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RAY'S

ALGEBRA

PART SECOND:

AN

ANALYTICAL TREATISE,

DESIGNED FOR

HIGH SCHOOLS AND COLLEGES.

By JOSEPH RAY, M. D.

PROFESSOR OF MATHEMATICS IN WOODWARD COLLEGE.

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PREFACE.

ALGEBRA is justly regarded one of the most interesting and useful branches of education, and an acquaintance with it is now sought by all who progress beyond the more common elements. To those who would know Mathematics, a knowledge not merely of its elementary principles, but also of its higher parts, is essential; while no one can lay claim to that discipline of mind which education confers, who is not familiar with the logic of Algebra.

It is both a demonstrative and a practical science — a system of truths and reasoning, from which is derived a collection of Rules that may be used in the solution of an endless variety of problems, not only interesting to the student, but many of which are of the highest possible utility in the arts of life.

The object of the present treatise is to present an outline of this science in a brief, clear, and practical form. The aim throughout has been to demonstrate every principle, and to furnish the student the means of understanding clearly the rationale of every process he is required to perform. No effort has been made to simplify subjects by omitting that which is difficult, but rather to present them in such a light as to render their acquisition within the reach of all who will take the pains to study.

To fix the principles in the mind of the student, and to show their bearing and utility, great attention has been paid to the preparation of practical exercises. These are intended rather to illustrate the principles of the science, than as difficult problems to torture the ingenuity of the learner, or amuse the already skillful Algebraist.

An effort has been made throughout the work, to observe a natural and strictly logical connection between the different parts, so that the learner may not be required to rely on a principle, or employ a process, with the rationale of which he is not already acquainted. The reference by Articles will always enable him to trace any subject back to its first principles.

The limits of a preface will not permit a statement of the peculiarities of the work, nor is it necessary, as those who are interested to know, will examine it for themselves. It is, however, proper to remark, that Equations of the Second Degree have received more than usual attention. The same may be said of Radicals, of the Binomial Theorem, and of Logarithms, all of which are so useful in other branches of mathematics.

On some subjects it was necessary to be brief, to bring the work within suitable limits. For example, what is here given of the Theory of Equations, is to be regarded merely as an outline of the more practical and interesting parts of the subject, which alone is sufficient for a distinct treatise, as may be seen by referring to the works of Young or Hymers in English, or of DeFourcy or Reynaud in French.

Some topics and exercises deemed both useful and interesting will be found here, not hitherto presented to the notice of students. But these, as well as the general manner of treating the subject, are now submitted, with deference, to the intelligent educational public, to whom the author is already greatly indebted for the favor with which his previous works have been received.

WOODWARD SCHOOL, May, 1852.

NOTICE.

☞ A KEY to this work, containing solutions to the more difficult problems, with remarks and suggestions, intended principally for private students, is now published.

The Key also embraces the DIOPHANTINE and INDETERMINATE ANALYSIS, with the NOTATION OF NUMBERS, &c., subjects not usually contained in the ordinary course of instruction in Higher Algebra.

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RAY'S

ALGEBRA:

PART SECOND.

CHAPTER I.

DEFINITIONS AND NOTATION.

ARTICLE **1.** Quantity is anything capable of being increased or diminished; such as numbers, lines, space, time, motion, &c.

REMARK.—If the pupil does not already know, let the instructor here explain to him what is meant by *unit of quantity*, the *numerical value* of quantity, &c. See Ray's Algebra, Part First, Arts. 2 to 10.

ART. **2.** Mathematics is the science of quantity, and of the symbols by which quantity is represented.

ART. **3.** Algebra is that branch of Mathematics which relates to the solution of problems and the demonstration of theorems, when any of the quantities employed are designated by letters.

ART. **4.** A *problem* is a question proposed for solution; a *theorem* is a proposition to be proved by a chain of reasoning.

ART. **5.** The operations of Algebra are conducted by means of figures, letters, and signs. The letters and signs are often called *symbols*.

ART. **6.** Known quantities are generally represented by the first letters of the alphabet, as a, b, c, &c.; and unknown quantities by the last letters, as t, v, x, y, z.

ART. **7.** The principal signs used in Algebra are the following:

$$=,\ +,\ -,\ \times,\ \div,\ (\,),\ >,\ \surd.$$

Each sign is the representative of certain words; and is used for the purpose of expressing the various operations in the most clear and brief manner.

ART. 8. The sign =, is termed the sign of *equality*. It is read *equal to*, or *equals*, and denotes that the quantities between which it is placed are equal to each other. Thus, $x=5$, denotes that the quantity represented by x is equal to 5.

ART. 9. The sign +, is termed the sign of *addition*. It is read *plus*, and denotes that the quantity to which it is prefixed is to be added.

Thus, $a+b$ denotes that b is to be added to a. If $a=3$, and $b=5$, then $a+b=3+5$, which are $=8$.

ART. 10. The sign —, is termed the sign of *subtraction*. It is read *minus*, and denotes that the quantity to which it is prefixed is to be subtracted. Thus, $a-b$ denotes that b is to be subtracted from a. If $a=7$ and $b=4$, then $7-4=3$.

ART. 11. The signs + and — are called *the* signs; the former is called the *positive*, and the latter the *negative* sign; they are said to be *contrary*, or *opposite*.

ART. 12. Every quantity is supposed to have either the positive or negative sign. Quantities having the positive sign are called *positive*; those having the negative sign are called *negative*. When a quantity has no sign prefixed to it, the sign + is understood, and it is to be considered positive.

ART. 13. Quantities having the same sign are said to have *like* signs; those having different signs are said to have *unlike* signs. Thus, $+a$ and $+b$, or $-a$ and $-b$ have like signs; while $+c$ and $-d$ have unlike signs.

ART. 14. The sign ×, is termed the sign of *multiplication*. It is read *into*, or *multiplied by*, and denotes that the quantities between which it is placed are to be multiplied together.

A dot, or period, is sometimes used to denote multiplication. Thus, $a\times b$, and $a.b$, each denote that a and b are to be multiplied together. The dot is not often employed to denote the multiplication of figures, since it is used, by some authors, to separate whole numbers and decimals.

The product of two or more letters is generally denoted by writing them in close succession. Thus, ab denotes the same as $a\times b$, or $a.b$; and abc means the same as $a\times b\times c$, or $a.b.c$; each signifying the continued product of the numbers designated by a, b, and c.

ART. 15. When two or more quantities are to be multiplied together, each of them is called a *factor*. Thus, in the product ab there are two factors, a and b: in the product $3\times5\times7$ there are three factors, 3, 5, and 7.

ART. **16.** The sign $\div$, is termed the sign of *division*. It is read *divided by*, and denotes that the quantity preceding it is to be divided by that following it. The most common method of representing the division of two quantities is to place the dividend as the numerator, and the divisor as the denominator of a fraction. Thus, $a \div b$, or $\frac{a}{b}$, signifies that a is to be divided by b.

Division is also represented thus, $a\underline{|b}$, where a denotes the dividend, and b the divisor.

ART. **17.** The sign $>$, is termed the sign of *inequality*. It denotes that one of the two quantities between which it is placed is greater than the other, the opening of the sign being turned toward the greater quantity.

Thus, $a>b$, denotes that a is greater than b. It is read, *a greater than b*. If $a=7$ and $b=2$, then $7>2$.

Also $c<d$, denotes that c is less than d. It is read, *c less than d*. If $c=3$ and $d=5$, then $3<5$.

ART. **18.** A *coëfficient* is a number or letter prefixed to a quantity, to show how often it is taken.

Thus, if a is to be taken 4 times, instead of writing $a+a+a+a$, we write $4a$; in like manner, $az+az+az=3az$.

The coëfficient is called *numeral*, or *literal*, according as it is a number or a letter. Thus, in the quantities $5x$ and mx, 5 is a numeral and m a literal coëfficient.

In $3az$, 3 may be considered as the coëfficient of az, or $3a$ may be considered as the coëfficient of z.

When no numeral coëfficient is expressed, 1 is always understood. Thus, a is the same as $1a$, and ax is the same as $1ax$.

ART. **19.** A *power* of a quantity, is the product arising from multiplying the quantity by itself one or more times. When the quantity is taken *twice* as a factor, the product is called the *second* power; when taken *three* times, the *third* power, and so on. Thus,

$2\times2=4$, is the second power of 2.

$2\times2\times2=8$, is the third power of 2.

$2\times2\times2\times2=16$, is the fourth power of 2.

Also, $a\times a=aa$, is the second power of a.

$a\times a\times a=aaa$, is the third power of a; and so on.

The second power is often termed the *square*, and the third power, the *cube*. To avoid repeating the same quantity as a factor, a small figure, termed an *exponent*, is placed on the right, and a little above it, to denote the number of times the quantity is taken as a factor, Thus, aa is written a^2; aaa is written a^3; $aabbb$ is written a^2b^3.

When no exponent is expressed, 1 is always understood. Thus, a is the same as a^1, each signifying the *first* power of a.

ART. **20.** A *root* of a quantity is a factor, which multiplied by itself a certain number of times, will produce the given quantity.

The root is called the *square* or *second* root, the *cube* or *third* root, the *fourth* root, &c., according to the number of times it must be taken as a factor to produce the given quantity. Thus, 2 is the second or square root of 4, since $2\times2=4$. In like manner, a is the fourth root of a^4, since $a\times a\times a\times a=a^4$.

ART. **21.** The sign $\surd$, or $\sqrt{\ }$, is called the *radical* sign. When prefixed to a quantity, it denotes that its root is to be extracted. Thus,

$\surd a$, or $\sqrt{a}$, denotes the square root of a.

$\sqrt[3]{}8$, or $\sqrt[3]{8}$, denotes the cube root of 8, which is 2.

$\sqrt[4]{}a$, or $\sqrt[4]{a}$, denotes the fourth root of a.

The number placed over the radical sign is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, and so on. When the radical sign has no index over it, 2 is understood; thus $\surd 9$ and $\sqrt[2]{}9$ signify the same thing.

ART. **22.** An *algebraic quantity*, or an *algebraic expression*, is any quantity written in algebraic language, that is, by means of symbols. Thus,

$5a$, is the algebraic expression of 5 times the number a;

$3b+4c$, is the algebraic expression for 3 times the number b increased by 4 times the number c;

$3a^2-7ab$, is the algebraic expression for 3 times the square of a, diminished by 7 times the product of the number a by the number b.

ART. **23.** A *monomial* is a quantity not united to any other by the sign of addition or subtraction; as $4a$, a^2bc, $-4xy$, &c.

A monomial is often called a *simple quantity*, or *term*.

ART. **24.** A *polynomial*, or *compound quantity*, is an algebraic expression composed of two or more terms; as $a+b$, $c-x+y$, &c.

ART. **25.** A *binomial* is a quantity having two terms; as $a+b$, x^2+y, &c.

A binomial, of which the second term is negative, as $a-b$, is sometimes called a *residual*.

ART. **26.** A *trinomial* is a quantity consisting of three terms; as, $a+b-c$.

Binomials and trinomials are polynomials.

ART. **27.** The *numerical value* of an algebraic expression is the number obtained by giving a particular value to each letter, and then performing the operations indicated. Thus, in the algebraic expression $4a-3c$, if $a=5$ and $c=6$, the numerical value is $4\times5-3\times6=20-18=2$.

ART. **28.** The value of a polynomial is not affected by changing the order of the terms, provided each term retains its respective sign. Thus $b^2-2ab+c$ is the same as $b^2+c-2ab$. This is self-evident.

ART. **29.** Each of the *literal* factors of any term is called a *dimension* of that term. The *degree* of any term is equal to the number of literal factors which it contains. Thus,

$5a$ is of the *first* degree ; it contains *one* literal factor.

ax is of the *second* degree ; it contains *two* literal factors.

$3a^3b^2c=3aaabbc$ is of the *sixth* degree ; it contains *six* literal factors.

It is obvious, that *the degree of any term is equal to the sum of the exponents of the letters which compose that term.*

Thus, $5a^2bc^4$ is of the seventh degree, since $2+1+4=7$.

ART. **30.** A polynomial is said to be *homogeneous*, when each of its terms is of the same degree. Thus,

$a+b-3c$ is of the first degree ; it is homogeneous.

x^3-7xy^2 is of the second degree ; it is homogeneous.

x^2-3xy^2 is not homogeneous.

ART. **31.** An algebraic quantity is said to be *arranged* according to the dimensions of any letter it contains, when the exponents of that letter occur in the order of their magnitudes, either *increasing* or *decreasing*. Thus,

$ax^2+a^2x-a^3x^3$, is arranged according to the ascending powers of a. $bx^3-b^3x^2+b^2x$, is arranged according to the descending powers of x.

ART. **32.** A parenthesis, ( ), is used to show that all the terms of a polynomial are to be considered together, as a single term. Thus,

$10-(a-b)$ means that $a-b$ is to be subtracted from 10 ;

$5(a+b-c)$ means that $a+b-c$ is to be multiplied by 5 ; while

$5a+(b-c)$ means that $b-c$ is to be added to $5a$.

When the parenthesis is used, the sign of multiplication is generally omitted. Thus, $(a-b)\times(a+b)$ is the same as $(a-b)(a+b)$.

A vinculum, ———, is sometimes used instead of a parenthesis. Thus, $\overline{a+b}\times5$ means the same as $5(a+b)$. Sometimes the vinculum is placed vertically ; it is then called a *bar*. Thus,

$\left.\begin{array}{r} a \\ -b \\ +c \end{array}\right| x^2$, is the same as $(a-b+c)x^2$.

ART. **33.** *Similar*, or *like* quantities, are such as differ only in their signs, or numerical coëfficients, or both. Thus, $2ab$ and $-2ab$, $3a^2b$ and $5a^2b$, $3a^2b$ and $-5a^2b$, are respectively similar.

Unlike quantities are different combinations of letters. Thus, $3ab^2$, and $3a^2b$, are unlike or dissimilar.

REMARK.—An exception, however, must be made in those cases where letters are taken to represent coëfficients. Thus, ax^2 and bx^2 are like quantities, when a and b are taken as coëfficients of x^2

ART. **34.** The *reciprocal* of a quantity is unity divided by that quantity. Thus,

The reciprocal of $a+b$ is $\frac{1}{a+b}$, and of 3, is $\frac{1}{3}$.

ART. **35.** The same letter, accented, is often used to denote quantities which occupy similar positions in different equations or investigations. Thus, a, a', a'', a''', represent four different quantities, of which a' is read, a prime; a'' is read, a second; a''' is read, a third; and so on.

ART. **36.** The following signs are also used, for the sake of brevity:

∞, a quantity indefinitely great, or infinity,

$\therefore$, signifies *therefore*, or *consequently*.

$\because$, signifies *since*, or *because*.

$\sim$, is used to represent the difference between two quantities, as $c \sim d$, when it is not known which is the greater.

EXERCISES

ON THE DEFINITIONS AND NOTATION.

Each example is intended to furnish to the pupil two distinct exercises. First, to be copied on the slate, or blackboard; and then *read*, that is, expressed in common language. Second, the numerical value to be found, supposing $a=2$, $b=3$, $c=4$, $x=5$, $y=6$.

1. $b+c-x$. *Ans.* 2.
2. $7b+x-y$. *Ans:* 20.
3. $abx-cy$. *Ans.* 6.
4. a^2by-3x^2. *Ans.* -3.
5. $c+a\times c-a$. *Ans.* 10.
6. $(c+a)(c-a.)$ *Ans.* 12.
7. $\frac{a^2+b+c-y}{2}$ *Ans.* $2\frac{1}{2}$.
8. $\frac{cx-ay}{x-b}$ *Ans.* 4.
9. $\frac{bc}{y}+\frac{ay}{c}$ *Ans.* 5.
10. $2c^2-a(x+y)(y-x)$ *Ans.* 10.
11. $\frac{ab(c-a)}{y-c}-\sqrt{aby}$ *Ans.* 0.

12. Find the difference between abx, and $a+b+x$, when $a=4$, $b=\frac{1}{2}$, $x=3$; and when $a=5$, $b=7$, $x=12$. *Ans.* $1\frac{1}{2}$ and 396.

13. Required the values of $a^2+2ab+b^2$, and $a^2-2ab+b^2$, when $a=7$ and $b=4$. *Ans.* 121 and 9.

14. What is $n(n-1)(n-2)(n-3)$, when $n=4$, and when $n=10$? *Ans.* 24, and 5040.

15. Find the difference between $6abc-2ab$, and $6abc\div2ab$, when a, b, c, are 3, 5, and 6 respectively. *Ans.* 492.

16. When $a=3$, what is the difference between a^2 and $2a$; a^3 and $3a$; a^4 and $4a$; a^5 and $5a$? *Ans.* 3, 18, 69, and 228.

17. Find the value of $\frac{a^2-b^2}{a^3+b^3}$, when $a=5$ and $b=3$. *Ans.* $\frac{2}{19}$.

18. Find the difference between each pair of the following expressions, when $a=6$ and $b=8$: $(a+b)^2$ and a^2+b^2; $5(a+b)$ and $5a+b$; $a+b$ and $\sqrt{(a^2+b^2)}$; $(a+b)^3$ and $a+b^3$; $\sqrt{(a^2+b^2)}$ and $\sqrt{a^2}+\sqrt{b^2}$. *Ans.* 96, 32, 4, 2226, and 4.

19. What is $3\sqrt{c}+2a\sqrt{(2a+b-x)}$ when $a=6$, $b=5$, $c=4$, and $x=1$? *Ans.* 54.

20. What is $n(n-1)(n-2)(n-3)(n-4)$, if n be 1, 2, 3, 4, 5, and 6, in succession? *Ans.* 0, 0, 0, 0, 120, and 720.

The pupil may verify the following expressions, by giving to each letter any value whatever.

21. $a(m+n)(m-n)=am^2-an^2$.

22. $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$.

23. $(x^4+x^2+1)(x^2-1)=x^6-1$.

24. $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{4ab}{a^2-b^2}$.

EXAMPLES

TO BE EXPRESSED IN ALGEBRAIC SYMBOLS.

1. Five times a, plus the second power of b.
2. x, plus y divided by $3z$.
3. x plus y, divided by $3z$.
4. 3 into x minus n times y, divided by m minus n.
5. m into x squared, minus m plus x squared, plus c into the cube of x.
6. a third power minus x third power, divided by a second power minus x second power.
7. a minus x third power into a plus x second power, divided by a second power plus x second power.

8. m squared plus n squared, into the square of m plus n.
9. The square root of m minus the square root of n.
10. The square root of m minus n.
11. The square root of m, minus n.
12. The square root of b squared, minus m plus n into c.

ANSWERS.

1. $5a+b^2$.
2. $x+\frac{y}{3z}$.
3. $\frac{x+y}{3z}$.
4. $\frac{3x-ny}{m-n}$.
5. $mx^2-(m+x)^2+cx^3$.
6. $\frac{a^3-x^3}{a^2-x^2}$.
7. $\frac{(a-x)^3(a+x)^2}{a^2+x^2}$.
8. $(m^2+n^2)(m+n)^2$.
9. $\sqrt{m}-\sqrt{n}$.
10. $\sqrt{(m-n)}$.
11. $\sqrt{m}-n$.
12. $\sqrt{\{b^2-(m+n)c\}}$.

ADDITION.

ART. **37.** ADDITION in Algebra, is the process of finding the simplest expression for the sum of two or more algebraic quantities.

There are three cases of algebraic addition:

1st. When the quantities are all similar, and have the same sign, either positive or negative.

2d. When the quantities are all similar, but part positive, and part negative.

3d. When the quantities are dissimilar, or part similar and part dissimilar.

ART. **38.** FIRST CASE.—Let it be required to find the sum of $3x^2y$, $5x^2y$, and $7x^2y$.

Here x^2y is taken, in the first term, 3 times; in the second, 5 times; and in the third, 7 times; hence, in all, it is taken 15 times. Since adding the quantities together cannot change their character, and since each term is positive, therefore their sum is positive.

OPERATION.

$$\begin{array}{r} +\ 3x^2y \\ +\ 5x^2y \\ +\ 7x^2y \\ \hline +15x^2y \end{array}$$

Next, let it be required to find the sum of $-3x^2y$, $-5x^2y$, and $-7x^2y$.

Here x^2y is taken, in the first term, -3 times; in the second, -5 times; and in the third, -7 times; hence, in all, it is taken -15 times.

OPERATION.

$$\begin{array}{r} -3x^2y \\ -5x^2y \\ -7x^2y \\ \hline -15x^2y \end{array}$$

Therefore, *To add together quantities having the same sign; find the sum of their coëfficients, and prefix it, with the common sign, to the literal part.*

ART. **39.** SECOND CASE.—Let it be required to find the sum of $+9a$, $-5a$, $+4a$, and $-2a$.

Before solving this example, the pupil must understand the following principle. Since the sign *plus* denotes that the quantity before which it is placed is to be added, and the sign *minus*, that the quantity before which it is placed is to be subtracted; therefore, *the sum of two equal quantities, of which one is positive and the other negative, is zero, or* 0. Thus, $+a-a=0$; $+5a^2-5a^2=0$; and so on.

Here $+9a+4a$ is $13a$; and $-5a-2a$ is $-7a$. Now, $-7a$ will cancel $+7a$ in the quantity $+13a$, and leave $+6a$ for the *aggregate*, or result of the four quantities.

OPERATION

$$\begin{array}{r} +9a \\ -5a \\ +4a \\ -2a \\ \hline +6a \end{array}$$

In like manner, if it be required to obtain the sum of $-9a$, $+5a$, $-4a$, and $+2a$, we find the sum of $-9a$ and $-4a$ is $-13a$, and the sum of $+5a$ and $+2a$ is $+7a$. Now, $+7a$ will cancel $-7a$ in the quantity $-13a$; which leaves $-6a$ for the aggregate, or sum of the quantities.

OPERATION.

$$\begin{array}{r} -9a \\ +5a \\ -4a \\ +2a \\ \hline -6a \end{array}$$

Therefore, *To add similar quantities having different signs; find the sum of the positive quantities and the sum of the negative quantities, then take the difference of their coëfficients, and prefix it, with the sign of the greater quantity, to the literal part.*

ART. **40.** THIRD CASE.—Let it be required to find the sum of $5a^2-8b+c$, $+b-a^2$, and $5b+3a^2$.

In writing the quantities, we place those which are similar under each other, for the sake of *convenience* in performing the operation. We then find, as in the preceding case, that the sum of the quantities in the first column is $+7a^2$, and in the second, $-2b$; and there being no term similar to c, it is connected to the other quantities by its proper sign.

OPERATION.

$$\begin{array}{r} 5a^2-8b+c \\ -a^2+b \\ 3a^2+5b \\ \hline 7a^2-2b+c \end{array}$$

ART. **41.** From the preceding, we derive the following

GENERAL RULE FOR THE ADDITION OF ALGEBRAIC QUANTITIES.— *Write the quantities to be added, placing those that are similar under each other; then reduce each set of similar terms, by taking the difference of the positive and negative coëfficients, and prefixing it, with the sign of the greater, to the literal part; after this, annex the other terms with their proper signs.*

REMARKS.—1. It is immaterial in what order the quantities are set down, if care is taken to prefix to each its proper sign.

2. It will often happen that the sum of two or more quantities is less than either. See Observations on Addition and Subtraction, page 24.

EXAMPLES.

1. Add together $4ax+3by$, $5ax+8by$, $8ax+6by$, and $20ax+by$. *Ans.* $37ax+18by$.

2. Add together $10cz-2ax^2$, $15cz-3ax^2$, $24cz-9ax^2$, and $3cz-8ax^2$. *Ans.* $52cz-22ax^2$.

3. Find the sum of $3x^2y^2-10y^4$, $-x^2y^2+5y^4$, $8x^2y^2-6y^4$, and $4x^2y^2+2y^4$. *Ans.* $14x^2y^2-9y^4$.

4. Add together $a+b+c+d$, $a+b+c-d$, $a+b-c+d$, $a-b+c+d$, and $-a+b+c+d$. *Ans.* $3a+3b+3c+3d$.

5. Add together $3(x^2-y^2)$, $8(x^2-y^2)$, and $-5(x^2-y^2)$. *Ans.* $6(x^2-y^2)$.

6. Required the sum of $10a^2b-12a^3bc-15b^2c^4+10$, $-4a^2b+8a^3bc-10b^2c^4-4$, $-3a^2b-3a^3bc+20b^2c^4-3$, and $2a^2b+12a^3bc+5b^2c^4+2$. *Ans.* $5a^2b+5a^3bc+5$.

7. Add together $a^2+b^2+c^2+d^2$, $ab-2a^2+ac-2c^2+ad-2d^2$, $a^3-3ab+b^3-3ac+c^3-3ad$, and $2ab-a+2ac-b+2ad-c$. *Ans.* $a^3+b^3+c^3+b^2-a^2-c^2-d^2-a-b-c$.

8. Add together $a^m-b^n+3x^p$, $2a^m-3b^n-x^p$, and $a^m+4b^n-x^q$. *Ans.* $4a^m+2x^p-x^q$.

SUBTRACTION.

ART. **42.** Subtraction, in Algebra, is the process of finding the simplest expression for the difference between two algebraic quantities.

The quantity to be subtracted is called the *Subtrahend*.

The quantity from which the subtraction is to be made is called the *Minuend*.

The quantity left, after the subtraction is performed, is called the *Difference*, or *Remainder*.

The explanation of the principles on which the operations depend, may be divided into two cases.

1st. Where all the terms of the quantity to be subtracted are positive.

2d. Where the quantity to be subtracted is either partly or wholly negative.

ART. **43.** To explain the first case, let it be required to subtract $4a$ from $7a$.

It is evident that 7 times any quantity, less 4 times that quantity, is equal to 3 times the quantity; therefore, $7a$ *less* $4a$ is equal to $3a$. Hence, to find the difference between two similar quantities, *we take the difference between their coëfficients, and prefix it to the common letter or letters.*

OPERATION.

$7a$	Minuend
$4a$	Subtrahend
$3a$	Remainder

If it be required to subtract b from a, unless we know the number of units represented by each, we can only *indicate* the operation, which is done by placing the sign *minus* before the quantity to be subtracted.

OPERATION.

a	Minuend
b	Subtrahend
$a-b$	Remainder

ART. **44.** To explain the second case, let it be required to subtract $b-c$ from a.

If we subtract b from a, the result, $a-b$, is obviously too little, for the quantity b, taken from a, ought to be diminished by c before the subtraction is effected. We have, in fact, subtracted a quantity too great by c, and, therefore, to obtain a true result, the difference $a-b$ must be increased by c; this gives, for the true remainder, $a-b+c$.

OPERATION.

a	Minuend
$b-c$	Subtrahend
$a-b+c$	Remainder

This operation may be explained by figures, thus:

Let $a=9$, $b=5$, and $c=3$; and let it be required to subtract $5-3$ from 9.

If we subtract 5 from 9, the remainder is $9-5$; but the quantity to be subtracted is 3 *less* than 5, therefore we have subtracted 3 too much; hence, we must add 3 to $9-5$, which gives $9-5+3$, or 7, for the true remainder.

The operation and illustration may be compared, thus:

From a	From 9	. . . . . $=9$
Take $b-c$	Take $5-3$	. . . . . $=2$
Rem. $a-b+c$	Rem. $9-5+3$	. . . . . $=7$

The same principle may be further illustrated by the following examples:

$$a-(c-a)=a-c+a=2a-c.$$
$$a-(a-c)=a-a+c=c.$$
$$a+c-(a-c)=a+c-a+c=2c.$$

In all these cases, we see that the same remainder would have been obtained, by changing the signs of the quantity to be subtracted, and then adding it.

ART. **45.** Hence we have the following

RULE FOR THE SUBTRACTION OF ALGEBRAIC QUANTITIES.—*Write the quantity to be subtracted under that from which it is to be taken, placing similar terms under each other.*

Conceive the signs of all the terms of the subtrahend to be changed, from + to —, or from — to +, and then reduce the result to its simplest form.

REMARKS.—1. Beginners may solve a few examples by *actually* changing the signs of the subtrahend. After this, it is better merely to conceive the signs to be changed; that if it becomes necessary to refer to the operation, we may be in no doubt with regard to the signs of the terms originally.

2. Subtraction in Algebra may be proved in the same manner as in Arithmetic, by adding together the remainder and the subtrahend; the sum should equal the minuend.

(1)

From	$8a^2b-3cx-z^2$
Take	$3a^2b+4cx-3z^2$
Rem.	$5a^2b-7cx+2z^2$

The same, with the signs of the subtrahend changed.

(1)

	$8a^2b-3cx-z^2$
	$-3a^2b-4cx+3z^2$
Rem.	$5a^2b-7cx+2z^2$

(2)

From	$5a^3-3mz+5y^4$
Take	$-2a^3+3mz+6y^4$
Rem.	$7a^3-6mz-y^4$

(3)

From	$ax^2-3cy^2-z^2$
Take	$bx^2-3cy^2+y^3$
Rem.	$(a-b)x^2-y^3-z^2$

EXAMPLES FOR PRACTICE.

4. From $4a-2b+3c$ take $3a+4b-c$. *Ans.* $a-6b+4c$
5. From $9x^2-4y+9$ take $7x^2+5y-14$. *Ans.* $2x^2-9y+23$.
6. From $23xy^2-7y+11x^2$ take $11xy^2-5y-9x^2$. *Ans.* $12xy^2-2y+20x^2$
7. From $12x+18$ take $12x-18+y$. *Ans.* $36-y$
8. From x^2-y^3 take $-4-y^3+4x^2$. *Ans.* $4-3x^2$
9. From $4ax^3+bx+c$ take $3x^3-2x+5$. *Ans.* $(4a-3)x^3+(2+b)x+c-5$.

10. From $-17x^3+9ax^2-7a^2x+15a^3$ take $-19x^3+9ax^2-9a^2x+17a^3$. *Ans.* $2x^3+2a^2x-2a^3$.

11. From x^3+3x^2+3x+1 take x^3-3x^2+3x-1. *Ans.* $6x^2+2$.

12. From $9a^mx^2-13+20ab^3x-4b^mcx^2$ take $3b^mcx^2+9a^mx^2-6+3ab^3x$. *Ans.* $17ab^3x-7b^mcx^2-7$.

13. From $a-x-(x-2a)+2a-x$ take $a-2x-(2a-x)+(x-2a)$. *Ans.* $8a-3x$.

14. From $4a^m+2x^p-x^q$ take $a^m-b^n+3x^p$ and $2a^m-3b^n-x^p$. *Ans.* $a^m+4b^n-x^q$.

REMARK.—The number of exercises in both Addition and Subtraction is purposely small, as ample practice of the best kind will be found in the operations of Multiplication and Division.

THE BRACKET, OR VINCULUM.

As the Bracket, or Vinculum, is frequently employed, it is proper that the pupil should become acquainted with the rules which govern its use in relation to Addition and Subtraction.

ART. **46.** 1st. *Where the sign plus precedes a vinculum, it may be omitted without affecting the expression.* This principle is self-evident.

Thus, $a+(b-c)$ is the same as $a+b-c$. The first shows that b is to be diminished by the number of units in c, and the remainder added to a; the second shows that a is to be increased by the number of units in b, and the result diminished by the number of units in c. Or, if $a=6$, $b=5$, and $c=3$,

Then $6+(5-3)=6+2=8$;

And $6+5-3=11-3=8$.

From this it follows, that any number of terms of an algebraic expression may be included within a vinculum, if it be preceded by the sign plus.

Thus, $x+y-z=x+(y-z)$.

2d. *Where the sign minus precedes a vinculum, it may be omitted if the signs of all the terms within it be changed.* This is evident, because the sign minus indicates subtraction, which is effected by changing the signs of all the terms of the quantity to be subtracted. Thus,

$$a-(b-c)=a-b+c.$$
$$a-(x-y+z)=a-x+y-z.$$

Sometimes several brackets, or vinculums, are employed in the same expression; by this principle they may all be removed. Thus,

$$a-\{a+b-[a+b-c-(a-b+c)]\}.$$
$$=a-\{a+b-[a+b-c-a+b-c]\}.$$
$$=a-\{a+b-a-b+c+a-b+c\}.$$
$$=a-a-b+a+b-c-a+b-c=b-2c.$$

3d. *Any quantity may be inclosed in a vinculum, and preceded by the sign minus, provided the signs of all the terms in the vinculum be changed.* This is evident from the preceding principle. Thus,

$$a-b+c=a-(b-c)=c-(b-a).$$

This principle often enables us to express the same quantity under several different forms. Thus,

$$a-b+c+d=a-\{b-c-d\}.$$
$$=a-\{b-(c+d)\}.$$

EXAMPLES FOR PRACTICE.

Simplify, as much as possible, the following expressions.

1. $(1-2x+3x^2)+(3+2x-x^2)$. *Ans.* $4+2x^2$.
2. $5a-4b+3c+(-3a+2b-c)$. *Ans.* $2a-2b+2c$.
3. $(a-b-c)+(b+c-d)+(d-e+f)+(e-f-g)$. *Ans.* $a-g$.
4. $3(x^2+y^2)-\{(x^2+2xy+y^2)-(2xy-x^2-y^2)\}$. *Ans.* x^2+y^2.
5. $a-(x-a)-\{x-(a-x)\}$. *Ans.* $3a-3x$.
6. $1-\{1-[1-(1-x)]\}$. *Ans.* x.
7. $a-(b-c)-(a-c)+c-(a-b)$. *Ans.* $3c-a$.
8. $a-\{a+b-[a+b+c-(a+b+c+d)]\}$. *Ans.* $-b-d$.

OBSERVATIONS ON ADDITION AND SUBTRACTION.

In order that the pupil may have clear and precise ideas concerning the various operations in Algebra, it is important to understand the meaning of the signs plus (+) and minus (—), and their relation to each other.

ART. **47.** All quantities are to be regarded as positive, unless, for some special reason, they are otherwise designated. Negative quantities embrace those that are, in some particular respect, the *opposite* of positive quantities.

Thus: If the sums of money *put into* a drawer be considered positive, those *taken out* would be negative; if a merchant's *gains* are positive, his *losses* are negative; if latitude *north* of the equator is reckoned +, then latitude *south* is —; if distance to the *right* of a certain line be reckoned +, then distance to the *left* would be —; if elevation *above* a certain point, or plane, be regarded as +, then distance *below* would be —; if time *after* a certain hour is +, then time *before* that hour is —; if motion in any given direction be +, then motion in the opposite direction would be —; and so on.

Art. **48.** This relation of the signs gives rise to some important particulars.

1st. The addition, to any quantity, of a negative number, produces a *less* result than adding zero. Thus, if we take any number, for example, 10, and add to it the numbers 3, 2, 1, 0, —1, —2, —3, we have

10	10	10	10	10	10	10
3	2	1	0	—1	—2	—3
13	12	11	10	9	8	7

We see from this, that adding a negative number produces the same result as subtracting an equal positive number. Thus, adding —3 to 10 gives the same result as subtracting +3 from 10.

2d. The subtraction of a negative quantity produces a greater result than subtracting zero. Thus, take any number, for example, 10, and subtract from it the numbers 3, 2, 1, 0, —1, —2, —3, we have

10	10	10	10	10	10	10
3	2	1	0	—1	—2	—3
7	8	9	10	11	12	13

We see, also, from this, that subtracting a negative number produces the same result as adding an equal positive number.

Art. **49.** When two negative quantities are considered algebraically, that is called the *least* which contains the greatest number of units. Thus, —3 is said to be less than —2. But, of two negative quantities, that which contains the greatest number of units is said to be *numerically* the greatest; thus, —3 is numerically greater than —2.

Art. **50.** The sum of two positive quantities is always *greater* than either of them; and the sum of two negative quantities, algebraically considered, is *less* than either of them. But the sum of a positive and negative quantity is always less than the positive quantity.

Ex. A merchant gains $3a$ (+30) dollars, but soon after loses $2a$ (—20) dollars; how much will his property be increased by the two operations?

$$3a+(-2a)=+a;$$

$$\text{Or, } +30+(-20)=+10.$$

That is, his property will be increased a (10) dollars.

Had he gained $3a$, and lost $5a$ dollars, then the sum would have been —20 dollars, and his property would, evidently, have been diminished.

Art. **51.** The difference of two positive quantities, as in Arithmetic, is always less than the greater quantity. Thus, $5a-(+2a)=+3a$.

The difference of two negative quantities is always greater, algebraically considered, than the minuend. Thus, $-5a-(-2a)=-3a$.

The difference between a positive and a negative quantity, found by subtracting the latter from the former, is always greater than either of them. Thus, $2a-(-a)=3a$.

Examples: 1. The latitude of A is 10° N. (+); the latitude of B is 5° S. (—); what is their difference of latitude? *Ans.* 15°.

2. At 7 A. M. of a certain day, the thermometer stood at —9°. that is, 9 degrees below zero; at 2 P. M., at +15°, that is, 15 degrees above zero; what was the change of temperature between these hours? *Ans.* 24°.

MULTIPLICATION.

Art. **52.** Multiplication, in Algebra, is the process of taking one algebraic quantity as often as there are units in another.

The quantity to be multiplied is called the *Multiplicand.*

The quantity by which we multiply is called the *Multiplier.*

The result of the operation is called the *Product.*

The multiplicand and multiplier are generally called *factors.*

Art. **53.** To understand the subject of algebraic multiplication, it is necessary that the pupil should be made acquainted with the following *preliminary principle:*

The product of two factors is the same, whichever be made the multiplier.

To prove this, suppose we have a sash containing a vertical and b horizontal rows. It is evident that the whole number of panes in the sash will be equal to the number in one row, taken as many times as there are rows.

Since there are a vertical rows and b panes in each row, the whole number of panes will be represented by b taken a times; that is, by ab.

Again, since there are b horizontal rows, and a panes in each row, the whole number of panes will be represented by a taken b times; that is, by ba. Hence, ab is equal to ba; that is, *the product of two factors is the same, whichever be made the multiplier.*

If we have three factors, *a*, *b*, and *c*, by the preceding principle, the product of two of them, as *a* and *b*, will be either *ab* or *ba*; if, then, we regard this product as a single factor, and multiply it by *c*, the product may be written either *abc*, *cab*, *bac*, or *cba*, all of which, by the preceding principle, are equal to each other. From this, it is evident that *the product of three factors is the same, in whatever order they are taken.* Thus, $2\times3\times4=4\times2\times3=3\times2\times4=4\times3\times2$; the product in each case being 24.

In a similar manner, it may be shown that the product of any number of factors is the same, in whatever order they are taken.

It follows from this principle, that $ac\times6=6ac$, $zyx^2\times5=5x^2yz$, and so on.

It also follows from this principle, that *when either of the factors of a product is multiplied, the product itself is multiplied.* Thus, if we take the product of two factors, as 2×3, and multiply it by 5, the product may be written $5\times2\times3$, or $5\times3\times2$; that is, 10×3, or 15×2, either of which is equal to 30.

REMARK. — In the multiplication of numbers, since each figure of the multiplicand is multiplied by the multiplier, pupils sometimes suppose that in multiplying the product of two or more factors, as *ab*, by a third factor, that *each of the factors* ought to be multiplied. That this would be erroneous is evident from the preceding principle.

ART. **54.** In multiplication there are four things to be considered in relation to each term, viz:

The sign;
The coëfficient;
The exponent;
The literal part.

REMARK. — In writing a monomial product, we generally write, first the sign, then the coëfficient, and then the literal part; but, in explaining the principles, it is most convenient to consider the sign last.

ART. **55.** OF THE COEFFICIENT. — To determine the rule of the coëfficients, let it be required to find the product of $2a$ by $3b$.

To indicate the multiplication, we may write the product thus, $2a\times3b$. But, by Art. 53, this is the same as $2\times3\times ab$, and $2\times3=6$, therefore the product is $6ab$. Hence, *the coëfficient of the product is obtained by multiplying together the coëfficients of the factors.* This is termed, *the rule of the coëfficients.*

OPERATION

$$\begin{array}{l} 2a \\ 3b \\ \hline 6ab \text{ product.} \end{array}$$

From this example we see, also, that *the literal part of the product is obtained by annexing to the coëfficient all the letters in the two factors.*

EXAMPLES.

2. $3ac\times5b=$	$15abc$.	4. $5a\times4ax=$	$20aax$.
3. $2am\times cn=$	$2acmn$.	5. $7cy\times3yz=$	$21cyyz$.

ART. **56.** OF THE EXPONENT.—To determine the rule of the exponents, let it be required to find the product of $2a^2$ by $3a^3$.

OPERATION.

$$2a^2=2aa$$
$$3a^3=3aaa$$
$$6a^5=6aaaaa$$

Since $2a^2$ is the same as $2aa$, and $3a^3$ the same as $3aaa$, the product will be $2aa\times3aaa$, or $6aaaaa$, which, for the sake of brevity, is written $6a^5$. Hence, *the exponent of a letter in the product is equal to the sum of its exponents in the two factors.* This is termed *the rule of the exponents.*

EXAMPLES.

2. $ab\times a=$	a^2b.	5. $a^m\times a^n=$	a^{m+n}.
3. $x^2y\times xy=$	x^3y^2.	6. $c^{n+1}\times c^{n-1}=$	c^{2n}.
4. $a^3x^2z\times axz^2=$	$a^4x^3z^3$.	7. $x^{m+p}\times x^{n-p}=$	x^{m+n}.

ART. **57.** From the two preceding articles we derive the following

RULE FOR MULTIPLYING ONE POSITIVE MONOMIAL BY ANOTHER.—*Multiply the coëfficients of the two terms together, and to the product annex all the letters in both quantities, giving to each letter an exponent equal to the sum of its exponents in the two factors.*

NOTE.—Although the product is the same, in whatever order the letters are placed (Art. 53), yet, for the sake of *convenience,* they are generally written alphabetically.

EXAMPLES.

1. Multiply bc by z. *Ans.* bcz
2. Multiply $3ax$ by by. *Ans.* $3abxy$.
3. Multiply $4am$ by $3bn$. *Ans.* $12abmn$.
4. Multiply $5a^2x$ by $7ax^3y$. *Ans.* $35a^3x^4y$.
5. Multiply $3a^mx^n$ by $9a^nx^m$. *Ans.* $27a^{m+n}x^{m+n}$.

ART. **58.** Let it be required to find the product of $a+b$ by m

OPERATION

$$a+b$$
$$m$$
$$ma+mb$$

Here, the sum of the units in a and b is to be taken m times. The units in a taken m times $=ma$, and the units in b taken m times $=mb$; hence, both together taken m times $=ma+mb$. Hence, when the sign of each term is positive, we have the following

RULE FOR MULTIPLYING A POLYNOMIAL BY A MONOMIAL.—*Multiply each term of the multiplicand by the multiplier.*

EXAMPLES.

2. Multiply $x+y$ by n. *Ans.* $nx+ny$.
3. Multiply ax^2+cz by $3ac$. *Ans.* $3a^2cx^2+3ac^2z$.
4. Multiply $2a^2+3b^2$ by $5ab$. *Ans.* $10a^3b+15ab^3$.
5. Multiply $mx+ny+vz$ by m^2n. *Ans.* $m^3nx+m^2n^2y+m^2nvz$.

ART. **59.** Let it be required to find the product of $a+b$ by $m+n$. Here, $a+b$ is to be taken as many times as there are units in $m+n$, which is evidently as many times as there are units in m, plus as many times as there are units in n. Thus,

$a+b$
$m+n$
$ma+mb$=the multiplicand taken m times.
$na+nb$=the multiplicand taken n times.
$ma+mb+na+nb$=the multiplicand taken $(m+n)$ times.

Hence, when all the terms in each are positive, we have the following

RULE FOR MULTIPLYING ONE POLYNOMIAL BY ANOTHER.—*Multiply each term of the multiplicand by each term of the multiplier, and add the products together.*

EXAMPLES.

2. Multiply $x+y$ by $a+c$. *Ans.* $ax+ay+cx+cy$.
3. Multiply $2x+3z$ by $3x+2z$. *Ans.* $6x^2+13xz+6z^2$.
4. Multiply $2a+c$ by $a+2c$. *Ans.* $2a^2+5ac+2c^2$.
5. Multiply x^2+xy+y^2 by $x+y$. *Ans.* $x^3+2x^2y+2xy^2+y^3$
6. Multiply $a^2+2ab+b^2$ by $a+b$. *Ans.* $a^3+3a^2b+3ab^2+b^3$.

ART. **60.** OF THE SIGNS.—In the preceding article it was assumed that the product of two positive quantities is also positive. It may, however, be shown, as follows:

1st. Let it be required to find the product of $+b$ by a.

The quantity b, taken once, is $+b$; taken twice, is evidently $+2b$; taken 3 times, is $+3b$, and so on. Therefore, taken a times, it is $+ab$. Hence, the product of two positive quantities is positive; or, as it may be more briefly expressed, *plus* multiplied by *plus* gives *plus*.

2d. Let it be required to find the product of $-b$ by a.

The quantity $-b$, taken once, is $-b$; taken twice is $-2b$; taken 3 times, is $-3b$; and hence, taken a times, is $-ab$; that is, a *negative* quantity, multiplied by a *positive* quantity, gives a *negative* product. This is generally expressed by saying, that *minus*, multiplied by *plus*, gives *minus*.

3d. Let it be required to multiply b by $-a$.

Since, when two quantities are to be multiplied together, either may be made the multiplier (Art. 53), this is the same as to multiply $-a$ by b, which gives $-ab$. That is, a *positive* quantity multiplied by a *negative* quantity, gives a *negative* product; or more briefly, *plus* multiplied by *minus*, gives *minus*.

4th. Let it be required to multiply $-b$ by $-a$.

The negative multiplier signifies that the multiplicand is to be taken positively as many times as there are units in the multiplier, and then subtracted.

The product of $-b$ by a, is $-ab$, and then changing the sign to subtract, it becomes $+ab$. Hence, the product of two negative quantities is positive; or, more briefly, *minus* multiplied by *minus*, gives *plus*.

NOTE.—The following proof of the last principle, that the product of two negative quantities is positive, is generally regarded by mathematicians as more satisfactory than the preceding, though it is not quite so simple. Either method may be used.

5th. To find the product of two negative quantities.

To do this, let it be required to find the product of $c-d$ by $a-b$. Here it is required to take $c-d$ as many times as there are units in $a-b$. It is obvious that this will be done by taking $c-d$ as many times as there are units in a, and then subtracting, from this product, $c-d$ taken as many times as there are units in b.

Since plus, multiplied by plus, gives plus, and minus, multiplied by plus, gives minus, the product of $c-d$ by a is $ac-ad$. In the same manner, the product of $c-d$ by b is $bc-bd$; changing the signs of the last product to subtract it, it becomes $-bc+bd$; hence, the product of $c-d$ by $a-b$, is $ac-ad-bc+bd$. But the last term, $+bd$, is the product of $-d$ by $-b$; hence, the product of two negative quantities is positive; or, more briefly, *minus* multiplied by *minus*, produces *plus*.

The multiplication of $c-d$ by $a-b$ may be written thus:

$$\begin{array}{l} c-d \\ \underline{a-b} \\ ac-ad = c-d \text{ taken } a \text{ times.} \\ \quad -bc+bd = c-d \text{ taken } b \text{ times, and then subtracted.} \\ \hline ac-ad-bc+bd \end{array}$$

To illustrate the operation by figures, let it be required to find the product of $9-4$ by $5-3$.

OPERATION.
9—4
5—3
45—20
—27+12
45—47+12

We first take 5 times 9—4; this gives a product too great by 3 times 9—4, or 27—12; subtracting this from the first product, we have, for the true result, 45—47+12, which reduces to +10. This is evidently correct, for 9—4=5, and 5—3=2, and the product of 5 by 2 is 10.

From the preceding illustrations, we derive the following

GENERAL RULE FOR THE SIGNS. — *Plus multiplied by plus, or minus multiplied by minus, gives plus. Plus multiplied by minus, or minus multiplied by plus, gives minus.*

OR, *the product of like signs gives plus, and of unlike signs gives minus.*

ART. **61.** From the preceding, we derive the following

GENERAL RULE FOR THE MULTIPLICATION OF ALGEBRAIC QUANTITIES. — *Multiply every term of the multiplicand by each term of the multiplier, observing,*

1st. *That the coëfficient of any term is equal to the product of the coëfficients of its factors.*

2d. *That the exponent of any letter in the product is equal to the sum of its exponents in the two factors.*

3d. *That the product of like signs gives plus in the product, and unlike signs gives minus. Then, add the several partial products together.*

NUMERICAL EXAMPLES TO VERIFY THE RULE OF THE SIGNS.

1. Multiply 7—4 by 5. *Ans.* 35—20=15=3×5.
2. Multiply 8+3 by 6—4. *Ans.* 48—14—12=22=11×2.
3. Multiply 5—8 by 4—9.
Ans. 20—77+72=+15=—3×—5.
4. Multiply 7—3 by 8—2. *Ans.* 56—38+6=24=6×4.

GENERAL EXAMPLES.

1. Multiply $4a^2-3ac+2$ by $5ax$. *Ans.* $20a^3x-15a^2cx+10ax$.
2. Multiply $5a-2ab+10$ by $-9ab$.
Ans. $-45a^2b+18a^2b^2-90ab$.
3. Multiply $2x+3z$ by $2x-3z$. *Ans.* $4x^2-9z^2$.
4. Multiply $4a^2-6a+9$ by $2a+3$. *Ans.* $8a^3+27$.
5. Multiply $a-b+c-d$ by $a+b-c-d$.
Ans. $a^2-b^2-c^2+d^2-2ad+2bc$.
6. Multiply $x^3+y^2+z^2$ by x^2+y^2.
Ans. $x^5+x^2y^2+x^2z^2+x^3y^2+y^4+y^2z^2$.

7. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$

8. Multiply $12x^3-8x^2y+15xy^2-10y^3$ by $3x+2y$.
Ans. $36x^4+29x^2y^2-20y^4$.

9. Multiply a^2+ax+x^2 by a^2-ax+x^2. *Ans.* $a^4+a^2x^2+x^4$.

10. Multiply $a^2+2ab+2b^2$ by $a^2-2ab+2b^2$. *Ans.* a^4+4b^4.

11. Multiply $x^2+2xy-3y^2$ by $x^2-5xy+4y^2$.
Ans. $x^4-3x^3y-9x^2y^2+23xy^3-12y^4$.

12. Multiply $1+x+x^2+x^3+x^4$ by $1-x$. *Ans.* $1-x^5$.

13. Multiply $27x^3+9x^2y+3xy^2+y^3$ by $3x-y$. *Ans.* $81x^4-y^4$.

14. Multiply $x^4+2x^3y+4x^2y^2+8xy^3+16y^4$ by $x-2y$.
Ans. x^5-32y^5.

15. Multiply $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $a+2b$.
Ans. a^5+32b^5.

16. Multiply $a^3+2a^2b+2ab^2+b^3$ by $a^3-2a^2b+2ab^2-b^3$.
Ans. a^6-b^6.

17. Multiply $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.
Ans. $a^3+b^3+c^3-3abc$.

18. Multiply $x^4-x^3+x^2-x+1$ by x^2+x-1.
Ans. $x^6-x^4+x^3-x^2+2x-1$.

19. Multiply $1+x+x^4+x^5$ by $1-x+x^2-x^3$. *Ans.* $1-x^8$.

20. Multiply $1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7$ by $1+2x+x^2$
Ans. $1-9x^8-8x^9$.

21. Multiply together $x-3$, $x+4$, $x-5$, and $x+6$.
Ans. $x^4+2x^3-41x^2-42x+360$.

22. Multiply together a^2+ab+b^2, $a^3-a^2b+b^3$, and $a-b$.
Ans. $a^6-a^5b+a^2b^4-b^6$.

23. Multiply together $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2.
Ans. a^6-b^6.

24. Prove that

$$x(x+1)(x+2)+x(x-1)(x-2)+4(x-1)x(x+1)=6x^3.$$

25. Find the value of the expression

$$(x+a)(x+b)(x+c)-(a+b+c)(x+a)(x+b)+(a^2+ab+b^2)(x+a).$$

Ans. x^3+a^3.

MULTIPLICATION BY DETACHED COEFFICIENTS.

Art. 62. In the multiplication of polynomials, it is evident that the coëfficients of the product depend on the coëfficients of the factors, and not upon the literal parts of the terms.

Hence, by *detaching* the coëfficients of the factors from the literal parts, and multiplying them together, we shall obtain the coëfficients of the product. If to these coëfficients the proper letters are then annexed, the whole product will be obtained. This method is applicable where the powers of the same letter increase or decrease regularly.

1. Multiply $a^2-2ab+b^2$ by $a+b$.

After finding the coëfficients, it is obvious that a^3 will be the first term and b^3 the last term; hence, the entire product is $a^3-a^2b-ab^2+b^3$.

OPERATION.

$$\begin{array}{r} 1-2+1 \\ 1+1 \\ \hline 1-2+1 \\ +1-2+1 \\ \hline 1-1-1+1 \end{array}$$

2. Multiply $a^3-3a^2b+b^3$ by a^2-b^2.

In this example, supposing the powers of a to decrease regularly toward the left, it is obvious that there is a term wanting in each factor. In such cases the coëfficient of each absent term must be considered zero, and supplied before commencing the operation.

OPERATION.

$$\begin{array}{r} 1-3+0+1 \\ 1+0-1 \\ \hline 1-3+0+1 \\ -1+3-0-1 \\ \hline 1-3-1+4-0-1 \end{array}$$

This method, termed *multiplication by detached coëfficients*, is useful in leading the pupil to consider the properties of coëfficients by themselves.

EXAMPLES.

3. Multiply $m^3+m^2n+mn^2+n^3$ by $m-n$ *Ans.* m^4-n^4.

4. Multiply $1+2z+3z^2+4z^3+5z^4$ by $1-z$.

Ans. $1+z+z^2+z^3+z^4-5z^5$.

The pupil may also solve, by this method, the general examples, Art. 61, from 7 to 20 inclusive, except example 17.

REMARKS ON ALGEBRAIC MULTIPLICATION.

ART. **63.** The *degree* of the product of any two monomials, is equal to the *sum* of the degrees of the multiplicand and multiplier. This is evident, since all the factors of both quantities appear in the product. Thus, $2a^2b$, which is of the 3d degree, multiplied by $3ab^3$ which is of the 4th degree, gives $6a^3b^4$, which is of the 7th degree. Hence, if two polynomials are *homogeneous*, their product will be homogeneous. Thus, in example 7, Art. 61, both multiplicand and multiplier are homogeneous, each term being of the 3d degree, and the product is homogeneous, each term being of the 6th degree.

ART. **64.** In the multiplication of two polynomials, when the partial products do not contain *similar terms*, the *whole number* of terms in the final product will be equal to the number of terms in the multiplicand, multiplied by the number of terms in the multiplier. Thus, if there be m terms in the multiplicand, and n terms in the multiplier, the number of terms in the product will be $m\times n$. Thus, in example 6, Art. 61, there are 3 terms in the multiplicand, 2 in the multiplier, and $3\times2=6$ in the product.

ART. **65.** If the partial products contain *similar terms,* the number of terms in the reduced product will evidently be less than $m\times n$; see examples 7 to 21 inclusive, Art. 61. It is important to note that there are two terms which can never be reduced with any others; these are,

1. That term which is the product of the two terms in the factors which contain the *highest* power of the same letter.

2. That term which is the product of the two terms in the factors which contain the *lowest* power of the same letter.

ART. **66.** The multiplication of two polynomials is *indicated* by inclosing each in a parenthesis, and writing them in succession. Thus, the multiplication of the polynomials $m+n$ and $p-q$, is indicated by $(m+n)(p-q.)$

When the operation is actually performed, the expression is said to be *expanded,* or *developed.*

DIVISION.

ART. **67.** Division, in Algebra, is the process of finding how often one algebraic quantity is contained in another. Or, *having the product of two factors, and one of them given, division teaches the method of finding the other.*

As in Arithmetic, the quantity by which we divide is called the *divisor;* the quantity to be divided, the *dividend;* the result of the operation, the *quotient.*

ART. **68.** In division, as in multiplication, there are four things to be considered, viz:

The sign;
The coëfficient;
The exponent;
The literal part.

ART. **69.** To ascertain the rule of the signs.

Since $\left.\begin{array}{l}+a\times+b=+ab\\-a\times+b=-ab\\+a\times-b=-ab\\-a\times-b=+ab\end{array}\right\}$ therefore $\left\{\begin{array}{l}+ab\div+b=+a\\-ab\div+b=-a\\-ab\div-b=+a\\+ab\div-b=-a\end{array}\right.$

From which we derive the following

RULE OF THE SIGNS.—*When both divisor and dividend have the same sign, the quotient will have the sign* $+$; *when they have different signs, the quotient will have the sign* $-$.

ART. **70.** To ascertain the *rule of the coëfficients*, the *rule of the exponents*, and the *rule of the literal part*. These may all be derived from the solution of a single example.

Let it be required to find how often $2a^2$ is contained in $6a^5b$.

$$\frac{6a^5b}{2a^2}=\frac{6}{2}a^{5-2}b=3a^3b.$$

Since division is the reverse of multiplication, the quotient, multiplied by the divisor, must produce the dividend; hence, to obtain this quotient, it is obvious,

1st. That the coëfficient of the quotient must be such a number, that when multiplied by 2 the product shall be 6; therefore, to obtain it, we divide 6 by 2. Hence, we have the following

RULE OF THE COEFFICIENTS.—*To obtain the coëfficient of the quotient, divide the coëfficient of the dividend by the coëfficient of the divisor.*

2d. The exponent of a in the quotient must be such a number, that when 2, the exponent of a in the divisor, is added to it, the sum shall be 5; hence, to obtain it, we must subtract 2 from 5; that is $5-2=3$, is the exponent of a in the quotient. This gives the following

RULE OF THE EXPONENTS.—*From the exponent of any letter in the dividend subtract its exponent in the divisor, the remainder will be its exponent in the quotient.*

3d. The letter b, which is a factor of the dividend, but not of the divisor, must be found in the quotient, in order that the product of the divisor and quotient may equal the dividend. Hence, *every letter found in the dividend, and not in the divisor, must be found in the quotient, with the same exponent as in the dividend.* This, in connection with the rule of the exponents, furnishes the *rule of the literal parts.*

ART. **71.** The preceding rules taken together, give the following

RULE FOR DIVIDING ONE MONOMIAL BY ANOTHER.—*Divide the coëfficient of the dividend by that of the divisor; observing, that like signs give plus and unlike signs give minus.*

After the coëfficient, write the letters common to both divisor and dividend, giving to each an exponent, equal to the excess of the exponent of the same letter in the dividend, over that in the divisor.

In the quotient, write the letters, with their respective exponents, that are found in the dividend, but not in the divisor.

EXAMPLES.

1. Divide $4a^5$ by $2a^2$ and by $-2a^2$. *Ans.* $2a^3$ and $-2a^3$
2. Divide $30a^4b^2$ by $5a^2b$. *Ans.* $6a^2b$
3. Divide $-28x^3y^7z^4$ by $-7xy^2z$. *Ans.* $4x^2y^5z^3$.
4. Divide $-35a^2b^3c$ by $5ab^2$. *Ans.* $-7abc$.
5. Divide $32xyz$ by $-8xy$. *Ans.* $-4z$.
6. Divide $42c^3m^2n$ by $-3cmn$. *Ans.* $-14c^2m$.
7. Divide x^{m+n} and x^{m-n} each by x^n. *Ans.* x^m and x^{m-2n}.
8. Divide v^{m+n} by v^{m+p}. *Ans.* v^{n-p}.

NOTE.—In solving the following examples, the pupil must recollect, that the quantities included within the vinculum are to be considered together, as a single quantity.

9. Divide $(a+b)^3$ by $(a+b)^2$. *Ans.* $(a+b)$.
10. Divide $(m-n)^5$ by $(m-n)^2$. *Ans.* $(m-n)^3$.
11. Divide $8(a-b)^3x^2y$ by $2(a-b)xy$. *Ans.* $4(a-b)^2x$.
12. Divide $6(x+z)^3(a-b)^2$ by $3(x+z)(a-b)^2$. *Ans.* $2(x+z)^2$.
13. Divide $a^2b^3(x-y)(y-z)^2$ by $ab^2(y-z)^2$. *Ans.* $ab(x-y)$.
14. Divide $(a+bx^2)^{p+1}$ by $(a+bx^2)^{p-1}$. *Ans.* $(a+bx^2)^2$.

ART. **72.** It is evident that one monomial cannot be divided by another in the following cases:

1st. When the coëfficient of the dividend is not exactly divisible by the coëfficient of the divisor.

2d. When the same literal factor has a greater exponent in the divisor than in the dividend.

3d. When the divisor contains one or more literal factors not found in the dividend.

In each of these cases the division is to be indicated by writing the divisor under the dividend, in the form of a fraction. This fraction may often be reduced to lower terms. See Art. 119.

ART. **73.** It has been shown, in Art. 53, that any product is multiplied by multiplying either of its factors; hence, conversely, *any dividend will be divided by dividing either of its factors.*

Thus, $\frac{6\times9}{3}=2\times9=18$, by dividing the factor 6.

Or, $\frac{6\times9}{3}=6\times3=18$, by dividing the factor 9.

DIVISION OF POLYNOMIALS BY MONOMIALS.

ART. **74.** In multiplying a polynomial by a monomial, we multiply each term of the multiplicand by the multiplier. Thus,

$(ab-b^2)\times a=a^2b-ab^2$; hence, $(a^2b-ab^2)\div a=ab-b^2$; therefore, we have the following

Rule for dividing a polynomial by a monomial.—*Divide each term of the dividend by the divisor, according to the rule for the division of monomials.*

EXAMPLES FOR PRACTICE.

1. Divide a^2+ab by a. *Ans.* $a+b$
2. Divide $3xy+2x^2y$ by $-xy$. *Ans.* $-3-2x$.
3. Divide $10a^2z-15z^2-25z$ by $5z$. *Ans.* $2a^2-3z-5$.
4. Divide $3ab+12abx-9a^2b$ by $-3ab$. *Ans.* $-1-4x+3a$.
5. Divide $5x^3y^3-40a^2x^2y^2+25a^4xy$ by $-5xy$.
   Ans. $-x^2y^2+8a^2xy-5a^4$.
6. Divide $4abc-24ab^2-32abd$ by $-4ab$. *Ans.* $-c+6b+8d$.
7. Divide $a^mb^3+a^{m+1}b^2+a^{n-2}b$ by ab. *Ans.* $a^{m-1}b^2+a^mb+a^{n-3}$.
8. Divide $4a^2(a+x)+6a(x+y)$ by $2a$.
   Ans. $2a(a+x)+3(x+y)$.
9. Divide $3a(x+y)+c^2(x+y)^2$ by $x+y$. *Ans.* $3a+c^2(x+y)$.
10. Divide $b^2c(m+n)-bc^2(m+n)$ by $bc(m+n)$. *Ans.* $b-c$.
11. Divide $(b+c)(b-c)^2-(b-c)(b+c)^2$ by $(b+c)(b-c)$.
    Ans. $(b-c)-(b+c)=-2c$
12. Divide $(m-n)^2(x+z)^3-(m-n)^3(x+z)^2$ by $(m-n)^2(x+z)^2$.
    Ans. $(x+z)-(m-n)$.

DIVISION OF ONE POLYNOMIAL BY ANOTHER.

Art. **75.** To explain the method of dividing one polynomial by another, we may regard the dividend as a product, of which the divisor and the quotient are the two factors. We shall first form a product, and then, by a reverse operation, explain the process of division.

Multiplication, or formation of a product.	Division, or decomposition of a product.		
a^3-5a^2b		$a^5-3a^4b-11a^3b^2+5a^2b^3$	a^3-5a^2b
$a^2+2ab-b^2$		a^5-5a^4b	$a^2+2ab-b^2$
a^5-5a^4b	1st r.	$+2a^4b-11a^3b^2+5a^2b^3$	Quotient.
$+2a^4b-10a^3b^2$		$+2a^4b-10a^3b^2$	
$-a^3b^2+5a^2b^3$	2d remainder,	$-a^3b^2+5a^2b^3$	
		$-a^3b^2+5a^2b^3$	
$a^5-3a^4b-11a^3b^2+5a^2b^3$	3d remainder,	0	

By comparing the product with the two factors, each being *arranged* according to the decreasing powers of the letter a, we see that the 1st term a^5 of the total product, is the product of the 1st term a^3 of the multiplicand, by a^2 the 1st term of the divisor;

that the last term $+5a^2b^3$ of the total product, is the result of the product of $-5a^2b$ the last term of the multiplicand, by $-b^2$ the last term of the multiplier; and that the other terms of the total product are the result of the reduction of the similar terms of the partial products. See Arts. 64 and 65.

Consequently, the division of a^5, the first term of the dividend by a^3, the first term of the divisor, will give a^2, the first term of the quotient.

The dividend expresses the sum of the partial products of the divisor by the different terms of the quotient; therefore, if we subtract from the dividend a^5-5a^4b, which is the product of the divisor a^3-5a^2b by a^2 the first term of the quotient, the *remainder* $+2a^4b-11a^3b^2+5a^2b^3$, will be the product of the divisor by the other terms of the quotient.

Knowing the 1st term a^2 of the quotient, and the 1st remainder, it is now required to find the other terms of the quotient. We remark, that the 1st remainder expresses the product of the divisor by the unknown terms of the quotient, and that, consequently, the 1st term $+2a^4b$ of the 1st remainder, is the product of the 1st term a^3 of the divisor by the 1st of the unknown terms of the quotient; therefore we shall obtain the 1st of these terms, that is, the 2d term of the required quotient, by dividing the 1st term $+2a^4b$ of the 1st remainder, by the 1st term a^3 of the divisor; this gives $+2ab$ the 2d term of the required quotient.

Lastly, to find the 3d term of the quotient, we subtract from the 1st remainder, the product of the divisor by $+2ab$, the 2d term of the quotient; the 2d remainder is the product of the divisor by the 3d term of the quotient; hence, the division of the 1st term $-a^3b^2$ of this 2d remainder, by the 1st term a^3 of the divisor, must give the 3d term of the quotient, which is thus found to be $-b^2$.

Subtracting from the 2d remainder, the product of the divisor by $-b^2$, the *remainder* zero, shows that the quotient $a^2+2ab-b^2$ is exact; for we have arrived at this remainder by subtracting from the dividend and the several remainders, the partial products of the divisor by the terms a^2, $+2ab$, $-b^2$ of the quotient.

Since there is no remainder when we subtract from the dividend the product of the divisor by $a^2+2ab-b^2$, therefore the dividend is the exact product of the divisor by $a^2+2ab-b^2$, which is, therefore, the required quotient.

Since each term of the quotient is found, by dividing that term of the dividend containing the highest power of a particular letter, by the term of the divisor containing the highest power of the same letter, the *divisor and dividend should always be arranged* (Art. 31) *with reference to a certain letter.*

The situation of the divisor in regard to the dividend, is a matter of arbitrary arrangement; by placing it on the right it is more easily multiplied by the respective terms of the quotient.

ART. 76. From the preceding we derive the following

RULE FOR THE DIVISION OF ONE POLYNOMIAL BY ANOTHER.— *Arrange the dividend and divisor with reference to a certain letter, and place the divisor on the right of the dividend.*

Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient. Multiply the divisor by this term, and subtract the product from the dividend.

Divide the first term of the remainder by the first term of the divisor; the result will be the second term of the quotient. Multiply the divisor by this term, and subtract the product from the last remainder.

Proceed in the same manner, and if you obtain 0 for a remainder, the division is said to be exact.

REMARKS.—1st. When there are more than two terms in the quotient, it is not necessary to bring down any more terms of the remainder, at each successive subtraction, than have corresponding terms in the quantity to be subtracted.

2d. It is evident that the exact division of one polynomial by another will be impossible, when the first term of the arranged dividend is not exactly divisible by the first term of the arranged divisor; or when any remainder is not divisible by the first term of the divisor.

1. Divide $15x^2+16xy-15y^2$ by $5x-3y$.

OPERATION.

$$\begin{array}{l|l} 15x^2+16xy-15y^2 & 5x-3y \\ \underline{15x^2-\ \ 9xy} & 3x+5y \text{ Quotient.} \\ \quad +25xy-15y^2 & \\ \quad \underline{+25xy-15y^2} & \end{array}$$

2. Divide m^2-n^2 by $m+n$.

OPERATION.

$$\begin{array}{l|l} m^2-n^2 & m+n \\ \underline{m^2+mn} & m-n \text{ Quotient.} \\ \quad -mn-n^2 & \\ \quad \underline{-mn-n^2} & \end{array}$$

3. Divide x^3+y^3 by $x+y$.

OPERATION.

$$\begin{array}{l|l} x^3+y^3 & x+y \\ \underline{x^3+x^2y} & x^2-xy+y^2 \text{ Quot.} \\ \quad -x^2y+\ y^3 & \\ \quad \underline{-x^2y-xy^2} & \\ \qquad\quad +xy^2+y^3 & \\ \qquad\quad \underline{xy^2+y^3} & \end{array}$$

4. Divide $7x^2y+5xy^2+2x^3+y^3$ by $3xy+x^2+y^2$.

OPERATION.

$$\begin{array}{l|l}
2x^3+7x^2y+5xy^2+y^3 & x^2+3xy+y^2 \\
\underline{2x^2+6x^2y+2xy^2} & 2x+y \text{ Quotient.} \\
\quad x^2y+3xy^2+y^3 & \\
\quad \underline{x^2y+3xy^2+y^3} &
\end{array}$$

In this example, neither divisor nor dividend being arranged with reference to either x or y, we arrange them with reference to x, and then proceed to perform the division.

5. Divide $x^2+x^3-7x^4+5x^5$ by $x-x^2$.

Division performed, by arranging both quantities according to the ascending powers of x.

$$\begin{array}{l|l}
x^2+x^3-7x^4+5x^5 & x-x^2 \\
\underline{x^2-x^3} & x+2x^2-5x^3 \\
\quad 2x^3-7x^4 & \text{Quotient.} \\
\quad \underline{2x^3-2x^4} & \\
\quad\quad -5x^4+5x^5 & \\
\quad\quad \underline{-5x^4+5x^5} &
\end{array}$$

Division performed, by arranging both quantities according to the descending powers of x.

$$\begin{array}{l|l}
5x^5-7x^4+x^3+x^2 & -x^2+x \\
\underline{5x^5-5x^4} & -5x^3+2x^2+x \\
\quad -2x^4+x^3 & \text{Quotient.} \\
\quad \underline{-2x^4+2x^3} & \\
\quad\quad -x^3+x^2 & \\
\quad\quad \underline{-x^3+x^2} &
\end{array}$$

The learner will perceive that the two quotients are the same, but differently arranged.

EXAMPLES FOR PRACTICE.

6. Divide $6x^2+5xy-4y^2$ by $3x+4y$. *Ans.* $2x-y$.
7. Divide $x^3-40x-63$ by $x-7$. *Ans.* x^2+7x+9.
8. Divide $3h^5+16h^4k-33h^3k^2+14h^2k^3$ by h^2+7hk.
Ans. $3h^3-5h^2k+2hk^2$.
9. Divide a^5-243 by $a-3$. *Ans.* $a^4+3a^3+9a^2+27a+81$.
10. Divide $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.
Ans. $x^4+2ax^3+3a^2x^2+2a^3x+a^4$.
11. Divide $1-6x^5+5x^6$ by $1-2x+x^2$.
Ans. $1+2x+3x^2+4x^3+5x^4$.
12. Divide $p^2+pq+2pr-2q^2+7qr-3r^2$ by $p-q+3r$.
Ans. $p+2q-r$.
13. Divide $4x^5+4x-x^3$ by $3x+2x^2+2$. *Ans.* $2x^3-3x^2+2x$.
14. Divide x^6-a^6 by $x^3+2ax^2+2a^2x+a^3$.
Ans. $x^3-2ax^2+2a^2x-a^3$.
15. Divide $m^2+2mp-n^2-2nq+p^2-q^2$ by $m-n+p-q$.
Ans. $m+n+p+q$.
16. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.
Ans. $a^2+b^2+c^2-ab-ac-bc$.

17. Divide $x^{m+n}+x^ny^n+x^my^m+y^{m+n}$ by x^n+y^m. *Ans.* x^m+y^n.
18. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$. *Ans.* x^2-ax-b.
19. Divide $mpx^3+(mq-np)x^2-(mr+nq)x+nr$ by $mx-n$.
Ans. px^2+qx-r.
20. Divide $a^{2m}-3a^mc^n+2c^{2n}$ by a^m-c^n. *Ans.* a^m-2c^n.
21. Divide $x^4+x^{-4}-x^2-x^{-2}$ by $x-x^{-1}$. *Ans.* x^3-x^{-3}.
22. Divide $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$ by $a^4+a^3b+a^2b^2+ab^3+b^4$.
Ans. $a^4-a^3b+a^2b^2-ab^3+b^4$.
23. Divide $a^2+(a-1)x^2+(a-1)x^3+(a-1)x^4-x^5$ by $a-x$.
Ans. $a+x+x^2+x^3+x^4$.
24. Divide $x(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$. *Ans.* $(x-1)a+x^2$.
25. Divide $x^3-8y^3+125z^3+30xyz$ by $x-2y+5z$.
Ans. $x^2+2xy-5xz+4y^2+10yz+25z^2$.
26. Divide $1-9x^8-8x^9$ by $1+2x+x^2$.
Ans. $1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7$.
27. Divide $1+2x$ by $1-3x$ to 5 terms in the quotient.
Ans. $1+5x+15x^2+45x^3+135x^4+$&c.
28. Divide $1-3x-2x^2$ by $1-4x$ to 6 terms in the quotient.
Ans. $1+x+2x^2+8x^3+32x^4+128x^5+$&c.

DIVISION BY DETACHED COEFFICIENTS.

Art. 77. From Art. 62, it is evident that division sometimes may be conveniently performed, by operating on the coëfficients *detached* from the letters, and afterwards supplying the letters. Thus, if it be required to divide $x^2+2xy+y^2$ by $x+y$, we may perform the operation as follows:

$$\begin{array}{l|l} 1+2+1 & 1+1 \\ \underline{1+1} & \overline{1+1} \\ \quad +1+1 & \\ \quad \underline{\;\;1+1} & \end{array}$$

Hence the coëfficients of the quotient are 1 and 1. Also, $x^2\div x=x$, and $y^2\div y=y$; therefore the quotient is $1x+1y$, or $x+y$.

2. Divide $12a^4-26a^3b-8a^2b^2+10ab^3-8b^4$ by $3a^2-2ab+b^2$.

$$\begin{array}{l|l} 12-26-8+10-8 & 3-2+1 \\ \underline{12-\;\;8+4} & \overline{4-6-8} \\ \quad -18-12+10 & \\ \quad \underline{-18+12-\;\;6} & \\ \qquad\quad -24+16-8 & \\ \qquad\quad \underline{-26+16-8} & \end{array}$$

Hence the coëfficients of the quotient are $4-6-8$. Also, $a^4\div a^2=a^2$, and $b^4\div b^2=b^2$; therefore the quotient is $4a^2-6ab-8b^2$.

When any of the intermediate powers of the letters are wanting, the coëfficients of the corresponding terms must be supplied with zero, as in the following example.

3. Divide a^3+x^3 by $a+x$.

$$\begin{array}{rl} 1+0+0+1 & |\underline{1+1} \\ \underline{1+1} & 1-1+1 \\ -1 & a^2-ax+x^2 \text{ Quotient.} \\ \underline{-1-1} & \\ +1+1 & \\ \underline{+1+1} & \end{array}$$

EXAMPLES.

4. Divide $6x^4+4x^3y-9x^2y^2-3xy^3+2y^4$ by $2x^2+2xy-y^2$.
Ans. $3x^2-xy-2y^2$.

5. Divide $m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$ by $m^2-2mn+n^2$.
Ans. $m^3-3m^2n+3mn^2-n^3$.

6. Divide $a^6-3a^4b^2+3a^2b^4-b^6$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^3+3a^2b+3ab^2+b^3$.

Most of the examples in the preceding article may be solved by this method.

CHAPTER II.

ALGEBRAIC THEOREMS,

DERIVED FROM MULTIPLICATION AND DIVISION.

REMARK.—One of the chief objects of Algebra is to establish certain general truths. The pupil has now obtained the necessary knowledge to prove the following theorems, which may be regarded as the simplest application of Algebra.

ART. **78.** THEOREM I.—*The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Let a represent one of the quantities, and b the other;
then $a+b=$their sum;
and $(a+b)\times(a+b)$, or $(a+b)^2=$the square of their sum.
But $(a+b)\times(a+b)=a^2+2ab+b^2$, which proves the theorem.

APPLICATION.

1. $(2+5)^2=4+20+25=49$.
2. $(2m+3n)^2=4m^2+12mn+9n^2$.

3. $(ax+by)^2=a^2x^2+2abxy+b^2y^2$.
4. $(ax^2+3xz^3)^2=a^2x^4+6ax^3z^3+9x^2z^6$.

ART. 79. THEOREM II.—*The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Let a represent one of the quantities, and b the other;
then $a-b$=their difference;
and $(a-b)\times(a-b)$, or $(a-b)^2$=the square of their difference.
But $(a-b)\times(a-b)=a^2-2ab+b^2$, which proves the theorem.

APPLICATION.

1. $(5-3)^2=25-30+9=4$.
2. $(2x-y)^2=4x^2-4xy+y^2$.
3. $(3x-5z)^2=9x^2-30xz+25z^2$.
4. $(az-3cx)^2=a^2z^2-6acxz+9c^2x^2$.

ART. 80. THEOREM III.—*The product of the sum and difference of two quantities, is equal to the difference of their squares.*

Let a represent one of the quantities, and b the other;
then $a+b$=their sum,
and $a-b$=their difference.
And $(a+b)(a-b)=a^2-b^2$, which proves the theorem.

APPLICATION.

1. $(7+4)(7-4)=49-16=33=11\times3$.
2. $(2x+y)(2x-y)=4x^2-y^2$.
3. $(3a^2+4b^2)(3a^2-4b^2)=9a^4-16b^4$.
4. $(3ax+5by)(3ax-5by)=9a^2x^2-25b^2y^2$.

ART. 81. THEOREM IV.—*The reciprocal of a quantity, is equal to the same quantity with the sign of its exponent changed.*

If we divide a^3 by a^5, the quotient is expressed by $\frac{a^3}{a^5}$, or by $a^{3-5}=a^{-2}$, since the rule for the exponents in division (Art. 70) requires that the exponent of the same letter in the divisor should be subtracted from that of the dividend. But $\frac{a^3}{a^5}$ is a fraction, and if we divide both terms by a^3, which does not alter its value (Ray's Arith., part 3d, Art. 147), it becomes $\frac{1}{a^2}$; hence $a^{-2}=\frac{1}{a^2}$, since each is equal to $\frac{a^3}{a^5}$.

In the same manner, $\frac{a^m}{a^n}=a^{m-n}$, by subtracting the exponents; or $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$, by dividing both terms by a^m; hence, $a^{m-n}=\frac{1}{a^{n-m}}$, which proves the theorem.

EXAMPLES.

1. $a^m=\frac{1}{a^{-m}}$.
2. $a^{-m}=\frac{1}{a^m}$.
3. $\frac{a}{b^m}=ab^{-m}$.
4. $\frac{a^m}{b^n}=a^mb^{-n}$.
5. $\frac{a}{b}=ab^{-1}$.
6. $\frac{1}{ab^2}=a^{-1}b^{-2}$.

