquad \text{or}$$

The *conjugate of the product of two quaternions is the product of their conjugates in a reverse order.*

$$K(\alpha\beta)=-\beta\cdot-\alpha=\beta\alpha. \qquad \text{Conf. § 67.}$$

92. Coplanar quaternions. Using arc strokes for q and r show that

$$qr=rq. \qquad \text{Conf. § 80.}$$

Therefore, since a quaternion and its reciprocal, opposite, conjugate, and any power are coplanar,

$$qKq = Kq \cdot q. \qquad q^{-1} \cdot -q = -q \cdot q^{-1}. \qquad Kq \cdot q^n = q^n Kq.$$

93. Using $q = Sq + Vq$, $r = Sr + Vr$, show that qr is a quaternion, the sum of a scalar and a vector.

Show that

$$Sqr = SqSr + S \cdot VqVr. \qquad V \cdot VqVr = -V \cdot VrVq.$$

$$Vqr = SqVr + SrVq + V \cdot VqVr. \qquad qr \neq rq. \quad \text{unless coplanar.}$$

$$Srq = SrSq + S \cdot VrVq. \qquad T \cdot qr = TqTr.$$

$$Vrq = SrSq + SqSr + V \cdot VrVq. \qquad V \cdot qr = VqVr.$$

$$Srq = Sqr. \qquad S \cdot qr \neq SqSr,$$

unless $S \cdot VqVr = 0$, i.e., unless the planes of the two quaternions are perpendicular.

$$Vqr \neq Vrq,$$

unless $V \cdot VqVr = 0$, i.e., unless the quaternions are coplanar.

Since

$$(p+q)(r+s) = pr + qr + ps + qs = pr + ps + qr + qs = \text{etc.},$$

therefore the distributive and associative law applies to quaternions, verified by resolving the quaternions into their vector factors and applying the results of § 52.

CHAPTER IX

VERSORS

94. $U\alpha\beta = -\cos\theta + \varepsilon\sin\theta = \cos D + \varepsilon\sin D$, turns any quaternion to which it is applied as a multiplier, horizontally through the angle D.

If $D = \frac{\pi}{2}$, then $U\alpha\beta = \varepsilon^1$ turns the multiplicand quaternion through the angle $\frac{\pi}{2}$. Applied again it turns it through another right angle, or in all through the angle π. That is ε turns the multiplicand through one rt. $\angle$, ε^2 turns it through two rt. $\angle$'s, and so on.* Hence, by the law of mathematical continuity, $\varepsilon^{\frac{1}{2}}$ should be the symbol of turning through $\frac{1}{2}$ rt. $\angle$, etc. If ϕ is expressed in radians, $\varepsilon^{\frac{2}{\pi}\phi}$ turns through the angle ϕ.

If ε is perpendicular to its operand, we recognize the familiar equations,

$$ij = k, \quad i^2 j = -j, \quad i^3 j = -k, \text{ etc.}$$

According to this notation then,

$$U\alpha\beta = \cos D + \varepsilon\sin D = \varepsilon^{\frac{2}{\pi}D}.$$

---

* Remember that ε^2 does not mean here ε multiplied by itself, but ε applied twice to some operand. See Appendix.

95. Since ε and $-\varepsilon$ are opposed, $\varepsilon^{-\frac{2}{\pi}D}$, which turns **clock-wise** from the upper side of the plane, must have the same effect as $(-\varepsilon)^{\frac{2}{\pi}D}$, which turns counterclockwise from the lower side of the plane. Hence

$$\varepsilon^{-\frac{2}{\pi}D}=(-\varepsilon)^{\frac{2}{\pi}D}, \qquad \text{or} \qquad e^{-\beta}=(-\varepsilon)^{\beta}.$$

If $D=\frac{\pi}{2}$, then $\varepsilon^{-1}=-\varepsilon$, or to put it in the more familiar form,

$$i^{-1}=-i.$$

96. Again, § 85,

$$U\frac{\beta}{\alpha}=\cos\theta+\varepsilon\sin\theta=\varepsilon^{\frac{2}{\pi}\theta},$$

and

$$U\frac{\alpha}{\beta}=\cos(-\theta)+\varepsilon\sin(-\theta)=\varepsilon^{-\frac{2}{\pi}\theta};$$

therefore,

$$\varepsilon^{-\theta}=\frac{1}{e^{\theta}}.$$

From these $\quad \varepsilon^{-\theta}=\cos(-\theta)+\varepsilon\sin(-\theta)=(-\varepsilon)^{\theta},$

$$-(\varepsilon^{\theta})=-\cos\theta-\varepsilon\sin\theta.$$

Hence in general,

$$(-\varepsilon)^{\theta}=\varepsilon^{-\theta}\neq-(\varepsilon^{\theta}),$$

unless $\theta=1$, in which case, as before, $\varepsilon^{-1}=-\varepsilon$.

97. The unit portion of a *quaternion* (considered as the symbol of an operation to be performed by the reader) results in the turning of the operand, and is therefore aptly termed a **versor.**

It must not be forgotten that *all vectors* (considered as symbols of an operation to be performed by the reader) *are versors,* as indeed all vectors are special cases of quaternions. The evanescence of the scalar part does not affect its versorial character.

i, j, and k are called **quadrantal versors.**

98. The method of expressing the unit part of a quaternion as the power of a vector gives a convenient method of indicating multiplication and division of *coplanar quaternions.*

$$Uq = \cos\phi + \varepsilon \sin\phi = \varepsilon^{\frac{2}{\pi}\phi}. \qquad Uq' = \cos\psi + \varepsilon \sin\psi = \varepsilon^{\frac{2}{\pi}\psi}.$$

$$\therefore \quad UqUq' = (\cos\phi + \varepsilon \sin\phi)(\cos\psi + \varepsilon \sin\psi) = \varepsilon^{\frac{2}{\pi}\phi}\varepsilon^{\frac{2}{\pi}\psi}$$

$$= \cos(\phi + \psi) + \varepsilon \sin(\phi + \psi) \qquad = \varepsilon^{\frac{2}{\pi}(\phi + \psi)}.$$

Similarly, $$\varepsilon^{\phi} \cdot \varepsilon^{-\psi} = \frac{\varepsilon^{\phi}}{\varepsilon^{\psi}} = \varepsilon^{\phi - \psi},$$

and $$(\varepsilon^{\phi})^m = \varepsilon^{m\phi}.$$

Hence, *The algebraic law of indices holds good for versors.*

Hence, *The angle of the product of two coplanar quaternions is the sum of the angles of the two factors.*

99. According to this notation,

$$\alpha\beta = 4(-\cos 30° + \varepsilon \sin 30°)$$

$$= 4(\cos 150° + \varepsilon \sin 150°) = 4\varepsilon^{\frac{5}{3}}.$$

$$\alpha\beta = 2(\cos 60° - \varepsilon \sin 60°)$$

$$= 2(\cos(-60°) + \varepsilon \sin(-60°)) = 2\varepsilon^{-\frac{2}{3}}.$$

Show that $q^2 = \{Tq(\cos\phi + \varepsilon\sin\phi)\}^2 = Tq)^2\varepsilon^{\frac{4\phi}{\pi}}$

$$= (Tq)^2(\cos 2\phi + \varepsilon\sin 2\phi).$$

$$q^n = (Tq)^n(\cos n\phi + \varepsilon\sin n\phi).$$

Hence any quaternion can be written, where ρ is some vector,

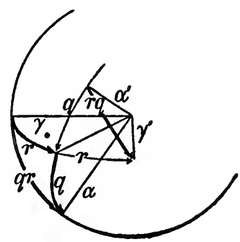
$$\rho^t = \rho^{\frac{2\phi}{\pi}}.$$

100. In the diagram to § 57,

$$\varepsilon^{\frac{2A}{\pi}}\gamma = \beta. \qquad \varepsilon^{\frac{2B}{\pi}}\alpha = \gamma. \qquad \varepsilon^{\frac{2C}{\pi}}\beta = -\alpha.$$

$$\therefore\quad \varepsilon^{\frac{2(A+B+C)}{\pi}} = -1 \quad\text{or}\quad A+B+C = \pi.$$

101. *What will convert rq into qr?* Assume for the moment that the tensors are unity. Let the strokes of q and r be as shown. Then the strokes of qr and rq will take the positions shown. From the table of § 83, we find



$$q = \frac{\alpha}{\beta}, \quad r = \frac{\gamma}{\beta}, \quad qr = \frac{\alpha}{\gamma}, \quad rq = \frac{\gamma'}{\alpha'},$$

and the problem becomes: To convert $\frac{\gamma'}{\alpha'}$ into $\frac{\alpha}{\gamma}$. Now

$$\frac{\alpha}{\gamma} = \frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma}, \qquad = \frac{\alpha}{\beta}\cdot\frac{\gamma'}{\beta'}, \qquad \text{§ 71.}$$

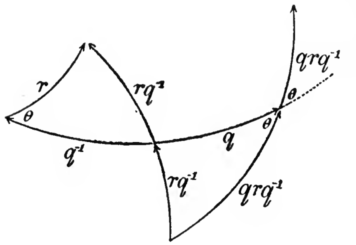
$$= \frac{\alpha}{\beta}\cdot\frac{\gamma'}{\alpha'}\cdot\frac{\alpha'}{\beta'}, \qquad = \frac{\alpha}{\beta}\cdot\frac{\gamma'}{\alpha'}\cdot\frac{\beta}{\alpha}. \qquad \text{§ 71.}$$

Or substituting for these their equivalents, § 69,

$$qr = q(rq)q^{-1}.$$

102. Meaning of $q(r)q^{-1}$. Indicate q and r by their arc strokes, as shown in the diagram. The other strokes will have the values indicated.

The two triangles have two sides and the included angle respectively equal, and, as shown in the figure, by the angle θ, qrq^{-1} is r revolved in the plane of q through $2\angle q$. This amounts to revolving the axis of r conically around the axis of q through $2\angle q$.