We see also from this, that *any factor may be transferred from one term of a fraction to the other, if at the same time the sign of its exponent be changed.* Thus,

$$\frac{x}{z}=xz^{-1}=\frac{z^{-1}}{x^{-1}}=\frac{1}{x^{-1}z}. \qquad \frac{a^2}{b^3}=a^2b^{-3}=\frac{1}{a^{-2}b^3}=\frac{b^{-3}}{a^{-2}}.$$

ART. **82.** THEOREM V.—*Any quantity, whose exponent is* 0, *is equal to unity.*

If we divide a^2 by a^2, and apply the rule for the exponents (Art. 70), we find $\frac{a^2}{a^2}=a^{2-2}=a^0$; but, since any quantity is contained in itself once, $\frac{a^2}{a^2}=1$.

Similarly, $\frac{a^m}{a^m}=a^{m-m}=a^0$. But, $\frac{a^m}{a^m}=1$.

Hence, $a^0=1$, since each is equal to $\frac{a^m}{a^m}$; which proves the theorem.

This notation is used, when we wish to preserve the trace of a letter, which has disappeared in the operation of division. Thus, if we divide a^2b by ab, the quotient is $\frac{a^2b}{ab}=a^{2-1}b^{1-1}=a^1b^0=a$.

Now the quotient is expressed correctly, either by a^1b^0, or by a, since both have the same value. The first form is used when we wish to show that the letter b was originally a factor, both of the dividend and divisor.

ART. **83.** THEOREM VI.—*The difference of the same power of two quantities is always divisible by the difference of the quantities.*

1. If we divide a^2-b^2 by $a-b$, the quotient is $a+b$.
2. If we divide a^3-b^3 by $a-b$, the quotient is a^2+ab+b^2.

In the same manner, we would find by trial, that the difference of the same power of any two quantities is divisible by the difference of the quantities. The general and direct proof of this theorem is as follows:

Let us divide $a^m - b^m$ by $a - b$.

$$\begin{array}{l|l} a^m - b^m & a - b \\ \hline a^m - a^{m-1}b & a^{m-1} + \dfrac{b(a^{m-1} - b^{m-1})}{a - b} \text{ Quotient.} \\ \quad a^{m-1}b - b^m & \\ = b(a^{m-1} - b^{m-1}). & \end{array}$$

In performing this division, we see that the first term of the quotient is a^{m-1}, and the first remainder, $b(a^{m-1} - b^{m-1})$.

The remainder consists of two factors, b and $a^{m-1} - b^{m-1}$. Now it is evident, that *if* the second of these factors is divisible by $a - b$, then will the quantity $a^m - b^m$ be divisible by $a - b$. Thus, if $a - b$ is contained c times in $a^{m-1} - b^{m-1}$, the entire quotient of $a^m - b^m$, divided by $a - b$, would be $a^{m-1} + bc$.

This proves, that IF $a^{m-1} - b^{m-1}$ is divisible by $a - b$, then will $a^m - b^m$, be divisible by $a - b$. That is, IF *the difference of the same powers of two quantities is divisible by the difference of the quantities themselves, then will the difference of the next higher powers of the same quantities, be divisible by the difference of the quantities.*

But we have seen, already, that $a^2 - b^2$ is divisible by $a - b$; hence, it follows, that $a^3 - b^3$ is also divisible by $a - b$. Again, since $a^3 - b^3$ is divisible by $a - b$, it follows that $a^4 - b^4$ is divisible by it. And so on, without limit, which proves the truth of the theorem generally.

NOTE.—In dividing the difference of the same powers of two quantities by the difference of those quantities, the quotients follow a simple law. Thus,

$$(a^2 - b^2) \div (a - b) = a + b;$$
$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2;$$
$$(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3;$$
$$(a^5 - b^5) \div (a - b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

The law is, that the exponent of the first letter decreases by unity, while that of the second increases by unity.

ART. 84. LEMMA.—In proving the next two theorems, it is necessary to remind the student, that the *even* powers of a negative quantity are *positive*, and the *odd* powers *negative*. Thus,

$-a$, the 1st power of $-a$, is negative.

$-a \times -a = a^2$, the 2d power, is positive.

$-a \times -a \times -a = -a^3$, the 3d power, is negative.

$-a \times -a \times -a \times -a = a^4$, the 4th power, is positive; and so on.

ART. 85. THEOREM VII.— *The difference of the even powers of the same degree of two quantities, is always divisible by the sum of the quantities.*

If we take the quantities $a-b$, and a^m-b^m, and put $-c$ instead of b, $a-b$ will become $a-(-c)=a+c$, and, when m is *even*, b^m will become c^m, and a^m-b^m will become $a^m-(+c^m)=a^m-c^m$: but a^m-b^m is always divisible by $a-b$; therefore,

a^m-c^m is always divisible by $a+c$ when m is *even*, which is the theorem.

EXAMPLES.

1. $(a^2-b^2)\div(a+b)=a-b$.
2. $(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3$.
3. $(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.

ART. 86. THEOREM VIII.— *The sum of the odd powers of the same degree of two quantities, is always divisible by the sum of the quantities.*

If we take the quantities $a-b$, and a^m-b^m, and put $-c$ instead of b, $a-b$ will become $a-(-c)=a+c$, and when m is *odd*, b^m will become $-c^m$ (Art. 84), and a^m-b^m will become $a^m-(-c^m)$ $=a^m+c^m$: but a^m-b^m is always divisible by $a-b$; therefore,

a^m+c^m is always divisible by $a+c$ when m is *odd*, which is the theorem.

EXAMPLES.

1. $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.
2. $(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$.
3. $(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

FACTORING.

NOTE.—Previous to studying the factoring of algebraic quantities, the pupil should be well acquainted with factoring numbers. See Ray's Arith., Part 3d, Arts. 121 to 124.

ART. 87. The following is a summary of the principles and the most useful rules employed in factoring numbers.

1st PRINCIPLE. A factor of a number is a factor of any multiple of that number.

2d Principle. A factor of any two numbers is also a factor of their sum.

Propositions deduced from these principles:

1. Every number ending with 0, 2, 4, 6, or 8, is divisible by 2.

2. Every number is divisible by 4, when the number denoted by its two right hand digits is divisible by 4.

3. Every number is divisible by 5, whose right hand digit is 0 or 5.

4. Every number whose first digits are 0, 00, &c., is divisible by 10, 100, &c.

The converse of each of the preceding propositions is also true. Thus, no number is divisible by 2, unless it ends with 0, 2, 4, 6, or 8.

5. Every composite number is divisible by the product of any two or more of its prime factors.

6. Every prime number, except 2 or 5, ends with 1, 3, 7, or 9.

Rule for resolving a composite number into its prime factors. — *Divide the given number by any prime number that will exactly divide it; divide the quotient again in the same manner, and so continue to divide until a quotient is obtained which is a prime number; then the last quotient and the several divisors are the prime factors of the given number.*

FACTORING OF ALGEBRAIC QUANTITIES.

Art. **88.** A *divisor*, or *factor* of a quantity, is a quantity that will exactly divide it; that is, without a remainder. Thus, a is a factor or divisor of ab, and $a+x$ is a divisor or factor of a^2-x^2.

Art. **89.** A *prime quantity* is one which is exactly divisible only by itself and by unity. Thus, x, y, and $x+z$, are prime quantities; while xy, and $ax+az$, are not prime.

Art. **90.** Two quantities, like two numbers, are said to be *prime to each other*, or *relatively prime*, when no quantity except unity will exactly divide them both. Thus, ab and cd are prime to each other.

Art. **91.** A *composite quantity*, is one which is the product of two or more factors, neither of which is unity. Thus, a^2-x^2 is a composite quantity, of which the factors are $a+x$ and $a-x$.

Art. **92.** To separate a monomial into its prime factors.

Rule. — *Resolve the coëfficient into its prime factors; then, these with the literal factors of the monomials, will be the prime factors of the given quantity.* The reason of this rule is self-evident.

Find the prime factors of the following monomials.

1. $18ab^2$. *Ans.* $2\times3\times3\times a.b.b.$
2. $28x^2yz^3$. *Ans.* $2\times2\times7\times x.x.y.z.z.z.$
3. $36a^2b^2y^3z$. *Ans.* $2\times2\times3\times3.a.a.b.b.y.y.y.z.$
4. $210ax^3yz^2$. *Ans.* $2\times3\times5\times7.a.x.x.x.y.z.z.$

ART. **93.** To separate a polynomial into its factors, when one of them is a monomial.

RULE.—*Divide the given quantity by the greatest monomial that will exactly divide each of its terms. Then the monomial divisor will be one factor, and the quotient the other.* The reason of this rule is self-evident.

Separate the following expressions into factors.

1. $a+ax$. *Ans.* $a(1+x)$.
2. $xz+yz$. *Ans.* $z(x+y)$.
3. x^2y+xy^2. *Ans.* $xy(x+y)$.
4. $6ab^2+9a^2bc$. *Ans.* $3ab(2b+3ac)$.
5. $4a^2bc+6ab^2c-10abc^2$. *Ans.* $2abc(2a+3b-5c)$.
6. $a^2bx^3y-ab^2xy^2+abcxyz^2$. *Ans.* $abxy(ax^2-by+cz^2)$.
7. $3x^2y-6x^3y^2+9x^2y^3$. *Ans.* $3x^2y(1-2xy+3y^2)$.
8. $12am^3n-18am^2n^2+30amn^3$. *Ans.* $6amn(2m^2-3mn+5n^2)$

ART. **94.** To separate any binomial or trinomial which is the product of two or more polynomials, into its prime factors.

1st. Any trinomial can be separated into two binomial factors, when the extremes are squares and positive, and the middle term is twice the product of the square roots of the extreme terms. The factors will be the sum or difference of the square roots of the extreme terms, according as the sign of the middle term is plus or minus. (See Arts. 78, 79.)

Thus, $a^2+2ab+b^2=(a+b)(a+b)$;
$a^2-2ab+b^2=(a-b)(a-b)$.

2d. Any binomial, which is the difference of two squares, can be separated into factors, one of which is the sum and the other the difference of their roots. (See Art. 80.)

Thus, $a^2-b^2=(a+b)(a-b)$.

3d. Any binomial which is the difference of the same powers of two quantities, can be separated into at least *two* factors, one of which is the difference of the two quantities. (See Art. 83).

Thus, $a^m-b^m=(a-b)(a^{m-1}+a^{m-2}b \ldots . +ab^{m-2}+b^{m-1}$, where a, b, and m, may be any quantities whatever.

In this case, one of the factors is the difference of the quantities, and the other may be found by dividing the given expression by this difference.

Thus, to find the other factor of x^3-y^3, we divide by $x-y$, and the quotient is x^2+xy+y^2; hence,

$$x^3-y^3=(x-y)(x^2+xy+y^2).$$

Similarly, $x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$.

4th. Any binomial which is the *difference of the even powers* of two quantities, higher than the second degree, can be separated into at least *three* factors, one of which is the *sum*, and another the difference of the quantities. (See Art. 85.)

Thus, by Art. 84, a^4-b^4, is divisible by $a+b$, and, by Art. 85, it is divisible by $a-b$; hence it is divisible by both $a+b$ and $a-b$, and the other factor will be found by dividing by their product. Or, it may be separated into factors, thus,

$$a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b).$$

5th. Any binomial which is the *sum of the odd powers* of two quantities, can be separated into at least two factors, one of which is the sum of the quantities. (Art. 86.) The other factor will be found by dividing the given binomial by this sum. Thus,

$$a^3+b^3=(a+b)(a^2-ab+b^2).$$

Separate the following expressions into their simplest factors.

1. $c^2+2cd+d^2$.
2. $a^2x^4+2ax^2y+y^2$.
3. $25x^2y^4+20xy^2z+4z^2$.
4. $9x^4-6x^2z^2+z^4$.
5. $4m^2x^2-4mn^2x+n^4$.
6. x^2-z^2.
7. $9a^2x^4-25$.
8. $16-a^2b^4z^6$.
9. $4m^2x^2-9n^2z^4$.
10. a^4-x^4.
11. z^3+1.
12. y^3-1.
13. $1+c^3$.
14. $a^3x^3-b^3y^3$.
15. x^5+y^5.
16. x^6-y^6.

ANSWERS.

1. $(c+d)(c+d)$.
2. $(ax^2+y)(ax^2+y)$.
3. $(5xy^2+2z)(5xy^2+2z)$.
4. $(3x^2-z^2)(3x^2-z^2)$.
5. $(2mx-n^2)(2mx-n^2)$.
6. $(x+z)(x-z)$.
7. $(3ax^2+5)(3ax^2-5)$.
8. $(4+ab^2z^3)(4-ab^2z^3)$.
9. $(2mx+3nz^2)(2mx-3nz^2)$.
10. $(a^2+x^2)(a^2-x^2)$ $=(a^2+x^2)(a+x)(a-x)$.
11. $(z+1)(z^2-z+1)$.
12. $(y-1)(y^2+y+1)$.
13. $(1+c)(1-c+c^2)$.
14. $(ax-by)(a^2x^2+abxy+b^2y^2)$.
15. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$.
16. $(x^3+y^3)(x^3-y^3)=(x^3+y^3)(x-y)(x^2+xy+y^2)=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)=(x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2)=(x^2-y^2)(x^4+x^2y^2+y^4.)$

Art. 94. To separate a quadratic trinomial into its factors.

A quadratic trinomial is of the form, x^2+ax+b, in which the sign of the second term may be either plus or minus.

To explain the method of performing this operation, let us examine the relation that exists between two binomial factors and their product.

1. $(x+a)(x+b)=x^2+(a+b)x+ab$.
2. $(x-a)(x-b)=x^2-(a+b)x+ab$.
3. $(x+a)(x-b)=x^2+(a-b)x-ab$.
4. $(x-a)(x+b)=x^2+(b-a)x-ab$.

From this we see that any trinomial may be resolved into two factors, when the first term is a square, and the coëfficient of the second term equal to the *sum* of any two quantities, whose *product* is equal to the third term.

Remark.—In Equations of the Second Degree (Art. 234), it will be shown how to perform this operation by a direct method; it is, however, a useful exercise for the pupil to do it by inspection, the only difficulty being to find two quantities whose sum is equal to the coëfficient of the second term, and product equal to the third term.

Trinomials to be decomposed into binomial factors.

1. x^2+3x+2. *Ans.* $(x+1)(x+2)$.
2. a^2+5a+6. *Ans.* $(a+2)(a+3)$.
3. $x^2-7x+12$. *Ans.* $(x-3)(x-4)$.
4. $x^2-8x+15$. *Ans.* $(x-3)(x-5)$.
5. x^2-x-2. *Ans.* $(x+1)(x-2)$.
6. x^2+x-12. *Ans.* $(x-3)(x+4)$.
7. x^2-x-12. *Ans.* $(x+3)(x-4)$.
8. x^2-5x+6. *Ans.* $(x-2)(x-3)$.
9. $x^2+2x-35$. *Ans.* $(x-5)(x+7)$.
10. x^2+x-56. *Ans.* $(x-7)(x+8)$.

Art. 95. Examples of binomials and trinomials that may be separated into factors, by first separating the monomial factor, and then applying the principles in Art. 93.

Ex. 1. $ax^3y-axy^3=axy(x^2-y^2)=axy(x+y)(x-y)$.

2. $3ax^2+6axy+3ay^2$. *Ans.* $3a(x+y)(x+y)$.
3. $2cx^2-12cx+18c$. *Ans.* $2c(x-3)(x-3)$.
4. $27a-18ax+3ax^2$. *Ans.* $3a(3-x)(3-x)$.
5. $3m^3n-3mn^3$. *Ans.* $3mn(m+n)(m-n)$.
6. $8z-2z^3$. *Ans.* $2z(2+z)(2-z)$.
7. $2x^5y-2xy^5$. *Ans.* $2xy(x^2+y^2)(x+y)(x-y)$.
8. $2x^2+6x-8$. *Ans.* $2(x+4)(x-1)$.

9. $2x^3+4x^2-70x$. *Ans.* $2x(x+7)(x-5)$.

10. $3a^3b-3a^2b-60ab$. *Ans.* $3ab(a-5)(a+4)$.

Solve the following questions by first indicating the operations to be performed, and then canceling the factors common to the dividend and divisor.

11. Multiply $4x-12$ by $1-x^2$, and divide the product by $2+2x$.

$$\frac{(4x-12)(1-x^2)}{2+2x}=\frac{4(x-3)(1+x)(1-x)}{2(1+x)}=2(x-3)(1-x)=$$
$$8x-6-2x^2.$$

12. Multiply $x^2+2xy+y^2$ by $x-y$, and divide the product by x^2-y^2. *Ans.* $x+y$.

13. Multiply $6am^2-6an^2$ by $m+n$, and divide the product by $2m^2+4mn+2n^2$. *Ans.* $3a(m-n)$.

14. Multiply together $1-c$, $1-c^2$, and $1+c^2$, and divide the product by $1-2c+c^2$. *Ans.* $1+c+c^2+c^3$.

15. Multiply together x^2+x-2 and x^2-x-6, and divide the product by x^2+4x+4. *Ans.* x^2-4x+3.

16. Multiply together x^3-x^2-30x and $x^2+11x+30$, and divide the product by the product of x^2-36 and $x^2+10x+25$.

Ans. x.

GREATEST COMMON DIVISOR.

ART. **96.** Any quantity that will exactly divide two or more quantities, is called a *common divisor*, or *common measure*, of those quantities. Thus, ab is a common divisor of ab^2 and abx.

REMARK.—Two quantities, like two numbers, often have more than one common divisor. Thus, a^2cx and $abdx$ have three common divisors, a, x, and ax.

ART. **97.** That common divisor of two quantities which is the greatest, both with regard to the coëfficients and exponents, is called their *greatest common divisor*, or *greatest common measure*. Thus, the greatest common divisor of $6a^2bx^2$ and $9a^3cxz$ is $3a^2x$.

ART. **98.** Quantities that have a common divisor are said to be *commensurable*, and those that have no common divisor are said to be *incommensurable*.

ART. **99.** To find the greatest common divisor of two or more monomials.

1. Let it be required to find the greatest common divisor of the two monomials, $14a^3cx$ and $21a^2bx$.

By separating each quantity into its prime factors, we have $14a^3cx=7\times2\times aaacx$, and $21a^2bx=7\times3\times aabx$.

By examining these quantities we find that 7, aa or a^2, and x, are the only factors *common* to both; hence, both the quantities can be exactly divided by either of these factors, or by their product, $7a^2x$, and by no other quantity whatever; therefore, $7a^2x$ is their greatest common divisor. This gives the following

RULE FOR FINDING THE GREATEST COMMON DIVISOR OF TWO OR MORE MONOMIALS.—*Resolve the quantities into their prime factors; then the product of those factors that are common to all the terms, will form their greatest common divisor.*

NOTE.—The greatest common divisor of the literal parts of the quantities may generally be found most easily by inspection, by taking each letter that is common to two or more of the quantities, with its *least* exponent.

2. Find the greatest common divisor of $6a^2xy$, $9a^3x^3$, and $15a^4x^4y^3$.

OPERATION.

$$6a^2xy=3\times2a^2xy$$
$$9a^3x^3=3\times3a^3x^3$$
$$15a^3x^4y^3=3\times5a^4x^4y^3$$

Here we find that 3 is the only numerical factor, and a and x the only letters common to all the quantities. The least powers of a and x, in either of the quantities, are a^2 and x; hence, the greatest common divisor is $3a^2x$.

Find the greatest common divisor of the following quantities.

3. $15abc^2$, and $21b^2cd$. *Ans.* $3bc$.

4. $4a^3b$, $10a^3c$, and $14a^2bc$. *Ans.* $2a^2$.

5. $15ax^3y$, $18x^3y^2$, and $21x^2y^2$. *Ans.* $3x^2y$.

6. $4ax^4y^3$, $20x^4y^2z$, and $12x^3y^3z^2$. *Ans.* $4x^3y^2$.

7. $2a^3b^4cx$, $3a^3b^3c^2z$, and $5a^2b^4cy$. *Ans.* a^2b^3c.

8. $12a^2x^2z^2$, $18ax^3z^2$, $30a^2x^3z$, and $6ax^3z^2$. *Ans.* $6ax^2z$.

ART. **100.** Previous to investigating the rule for finding the greatest common divisor of two polynomials, it is necessary to demonstrate the following

PROPOSITION.—*Any common divisor of two quantities, will always exactly divide their remainder after division.*

Let AD and BD be either two monomials, or polynomials, of which D is a common divisor, and let AD be greater than BD.

Divide AD by BD, and if BD is not contained an exact number of times in AD, suppose it is contained Q times with a remainder, which may be called R. Then, since the remainder is found, by subtracting the product of the divisor by the quotient from the dividend, we have, R=AD—BDQ. Dividing both sides by D, we get $\frac{R}{D}$=A—BQ; but A and BQ are each entire quantities, therefore their difference, $\frac{R}{D}$, must be an entire quantity. Hence, *any common divisor of two quantities* (and of course the greatest common divisor), *will always exactly divide their remainder after division.*

BD)AD(Q
BDQ
AD—BDQ=R

REMARK.—In the preceding demonstration it is assumed that the pupil understands the following axioms:

First. *If two equal quantities be divided by the same quantity the quotients will be equal.*

Second. *The difference of two entire quantities is also an entire quantity.*

ART. **101.** Let it be required to find the greatest common divisor of two polynomials, A and B, of which A is the greater.

If we divide A by B, and there is no remainder, B is, evidently, the greatest common divisor, since it can have no divisor greater than itself.

B)A(Q
BQ
A—BQ=R, 1st Rem.

Divide A by B, and call the quotient Q, then if there is a remainder R, it is evidently less than either of the quantities A and B; and by the preceding theorem it is also exactly divisible by the greatest common divisor; hence, the greatest common divisor must divide A, B, and R, and cannot be greater than R. But if R will exactly divide B, it will also exactly divide A, since A=BQ+R, and therefore will be the greatest common divisor sought.

R)B(Q′
RQ′
B—RQ′=R′, 2d Rem.

A=BQ+R
B=RQ′+R′

Since the dividend is equal to the product of the divisor by the quotient, plus the remainder.

Suppose, however, that when we divide R into B, to ascertain if it will exactly divide it, we find that the quotient is Q′, with a remainder R′.. Now, it has been shown that whatever exactly divides two quantities, will divide their remainder after division (Art. 100); and since the greatest common divisor of A and B, has been shown to divide B and R, it must also divide their remainder

R′, and therefore cannot be greater than R′. And, if R′ exactly divides R, it will also divide B, since B=RQ′+R′; and whatever exactly divides B and R, will also exactly divide A, since A=BQ +R; therefore, if R′ exactly divides R, it will exactly divide both A and B, and will be their greatest common divisor.

In the same manner, by continuing to divide the last divisor by the last remainder, it may always be shown, that the greatest common divisor of A and B will exactly divide every new remainder, and, of course, cannot be greater than either of them. It may also always be shown, as above, in the case of R′, that any remainder, which exactly divides the preceding divisor, will also exactly divide A and B. Then, since the greatest common divisor of A and B cannot be greater than this remainder, and as this remainder is a common divisor of A and B, it will be their greatest common divisor sought.

The same principle may be illustrated by numbers, by calling A, 55, and B, 15, and proceeding to find their greatest common divisor.

ART. **102.** When the remainders decrease to unity, or when we arrive at a remainder which does not contain the letter of arrangement, it is evident that there is no common divisor of the two quantities.

ART. **103.** If either quantity contains a factor not found in the other, that factor may be canceled without affecting the common divisor. Thus, in the two quantities, $x(x^2-y^2)$ and $y(x^2+2xy+y^2)$, of which the greatest common divisor is $x+y$, we may cancel x in the first, or y in the second, or both of them, and the greatest common divisor of the resulting quantities will still be $x+y$.

ART. **104.** We may multiply either quantity by a factor *not found* in the other, without changing the greatest common divisor. Thus, in the two quantities, $x(x^2-y^2)$ and $y(x^2+2xy+y^2)$, if we multiply the first by m and the second by n, we have $mx(x^2-y^2)$ and $ny(x^2+2xy+y^2)$, of which the greatest common divisor is still $x+y$.

ART. **105.** But if we multiply either quantity by a factor found in the other, we change the greatest common divisor. Thus, in the two quantities, $x(x^2-y^2)$ and $y(x^2+2xy+y^2)$, if we multiply the second by x, the two quantities become $x(x^2-y^2)$ and $xy(x^2+2xy+y^2)$, of which the greatest common divisor is $x(x+y)$ instead of $x+y$ as before. In like manner, if we multiply the first quantity by y, the greatest common divisor of the two resulting quantities will be $y(x+y)$

ART. **106.** From Art. 101 it is evident that the greatest common divisor of two quantities will exactly divide each of the successive remainders; therefore, the principles of the three preceding articles apply to the successive remainders that arise in finding the greatest common divisor.

ART. **107.** It is evident that any common factor of two quantities, must also be a factor of their greatest common divisor. Where such common factor is easily seen, as when it is a monomial, it simplifies the operation to set it aside, and find the greatest common divisor of the remaining quantities.

We shall now show the application of these principles.

1. Find the greatest common divisor of x^3-z^3 and $x^4-x^2z^2$.

Here the second quantity contains x^2 as a factor, but it is not a factor of the first; we may therefore cancel it (Art. 103), and the second quantity becomes x^2-z^2. Then divide the first by it. After dividing, we find that z^2 is a factor of the remainder, but not of x^2-z^2, the next dividend. We therefore cancel it (Art. 103), and the second divisor becomes $x-z$. Then, dividing by this, we find there is no remainder; therefore $x-z$ is the greatest common divisor.

OPERATION.

$$\begin{array}{l|l} x^3-z^3 & x^2-z^2 \\ x^3-xz^2 & x \\ \hline xz^2-z^3 & \\ \text{or } (x-z)z^2 & \end{array}$$

$$\begin{array}{l|l} x^2-z^2 & x-z \\ x^2-xz & x+z \\ \hline xz-z^2 & \\ xz-z^2 & \\ \hline \end{array}$$

2. Find the greatest common divisor of $x^5+x^2z^3$ and $x^5-x^3z^2$.

The factor x^2 is common to both quantities; it is, therefore, a factor of the greatest divisor (Art. 107), and may be taken out and reserved. Doing this, the quantities become x^3+z^3 and x^3-xz^2. The second quantity still contains a common factor, x, which the first does not; canceling this, it becomes x^2-z^2. Then proceeding as in the first example, we find that $x+z$ divides without a remainder; therefore, $x^2(x+z)$ is the required greatest common divisor.

OPERATION.

$$\begin{array}{l|l} x^3+z^3 & x^2-z^2 \\ x^3-xz^2 & x \\ \hline xz^2+z^3 & \\ \text{or } (x+z)z^2 & \end{array}$$

$$\begin{array}{l|l} x^2-z^2 & x+z \\ x^2+xz & x-z \\ \hline -xz-z^2 & \\ -xz-z^2 & \\ \hline \end{array}$$

3. Find the greatest common divisor of $10a^2x^2-4a^2x-6a^2$, and $5bx^2-11bx+6b$.

By separating the monomial factors, we find

$$10a^2x^2-4a^2x-6a^2=2a^2(5x^2-2x-3),$$
$$\text{and } 5bx^2-11bx+6b=b(5x^2-11x+6).$$

[OVER.]

The factors $2a$ and b have no common measure, and therefore are not factors of the common divisor. We may therefore suppress them (Art. 103), and proceed to find the greatest common divisor of the remaining quantities, which is found to be $x-1$.

OPERATION.

$$\begin{array}{l|l} 5x^2-11x+6 & \underline{5x^2-2x-3} \\ \underline{5x^2-\ 2x-3} & \underline{1} \\ \quad -9x+9 & \\ \text{or } -9(x-1) & \end{array}$$

$$\begin{array}{l|l} 5x^2-2x-3 & \underline{x-1} \\ \underline{5x^2-5x} & \underline{5x+3} \\ \quad 3x-3 & \\ \quad \underline{3x-3} & \end{array}$$

4. Find the greatest common divisor of $4a^2-5ay+y^2$, and $3a^3-3a^2y+ay^2-y^3$.

In solving this example, there are two instances in which it is necessary to multiply the dividend, in order that the coëfficient of the first term may be exactly divisible by the first term of the divisor (Art. 104).

We find $19y^2$ is a factor of the first remainder, but it is not a factor of the first divisor, and, therefore, cannot be a factor of the greatest common divisor; it must, therefore, be suppressed.

OPERATION.

$$\begin{array}{r|l} 3a^3-3a^2y+ay^2-y^3 & \underline{4a^2-5ay+y^2} \\ \underline{4} & \\ 12a^3-12a^2y+4ay^2-4y^3 & \underline{3a+3y} \\ \underline{12a^3-15a^2y+3ay^2\qquad} & \\ 3a^2y+ay^2-4y^3 & \\ \underline{4} & \\ 12a^2y+\ 4ay^2-16y^3 & \\ \underline{12a^2y-15ay^2+\ 3y^3} & \\ 19ay^2-19y^3 & \\ \text{or } 19y^2(a-y) & \end{array}$$

$$\begin{array}{l|l} 4a^2-5ay+y^2 & \underline{a-y} \text{ greatest com. divisor.} \\ \underline{4a^2-4ay} & \underline{4a-y} \\ \quad -ay+y^2 & \\ \quad \underline{-ay+y^2} & \end{array}$$

ART. **108.** From the preceding demonstrations and examples, we derive the following

RULE FOR FINDING THE GREATEST COMMON DIVISOR OF TWO POLYNOMIALS.—1. *Divide the greater polynomial by the less, and if there is no remainder, the less quantity will be the divisor sought.*

2. *If there be a remainder, divide the first divisor by it, and continue to divide the last divisor by the last remainder, until a divisor is obtained which leaves no remainder; this will be the greatest common divisor of the two given polynomials.*

NOTES.—1. When the highest power of the *leading* letter is the same in both, it is immaterial which of the quantities is made the dividend.

2. If both quantities contain a common factor, let it be set aside, as forming a factor of the common divisor, and proceed to find the greatest common divisor of the remaining factors, as in Example 2.

3. If either quantity contains a factor not found in the other, it may be canceled before commencing the operation, as in Example 3. See Art. 103.

4. Whenever it is necessary, the dividend may be multiplied by any quantity which will render the first term exactly divisible by the first term of the divisor. See Art. 104.

5. If, in any case, the remainder does not contain the leading letter, there is no common divisor.

6. To find the greatest common divisor of three or more quantities, first find the greatest common divisor of two of them; then of that divisor and one of the other quantities, and so on. The last divisor thus found will be the greatest common divisor sought.

7. Since the greatest common divisor of two quantities contains all the factors common to both, it may be found most easily by separating the quantities into factors, where this can be done by the rules for factoring. Arts. 92 to 95.

Find the greatest common divisor of the quantities in each of the following

EXAMPLES.

1. $5x^2-2x-3$ and $5x^2-11x+6$. *Ans.* $x-1$.
2. $9x^2-4$ and $9x^2-15x-14$. *Ans.* $3x+2$.
3. $a^2-ab-12b^2$ and $a^2+5ab+6b^2$. *Ans.* $a+3b$.
4. $4a^2-b^2$ and $4a^2+2ab-2b^2$. *Ans.* $2a-b$.
5. a^4-x^4 and $a^3+a^2x-ax^2-x^3$. *Ans.* a^2-x^2.
6. $x^3-5x^2+13x-9$ and x^3-2x^2+4x-3. *Ans.* $x-1$.
7. $x^3-5x^2+16x-12$ and $x^3-2x^2-15x+16$. *Ans.* $x-1$.
8. $21x^3-26x^2+8x$ and $6x^2-x-2$. *Ans.* $3x-2$.
9. $2x^4+11x^3-13x^2-99x-45$ and $2x^3-7x^2-46x-21$. *Ans.* $2x^2+7x+3$.
10. x^4+2x^2+9 and $7x^3-11x^2+15x+9$. *Ans.* x^2-2x+3.
11. $48x^2+16x-15$ and $24x^3-22x^2+17x-5$. *Ans.* $12x-5$.
12. x^2+5x+4, x^2+2x-8, and $x^2+7x+12$. *Ans.* $x+4$.
13. $x^4+a^2x^2+a^4$ and $x^4+ax^3-a^3x-a^4$. *Ans.* x^2+ax+a^2.
14. $2b^3-10ab^2+8a^2b$ and $9a^4-3ab^3+3a^2b^2-9a^3b$. *Ans.* $a-b$.
15. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$. *Ans.* x^2-1.

LEAST COMMON MULTIPLE.

ART. **109.** A multiple of a quantity is any quantity that contains it exactly. Thus, 6 is a multiple of 2 or of 3; and ab is a multiple of a or of b; also, $a(b—c)$ is a multiple of a or of $(b—c)$.

ART. **110.** A *common multiple* of two or more quantities, is a quantity that contains either of them exactly. Thus, 12 is a common multiple of 2 and 3; and $20xy$ is a common multiple of $2x$ and $5y$.

ART. **111.** The *least common multiple* of two or more quantities, is the least quantity that will contain them exactly. Thus, 6 is the least common multiple of 2 and 3; and $10xy$ is the least common multiple of $2x$ and $5y$.

REMARK.—Two or more quantities can have but *one least* common multiple, while they may have an unlimited number of common multiples.

ART. **112.** To find the least common multiple of two or more quantities.

From the nature of the least common multiple of two or more quantities, it is evident that it contains all the prime factors of each of the quantities once, and does not contain any prime factor besides; for, if it did not contain all the prime factors of any quantity, it would not be divisible by that quantity; and if it contained any prime factor not found in either of the quantities, it would not be the *least* common multiple. Thus, the least common multiple of ab and bc must contain the factors a, b, c, and no other factor. Hence,

The least common multiple of two or more quantities, contains all the prime factors of those quantities once, and does not contain any other factor.

With this principle let us find the least common multiple of mx, nx, and m^2nz.

OPERATION.

$$\begin{array}{r|rrr} m & mx & nx & m^2nz \\ \hline n & x & nx & mnz \\ \hline x & x & x & mz \\ \hline & 1 & 1 & mz \end{array}$$

$$m\times n\times x\times mz=m^2nxz$$

Arranging the quantities as in the margin, we see that m is a prime factor common to two of them. It must, therefore, even if found in only one of the quantities, be a factor of the least common multiple, and we place it on the left of the quantities. Then, since the same factor can occur but once in the least common multiple, we cancel m in each of the quantities in which it is found, which is done by dividing by it.

We next observe that n is a factor common to two of the remaining quantities; we therefore place it on the left, as another factor of the least common multiple, and cancel it in each of the terms in which it is found.

By examining the remaining quantities, we find that x is a factor common to two of them. We then place it on the left, as another factor of the least common multiple, and cancel it in each of the terms in which it is found.

We thus find that the least common multiple must contain the factors m, n, and x; it must also contain the factor mz, otherwise it would not contain all the prime factors found in one of the quantities. Hence the products, $m \times n \times x \times mz = m^2nxz$, contains all the prime factors of the quantities once, and does not contain any other factor; it is, therefore, the required least common multiple. Hence we have the following

RULE FOR FINDING THE LEAST COMMON MULTIPLE OF TWO OR MORE QUANTITIES.—1. *Arrange the quantities in a horizontal line, and divide them by any prime factor that will divide two or more of them without a remainder, and set the quotients and the undivided quantities in a line beneath.*

2. *Continue dividing as before, until no prime factor, except unity, will divide two or more of the quantities without a remainder.*

3. *Multiply the divisors and the quantities in the last line together, and the product will be the least common multiple required.*

OR, *Separate the given quantities into their prime factors, and then multiply together such of these factors as are necessary to form a product that will contain all the prime factors in each quantity: this product will be the least common multiple required.*

ART. **113.** Since the greatest common divisor of two quantities contains all the factors common to both, it follows, that *if we divide the product of two quantities by their greatest common divisor, the quotient will be their least common multiple.*

Find the least common multiple of the quantities in each of the following

EXAMPLES.

1. $6a^2$, $9ax^3$, and $24x^5$. *Ans.* $72a^2x^5$.
2. $32x^2y^2$, $40ax^5y$, and $5a^2x(x-y)$. *Ans.* $160a^2x^5y^2(x-y)$.

3. $3x+6y$ and $2x^2-8y^2$. *Ans.* $6x^2-24y^2$.

4. a^3+x^3 and a^2-x^2. *Ans.* $a^4-a^3x+ax^3-x^4$.

5. $4(a^2+ax)$, $12(ax^2-x^3)$, and $18(a^2-x^2)$.
Ans. $36ax^2(a^2-x^2)$.

6. $2x-1$, $4x^2-1$, and $4x^2+1$. *Ans.* $16x^4-1$.

7. $x-1$, x^2-1, $x-2$, and x^2-4. *Ans.* x^4-5x^2+4.

8. x^2-1, x^2+1, $(x-1)^2$, $(x+1)^2$, x^3-1, and x^3+1.
Ans. $x^{10}-x^6-x^4+1$.

9. $4(1-x)^2$, $8(1-x)$, $8(1+x)$, and $4(1+x^2)$.
Ans. $8(1-x)(1-x^4)$.

10. $3x^2-11x+6$, $2x^2-7x+3$, and $6x^2-7x+2$. (See Art. 113.) *Ans.* $6x^3-25x^2+23x-6$.

CHAPTER III.

ALGEBRAIC FRACTIONS.

DEFINITIONS.

ART. **114.** Algebraic fractions are represented in the same manner as common fractions in Arithmetic. The quantity below the line is called the *denominator*, because it *denominates*, or shows the number of parts into which the unit is divided; and the quantity above the line is called the *numerator*, because it *numbers*, or shows how many parts are taken. Thus, in the fraction, $\frac{a-b}{c+d}$, if $a=5$ $b=3$, $c=2$, and $d=1$, the denominator $c+d$ shows that a unit is divided into 3 equal parts, and $a-b$ shows that 2 of those parts are taken.

ART. **115.** The terms *proper*, *improper*, *simple*, *compound*, and *complex*, have the same meaning when applied to algebraic fractions, as to common numerical fractions.

ART. **116.** Every quantity not expressed under the form of a fraction, is called an *entire* algebraic quantity. Thus, $cx-d$ is an entire quantity.

ART. **117.** Every quantity composed partly of an entire quantity and partly of a fraction, is called a *mixed* quantity. Thus, $a+\frac{x}{b}$, is a mixed quantity.

NOTE.—The same *principles* and *rules* are applicable to algebraic and to common numerical fractions. However, as a good knowledge of fractions is of great importance to the student, we shall present a concise demonstration of the fundamental principles and rules of operation. In these demonstrations the pupil is supposed to be acquainted with this self-evident principle: *If we perform the same operations on two equal quantities the results will be equal.*

ART. **118.** PROPOSITION.—*The value of a fraction is not altered, if we multiply or divide both terms by the same quantity.*

Let $\frac{A}{B}$ be a fraction whose value is Q.

Then $\frac{A}{B}=Q$; but, from the nature of fractions, A represents a dividend, B the divisor, and Q the quotient; and by the nature of division,

$A=BQ.$

If m represents any number, then

$mA=mBQ$; dividing these equals by mB, we have

$\frac{mA}{mB}=Q$; which proves the 1st part of the proposition.

Again, take the equals

$A=BQ$, and divide each by m, we have (Art. 73),

$\frac{A}{m}=\frac{B}{m}Q$; divide each of these equals by $\frac{B}{m}$, then

$\frac{\frac{A}{m}}{\frac{B}{m}}=Q$; which proves the 2d part of the proposition.

CASE I.—TO REDUCE A FRACTION TO ITS LOWEST TERMS.

ART. **119.** Since the value of a fraction is not changed by dividing both terms by the same quantity (Art. 118), we have the following

RULE.—*Divide both terms by their greatest common divisor.*

OR, *Resolve both terms into their prime factors, and then cancel those factors which are common.*

EXAMPLES.

1. Reduce $\frac{10acx^2}{15bcx^3}$ to its lowest terms.

$$\frac{10acx^2}{15bcx^3}=\frac{2a\times5cx^2}{3bx\times5cx^2}=\frac{2a}{3bx}. \quad Ans.$$

Fractions to be reduced to their lowest terms.

2. $\frac{a^2x}{3a^3bx}$. *Ans.* $\frac{1}{3ab}$.

3. $\frac{ax+x^2}{3bx-cx}$. *Ans.* $\frac{a+x}{3b-c}$.

4. $\frac{3a^2+3ab}{3a^2-3ab}$. *Ans.* $\frac{a+b}{a-b}$.

5. $\frac{a^n}{a^{n+1}}$. *Ans.* $\frac{1}{a}$.

6. $\frac{mnp-m^2p}{m^2p+mp^2}$. *Ans.* $\frac{n-m}{m+p}$.

7. $\frac{2ax-4ax^2}{6ax}$. *Ans.* $\frac{1-2x}{3}$.

8. $\frac{21a^3b^2x-9ab^3x^2}{15a^2b^2x-3a^5b^4x^2}$. *Ans.* $\frac{7a^2-3bx}{5a-a^4b^2x}$.

9. $\frac{1-x}{1-x^2}$. *Ans.* $\frac{1}{1+x}$.

10. $\frac{5a^2+5ax}{a^2-x^2}$. *Ans.* $\frac{5a}{a-x}$.

11. $\frac{x^2+2x-3}{x^2+5x+6}$. *Ans.* $\frac{x-1}{x+2}$.

12. $\frac{x^2-3x-70}{x^3-39x+70}$. *Ans.* $\frac{x-10}{x^2-7x+10}$.

13. $\frac{x^3-4x^2+5}{x^3+1}$. *Ans.* $\frac{x^2-5x+5}{x^2-x+1}$.

14. $\frac{4x^2-12ax+9a^2}{8x^3-27a^3}$. *Ans.* $\frac{2x-3a}{4x^2+6ax+9a^2}$.

15. $\frac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$. *Ans.* $\frac{5x^2+1}{9x^3-4x}$.

16. $\frac{2x^3+8x^2y+16xy^2+16y^3}{8x^2+4xy-24y^2}$. *Ans.* $\frac{x^2+2xy+4y^2}{2(2x-3y)}$.

Remark.—Instead of finding the greatest common divisor by the rule, Art. 119, it is often preferable to separate the quantities into factors by the rules for factoring (Arts. 87 to 95), and then cancel those factors common to both terms. The following examples should be solved in this manner.

17. Reduce $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$ to its lowest terms.

$$x^2+(a+c)x+ac=x^2+ax+cx+ac$$
$$=x(x+a)+c(x+a)=(x+c)(x+a).$$

Also, $x^2+(b+c)x+bc=(x+c)(x+b)$;

$\therefore$ the fraction becomes $\frac{(x+c)(x+a)}{(x+c)(x+b)}=\frac{x+a}{x+b}$. *Ans.*

18. $\frac{ac+by+ay+bc}{af+2bx+2ax+bf}$. *Ans.* $\frac{c+y}{f+2x}$.

19. $\frac{6ac+10bc+9ax+15bx}{6c^2+9cx-2c-3x}$. *Ans.* $\frac{3a+5b}{3c-1}$.

20. $\frac{x^8+x^6y^2+x^2y+y^3}{x^4-y^4}$. *Ans.* $\frac{x^6+y}{x^2-y^2}$.

21. $\frac{a^3+(a+b)ax+bx^2}{a^4-b^2x^2}$. *Ans.* $\frac{a+x}{a^2-bx}$.

22. $\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$. *Ans.* $\frac{x^{m-1}}{b(a+bx)}$.

23. $\frac{acx^2+(ad+bc)x+bd}{a^2x^2-b^2}$. *Ans.* $\frac{cx+d}{ax-b}$.

24. $\frac{a^3+ab^2-a^2b-b^3}{4a^4-2a^2b^2-4a^3b+2ab^3}$. *Ans.* $\frac{a^2+b^2}{2a(2a^2-b^2)}$.

25. $\frac{2a^2+ab-b^2}{a^3+a^2b-a-b}$. *Ans.* $\frac{2a-b}{a^2-1}$.

ART. **120.** Exercises in Division (see Art. 72), in which the quotient is a fraction, and capable of being reduced to lower terms.

1. Divide $2a^3x^2$ by $5a^2x^2b$. *Ans.* $\frac{2a}{5b}$.

2. Divide $16bc^3x^2$ by $20ac^2x^3$. *Ans.* $\frac{4bc}{5ax}$

3. Divide $ax+x^2$ by $3bx-cx$. *Ans.* $\frac{a+x}{3b-c}$.

4. Divide a^3-b^3 by a^2-b^2. *Ans.* $\frac{a^2+ab+b^2}{a+b}$.

5. Divide a^3-b^3 by $(a-b)^2$. *Ans.* $\frac{a^2+ab+b^2}{a-b}$.

6. Divide n^3-2n^2 by n^2-4n+4. *Ans.* $\frac{n^2}{n-2}$.

7. Divide $3x^3-3x^2-63x+135$ by $3x^2-2x-21$.
Ans. $\frac{3x^2+6x-45}{3x+7}$.

CASE II.—TO REDUCE A FRACTION TO AN ENTIRE OR MIXED QUANTITY.

ART. **121.** Since the numerator of the fraction may be regarded as a dividend, and the denominator the divisor, this is merely a case of division. Hence we have the following

RULE.—*Divide the numerator by the denominator, for the entire part, and if there be a remainder, place it over the denominator, for the fractional part.*

NOTE.—The fractional part should be reduced to its lowest terms.

1. Reduce $\frac{a^3+a^2-ax^2}{a^2-ax}$ to an entire or mixed quantity.

$$\frac{a^3+a^2-ax^2}{a^2-ax}=a+x+\frac{a^2}{a^2-ax}=a+x+\frac{a}{a-x}\ \textit{Ans.}$$

Reduce the following fractions to entire or mixed quantities.

2. $\frac{ax-x^2}{a}$. *Ans.* $x-\frac{x^2}{a}$.

3. $\frac{1-2x^2}{1+x}$. *Ans.* $1-x-\frac{x^2}{1+x}$

4. $\frac{a^2+2b^2}{a-b}$. *Ans.* $a+b+\frac{3b^2}{a-b}$.

5. $\frac{1+2x}{1-3x}$. *Ans.* $1+5x+\frac{15x^2}{1-3x}$.

6. $\frac{x^3+bx^2}{x^2-bx}$. *Ans.* $x+\frac{2bx}{x-b}$.

7. $\frac{ax^2-ax-x+1}{ax-a}$. *Ans.* $x-\frac{1}{a}$.

8. $\frac{x^2z^2-z^2+xz-z-x+1}{x^2-1}$. *Ans.* $z^2+\frac{z-1}{x+1}$.

CASE III. — TO REDUCE A MIXED QUANTITY TO THE FORM OF A FRACTION.

ART. **122.** Let it be required to reduce $a+\frac{b}{c}$ to the form of a fraction.

It is evident that a is the same as $\frac{a}{1}$, and $\frac{a}{1}=\frac{a\times c}{1\times c}=\frac{ac}{c}$. Art. 118.

Hence, $a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}$.

Similarly, $a-\frac{b}{c}=\frac{ac-b}{c}$. Hence we have the following

RULE. — *Multiply the entire part by the denominator of the fraction; then add the numerator to the product, and place the result over the denominator.*

Before proceeding to the application of this rule, it is necessary for the learner to consider

THE SIGNS OF FRACTIONS.

ART. **123.** Each of the several terms of the numerator and denominator of a fraction, is preceded by the sign plus or minus, expressed or understood, and the fraction taken as a whole, is also preceded by the sign plus or minus, expressed or understood.

Thus, in the fraction $-\frac{a^2-b^2}{x+y}$, the sign of a^2, the first term of the numerator, is plus; of the second, b^2, minus; while the sign of each term of the denominator is plus; but the sign of the fraction, taken as a whole, is minus. The pupil must always recollect, that the signs of the several terms relate only to those terms to which they are prefixed, while the sign placed before the fraction relates to it as a whole.

ART. **124.** It is often convenient to change the signs of the numerator or denominator of a fraction, or of both. We will now show the law regulating these changes.

By the rule for the signs, in Division (Art. 69), we have, $\frac{+ab}{+a}=+b$; or, changing the signs of *both* terms, $\frac{-ab}{-a}=+b$.

But, if we change the sign of the *numerator*, we have $\frac{-ab}{+a}=-b$.

And, changing the sign of the *denominator*, we have $\frac{+ab}{-a}=-b$.

Hence, *The signs of both terms of a fraction may be changed, without altering its value or changing its sign, as a whole; but, if the sign of either term be changed, the sign of the fraction will be changed.*

Hence, also, *The signs of either term of a fraction may be changed, without altering its value, if the sign of the fraction be changed at the same time.*

Thus, $\frac{a^2-x^2}{a-x}=-\frac{-a^2+x^2}{a-x}=-\frac{a^2-x^2}{-a+x}=-(-a-x)=a+x.$

And, $a-\frac{a^2-x^2}{a-x}=a+\frac{-a^2+x^2}{a-x}=a+\frac{a^2-x^2}{-a+x}=-x.$

EXAMPLES.

Reduce the following quantities to a fractional form.

1. $a+x+\frac{a^2-ax}{x}$. *Ans.* $\frac{a^2+x^2}{x}$.

2. $a^2-ax+x^2-\frac{2x^3}{a+x}$. *Ans.* $\frac{a^3-x^3}{a+x}$.

3. $2a-x+\frac{(a-x)^2}{x}$. *Ans.* $\frac{a^2}{x}$.

4. $a-\frac{a^2}{a+b}$. *Ans.* $\frac{ab}{a+b}$.

5. $x+y+\frac{y^2}{x-y}$. *Ans.* $\frac{x^2}{x-y}$.

6. $a-x-\frac{a^2+x^2}{a+x}$. *Ans.* $-\frac{2x^2}{a+x}$.

7. $a^3+ax^2+\frac{ax^4}{a^2-x^2}$. *Ans.* $\frac{a^5}{a^2+x^2}$.

8. $1-\frac{(x-y)^2}{x^2+y^2}$. *Ans.* $\frac{2xy}{x^2+y^2}$.

CASE IV. — TO REDUCE FRACTIONS OF DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING A COMMON DENOMINATOR.

ART. **125.** 1. Let it be required to reduce $\frac{a}{m}$, $\frac{b}{n}$, and $\frac{c}{r}$, to a common denominator.

It is evident that we may multiply both terms of each fraction by the same quantity, since this (Art. 118) will not change its value. Now, if we multiply both terms of each fraction by the denominators of the other two fractions, the new denominators of each will be the same, since, in each case, they will consist of the product of the same factors; that is, of all the denominators.

Thus, $\frac{a\times n\times r}{m\times n\times r}=\frac{anr}{mnr}$.

$$\frac{b\times m\times r}{m\times m\times r}=\frac{bmr}{mnr}.$$

$$\frac{c\times m\times n}{r\times m\times n}=\frac{cmn}{mnr}.$$

It is evident that the value of each fraction is not changed, and that they have the same denominator. Hence, we have the following

RULE FOR REDUCING FRACTIONS TO A COMMON DENOMINATOR. — *Multiply both terms of each fraction by the product of all the denominators, except its own.*

REMARK. — Since each denominator of the new fractions will consist of the product of all the denominators of the given fractions, it is unnecessary to perform the multiplication more than once.

EXAMPLES.

Reduce the fractions in each of the following examples, to others having a common denominator.

2. $\frac{1}{x}$, $\frac{2}{y}$, and $\frac{3}{z}$. *Ans.* $\frac{yz}{xyz}$, $\frac{2xz}{xyz}$, $\frac{3xy}{xyz}$.

3. $\frac{a}{b}$ and $\frac{b}{a}$. *Ans.* $\frac{a^2}{ab}$ and $\frac{b^2}{ab}$.

4. $\frac{x}{x-a}$ and $\frac{a}{x+a}$. *Ans.* $\frac{x^2+ax}{x^2-a^2}$ and $\frac{ax-a^2}{x^2-a^2}$.

ART. **126.** It frequently happens, that the denominators of the fractions to be reduced, contain a common factor. In such cases the preceding rule does not give the *least* common denominator.

1. Let it be required to reduce $\frac{a}{m}$, $\frac{b}{mn}$, and $\frac{c}{nr}$, to their least common denominator.

Since both terms of a fraction may be multiplied by the same quantity without altering its value, the first fraction may have any denominator that is a multiple of m; the second, any denominator that is a multiple of mn; and the third, any denominator that is a multiple of nr. Hence, any common denominator of the three fractions must be a multiple of m, mn, and nr, and their *least* common denominator must be the *least* common multiple of the three given denominators.

The least common multiple of the three denominators is easily found (Art. 112) to be mnr. It now remains to reduce each fraction to another whose denominator shall be mnr.

The first fraction is $\frac{a}{m}$; in order to change this to another, whose denominator shall be mnr, we must multiply both terms by the same quantity, and by such a quantity that when multiplied by m the product shall be mnr. But this multiplier will evidently be obtained by dividing mnr by m; that is, by dividing the least common multiple of the given denominators, by the denominator of the first fraction. It is evident that the other fractions may be reduced in the same manner; the operation is as follows:

$mnr \div m = nr$, and $\frac{a \times nr}{m \times nr} =$ . . . . $\frac{anr}{mnr}$.

$mnr \div mn = r$, and $\frac{b \times r}{mn \times r} =$ . . . . $\frac{br}{mnr}$.

$mnr \div nr = m$, and $\frac{c \times m}{nr \times m} =$ . . . . $\frac{mc}{mnr}$.

The process of multiplying the denominators by the quotients may be omitted, since the product in each case will be equal to the least common multiple. This gives the following

RULE FOR REDUCING FRACTIONS OF DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING THE LEAST COMMON DENOMINATOR. — 1. *Find the least common multiple of all the denominators; this will be the common denominator.*

2. *Divide the least common multiple by the first of the given denominators, and multiply the quotient by the first of the given numerators; the product will be the first of the required numerators.*

3. *Proceed, in a similar manner, to find each of the other numerators.*

NOTE.—Before commencing the operation, each fraction must be in its lowest terms.

EXAMPLES.

Reduce the fractions, in each of the following examples, to equivalent fractions having the least common denominator.

2. $\frac{a}{6xy}, \frac{b}{3x}, \frac{c}{2y}$. *Ans.* $\frac{a}{6xy}, \frac{2by}{6xy}, \frac{3cx}{6xy}$.

3. $\frac{x}{a+b}, \frac{y}{a-b}, \frac{z}{a^2-b^2}$. *Ans.* $\frac{x(a-b)}{a^2-b^2}, \frac{y(a+b)}{a^2-b^2}, \frac{z}{a^2-b^2}$.

4. $\frac{m-n}{m+n}, \frac{m+n}{m-n}, \frac{m^2n^2}{m^2-n^2}$. *Ans.* $\frac{(m-n)^2}{m^2-n^2}, \frac{(m+n)^2}{m^2-n^2}, \frac{m^2n^2}{m^2-n^2}$.

Other exercises will be found in the Addition of Fractions.

NOTE.—The two following articles depend on the principles explained in the preceding article, and are therefore introduced here. They will both be found of frequent use, especially in completing the square in the solution of equations of the second degree.

ART. **127.** To reduce an entire quantity to the form of a fraction having a given denominator.

RULE.—*Multiply the entire quantity by the given denominator, and write the product over it.*

EXAMPLES.

1. Reduce x to a fraction whose denominator is a. *Ans.* $\frac{ax}{a}$.

2. Reduce $2az$ to a fraction whose denominator is z^2. *Ans.* $\frac{2az^3}{z^2}$.

3. Reduce $x+y$ to a fraction whose denominator is $x-y$. *Ans.* $\frac{x^2-y^2}{x-y}$.

4. Reduce $m-n$ to a fraction whose denominator is $a(m-n)^2$. *Ans.* $\frac{a(m-n)^3}{a(m-n)^2}$.

ART. **128.** To convert a fraction to an equivalent one, having a given denominator.

RULE.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms by the quotient.*

REMARK.—This rule is perfectly general, but it is never applied except when the required denominator is a multiple of the given one. In other cases it would produce a complex fraction. Thus, if it were required to reduce $\frac{2}{3}$ to an equivalent fraction with a denominator 5, the numerator of the new fraction would be $3\frac{1}{3}$.

EXAMPLES.

1. Convert $\frac{3}{7}$ to an equivalent fraction, having for its denominator, 49. *Ans.* $\frac{21}{49}$.

2. Convert $\frac{a}{3}$ and $\frac{5}{c}$ to equivalent fractions having the denominator $9c^2$. *Ans.* $\frac{3ac^2}{9c^2}, \frac{45c}{9c^2}$.

3. Convert $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ to equivalent fractions having the denominator a^2-b^2. *Ans.* $\frac{(a+b)^2}{a^2-b^2}, \frac{(a-b)^2}{a^2-b^2}$.

CASE V. — ADDITION AND SUBTRACTION OF FRACTIONS.

ART. **129.** It is self-evident that two algebraic fractions, like two arithmetical fractions, must have a common denominator, before we can find either their sum or their difference.

1. Let it be required to find the value of $\frac{a}{d}, \frac{b}{d}$, and $\frac{c}{d}$.

Let $\frac{a}{d}=m$, $\frac{b}{d}=n$, and $\frac{c}{d}=r$.

Then $a=md$, $b=nd$, and $c=rd$;

and $a+b+c=md+nd+rd$;

or, $a+b+c=(m+n+r)d$;

hence $\frac{a+b+c}{d}=m+n+r$.

This gives the following

RULE FOR THE ADDITION OF FRACTIONS. — *Reduce the fractions, if necessary, to a common denominator; add the numerators together, and place their sum over the common denominator.*

ART. **130.** 2. Let it be required to subtract $\frac{b}{d}$ from $\frac{a}{d}$.

Let $\frac{a}{d}=m$, and $\frac{b}{d}=n$.

Then $a=md$, and $b=nd$;

and $a-b=md-nd, =(m-n)d$;

hence $\frac{a-b}{d}=m-n$.

This gives the following

RULE FOR THE SUBTRACTION OF FRACTIONS. — *Reduce the fractions, if necessary, to a common denominator; then subtract the numerator of the fraction to be subtracted from the numerator of the other, and place the remainder over the common denominator.*

EXAMPLES IN ADDITION OF FRACTIONS.

1. Add $\frac{a}{b}$ and $\frac{3a}{4b}$ together. *Ans.* $\frac{7a}{4b}$

2. Add $\frac{a}{b}$ and $\frac{b}{a}$ together. *Ans.* $\frac{a^2+b^2}{ab}$

3. Add $\frac{1}{1+x}$ and $\frac{1}{1-x}$ together. *Ans.* $\frac{2}{1-x^2}$.

Find the value

4. Of $\frac{c}{x}+\frac{b}{x^3}+\frac{a}{x^5}$. *Ans.* $\frac{a+bx^2+cx^4}{x^5}$.

5. Of $\frac{1}{3(1-x)}+\frac{2+x}{3(1+x+x^2)}$. *Ans.* $\frac{1}{1-x^3}$.

6. Of $\frac{b}{d}+\frac{ad-bc}{d(c+dx)}$. *Ans.* $\frac{a+bx}{c+dx}$.

7. Of $\frac{p}{ab}+\frac{q}{ac}+\frac{r}{bc}$. *Ans.* $\frac{pc+qb+ra}{abc}$.

8. Of $\frac{1}{x}+\frac{1}{x+1}+\frac{2}{x-3}$. *Ans.* $\frac{4x^2-3x-3}{x^3-2x^2-3x}$.

9. Of $\frac{x}{x+y}+\frac{y}{x-y}$. *Ans.* $\frac{x^2+y^2}{x^2-y^2}$.

10. Of $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}$. *Ans.* $\frac{1}{1-x^4}$.

11. Of $\frac{p-q}{pq}+\frac{r-p}{pr}+\frac{q-r}{qr}$. *Ans.* 0.

12. Of $\frac{2}{x+a}+\frac{3a}{(x+a)^2}+\frac{3a-2x}{x^2-2ax+3a^2}$. *Ans.* $\frac{18a^3}{x^4+4a^3x+3a^4}$.

13. Of $\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$. *Ans.* $\frac{1}{a^4-x^4}$.

14. Of $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$. *Ans.* $\frac{1}{abc}$.

EXAMPLES IN SUBTRACTION OF FRACTIONS.

In the first ten examples the second fraction is to be subtracted from the first.

1. $\frac{5x}{7a}$ and $\frac{3y}{7}$. *Ans.* $\frac{5x-3ay}{7a}$.

2. $\frac{1}{a-b}$ and $\frac{1}{a+b}$. *Ans.* $\frac{2b}{a^2-b^2}$.

3. $\frac{p+q}{p-q}$ and $\frac{p-q}{p+q}$. *Ans.* $\frac{4pq}{p^2-q^2}$.

4. $\frac{n-1}{n}$ and $\frac{n}{n-1}$. *Ans.* $\frac{1-2n}{n^2-n}$.

5. $\frac{1}{1-x}$ and $\frac{2}{1-x^2}$. *Ans.* $\frac{x-1}{1-x^2}=-\frac{1}{1+x}$.

6. $\frac{1}{(x+1)(x+2)}$ and $\frac{1}{(x+1)(x+2)(x+3)}$. *Ans.* $\frac{1}{(x+1)(x+3)}$.

7. $\frac{1}{(x+1)(x+2)}$ and $\frac{3}{(x+1)(x+2)(x+3)}$.

Ans. $\frac{x}{(x+1)(x+2)(x+3)}$.

8. $\frac{a}{c}$ and $\frac{(ad-bc)x}{c(c+dx)}$. *Ans.* $\frac{a+bx}{c+dx}$.

9. $\frac{1}{2}\cdot\frac{3m+2n}{3m-2n}$ and $\frac{1}{2}\cdot\frac{3m-2n}{3m+2n}$. *Ans.* $\frac{6mn}{9m^2-4n^2}$.

10. $\frac{a+c}{(a-b)(x-a)}$ and $\frac{b+c}{(a-b)(x-b)}$. *Ans.* $\frac{x+c}{(x-a)(x-b)}$.

Find the value

11. Of $\frac{4m-3n}{3(1-n)}-\frac{m+3n}{3(1-n)}+\frac{2n}{1-n}$. *Ans.* $\frac{m}{1-n}$.

12. Of $\frac{a-b}{ab}-\frac{a-c}{ac}+\frac{b-c}{bc}$. *Ans.* 0.

13. Of $\frac{1}{2x+y}+\frac{1}{2x-y}-\frac{3x}{4x^2-y^2}$. *Ans.* $\frac{x}{4x^2-y^2}$.

14. Of $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^3-x^2y}{x^2y-y^3}$. *Ans.* 1.

15. Of $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$. *Ans.* $\frac{x+3}{x^4-1}$.

16. Of $\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}+\frac{x-1}{x^2+1}-\frac{1}{(x^2+1)^2}$. *Ans.* $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

CASE VI.—MULTIPLICATION OF FRACTIONS.

ART. **131**. 1. Let it be required to find the product of $\frac{a}{b}$ by $\frac{c}{d}$.

Let $\frac{a}{b}=m$, and $\frac{c}{d}=n$.

Then $a=bm$, and $c=dn$

$\therefore$ $ac=bmdn,=bd\times mn$; or, dividing by bd, $\frac{ac}{bd}=mn$.

Hence, to find the product of two or more fractions, we have the following

RULE.—*Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

REMARKS.—1st. This rule is general, and embraces all the cases in which a fraction is a factor. Thus, if it be required to multiply a fraction by an integral quantity, the latter may be placed under the form of a fraction, by writing unity beneath it.

2d. If either of the factors is a mixed quantity, it is best to reduce it to an improper fraction, before commencing the operation.

3d. When the numerators and denominators have common factors, the process may be abbreviated by indicating the operation, and then canceling the factors common to both terms.

Thus $\frac{2a^2}{a^2-b^2}\times\frac{(a+b)^2}{4a^2b}=\frac{2a^2\times(a+b)(a+b)}{(a+b)(a-b)4a^2b}=\frac{a+b}{2b(a-b)}$.

EXAMPLES.

Find the products of the fractions in each of the following exercises, expressed in their simplest forms.

1. $\frac{3x}{4}$ by $\frac{4x}{3y}$ and $\frac{8a^2b}{c}$ by $\frac{c^2d}{8a^3}$. *Ans.* $\frac{x^2}{y}$, and $\frac{bcd}{a}$.

2. $\frac{4ax}{cy}$, $\frac{cxy-y^2}{6x^2+6xy}$. *Ans.* $\frac{2a(cx-y)}{3c(x+y)}$.

3. $a-\frac{x^2}{a}$, $\frac{a}{x}+\frac{x}{a}$. *Ans.* $\frac{a^4-x^4}{a^2x}$.

4. $1-\frac{x-y}{x+y}$ and $2+\frac{2y}{x-y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

5. $\frac{1+a+a^2}{1-b+b^2}$ and $\frac{1-a}{1+b}$. *Ans.* $\frac{1-a^3}{1+b^3}$.

6. $\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3}$ and $\frac{a^2-ax+x^2}{a+x}$. *Ans.* $\frac{a^4+a^2x^2+x^4}{a^4-x^4}$.

7. $\frac{x^2-9x+20}{x^2-6x}$ and $\frac{x^2-13x+42}{x^2-5x}$. *Ans.* $\frac{x^2-11x+28}{x^2}$.

8. $\frac{x^2+3x+2}{x^2+2x+1}$ and $\frac{x^2+5x+4}{x^2+7x+12}$. *Ans.* $\frac{x+2}{x+3}$.

9. $\frac{4ax}{3by}$, $\frac{a^2-x^2}{c^2-x^2}$, $\frac{bc+bx}{a^2-ax}$. *Ans.* $\frac{4x(a+x)}{3y(c-x)}$.

10. $\frac{a^2-b^2}{x+y}$, $\frac{x^2-y^2}{a-b}$, $\frac{a^2}{(x-y)^2}$. *Ans.* $\frac{a^2(a+b)}{x-y}$.

11. x^2+x+1 by $\frac{1}{x^2}-\frac{1}{x}+1$. *Ans.* $x^2+1+\frac{1}{x^2}$.

12. $x+1+\frac{1}{x}$ by $x-1+\frac{1}{x}$. *Ans.* $x^2+1+\frac{1}{x^2}$.

13. $\frac{4a}{3x}+\frac{3x}{2b}$ by $\frac{2b}{3x}+\frac{3x}{4a}$. *Ans.* $\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}$.

14. $\frac{pr+(pq+qr)x+q^2x^2}{p-qx}$ by $\frac{ps+(pt-qs)x-qtx^2}{p+qx}$.

Ans. $rs+(rt+qs)x+qtx^2$.

Find the value

15. Of $\left(\frac{x}{a}-\frac{y}{b}\right)\frac{z}{c}+\left(\frac{x}{a}-\frac{z}{c}\right)\frac{y}{b}+\left(\frac{y}{b}-\frac{z}{c}\right)\frac{x}{a}$.

Ans. $\frac{2y}{b}\left(\frac{x}{a}-\frac{z}{c}\right)$.

16. Of $\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{c}{d}-\frac{d}{c}\right)$.

Ans. $\frac{2bc}{ad}+\frac{2ad}{bc}$.

CASE VII.—DIVISION OF FRACTIONS.

ART. **132.** 1. Let it be required to find the quotient of $\frac{a}{b}$ by $\frac{c}{d}$.

$$\text{Let } \frac{a}{b}=m, \text{ and } \frac{c}{d}=n. \text{ Then,}$$

$$a=bm, \text{ and } c=dn.$$

Multiplying both terms of the first equality by d, and of the second by b, we find

$$ad=bdm, \text{ and } bc=bdn.$$

$$\text{therefore } \frac{ad}{bc}=\frac{bdm}{bdn}=\frac{m}{n};$$

$$\text{that is, } \frac{m}{n}=\frac{a}{b}\times\frac{d}{c}.$$

Hence, to find the quotient of one fraction divided by another, we have the following

RULE.—*Invert the divisor, and proceed as in multiplication of fractions.*

REMARK.—This rule is general, and embraces not only all the cases in which either divisor or dividend is a fraction, but is also applicable when both are integral quantities, since any integral quantity may be placed under the form of a fraction, by writing unity beneath it.

$$\text{Thus } a\div b=\frac{a}{1}\times\frac{1}{b}=\frac{a}{b}.$$

Remarks 2 and 3, Art. 131, apply equally well to division as to multiplication of fractions.

EXAMPLES.

Required, in their simplest forms, the quotients

1. Of $\frac{ab^2c^3}{x^2y}\div\frac{a^3b^2c}{xy^2}$. *Ans.* $\frac{c^2y}{a^2x}$.

2. Of $\frac{a+b}{a+c}\div\frac{a-c}{a-b}$. *Ans.* $\frac{a^2-b^2}{a^2-c^2}$.

3. Of $\frac{x^3-a^2x}{a^2}\div\frac{ax-a^2}{x}$. *Ans.* $\frac{x^3+ax^2}{a^3}$.

4. Of $\left(1+\frac{1}{a}\right)\div\left(1-\frac{1}{a^2}\right)$. *Ans.* $\frac{a}{a-1}$.

5. Of $\frac{x^3+y^3}{x^2-y^2}\div\frac{x^2-xy+y^2}{x-y}$. *Ans.* 1.

6. Of $\frac{a^4-x^4}{a^2-2ax+x^2}\div\frac{a^2x+x^3}{a^3-x^3}$. *Ans.* $\frac{a+x}{x}(a^2+ax+x^2)$.

7. Of $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\left(\frac{1}{1-x}-\frac{x}{1+x}\right)$. *Ans.* 1.

8. Of $\left(\frac{x}{1+x}+\frac{1-x}{x}\right)\div\left(\frac{x}{1+x}-\frac{1-x}{x}\right)$. *Ans.* $\frac{1}{2x^2-1}$.

9. Of $\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$. *Ans.* $\frac{x^2+y^2}{2xy}$.

10. Of $\frac{3x}{2x-2}\div\frac{2x}{x-1}$. *Ans.* $\frac{3}{4}$.

11. Of $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$. *Ans.* $\frac{x-1}{x-5}$.

12. Of $\frac{4a(a^2-x^2)}{3b(c^2-x^2)}\div\frac{a^2-ax}{bc+bx}$. *Ans.* $\frac{4(a+x)}{3(c-x)}$.

13. Of $\left(x^4-\frac{1}{x^4}\right)\div\left(x-\frac{1}{x}\right)$. *Ans.* $x^3+\frac{1}{x^3}+x+\frac{1}{x}$.

ART. **133.** To reduce a complex fraction to a simple one.

This is merely a case of division, in which the dividend and divisor are either fractions or mixed quantities.

Thus $\dfrac{a+\frac{b}{c}}{m-\frac{n}{r}}$ is the same as to divide $a+\frac{b}{c}$ by $m-\frac{n}{r}$.

$$\left(a+\frac{b}{c}\right)\div\left(m-\frac{n}{r}\right)=\frac{ac+b}{c}\div\frac{mr-n}{r}=\frac{ac+b}{c}\times\frac{r}{mr-n}=\frac{acr+br}{cmr-cn}.$$

Let the following examples be solved in the same manner.

1. $\dfrac{\frac{3x}{2x-2}}{\frac{2x}{x-1}}$. *Ans.* $\frac{3}{4}$.

2. $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}-\frac{g}{h}}$. *Ans.* $\frac{fh(ad+bc)}{bd(eh-fg)}$.

3. $\dfrac{\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}$. *Ans.* $\dfrac{a^2+1}{2a}$.

4. $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}$. *Ans.* $\dfrac{b}{a}$.

5. $\dfrac{\dfrac{1}{a}+\dfrac{1}{ab^3}}{b-1+\dfrac{1}{b}}$. *Ans.* $\dfrac{b+1}{ab^2}$.

Art. 134. Resolution of fractions into series.

Def.—An *infinite* series consists of an unlimited number of terms which observe the same law.

The *law* of a series is a relation existing between its terms, so that, when some of them are known, the succeeding terms may be easily obtained.

Thus, in the infinite series $1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+$ &c., any term may be found by multiplying the preceding term by $-\frac{1}{x}$.

Any proper algebraic fraction, whose denominator is a polynomial, may, by division, be resolved into an infinite series; for the numerator is a dividend, and the denominator a divisor, so related to each other that the division can never terminate, and the quotient will therefore be an infinite series. After finding a few terms of the series, the law of continuation is, in general, easily seen, and the succeeding terms may be found without continuing the division.

EXAMPLES.

1. Convert the fraction $\frac{1-x}{1+x}$ into an infinite series.

$$
\begin{array}{l|l}
1-x & 1+x \\
\hline
\underline{1+x} & 1-2x+2x^2-2x^3+\text{\&c.} \\
-2x & \\
\underline{-2x-2x^2} & \\
\quad +2x^2 & \\
\quad \underline{+2x^2+2x^3} & \\
\qquad -2x^3 &
\end{array}
$$

It is evident that the law of this series is, that each term, after the second, is equal to the preceding term, multiplied by $-x$.

In a similar manner, let the fractions in each of the following examples be resolved into an infinite series.

2. $\frac{1}{1+r^2}=1-r^2+r^4-r^6+r^8-$ &c., to infinity.

3. $\frac{1}{1-r+r^2}=1+r-r^3-r^4+r^6+r^7-r^9-r^{10}+$ &c.

4. $\frac{1}{1+r+r^2}=1-r+r^3-r^4+r^6-r^7+r^9-r^{10}+$ &c.

5. $\frac{a}{a+b}=1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+$ &c.

MISCELLANEOUS PROPOSITIONS IN FRACTIONS.

Of the forms $\frac{0}{b}$, $\frac{b}{0}$, and $\frac{0}{0}$.

When the two terms of a fraction $\frac{a}{b}$ are finite determinate quantities, the fraction has necessarily a finite determinate value, which is, the quotient of a divided by b.

Let us now examine the cases where the numerator or denominator, or both, reduce to zero.

Art. 135. To prove that $\frac{0}{b}=0$.

While the denominator b is a constant number, if the numerator a diminishes, the value of the fraction diminishes. Thus, in the fractions $\frac{7}{8}$, $\frac{5}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$, each is less than the preceding. Hence, as the numerator a diminishes, and approaches to zero, the value of $\frac{a}{b}$ diminishes and approaches to zero; and finally, when $a=0$, the expression $\frac{0}{b}$ reduces to zero.

Or thus: Since the product of zero, by any number, is *zero*, therefore the quotient of *zero*, divided by any number, is zero. That is, since $0\times b=0$, therefore $\frac{0}{b}=0$.

Art. 136. To prove that $\frac{a}{0}=\infty$.

If the numerator a, of a fraction, remains constant, and the denominator diminishes, the value of the fraction increases.

Thus: 1st. Suppose the denominator 1; then $\frac{a}{1}=a$.

2nd. Suppose the denominator $\frac{1}{10}$; then $\frac{a}{.1}=10a$.

3rd. Suppose the denominator $\frac{1}{100}$; then $\frac{a}{.01}=100a$.

4th. Suppose the denominator $\frac{1}{1000}$; then $\frac{a}{.001}=1000a$.

From this it is evident, that if the denominator is *less* than any assignable quantity, that is 0, the value of the fraction is *greater* than any assignable quantity, that is *infinitely great*, or *infinity*. This is designated by the sign ∞; that is

$$\frac{a}{0}=\infty.$$

ART. **137.** To prove that $\frac{0}{0}$ is *indeterminate* in value.

When both numerator and denominator are zero, the fraction $\frac{a}{b}$ becomes $\frac{0}{0}$. Now since the divisor zero, multiplied by *any number whatever*, produces the dividend zero; therefore the quotient of zero, divided by zero, may be taken any number whatever; that is, the fraction $\frac{0}{0}$ is *indeterminate*.

It is important, however, for the pupil to know, that the form $\frac{0}{0}$ is often the result of a particular supposition, when both terms of a fraction contain a common factor.

Thus, if $x=\frac{a^2-b^2}{a-b}$, and we make $b=a$, it becomes $\frac{a^2-a^2}{a-a}=\frac{0}{0}$; but if we cancel the common factor, $a-b$, and then make $b=a$, we have $x=2a$.

Similarly, the fraction $x=\frac{a^2-1}{a^2+a-2}$ becomes $\frac{0}{0}$ when $a=1$; but if we divide both terms by their common factor, $a-1$, we have $x=\frac{a+1}{a+2}$, which reduces to $\frac{2}{3}$ when $a=1$.

These examples show, that if the value of any quantity is $\frac{0}{0}$, before we decide that it is really indeterminate, we must see that the apparent indetermination has not arisen from the existence of a factor, which, by a particular supposition, became equal to zero.

ART. **138.** THEOREM.—*If the same quantity be added to both terms of a proper fraction, the new fraction resulting will be greater than the first; but if the same quantity be added to both terms of an improper fraction, the new fraction resulting will be less than the first.*

Let $\frac{a}{b}$ be a proper fraction, a being less than b.

Let m represent the quantity to be added to each term, then the resulting fraction is $\frac{a+m}{b+m}$.

To determine which of the fractions, $\frac{a}{b}$ and $\frac{a+m}{b+m}$, is the greater, we must reduce them to a common denominator;

$$\text{this gives } \frac{a}{b}=\frac{ab+am}{b^2+bm},$$

$$\text{and } \frac{a+m}{b+m}=\frac{ab+bm}{b^2+bm}.$$

Since the denominators are the same, that fraction is the greatest which has the greatest numerator.

When $\frac{a}{b}$ is a proper fraction, a is less than b;

therefore am is less than bm,
and $ab+am<ab+bm$;

that is, the resulting fraction is greater than the first.

But if $\frac{a}{b}$ is an improper fraction, it is evident that

$$ab+am>ab+bm;$$

that is, the resulting fraction is less than the first.

ART. **139.** THEOREM.—*If the same quantity be subtracted from both terms of a proper fraction, the new fraction resulting will be less than the first; but if the same quantity be subtracted from both terms of an improper fraction, the new fraction resulting will be greater than the first.*

Let $\frac{a}{b}$ be a proper fraction, a being less than b. Let m represent the quantity to be subtracted from each term, then the resulting fraction is $\frac{a-m}{b-m}$. To determine which fraction is the greater, we reduce them to a common denominator, and compare their numerators;

$$\text{this gives } \frac{a}{b}=\frac{ab-am}{b^2-bm},$$

$$\text{and } \frac{a-m}{b-m}=\frac{ab-bm}{b^2-bm}.$$

If $a<b$, then $am<bm$; and if $am<bm$, then

$$ab-am>ab-bm;$$

that is, the resulting fraction is less than the first.

But if $a>b$, then $am>bm$; and if $am>bm$, then

$$ab-am<ab-bm;$$

that is, the resulting fraction is greater than the first.

MISCELLANEOUS EXERCISES IN FRACTIONS.

1. Prove that $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{18}{x^2-9}$.

2. Prove that $\frac{a^2+a+1}{(a-b)(a-c)}+\frac{b^2+b+1}{(b-a)(b-c)}+\frac{c^2+c+1}{(c-a)(c-b)}=1$.

3. Find the value of $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$, when $x=5\frac{1}{2}$.

Ans. 9.

4. Find the value of $\frac{x}{2}-\left\{\frac{2x-3}{3}-\frac{3x-1}{4}\right\}\div\frac{x-1}{2}$, when $x=4\frac{1}{3}$.

Ans. $2\frac{5}{6}$.

5. Find the value of $ax+by$, when $x=\frac{cq-br}{aq-bp}$ and $y=\frac{ar-cp}{aq-bp}$.

Ans. c.

6. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$. *Ans.* 2.

7. Find the value of $\frac{a^n}{2na^n-2nx}+\frac{b^n}{2nb^n-2nx}$, when $x=\frac{a^n+b^n}{2}$.

Ans. $\frac{1}{n}$.

8. Prove that the sum or difference of any two quantities divided by their product, is equal to the sum or difference of their reciprocals.

9. If two fractions are together equal to 1, prove that their difference is the same as the difference of their squares.

10. If the difference of two fractions is equal to $\frac{p}{q}$, show that p times their sum is equal to q times the difference of their squares.

11. Prove that $\frac{a^2+h^2}{(a-b)(a-c)}+\frac{b^2+h^2}{(b-a)(b-c)}+\frac{c^2+h^2}{(c-a)(c-b)}=1$: that when the terms are multiplied respectively by $b+c$, $a+c$, and $a+b$, the sum $=0$: and that when multiplied respectively by bc, ac, and ab, it is $=h^2$.

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

ART. **140.** An equation is an algebraic expression, stating the equality between two quantities. Thus

$$x-5=3,$$

is an equation stating that if 5 be subtracted from x, the remainder will be 3.

ART. **141.** Every equation is composed of two parts, separated from each other by the sign of equality. The quantity on the left of the sign of equality, is called the *first member* or side of the equation. The quantity on the right, is called the *second member* or side. The members or quantities are composed of one or more terms.

ART. **142.** There are generally two classes of quantities in an equation, the *known* and the *unknown*. The known quantities are represented either by numbers, or the first letters of the alphabet, as a, b, c, &c.; and the unknown quantities by the last letters of the alphabet, as x, y, z, &c.

ART. **143.** Equations are divided into degrees, called *first*, *second*, *third*, and so on. The degree of an equation depends on the highest power of the unknown quantity which it contains. Thus, an equation which contains no power of the unknown quantity higher than the first, is termed *an equation of the first degree*, or a simple equation.

An equation in which the highest power of the unknown quantity is of the second degree, is called *an equation of the second degree*, or a *quadratic equation.*

Similarly, we have equations of the third degree, fourth degree, and so on; those of the third degree are generally called *cubic* equations, and those of the *fourth* degree, *biquadratic* equations.

Thus,

$ax-b=c$, is an equation of the 1st degree.

$x^2+2px=q$, " " " 2d " or quadratic equation.

$x^3-px=q$, " " " 3d " or cubic equation.

$x^4+ax^3+px=q$, " " 4th " or biquadratic eq.

$x^n+ax^{n-1}+bx^{n-2}=c$, " nth degree.

When any equation contains more than one unknown quantity, its degree is equal to the greatest sum of the exponents of the unknown quantity, in any of its terms. Thus,

$xy+ax-by=c$, is an equation of the 2nd degree.

$x^2y+x^2-cx=a$, is an equation of the 3rd degree.

ART. **144.** An equation of any degree is said to be *complete*, when it contains all the powers of the unknown quantity, from 0 up to the given degree. When one or more terms are wanting, the equation is said to be *incomplete*.

Thus, $x^2+px+q=0$, is a complete equation of the second degree, the term q being equivalent to qx^0, since $x^0=1$. (Art. 82.)

$x^3+px^2+qx+r=0$, is a complete equation of the third degree.

$ax^2=q$, is an incomplete equation of the second degree.

$x^3+px=q$, is an incomplete equation of the third degree.

ART. **145.** An *identical equation*, is one in which the two members are identical; or, one in which one of the members is the result of the operations indicated in the other.

Thus, $ax-b=ax-b$,

$8x-3x=5x$,

$(x+3)(x-3)=x^2-9$, are identical equations.

Equations are also distinguished as *numerical* and *literal*.

A *numerical* equation is one in which all the known quantities are expressed by numbers.

Thus $2x^2+3x=10x+15$, is a numerical equation.

A *literal* equation is one in which the known quantities are represented by letters, or by letters and numbers.

Thus, $ax+b=cx+d$,

and $ax+b=3x-5$, are literal equations.

ART. **146.** Every equation may be regarded as the statement, in algebraic language, of a particular question.

Thus, $x-5=9$, may be regarded as the statement of the following question:—To find a number from which, if 5 be subtracted, the remainder shall be 9.

If we add 5 to each member, we shall have

$$x-5+5=9+5, \text{ or } x=14.$$

To *solve* an equation, is *to find the value of the unknown quantity*; or, to find a number or expression, which, being substituted for the unknown quantity, will render the two members identical.

REMARK.—The solution of equations is the most useful and interesting part of algebra.

An equation is said to be *verified*, when the value of the unknown quantity being substituted for it, the two members are rendered equal to each other.

Thus, in the equation $x-5=9$, if 14, the value of x, be substituted instead of it, we have

$$14-5=9;$$
$$\text{or, } 9=9.$$

ART. **147.** The value of the unknown quantity, in any equation, is called the *root* of that equation.

EQUATIONS OF THE FIRST DEGREE, CONTAINING BUT ONE UNKNOWN QUANTITY.

ART. **148.** The operations employed to find the value of the unknown quantity in any equation, are founded on this evident principle:

If we perform the same operation on two equal quantities, the results will be equal.

This principle or axiom may be otherwise stated, as follows:

1. *If, to two equal quantities, the same quantity be added, the sums will be equal.*
2. *If, from two equal quantities, the same quantity be subtracted, the remainders will be equal.*
3. *If two equal quantities be multiplied by the same quantity, the products will be equal.*
4. *If two equal quantities be divided by the same quantity, the quotients will be equal.*
5. *If two equal quantities be raised to the same power, the results will be equal.*
6. *If the same root of two equal quantities be extracted, the results will be equal.*

REMARK.—An axiom is a self-evident truth. The preceding axioms are the foundation of a large part of the reasoning in mathematics.

ART. **149.** There are two operations of frequent use in the solution of equations. These are, first, *to clear an equation of fractions;* and second, *to transpose the terms in order to find the value of the unknown quantity.*

These are named in the order in which they are used in the solution of an equation; we shall, however, first consider the subject of

TRANSPOSITION.

ART. **150.** Suppose we have the equation

$$ax+b=c-dx.$$

Since, by the preceding principle, the equality will not be affected by adding the same quantity to both members; or, by subtracting the same quantity from both members; if we add dx to each side, we have

$$ax+b+dx=c-dx+dx.$$

If we subtract b from each member, we have

$$ax+b-b+dx=c-dx+dx-b.$$

But $+b-b$ cancel each other, so do $-dx+dx$; omitting these, we have $ax+dx=c-b$.

But this result is the same as if we had removed the terms $+b$ and $-dx$ to the opposite members of the equation, and at the same time changed their signs. Hence,

Any quantity may be transposed from one side of an equation to the other, if, at the same time, its sign be changed.

This is termed the *Rule of Transposition.*

TO CLEAR AN EQUATION OF FRACTIONS.

ART. **151.** 1. Let it be required to clear the following equation of fractions.

$$\frac{x}{ab}-\frac{x}{bc}=d.$$

Since the first term is divided by ab, if we multiply it by ab, the divisor will be removed; but if we multiply the first term by ab, we must multiply all the other terms by ab, in order to preserve the equality of the members. Again, since the second term is divided by bc, if we multiply it by bc, the divisor will be removed; but if we multiply the second term by bc, we must multiply all the other terms by bc, in order to preserve the equality of the members. Hence, if we multiply all the terms on both sides by $ab\times bc$, the equation will be cleared of fractions.

Instead, however, of multiplying every term by $ab\times bc$, it is evident, that if each term be multiplied by such a quantity as will contain the denominators without a remainder, that all the denominators will be removed. This quantity is evidently the *least common multiple of the denominators*, which, in this case, is abc; then, multiplying both sides of the equation by abc, we have

$$cx-ax=abcd.$$

From which we derive the following

RULE FOR CLEARING AN EQUATION OF FRACTIONS.—*Find the least common multiple of all the denominators, and multiply each term of the equation by it.*

EXAMPLES FOR PRACTICE,

In clearing equations of fractions.

2. $\frac{x}{3}-\frac{x}{4}=1$. *Ans.* $4x-3x=12$.

3. $\frac{x}{4}+\frac{x}{6}=5$. *Ans.* $3x+2x=60$.

4. $\frac{x}{4}-\frac{x}{8}+\frac{x}{12}=3\frac{1}{2}$. *Ans.* $6x-3x+2x=84$.

5. $2x+\frac{x-3}{5}=\frac{x+9}{2}$. *Ans.* $20x+2x-6=5x+45$.

6. $2x-\frac{x-3}{5}=\frac{x-3}{2}$. *Ans.* $20x-2x+6=5x-15$.

7. $x-\frac{x-2}{4}=5-\frac{x+2}{6}$. *Ans.* $12x-3x+6=60-2x-4$.

8. $\frac{x}{ab}+\frac{ax}{bc}-\frac{bx}{ac}=m$. *Ans.* $cx+a^2x-b^2x=abcm$.

9. $a-\frac{b-c}{x}=d+\frac{b-c}{z}$. *Ans.* $axz-bz+cz=dxz+bx-cx$.

10. $\frac{x-a}{a+b}-\frac{x-a}{a-b}=\frac{2nb}{a^2-b^2}$.

Ans. $ax-a^2-bx+ab-ax+a^2-bx+ab=2nb$.

SOLUTION OF EQUATIONS OF THE FIRST DEGREE, CONTAINING ONLY ONE UNKNOWN QUANTITY.

ART. **152.** The unknown quantity in an equation may be combined with the known quantities, either by addition, subtraction, multiplication, or division; or by two or more of these different methods.

1. Let it be required to find the value of x, in the equation

$$a+x=b,$$

where the unknown quantity is connected by *addition.*

By *subtracting* a from each side (Art. 148), we have

$$x=b-a.$$

2. Let it be required to find the value of x, in the equation

$$x-a=b,$$

where the unknown quantity is connected by *subtraction.*

By *adding* a to each side (Art. 148), we have

$$x=b+a.$$

3. Let it be required to find the value of x, in the equation

$$ax=b,$$

where the unknown quantity is connected by *multiplication.*

By *dividing* each side by a, we have

$$x=\frac{b}{a}.$$

4. Let it be required to find the value of x, in the equation

$$\frac{x}{a}=b,$$

where the unknown quantity is connected by *division*.

By *multiplying* each side by a, we have

$$x=b\times a=ab$$

From the solution of these examples, we see that

When the unknown quantity is connected by addition, it is to be separated by subtraction. When it is connected by subtraction, it is to be separated by addition. When it is connected by multiplication, it is to be separated by division. And, when it is connected by division, it is to be separated by multiplication.

5. Let it be required to find the value of x, in the equation

$$3x-\frac{24-2x}{7}=x+8.$$

Clearing the equation of fractions, we have

$$21x-(24-2x)=7x+56,$$

$$\text{or } 21x-24+2x=7x+56.$$

Transposing the terms $7x$ and -24, we have

$$21x+2x-7x=56+24;$$

reducing, $16x=80$;

dividing by 16, $x=\frac{80}{16}=5$.

It will be readily seen that this solution consists of three steps, viz.:

1st. Clearing the equation of fractions.

2nd. Transposition.

3rd. Reducing like terms, and dividing by the coëfficient of x.

Let this value of x be substituted instead of x in the original equation, and, if it is the true value, the two members will be equal to each other.

Original equation, $3x-\frac{24-2x}{7}=x+8$.

Substituting 5 in place of x, it becomes

$$3\times5-\frac{24-2\times5}{7}=5+8,$$

or $15-2=5+8$,

or $13=13$.

The operation of substituting the value of the unknown quantity instead of itself, in the original equation, to see if it will render the two members equal to each other, is called *verification.*

6. Find the value of x, in the equation

$$x-\frac{x+a}{ab}=d+\frac{x}{bc}.$$

1st *step* . . . $abcx-cx-ac=abcd+ax.$
2nd *step* . . . $abcx-cx-ax=abcd+ac.$
Factoring . . . $(abc-c-a)x=ac(bd+1).$
3rd *step* . . . . $x=\frac{ac(bd+1)}{abc-c-a}.$

Art. **153.** From the solution of the preceding examples, we derive the following

Rule for the solution of an equation of the first degree.—
1. *If necessary, clear the equation of fractions; and perform all the operations indicated.*
2. *Transpose all the terms containing the unknown quantity to one side, and the known quantities to the other.*
3. *Reduce each member to its simplest form, and divide both sides by the coëfficient of the unknown quantity.*

Remark.—This rule gives the method of proceeding most generally advantageous, but in some cases it is best to perform the operations indicated, and transpose the necessary terms, before clearing of fractions. Experience can alone determine the best method in particular cases.

EXAMPLES FOR PRACTICE.

Note.—Let the pupil verify the value of the unknown quantity in each example.

Find the value of the unknown quantity in each of the following examples.

7. $\frac{3x+7}{14}-\frac{2x-7}{21}+2\frac{3}{4}=\frac{x-4}{4}.$

1st *step* . . . $18x+42-8x+28+231=21x-84;$
2nd *step* . . . $18x-8x-21x=-231-42-28-84;$
3rd *step* . . . $-11x=-385,$
$x=35.$

Verification. $\frac{3\times35+7}{14}-\frac{2\times35-7}{21}+2\frac{3}{4}=\frac{35-4}{4},$

$8-3+2\frac{3}{4}=7\frac{3}{4},$

$7\frac{3}{4}=7\frac{3}{4}.$

8. $5(x+1)-2=3(x+5).$ *Ans.* $x=6.$
9. $3(x-2)+4=4(3-x).$ *Ans.* $x=2.$

10. $5-3(4-x)+4(3-2x)=0$. *Ans.* $x=1$.

11. $3(x-3)-2(x-2)+x-1=x+3+2(x+2)+3(x+1)$. *Ans.* $x=-4$.

12. $5(5x-6)-4(4x-5)+3(3x-2)-2x-16=0$. *Ans.* $x=2$.

13. $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$ *Ans.* $x=12$.

14. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=7\frac{5}{6}$. *Ans.* $x=10$.

15. $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{7}{3}$. *Ans.* $x=\frac{1}{2}$.

16. $\frac{3x+1}{2}-\frac{2x}{3}=10+\frac{x-1}{6}$. *Ans.* $x=14$.

17. $\frac{x-7\frac{1}{2}}{2}=\frac{3x-9}{4}+\frac{27-5x}{3}$. *Ans.* $x=7\frac{7}{17}$.

18. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$. *Ans.* $x=8$.

19. $\frac{3x-1}{7}+\frac{6-x}{4}-\frac{2x-4}{12}=2-\frac{x+2}{28}$. *Ans.* $x=5$.

20. $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}$. *Ans.* $x=9$.

21. $\frac{1}{3}(2x-10)-\frac{1}{11}(3x-40)=15-\frac{1}{5}(57-x)$. *Ans.* $x=17$.

22. $\frac{1}{7}(x-\frac{1}{2})-\frac{1}{5}(\frac{2}{3}-x)=1\frac{13}{30}$. *Ans.* $x=4\frac{7}{9}$.

23. $\frac{1}{4}(4+\frac{3}{2}x)-\frac{1}{7}(2x-\frac{1}{8})=\frac{31}{28}$. *Ans.* $x=\frac{2}{3}$.

24. $\frac{1}{2}(\frac{2}{3}x+4)-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left(\frac{6}{x}-1\right)$ *Ans.* $x=3$.

25. $3\frac{1}{8}\times\left\{28-\left(\frac{x}{8}+24\right)\right\}=3\frac{1}{2}\times\left\{2\frac{1}{3}+\frac{x}{4}\right\}$. *Ans.* $x=4$.

26. $1-\frac{x}{2}\left(1-\frac{3}{4x}\right)=\frac{2}{3}\left(3-\frac{5x}{2}\right)+5\frac{13}{40}$. *Ans* $x=5.1$.

27. $\frac{1}{2}(x-\frac{51}{26})-\frac{2}{13}(1-3x)=x-\frac{1}{39}\left(5x-\frac{1-.3x}{.4}\right)$. *Ans.* $x=11$.

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28. $bx+2x-a=3x-2c$ *Ans.* $x=\frac{a-2c}{b-1}$.

29. $a^2x+b^3=b^2x+a^3$. *Ans.* $x=\frac{a^2+ab+b^2}{a+b}$.

30. $ax+b^2=a^2+bx$. *Ans.* $x=a+b$.

31. $\frac{bx}{a}-\frac{d}{c}=\frac{a}{b}-\frac{cx}{d}$. *Ans.* $x=\frac{ad}{bc}$.

32. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$. *Ans.* $x=\frac{1}{ab}$.

33. $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$. *Ans.* $x=\frac{c}{2b}(3a-b)$.

34. $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$. *Ans.* $x=\frac{ac(1-3ab)}{c-ad}$.

35. $\frac{ax}{b}+\frac{cx}{f}+g=qx+\frac{1}{f}(fh-cx)$. *Ans.* $x=\frac{bf(h-g)}{af+2bc-bfq}$.

36. $\frac{2x}{a-2b}=3+\frac{x}{2a-b}$ *Ans.* $x=2a-5b+\frac{2b^2}{a}$.

37. $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=10a+11b$.
Ans. $x=25a+24b$.

38. $\frac{3x-a}{b}+\frac{x+2b}{c}=\frac{7x}{c}-\frac{a}{4}$. *Ans.* $x=\frac{8b^2-4ac+abc}{12(2b-c.)}$.

39. $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$. *Ans.* $x=\frac{b(a-b+c)}{a}$.

QUESTIONS PRODUCING EQUATIONS OF THE FIRST DEGREE, CONTAINING ONLY ONE UNKNOWN QUANTITY.

ART. **154.** The solution of a problem by algebra, consists of two distinct parts:

1st. *To express the conditions of the problem in algebraic language; that is, to form the equation.*

2nd. *To solve the equation; that is, to find the value of the unknown quantity.*

Sometimes the statement of the question proposed, furnishes the equation directly; and sometimes it is necessary, from the conditions given, to deduce others, from which to form the equation. When the conditions furnish the equation directly, they are called *explicit* conditions. When the conditions are deduced from those given in the question, they are called *implied* conditions.

It is impossible to give a precise rule by means of which every question may be readily stated in the form of an equation. The first step is, to understand fully the nature of the question, so as to be able to prove the correctness or incorrectness of any proposed answer. After this, the equation, by the solution of which the value of the unknown quantity is to be found, may generally be formed by the following

RULE. — *Denote the required quantity by one of the final letters of the alphabet; then, by means of signs, indicate the same operations that it would be necessary to make on the answer, to verify it.*

EXAMPLES.

1. Find two numbers such, that their sum shall be 50, and their difference 12.

Let x denote the least of the two required numbers.

Then will . . $x+12=$ the greater,
And . . . . . $x+x+12=50$, by the question.
Transposing, . $x+x=50-12$.
Reducing, . . $2x=38$.
Dividing, . . . $x=19$, the less number;
And . . . . . $x+12=19+12=31$, the greater number.

VERIFICATION. $31+19=50$, and $31-19=12$.

2. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{5}$ part by 6?

Let $x=$ the required number.

Then will its $\frac{1}{3}$ part be denoted by $\frac{x}{3}$, and its $\frac{1}{5}$ part, by $\frac{x}{5}$.

Therefore, . . . $\frac{x}{3}-\frac{x}{5}=6$.
Clearing, . . . $5x-3x=90$.
Reducing, . . . $2x=90$.
Dividing, . . . $x=45$, the number required.

VERIFICATION. $\frac{1}{3}$ of $45=15$, $\frac{1}{5}$ of $45=9$; $15-9=6$.

3. Divide \$500 among A, B, and C, so that B shall have \$20 more than A, and C \$75 more than A.

Let . . . $x=$A's share,
Then . . $x+20=$B's share,
And . . . $x+75=$C's share.

Then $x+\overline{x+20}+\overline{x+75}=500$, by the question.

Reducing, . . . $3x+95=500$.
Transposing, . . $3x=500-95=405$.
Dividing, ... $x=135$, A's share.
$x+20=155$, B's share.
$x+75=210$, C's share.

VERIFICATION. $135+155+210=500$.

4. Out of a cask of wine which had leaked away $\frac{1}{5}$, 35 gallons were drawn, and then, being guaged it was $\frac{1}{2}$ full; how much did it hold?

Let $x=$ the number of gallons it held;

then $\frac{x}{5}=$ " " " leaked out.

There had been taken away $\frac{x}{5}+35$ gallons, and there remained $x-\left(\frac{x}{5}+35\right)$ gallons.

$$\therefore x-\left(\frac{x}{5}+35\right)=\frac{x}{3}.$$

Clearing, ... $15x-3x-35\times15=5x$;
Transposing, . . $15x-3x-5x=35\times15$;
Reducing, . . . $7x=35\times15$;
$\therefore x=5\times15=75$.

5. A laborer was engaged for 20 days. For each day that he worked, he received 50 cents and his boarding; and for each day that he was idle, he paid 25 cents for his boarding. At the expiration of the time, he received $4; how many days did he work, and how many days was he idle?

Let . . $x=$ the number of days he worked;
Then, . $20-x=$ " " " " was idle.
Also, . $50x=$ wages due for work.
And . $25(20-x)=$ the amount to be deducted for boarding.

$\therefore 50x-25(20-x)=400$;
$50x-500+25x=400$;
$75x=400+500=900$;
$x=12=$ the number of days he worked.
$20-x=8=$ " " " " " was idle.

PROOF. $50\times12=600$ cents $=$ wages;
$25\times\ 8=200$ " $=$ boarding.
Diff. $=400$ " or $4.

In solving this example, we reduce the $4 to cents, in order that all the quantities on both sides of the equation, may be of the same denomination, it being regarded as a self-evident principle, that we can only compare quantities of the same name. Hence, *all the quantities in both members of an equation, must be of the same denomination.*

6. What two numbers are as 3 to 5, to each of which, if 9 be added, the sums shall be to each other as 6 to 7.

If we put x to represent the first number, the second will be $\frac{5x}{3}$. But we may avoid fractions by putting $3x$ for the first number, and $5x$ for the second, which fulfills the first condition.

Then, $3x+9 : 5x+9 :: 6 : 7$.

But in every proportion, the product of the means is equal to the product of the extremes. (Arith. Part 3rd, Art. 209.)

Hence, $6(5x+9)=7(3x+9)$.
$30x+54=21x+63$,
$30x-21x=63-54$,

$$9x=9,$$
$$x=1,$$
$$\therefore\ 3x=3, \text{ and } 5x=5.$$

The method of representing the quantities by $3x$ and $5x$, so as to avoid fractions, is of general application, and may be expressed thus:— *When two or more unknown quantities, in any problem, have to each other a given ratio, it is best to assume each of them a multiple of some other unknown quantity, so that they shall have to each other the given ratio.*

7. A courier who traveled at the rate of $31\frac{1}{2}$ miles in 5 hours, was dispatched from a certain city; 8 hours after his departure, another courier was sent to overtake him. The second courier traveled at the rate of $22\frac{1}{2}$ miles in 3 hours. In what time did he overtake the first, and at what distance from the place of departure?

Let $x=$ the number of hours that the second courier travels. Then, since the first courier travels at the rate of $31\frac{1}{2}$ miles in 5 hours, that is, $\frac{63}{10}$ miles in 1 hour, he will travel $\frac{63x}{10}$ miles in x hours, and since he started 8 hours before the second courier, the whole distance traveled by him will be $(8+x)\frac{63}{10}$.

Again, since the second courier travels at the rate of $22\frac{1}{2}$ miles in 3 hours, that is, $\frac{45}{6}$ miles in 1 hour, he will travel $\frac{45}{6}x$ miles in x hours.

But the couriers are supposed to be together at the end of the time x, and, therefore, the distance traveled by each must be the same; hence

$$\frac{45x}{6}=(8+x)\frac{63}{10};$$
$$450x=(8+x)378;$$
$$\therefore\ 72x=378\times8; \text{ divide each side by } 8;$$
$$9x=378;$$
$$x=42.$$

Hence the second courier will overtake the first in 42 hours, and the whole distance traveled by each is $\frac{45}{6}\times42=315$ miles.

8. A smuggler had a quantity of brandy, which he expected would sell for 198 shillings; after he had sold 10 gallons, a revenue officer seized one third of the remainder, in consequence of which he sells the whole for only 162 shillings. Required the number of gallons he had, and the price per gallon.

Let $x=$ the number of gallons;

then $\frac{198}{x}$ is the price per gallon, in shillings;

$\frac{x-10}{3}$ is the quantity seized, the value of which is

$$198-162=36 \text{ shillings.}$$

$$\therefore \frac{x-10}{3}\times\frac{198}{x}=36.$$

$(x-10)66=36x$, by clearing of fractions;

$66x-660=36x$;

$30x=660$;

$\therefore x=22$, the number of gallons;

and $\frac{198}{x}=\frac{198}{22}=9$ shillings, the price per gallon.

9. There are three numbers whose sum is 133; the second is twice the first, and the third twice the second. Required the numbers. *Ans.* 19, 38, and 76.

10. There are three numbers whose sum is 187; the second is 3 times, and the third $4\frac{1}{2}$ times, the first. Required the numbers. *Ans.* 22, 66, and 99.

11. There are two numbers, of which the first is $3\frac{1}{2}$ times the second, and their difference is 100. Required the numbers. *Ans.* 40 and 140.

12. There are three numbers, whose sum is 156; the second is $3\frac{1}{2}$ times the first, and the third is equal to the remainder left, after subtracting the difference of the first and second from 100. Required the numbers. *Ans.* 28, 98, and 30.

13. What number is that, whose half, third, and fourth parts, taken together, are equal to 52? *Ans.* 48.

14. What number is that, which being increased by its six-sevenths, and diminished by 20, shall be equal to 45? *Ans.* 35.

15. What number is that, to which if its third and fourth parts be added, the sum will exceed its sixth part by 51? *Ans.* 36.

16. Find a number which, being multiplied by 4, becomes as much above 40 as it is now below it. *Ans.* 16.

17. What number is that, to which if 16 be added, 4 times the sum will be equal to 10 times the number increased by 1? *Ans.* 9.

18. The sum of two numbers is 30; and if the less be subtracted from the greater, one-fourth of the remainder will be 3. Required the numbers. *Ans.* 9 and 21.

19. A laborer was engaged for 28 days, upon the condition that for every day he worked he was to receive 75 cents, and for

every day he was absent, to forfeit 25 cents. At the end of his time he received \$12. How many days did he work? *Ans.* 19.

20. A has three times as much money as B, but if B give A \$50, then A will have four times as much as B. Find the money of each. *Ans.* A, \$750; B, \$250.

21. From a bag of money which contained a certain sum, there was taken \$20 more than its half; from the remainder, \$30 more than its third part; and from the remainder, \$40 more than its fourth part, and then there was nothing left. What sum did it contain? *Ans.* \$290.

22. A merchant gains the first year, 15 per cent. on his capital; the second year, 20 per cent. on the capital at the close of the first; and the third year, 25 per cent. on the capital at the close of the second; when he finds that he has cleared \$1000.50. Required his capital. *Ans.* \$1380.

23. A is twice as old as B; 22 years ago, he was three times as old. What is A's age? *Ans.* 88.

24. A person buys 4 houses; for the second, he gives half as much again as for the first; for the third, half as much again as for the second; and for the fourth, as much as for the first and third together: he pays \$8000 for them all. Required the cost of each. *Ans.* \$1000, \$1500, \$2250, and \$3250.

25. A cistern is filled in 24 minutes by 3 pipes, the first of which conveys 8 gallons more, and the second 7 gallons less, than the third every 3 minutes. The cistern holds 1050 gallons. How much flows through each pipe in a minute?

Ans. $17\frac{5}{36}$, $12\frac{5}{36}$, $14\frac{17}{36}$.

26. A can do a piece of work in three days, B in 6 days, and C in 9 days. Find the time in which all together can perform it.

Let $x=$ the required number of days;

then $\frac{1}{x}=$ part of the work performed by all in one day.

But A does $\frac{1}{3}$, B $\frac{1}{6}$, and C $\frac{1}{9}$, in one day.

$\therefore \frac{1}{3}+\frac{1}{6}+\frac{1}{9}=\frac{1}{x}$. *Ans.* $1\frac{7}{11}$ days.

27. If A does a piece of work in 10 days, which A and B can do together in 7 days; how long would B take to do it alone?

Ans. $23\frac{1}{3}$ days.

28. A performs $\frac{2}{7}$ of a piece of work in 4 days; he then receives the assistance of B, and the two together finish it in 6 days. Required the time in which each can do it alone.

Ans. A, 14 days; B, 21 days.

29. A person bought an equal number of sheep, cows, and oxen, for \$330; each sheep cost \$3, each cow \$12, and each ox \$18. Required the number of each. *Ans.* 10.

30. A sum of money is to be divided among five persons; A, B, C, D, and E. B received \$10 less than A; C, \$16 more than B; D, \$5 less than C; E, \$15 more than D; and the shares of the last two are equal to the sum of the shares of the other three. Required the share of each.

Ans. A, \$21; B, \$11; C, \$27; D, \$22; E, \$37.

31. A bought eggs at 18 cents a dozen, but had he bought 5 more for the same money, they would have cost him $2\frac{1}{2}$ cents a dozen less. How many eggs did he buy? *Ans.* 31.

32. A person bought a certain number of sheep for \$94; having lost 7 of them, he sold one-fourth of the remainder at prime cost, for \$20. How many sheep had he at first?

Ans. 47.

33. There are two places, 154 miles distant from each other, from which two persons, A and B, set out at the same instant, to meet on the road. A travels at the rate of 3 miles in 2 hours, and B at the rate of 5 miles in 4 hours. How long, and how far, did each travel before they met?

Ans. 56 hours, and A traveled 84, and B, 70 miles.

34. Find that number, which, multiplied by 5, and 24 taken from the product, the remainder divided by 6, and 13 added to the quotient, will still give the same number. *Ans.* 54.

35. In a bag containing eagles and dollars, there are three times as many eagles as dollars; but if 8 eagles and as many dollars be taken away, there will be left five times as many eagles as dollars. How many were there of each?

Ans. 48 eagles, 16 dollars.

36. If 10 apples cost a cent, and 25 pears cost 2 cents, and you buy 100 apples and pears for $9\frac{1}{2}$ cents, how many of each will you have? *Ans.* 75 apples and 25 pears.

37. Suppose that for every 8 sheep a farmer keeps, he should plough an acre of land, and allow one acre of pasture for every 5 sheep, how many sheep may he keep on 325 acres?

Ans. 1000.

38. A person has just 2 hours spare time; how far may he ride in a stage which travels 12 miles an hour, so as to return home in time, walking back at the rate of 4 miles an hour.

Ans. 6 miles.

39. If 65℔ of sea-water contain 2℔ of salt, how much fresh water must be added to these 65℔, in order that the quantity of

salt contained in 25℔ of the new mixture, shall be reduced to 4 ounces, or $\frac{1}{4}$ of a ℔. *Ans.* 135℔.

40. A mass of copper and tin weighs 80℔; and for every 7℔ of copper, there are 3℔ of tin. How much copper must be added to the mass, that for every 11℔ of copper, there may be 4℔ of tin? *Ans.* 10℔.

41. A merchant maintained himself for 3 years, at a cost of $250 a year; and in each of those years, augmented that part of his stock which was not so expended, by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was that stock? *Ans.* $3700.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

ART. **155.** From what we have already seen, it is evident that the value of any one of the symbols concerned in an equation, is entirely dependent on the rest, and it can become *known*, only when the values of the rest are *given*, or *known* Thus, in the equation

$$x+y=a,$$

the value of x depends on the values of y and a, and can only become known when they are known; therefore, *to find the value of any unknown quantity, we must obtain a single equation containing it and known quantities.* Hence, when we have two or more equations containing two or more unknown quantities, we must obtain from them a single equation containing only one unknown quantity. The method of doing this is termed *elimination*, which may be defined briefly, thus:—Elimination is the process of deducing, from two or more equations containing two or more unknown quantities, a single equation containing only one unknown quantity.

There are three principal methods of elimination:

1st. Elimination by substitution.

2nd. Elimination by comparison.

3rd. Elimination by addition and subtraction.

ELIMINATION BY SUBSTITUTION.

ART. **156.** Elimination by substitution consists in finding the value of one of the unknown quantities in one of the equations, in terms of the other unknown quantity and known terms, and substituting this, instead of the quantity, in the other equation.

To explain this method, let it be required to find the values of x and y, in the following equations.

$$2x+3y=33, \qquad (1)$$
$$4x+5y=59. \qquad (2)$$

From eq. (1), by transposing $3y$ and dividing by 2, we have

$$x=\frac{33-3y}{2}.$$

Substituting this value of x, instead of x in eq. (2), we have

$$4\left(\frac{33-3y}{2}\right)+5y=59;$$

$$\text{or, } 66-6y+5y=59;$$
$$-y=-7;$$
$$y=7;$$
$$\text{and } x=\tfrac{33-3\times7}{2}=6.$$

The following is the general form to which two equations of the first degree, containing two unknown quantities, may always be reduced. The signs of the known quantities, a, b, c, &c., may be either plus or minus.

$$ax+by=c, \qquad (1)$$
$$a'x+b'y=c'. \qquad (2)$$

From eq. (1), by transposing by, and dividing by a, we have

$$x=\frac{c-by}{a}.$$

Substituting this value of x in eq. (2), we have

$$a'\left(\frac{c-by}{a}\right)+b'y=c';$$

$$a'c-a'by+ab'y=ac';$$
$$(ab'-a'b)y=ac'-a'c;$$
$$y=\frac{ac'-a'c}{ab'-a'b}.$$

$$\text{But } x=\frac{c-by}{a}=\frac{c-b\left(\frac{ac'-a'c}{ab'-a'b}\right)}{a}=\frac{ab'c-a'bc-abc'+a'bc}{a(ab'-a'b)}$$
$$=\frac{b'c-bc'}{ab'-a'b}.$$

Hence, when we have two equations, containing two unknown quantities, we have the following

Rule for elimination by substitution.—*Find an expression for the value of one of the unknown quantities in either equation, and substitute this value, instead of the same unknown quantity, in the other equation; there will thus be formed a new equation, containing only one unknown quantity.*

ELIMINATION BY COMPARISON.

Art. 157. Elimination by comparison consists in finding the value of the same unknown quantity in two different equations, and then placing these values equal to each other.

To illustrate this method, we will take the same equations as in the preceding article.

$$2x+3y=33, \qquad (1)$$
$$4x+5y=59. \qquad (2)$$

From eq. (1), by transposing and dividing, we have $x=\frac{33-3y}{2}$.

From eq. (2), by transposing and dividing, we have $x=\frac{59-5y}{4}$

Placing these values of x equal to each other,

$$\frac{59-5y}{4}=\frac{33-3y}{2};$$

$59-5y=66-6y$, by clearing of fractions;

$y=7$, by transposition.

The value of x may be found similarly, by first finding the values of y, and placing them equal to each other. But after finding the value of one of the unknown quantities, that of the other may generally be found most readily by substitution. Thus,

$$4x+5\times7=59;$$

whence $x=\frac{59-35}{4}=6$.

General equations, $ax+by=c, \qquad (1)$

$a'x+b'y=c'. \qquad (2)$

From eq. (1), by transposing and dividing, $x=\frac{c-by}{a}$,

From eq. (2), by transposing and dividing, $x=\frac{c'-b'y}{a'}$;

equating these values of x,

$$\frac{c-by}{a}=\frac{c'-b'y}{a'};$$

$a'c-a'by=ac'-ab'y$, by clearing of fractions;

$(ab'-a'b)y=ac'-a'c$, by transposing;

$$y=\frac{ac'-a'c}{ab'-a'b}.$$

From eq. (1), $y=\frac{c-ax}{b}$;

From eq. (2), $y=\frac{c'-a'x}{b'}$;

equating these values of y,

$$\frac{c'-a'x}{b'}=\frac{c-ax}{b};$$

$$bc'-a'bx=b'c-ab'x;$$

$$(ab'-a'b)x=b'c-bc';$$

$$x=\frac{b'c-bc'}{ab'-a'b}.$$

Hence, when we have two equations, containing two unknown quantities, we have the following

RULE FOR ELIMINATION BY COMPARISON.—*Find an expression for the value of the same unknown quantity in each of the given equations, and place these values equal to each other; there will thus be formed a new equation, containing only one unknown quantity.*

ELIMINATION BY ADDITION AND SUBTRACTION.

ART. **158.** Elimination by addition and subtraction consists in multiplying or dividing two equations, so as to render the coëfficient of one of the unknown quantities, the same in both; and then, by adding or subtracting, to cause the terms containing it to disappear.

Taking the same equations as in the preceding articles,

$$2x+3y=33, \quad (1)$$
$$4x+5y=59. \quad (2)$$

It is evident that if we multiply eq. (1) by 2, that the coëfficient of x will be the same in the two equations.

$4x+6y=66$ (3), by $\times$ing eq. (1) by 2.
$4x+5y=59$, eq. (2) brought down.
$y=7$.

Since the coëfficients of x have the *same* sign in these equations, if we *subtract*, the terms containing x will cancel each other, and the resulting equation will contain only y, the value of which may then be found. It is evident that if the signs of the coëfficients of x had been *different*, that by *adding*, it would have been canceled.

Having obtained the value of y, that of x may be obtained by substitution, or similar to that of y, as follows:

It is evident that if we multiply eq. (1) by 5, and eq. (2) by 3, that the coëfficients of y will be the same in both.

$10x+15y=165$, (4) by $\times$ing eq. (1) by 5.
$12x+15y=177$, (5) by $\times$ing eq. (2) by 3.
$2x=12$, by subtracting eq. (4) from (5).
$x=6$.

General equations, $ax+by=c, \qquad (1)$

$a'x+b'y=c'. \qquad (2)$

It is evident that we shall render the coëfficients of x the same in both equations, by multiplying eq. (1) by a', and eq. (2) by a.

$$aa'x+a'by=a'c, \quad (3) \quad \text{by} \times\text{ing eq. (1) by } a';$$
$$aa'x+ab'y=ac', \quad (4) \quad \text{by} \times\text{ing eq. (2) by } a;$$
$$(ab'-a'b)y=ac'-a'c, \quad \text{by subtracting};$$
$$y=\frac{ac'-a'c}{ab'-a'b}.$$

The coëfficients of y in the two equations will evidently become equal by multiplying eq. (1) by b', and eq. (2), by b.

$$ab'x+bb'y=b'c, \quad (5) \quad \text{by} \times\text{ing eq. (1) by } b';$$
$$a'bx+bb'y=bc', \quad (6) \quad \text{by} \times\text{ing eq. (2) by } b;$$
$$(ab'-a'b)x=b'c-bc', \quad \text{by subtracting};$$
$$x=\frac{b'c-bc'}{ab'-a'b}.$$

It is evident that after we have rendered the coëfficients of the quantity to be eliminated the same in both equations, if the signs are *alike* we must *subtract;* but if they are *unlike* we must *add.*

Hence, when we have two equations containing two unknown quantities, we have the following

RULE, FOR ELIMINATION BY ADDITION AND SUBTRACTION. — *Multiply, or divide the equations, if necessary, so that one of the unknown quantities will have the same coëfficient in both. Then take the difference, or the sum of the equations, according as the signs of the equal terms are alike or unlike, and the resulting equation will contain only one unknown quantity.*

REMARK. — When the coëfficients of the quantity to be eliminated are prime to each other, they may be equated by multiplying each equation by the coëfficient of the unknown quantity in the other. When the coëfficients are not prime, find their least common multiple, and multiply each equation by the quotient obtained by dividing the least common multiple by the coëfficient of the unknown quantity to be eliminated in the other equation.

If the equations have fractional coëfficients, they ought to be cleared, before applying the rule.

EXAMPLES FOR PRACTICE.

NOTE. — It is recommended to the pupil to solve several of the following examples, by each of the preceding rules.

1. $x+3y=10$, $3x+2y=9$. *Ans.* $x=1$, $y=3$.

2. $2x+3y=18$, $3x-2y=1$. *Ans.* $x=3$, $y=4$.

3. $2x-9y=11$, *Ans.* $x=1$,
$3x-12y=15$. $y=-1$.

4. $3x-7y=7$, *Ans.* $x=7$,
$11x+5y=87$. $y=2$.

5. $9x-4y=8$, *Ans.* $x=4$,
$13x+7y=101$. $y=7$.

6. $x-\frac{1}{7}(y-2)=5$, *Ans.* $x=5$,
$4y-\frac{1}{3}(x+10)=3$. $y=2$.

7. $\frac{x}{3}+\frac{y}{5}=8$, *Ans.* $x=18$,
$\frac{x}{9}-\frac{y}{10}=1$. $y=10$.

8. $\frac{4x+5y}{40}=x-y$, *Ans.* $x=\frac{1}{4}$,
$\frac{2x-y}{3}+2y=\frac{1}{2}$. $y=\frac{1}{5}$.

9. $\frac{1}{3}(x+y)+\frac{1}{4}(x-y)=59$,
$5x-33y=0$.
Ans. $x=99$,
$y=15$.

10. $\frac{4}{5+y}=\frac{5}{12+x}$, *Ans.* $x=2$,
$2x+5y=35$. $y=\frac{31}{5}$.

11. $\frac{2x+6}{3y+2}=\frac{8}{7}$, *Ans.* $x=5$,
$8x-4=9y$. $y=4$

12. $2x-\frac{y+3}{4}=7+\frac{3y-2x}{5}$, *Ans.* $x=5$,
$4y+\frac{x-2}{3}=26\frac{1}{2}-\frac{2y+1}{2}$. $y=5$

13. $\frac{3x+4y+3}{10}-\frac{2x+7-y}{15}=5+\frac{y-8}{5}$, *Ans.* $x=7$
$\frac{9y+5x-8}{12}-\frac{x+y}{4}=\frac{7x+6}{11}$. $y=9$.

14. $ax=by$, *Ans.* $x=\frac{bc}{a+b}$,
$x+y=c$. $y=\frac{ac}{a+b}$.

15. $x+ay=b$, *Ans.* $x=\frac{ac+b^2}{a^2+b}$,
$ax-by=c$. $y=\frac{ab-c}{a^2+b}$.

16. $3ax-2by=c$, *Ans.* $x=\frac{11bc}{2a^2+3ab}$,
$a^2x+b^2y=5bc$. $y=\frac{c}{b}\left(\frac{15b-a}{2a+3b}\right)$.

17. $(a-b)x+(a+b)y=c$, *Ans.* $x=\frac{1}{2b}\left(\frac{n}{a-b}-c\right)$,
$(a^2-b^2)(x+y)=n$, $y=\frac{1}{2b}\left(c-\frac{n}{a+b}\right)$.

18. $\frac{m}{x}+\frac{n}{y}=a$, *Ans.* $x=\frac{m^2-n^2}{ma-nb}$,
$\frac{n}{x}+\frac{m}{y}=b$. $y=\frac{m^2-n^2}{mb-na}$.
{ Apply Rule, Art, 158. }

19. $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$, $\qquad$ *Ans.* $x=\frac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$,

$\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$. $\qquad$ $y=\frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

20. $(a^2-b^2)(5x+3y)=(4a-b)2ab$, $\qquad$ *Ans.* $x=\frac{ab}{a+b}$,

$a^2y-\frac{ab^2c}{a+b}+(a+b+c)bx=b^2y+(a+2b)ab$. $\qquad$ $y=\frac{ab}{a-b}$.

REMARK.—Transpose b^2y in eq. (2), multiply by 3 and subtract, there will then result an equation involving x.

QUESTIONS PRODUCING SIMULTANEOUS EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

ART. **159.** The questions contained in Art. 154, are all capable of being solved by using one unknown quantity; although in some of the examples, the number of unknown quantities was two or more. But in those questions where there was more than one unknown quantity, there was such a relation existing between the several quantities, that it was easy to express each one in terms of the other. It frequently happens, however, that in a problem containing more than one unknown quantity, there may be no direct relation existing between them, by means of which either may be found in terms of the other. In such a case it becomes necessary to use a separate symbol for each unknown quantity, and then to find equations containing these symbols, on the same principle as when there was but one unknown quantity; that is, in brief, *regard the symbols as the answer to the question, and then proceed in the same manner which it would be necessary to do to prove the answer.* After the equations are obtained, the values of the unknown quantities may be found, by either of the three different methods of elimination.

We shall now give an example, to show that the same question may sometimes be solved by using either one or two unknown quantities.

1. The difference of two numbers is a, and the less is to the greater as m to n; required the numbers.

Solution by using one unknown quantity.

Let $mx=$ the less number, then $nx=$ the greater;

And $nx-mx=a$.

$$x=\frac{a}{n-m};$$

$$\therefore mx=\frac{ma}{n-m}, \text{ the less number;}$$

$$nx=\frac{na}{n-m}, \text{ the greater number.}$$

Solution by using two unknown quantities.

Let $x=$ the less number, and $y=$ the greater.

Then, $y-x=a$, (1)

and $x:y::m:n$; or $my=nx$. (2)

Since $my=nx$, we have $y=\frac{nx}{m}$;

substituting this value of x in eq. (1),

$$\frac{nx}{m}-x=a,$$

$$nx-mx=ma;$$

$$\text{whence } x=\frac{ma}{n-m},$$

$$\text{and } y=\frac{n}{m}x=\frac{mna}{m(n-m)}=\frac{na}{n-m}.$$

2. The hour and minute hands of a watch are opposite at 6 o'clock; when are they next opposite?

Let $x=$ minute spaces moved over by the hour hand, and $y=$ minute spaces moved over by the minute hand. Then since the minute hand moves 12 times as fast as the hour hand,

$x:y::1:12$, or $y=12x$. (1)

But the minute hand must evidently pass over 60 minutes more than the hour hand; hence

$y=x+60$. (2)

Substituting, $12x=x+60$,

$11x=60$,

$x=5\frac{5}{11}$ min.

$y=65\frac{5}{11}$ min.$=1$h., $5\frac{5}{11}$m.

Hence, the hands are next opposite at $5\frac{5}{11}$m. past 7.

In a similar manner the period of coincidence of the hands may be found.

3. There is a number consisting of two digits, which divided by the sum of its digits, gives a quotient 7; but if the digits be written in an inverse order, and the number thence arising be divided by the sum of the digits increased by 4, the quotient $=3$. Required the number. *Ans.* 84.

In solving questions of this kind, the pupil must observe that any number consisting of two places of figures, is équal to 10 times the figure in the ten's place plus the figure in the unit's place. Thus, 35 is equal to $10\times3+5$. In a similar manner, 456 is equal to $100\times4+10\times5+6$.

Let $x=$ the digit in ten's place, and $y=$ the digit in unit's place.

Then $10x+y=$the number.

And $10y+x=$the number when the digits are reversed.

$$\text{Also, } \frac{10x+y}{x+y}=7. \qquad \frac{10y+x}{x+y+4}=3.$$

From these equations we readily find $x=8$, and $y=4$.

4. A farmer sells to one man 5 sheep and 7 cows for $111, and to another, at the same rate, 7 sheep and 5 cows for $93. Required the price of a sheep and that of a cow.

Ans. Sheep, $4; cow, $13.

5. If 7℔ of tea and 9℔ of coffee cost $5.20, and at the same rate 4℔ of tea and 11℔ of coffee cost $3.85; it is required to find the price of a pound of each. *Ans.* Tea, 55c.; coffee, 15c.

6. A and B are in trade together with different sums; if $50 be added to A's money, and $20 be taken from B's, they will have the same sum; but if A's money were 3 times, and B's 5 times as great as each really is, they would have together $2350. How much has each? *Ans.* A, $250; B, $320.

7. A and B have together $9800; A invests the sixth part of his money in business, and B the fifth part, and then each has the same sum remaining. How much has each?

Ans. A, $4800; B, $5000.

Let $6x=$A's money, and $5y=$B's.

8. What fraction is that, such that if the numerator and denominator be each increased by 1, the value is $\frac{1}{2}$; but if each be diminished by 1, the value is $\frac{1}{3}$? *Ans.* $\frac{3}{7}$.

9. Find two numbers, such that one-third of the first exceeds one-fourth of the second by 3, and one-fourth of the first and one-fifth of the second are together equal to 10. *Ans.* 24 and 20.

10. A grocer knows neither the weight nor the first cost of a box of tea which he had purchased. He only recollects that if he had sold the whole at 30 cts. per ℔, he would have gained $1, but if he had sold it at 22 cts. per ℔, he would have lost $3. Required the number of pounds in the box, and the first cost per ℔. *Ans.* 50℔ at 28 cts.

11. The rent of a field is a certain fixed number of bushels of wheat, and a fixed number of bushels of corn. When wheat is 55 cents, and corn 33 cents per bushel, the portions of rent by wheat and corn are equal; but when wheat is 65 cents and corn

41 cents per bushel, the rent is increased by $1.40. What is the grain rent?

Ans. 6 bushels of wheat and 10 bushels of corn.

12. The quantity of water which flows from an orifice is proportional to the area of the orifice, and the velocity of the water. Now there are two orifices in a reservoir, the areas being as 5 to 13, and the velocities as 8 to 7, and from one there issued in a certain time 561 cubic feet more than from the other. How much water did each orifice discharge in this time?

Ans. 440 and 1001 cubic feet.

13. Find two numbers in the ratio of 5 to 7, to which two other required numbers in the ratio of 3 to 5 being respectively added, the sums shall be in the ratio of 9 to 13; and the difference of those sums =16. *Ans.* 30 and 42, and 6 and 10.

14. A boy spends 30 cents in apples and pears, buying his apples at 4 and his pears at 5 for a cent; he then finds that half his apples and one-third of his pears cost 13 cents. How many of each did he buy? *Ans.* 72 apples and 60 pears.

15. A farmer rents a farm for $245 per year; the tillable land being valued at $2 per acre, and the pasture at $1.40; now the number of acres of tillable, is to half the excess of the tillable above the pasture, as 28 to 9. How many acres are there of each? *Ans.* 98 acres tillable, and 35 of pasture.

16. Find that number of 2 figures to which, if the number formed by changing the places of the digits be added, the sum is 121; and if the less of the same two numbers be taken from the greater, the remainder is 9. *Ans.* 65.

17. To determine three numbers such that if 6 be added to the first and second, the sums will be in the ratio of 2 to 3; if 5 be added to the first and third, the sums will be in the ratio of 7 : 11; but if 36 be subtracted from the second and third, the remainders will be as 6 to 7. *Ans.* 30, 48, 50.

SUGGESTION.—Let $2x-6$, $3x-6$, and y be the numbers.

18. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time could each do it separately? *Ans.* A in 24, B in 48 days.

19. A and B drink from a cask of beer for 2 hours, after which A falls asleep, and B drinks the remainder in 2 hours and 48 minutes; but if B had fallen asleep and A had continued to drink, it would have taken him 4 hours and 40 minutes to finish the cask. In what time could each singly drink the whole?

Ans. A in 10 hrs., B in 6 hrs.

20. Divide the fraction $\frac{8}{5}$ into two parts, so that the numerators of the two parts taken together shall be equal to their denominators taken together. *Ans.* $\frac{1}{2}$ and $\frac{11}{10}$.

21. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many of each will it hold?

Ans. 21 crowns or 63 guineas.

22. When wheat was 5 shillings a bushel, and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Ans. He must buy 9 bushels of wheat, and 12 bushels of rye.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

ART. **160.** Simultaneous equations of the first degree involving three or more unknown quantities, may be solved by either of the three methods of elimination, explained in Arts. 155 to 159; but the method most generally applicable, is that of elimination by addition and subtraction, which we shall now proceed to apply in the solution of a problem containing three unknown quantities.

1. Given
$$5x-4y+2z=48, \quad (1)$$
$$3x+3y-4z=24, \quad (2)$$
$$2x-5y+3z=19, \quad (3) \quad \text{to find } x, y, \text{ and } z.$$

To eliminate z from the first two equations, we may multiply eq. (1) by 2, and then add this to eq. (2); thus,

$$10x-8y+4z=96, \text{ by } \times\text{ing eq. (1) by 2,}$$
$$3x+3y-4z=24, \quad (2)$$
$$13x-5y \quad =120, \quad (5) \quad \text{by adding.}$$

We may eliminate z from equations (1) and (3), by multiplying eq. (1) by 3, and eq. (3) by 2, and then subtracting; thus,

$$15x-12y+6z=144, \text{ by } \times\text{ing eq. (1) by 3,}$$
$$4x-10y+6z=\ 38, \text{ by } \times\text{ing eq. (3) by 2,}$$
$$11x-\ 2y \quad =106, \quad (6) \quad \text{by subtracting.}$$

We may now eliminate y from equations (5) and (6), by multiplying eq. (5) by 2, and eq. (6) by 5, and then subtracting; thus,

$$26x-10y=240, \text{ by } \times\text{ing eq. (5) by 2,}$$
$$55x-10y=530, \text{ by } \times\text{ing eq. (6) by 5,}$$
$$29x \quad =290;$$
$$x=10.$$

$110-2y=106$, by substituting 10 for x in eq. (6); whence $y=2$.

$50-8+2z=48$, by substituting for x and y in eq. (1); whence $z=3$.

It is evident that the same method may be applied when the number of equations is four or more. Hence we derive the following

GENERAL RULE FOR ELIMINATION BY ADDITION AND SUBTRACTION.—1st. *Combine any one of the equations with each of the others, so as to eliminate the same unknown quantity; there will thus arise a new class of equations, containing one less unknown quantity.*

2nd. *Combine any one of these new equations with each of the others, so as to eliminate another unknown quantity; there will thus arise another class of equations, containing two less unknown quantities.*

3rd. *Continue this series of operations until a single equation is obtained, containing but one unknown quantity, from which its value may be easily found; then by going back, and substituting this value in the derived equations, the values of the other unknown quantities may be readily found.*

REMARK.—Although the method of elimination by addition and subtraction is generally the best when the number of unknown quantities is three or more, yet in some particular instances, solutions may be obtained more easily and elegantly by other means, which the pupil must acquire by experience and tact. As a specimen, we present the following question and solution.

2. Given $-x+y+z=a$, (1)

$x-y+z=b$, (2)

$x+y-z=c$, (3) to find x, y, and z.

By adding the three equations together, and calling $a+b+c=s$, we find

$$x+y+z=s. \quad (4)$$

Then by subtracting eqs. (1), (2), and (3), respectively from (4), and dividing by 2, we find.

$$x=\tfrac{1}{2}(s-a),$$
$$y=\tfrac{1}{2}(s-b),$$
$$z=\tfrac{1}{2}(s-c).$$

EXAMPLES,

To be solved by either of the different methods of elimination.

3. $\left.\begin{array}{l} x+y+z=6, \\ 3x-y+2z=7, \\ 4x+3y-z=7. \end{array}\right\}$ *Ans.* $x=1$, $y=2$, $z=3$.

4. $\left.\begin{array}{l}3x+4y-5z=32,\\4x-5y+3z=18,\\5x-3y-4z=2.\end{array}\right\}$ *Ans.* $x=10$, $y=8$, $z=6$.

5. $\left.\begin{array}{l}x-9y+3z-10u=21,\\2x+7y-z-u=683,\\3x+y+5z+2u=195,\\4x-6y-2z-9u=516.\end{array}\right\}$ *Ans.* $x=100$, $y=60$, $z=-13$, $u=-50$.

6. $\left.\begin{array}{l}x+\frac{1}{2}y=10-\frac{1}{3}z,\\\frac{1}{2}(x+z)=9-y,\\\frac{1}{4}(x-z)=2y-7.\end{array}\right\}$ *Ans.* $x=7$, $y=4$, $z=3$.

7. $\left.\begin{array}{l}y+\frac{z}{3}=\frac{x}{5}+5,\\\frac{x-1}{4}-\frac{y-2}{5}=\frac{z+3}{10},\\x-\frac{2y-5}{3}=1\frac{3}{4}-\frac{z}{12}.\end{array}\right\}$ *Ans.* $x=5$, $y=7$, $z=-3$.

8. $\left.\begin{array}{l}9x-2z+u=41,\\7y-5z-t=12,\\4y-3x+2u=5,\\3y-4u+3t=7,\\7z-5u=11.\end{array}\right\}$ *Ans.* $x=5$, $y=4$, $z=3$, $u=2$, $t=1$.

Examples, to be solved by special methods.

9. $\left.\begin{array}{l}\frac{1}{x}+\frac{1}{y}=a,\\\frac{1}{x}+\frac{1}{z}=b,\\\frac{1}{y}+\frac{1}{z}=c.\end{array}\right\}$ *Ans.* $x=\frac{2}{a+b-c}$, $y=\frac{2}{a-b+c}$, $z=\frac{2}{b+c-a}$.

SUGGESTION. — Subtract eq. (3) from (2), then combine the resulting equation with (1), to find x and y; z may be found similarly.

10. $\left.\begin{array}{l}\frac{2}{x}+\frac{3}{y}-\frac{4}{z}=\frac{1}{12},\\\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=\frac{19}{24},\\-\frac{4}{x}+\frac{5}{y}+\frac{6}{z}=\frac{1}{2}.\end{array}\right\}$ *Ans.* $x=6$, $y=12$, $z=8$.

11. $\left.\begin{array}{l}\frac{2}{x}-\frac{5}{3y}+\frac{1}{z}=\frac{85}{27}, \\ \frac{1}{4x}+\frac{1}{y}+\frac{2}{z}=\frac{443}{72}, \\ \frac{5}{6x}-\frac{1}{y}+\frac{4}{z}=\frac{433}{36}.\end{array}\right\}$ *Ans.* $x=6$, $y=9$, $z=\frac{1}{3}$.

12. $\left.\begin{array}{l}-x+y+z+v=a, \\ x-y+z+v=b, \\ x+y-z+v=c, \\ x+y+z-v=d.\end{array}\right\}$ *Ans.* $x=\frac{1}{2}(s-a)$, $y=\frac{1}{2}(s-b)$, $z=\frac{1}{2}(s-c)$, $v=\frac{1}{2}(s-d)$, where $s=\frac{1}{2}(a+b+c+d)$.

QUESTIONS PRODUCING SIMULTANEOUS EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

ART. **161.** When a question contains *three* or more unknown quantities, equations involving them can be found on the same principle as in questions containing *one* or *two* unknown quantities. (See Arts. 154 and 159.) The values of the unknown quantities may then be found, in the same manner as in the preceding examples.

1. The stock of three traders amounts to \$760; the shares of the first and second exceed that of the third by \$240; and the sum of the second and third exceeds the first by \$360: what is the share of each? *Ans.* \$200, \$300, and \$260.

2. What three numbers are there, each greater than the preceding, whose sum is 20, and such that the sum of the first and second is to the sum of the second and third, as 4 is to 5; and the difference of the first and second, is to the difference of the first and third, as 2 to 3? *Ans.* 5, 7, and 8.

3. Find four numbers, such that the sum of the first, second, and third, shall be 13; the sum of the first, second, and fourth, 15; the sum of the first, third, and fourth, 18; and lastly, the sum of the second, third, and fourth, 20. *Ans.* 2, 4, 7, 9.

4. The sum of three digits composing a certain number is 16: the sum of the left and middle digits, is to the sum of the middle and right ones as 3 to $3\frac{2}{3}$; and if 198 be added to the number, the digits will be inverted. Required the number. *Ans.* 547.

5. At an election where each elector may give two votes to different candidates, but only one to the same, it is found on counting the votes, that of the candidates A, B, C, A had 158 votes, B had 132, and C 58. Now 26 voted for A only, 30 for

B only, and 28 for C only. How many voted for A and B jointly; how many for A and C; and how many for B and C?

Ans. For A and B, 102; A and C, 30; B and C, 0.

6. It is required to find three numbers such, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall together make 46; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, shall together make 35; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, shall together make $28\frac{1}{3}$. *Ans.* 12, 60, and 80.

7. The sum of three numbers, taken two and two, are a, b, and c. What are the numbers?

Ans. $\frac{1}{2}(a+b-c)$, $\frac{1}{2}(a+c-b)$, and $\frac{1}{2}(b+c-a)$.

8. A person has four casks, the second of which being filled from the first, leaves the first four-sevenths full. The third being filled from the second, leaves it one-fourth full; and when the third is emptied into the fourth, it is found to fill only nine-sixteenths of it. But the first will fill the third and fourth and have fifteen quarts remaining. How many quarts does each hold?

Ans. 140, 60, 45, and 80, respectively.

9. In the crew of a ship consisting of sailors and soldiers, there were 22 sailors to every 3 guns, and 10 sailors over; also the whole number of hands was 5 times the number of soldiers and guns together; but after an engagement, in which the slain were one-fourth of the survivors, there wanted 5 men to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors. *Ans.* 90 guns, 55 soldiers, 670 sailors.

CHAPTER V.

SUPPLEMENT TO EQUATIONS OF THE FIRST DEGREE.

I. GENERALIZATION.

Art. 162. Equations are termed *literal* when the known quantities are represented, either entirely or partly, by letters. Quantities represented by letters are termed *general* values — because the solution of one problem furnishes a *general* solution which embraces all others, where the letters have specified numerical values.

The answer to a problem, where the known quantities are represented by letters, is termed a *formula;* and a formula expressed in ordinary language, furnishes a *rule.*

By the application of algebra to the solution of *general* questions, a great number of useful and interesting truths and rules may be established. We shall now illustrate this subject by an example.

ART. **163.** It is required to divide a given number a into three parts, having to each other the same ratio as the numbers m, n, and p.

Let mx, nx, and px, represent the required parts, since these are evidently to each other as m, n, and p.

$$\text{Then } mx+nx+px=a,$$

$$\text{and} \quad x=\frac{a}{m+n+p},$$

$$\therefore mx=\frac{ma}{m+n+p},$$

$$nx=\frac{na}{m+n+p},$$

$$px=\frac{pa}{m+n+p}.$$

This formula translated into ordinary language, gives the following

RULE FOR DIVIDING A GIVEN NUMBER INTO PARTS HAVING TO EACH OTHER A GIVEN RATIO. — *Multiply the given number by each term of the ratios respectively, and divide the products by the sum of the numbers expressing the ratios. The respective quotients will be the required parts.*

The pupil may solve the following examples by this rule, and test its accuracy by verifying the results.

2. Divide 69 into three parts, having to each other the same ratio as the numbers 5, 7, and 11. *Ans.* 15, 21, and 33.

3. Divide $38\frac{1}{2}$ into four parts, having to each other the same ratio as the fractional numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

Ans. 15, 10, $7\frac{1}{2}$, and 6.

The pupil may now solve the following general examples, and express the formula in ordinary language, so as to form a general rule.

4. The sum of two numbers is a, and their difference b. Required the numbers. *Ans.* Greater, $\frac{a}{2}+\frac{b}{2}$; less, $\frac{a}{2}-\frac{b}{2}$.

5. The difference of two numbers is a, and the greater is to the less as m to n: find the numbers. *Ans.* $\frac{ma}{m-n}$ and $\frac{na}{m-n}$.

6. The sum of two numbers is a, and their sum is to their difference as m to n: required the numbers.

Ans. Greater $=\frac{(m+n)a}{2m}$, less $\frac{(m-n)a}{2m}$.

7. Divide the number a into three such parts, that the second shall exceed the first by b, and the third exceed the second by c.

Ans. $\frac{a-2b-c}{3}$, $\frac{a+b-c}{3}$, $\frac{a+b+2c}{3}$.

8. Divide the number a into four such parts, that the first increased by m, the second diminished by m, the third multiplied by m, and the fourth divided by m, shall be all equal to each other.

Suggestion.—The simplest method of solving questions of this kind, is to make such a supposition for the values of the unknown quantities as will fulfill one or more of the conditions. In this question the last four conditions will be fulfilled by representing the four parts by $x-m$, $x+m$, $\frac{x}{m}$, and mx.

Ans. $\frac{ma}{(m+1)^2}-m$, $\frac{ma}{(m+1)^2}+m$, $\frac{a}{(m+1)^2}$, $\frac{m^2a}{(m+1)^2}$.

9. A person has just a hours at his disposal; how far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

Ans. $\frac{abc}{b+c}$ miles.

10. Given the sum of two numbers $=a$, and the quotient of the greater divided by the less $=b$; required the numbers.

Ans. Less $=\frac{a}{b+1}$, greater $=\frac{ab}{b+1}$.

This formula gives the following simple rule:—*To find the less number, divide the sum of the numbers by their quotient increased by unity.*

11. A person distributed a cents among n beggars, giving b cents to some, and c to the rest. How many were there of each?

Ans. $\frac{a-nc}{b-c}$ at b cts., and $\frac{nb-a}{b-c}$ at c cts.

12. Divide the number n into two such parts, that the quotient of the greater divided by the less shall be q, with a remainder r.

Ans. $\frac{nq+r}{1+q}$, $\frac{n-r}{1+q}$.

13. If A and B together can perform a piece of work in a days, A and C together the same in b days, and B and C together in c days: find the time in which each can perform it separately.

Ans. A in $\frac{2abc}{ac+bc-ab}$, B in $\frac{2abc}{ab+bc-ac}$, C in $\frac{2abc}{ab+ac-bc}$ days.

14. A, B, and C, hold a pasture in common, for which they pay P\$ a year. A puts in a oxen for m months; B, b oxen for n months; and C, c oxen for p months: required each one's share of the rent.

Ans. A's, $\frac{ma}{ma+nb+pc}P$\$; B's, $\frac{nb}{ma+nb+pc}P$\$;

C's, $\frac{pc}{ma+nb+pc}P$\$.

From these formula is derived the rule of *Compound Fellowship.*

15. A mixture is made of a ℔ of tea at m shillings per ℔, b ℔ at n shillings, and c ℔ at p shillings: what will be its cost per ℔.

Ans. $\frac{ma+nb+pc}{a+b+c}$.

From this formula is derived the rule termed *Alligation Medial.*

16. A waterman rows a given distance a and back again in b hours, and finds that he can row c miles with the current for d miles against it: required the times of rowing down and up the stream, also the rate of the current and the rate of rowing.

Ans. Time down, $\frac{bd}{c+d}$; time up, $\frac{bc}{c+d}$;

rate of current, $\frac{a(c^2-d^2)}{2bcd}$; rate of rowing, $\frac{a(c+d)^2}{2bcd}$.

II. NEGATIVE SOLUTIONS.

ART. **164.** It has been stated already (Art. 12), that when a quantity has no sign prefixed, the sign *plus* is understood; and also (Art. 47), that all numbers or quantities are regarded as positive, unless they are otherwise designated. Hence, in all problems it is understood that the results are required in positive numbers. It sometimes happens, however, in the solution of a problem, that the result has the *minus* sign. Such a result is termed a negative solution. We shall now examine a question of this kind.

1. What number must be subtracted from 3 that the remainder shall be 7?

Let $x=$ the number.

$$\text{Then } 3-x=7,$$

$$\text{whence } -x=7-3,$$
$$\text{or } \quad x=-4.$$

Now -4 subtracted from 3, according to the rule for algebraic subtraction, gives a remainder equal to 7; thus $3-(-4)=7$. The result, -4, is said to satisfy the question in an *algebraic sense:* but the problem is evidently impossible in an *arithmetical sense,* since any positive number subtracted from 3 must *diminish* instead of *increasing* it; and this *impossibility* is shown by the result being *negative.* But, since subtracting -4 is the same as adding $+4$ (Art. 48), the result is the answer to the following question:—What number must be *added* to 3, that the *sum* shall be equal to 7?

Let the question now be generalized, thus:

What number must be subtracted from a, that the remainder shall be b?

Let $x=$ the number,

$$\text{Then } \quad a-x=b,$$
$$\text{whence } x=a-b.$$

Now, since $a-(a-b)=b$, this value of x will always satisfy the question in an algebraic sense.

While b is *less* than a, the value of x will be *positive;* and whatever values are given to a and b, the question will be consistent, and can be answered in an *arithmetical sense.* Thus, if $a=10$, and $b=4$, then $x=6$.

But if b becomes greater than a, the value of x will be *negative;* and whatever values are given to a and b, the result obtained will satisfy the question in its *algebraic,* but not in its *arithmetical sense.*

Thus if $a=8$ and $b=10$, then $x=-2$. Now $8-(-2)=8+2=10$; that is, if we *add* 2 to 8, the *sum* will be 10. We thus see that when b becomes greater than a, the question, to be consistent in an *arithmetical sense,* should read:—What number must be *added* to a that the *sum* shall be equal to b?

From this we derive the following important general principles:

1st. *A negative solution indicates some inconsistency or absurdity in the question from which the equation was derived.*

2nd. *When a negative solution is obtained, the question, to which it is the answer, may be so modified as to be consistent.*

The pupil may now read carefully the "OBSERVATIONS ON ADDITION AND SUBTRACTION," page 24, and then modify the following questions so that they shall be consistent, and the results true in an arithmetical sense.

2. What number must be *added* to the number 30, that the *sum* shall be 19? ($x=-11$).

3. The *sum* of two numbers is 9, and their *difference* 25; required the numbers. *Ans.* 17 and -8.

4. What number is that whose half subtracted from its third leaves a remainder 15? ($x=-90$).

5. A father's age is 40 years; his son's age is 13 years; *in how many years* will the age of the father be 4 times that of the son? ($x=-4$).

6. The triple of a certain number *diminished* by 100, is equal to 4 times the number *increased* by 200. Required the number ($x=-300$).

III. DISCUSSION OF PROBLEMS.

ART. **165.** After a question has been *generalized* and solved, we may inquire what values the results will have, when particular suppositions are made with regard to the known quantities. The determination of these values, and the examination of the various results, to which different suppositions give rise, constitute the *discussion of the problem.*

The various forms which the value of the unknown quantity may assume, are shown in the discussion of the following question.

1. After subtracting b from a, what number, multiplied by the remainder, will give a product equal to c?

Let $x=$ the number.

Then $(a-b)x=c$,

and $$x=\frac{c}{a-b}.$$

Now, this result may have five different forms, depending on the different values that may be given to a, b, and c.

REMARK.—In the following forms, A denotes merely *some quantity.*

1st. When b is less than a. This gives positive values of the form $+$A.

2nd. When b is greater than a. This gives negative values of the form $-$A.

3rd. When b is equal to a. This gives values of the form $\frac{A}{0}$.

4th. When c is 0, and b either greater or less than a. This gives values of the form $\frac{0}{A}$.

5. When b is equal to a, and c is equal to 0. This gives values of the form $\frac{0}{0}$.

We shall now examine each of these cases.

I. Values of the form $+A$, or when b is less than a.

In this case, $a-b$ is positive, and the value of x is positive.

To illustrate this form, let $a=10$, $b=3$, and $c=35$, then $x=5$.

II. Values of the form $-A$, or when b is greater than a.

In this case, $a-b$ is a negative quantity, and the value of x will be negative. This evidently should be so, since minus multiplied by minus, gives plus; that is, if $a-b$ is minus, x must be minus, in order that their product shall be equal to c, a positive quantity.

To illustrate this form by numbers, let $a=5$, $b=8$, and $c=12$; then $a-b=-3$, $x=-4$, and $-3\times-4=+12$.

III. Values of the form $\frac{A}{0}$, or when b is equal to a.

In this case x becomes equal to $\frac{c}{0}$. But the value of a fraction of which the numerator is any finite quantity, and the denominator zero (Art. 136), is infinite; that is, $\frac{c}{0}=\infty$.

This is interpreted by saying, that no finite value of x will satisfy the equation; that is, there is no number, which being multiplied by 0, will give a product equal to c.

IV. Values of the form $\frac{0}{A}$, that is, when c is 0 and b is either greater or less than a.

If we put $a-b=d$, then $x=\frac{0}{d}=0$, since $d\times0=0$; that is, when the product is zero, one of the factors must be zero.

V. Values of the form $\frac{0}{0}$, that is, when $b=a$, and $c=0$.

In this case we have $x=\frac{c}{a-b}=\frac{0}{0}$, or $x\times0=0$. But $\frac{0}{0}$ is the symbol of indetermination (Art. 137); hence *any finite value* of x whatever will satisfy this equation; that is, x is indeterminate.

The discussion of the following problem, originally proposed by Clairaut, will serve to illustrate further the preceding principles, and show that the results of every correct solution correspond to the circumstances of the problem.

PROBLEM OF THE COURIERS.

ART. 166. Two couriers depart at the same time, from two places, A and B, distant a miles from each other; the former travels m miles an hour, and the latter n miles: where will they meet?

There are two cases of this problem, according as the couriers are traveling *toward* each other, or in the *same* direction.

I. When the couriers travel toward each other.

Let P be the point where they meet, and a=AB, the distance between the two places.

Let x=AP, the distance which the first travels.

Then $a-x$=BP, the distance which the second travels.

But the distance each travels, divided by the number of miles traveled per hour, will give the number of hours he was traveling.

Therefore, $\frac{x}{m}=$ the number of hours the first travels.

And $\frac{a-x}{n}=$ " " " " second travels.

But they both travel the same number of hours, therefore

$$\frac{x}{m}=\frac{a-x}{n};$$

$nx=ma-mx$, by clearing of fractions;

whence $x=\frac{ma}{m+n}$;

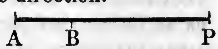
and $a-x=\frac{na}{m+n}$.

1st. Suppose $m=n$, then $x=\frac{ma}{2m}=\frac{a}{2}$, and $a-x=\frac{a}{2}$; that is, if the couriers travel at the same rate, each travels precisely half the distance.

2nd. Suppose $n=0$, then $x=\frac{ma}{m}=a$; that is, if the second courier remains at rest, the first travels the whole distance from A to B.

Both these results are evidently true, and correspond to the circumstances of the problem.

II. When the couriers travel in the same direction.



As in the first case, let P be the point of meeting, each traveling from A toward P, and let a=AB, the distance between the places;

x=AP, " " the first travels;

then $x-a$=BP, " " " second travels.

Then, reasoning as in the first case, we have

$$\frac{x}{m}=\frac{x-a}{n};$$

$$nx=mx-ma;$$

$$\text{whence } x=\frac{ma}{m-n};$$

$$\text{and } x-a=\frac{na}{m-n}.$$

1st. If we suppose m greater than n, the value of x will be positive; that is, the couriers will meet on the right of B. This evidently corresponds to the circumstances of the problem.

2nd. If we suppose n greater than m, the value of x, and also that of $x-a$, will be negative. This value of x being negative, shows that there is some inconsistency in the question (Art. 164). Indeed, where m is less than n, it is evident that the couriers cannot meet, since the forward courier is traveling faster than the hindmost.

Let us now inquire how the question may be modified, that the value obtained for x shall be consistent.

If we suppose the *direction changed* in which the couriers travel, that is, that the first travels from A, and

the second from B, toward P′; and that a=AB,

x=AP′,

$a+x$=BP′, we have,

reasoning as before, $\frac{x}{m}=\frac{a+x}{n}$;

whence $x=\frac{ma}{n-m}$, and $a+x=\frac{na}{n-m}$.

The distances traveled are now both positive, and the question will be consistent, if we regard the couriers, instead of traveling *toward* P, as traveling in the opposite direction, toward P′. The change of sign thus indicating a change of direction (Art 47).

3rd. Let us suppose m equal to n.

In this case x is equal to $\frac{ma}{0}$, and $x-a=\frac{na}{0}$.

But it has been shown already (Art. 136), that when the unknown quantity takes this form, it is not satisfied by any finite value, or, it is infinitely great. This evidently corresponds to the circumstances of the problem; for, if the couriers travel at the same rate, the one can *never* overtake the other. This is sometimes otherwise expressed, by saying they only meet at an infinite distance from the point of starting.

4th. Let us suppose $a=0$.

Then $x=\frac{0}{m-n}$, and $x-a=\frac{0}{m-n}$.

When the unknown quantity takes this form, it has been shown already (Art. 135) that its value is 0. This corresponds to the circumstances of the problem; for, if the couriers are *no* distance apart, they will have to travel *no* (0) distance to be together.

5th. Let us suppose $m=n$, and $a=0$.

In this case, $x=\frac{0}{0}$, and $x-a=\frac{0}{0}$. But when the unknown quantity takes this form, it has been shown (Art. 137) that it may have *any finite value* whatever. This, also, evidently corresponds to the circumstances of the problem; for, if the couriers are *no* distance apart, and travel at the *same* rate, they will be always together; that is, at *any distance whatever* from the point of starting.

6th. Let us suppose $n=0$.

In this case $x=\frac{am}{m}=a$; that is, the first courier travels from A to B, overtaking the second at B.

7th. Lastly, let us suppose $n=\frac{m}{2}$.

In this case $x=\frac{2ma}{m}=2a$; that is, the first travels twice the distance from A to B, before overtaking the second. The results in the last two cases evidently correspond to the circumstances of the problem.

IV. CASES OF INDETERMINATION IN EQUATIONS OF THE FIRST DEGREE, AND IMPOSSIBLE PROBLEMS.

ART. **167.** An equation is termed *independent*, when the relation of the quantities which it contains, cannot be obtained directly from others with which it is compared. Thus, the equations

$$x+3y=19,$$
$$\text{and } 2x+5y=33,$$

are independent of each other, since the one cannot be obtained from the other in a direct manner.

The equations $x+3y=19$,

$$2x+6y=38,$$

are not independent of each other, the second being derived directly from the first, by multiplying both sides by 2.

ART. 168. An equation is said to be *indeterminate*, when it can be verified by different values of the same unknown quantity. Thus, if we have the equation

$x-y=3$, by transposition we find

$x=3+y$.

If we make $y=1$, then $x=4$; if we make $y=2$, then $x=5$ and so on; from which it is evident, that an unlimited number of values may be given to x and y, that will verify the equation.

If we have two equations containing three unknown quantities, we may eliminate one of them; this will leave one equation containing two unknown quantities, which, as in the preceding example, will be indeterminate. Thus, if we have the following equations,

$$x+3y-5z=20,$$
$$x-y+3z=16.$$

If we eliminate x, we have, after reducing,

$$y-2z=1;$$

whence $y=1+2z$.

If we make $z=1$, then $y=3$, and $x=20+5z-3y=16$. If we make $z=2$, then $y=5$, and $x=15$.

In the same manner, an unlimited number of values of the three unknown quantities may be found, that will verify both equations. Other examples might be given, but these are sufficient to establish the following general principle.

When the number of unknown quantities exceeds the number of independent equations, the problem is indeterminate.

A question is sometimes indeterminate that involves only one unknown quantity; the equation deduced from the conditions being of that class denominated identical. (Art. 145.) The following is an example:

What number is that, whose $\frac{1}{4}$ increased by the $\frac{1}{6}$ is equal to the $\frac{11}{20}$ diminished by the $\frac{2}{15}$?

Let $x=$ the number.

Then $\frac{x}{4}+\frac{x}{6}=\frac{11x}{20}-\frac{2x}{15}$;

clearing of fractions, $15x+10x=33x-8x$;

or, $25x=25x$;

which will be verified by *any value whatever* of x.

A more simple example is the following:—What number is that whose half, third, and fourth, taken together, is equal to the number itself increased by its one-twelfth.

ART. **169.** The reverse of the preceding case requires to be considered; that is, when the number of equations is greater than the number of unknown quantities. Thus, we may have

$$2x+3y=23, \quad (1)$$
$$3x-2y=2, \quad (2)$$
$$5x+4y=40. \quad (3)$$

Each of these equations being independent of the other two, one of them is unnecessary, since the values of x and y, which are 4 and 5, may be found from either two of them.

When a problem contains more conditions than are necessary for determining the values of the unknown quantities, those that are unnecessary are termed *redundant*.

The number of equations may exceed the number of unknown quantities, so that the values of the unknown quantities shall be incompatible with each other. Thus, if we have

$$x+y=12, \quad (1)$$
$$2x+y=17, \quad (2)$$
$$3x+2y=30. \quad (3)$$

The values of x and y, found from equations (1) and (2), are $x=5$, $y=7$; from equations (1) and (3), $x=6$, and $y=6$; and from equations (2) and (3), $x=4$, and $y=9$. From this, it is manifest that only two of these equations can be true at the same time.

A question is sometimes impossible that involves only one unknown quantity. The following is an example:

What number is that whose $\frac{7}{12}$ diminished by 5 is equal to the difference between its $\frac{3}{4}$ and $\frac{1}{6}$ increased by 7.

Let x=the number, then $\frac{7x}{12}-5=\frac{3x}{4}-\frac{x}{6}+7$;

clearing of fractions, $7x-60=9x-2x+84$,

reducing, $0=144$,

which shows that the question is absurd.