Hence qrq^{-1} differs from r only in being rotated through a certain angle. Hence $q(\ )q^{-1}$ may be aptly called a **rotator,** since it rotates any quaternion inserted in the parenthesis.

This is a special case of a more general function, called a nonion, which we shall meet with farther on.

103. If r is some multiple of q, say $q'q$, then

$$q(q'q)q^{-1} = qq'qq^{-1} = qq'.$$

Hence qq' is $q'q$ rotated through the angle $2\angle q$ in the plane of q.

Compare this with the figure of § 80, and see how they agree.

Similarly, $q^{-1}rq$ *rotates* r *negatively through* $2\angle q$.

Since, $\quad Tq\cdot T\dfrac{1}{q} = 1,\quad$ therefore, $\quad T\cdot qrq^{-1} = Tr,$

so that in the discussion above only the unit or versor parts needed to be considered.

104. $$q\beta q^{-1} = \beta',$$

revolves the plane of β, or β, conically around the axis of q through $2\angle q$.

105. $$\alpha\beta\alpha^{-1} = \beta',$$

revolves β conically around α through the angle 2π, that is, turns β in the plane of $\alpha\beta$ to a corresponding position on the other side of α.

106. *Exercises.* If $\alpha\gamma = q$, show that $\alpha q \alpha^{-1} = Kq$.

If α, β, γ are coplanar, show that $\alpha \cdot \beta\gamma \cdot \alpha^{-1} = \alpha^{-1}\beta\gamma\alpha = K\beta\gamma$.

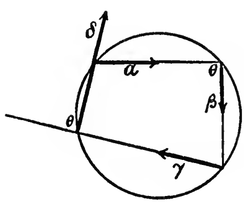
$$(qrq^{-1})^x = qr^xq^{-1}. \qquad (jkj^{-1})^2 = jk^2j^{-1}.$$

$$rqBq^{-1}r^{-1} = rq \cdot B \cdot (rq)^{-1}.$$

CHAPTER X

INTERPRETATION OF VECTOR EQUATIONS

107. $\alpha\beta\gamma = \alpha\beta^2\beta^{-1}\gamma = -b^2\frac{\alpha}{\beta}\gamma = \delta$, α, β, γ coplanar, i.e., $S\alpha\beta\gamma = 0$. $U \cdot -\frac{\alpha}{\beta}$ acting on β turns it through an angle θ into $-\alpha$. Acting on γ it turns it through the same angle into δ. If α, β, γ represent in direction the successive sides of a polygon (which is always possible) then $U \cdot -\frac{\alpha}{\beta}$ must turn γ into a direction coinciding with the fourth side of a polygon inscribed in a circle drawn through the intersection of α, β, and γ. The diagram shows this. Hence δ is in direction the fourth side of the inscribed polygon of which α, β, and γ are the other three.



If the circle passes through three intersections, that circumscribes α β, γ as a triangle, then δ becomes tangent to this circle.

108. Locus of ξ in $\beta = \alpha\xi$.

$$\xi = \alpha^{-1}\beta = S\alpha^{-1}\beta + V\alpha^{-1}\beta. \qquad \therefore \quad S\alpha^{-1}\beta = 0, \quad \xi = V\alpha^{-1}\beta,$$

and ξ is a constant vector perpendicular to α and to β, and therefore locates a point.

109. Locus of ξ in $V\alpha\xi=\beta$.

$$\beta\perp\alpha. \qquad \alpha\xi=S\alpha\xi+V\alpha\xi=x+\beta.$$

$$\therefore \quad \xi=\alpha^{-1}x+\alpha^{-1}\beta=z\alpha+\gamma. \qquad [\text{since } \alpha\perp\beta,$$

and the locus of ξ is a vector through the point determined by $\gamma(\perp\alpha, \perp\beta)$, and parallel to α.

Notice the difference in the reading and interpretation of the equations of this and the previous section. In one β is the whole of $\alpha\xi$, in the other only the vector part.

110. Locus of ξ in $S\alpha\xi=0$, $S\alpha\beta=0$, $S\beta\xi=-c$.

$S\alpha\xi=0$ limits ξ to the plane of α.

$S\alpha\beta=0$ puts β in the same plane.

$S\beta\xi=-c$ or $bx\cos\theta=c$,

where b and x are the tensors of β, ξ respectively, makes the projection of ξ on β a constant, viz., $x\cos\theta=\frac{c}{b}$. Therefore the locus of ξ is a line $\perp$ to β in the plane of α and through the point of β distant from the origin $\frac{c}{b}$.

111. From the table of § 85, we find

$$\alpha\beta=\cos\phi+\varepsilon\sin\phi. \qquad (\alpha+\beta)(\alpha+\beta)=(\alpha+\beta)^2$$

$$\beta\alpha=\cos\phi-\varepsilon\sin\phi. \qquad =\alpha^2+\alpha\beta+\beta\alpha+\beta^2.$$

$$\therefore \quad \alpha\beta+\beta\alpha=2S\alpha\beta=2S\beta\alpha. \qquad =\alpha^2+2S\alpha\beta+\beta^2.$$

$$\alpha\beta-\beta\alpha=2V\alpha\beta=-2V\beta\alpha.$$

112. By §§ 53, 54, the last equation of § 111 becomes the well-known formula for triangle,

$$c^2=a^2+b^2-2ab\cos\theta.$$

113. As in § 111, $(\alpha-\beta)^2=\alpha^2-2S\alpha\beta+\beta^2.$

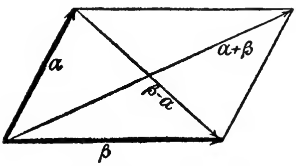
$$\begin{aligned}(\alpha+\beta)(\alpha-\beta)&=\alpha^2-\alpha\beta+\beta\alpha-\beta^2\\&=\alpha^2-(\alpha\beta-\beta\alpha)-\beta^2\\&=\alpha^2-2V\alpha\beta-\beta^2.\end{aligned}$$

$$\begin{aligned}\alpha\beta\cdot\beta\alpha&=(S\alpha\beta+V\alpha\beta)(S\alpha\beta-V\alpha\beta)\\&=(S\alpha\beta)^2-(V\alpha\beta)^2\\&=a^2b^2\cos^2 D-\varepsilon^2\sin^2 D\cdot a^2b^2\\&=a^2b^2(\cos^2 D+\sin^2 D)\\&=(T\alpha\beta)^2.\end{aligned}$$

$$\begin{aligned}\alpha\beta\cdot\beta\alpha&=\alpha\beta^2\alpha && \text{by the associative law.}\\&=\alpha^2\beta^2 && \text{since } \beta^2 \text{ is a scalar.}\\&\neq(\alpha\beta)^2. && \text{Conf. § 88, (5).}\end{aligned}$$

$$\begin{aligned}\alpha\beta\cdot\alpha\beta&=(\alpha\beta)^2\\&=(S+V)^2.\end{aligned}\qquad\begin{aligned}\alpha\beta\cdot\alpha\beta&=\cos^2 D+\varepsilon\, 2\sin D\cos D-\sin^2 D\\&=\cos 2D+\varepsilon\sin 2D. \quad \text{Conf. § 98.}\end{aligned}$$

114. Applications. *If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.*

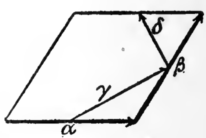


By hypoth.,

$$\begin{aligned}S(\alpha+\beta)(\beta-\alpha)&=0 && \text{§ 56.}\\&=\beta^2-\alpha^2. && \text{§ 113.}\end{aligned}$$

$$\therefore\quad (T\beta)^2=(T\alpha)^2. \qquad \text{Q.E.D.}$$

115. *The joins* of the mid-points of the sides of a rhombus are at right angles.*



$$\begin{aligned}\gamma&=\tfrac{1}{2}(\alpha+\beta).\\\delta&=\tfrac{1}{2}(\beta-\alpha).\\\gamma\delta&=\tfrac{1}{4}(\alpha\beta-\alpha^2+\beta^2-\beta\alpha)\\&=\tfrac{1}{4}(2V\alpha\beta+\beta^2-\alpha^2). \qquad \text{§ 113.}\end{aligned}$$

$$\therefore\quad S\gamma\delta=\tfrac{1}{4}(\beta^2-\alpha^2)=0. \qquad [\text{since } T\alpha=T\beta.$$

$$\therefore\quad \gamma\perp\delta. \qquad \text{Q.E.D.}$$

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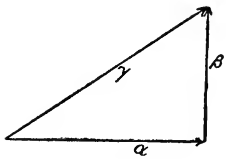
* Join = the line joining.

116. *In any plane triangle to find a side in terms of the other two sides and the opposite angles.*

$$\alpha+\beta=\gamma.$$



Multiplying by (or into) α, (or any vector, in order to get a quaternion)

$$\alpha^2+\alpha\beta=\alpha\gamma.$$

Taking the scalar parts, we have

$-a^2+S\alpha\beta=S\alpha\gamma.$ $\quad -a^2-ab\cos(180°-C)=-ac\cos B.$

$\therefore\quad a=b\cos C+c\cos B.$ (Conf. Trigonom.) Q.E.I.

117. Had we taken the vector parts in § 116, we would have found

$V\alpha\beta=V\alpha\gamma.$ $\quad ab\sin C\cdot\varepsilon=ac\sin B\cdot\varepsilon,$

or $\quad b\sin C=c\sin B.$ (Conf. Trigonom.)

118. In § 116, had we divided by some vector in order to get a quaternion, say γ, then we would have had the same result, $c=a\cos B+b\cos A$.

119. Had we divided by α or β, say α, then if the triangle were right-angled at C, since $S\frac{\beta}{\alpha}=0$, we would get

$$1=S\frac{\gamma}{\alpha}=\frac{c}{a}\cos B, \qquad \text{or} \qquad \cos B=\frac{a}{c}.$$

120. Had we taken the vector parts in § 118, we should have had $\quad a\sin B=b\sin A.$ (Law of Sines, Trig.)

Had we taken vector parts in § 119, then since $\sin C=1$,

$$\sin B=\frac{b}{c}.$$

121. $\sqrt{i}$ is representable by a prolate spheroidal shell with its major axis i and its two minor axes $\frac{1}{2}j$ and $\frac{1}{2}k$.

Repetitions of this process tend toward the unit shell and $i^\circ = 1$.

In the same way $\sqrt{\alpha i} = \sqrt{\alpha}\sqrt{i}$ is representable by a prolate shell whose major axis is i and whose other axes are $\frac{1}{4}j$, $\frac{1}{4}k$, leading similarly to $(\alpha\beta)^\circ = 1$.

From § 113, since,

$$\sqrt{(\alpha\beta)^2} = \sqrt{\cos 2D + \varepsilon \sin 2D} = \cos D + \varepsilon \sin D.$$

$$\therefore \quad (\alpha\beta)^\circ = \cos 0^\circ = 1,$$

the same as above, and agreeing with § 89.

122. Formulæ for reference and practice.

θ = angle from α to β.
D = angle of the quaternion. [§ 76.
$p, q, r \ldots$ = quaternions.

1. $q = Tq(-\cos\theta + \varepsilon \sin\theta) = \alpha\beta$.
2. $\quad = Tq(\cos D + \varepsilon \sin D)$.
3. $\quad = Tq \cdot Uq = Tq\varepsilon^{\frac{2}{\pi}D}$.
4. $q = Sq + Vq$.
5. $Kq = Sq - Vq = Tq(\cos D - \varepsilon \sin D) = Tq\varepsilon^{-\frac{2}{\pi}D}$.
6. $Sq = Tq \cos D = Tq(-\cos\theta)$.
7. $TVq = Tq \sin\theta = Tq \sin D$.
8. $Vq = TVq \cdot UVq = Tq \sin D \cdot \varepsilon$.
9. $\varepsilon^{\theta + \theta'}$.
10. $Sq = \frac{1}{2}(q + Kq)$.
11. $Vq = \frac{1}{2}(q - Kq)$.
12. $(Tq)^2 = qKq = Kq \cdot q = (Sq)^2 - (Vq)^2$.
13. $(Tq)^2 = (Sq)^2 + (TVq)^2$.
14. $q^{-1} = \dfrac{Kq}{(Tq)^2}$.
15. $KKq = q$.
16. $Kx = x$.
17. $Rq = q^{-1}$.
18. $K(-q) = -Kq$.
19. $K\alpha = -\alpha\alpha$.
20. $T\alpha^2 = -a^2$.
21. $S\alpha = 0$.
22. $V\alpha = \alpha$.
23. $K\alpha\beta = \beta\alpha$.
24. $S\alpha\beta = S\beta\alpha$.
25. $V\alpha\beta = -V\beta\alpha$.
26. $\alpha\beta + \beta\alpha = 2S\alpha\beta$.
27. $\alpha\beta - \beta\alpha = 2V\alpha\beta$.

28. $(\alpha \pm \beta)^2 = \alpha^2 \pm 2S\alpha\beta + \beta^2$.
29. $T(pqr \ldots) = Tp \cdot Tq \cdot Tr \ldots$
30. $U(pqr \ldots) = Up \cdot Uq \cdot Ur \ldots$
31. $S(pqr \ldots) = S(qr \ldots p) = S(r \ldots pq) = \ldots$
32. $K(pqr \ldots) = \ldots Kr \cdot Kq \cdot Kp$.
33. $S(xp + yq + \ldots)^* = xSp + ySq + \ldots$
34. $V(xp + yq + \ldots)^* = xVp + yVq + \ldots$
35. $S(q + r + s + \ldots) = Sq + Sr + Ss + \ldots$
36. $V(q + r + s + \ldots) = Vq + Vr + Vs + \ldots$
37. $Vq'q = Sq'Vq + SqVq' + V(Vq'Vq)$.
38. $S(\alpha \cdot \beta\gamma) = S\alpha(S\beta\gamma + V\beta\gamma) = S\alpha V\beta\gamma$.
39. $S\alpha VM\beta = S\alpha M\beta$. [$M = m + \mu$.
40. $= S\alpha(m + \mu)\beta = mS\alpha\beta + S \cdot \alpha\mu\beta$.
41. $= mS\beta\alpha - S\beta\mu\alpha$. [§ 51.
42. $= S\beta m\alpha - S\beta\mu\alpha = S \cdot \beta KM\alpha$. [§ 122 (5).
43. $S(a + \alpha)(b + \beta) = S(b + \beta)(a + \alpha)$.
44. $Spq = Sqp$.
45. $K\alpha\beta\gamma = K(\alpha\beta \cdot \gamma) = K\gamma \cdot K\alpha\beta = -\gamma \cdot \beta\alpha$.
46. $\alpha\beta \cdot \gamma - \gamma \cdot \beta\alpha = 2S\alpha\beta\gamma = -2S\gamma\beta\alpha$.
47. $S\alpha\beta\gamma = S(\alpha\beta \cdot \gamma) = S(p \cdot q) = S(q \cdot p)$.
48. $= S(\gamma\alpha\beta) = S\gamma\alpha\beta = S\beta\gamma\alpha$.
49. $S(\alpha\beta \cdot \gamma) = S(S\alpha\beta + V\alpha\beta)\gamma$.
50. $= S \cdot \gamma V\alpha\beta = -S \cdot \gamma V\beta\alpha$.
51. $= -S \cdot \gamma(V\beta\alpha + S\beta\alpha)$.
52. $= -S\gamma\beta\alpha$.
53. $= -S\beta\alpha\gamma$.
54. $S(\alpha_1\alpha_2 \ldots \alpha)_n = (-1)^n S(\alpha_n \ldots \alpha_2\alpha_1)$.
55. $S\alpha\beta\gamma = S\alpha V\beta\gamma$.
56. $= S \cdot \alpha V\beta\gamma = S \cdot \beta V\gamma\alpha = S\gamma V\alpha\beta$.
57. $\alpha\beta\gamma + \gamma\beta\alpha = 2V\alpha\beta\gamma = 2V\gamma\beta\alpha$.
58. $2V\alpha\beta\gamma = \alpha\beta\gamma + (\alpha\gamma\beta - \alpha\gamma\beta - \gamma\alpha\beta + \gamma\alpha\beta) + \gamma\beta\alpha$. [57.

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* See Appendix.

59. $2V\alpha\beta\gamma = \alpha(\beta\gamma+\gamma\beta) - (\alpha\gamma+\gamma\alpha)\beta + \gamma(\alpha\beta+\beta\alpha).$
60. $\qquad = 2(\alpha S\beta\gamma - \beta S\alpha\gamma + \gamma S\alpha\beta).$ [26.
61. $V\alpha\beta\gamma - \alpha S\beta\gamma = V\alpha\beta\gamma - V\cdot\alpha S\beta\gamma.$
62. $\qquad = V\alpha(\beta\gamma - S\beta\gamma).$
63. $\qquad = V\cdot\alpha V\beta\gamma.$
64. $\qquad = -\beta S\gamma\alpha + \gamma S\alpha\beta.$ [60.
65. $\qquad = -V\cdot V(\beta\gamma)\alpha.$ [62, 25.
66. $\therefore\ V\cdot V(\beta\gamma)\alpha = \beta S\gamma\alpha - \gamma S\alpha\beta.$ [64.
67. $V\cdot(V\alpha\beta)V\gamma\delta = -\gamma S\delta V\alpha\beta + \delta S\cdot(V\alpha\beta)\gamma.$ [63, 64.
68. $\qquad = -\gamma S\alpha\beta\delta + \delta S\alpha\beta\gamma.$ [56, 24.
69. $V\cdot(V\alpha\beta)V\gamma\delta = +\alpha S\cdot\beta V\gamma\delta - \beta S\cdot(V\gamma\delta)\alpha.$ [66.
70. $\qquad = \alpha S\beta\gamma\delta - \beta S\alpha\gamma\delta.$ [56.
71. $\delta S\alpha\beta\gamma = \alpha S\beta\gamma\delta + \beta S\gamma\alpha\delta + \gamma S\alpha\beta\delta.$ [68, 70.
72. $V\cdot V\alpha\beta V\beta\gamma = \gamma S\cdot(V\alpha\beta)\beta - \beta S\cdot\gamma V\alpha\beta.$ [67.
73. $\qquad = \gamma S\beta V\alpha\beta - \beta S\gamma\alpha\beta.$ [24, 56.
74. $\qquad = \beta S\beta\alpha\gamma.$ [$S\beta V\alpha\beta = 0$, 60.
75. $\alpha S\alpha\beta\gamma = V\cdot V\beta\alpha V\alpha\gamma.$ [74.
76. $\beta S\alpha\beta\gamma = V\cdot V\gamma\beta V\beta\alpha.$ [48, 75.
77. $\gamma S\alpha\beta\gamma = V\cdot V\alpha\gamma V\gamma\beta.$
78. $\delta = -(iSi\delta + jSj\delta + kSk\delta).$ [71.

123. In (63), (64), $V\cdot\alpha V\beta\gamma$ is perpendicular to α and coplanar with β, γ. That it is perpendicular to α can be shown by multiplying by α and taking scalars thus,

$$S\alpha(V\cdot\alpha V\beta\gamma) = S\alpha\gamma S\alpha\beta - S\alpha\beta S\alpha\gamma = 0. \quad [63, 64.$$

It is in the plane of $\beta\gamma$ since

$$V\cdot\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma. \quad [63, 64.$$

124. In (58), (60) $V\alpha\beta\gamma$ is of the form $x\alpha + y\beta + z\gamma$ and is therefore the intermediate diagonal of the parallelopiped of which the edges are $\alpha S\beta\gamma$, $-\beta S\alpha\gamma$, $\gamma S\alpha\beta$.