REMARK.—Problems from which contradictory equations are deduced, are termed *irrational* or *impossible*. The pupil should be able to detect the character of such questions when they occur, in order that his efforts may not be wasted in attempting to perform an impossibility.

EXAMPLES TO ILLUSTRATE THE PRECEDING PRINCIPLES.

1. What number is that, which being divided successively by m and n, and the first quotient subtracted from the second, the remainder shall be q?

Ans. $x=\frac{mnq}{m-n}$.

What supposition will give a negative solution? An infinite solution? An indeterminate solution? Illustrate by numbers.

2. Two boats, A and B, set out at the same time, one from C to L, and the other from L to C; the boat A runs m miles, and the boat B, n miles per hour. Where will they meet, supposing it to be a miles from C to L?

Ans. $\frac{ma}{m+n}$ miles from C, or $\frac{na}{m+n}$ miles from L.

Under what circumstances will the boats meet half way between C and L? Under what circumstances will they meet at C? At L? Under what circumstances will they meet above C? Below L? Under what circumstances will they never meet? Under what circumstances will they sail together? Illustrate each of these questions by using numbers.

3. What number is that which, being multiplied by 8, the product increased by 16, and the sum divided by 4, will give a quotient equal to twice the number diminished by 7?

Resulting eq. $11=0$.

What does this result show? (Art. 169.)

4. There are three persons, A, B, and C, whose ages are so related, that B is 6 years younger than A and 4 years older than C; and $\frac{1}{3}$ of A's age increased by $\frac{1}{4}$ of C's, is equal to $\frac{7}{12}$ of B's, increased by one year. Required their ages.

Resulting equation. $0=0$.

If A's age was 15 years, B's was 9, and C's 5; if A's was 16, B's was 10, and C's 6.

5. Given $2x-y=2$,
$5x-3y=3$,
$3x+2y=17$,
$4x+3y=24$;

Ans. $x=3$, $y=4$.

to find x and y, and show how many equations are redundant. (Art. 169.)

6. Given $x+2y=11$,
$2x-y=7$,
$3x-y=17$,
$x+3y=19$;

to show that the equations are incompatible. (Art. 169.)

V. AN EQUATION OF THE FIRST DEGREE HAS BUT ONE ROOT.

Art. 170. In any equation of the first degree involving only one unknown quantity (x), if a represents the sum of the positive, and $-c$ the sum of the negative coëfficients of x; b the sum of the positive, and $-d$ the sum of the negative known quantities, it will evidently reduce to the following form:

$$ax-cx=b-d,$$

$$\text{or } (a-c)x=b-d.$$

Let $a-c=m$, and $b-d=n$, we then have

$$mx=n,$$

$$\text{whence } x=\frac{n}{m}.$$

Now since n divided by m can give but one quotient, we infer that, *an equation of the first degree has but one root;* that is, in an equation of the first degree, involving but one unknown quantity, there is but *one* value that will verify the equation.

VI. EXAMPLES INVOLVING THE SECOND POWER OF THE UNKNOWN QUANTITY.

ART. **171.** It sometimes happens in the solution of an equation of the first degree, that the second or some higher power of the unknown quantity occurs, but in such a manner that it is easily removed, after which the equation may be solved in the usual manner.

The following equations and problems belong to this calss.

1. $(4+x)(x-5)=(x-2)^2$.

Performing the operations indicated, we have

$$x^2-x-20=x^2-4x+4.$$

Omitting x^2 on each side and transposing, we have

$$3x=24, \text{ or } x=8.$$

2. $\frac{(2x+3)x}{2x+1}+\frac{1}{3x}=x+1$. *Ans.* $x=1$.

3. $\frac{4x}{5-x}-\frac{20-4x}{x}=\frac{15}{x}$. *Ans.* $x=3\frac{2}{11}$.

4. $\frac{3x^2-2x+1}{5}=\frac{(7x-2)(3x-6)}{35}+\frac{9}{10}$. *Ans.* $x=1\frac{5}{68}$.

5. $\frac{3+2x}{1+2x}-\frac{5+2x}{7+2x}=1-\frac{4x^2-2}{7+16x+4x^2}$. *Ans.* $x=\frac{7}{8}$.

6. $\frac{4x+3}{6x-43}(3x-19)=2x+19$. *Ans.* $x=8$.

7. $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$. *Ans.* $x=3$.

8. $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$. *Ans.* $x=\frac{b}{c}$.

9. $\frac{cx^m}{a+bx}=\frac{dx^m}{e+fx}$. *Ans.* $x=\frac{ad-ce}{cf-bd}$.

10. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$. *Ans.* $x=\frac{ac}{b}$.

11. It is required to find a number which being divided into two, and into three equal parts, four times the product of the two equal parts, shall be equal to the continued product of the three equal parts. *Ans.* 27.

12. A rectangular floor is of a certain size. If it were 5 feet broader and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 feet more. Required its length and breadth.

Ans. Length, 12 feet; breadth, 9 feet.

CHAPTER VI.

FORMATION OF POWERS—EXTRACTION OF ROOTS—RADICALS—INEQUALITIES.

I. INVOLUTION OR FORMATION OF POWERS.

ART. **172.** *The power of a number* is the product obtained by multiplying it a certain number of times by itself.

Any number is the *first power* of itself.

When the number is taken *twice* as a factor, the product is called the *second power* or *square* of the number.

When the number is taken *three times* as a factor, the product is called the *third power* or *cube* of the number.

In like manner, the *fourth, fifth,* &c., *powers* of a number, are the products arising from taking the number *four* times, *five* times, &c., as a factor, the power being always denoted by the number of times the number is taken as a factor.

The number which denotes the power is called the *index* or *exponent* of the power, and is written to the right of the number and a little above it.

Thus, $3=3^1=3$, is the 1st power of 3.

$3\times3=3^2=9$, " " 2nd " " 3.

$3\times3\times3=3^3=27$, " " 3rd " " 3.

$3\times3\times3\times3=3^4=81$, " " 4th " " 3.

$3\times3\times3\times3\times3=3^5=243$, " " 5th " " 3.

$\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}=(\frac{3}{5})^4=\frac{81}{625}$, " " 4th " " $\frac{3}{5}$.

From the preceding we see, that *the* nth *power of a quantity is the product of* n *factors each equal to the quantity.* Hence we have the following

RULE FOR RAISING A QUANTITY TO ANY REQUIRED POWER. — *Multiply the given quantity by itself, until it is taken as a factor as many times as there are units in the exponent of the required power.*

REMARK. — This rule is perfectly general, and applies either to monomials or polynomials, whether integral or fractional.

EXAMPLES FOR PRACTICE.

1. Find the square of $5ax^2z^3$. *Ans.* $25a^2x^4z^6$.
2. Find the square of $-3b^2cd$. *Ans.* $9b^4c^2d^2$.
3. Find the cube of $2x^2z$. *Ans.* $8x^6z^3$.
4. Find the cube of $-3a^3c^2$. *Ans.* $-27a^9c^6$.
5. Find the fourth power of $-2xz^2$. *Ans.* $16x^4z^8$.
6. Find the fifth power of $-3a^2b^3$. *Ans.* $-243a^{10}b^{15}$.
7. Find the seventh power of $-m^2n$. *Ans.* $-m^{14}n^7$.
8. Find the square and the cube of $\frac{2}{5}a^3x^{m+2}y^{p-1}$.

   (1) *Ans.* $\frac{4}{25}a^6x^{2m+4}y^{2p-2}$. (2) $\frac{8}{125}a^9x^{3m+6}y^{3p-3}$.

9. Find the square of $a-x$. *Ans.* $a^2-2ax+x^2$.
10. Find the square of $mx-nx^2$. *Ans.* $m^2x^2-2mnx^3+n^2x^4$.
11. Find the cube of $2x-z$. *Ans.* $8x^3-12x^2z+6xz^2-z^3$.
12. Find the cube of $3x+2y$.

    Ans. $27x^3+54x^2y+36xy^2+8y^3$.
13. Find the fourth power of $m-n$.

    Ans. $m^4-4m^3n+6m^2n^2-4mn^3+n^4$.
14. Find the square of $a+b-c$.

    Ans. $a^2+2ab+b^2-2ac-2bc+c^2$.
15. Find the cube of $a-b+c$.

    Ans. $a^3-3a^2b+3ab^2-b^3+3a^2c-6abc+3b^2c+3ac^2-3bc^2+c^3$.
16. Find the square of $a+b-c+d$.

    Ans. $a^2+2ab+b^2-2ac-2bc+c^2+2ad+2bd-2cd+d^2$.
17. Find the square of $\frac{ax}{bz^2}$. *Ans.* $\frac{a^2x^2}{b^2z^4}$.
18. Find the square of $\frac{a-x}{a+x}$. *Ans.* $\frac{a^2-2ax+x^2}{a^2+2ax+x^2}$.
19. Find the cube of $\frac{ab}{cd^2}$. *Ans.* $\frac{a^3b^3}{c^3d^6}$.
20. Find the cube of $\frac{2a^2}{3c^3}$. *Ans.* $\frac{8a^6}{27c^9}$.

21. Find the cube of $\frac{m-n}{m-2n}$. *Ans.* $\frac{m^3-3m^2n+3mn^2-n^3}{m^3-6m^2n+12mn^2-8n^3}$.

22. Find the square of $\frac{2}{5}a-\frac{1}{2}b$. *Ans.* $\frac{4}{25}a^2-\frac{2}{5}ab+\frac{1}{4}b^2$.

23. Find the cube of $\frac{1}{2}a-\frac{2}{3}b$. *Ans.* $\frac{1}{8}a^3-\frac{8}{27}b^3-\frac{1}{2}a^2b+\frac{2}{3}ab^2$.

24. Find the square of $x-\frac{1}{x}-1$. *Ans.* $x^2-2x+\frac{1}{x^2}+\frac{2}{x}-1$.

25. Find the cube of $x-\frac{1}{x}$. *Ans.* $x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)$.

26. Find the cube of $x^{pq}-1$. *Ans.* $x^{3pq}-3x^{2pq}+3x^{pq}-1$.

27. Find the cube of e^x-e^{-x}. *Ans.* $e^{3x}-e^{-3x}-3(e^x-e^{-x})$.

28. If $x+\frac{1}{x}=p$, show that $x^3+\frac{1}{x^3}=p^3-3p$.

29. If two numbers differ by unity, prove that the difference of their squares is equal to the sum of the numbers.

30. Show that the sum of the cubes of any three consecutive integral numbers is divisible by the sum of those numbers.

NOTE.—For a more general method of raising a binomial to any required power, see the Binomial Theorem, Art. 310, page 264.

II. EXTRACTION OF THE SQUARE ROOT.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

ART. **173.** The *root* of a number, is a factor which multiplied by itself a certain number of times will produce the given number.

The *second root* or *square root* of a number, is that number which multiplied by itself, that is, taken *twice* as a factor, will produce the given number.

The process of finding the second root of a number, is called the *extraction of the square root.*

ART. **174.** To show the relation that exists between the number of figures in any given number, and the number of figures in its square root.

The first ten numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

and their squares are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

The numbers in the first line are also the square roots of the numbers in the second line.

We see, from this, that the square root of a number between 1 and 4, is a number between 1 and 2; the square root of a number between 4 and 9, is a number between 2 and 3; the square root of a number between 16 and 25, is a number between 4 and 5, and so on.

Since the square root of 1 is 1, and the square root of any number less than 100 is either one figure, or one figure and a fraction, it is evident that *when the number of places of figures in a number, is not more than* TWO, *the number of places of figures in the square root will be* ONE.

Again, take the numbers

10, 20, 30, 40, 50, 60, 70, 80, 90, 100;

their squares are

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000.

From this we see, that the square root of 100 is 10; and of any number greater than 100 and less than 10000, the square root will be less than 100, that is, it will consist of *two* places of figures; hence, *when the number of places of figures is more than* TWO, *and not more than* FOUR, *the number of places of figures in the square root will be* TWO.

In the same manner it may be shown, that when the number of places of figures in a given number is more than *four*, and not more than *six*, *the* number of places in the square root will be *three*, and so on.

Or, as the same principle may be expressed otherwise, thus:— When the number of places in the given number is either *one* or *two*, there will be *one* figure in the root; when the number of places is either *three* or *four*, there will be *two* figures in the root; when the number of places is either *five or six*, there will be *three* figures in the root, and so on.

ART. **175.** Investigation of a rule for extracting the square root.

Every number may be regarded as being composed of tens and units. Thus, 76 consists of 7 tens and 6 units; and 576 consists of 57 tens and 6 units. Therefore, if we represent the tens by t, and the units by u, any number will be represented by $t+u$, and its square by the square of $t+u$, or $(t+u)^2$.

$$(t+u)^2=t^2+2tu+u^2=t^2+(2t+u)u.$$

Hence, *the square of any number is composed of two quantities, one of which is the square of the tens, and the other twice the tens plus the units multiplied by the units.*

Thus, the square of 25, which is equal to 2 tens and 5 units, is

$$\begin{array}{rr} \text{2 tens squared} & =(20)^2=400 \\ \text{(4 tens + 5 units) multiplied by } 5 & =(40+5)5=\underline{225} \\ & 625 \end{array}$$

1. Let it now be required to extract the square root of 625.

Since the number consists of three places of figures, its root will consist of two places, according to the principle established in Art. 174; we, therefore, separate it into two periods, as in the margin.

```
         6̇2̇5|25
         400|
20×2=40 |225
      5 |225
     45 |
```

Since the square of 2 tens is 400, and of 3 tens is 900, it is evident that the greatest square contained in 600 is the square of 2 tens (20); the square of 2 tens (20) is 400; and subtracting this from 625, the remainder is 225.

Now, according to the preceding theorem, the remainder 225, consists of twice the tens plus the units, multiplied by the units; that is, by the formula, it is $(2t+u)u$, of which t is already found to be 2, and it remains to find u. Now the product of the tens by the units cannot give a product less than tens; therefore, the unit's figure (5) forms no part of the double product of the tens by the units. Hence, if we divide the remaining figures (22) by the double of the tens (4), the quotient will be the unit's figure, or a figure greater than it.

We, therefore, double the tens, which makes 4 ($2t$), and divide this into 22, which gives 5 (u) for a quotient; this is the unit's figure of the root. This unit's figure (5) is to be added to the double of the tens (40), and the sum multiplied by the unit's figure. The double of the tens plus the units is $40+5=45$ $(2t+u)$; multiplying this by 5 (u), the product is 225, which is the double of the tens plus the units, multiplied by the units. As there is nothing left after subtracting this from the first remainder, we conclude that 25 is the exact square root of 625.

In squaring the tens, and also in doubling them, it is customary to omit the ciphers, though they are understood. Also, the unit's figure is added to the double of the tens, by merely writing it in the unit's place. The actual operation is usually performed as in the margin.

```
    6̇2̇5|25
    400
 45|225
   |225
```

2. Let it be required to extract the square root of 59049. Since this number consists of five places of figures, its square root will consist of three places. (Art. 174.) We, therefore, separate it into *three* periods.

```
    5̇90̇49̇|243
    4
 44|190
   |176
483|1449
   |1449
```

In performing this operation, we find the square root of the number 590, on the same principle as in the preceding example.

We next consider 24 as so many tens, and proceed to find the unit's figure (3) in the same manner as in the preceding example.

From these illustrations, we derive the following

RULE FOR THE EXTRACTION OF THE SQUARE ROOT OF NUMBERS.—

1st. *Separate the given number into periods of two places each, beginning at the unit's place.* (The left period will often contain but one figure.)

2nd. *Find the greatest square in the left period, and place its root on the right, after the manner of a quotient in division. Subtract the square of the root from the left period, and to the remainder bring down the next period for a dividend.*

3rd. *Double the root already found, and place it on the left for a divisor. Find how many times the divisor is contained in the dividend, exclusive of the right hand figure, and place the figure in the root and also on the right of the divisor.*

4th. *Multiply the divisor thus increased by the last figure of the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

5th. *Double the whole root already found for a new divisor, and continue the operation as before, until all the periods are brought down.*

NOTE.—If, in any case, the dividend will not contain the divisor, the right hand figure of the former being omitted, place a cipher in the root and also at the right of the divisor, and bring down the next period.

ART. **176.** In division of numbers, when the remainder is greater than the divisor, the last quotient figure may be increased by at least 1; but in extracting the square root of numbers, the remainder may, sometimes, be greater than the divisor, while the last figure of the root cannot be increased. To know when any figure may be increased, the pupil must be acquainted with the relation that exists between the squares of two consecutive numbers.

Let a and $a+1$, be two consecutive numbers.

Thus, $(a+1)^2=a^2+2a+1$, is the square of the greater,

and $(a)^2=a^2$, " " " " " less.

Their difference is $2a+1$.

From which we see, that *the difference of the squares of two consecutive numbers, is equal to twice the less number, increased by unity.*

Consequently, when any remainder is less than twice the part of the root already found, plus unity, the last figure cannot be increased.

EXAMPLES,

In extracting the square root of whole numbers.

1. 2601.	*Ans.* 51.	7. 553536.	*Ans.* 744.
2. 7225.	*Ans.* 85.	8. 5764801.	*Ans.* 2401.
3. 9801.	*Ans.* 99.	9. 43046721.	*Ans.* 6561.
4. 47089.	*Ans.* 217.	10. 49042009.	*Ans.* 7003.
5. 138384.	*Ans.* 372.	11. 1061326084.	*Ans.* 32578.
6. 390625.	*Ans.* 625.	12. 943042681.	*Ans.* 30709.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

ART. **177.** Since $\frac{2}{5}\times\frac{2}{5}=\frac{4}{25}$, therefore, the square root of $\frac{4}{25}$ is $\frac{2}{5}$; that is $\sqrt{\frac{4}{25}}=\frac{\sqrt{4}}{\sqrt{25}}=\frac{2}{5}$. Hence,

When both terms of a fraction are perfect squares, its square root will be found by extracting the square root of both terms.

Before attempting to extract the square root of a fraction, it should be reduced to its lowest terms, unless both numerator and denominator are perfect squares. The reason for this will be seen by the following example.

Find the square root of $\frac{20}{45}$.

Here $\frac{20}{45}=\frac{4\times5}{9\times5}$. Now, neither 20 nor 45 are perfect squares; but, by canceling the common factor 5, the fraction becomes $\frac{4}{9}$, of which the square root is $\frac{2}{3}$.

When both terms are perfect squares, and contain a common factor, the reduction may be made either before, or after the square root is extracted.

Thus, $\sqrt{\frac{36}{81}}=\frac{6}{9}=\frac{2}{3}$; or, $\frac{36}{81}=\frac{4}{9}$, and $\sqrt{\frac{4}{9}}=\frac{2}{3}$.

EXAMPLES,

In extracting the square root of fractions.

1. $\frac{25}{49}$.	*Ans.* $\frac{5}{7}$.	4. $\frac{833}{2057}$.	*Ans.* $\frac{7}{11}$.
2. $\frac{64}{121}$.	*Ans.* $\frac{8}{11}$.	5. $\frac{9747}{10092}$.	*Ans.* $\frac{57}{58}$.
3. $\frac{225}{400}$.	*Ans.* $\frac{3}{4}$.	6. $\frac{56169}{1000000}$.	*Ans.* $\frac{237}{1000}$.

ART. **178.** A number whose square root can be ascertained exactly, is termed a *perfect square*. Thus, 4, 9, 16, &c., are perfect squares. Such numbers are comparatively few.

A number whose square root cannot be ascertained exactly, is termed an *imperfect square*. Thus, 2, 3, 5, 6, &c., are imperfect squares.

Since the difference of two consecutive square numbers, a^2 and a^2+2a+1, is $2a+1$; therefore, there are always $2a$ imperfect squares between them. Thus, between the square of 5 (25) and the square of 6 (36), there are 10 ($2a=2\times5$) imperfect squares.

A root which cannot be expressed exactly, is called a *radical*, or *surd*, or *irrational root*. The root obtained is also called an *approximate value*, or *approximate root*. Thus, $\sqrt{2}$ is an irrational root; it is 1.414+.

The sign + is sometimes placed after an approximate root, to denote that it is less than the true root; and the sign —, that it is greater than the true root.

ART. **179.** To prove that the square root of an imperfect square cannot be a fraction.

REMARK.—It might be supposed, that when the square root of a whole number cannot be expressed by a whole number, that it might be found exactly equal to some fraction. That it cannot, will now be shown.

Let c be an imperfect square, such as 2, and, *if possible*, let its square root be equal to a fraction $\frac{a}{b}$, which is supposed to be in its *lowest* terms.

Then $\sqrt{c}=\frac{a}{b}$; and $c=\frac{a^2}{b^2}$, by squaring both sides.

Now, by supposition, a and b have no common factor, therefore their squares, a^2 and b^2, can have no common factor, since to square a number, we merely repeat its factors. Consequently, $\frac{a^2}{b^2}$ must be in its lowest terms, and cannot be equal to a whole number. Therefore, the equation $c=\frac{a^2}{b^2}$, is not true; and hence the *supposition* on which it is founded is *false*, that is, the supposition that $\sqrt{c}=\frac{a}{b}$ is not *true;* therefore, the square root of an imperfect square cannot be a fraction.

APPROXIMATE SQUARE ROOTS.

ART. **180.** To explain the method of finding the approximate square root of an imperfect square, let it be required to find the square root of 5 to within $\frac{1}{3}$.

If we reduce 5 to a fraction whose denominator is 9 (the square of 3, the denominator of the fraction $\frac{1}{3}$), we have $5=\frac{45}{9}$.

Now the square root of 45 is greater than 6 and less than 7; therefore the square root of $\frac{45}{9}$ is greater than $\frac{6}{3}$, and less than $\frac{7}{3}$; hence $\frac{6}{3}$, or 2, is the square root of 5 to within $\frac{1}{3}$.

To generalize this explanation, let it be required to extract the square root of a to within a fraction $\frac{1}{n}$.

We may write a (Art. 127) under the form $\frac{an^2}{n^2}$, and if we denote the entire part of the square root of an^2 by r, the number an^2 will be comprised between r^2 and $(r+1)^2$; therefore $\frac{an^2}{n^2}$ will be comprised between $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$; hence the square root of $\frac{an^2}{n^2}$ will be comprised between $\frac{r}{n}$ and $\frac{r+1}{n}$.

But the difference between $\frac{r}{n}$ and $\frac{r+1}{n}$ is $\frac{1}{n}$, therefore $\frac{r}{n}$ represents the square root of a to within $\frac{1}{n}$. From this we derive the following

Rule for extracting the square root of a whole number to within a given fraction. — *Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation; extract the square root of this product to the nearest unit, and divide the result by the denominator of the fraction.*

EXAMPLES FOR PRACTICE.

1. Find the square root of 3 to within $\frac{1}{3}$. *Ans.* $1\frac{2}{3}$.
2. Find the square root of 10 to within $\frac{1}{4}$. *Ans.* 3.
3. Find the square root of 19 to within $\frac{1}{6}$. *Ans.* $4\frac{1}{3}$.
4. Find the square root of 30 to within $\frac{1}{10}$. *Ans.* 5.4.
5. Find the square root of 75 to within $\frac{1}{100}$. *Ans.* 8.66.

Since the square of 10 is 100, the square of 100, 10000, and so on, the number of ciphers in the square of the denominator of a decimal fraction, is equal to *twice* the number in the denominator itself. Therefore, *when the fraction which determines the degree of approximation is a decimal, it is merely necessary to add two ciphers for each decimal place required; and, after extracting the square root, to point off from the right one place of decimals for each two ciphers added.*

6. Find the square root of 3 to five places of decimals.
Ans. 1.73205.

7. Find the square root of 7 to five places of decimals.
Ans. 2.64575.

3. Find the square root of 50. *Ans.* 7.071067+.

9. Find the square root of 500. *Ans.* 22. 360679+.

ART. **181.** To find the approximate square root of a fraction.

1. Let it be required to find the square root of $\frac{4}{7}$ to within $\frac{1}{7}$.

$$\frac{4}{7}=\frac{4}{7}\times\frac{7}{7}=\frac{28}{49}.$$

Now, since the square root of 28 is greater than 5 and less than 6, the square root of $\frac{28}{49}$ is greater than $\frac{5}{7}$ and less than $\frac{6}{7}$; therefore $\frac{5}{7}$ is the square root of $\frac{4}{7}$ to within less than $\frac{1}{7}$.

From this it is evident, that *if we multiply the numerator of a fraction by its denominator, then extract the square root of the product to the nearest unit, and divide the result by the denominator, the quotient will be the square root of the fraction to within one of its equal parts.*

2. Find the square root of $\frac{7}{11}$ to within $\frac{1}{11}$. *Ans.* $\frac{8}{11}$.

3. Find the square root of $\frac{11}{16}$ to within $\frac{1}{16}$. *Ans.* $\frac{13}{16}$.

4. Find the square root of $\frac{17}{20}$ to within $\frac{1}{20}$. *Ans.* $\frac{9}{10}$.

It is obvious that any decimal may be written in the form of a common fraction, and having its denominator a perfect square, by adding ciphers to both terms. Thus $.3=\frac{3}{10}=\frac{30}{100}$; $.156=\frac{1560}{10000}$, and so on. Therefore, the square root of a decimal may be found, as in the method of finding the approximate square root of a whole number (Art. 180), by *annexing ciphers to the given decimal, until the number of decimal places shall be equal to double the number required in the root. Then, after extracting the root, pointing off from the right the required number of decimal places.*

5. Find the square root of .4 to six places. *Ans.* .632455+.

6. Find the square root of .35 to six places. *Ans.* .591607+.

The square root of a whole number and a decimal may be found in the same manner. Thus, the square root of 1.2 is the same as the square root of $1.20=\frac{120}{100}$, which, extracted to five places, is 1.09544+.

7. Find the square root of 7.532 to five places.
Ans. 2.74444+.

When the denominator of a fraction is a perfect square, its square root may be found by extracting the square root of the numerator to as many places of decimals as are required, and dividing the result by the square root of the denominator.

Or, by reducing the fraction to a decimal, and then extracting its square root. When the denominator of the fraction is not a perfect square, the latter method should be used.

8. Find the square root of $\frac{5}{16}$ to five places.

$\sqrt{5}=2.23606+$, $\sqrt{16}=4$, $\sqrt{\frac{5}{16}}=\frac{2.23606+}{4}=.55901+$.

Or, $\frac{5}{16}=.3125$, and $\sqrt{.3125}=.55901+$.

9. Find the square root of $\frac{3}{5}$. *Ans.* .774596+.
10. Find the square root of $1\frac{1}{4}$. *Ans.* 1.11803+.
11. Find the square root of $3\frac{5}{8}$. *Ans.* 1.903943+.
12. Find the square root of $11\frac{2}{9}$. *Ans.* 3.349958+.
13. Find the square root of $\frac{5}{12}$. *Ans.* 0.645497+.
14. Find the square root of $17\frac{3}{8}$. *Ans.* 4.168333+.

EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.

EXTRACTION OF THE SQUARE ROOT OF MONOMIALS.

ART. **182.** From Art. 172, it is evident that to square a monomial, we must square its coëfficient, and multiply the exponent of each letter by 2. Thus,

$$(3mn^2)^2=3mn^2\times3mn^2=9m^2n^4.$$

Therefore $\sqrt{9m^2n^4}=3mn^2$. Hence, we have the following

RULE FOR EXTRACTING THE SQUARE ROOT OF A MONOMIAL.—*Extract the square root of the coëfficient and divide the exponent of each letter by* 2.

Since $+a\times+a=+a^2$, $-a\times-a=+a^2$;
therefore, $\sqrt{a^2}=+a$, or $-a$.

Hence the square root of any positive quantity is either *plus*, or *minus*. This is generally expressed by writing the double sign before the square root. Thus, $\sqrt{4a^2}=\pm2a$; which is read *plus or minus* $2a$.

If a monomial is *negative*, the extraction of the square root is impossible, since the square of any quantity, either positive or negative, is necessarily positive. Thus $\sqrt{-4}$, $\sqrt{-a^2b^2}$, $\sqrt{-b}$, are algebraic symbols, which indicate impossible operations. Such expressions are termed *imaginary quantities*. An example of their occurrence always arises, in proceeding to find the value of the unknown quantity in an equation of the second degree, when some absurdity, or impossibility exists in the equation, or in the problem from which it was derived.

EXAMPLES FOR PRACTICE.

1. $16x^2y^4$. *Ans.* $\pm 4xy^2$.
2. $25m^2n^2$. *Ans.* $\pm 5mn$.
3. $36x^4z^6$. *Ans.* $\pm 6x^2z^3$.
4. $81a^2b^2c^4$. *Ans.* $\pm 9abc^2$.
5. $m^2x^4y^6z^8$. *Ans.* $\pm mx^2y^3z^4$.
6. $1024a^2b^6z^{10}$. *Ans.* $32ab^3z^5$.

Since $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$; therefore $\sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \pm\frac{a}{b}$;

Hence, *to find the square root of a monomial fraction, extract the square root of both terms.*

7. Find the square root of $\frac{a^2b^4}{c^2d^6}$. *Ans.* $\pm\frac{ab^2}{cd^3}$.

8. Find the square root of $\frac{4x^2y^2}{25a^2b^4}$. *Ans.* $\pm\frac{2xy}{5ab^2}$.

EXTRACTION OF THE SQUARE ROOT OF POLYNOMIALS.

ART. **183.** In order to deduce a rule for extracting the square root of polynomials, let us first find the relation that exists between the several terms of any quantity and its square.

$(a+b)^2 = a^2+2ab+b^2 = a^2+(2a+b)b$.

$(a+b+c)^2 = a^2+2ab+b^2+2ac+2bc+c^2 = a^2+(2a+b)b+(2a+2b+c)c$.

$(a+b+c+d)^2 = a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2 = a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d$.

Or, by calling the successive terms of a polynomial, r, r', r'', r''', and so on, we shall have $(r+r'+r''+r''')^2 = r^2+(2r+r')r'+(2r+2r'+r'')r''+(2r+2r'+2r''+r''')r'''$, where the law is manifest.

In this formula, r, r', r'', r''', may represent any algebraic quantities whatever, either integral or fractional, positive or negative.

This formula gives the following *law:*

The square of any polynomial is equal to the square of the first term — plus twice the first term, plus the second, multiplied by the second — plus twice the first and second terms, plus the third, multiplied by the third — plus twice the first, second, and third terms, plus the fourth, multiplied by the fourth, and so on. Hence, by reversing the operation, we have the following

RULE FOR EXTRACTING THE SQUARE ROOT OF A POLYNOMIAL.

1st. *Arrange the polynomial with reference to a certain letter; then find the first term of the root by extracting the square root of the first term of the polynomial; place the result on the right, and subtract its square from the given quantity.*

2nd. *Divide the first term of the remainder by double the part of the root already found, and annex the result to both the root and the divisor. Multiply the divisor thus increased by the second term of the root, and subtract the product from the remainder.*

3rd. *Double the terms of the root already found for a partial divisor, then divide the first term of the remainder by the first term of the divisor, and annex the result to both the root and the partial divisor. Multiply the divisor thus increased by the third term of the root, and subtract the product from the last remainder. Then proceed in a similar manner to find the other terms.*

REMARK. — If in the course of the operations in any example, we find a remainder of which the first term is not exactly divisible by double the first term of the root, we may conclude that the polynomial is not a perfect square.

1. Find the square root of $4x^2y^2+12x^2y+9x^2-30xy^2-20xy^3+25y^4$.

Arranging the polynomials with reference to y, we have

$$\begin{array}{lll}
 & & \text{ROOT.} \\
 & 25y^4-20xy^3+4x^2y^2-30xy^2+12x^2y+9x^2 & \underline{|5y^2-2xy-3x} \\
 & 25y^4 & \\
10y^2-2xy & |-20xy^3+4x^2y^2 & \\
 & |-20xy^3+4x^2y^2 & \\
10y^2-4xy-3x & |-30xy^2+12x^2y+9x^2 & \\
 & |-30xy^2+12x^2y+9x^2 &
\end{array}$$

The square root of the first term is $5y^2$, which we write as the first term of the root. We next subtract the square of $5y^2$ from the given polynomial, and then divide the first term of the remainder, $-20xy^3$, by $10y^2$, which gives $-2xy$, the second term of the root. We then place $-2xy$ in the root and also in the divisor, and multiply the divisor, thus increased, by $-2xy$, and subtract the product from the first remainder. We then double $5y^2-2xy$, the terms of the root already found, for a partial divisor, and divide the first term, $10y^2$ of the divisor, into $-30xy^2$, the first term of the remainder, which gives $-3x$ for the third term of the root. Completing the division, multiplying by $-3x$, and subtracting, we find there is nothing left.

NOTE. — The first remainder consists of all the terms after the first subtraction, and the second, of all the terms after the second subtraction. It is useless to bring down more terms than have corresponding terms in the quantity to be subtracted.

If the preceding example be arranged according to the powers of x, the root found will be $3x+2xy-5y^2$. This is correct, also as may be shown generally, thus,

$$\sqrt{(\text{polynomial})^2}=\pm(\text{polynomial}).$$

Thus, $\sqrt{(a^2+2ax+x^2)}=\sqrt{(a+x)^2}=\pm(a+x)=a+x$ or $-a-x$.

EXAMPLES FOR PRACTICE.

2. $x^2+6ax+9a^2$. *Ans.* $x+3a$.
3. $16x^2-40xy+25y^2$. *Ans.* $4x-5y$.
4. $4x^2z^2-12xyz+9y^2$. *Ans.* $2xz-3y$.
5. $49a^{4m-6}-42a^{6m-2}+9a^{8m+2}$. *Ans.* $7a^{2m-3}-3a^{4m+1}$.
6. $1+2x+7x^2+6x^3+9x^4$. *Ans.* $1+x+3x^2$.
7. $9a^4-12a^3b+34a^2b^2-20ab^3+25b^4$. *Ans.* $3a^2-2ab+5b^2$.
8. $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.
   Ans. x^3+2x^2+3x+4.
9. $9x^2-6xy+30xz+6xt+y^2-10yz-2yt+25z^2+10zt+t^2$.
   Ans. $3x-y+5z+t$.
10. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. *Ans.* $x^2-x+\frac{1}{4}$.
11. $\frac{25a^2b^2}{4}-\frac{5abc^2}{3}+\frac{c^4}{9}$. *Ans.* $\frac{5ab}{2}-\frac{c^2}{3}$.
12. $x^2+\frac{2}{3}ax+\frac{a^2}{9}-bx-\frac{1}{3}ab+\frac{b^2}{4}$. *Ans.* $x+\frac{a}{3}-\frac{b}{2}$.
13. $\frac{1051x^2}{25}-\frac{6x}{5}-\frac{14x^3}{5}+9+49x^4$. *Ans.* $7x^2-\frac{x}{5}+3$.
14. $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$. *Ans.* $\frac{a}{b}-\frac{b}{a}$.
15. $\frac{a^2}{b^2}+\frac{2a}{b}+\frac{b^2}{a^2}+\frac{2b}{a}+3$. *Ans.* $\frac{a}{b}+\frac{b}{a}+1$.
16. Reduce the following expression to its simplest form, and extract the square root.
    $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$. *Ans.* a^2+b^2.
17. Find the square root of $1-x^2$ to five terms.
    Ans. $1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-$ &c.
18. Find the first five terms of the square roots of x^2+a^2, and of x^2-2ax.
    Ans. $x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-\frac{5a^8}{128x^7}+$ &c.
    and $x-a-\frac{a^2}{2x}-\frac{a^3}{2x^2}-\frac{5a^4}{8x^3}-$ &c.

ART. **184.** The following remarks will be found useful.

1st. *No binomial can be a perfect square;* for the square of a monomial is a monomial, and the square of a binomial is a trinomial. Thus a^2+b^2 is not a perfect square, but if we add to it, or subtract from it, $2ab$, it becomes the square of $a+b$ or of $a-b$.

2nd. In order that a trinomial may be a perfect square, the two extreme terms must be perfect squares, and the middle term the double product of the square roots of the extreme terms. Hence, to find the square root of a trinomial when it is a perfect square, *extract the square roots of the extreme terms, and unite them by the sign plus or minus, according as the second term is plus or minus.* Thus, $a^{2m}-4a^{m+n}+4a^{2n}$ is a perfect square, since $\sqrt{a^{2m}}=a^m$, $\sqrt{4a^{2n}}=2a^n$, and $+a^m\times-2a^n\times2=-4a^{m+n}$. But $4x^2+8xy+9y^2$ is not a perfect square, since $\sqrt{4x^2}=2x$, $\sqrt{9y^2}=3y$, and $2x\times3y\times2=12xy$, which is not equal to the middle term $8xy$.

III. EXTRACTION OF THE CUBE ROOT.

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

ART. **185.** The *cube* or *third* power of a number, is the product arising from taking it *three* times as a factor. (Art. 172.) The *cube root*, or *third root* is one of three equal factors into which it may be resolved; hence, to extract the cube root of a number, is to find a number which, taken *three* times as a factor, will produce the given number.

ART. **186.** To show the relation that exists between the number of figures in any given number, and the number of figures in its cube root.

The first ten numbers and their cubes are:

roots,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10;
cubes,	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000.

The numbers in the second line are the cubes of those in the first; and, reciprocally, the numbers in the first line are the cube roots of those in the second. We see from this that the cube of a number consisting of one place of figures, does not exceed three places.

Again, comparing the numbers 10 and 100 with their cubes, we have,

numbers,	10,	100;
cubes,	$\dot{1}00\dot{0}$,	$\dot{1}00\dot{0}00\dot{0}$.

Since the cube of 10 is $\dot{1}00\dot{0}$, and the cube of 99, which is less than 100, is less than $\dot{1}00\dot{0}00\dot{0}$; therefore, the cube of a number consisting of two places of figures, has more than *three* places and not more than *six* places of figures.

Again, since the cube of 100 is $\dot{1}00\dot{0}00\dot{0}$, and the cube of 1000 is $\dot{1}00\dot{0}00\dot{0}00\dot{0}$; therefore, the cube of a number consisting of three places of figures has more than *six* places, and not more than *nine* places of figures. Therefore, if we begin at the unit's place of a number, and separate it into periods of three places each, the number of periods will show the number of places of figures in the root. The left period will often contain only one or two figures.

ART. **187.** To investigate a rule for the extraction of the cube root.

The first step in this investigation is to show the relation that exists between any number composed of units and tens, and its cube.

Let $t=$ the tens and $u=$ the units of a given number.

Then $t+u=$ the number,

and $(t+u)^3=$ the cube of the number.

But $(t+u)^3=t^3+3t^2u+3tu^2+u^3=t^3+(3t^2+3tu+u^2)u$.

Hence, *the cube of any number consisting of tens and units, is equal to the cube of the tens, — plus three times the square of the tens, plus three times the product of the tens and units, plus the square of the units, all three multiplied by the units.*

With this principle, let us proceed to extract the cube root of 13824.

$$
\begin{array}{rr|l}
 & 1\dot{3}82\dot{4} & \underline{24}\ (tu) \\
 & 8 & \\
\hline
3t^2= & 1200 & 5824 \\
3tu= & 240 & \\
u^2= & 16 & \\
\hline
 & 1456 & 5824 \\
\end{array}
$$

We commence, by separating the number into periods, by placing points over the figures in unit's and thousand's places; and as there are two periods, there will be two figures in the root. We find the greatest cube in 13 (thousand), which is 8 (thousand); the cube root of this is 2 (t); and its cube, 8 (thousand), corresponds to t^3 in the formula.

We then subtract this from the given number, and find a remainder 5824, which corresponds to $(3t^2+3tu+u^2)u$ in the formula. The first term, $3t^2$, of this formula, is sometimes termed the trial divisor, as it is used to find the unit's figure u.

If the remaining terms were only $3t^2u$, we could readily find u by dividing by $3t^2$, but if we divide by $3t^2$, we may obtain a figure too large, on account of omitting the terms $3tu+u^2$, of which u is as yet unknown. But if we first obtain a figure too large, at a second trial we must take one that is less.

Since the square of tens is hundreds, therefore, in using three times the square of the ten's figure as a trial divisor, we must omit the figures (24) in the unit's and ten's places of the dividend.

In this case we find 12 is contained in 58 four times. This gives 4 (u) for the required unit's figure, and we now proceed to complete the divisor by first adding to $3t^2$, three times the product of the tens by the units ($3tu$), and writing the product in ten's place, since the product of tens by units gives a product of tens. We next write the square of the unit's figure (u^2), and then taking the sum, find the complete divisor 1456, which corresponds to $3t^2+3tu+u^2$. Multiply this by 4 (u) the product is 5824, which subtracted from the first remainder leaves zero (0), and shows that 24 is the exact cube root required.

In cubing the tens, it is customary to omit the ciphers; but in taking three times the square of the tens, also in taking three times the product of the tens by the units, it is best to write ciphers in the vacant orders.

2. Let it be required to find the cube root of 44361864.

	$\dot{4}4\dot{3}61\dot{8}64$	$h\,t\,u$ 354
	27	
$3h^2=2700$	17361	
$3ht=\ \ 450$		
$t^2=\ \ \ 25$		
3175	15875	
$3(h+t)^2=367500$	1486864	
$3(h+t)u=\ \ 4200$		
$u^2=\ \ \ 16$		
371716	1486864	

After separating the number into periods, we find the cube root (35) of 44361 on the same principles as in the preceding example. Then considering 35 ($10h+t$) as so many tens, we find the unit's figure (4), as in the preceding example.

In dividing by the trial divisor 27, to find the second figure (5), we first obtain 6, but as this is found by trial to be too large, we take a less number.

From the preceding we derive the following

RULE FOR THE EXTRACTION OF THE CUBE ROOT OF NUMBERS.—

1st. *Separate the given number into periods of three places each, beginning at the unit's place.* (The left period will often contain but one or two figures.)

2nd. *Find the greatest cube in the left period, and place its root on the right, as in division. Subtract the cube of the root from the left period, and to the remainder bring down the next period for a dividend.*

3rd. *Square the root already found, and multiply it by 3 for a trial divisor. Find how many times this divisor is contained in the dividend, omitting the unit's and ten's figures, and write the result in the root. Add together, the trial divisor with two ciphers annexed; three times the product of the last figure of the root by the rest, with one cipher annexed; and the square of the last figure; the sum will be the complete divisor.*

4th. *Multiply the complete divisor by the last figure of the root, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend, and so proceed until all the periods are brought down.*

Extract the cube root of the following numbers.

3. 12167. *Ans.* 23.
4. 39304. *Ans.* 34.
5. 493039. *Ans.* 79.
6. 2097152. *Ans.* 128.
7. 14348907. *Ans.* 243.
8. 43614208. *Ans.* 352.
9. 127263527. *Ans.* 503.
10. 403583419. *Ans.* 739.
11. 1883652875. *Ans.* 1235.
12. 158252632929. *Ans.* 5409.

Since a fraction is cubed by cubing its numerator and denominator, therefore, the cube root of a fraction may be found by extracting the cube root of both terms, the fraction being in its *lowest terms* before commencing the operation, for reasons similar to those given in Art. 177.

13. Find the cube root of $\frac{64}{125}$. *Ans.* $\frac{4}{5}$.

14. Find the cube root of $\frac{216}{2744}$. *Ans.* $\frac{3}{7}$.

ART. 188. A number whose cube root can be exactly ascertained is a *perfect cube*. Thus, 8, 27, 64, &c., are perfect cubes. These numbers, like perfect squares, are comparatively few.

A number whose cube root cannot be exactly ascertained is termed an *imperfect cube*. Thus, 2, 3, 4, &c., are imperfect cubes.

It may be shown, by a course of reasoning precisely similar to that employed in Art. 179, that *the cube root of an imperfect cube cannot be a fraction.*

APPROXIMATE CUBE ROOTS.

ART. **189.** To illustrate the method of finding the approximate cube root of an imperfect cube, let it be required to find the cube root of 6 to within $\frac{1}{4}$.

Reducing 6 to a fraction whose denominator is 64 (the cube of 4 the denominator of the fraction $\frac{1}{4}$), we have $6=\frac{384}{64}$.

Now the cube root of 384 is greater than 7 and less than 8; therefore the cube root of $\frac{384}{64}$ is greater than $\frac{7}{4}$ and less than $\frac{8}{4}$; hence $\frac{7}{4}$ is the cube root of 6 to within less than $\frac{1}{4}$.

To generalize this method, let it be required to extract the cube root of a number a, to within a fraction $\frac{1}{n}$.

$$a=\frac{a\times n^3}{1\times n^3}=\frac{an^3}{n^3}.$$

Let r be the root of the greatest cube contained in an^3; then $\frac{an^3}{n^3}$ is comprised between $\frac{r^3}{n^3}$ and $\frac{(r+1)^3}{n^3}$; hence its cube root is comprised between $\frac{r}{n}$ and $\frac{r+1}{n}$; and since the difference of these fractions is $\frac{1}{n}$, therefore $\frac{r}{n}$ is the cube root of a to within $\frac{1}{n}$.

From this we derive the following

RULE FOR EXTRACTING THE CUBE ROOT OF A WHOLE NUMBER TO WITHIN A GIVEN FRACTION. — *Multiply the given number by the cube of the denominator of the fraction which determines the degree of approximation; extract the cube root of this product to the nearest unit, and divide the result by the denominator of the fraction.*

2. Find the cube root of 5 to within $\frac{1}{5}$. *Ans.* $1\frac{3}{5}$.

3. Find the cube root of 10 to within $\frac{1}{8}$. *Ans.* $2\frac{1}{8}$.

Since the cube of 10 is 1000, the cube of 100, 1000000, and so on, the number of ciphers in the cube of the denominator of a decimal fraction, is equal to three times the number in the denominator itself. Therefore, *when the fraction which determines the degree of approximation is a decimal, it is merely necessary to add three ciphers for each decimal place required; and after extracting the root, to point off from the right one place of decimals for each three ciphers added.*

4. Find the cube root of 2 to five places. *Ans.* 1.25992.

5. Find the cube root of 9 to five places. *Ans.* 2.08008.

6. Find the cube root of 37 to six places. *Ans.* 3.332222.

By adding ciphers to both terms, any decimal may be written in the form of a fraction, having its denominator a perfect cube; thus, $.2=\frac{200}{1000}$, $.25=\frac{250}{1000}$, and so on. Therefore the cube root may be found, as in the preceding examples, *by annexing ciphers to the given decimal, until the number of decimal places shall be equal to three times the number of decimal places required in the root. Then, after extracting the root, pointing off from the right the required number of decimal places.*

7. Find the cube root of .4 to four places. *Ans.* .7368.
8. Find the cube root of .25 to five places. *Ans.* .62996.

The cube root of a whole number and a decimal may be found in the same manner. Thus, the cube root of 6.4, is the same as the cube root of $\frac{6400}{1000}$, which is 1.85663+.

9. Find the cube root of 12.5 to five places. *Ans.* 2.32079.
10. Find the cube root of 34.3 to six places. *Ans.* 3.249112.

To find the cube root of a fraction or a mixed number, reduce the fraction to a decimal, and then proceed as in the preceding examples.

11. Find the cube root of $\frac{5}{9}$. *Ans.* .82207+.
12. Find the cube root of $5\frac{104}{105}$. *Ans.* 1.816+.
13. Divide the cube root of $\frac{2515.456}{32768}$ by the square root of the square root of 8.3521. *Ans.* .25.
14. Add together the cube roots of .059319 and 4.173281; and multiply the sum by the square root of $105\frac{1}{16}$. *Ans.* 20.5.

EXTRACTION OF THE CUBE ROOT OF ALGEBRAIC QUANTITIES.

EXTRACTION OF THE CUBE ROOT OF MONOMIALS.

ART. **190.** If we cube a monomial, for example, $2ax^2$, we have

$$(2ax^2)^3=2ax^2\times2ax^2\times2ax^2=2^3a^3x^{2\times3}=8a^3x^6.$$

That is, to cube a monomial, we must cube the coëfficient, and multiply the exponent of each letter by 3. Hence, by a converse operation we have the following

RULE FOR EXTRACTING THE CUBE ROOT OF A MONOMIAL.—*Extract the cube root of the coëfficient, and divide the exponent of each letter by* 3.

Find the cube root of each of the following monomials.

1. $8x^3z^6$. *Ans.* $2xz^2$.
2. $27x^6y^{15}$ *Ans.* $3x^2y^5$.
3. $-8a^6$. *Ans.* $-2a^2$.
4. $-64a^3m^6$. *Ans.* $-4am^2$.
5. $a^{3m+3c}x^6$. *Ans.* $a^{m+c}x^2$.
6. $-x^{3pq}z^6$. *Ans.* $-x^{pq}z^2$.

Since $\left(\frac{a}{b}\right)^3=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3}$; therefore, $\sqrt[3]{\frac{a^3}{b^3}}=\frac{a}{b}$.

Hence, *to find the cube root of a monomial fraction, extract the cube root of both terms.*

7. Find the cube root of $\frac{8a^3}{27x^6}$. *Ans.* $\frac{2a}{3x^2}$.

8. Find the cube root of $-\frac{64x^3y^6}{125m^3n^9}$. *Ans.* $-\frac{4xy^2}{5mn^3}$.

EXTRACTION OF THE CUBE ROOT OF POLYNOMIALS.

ART. **191.** To investigate a rule for extracting the cube root of polynomials.

Let us first examine the relation that exists between a polynomial and its cube.

$(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+(3a^2+3ab+b^2)b.$

$(a+b+c)^3=\{(a+b)+c\}^3=(a+b)^3+\{3(a+b)^2+3(a+b)c+c^2\}c.$

$(a+b+c+d)^3=\{(a+b+c)+d\}^3=(a+b+c)^3+\{3(a+b+c)^2+3(a+b+c)d+d^2\}d.$

Hence, the cube of a polynomial is formed according to the following law:

The cube of a polynomial is equal to the cube of the first term — plus three times the square of the first term, plus three times the product of the first term by the second, plus the square of the second, all three multiplied by the second — plus three times the square of the first two terms, plus three times the product of the first two terms by the third, plus the square of the third, all three multiplied by the third, and so on.

From this law, by reversing the process, we derive the following

RULE FOR THE EXTRACTION OF THE CUBE ROOT OF A POLYNOMIAL.—1st. *Arrange the polynomial with reference to a certain letter. Extract the cube root of the first term, this will give the first term of the root, and subtract its cube from the given polynomial.*

2nd. *Take three times the square of the first term of the root, and call it a trial divisor for finding each of the remaining terms of the root. Find how often the trial divisor is contained in the first term of the remainder, this will give the second term of the root. Then form a complete divisor by adding together three times the square of the first term of the root, plus three times the product of the first term by the second, plus the square of the second. Multiply these by the* *second term of the root and subtract the product from the first remainder.*

3rd. *Again find how often the trial divisor is contained in the first term of the remainder, this will give the third term of the root. Then form a complete divisor as before, by adding together three times the square of the first and second terms, plus three times the product of the first and second terms by the third, plus the square of the third. Multiply these by the third term of the root and subtract the product from the last remainder.*

4th. *Continue this process till all the terms of the root are found.*

NOTE.— The remainder in each case, is all the terms left after each subtraction.

1. Find the cube root of $x^6-6x^5+12x^4+3a^2x^4-8x^3-12a^2x^3+12a^2x^2+3a^4x^2-6a^4x+a^6$.

$$x^6-6x^5+12x^4+3a^2x^4-8x^3-12a^2x^3+12a^2x^2+3a^4x^2-6a^4x+a^6 \;\;|\, x^2-2x+a^2$$

$$x^6$$

$$3x^4-6x^3+4x^2)\;-6x^5+12x^4-8x^3$$

$$-6x^5+12x^4-8x^3$$

$$3x^4-12x^3+12x^2+3a^2x^2-6a^2x+a^4)\;+3a^2x^4-12a^2x^3+12a^2x^2$$

$$+3a^4x^2-6a^4x+a^6.$$

$$+3a^2x^4-12a^2x^3+12a^2x^2$$

$$+3a^4x^2-6a^4x+a^6.$$

To bring the work within the page, the last remainder and subtrahend are each written in two lines.

We first extract the cube root of x^6, which gives x^2 for the first term of the required root. Then 3 times the square of this, $3(x^2)^2=3x^4$, constitutes the *trial divisor* for finding the remaining terms. To find the second term of the root we divide $3x^4$ into $-6x^5$, the first term of the remainder, which gives $-2x$, the second term of the root. We then form the complete divisor by adding together $3(x^2)^2+3(x\times-2x)+(-2x)^2=3x^4-6x^3+4x^2$. Multiplying this by the second term, $-2x$, and subtracting the product from the first remainder, the first term of the second remainder is $+3a^2x^4$, which, divided by the trial divisor, gives $+a^2$, for the third term of the root. We next find the complete divisor by adding together $3(x^2-2x)^2+3(x^2-2x)a^2+(a^2)^2=3x^4-12x^3+12x^2+3a^2x^2-6a^2x+a^4$. Multiplying this by a^2 and subtracting, there is no remainder; hence the root obtained is exact.

Find the cube root

2. Of $a^3+24a^2b+192ab^2+512b^3$. *Ans.* $a+8b$.

3. Of $8a^3-84a^2x+294ax^2-343x^3$. *Ans.* $2a-7x$.

4. Of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

Ans. a^2-2a+1.

5. Of $x^6-9x^5+39x^4-99x^3+156x^2-144x+64$.

Ans. x^2-3x+4.

6. Of $(a+1)^{6n}x^3-6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x-8c^3a^{3p}$

Ans. $(a+1)^{2n}x-2ca^p$.

7. Find the first three terms of the cube root of $1-x$.

Ans. $1-\frac{x}{3}-\frac{x^2}{9}-$&c.

IV. EXTRACTION OF THE FOURTH ROOT, SIXTH ROOT, NTH ROOT, &C.

Art. 192. The fourth root of a number is one of four equal factors, into which the number may be resolved; and in general, the n^{th} root of a number is one of the n equal factors into which the number may be resolved.

When the degree of the root to be extracted is a multiple of two or more numbers, as 4, 6, &c., *the root can be obtained by extracting the roots of more simple degrees.*

To explain this we remark, that

$$(a^3)^4=a^3\times a^3\times a^3\times a^3=a^{3+3+3+3}=a^{3\times 4}=a^{12}$$

and in general

$$(a^m)^n=a^m\times a^m\times a^m\times a^m \ldots = a^{m\times n}=a^{mn}.$$

Hence, *the* n^{th} *power of the* m^{th} *power of a number, is equal to the* mn^{th} *power of the number.*

Reciprocally, *the* mn^{th} *root of a number, is equal to the* n^{th} *root of the* m^{th} *root of that number;* that is

$$\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}.$$

To prove this, let $\sqrt[n]{\sqrt[m]{a}}=a'$;

raising both members to the n^{th} power, we have

$$\sqrt[m]{a}=a'^n;$$

and by raising both members of the last equation to the m^{th} power, $a=a'^{mn}$;

extracting the mn^{th} root of both members,

$$\sqrt[mn]{a}=a'.$$

But, by supposition $\sqrt[n]{\sqrt[m]{a}}=a'$;

therefore, $\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}$.

It may be proved similarly, that $\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}$.

From this it follows that $\sqrt[4]{a}=\sqrt{\sqrt{a}}$; and $\sqrt[6]{a}=\sqrt[3]{\sqrt{a}}$, or $\sqrt{\sqrt[3]{a}}$; in like manner $\sqrt[8]{a}=\sqrt{\sqrt{\sqrt{a}}}$, and so on.

EXAMPLES FOR PRACTICE.

1. Find the 4th root of 65536. *Ans.* 16.
2. Find the 4th root of 1500625. *Ans.* 35.
3. Find the 4th root of 13107,9601. *Ans.* 10·7.
4. Find the 6th root of 2985984. *Ans.* 12.
5. Find the 6th root of 11390625. *Ans.* 15.
6. Find the 8th root of 214358881. *Ans.* 11.
7. Find the 4th root of $81a^4x^8$. *Ans.* $3ax^2$.
8. Find the 4th root of $\frac{16x^4y^{12}}{625a^8c^4}$. *Ans.* $\frac{2xy^3}{5a^2c}$.
9. Find the 4th root of $a^4+4a^3bx+6a^2b^2x^2+4ab^3x^3+b^4x^4$. *Ans.* $a+bx$.
10. Find the 4th root of $\frac{x^4}{y^4}+\frac{4x^2}{y^2}+\frac{y^4}{x^4}+\frac{4y^2}{x^2}+6$. *Ans.* $\frac{x}{y}+\frac{y}{x}$.
11. Find the 4th root of $x^8-4x^6+10x^4-16x^2+19-\frac{16}{x^2}+\frac{10}{x^4}-\frac{4}{x^6}+\frac{1}{x^8}$. *Ans.* $x^2-1+\frac{1}{x^2}$.
12. Find the 6th root of $a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20$. *Ans.* $a-\frac{1}{a}$.

ART. **193.** It has been shown already (Arts. 182, 183,) that the square root of a monomial, or a polynomial, may be preceded either by the sign $+$, or $-$; we shall now explain the law in regard to the roots generally.

If we take the successive powers of $+a$, we have

$$+a, \quad +a^2, \quad +a^3, \quad +a^4, \ldots\ldots$$

the successive powers of $-a$, are

$$-a, \quad +a^2, \quad -a^3, \quad +a^4, \ldots +a^{2n}, \quad -a^{2n+1}.$$

From this we see that every *even* power is positive, and that an *odd* power has the same sign as the root.

In general, let n be any whole number, then every power of an even degree, as $2n$, may be considered as the n^{th} power of the square, that is, $a^{2n}=(a^2)^n$.

Hence, *every power of an even degree is essentially positive, whether the quantity itself be positive or negative.*

Thus, $(\pm 3a)^4=+81a^4$; $(\pm 2b^2)^6=+64b^{12}$.

Again, as every power of an odd degree $(2n+1)$ is the product of a power of an even degree, $2n$, by the first power, it follows, that *every power of an uneven degree of a monomial has the same sign as the monomial itself.*

Thus, $(+2a)^3=+8a^3$, $(-2a)^3=-8a^3$.
Hence, it is evident,

1st. That *every odd root of a monomial must have the same sign as the monomial itself.*

Thus, $\sqrt[3]{+8a^3}=+2a$, $\sqrt[3]{-8a^3}=-2a$, $\sqrt[5]{-32a^{10}}=-2a^2$.

2nd. That *an even root of a positive monomial may be either positive or negative.*

Thus, $\sqrt[4]{81a^4b^8}=\pm3ab^2$, $\sqrt[6]{64a^{12}}=\pm2a^2$.

3rd. That *every even root of a negative monomial is impossible;* since no quantity raised to a power of an even degree can give a negative result. Thus, $\sqrt[4]{-a^4}$, $\sqrt[6]{-b}$, $\sqrt[8]{-c}$, are symbols of operations which cannot be performed. They are imaginary expressions like $\sqrt{-a}$, $\sqrt{-b}$. (Art. 182.)

TO EXTRACT THE N^{th} ROOT OF A MONOMIAL.

ART. **194.** In raising any monomial to the n^{th} power according to the rule, Art. 172, it is obvious that the process consists in raising the numeral coëfficient to the n^{th} power, and multiplying the exponent of each letter by n, thus, $(2a^2b^4)^3=2a^2b^4\times2a^2b^4\times2a^2b^4=2^3a^{2\times3}b^{4\times3}=8a^6b^{12}$.

Hence, conversely, to find the n^{th} root of a monomial,

Extract the nth *root of the coëfficient, and divide the exponent of each letter by* n.

REMARK.—In the following examples, the pupil is expected to find the root of the numeral coëfficient by inspection, as we have given no rules for extracting the 5th, 7th, &c., roots of numbers. Indeed, in the present state of science such rules are useless, for when the operations are required they are readily performed by means of Logarithms.

1. Find the 5th root of $-32a^5x^{10}$. *Ans.* $-2ax^2$.
2. Find the 6th root of $729b^6c^{18}$. *Ans.* $\pm3bc^3$.
3. Find the 7th root of $128x^7y^{14}$. *Ans.* $2xy^2$.
4. Find the 8th root of $6561a^8b^{16}$. *Ans.* $\pm3ab^2$.
5. Find the 9th root of $-512x^9z^{18}$. *Ans.* $-2xz^2$.
6. Find the 10th root of $1024b^{10}z^{30}$. *Ans.* $\pm2bz^3$.
7. Find the m^{th} root of ab^mc^{mq}. *Ans.* $bc^q(\sqrt[m]{a})$.
8. Extract $\sqrt[n]{a^{4n}b^nc^{2n}}$. *Ans.* a^4bc^2.

V. RADICAL QUANTITIES.

NOTE.—These quantities are generally called *surds* by English writers; while the French more properly term them *radicals*, from the Latin word *radix*, a root, because they express the *roots* of quantities. The Germans also distinguish them by a synonymous term, *wurzel grössen*, (*root quantities*).

ART. **195.** A *rational quantity* is one either not affected by the radical sign, or of which the root indicated can be exactly ascertained; thus, 2, a, $\sqrt{4}$, and $\sqrt[3]{8}$ are rational quantities.

A *radical quantity* is one of which the root indicated cannot be exactly expressed in numbers; thus, $\sqrt{5}$ is a radical; its value is 2.23606797 nearly.

Radicals are frequently called *irrational quantities*, or *surds*.

ART. **196.** From Art. 193 it is evident that when a monomial is a perfect power of the n^{th} degree, its numeral coëfficient is a perfect power of that degree, and the exponent of each letter is divisible by n. Thus $4a^2$ is a perfect square, and $8a^6$ is a perfect cube; but $6a^3$ is not a perfect square, because 6 is not a perfect square, and 3 is not divisible by 2; also, $8a^4$ is not a perfect cube, for, although 8 is a perfect cube, the exponent 4 is not divisible by 3.

In extracting any root, when the exact division of the exponent cannot be performed, it may be indicated by writing the divisor under it in the form of a fraction. Thus, $\sqrt{a^3}$ may be written $a^{\frac{3}{2}}$, and $\sqrt[3]{a^4}$ may be written $a^{\frac{4}{3}}$; and in general the n^{th} root of the m^{th} root of any quantity, is expressed either by $\sqrt[n]{a^m}$, or $a^{\frac{m}{n}}$.

Since a is the same as a^1 (Art. 19), the square root of a may be expressed thus, $a^{\frac{1}{2}}$; the cube root thus, $a^{\frac{1}{3}}$; and the n^{th} root thus, $a^{\frac{1}{n}}$. Hence, the following expressions are to be considered equivalent:

$$\sqrt{a} \text{ and } a^{\frac{1}{2}},$$

$$\sqrt[3]{a} \text{ and } a^{\frac{1}{3}},$$

$$\sqrt[n]{a} \text{ and } a^{\frac{1}{n}}.$$

$$\text{Also, } \sqrt[3]{a^2} \text{ and } a^{\frac{2}{3}},$$

$$\sqrt[n]{a^m} \text{ and } a^{\frac{m}{n}}.$$

From this we see, that *the numerator of the fractional exponent denotes the power of the quantity, and the denominator the root of that power to be extracted.*

Art. 197. Theorem. *Any quantity affected with a fractional exponent, may be transferred from one term of a fraction to the other, if, at the same time, the sign of its exponent be changed.* This proposition has already been established (Art. 81) when the exponent is integral; we will now prove it when the exponent is fractional.

Let it be required to extract the cube root of $\frac{1}{a^2}$, and of its equivalent a^{-2}.

To extract the cube root of a fraction, we extract the cube root of each term (Art. 190), hence, $\sqrt[3]{\frac{1}{a^2}}=\frac{1}{a^{\frac{2}{3}}}$. But, to extract the cube root of a^{-2} we must divide the exponent -2 by 3 (Art. 194), hence,

$$\sqrt[3]{a^{-2}}=a^{-\frac{2}{3}}$$

$$\therefore \frac{1}{a^{\frac{2}{3}}}=a^{-\frac{2}{3}}.$$

Similarly, $\frac{1}{a^m}=a^{-m}$.

Extracting the n^{th} root of each side,

$$\frac{1}{a^{\frac{m}{n}}}=a^{-\frac{m}{n}};$$

which establishes the theorem.

Art. 198. The quantity which stands before the radical sign is called the *coëfficient* of the radical. Thus, in the expressions $a\sqrt{b}$, and $2\sqrt[3]{c}$, the quantities a and 2 are called coëfficients.

Radicals are said to be of the *same degree* when they have the same index; thus, $a^{\frac{2}{3}}$ and $5^{\frac{2}{3}}$, or $\sqrt[3]{a^2}$ and $\sqrt[3]{5^2}$, are of the same degree.

Similar radicals are those which have the same index, and the same quantity under the radical sign; thus, $a\sqrt{b}$ and $c\sqrt{b}$ are similar radicals; so, also, are $3\sqrt[3]{a^2}$ and $5\sqrt[3]{a^2}$.

REDUCTION OF RADICALS

Case I.—To reduce radicals to their most simple form.

Art. 199. Reduction of radicals consists in changing the form of the quantities, without altering their value. Reduction of radicals of the second degree is founded on the following principle:

The square root of the product of two or more factors is equal to the product of the square roots of those factors:

That is, $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$; which is thus proved;

$(\sqrt{ab})^2=ab.$

And $(\sqrt{a}\times\sqrt{b})^2=(\sqrt{a}\times\sqrt{b})\times(\sqrt{a}\times\sqrt{b})=\sqrt{a}\times\sqrt{a}\times\sqrt{b}\times\sqrt{b}=ab.$

Hence, $\sqrt{ab}$ and $\sqrt{a}\sqrt{b}$, are equal to each other; since the square of each is equal to ab.

By this principle, $\sqrt{36}=\sqrt{4\times9}=2\times3$; $\sqrt{144}=\sqrt{9\times16}=3\times4$.

Any radical of the second degree can be reduced to a more simple form when it can be separated into factors, one of which is a perfect square.

Thus, $\sqrt{8}=\sqrt{4\times2}=\sqrt{4}\times\sqrt{2}=2\sqrt{2}$.

$\sqrt{m^5n^3}=\sqrt{m^4n^2\times mn}=\sqrt{m^4n^2}\times\sqrt{mn}=m^2n\sqrt{mn}.$

$\sqrt{28a^3c^2}=\sqrt{4a^2c^2\times7a}=\sqrt{4a^2c^2}\times\sqrt{7a}=2ac\sqrt{7a}.$

Hence, we have the following

Rule for the reduction of a radical of the second degree to its simplest form. 1st. *Separate the quantity to be reduced, into two parts, one of which shall contain all the factors that are perfect squares, and the other the remaining factors.*

2nd. *Extract the square root of the part that is a perfect square, and prefix it as a coëfficient to the other part placed under the radical sign.*

To determine if any quantity contains a numeral factor that is a perfect square, ascertain if it is divisible by either of the perfect squares 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, &c. If not thus divisible, it contains no factor that is a perfect square, and the numeral factor cannot be reduced.

Reduce to their simplest forms, the radicals in each of the following

EXAMPLES FOR PRACTICE.

1. $\sqrt{12}$, $\sqrt{18}$, $\sqrt{45}$, $\sqrt{32}$, $\sqrt{50a^3}$, $\sqrt{72a^2b^3}$.

Ans. $2\sqrt{3}$, $3\sqrt{2}$, $3\sqrt{5}$, $4\sqrt{2}$, $5a\sqrt{2a}$, $6ab\sqrt{2b}$.

2. $\sqrt{245}$ $\sqrt{448}$, $\sqrt{810}$, $\sqrt{507b^3c^2}$, $\sqrt{1805a^4b^2}$.

Ars $7\sqrt{5}$, $8\sqrt{7}$, $9\sqrt{10}$, $13bc\sqrt{3b}$, $19a^2b\sqrt{5}$

In a similar manner polynomials may sometimes be simplified. Thus,

$$\sqrt{(3a^3-6a^2c+3ac^2)}=\sqrt{3a(a^2-2ac+c^2)}=(a-c)\sqrt{3a}.$$

3. $\sqrt{(a^3-a^2b)}$, $\sqrt{ax^2-6ax+9a}$, $\sqrt{(x^2-y^2)(x+y)}$.

Ans. $a\sqrt{(a-b)}$, $(x-3)\sqrt{a}$, $(x+y)\sqrt{(x-y)}$.

To reduce a fractional radical to its most simple form by the same principle: Render the denominator of the fraction a perfect square by multiplying or dividing both terms by the same quantity. Then separate the fraction into two factors, one of which is a perfect square. Extract the square root of this factor, and write it as a coëfficient to the other factor placed under the radical sign.

4. Reduce $\sqrt{\frac{4}{5}}$, and $\sqrt{\frac{a}{b}}$, to their simplest forms.

$$\sqrt{\tfrac{4}{5}}=\sqrt{\tfrac{4}{5}\times\tfrac{5}{5}}=\sqrt{\tfrac{20}{25}}=\sqrt{\tfrac{4}{25}\times 5}=\sqrt{\tfrac{4}{25}}\times\sqrt{5}=\tfrac{2}{5}\sqrt{5}.$$

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{a}{b}\times\frac{b}{b}}=\sqrt{\frac{ab}{b^2}}=\sqrt{\frac{1}{b^2}\times ab}=\sqrt{\frac{1}{b^2}}\times\sqrt{ab}=\frac{1}{b}\sqrt{ab}.$$

5. $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{3}{8}}$, $\sqrt{\frac{18}{25}}$, $6\sqrt{\frac{1}{12}}$, $30\sqrt{\frac{3}{10}}$, $18\sqrt{\frac{5}{72}}$.

Ans. $\frac{1}{2}\sqrt{2}$, $\frac{1}{4}\sqrt{6}$, $\frac{3}{5}\sqrt{2}$, $\sqrt{3}$, $3\sqrt{30}$, $\frac{3}{2}\sqrt{10}$.

6. $\sqrt{\frac{c^2}{b}}$, $\sqrt{\frac{3a}{5b}}$, $\sqrt{\frac{a^3x^2}{4c^2y}}$, $\left(\frac{3xy^3}{98z^4}\right)^{\frac{1}{2}}$.

Ans. $\frac{c}{b}\sqrt{b}$, $\frac{1}{5b}\sqrt{15ab}$, $\frac{ax}{2cy}\sqrt{ay}$, $\frac{y}{14z^2}(6xy)^{\frac{1}{2}}$.

Art. **200.** To reduce radicals of any degree to the most simple form.

The principle of Art. 199 may be generalized thus:

The n*th root of the product of two or more factors, is equal to the product of the* n*th roots of those factors.*

That is, $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$; which is thus proved:

$$(\sqrt[n]{ab})^n=ab;$$

and $(\sqrt[n]{a}\times\sqrt[n]{b})^n=(\sqrt[n]{a})^n\times(\sqrt[n]{b})^n=ab.$

Since the n^{th} powers of these expressions are equal, the quantities themselves are equal, that is,

$$\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}.$$

1. Reduce $\sqrt[3]{54}$ to its most simple form.

$$\sqrt[3]{54}=\sqrt[3]{27\times2}=\sqrt[3]{27}\times\sqrt[3]{2}=3\sqrt[3]{2}.$$

Similarly, $\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{2}{3}\times\frac{3}{3}\times\frac{3}{3}}=\sqrt[3]{\frac{18}{27}}=\sqrt[3]{\frac{1}{27}\times18}=\frac{1}{3}\sqrt[3]{18}.$

Hence, for the simplification of monomial radicals, we have the following

GENERAL RULE.—*Separate the quantity into two factors, one of which is a perfect power of the given degree; extract its root and prefix the result as a coëfficient to the other factor placed under the radical sign.*

Reduce the radicals in each of the following examples to the most simple form:

2. $\sqrt[3]{40}$, $\sqrt[3]{80a^5b^3}$, $\sqrt[3]{81c^4}$, $\sqrt[3]{128a^6c^5}$, $\sqrt[3]{162m^4n^5}$.

Ans. $2\sqrt[3]{5}$, $2ab\sqrt[3]{10a^2}$, $3c\sqrt[3]{3c}$, $4a^2c\sqrt[3]{2c^2}$, $3mn\sqrt[3]{6mn^2}$.

3. $\sqrt[3]{320}$, $\sqrt[3]{2808}$, $\sqrt[3]{a^5b^3}$, $\sqrt[4]{32}$, $\sqrt[4]{144}$, $\sqrt[5]{128}$.

Ans. $4\sqrt[3]{5}$, $6\sqrt[3]{13}$, $ab\sqrt[3]{a^2}$, $2\sqrt[4]{2}$, $2\sqrt[4]{9}$, $2\sqrt[5]{4}$.

4. $\sqrt[3]{\frac{1}{2}}$, $\sqrt[3]{\frac{3}{4}}$, $\sqrt[3]{\frac{1}{6}}$, $\sqrt[3]{\frac{5}{9}}$, $\sqrt[3]{\frac{7}{8}}$, $\sqrt[3]{\frac{9}{25}}$.

Ans. $\frac{1}{2}\sqrt[3]{4}$, $\frac{1}{2}\sqrt[3]{6}$, $\frac{1}{6}\sqrt[3]{36}$, $\frac{1}{3}\sqrt[3]{15}$, $\frac{1}{2}\sqrt[3]{7}$, $\frac{1}{5}\sqrt[3]{45}$.

5. $\sqrt[4]{162}$, $\sqrt[4]{768}$, $\sqrt[4]{1250}$, $\sqrt[4]{3888}$.

Ans. $3\sqrt[4]{2}$, $4\sqrt[4]{3}$, $5\sqrt[4]{2}$, $6\sqrt[4]{3}$.

6. $\sqrt[4]{64a^5}$, $\sqrt[4]{32a^5b^7}$, $\sqrt[4]{48m^8n^5}$, $\sqrt[4]{\frac{2}{3}}$.

Ans. $2a\sqrt[4]{4a}$, $2ab\sqrt[4]{2ab^3}$, $2m^2n\sqrt[4]{3n}$, $\frac{1}{3}\sqrt[4]{54}$.

7. $\sqrt[5]{64}$, $\sqrt[5]{729a^6}$, $\sqrt[6]{\frac{1}{2}}$, $\sqrt[6]{\frac{2}{3}}$, $\sqrt[5]{\frac{3}{4}}$.

Ans. $2\sqrt[5]{2}$, $3a\sqrt[5]{3a}$, $\frac{1}{2}\sqrt[6]{32}$, $\frac{1}{3}\sqrt[6]{486}$, $\frac{1}{4}\sqrt[5]{768}$.

ART. **201.** Since the mn^{th} root of a is equal to the m^{th} root of the n^{th} root, or the n^{th} root of the m^{th} root (Art. 192), therefore the mn^{th} root of any algebraic expression may be simplified when it is a complete power of the m^{th} or n^{th} degree.

Thus, $\sqrt[4]{9a^2}=\sqrt{\sqrt{9a^2}}=\sqrt{3a}$.

Also, $\sqrt[6]{a^2-2ab+b^2}=\sqrt[3]{\sqrt{a^2-2ab+b^2}}=\sqrt[3]{a-b}$.

In general, $\sqrt[mn]{a^n}=\sqrt[m]{\sqrt[n]{a^n}}=\sqrt[m]{a}$.

Reduce each of the following radicals to its most simple form:

1. $\sqrt[4]{36a^2c^2}$, $\sqrt[4]{81m^2n^4}$, $\sqrt[4]{25a^4b^2}$, $\sqrt[6]{4a^2}$.

Ans. $\sqrt{6ac}$, $3n\sqrt{m}$, $a\sqrt{5b}$, $\sqrt[3]{2a}$.

2. $\sqrt[6]{16a^2c^4}$, $\sqrt[6]{27a^3}$, $\sqrt[6]{125b^3}$, $\sqrt[6]{64a^3}$.

Ans. $\sqrt[3]{4ac^2}$, $\sqrt{3a}$, $\sqrt{5b}$, $2\sqrt{a}$.

Case II. To reduce a rational quantity to the form of a radical.

Art. **202.**—If we square a, and then extract the square root of the square, the result is evidently a; that is, $a=\sqrt{a^2}=a^{\frac{2}{2}}$.

In like manner the cube root of the cube of a, is a; that is,

$$a=\sqrt[3]{a^3}=a^{\frac{3}{3}}.$$

Generally, $a=\sqrt[m]{a^m}=a^{\frac{m}{m}}$.

Hence, to reduce a rational quantity to the form of a radical of any degree, we have the following

Rule.—*Raise the quantity to a power corresponding to the given root, and write it under the radical sign.*

EXAMPLES.

1. Reduce 6 to the form of the square root. *Ans.* $\sqrt{36}$.
2. Reduce 2 to the form of the cube root. *Ans.* $\sqrt[3]{8}$.
3. Reduce $3ax$ to the form of the square root. *Ans.* $\sqrt{9a^2x^2}$.
4. Reduce $-3a$ to the form of the cube root.

Ans. $\sqrt[3]{-27a^3}$, or $(-27a^3)^{\frac{1}{3}}$.

5. Reduce $m-n$ to the form of the square root.

Ans. $\sqrt{m^2-2mn+n^2}$.

By the same principle the coëfficient of a radical may be passed under the radical sign.

Thus, $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{12}$.

So, also, $a\sqrt{b}=\sqrt{a^2}\times\sqrt{b}=\sqrt{a^2b}$.

Generally $a\sqrt[m]{b}=\sqrt[m]{a^m}\times\sqrt[m]{b}=\sqrt[m]{a^mb}$.

6. Express $3\sqrt{6}$ entirely under the radical sign. *Ans.* $\sqrt{54}$.
7. Express $5\sqrt{7}$, and $a^2\sqrt{b}$, entirely under the radical sign.

Ans. $\sqrt{175}$, and $\sqrt{a^4b}$.

8. Pass the coëfficient of the quantity $2\sqrt[3]{5}$, under the radical sign. *Ans.* $\sqrt[3]{40}$.

CASE III.—To reduce radicals having different indices to equivalent radicals having a common index.

ART. **203.** Since $\sqrt[m]{a}=\sqrt[mn]{a^n}$ (Art. 192), or $a^{\frac{1}{m}}=a^{\frac{n}{mn}}$ (Art. 118); therefore, *we may multiply the index of a radical by any number, provided we elevate the quantity under the sign to a power of the same degree denoted by the radical.* This is really only multiplying both terms of the fractional exponent by the same number, which does not change its value. (Art. 118.)

Let it be required to reduce $\sqrt[3]{2a}$, and $\sqrt[4]{3b}$, or $(2a)^{\frac{1}{3}}$ and $(3b)^{\frac{1}{4}}$ to quantities of equal value, having the same index.

Reducing the fractional exponents to the same denominator, we have $\frac{1}{3}=\frac{4}{12}$, and $\frac{1}{4}=\frac{3}{12}$; hence, $(2a)^{\frac{1}{3}}=(2a)^{\frac{4}{12}}$, or $\sqrt[12]{(2a)^4}$, and $(3b)^{\frac{1}{4}}=(3b)^{\frac{3}{12}}$, or $\sqrt[12]{(3b)^3}$.

Hence, we have the following

RULE.—*Reduce the fractional exponents to a common denominator; then the numerator of each fraction will represent the power to which the corresponding quantity is to be raised, and the common denominator the index of the root to be extracted.*

EXAMPLES FOR PRACTICE.

1. Reduce $\sqrt{3}$ and $\sqrt[3]{2}$, or $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$, to a common index.

Ans. $\sqrt[6]{27}$ and $\sqrt[6]{4}$, or $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$.

2. Reduce $\sqrt[3]{5}$ and $\sqrt{4}$ to a common index.

Ans. $\sqrt[6]{25}$, and $\sqrt[6]{64}$.

3. Reduce a^2 and $b^{\frac{1}{2}}$ to a common index. *Ans.* $\sqrt{a^4}$ and $\sqrt{b}$.

4. Reduce $\sqrt[4]{a}$, $\sqrt[6]{5b}$, and $\sqrt[8]{6c}$, to a common index.

Ans. $\sqrt[24]{a^6}$, $\sqrt[24]{625b^4}$, and $\sqrt[24]{216c^3}$.

5. Reduce $\sqrt{a^3}$, $\sqrt[3]{a^2}$, and $\sqrt[4]{a^3}$ to a common index.

Ans. $\sqrt[12]{a^{18}}$, $\sqrt[12]{a^8}$, and $\sqrt[12]{a^9}$.

6. Reduce $3^{\frac{2}{3}}$, $2^{\frac{3}{4}}$, and $5^{\frac{1}{2}}$ to a common index.

Ans. $3^{\frac{8}{12}}$, $2^{\frac{9}{12}}$, and $5^{\frac{6}{12}}$, or $\sqrt[12]{6561}$, $\sqrt[12]{512}$, and $\sqrt[12]{15625}$

ADDITION AND SUBTRACTION OF RADICALS.

ART. **204.** Let it be required to find the sum of $3\sqrt[3]{a}$, and $5\sqrt[3]{a}$.

It is evident that 3 times and 5 times any quantity whatever, must make 8 times that quantity; therefore,

$$3\sqrt[3]{a}+5\sqrt[3]{a}=8\sqrt[3]{a}.$$

But, if it were required to find the sum of $3\sqrt{a}$ and $5\sqrt[3]{a}$, since the *square* root of a and the *cube* root of a are *different* quantities, we cannot add them together and call them by the *same* name. Therefore, we can only *indicate* their addition; thus,

$$3\sqrt{a}+5\sqrt[3]{a}.$$

From this we see that to add similar radicals we must find the sum of their coëfficients, and place it before the common radical, and that to add radicals which are not similar, they must be connected by their proper signs.

Radicals that are not similar may often be rendered similar; thus, $\sqrt{12}$ and $\sqrt{27}$ are equal to $2\sqrt{3}$ and $3\sqrt{3}$, and their sum is $5\sqrt{3}$.

It is evident that the subtraction of radicals may be performed in the same manner as addition, except that the signs of the subtrahend must be changed. (Art. 44.)

From the preceding we derive the following

RULE FOR THE ADDITION OF RADICALS.—1st. *Reduce the radicals to their simplest forms.*

2nd. *If the radicals are similar, find the sum of their coëfficients and prefix it to the common radical; but if they are not similar, connect them by their proper signs.*

RULE FOR THE SUBTRACTION OF RADICALS.—*Change the sign of the subtrahend and proceed as in addition of radicals.*

EXAMPLES FOR PRACTICE.

1. Find the sum of $\sqrt{448}$ and $\sqrt{112}$. *Ans.* $12\sqrt{7}$.

2. Find the sum of $\sqrt[3]{24}$ and $\sqrt[3]{81}$. *Ans.* $5\sqrt[3]{3}$.

3. Find the sum of $\sqrt[3]{48}$ and $\sqrt[3]{162}$. *Ans.* $5\sqrt[3]{6}$.

4. Find the sum of $\sqrt{18a^5b^3}$ and $\sqrt{50a^3b^3}$.

Ans. $(3a^2b+5ab)\sqrt{2ab}$.

5. Subtract $\sqrt{180}$ from $\sqrt{405}$. *Ans.* $3\sqrt{5}$.

6. Subtract $\sqrt[3]{40}$ from $\sqrt[3]{135}$. *Ans.* $\sqrt[3]{5}$.

Perform the operations indicated in each of the following examples:

7. $\sqrt{243}+\sqrt{27}+\sqrt{48}$. *Ans.* $16\sqrt{3}$.

8. $\sqrt{24}+\sqrt{54}-\sqrt{96}$. *Ans.* $\sqrt{6}$.

9. $\sqrt{128}-2\sqrt{50}+\sqrt{72}-\sqrt{18}$. *Ans.* $\sqrt{2}$.

10. $2\sqrt{8}-7\sqrt{18}+5\sqrt{72}-\sqrt{50}$. *Ans.* $8\sqrt{2}$.

11. $\sqrt{48ab^2}+b\sqrt{75a}+\sqrt{3a(a-9b)^2}$. *Ans.* $a\sqrt{3a}$.

12. $2\sqrt{\frac{5}{3}}+\frac{1}{6}\sqrt{60}+\sqrt{15}+\sqrt{\frac{3}{5}}$. *Ans.* $\frac{11}{5}\sqrt{15}$.

13. $\sqrt[3]{128}-\sqrt[3]{686}-\sqrt[3]{16}+4\sqrt[3]{250}$. *Ans.* $15\sqrt[3]{2}$.

14. $2\sqrt[3]{\frac{1}{4}}+8\sqrt[3]{\frac{1}{32}}$. *Ans.* $3\sqrt[3]{2}$.

15. $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2}-5\sqrt[3]{128}$. *Ans.* $8\sqrt[3]{2}$.

16. $6\sqrt[6]{4a^2}+2\sqrt[3]{2a}+\sqrt[9]{8a^3}$. *Ans.* $9\sqrt[3]{2a}$.

17. $3\sqrt{\frac{2}{3}}+7\sqrt{\frac{27}{50}}-\sqrt{54}$. *Ans.* $\frac{1}{10}\sqrt{6}$.

18. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$. *Ans.* $\frac{25}{2}\sqrt{3}$

19. $\sqrt[3]{40}-\frac{1}{2}\sqrt[3]{320}+\sqrt[3]{135}$. *Ans.* $3\sqrt[3]{5}$.

20. $\sqrt[4]{16}+\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}-7\sqrt[6]{9}$. *Ans.* 10.

21. $8(\frac{3}{4})^{\frac{1}{2}}+\frac{1}{2}\times12^{\frac{1}{2}}-\frac{4}{3}\times27^{\frac{1}{2}}-2(\frac{3}{16})^{\frac{1}{2}}$. *Ans.* $\frac{1}{2}\sqrt{3}$.

22. $b(8a^6b)^{\frac{1}{3}}+4a(a^3b^4)^{\frac{1}{3}}-(125a^6b^4)^{\frac{1}{3}}$. *Ans.* $a^2b^{\frac{4}{3}}$.

23. $\sqrt{\dfrac{ab^3}{c^2}}+\dfrac{1}{2c}\sqrt{(a^3b-4a^2b^2+4ab^3)}$. *Ans.* $\dfrac{a}{2c}\sqrt{ab}$.

MULTIPLICATION AND DIVISION OF RADICALS.

ART. **205.** The rule for the multiplication of radicals is founded on the principle (Art. 200) that *the product of the* n^{th} *root of two or more quantities, is equal to the* n^{th} *root of their product*; that is,

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}.$$

Hence, (Art. 53), $a\sqrt[n]{b}\times c\sqrt[n]{d}=a\times c\times\sqrt[n]{b}\times\sqrt[n]{d}=ac\sqrt[n]{bd}$.

The rule for division is founded on the principle that *the quotient of the* n[th] *roots of two quantities is equal to the* n[th] *root of their quotient*; that is,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}};\text{ which is thus proved.}$$

If we raise $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ to the n^{th} power, we have

$$\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n}=\frac{a}{b};$$

and if we raise $\sqrt[n]{\frac{a}{b}}$ to the n^{th} power, we have

$$\left(\sqrt[n]{\frac{a}{b}}\right)^n=\frac{a}{b}.$$

Since the same powers of the two quantities are equal, we infer that the quantities themselves are equal; that is,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}.$$

Hence, $ac\sqrt{bd}\div a\sqrt{b}=\frac{ac\sqrt{bd}}{a\sqrt{b}}=\frac{ac}{a}\sqrt{\frac{bd}{b}}=c\sqrt{d}$.

Therefore, we have the following

RULES FOR THE MULTIPLICATION AND DIVISION OF RADICALS.—

I. *If the radicals have different indices, reduce them to the same index.*

II. TO MULTIPLY.— *Multiply the coëfficients together for the coëfficient of the product, and also the parts under the radical for the radical part of the product.*

III. TO DIVIDE.— *Divide the coëfficient of the dividend by the coëfficient of the divisor for the coëfficient of the quotient, and the radical part of the dividend by the radical part of the divisor for the radical part of the quotient.*

EXAMPLES FOR PRACTICE.

1. Multiply $2\sqrt{ab}$ by $3a\sqrt{abc}$.

$$2\sqrt{ab}\times3a\sqrt{abc}=2\times3a\sqrt{ab\times abc}=6a\sqrt{a^2b^2c}=6a^2b\sqrt{c}.$$

2. Divide $4a\sqrt{ab}$ by $2\sqrt{ac}$.

$$\frac{4a\sqrt{ab}}{2\sqrt{ac}}=\frac{4a}{2}\sqrt{\frac{ab}{ac}}=2a\sqrt{\frac{b}{c}}=2a\sqrt{\frac{bc}{c^2}}=\frac{2a}{c}\sqrt{bc}.$$

3. Multiply $2\sqrt[3]{3}$ by $3\sqrt{2}$.

$$2\sqrt[3]{3}=2\sqrt[6]{3^2};\ 3\sqrt{2}=3\sqrt[6]{2^3}$$

$$2\sqrt[6]{3^2}\times3\sqrt[6]{2^3}=2\times3\sqrt[6]{3^2\times2^3}=6\sqrt[6]{72}.$$

4. Divide $6\sqrt{2}$ by $3\sqrt[3]{2}$.

$$6\sqrt{2}=6\sqrt[6]{2^3}=6\sqrt[6]{8}$$

$$3\sqrt[3]{2}=3\sqrt[6]{2^2}=3\sqrt[6]{4}.$$

$$\frac{6\sqrt[6]{8}}{3\sqrt[6]{4}}=\tfrac{6}{3}\sqrt[6]{\tfrac{8}{4}}=2\sqrt[6]{2}.$$

5. Multiply $3\sqrt{12}$ by $5\sqrt{18}$. *Ans.* $90\sqrt{6}$.

6. Multiply $4\sqrt[3]{12}$ by $3\sqrt[3]{4}$. *Ans.* $24\sqrt[3]{6}$.

7. Multiply $\frac{1}{3}\sqrt[3]{18}$ by $7\sqrt[3]{15}$. *Ans.* $7\sqrt[3]{10}$.

8. Multiply together $5\sqrt{3}$, $7\sqrt{\frac{8}{3}}$, and $\sqrt{2}$. *Ans.* 140.

9. Multiply $\sqrt{3}$ by $\sqrt[3]{2}$. *Ans.* $\sqrt[6]{108}$.

10. Multiply $3\sqrt[3]{b}$ by $4\sqrt[4]{a}$. *Ans.* $12\sqrt[12]{a^3b^4}$.

11. Multiply together $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$. *Ans.* $\sqrt[12]{648000}$.

12. Multiply $\sqrt{2}\times\sqrt[3]{3}$ by $\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{3}}$. *Ans.* $\sqrt[4]{2}$.

13. Multiply together $\sqrt[2n]{x}$, $\sqrt[n]{x^2}$, and $\sqrt[3n]{x^3}$. *Ans.* $\sqrt[2n]{x^7}$.

14. Divide $\sqrt{40}$ by $\sqrt{2}$. *Ans.* $2\sqrt{5}$.

15. Divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

16. Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$. *Ans.* 6.

17. Divide $70\sqrt[3]{9}$ by $7\sqrt[3]{18}$. *Ans.* $5\sqrt[3]{4}$.

18. Divide $\sqrt[6]{72}$ by $\sqrt{2}$. *Ans.* $\sqrt[3]{3}$.

19. Divide $4\sqrt[3]{9}$ by $2\sqrt{3}$. *Ans.* $2\sqrt[6]{3}$.

20. Divide $20\sqrt[6]{200}$ by $4\sqrt{2}$. *Ans.* $5\sqrt[3]{5}$.

21. Divide $\sqrt[6]{72}$ by $\sqrt[3]{3}$. *Ans.* $\sqrt{2}$.

22. Divide $\sqrt[3]{4}$ by $\frac{1}{2}\sqrt[3]{6}$. *Ans.* $\frac{2}{3}\sqrt[3]{18}$.

23. Divide $\sqrt[4]{\frac{b}{a}}$ by $\sqrt[4]{\frac{a}{b}}$. *Ans.* $\sqrt{\frac{b}{a}}$.

24. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$. *Ans.* $\frac{1}{10}$.

When one or more of several polynomial factors contains radicals, they may be multiplied together by observing the rule for the exponents in the case of monomial factors.

25. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$. *Ans.* $1-\sqrt{5}$.

26. Multiply $\sqrt{2}+1$ by $\sqrt{2}-1$. *Ans.* 1.

27. Multiply $11\sqrt{2}-4\sqrt{15}$ by $\sqrt{6}+\sqrt{5}$. *Ans.* $2\sqrt{3}-\sqrt{10}$.

28. Raise $\sqrt{2}+\sqrt{3}$ to the fourth power. *Ans.* $49+20\sqrt{6}$.

29. Multiply $3\sqrt{4+6\sqrt{2}}$ by $5\sqrt{2}$. *Ans.* $30\sqrt{2+3\sqrt{2}}$.

30. Multiply $\sqrt[3]{12+\sqrt{19}}$ by $\sqrt[3]{12-\sqrt{19}}$. *Ans.* 5.

31. Multiply $x^2-x\sqrt{2}+1$ by $x^2+x\sqrt{2}+1$. *Ans.* x^4+1.

32. $(x^2+1)(x^2-x\sqrt{3}+1)(x^2+x\sqrt{3}+1)$. *Ans.* x^6+1.

33. $(2\sqrt{8}+3\sqrt{5}-7\sqrt{2})(\sqrt{72}-5\sqrt{20}-2\sqrt{2})$.

Ans. $42\sqrt{10}-174$.

ART. **206.** To reduce a fraction whose denominator contains radicals, to an equivalent fraction having a rational denominator.

When the denominator of the fraction is a monomial, as $\frac{a}{\sqrt{b}}$, if we multiply both terms by $\sqrt{b}$, the denominator will be rational.

$$\text{Thus, } \frac{a}{\sqrt{b}}=\frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}}=\frac{a\sqrt{b}}{b}.$$

Again, if the denominator is $\sqrt[3]{a}$, if we multiply both terms by $\sqrt[3]{a^2}$, the denominator will become $\sqrt[3]{a}\times\sqrt[3]{a^2}=a$.

In like manner, if the denominator is $\sqrt[m]{a^n}$, it will become rational by multiplying it by $\sqrt[m]{a^{m-n}}$, since

$$\sqrt[m]{a^n}\times\sqrt[m]{a^{m-n}}=\sqrt[m]{a^n\times a^{m-n}}=\sqrt[m]{a^m}=a.$$

Therefore, *when the denominator of the fraction is a monomial, multiply both terms by such a factor as will render the exponent of the given radical equal to unity.*

Since the sum of two quantities, multiplied by their difference, is equal to the difference of their squares (Art. 80); if the fraction is of the form $\frac{a}{b+\sqrt{c}}$, and we multiply both terms by $b-\sqrt{c}$, the denominator will be rational.

$$\text{Thus, } \frac{a}{b+\sqrt{c}}=\frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})}=\frac{ab-a\sqrt{c}}{b^2-c}.$$

For the same reason, if the denominator is $b-\sqrt{c}$, the multiplier will be $b+\sqrt{c}$. If the denominator is $\sqrt{b}+\sqrt{c}$, the multiplier will be $\sqrt{b}-\sqrt{c}$; and if the denominator is $\sqrt{b}-\sqrt{c}$, the multiplier will be $\sqrt{b}+\sqrt{c}$.

If the denominator is of the form $\sqrt{a}+\sqrt{b}+\sqrt{c}$, it may be rendered rational by two successive multiplications. Thus, $(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})=(\sqrt{a}+\sqrt{b})^2-c=a+b-c+2\sqrt{ab}$.

Let $2d=a+b-c$, then $2d+2\sqrt{ab}$ will become rational if it be multiplied by $d-\sqrt{ab}$;

$\therefore$ the whole multiplier is $(\sqrt{a}+\sqrt{b}-\sqrt{c})\left(\frac{a+b-c}{2}-\sqrt{ab}\right)$.

Reduce the following fractions to equivalent ones having rational denominators.

1. $\frac{1}{\sqrt{3}}$. *Ans.* $\frac{\sqrt{3}}{3}=\frac{1}{3}\sqrt{3}$.
2. $\frac{\sqrt{3}}{\sqrt{6}}$. *Ans.* $\frac{\sqrt{18}}{6}=\frac{1}{2}\sqrt{2}$.
3. $\frac{2}{\sqrt[3]{3}}$. *Ans.* $\frac{2}{3}\sqrt[3]{9}$.
4. $\frac{6}{\sqrt[5]{4^3}}$. *Ans.* $\frac{3}{2}\sqrt[5]{16}$.
5. $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$. *Ans.* $4+\sqrt{2}$.
6. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. *Ans.* $5+2\sqrt{6}$.
7. $\frac{\sqrt{3}+1}{2-\sqrt{3}}$. *Ans.* $5+3\sqrt{3}$.

8. $\dfrac{1-\sqrt{5}}{3+\sqrt{5}}$. *Ans.* $2-\sqrt{5}$.

9. $\dfrac{3\sqrt{5}-2\sqrt{2}}{2\sqrt{5}-\sqrt{18}}$. *Ans.* $9+\frac{5}{2}\sqrt{10}$.

10. $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$. *Ans.* $\dfrac{\sqrt{30}+3\sqrt{2}+2\sqrt{3}}{12}$.

11. $\dfrac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}$. *Ans.* $\sqrt{6}+\sqrt{2}+\sqrt{5}$.

12. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$. *Ans.* $2x$.

13. $\dfrac{\sqrt{(x+a)}+\sqrt{(x-a)}}{\sqrt{(x+a)}-\sqrt{(x-a)}}$. *Ans.* $\dfrac{x+\sqrt{(x^2-a^2)}}{a}$.

14. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$. *Ans.* $2x^2$.

REMARK.—The utility of most of the preceding transformations consists in shortening the calculations necessary to find the numerical value of a fractional radical. Thus, if it be required to find the value of $\dfrac{2}{\sqrt{5}}$, we may divide 2 by the square root of 5. But $\dfrac{2}{\sqrt{5}}$ is equal to $\frac{2}{5}\sqrt{5}$, where it is merely necessary to extract the square root of 5 and take two-fifths of the result. A comparison of the two methods of operation will show that the latter is much shorter than the former.

Reduce each of the following fractions to its simplest form, and find the numerical value of the result.

15. $\dfrac{2}{\sqrt{5}}$, and $\dfrac{1}{\sqrt{2}}$. *Ans.* .894427+, and .707106+.

16. $\dfrac{1}{2+\sqrt{3}}$. *Ans.* .267949+.

17. $\dfrac{1+\sqrt{2}}{2-\sqrt{2}}$. *Ans.* 4.12132+.

18. $\dfrac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$. *Ans.* 15.745966+

POWERS OF RADICALS.

ART. **207.** Let it be required to raise $\sqrt[m]{a}$ to the n^{th} power.

By the rule for the multiplication of radicals (Art. 205), we have $(\sqrt[m]{a})^n=\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}$ . . . . to n factors, $=\sqrt[m]{a\times a\times a \ldots\ldots \text{ to } n \text{ factors}}=\sqrt[m]{a^n}$.

Hence, to raise a radical quantity to any power, we have the following

RULE.—*Raise the quantity under the radical to the given power, and affect the result with the primitive radical sign.*

$$\text{Thus, } (\sqrt[4]{4a^3})^2=\sqrt[4]{(4a^3)^2}=\sqrt[4]{16a^6}=2a\sqrt[4]{a^2}=2a\sqrt{a}.$$

If the quantity have a coëfficient, it must also be raised to the given power. Thus,

$$(2\sqrt[4]{3a})^3=2^3\times\sqrt[4]{(3a)^3}=8\sqrt[4]{27a^3}.$$

If the index of the radical is a multiple of the exponent of the power, the operation may be simplified. Thus,

$$(\sqrt[4]{2a})^2=(\sqrt{\sqrt{2a}})^2 \text{ (Art. 192)};$$

and since the operation of squaring removes the first radical, we have

$$(\sqrt[4]{2a})^2=\left(\sqrt{\sqrt{2a}}\right)^2=\sqrt{2a}.$$

In general, $(\sqrt[mn]{a})^n=\left(\sqrt[n]{\sqrt[m]{a}}\right)^n=\sqrt[m]{a}.$

Hence, *if the index of the radical is divisible by the exponent of the power, we may perform this division, and leave the quantity under the radical sign unchanged.*

EXAMPLES.

1. Raise $\sqrt[3]{2a}$ to the 4th power. *Ans.* $2a\sqrt[3]{2a}$.
2. Raise $3\sqrt[3]{2ab^2}$ to the 4th power. *Ans.* $162ab^2\sqrt[3]{2ab^2}$.
3. Raise $2\sqrt{xy^3}$ to the 5th power. *Ans.* $32x^2y^7\sqrt{xy}$.
4. Raise $\sqrt[4]{ac^2}$ to the 2nd power. *Ans.* $c\sqrt{a}$.
5. Raise $\sqrt{ac^2}$ to the 4th power. *Ans.* a^2c^4.
6. Raise $3\sqrt[3]{2a}$ to the 5th power. *Ans.* $486a\sqrt[3]{4a^2}$.
7. Raise $\sqrt[6]{3c^2}$ to the 3rd power. *Ans.* $c\sqrt{3}$.
8. Raise $\sqrt{x-y}$ to the 3rd power. *Ans.* $(x-y)\sqrt{x-y}$.

ROOTS OF RADICALS.

Art. 208. Since $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$ (Art. 192), therefore, to extract the roots of radicals, we have the following

Rule.—*Multiply the index of the radical by the index of the root to be extracted, and leave the quantity under the radical sign unchanged.*

Thus, the square root of $\sqrt[3]{2a}$ is $\sqrt{\sqrt[3]{2a}}=\sqrt[6]{2a}$.

If the radical has a coëfficient, its root must be extracted by the rule (Art. 194). Thus,

$$\sqrt{9a^2\sqrt[3]{3c}}=\sqrt{9a^2}\times\sqrt{\sqrt[3]{3c}}=3a\sqrt[6]{3c}.$$

If the quantity under the radical is a perfect power of the same degree as the root to be extracted, the process may be simplified. Thus,

$$\sqrt[3]{\sqrt[4]{8a^3}} \text{ is equal (Art. 192) to } \sqrt[4]{\sqrt[3]{8a^3}}=\sqrt[4]{2a}.$$

EXAMPLES.

1. Extract the cube root of $\sqrt{a^2b}$. *Ans.* $\sqrt[6]{a^2b}$
2. Extract the 4th root of $16a^8\sqrt[3]{2c}$. *Ans.* $2a^2\sqrt[12]{2c}$.
3. Extract the cube root of $\sqrt[4]{27a^3}$. *Ans.* $\sqrt[4]{3a}$.
4. Extract the square root of $\sqrt[3]{49a^2}$. *Ans.* $\sqrt[3]{7a}$.
5. Extract the cube root of $64\sqrt[4]{8a^6}$. *Ans.* $4\sqrt[4]{2a^2}$.
6. Extract the cube root of $(m+n)\sqrt{m+n}$. *Ans.* $\sqrt{m+n}$.

IMAGINARY, OR IMPOSSIBLE QUANTITIES.

Art. 209. An imaginary quantity (Arts. 182, 193,) is an even root of a negative quantity.

Thus, $\sqrt{-a}$, and $\sqrt[4]{-b^4}$, are imaginary quantities.

The rules for the multiplication and division of radicals (Art 205) require some modification when imaginary quantities are to be multiplied or divided.

Thus, by the rule (Art. 205), $\sqrt{-a}\times\sqrt{-a}=\sqrt{-a\times-a}=\sqrt{a^2}=\pm a$. But, since the square root of any quantity multiplied by the square root itself, must give the original quantity, therefore, $\sqrt{-a}\times\sqrt{-a}=-a$.

ART. **210.** *Every imaginary quantity may be resolved into two factors, one a real quantity, and the other the imaginary expression,* $\sqrt{-1}$; *or an expression containing it.*

This is evident if we consider that every *negative* quantity may be regarded as the product of two factors, one of which is -1. Thus, $-a=a\times-1$, $-b^2=b^2\times-1$, and so on.

$$\text{Hence, } \sqrt{-a}=\sqrt{a\times-1}=\sqrt{a}\times\sqrt{-1}.$$

$$\sqrt{-a^2}=\sqrt{a^2\times-1}=\sqrt{a^2}\times\sqrt{-1}=\pm a\sqrt{-1}.$$

$$\sqrt[4]{-a}=\sqrt[4]{a\times-1}=\sqrt[4]{a}\times\sqrt[4]{-1}=\pm\sqrt[4]{a}.\sqrt{\sqrt{-1}}.$$

Since the square root of any quantity, multiplied by the square root itself, must give the original quantity;

$$\text{therefore, } (\sqrt{-1})^2=\sqrt{-1}\times\sqrt{-1}=-1.$$

$$\text{also, } (\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=-1\sqrt{-1}=-\sqrt{-1}.$$

$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=+1.$$

Attention to this principle will render all the algebraic operations, with imaginary quantities, easily performed.

Thus, $\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}=\sqrt{ab}\times(\sqrt{-1})^2=-\sqrt{ab}$.

If it be required to find the product of $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$, the operation is performed as in the margin.

OPERATION.

$$\begin{array}{r} a+b\sqrt{-1} \\ a-b\sqrt{-1} \\ \hline a^2+ab\sqrt{-1} \\ -ab\sqrt{-1}+b^2 \\ \hline a^2+b^2. \end{array}$$

Since $(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2+b^2$; therefore, $a^2+b^2=(a+b\sqrt{-1})(a-b\sqrt{-1})$; hence, any binomial whose terms are positive, may be resolved into two factors, one of which is the sum and the other the difference of a real and an imaginary quantity. Thus,

$$m+n=(\sqrt{m}+\sqrt{n}\sqrt{-1})(\sqrt{m}-\sqrt{n}\sqrt{-1}).$$

EXAMPLES.

1. Find the sum and difference of $a+b\sqrt{-1}$, and $a-b\sqrt{-1}$.

   Ans. $2a$, and $2b\sqrt{-1}$.

2. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$. *Ans.* $-ab$.

3. Find the 3rd and 4th powers of $a\sqrt{-1}$.

   Ans. $-a^3\sqrt{-1}$, and a^4.

4. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$. *Ans.* $-6\sqrt{6}$.

5. Find the cube of $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$. *Ans.* 1.

6. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.

7. Simplify the fraction $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$. *Ans.* $\sqrt{-1}$.

8. Find the continued product of $x+a$, $x+a\sqrt{-1}$, $x-a$, and $x-a\sqrt{-1}$. *Ans.* x^4-a^4.

9. Of what number are $24+7\sqrt{-1}$, and $24-7\sqrt{-1}$, the imaginary factors? *Ans.* 625.

VI. THEORY OF FRACTIONAL EXPONENTS.

ART. **211.** The rules for the exponents in multiplication and division (Arts. 56 and 70), have been demonstrated, under the supposition that the exponents were *integral.* These rules, as well as those which relate to the formation of powers (Art. 172), and the extraction of roots (Art. 194), are equally applicable when the exponents are *fractional.*

Fractional exponents have their origin (Art. 196) in the extraction of roots, when the exponent of the power is not divisible by the index of the root. Thus, in extracting the n^{th} root of a^m, the operation requires that the exponent m should be divided by the index n. When m is divisible by n the exact root of a^m is obtained, but when m is not divisible by n, the operation is indicated by indicating the division of the exponents. Thus,

$$\sqrt[n]{a^m}=a^{\frac{m}{n}}.$$

As has been shown already (Art. 196), every radical quantity may be represented by the same quantity with a fractional exponent, the numerator of the exponent denoting the power of the given quantity, and the denominator the index of the required root.

$$\text{Thus, } \sqrt[3]{a^2}=a^{\frac{2}{3}},\ \sqrt[3]{a^4}=a^{\frac{4}{3}},\ \sqrt[n]{\frac{1}{a^m}}=\sqrt[n]{a^{-m}}=a^{-\frac{m}{n}}.$$

As a^p is called a to the p power, when p is a positive whole number; so, by analogy, $a^{\frac{2}{3}}$, $a^{\frac{4}{3}}$ and $a^{-\frac{m}{n}}$, are called respectively, a to the $\frac{2}{3}$ power, a to the $\frac{4}{3}$ power, and a to the minus $\frac{m}{n}$ power. But it would, perhaps, be more accurate to say, a exponent $\frac{2}{3}$, a exponent $\frac{4}{3}$, a exponent $-\frac{m}{n}$; and reserve the term *power* to denote the product arising from multiplying a quantity by itself one or more times (Art. 19).

MULTIPLICATION AND DIVISION OF QUANTITIES WITH FRACTIONAL EXPONENTS.

ART. **212.** It has been shown (Art. 56) that *the exponent of any letter in the product is equal to the sum of its exponents in the two factors.* It will now be shown that the same rule applies when the exponents are fractional.

1. Let it be required to multiply $a^{\frac{2}{3}}$ by $a^{\frac{4}{5}}$.

$$a^{\frac{2}{3}}=\sqrt[3]{a^2}=\sqrt[15]{a^{10}};\ a^{\frac{4}{5}}=\sqrt[5]{a^4}=\sqrt[15]{a^{12}},\ \text{(Art. 205.)}$$

$$a^{\frac{2}{3}}\times a^{\frac{4}{5}}=\sqrt[15]{a^{10}}\times\sqrt[15]{a^{12}}=\sqrt[15]{a^{10+12}}=\sqrt[15]{a^{22}}=a^{\frac{22}{15}}.$$

But this result is the same as that obtained by adding the exponents together. Thus,

$$a^{\frac{2}{3}}\times a^{\frac{4}{5}}=a^{\frac{2}{3}+\frac{4}{5}}=a^{\frac{10}{15}+\frac{12}{15}}=a^{\frac{22}{15}}.$$

2. Let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$.

$$a^{-\frac{3}{4}}=\sqrt[4]{\frac{1}{a^3}}=\sqrt[12]{\frac{1}{a^9}};\ a^{\frac{5}{6}}=\sqrt[6]{a^5}=\sqrt[12]{a^{10}}.$$

$$a^{-\frac{3}{4}}\times a^{\frac{5}{6}}=\sqrt[12]{\frac{1}{a^9}}\times\sqrt[12]{a^{10}}=\sqrt[12]{\frac{1}{a^9}\times a^{10}}=\sqrt[12]{a^{-9+10}}=\sqrt[12]{a}=a^{\frac{1}{12}}.$$

And in general, the product of $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$ is,

$$a^{-\frac{m}{n}}\times a^{\frac{p}{q}}=a^{-\frac{m}{n}+\frac{p}{q}}=a^{\frac{np-mq}{nq}}.$$

Hence, to multiply quantities affected with *fractional* exponents, *apply the rule given* (Art. 56) *in the case of entire exponents.*

ART. **213.** In the preceding article (212) it has been shown that when the exponents are *fractional*, the exponent of any letter in the product is equal to the *sum* of its exponents in the two factors; and since division is the reverse of multiplication, therefore the exponent of any letter in the quotient must be equal to the excess of its exponent in the dividend over that in the divisor.

That is, $a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-np}{nq}}$.

Perform the operations indicated in each of the following

EXAMPLES.

1. $a^{\frac{1}{2}} \times a^{\frac{2}{3}}$, and $a^{-\frac{1}{2}} \times a^{\frac{2}{3}}$. *Ans.* $a^{\frac{7}{6}}$, and $a^{\frac{1}{6}}$.

2. $a^{\frac{3}{4}}c^{-1} \times a^2c^{\frac{3}{5}}$. *Ans.* $a^{\frac{11}{4}}c^{-\frac{2}{5}}$.

3. $\left(\frac{ay}{x}\right)^{\frac{1}{2}} \times \left(\frac{bx}{y^2}\right)^{\frac{1}{3}} \times \left(\frac{y^2}{a^3b^2}\right)^{\frac{1}{6}}$. *Ans.* $\left(\frac{y}{x}\right)^{\frac{1}{6}}$.

4. $(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}})$. *Ans.* $a-b$.

5. $(x^{\frac{1}{4}}y + y^{\frac{2}{3}})(x^{\frac{1}{4}} - y^{-\frac{1}{3}})$. *Ans.* $x^{\frac{1}{2}}y - y^{\frac{1}{3}}$.

6. $(a+b)^{\frac{1}{m}} \times (a+b)^{\frac{1}{n}} \times (a-b)^{\frac{1}{m}} \times (a-b)^{\frac{1}{n}}$. *Ans.* $(a^2-b^2)^{\frac{m+n}{mn}}$.

7. $x^{\frac{2}{3}} \div x^{\frac{1}{4}}$, and $x^{\frac{3}{m}}y^n \div x^{\frac{2}{n}}y^m$. *Ans.* $x^{\frac{5}{12}}$, and $x^{\frac{3n-2m}{mn}}y^{n-m}$.

8. $(a^{\frac{3}{4}} - b^{\frac{3}{4}}) \div (a^{\frac{1}{4}} - b^{\frac{1}{4}})$. *Ans.* $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

9. $(a-b^2) \div (a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}})$. *Ans.* $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

POWERS AND ROOTS OF QUANTITIES WITH FRACTIONAL EXPONENTS.

ART. **214.** Since the m^{th} power of a quantity is the product of m factors, each equal to the quantity (Art. 172); therefo[re] to raise $a^{\frac{1}{n}}$ to the m^{th} power, we must find the product

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \ldots . \text{ to } m \text{ factors.}$$

Here it is evident the exponent $\frac{1}{n}$ must be taken m times, hence, $(a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$.

Therefore, to raise a quantity affected with a fractional exponent to any power, *multiply the exponent of the quantity by the exponent of the power.*

Thus, $(a^{\frac{1}{2}}b^{\frac{1}{3}})^4 = a^{\frac{4}{2}}b^{\frac{4}{3}} = a^2b^{\frac{4}{3}}$.

ART. **215.** We have just seen in the preceding article, that in finding the m^{th} power of any quantity, we must *multiply* the exponent of the quantity by the exponent of the power. Hence, conversely, in extracting the m^{th} root, we must *divide* the exponent of the quantity by the exponent of the root; that is,

Since $(a^{\frac{1}{n}})^m = a^{\frac{1}{n}\times m} = a^{\frac{m}{n}}$; therefore,

$$\sqrt[m]{a^{\frac{m}{n}}} = a^{\frac{m}{n}\div m} = a^{\frac{m}{n}\times\frac{1}{m}} = a^{\frac{1}{n}}.$$

From the preceding it is obvious that the rules in Arts. 172 and 194 apply, without any change, to quantities having fractional exponents.

EXAMPLES.

1. Raise $a^{\frac{1}{2}}b^{\frac{1}{3}}$ to the 4th power. *Ans.* $a^2b^{\frac{4}{3}}$.

2. Raise $-2x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$ to the 3rd, 4th, and 6th powers.

   Ans. $-8x^{\frac{3}{2}}yz^{\frac{3}{4}}$; $16x^2y^{\frac{4}{3}}z$; $64x^3y^2z^{\frac{3}{2}}$.

3. Find the square of $a-(ax-a^2)^{\frac{1}{2}}$. *Ans.* $ax-2a(ax-a^2)^{\frac{1}{2}}$.

4. Find the square of $\left(\frac{1+m}{2}\right)^{\frac{1}{2}}+\left(\frac{1-m}{2}\right)^{\frac{1}{2}}$.

   Ans. $1+(1-m^2)^{\frac{1}{2}}$.

5. Find the cube of $a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x$.

   Ans. $ax^{-3}+3a^{\frac{1}{3}}x^{-1}+3a^{-\frac{1}{3}}x+a^{-1}x^3$.

6. Find the square roots of $3(5)^{\frac{1}{3}}$; and $\dfrac{7a^2(a)^{\frac{1}{3}}}{9(343b^2)^{\frac{1}{6}}}$.

   Ans. $(135)^{\frac{1}{6}}$; $\dfrac{7^{\frac{1}{4}}a^{\frac{7}{6}}}{3b^{\frac{1}{6}}}$.

7. Find the cube roots of $(27a^3x)^{\frac{1}{2}}$ and $(27a^3x)^{\frac{1}{3}}$.

   Ans. $3^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{1}{6}}$ or $(3ax^{\frac{1}{3}})^{\frac{1}{2}}$; and $(3ax^{\frac{1}{3}})^{\frac{1}{3}}$.

8. Find the square root of $5x^3-4x(5cx)^{\frac{1}{2}}+4c$.

Ans. $5^{\frac{1}{2}}x^{\frac{3}{2}}-2c^{\frac{1}{2}}$.

9. Find the square root of $1+\frac{41}{16}a-\frac{3+3a}{2}a^{\frac{1}{2}}+a^2$.

Ans. $1-\frac{3a^{\frac{1}{2}}}{4}+a$.

10. Find the cube root of $\frac{1}{8}a^3-\frac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}}$.

Ans. $\frac{1}{2}a-2b^{\frac{1}{2}}$.

Remark.—In solving examples 8, 9, and 10, the pupil is expected to combine the rules, Arts. 183 and 191, with those for fractional exponents.

VII. EQUATIONS CONTAINING RADICALS.

Note to Teachers.—This part of the subject of Equations of the First Degree could not be treated till after Radicals, as the operations necessarily involve the formation of powers and the multiplication of radicals.

Art. **216.** In the solution of questions containing radicals, the method to be pursued will often depend on the judgment of the pupil, as many of the questions can be solved in different ways, and the shortest processes can only be learned from *practice.*

1st. When the equation to be solved contains only one radical expression, transpose it to one side of the equation and the rational terms to the other; then involve both sides to a power corresponding to the radical sign.

Ex. Given, $\sqrt[3]{(a^3+x)}-a=c$, to find x.

Transposing, $\sqrt[3]{(a^3+x)}=c+a$;

Cubing, $a^3+x=c^3+3ac^2+3a^2c+a^3$;

Whence, $x=c^3+3ac^2+3a^2c$.

2nd. When a radical expression occurs under the radical sign, the operation of involution must be repeated.

Ex. Given $\sqrt{x-\sqrt{1-x}}=1-\sqrt{x}$, to find x.

Squaring, $x-\sqrt{1-x}=1-2\sqrt{x}+x$;

Canceling x on each side and squaring again,

$$1-x=1-4\sqrt{x}+4x.$$

Canceling 1 on each side, transposing, squaring, and reducing

We find, $x=\frac{16}{25}$.

3rd. When there are two or more radical expressions, it is generally preferable to make one of them stand alone, before performing the process of involution.

Ex. Given, $\sqrt{x+9}-\sqrt{x}=1$, to find x.

Transposing $-\sqrt{x}$, we have $\sqrt{x+9}=1+\sqrt{x}$.

Squaring each side, $x+9=1+2\sqrt{x}+x$;

Canceling x on each side, transposing and dividing by 2,

$\sqrt{x}=4$; hence, $x=16$.

In some cases, however, it is preferable, when an equation contains two radical expressions, to retain them both on the same side. Thus, the equation

$$\sqrt{\left(\frac{x+a}{x-a}\right)}+\sqrt{\left(\frac{x-a}{x+a}\right)}=b,$$

will be cleared of radicals at once, by squaring each side, the value of x being $\frac{ab}{\sqrt{b^2-4}}$.

EXAMPLES FOR PRACTICE.

4. $\sqrt{(x+5)}+3=8-\sqrt{x}$. *Ans.* $x=4$.

5. $\sqrt{1+\sqrt{(3+\sqrt{6x})}}=2$. *Ans.* $x=6$.

6. $\sqrt{x+a}=\sqrt{x}+a$. *Ans.* $x=\frac{(a-1)^2}{4}$.

7. $\sqrt{2x-3a}+\sqrt{2x}=3\sqrt{a}$. *Ans.* $x=2a$.

8. $\sqrt{\{13+\sqrt{[7+\sqrt{(3+\sqrt{x})}]}\}}=4$. *Ans.* $x=1$.

9. $\sqrt{2+x}+\sqrt{x}=\frac{4}{\sqrt{2+x}}$. *Ans.* $x=\frac{2}{3}$.

10. $\sqrt{a+x}+\sqrt{\frac{a}{x}}=\sqrt{x}$. *Ans* $x=\frac{a}{a+2\sqrt{a}}$.

11. $\sqrt{x+13}-\sqrt{x-11}=2.$ *Ans.* $x=36.$

12. $a\sqrt{x}+b\sqrt{x}-c\sqrt{x}=d.$ *Ans.* $x=\frac{d^2}{(a+b-c)^2}.$

13. $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}.$ *Ans.* $x=\frac{1}{1-a}.$

14. $x+a=\sqrt{a^2+x\sqrt{(b^2+x^2)}}.$ *Ans.* $x=\frac{b^2-4a^2}{4a}.$

15. $\frac{x-4}{\sqrt{x}+2}=5\sqrt{x}-8+\frac{3\sqrt{x}}{2}.$ *Ans.* $x=\frac{144}{121}.$

16. $\frac{x-a}{\sqrt{x}+\sqrt{a}}=\frac{\sqrt{x}-\sqrt{a}}{3}+2\sqrt{a}.$ *Ans.* $x=16a.$

17. $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}.$ *Ans.* $x=3.$

18. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}.$ *Ans.* $x=\frac{(b-a)^2}{2a-b}.$

19. $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}.$ *Ans.* $x=\frac{a(b+c)}{b-c}.$

20. $\frac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=c.$ *Ans.* $x=\frac{a(c-1)^2}{4c}.$

21. $\sqrt{\sqrt{x}+3}-\sqrt{\sqrt{x}-3}=\sqrt{2\sqrt{x}}.$ *Ans.* $x=9.$

22. $\frac{1}{x}+\frac{1}{a}=\sqrt{\left\{\frac{1}{a^2}+\sqrt{\left(\frac{1}{b^2x^2}+\frac{1}{x^4}\right)}\right\}}.$ *Ans.* $x=\frac{4ab^2}{a^2-4b^2}.$

23. $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a.$

Ans. $x=8.$

VIII. INEQUALITIES.

ART. **217.** In the discussion of problems it often becomes necessary to compare quantities that are *unequal*, and to operate upon them so as to determine the values of the unknown quantities, or to establish certain relations between them.

In most cases the methods of operating on equations apply to inequalities; still there are some exceptions, which render it expedient to present the principles and rules of operation in one view.

ART. **218.** DEF.—In the theory of inequalities, it is convenient to consider negative quantities less than zero. Also, in comparing two negative quantities, that is considered the least which contains the greatest number of units; thus, $0>-1$, and $-3>-5$.

Two inequalities are said to subsist in the *same* sense when the greater quantity stands on the right in both, or on the left in both; thus,

$$5>3, \quad \text{and} \quad 2<5,$$
$$7>4, \quad\quad\quad 3<8.$$

are said to subsist in the same sense.

Two inequalities are said to subsist in a *contrary* sense, when the greater stands on the *right* in one and on the *left* in the other; thus, $5>1$ and $4<8$ are inequalities which subsist in a contrary sense.

ART. **219.** PROP. I.—*The same quantity, or equal quantities, may be added to or subtracted from both members of an inequality, and the resulting inequality will continue in the same sense.*

Thus, $7>5$, and by adding 4 to each member,
$11>9$; or by subtracting 4 from each member,
$3>1$.
Also, $-5<-3$, and by adding 4 to each member,
$-1<+1$; or by subtracting 4 from each member,
$-9<-7$.
Similarly, if $a>b$, then
$a+c>b+c$, or $a-c>b-c$.

It follows from this proposition, that *any quantity may be transposed from one side of an inequality to the other, if at the same time its sign be changed.* Thus, if

$$a^2+b^2>2ab+c^2,$$
$$a^2+b^2-2ab>2ab-2ab+c^2,$$
$$\text{or } a^2-2ab+b^2>c^2.$$

ART. **220.** PROP. II.—*If two inequalities exist in the same sense, the corresponding members may be added together, and the resulting inequality will exist in the same sense.* Thus,

$$7>6, \text{ and } 5>4, \text{ and}$$
$$7+5>6+4, \text{ or } 12>10.$$

But when two inequalities exist in the same sense, if we *subtract* the corresponding members, the resulting inequality will sometimes exist in the *same* sense, at other times in a *contrary* sense.

First, $\begin{array}{r} 7>3 \\ \underline{4>1} \\ 3>2 \end{array}$ By subtracting, we find the resulting inequality exists in the *same* sense.

Second, $\begin{array}{r} 10>9 \\ \underline{8>3} \\ 2<6 \end{array}$ In this case, after subtracting, we find the resulting inequality exists in a *contrary* sense.

In general, if $a>b$ and $c>d$, then, according to the particular values of a, b, c, and d, we may have $a-c>b-d$, $a-c<b-d$, or $a-c=b-d$.

Art. **221.** Prop. III.— *If the two members of an inequality be multiplied or divided by a positive number, the resulting inequality will exist in the same sense.* Thus,

$$8>4 \text{ and } 8\times3>4\times3, \text{ or } 24>12.$$
$$\text{Also, } 8\div2>4\div2, \text{ or } 4>2.$$

This principle enables us to clear an inequality of fractions by multiplying both sides by the least common multiple of the denominators.

But, if the two members of an inequality be multiplied or divided by a negative number, the resulting inequality will exist in a contrary sense. Thus, $-3<-1$, but $-3\times-2>-1\times-2$, or $6>2$.

From this principle we derive

Art. **222.** Prop. IV.— *The signs of all the terms of both members of an inequality may be changed, if at the same time we establish the resulting inequality in a contrary sense, because this is the same as multiplying both members by* -1.

Art. **223.** Prop. V.— *Both members of a positive inequality may be raised to the same power, or have the same root extracted, and the resulting inequality will exist in the same sense.* Thus,

$$2<3 \text{ and } 2^2<3^2,\ 2^3<3^3; \text{ or } 4<9,\ 8<27; \text{ and so on.}$$

$$\text{Also, } 25>16, \text{ and } \sqrt{25}>\sqrt{16}, \text{ or } 5>4; \text{ and so on.}$$

But if the signs of both members of an inequality are not positive, after raising both members to the same power, or extracting the same root, the resulting inequality will sometimes exist in the same sense, and at others in a contrary sense.

Thus, $3>-2$, and $3^2>(-2)^2$, or $9>4$.
But, $-3<-2$, and $(-3)^2>(-2)^2$, or $9>4$.

EXAMPLES INVOLVING THE PRINCIPLES OF INEQUALITIES.

1. Five times a certain whole number added to 4 is greater than twice the number added to 19; and 5 times the number diminished by 4 is less than 4 times the number increased by 4 Required the number.

Let $x=$ the number.

Then, $5x+4>2x+19$, (1)

and $5x-4<4x+4$. (2)

$5x-2x>19-4$, from eq. (1) by transposing,

$3x>15$, by reducing,

$x>5$, by dividing both members by 3.

$5x-4x<4+4$, from eq. (2) by transposing,

$x<8$, by reducing.

Hence, the number is greater than 5 and less than 8, consequently either 6 or 7 will fulfill the conditions.

2. If $4x-7<2x+3$, and $3x+1>13-x$, find x.

Ans. $x=4$.

3. Find the limit of x in the equation $7x-3>32$.

Ans. $x>5$.

4. Find the limit of x in the equation $5+\frac{1}{3}x<8+\frac{1}{4}x$.

Ans. $x<36$.

5. Show that $\frac{a+c+e}{b+d+f}>$ the least, and $<$ the greatest of the fractions, $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, each letter representing a positive quantity.

Let G be a quantity greater, and g a quantity less than any of the fractions, $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$. Then,

$$\frac{a}{b}<G,\ \frac{c}{d}<G,\ \frac{e}{f}<G.$$

$$\frac{a}{b}>g,\ \frac{c}{d}>g,\ \frac{e}{f}>g.$$

$$\therefore\ a<b\text{G},\ c<d\text{G},\ e<f\text{G}.$$

$$a>bg,\ c>dg,\ e>fg.$$

$$\therefore\ a+c+e<(b+d+f)\text{G},$$

$$\text{and } a+c+e>(b+d+f)g.$$

$$\therefore\ \frac{a+c+e}{b+d+f}<\text{G and }>g.$$

6. It is required to prove that the sum of the squares of any two *unequal* magnitudes is always greater than twice their product.

Since the square of every quantity, whether positive or negative, is positive, it follows that

$$(a-b)^2,\text{ or } a^2-2ab+b^2>0\ ;$$

Adding, $+2ab$ to each side (Art. 219),

$$a^2-2ab+b^2+2ab>0+2ab,$$

or $a^2+b^2>2ab$, which was required to be proved.

Most of the inequalities usually met with, are made to depend ultimately upon this principle.

7. Which is greater, $\sqrt{5}+\sqrt{14}$ or $\sqrt{3}+3\sqrt{2}$?

Ans. the former.

8. Given $\frac{1}{4}(x+2)+\frac{1}{3}x<\frac{1}{2}(x-4)+3$ and $>\frac{1}{2}(x+1)+\frac{1}{3}$, to find x. *Ans.* $x=5$.

9. The double of a certain number increased by 7 is not greater than 19, and its triple diminished by 5 is not less than 13. Required the number. *Ans.* 6.

10. Show that n^3+1 is greater than n^2+n, unless $n=1$.

11. Show that every fraction $+$ the fraction inverted is greater than 2; that is, that $\frac{a}{b}+\frac{b}{a}>2$.

12. If $x>y$, show that $x-y>(\sqrt{x}-\sqrt{y})^2$.

13. Show that $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$, unless $a=b$.

14. Show that $a^2+b^2+c^2>ab+ac+bc$, unless $a=b=c$.

15. Show that the ratio of a^2+b^2 to a^3+b^3 is greater than the ratio of $a+b$ to a^2+b^2.

16. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, which is greater, xy, or $ac+bd$? *Ans.* xy.

17. Show that $abc>(a+b-c)(a+c-b)(b+c-a)$, unless $a=b=c$.

CHAPTER VII.

EQUATIONS OF THE SECOND DEGREE

ARTICLE **224.** An Equation of the Second Degree (see Art. 143) is one in which the greatest exponent of the unknown quantity is 2. Thus,

$x^2=a$, and

$ax^2+bx=c$, are equations of the second degree.

An equation containing two or more unknown quantities, in which the greatest exponent, or the greatest sum of the exponents of the unknown quantities in one term is 2, is also an equation of the second degree.

Thus, $xy=a$, $x^2+xy=b$, $xy-x-y=c$, are equations of the second degree.

Equations of the Second Degree are frequently termed *Quadratic Equations.*

ART. **225.** Equations of the second degree, containing only one unknown quantity, are divided into two classes; viz.: *incomplete,* and *complete.*

An incomplete equation of the second degree contains only the second power of the unknown quantity and known terms. Thus,

$$x^2+2=47-4x^2, \text{ and}$$

$$ax^2+b=cx^2-d,$$

are incomplete equations of the second degree.

A complete equation of the second degree contains the first as well as the second power of the unknown quantity, and known terms. Thus,

$$5x^2+7x=34, \text{ and}$$

$$ax^2-bx^2+cx-dx=e-f,$$

are complete equations of the second degree.

REMARK.— Incomplete equations are sometimes termed *Pure Quadratics;* and complete equations, *Affected,* or *Adfected Quadratics.*

ART. **226.** The general form of an incomplete equation of the second degree is $ax^2=b$.

The general form of a complete equation of the second degree is $ax^2+bx=c$.

Every equation of the second degree containing only one unknown quantity may be reduced to one of these forms. For, in the case of an incomplete equation, all the terms containing x^2 may be collected together, and then, if the coëfficient of x^2 contains more than one term, it may be assumed equal to a single quantity, as a, and the sum of the known quantities to another quantity, b; and the equation then becomes,

$$ax^2=b, \text{ or } ax^2-b=0.$$

A complete equation may be reduced in like manner; for, all the terms containing x^2 may be reduced to one term, as ax^2; and those containing x to one, as bx; and the known terms to one, as c; the equation then is,

$$ax^2+bx=c, \text{ or } ax^2+bx-c=0.$$

Hence, we infer, that *every equation of the second degree containing only one unknown quantity, may be reduced to an incomplete equation containing two terms, or to a complete equation containing three terms.*

Frequent illustrations of these principles will occur hereafter.

INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

ART. **227.** 1. Let it be required to find the value of x in the equation,

$$\tfrac{1}{3}x^2-3+\tfrac{5}{12}x^2=12\tfrac{3}{4}-x^2.$$

Clearing of fractions, $4x^2-36+5x^2=153-12x^2$;

Transposing and reducing, $21x^2=189$;

Dividing, $x^2=9$;

Extracting the square root of both members;

$$x=\pm3, \text{ that is, } x=+3, \text{ or } x=-3.$$

Verification. $\tfrac{1}{3}(+3)^2-3+\tfrac{5}{12}(+3)^2=12\tfrac{3}{4}-(+3)^2.$

$$\text{or, } 3-3+3\tfrac{3}{4}=12\tfrac{3}{4}-9;$$

$$3\tfrac{3}{4}=3\tfrac{3}{4}.$$

Since the square of -3 is the same as the square of $+3$, the value $x=-3$, will give the same result as $x=+3$.

2. Given $ax^2+b=d+cx^2$, to find the value of x.

Transposing, $ax^2-cx^2=d-b$;

Factoring, $(a-c)x^2=d-b$;

Dividing, $x^2=\dfrac{d-b}{a-c}$;

$$x=\pm\sqrt{\frac{d-b}{a-c}}.$$

From the preceding examples, we derive the following

RULE FOR THE SOLUTION OF AN INCOMPLETE EQUATION OF THE SECOND DEGREE.—*Reduce the equation to the form* $ax^2=b$. *Divide by the coëfficient of* x^2, *and extract the square root of both members.*

ART. **228.** If we solve the equation $ax^2=b$, we have,

$$x^2=\frac{b}{a};$$

$$\text{and } x=\pm\sqrt{\frac{b}{a}}; \text{ that is,}$$

$$x=+\sqrt{\frac{b}{a}}, \text{ and } x=-\sqrt{\frac{b}{a}}.$$

If we substitute each of these values of x in the equation $ax^2=b$, we find,

$$a\times\left(+\sqrt{\frac{b}{a}}\right)^2=b, \text{ or } a\times\frac{b}{a}=b;$$

$$\text{and } a\times\left(-\sqrt{\frac{b}{a}}\right)^2=b, \text{ or } a\times\frac{b}{a}=b.$$

Since each of these *roots* or *values* of x, verifies the equation, and since the square root of $\frac{b}{a}$ can *only* be $+\sqrt{\frac{b}{a}}$, or $-\sqrt{\frac{b}{a}}$, therefore we infer,

1st. *That every incomplete equation of the second degree has two roots, and only two.*

2nd. *That these roots are equal in value, but have contrary signs.*

NOTE.—Let the pupil recollect that the term *root*, in reference to an equation, is equivalent to the *value of the unknown quantity.*

EXAMPLES FOR PRACTICE.

1. $11x^2-44=5x^2+10$. *Ans.* $x=\pm3$.
2. $\frac{1}{3}(x^2-12)=\frac{1}{4}x^2-1$. *Ans.* $x=\pm6$.
3. $(x+2)^2=4x+5$. *Ans.* $x=\pm1$.
4. $\frac{3}{4}x^2-(2x^2-3)=\frac{16x^2+9}{5}$. *Ans.* $x=\pm\sqrt{\frac{24}{89}}$.
5. $8x+\frac{7}{x}=\frac{65x}{7}$. *Ans.* $x=\pm2\frac{1}{3}$.

6. $\frac{8}{1-2x}+\frac{8}{1+2x}=25.$ *Ans.* $x=\pm.3.$

7. $(5x+\frac{1}{2})^2=756\frac{1}{2}+5x.$ *Ans.* $x=\pm5\frac{1}{2}.$

8. $\frac{7x^2+8}{21}-\frac{x^2+4}{8x^2-11}=\frac{x^2}{3}.$ *Ans.* $x=\pm2.$

9. $\frac{x+7}{x^2-7x}-\frac{x-7}{x^2+7x}=\frac{7}{x^2-73}.$ *Ans.* $x=\pm9.$

10. $\frac{a}{b+x}+\frac{a}{b-x}=c.$ *Ans.* $x=\pm\frac{1}{c}\sqrt{b^2c^2-2abc}.$

11. $x\sqrt{6+x^2}=1+x^2.$ *Ans.* $x=\pm\frac{1}{2}.$

The methods of clearing an equation of radicals have been already explained in Art. 216.

12. $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}.$ *Ans.* $x=\pm\frac{a}{3}\sqrt{3}.$

13. $\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}.$ *Ans.* $\pm\sqrt{2ab-b^2}.$

14. $\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x.$ *Ans.* $x=\pm\sqrt{3}.$

15. $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b.$ *Ans.* $x=\pm\frac{2a\sqrt{b}}{b+1}.$

QUESTIONS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

Art. **229.** In the solution of problems producing equations of the second degree, the equation is found on the same principle as in questions producing equations of the first degree. See Art. 154.

1. What two numbers have the ratio of 2 to 5, and the sum of whose squares is 261?

Let $2x$ and $5x=$ the numbers.
Then, $4x^2+25x^2=29x^2=261$;
Whence, $x^2=9$ and $x=3$.
Hence, $2x=6$ and $5x=15$, the required numbers.

2. The square of a certain number diminished by 17, is equal to 130 diminished by twice the square of the number. Required the number. *Ans.* 7.

3. There is a certain number, which being subtracted from 10 and the remainder multiplied by the number itself, gives the same product as 10 times the remainder left after subtracting $6\frac{2}{5}$ from the number. Required the number. *Ans.* 8.

4. What number is that, the third part of whose square being subtracted from 30, leaves the same remainder as one-fourth of the square increased by 9? *Ans.* 6.

5. There are two numbers whose difference is $\frac{2}{9}$ths of the greater, and the difference of their squares is 128; find them.

Ans. 18 and 14.

6. Divide the number 21 into two such parts, that the square of the less shall be to that of the greater as 4 to 25.

Let x and $21-x=$ the parts.

Then, $x^2 : (21-x)^2 :: 4 : 25$;

or, (Arith., Art. 209,) $25x^2=4(21-x)^2$;

Extracting the square root of both sides,

$$5x=2(21-x);$$

Whence, $x=6$, and $21-x=15$.

7. Divide the number 14 into two such parts, that the quotient of the greater divided by the less, shall be to the quotient of the less divided by the greater, as 16 to 9. *Ans.* 6 and 8.

8. What number is that which being added to 20 and subtracted from 20, the product of the sum and difference shall be 319?

Ans. 9.

9. What two numbers are they, whose product is 126, and the quotient of the greater divided by the less, $3\frac{1}{2}$?

Ans. 6 and 21.

10. The product of two numbers is p, and their quotient q. Required the numbers. *Ans.* $\sqrt{pq}$ and $\sqrt{\frac{p}{q}}$.

11. The sum of the squares of two numbers is 370, and the difference of their squares 208. Required the numbers.

Ans. 9 and 17.

12. The sum of the squares of two numbers is c, and the difference of their squares, d. Required the numbers.

Ans. $\frac{1}{2}\sqrt{2(c+d)}$, and $\frac{1}{2}\sqrt{2(c-d)}$.

13. A certain sum of money is lent at 5 per cent. per annum. If we multiply the number of dollars in the principal by the

number of dollars in the interest for 3 months, the product is 720. What is the sum lent? *Ans.* \$240.

14. It is required to find three numbers, such that the product of the first and second $=a$, the product of the first and third $=b$, and the sum of the squares of the second and third $=c$.

Ans. $\sqrt{\left(\frac{a^2+b^2}{c}\right)}$, $a\sqrt{\left(\frac{c}{a^2+b^2}\right)}$, and $b\sqrt{\left(\frac{c}{a^2+b^2}\right)}$.

15. The spaces through which a body falls in different periods of time, being to each other as the squares of those times, in how many seconds will a body fall through 400 feet, the space it falls through in one second being 16.1 feet?

Let $x=$ the required number of seconds, then

$16.1 : 400 :: 1^2 : x^2$; whence, $x=4.97+$ sec.

In what time will a body fall through a hight of 1000 feet?

Ans. 7.88+ sec.

16. What two numbers are as 3 to 5, and the sum of whose cubes is 1216?

Let $3x$ and $5x=$ the numbers;

Then $27x^3+125x^3=152x^3=1216$,

whence, $x^3=8$,

and $x=\sqrt[3]{8}=2$.

Hence, the numbers are 6 and 10.

REMARK.—This is properly a pure equation of the *third* degree; but questions producing such equations are generally arranged with those of the second degree.

17. A money safe contains a certain number of drawers. In each drawer there are as many divisions as there are drawers, and in each division there are four times as many dollars as there are drawers. The whole sum in the safe is \$5324; what is the number of drawers? *Ans.* 11.

18. Two travelers, A and B, set out to meet each other; A leaving the town C at the same time that B left D. They traveled the direct road from C to D, and on meeting it appeared that A had traveled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What is the distance between C and D? *Ans.* 126 miles.

19. Two men, A and B, engaged to work for a certain number of days at different rates. At the end of the time, A, who had

played 4 of those days, received 75 shillings; but B, who had played 7 of those days, received only 48 shillings. Now had B played only 4 days, and A played 7 days, they would have received the same sum. For how many days were they engaged?

Ans. 19 days.

20. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same number of gallons as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Ans. 64, 48, 36, and 27 gallons.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

ART. **230.** 1. Let it be required to find the value of x in the equation,

$$x^2-6x+9=4.$$

It is evident, from Art. 184, that the first member of this equation is a perfect square. By extracting the square root of both members, we find

$$x-3=\pm2;$$

Whence, $x=3\pm2=3+2=5$, or $3-2=1$.

Verification. $(5)^2-6(5)+9=4$, that is, $25-30+9=4$.

$(1)^2-6(1)+9=4$, that is, $1-6+9=4$.

Hence, x has *two values*, $+5$, and $+1$, either of which verifies the equation.

2. Let it be required to find the value of x in the equation,

$$x^2-6x=27.$$

If the left member of this equation were a perfect square, we might find the value of x by extracting the square root, as in the preceding example. To ascertain what is necessary to render the first member a perfect square, let us compare it with the square of $x-a$, which is,

$$x^2-2ax+a^2.$$

We have, $x^2-6x \qquad =27.$

From this we see that $2a$ corresponds to 6; hence, a corresponds to 3, and a^2 to 9. Hence, by adding 9, which is the square of half the coëfficient of x, to each member, the equation becomes

$$x^2-6x+9=36.$$

Extracting the square root, $x-3=\pm6$.

Whence, $x=3\pm6=+9$, or -3, either of which values of x will verify the equation.

ART. **231.** We will now proceed to explain the method of completing the square.

Since every complete equation of the second degree (Art. 226) may be reduced to the form,

$ax^2+bx=c$; if we divide both sides by a, we have $$x^2+\frac{b}{a}x=\frac{c}{a}$$

For the sake of simplicity, let $\frac{b}{a}=2p$, and $\frac{c}{a}=q$. The equation then becomes

$$x^2+2px=q. \qquad (1)$$

If $\frac{b}{a}$ is negative, and $\frac{c}{a}$ positive, the equation becomes

$$x^2-2px=q. \qquad (2)$$

If $\frac{b}{a}$ is positive, and $\frac{c}{a}$ negative, the equation becomes

$$x^2+2px=-q. \qquad (3)$$

Lastly, if $\frac{b}{a}$ and $\frac{c}{a}$ are both negative, the equation becomes

$$x^2-2px=-q. \qquad (4)$$

Hence, *every complete equation of the second degree, may be reduced to the form* $x^2+2px=q$, *in which* $2p$ *and* q *may be either positive or negative, integral or fractional quantities.*

We will now proceed to explain the principle by which the first member of this equation may always be made a perfect square.

Since the square of a binomial is equal to the square of the first term, plus twice the product of the first term by the second, plus the square of the second; if we consider x^2+2px as the first two terms of the square of a binomial, we find x^2 is the *square* of the first term; hence, the first term must be x; we next observe that $2px$ is the *double* of the product of the first term by the second; therefore, if we divide $2px$ by x, the quotient $2p$ is double the second term. Hence p, which is *half the coëfficient of* x, is the second term of the binomial; therefore, its square, p^2, added to x^2+2px, will render it a perfect square. But, to preserve the equality, we must add the same quantity to both sides. This gives,

$$x^2+2px+p^2=q+p^2;$$

Extracting the square root, $x+p=\pm\sqrt{q+p^2}$;

Transposing, $x=-p\pm\sqrt{q+p^2}$.

It is obvious that in each of the remaining three forms, the square may be completed on the same principle; that is, by taking half the coëfficient of the first power of x, squaring it, and adding it to each member.

Solving equations (2), (3), and (4), and collecting together the four different forms, and the values of x in each, we have the following table.

(1)	$x^2+2px=q$.	$x=-p\pm\sqrt{q+p^2}$.
(2)	$x^2-2px=q$.	$x=+p\pm\sqrt{q+p^2}$.
(3)	$x^2+2px=-q$.	$x=-p\pm\sqrt{-q+p^2}$.
(4)	$x^2-2px=-q$.	$x=+p\pm\sqrt{-q+p^2}$.

Although the method of finding the values of x is the same in each of these forms, it is convenient to distinguish between them. See Art. 235.

From the preceding we derive the following

Rule for the solution of a complete equation of the second degree.—1st. *Reduce the equation, by clearing of fractions and transposition, to the form* $ax^2+bx=c$.

2nd. *Divide each side of the equation by the coëfficient of* x^2, *and add to each member the square of half the coëfficient of the first power of* x.

3rd. *Extract the square root of both sides, and transpose the known term to the second member.*

Remarks.—1st. When the coëfficient of x^2 is negative, as in the equation $-x^2+mx=n$, the pupil may not perceive that it is embraced in the four general forms. This difficulty is obviated by multiplying both sides of the equation by -1.

2nd. Since the sign of the square root of x^2, or of $(x+p)^2$, is $\pm$, it might seem that when $x^2=m^2$, we should have $\pm x=\pm m$, that is, $+x=+m$(1), $+x=-m$(2), $-x=+m$(3), and $-x=-m$(4). But it is evident that equations (1) and (4) are the same equation, as also (2) and (3). Hence, $+x=\pm m$, embraces all the values of x. For the same reason it is necessary to take only the plus sign of the square root of $(x+p)^2$.

1. Given $17x-2x^2=32-3x$, to find x.

Transposing, $-2x^2+20x=32$;

Reducing, $x^2-10x=-16$;

Completing the square by adding $(\frac{10}{2})^2=25$ to both sides of the equation,

$$x^2-10x+25=-16+25=9;$$

Extracting the root, $x-5=\pm 3$;

Whence, $x=5\pm 3=8$, or 2.

Verification. $17(8)-2(8)^2=32-3(8)$, or $+8=+8$.

$17(2)-2(2)^2=32-3(2)$, or $+26=+26$.

2. Given $3x^2-2x=65$, to find x.

Dividing by 3, $x^2-\frac{2}{3}x=\frac{65}{3}$;

Completing the square, $x^2-\frac{2}{3}x+(\frac{1}{3})^2=\frac{65}{3}+(\frac{1}{3})^2=\frac{196}{9}$.

Extracting the root, $x-\frac{1}{3}=\pm\frac{14}{3}$.

Whence, $x=\frac{1}{3}\pm\frac{14}{3}=5$, or $-4\frac{1}{3}$.

Both of which values verify the equation.

3. Given $4a^2-2x^2+2ax=18ab-18b^2$, to find x.

Transposing, $-2x^2+2ax=-4a^2+18ab-18b^2$;

Dividing by -2, $x^2-ax=2a^2-9ab+9b^2$;

Completing the square, $x^2-ax+\frac{a^2}{4}=\frac{9a^2}{4}-9ab+9b^2$;

$$\text{or, } \left(x-\frac{a}{2}\right)^2=\left(\frac{3a}{2}-3b\right)^2;$$

$$x-\frac{a}{2}=\pm\left(\frac{3a}{2}-3b\right);$$

$$\text{Whence, } x=\frac{a}{2}\pm\left(\frac{3a}{2}-3b\right);$$

$$x=\frac{a}{2}+\left(\frac{3a}{2}-3b\right)=2a-3b;$$

$$x=\frac{a}{2}-\left(\frac{3a}{2}-3b\right)=-a+3b.$$

4. Given $x+\sqrt{(5x+10)}=8$, to find x.

By transposition, $\sqrt{(5x+10)}=8-x$;

By squaring, $5x+10=64-16x+x^2$;

or, $x^2-21x=-54$;

Completing the square, $x^2-21x+(\frac{21}{2})^2=\frac{441}{4}-54=\frac{225}{4}$,

Extracting the root, $x-\frac{21}{2}=\pm\frac{15}{2}$;

Whence, $x=\frac{21}{2}\pm\frac{15}{2}=\frac{36}{2}=18$, or $\frac{6}{2}=3$.

These two values of x are the roots of the quadratic equation, $x^2-21x=-54$; but they will not both verify the proposed equation $x+\sqrt{(5x+10)}=8$, from which the former was derived, for the following reasons. Since the square root of a quantity may have either the sign $+$ or $-$ prefixed to it, the proposed equation might have been $x\pm\sqrt{(5x+10)}=8$; because by the operations which have been employed, the same resulting equation, $x^2-21x=-54$, would be obtained, whether the sign of the radical part be $+$ or $-$.

Hence, in the equation $x+\sqrt{(5x+10)}=8$, the value of x is 3; but in the equation $x-\sqrt{(5x+10)}=8$, the value of x is 18.

EXAMPLES FOR PRACTICE.

5. $x^2+4x=60$. *Ans.* $x=6$, or -10.
6. $x^2-4x=60$. *Ans.* $x=10$, or -6.
7. $x^2+16x=-60$. *Ans.* $x=-6$, or -10.
8. $x^2-16x=-60$. *Ans.* $x=6$, or 10.
9. $x^2-6x=6x+28$. *Ans.* $x=14$, or -2.
10. $\frac{x^2}{10}+350-12x=0$. *Ans.* $x=70$, or 50.
11. $\frac{4x^2}{5}+8x-50\frac{1}{4}=429\frac{3}{4}$. *Ans.* $x=20$, or -30.
12. $2x=4+\frac{6}{x}$. *Ans.* $x=3$, or -1
13. $3x^2+10x=57$. *Ans.* $x=3$, or $-6\frac{1}{3}$
14. $(x-1)(x-2)=1$ *Ans.* $x=\frac{1}{2}(3\pm\sqrt{5})$.
15. $4x^2-3x-5=80$. *Ans.* $x=5$, or $-4\frac{1}{4}$.
16. $\frac{1}{2}x^2-\frac{1}{4}x+2=9$. *Ans.* $x=4$, or $-3\frac{1}{2}$.
17. $x=1+\frac{110}{x}$. *Ans.* $x=11$, or -10.
18. $3(x-2)^2=8(x-2)+3$. *Ans.* $x=5$, or $1\frac{2}{3}$.
19. $\frac{x+22}{3}-\frac{4}{x}=\frac{9x-6}{2}$. *Ans.* $x=2$, or $\frac{12}{25}$.

20. $\frac{2x^2}{3}+3\frac{1}{2}=\frac{x}{2}+8.$ *Ans.* $x=3$, or $-2\frac{1}{4}$.

21. $17x^2+19x=1848.$ *Ans.* $x=9\frac{15}{17}$, or -11.

22. $\frac{z^2}{3}-\frac{z}{10}+\frac{1}{6}=\frac{1}{5}.$ *Ans.* $z=\frac{1}{2}$, or $-\frac{1}{5}$.

23. $3x-\frac{1}{4}x^2=10.$ *Ans.* $x=6\pm2\sqrt{-1}$.

24. $\frac{2x-10}{8-x}-\frac{x+3}{x-2}=2.$ *Ans.* $x=7$, or $\frac{4}{5}$.

25. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}.$ *Ans.* $x=6$, or $3\frac{1}{13}$.

26. $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35}.$ *Ans.* $x=11$, or -13.

27. $\frac{1}{x^2-3x}+\frac{1}{x^2+4x}=\frac{9}{8x}.$ *Ans.* $x=4$, or $-3\frac{2}{9}$.

28. $\frac{48}{x+3}=\frac{165}{x+10}-5.$ *Ans.* $x=5\frac{2}{5}$, or 5.

29. $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1.$ *Ans.* $x=21$, or 5.

30. $\frac{4x+7}{19}+\frac{5-x}{3+x}=\frac{4x}{9}.$ *Ans.* $x=3$, or -8.7.

31. $x+\frac{1}{x}=\frac{4}{\sqrt{3}}.$ *Ans.* $x=\sqrt{3}$, or $\frac{1}{3}\sqrt{3}$.

32. $\frac{x+\frac{1}{x}}{x-\frac{1}{x}}+\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{13}{4}.$ *Ans.* $x=3$, or $-\frac{7}{5}$.

33. $\frac{x}{a}+\frac{a}{x}-\frac{2}{a}=0.$ *Ans.* $x=1\pm\sqrt{(1-a^2)}$.

34. $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}.$ *Ans.* $x=\frac{3}{4}a$, or $\frac{1}{2}a$.

35. $\frac{a^2-b^2}{c}=2ax-cx^2.$ *Ans.* $x=\frac{a\pm b}{c}$.

36. $x^2-(a+b)x+ab=0.$ *Ans.* $x=a$, or b.

37. $(a-b)x^2-(a+b)x+2b=0.$ *Ans.* $x=1$, or $\frac{2b}{a-b}$.

38. $mqx^2-mnx+pqx-np=0$. *Ans.* $x=\frac{n}{q}$, or $-\frac{p}{m}$.

39. $\frac{x}{a}+bx^{-1}=c$. *Ans.* $x=\frac{ac\pm\sqrt{(a^2c^2-4ab)}}{2}$.

40. $\frac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})x=\frac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}$.

Ans. $x=a$, or $-b$.

41. $adx-acx^2=bcx-bd$. *Ans.* $x=\frac{d}{c}$, or $-\frac{b}{a}$.

42. $\sqrt{(x+5)}=\frac{12}{\sqrt{(x+12)}}$. *Ans.* $x=4$, or -21.

43. $\sqrt{x}+\sqrt{(a-x)}=\sqrt{b}$. *Ans.* $x=\frac{a\pm\sqrt{2ab-b^2}}{2}$.

44. $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. *Ans.* $x=4$.

45. $\sqrt{x^3}-2\sqrt{x}=x$. *Ans.* $x=4$.

46. $\sqrt{x+a}-\sqrt{x+b}=\sqrt{2x}$. *Ans.* $x=-\frac{a+b}{2}\pm\frac{1}{2}\sqrt{2a^2+2b^2}$.

47. $(x-c)\sqrt{ab}-(a-b)\sqrt{cx}=0$. *Ans.* $x=\frac{ac}{b}$, or $\frac{bc}{a}$.

48. $\sqrt{a+x}+\sqrt{a-x}=\frac{12a}{5\sqrt{a+x}}$. *Ans.* $x=\frac{4a}{5}$, or $\frac{3a}{5}$.

ART. **232.** HINDOO METHOD OF SOLVING QUADRATICS.—When an equation is brought to the form $ax^2+bx=c$, it may be reduced to an equation of the first degree, without dividing by the coëfficient of x^2; thus avoiding fractions.

If we multiply every term of the equation $ax^2+bx=c$, by *four* times the coëfficient of the *first* term, and add to both sides the *square* of the coëfficient of the *second* term, we shall have,

$$4a^2x^2+4abx+b^2=4ac+b^2.$$

Now the first member of this equation is a perfect square, and by extracting the square root of both sides, we have

$2ax+b=\pm\sqrt{4ac+b^2}$, which is an equation of the *first* degree. This gives the following, called the

HINDOO RULE FOR THE SOLUTION OF EQUATIONS OF THE SECOND DEGREE.—*Reduce the equation to the form* $ax^2+bx=c$. *Multiply both sides by four times the coëfficient of* x^2. *Add the square of the coëfficient of* x *to each side, and then extract the square root. This will give an equation of the first degree, from which the value of* x *is easily found.*

1. Given $2x^2-5x=3$, to find x.

Multiplying both sides by 8, which is *four* times the coëfficient of x^2, we have $16x^2-40x=24$.

Adding to each side 25, which is the square of the coëfficient of x, we have

$$16x^2-40x+25=49\;;$$

Extracting the root, $4x-5=\pm7$;

Whence, $x=3$, or $-\frac{1}{2}$.

Find the value of the unknown quantity in each of the following examples by the *Hindoo* rule.

2. $3x^2+5x=2$. *Ans.* $x=\frac{1}{3}$, or -2.
3. $x^2+x=30$. *Ans.* $x=5$, or -6.
4. $x^2-x=72$. *Ans.* $x=9$, or -8.
5. $\frac{40}{x-5}+\frac{27}{x}=13$. *Ans.* $x=9$, or $\frac{15}{13}$.

PROBLEMS PRODUCING COMPLETE EQUATIONS OF THE SECOND DEGREE.

ART. **233.** 1. A person bought a certain number of sheep for 40 dollars, and if he had bought 2 more for the same sum they would have cost a dollar apiece less; required the number of sheep, and the price of each.

Let x be the number of sheep, then $\frac{40}{x}$ is the price of one, and $\frac{40}{x+2}$ is the price of one on the second supposition.

$$\therefore\quad \frac{40}{x+2}=\frac{40}{x}-1, \text{ by the question.}$$

Solving this equation, we find $x=-1\pm9=8$, or -10, the number of sheep; and $\frac{40}{x}=\frac{40}{8}=5$ dollars, the price of each.

$$\text{Also, } \frac{40}{x}=\frac{40}{-10}=-4.$$

Now either of these values of x satisfies the equation, but the negative value, -10, does not fulfill the conditions of the question in an arithmetical sense. But, since the subtraction of a negative quantity produces the same result as the addition of a positive quantity of the same numerical value, the question may be so modified that the value, -10, will be a correct answer to it, the 10 being reckoned positive. The question thus modified is: A person *sells* a certain number of sheep for 40 dollars. If he had sold 2 *fewer* for the same sum he would have received a dollar apiece *more* for them; required the number sold.

REMARK.—In the preceding, and in many other cases, especially in the solution of philosophical questions, we derive answers which do not correspond with the conditions. The reason is that the algebraical expression is more general than the common language; and the equation, which is a proper representative of the conditions of the given question, also expresses other conditions; and hence, when it is solved, answers should be obtained, fulfilling *all* the conditions expressed by the equation.

2. Find a number such, that if 17 times the number be diminished by its square, the remainder shall be 70.

Let $x=$ the number.

Then $17x-x^2=70$.

or, $x^2-17x=-70$.

Whence, $x=7$, or 10.

In this case *both* values of x satisfy the question in its arithmetical sense. Thus,

$$17\times7-7^2=119-49=70.$$

$$\text{or, } 17\times10-10^2=170-100=70.$$

3. Of a number of bees, after eight-ninths, and the square root of half of them, had flown away, there were two remaining; what was the number at first?

To avoid radicals, let $2x^2$ represent the number of bees at first; then,

$$\frac{16x^2}{9}+x+2=2x^2;$$

Whence, $x=6$, or $-1\frac{1}{2}$; but the latter value, being *fractional*, is excluded by the nature of the question; the number of bees is $2\times6^2=72$.

4. Divide a into two parts, whose product shall be b^2.

Let $x=$ one part, then $a-x=$ the other;

$\therefore$ $x(a-x)$, or $ax-x^2=b^2$.

Whence, $x=\frac{1}{2}(a\pm\sqrt{a^2-4b^2})$; that is,

$x=\frac{1}{2}(a\pm\sqrt{a^2-4b^2})$, and $a-x=\frac{1}{2}(a\mp\sqrt{a^2-4b^2})$, are the parts required, and the two parts are the same, whether the upper or lower sign of the radical quantity be used. Thus, if the number a is 20, and b 8, the parts are 16 and 4, or 4 and 16.