125. In (67), since $S\cdot\gamma S\alpha\beta = S\cdot\delta S\alpha\beta = 0$,

$$\begin{aligned} V\cdot V\alpha\beta V\gamma\delta &= \delta S\{S\alpha\beta\cdot\gamma + V\alpha\beta\cdot\gamma\} - \gamma S\{\delta S\alpha\beta + \delta V\alpha\beta\} \\ &= \delta S(S\alpha\beta + V\alpha\beta)\gamma - \gamma S\delta(S\alpha\beta + V\alpha\beta) \\ &= \delta S\alpha\beta\gamma - \gamma S\delta\alpha\beta, \end{aligned}$$

and it is evident that $V\cdot V\alpha\beta V\gamma\delta$ is coplanar with δ and γ. Moreover, since

$$V\cdot V\alpha\beta V\gamma\delta = V\cdot V\delta\gamma V\alpha\beta = \beta S\delta\gamma\alpha - \alpha S\beta\delta\gamma, \quad [25, 52, 67.$$

it is also coplanar with α and β, and therefore must be along the intersection of the planes of α, β and γ, δ.

126. In (71), $\delta S\alpha\beta\gamma$ is the intermediate diagonal of the parallelopiped of which the three edges are $\alpha S\beta\gamma\delta$, $\beta S\gamma\alpha\delta$, $\gamma S\alpha\beta\delta$.

CHAPTER XI

QUATERNION EQUATIONS OF THE FIRST DEGREE

127. An equation of the first degree with respect to an unknown quaternion X, is that which contains the quaternion to the first power together with known quaternions, either isolated or under the symbols S or V.

The general equation will then have the form

$$\Sigma AXB + \Sigma C \cdot SAXB + \Sigma D \cdot VAXB \cdot E = F.$$

The third term assumes the form of the first two if we replace $VAXB$ by $AXB - SAXB$, so that the general equation reduces to the form

$$\Sigma AXB + \Sigma C \cdot SAXB = F.$$

128. To resolve this equation we decompose the quaternions into their scalar and vector parts. Thus, putting $A = a + \alpha$, $B = b + \beta$, etc., we have

$$\Sigma (a+\alpha)(x+\xi)(b+\beta) + \Sigma (c+\gamma) S \cdot (a+\alpha)(x+\xi)(b+\beta) = d + \delta.$$

The sum of the scalar parts of this will be found to be embraced in the general term $S\alpha\xi$, therefore the scalar part of the equation is $S\alpha\xi = d$.

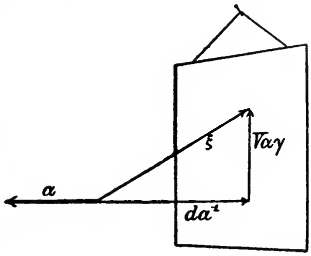
129. Solution of the linear scalar equation $S\alpha\xi = d$.

This may be written, § 122, (33),

$$S\alpha(\xi - d\alpha^{-1}) = 0, \qquad [\alpha d \alpha^{-1} = d.$$

where, § 56, (2), evidently $(\xi - d\alpha^{-1})$ is some vector β, $\perp \alpha$, or $\beta = V\alpha\gamma$, where γ is an arbitrary vector. Therefore,

$$\xi = d\alpha^{-1} + \beta = d\alpha^{-1} + V\alpha\gamma.$$



The geometrical interpretation of this is shown in the diagram, where since γ is arbitrary, the locus of the extremity of ξ must be the plane $\perp \alpha$ and through the point $d\alpha^{-1}$.

130. The vector part of the *second* term in § 128, is, neglecting the Σ,

$$abx\gamma + aS\xi\beta \cdot \gamma + xS\alpha\beta \cdot \gamma + bS\alpha\xi + S\alpha\xi\beta \cdot \gamma.$$

Multiplying out the *first* term we find the vector part to be

$$ax\beta + ab\xi + a\xi\beta + bx\alpha + x\alpha\beta + b\alpha\xi + \alpha\xi\beta.$$

Of these forms

$$a\xi\beta = a(S\xi\beta + V\xi\beta), \qquad b\alpha\xi = b(S\alpha\xi + V\alpha\xi),$$

$$x\alpha\beta = x(S\alpha\beta + V\alpha\beta), \qquad \alpha\xi\beta = S\alpha\xi\beta + V\alpha\xi\beta$$

$$= S\alpha\xi\beta + \alpha S\xi\beta - \xi S\alpha\beta + \beta S\alpha\xi. \quad [\S\ 122, (61), (64).$$

Of the final forms, $abx \cdot \gamma$, $aS\xi\beta \cdot \gamma$, $xS\alpha\beta \cdot \gamma$, $b \cdot S\alpha\xi \cdot \gamma$, $S\alpha\xi\beta \cdot \gamma$, $abx\gamma$, $ax\beta$, $bx\alpha$, $aS\xi\beta$, $xV\alpha\beta$, $bS\alpha\xi$,* $\alpha S\xi\beta$, $\beta S\alpha\xi$, are comprehended under the general form,

$$\alpha S\beta\xi;$$

$ab\xi$, $\xi S\alpha\beta$ are comprehended under the general form,

$$Vm\xi = V(ab + S\alpha\beta + \ldots);$$

---

* Or neglected as purely scalar.

$aV\xi\beta$, $bV\alpha\xi$ are comprehended under the form,

$$V\mu\xi = (V\beta\xi + V\alpha\xi + V\gamma\xi + \ldots).$$

Hence the vector part of the general equation becomes

$$\Sigma\alpha S\beta\xi + Vm\xi + V\mu\xi = \delta,$$

or $$\Sigma\alpha S\beta\xi + V(m+\mu)\xi = \delta. \qquad [§ 122, (34).$$

or $$\Sigma\alpha S\beta\xi + V\cdot M\xi = \delta. \qquad [§ 61.$$

This is generally abbreviated under the functional sign,

$$\Sigma\alpha S\beta\xi + V\cdot M\xi = \phi\xi = \delta.$$

131. Hamilton called this a *linear vector function of the vector* ξ. Considered as an operator its application to ξ has many interesting results, some of which will be investigated in the following pages.

We will consider first the properties of ϕ itself, and then the result of its application to the vector ξ.

For reasons given later this function is called a *strain function.*

132. Properties of ϕ. Since § 122 (35),

$$\beta S\alpha(\xi + \eta + \ldots) = \beta S\alpha\xi + \beta S\alpha\eta + \ldots,$$

and § 122, (36),

$$V\cdot M(\xi + \eta + \ldots) = V\cdot M\xi + V\cdot M\eta + \ldots;$$

$$\therefore \quad \Sigma\beta S\alpha(\xi + \eta + \ldots) + V\cdot M(\xi + \eta + \ldots)$$
$$= (\Sigma\beta S\alpha\xi + V\cdot M\xi) + (\Sigma\beta S\alpha\eta + V\cdot M\eta) + \ldots,$$

or $$\phi(\xi + \eta + \ldots) = \phi\xi + \phi\eta + \ldots,$$

that is, *ϕ is distributive over a sum.*

133. If $\xi=\eta=\ldots$ to n terms,

then $$\phi n\xi=n\phi\xi,$$

that is, ϕ *is commutative with a scalar factor.*

134. If we define ϕ^{-1} by the equation, $\phi^{-1}\phi=1$,* then

$$\phi^{-1}\phi\xi=\phi^{-1}\delta=\xi,$$

and from § 132,

$$\phi^{-1}\phi(\xi+\eta+\ldots)=\phi^{-1}(\phi\xi+\phi\eta+\ldots),$$

or $$\xi+\eta+\ldots\ =\phi^{-1}(\phi\xi+\phi\eta+\ldots).$$

But $\phi\xi=\delta$, etc., and $\phi^{-1}\delta=\xi$, etc., and therefore,

$$\phi^{-1}\delta+\phi^{-1}\delta_1+\ldots=\phi^{-1}(\delta+\delta_1+\ldots).$$

Hence ϕ^{-1} *has the same properties as* ϕ.

By repeated applications we can easily get, for all integral values of k,

$$\phi^k\phi^{-k}\xi=\xi.$$

135. Conjugate strain functions. Operating on $\phi\xi=\Sigma\alpha S\beta\xi+V\cdot M\xi$ with $S\cdot\eta$, we get

$$\begin{aligned}S\cdot\eta\phi\xi&=S\eta(\Sigma\alpha S\beta\xi+V\cdot M\xi)\\&=\Sigma S\cdot\eta\alpha S\beta\xi+S\cdot\eta V\cdot M\xi\\&=\Sigma S\xi\beta S\alpha\eta+S\eta(m+\mu)\xi\\&=\Sigma S\xi\beta S\alpha\eta+S\xi(m+\mu)\eta &&[\text{§ 122, (24), (38).}\\&=\Sigma S\xi\beta S\alpha\eta+S\xi(m-\mu)\eta &&[\text{§ 122 (52).}\\&=S\xi(\Sigma\beta S\alpha\eta+VKM\eta) &&[\text{§ 122 (39), (5).}\\&=S\xi\phi'\eta.\end{aligned}$$

ϕ and ϕ' are called **conjugate strain functions.** They evidently differ in the interchange of the known vector α, β, and of the quaternion M and its conjugate KM.

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* See Appendix.

136. When $\phi = \phi'$, that is, the function conjugates into itself, the functions are called **self-conjugate strain functions,** in which case $S\eta\phi\xi = S\xi\phi\eta$.

137. Types of self-conjugate functions. Since

$$S\xi\phi\eta = S\eta\phi'\xi. \qquad \therefore \quad S\eta\phi\xi = S\xi\phi'\eta,$$

and writing * $(\phi+\phi')\xi$ for $\phi\xi+\phi'\xi$,

$$S\eta(\phi+\phi')\xi = S\xi(\phi+\phi')\eta,$$

which shows that the *operator* $\phi+\phi'$ *is always self conjugate.*

138. Furthermore, $S\xi\phi\xi = S\xi\phi'\xi$,

whence $\qquad S\xi(\phi-\phi')\xi = 0,$

and therefore

the vector $(\phi-\phi')\xi$ *is perpendicular to* ξ

or $\qquad (\phi-\phi')\xi = V\varepsilon\xi,$

where ε is some unknown vector.