The *forms* of these results enable us to determine the limits under which the problem is *possible;* for it is evident that if $4b^2$ be greater than a^2, $\sqrt{a^2-4b^2}$ becomes *imaginary*, and thus the two parts are unassignable, according to the principles of arithmetic; that is, no such parts can be found. It is also easily seen that the extreme possible case will be, when $\sqrt{a^2-4b^2}=0$, in which case $x=\frac{1}{2}a$, and $a-x=\frac{1}{2}a$; also, $b^2=\frac{1}{4}a^2$.

Remark.— In the following examples, that value of the unknown quantity only is given, which satisfies the conditions of the question in an arithmetical sense.

5. What two numbers are those whose sum is 20 and product 36? *Ans.* 2 and 18.

6. Divide the number 15 into two such parts that their product shall be to the sum of their squares, in the ratio of 2 to 5.
Ans. 5 and 10.

7. Find a number such, that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21. *Ans.* 7 or 3.

8. It is required to divide the number 24 into two such parts that their product shall be equal to 35 times their difference.
Ans. 10 and 14.

9. Divide the number 346 into two such parts that the sum of their square roots shall be 26. *Ans.* 11^2 and 15^2.

Suggestion.— Let $x=$ the square root of one of the parts, and $26-x$, the square root of the other part.

10. What number added to its square root gives 132?
Ans. 121.

11. What number exceeds its square root by $48\frac{3}{4}$?
Ans. $56\frac{1}{4}$.

12. What two numbers are those, whose sum is 41, and the sum of whose squares is 901? *Ans.* 15 and 26.

13. What two numbers are those, whose difference is 8, and the sum of whose squares is 544? *Ans.* 12 and 20.

14. A merchant sold a piece of cloth for 24 dollars, and gained as much per cent. as the cloth cost him. Required the first cost.

Ans. 20 dollars.

15. Two persons, A and B, had a distance of 39 miles to travel, and they started at the same time; but A, by traveling $\frac{1}{4}$ of a mile an hour more than B, arrived one hour before him; find their rates of traveling. *Ans.* A $3\frac{1}{4}$, B 3 miles per hour.

16. A and B distribute 1200 dollars each among a number of persons; A gives to 40 persons more than B, and B gives 5 dollars apiece to each person more than A; find the number of persons. *Ans.* 120 and 80.

17. From two towns, distant from each other 320 miles, two persons, A and B, set out at the same instant to meet each other A traveled 8 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day; how many miles did each travel per day?

Ans. A 24, and B 16 miles.

18. A set out from C towards D, and traveled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in one day, he met A. Required the distance of the places C and D.

Ans. 76, or 152 miles.

19. A grazier bought a certain number of oxen for $240 and after losing 3, sold the remainder for $8 a head more than they cost him, thus gaining $59 by his bargain. What number did he buy? *Ans.* 16.

20. Divide the number 100 into two such parts that their product may be equal to the difference of their squares.

Ans. 38.197, and 61.803 nearly.

21. Two persons, A and B, jointly invested $500 in business, each contributing a certain sum; A let his money remain 5 months, and B only 2, and each received back $450, capital and profit. How much did each advance?

Ans. A $200, B $300.

22. It is required to divide each of the numbers 11 and 17 into two parts, so that the product of the first parts of each may be 45, and of the second 48. *Ans.* 5, 6, and 9, 8.

23. Divide each of the numbers 21 and 30 into two parts, so that the first part of 21 may be three times as great as the first

part of 30; and that the sum of the squares of the remaining parts may be 585. *Ans.* 18, 3, and 6, 24.

24. Divide each of the numbers 19 and 29 into two parts, so that the difference of the squares of the first parts of each may be 72, and the difference of the squares of the remaining parts 180. *Ans.* 7, 12, and 11, 18.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND DEGREE.

ART. **234.** The discussion of the general equation of the second degree, consists in investigating the *general properties* of the equation, and in *interpreting the results,* which are derived from making particular suppositions on the different quantities which it contains.

The general form, to which every complete equation of the second degree, containing one unknown quantity, may be reduced (Art. 226), is

$$x^2+2px=q,$$

in which $2p$ and q may be either both positive or both negative, or one positive and the other negative.

Completing the square, we have

$$x^2+2px+p^2=q+p^2.$$

Now, $x^2+2px+p^2=(x+p)^2$. For the sake of simplicity, put $q+p^2=m^2$, that is, $\sqrt{q+p^2}=m$, then

$$(x+p)^2=m^2;$$

Transposing, $$(x+p)^2-m^2=0.$$

But, since the left member of this equation is the difference of two squares, it may be resolved into two factors (Art. 93); this gives $$(x+p+m)(x+p-m)=0.$$

Now this equation can be satisfied in *two* ways, and in *only* two; that is, by making either of the factors equal to 0. If we make the second factor equal to zero, we have

$$x+p-m=0;$$

Or, by transposing, $x=-p+m=-p+\sqrt{q+p^2}$.

If we make the first factor equal to zero, we have

$$x+p+m=0;$$

Or, by transposing, $x=-p-m=-p-\sqrt{q+p^2}$.

Hence, we have

PROPERTY 1ST. *Every equation of the second degree has two roots (or values of the unknown quantity), and only two.*

From the equation $(x+p+m)(x+p-m)=0$, we derive

PROPERTY 2ND. *Every complete equation of the second degree, reduced to the form* $x^2+2px=q$, *may be decomposed into two binomial factors, of which the first term in each is* x, *and the second, the two roots with the signs changed.*

Thus, the two roots of the equation, $x^2-7x+10=0$, are $x=2$, and $x=5$; hence, $x^2-7x+10=(x-2)(x-5)$.

It is now evident that the *direct* method of resolving a quadratic trinomial into its factors, is, *to place it equal to zero, and then find the roots of the resulting equation.*

In this manner let the learner solve the examples in Art. 94, page 50.

By reversing the operation, we can readily form an equation whose roots shall have any given values.

Thus, let it be required to form an equation whose roots shall be -3 and 4.

We must have $x=-3$, or $x+3=0$,

And $x=4$, or $x-4=0$.

Hence, $(x+3)(x-4)=0$;

Or, $x^2-x-12=0$;

Or, $x^2-x=12$, which is an equation whose roots are -3 and $+4$.

1. Find an equation whose roots are 4 and 5.
Ans. $x^2-9x=-20$.

2. Find an equation whose roots are $-\frac{1}{2}$ and $+\frac{1}{3}$.
Ans. $x^2+\frac{1}{6}x=\frac{1}{6}$.

3. Find an equation, without fractional coëfficients, whose roots are $\frac{2}{3}$ and $\frac{4}{5}$. *Ans.* $15x^2-22x=-8$.

4. Find an equation whose roots are $m+n$, and $m-n$.
Ans. $x^2-2mx=n^2-m^2$.

Resuming the equation $x^2+2px=q$, and denoting the two roots by x' and x'', we have

$$x' = -p + \sqrt{q+p^2},$$
$$x'' = -p - \sqrt{q+p^2}.$$

Adding, $x'+x''=-2p$. But, $-2p$ is the coëfficient of x, taken with a *contrary* sign. Hence, we have

PROPERTY 3RD. *The sum of the two roots of an equation of the second degree, reduced to the form* $x^2+2px=q$, *is equal to the coëfficient of the first power of* x, *taken with a contrary sign.*

If we take the product of the roots, we have

$$\begin{array}{l} x' = -p + \sqrt{q+p^2}, \\ x'' = -p - \sqrt{q+p^2} \\ \hline \quad p^2 - p\sqrt{q+p^2} \\ \qquad + p\sqrt{q+p^2} - (q+p^2) \\ \hline x'x'' = p^2 \;.\;.\;.\;.\; -(q+p^2) = -q. \end{array}$$

But $-q$ is the known term of the equation, taken with a *contrary* sign. Hence, we have

PROPERTY 4TH. *The product of the two roots of an equation of the second degree, reduced to the form* $x^2+2px=q$, *is equal to the known term taken with a contrary sign.*

NOTE.— In the preceding demonstrations, we have regarded $2p$ and q as both positive; but the same conclusions will be obtained by taking them both negative, or one positive and the other negative.

ART. **235.** We shall now proceed to determine the *essential* sign of the roots in each of the four different forms, and to compare the two roots in each form, in regard to their numerical magnitude.

To do this, it is necessary first to compare p with $\sqrt{q+p^2}$, and also with $\sqrt{-q+p^2}$.

If we examine $\sqrt{q+p^2}$, we see that its value must be a quantity *greater* than p, since the square root of p^2 alone, is p.

But the value of $\sqrt{-q+p^2}$, must be *less* than p, since it is the square root of a quantity less than p^2.

With these principles, a careful consideration of the roots, or values of x in each of the four different forms will render the following conclusions evident:

1st form $x^2+2px=q$.

$$x'=-p+\sqrt{q+p^2};$$
$$x''=-p-\sqrt{q+p^2}.$$

The first root is essentially positive, and the second essentially negative; and the first root is numerically less than the second.

2nd form, $x^2-2px=q$.

$$x'=p+\sqrt{q+p^2};$$
$$x''=p-\sqrt{q+p^2}.$$

The first root is essentially positive, and the second essentially negative; and the first root is numerically greater than the second.

3rd form, $x^2+2px=-q$.

$$x'=-p+\sqrt{-q+p^2},$$
$$x''=-p-\sqrt{-q+p^2}.$$

Both roots are essentially negative, and the first root is numerically less than the second.

4th form, $x^2-2px=-q$.

$$x'=p+\sqrt{-q+p^2};$$
$$x''=p-\sqrt{-q+p^2}.$$

Both roots are essentially positive, and the first root is numerically greater than the second.

It is obvious that in each of the forms, the *exact* numerical value of the roots can be found, only when $\sqrt{q+p^2}$, or $\sqrt{-q+p^2}$ is a perfect square.

NOTE.— Questions 5, 6, 7, 8, page 186, are specially adapted to illustrate the four different forms. See, also, Ray's Algebra, Part 1st, Art. 217.

ART. **236.** We shall now proceed to show *when* the roots become imaginary, and *why*.

In the third and fourth forms, the radical part is $\sqrt{-q+p^2}$. Now when q is *greater* than p^2, this is essentially negative, and we are required to extract the square root of a negative quantity, which is impossible (Art. 193). Hence, when q is greater than p^2, that is, *when the known term is negative, and greater than the square of half the coëfficient of the first power of* x, *the roots are imaginary.*

To show why the roots are imaginary, we must inquire, into what two parts a number must be divided, that the product of the parts shall be the *greatest* possible.

Let $2p$ represent any number, and let the parts, into which it is supposed to be divided, be $p+z$, and $p-z$. The product of these parts is

$$(p+z)(p-z)=p^2-z^2.$$

Now this product is evidently the greatest, when z^2 is the least; that is, when $z^2=0$, or $z=0$. But when z is 0, the parts are p and p. Hence,

When a number is divided into two equal parts, their product is greater than that of any other two parts into which the number can be divided. Or, as the same principle may be otherwise expressed,

The product of any two unequal numbers, is less than the square of half their sum.

Now it has been shown (Art. 234, Properties 3rd and 4th), that $2p$, the coëfficient of the first power of x, is equal to the *sum* of the two roots, and that q is equal to their *product.* But, when q is greater than p^2, we have the product of two numbers, *greater* than the square of half their sum, which, by the preceding principle, is *impossible.* If, then, any problem furnishes an equation of the form $x^2 \pm 2px = -q$, in which the known term is negative and *greater* than the square of half the coëfficient of the first power of the unknown quantity, we infer, that the conditions of the problem are *incompatible* with each other. The following is an example.

Let it be required to divide the number 8 into two parts, whose product shall be 18.

Let x and $8-x$ represent the parts.

Then, $x(8-x)=18$; or $x^2-8x=-18$;

Whence, $x=4+\sqrt{-2}$, or $4-\sqrt{-2}$.

These expressions for the values of x, show that the problem is *impossible.* This we also know from the preceding theorem, since the number 8 cannot be divided into any two parts whose product will be greater than 16. Thus, the algebraic solution renders it manifest that the problem is impossible.

ART. **237.** Examination of particular cases.

1st. If, in the third and fourth forms, where q is negative, we suppose $q=p^2$, the radical, $\sqrt{-q+p^2}$, becomes 0, and $x=-p$, in the third form, or $+p$ in the fourth form. It is then said, *the two roots are equal.*

In fact, if we substitute p^2 for q, the equation in the third form becomes $x^2+2px+p^2=0$.

Hence, $(x+p)^2$, or, $(x+p)(x+p)=0$.

In this case, the first member is the *product of two equal factors.* Hence, the roots of the equation are equal, since either of the two factors, being placed equal to zero, gives the same value for x. A similar conclusion is obtained by substituting p^2 for q in the fourth form.

2nd. If, in the general equation, $x^2+2px=q$, we suppose $q=0$, the two values of x reduce to,

$$x=-p+p=0, \text{ and } x=-p-p=-2p.$$

In fact, the equation is then of the form

$$x^2+2px=0, \text{ or } x(x+2p)=0,$$

which can be satisfied only by making

$$x=0, \text{ or } x+2p=0;$$

Whence, $x=0$, or $x=\ -2p$.

3rd. If, in the general equation, $x^2+2px=q$, we suppose $2p=0$, we have $x^2=q$,

Whence, $x=\pm\sqrt{q}$.

In this case, *the two values of* x *are equal and have contrary signs, real,* if q is *positive,* as in the first and second forms, and *imaginary,* if q is *negative,* as in the third and fourth forms.

Under this supposition the equation contains only two terms, and belongs to the class treated of Art. 228.

4th. If we suppose $2p=0$, and $q=0$, the equations reduce to $x^2=0$, and give the two values of x, in all the forms, each equal to 0.

ART. **238.** There remains a singular case to be examined, which is sometimes met with in the solution of problems producing equations of the second degree.

To discuss it, take the equation

$$ax^2+bx=c.$$

Solving this equation, the values of x are

$$x=\frac{-b+\sqrt{b^2+4ac}}{2a} \quad x=\frac{-b-\sqrt{b^2+4ac}}{2a}.$$

If, now, we suppose $a=0$, these values become

$$x=\frac{-b+b}{0}=\frac{0}{0}, \quad x=\frac{-b-b}{0}=\frac{-2b}{0}=\infty.$$

That is, one value of x is *indeterminate* and the other *infinite* (Arts. 136, 137).

But if we suppose $a=0$ in the given equation, we have

$$bx=c, \text{ and } x=\frac{c}{b}.$$

But x can have only two values, (Art. 234); hence, there is at least an apparent contradiction; how can it be reconciled?

Let us first examine the value of $x=\frac{0}{0}$.

If we multiply both terms of the second member of the equation $x=\frac{-b+\sqrt{b^2+4ac}}{2a}$ by $-b-\sqrt{b^2+4ac}$, we have

$$x=\frac{b^2-(b^2+4ac)}{2a(-b-\sqrt{b^2+4ac})}=\frac{-4ac}{2a(-b-\sqrt{b^2+4ac})},$$

or, by dividing both terms by $2a$

$$x=\frac{-2c}{-b-\sqrt{b^2+4ac}};$$

Whence, $x=\frac{c}{b}$, by making $a=0$.

Hence we see, that the value of $x=\frac{0}{0}$, is really $\frac{c}{b}$, and arises from having made a factor zero, that was common to the numerator and denominator.

We shall now examine the value of $x=-\frac{2b}{0}=\infty$.

By supposing $a=0$, the equation $ax^2+bx=c$, reduces to $bx=c$, an equation of the *first* degree.

It is, therefore, *impossible* that it can have more than *one* root (Art. 170.) Hence, the supposition that it has *two*, gives one value *infinite*, which is equivalent to saying, the equation has but one *finite* root.

If we had at the same time

$a=0$, $b=0$, $c=0$, the equation would be altogether indeterminate. This is the only case of indetermination presented by the equation of the second degree.

ART. **239.** We shall now apply the preceding principles in the discussion of a problem, which presents most of the circumstances commonly met with, in problems producing equations of the second degree.

PROBLEM OF THE LIGHTS.

It is required to find, in a line BC, which joins two lights, B and C, of different intensities, a point which is illuminated equally by each.

It is a principle in optics that, *the intensity of the same light at different distances, is inversely as the square of the distance.*

Let a be the distance BC between the two lights.

Let b be the intensity of the light B at the distance of one foot from B.

Let c be the intensity of the light C at the distance of one foot from C.

Let P be the point required.

Let BP $=x$, then CP $=a-x$.

By the optical principle above stated, since the intensity of the light B at the distance of 1 foot, is b, its intensity at the distance of 2, 3, 4,. . . . feet, must be $\frac{b}{4}, \frac{b}{9}, \frac{b}{16}$ . . . . ; hence, the intensity of B, at the distance of x feet, must be $\frac{b}{x^2}$. In like manner, the intensity of the light C, at the distance of $a-x$ feet, must be $\frac{c}{(a-x)^2}$. But, by the conditions of the problem, these two intensities are equal; hence, we have for the equation of the problem,

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}, \text{ which easily reduces to } \frac{(a-x)^2}{x^2}=\frac{c}{b};$$

$$\text{Whence, } \frac{a-x}{x}=\frac{+\sqrt{c}}{\sqrt{b}}, \text{ or } \frac{-\sqrt{c}}{\sqrt{b}}.$$

This gives the following results:

1st. $x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, whence $a-x=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$.

2nd. $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, whence $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$.

We shall now proceed to discuss these values.

I. Let $b>c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, is positive and less than a, for $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a proper fraction; hence, this value gives for the point illuminated equally, a point P situated *between* B and C. We perceive, also, that the point P is nearer to C than B; for since $b>c$, we have $\sqrt{b}+\sqrt{b}>\sqrt{b}+\sqrt{c}$, or $2\sqrt{b}>\sqrt{b}+\sqrt{c}$, and $\therefore$ $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and, consequently, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. This is manifestly correct, for the required point must be nearer the light of less intensity. The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is positive, and evidently less than $\frac{a}{2}$.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is positive, and greater than a for $\sqrt{b}>\sqrt{b}-\sqrt{c}$; $\therefore$ $\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}>1$, and $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}>a$.

This value gives a point P′, situated on the prolongation of BC, and to the right of the two lights. In fact, we suppose that the two lights emit rays in all directions; there will, therefore, be a point P′, to the right of C, and nearer the light of less intensity, which is illuminated equally by the two lights.

It is easy to perceive, why the two values thus obtained, are expressed by the same equation. If, instead of assuming BP for the unknown quantity x, we take BP′, then $CP'=x-a$, and the equation of the problem is

$$\frac{b}{x^2}=\frac{c}{(x-a)^2};$$

but $(x-a)^2=(a-x)^2$. Hence, the new equation is the same as that already found, and, consequently, ought to give BP′, as well as BP.

The second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it ought to be, because $x>a$, but changing the signs of the equation

$a-x=\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, we find $x-a=\dfrac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, and this value of $x-a$, represents the distance CP′.

II. Let $b<c$.

The first value of x, $\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is positive, and less than $\dfrac{a}{2}$, for $\sqrt{b}+\sqrt{c}>\sqrt{b}+\sqrt{b}$, or $>2\sqrt{b}$; $\therefore$ $\dfrac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\frac{1}{2}$, and $\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\dfrac{a}{2}$.

The corresponding value of $a-x$, $\dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is positive, and greater than $\dfrac{a}{2}$. These values of x, and of $a-x$, show that the point P is situated between B and C, but nearer to B than C. This is evidently a true result, since, under the present supposition, the intensity of the light B is less than that of the light C.

The second value of x, $\dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, or $\dfrac{-a\sqrt{b}}{\sqrt{c}-\sqrt{b}}$, is essentially negative. To interpret this result, we must recollect that if distance to the *right* of a certain point is reckoned *positive*, then distance to the *left* is *negative* (Art. 47); hence, if we consider P″ on the *left* of B, as the point illuminated equally, we ought to represent the distance BP″ by $-x$, and then the distance CP″ would be represented by BP″+BC$=-x+a=a-x$. Under this supposition, the equation of the problem would be

$$\frac{b}{(-x)^2}=\frac{c}{(a-x)^2},\ \text{that is,}\ \frac{b}{x^2}=\frac{c}{(a-x)^2},$$

and the solution of this equation, which is the same as that obtained for P, ought to give the point P″.

The corresponding value of $a-x$, is $\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}=\dfrac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}$. It is positive and greater than a, for $\sqrt{c}>\sqrt{c}-\sqrt{b}$ $\therefore$ $\dfrac{\sqrt{c}}{\sqrt{c}-\sqrt{b}}>1$.

and $\dfrac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}>a$. This represents the distances CP″, and is merely the *sum* of the distances CB and BP″. These results are manifestly correct, and correspond to the circumstances of the problem.

III. Let $b=c$.

The first values of x, and of $a-x$, reduce to $\frac{a}{2}$, which shows that the point illuminated equally, is at the middle of the line BC, a result manifestly true, upon the supposition that the intensities of the two lights are *equal.*

The other two values are reduced to $\frac{a\sqrt{b}}{0}=\infty$. (Art. 136). This result is manifestly true, for the intensities of the two lights being supposed equal, there is no point at any *finite* distance, except the point P, which is equally illuminated by both.

IV. Let $b=c$, and $a=0$.

The first system of values of x and $a-x$, become 0. This is evidently correct, for when the distance BC becomes 0, the distances BP and CP also become 0.

The second system of values of x and $a-x$, become $\frac{0}{0}$; this is the symbol of indetermination (Art. 137).

This result is also correct, for if the two lights are *equal,* and placed at the same point, *every point* on either side of them will be illuminated equally by each.

V. Let $a=0$, b not being $=c$.

In this case, all the values of x and of $a-x$ reduce to 0, which shows that there is only one point equally illuminated by each; viz: the point in which the two lights are placed.

The preceding discussion, affords an example of the precision with which algebra answers to all the circumstances included in the enunciation of a problem.

ART. **239**[a]. Examples for the discussion and illustration of principles.

1. Required a number such, that twice its square, increased by 8 times the number itself, shall be 90. *Ans.* 5, or -9.

How may the question be changed, that the negative answer, taken positively, shall be correct in an arithmetical sense?

2. The difference of two numbers is 4, and their product 21. Required the numbers. *Ans.* $+3$, $+7$, or -3 and -7.

3. A man bought a watch, which he afterward sold for 16 dollars. By the sale his loss per cent. on the first cost of the watch, was the same as the number of dollars which he paid for it. What did he pay for the watch?

Ans. 20 dollars, or 80 dollars.

4. Required a number such, that the square of the number increased by 6 times the number, and this sum increased by 7, the result shall be 2. *Ans.* $x=-1$, or -5.

What do the values of x show? How may the question be changed to be possible in an arithmetical sense?

5. Divide the number 10 into two such parts, that the product shall be 24. *Ans.* 4 and 6, or 6 and 4.

Is there more than one solution? Why?

6. Divide the number 10 into two such parts that the product shall be 26. *Ans.* $5+\sqrt{-1}$, and $5-\sqrt{-1}$.

What do these results show?

7. Divide the number a into two such parts, that their squares shall be to each other as 1 to n.

$$Ans.\ x=\frac{a\sqrt{n}}{1+\sqrt{n}},\ \text{and}\ a-x=\frac{a}{1+\sqrt{n}}.$$

$$\text{Or}\ x=-\frac{a\sqrt{n}}{1-\sqrt{n}},\ \text{and}\ a-x=\frac{a}{1-\sqrt{n}}.$$

What are the parts when $a=12$ and $n=4$? When $a=10$ and $n=1$?

8. The mass of the earth, according to astronomers, is 80 times that of the moon, and their mean distance asunder 240000 miles. Now the attraction of gravitation being directly as the quantity of matter, and inversely as the square of the distance from the center of attraction, it is required to find at what point on the line passing through the centers of these bodies the forces of attraction are equal?

Ans. 215865.5+ miles from the earth,
and 24134.5— " " " moon.
Or, 270210.4+ " " " earth,
and 30210.4+ " beyond the moon from the earth.

This question involves the same principles as the Problem of the lights, and may be discussed in a similar manner. The required results, however, may be obtained directly from the values of x, page 200, calling $a=240000$, $b=80$, and $c=1$.

TRINOMIAL EQUATIONS.

ART. **240.** A trinomial equation is one consisting of *three* terms. The general form is $ax^m+bx^n=c$, in which all the quantities are supposed to be known, except x.

Every trinomial equation of the form

$x^{2n}+2px^n=q$, that is, every equation of three terms containing only *two* powers of the unknown quantity, and in which one of the exponents is *double* the other, can be solved in the same manner as a complete equation of the second degree. Thus,

let $x^n=y$, then $x^{2n}=y^2$, and the equation becomes

$$y^2+2py=q;$$

Whence (Art. 231), $y=-p\pm\sqrt{q+p^2}$;

Substituting x^n for y, and extracting the n^{th} root of both sides,

$$x=\sqrt[n]{-p\pm\sqrt{q+p^2}}.$$

As an example, let it be required to find the value of x in the equation $x^4-2px^2=q$.

Let $x^2=y$, the equation then becomes

$$y^2-2py=q.$$

Whence, $y=+p\pm\sqrt{q+p^2}=x^2$.

$$\therefore\quad x=\pm\sqrt{p\pm\sqrt{q+p^2}}.$$

ART. **241.** BINOMIAL SURDS.—Expressions of the form $A\pm\sqrt{B}$, or $\sqrt{A}\pm\sqrt{B}$, like the value of x just found, are called *Binomial surds*; they are sometimes found in the solution of Trinomial equations of the *fourth* degree, and as it is sometimes possible to reduce them to a more simple form by extracting the square root, it is necessary to consider them here.

We shall first show that it is sometimes possible to extract the square root of $A\pm\sqrt{B}$, or $\sqrt{A}\pm\sqrt{B}$.

$$(2\pm\sqrt{3})^2=7\pm4\sqrt{3};\ \therefore\ \sqrt{7\pm4\sqrt{3}}=2\pm\sqrt{3}.$$

$$(\sqrt{2}\pm\sqrt{3})^2=5\pm2\sqrt{6};\ \therefore\ \sqrt{5\pm2\sqrt{6}}=\sqrt{2}\pm\sqrt{3}.$$

We shall now proceed to show that it is always possible to extract the square root of $A\pm\sqrt{B}$, or $\sqrt{A}\pm\sqrt{B}$, if A^2-B is a perfect square. To do this it is necessary to prove the following theorems.

THEOREM I. — *The square root of no quantity can be partly rational and partly a radical of the second degree.*

For, if possible, let $\sqrt{x}=a+\sqrt{b}$; $\therefore$ squaring both sides,

$x=a^2+2a\sqrt{b}+b$; $\therefore$ $\sqrt{b}=\frac{x-a^2-b}{2a}$, that is, an irrational quantity is equal to a rational quantity, which is *impossible*; hence, the supposition is *impossible* and the theorem is true.

THEOREM II.—*In any equation consisting of rational quantities and radicals of the second degree, the rational quantities on each side are equal, and also the irrational quantities.*

If $x+\sqrt{y}=a+\sqrt{b}$, then $x=a$, and $\sqrt{y}=\sqrt{b}$.

For if x does not $=a$, let $x=a+m$;

$$\therefore a+m+\sqrt{y}=a+\sqrt{b}; \therefore m+\sqrt{y}=\sqrt{b};$$

that is, the square root of a quantity is partly rational and partly irrational, which has been shown by Th. I, to be impossible; hence, $x=a$, and $\sqrt{y}=\sqrt{b}$.

We shall now proceed to find a formula for extracting the square root of $A+\sqrt{B}$.

Assume $\sqrt{A+\sqrt{B}}=\sqrt{x}+\sqrt{y}$,

$A+\sqrt{B}=x+y+2\sqrt{xy}$, by squaring.

By Th. II, $x+y=A(1)$; and $2\sqrt{xy}=\sqrt{B}(2)$;

Squaring equations (1) and (2), we have

$$x^2+2xy+y^2=A^2,$$
$$4xy \quad =B;$$

Subtracting, $x^2-2xy+y^2=A^2-B$; or $(x-y)^2=A^2-B$

Let A^2-B be a perfect square $=C^2$, then $C=\sqrt{A^2-B}$.

$$\therefore (x-y)^2=C^2, \text{ or } x-y=C;$$

But $x+y=A$;

Whence, $x=\frac{A+C}{2}$; and $y=\frac{A-C}{2}$.

and $\sqrt{x}=\pm\sqrt{\frac{A+C}{2}}$; and $\sqrt{y}=\pm\sqrt{\frac{A-C}{2}}$.

$$\therefore \sqrt{x}+\sqrt{y}=\sqrt{A+\sqrt{B}}=\pm\sqrt{\frac{A+C}{2}}\pm\sqrt{\frac{A-C}{2}}.$$

Similarly, $\sqrt{x}-\sqrt{y}=\sqrt{A-\sqrt{B}}=\pm\sqrt{\frac{A+C}{2}}\mp\sqrt{\frac{A-C}{2}}$.

$$\text{or}\quad \sqrt{A+\sqrt{B}}=\pm\left(\sqrt{\frac{A+C}{2}}+\sqrt{\frac{A-C}{2}}\right). \quad \text{(K).}$$

$$\text{and}\quad \sqrt{A-\sqrt{B}}=\pm\left(\sqrt{\frac{A+C}{2}}-\sqrt{\frac{A-C}{2}}\right). \quad \text{(L).}$$

These formulas are easily verified by squaring each side, and substituting A^2-B for C^2.

EXAMPLES FOR PRACTICE.

1. Extract the square root of $31+10\sqrt{6}$.

Here $A=31$, $\sqrt{B}=10\sqrt{6}$; $\therefore A^2-B=C^2=961-600=361$, and $C=19$.

$\therefore A+C=50$, $A-C=12$; $\therefore x=25$, $y=6$;

$\therefore \sqrt{x}+\sqrt{y}=\sqrt{A+\sqrt{B}}=\sqrt{25}+\sqrt{6}=5+\sqrt{6}$.

2. Reduce $\sqrt{np+2m^2-2m\sqrt{np+m^2}}$, to its simplest form.

Here $A=np+2m^2$, and $B=4m^2(np+m^2)$.

$A^2-B=n^2p^2$, and $C=np$, (formula L).

$\therefore A+C=2np+2m^2$, $A-C=2m^2$ $\therefore x=np+m^2$, $y=m^2$.

Formula (L) gives $\sqrt{x}-\sqrt{y}=\pm(\sqrt{np+m^2}-m)$.

PROOF. $\pm(\sqrt{np+m^2}-m)^2=np+2m^2-2m\sqrt{np+m^2}$.

3. Find the square root of $11+6\sqrt{2}$. *Ans.* $3+\sqrt{2}$.

4. Find the square root of $7-4\sqrt{3}$. *Ans.* $2-\sqrt{3}$.

5. Find the square root of $3\pm2\sqrt{2}$. *Ans.* $\sqrt{2}\pm1$.

6. Find the square root of $13+2\sqrt{30}$. *Ans.* $\sqrt{10}+\sqrt{3}$.

7. Find the square root of $17+2\sqrt{60}$. *Ans.* $2\sqrt{3}+\sqrt{5}$.

8. Find the square root of $x-2\sqrt{x-1}$. *Ans.* $\sqrt{x-1}-1$.

9. Find the square root of $2a\sqrt{-1}$. $(A=0)$.

Ans. $\sqrt{a}(1+\sqrt{-1})$.

10. Find the square root of $x+y+z+2\sqrt{xz+yz}$.

Ans. $\sqrt{x+y}+\sqrt{z}$.

11. Reduce to its simplest form $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}$. *Ans.* $\pm 2\sqrt{bc-b^2}$.

ART. **242.** We shall now resume the subject of Trinomial Equations. The general form of Trinomial equations is $x^{2n}+2px^n=q$; but there are several varieties of this form, of which the following are the principal: viz: $x+\sqrt{x}=q$; $x^4+px^2=q$, $x^n+px^{\frac{n}{2}}=q$, $x^{3n}+px^{\frac{3n}{2}}=q$, $x^{4n}+px^{2n}=q$, $(x^2+px+q)^2+b(x^2+px+q)=r$, and $(x^2+px+q)^{2n}+b(x^2+px+q)^n=k$.

It is easily seen that some of these varieties, if developed, would produce very complicated expressions. Yet they may all be solved by the general method given above, that is, by placing the lowest power of the compound term equal to an unknown quantity, and then *substituting* the latter in the given equation. In some cases it is necessary to substitute more than once, as in the following, which is one of the most complicated forms:

$$(x^2+px+q)^{2n}+b(x^2+px+q)^n=k.$$

Let $x^2+px+q=y$, then $(x^2+px+q)^{2n}=y^{2n}$, and the equation becomes $\quad y^{2n}+by^n=k$.

Let $y^n=z$, the equation then becomes $z^2+bz=k$, from which z is found $=\frac{-b\pm\sqrt{b^2+4k}}{2}$; whence $y=\sqrt[n]{\frac{-b\pm\sqrt{b^2+4k}}{2}}$.

Calling this last expression h, for simplicity, and we have

$$x^2+px+q=h,$$

Whence, $\quad x=-\frac{p}{2}\pm\sqrt{h-q+\frac{p^2}{4}}$;

$$\therefore\ x=-\frac{p}{2}\pm\sqrt{\left\{\sqrt[n]{\frac{-b\pm\sqrt{b^2+4k}}{2}}-q+\frac{p^2}{4}\right\}}.$$

EXAMPLES.

1. Given, $x^6-6x^3=16$, to find the value of x.

Assume, $\quad x^3=y$, then $x^6=y^2$, and

$y^2-6y \quad =16$;

Whence, $\quad y=8$, or -2.

Therefore, $\quad x^3=8$, or -2.

and $\quad x=2$, or $-\sqrt[3]{2}$.

It will be shown hereafter, (Art. 396), that every equation has as many roots as there are units in the exponent of the highest power of the unknown quantity. We do not, therefore, by this method, in all cases, obtain all the values of the unknown quantity. Thus, in the preceding example, there are four values of x not determined.

2. Given $5x-4\sqrt{x}=33$, to find the value of x.

Assume, $\sqrt{x}=y$, then $x=y^2$, and

$$5y^2-4y=33;$$

Whence, $y=3$, or $-\frac{11}{5}$;

Consequently, $x=9$, or $\frac{121}{25}$.

3. Given, $\sqrt{x+12}+\sqrt[4]{x+12}=6$, to find the value of x.

Assume, $\sqrt[4]{x+12}=y$; then $\sqrt{x+12}=y^2$, and

$$y^2+y=6;$$

Whence, $y=2$, or -3;

$$\therefore\ \sqrt[4]{x+12}=2, \text{ or } -3;$$

Whence, $x+12=16$, or 81;

and $x=4$, or 69.

Or, without introducing a new letter y, we may consider the whole expression under the radical as the unknown quantity, and proceed to complete the square thus,

$$\sqrt{x+12}+\sqrt[4]{x+12}+\tfrac{1}{4}=6+\tfrac{1}{4}=\tfrac{25}{4};$$

Extracting the root, $\sqrt[4]{x+12}+\frac{1}{2}=\pm\frac{5}{2}$;

$$\sqrt[4]{x+12}=-\tfrac{1}{2}\pm\tfrac{5}{2}=+2, \text{ or } -3.$$

$$x+12=16, \text{ or } 81.$$

Whence, $x=4$, or 69.

The principle of both operations is the same, but the use of the new letter renders the process more easily understood by beginners.

4. Given $3x^2+\sqrt{3x^2+1}=55$, to find the value of x.

Adding 1 to each member, the equation becomes

$$3x^2+1+\sqrt{3x^2+1}=56.$$

The equation may now be solved like the preceding.

The values of x are $+4$, -4, $+\sqrt{21}$, and $-\sqrt{21}$.

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Find the values of x in each of the following examples.

5. $x^4-25x^2=-144$. *Ans.* $x=\pm3$, or ±4.

6. $5x^4+7x^2=6732$. *Ans.* $x=\pm6$, or $\pm\frac{1}{10}\sqrt{-3740}$.

7. $9x^6-11x^3=488$. *Ans.* $x=2$, or $\frac{1}{3}\sqrt[3]{-183}$.

8. $x^3-x^{\frac{3}{2}}=15500$. *Ans.* $x=25$, or $(-124)^{\frac{2}{3}}$.

9. $x^{\frac{5}{6}}+x^{\frac{5}{3}}=1056$. *Ans.* $x=64$, or $(-33)^{\frac{6}{5}}$.

10. $x+5=\sqrt{x+5}+6$. *Ans.* $x=4$, or -1.

11. $2\sqrt{x^2-3x+11}=x^2-3x+8$.

Ans. $x=2$, 1, or $\frac{3}{2}\pm\frac{1}{2}\sqrt{-31}$.

12. $x^2-7x+\sqrt{x^2-7x+18}=24$.

Ans. $x=9$, -2, or $\frac{1}{2}(7\pm\sqrt{173})$.

13. $(x^2-9)^2=3+11(x^2-2)$. *Ans.* $x=\pm5$, or ±2.

14. $\left(x+\frac{8}{x}\right)^2+x=42-\frac{8}{x}$.

Ans. $x=4$, 2, or $\frac{1}{2}(-7\pm\sqrt{17})$.

15. $x^4\left(1+\frac{1}{3x}\right)^2-(3x^2+x)=70$.

Ans. $x=3$, $-3\frac{1}{3}$, or $\frac{1}{6}(-1\pm\sqrt{-251})$.

16. $x\sqrt{\left(\frac{6}{x}-x\right)}=\frac{1+x^2}{\sqrt{x}}$. *Ans.* $x=\pm\sqrt{(1\pm\frac{1}{2}\sqrt{2})}$.

Art. **243.** In the preceding examples the form of the trinomial equation, is either given or easily ascertained; but it sometimes happens that questions are given, in which the *compound term* is not presented to view, but which may be reduced to the form of a trinomial equation by the following method:

If the greatest exponent of the unknown quantity is not *even*, it must be made *even*, by multiplying both members of the equation by the unknown quantity. Then extract the square root to two or three terms, and if we find a remainder (omitting known terms if necessary), which is any *multiple* or any *part* of the root already found, the given equation may be reduced to a trinomial, of which the *compound term* will be the root already found.

Example. Given, $x^3-4ax^2-2a^2x+12a^3=\frac{16a^4}{x}$, to find x.

Multiplying both sides by x, and transposing, we have

$$x^4-4ax^3-2a^2x^2+12a^3x-16a^4=0.$$

Proceeding to extract the square root, we have the following

OPERATION.

$$\begin{array}{r|l} x^4-4ax^3-2a^2x^2+12a^3x-16a^4 & x^2-2ax. \\ x^4 & \\ 2x^2-2ax \quad -4ax^3-2a^2x^2 & \\ -4ax^3+4a^2x^2 & \end{array}$$

Remainder $\quad -6a^2x^2+12a^3x-16a^4$;

or, $\quad -6a^2(x^2-2ax)-16a^4$.

Hence, the given equation may be written thus:

$$(x^2-2ax)^2-6a^2(x^2-2ax)-16a^4=0.$$

Assuming $x^2-2ax=y$, we find $y=8a^2$, or $-2a^2$; then from the equation $x^2-2ax=8a^2$, or $-2a^2$, we find

$$x=4a,\ -2a, \text{ or } a\pm a\sqrt{-1}.$$

2. $x^4-2x^3-2x^2+3x=108$.

Ans. $x=4$, -3, or $\frac{1}{2}(1\pm\sqrt{-35})$.

3. $x^4-2x^3+x=30$. *Ans.* $x=3$, -2, or $\frac{1}{2}(1\pm\sqrt{-19})$.

4. $x^3-6x^2+11x-6=0$. *Ans.* $x=1$, 2, or 3.

5. $x^4-6x^3+5x^2+12x=60$.

Ans. $x=5$, -2, or $\frac{1}{2}(3\pm\sqrt{-15})$.

6. $x^4-8x^3+10x^2+24x=-5$. *Ans.* $x=5$, -1, or $2\pm\sqrt{5}$.

7. $4x^4+\frac{x}{2}=4x^3+33$. *Ans.* $x=2$, $-\frac{3}{2}$, or $\frac{1}{4}(1\pm\sqrt{-43})$.

8. $\frac{x}{14}-\frac{30}{7x^3}+\frac{12+\frac{1}{2}x}{3x}=\frac{7}{2x^2}+1\frac{1}{8}$.

Ans. $x=4$, 3, or $\frac{1}{2}(7\pm\sqrt{69})$.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

ART. **244.** Equations of the second degree, containing two or more unknown quantities, may be divided into two classes.

1st. Pure Equations.

2nd. Adfected Equations.

The first class embraces those equations that may be solved without completing the square; the second, those in the solution of which it is necessary to complete the square. The same equations, however, may sometimes be solved by both methods.

Art. 245. Pure Equations.—Pure equations may in general be reduced to the solution of one of the following forms, or pairs of equations.

(1.) $\left.\begin{matrix} x+y=a \\ xy=b \end{matrix}\right\}$. (2.) $\left.\begin{matrix} x-y=c \\ xy=b \end{matrix}\right\}$. (3.) $\left.\begin{matrix} x^2+y^2=a \\ x^2-y^2=b \end{matrix}\right\}$.

We shall explain the general method of solution in each of these cases.

1. To solve $x+y=a$ (1), and $xy=b$ (2), we must find $x-y$.

Squaring Eq. (1), $x^2+2xy+y^2=a^2$;

Multiplying Eq. (2) by 4, $4xy \quad =4b$;

Subtracting, $x^2-2xy+y^2=a^2-4b$,

or, $(x-y)^2=a^2-4b$;

Whence, $x-y=\pm\sqrt{a^2-4b}$;

But, $x+y=a$;

Adding and dividing by 2, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}$.

Subtracting and dividing by 2, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{a^2-4b}$.

The pair of equations (2) is solved in the same manner, except that in finding $x+y$ we must *add* 4 times the second equation to the square of the first.

The pair of equations (3) is solved merely by adding and subtracting, then dividing by 2 and extracting the square root.

EXAMPLES IN PURE EQUATIONS.

1. Given, $x^2+y^2=d$ (1), and $x+y=a$ (2), to find x and y.

Squaring Eq. (2), $x^2+2xy+y^2=a^2$;

But, $x^2 \quad +y^2=d$ (1).

Subtracting, $2xy=a^2-d$, (3).

Take (3) from (1); $x^2-2xy+y^2=2d-a^2$,

$\therefore\ x-y=\pm\sqrt{2d-a^2}$.

Whence, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{2d-a^2}$, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{2d-a^2}$.

2. Given, $x^2+xy+y^2=91$(1), and $x+\sqrt{xy}+y=13$(2), to find x and y.

Divide Eq. (1) by (2), $x-\sqrt{xy}+y=7$. (3).

But, $x+\sqrt{xy}+y=13$. (2).

By subtracting, $2\sqrt{xy}=6$. (4).

By adding, $x+y=10.$ (5)

Squaring, (5), $x^2+2xy+y^2=100:$

Squaring, (4), $4xy \quad = 36;$

$x^2-2xy+y^2=64, \therefore\ x-y=\pm 8.$

But, $x+y=10$, whence, $x=9$ or 1, and $y=1$ or 9.

Equations of higher degrees than the second, that can be solved by simple methods, are usually classed with pure equations of the second degree.

3. Given, $x^{\frac{1}{4}}+y^{\frac{1}{5}}=6$, and $x^{\frac{3}{4}}+y^{\frac{3}{5}}=126$, to find x and y.

In all cases of fractional exponents, it renders the operations more simple to learners, to make such substitutions as will render the exponents *integral*. In this example, let $x^{\frac{1}{4}}=P$, and $y^{\frac{1}{5}}=Q$; then $x^{\frac{3}{4}}=P^3$, and $y^{\frac{3}{5}}=Q^3$. The given equations then become,

$P+Q=6$ (1),

$P^3+Q^3=126$ (2).

Dividing Eq. (2) by (1), $P^2-PQ+Q^2=21;$

Squaring Eq. (1), $P^2+2PQ+Q^2=36;$

Subtracting, $3PQ=15, \therefore\ PQ=5.$

Having $P+Q=6$, and $PQ=5$, by the method explained in form (1), we readily find $P=5$ or 1, and $Q=1$ or 5.

Whence, $x=625$ or 1, and $y=1$ or 3125.

4. Given, $(x-y)(x^2-y^2)=160$ (1),

$(x+y)(x^2+y^2)=580$ (2), to find x and y.

$x^3-x^2y-xy^2+y^3=160$ (1), by multiplying.

$x^3+x^2y+xy^2+y^3=580$ (2), " "

$2x^2y+2xy^2=420$ (3), by subtracting.

Add (3) to (2), $x^3+3x^2y+3xy^2+y^3=1000.$

Extract cube root, $x+y=10.$

From (3), $xy(x+y)=210; \therefore\ xy=21.$

From $x+y=10$, and $xy=21$, we readily find $x=7$ or 3, and $y=3$, or 7.

Let the following examples be solved by the preceding or similar methods.

5. $x-y=2,$ *Ans.* $x=15$, or -13;
$x^2+y^2=394.$ $y=13$, or -15.

6. $x^2+y^2=13,$ *Ans.* $x=\pm 3$;
$xy = 6.$ $y=\pm 2.$

7. $2x+y=7$, $4x^2+y^2=25$. *Ans.* $x=2$, or $\frac{3}{2}$; $y=3$, or 4.

8. $x^2-y^2=16$, $x-y=2$. *Ans.* $x=5$; $y=3$.

9. $x+y=11$, $x^3+y^3=407$. *Ans.* $x=7$, or 4; $y=4$, or 7.

10. $7(x^3+y^3)=9(x^3-y^3)$, $x^2y-y^2x=16$. *Ans.* $x=4$; $y=2$.

11. $x^2+xy=84$, $x^2-y^2=24$. *Ans.* $x=\pm7$; $y=\pm5$.

12. $x^3+y^3=152$, $x^2-xy+y^2=19$. *Ans.* $x=5$, or 3; $y=3$, or 5.

13. $x^2+y^2+xy=208$, $x+y=16$. *Ans.* $x=12$, or 4; $y=4$, or 12.

14. $x^3-y^3=7xy$, $x-y=2$. *Ans.* $x=4$, or -2; $y=2$, or -4.

15. $x^4+x^2y^2+y^4=91$, $x^2+xy+y^2=13$. *Ans.* $x=\pm3$, or ±1; $y=\pm1$, or ±3.

16. $x-y=\sqrt{x}+\sqrt{y}$, $x^{\frac{3}{2}}-y^{\frac{3}{2}}=37$. *Ans.* $x=16$, or 9; $y=9$, or 16.

17. $x^{\frac{1}{4}}+y^{\frac{1}{3}}=5$, $x^{\frac{2}{4}}+y^{\frac{2}{3}}=13$. *Ans.* $x=16$, or 81; $y=27$, or 8.

18. $x^{\frac{1}{3}}+y^{\frac{1}{3}}=5$, $x+y=35$. *Ans.* $x=8$, or 27; $y=27$, or 8.

19. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=4$, $x^{\frac{3}{2}}+y^{\frac{3}{2}}=28$. *Ans.* $x=9$, or 1; $y=1$, or 9.

20. $x^3+y^3=351$, $xy=14$. *Ans.* $x=7$, or 2; $y=2$, or 7.

21. $x+y=4$, $x^4+y^4=82$. *Ans.* $x=3$, or 1; $y=1$, or 3.

22. $x(y+z)=a$, $y(x+z)=b$, $z(x+y)=c$.

Ans. $x=\pm\sqrt{\dfrac{(a+c-b)(a+b-c)}{2(b+c-a)}}$,

$y=\pm\sqrt{\dfrac{(a+b-c)(b+c-a)}{2(a+c-b)}}$,

$z=\pm\sqrt{\dfrac{(b+c-a)(a+c-b)}{2(a+b-c)}}$.

ART. **246.** ADFECTED EQUATIONS.—The most general form of an equation of the second degree, containing two unknown quantities, is,

$$ax^2+bxy+cx+dy^2+ey+f=0.$$

By arranging the terms according to the powers of x, and dividing by the coëfficient of the first term, two equations of the second degree containing two unknown quantities, may be reduced to the following forms:

$$x^2+(a\,y+b\,)x+c\,y^2+d\,y+e=0 \qquad (1),$$
$$x^2+(a'y+b')x+c'y^2+d'y+e'=0 \qquad (2).$$

To find the values of either of the unknown quantities, we must eliminate the other. We shall now show that this operation produces an equation of the *fourth* degree.

By subtracting the second equation from the first, and making $a-a'=a''$, $b-b'=b''$, &c., we have

$$(a''y+b'')x+c''y^2+d''y+e''=0.$$

Whence, $$x=-\frac{c''y^2+d''y+e''}{a''y+b}.$$

Substituting this value of x in the first equation, we get

$$\frac{(c''y^2+d''y+e'')^2}{(a''y+b'')^2}-(ay+b)\frac{c''y^2+d''y+e''}{a''y+b''}+cy^2+dy+e=0;$$

and multiplying by $(a''y+b'')^2$,

$$(c''y^2+d''y+e'')^2-(ay+b)(a''y+b'')(c''y^2+d''y+e'')+(cy^2+dy+e)(a''y+b'')^2=0,$$

an equation of the *fourth* degree.

Hence, in general, *the solution of two equations of the second degree, containing two unknown quantities, depends upon the solution of an equation of the fourth degree, containing one unknown quantity.*

ART. **247.** There are, however, two cases in which two equations of the second degree, containing two unknown quantities, may always be solved as equations of the second degree; viz:

CASE I.—When one of the equations, containing two unknown quantities, rises only to the *first* degree, and the other to the *second* degree, the values of the unknown quantities may be found by the solution of an equation of the second degree.

Given, $$ax+by=c \quad (1),$$
$$dx^2+exy+fy^2+gx+hy=k \quad (2),$$ to find x and y.

Here equation (1) is evidently the general equation of the first degree between x and y, and equation (2) the general equation of the second degree between x and y.

From eq. (1), we have $x=\frac{c-by}{a}$; and substituting this value of x in equation (2), and multiplying every term by a^2, we have

$$d(c^2-2bcy+b^2y^2)+aey(c-by)+a^2fy^2+ag(c-by)+a^2hy=a^2k,$$

an equation of the second degree, from which the value of y may be found by the rule (Art. 231). Having the value of y, that of x may be found from the equation $x=\frac{c-by}{a}$.

CASE II.—When both equations of the second degree are *homogeneous*, that is, have the sum of the indices of the unknown quantities the same in every term which contains unknown quantities, they may be solved by an equation of the second degree.

Given, $ax^2+bxy+cy^2=d \quad (1),$

$a'x^2+b'xy+c'y^2=d' \quad (2)$, to find x and y.

Let $y=tx$, where t is a third unknown quantity, termed an *auxiliary* quantity. Substituting this value of x in the two equations, we have

$$ax^2+btx^2+ct^2x^2=x^2(a+bt+ct^2)=d \quad (3),$$
$$a'x^2+b'tx^2+c't^2x^2=x^2(a'+b't+c't^2)=d' \quad (4).$$

From eq. (3) we find $x^2=\frac{d}{a+bt+ct^2}$.

and from eq. (4) $x^2=\frac{d'}{a'+b't+c't^2}$.

$$\therefore \frac{d}{a+bt+ct^2}=\frac{d'}{a'+b't+c't^2}.$$

or, $d(a'+b't+c't^2)=d'(a+bt+ct^2)$,

an equation of the second degree, from which the value of t may be found, (Art. 231), and thence x from the equation

$x^2=\frac{d}{a+bt+ct^2}$, and thence y from the equation $y=tx$.

ART. **248.** When two equations of the second degree are *symmetrical* with respect to the two unknown quantities, that is, when the two unknown quantities are *similarly* involved, they may frequently be solved by substituting for the unknown quantities, the sum and difference of two others.

Example. Given $x+y=a$ (1),

$x^5+y^5=b$ (2), to find x and y,

Let $x=s+z$, and $y=s-z$, then $s=\frac{a}{2}$.

$$x^5=s^5+5s^4z+10s^3z^2+10s^2z^3+5sz^4+z^5,$$
$$y^5=s^5-5s^4z+10s^3z^2-10s^2z^3+5sz^4-z^5\,;$$
$$x^5+y^5=2s^5+20s^3z^2+10sz^4=b.$$

By substituting the value of $s=\frac{a}{2}$, and reducing, we find

$$z^4+\frac{a^2}{2}z^2=\frac{16b-a^5}{80a}.$$

From this equation, by completing the square, we find

$$z=\pm\sqrt{\left\{-\frac{a^2}{4}\pm\sqrt{\left(\frac{a^5+4b}{20a}\right)}\right\}}.$$

$$x=s+z=\frac{a}{2}\pm\sqrt{\left\{-\frac{a^2}{4}\pm\sqrt{\left(\frac{a^5+4b}{20a}\right)}\right\}}.$$

$$y=s-z=\frac{a}{2}\mp\sqrt{\left\{-\frac{a^2}{4}\pm\sqrt{\left(\frac{a^5+4b}{20a}\right)}\right\}}.$$

ART. **249.** An artifice that is often used with advantage, consists in adding such a number to both members of an equation, as will render the side containing the unknown quantities, a trinomial equation that can be resolved by completing the square, (Art. 240). The following is an example :

2. Given, $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}=\frac{27}{4}$ (1),

and $x^2+y^2=20$ (2), to find x and y.

Since, $\left(\frac{x}{y}+\frac{y}{x}\right)^2=\frac{x^2}{y^2}+2+\frac{y^2}{x^2}$; add 2 to each side of eq. (1), and then $\frac{1}{4}$ to complete the square.

$$\therefore\ \left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)+\tfrac{1}{4}=\frac{27}{4}+2+\tfrac{1}{4}=9\,;$$

Whence, $\frac{x}{y}+\frac{y}{x}=\pm3-\frac{1}{2}=\frac{5}{2}$ or $-\frac{7}{2}$.

Let $\frac{x}{y}+\frac{y}{x}=\frac{5}{2}$; then, $\frac{x^2+y^2}{xy}$, or $\frac{20}{xy}=\frac{5}{2}$;

whence, $xy=8$, and $2xy=16$

From the equation $x^2+y^2=20$, and $2xy=16$, we readily find $x=\pm4$, and $y=\pm2$.

By taking $\frac{x}{y}+\frac{y}{x}=-\frac{7}{2}$, two other values of x and y may be found.

ART. **250.** Another artifice consists in considering the sum, difference, product, or quotient of the two unknown quantities, as a *single unknown quantity*, and first finding its value. Thus in example 9 following, the value of xy should be found from the first equation; and in example 10, that of $\frac{x}{y}$.

Other unknown *auxiliaries* may also sometimes be employed with advantage, but their use, as well as that of various other expedients that may be employed, can only be learned by experience, while much will always depend on the judgment and tact of the operator.

EXAMPLES FOR PRACTICE.

NOTE.—In some of the examples all the values of the unknown quantities are not given; those omitted are generally imaginary.

3. $x^2+y^2+x+y=330$, $x^2-y^2+x-y=150$. *Ans.* $x=15$, or -16; $y=9$, or -10.

4. $x+4y=14$, $y^2+4x=2y+11$. *Ans.* $x=2$, or -46; $y=3$, or 15.

5. $2y-3x=14$, $3x^2+2(y-11)^2=14$. *Ans.* $x=2$, or $1\frac{1}{5}$; $y=10$, or $8\frac{4}{5}$.

6. $x-y=2$, $\frac{x}{y}-\frac{y}{x}=1\frac{1}{15}$. *Ans.* $x=5$, or $\frac{3}{4}$; $y=3$, or $-1\frac{1}{4}$.

7. $3x^2+xy=18$, $4y^2+3xy=54$. (Art. 247, Case II.) *Ans.* $x=\pm2$, or $\pm2\sqrt{3}$; $y=\pm3$, or $\mp3\sqrt{3}$.

8. $x^2+xy=10$, $xy+2y^2=24$. *Ans.* $x=\pm2$, or $\pm5\sqrt{2}$; $y=\pm3$, or $\pm4\sqrt{2}$.

9. $4xy=96-x^2y^2$, $x+y=6$. *Ans.* $x=2, 4$, or $3\pm\sqrt{21}$; $y=4, 2$, or $3\mp\sqrt{21}$.

10. $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9}$, $x-y=2$. *Ans.* $x=5$, or $\frac{17}{10}$; $y=3$, or $-\frac{3}{10}$.

11. $x^2y^2=180-8xy,$ *Ans.* $x=5$, or 6;
$x+3y=11.$ $y=2$, or $\frac{5}{3}$.

12. $x+y+\sqrt{x+y}=12,$ *Ans.* $x=5$, or 4;
$x^2+y^2=41.$ $y=4$, or 5.

13. $x+y+x^2+y^2=18,$ *Ans.* $x=3, 2$, or $-3\pm\sqrt{3}$;
$xy=6.$ $y=2, 3$, or $-3\mp\sqrt{3}$.

14. $x^2+3x+y=73-2xy,$ *Ans.* $x=4$, or 16;
$y^2+3y+x=44.$ $y=5$, or -7.

15. $\frac{1}{x}+\frac{1}{y}=a,$ *Ans.* $x=\frac{2}{a\pm\sqrt{2b-a^2}}$;
$\frac{1}{x^2}+\frac{1}{y^2}=b.$ $y=\frac{2}{a\mp\sqrt{2b-a^2}}$.

16. $xy+xy^2=12,$ *Ans.* $x=2$, or 16;
$x+xy^3=18.$ $y=2$, or $\frac{1}{2}$.

17. $x^2+y^2-x-y=78,$ *Ans.* $x=9$, or 3;
$x+y+xy=39.$ $y=3$, or 9.

18. $(x+y)^2-3y=28+3x,$ *Ans.* $x=5$, or $3\frac{1}{2}$;
$2xy+3x=35.$ $y=2$, or $3\frac{1}{2}$.

19. $\left(\frac{3x}{x+y}\right)^{\frac{1}{2}}+\left(\frac{x+y}{3x}\right)^{\frac{1}{2}}=2,$ *Ans.* $x=6$, or $-4\frac{1}{2}$;
$xy-(x+y)=54.$ $y=12$, or -9.

20. $x^2+4(x^2+3y+5)^{\frac{1}{2}}=55-3y,$ *Ans.* $x=5$, or $-\frac{5}{7}$.
$6x-7y=16.$ $y=2$, or $-\frac{430}{49}$.

21. $\frac{y}{(x+y)^{\frac{3}{2}}}+\frac{\sqrt{x+y}}{y}=\frac{17}{4\sqrt{x+y}},$ *Ans.* $x=6$, or 3;
$x=y^2+2.$ $y=2$, or 1.

QUESTIONS PRODUCING SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE, INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

ART. **251.** 1. There are two numbers, whose sum multiplied by the less, is equal to 4 times the greater, but whose sum multiplied by the greater is equal to 9 times the less. What are the numbers? *Ans.* 3.6, and 2.4.

2. There is a number consisting of two digits, which being multiplied by the digit in tens place, the product is 46; but if

the sum of the digits be multiplied by the same digit, the product is only 10. Required the number. *Ans.* 23.

3. What two numbers are those whose difference multiplied by the difference of their squares, will produce 32, and whose sum multiplied by the sum of their squares, is 272? *Ans.* 5 and 3.

4. The product of two numbers is 10, and the sum of their cubes 133. Required the numbers. *Ans.* 2 and 5.

5. If the sum of two numbers be multiplied by the greater, and that product be divided by the less, the quotient will be 24; but if their sum be multiplied by the less, and that product be divided by the greater, the quotient will be 6. Required the numbers. *Ans.* 4 and 8.

NOTE.—The preceding problems may be solved by pure equations.

6. The difference of two numbers is 15, and half their product is equal to the cube of the less number; find them.

Ans. 18 and 3.

7. The product of two numbers is 24, and their sum multiplied by their difference is 20; find them. *Ans.* 4 and 6.

8. What two numbers are those whose sum multiplied by the greater is 120, and whose difference multiplied by the less is 16?

Ans. 2 and 10.

9. What two numbers are those whose sum added to the sum of their squares is 42, and whose product is 15?

Ans. 3 and 5.

10. Find two numbers such, that their product added to their sum shall be 47, and their sum taken from the sum of their squares shall leave 62. *Ans.* 5 and 7.

11. Find two numbers such, that their sum, their product, and the difference of their squares shall be all equal to each other.

Ans. $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

12. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. *Ans.* $\frac{1}{2}\sqrt{5}$, and $\frac{1}{4}(5+\sqrt{5})$.

13. A grocer sold 80 pounds of mace and 100 pounds of cloves for 65 dollars; but he sold 60 pounds more of cloves for 20 dollars than he did of mace for 10 dollars. Required the price of a pound of each. *Ans.* Mace 50 cts, cloves 25 cts.

14. A and B gained by trading 100 dollars. Half of A's stock was less than B's by 100 dollars, and A's gain was three twen-

tieth's of B's stock. Supposing the gains in proportion to the stock, required the stock and gain of each.

Ans. A's stock \$600, B's \$400;
A's gain \$60, B's \$40.

15. The product of two numbers added to their sum is 23; and 5 times their sum taken from the sum of their squares leaves 8. Required the numbers. *Ans.* 2 and 7.

16. There are three numbers, the difference of whose differences is 5; their sum is 44, and continued product 1950; find the numbers. *Ans.* 25, 13, 6.

17. Divide the number 26 into three such parts that their squares shall have equal differences, and that the sum of those squares shall be 300. *Ans.* 14, 10, 2.

18. The number of men in both fronts of two columns of troops, A and B, where each consisted of as many ranks as it had men in front, was 84; but when the columns changed ground, and A was drawn up with the front that B had, and B with the front that A had, then the number of ranks in both columns was 91. Required the number of men in each column.

Ans. 2304, and 1296.

ART. **252.** FORMULÆ — GENERAL SOLUTIONS.— A *general* solution to a problem producing an equation of the second degree, like one of the first degree, gives rise to a *formula* (Art. 162), which expressed in ordinary language, furnishes a *rule.* We shall illustrate the subject by a few examples.

1. Two men, A and B, bought 300 (a) acres of land for 600 (b) dollars, of which A paid 300 (c) dollars, and B 300 ($b-c$) dollars. For certain reasons they agreed to divide the land so that B should pay 75 cents (d *dollars*) per acre more than A. How much land did each man get, and what did he pay per acre?

GENERAL SOLUTION. Let $x=$ cost of A's land per acre,

then $x+d=$ cost of B's land per acre;

also, $\frac{c}{x}=$ acres A got, and $\frac{b-c}{x+d}=$ acres B got.

$$\therefore \frac{c}{x}+\frac{b-c}{x+d}=a, \text{ by the problem.}$$

Clearing of fractions and reducing,

$$x^2-\frac{b-ad}{a}x=\frac{cd}{a}.$$

$$x^2-\frac{b-ad}{a}x+\frac{(b-ad)^2}{4a^2}=\frac{(b-ad)^2+4acd}{4a^2};$$

$$x=\frac{b-ad}{2a}\pm\frac{\sqrt{(b-ad)^2+4acd}}{2a};$$

$$\text{or, } x=\frac{1}{2a}\{\sqrt{(b-ad)^2+4cad}+b-ad\}.$$

This formula gives the following rule for finding the amount paid per acre, by him who paid least per acre:

RULE.—*Find the cost of the whole number of acres at the difference between the prices per acre of the different pieces of land; subtract this from the amount paid for the whole land; square this remainder and add to it the cost of the whole number of acres at the difference between the prices per acre, multiplied by four times the sum of money paid by him who paid least per acre; extract the square root of this sum, add to the square root thus found, the remainder that was squared, and divide the sum by twice the whole number of acres; the quotient will be the amount paid per acre by him who paid least per acre. Having this, every other requirement in the question is easily found.*

For the particular case the results are,

A paid \$2.443 per acre, and got 122.8 acres nearly.

B paid \$1.693 per acre, and got 177.2 acres nearly.

2. Investigate a formula for finding two numbers, x and y, of which the sum of the squares is s, and difference of the squares d.

Ans. $x=\frac{1}{2}\sqrt{2(s+d)}$; $y=\frac{1}{2}\sqrt{2(s-d)}$.

3. Investigate a formula for finding two numbers, x and y, of which the difference is d, and the product p.

Ans. $x=\frac{1}{2}(d+\sqrt{d^2+4p})$;
$y=\frac{1}{2}(-d+\sqrt{d^2+4p})$.

4. Investigate a formula for finding a number, x, of which the sum of the number and its square root is s.

Ans. $x=s+\frac{1}{2}-\sqrt{s+\frac{1}{4}}$.

5. The same when the difference of the number x, and its square root is d.

Ans. $x=d+\frac{1}{2}+\sqrt{d+\frac{1}{4}}$.

6. Given $x+y=s$, and $xy=p$, to find the value of x^2+y^2 x^3+y^3, and x^4+y^4, in terms of s and p.

Ans. $x^2+y^2=s^2-2p$;
$x^3+y^3=s^3-3ps$;
$x^4+y^4=s^4-4ps^2+2p^2$.

Let the student express each of the preceding formulæ in the form of a RULE, and exemplify its use, by forming examples with particular numbers, and then solving them.

NOTE.— In the great variety of equations that occur, which may be solved as equations of the second degree, it is not to be supposed that Rules can be given for every operation necessary for their solution. The artifices by which algebraic calculations are abridged, are numerous, and their successful application can be learned only by practice. In the following article, which is intended only for advanced students, we shall exhibit some of these artifices.

ART. **253.** SPECIAL SOLUTIONS AND EXAMPLES.—If an equation can be placed under the form

$$(x+a)X=0,$$

in which X represents an expression involving x, the unknown quantity; since the equation will be satisfied by making either factor $=0$, we have $x+a=0$, and $X=0$. Therefore, $x=-a$, is one solution of the equation, and the other values of x will be found by solving the equation $X=0$. Hence, whenever an equation is simplified by division, or the omission of a factor, if the divisor or factor contains the unknown quantity, one solution, at least, of the equation will be found by putting that divisor or factor equal to 0. Thus, the equation $x^3-x^2-4x+4=0$, may be placed under the form $(x-2)(x^2+x-2)=0$. Hence, $x-2=0$, or $x=+2$, and from the other factor we find $x=+1$, or -2.

The difficulty to be overcome in applying this artifice, consists in finding the factors of the given equation, or in transforming it so that it can be readily separated into factors.

Ex. 1. Given, $x-1=2+\frac{2}{\sqrt{x}}$, to find x.

Since, $x-1=(\sqrt{x}+1)(\sqrt{x}-1)$ and $2+\frac{2}{\sqrt{x}}=\frac{2}{\sqrt{x}}(\sqrt{x}+1)$,

$$\therefore\ (\sqrt{x}+1)(\sqrt{x}-1)=\frac{2}{\sqrt{x}}(\sqrt{x}+1);$$

$$\therefore\ \sqrt{x}+1=0, \text{ and } x=(-1)^2=1.$$

Also, $\sqrt{x}-1=\frac{2}{\sqrt{x}}$, by dividing by $\sqrt{x}+1$.

Whence, $\sqrt{x}=2$, or -1; and $x=4$, or 1.

2. $x^3-3x=2$. (Add $2x$ to each side.) *Ans.* $x=-1$, or 2.

3. $x^2-\frac{2}{3x}=1\frac{4}{9}$. $\left(\text{Transpose } \frac{4}{9} \text{ and } \frac{2}{3x}.\right)$

Ans. $x=-\frac{2}{3}$, or $\frac{1}{3}(1\pm\sqrt{10})$.

4. $2x^3-x^2=1$. *Ans.* $x=1$, or $\frac{1}{4}(-1\pm\sqrt{-7})$.

5. $x^3-3x^2+x+2=0$. *Ans.* $x=2$, or $\frac{1}{2}(1\pm\sqrt{5})$.

6. $x^3=6x+9$. *Ans.* $x=3$, or $\frac{1}{2}(-3\pm\sqrt{-3})$.

7. $x+7x^{\frac{1}{3}}=22$. *Ans.* $x=8$, or $29\pm7\sqrt{-10}$.

$x+7x^{\frac{1}{3}}-22=(x-8)+7(x^{\frac{1}{3}}-2)$. $x^{\frac{1}{3}}-2$ is a divisor.

8. $x^4+\frac{13}{3}x^3-39x=81$.

Ans. $x=\pm3$, or $\frac{1}{6}(-13\pm\sqrt{-155}$.

An artifice that is frequently employed, consists in adding to each side of the equation, such a number or quantity as will render both sides perfect squares.

9. Given, $x=\frac{12+8\sqrt{x}}{x-5}$, to find x.

Clearing of fractions, $x^2-5x=12+8\sqrt{x}$.

Add $x+4$ to each side, and extract the square root.

$$x-2=\pm(4+\sqrt{x}).$$

From which we easily find $x=9$, 4, or $\frac{1}{2}(-3\pm\sqrt{-7})$.

10. $x-3=\frac{3+4\sqrt{x}}{x}$. *Ans.* $x=\frac{1}{2}(7\pm\sqrt{13})$, $\frac{1}{2}(-1\pm\sqrt{-3})$.

11. $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$. Add $\frac{1}{x^2}$ to each side.

Ans. $x=2$, $-\frac{8}{7}$, or $\frac{1}{7}(-3\pm\sqrt{93})$

12. $x^4+\frac{17x^3}{2}-34x=16$. *Ans.* $x=\pm2$, -8, or $-\frac{1}{2}$.

Transpose $34x$, and add $\left(\frac{17x}{4}\right)^2$ to each side.

13. $x^4\left(1+\frac{1}{3x}\right)^2-(3x^2+x)=70$.

Ans. $x=3$, $-3\frac{1}{3}$, or $\frac{1}{6}(-1\pm\sqrt{-251})$.

Divide by x^4 and add $\frac{9}{4x^4}$ to each side.

14. $\frac{18}{x^2}+\frac{81-x^2}{9x}=\frac{x^2-65}{72}$. Multiply by 2, and add $\frac{8x}{36}+\frac{81}{36}$ to each side. *Ans.* 9, —4, or —9.

15. $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$.

Multiply both sides by 3, transpose $\frac{841}{x^2}$ and $\frac{1}{x^2}$, and add 1 to each side to complete the square.

Ans. $x=2$, $-\frac{14}{9}$, or $\frac{1}{9}(-2\pm\sqrt{-266})$.

We shall now present a few solutions giving examples of other artifices.

16. $\frac{1+x^4}{(1+x)^4}=a$; to find x.

$$1+x^4=a(1+x)^4=a(1+4x+6x^2+4x^3+x^4),$$

$$(1-a)(1+x^4)=4a(x+x^3)+6ax^2,$$

dividing by x^2, $(1-a)\left(x^2+\frac{1}{x^2}\right)=4a\left(x+\frac{1}{x}\right)+6a$,

$$x^2+\frac{1}{x^2}-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a},$$

$$\left(x+\frac{1}{x}\right)^2-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a}+2=\frac{2+4a}{1-a}.$$

Complete the square, and find the value of $x+\frac{1}{x}$, which is $\frac{2a\pm\sqrt{2(1+a)}}{1-a}$, call this $2p$, and we then find $x=p\pm\sqrt{p^2-1}$.

17. $x^{x+y}=y^{4a}$, and $y^{x+y}=x^a$, to find x and y.

From 1st equation $y=x^{\frac{x+y}{4a}}$,

" 2nd " $y=x^{\frac{a}{x+y}}$;

$$\therefore x^{\frac{x+y}{4a}}=x^{\frac{a}{x+y}}, \text{ and } \frac{x+y}{4a}=\frac{a}{x+y};$$

$$(x+y)^2=4a^2, \text{ or } x+y=2a;$$

but $x^a=y^{2a}$, since $x+y=2a$,

$\therefore$ $x=y^2$, and $y^2+y=2a$.

Whence, $y=\frac{1}{2}(-1\pm\sqrt{8a+1})$, and $x=\frac{1}{2}(4a+1\mp\sqrt{8a+1})$.

When two unknown quantities are found in an equation, in the form of $x+y$ and xy, it is generally expedient to put their sum $x+y=s$, and their product $xy=p$.

18. Given $(x+y)(xy+1)=18xy$ (1),

$(x^2+y^2)(x^2y^2+1)=208x^2y^2$ (2), to find x and y.

Let $x+y=s$, and $xy=p$, then

$$s(p+1)=18p \quad (1),$$

and $$(s^2-2p)(p^2+1)=208p^2 \quad (2).$$

From the square of (1) take (2), and after dividing by $2p$, we have $$s^2+p^2+1=58p. \quad (3),$$

but $$2s(p+1)=36p \text{ for } (2),$$

and $$2p=2p.$$

Adding, $$(s+p+1)^2=96p,$$

$$s+p+1=4\sqrt{6p};$$

but $$p+1=\frac{18p}{s};$$

$\therefore\ s=4\sqrt{6p}-\frac{18p}{s}$, or $s^2-4s\sqrt{6p}=-18p$, from

which, $s=3\sqrt{6p}$, or $\sqrt{6p}$.

But, $p+1=s$, $=3\sqrt{6p}$, or $\sqrt{6p}$,

Whence, $p=26\pm\sqrt{675}$, or $2\pm\sqrt{3}$,

and $s=\pm3\sqrt{\{6(26\pm\sqrt{675})\}}$, or $\pm3\sqrt{\{6(2\pm\sqrt{3})\}}$.

Having $x+y$, and xy, the values of x and y are easily found (Art. 246); two of the values are $x=7\pm4\sqrt{3}$, $y=2\mp\sqrt{3}$.

A similar substitution may be used in solving the following example:

19. $2(x+y)^3+1=(x^2+y^2)(xy+x^3+y^3)$ (1),
$x+y=3$ (2).

Ans. $x=2$, $y=1$.

20. $1+x^3=a(1+x)^3$. *Ans.* $x=\frac{1+2a\pm\sqrt{12a-3}}{2(1-a)}$.

21. $\frac{a}{x^2}-\frac{1}{x}\sqrt{x-2a-\frac{a}{x}}=1$.

Ans. $x=\frac{1}{4}\{1\pm\sqrt{1-8a}\pm\sqrt{2\pm2(1-8a)^{\frac{1}{2}}+8a}\}$.

22. $x+y+xy(x+y)+x^2y^2=85$, *Ans.* $x=6$, or 1.
$xy+(x+y)+xy(x+y)=97$. $y=1$, or 6.

23. $\frac{2c^2}{d^2}+\frac{ac}{d}-(a-b)(2c+ad)\frac{x}{d}=(a+b)\frac{cx}{d}-(a^2-b^2)x^2.$

Ans. $x=\frac{2c+ad}{(a+b)d}$, or $\frac{c}{(a-b)d}$.

24. $(x^3+1)(x^2+1)(x+1)=30x^3.$ *Ans.* $x=\frac{1}{2}(3\pm\sqrt{5}).$

25. $x^3+y^3=35,$ *Ans.* $x=3$, 2, or $1\pm\frac{1}{2}\sqrt{22}$;
$x^2+y^2=13.$ $y=2$, 3, or $1\mp\frac{1}{2}\sqrt{22}$.

26. $\frac{xyz}{x+y}=a,$ *Ans.* $x=\sqrt{\left\{\frac{2abc(ac+bc-ab)}{(ab+ac-bc)(ab+bc-ac)}\right\}}$;
$\frac{xyz}{x+z}=b,$ $y=\sqrt{\left\{\frac{2abc(ab+bc-ac)}{(ac+bc-ab)(ab+ac-bc)}\right\}}$;
$\frac{xyz}{y+z}=c.$ $z=\sqrt{\left\{\frac{2abc(ab+ac-bc)}{(ac+bc-ab)(ab+bc-ac)}\right\}}$.

27. $(x^6+1)y=(y^2+1)x^3,$
$(y^6+1)x=9(x^2+1)y^3.$

Ans. $x=\frac{1}{2}\{\sqrt{\sqrt[3]{3}+3}+\sqrt{\sqrt[3]{3}-1}\}$;
$y=\frac{1}{2}\{\sqrt[3]{3}.\sqrt{\sqrt[3]{3}+3}\pm\sqrt{3\sqrt[3]{9}-1}\}$.

CHAPTER VIII.

RATIO, PROPORTION, AND PROGRESSIONS.

ART. **254.** Two quantities of the same kind, may be compared in two ways:

1st. By considering *how much* the one exceeds the other.

2nd. By considering *how many times* the one is *contained* in the other.

The first method is termed comparison by *Difference;* the second, comparison by *Quotient.* The first is sometimes called *Arithmetical ratio,* the second, *Geometrical ratio.*

If we compare 2 and 6, we find that 2 is *four less* than 6, or that 2 is *contained* in 6 *three* times.

Also, the arithmetical ratio of a to b is $b—a$, the geometrical ratio of a to b is $\frac{b}{a}$. The term Ratio, unless it is otherwise stated, always signifies geometrical ratio.

ART. **255.** *Ratio* is the quotient which arises from dividing one quantity by another of the *same* kind. Thus, the ratio of 2 to 6 is 3, and the ratio of a to ma is m.

ART. **256.** When two numbers, as 2 and 6, are compared, the *first* is called the *antecedent*, and the *second* the *consequent*. An antecedent and consequent, when spoken of as *one*, are called a *couplet*. When spoken of as *two*, they are called the *terms* of the ratio. Thus, 2 and 6 *together* form a couplet, of which 2 is the *first term*, and 6 the *second term*.

ART. **257.** Ratio is expressed in two ways:

1st. In the form of a fraction, of which the *antecedent* is the *denominator*, and the *consequent* the *numerator*. Thus, the ratio of 2 to 6 is expressed by $\frac{6}{2}$; the ratio of a to b, by $\frac{b}{a}$.

2nd. By placing two points vertically between the terms of the ratio. Thus, the ratio of 2 to 6, is written 2 : 6; the ratio of a to b, $a : b$, &c.

ART. **258.** The ratio of two quantities, may be either a whole number, a common fraction, or an *interminate* decimal.

Thus, the ratio of 2 to 6 is $\frac{6}{2}$, or 3.

The ratio of 10 to 4 is $\frac{4}{10}$, or $\frac{2}{5}$.

The ratio of 2 to $\sqrt{5}$ is $\frac{\sqrt{5}}{2}$, or $\frac{2.236+}{2}$, or 1.118+.

We see, from this, that the ratio of two quantities cannot always be expressed exactly, except by symbols; but, by taking a sufficient number of decimal places, it may be found to any required degree of exactness.

ART. **259.** Since the ratio of two numbers is expressed by a fraction, of which the antecedent is the denominator, and the consequent the numerator, it follows, that whatever is true with regard to a fraction, is true with regard to the terms of a ratio. Hence,

1st. *To multiply the consequent, or divide the antecedent of a ratio by any number, multiplies the ratio by that number.* (Arith., Part 3rd, Arts. 142, 145.)

2nd. *To divide the consequent, or to multiply the antecedent of a ratio by any number, divides the ratio by that number.* (Arith., Part 3rd., Arts. 143, 144.)

3rd. *To multiply, or divide, both the antecedent and consequent of a ratio by any number, does not alter the ratio.* (Art. 118.)

ART. **260.** When the terms of a ratio are equal to each other, the ratio is said to be a ratio of *equality.* When the second term is greater than the first, the ratio is said to be a ratio of *greater inequality,* and when it is less, the ratio is said to be a ratio of *less inequality.*

Thus, the ratio of 2 to 2 is a ratio of equality.

The ratio of 2 to 3 is a ratio of greater inequality.

The ratio of 3 to 2 is a ratio of less inequality.

Hence, a ratio of equality may be expressed by 1; a ratio of greater inequality, by a number greater than 1; and a ratio of less inequality, by a number less than 1.

ART. **261.** When the corresponding terms of two or more ratios are multiplied together, the ratios are said to be *compounded,* and the result is termed a *compound ratio.* Thus, the ratio of a to b, compounded with the ratio of c to d is $\frac{b}{a}\times\frac{d}{c}=\frac{bd}{ac}$.