Consequently,
$$\phi\xi = \tfrac{1}{2}(\phi+\phi')\xi + \tfrac{1}{2}(\phi-\phi')\xi$$
$$= \tfrac{1}{2}(\phi+\phi)'\xi + \tfrac{1}{2}V\varepsilon\xi,$$

which shows that a *linear vector function of* ξ *differs from a self-conjugate function only by a term of the form* $V\varepsilon\xi$. If it is already self conjugate the vector $\varepsilon = 0$.

139. Since

$$S\cdot\xi\phi\phi'\eta = S\xi\phi(\phi'\eta) \qquad S\cdot\xi\phi\phi'\eta = S\cdot\eta\phi(\phi'\xi) \quad [\S\ 135.$$
$$= S\cdot\phi'\eta\phi'\xi \quad [\S\ 135. \qquad = S\eta\phi\phi'\xi,$$
$$= S\cdot\phi'\xi\phi'\eta; \quad [\S\ 122\ (24).$$

therefore $\phi\phi'$ *is a self-conjugate strain function.*

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* This is allowable because the functional symbol ϕ has the same properties as an algebraic factor (distributive over a sum and commutative with scalar factors) and can be treated like one. See **Appendix.**

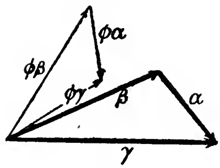
140. If by $(\phi+g)\xi$ we understand $\phi\xi+g\xi$, where g is a scalar, $\phi+g$ is also a linear vector function, since it has all the properties of ϕ as the reader can easily demonstrate for himself. Hence

$$S\cdot\xi(\phi+g)\eta=S\cdot\xi(\phi\eta+g\eta) \qquad =S\cdot\xi\phi\eta+S\cdot\xi g\eta,$$

$$=S\cdot\eta\phi'\xi+gS\cdot\eta\xi \qquad =S\cdot\eta(\phi'+g)\xi,$$

or $\phi'+g$ *is a conjugate function to* $\phi+g$.

141. Application of ϕ to a vector ξ. If an elastic solid, that is, the vector connecting the several elements, be subjected to the operation ϕ, then all its particles, for instance, those determined by the vectors α, β, γ, are displaced to positions determined by the vectors $\phi\alpha$, $\phi\beta$, $\phi\gamma$. In general, any particle whose vector is ξ occupies after the operation the position whose vector is $\phi\xi$.



Also any vector α is displaced to the position $\phi\alpha$, for since

$$\alpha=\gamma-\beta,$$

$$\phi\alpha=\phi\gamma-\phi\beta.$$

Hence, *Any straight line of particles parallel to α, say $x\alpha$, is homogeneously stretched and turned by the operation ϕ into a straight line of particles parallel to $\phi\alpha$, and the ratio of extension is* $\dfrac{x\phi\alpha}{x\alpha}=\dfrac{\phi\alpha}{\alpha}$.

The operation ϕ is called a **strain** and the property that parallel lengths are strained into parallel lengths and stretched proportionally is the physical definition of **linear homogeneous strain.** Portions of the body originally equal, similar and similarly placed remain after the strain equal, similar and similarly placed.

142. If in the general equation of the first degree with respect to an unknown quaternion, § 127, instead of separating the scalar and vector parts, we had merely indicated the vector part, thus

$$\Sigma VQXR = VF,$$

this operator $VQ(\ )R$ must of course be the undeveloped form of ϕ.

If $R = Q^{-1}$, we have the *rotator* of § 102 as a special case of ϕ.

If Q and R degrade into scalars, we have $\Sigma VmX = nX$, or dilatation merely.

A combination of rotation and dilatation makes the strain just defined.

143. Properties of ϕ. To get a slightly different view, let us suppose α, β, γ to be *unit vectors at right angles* to each other. Hence, § 122 (78),

$$\xi = -(\alpha S\alpha\xi + \beta S\beta\xi + \gamma S\gamma\xi).$$

By the definition of homogeneous strain this is changed into

$$\xi' = \phi\xi = -(\alpha' S\alpha\xi + \beta' S\beta\xi + \gamma' S\gamma\xi),$$

where α', β', γ' are three vectors upon which the same proportional distances $S\alpha\xi$, $S\beta\xi$, $S\gamma\xi$ are laid off. This is necessary in order to preserve the similarity required by homogeneous strain. By substituting α, β, γ for ξ, we find

$$\alpha' = \phi\alpha, \quad \beta' = \phi\beta, \quad \gamma' = \phi\gamma.$$

But

$$\phi\alpha = \alpha' = -(\alpha S\alpha\alpha' + \beta S\beta\alpha' + \gamma S\gamma\alpha') = \alpha A_\alpha + \beta B_\alpha + \gamma C_\alpha,$$

$$\phi\beta = \beta' = -(\alpha S\alpha\beta' + \beta S\beta\beta' + \gamma S\gamma\beta') = \alpha A_\beta + \beta B_\beta + \gamma C_\beta,$$

$$\phi\gamma = \gamma' = -(\alpha S\alpha\gamma' + \beta S\beta\gamma' + \gamma S\gamma\gamma') = \alpha A_\gamma + \beta B_\gamma + \gamma C_\gamma,$$

which equations contain *nine* arbitrary constants, $S\alpha\alpha'$, $S\beta\alpha'$, etc., since α', β', γ' are entirely independent of α, β, γ.

Operating on $\phi\xi$ with $S\cdot\eta$ we have

$$S\cdot\eta\phi\xi = -(S\alpha'\eta S\alpha\xi + S\beta'\eta S\beta\xi + S\gamma'\eta S\gamma\xi)$$

$$= \quad -S\cdot\xi(\alpha S\alpha'\eta + \beta S\beta'\eta + \gamma S\gamma'\eta), \qquad [\S\ 122\ (24).$$

where the expression in the parenthesis is a linear vector function of η, which will be shown (§ 144) to depend upon the same nine scalars A_α, B_α, etc., as those in ϕ, and which we may therefore appropriately designate by ϕ', thus

$$\phi'\eta = -(\alpha S\alpha'\eta + \beta S\beta'\eta + \gamma S\gamma'\eta),$$

whence obviously $\qquad S\cdot\eta\phi\xi = S\cdot\xi\phi'\eta,$

as before.

144. Substituting α, β, γ for η in the expression for $\phi'\eta$, we find

$$\phi'\alpha = -(\alpha S\alpha'\alpha + \beta S\beta'\alpha + \gamma S\gamma'\alpha);$$

$$\phi'\beta = -(\alpha S\alpha'\beta + \beta S\beta'\beta + \gamma S\gamma'\beta);$$

$$\phi'\gamma = -(\alpha S\alpha'\gamma + \beta S\beta'\gamma + \gamma S\gamma'\gamma),$$

or using the notation of § 143,

$$\phi'\alpha = \alpha A_\alpha + \beta A_\beta + \gamma A_\gamma;$$

$$\phi'\beta = \alpha B_\alpha + \beta B_\beta + \gamma B_\gamma;$$

$$\phi'\gamma = \alpha C_\alpha + \beta C_\beta + \gamma C_\gamma;$$

which shows that ϕ' depends upon the same nine scalars as ϕ.

145. If α, β, γ are given non-coplanar vectors, then § 29,

$$\xi = x\alpha + y\beta + z\gamma,$$

whence $$\phi\xi = x\phi\alpha + y\phi\beta + z\phi\gamma.$$

The *vector* $\phi\xi$ *is known when the three vectors* $\phi\alpha$, $\phi\beta$, $\phi\gamma$, *are known.* Each of these vectors involves three scalar constants as in the case of ξ, multiples of the reference vectors α, β, γ. Therefore the value of ϕ depends upon nine scalar constants. It has therefore been called a **nonion.**

CHAPTER XII

APPLICATIONS OF ϕ

146. Changes of volume due to ϕ. ρ, $\phi\rho$, $\phi^2\rho$ are, in general, not in one plane, and hence, § 29,

$$\phi^3\rho = m_2\phi^2\rho - m_1\phi\rho + m\rho, \quad . \quad . \quad . \quad . \quad (1)$$

where m_2, m_1, m are scalars independent of ρ. The independence is obvious, since we may put α, β, γ in succession for ρ and thus obtain three equations from which they can be obtained.

From § 59 the volume of the parallelopiped whose three conterminous edges are ρ, $\phi\rho$, $\phi^2\rho$ is

$$-S\cdot\rho\phi\rho\phi^2\rho.$$

After the strain this volume is

$$-S\cdot\phi\rho\phi^2\rho\phi^3\rho.$$

Multiplying Eq. (1) by $\phi\rho\phi^2\rho$ and dividing by $\rho\phi\rho\phi^2\rho$, we have

$$\frac{S\cdot\phi\rho\phi^2\rho\phi^3\rho}{S\cdot\rho\phi\rho\phi^2\rho} = m. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This ratio of dilatation m due to ϕ, the ratio between the volumes after and before the strain, is called the **modulus** of the strain ϕ.

147. By referring to the equations of § 143 we see that the scalar part of the product $\phi\alpha\phi\beta\phi\gamma$ is confined to those terms in which all three of the vectors α, β, γ appear, that is, taking $S\cdot\alpha\beta\gamma=1$,

$$m=\frac{S\cdot\phi\alpha\phi\beta\phi\gamma}{S\cdot\alpha\beta\gamma}$$

$$=A_\alpha B_\beta C_\gamma-A_\alpha B_\gamma C_\beta-B_\alpha A_\beta C_\gamma+B_\alpha A_\gamma C_\beta+$$
$$+C_\alpha(A_\beta B_\gamma-A_\gamma B_\beta).$$

In the same way from § 144,

$$\frac{S\cdot\phi'\alpha\phi'\beta\phi'\gamma}{S\cdot\alpha\beta\gamma}=A_\alpha B_\beta C_\gamma-A_\alpha B_\gamma C_\beta-,\text{ etc.}$$
$$=m.$$

Hence, *Conjugate strains produce equal changes of volume.*

148. Special values of *m*. If $m=1$ there is no change of volume caused by the strain.

If $m=0$, there results a zero volume due to the solid becoming strained into a plane, a line, or a point, and in this case ϕ is called a **null function, singly, doubly**, or **triply null** according as the strain results in a plane, a line or a point. The plane, line, or point into which the solid is strained is called the **strain plane** of ϕ, strain line, etc.