A ratio compounded of two equal ratios is called a *duplicate ratio.* Thus, the duplicate ratio of $\frac{b}{a}$ is $\frac{b}{a}\times\frac{b}{a}=\frac{b^2}{a^2}$.

A ratio compounded of three equal ratios is called a *triplicate ratio.* Thus, the triplicate ratio of $\frac{b}{a}$ is $\frac{b}{a}\times\frac{b}{a}\times\frac{b}{a}=\frac{b^3}{a^3}$.

The ratio of the *square roots* of two quantities is called a *subduplicate ratio.* Thus, the subduplicate ratio of 4 to 9 is $\frac{3}{2}$; and the subduplicate ratio of a to b is $\frac{\sqrt{b}}{\sqrt{a}}$.

The ratio of the *cube roots* of two quantities is called a *subtriplicate ratio.* Thus, the subtriplicate ratio of 8 to 27 is $\frac{3}{2}$; and the subtriplicate ratio of a to b is $\frac{\sqrt[3]{b}}{\sqrt[3]{a}}$.

ART. **262.** Ratios may be compared with each other by reducing the fractions which represent them to a common denominator. Thus, to ascertain whether the ratio of 2 to 7 is less or greater than the ratio of 3 to 10, we have the two fractions $\frac{7}{2}$, and $\frac{10}{3}$, which being reduced to a common denominator are $\frac{21}{6}$ and $\frac{20}{6}$; and since the first is greater than the second, we conclude that the ratio of 2 to 7 is *greater* than the ratio of 3 to 10.

PROPORTION.

ART. **263.** Proportion is an equality of ratios. That is, *when two ratios are equal, their terms are said to be proportional.* Thus, if the ratio of a to b, is equal to the ratio of c to d, that is, if $\frac{b}{a}=\frac{d}{c}$, then a, b, c, d, form a proportion, and we say that a is to b as c is to d.

Proportion is written in two ways:

1st. By placing a double colon between the ratios; thus,

$$a:b::c:d.$$

2nd. By placing the sign of equality between the ratios; thus,

$$a:b=c:d.$$

The first method is the one commonly used.

From the preceding definition, it follows, that when four quantities are in proportion, the second, divided by the first, must give the same quotient as the fourth divided by the third. This is the primary *test* of the proportionality of four quantities. Thus, if 3, 5, 6, 10, are the four terms of a proportion, so that $3:5::6:10$, we must have $\frac{5}{3}=\frac{10}{6}$.

If these fractions are equal to each other, the proportion is *true;* if they are not equal to each other, it is *false.* Thus, let it be required to determine whether $3:8::2:5$. The ratios are $\frac{8}{3}$ and $\frac{5}{2}$, or $\frac{16}{6}$ and $\frac{15}{6}$; hence, the proportion is false.

REMARK.— In common language the words *ratio* and *proportion* are sometimes confounded with each other. Thus, two quantities are said to be in the proportion of 2 to 3, instead of in the ratio of 2 to 3. A ratio subsists between *two* quantities, a proportion only between *four.* It requires *two equal ratios* to form a proportion.

ART. **264.** Each of the four quantities in a proportion is called a *term.* The first and last terms are called the *extremes;* and the second and third terms, the *means.*

The *terms* of a proportion may be either monomials or polynomials.

ART. **265.** Of four quantities in proportion, the first and third are called *the antecedents,* and the second and fourth, *the conse-*

quents (Art. 257); and the last is said to be a fourth proportional to the other three taken in their order.

Art. **266.** Three quantities are in proportion when the first has the same ratio to the second, that the second has to the third. In this case the middle term is called a *mean proportional* between the other two. Thus, if we have the proportion

$$a : b :: b : c,$$

then b is called a *mean proportional* between a and c; and c is called a *third proportional* to a and b.

When *several* quantities have the same ratio between each two that are consecutive, they are said to form a *continued proportion.*

Art. **267.** Proposition I.—*In every proportion, the product of the means is equal to the product of the extremes.*

Let $\quad a : b :: c : d.$

Since this is a true proportion, the ratio of the first term to the second, is equal to the ratio of the third term to the fourth (Art. 263). Therefore, we must have

$$\frac{b}{a}=\frac{d}{c}.$$

Multiplying both sides of this equality by ac, to clear it of fractions, we have $\quad \frac{abc}{a}=\frac{adc}{c};$

or, $\quad bc=ad.$

Illustration by numbers. $2:6:5:15$; and $6\times5=2\times15$.

From the equation $bc=ad$, we find $d=\frac{bc}{a}$, $c=\frac{ad}{b}$, $b=\frac{ad}{c}$, and $a=\frac{bc}{d}$. Hence, if any three terms of a proportion are given, the fourth may be found.

Ex. 1. The first three terms of a proportion are $x+y$, x^2-y^2, and $x-y$; what is the fourth? *Ans.* $x^2-2xy+y^2$.

2. The first, third, and fourth terms of a proportion are $(m-n)^2$, m^2-n^2, and $m+n$; required the second.

Ans. $m-n$.

Remark.—This proposition furnishes a more convenient *test* of the proportionality of four quantities, than the method given in Art. 263. Thus, to ascertain whether $2 : 3 :: 5 : 8$, it is merely necessary to compare the product of the means and extremes; and since 3×5 is not equal to 2×8, we infer that 2, 3, 5, and 8, are *not* in proportion.

ART. **268.** PROPOSITION II. Conversely, *If the product of two quantities is equal to the product of two others, two of them may be made the means, and the other two the extremes of a proportion.*

Let $bc=ad$.

Dividing each of these equals by ac, we have

$$\frac{bc}{ac}=\frac{ad}{ac},$$

$$\text{or}\quad \frac{b}{a}=\frac{d}{c}.$$

That is, (Art. 263), $a:b::c:d$.

By dividing each of the equals by ab, we may prove that

$$a:c::b:d.$$

Illust. $3\times12=4\times9$, and $3:4::9:12$; also, $3:9::4:12$.

ART. **269.** PROPOSITION III. *If three quantities are in proportion, the product of the extremes is equal to the square of the mean.*

If $a:b::b:c$,

then (Art. 267), $ac=bb=b^2$.

It follows from Art. 268, that the converse of this proposition is also true. Thus, if $ac=b^2$,

then, $a:b::b:c$.

That is, *if the product of the first and third of three quantities is equal to the square of the second, the first is to the second, as the second to the third.*

NOTE.— It is recommended to the teacher to require the pupils to illustrate all the propositions by numbers. (See Ray's Algebra, Part 1st., Proportion.)

ART. **270.** PROPOSITION IV.— *If four quantities are in proportion, they will be in proportion by* ALTERNATION; *that is, the first will be to the third, as the second to the fourth.*

Let $a:b::c:d$.

Then, (Art. 263), $\frac{b}{a}=\frac{d}{c}$.

Multiply both sides by c, $\frac{bc}{a}=d$;

divide both sides by b, $\frac{c}{a}=\frac{d}{b}$.

That is (Art. 263), $a:c::b:d$.

NOTE.— This proposition is true, only when the four quantities are of the *same* kind.

ART. 271. PROPOSITION V.—*If four quantities are in proportion, they will be in proportion by* INVERSION; *that is, the second will be to the first, as the fourth to the third.*

Let $a:b::c:d.$

Then (Art. 263), $\frac{b}{a}=\frac{d}{c}$;

dividing 1 by each side, $\frac{1}{\frac{b}{a}}=\frac{1}{\frac{d}{c}}$;

or, $\frac{a}{b}=\frac{c}{d}.$

That is, (Art. 263), $b:a::d:c.$

ART. 272. PROPOSITION VI.—*If two sets of proportions have an antecedent and consequent in the one, equal to an antecedent and consequent in the other, the remaining terms will be proportional.*

Let $a:b::c:d$ (1),

and $a:b::e:f$ (2);

then will $c:d::e:f.$

From the 1st proportion $\frac{b}{a}=\frac{d}{c}$, from the 2nd, $\frac{b}{a}=\frac{f}{e}.$

Hence, $\frac{d}{c}=\frac{f}{e}$; which gives $c:d::e:f.$

ART. 273. PROPOSITION VII.—*If four quantities are in proportion, they will be in proportion by* COMPOSITION; *that is, the sum of the first and second will be to the second, as the sum of the third and fourth is to the fourth.*

Let $a:b::c:d,$

Then will $a+b:b::c+d:d.$

From the 1st proportion, $bc=ad$, (Art. 267);

$bd=bd$;

Adding the two equations together, $bc+bd=ad+bd$;

factoring, $b(c+d)=d(a+b)$;

dividing each side by $c+d$ $b=\frac{d(a+b)}{c+d}$;

by $a+b$ $\frac{b}{a+b}=\frac{d}{c+d}.$

This gives, $a+b:b::c+d:d.$

NOTE.—In a similar manner let the student prove that the sum of the first and second of two quantities is to the *first*, as the sum of the third and fourth is to the *third*.

ART. 274. PROPOSITION VIII.—*If four quantities are in proportion, they will be in proportion by* DIVISION; *that is, the difference of the first and second will be to the second, as the difference of the third and fourth is to the fourth.*

Let $a:b::c:d$ (1),

then will $a-b:b::c-d:d$.

From the 1st proportion, $bc=ad$ (Art. 267).

$$bd=bd;$$

subtracting, $bc-bd=ad-bd$;

factoring, $b(c-d)=d(a-b)$.

Dividing each side by $c-d$, $b=\frac{d(a-b)}{c-d}$;

by $a-b$ $\frac{b}{a-b}=\frac{d}{c-d}$.

This gives $a-b:b::c-d:d$.

NOTE.—In a similar manner, let the student prove that the difference of the first and second is to the *first*, as the difference of the third and fourth is to the *third*.

ART. 275. PROPOSITION IX.—*If four quantities are in proportion, the sum of the first and second will be to their difference, as the sum of the third and fourth is to their difference.*

Let $a:b::c:d$ (1),

then will $a+b:a-b::c+d:c-d$.

From the 1st, by Composition, (Art. 273),

$$a+b:b::c+d:d;$$

By Alternation, $a+b:c+d::b:d$;

this gives, $\frac{c+d}{a+b}=\frac{d}{b}$.

From the 1st, by Division,

$$a-b:b::c-d:d;$$

by Alternation, $a-b:c-d::b:d$;

this gives, $\frac{c-d}{a-b}=\frac{d}{b}$; hence, $\frac{c+d}{a+b}=\frac{c-d}{a-b}$.

That is, $a+b:c+d::a-b:c-d$,

or by Alternation, $a+b:a-b::c+d:c-d$.

ART. **276.** PROPOSITION X.—*If four quantities are in proportion, like powers or roots of those quantities will also be in proportion.*

Let $a:b::c:d,$

then will $a^n:b^n::c^n:d^n.$

From the 1st, $\frac{b}{a}=\frac{d}{c}.$ Raising each of these equals to the n^{th} power, $\frac{b^n}{a^n}=\frac{d^n}{c^n}.$

That is, $a^n:b^n::c^n:d^n,$

where n may be either a whole number or a fraction.

ART. **277.** PROPOSITION XI.—*If two sets of quantities are in proportion, the products of the corresponding terms will also be in proportion.*

Let $a:b::c:d,$ (1),

and $m:n::r:s$ (2),

then will $am:bn::cr:ds.$

For from the 1st, $\frac{b}{a}=\frac{d}{c};$

and from the 2nd, $\frac{n}{m}=\frac{s}{r}.$ Multiplying these equals together, $\frac{b}{a}\times\frac{n}{m}=\frac{d}{c}\times\frac{s}{r}$, or $\frac{bn}{am}=\frac{ds}{cr};$

this gives $am:bn::cr:ds.$

ART. **278.** PROPOSITION XII.—*In any number of proportions having the same ratio, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Let $a:b::c:d::m:n,$ &c.

Then $a:b::a+c+m:b+d+n.$

Since $a:b::c:d,$ we have $bc=ad$ (Art. 267).

Since $a:b::m:n,$ we have $bm=an$

$ab=ab.$ The sum of these equalities gives $ab+bc+bm=ab+ad+an.$

Factoring, $b(a+c+m)=a(b+d+n).$

Dividing by $a+c+m,$ $b=\frac{a(b+d+n)}{a+c+m}.$

Dividing both sides by $a,$ $\frac{b}{a}=\frac{b+d+n}{a+c+m}.$

This gives $a:b::a+c+m:b+d+n.$

REMARK.—In most of the preceding demonstrations, the conclusion has been derived directly from the *equality of ratios*. In several cases, however, it may be derived more easily from Art. 268, Proposition II; but the method here given is considered the most satisfactory, as it keeps before the student, the *principle* on which proportion depends.

EXERCISES IN RATIO AND PROPORTION.

1. Which is the greater ratio, that of 3 to 4, or 3^2 to 4^2? *Ans.* last.

2. Compound the duplicate ratio of 2 to 3; the triplicate ratio of 3 to 4; and the subduplicate ratio of 64 to 36. *Ans.* 1 to 4.

3. What quantity must be added to each of the terms of the ratio $m:n$, that it may become equal to $p:q$? *Ans.* $\frac{mq-np}{p-q}$.

4. If the ratio of a to b is $2\frac{2}{3}$, what is the ratio of $2a$ to b, and of $3a$ to $4b$? *Ans.* $1\frac{1}{3}$, and $3\frac{5}{9}$.

5. If the ratio of a to $7b$ is $5\frac{1}{4}$, what is the ratio of a to b, and of $5a$ to $4b$? *Ans.* $\frac{3}{4}$, and $\frac{3}{5}$.

6. If the ratio of a to b is $1\frac{2}{3}$, what is the ratio of $a+b$ to b, and of $b-a$ to a? *Ans.* $\frac{5}{8}$, and $\frac{3}{2}$.

7. If the ratio of m to n is $\frac{4}{7}$, what is the ratio of $m-n$ to $6m$, and also to $5n$? *Ans.* 14, and $6\frac{2}{3}$.

8. If the ratio of m to $2m+3n$ is $2\frac{3}{5}$, what is the ratio of m to n? *Ans.* 5 to 1.

9. If the ratio of m to n is $3\frac{1}{2}$, what is the ratio of $12m$ to $m+n$, and of $12n$ to $n-2m$. *Ans.* $\frac{3}{8}$, and $\frac{1}{28}$.

10. If the ratio of $5y-8x$ to $7x-5y$ is 6, what is the ratio of x to y? *Ans.* 7 to 11.

11. What is the proportion deducible from the equation $ab=a^2-x^2$. *Ans.* $a:a+x::a-x:b$.

12. What is the proportion deducible from the equation $x^2+y^2=2ax$? *Ans.* $x:y::y:2a-x$.

13. Four given numbers are represented by a, b, c, d; what quantity added to each will make them proportionals?

Ans. $\frac{bc-ad}{a-b-c+d}$.

14. If four numbers are proportionals, show that there is no number which, being added to each, will leave the resulting four numbers proportionals.

15. Find x in terms of y from the proportions $x : y :: a^3 : b^3$, and $a : b :: \sqrt[3]{c+x} : \sqrt[3]{d+y}$. *Ans.* $x=\frac{cy}{d}$.

16. Prove that equal multiples of two quantities are to each other as the quantities themselves.

17. Prove that like parts of two quantities are to each other as the quantities themselves.

18. Prove that in any proportion, if there be taken equal multiples of the antecedents, and equal multiples of the consequents, the resulting quantities and the antecedents will be proportional.

19. If $a : b :: c : d$, prove that $ma : mb :: nc : nd$, and also that $ma : nb :: mc : nd$, m and n being any multiples.

20. If $a : b :: c : d$, prove that $\frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$; and also that

$$\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}.$$

21. Prove that the quotients of the corresponding terms of two proportions are proportional.

22. Prove that if two sets of proportions have their antecedents proportional, their consequents will also be proportional.

23. Prove that if the antecedent and consequent of a ratio be increased or diminished by like parts of each, the resulting quantities and the antecedent and consequent will be proportional.

24. If $(a+b)^2 : (a-b)^2 :: b+c : b-c$, show that

$$a : b :: \sqrt{2a-c} : \sqrt{c}.$$

ART. **279.** The preceding exercises are designed merely to make the student acquainted with the principles of ratio and proportion. The following are intended as exercises in the application of the principles of proportion to the solution of problems.

1. Resolve the number 24 into two factors, so that the sum of their cubes may be to the difference of their cubes, as 35 to 19.

Let x and y denote the required factors; then $xy=24$, and

$$x^3+y^3 : x^3-y^3 :: 35 : 19;$$

$\therefore$ (Art. 275), $\quad 2x^3 : 2y^3 :: 54 : 16;$

or, $\quad x^3 : y^3 :: 27 : 8;$

or, (Art. 276), $\quad x : y :: 3 : 2.$

From which $y=\frac{2}{3}x$; then substituting the value of y in the equation $xy=24$, we find $x=\pm 6$; hence, $y=\pm 4$.

2. Given, $\frac{\sqrt[3]{x+1}+\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}}=2$, to find x.

Resolving this equation into a proportion, we have

$$\sqrt[3]{x+1}-\sqrt[3]{x-1} : \sqrt[3]{x+1}+\sqrt[3]{x-1} : : 1 : 2;$$

$\therefore$ (Art 275), $2\sqrt[3]{x+1} : 2\sqrt[3]{x-1} : : 3 : 1$;

or, $\sqrt[3]{x+1} : \sqrt[3]{x-1} : : 3 : 1$;

or, (Art. 276), $x+1 : x-1 : : 27 : 1$;

(Art. 275), $2x : 2 : : 28 : 26$;

whence, $52x=56$, or $x=1\frac{1}{13}$.

3. $x+y : x-y : : 3 : 1$,
$x^3-y^3=56$. *Ans.* $x=4$, $y=2$.

4. $x-y : x : : 5 : 6$,
$xy^2=384$. *Ans.* $x=24$, $y=4$.

5. $x+y : x : : 7 : 5$,
$xy+y^2=126$. *Ans.* $x=\pm15$, $y=\pm6$.

6. $(x+y)^2 : (x-y)^2 : : 64 : 1$,
$xy=63$. *Ans.* $x=\pm9$, $y=\pm7$.

7. $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$. *Ans.* $x=\pm\frac{2a\sqrt{b}}{b+1}$.

8. $\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\frac{1}{b}$. *Ans.* $x=\frac{2ab}{b^2+1}$.

9. $\frac{a+x}{a+x+\sqrt{2ax+x^2}}=\frac{1}{b}$. *Ans.* $x=\pm a\left(\frac{+1\mp\sqrt{2b-b^2}}{\pm\sqrt{2b-b^2}}\right)$.

10. It is required to find two numbers whose product is 320, and the difference of whose cubes is to the cube of their difference, as 61 is to 1. *Ans.* 20 and 16.

ART. **280.** HARMONICAL PROPORTION.—Three or four quantities are said to be in *harmonical proportion*, when the first has the same ratio to the last, that the difference betweeen the first and second has to the difference between the last, and the last except one. Thus, a, b, c, are in harmonical proportion when $a : c : : a-b : b-c$; and a, b, c, d, are in harmonical proportion when $a : d : : a-b : c-d$.

1. Let it be required to find a third harmonical proportional x to two given numbers a and b.

We have, $a:x::a-b:b-x$;

$\therefore$ (Art. 267), $a(b-x)=x(a-b)$;

whence, $x=\dfrac{ab}{2a-b}$.

2. Find a third harmonical proportional to 3 and 5. *Ans.* 15.

3. Find a fourth harmonical proportional x, to three given numbers, a, b, and c. *Ans.* $x=\dfrac{ac}{2a-b}$.

VARIATION.

ART. **281.** *Variation*, or as it is sometimes termed, *General Proportion*, is merely an abridged form of common Proportion.

Variable quantities are such as admit of various values in the same computation. *Constant*, or *invariable* quantities have only one fixed value.

One quantity is said to *vary directly* as another, when the two quantities depend upon each other in such a manner, that if one be changed the other is changed *in the same proportion.*

Thus, if A and B are two variable quantities, mutually dependent on each other, in such a way, that if A be changed to any other value a, B must be changed to another value b, such that $A:a::B:b$, then A is said to *vary directly* as B.

This relation is expressed thus, $A \propto B$, the symbol $\propto$ being used instead of *varies*, or *varies as.*

From this it will be seen that variation is merely an abridgment of Proportion, and that *four* quantities are understood, although only *two* are expressed.

NOTE.—When it is simply stated that one quantity *varies* as another, it is always meant that the one varies *directly* as the other.

ART. **282.** There are four different kinds of Variation, which are distinguished as follows:

(1). $A\propto B$. Here A is said to vary *directly* as B.

Ex. If a man works for a certain sum per day, the amount of his wages *varies* as the number of days in which he works.

(2). $A\propto \frac{1}{B}$. Here A is said to vary *inversely* as B.

Ex. If a man has to perform a journey of a certain number of miles, the *time* in which he performs it will vary *inversely* as the *rate* of traveling. Thus, if he *doubles* his speed, he will perform the journey in *half* the time.

(3). $A \propto BC$. Here A is said to vary as B and C *jointly.*

Ex. The wages to be received by a workman will vary jointly as the *number* of days he works, and the *wages per day.*

(4). $A \propto \frac{B}{C}$. Here A is said to vary *directly* as B, and *inversely* as C.

Ex. The base of a triangle varies as the area *directly,* and the altitude *inversely.*

Let the pupil give other examples of each kind of Variation.

In the following articles, A, B, C, represent corresponding values of any variable quantities, and *a, b, c,* any other corresponding values of the *same* quantities.

ART. **283.** *If one quantity vary as a second, and that second as a third, the first varies as the third.*

Let $A \propto B$, and $B \propto C$, then shall $A \propto C$. For $A : a :: B : b$, and $B : b :: C : c$, therefore, (Art. 272), $A : a :: C : c$; that is, $A \propto C$.

In a similar manner it may be proved that if $A \propto B$, and $B \propto \frac{1}{C}$, that $A \propto \frac{1}{C}$.

ART. **284.** *If each of two quantities vary as a third, their sum, or their difference, or the square root of their product, will vary as the third.*

Let $A \propto C$, and $B \propto C$, then $A \pm B \propto C$; also, $\sqrt{AB} \propto C$.

By the supposition, $A : a :: C : c :: B : b$;

$$\therefore \quad A : a :: B : b;$$

alternately, (Art. 270), $A : B :: a : b$;

by Composition or Division, $A \pm B : B :: a \pm b : b$;

alternately, $A \pm B : a \pm b :: B : b :: C : c$;

that is, $A \pm B \propto C$.

Again, $A : a :: C : c$;

and $B : b :: C : c$;

$\therefore$ (Art. 277), $AB : ab :: C^2 : c^2$;

and, (Art. 276), $\sqrt{AB} : \sqrt{ab} :: C : c$;

that is, $\sqrt{AB} \propto C$.

ART. **285.** *If one quantity vary as another, it will also vary as any multiple, or any part of the other.*

Let $A \propto B$, and m be any constant quantity, then since $A : a :: B : b$, $A : a : mB : mb$, or $A : a :: \frac{B}{m} : \frac{b}{m}$, (Art. 260, 3rd); that is, $A \propto mB$, or $\propto \frac{B}{m}$.

ART. **286.** *If one quantity vary as another, any power or root of the former will vary as the same power or root of the latter.*

Let $A \propto B$, then $A : a :: B : b$, and (Art. 276), $A^n : a^n :: B^n : b^n$; that is, $A^n \propto B^n$, where n may be integral or fractional.

ART. **287.** *If one quantity vary as another, and each of them be multiplied or divided by any quantity, variable or invariable, the products, or quotients, will vary as each other.*

Let A vary as B, and let T be any other quantity. Then, by the supposition, $A : a :: B : b$;

$$\therefore \ AT : at :: BT : bt; \text{ that is, } AT \propto BT.$$

$$\text{Also, } \frac{A}{T} : \frac{a}{t} :: \frac{B}{T} : \frac{b}{t}; \text{ that is, } \frac{A}{T} \propto \frac{B}{T}.$$

ART. **288.** *If one quantity vary as two others jointly, either of the latter varies as the first directly, and the other inversely.*

Let $V \propto FT$, then (Art. 287), $\frac{V}{T} \propto \frac{FT}{T}$, or $F \propto \frac{V}{T}$, and similarly, $T \propto \frac{V}{F}$.

ART. **289.** *If* A *vary as* B, A *is equal to* B *multiplied by some constant quantity.*

Since, by supposition, $A : a :: B : b$, therefore, $A = \frac{a}{b}B$; but a and b are supposed to be constant, being certain corresponding values of A and B. Hence, if we denote $\frac{a}{b}$ by m, we have $A = mB$.

It is evident that if we know any corresponding values of A and B, that the constant quantity m may be found.

ART. **290.** In general the simplest method of treating *variations,* is to convert them into *equations.*

Ex. 1. Given, that $y \propto$ the sum of two quantities one of which varies as x, and the other as x^2, to find the corresponding equation.

Because one part $\propto x$, let this $=mx$,

and the other " $\propto x^2$, " " $=nx^2$;

$\therefore$ $y=mx+nx^2$, where m and n are two unknown invariable quantities which can only be found when we know two pairs of corresponding values of x and y.

2. If $y=r+s$, where $r\propto x$ and $s\propto\frac{1}{x}$, and if, when $x=1$, $y=6$, and when $x=2$, $y=9$, what is the equation between x and y?

Let $r=mx$, and $s=\frac{n}{x}$ $\therefore$ $y=mx+\frac{n}{x}$.

But if $x=1$, $y=6$, $\therefore$ $6=m+n$;

and if $x=2$, $y=9$, $\therefore$ $9=2m+\frac{n}{2}$.

Hence, $m=4$; $n=2$, and $y=4x+\frac{2}{x}$.

EXAMPLES FOR PRACTICE.

3. If $y\propto x$; and when $x=2$, $y=4a$; find the equation between x and y. *Ans.* $y=2ax$.

4. If $y\propto\frac{1}{x}$; and when $x=\frac{1}{2}$, $y=8$; find the equation between x and y. *Ans.* $y=\frac{4}{x}$.

5. If $y^2\propto a^2-x^2$; and when $x=\sqrt{a^2-b^2}$, $y=\frac{b^2}{a}$; find tho equation between x and y. *Ans.* $y=\frac{b}{a}\sqrt{a^2-x^2}$.

6. If y is equal to the sum of two quantities, one of which varies as x, and the other varies inversely as x^2; and when $x=1$, $y=6$, and when $x=2$, $y=5$; find the equation between x and y. *Ans.* $y=2x+\frac{4}{x^2}$.

7. Given that y is equal to the sum of three quantities, of which the 1st is invariable, the second varies as x, and the third varies as x^2. Also when $x=1, 2, 3$, $y=6, 11, 18$, respectively; find y in terms of x. *Ans.* $y=3+2x+x^2$.

8. Given that $s \propto t^2$, when f is constant; and $s \propto f$, when t is constant; also, $2s=f$, when $t=1$. Find the equation between f, s, and t. *Ans.* $s=\frac{1}{2}ft^2$.

9. If $y=r+s$ where $r \propto x$, and $s \propto \sqrt{x}$; and if when $x=4$, $y=5$, and when $x=9$, $y=10$; find y in terms of x.

Ans. $y=\frac{5}{6}(x+\sqrt{x})$.

10. Given that $x \propto \frac{1}{y^m}$, and $y \propto \frac{1}{z^n}$; also, when $x=a$, $z=c$; find the equation between x and z. *Ans.* $az^{mn}=c^{mn}x$.

ARITHMETICAL PROGRESSION.

ART. 291. Quantities are said to be in *Arithmetical Progression*, when they increase or decrease by a *Common Difference.*

Thus, 1, 3, 5, 7, 9, &c., a, $a+d$, $a+2d$, &c., a, $a-d$, $a-2d$, &c., are quantities respectively in Arithmetical Progression.

The series is said to be *increasing* or *decreasing*, according as d is positive or negative.

ART. 292. To investigate a rule for finding any term of an arithmetical progression, take the following series, in which the first line denotes the number of each term, the second an *increasing* arithmetical series, and the third a *decreasing* arithmetical series.

1	2	3	4	5	
a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,	&c.,
a,	$a-d$,	$a-2d$,	$a-3d$,	$a-4d$,	&c.

It is manifest that the coëfficient of d in any term is less by *unity* than the number of that term in the series; therefore, the n^{th} term $=a+(n-1)d$.

If we designate the n^{th} term by l, we have

$$l=a+(n-1)d, \text{ when the series is increasing,}$$

and

$$l=a-(n-1)d, \text{ when the series is decreasing.}$$

Hence, we have the following

RULE, FOR FINDING ANY TERM OF AN ARITHMETICAL SERIES.— *Multiply the common difference by the number of terms less one and add the product to the first term when the series is increasing, but subtract it from the first term when the series is decreasing.*

The equation $l=a+(n-1)d$, contains four variable quantities, any one of which may be found when the other three are known

ART. **293.** Having given the first term a, the common difference d, and the number of terms n, to find S, the sum of the series.

If we take any arithmetical series, as the following, and write the same series under it in an inverted order, we have

$$S = 1+3+5+7+9+11,$$
$$S = 11+9+7+5+3+1.$$

Adding, $2S = 12+12+12+12+12+12.$

$2S = 12\times$ the number of terms.

$2S = 12\times 6 = 72.$

Whence, $S = \frac{1}{2}$ of $72 = 36$, the sum of the series.

To render this method general, let $l=$ the last term, and write the series both in a direct and inverted order.

Then, $S = a+(a+d)+(a+2d)+(a+3d) \ldots +l,$

and $S = l+(l-d)+(l-2d)+(l-3d) \ldots +a.$

By adding the corresponding terms, we have

$2S = (l+a)+(l+a)+(l+a)+(l+a) \ldots +(l+a),$

$2S = (l+a)$ taken as many times as there are terms (n) in the series.

Hence, $2S = (l+a)n$;

$$S = (l+a)\frac{n}{2} = \left(\frac{l+a}{2}\right)n.$$

This formula gives the following

RULE, FOR FINDING THE SUM OF AN ARITHMETICAL SERIES.—*Multiply half the sum of the two extremes, by the number of terms.*

From the preceding it appears, that *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.*

ART. **294.** The equations $l = a+(n-1)d$,

$$\text{and } S = (a+l)\frac{n}{2},$$

furnish the means of solving this general problem: *Knowing any three of the five quantities, a, d, l, n, S, which enter into an arithmetical series, to determine the other two.*

The following table contains the results of the solution of all the different cases. These formulæ should be verified by the student.

No.	Given.	Required.	Formulæ.
1.	a, d, n	l	$l=a+(n-1)d,$
2.	a, d, S		$l=-\frac{1}{2}d\pm\sqrt{\{2dS+(a-\frac{1}{2}d)^2\}},$
3.	a, n, S		$l=\frac{2S}{n}-a,$
4.	d, n, S		$l=\frac{S}{n}+\frac{(n-1)d}{2}.$
5.	a, d, n	S	$S=\frac{1}{2}n\{2a+(n-1)d\},$
6.	a, d, l		$S=\frac{l+a}{2}+\frac{l^2-a^2}{2d},$
7.	a, n, l		$S=(l+a)\frac{n}{2},$
8.	d, n, l		$S=\frac{1}{2}n\{2l-(n-1)d\}.$
9.	a, n, l	d	$d=\frac{l-a}{n-1},$
10.	a, n, S		$d=\frac{2(S-an)}{n(n-1)},$
11.	a, l, S		$d=\frac{l^2-a^2}{2S-l-a},$
12.	n, l, S		$d=\frac{2(nl-S)}{n(n-1)}.$
13.	a, d, l	n	$n=\frac{l-a}{d}+1,$
14.	a, d, S		$n=\frac{\pm\sqrt{(2a-d)^2+8dS}-2a+d}{2d},$
15.	a, l, S		$n=\frac{2S}{l+a},$
16.	d, l, S		$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}.$
17.	d, n, l	a	$a=l-(n-1)d,$
18.	d, n, S		$a=\frac{S}{n}-\frac{(n-1)d}{2},$
19.	d, l, S		$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS},$
20.	n, l, S		$a=\frac{2S}{n}-l.$

EXAMPLES FOR PRACTICE.

1. Find the 15[th] term of the series 3, 7, 11, &c. *Ans.* 59.

2. Find the 20[th] term of the series 5, 1, —3, &c.
Ans. —71.

3. Find the 8^{th} term of the series $\frac{2}{3}$, $\frac{7}{12}$, $\frac{1}{2}$, &c. *Ans.* $\frac{1}{12}$.

4. Find the 30^{th} term of the series —27, —20, —13, &c. *Ans.* 176.

5. Find the n^{th} term of $1+3+5+7$. . . . *Ans.* $2n-1$.

and of $2+2\frac{1}{3}+2\frac{2}{3}+$. . . . *Ans.* $\frac{1}{3}(n+5)$.

and of $13+12\frac{2}{3}+12\frac{1}{3}+$. . . . *Ans.* $\frac{1}{3}(40-n)$.

Find the sum of

6. $1+2+3+4$ &c., to 50 terms. (See Formula 5). *Ans.* 1275.

7. $7+\frac{29}{4}+\frac{15}{2}+$, &c., to 16 terms. *Ans.* 142.

8. $12+8+4+$, &c., to 20 terms. *Ans.* —520.

9. $2+2\frac{1}{3}+2\frac{2}{3}+$, &c., to n terms. *Ans.* $\frac{n}{6}(n+11)$.

10. $13+12\frac{2}{3}+12\frac{1}{3}+$, &c., to n terms. *Ans.* $\frac{n}{6}(79-n)$.

11. $\frac{1}{2}-\frac{2}{3}-1\frac{1}{6}-$, &c., to n terms. *Ans.* $\frac{n}{12}(13-7n)$.

12. $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+$, &c., to n terms.

Ans. $\frac{n}{a+b}\left\{na-\frac{(n+1)b}{2}\right\}$.

13. $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+$, &c., to n terms. *Ans.* $\frac{n-1}{2}$.

14. If a falling body descends $16\frac{1}{12}$ feet the first second, three times this distance the next, five times the next, and so on, how far will it fall the 30th second, and how far altogether in half a minute? *Ans.* $948\frac{11}{12}$, and 14475 ft.

15. Two hundred stones being placed on the ground in a straight line, at the distance of 2 feet from each other; how far will a person travel who shall bring them separately to a basket, which is placed 20 yards from the first stone, if he starts from the spot where the basket stands? *Ans.* 19 miles, 4 fur., 640 ft.

16. Insert 3 arithmetical means between 2 and 14.

To solve this problem generally, let it be required to insert m arithmetical means between a and b.

Since the required terms, and those which are given, form an arithmetical series, if we insert m terms between a and b, we shall have a series consisting of $(m+2)$ terms. Then to find d, the common difference, substitute $m+2$ instead of n in Formula 9, page 245, and we have $d=\frac{l-a}{n-1}=\frac{b-a}{m+2-1}=\frac{b-a}{m+1}$. Therefore, *the common difference* (d) *will be equal to the difference of the extremes* $(b-a)$ *divided by the number of terms plus one.*

In the particular example we find $d=3$; hence, the terms are 5, 8, and 11.

17. Insert 4AR. means between 3 and 18.

Ans. 6, 9, 12, 15.

18. Insert 9AR. means between 1 and -1.

Ans. $\frac{4}{5}$, $\frac{3}{5}$, &c., to $-\frac{4}{5}$.

19. How many terms of the series 19, 17, 15, &c., amount to 91? *Ans.* 13, or 7.

Explain this result.

20. How many terms of the series .034, .0344, .0348, &c., amount to 2.748? *Ans.* 60.

21. The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9; find the series.

Ans. 1, 3, 5, 7, 9, &c.

22. The first two terms of an arithmetical progression being together $=18$, and the next three terms $=12$, how many terms must be taken to make 28? *Ans.* 4, or 7.

23. The n^{th} term of an arithmetical series is $\frac{1}{6}(3n-1)$, find the first term, the common difference, and the sum of n terms.

Ans. $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{n}{12}(3n+1)$.

24. In the series 1, 3, 5, &c., the sum of $2r$ terms : the sum of r terms : : x : 1; determine the value of x. *Ans.* 4.

25. Find the ratio of the latter half of $2n$ terms of any AR. series, to the sum of $3n$ terms of the same series. *Ans.* 3.

26. The sum of n arithmetical means between 1 and 19 : sum of the first $n-2$ of them : : 5 : 3; required n. *Ans.* 8.

27. A traveler sets out for a certain place, and travels 1 mile the first day, 2 the second, and so on. In 5 days afterward another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

Ans. 3 days or 10 days, and travel 36 miles, or 120 miles.

Explain these results.

GEOMETRICAL PROGRESSION.

ART. **295.** A Geometrical Progression is a series of terms each of which is derived from the preceding, by multiplying it by a constant quantity termed the *ratio.*

Thus, 1, 2, 4, 8, 16, &c., is an *increasing* geometrical progression, whose common ratio is 2.

54, 18, 6, 2, &c., is a *decreasing* geometrical progression, whose common ratio is $\frac{1}{3}$.

In general, a, ar, ar^2, ar^3, &c., is a geometrical progression, whose common ratio is r, and which is an *increasing* series when r is *greater* than 1; but a *decreasing* series when r is *less* than 1.

It is evident that *in any given geometrical series, the common ratio will be found by dividing any term by the term next preceding.*

REMARK.— An Arithmetical Progression may be defined to be a series in which the *difference* between any two consecutive terms is the *same ;* and a Geometrical Progression a series in which the *ratio* of any two consecutive terms is the *same.* Hence, a geometrical progression is a *continued proportion.* (Art. 266.)

ART. **296.** To find the last term of a geometrical progression.

Let a denote the first term, r the common ratio, l the n^{th} term; and S the sum of n terms ; then the respective terms of the series will be

$$\begin{array}{ccccccccc} 1, & 2, & 3, & 4, & 5 \ldots\ldots & n-3, & n-2, & n-1, & n, \\ a, & ar, & ar^2, & ar^3, & ar^4 \ldots\ldots & ar^{n-4}, & ar^{n-3}, & ar^{n-2}, & ar^{n-1}. \end{array}$$

That is, the exponent of r, in the *second* term is 1, in the third term 2, in the *fourth* term 3, and so on ; hence, the n^{th} term of the series will be $l=ar^{n-1}$; that is, *any term of a geometrical series is equal to the product of the first term, by the ratio raised to a power, whose exponent is one less than the number of terms.*

Ex. Let it be required to find the 6^{th} term of the geometrical progression whose first term is 7, and common ratio 2.

$2^5=2\times2\times2\times2\times2=32$; and $7\times32=224$, the 6^{th} term.

ART. **297.** To find the sum of all the terms of a geometrical progression.

If we multiply any geometrical series by the ratio, the product will be a new series, of which every term, except the last, will have a corresponding term in the first series.

Thus, take the series, 1, 3, 9, 27, 81, and represent its sum by S; then

$$S=1+3+9+27+81.$$

Multiplying each term by the ratio 3, we have

$$3S=\quad 3+9+27+81+243.$$

The terms of the two series are identical, except the *first* term of the first series, and the *last* term of the second series. If, then, we subtract the first equation from the second, all the other terms will disappear, and we shall have

$$3S-S=243-1;$$

whence, $S=121.$

To generalize this method, let a, ar, ar^2, ar^3, &c., be any geometrical series, and S its sum; then,

$$S=a+ar+ar^2+ar^3 \ldots +ar^{n-2}+ar^{n-1}.$$

Multiplying this equation by r, we have

$$rS=\quad ar+ar^2+ar^3 \ldots\ldots +ar^{n-1}+ar^n.$$

Subtracting, $rS-S=ar^n-a;$

or, $S(r-1)=a(r^n-1);$

whence, $S=\dfrac{a(r^n-1)}{r-1}.$

Since, $l=ar^{n-1}$, we have $rl=ar^n$;

therefore, $S=\dfrac{ar^n-a}{r-1}=\dfrac{rl-a}{r-1}.$

This formula gives the following

RULE, FOR FINDING THE SUM OF A GEOMETRICAL SERIES.—*Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.*

Ex. Find the sum of 6 terms of the progression 3, 12, 48, &c.

The last term $=3\times4^5=3\times1024=3072.$

$$S=\frac{lr-a}{r-1}=\frac{3072\times4-3}{4-1}=4095. \quad \textit{Ans.}$$

ART. **298.** If the ratio r is less than 1, the progression is decreasing, and the last term l, or ar^{n-1}, is less than a. In order that both terms of the fraction $\dfrac{rl-a}{r-1}$, or $\dfrac{ar^n-a}{r-1}$ may be positive, the signs of the terms may be changed, (Art. 124), and we have $S=\frac{a-rl}{1-r}$, or $=\frac{a-ar^n}{1-r}$. Therefore, the sum of the series, when the progression is decreasing, is found by the same rule, as when it is increasing, except that the product of the last term by the ratio, is to be subtracted from the first term, and the ratio subtracted from unity.

ART. **299.** The formula $S=\frac{a-ar^n}{1-r}$, by separating the numerator into two parts, may be placed under the form

$$S=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now when r is less than 1, it must be a *proper* fraction, which may be represented by $\frac{p}{q}$; then $r^n=\left(\frac{p}{q}\right)^n=\frac{p^n}{q^n}$.

Since p is less than q, the higher the power to which the fraction is raised, the less will be the numerator compared with the denominator; that is, the less will be the value of the fraction; therefore, when n becomes *very large*, the value of $\frac{p^n}{q^n}$, or r^n, will be *very small;* and when n becomes *infinitely great*, the value of $\frac{p^n}{q^n}$, or r^n, will be *infinitely small;* that is, 0. But, when the numerator of a fraction is zero (Art. 135) its value is 0. This reduces the value of S to $\frac{a}{1-r}$.

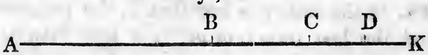
Hence, *when the number of terms of a decreasing geometrical series is infinite, the last term is zero, and the sum is equal to the first term divided by one minus the ratio.*

Ex. Find the sum of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$, &c.

Here $a=1$, $r=\frac{1}{2}$, and $S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2$. *Ans.*

That the sum of an *infinite* number of terms of a geometrical progression may be *finite,* will easily appear from the following illustration:

Take a straight line, AK, and bisect it in B; bisect BK in C; CK in D, and so on continually; then will



$AK=AB+BC+CD+$, &c., *in infinitum,* $=AB+\frac{1}{2}AB+\frac{1}{4}AB$ &c., *in infinitum,* $=2AB$, which agrees with the example.

Art. 300. The two equations, $l=ar^{n-1}$, and $S=\frac{ar^n-a}{r-1}$, furnish this general problem: *Knowing any three of the five quantities a, r, n, l, and* S, *of a geometrical progression, to determine the other two.* The following table contains all the values of each unknown quantity, or the equations from which it may be derived.

No.	Given.	Required.	Formulæ.
1.	a, r, n		$l=ar^{n-1}$,
2.	a, r, S		$l=\frac{a+(r-1)S}{r}$,
3.	a, n, S	l	$l(S-l)^{n-1}-a(S-a)^{n-1}=0$,
4.	r, n, S		$l=\frac{(r-1)Sr^{n-1}}{r^n-1}$.
5.	a, r, n		$S=\frac{a(r^n-1)}{r-1}$,
6.	a, r, l	S	$S=\frac{rl-a}{r-1}$,
7.	a, n, l		$S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$,
8.	r, n, l		$S=\frac{lr^n-l}{r^n-r^{n-1}}$.
9.	r, n, l		$a=\frac{l}{r^{n-1}}$,
10.	r, n, S	a	$a=\frac{(r-1)S}{r^n-1}$,
11.	r, l, S		$a=rl-(r-1)S$,
12.	n, l, S		$a(S-a)^{n-1}-l(S-l)^{n-1}=0$.
13.	a, n, l		$r=\sqrt[n-1]{\frac{l}{a}}$,
14.	a, n, S	r	$r^n-\frac{S}{a}r+\frac{S-a}{a}=0$.
15.	a, l, S		$r=\frac{S-a}{S-l}$,
16.	n, l, S		$r^n-\frac{S}{S-l}r^{n-1}+\frac{l}{S-l}=0$.
17.	a, r, l		$n=\frac{\log. l-\log. a}{\log. r}+1$,
18.	a, r, S	n	$n=\frac{\log. [a+(r-1)S]-\log. a}{\log. r}$,
19.	a, l, S		$n=\frac{\log. l-\log. a}{\log. (S-a)-\log. (S-l)}+1$.
20.	r, l, S		$n=\frac{\log. l-\log. [lr-(r-1)S]}{\log. r}+1$.

REMARK.— To determine the value of the unknown quantity in Nos. 3, 12, 14, and 16, may require the solution of an equation higher than the second degree. The values of n in the last four Nos. are obtained from the solution of an exponential equation (see Art. 382). Although the method of solving these equations has not been given, it was deemed proper to complete the table for the convenience of reference. The pupil should be required to verify all the values except those here referred to.

EXAMPLES FOR PRACTICE.

1. Find the 8^{th} term of the series 5, 10, 20, &c. *Ans.* 640.

2. Find the 7^{th} term of the series 54, 27, $13\frac{1}{2}$, &c.
Ans. $\frac{27}{32}$.

3. Find the 6^{th} term of the series $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c. *Ans.* $\frac{4}{9}$.

4. Find the 7^{th} term of the series —21 14, —$9\frac{1}{3}$, &c.
Ans. $-\frac{448}{243}$.

5. Find the n^{th} term of the series $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, &c. *Ans.* $\frac{3^{n-2}}{2^{n-1}}$.

Find the sum

6. Of 1+3+9+, &c., to 9 terms. *Ans.* 9841.
7. Of 1+4+16+, &c., to 8 terms. *Ans.* 21845.
8. Of 8+20+50+, &c., to 7 terms. *Ans.* $3249\frac{7}{8}$.
9. Of 5+20+80+, &c., to 8 terms. *Ans.* 109225.
10. Of 1+3+9+, &c., to n terms. *Ans.* $\frac{1}{2}(3^n-1)$.
11. Of 1—2+4—8+, &c., to n terms. *Ans.* $\frac{1}{3}(1\mp 2^n)$.
12. Of $x-y+\frac{y^2}{x}-\frac{y^3}{x^2}+$, &c., to n terms.

Ans. $\frac{x^2}{x+y}\left\{1-\left(-\frac{y}{x}\right)^n\right\}$.

13. The first term is 4, the last term 12500, and the number of terms 6. Required the ratio and the sum of all the terms.
Ans. Ratio =5; Sum =15624.

4. Find the geometrical progression, when the sum of the first and second terms is 9, and the sum of the first and third 15.
Ans. 3+6+12+, &c., $13\frac{1}{2}-4\frac{1}{2}+1\frac{1}{2}-$, &c.

Find the sum of an infinite number of terms of each of the following series:

15. Of $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+$, &c. *Ans.* $\frac{4}{3}$.

16. Of $9+6+4+$, &c. *Ans.* 27.

17. Of $6+2+\frac{2}{3}+$, &c. *Ans.* 9.

18. Of $\frac{2}{3}-\frac{1}{3}+\frac{1}{6}-$, &c. *Ans.* $\frac{4}{9}$.

19. Of $100+40+16+$, &c. *Ans.* $166\frac{2}{3}$.

20. Of $a+b+\frac{b^2}{a}+\frac{b^3}{a^2}+$, &c *Ans.* $\frac{a^2}{a-b}$.

21. Of $1+2a+2a^2+2a^3+$, &c. *Ans.* $\frac{1+a}{1-a}$.

22. The sum of an infinite geometric series is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

Ans. $2+\frac{2}{3}+\frac{2}{9}+$ . . or $4-\frac{4}{3}+\frac{4}{9}-$. . .

23. Find a geometric mean between 4 and 16. *Ans.* 8.

Let $a=4$, $c=16$, and m the required mean; then $a:m::m:c$; whence $m=\sqrt{ac}$.

24. The first term of a geometric series is 3, the last term 96, and the number of terms 6; find the ratio, and the intermediate terms.

By formula 13, page 251, we find $r=\sqrt[n-1]{\frac{l}{a}}$, which in this case becomes $r=\sqrt[5]{32}=2$; hence, the intermediate terms are 6, 12, 24, 48.

If it be required to insert m geometrical means between two numbers a and b, we have n, the whole number of terms, $=m+2$; hence, $n-1=m+1$, and $r=\sqrt[m+1]{\frac{l}{a}}$.

25. Insert two geometric means between $\frac{16}{27}$, and 2.

Ans. $\frac{8}{9}$, $\frac{4}{3}$.

26. Insert seven geometric means between 2 and 13122.

Ans. 6, 18, 54, 162, 486, 1458, 4374.

ART. **301.** To find the value of *Circulating Decimals*, that is, decimals in which one or more figures are continually repeated.

Circulating decimals are quantities in geometrical progression, where the common ratio is $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., according as one, two, or three figures recur; thus the circulating decimal .253131. . . . is equal to $\frac{25}{100}+\left(\frac{31}{10^4}+\frac{31}{10^6}+\frac{31}{10^8}+\text{, \&c.}\right)$; and the part within the bracket is a geometrical series, of which the common ratio is $\frac{1}{10^2}=\frac{1}{100}$; we have, therefore, $a=\frac{31}{10^4}$; $r=\frac{1}{100}$; $\frac{a}{1-r}=\frac{31}{10^4}\div\frac{99}{100}=\frac{31}{9900}$; and the sum of the whole series $=\frac{25}{100}+\frac{31}{9900}=\frac{2506}{9900}=\frac{1253}{4950}$.

This operation may be performed more simply, as follows:

Let S=.253131. . . . . Multiply by 100, in order to remove the decimal point to the commencement of the *first* period of decimals, we have

$$100S=25.3\dot{1}3\dot{1}\ldots$$

Again, multiplying by 100 to remove the decimal point to the commencement of the *second* period of decimals, we have

$$10000S=2531.3131\ldots$$

Subtracting the preceding equation from the last, we get

$$9900S=2506;\ \therefore\ S=\tfrac{2506}{9900}.$$

1. Find the value of .636363. . . . *Ans.* $\frac{7}{11}$.

2. Find the value of .54123123. . . . *Ans.* $\frac{18023}{33300}$

HARMONICAL PROGRESSION.

ART. **302.** Three or more quantities are said to be in Harmonical Progression, when their reciprocals are in arithmetical progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, &c.; and $\frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}$, &c.
are in harmonical progression, because their reciprocals

$$1, 3, 5, 7, \text{\&c.; and } 4, 3\tfrac{1}{2}, 3, 2\tfrac{1}{2}, \text{\&c.}$$

are in arithmetical progression.

ART. **303.** PROPOSITION.—*If three quantities are in harmonical progression, the first term is to the third, as the difference of the first and second, is to the difference of the second and third.*

For if a, b, c, are in harmonical progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, are in arithmetical progression,

$$\therefore \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}.$$ Hence, multiplying by abc,

$$ac-bc=ab-ac;\ \text{or}\ c(a-b)=a(b-c).$$

Dividing both sides by $a-b$, and by a, we have

$$\frac{c}{a}=\frac{b-c}{a-b};$$

this gives $\quad a:c::a-b:b-c.$

Therefore, a Harmonical Progression is a series of quantities in harmonical proportion (Art. 280); or such that if any three consecutive terms be taken, the first is to the third, as the difference of the first and second is to the difference of the second and third.

From this proposition it follows, that all problems with respect to numbers in harmonical progression, may be solved by *inverting* them, and considering the reciprocals as quantities in arithmetical progression. This renders it unnecessary to give any special rules for the solution of problems in harmonical progression.

EXAMPLES FOR PRACTICE.

1. Given the first two terms of a harmonical progression, a and b, to find the n^{th} term.

Let l be the n^{th} term, then (Art. 302), $\frac{1}{a}$ and $\frac{1}{b}$ are the first two terms of an arithmetical progression, and it is required to find $\frac{1}{l}$, the n^{th} term.

$$d=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}, \text{ and } \frac{1}{l}=\frac{1}{a}+(n-1)d, \text{ (Art. 292)};$$

$$\therefore\ \frac{1}{l}=\frac{1}{a}+(n-1)\frac{a-b}{ab}=\frac{(n-1)a-(n-2)b}{ab};$$

whence, $\quad l=\frac{ab}{(n-1)a-(n-2)b}.$

By means of this formula, when any two successive terms of a harmonical progression are given, any other term may be found.

2. Insert m harmonic means between a and b.

Here, if d be the common difference of the reciprocals of the terms, we have

$$\frac{1}{b}=\frac{1}{a}+(n-1)d, \text{ and } d=\frac{a-b}{(n-1)ab}=\frac{a-b}{(m+1)ab};$$

whence the arithmetical progression is found; and by inverting its terms, the harmonicals are also found.

3. Insert two harmonic means between 3 and 12.

Ans. 4 and 6.

4. Insert two harmonic means between 2 and $\frac{1}{5}$.

Ans. $\frac{1}{2}$ and $\frac{2}{7}$.

5. The first term of a harmonic series is $\frac{1}{2}$, and the 6^{th} $\frac{1}{12}$; find the intermediate terms. *Ans.* $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$.

6. *a*, *b*, *c*, are in arithmetical progression, and *b*, *c*, *d*, are in harmonical progression; prove that $a : b :: c : d$.

PROBLEMS IN ARITHMETICAL AND GEOMETRICAL PROGRESSION.

ART. **304.** The sum of five numbers in arithmetical progression is 35, and the sum of their squares 335; find the numbers.

Ans. 1, 4, 7, 10, 13.

Let $x-2y$, $x-y$, x, $x+y$, $x+2y$, be the numbers.

2. There are four numbers in arithmetic progression, and the sum of the squares of the extremes is 68, and of the means 52; find them. *Ans.* 2, 4, 6, 8.

Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers.

SUGGESTION.—When the number of terms in an arithmetic progression is *odd*, the common difference should be called y, and the middle term x; but when the number of terms is *even*, the common difference must be $2y$, and the two middle terms $x-y$, and $x+y$.

3. The sum of 3 numbers in arithmetical progression is 30, and the sum of their squares 308; find them. *Ans.* 8, 10, 12.

4. There are 4 numbers in arithmetical progression, their sum is 26, and their product 880; find them. *Ans.* 2, 5, 8, 11.

5. There are 3 numbers in geometrical progression, whose sum is 31; and the sum of the 1st and 2nd : sum of 1st and 3rd :: 3 : 13; find them. *Ans.* 1, 5, 25.

6. The sum of the squares of three numbers in arithmetic progression is 83; and the square of the mean is greater by 4 than the product of the extremes. Required the numbers.

Ans. 3, 5, 7

7. Find 4 numbers in arithmetical progression, such that the product of the extremes $=27$; of the means $=35$.

Ans. 3, 5, 7, 9.

8. There are 3 numbers in arithmetical progression, whose

sum is 18; but if you multiply the first term by 2, the second by 3, and the third by 6, the products will be in geometrical progression; find them. *Ans.* 3, 6, 9.

9. The sum of the 4^{th} powers of three successive natural numbers is 962; find them. *Ans.* 3, 4, 5.

10. The product of four successive natural numbers is 840; find them. *Ans.* 4, 5, 6, 7.

11. The product of four numbers in arithmetical progression is 280, and the sum of their squares 166; find them.

Ans. 1, 4, 7, 10.

12. The sum of 9 numbers in arithmetical progression is 45, and the sum of their squares 285; find them.

Ans. 1, 2, 3, &c., to 9.

13. The sum of 7 numbers in arithmetical progression is 35, and the sum of their cubes 1295; find them.

Ans. 2, 3, &c., to 8.

14. Prove that when the arithmetical mean of two numbers is to the geometric mean : : 5 : 4; that one of them is 4 times the other.

15. The sum of 3 numbers in geometrical progression is 7; and the sum of their reciprocals is $\frac{7}{4}$; find them. *Ans.* 1, 2, 4.

16. There are 4 numbers in geometrical progression, the sum of the first and third is 10, and the sum of the second and fourth is 30; find them. *Ans.* 1, 3, 9, 27.

17. There are 4 numbers in geometrical progression, the sum of the extremes is 35, the sum of the means is 30; find them.

Ans. 8, 12, 18, 27.

18. There are 4 numbers in arithmetical progression, which being increased by 2, 4, 8, and 15 respectively, the sums are in geometrical progression; find them. *Ans.* 6, 8, 10, 12.

19. There are 3 numbers in geometrical progression whose continued product is 64, and the sum of their cubes 584; find them. *Ans.* 2, 4, 8.

SUGGESTION.— In solving difficult problems in geometrical progression, instead of denoting the terms by x, xy, xy^2, &c., it is sometimes preferable to express them by other forms. Thus, three terms may be expressed by x, $\sqrt{xy}$, y, or, x^2, xy, y^2; four terms by $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$; five terms by $\frac{x^3}{y}$, x^2, xy, y^2, $\frac{y^3}{x}$; six terms by $\frac{x^3}{y^2}$, $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$, $\frac{y^3}{x^2}$. In all these cases the product of the first and third of any three consecutive terms, is equal to the square of the middle term.

CHAPTER IX.

PERMUTATIONS, COMBINATIONS, AND BINOMIAL THEOREM.

ART. **305.** The different orders in which quantities can be arranged, are called their *Permutations*. Quantities may be arranged in sets of one and one, two and two, three and three, and so on. Thus, if we have three quantities, a, b, c, we may arrange them in sets of *one*, of *two*, or of *three*, thus:

Of one	a,	b,	c.
Of two	ab, ac;	ba, bc;	ca, cb.
Of three	abc, acb;	bac, bca;	cab, cba.

REMARK.—Some writers, confine the term *permutations* to the class where the quantities are taken *all* together, and give the title of *arrangements*, or *variations*, to those groups of one and one, two and two, three and three, &c., in which the number of quantities in each group is *less* than the whole number of quantities.

ART. **306.** To find the number of permutations that can be formed out of n letters, taken *singly*, taken *two* together, *three* together. . . . and r together.

Let $a, b, c, d, \ldots \ldots k$, be the n letters; and let P_1 denote the whole number of permutations where the letters are taken *singly;* P_2 the whole number of permutations taken 2 together . . . . and P_r the whole number of permutations taken r together.

The number of permutations of n letters taken singly, or one and one, is evidently equal to the number of letters, that is n; therefore,

$$P_1=n.$$

The number of permutations of n letters, taken two together, is $n(n-1)$. For since there are n quantities

$$a, b, c, d, \ldots \ldots k,$$

if we remove a, there will remain $(n-1)$ quantities,

$$b, c, d, \ldots \ldots k.$$

Writing a before each of these $(n-1)$ quantities, we shall have

$$ab, ac, ad \ldots \ldots ak.$$

That is, $(n-1)$ permutations in which a stands *first*.

In the same manner there are $(n-1)$ permutations in which b stands first, and so of each of the remaining letters $c, d \ldots k$. And since there are n letters, there are $n(n-1)$ permutations taken *two* together; that is,

$$P_2=n(n-1).$$

Hence, *the number of permutations of* n *letters taken two together, is equal to the number of letters, multiplied by the number less one.*

For example, if $n=4$, the number of permutations of the four letters, a, b, c, d, taken two together, is $4\times(4-1)=4\times3=12$. Thus, *ab, ac, ad,* || *ba, bc, bd,* || *ca, cb, cd,* || *da, db, dc.*

The number of permutations of n letters, taken *three* together, is $n(n-1)(n-2)$. For if we take $(n-1)$ letters $b, c, d, \ldots k$, the number of permutations taken *two* together, by the last paragraph, is

$$(n-1)(n-2).$$

Let a be placed before each of these permutations; then there are $(n-1)(n-2)$ permutations of n letters, taken *three* together, in which a stands first. Proceeding in the same manner with b, there are $(n-1)(n-2)$ permutations in which b stands *first;* and so for each of the n letters. Hence, the whole number of permutations of n letters, taken *three* together, is $n(n-1)(n-2)$; that is,

$$P_3=n(n-1)(n-2).$$

Hence, *the number of permutations of* n *letters taken three together, is equal to the number of letters, multiplied by the number less one, multiplied by the number less two.*

For example, if $n=4$, the number of permutations of the four letters, a, b, c, d, taken *three* together, is $4(4-1)(4-2)=4\times3\times2=24$. Thus,

abc, abd, acb, acd, adb, adc, bac, bad, bca, bcd, bda, bdc, cab, cad, cba, cbd, cda, cdb, dab, dac, dba, dbc, dca, dcb.

By following the same method, we can prove that the number of permutations of n letters taken *four* together, is

$$P_4=n(n-1)(n-2)(n-3).$$

By examining each of the preceding results, we see that the negative number in the last factor is *less by unity*, than the number of letters in each permutation. Thus,

$$P_1=n= \ldots \ldots \ldots \ldots \ldots \ldots \ldots n-\overline{1-1}.$$

$$P_2=n(n-1)= \ldots \ldots \ldots \ldots \ldots n(n-\overline{2-1}).$$

$$P_3=n(n-1)(n-2)= \ldots\ldots\ n(n-1)(n-\overline{3-1}).$$

$$P_4=n(n-1)(n-2)(n-3)=\ n(n-1)(n-2)(n-\overline{4-1}).$$

Hence, from *analogy*, we conclude, that the number of permutations of n things taken r together, is

$$P_r=n(n-1)(n-2)\ \ldots\ldots\ldots\ldots\ldots\ (n-\overline{r-1}).$$

ART. **306***a*, *Corollary*. If *all* the letters be taken together, then r becomes equal to n, and the last factor becomes **1**; that is,

$$P_n=n(n-1)(n-2)\ \ldots\ldots\ldots\ldots\ldots\ (n-\overline{n-1}),$$

or $$P_n=n(n-1)(n-2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 1.$$

Or, inverting the order of the factors,

$$P_n=1\times2\times3\ \ldots\ldots\ldots\ldots\ldots\ldots\ (n-1)n.$$

Hence, *the number of permutations of* n *letters taken* n *together, is equal to the product of the natural numbers from* 1 *up to* n.

Ex. The permutations of three letters, a, b, c, taken *three* together, is $1\times2\times3=6$.

ART. **307.** If the *same* letter occur p times, the number of permutations in n letters, taken *all* together, is

$$\frac{1\times2\times3\ \ldots\ldots\ (n-1)n}{1\times2\times3\ \ldots\ p}.$$

Suppose these p letters to be all different. Then for any particular position of the other letters, these p quantities, taken p together, will form $(1\times2\times3\ \ldots\ p)$ permutations from their interchange with each other; and when these letters are *alike*, these permutations are all reduced to *one*. And as this is true for every position of the other letters, there will be altogether $(1\times2\times3\ \ldots\ p)$ times fewer permutations when they are alike than when they are all different.

Thus, in the letters A, I, D, there are $1\times2\times3=6$ permutations taken *all* together, but if I becomes D, then three of these permutations become identical with the remaining three, and the whole number of permutations of the letters ADD taken all together, is $\dfrac{1\times2\times3}{1\times2}=3$.

ART. **307***a*, *Corollary*. In like manner, if the same letter occur p times, another letter q times, a third letter r times, and so on, the number of permutations taken *all* together, is

$$\frac{1\times2\times3\ \ldots\ldots\ldots\ldots\ (n-1)n}{(1\times2\ \ldots\ p)(1\times2\ \ldots\ q)(1\times2\ \ldots\ r)\times,\ \&c}.$$

For by the last article, if p letters be alike, there will be $(1\times2\times3\ .\ .\ .\ p)$ fewer permutations than when they are all different; also if q other letters be alike, but different from the first, there will be $(1\times2\times3\ .\ .\ q)$ times fewer permutations, and so on; hence, there will be altogether $(1\times2\times3\ .\ .\ .\ p)(1\times2\times3\ .\ .\ .\ q)$, &c., times fewer permutations than when the letters are all different, and consequently the general expression will be as announced.

ART. **308.** COMBINATIONS.—The *Combinations* of quantities are the *different* collections that can be formed out of them, without reference to the *order* in which they are placed. Thus, *ab*, *ac*, *bc*, are the combinations of the letters *a*, *b*, *c*, taken *two* together; *ab* and *ba*, though different permutations, forming the same combination.

PROPOSITION.—*To find the number of combinations that can be formed out of* n *letters, taken singly, taken two together, three together, ... and* r *together.*

Let C_1 denote the number of combinations of n things taken singly; C_2 the number of combinations taken two together, ... and C_r the number of combinations taken r together.

The number of combinations of n letters taken singly is evidently n; that is,

$$C_1=n.$$

The number of *permutations* of n letters, taken *two* together, is $n(n-1)$; but each combination, as *ab*, admits of (1×2) permutations, *ab*, *ba*; therefore there are (1×2) times as many permutations as combinations. Hence,

$$C_2=\frac{n(n-1)}{1\times2}.$$

Again, in n letters taken *three* together, the number of permutations is $n(n-1)(n-2)$; but each combination of three letters, as *abc*, admits of $1\times2\times3$ permutations; therefore, there are $1\times2\times3$ times as many permutations as combinations. Hence,

$$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}.$$

And in the same manner it appears that in n letters, the number of combinations, each of which contains r of them, is

$$C_r=\frac{n(n-1)(n-2)\ .\ .\ .\ [n-(r-1)]}{1\times2\times3\ .\ .\ .\ .\ .\ .\ .\ r}.$$

Ex. The number of combinations of 5 letters, taken *three* together, is $\frac{5\times4\times3}{1\times2\times3}=10.$

ART. **309.** *The number of combinations of* n *things taken* r *together, is the same as the number of combinations of n things taken* n—r *together.*

The truth of this proposition is evident from the following consideration: if out of n things r be taken, $(n-r)$ things will always be left; and for every different parcel containing r things, there will be a different one left containing $(n-r)$; therefore, the *number* of parcels containing r things, must be equal to the number containing $(n-r)$.

For example, in the letters *abcde,* for each combination of *three* letters, there is a different one of *two* letters. Thus,

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.
de, ce, cd, be, bd, bc, ae, ad, ac, ab.

Hence, in finding the number of combinations taken r together, when $r>\frac{1}{2}n$, the shorter method is to find the number taken $(n-r)$ together.

EXAMPLES FOR PRACTICE.

1. How many permutations of two letters each, can be formed out of the letters a, b, c, d, e? How many of three? How many of four? *Ans.* (1) 20. (2) 60. (3) 120.

2. How many combinations of two letters each, can be formed out of the letters a, b, c, d, e? How many of three? How many of four? How many of five?
Ans. (1) 10. (2) 10. (3) 5. (4) 1.

3. In how many ways, taken all together, may the letters in the word NOT be written? In the word HOME.
Ans. 6, and 24.

4. How often can 6 persons change their places at dinner so as not to sit twice in the same order? *Ans.* 720.

5. In how many different ways, taken all together, can the seven prismatic colors be arranged? *Ans.* 5040.

6. In how many different ways can six letters be arranged when taken singly, two by two, three by three, and so on, till they are all taken? *Ans.* 1956.

SUGGESTION.— Take the sum of the different permutations.

7. How many different products can be formed with any two of the figures 3, 4, 5, 6? *Ans.* 6.

8. How many different products can be formed with any three of the figures 1, 3, 5, 7, 9? *Ans.* 10.

9. The number of permutations of n things taken four together = six times the number taken three together; find n.
Ans. $n=9$.

10. The number of permutations of 15 things taken r together = ten times the number taken $(r-1)$ together; find r.
Ans. $r=6$.

11. How many different sums of money can be formed with a cent, a three cent piece, a half dime, and a dime?

SUGGESTION.—Take the sum of the different combinations of four things taken singly, two together, three together, and four together.
Ans. 15.

12. With the addition of a twenty-five cent piece, and a half dollar, to the coins in the last example, how many different sums of money may be formed? *Ans.* 63.

13. At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates and only 3 persons to be chosen; in how many ways may a man vote? *Ans.* 14.

14. Of the combinations of 5 letters, a, b, c, d, e, taken three together, in how many will a occur?

SUGGESTION.—First find the combinations of *four* letters taken *two* together?
Ans. 6.

15. On how many nights may a different guard be posted of 4 men out of 16? and on how many of these will any particular man be on guard? *Ans.* 1820, and 455.

16. The number of combinations of n quantities *four* together, is to the number *two* together, as 15 to 2; find n. *Ans.* $n=12$.

17. How many changes may be rung with 5 bells out of 8, and how many with the whole peal? *Ans.* 6720, and 40320.

18. Find the number of permutations taken *all* together, that can be made out of the letters of the word Algebra. (See Art. 307.) *Ans.* 2520.

19. In how many ways can we write the term $a^3b^4c^2$?

SUGGESTION.—There are 3 a's 4 b's, and 2 c's. (See Art. 307*a*.)
Ans. 1260.

20. In how many terms in the preceding example, will a^3 stand first?

Suggestion.— The number will be equal to the permutations taken all together, of the letters in b^4c^2.

Ans. 15.

21. In the permutations formed out of a, b, c, d, e, f, g, taken all together, how many begin with ab? How many with abc? How many with $abcd$? *Ans.* (1)120. (2)24. (3)6.

22. Out of 17 consonants and 5 vowels, how many words can be formed, having two consonants and one vowel in each?

Ans. 4080.

23. Find the number of combinations that can be formed out of the letters of the word "*Notation*," taken 3 together.

Ans. 22.

BINOMIAL THEOREM,

WHEN THE EXPONENT IS A POSITIVE INTEGER.

Art. **310.** We have already explained (Art. 172) the method of finding any power of a binomial, by repeated multiplication; and we shall now proceed from the theory of Combinations (Art. 308), to derive a general rule, which is called the *Binomial Theorem*, and sometimes *Sir Isaac Newton's Theorem*, from the name of the inventor.

In its most general form the Binomial Theorem teaches the method of developing into a series any binomial whose index is either integral or fractional, positive or negative; that is, quantities of the form

$$(a+x)^n,\ (a+x)^{-n},\ (a+x)^{\frac{n}{m}},\ (a+x)^{-\frac{n}{m}},$$

where a or x may be either plus or minus.

The following investigation applies only to the case where the exponent is *positive* and *integral*, the other cases will be considered hereafter. (See Art. 319.)

By actual multiplication it appears that

$$(x+a)(x+b)=x^2\left.\begin{array}{l}+a\\+b\end{array}\right|x+ab.$$

In like manner $(x+a)(x+b)(x+c)$

$$=x^3\left.\begin{array}{l}+a\\+b\\+c\end{array}\right|x^2\left.\begin{array}{l}+ab\\+ac\\+bc\end{array}\right|x+abc.$$

Also, $(x+a)(x+b)(x+c)(x+d)$

$$=x^4+\left.\begin{matrix}a\\+b\\+c\\+d\end{matrix}\right|x^3+\left.\begin{matrix}ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right|x^2+\left.\begin{matrix}abc\\+abd\\+acd\\+bcd\end{matrix}\right|x+abcd.$$

An examination of either of these products, shows that it is composed of a series of descending powers of x, and of certain coëfficients, formed according to the following law:

1st. The *exponent* of the highest power of x is the same as the *number* of binomial factors, and the other exponents of x decrease by 1 in each succeeding term.

2nd. The *coëfficient* of the first term is 1; of the second, the sum of the quantities a, b, c, &c.; of the third, the sum of the products of every two of the quantities a, b, c, &c.; of the fourth, the sum of the products of every three, and so on; and of the last, the product of all the n quantities a, b, c, &c.

Suppose, then, this law to hold for the product of n binomial factors $x+a$, $x+b$, $x+c$, ... $x+k$; so that $(x+a)(x+b)(x+c)\,.....\,(x+k)=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\,....\,+K$,

where
$$A=a+b+c+\,....\,+k.$$
$$B=ab+ac+ad+\,.....$$
$$C=abc+abd+\,......$$
$$\&c. = \&c. ....$$
$$K=abcd\,......\,k.$$

If we multiply both sides of this equation by a new factor $x+l$, we have

$$(x+a)(x+b)(x+c)\,.....\,(x+k)(x+l)$$
$$=x^{n+1}+\left.\begin{matrix}A\\+l\end{matrix}\right|x^n+\left.\begin{matrix}B\\+Al\end{matrix}\right|x^{n-1}+\left.\begin{matrix}C\\Bl\end{matrix}\right|x^{n-2}\,....\,....+Kl.$$

Here
$$A+l=a+b+c+\,....\,+k+l;$$
$$B+Al=ab+ac+ad\,...\,+al+bl\,...\,+kl.$$
$$\&c. = \&c. .....$$
$$Kl=abcd\,....\,kl.$$

It is evident the same law still holds; that is,

1st. The *exponent* of the highest power of x is the same as the *number* of binomial factors; and the other exponents of x decrease by 1 in each succeeding term.

2nd. The coëfficient of the first term is 1.

$A+l$, the coëfficient of the second term, is the sum of the second terms, $a, b, c, \dots k$, and l of the binomial factors.

$B+Al$, the coëfficient of the third term, is the sum of the products of the second terms of the binomial factors taken *two* together; for by hypothesis, B is the sum of the products of n binomial factors, taken *two* together, and Al is the product of the second terms of the preceding n binomials by the second term l of the new binomial; therefore, $B+Al$ is the product of the second terms of all the binomial factors taken *two* together.

Kl, the last term, is the product of *all* the second terms of the $n+1$ binomial factors.

Hence, if the law holds when n binomial factors are multiplied together, it will hold when $n+1$ factors are multiplied together; but it has been shown by actual multiplication to hold up to 4 factors; therefore it is true for $4+1$, that is 5; and if for 5, then also for $5+1$, that is 6; and so on generally, for any number whatever.

Now let b, c, d, &c., each $=a$,

then $A=a+a+a+a+$, &c., to n terms $=na$.

$B=a^2+a^2+$&c., $=a^2$ taken as many times as is equal to the No. of combinations of n things taken *two* together, which is (Art. 308), $=\dfrac{n(n-1)a^2}{1\cdot 2}$.

$C=a^3+a^3+$, &c., $=a^3$ taken as many times as is equal to the No. of combinations of the things taken *three* together, which is (Art. 308), $=\dfrac{n(n-1)(n-2)a^3}{1\cdot 2\cdot 3}$.

&c. = &c.

$K=aaa \dots$ to n factors $=a^n$.

Also, $(x+a)(x+b)(x+c) \dots\dots (x+l)$ becomes

$$(x+a)(x+a)(x+a) \dots\dots (x+a)=(x+a)^n.$$

$$\therefore (x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3x^{n-3}$$
$$+\dots\dots+a^n.$$

By changing x to a and a to x, we have

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+$$
$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 \dots\dots +x^n.$$

Let $a=1$, then since every power of 1 is 1,

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot2}x^2+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3+\ldots\ldots+x^n.$$

Cor. 1.—It is obvious that the sum of the exponents of a and x in each term $=n$.

Cor. 2.—If either term of the binomial is negative, every *odd* power of that term will be negative (Art. 193); therefore the signs of the terms in which the odd powers are found will be *negative*.

$$\therefore\ (1-x)^n=1-nx+\frac{n(n-1)}{1\cdot2}x^2-\frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3+, \text{ \&c.}$$

Cor. 3.—The general term of the series is

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-r+2)}{1\cdot2\cdot3\ \ \ldots\ldots\ (r-1)}a^{n-r+1}x^{r-1}.$$

For the 1st term is . . . . . . a^n,

2nd " " . . . . . . $na^{n-1}x$,

3rd " " . . . . . . $\frac{n(n-1)}{1\cdot2}a^{n-2}x^2$,

4th " " . . . . . . $\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^{n-3}x^3$,

&c., &c.

Here it is evident the coëfficient of any term is formed of the product of the factors $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, &c., in number, *one less* than the number which denotes the place of the term; therefore, the coëfficient of the r^{th} term will be

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ [n-(r-2)]}{1\cdot2\cdot3\ \ \ldots\ldots\ (r-1)}.$$

Also, the exponent of x is the same as the denominator of the last factor of the coëfficient; and the exponent of a is equal to n minus the exponent of x, (Cor. 1); therefore, the whole r^{th} term is,

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-r+2)}{1\cdot2\cdot3\ \ \ldots\ldots\ (r-1)}a^{n-r+1}x^{r-1}.$$

This is called *the general term*, because by making $r=2, 3, 4$ &c., all the others can be deduced from it.

Ex. Required the 5^{th} term of $(a-x)^7$.

Here $r=5$, and $n=7$;

$$\therefore \text{ term required } = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}(a)^3(-x)^4 = 35a^3x^4.$$

Cor. 4.—If n be a positive integer, and $r=n+2$, then $(n-r+2)$ becomes 0, and the $(n+2)$ term vanishes; therefore, the series consists of $(n+1)$ terms altogether; that is, *in raising a binomial to any given power, the number of terms is one greater than the exponent of the power to which the binomial is to be raised.*

Cor. 5.—When the index of the binomial is a positive integer, the coëfficients of the terms taken in an inverse order from the end of the series, are equal to the coëfficients of the corresponding terms taken in a direct order from the beginning.

If we compare the expansion of $(a+x)^n$, and $(x+a)^n$, we have

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 +, \&c$$

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{1 \cdot 2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}a^3 +, \&c$$

Since the binomials are the same, the series resulting from their expansion must be the same, except that the order of the terms will be inverted. It is clearly seen that the coëfficients of the corresponding terms are equal.

Hence, in expanding a binomial, whose index is a positive integer, the latter half of the expansion may be taken from the first half.

Ex. Expand $(a-b)^5$.

Here the number of terms $(n+1)$ is equal to 6; therefore, it will only be necessary to calculate the coëfficients of the first three, thus:

$$(a-b)^5 = a^5 - 5a^4b + \frac{5 \cdot 4}{1 \cdot 2}a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

Cor. 6.—The sum of the coëfficients of any expanded binomial whose index is n, and where both terms are positive, is always equal to 2^n.

For if $x=a=1$, then $(x+a)^n = (1+1)^n = 2^n$

$$=1 + \frac{n}{1} + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} +, \&c.$$

Thus, the coëfficients of

$$a+x=1+1=2=2^1,$$
$$(a+x)^2=1+2+1=4=2^2,$$
$$(a+x)^3=1+3+3+1=8=2^3,$$
$$(a+x)^4=1+4+6+4+1=16=2^4,$$
&c., &c.

ART. **311.** In the application of the Binomial Theorem, it is convenient to observe, that *if the coëfficient of any term be multiplied by the exponent of the first letter of the binomial in that term, and the product be divided by the number of the term, the quotient will be the coëfficient of the next term.* Thus, in raising $a-x$ to the 7^{th} power, the terms without the coëfficients, are

$$a^7,\ a^6x,\ a^5x^2,\ a^4x^3,\ a^3x^4,\ a^2x^5,\ ax^6,\ x^7;$$

and the coëfficients are

$$1,\ \frac{1\times7}{1},\ \frac{7\times6}{2},\ \frac{21\times5}{3},\ \frac{35\times4}{4},\ \frac{35\times3}{5},\ \frac{21\times2}{6},\ \frac{7\times1}{7}.$$

And since the signs of the terms are alternately plus and minus, (Art. 310, Cor. 2), we have

$$(a-x)^7=a^7-7a^6x+21a^5x^2-35a^4x^3+35a^3x^4-21a^2x^5$$
$$+7ax^6-x^7.$$

ART. **312.** If the terms of the given binomial are affected with coëfficients, or exponents, they must be raised to the required powers, by the rule for the involution of monomials (Art. 172). Thus,

$$(2a^2-3b^3)^4=(2a^2)^4-\frac{4}{1}(2a^2)^3(3b^3)+\frac{4\cdot3}{1\cdot2}(2a^2)^2(3b^3)^2,$$

$$-\frac{4\cdot3\cdot2}{1\cdot2\cdot3}(2a^2)(3b^3)^3+\frac{4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4}(3b^3)^4.$$

$$=16a^8-4\times8a^6\times3b^3+6\times4a^4\times9b^6-4\times2a^2\times27b^9+81b^{12}$$
$$=16a^8-96a^6b^3+216a^4b^6-216a^2b^9+81b^{12}.$$

ART. **313.** By means of the Binomial Theorem we can raise any polynomial to any power. Thus, let it be required to raise $a-b+c$ to the third power.