When $m=0$, then $S\phi\alpha\phi\beta\phi\gamma$, § 147, becomes zero and, § 60, $\phi\alpha$, $\phi\beta$, $\phi\gamma$ are coplanar vectors, and we can have a relation,

$$x\phi\alpha+y\phi\beta+z\phi\gamma=0,$$

or § 132, $$\phi(x\alpha+y\beta+z\gamma)=0.$$

The vector, $x\alpha+y\beta+z\gamma$, whose strain, i.e., the result of the application of the strain function ϕ, is zero is called a **null direction** of ϕ. In this case ϕ is **singly null,** unless $\phi\alpha$, $\phi\beta$, $\phi\gamma$ becomes collinear.

149. If there is only one vector, say α, whose strain is zero, i.e., $\phi\alpha=0$, then

$$S\cdot\phi\alpha\phi\beta\phi\gamma=0,$$

and therefore $m=0$, and $\phi\beta$ cannot be parallel to $\phi\gamma$ (i.e., $\phi\beta\neq x\phi\gamma$), for otherwise $\phi(\beta-x\gamma)$ would be equal to zero and there would be a second vector, $\beta-x\gamma$, with a strain of zero, which is contrary to the hypothesis. Hence, if there is only one null vector and $\rho=x\alpha+y\beta+z\gamma$, then

$$\phi\rho=y\phi\beta+z\phi\gamma,$$

and $\phi\rho$ is a vector in a plane parallel to $\phi\beta$, $\phi\gamma$, that is, ϕ is singly null.

150. If there are two (only) vectors whose strain is zero, say $\phi\alpha=0$, $\phi\beta=0$, then $\phi\rho=z\phi\gamma$, which confines $\phi\rho$, the strain of the general vector ρ, to the line parallel to $\phi\gamma$, and in this case ϕ is doubly null.

151. Similarly, if $\phi\alpha=0$, $\phi\beta=0$, $\phi\gamma=0$, then $\phi\rho=0$ for all values of ρ and ϕ is a triply-null function.

152. When ϕ is singly null, say $\phi\alpha=0$, then the general vector of any point in the strain plane of ϕ is, § 149,

$$\phi\rho=y\phi\beta+z\phi\gamma,$$

and all particles that strain into this plane have the vectors,

$$\rho=x\alpha+y\beta+z\gamma,$$

where x is arbitrary, since $\phi\alpha=0$, and therefore the locus of ρ is a line parallel to α. Hence,

A singly null function strains each of its null lines into a definite point of its strain plane. Similarly,

A doubly-null funcion strains each of its null planes into a definite point of its strain line.

153. Special applications of ϕ. *What vectors, if any, are unchanged in position by the strain ϕ?*

If ρ is unchanged by the strain, then

$$\phi\rho = g\rho, \qquad \phi^2\rho = g^2\rho, \qquad \phi^3\rho = g^3\rho,$$

and the equation of § 146 becomes

$$(g^3 - m_2 g^2 + m_1 g - m)\rho = 0, \quad . \quad . \quad . \quad . \quad (1)$$

which must have one real root g_1 and may have three. In other words, the curious fact that however a body may be homogeneously strained, there is always at least one vector whose direction remains unchanged. Hence

$$\phi\rho - g_1\rho = 0,$$

and $$S \cdot \mu\phi\rho - g_1 S\mu\rho = 0, \qquad [\mu = \text{any vector}.$$

and $$S \cdot \rho(\phi'\mu - g_1\mu) = 0 = S \cdot \rho(\phi' - g)\mu. \qquad [§ 135.$$

Hence, § 56 (2), $(\phi' - g_1)\mu$ is perpendicular to ρ.

From this it appears that the operator $(\phi' - g)$ applied to any vector μ throws it into a plane perpendicular to ρ, or, in other words, cuts off the component of the strained μ which is parallel to ρ.

Also, $$\rho \| V \cdot (\phi' - g_1)\mu(\phi' - g_1)\nu, \quad [\mu, \nu \text{ any vectors}.$$

or multiplying out

$$\rho \| V(\phi'\mu\phi'\nu - g_1(\phi'\mu \cdot \nu + \mu\phi'\nu) + g_1^2\mu\nu)$$
$$\| V\phi'\mu\phi'\nu - g_1(V\phi'\mu \cdot \nu + V\mu\phi'\nu) + g_1^2 V\mu\nu.$$

154. Properties of $(\phi' - g)$. Like ϕ, this operator produces a homogeneous strain, as can be shown in a manner similar to that in the case of ϕ. It has the same general properties and has a modulus m_g. Omitting the primes, which will not affect the discussion, we have, after expanding,

$$m_g=\frac{S\cdot(\phi-g)\alpha(\phi-g)\beta(\phi-g)\gamma}{S\cdot\alpha\beta\gamma}$$

$$=m-g\frac{S(\alpha\phi\beta\phi\gamma+\beta\phi\gamma\phi\alpha+\gamma\phi\alpha\phi\beta)}{S\cdot\alpha\beta\gamma}$$

$$+g^2\frac{S(\alpha\beta\phi\gamma+\gamma\alpha\phi\beta+\beta\gamma\phi\alpha)}{S\cdot\alpha\beta\gamma}-g^3$$

$$=m-m'_1g+m'_2g^2-g^3.$$

155. In § 153 we found the expression,

$$V\cdot(\phi'-g_1)\mu(\phi'-g_1)\nu,$$

the vector part of the product of two strained vectors. To investigate this we return to the last equation of § 146, which can be written,

$$m=\frac{S\cdot\phi\alpha\phi\beta\phi\gamma}{S\cdot\alpha\beta\gamma}=\frac{S\cdot(\phi\beta\phi\gamma)\phi\alpha}{S\cdot\alpha\beta\gamma}=\frac{S\alpha\phi'V(\phi\beta\phi\gamma)}{S\cdot\alpha V\beta\gamma}, \quad [\S\ 135.$$

or since α is any vector whatever,

$$\phi'V(\phi\beta\phi\gamma)=mV\beta\gamma, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or since, § 147, ϕ and ϕ' have the same modulus,

$$\phi V(\phi'\beta\phi'\gamma)=mV\beta\gamma,$$

Using $(\phi-g)$ in place of ϕ, we have

$$(\phi-g)V(\phi'-g)\beta(\phi'-g)\gamma=m_gV\beta\gamma$$

$$=(\phi-g)V[\phi'\beta\phi'\gamma-g(\phi'\beta\cdot\gamma+\beta\phi'\gamma)+g^2\beta\gamma]$$

$$=(\phi-g)[V\phi'\beta\phi'\gamma-g(V\phi'\beta\cdot\gamma+V\beta\phi'\gamma)+g^2V\beta\gamma]$$

$$=(\phi-g)[m\phi^{-1}V\beta\gamma-g(V\phi'\beta\cdot\gamma+V\beta\phi'\gamma)+g^2V\beta\gamma] \quad [(1)$$

$$=(m-m'_1g+m'_2g^2-g^3)V\beta\gamma.$$

156. In this equation and the last expression of § 153, we have the four vectors $V\beta\gamma$, $V\cdot\phi'\beta\cdot\gamma$, $V\cdot\beta\phi'\gamma$, $\phi V\beta\gamma$, and we can write

$$\phi V\beta\gamma = xV\beta\gamma + yV\cdot\phi'\beta\cdot\gamma + zV\beta\phi'\gamma.$$

Operate successively with $S\cdot\alpha$, $S\cdot\beta$, $S\cdot\gamma$ and we get

$$S\beta\gamma\phi'\alpha = xS\cdot\alpha\beta\gamma + yS\cdot\gamma\alpha\phi'\beta + zS\cdot\alpha\beta\phi'\gamma.$$

$$S\beta\gamma\phi'\beta = yS\cdot\gamma\beta\phi'\beta.$$

$$S\beta\gamma\phi'\gamma = zS\cdot\gamma\beta\phi'\gamma.$$

But § 122 (53), $y=-1$, $z=-1$, and

$$x = \frac{S\cdot\beta\gamma\phi'\alpha + S\cdot\gamma\alpha\phi'\beta + S\cdot\alpha\beta\phi'\gamma}{S\cdot\alpha\beta\gamma} = m'_2. \qquad [\S\ 154.$$

157. Hence, § 156,

$$\phi V\beta\gamma = m'_2 V\beta\gamma - V\phi'\beta\cdot\gamma - V\beta\phi'\gamma,$$

or substituting in the equation of § 155 and using ξ for $V\beta\gamma$, we get

$$(\phi-g)[m\phi^{-1} - g(m'_2-\phi) + g^2]\xi = (m - m'_1 g + m'_2 g^2 - g^3)\xi,$$

or expanding,

$$(m + g\phi^2 + m'_2 g\phi + g^2\phi - gm\phi^{-1} + g^2 m'_2 - g^2\phi - g^3)\xi$$
$$= (m - m'_1 g + m'_2 g^2 - g^3)\xi,$$

or $$(\phi^2 - m'_2\phi - m\phi^{-1})\xi = -m'_1\xi, \qquad . \quad . \quad . \quad (1)$$

or $$(\phi^3 - m'_2\phi^2 + m'_1\phi - m)\xi = 0,$$

where ξ is any vector whatever.

158. Comparing this with Eq. (1), § 146, we have

$$m_2 = m'_2 = \frac{S\cdot\alpha\beta\phi\gamma + S\cdot\gamma\alpha\phi\beta + S\cdot\beta\gamma\phi\alpha}{S\alpha\beta\gamma},$$

$$m_1 = m'_1 = \frac{S\cdot\alpha\phi\beta\phi\gamma + S\cdot\beta\phi\gamma\phi\alpha + S\cdot\gamma\phi\alpha\phi\beta}{S\alpha\beta\gamma}$$

The expression of § 153 now becomes, by §§ 155, 157, and the equation above,

$$\rho \| (m\phi^{-1} - g_1(m_2 - \phi) + g_1^2)\xi$$

$$\| \left(\frac{m}{g_1} - (m_2 - g_1)\phi + \phi^2\right)\xi,$$

[dividing by g_1 and operating with ϕ or, multiplying and dividing by $(\phi - g_1)$ and then substituting the value of

$$\frac{\phi m}{g_1} + m_2 g_1 \phi - g_1^2 \phi,$$

from (1) of § 153,

$$\rho \| \frac{\phi^3 - m_2\phi^2 + m\phi - m}{\phi - g_1}\xi. \qquad [\xi = \text{any vector}.$$

159. Since (1) of § 153 is true for all vectors, and since ϕ is commutative with scalars, it can be written,

$$(\phi - g_1)(\phi - g_2)(\phi - g_3)\xi$$

and, § 158, $\rho \| (\phi - g_2)(\phi - g_3)\xi,$

that is, the *operator* $(\phi - g_2)(\phi - g_3)$ *when applied to any vector whatever results in a vector parallel to* ρ, *where*

$$(\phi - g_1)\rho = 0,$$

or

$$\phi\rho = g_1\rho,$$

ρ being unchanged in direction by the strain ϕ.

160. We can fortify this by another line of attack. As before, § 153, under the condition, $\phi\xi = g\xi$, we have

$$(g^3 - m_2 g^2 + m_1 g - m)\xi = 0.$$

Supposing for the moment that the three roots are real, the solution of the problem will be given by one of the directions α, β, γ which satisfies the conditions,

$$(\phi - g_1)\alpha = 0, \quad (\phi - g_2)\beta = 0, \quad (\phi - g_3)\beta = 0. \quad . \quad (1)$$

Now
$$\xi = a\alpha + b\beta + c\gamma.$$

Operating with $(\phi - g_1)$, we get

$$(\phi - g_1)\xi = b(\phi - g_1)\beta + c(\phi - g_1)\gamma$$

$$= (b\phi - bg_1 + bg_2 - b\phi)\beta + (c\phi - cg_1 + cg_3 - c\phi)\gamma$$

[subtracting $b(\phi - g_2)\beta = 0$ and $c(\phi - g_3)\gamma = 0$

$$= b(g_2 - g_1)\beta + c(g_3 - g_1)\gamma.$$

161. Thus we see that the operator $(\phi - g_1)$ cuts off from the general vector ξ the component parallel to α.

Operating again, with $(\phi - g_2)$, this becomes

$$(\phi - g_1)(\phi - g_2)\xi = c(g_3 - g_1)(g_3 - g_1)\gamma.$$

In the same way,

$$(\phi - g_1)(\phi - g_3)\xi = b(g_2 - g_1)(g_2 - g_3)\beta,$$

$$(\phi - g_2)(\phi - g_3)\xi = a(g_1 - g_2)(g_1 - g_3)\alpha.$$

That is, the operator,

$$(\phi - g_2)(\phi - g_3),$$

operating on α, since ξ is any vector, leaves its direction unchanged, and

$$\alpha \| (\phi - g_2)(\phi - g_3)\xi.$$

If the three roots g_1, g_2, g_3 are unequal the three unchanged directions are given by the operators above.

162. Those lines that are unchanged in direction by the strain ϕ have been suggestively called **latent lines** of ϕ, and evidently from the relation $(\phi - g_1)\rho = 0$, the *latent lines of ϕ are the null directions of* $(\phi - g_1)$.

Hence $\phi - g_1$ is a null strain, or § 154,

$$\text{mod. } (\phi - g_1) = m_g = m - m_1 g + m_2 g^2 - g^3 = 0.$$

The roots of this cubic, g_1, g_2, g_3 are the ratios of dilatation or extension of the latent lines of ϕ and are called the **latent roots** of ϕ. Planes whose vectors are not strained out of the plane by ϕ are called the **latent planes** of ϕ.

163. *If α, β, γ are the latent directions of ϕ, with the latent roots g_1, g_2, g_3, then (β, γ), (γ, α), (α, β) determine the latent planes of ϕ, which are enlarged by the strain ϕ in the ratios g_2g_3, g_3g_1, g_1g_2.*

For if $x\beta + y\gamma$ be a vector in the plane $\beta\gamma$, then

$$\phi(x\beta + y\gamma) = x\phi\beta + y\phi\gamma = xg_2\beta + yg_3\gamma,$$

and evidently the strained vector remains in the plane, though its direction has been changed. Q.E.D.

Also for the two vectors in the plane β, γ,

$$TV\phi(x\beta + y\gamma)(x_1\beta + y_1\gamma) = TV(xg_2\beta + yg_3\gamma)(x_1g_2\beta + y_1g_3\gamma)$$

$$= g_2g_3 TV(x\beta + y\gamma)(x_1\beta + y_1\gamma.) \quad \text{Q.E.D.}$$

Conf. § 58.]

164. If $g_2=g_3$, and we operate upon upon ξ (§ 160) with $\phi-g_2$, remembering that $(\phi-g_2)\beta=(\phi-g_2)\gamma=0$, we get

$$(\phi-g_2)\xi=a(g_1-g_2)\alpha,$$

and the latent direction is given by

$$(\phi-g_2)\xi.$$

Operating upon ξ by $(\phi-g_1)$ we get

$$(\phi-g_1)\xi=(g_2-g_1)(b\beta+c\gamma).$$

Since $\quad (\phi-g_2)(b\beta+c\gamma)=(\phi\beta-g_2\beta)+c(\phi-g_2)\gamma,$

$$=0 \qquad [(\phi-g_2)\beta=0,\ (\phi-g_2)\gamma=0.$$

therefore every line of the plane $\beta_1\gamma$ is unchanged in direction, as well as kept in the plane.

165. $g_1=g_2=g_3$.

Operating on $\xi=a\alpha+b\beta+cg$ with $\phi-g_1$ we obtain

$$(\phi-g_1)\xi=0,$$

that is, *when the latent roots are all equal, all the vectors are latent vectors.*

166. There cannot be two latent directions for the same root g_1. For if $(\phi-g_1)\delta=0$ as well as $(\phi-g_1)\alpha=0$, we should get by the method of § 161,

$$(\phi-g_2)(\phi-g_3)\xi=d(g_1-g_2)(g_1-g_3)\delta,$$

or $\qquad a(g_1-g_2)(g_1-g_3)\alpha=d(g_1-g_2)(g_1-g_3)\delta.$

which cannot be unless

$$g_1=g_2, \qquad g_1=g_3,$$

that is, unless the roots are all equal.

167. Since the latent planes are strained in different ratios (the latent roots being unequal) they cannot coincide and their intersections, *the latent vectors*, α, β, γ, *cannot be coplanar.*

$\phi - g_1$ *strains all vectors into the plane* β, γ, *that is, into the plane determined by the other latent vectors.*

For if $$\xi = x\alpha + y\beta + z\gamma = \text{general vector},$$

then $$(\phi - g_1)\xi = x(\phi - g_1)\alpha + y(\phi - g_1)\beta + z(\phi - g_1)\gamma$$

$$= y(g_2 - g_1)\beta + z(g_3 - g_1)\gamma, \quad [(\phi - g_1)\alpha = 0.$$

$$= \text{vector in plane } \beta, \gamma. \qquad \text{Q.E.D.}$$

Repeating this operation we get

$$(\phi - g_1)(\phi - g_2)(\phi - g_3)\xi = 0,$$

as already shown in § 159.

The strain plane of $\phi - g_1$ *is the latent plane of* ϕ.

168. *Two conjugate strains have the same latent roots.* For since $$(\phi - g_1)\alpha = 0.$$

$$\therefore \quad 0 = S\xi(\phi - g_1)\alpha = S\alpha(\phi' - g_1)\xi, \qquad [\S\ 135.$$

and evidently $\phi' - g_1$ strains all vectors into a plane perpendicular to α, i.e., $\phi' - g_1$ is a null function and g_1 is a latent root of ϕ' (§ 162).

169. *The latent plane of one strain is perpendicular to the corresponding latent line of the conjugate strain.*

The strain plane of $\phi' - g_1$ is the latent plane of ϕ', § 167, and therefore the latent plane of ϕ' is perpendicular to α, the latent line of ϕ.

170. *If a strain is self conjugate its three latent roots are real.*

Suppose $g_1+t_1\sqrt{-1}$ to be one of the roots, and let the application of the other factors of the cubic be denoted by

$$(\phi-g_2)(\phi-g_3)\xi=\alpha+\alpha_1\sqrt{-1}.$$

Then, by § 160 (1),

$$\phi(\alpha+\alpha_1\sqrt{-1})=(g_1+t_1\sqrt{-1})(\alpha+\alpha_1\sqrt{-1},$$

or equating the reals and imaginaries,

$$\phi\alpha=g_1\alpha-t_1\alpha_1, \qquad \phi\alpha_1=g_1\alpha_1+t_1\alpha.$$

Whence $$S\alpha_1\phi\alpha=g_1S\alpha_1\alpha-t_1S\alpha_1\alpha_1,$$

$$S\alpha\phi\alpha_1=g_1S\alpha\alpha_1+t_1S\alpha\alpha,$$

or, since $$S\alpha_1\phi\alpha=S\alpha\phi\alpha_1,$$

$$0=t_1(\alpha^2+\alpha_1{}^2),$$

whence $$t_1=0.$$ Q.E.D.

171. If a strain is self conjugate, that is, if $\phi=\phi'$ or $S\eta\phi\xi=S\xi\phi\eta$, then by § 170 it must have three mutually perpendicular latent directions.

Conversely: If ϕ have three mutually perpendicular latent directions i, j, k, with the corresponding latent roots, a, b, c, then ϕ is self conjugate. For, § 122 (78),

$$\xi=-(iSi\xi+jSj\xi+kSk\xi),$$

$$\phi\xi=-(aiSi\xi+bjSj\xi+ckSk\xi); \qquad [\phi i=ai, \text{ etc.}$$

$$S\eta\phi\xi=-aS\eta iSi\xi-bS\eta jSj\xi-cS\eta kSk\xi;$$

$$\eta = -(iSi\eta + jSj\eta + kSk\eta);$$

$$\phi\eta = -aiSi\eta - bjSj\eta - ckSk\eta;$$

$$S\xi\phi\eta = -aSi\xi Si\eta - bSj\xi Sj\eta - cSk\xi Sk\eta$$

$$= S\eta\phi\xi,$$

and ϕ is self conjugate. Q.E.D.

172. *When the strain is self conjugate* ($\phi = \phi'$) *and the latent roots equal, the strain is non-rotational and is called* **pure.**

Since the strain is self conjugate, § 168, the latent lines are i, j, k, and

$$\xi = -iSi\xi - jSj\xi - kSk\xi,$$

$$\phi\xi = -aiSi\xi - ajSj\xi - akSk\xi$$

$$= a(-iSi\xi - jSj\xi - kSk\xi)$$

$$= a\xi, \qquad [a = \text{latent root.}$$

and ξ is not rotated.

That is, all vectors are latent lines. Q.E.D.

When ϕ is self conjugate ($\phi - \phi'$), by reference to §§ 143, 144, if the strain is given by

$$\phi i = xi + yj + zk;$$

$$\phi j = x'i + y'j + z'k;$$

$$\phi k = x''i + y''j + z''k;$$

then when the strain is pure,

$$x' = y, \qquad x'' = z, \qquad y'' = z',$$

and *a pure strain depends upon six instead of upon nine scalar constants.*

173. *If the strain is self conjugate ($\phi = \phi'$) and the three latent roots a, b, c are not equal, only the vectors i, j, k are non-rotated.*

As in the previous section,

$$\xi = -iSi\xi - jSj\xi - kSk\xi,$$

$$\phi\xi = -aiSi\xi - bjSj\xi - ckSk\xi$$

$$\neq x\xi. \qquad \text{Q.E.D.}$$

$$\phi i = -aiSii - bjSij - ckSik$$

$$= ai.$$

Similarly, $\phi j = bj, \quad \phi k = ck.$ Q.E.D.

174. Example. If ϕ have three latent directions α, β, γ with the three latent roots a, b, c, then

$$\xi = \frac{\alpha S\beta\gamma\xi + \beta S\alpha\gamma\xi + \gamma S\beta\alpha\xi}{S\alpha\beta\gamma}.$$

$$\phi\xi = \frac{a\alpha S\beta\gamma\xi + \dots}{S\alpha\beta\gamma}.$$

$$\phi'\xi = \frac{a\alpha S\beta\gamma\xi + \dots}{S\alpha\beta\gamma}, \qquad [\S\ 168.$$

and $\phi = \phi'$, or ϕ is self conjugate. Where is the error?

Ans. $\phi'\alpha \neq a\alpha$.

175. Examples. Solve $V\alpha\xi\beta = \gamma = \phi\xi$ for ξ. By § 146 (2),

$$m = \frac{1}{S \cdot \lambda\mu\nu} S(V\alpha\lambda\beta V\alpha\mu\beta V\alpha\nu\beta),$$

where λ, μ, ν are any three non-coplanar vectors. Putting for these α, β, γ, if they are not coplanar, we get

$$m = \frac{1}{S\cdot\alpha\beta\gamma} S(\alpha^2\beta\cdot\alpha\beta^2 V\alpha\gamma\beta) = \frac{\alpha^2\beta^2}{S\cdot\alpha\beta\gamma} S(\beta\alpha V\alpha\gamma\beta) = \alpha^2\beta^2 S\alpha\beta.$$

[$V\alpha\gamma\beta = \alpha S\gamma\beta - \gamma S\alpha\beta + \beta S\alpha\gamma$. [§ 122 (64).

By § 158,

$$m_1 = \frac{1}{S\alpha\beta\gamma} S(\alpha\cdot\alpha\beta^2\cdot V\alpha\gamma\beta + \beta V\alpha\gamma\beta\cdot\alpha^2\beta + \gamma\cdot\alpha^2\beta\cdot\alpha\beta^2)$$

$$= -\alpha^2\beta^2.$$

$$m_2 = -S\alpha\beta.$$

By § 157 (1), we get

$$\alpha^2\beta^2 S\alpha\beta\cdot\phi^{-1}\gamma = -\alpha^2\beta^2\gamma + S\alpha\beta V\alpha\gamma\beta + V(\alpha V\alpha\gamma\beta\cdot\beta)$$

$$= -\alpha^2\beta^2\gamma + S\alpha\beta V\alpha\gamma\beta + \alpha^2 S\gamma\beta\cdot\beta$$

$$- S\alpha\beta V\alpha\gamma\beta + S\alpha\gamma\cdot V\alpha\beta^2$$

[§ 122 (64).

$$= -\alpha^2\beta^2\gamma + \alpha^2\beta S\beta\gamma + \alpha\beta^2 S\alpha\gamma,$$

whence $$\phi^{-1}\gamma = \xi = \frac{-\gamma + \alpha^{-1}S\alpha\gamma + \beta^{-1}S\beta\gamma}{S\alpha\beta}.$$

APPENDIX

FUNCTIONAL SYMBOLS

As every discrete magnitude is of necessity derived from unity by some algebraic operation, so the symbol representing a discrete magnitude can be considered as the symbol of some operation whose operand is unity.

Symbolizing the operation of converting unity into the magnitude x by the symbol $x \cdot 1$, we have the functional equation (x operating upon 1),

$$x \cdot 1 = x.$$

A second application gives

$$x(x \cdot 1) = x^2 \cdot 1 = x^2,$$

where the 2 of $x^2 \cdot 1$ shows the number of operations, and the 2 of x^2 the result.

The inverse of the operation must be symbolized by

$$x^{-1} \cdot 1 = \frac{1}{x} \cdot 1.$$

In the expression $x \cdot 1$ occur two concepts, the *operation*, $x \cdot 1$ and the *effect* of the operation, x.

The inverse of the operation is $x^{-1}\cdot 1=\frac{1}{x}\cdot 1$, the inverse of the effect is $(x)^{-1}=\frac{1}{x}$, and in this case the two results coincide.

If we take a different operand, say y, then $x^{-1}\cdot y$ and $(xy)^{-1}$, the inverse of the operation and the inverse of the effect are not the same.

Every algebraic expression can be considered as a functional operation. Thus adding 1 to x can be considered as an operation symbolized thus,

$$f(x)=x+1.$$

The inverse operation would be whatever operation is necessary to change back from $x+1$ to x to original operand. Thus

$$f^{-1}(x)=x-1.$$

Here again $f^{-1}\neq\frac{1}{f}$, where, of course, f^{-1} symbolizes the inverse of the operation and $(f)^{-1}=\frac{1}{f}$, the inverse of the effect.

Similarly, $$\log^{-1} x\neq(\log x)^{-1},$$

$$\sin^{-1} x\neq(\sin x)^{-1}.$$

EXAMPLES. If $f(x)=x+1$, $f^{-1}(x)=x-1$, $f^2(x)=x+2$.

If $f(x)=\frac{1}{x^2+1}$, $f^{-1}(x)=\sqrt{\frac{1-x}{x}}$, $f^2(x)=\frac{x^4+2x^2+1}{x^4+2x^2+2}$.

If $f(x)=\frac{1}{x}+x$, $f^{-1}(x)=\frac{x-\sqrt{x^2-4}}{2}$, $f^2(x)=\frac{x}{1+x^2}+\frac{1+x^2}{x}$.

If $$f(x) = x^2 + 2x + 6, \quad f^{-1}(x) = -1 \pm \sqrt{x-5},$$

$$f^2(x) = (x^2+2x+6)^2 + 2(x^2+2x+6) + 6.$$

In these examples, omitting the operand, show that

$$ff^{-1} = f^\circ = 1, \quad f^2f^{-1} = f \quad f^{-1}f^2 = f, \quad f^{-1}f = f^\circ = 1,$$

$$f^3f^{-2} = f, \qquad f^{-3}f = f^{-2}.$$

In the same way prove

$$\sin \sin^{-1} x = x, \qquad \log \log^{-1} x = x,$$

$$\log^{-1} \log x = x, \qquad \sin^{-1} \sin^2 x = \sin x.$$

If $f(x) = x^2 + 3, \quad F(x) = 2 - \sqrt{x}$, then

$$fF(x) = (2 - \sqrt{x})^2 + 3, \qquad F \cdot f(x) = 2 - \sqrt{x^2+3}.$$

If $\phi(x) = \dfrac{2x-1}{3x-2} = y$, then $\phi(y) = x$, $\phi^2(x) = x$.

The algebraic symbol is distributive over and commutative with its operand, that is, $xy + xz + \ldots = x(y + z + \ldots)$, $xy = yx$.

Other symbols of operation which like these are distributive over and commutative with the operand will be subject to the same algebraic laws. Hence we can treat these symbols of operation just as we treat algebraic symbols of operation.

Hence we can write § 122 (33), (34), § 132, § 135,

$$Sp + Vp = (S + V)p; \qquad \phi\xi - \phi'\xi = (\phi - \phi')\xi,$$

$$\phi n\xi + \phi'\xi = (n\phi + \phi')\xi, \qquad \phi\xi + gn\xi = (\phi + gn)\xi.$$

We recall the similar results in Calculus, where, omitting the operand, we have

$$(D^2 - a^2) = (D + a)(D - a) \qquad \left[D = \frac{d}{dx}.\right.$$

$$(D^2 - D - 2) = (D + 1)(D - 2)$$

$$x^2 D^2 = x^2 D \cdot D = x^2 D \frac{\theta}{x} \qquad [xD = \theta.$$

$$= x^2 \frac{xd \cdot \theta - \theta}{x^2}$$

$$= \theta \cdot \theta - \theta = \theta(\theta - 1),$$

where θ is commutative with constants but not with either x or D.

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