Let $a-b=m$, then $(a-b+c)^3=(m+c)^3=m^3+3m^2c+3mc^2+c^3$.

Substituting for m its equal $a-b$, we find

$$(a-b+c)^3=(a-b)^3+3(a-b)^2c+3(a-b)c^2+c^3.$$

Developing the powers of $a-b$, and performing the operations indicated, we finally obtain

$$(a-b+c)^3=a^3-3a^2b+3ab^2-b^3+3a^2c-6abc+3b^2c+3ac^2 -3bc^2+c^3.$$

EXAMPLES FOR PRACTICE.

1. Expand $(a+b)^8$, $(a-b)^7$, $(2x-3y)^5$, and $(5-4x)^4$.

(1) *Ans.* $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

(2) *Ans.* $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$.

(3) *Ans.* $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$.

(4) *Ans.* $625-2000x+2400x^2-1280x^3+256x^4$.

2. Required the coëfficient of x^6 in the expansion of $(x+y)^{10}$. *Ans.* 210.

3. Find the 5^{th} term of the expansion of $(c^2-d^2)^{12}$. *Ans.* $495c^{16}d^8$.

SUGGESTION.—(See Cor. 3; Art. 310.) Instead of a, x, n, and r, substitute c^2, $-d^2$, 12, and 5.

4. Find the 7^{th} term of $(a^3+3ab)^9$. *Ans.* $61236a^{15}b^6$.

5. Find the 5^{th} term of $(3a^2-7x^3)^8$. *Ans.* $13613670a^8x^{12}$.

6. Find the 6^{th} term of $(ax+by)^{10}$. *Ans.* $252a^5b^5x^5y^5$.

7. Find the middle term of $(a^m+x^n)^{12}$. *Ans.* $924a^{6m}x^{6n}$.

8. Find the two middle terms of $(a+x)^{13}$. *Ans.* $1716a^7x^6$, and $1716a^6x^7$.

9. Find the 8^{th} term of $(1+x)^{11}$. *Ans.* $330x^7$.

10. Find the 6^{th} term of $(x-y)^{30}$. *Ans.* $-142506x^{25}y^5$.

11. Expand $(3ac-2bd)^5$. *Ans.* $243a^5c^5-810a^4c^4bd+1080a^3c^3b^2d^2-720a^2c^2b^3d^3+240acb^4d^4-32b^5d^5$.

12. Expand $(a+2b-c)^3$. *Ans.* $a^3+6a^2b+12ab^2+8b^3-3a^2c-12abc-12b^2c+3ac^2+6bc^2-c^3$.

13. Prove that the sum of the coëfficients of the odd terms of $(a+x)^n$, is equal to the sum of the coëfficients of the even terms.

CHAPTER X.

INDETERMINATE COEFFICIENTS: BINOMIAL THEOREM, GENERAL DEMONSTRATION: SUMMATION AND INTERPOLATION OF SERIES.

INDETERMINATE COËFFICIENTS.

ART. **314.** The method of developing algebraic expressions into series, by assuming a series with unknown coëfficients, and then, by equating the coëfficients of the like powers of x, finding the values of the assumed coëfficients, is termed the method of *Indeterminate Coëfficients.* It depends on the following

THEOREM.

If $A+Bx+Cx^2+Dx^3+$, &c., $=A'+B'x+C'x^2+D'x^3+$, &c., for every possible value of x (A, B, A', B', &c., not containing x); then shall $A=A'$, $B=B'$, $C=C'$, &c.; that is, *the coëfficients of the terms involving the same powers of x in the two series, are respectively equal.*

For, by transposing all the terms into the first member, we have $A-A'+(B-B')x+(C-C')x^2+(D-D')x^3+$, &c., $=0$.

If $A-A'$ is not equal to 0, let it be equal to some quantity p; then we have $(B-B')x+(C-C')x^2+(D-D')x^3+$, &c., $=-p$.

Now since A and A' are constant quantities, their difference, p, must be constant; but $-p=(B-B')x+(C-C')x^2+$, &c., a quantity which may evidently have various values, depending on the different values of the variable x; therefore, p must be variable; that is, we have proved the same quantity (p) to be both *fixed* and *variable,* which is *impossible.* Therefore, there is no *possible quantity* (p) which can express the difference $A-A'$; or in other words

$$A-A'=0 \quad \therefore \quad A=A'.$$

Hence, $(B-B')x+(C-C')x^2+(D-D')x^3+$, &c., $=0$.

By dividing each side by x, we have

$$B-B'+(C-C')x+(D-D')x^2+, \text{\&c.}, =0.$$

By a process of reasoning, exactly similar to that used in the case of A—A′, we may show that B=B′. And so on for the remaining coëfficients of the like powers of x.

Cor.— If we have an equation of the form

$A+Bx+Cx^2+Dx^3+Ex^4+$, &c., $=0$, which is true for *any value whatever of* x, then $A=0$, $B=0$, $C=0$, &c.; that is, *each coëfficient is separately equal to zero.*

For the right hand member may evidently be put under the form $0+0x+0x^2+0x^3+$, &c.; then comparing the coëfficients of the like powers of x, we have $A=0$, $B=0$, $C=0$, &c.

REMARK.— As the values of the coëfficients assumed are at first unknown, this method might more properly be termed, the method of *undetermined* coëfficients, or the method of *unknown* coëfficients.

ART. **315.** Let it be required to develope $\frac{a}{a+bx}$ into a series without a resort to division.

It is obvious that the series will consist of the powers of x multiplied by certain undetermined coëfficients, depending on either a or b, or both of them, and that x may not enter into the first term; therefore, let us assume $\frac{a}{a+bx}=A+Bx+Cx^2+Dx^3$ $+$, &c.

Multiply both sides by the denominator $a+bx$, and arrange the terms according to the powers of x; we thus obtain

$$a=Aa+\left.\begin{matrix}Ba\\+Ab\end{matrix}\right|x+\left.\begin{matrix}Ca\\+Bb\end{matrix}\right|x^2+\left.\begin{matrix}Da\\+Cb\end{matrix}\right|x^3+, \&c.$$

But by the preceding theorem, the coëfficients of the same powers of x in each term are equal to each other; therefore,

$$a=Aa; \text{ hence, } A=1;$$

$$Ba+Ab=0; \quad \text{“} \quad B=-\frac{b}{a};$$

$$Ca+Bb=0; \quad \text{“} \quad C=+\frac{b^2}{a^2};$$

$$Da+Cb=0; \quad \text{“} \quad D=-\frac{b^3}{a^3}, \&c.$$

Substituting these values of the coëfficients in the assumed series, we find

$\frac{a}{a+bx}=1-\frac{b}{a}x+\frac{b^2}{a^2}x^2-\frac{b^3}{a^3}x^3+\frac{b^4}{a^4}x^4$, &c., the same as would be obtained by actual division.

REMARK.— Lest the learner may not see that the left member a, is of the same form as the assumed series on the right, it is proper to observe that a is the same as $a+0x+0x^2+0x^3+$, &c.

ART. **316.** A series with indeterminate coëfficients, is generally assumed to proceed according to the ascending integral and positive powers of x, beginning with x^0; but in many series this is not the case; the error in the assumption will then be shown, either by an impossible result, or by the coëfficients of those terms which do not exist in the *actual* series, being found equal to zero.

Thus, if it be required to develope $\frac{1}{3x-x^2}$, and we assume the series to be $A+Bx+Cx^2+Dx^3+Ex^4+$, &c., we have, after clearing of fractions,

$$1=3Ax+(3B-A)x^2+(3C-B)x^3+, \text{\&c.};$$

from which, by equating the coëfficients of the same powers of x,

$$1=0;$$

$$3A=0, \text{\&c.}$$

But the first equation, $1=0$, is absurd, from which we infer that the expression cannot be developed under the assumed form.

But $\frac{1}{3x-x^2}=\frac{1}{x}\times\frac{1}{3-x}$, and if we put $\frac{1}{3-x}=A+Bx+Cx^2+Dx^3$ $+$, &c.; after clearing of fractions, and equating the coëfficients of the like powers of x, we find $A=\frac{1}{3}$, $B=\frac{1}{9}$, $C=\frac{1}{27}$, $D=\frac{1}{81}$, &c.

Therefore, $\frac{1}{3x-x^2}=\frac{1}{x}\left(\frac{1}{3}+\frac{x}{9}+\frac{x^2}{27}+\frac{x^3}{81}+, \text{\&c.}\right)$

$$=\frac{x^{-1}}{3}+\frac{x^0}{9}+\frac{x}{27}+\frac{x^2}{81}+, \text{\&c.};$$

that is, the development contains a term affected with a *negative* exponent. Hence, at the outset we ought to have assumed

$$\frac{1}{3x-x^2}=Ax^{-1}+B+Cx+Dx^2+, \text{\&c.}$$

Again, if we assume

$$\frac{1-x^2}{1+x^2-x^4}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+, \text{\&c.};$$

we shall find the true series to be

$$1-2x^2+3x^4-5x^6+8x^8- \text{\&c.},$$

the coëfficients B, D, F, &c., of the odd powers of x becoming zero. We might, therefore, have assumed

$$\frac{1-x^2}{1+x^2-x^4}=A+Cx^2+Ex^4+Gx^6+, \&c.$$

ART. 317. EVOLUTION BY INDETERMINATE COEFFICIENTS.

Ex. Extract the square root of a^2+x^2.

Assume $\sqrt{(a^2+x^2)}=A+Bx+Cx^2+Dx^3+Ex^4+, \&c.$

Squaring both sides so as to obtain quantities of the same form, we find

$$a^2+x^2=A^2+2ABx+\left.\begin{array}{l}2AC\\+B^2\end{array}\right|x^2+\left.\begin{array}{l}2AD\\+2BC\end{array}\right|x^3+\left.\begin{array}{l}2AE\\+2BD\\+C^2\end{array}\right|x^4+, \&c.$$

From which, by equating the corresponding coëfficients, we get

$$A^2=a^2,\ 2AB=0,\ 2AC+B^2=1,\ 2AD+2BC=0, \&c.$$

From which we find $A=a$, $B=0$, $C=\frac{1}{2a}$, $D=0$, $E=-\frac{1}{8a^3}$, &c.

Hence, $\sqrt{(a^2+x^2)}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+, \&c.$

ART. 318. DECOMPOSITION OF RATIONAL FRACTIONS.—Fractions whose denominators can be separated into certain factors, may often be decomposed into other fractions whose denominators shall consist of one or more of these factors. We shall illustrate the method of operation by an example.

Decompose $\frac{5x-14}{x^2-6x+8}$ into two other fractions whose denominators shall be the factors of x^2-6x+8.

Since $x^2-6x+8=(x-2)(x-4)$, (Art. 234, Prop. 2nd), assume

$$\frac{5x-14}{x^2-6x+8}=\frac{A}{x-2}+\frac{B}{x-4}.$$

Reducing the fractions to a common denominator,

we have $\frac{5x-14}{x^2-6x+8}=\frac{A(x-4)+B(x-2)}{(x-2)(x-4)}$;

or $5x-14=A(x-4)+B(x-2)=(A+B)x-4A-2B.$

Now since this equation is true for any value whatever of x, we may equate the coëfficients of the corresponding terms, (Art. 314); this gives

$$A+B=5;\ -4A-2B=-14;\ \text{whence, } A=2, \text{ and } B=3.$$

$$\therefore\ \frac{5x-14}{x^2-6x+8}=\frac{2}{x-2}+\frac{3}{x-4}.$$

EXAMPLES FOR PRACTICE.

By the method of Indeterminate Coëfficients show that,

1. $\frac{1+2x}{1-3x}=1+5x+15x^2+45x^3+$, &c.

2. $\frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+18x^5+$, &c.

3. $\frac{1-3x+2x^2}{1+x+x^2}=1-4x+5x^2-x^3-4x^4+$, &c.

4. $\frac{3+2x}{5+7x}=\frac{3}{5}-\frac{11}{5^2}x+\frac{7\cdot 11}{5^3}x^2-\frac{7^2\cdot 11}{5^4}x^3+\frac{7^3\cdot 11}{5^5}x^4+$, &c.

5. $\frac{1+x}{(1-x)^3}=1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+$, &c.

6. $\sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{2\cdot 4}-\frac{3x^3}{2\cdot 4\cdot 6}-\frac{3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8}-$, &c.

7. $\sqrt{(1+x+x^2)}=1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}+$, &c.

8. $\sqrt{(1+x+x^2+x^3+\text{, \&c.})}=1+\frac{x}{2}+\frac{3}{2\cdot 4}x^2+\frac{3\cdot 5}{2\cdot 4\cdot 6}x^3+$.

9. $\frac{1+x}{x-x^2}=\frac{1}{x}+\frac{2}{1-x}$.

10. $\frac{8x-4}{x^2-4}=\frac{5}{x+2}+\frac{3}{x-2}$.

11. $\frac{x+1}{x^2-7x+12}=\frac{5}{x-4}-\frac{4}{x-3}$.

12. $\frac{x^2}{(x^2-1)(x-2)}=\frac{4}{3(x-2)}-\frac{1}{2(x-1)}+\frac{1}{6(x+1)}$.

13. $\frac{1}{x^4-a^4}=\frac{1}{4a^3(x-a)}-\frac{1}{4a^3(x+a)}-\frac{1}{2a^2(x^2+a^2)}$.

14. $\frac{1}{x^6-1}=\frac{1}{6}\left\{\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right\}$.

BINOMIAL THEOREM,

WHEN THE EXPONENT IS FRACTIONAL OR NEGATIVE.

ART. **319.** We shall now proceed to prove the truth of the Binomial Theorem generally; that is, to show that

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+, \text{\&c.}$$

whether n be integral or fractional, positive or negative.

First, $$a+b=a\left(1+\frac{b}{a}\right);$$

$$\therefore\ (a+b)^n=a^n\left(1+\frac{b}{a}\right)^n=a^n(1+x)^n, \text{ if } x=\frac{b}{a}.$$

Hence, if we can find the law of the expansion of $(1+x)^n$, we may obtain that of $(a+b)^n$, by writing $\frac{b}{a}$ for x, and multiplying by a^n. We shall, therefore, first prove that

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+, \text{\&c.}$$

The proof may be divided into two parts:

1st. To show that $(1+x)^n=1+nx+$, &c.

2nd. To find the general law of the coëfficients.

FIRST. To prove that the coëfficient of the second term of the expansion of $(1+x)^n$ is n, whether n be integral or fractional, positive or negative.

Let the index be positive and integral; then, since by multiplication we know that

$$(1+x)^2=1+2x+, \text{\&c.},$$
$$(1+x)^3=1+3x+, \text{\&c.};$$

let us assume that $(1+x)^{n-1}=1+(n-1)x+$, &c.

Multiply both sides of this equality by $1+x$, then

$$(1+x)^{n-1}(1+x)=\{1+(n-1)x+, \text{\&c.}\}(1+x);$$

or, $$(1+x)^n=1+nx+, \text{\&c., by multiplication.}$$

Hence, if the proposition is true for any one index $n-1$, it it will be true for the next higher index n. Now, by multiplication, it is true for the index 3, it is therefore true for the index $3+1=4$; and therefore true for the index $4+1=5$, and so on.

Hence, by continued induction, it is always true for n when it is integral and positive.

Next let n be a fraction $=\frac{p}{q}$.

Also, let $(1+x)^{\frac{p}{q}}=1+ax+$, &c., $=1+Ax$, where Ax is put to represent all the terms after the first.

Since $(1+x)^{\frac{p}{q}}=1+Ax$, $\therefore$ by raising both sides to the q power $(1+x)^p=(1+Ax)^q$;

$$\therefore 1+px+, \&c., =1+qAx+, \&c.,$$
$$=1+q(ax+, \&c.,)+, \&c.,$$
$$=1+qax+, \&c.$$

By equating the coëfficients of the like powers of x (Art. 314),

$$p=qa \therefore a=\frac{p}{q},$$

and $$(1+x)^{\frac{p}{q}}=1+\frac{p}{q}x+, \&c.$$

Lastly, let n be negative; then, (Art. 81),

$$(1+x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+, \&c.}=1-nx+, \&c., \text{ by division.}$$

$$\therefore (1+x)^n=1+nx+, \&c., \text{ whatever be the value of } n$$

$$\therefore (a+b)^n=a^n\left(1+\frac{b}{a}\right)^n=a^n(1+n\frac{b}{a}+, \&c.),$$
$$=a^n+na^{n-1}b+, \&c.,$$

and the first two terms of the series are determined.

SECOND. To find the general law of the coëfficients.

Let $(1+x)^n=1+nx+Bx^2+Cx^3+Dx^4+$, &c., where B, C, D, &c., depend upon n.

For x, put $x+z$, and consider $(x+z)$ as *one term*, then

$$\{1+(x+z)\}^n=1+n(x+z)+B(x+z)^2+C(x+z)^3+, \&c.$$

But $(a+b)^n=a^n+na^{n-1}b+$, &c.;

$$\therefore (x+z)^2=x^2+2xz+, \&c.;$$
$$(x+z)^3=x^3+3x^2z+, \&c.;$$
$$(x+z)^4=x^4+4x^3z+, \&c.;$$

$$\therefore \{1+(x+z)\}^n=1+nx+Bx^2+Cx^3+Dx^4+, \&c.,$$
$$+(n+2Bx+3Cx^2+4Dx^3+, \&c.)z+, \&c.,$$
$$=(1+x)^n+(n+2Bx+3Cx^2+4Dx^3+, \&c.)z+, \&c., (A).$$

But, considering $(1+x)$ as *one term*, $(1+x+z)^n=\{(1+x)+z\}^n$; and $\{(1+x)+z\}^n=(1+x)^n+n(1+x)^{n-1}z+$, &c. (B).

Equating the coëfficients of z in (A) and (B),

$$n+2Bx+3Cx^2+4Dx^3+, \text{\&c.}, =n(1+x)^{n-1};$$

multiplying both sides by $1+x$, we have

$$\left.\begin{array}{l} n+2Bx+3Cx^2+4Dx^3+, \text{\&c.} \\ +\quad nx+2Bx^2+3Cx^3+, \text{\&c.} \end{array}\right\} \begin{array}{l} =n(1+x)^n \\ =n(1+nx+Bx^2+Cx^3+, \text{\&c}). \end{array}$$

By equating the coëfficients of the same powers of x, we have

$$2B+n=n^2 \;\therefore\; 2B=n^2-n=n(n-1).$$

$$B=\frac{n(n-1)}{1\cdot 2};$$

$$3C+2B=Bn \;\therefore\; 3C=B(n-2),$$

$$C=\frac{B(n-2)}{3}=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3};$$

also, $$4D+3C=nC \;\therefore\; 4D=C(n-3);$$

$$D=\frac{C(n-3)}{4}=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}.$$

Similarly $$5E=D(n-4);$$

$$E=\frac{D(n-4)}{5}=\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}.$$

Hence, generally, if N is any coëfficient, M the one which next precedes it, and $r-1$ the largest factor in the denominator of M, we have $$N=\frac{M\{n-(r-1)\}}{r}=\frac{M(n+1-r)}{r};$$

$$\therefore\; (1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+, \text{\&c.},$$

and $\therefore$ putting $\frac{b}{a}$ for x, $(a+b)^n=a^n\left(1+\frac{b}{a}\right)^n$,

$$=a^n\left\{1+n\frac{b}{a}+\frac{n(n-1)}{1\cdot 2}\frac{b^2}{a^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{b^3}{a^3}+\text{\&c.},\right\},$$

$$=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+, \text{\&c.}$$

If $-b$ be put for b, then since the *odd* powers of $-b$ are *negative* (Art. 193) and the *even* powers positive,

$$(a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+, \text{\&c.},$$

which establishes the Binomial Theorem in its most general form.

Remark.— From the preceding formula and demonstrations, corollaries, similar to those in Art. 310, may be drawn, but it is not necessary to repeat them. The following additional proposition is sometimes useful.

Art. 320. To find the greatest term in the expansion of $(a+b)^n$.

From Cor. 3, Art. 310, we have seen that the r^{th} term is

$$\frac{n(n-1)\ldots\ldots\ldots(n-r+2)}{1\cdot 2\ldots\ldots\ldots(r-1)}a^{n-r+1}b^{r-1}, \text{hence,}$$

from the general law, the $(r+1)^{th}$ term is

$$\frac{n(n-1)\ldots\ldots(n-r+1)}{1\cdot 2\ldots\ldots r}a^{n-r}b^{r}.$$

Therefore, the $(r+1)^{th}$ term is derived from the r^{th} by multiplying the latter by $\frac{n-r+1}{r}\cdot\frac{b}{a}$.

While this multiplier is *greater* than 1, each term must be greater than the *preceding*. Hence, the r^{th} term will be the greatest when $\frac{n-r+1}{r}\cdot\frac{b}{a}$ *first* <1;

or, $(n-r+1)b<ar$, (Art. 221);

or, $r(a+b)>(n+1)b$, (Art. 219);

or, $r>(n+1)\frac{b}{a+b}$. (Art. 221).

Take r, therefore, the *first* whole number greater than $(n+1)\frac{b}{a+b}$, and the r^{th} term will be the *greatest* of the series.

If $(n+1)\frac{b}{a+b}$ is a whole number, then two terms are equal, each of which is greater than any of the other terms.

Ex. 1. Find the greatest term in the expansion of $(1+\frac{4}{3})^{\frac{10}{3}}$.

Here $(n+1)\frac{b}{a+b}=(\frac{13}{3})\frac{\frac{4}{3}}{1+\frac{4}{3}}=\frac{52}{21}$; $\therefore$ $r>2$; hence $r=3$.

2. Find the greatest term in the expansion of $(1+\frac{9}{10})^{\frac{8}{3}}$.

Ans. 2^{nd}.

3. Find the greatest term in the expansion of $(3+5x)^8$, when $x=\frac{1}{2}$. *Ans.* 5^{th}.

Cor. 1. By finding the greatest term of a series, we determine the point at which the series begins to *converge*; that is, the point from which the terms become less and less.

Cor. 2. It is also evident that when $\frac{n-r+1}{r}$ is *first* less than 1, that the *coëfficient* of the preceding term is the greatest.

But when $\frac{n-r+1}{r}>1$,

$$n-r+1>r, \text{ (Art. 221)},$$

or, $$2r>n+1, \text{ (Art. 219)};$$

or, $$r>\frac{n+1}{2}, \text{ (Art. 221)}.$$

Hence, the whole number next greater than $\frac{n+1}{2}$, or next less than $\frac{n+1}{2}+1=\frac{n+3}{2}$, denotes the term having the greatest coëfficient.

If n is an *odd* integer, there will be two coëfficients, the $\left(\frac{n+1}{2}\right)^{th}$, and the $\left(\frac{n+3}{2}\right)^{th}$, each greater than any other.

Ex. Find the term having the greatest coëfficient in the expansion of $(a+b)^{10}$; and the two terms having the greatest coëfficients in the expansion of $(x-y)^7$. *Ans.* 6^{th}, and 4^{th} and 5^{th}.

ART. **321.** In the application of the Binomial Theorem, it is merely necessary to take the general formula $(a+b)^n=a^n+na^{n-1}b+$, &c., and substitute the given quantities instead of the symbols to which they correspond in the formula, and then reduce each term to its most simple form.

Ex. 1. Find the expansion of $(1+x)^{\frac{1}{2}}$.

Here $a=1$, $b=x$, $n=\frac{1}{2}$.

$$\therefore (1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}x^3+, \text{ \&c.}$$

$$=1+\tfrac{1}{2}x-\frac{1\cdot 1}{2\cdot 4}x^2+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4+, \text{ \&c.}$$

Ex. 2. Develope $(1-x)^{-\frac{1}{2}}$. Here $a=1$, $b=-x$, $n=-\frac{1}{2}$.

$$\therefore\ (1-x)^{-\frac{1}{2}}=1-\tfrac{1}{2}(-x)+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1\cdot 2}(-x)^2$$

$$+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1\cdot 2\cdot 3}(-x)^3+, \text{ \&c.}$$

$$=1+\tfrac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+, \text{ \&c.}$$

In making these developments it will assist the pupil, to recollect that every root and every power of 1, is 1.

Ex. 3. Develope $\sqrt{a+b}$ into a series.

Since $a+b=a\left(1+\frac{b}{a}\right)$, $\therefore\ \sqrt{a+b}=\sqrt{a}\left(1+\frac{b}{a}\right)^{\frac{1}{2}}$.

Here $\qquad a=1,\ b=\frac{b}{a},\ n=\frac{1}{2}$.

$$\therefore\ \left(1+\frac{b}{a}\right)^{\frac{1}{2}}=1+\tfrac{1}{2}\frac{b}{a}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}.\frac{b^2}{a^2}+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}.\frac{b^3}{a^3}+, \text{ \&c.}$$

$$=1+\tfrac{1}{2}.\frac{b}{a}-\frac{1}{2\cdot 4}.\frac{b^2}{a^2}+\frac{1\cdot 3}{2\cdot 4\cdot 6}.\frac{b^3}{a^3}-, \text{ \&c.}$$

Hence, $\sqrt{a+b}=\sqrt{a}(1+\frac{b}{2a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}+, \text{ \&c.})$.

EXAMPLES FOR PRACTICE.

1. $\frac{1}{1-x}=(1-x)^{-1}=1+x+x^2+x^3+x^4+$, &c.

2. $\frac{1}{(1-x)^2}=(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+$, &c.

3. $\frac{a^2}{(a+x)^2}=a^2(a+x)^{-2}=1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}-$, &c.

4. $\sqrt[3]{1-x^3}=1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}-$, &c.

5. $\sqrt{a^2+x}=a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\frac{5x^4}{128a^7}+$, &c.

6. $(a^3-x)^{\frac{1}{3}}=a-\frac{x}{3a^2}-\frac{x^2}{9a^5}-\frac{5x^3}{81a^8}-\frac{10x^4}{243a^{11}}-$, &c.

7. $(1+2x)^{\frac{1}{2}}=1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+$, &c.

8. $\sqrt{a^2-x^2}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-$, &c.

9. $\sqrt[3]{a+x}=\sqrt[3]{a}(1+\frac{1x}{3a}-\frac{1x^2}{9a^2}+\frac{5x^3}{81a^3}-\frac{10x^4}{243a^4}+$, &c.).

10. $(a^3+x^3)^{\frac{1}{3}}=a(1+\frac{x^3}{3a^3}-\frac{2x^6}{3\cdot 6a^6}+\frac{2\cdot 5x^9}{3\cdot 6\cdot 9a^9}-$, &c.).

11. $\sqrt[3]{9}=\sqrt[3]{8+1}=2+\frac{2}{3}\cdot\frac{1}{8}-\frac{2}{9}\cdot\frac{1}{8^2}+\frac{10}{81}\cdot\frac{1}{8^3}-$, &c.

12. $(a^3-x^3)^{\frac{1}{3}}=a(1-\frac{x^3}{3a^3}+\frac{2x^6}{3\cdot 6a^6}-\frac{2\cdot 5x^9}{3\cdot 6\cdot 9a^9}+$, &c.).

13. $\frac{a^3}{(a^3-x^3)^{\frac{2}{3}}}=a+\frac{2x^3}{3a^2}+\frac{2\cdot 5x^6}{3\cdot 6a^5}+\frac{2\cdot 5\cdot 8x^9}{3\cdot 6\cdot 9a^8}+$, &c.

ART. **322.** To find the approximate roots of numbers by the Binomial Theorem.

Let N represent any proposed number whose n^{th} root is required, take a such that a^n is the nearest perfect n^{th} power to N, so that $N=a^n\pm b$, b being small compared with a, and $+$ or $-$, according as $N>$ or $<a^n$;

then $\sqrt[n]{N}=a\left(1\pm\frac{b}{a^n}\right)^{\frac{1}{n}}=$, by writing $\frac{b}{a^n}$ for b in the general formula;

$$a\{1\pm\frac{1}{n}\cdot\frac{b}{a^n}-\frac{1}{n}\cdot\frac{n-1}{2n}\left(\frac{b}{a^n}\right)^2\pm\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\left(\frac{b}{a^n}\right)^3-,\&c.\}.$$

Of this series a few terms only, when b is small with regard to a^n, will give the required root to a considerable degree of accuracy.

Ex. Required the approximate cube root of 128.

Here $\sqrt[3]{128}=\sqrt[3]{5^3+3}=5\sqrt[3]{1+\frac{3}{125}}$;

$$=5\{1+\frac{1}{3}\cdot\frac{3}{125}-\frac{1}{3}\cdot\frac{1}{3}\left(\frac{3}{125}\right)^2+\frac{1}{3^2}\cdot\frac{5}{9}\left(\frac{3}{125}\right)^3-,\ldots\};$$

$$=5+\frac{1}{5^2}-\frac{1}{5^5}+\frac{1}{3}\cdot\frac{1}{5^7}-\ldots=5+\frac{2^2}{10^2}-\frac{2^5}{10^5}+\frac{1}{3}\cdot\frac{2^7}{10^7}-\ldots$$

$$=5+0.04-0.00032+0.0000042-\ldots$$

$$=5.0396842.$$

ART. **323.** In the preceding example, since the series continues to infinity, we obtain only an approximate value for the required root, and as the denominators increase more rapidly than the numerators, a few terms only need be taken for practical purposes; still it may be required to find what is the *limit* in the error occasioned by neglecting the remaining terms of the series. To do this let R be the true root, and as the terms are alternately positive and negative, let

$$R = a - b + c - d + e - f + g - h + k - l +, \text{ \&c., and let}$$
$$R' = a - b + c - d + e - f,$$
$$R'' = a - b + c - d + e - f + g.$$

Then since the terms continually decrease, $a-b$, $c-d$, $e-f$, $g-h$, &c., are all positive, and therefore R′, which contains three only of those differences, will be *less* than R. For the same reason all the pairs of terms after g, as $-h+k$, $-l+m$, &c., will be all negative, and R″ will be *greater* than R; therefore, the true value of the series lies between R′ and R″, or

$$a-b+c-d+e-f,$$
and
$$a-b+c-d+e-f+g.$$

Hence, *the error committed by the omission of any number of the terms of a converging series, is less than the first term of the omitted part of the series.*

Thus, in the preceding example, if we had stopped at the second term, the error would have been less than .0000042.

15. Find the 5^{th} root of 35. *Ans.* 2.036172+.

Here $$N = 35 = 32 + 3 = 2^5\left(1 + \frac{3}{2^5}\right).$$

16. The student may solve the following examples:

(1). $\sqrt{10} = \sqrt{9+1} = 3.16227$ . . . true to 0.00001.

(2). $\sqrt[3]{30} = \sqrt[3]{27+3} = 3.10723$ . . . true to 0.00001.

(3). $\sqrt[3]{24} = \sqrt[3]{27-3} = 2.88449$ . . . true to 0.00001.

(4). $\sqrt[4]{260} = \sqrt[4]{256+4} = 4.01553$ . . . true to 0.00001.

(5). $\sqrt[7]{108} = \sqrt[7]{128-20} = 1.95204$ . . . true to 0.00001.

REMARK.— Instead of extracting the nth root by the formula in Art. 322, the operation may be performed by the general formula of the preceding article, the number whose root is to be extracted being divided into any two parts whatever. The advantage of the formula in Art. 322 consists in the rapid convergence of its terms. Thus in finding the 4th root of 260 true to five places of decimals, it is only necessary to take two terms of the series.

THE DIFFERENTIAL METHOD OF SERIES.

ART. **324.** A *Series* consists of a number of terms, each of which is derived from one or more of the preceding terms, according to some determinate *law.* (Art. 134.)

The use of the differential method is, 1st, to find the successive differences of the terms of a series; 2nd, any particular term of the series; or, 3rd, the sum of a finite number of its terms.

If, in any series, we take the first term from the second, the second from the third, the third from the fourth, and so on, the new series thus formed is called the *First order of differences.*

If we proceed with this new series in the same manner, we shall obtain another series termed the *Second order of differences.*

In a similar manner we find the *third, fourth,* &c., orders of differences.

Thus, if we have the series

	1 , 8 , 27 , 64 , 125 , 216, . .
the 1st order of differences is	7 , 19 , 37 , 61 , 91 , . . .
" 2nd " " " "	12 , 18 , 24 , 30 , ...
" 3rd " " " "	, 6 , 6 , 6 , ...

ART. **325.** PROBLEM 1.—*To find the first term of any order of differences.*

Let the series be $a, b, c, d, e, \ldots\ldots.$

then the respective orders of differences are,

1st order, $b-a$, $c-b$, $d-c$, $e-d, \ldots.$

2nd order, $c-2b+a$, $d-2c+b$, $e-2d+c, \ldots.$

3rd order, $d-3c+3b-a$, $e-3d+3c-b, \ldots.$

4th order, $e-4d+6c-4b+a.$

Here each difference pointed off by commas, though a compound quantity, is called a *term.* Thus the first term in the 1st order is $b-a$; in the second order $c-2b+a$, &c.

If we denote the first terms in the 1st, 2nd, 3rd, 4th, &c., orders of differences by D_1, D_2, D_3, D_4, &c., and invert the order of the letters so that a shall stand first, we have

$$D_1=-a+b;$$
$$D_2=a-2b+c;$$
$$D_3=-a+3b-3c+d;$$
$$D_4=a-4b+6c-4d+e;$$
$$\&c., = \ldots.\ \&c.$$

Here the coëfficients of a, b, c, d, &c., in the n^{th} order of differences are evidently the coëfficients of the terms of a binomial raised to the n^{th} power; and their signs are alternately positive and negative; hence, when n is *even*, the first term of the n^{th} order of differences is

$$a-nb+\frac{n(n-1)}{1\cdot 2}c-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d+, \text{\&c., and}$$

$$-a+nb-\frac{n(n-1)}{1\cdot 2}c+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d-, \text{\&c., when } n \text{ is odd.}$$

Cor. It is evident from the coëfficients that when $n=1$, the value of D_n has only *two* terms, for then $n-1=0$; when $n=2$, this value has only *three* terms, for then $n-2=0$, and so on.

Ex. 1. Find the first term of the fourth order of differences of the series $1^3, 2^3, 3^3, 4^3, 5^3, \ldots$ or $1, 8, 27, 64$, &c., ...

Here $n=4$, hence take five terms of the first value of D_n, and $a=1$, $b=8$, $c=27$, $d=64$, $e=125$, and $D_4=$

$$1-4\times 8+\frac{4\times 3}{1\times 2}\times 27-\frac{4\times 3\times 2}{1\times 2\times 3}\times 64+\frac{4\times 3\times 2\times 1}{1\times 2\times 3\times 4}\times 125=$$

$$1-32+162-256+125=0, \textit{ Ans.}$$

Remark.—It is evident the first term of any particular order of differences may be found by continued subtraction. It is important, however, that the learner should be acquainted with the general law as expressed in the above series.

EXAMPLES FOR PRACTICE.

2. Find the first term of the second order of differences of the series $1^2, 2^2, 3^2, 4^2, \ldots$ or $1, 4, 9, 16, 25 \ldots$ *Ans.* 2.

3. What is the first term of the third order of differences of the series 1, 3, 6, 10, 15, &c.? *Ans.* 0.

4. Required the first term of the fifth order of differences of the series $1, 3, 3^2, 3^3, 3^4$, &c. *Ans.* 32.

5. Find the first term of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c. *Ans.* $-\frac{1}{32}$.

Art. **326**. Problem II.—*To find the n^{th} term of the series a, b, c, d, e, &c.*

From the preceding article we have seen that

$D_1=-a+b$; whence $b=a+D_1$;
$D_2=a-2b+c$; " $c=a+2D_1+D_2$;
$D_3=-a+3b-3c+d$; " $d=a+3D_1+3D_2+D_3$;
$D_4=a-4b+6c-4d+e$; " $e=a+4D_1+6D_2+4D_3+D_4$.
. . . .

It is evident from inspection that the *coëfficients* of the first terms of the different orders of differences, in the value of any term of the series, as of e the fifth term, are the *coëfficients* of the terms of a binomial involved to a power whose exponent is *one less* than the number denoting the place of the term; that is, the coëfficients of the n^{th} term of the series, are the coëfficients of the $(n-1)$ power of a binomial. Hence, writing $n-1$ instead of n, in the coëfficients of the n^{th} power of $a+b$, (Art. 319), the n^{th} term of the series is

$$a+(n-1)D_1+\frac{(n-1)(n-2)}{1\cdot2}D_2+\frac{(n-1)(n-2)(n-3)}{1\cdot2\cdot3}D_3+, \text{\&c.}$$

Ex. 1. Find the 12^{th} term of the series 1, 3, 6, 10, 15, 21, . .

1 , 3 , 6 , 10 , 15, ...
2 , 3 , 4 , 5, ... hence $D_1=2$;
1 , 1 , 1, ... " $D_2=1$;
0 , 0, ... " $D_3=0$;

and the succeeding orders of differences are also evidently 0; hence 12^{th} term

$$=a+(n-1)D_1+\frac{(n-1)(n-2)}{1\cdot2}D_2=1+11\times2+\frac{11\times10}{2}\times1$$
$$=1+22+55=78. \quad \textit{Ans.}$$

2. Find the n^{th} term of the series 2, 6, 12, 20, 30, ...

2 , 6 , 12 , 20 , 30, ...
4 , 6 , 8 10, ... hence $D_1=4$;
2 , 2 , 2 . . . . " $D_2=2$;
0 , 0, ... " $D_3=0$;

hence n^{th} term $=2+(n-1)4+\frac{(n-1)(n-2)}{1\cdot2}\times2=n^2+n$. *Ans.*

From the formula n^2+n, or $n(n+1)$, any term of the series is readily found; thus the 20^{th} term $=20(20+1)=420$.

It is also evident that the n^{th} term of a series can be found exactly only when some order of differences is zero.

EXAMPLES FOR PRACTICE.

3. Find tne 15^{th} term, and the n^{th} term of the series $1, 2^2, 3^2, 4^2$, . . . or, 1, 4, 9, 16, ... *Ans.* 225, and n^2.

4. Find the 12^{th} term of the series 1, 5, 15, 35, 70, 126, &c *Ans.* 1365.

5. Find the n^{th} term of the series 1, 3, 6, 10, &c. *Ans.* $\frac{n(n+1)}{2}$.

6. Find the n^{th} term of the series 1, 4, 10, 20, 35, 56, &c. *Ans.* $\frac{n(n+1)(n+2)}{2\times3}$.

7. Find the 9^{th} term of the series $2\cdot5\cdot7$, $4\cdot7\cdot9$, $6\cdot9\cdot11$, $8\cdot11\cdot13$, &c. *Ans.* 8694.

8. What is the n^{th} term of the series 1×2, 3×4, 5×6, &c.? *Ans.* $4n^2-2n$.

ART. **327.** PROBLEM III.—*To find the sum of n terms of the series a, b, c, d, e, &c.*

Assume the series $0, a, a+b, a+b+c, a+b+c+d, \ldots$

Subtracting each term from the next succeeding, we have

$$a, b, c, d, e, \&c.,$$

which is the series whose sum it is proposed to find. Hence, the sum of n terms of the proposed series, which it is now required to find, is the $(n+1)^{th}$ term of the assumed series.

It is evident the n^{th} order of differences in the given series, is equal to the $(n+1)^{th}$ order in the assumed series. Hence, if we compare the quantities in the assumed series, with those of the formula for finding the n^{th} term of a series (Art. 326), we have

$$0 \text{ for } a,$$
$$n+1 \text{ for } n,$$
$$a \text{ for } D_1,$$
$$D_1 \text{ for } D_2, \&c.$$

Substituting these values in the formula, we have $0+(n+1-1)a$

$$+\frac{(n+1-1)(n+1-2)}{1\cdot2}D_1+\frac{(n+1-1)(n+1-2)(n+1-3)}{1\cdot2\cdot3}D_2+,$$

or, $na+\frac{n(n-1)}{1\cdot 2}D_1+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}D_2+$, &c., which is the sum of n terms of the proposed series.

Ex. 1. Find the sum of n terms of the odd numbers 1, 3, 5, 7, 9. . . . .

Here $a=1$, $D_1=2$, $D_2=0$; hence,

$$\text{Sum }=na+\frac{n(n-1)}{1\cdot 2}D_1=n\times1+\frac{n(n-1)}{2}\times2=n+n^2-n=n^2.$$

2. Find the sum of n terms of the series $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$.

Here $a=1$, $D_1=3$, $D_2=2$, $D_3=0$; hence,

$$\text{Sum }=na+\frac{n(n-1)}{1\cdot 2}D_1+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}D_2=n+\frac{3n(n-1)}{2}$$

$$+\frac{n(n-1)(n-2)}{3}=\frac{n(n+1)(2n+1)}{6}.$$

EXAMPLES FOR PRACTICE.

3. Find the sum of n terms of the series $1+3+6+10+15$, &c. *Ans.* $\frac{n(n+1)(n+2)}{6}$.

4. Find the sum of 20 terms of the series $3+11+31+69+131$, &c. *Ans.* 44330.

5. Find the sum of 20 terms of the series

$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+$, &c. *Ans.* 53130.

6. Find the sum of n terms of the series of cube numbers $1^3+2^3+3^3+$, &c. *Ans.* $[\frac{1}{2}n(n+1)]^2$.

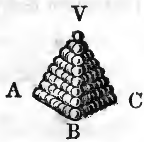
7. Find the sum of n terms of the series $1+4+10+20+35 \ldots$. *Ans.* $\frac{n(n+1)(n+2)(n+3)}{1\times2\times3\times4}$.

8. Find the sum of 25 terms of the series whose n^{th} term is $n^2(3n-2)$. *Ans.* 305825.

ART. **328.** PILING OF CANNON BALLS AND SHELLS.

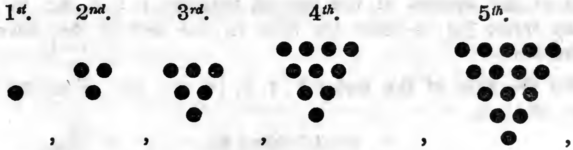
Balls and shells are usually piled by horizontal courses, either in the form of a pyramid or a wedge; the base being either an equilateral triangle, or a square, or a rectangle. In the triangle and square, the pile terminates in a single ball, but in the rectangle it finishes in a ridge, or single row of balls.

Art. 329. *To find the number of balls in a triangular pile.*



A triangular pile, as V—ABC, is formed of successive horizontal courses of the form of an equilateral triangle, such that the number of balls in the sides of these courses, decreases continually by unity, from the bottom to the single ball at the top.

If we commence at the top, the number of balls in the respective courses will be as follows:



and so on. Hence, the number of balls in the respective courses is 1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, and so on; or 1, 3, 6, 10, 15

Hence, to find the number of balls in a triangular pile, is to find the sum of the series 1, 3, 6, 10, 15, &c., to as many terms (n) as there are balls in one side of the lowest course.

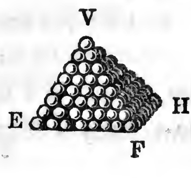
By applying the formula (Art. 327) to finding the sum of n terms of the series 1, 3, 6, 10, &c., we have $a=1$, $D_1=2$, $D_2=1$, and $D_3=0$.

Hence, the formula $na+\frac{n(n-1)}{1\cdot 2}D_1+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}D_2$ becomes

$$n+\frac{n(n-1)}{2}\times 2+\frac{n(n-1)(n-2)}{2\times 3}\times 1=n+n^2-n+\frac{n^3-3n^2+2n}{6}$$

$$=\frac{n^3+3n^2+2n}{6}=\frac{n(n^2+3n+2)}{6}=\frac{n(n+1)(n+2)}{6}. \quad \text{(A)}$$

Art. 330. *To find the number of balls in a square pile.*



A square pile, as V—EFH, is formed of successive square horizontal courses, such that the number of balls in the sides of these courses, decreases continually by unity, from the bottom to the single ball at the top.

If we commence at the top, the number of balls in the respective courses will be as follows:

1st. 2nd. 3rd. 4th.

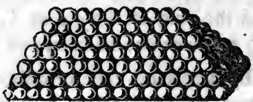
and so on. Hence, the number of balls in the respective courses is $1^2, 2^2, 3^2, 4^2, 5^2$, &c., or 1, 4, 9, 16, 25, and so on. Therefore, to find the number of balls in a square pile, is to find the sum of the squares of the natural numbers 1, 2, 3, &c., to as many terms (n) as there are balls in one side of the lowest course.

But the sum of the series 1, 4, 9, 16, &c. (see example 2, page 288), is

$$\frac{n(n+1)(2n+1)}{6}. \qquad \text{(B)}$$

Art. **331.** *To find the number of balls in a rectangular pile.*

A rectangular pile, as EFDBCA, is formed of successive rectangular courses, such that the number of balls in each of the sides of these courses, decreases continuously by unity, from the bottom to the single row of balls at the top.



If we commence at the top, the number of balls in the *breadth* of the first row is 1, of the second 2, of the third 3, and so on. Also, if $m+1$ denotes the number of balls in the top row, the number in the *length* of the second row will be $m+2$, in the third row $m+3$, and so on. Hence, the number of balls in the respective courses, commencing with the top, will be $1(m+1)$, $2(m+2)$, $3(m+3)$, and in the n^{th} course $n(m+n)$. Therefore, the number of balls (S) in a complete rectangular pile of n courses will be

$$S=1(m+1)+2(m+2)+3(m+3)+\ldots\ldots\ldots+n(m+n)$$
$$=m(1+2+3+4\ldots+n)+(1^2+2^2+3^2+4^2+\ldots+n^2);$$

but the sum of n terms of the series in the first parenthesis, (Art. 327,) is $\frac{n(n+1)}{2}$, and the sum of n terms of the series in the second parenthesis has just been found (Art. 330) to be $\frac{n(n+1)(2n+1)}{6}$; hence, by substitution, we have

$$S=\frac{mn(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)}{6}(3m+2n+1) \quad (C).$$

Here $m+n$ represents the number of balls in the length of the lowest course. If we put $m+n=l$, we have $3m+2n=3l-n$; substituting this for $3m+2n$, in the preceding formula, it becomes

$$S=\frac{n(n+1)}{6}(3l-n+1).$$

It is evident that the number of courses in a triangular or square pile, is equal to the number of balls in one side of the base course, and in the rectangular pile to the number of balls in the *breadth* of the base course.

Art. **332.** Collecting together the results of the three preceding articles, we have for the number of balls

in a Triangular pile $\frac{1}{6}n(n+1)(n+2)$ . . . . (A);

in a Square pile $\frac{1}{6}n(n+1)(2n+1)$ . . . . (B);

in a Rectangular pile $\frac{1}{6}n(n+1)(3l-n+1)$ . . . . . . (C)

In formulæ (A) and (B), n denotes the number of courses, or the number of balls in the base course. In formula (C) n denotes the number of balls in the breadth of the base course, and l the number in the length.

The number of balls in an *incomplete* pile is evidently found by subtracting the number in the pile which is wanting at the top, from the whole pile considered as complete.

EXAMPLES FOR PRACTICE.

1. Find the number of balls in a triangular pile of 15 courses.

Here $n=15$, and substituting this value instead of n in formula A, (Art. 332), we have the number

$$=\frac{15(15+1)(15+2)}{2\times3}=\frac{15\times16\times17}{6}=680. \quad \textit{Ans.}$$

2. Find the number of balls in an incomplete triangular pile of 15 courses, having 21 balls in the upper course.

Here we must first find the number of shot in one side of the upper course. From the illustrations in Art. 329, it is evident that the number of balls in any triangular course, is equal to the sum of the natural numbers 1, 2, 3, &c., to the number (n) in one side. Now the sum of the numbers 1, 2, 3, &c., to n, is (Art. 327) $\frac{n(n+1)}{2}$; hence, $\frac{n(n+1)}{2}=21$, or $n^2+n=42$, from which (Art. 231) we find $n=6$, and therefore 5 courses have been removed from the pile; hence, by formula A, (Art. 332), the number of balls in the pile considered as complete, is

$\frac{20\times21\times22}{2\times3}=1540$, and the number in the pile removed is

$\frac{5\times6\times7}{2\times3}=35$ $\therefore$ the number in the incomplete pile is $1540-35=1505$.

3. Find the number of balls in a square pile of 15 courses. *Ans.* 1240.

4. Find the number of balls in a rectangular pile, the length and breadth of the base containing 52 and 34 balls respectively. *Ans.* 24395.

5. Find the number of balls in an incomplete triangular pile, a side of the base course having 25 balls, and a side of the top 13. *Ans.* 2561.

6. Find the number of balls in an incomplete triangular pile of 15 courses, having 38 balls in a side of the base. *Ans.* 7580.

7. Find the number of balls in an incomplete square pile, a side of the base course having 44 balls, and a side of the top 22. *Ans.* 26059.

8. The number of balls in the base and top courses of a square pile are 1521 and 169 respectively; how many are in the incomplete pile. *Ans.* 19890.

9. The number of balls in a complete rectangular pile of 20 courses is 6440; how many balls are in its base? *Ans.* 740.

10. The number of balls in a triangular pile is to the number in a square pile having the same number of balls in the side of the base, as 6 to 11; required the number in each pile. *Ans.* 816, and 1496.

11. How many balls are in an incomplete rectangular pile of

8 courses, having 36 balls in the longer side, and 17 in the shorter side of the upper course. *Ans.* 6520.

ART. **333.** INTERPOLATION OF SERIES.

Each of the various tables employed in the different departments of science, may be regarded as the terms of a mathematical series. These tables are generally calculated from particular formulæ, but in many cases the computations are so very laborious, that only certain terms at regular intervals, are calculated, and the intermediate ones are derived from these by a process termed *Interpolation.* Also, in many investigations values of the quantities in the tables are required, intermediate between those given, or extending beyond them. These, likewise, are determined by Interpolation.

The principle on which Interpolation is founded is that explained in Art. 326; that is, having certain terms of a series given, to find the n^{th} term. To do this with entire accuracy, requires that we should have such a number of terms of the series given, that we can obtain an order of differences equal to zero. In most cases, however, the differences, D_1, D_2, D_3, &c., do not vanish, but become so small that their omission after D_2, or D_3, causes no sensible error in the result, and we obtain what is termed, approximate values of the required quantities.

ART. **334.** When the 3rd order of differences of any given series of quantities vanishes, or becomes very small, then (Art. 326) we have the equation $-a+3b-3c+d=0$, and any of the quantities a, b, c, or d, may be found, when the other three are given. Similarly, if the fourth differences vanish, then

$$a-4b+6c-4d+e=0.$$

Ex. Given $\sqrt[3]{25}=2.92401$, $\sqrt[3]{26}=2.96249$, $\sqrt[3]{27}=3$,

$\sqrt[3]{29}=3.07231$, to find the cube root of 28.

Here four quantities are given to find a fifth, therefore, supposing the fourth order of differences to vanish, we have

$a-4b+6c-4d+e=0$, where d is the term to be interpolated; hence,

$$4d=a+6c+e-4b=2.92401+18+3.07231-11.84996$$
$$=12.14636,$$

where d, or $\sqrt[3]{28}=3.03659$, which is true to .00001.

Art. 335. When the terms are equidistant, and it is required to interpolate a term intermediate to any two of them, we may put p to represent the distance of the required term (t) from a, the first term of the series, in which case $p=n-1$ in the formula Art. 326, and the required term is

$$t=a+pD_1+\frac{p(p-1)}{1\cdot 2}D_2+\frac{p(p-1)(p-2)}{1\cdot 2\cdot 3}D_3+, \&c.$$

The interval between the given numbers is always to be considered as *unity*, and p is to be reckoned in parts of this interval; hence, p will be fractional.

Interpolation is of extensive application in Astronomy; and in most instances sufficient accuracy is obtained by making use of first and second differences only. The correction to be applied to the first term then is

$$pD_1+\frac{p(p-1)}{2}D_2=p(D_1+\frac{p-1}{2}D_2).$$

In practice, however, the method generally adopted is, to take the two terms of the series which precede, and the two terms which follow the term required, and find from them the three first differences, and the two second differences. Then, taking the *second* of the three first differences and calling it d, and the *mean* of the two second differences and calling it d', and denoting the fractional part of the interval by t, the correction to be applied to this second term is

$$t(d+\frac{t-1}{2}d').$$

Ex. Having given the logarimths of 102, 103, 104, and 105 let it be required to find the logarithm of 103.55.

Nos.	Logarithms.	1st Diff.	2nd Diff.	Mean of 2nd Diff.
102	2.0086002			
103	2.0128372	42370		
		41961	—409	—405
104	2.0170333	41560	—401	
105	2.0211893			

Here $t=.55$, $d=41961$, $d'=-405$, and

$$t(d+\frac{t-1}{2}d')=.55(41961+\frac{.45}{2}\times 405), \quad = \quad 23129$$

$$\text{log. } 103=2.0128372$$
$$\text{log. } 103.55=2.0151501$$

EXAMPLES FOR PRACTICE.

1. Find the 2^{nd} term of the series of which the 4^{th} differences vanish, the 1^{st}, 3^{rd}, 4^{th}, and 5^{th} terms being 3, 15, 30, 55; and find the 6^{th}, 7^{th} and 8^{th} terms. *Ans.* 7 ; and 93, 147, and 220.

2. Find the 5^{th} term of the series of which the 6^{th} differences vanish, and the 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 6^{th}, and 7^{th} terms are 11, 18, 30, 50, 132, 209. *Ans.* 82.

3. Given the logarithms of 101, 102, 104, and 105; viz.: 2.0043214, 2.0086002, 2.0170333, and 2.0211893, to find the logarithm of 103. *Ans.* 2.0128372.

4. Given the cube roots of 60, 62, 64, and 66; viz.: 3.91487, 3.95789, 4, and 4.04124, to find the cube root of 63. *Ans.* 3.97905.

5. Having given the squares of any two consecutive whole numbers, show how the squares of the succeeding whole numbers may be obtained by addition.

INFINITE SERIES.

ART. **336.** An *infinite series* is a series consisting of an unlimited number of terms, each of which is derived from the preceding term or terms, according to some law. For examples see Art. 134, and page 253.

The *sum* of an infinite series, is the *limit* to which we approach more nearly by adding together more terms, but which cannot be exceeded by adding together any number of terms whatever.

A *convergent* series is one which has a *sum* or *limit*. Thus,

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{64}+, \text{ \&c.},$$

is a convergent series, whose limit is 2, since the sum of any number of terms whatever cannot exceed 2, but will approach it more nearly as the number of terms taken is greater.

A *divergent* series is one which has no *sum* or *limit*, as

$$1+2+4+8+16+32+, \text{ \&c.}$$

An *ascending* series is one in which the powers of the leading quantity continually increase; and a *descending* series is one in which the powers of the leading quantity continually diminish.

Thus, $a+bx+cx^2+dx^3+$, is an ascending series, and

$a+bx^{-1}+cx^{-2}+dx^{-3}+$,

or $a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}+$, is a descending series.

ART. **337.** There are *four* general methods of converting an algebraic expression into an infinite series of equivalent value, each of which has been already exemplified; viz.:

1st. By Division. See Art. 134.

2nd. By Extraction of Roots. See examples 17, 18, page 136.

3rd. By Indeterminate Coëfficients. See Art. 314, and examples, page 275.

4th. By the Binomial Theorem. See Art. 319, and examples, pages 281, 282.

ART. **338.** The summation of a series is the finding a finite expression equivalent to the series.

The *general term* of a series is an expression from which the several terms of the series may be derived according to some determinate law. Thus, in the series $\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}+$ . . . . . the general term is $\frac{a}{x}$, because by making $x=1, 2, 3$, &c., each term of the series is found.

Again, in the series $2\cdot2+2\cdot3+2\cdot4+2\cdot5+$ . . . . the general term is $2(x+1)$.

As different series are in general governed by different laws, the methods of finding the sum, which are applicable to one class, will not apply universally.

We shall now explain two of the methods of most general application.

FIRST METHOD.—If the series is a regular decreasing geometrical series, whose first term is a, and ratio r, its sum is $\frac{a}{1-r}$ (Art. 299.)

SECOND METHOD.—By subtraction. To find the sum of a series whose general term is $\frac{q}{n(n+p)}$.

Since $\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}$ $\therefore$ $\frac{q}{n(n+p)}=\frac{1}{p}\left\{\frac{q}{n}-\frac{q}{n+p}\right\}$;

or, any fraction of the form $\frac{q}{n(n+p)}$ is equal to $\frac{1}{p}^{th}$, the difference between the two fractions $\frac{q}{n}$ and $\frac{q}{n+p}$; that is, any term of the series whose general term is $\frac{q}{n(n+p)}$ is equal to the difference between the corresponding terms of the two series whose general terms are $\frac{q}{n}$ and $\frac{q}{n+p}$; hence, the sum of the former series is equal to the difference between the sums of the two latter. Therefore, if the sums of the two latter were known, by taking their difference the sum of the former series would be found. The sums of these two series, however, are not known, but their difference can be found, when, after a certain number of terms of the series $\frac{q}{n}$, the succeeding terms are identical with those of $\frac{q}{n+p}$. In general, this certain number is after p terms of the $n+p$ former series.

Ex. 1. Required the sum of the series $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+$, &c., ad infinitum, that is, *to infinity*.

Here $q=1$, $p=2$, and $n=1, 2, 3$, &c.; and the two series are

$$\left.\begin{matrix}1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+, \text{\&c., ad inf.} \\ -(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+, \text{\&c., ad inf.})\end{matrix}\right\} =1, \therefore \frac{1}{p}=\frac{1}{2}=\text{ sum.}$$

The sum of n terms of the same series is found in a manner nearly similar. Thus,

$$\left\{\begin{matrix}1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\cdots\frac{1}{2n-1} \\ -(\frac{1}{3}+\frac{1}{5}+\frac{1}{7})\cdots\frac{1}{2n-1}+\frac{1}{2n+1}\end{matrix}\right\}=1-\frac{1}{2n+1}$$

$$=\frac{2n}{2n+1}, \text{ and } \frac{1}{p} \text{ of this sum is } \frac{n}{2n+1}=\text{ sum.}$$

2. Find the sum of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+$&c., ad inf.

Here $q=1$, $p=1$, and $n=1, 2, 3$, &c. *Ans.* 1.

3. Find the sum of the above series to n terms.

Ans. $\frac{n}{n+1}$.

4. Find the sum of the series $\frac{1}{1\cdot4}+\frac{1}{2\cdot5}+\frac{1}{3\cdot6}+\frac{1}{4\cdot7}+$,&c. ad infinitum.

Here $q=1$, and $p=3$. *Ans.* $\frac{11}{18}$.

5. Find the sum of the series $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+$, &c., ad infinitum.

Here $q=1$, $p=2$, and $n=1, 2, 3$, &c. *Ans.* $\frac{3}{4}$.

6. Find the series whose general term is $\frac{1}{n(n+4)}$; also find its sum continued to infinity.

Ans. Series $=\frac{1}{1\cdot5}+\frac{1}{2\cdot6}+\frac{1}{3\cdot7}+\frac{1}{4\cdot8}+$, &c., sum $=\frac{25}{48}$.

The sums of series may often be found by reducing them, by multiplication or division, to the forms of other series whose sums are known.

7. Find the sum of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$, &c., ad infinitum. *Ans.* 2.

Suggestion.—By dividing by 2 this series becomes the same as that in example 2nd.

8. Find the sum of the series $\frac{1}{3\cdot8}+\frac{1}{6\cdot12}+\frac{1}{9\cdot16}+$, &c., ad infinitum. (Multiply by $3\cdot4$). *Ans.* $\frac{1}{12}$.

Remark.—The preceding examples afford an illustration of the manner in which the sums of certain classes of infinite series may be found. The sums of a great variety of series may be found by other and more complicated methods. But the subject is more curious than useful, and is too complex and extensive for an elementary work.

RECURRING SERIES.

Art. **339.** A *Recurring Series* is a series so constituted that every term is connected with one or more of the terms which precede it by an invariable law, usually dependent on the operations of addition, subtraction, &c. Thus, in the series

$$1+2x+3x^2+5x^3+8x^4+13x^5+21x^6+, \text{\&c.},$$

the sum of the coëfficients of any two consecutive terms is equa to the coëfficient of the next following term. If the series be expressed by

$$A+B+C+D+E+F+G+H+, \text{\&c., then}$$

the 1st term	$A=1$;
the 2nd "	$B=2x$;
the 3rd "	$C=3x^2=Bx+Ax^2$;
the 4th "	$D=5x^3=Cx+Bx^2$;
the 5th "	$E=8x^4=Dx+Cx^2$;
the 6th "	$F=13x^5=Ex+Dx^2$, &c.

That is, each term after the second is equal to the one next preceding, multiplied by x, plus the second next preceding, multiplied by x^2; hence, all the terms after the first two *recur* according to a definite law.

ART. **340.** The particular expression by means of which any term of the series may be found when the preceding terms are known, is called the *scale of the series*, and that by means of which the coëfficients may be found, the *scale of the coëfficients*. Recurring series are said to be of the *first order*, *second order*, &c., according to the *number* of terms contained in the scale. Thus in the expansion of $\frac{a}{a+bx}$, (Art. 315), we find

$$\frac{a}{a+bx}=1-\frac{b}{a}x+\frac{b^2}{a^2}x^2-\frac{b^3}{a^3}x^3+\frac{b^4}{a^4}x^4-\frac{b^5}{a^5}x^5+, \text{\&c.},$$

where each term after the first is equal to the preceding, multiplied by $-\frac{b}{a}x$. In this case $-\frac{b}{a}x$ is termed the scale of the series, $-\frac{b}{a}$ the scale of the coëfficients, and the series is said to be of the first order. This is the most simple form of a recurring series.

ART. **341.** *To find the scale of a series.*

When the series is of the first order, the scale is easily determined, being the ratio of any two consecutive terms.

When the series is of the *second order*, the law of the series depends on two terms, and the scale consists of two parts. Let $p+q$ represent the scale of the recurring series

$$A+B+C+D+E+F+, \text{\&c.}$$

Then the 3rd term $C=Bpx+Aqx^2$;

the 4th term $D=Cpx+Bqx^2$;

the 5th term $E=Dpx+Cqx^2$; &c.

The values of p and q may be found from any two of these equations. Taking the last two, and making $x=1$, since the scale of the series is the same, whatever be the value of x, we have

$$D=Cp+Bq,$$

$$E=Dp+Cq; \text{ whence, (Art. 158),}$$

$$p=\frac{CD-BE}{C^2-BD},\ q=\frac{CE-D^2}{C^2-BD}.$$

Since these formulæ were obtained by supposing $x=1$, therefore, in substituting the values of B, C, D, &c., x must be considered 1.

Ex. Find the scale of the series $1+2x+3x^2+4x^3+5x^4+$, &c.

Here $A=1$, $B=2x$, $C=3x^2$, $D=4x^3$, $E=5x^4$, &c.

Making $x=1$, and substituting the values of B, C, D, &c.,

$$p=\frac{3\times4-2\times5}{3\times3-2\times4}=2,\ q=\frac{3\times5-4\times4}{3\times3-2\times4}=-1.$$

Thus, the 4th term, $4x^3=2\times x\times3x^2+2x\times-1\times x^2$.

Other exercises will be had in finding the sums of recurring series.

ART. **342.** In a recurring series of the *third order* the law of the series depends on *three* terms. If we let $p+q+r$ represent the scale of the series

$$A+B+C+D+E+F+, \text{ \&c.,}$$

then the 4th term $D=Cpx+Bqx^2+Arx^3$;

the 5th term $E=Dpx+Cqx^2+Brx^3$;

the 6th term $F=Epx+Dqx^2+Crx^3$; &c.

Making $x=1$, the values of p, q and r, are readily found, (Art. 158); and in a similar manner the scale may be determined in the higher orders of recurring series.

In finding the scale of a series we may first make trial of two terms. If the results thus obtained do not reproduce the series we may try three terms, four terms, and so on, till a correct result is obtained. If in any case we assume too many terms, the redundant terms will be found equal to zero.

Art. 343. *To find the sum of an infinite recurring series whose scale of relation is known.*

Let $A+B+C+D+E+$, &c., be a recurring series whose scale of relation is $p+q$; then

the 1st term	$A=A$;
the 2nd "	$B=B$;
the 3rd "	$C=Bpx+Aqx^2$;
the 4th "	$D=Cpx+Bqx^2$;
the 5th "	$E=Dpx+Cqx^2$; &c.

If the series be continued to infinity, the last term may be considered zero. Then if S represent the required sum, by adding together the corresponding members of the preceding equalities, and observing that $B+C+D+$, &c., $=S-A$, we have

$$S=A+B+px(S-A)+qx^2\times S;$$

or, $$S-pxS-qx^2S=A+B-Apx;$$

or, $$S(1-px-qx^2)=A+B-Apx;$$

or, $$S=\frac{A+B-Apx}{1-px-qx^2}.$$

If we make $q=0$, the formula becomes

$S=\frac{A+B-Apx}{1-px}$, which is the formula for finding the sum of an infinite recurring series of the first order.

In a manner similar to the preceding, the sum may be found when the scale of the series consists of three, four, &c., terms.

Remark.— Every summable infinite series, of which recurring series are only a particular class, may be supposed to arise from the development of a rational fraction; hence, to find the sum of an infinite recurring series, is to find the generating fraction of the series.

EXAMPLES FOR PRACTICE.

1\. Find the sum of the infinite recurring series $1+3x+5x^2+7x^3+9x^4+11x^5+$, &c.

Here $A=1$, $B=3x$, $C=5x^2$, $D=7x^3$, $E=9x^4$, &c.

Making $x=1$, and substituting in the formula (Art. 341), we have $p=\frac{5\times7-3\times9}{5\times5-3\times7}=2$, $q=\frac{5\times9-7\times7}{5\times5-3\times7}=-1$.

$$S=\frac{A+B-Apx}{1-px-qx^2}=\frac{1+3x-2x}{1-2x+x^2}=\frac{1+x}{(1-x)^2}.$$

In each of the following series find the scale of relation, and the sum (S) of an infinite number of terms.

2. $1+6x+12x^2+48x^3+120x^4+$, &c.

Ans. $p=1$, $q=6$; $S=\frac{1+5x}{1-x-6x^2}$.

3. $1+2x+3x^2+4x^3+5x^4+6x^5+$, &c.

Ans. $p=2$, $q=-1$; $S=\frac{1}{(1-x)^2}$.

4. $\frac{a}{c}-\frac{abx}{c^2}+\frac{ab^2x^2}{c^3}-\frac{ab^3x^3}{c^4}+$, &c.

Ans. $p=-\frac{b}{c}$, $q=0$; $S=\frac{a}{c+bx}$.

5. $x+x^2+x^3+$, &c. *Ans.* $p=1$, $q=0$; $S=\frac{x}{1-x}$.

6. $x-x^2+x^3-x^4+$, &c. *Ans.* $p=-1$, $q=0$; $S=\frac{x}{1+x}$.

7. $1+2x+8x^2+28x^3+100x^4+356x^5+$, &c.

Ans. $p=3$, $q=2$; $S=\frac{1-x}{1-3x-2x^2}$.

8. $1+3x+5x^2+7x^3+9x^4+$, &c.

Ans. $p=2$, $q=-1$, $S=\frac{1+x}{1-2x+x^2}$.

9. $1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+6^2x^5+$, &c.

Ans. $p=3$, $q=-3$, $r=1$; $S=\frac{1+x}{(1-x)^3}$.

REVERSION OF SERIES.

ART. **344.** To revert a series is to express the value of the unknown quantity in it by means of another series involving the powers of some other quantity.

Let x and y represent two indeterminate quantities, and let the value of y be expressed by a series involving the powers of x; thus,

$$y=ax+bx^2+cx^3+dx^4+, \text{\&c.}, \quad (1).$$

in which a, b, c, d, &c., are known quantities; then to *revert* this series is to express the value of x in a series containing the known quantities a, b, c, d, &c., and the powers of y.

To revert this series, assume

$x = Ay + By^2 + Cy^3 + Dy^4$, &c. (2), in which the coëfficients A, B, C . . . . are undetermined.

Find the values of y^2, y^3, y^4 . . . . from (1), thus,

$$\begin{aligned} y^2 &= a^2x^2 + 2abx^3 + (b^2 + 2ac)x^4 + \ldots \\ y^3 &= \qquad\qquad a^3x^3 + 3a^2bx^4 + \ldots \\ y^4 &= \qquad\qquad\qquad\qquad a^4x^4 + \ldots \text{ \&c.} \end{aligned}$$

Substituting these values in (2), and arranging, we have

$$0 = \begin{array}{r|} Aa \\ -1 \end{array} x + \begin{array}{r|} Ab \\ Ba^2 \end{array} x^2 + \begin{array}{r|} Ac \\ +2Bab \\ +\ Ca^3 \end{array} x^3 + \begin{array}{r|} Ad \\ +\ Bb^2 \\ +\ 2Bac \\ +3Ca^2b \\ +\ Da^4 \end{array} x^4 +, \text{ \&c.}$$

and since this is universally true, whatever be the value of x, the coëfficients of x, x^2, x^3, &c., will each $=0$. (Art. 314, Cor.) Hence, we have

$$Aa - 1 \ldots\ldots\ldots\ldots = 0, \therefore A = \frac{1}{a},$$

$$Ab + Ba^2 \ldots\ldots\ldots\ldots = 0, \therefore B = -\frac{b}{a^3},$$

$$Ac + 2Bab + Ca^3 \ldots\ldots\ldots = 0, \therefore C = \frac{2b^2 - ac}{a^5},$$

$$Ad + Bd^2 + 2Bac + 3Ca^2b + Da^4 = 0, \therefore D = -\frac{a^2d - 5abc + 5b^3}{a^7}.$$

$$\text{Hence, } x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 - \frac{a^2d - 5abc + 5b^3}{a^7}y^4 +, \text{ \&c.} \quad (3)$$

ART. **345.** If the given series has a constant term prefixed, thus, $y = a' + ax + bx^2 + cx^3 + dx^4 + \ldots\ldots$

assume $y - a' = z$, and we have

$$z = ax + bx^2 + cx^3 + dx^4 +, \text{ \&c.}$$

But this is the same as (1) in the preceding article, except that z stands in the place of y; hence, if z be substituted for y in [(3), Art. 344], the result will be the required development of x; and then $y - a'$ being substituted for z, the result is

$$x = \frac{1}{a}(y - a') - \frac{b}{a^3}(y - a')^2 + \frac{2b^2 - ac}{a^5}(y - a')^3 -, \text{ \&c.}$$

ART. **346.** When the given series contains the odd powers of x, assume for x another series containing the odd powers of y. Thus, if

$$y=ax+bx^3+cx^5+dx^7+\ldots\ldots$$

to develope x in terms of y, assume

$$x=Ay+By^3+Cy^5+Dy^7+\ldots\ldots$$

Then by substituting the values of y, y^2, &c., derived from the former equation, in the latter, and equating the coëfficients to zero, we find

$$x=\frac{1}{a}y-\frac{b}{a^4}y^3+\frac{3b^2-ac}{a^7}y^5-\frac{a^3d-8abc+12b^3}{a^{10}}y^7+, \&c.$$

If both sides of the equation be expressed in a series, as

$$ay+by^2+cy^3+, \&c., =a'x+b'x^2+c'x^3+, \&c.,$$

and it be required to find y in terms of x, we must assume, as before,

$$y=Ax+Bx^2+Cx^3+Dx^4+, \&c.,$$

and substitute the values of y, y^2, y^3, &c., derived from this last equation, in the proposed equation; we shall then, by equating the coëfficients of the like powers of x, determine the values of A, B, C, &c., as before.

EXAMPLES FOR PRACTICE.

The following exercises may be solved either by substituting the values of a, b, c, &c., in the equations obtained in the preceding articles, or by proceeding according to the methods by which those equations were obtained.

1. Given the series $y=x-x^2+x^3-x^4+\ldots\ldots$ to find the value of x in terms of y. *Ans.* $x=y+y^2+y^3+y^4+$, &c.

Find the value of x, in an infinite series in terms of y:

2. When $y=x+x^2+x^3+$, &c.
Ans. $x=y-y^2+y^3-y^4+y^5-$, &c.

3. When $y=2x+3x^3+4x^5+5x^7+$, &c.
Ans. $x=\frac{1}{2}y-\frac{3}{16}y^3+\frac{19}{128}y^5-$, &c.

4. When $y=1-2x+3x^2$.
Ans. $x=-\frac{1}{2}(y-1)+\frac{3}{8}(y-1)^2-\frac{9}{16}(y-1)^3+$, &c.

5. When $y=1+x-2x^2+x^3$.
Ans. $x=y-1+2(y-1)^2+7(y-1)^3+30(y-1)^4$.

6. When $y=x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+$, &c.

Ans. $x=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+$, &c.

7. When $y+ay^2+by^3+cy^4 \ldots =gx+hx^2+kx^3+lx^4 \ldots$

Ans. $x=\frac{y}{g}+\frac{(ag^2-h)y^2}{g^3}+\frac{[bg^4-kg-2h\ (ag^2-h)]y^3}{g^5}+\ldots$

CHAPTER XI.

CONTINUED FRACTIONS: LOGARITHMS: EXPONENTIAL EQUATIONS: INTEREST, AND ANNUITIES.

CONTINUED FRACTIONS.

ART. **347.** A *continued fraction* is one whose denominator is continued by being itself a *mixed number*, and the denominator of the fractional part again continued as before, and so on; thus,

$$\frac{1}{a+\frac{1}{b}},\quad \frac{1}{a+\frac{1}{b+\frac{1}{c}}},\quad \frac{1}{a+\frac{1}{b+\frac{1}{c+\frac{1}{d}}}}$$

in which a, b, c, d, &c., are positive whole numbers, are called continued fractions.

Continued fractions are useful in *approximating* to the values of ratios expressed by *large numbers*, in resolving exponential equations, in resolving indeterminate equations of the first degree, &c.

ART. **348.** *To express a rational fraction in the form of a continued fraction.*

Let it be required to reduce $\frac{30}{157}$ to a continued fraction.

If we divide both terms of the fraction by the numerator, we find $\frac{30}{157}=\frac{1}{5+\frac{7}{30}}$.

Since the value of a fraction is the quotient arising from dividing the numerator by the denominator (Arith., Part 3rd, Art. 136), if we omit $\frac{7}{30}$, the denominator will be too *small*, and consequently $\frac{1}{5}$, the value of the fraction, will be too *large*.

Again, if we divide both terms of the fraction $\frac{7}{30}$ by the numerator, we find $\frac{30}{157}=\cfrac{1}{5+\cfrac{1}{4+\frac{2}{7}}}$.

If we omit $\frac{2}{7}$, the value will be expressed by $\cfrac{1}{5+\frac{1}{4}}=\frac{4}{21}$.

By omitting $\frac{2}{7}$, the denominator 4 will be *less* than the true denominator, and $\frac{1}{4}$ will be *larger* than the number which ought to be added to 5; hence, 1 divided by $5+\frac{1}{4}$, or $\frac{4}{21}$ will be *less* than the true value of the fraction.

We see from this, that by stopping at the *first* reduction, and omitting the fractional part, the result is too *great;* but by stopping at the *second* reduction and omitting the fractional part, the result is too *small.* Hence, generally,

By stopping at an odd reduction and neglecting the fractional part, the result is too great; but by stopping at an even reduction, and neglecting the fractional part, the result is too small.

Since $\frac{2}{7}=\cfrac{1}{3+\frac{1}{2}}$, we find

$$\frac{30}{157}=\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\frac{1}{2}}}}$$

. . . . 1st	reduction,	too great;
. . . . 2nd	"	too small;
. . . . 3rd	"	too great;
. . . . 4th	"	true value.

It is evident that the process of reducing a fraction to a continued fraction, is the same as that of finding the greatest common divisor of the two terms of the fraction. (See Arith., Part 3rd, Art. 128.)

By this process we find

$$\frac{13}{30}=\frac{1}{2+\frac{1}{3+\frac{1}{4}}}. \qquad \frac{49}{204}=\frac{1}{4+\frac{1}{6+\frac{1}{8}}}$$

ART. **349.** The different quantities

$$\frac{1}{a}, \quad \frac{1}{a+\frac{1}{b}}, \quad \frac{1}{a+\frac{1}{b+\frac{1}{c}}}, \text{ \&c.,}$$

are called *converging fractions*, because each one in succession, gives a nearer value of the given expression.

The fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c., are called *integral* fractions.

ART. **350.** *To explain the manner in which the converging fractions are found from the integral fractions.*

1. $\frac{1}{a}$ . . . . $=\frac{1}{a}$ 1st conv. fraction.

2. $\frac{1}{a+\frac{1}{b}}$ . . . . $=\frac{b}{ab+1}$ 2nd conv. fraction.

3. $\frac{1}{a+\frac{1}{b+\frac{1}{c}}}$ . . . . $=\frac{bc+1}{c(ab+1)+a}$. 3rd conv. fraction.

By examining the third converging fraction, we find it is formed from the 1st, and 2nd, and from the 3rd integral fraction as follows:

{ num. =3rd quot. × num. of 2nd conv. fract. + num. of 1st conv. fract.
{ denom. =3rd quot. × den. of 2nd conv. fract. + den. of 1st conv. fract.

To prove the general law of formation, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$ be the three converging fractions corresponding to the three integral fractions $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$, and, as has already been shown,

$$\frac{R}{R'}=\frac{Q\,c+P}{Q'c+P'}.$$

Let us now take the next integral fraction $\frac{1}{d}$, and let $\frac{S}{S'}$ express the 4^{th} converging fraction. Then it is obvious that $\frac{R}{R'}$ will become $\frac{S}{S'}$ by substituting $c+\frac{1}{d}$, instead of c; hence,

$$\frac{S}{S'}=\frac{Q\left(c+\frac{1}{d}\right)+P}{Q'\left(c+\frac{1}{d}\right)+P'}=\frac{(Q\,c+P\,)d+Q}{(Q'c+P')d+Q'}=\frac{R\,d+Q}{R'd+Q'}.$$

From this we see that the fourth converging fraction is deduced from the two immediately preceding it, according to the same law by which the third was deduced from the 1^{st} and 2^{nd}, and it is evident the fifth converging fraction may be deduced in the same manner. Hence, to find the n^{th} converging fraction,

Multiply the denominator of the n^{th} *integral fraction by the numerator of the* $(n-1)^{th}$ *converging fraction, and add to the product the numerator of the* $(n-2)^{th}$ *converging fraction. This will give the numerator of the* n^{th} *converging fraction.*

Multiply the denominator of the n^{th} *integral fraction by the denominator of the* $(n-1)^{th}$ *converging fraction, and add to the product the denominator of the* $(n-2)^{th}$ *converging fraction. This will give the denominator of the* n^{th} *converging fraction.*

Ex. To find a series of converging fractions for $\frac{84}{227}$.

The integral fractions are $\frac{1}{2}, \frac{1}{1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{1}, \frac{1}{3}, \frac{1}{2}$.

The converging fractions are $\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{7}{19}, \frac{10}{27}, \frac{37}{100}, \frac{84}{227}$.

ART. **351.** *To show that the difference between any two consecutive converging fractions is always a fraction having +1, or —1, for its numerator, according as the fraction subtracted is in an even or odd place.*

$$\frac{1}{a}-\frac{b}{ab+1}=\frac{ab+1-ab}{a(ab+1)}\cdots\cdots=\frac{+1}{a(ab+1)}$$

$$\frac{b}{ab+1}-\frac{bc+1}{c(ab+1)+a}=\frac{bc(ab+1)+ab-(ab+1)(bc+1)}{(ab+1)[c(ab+1)+a]}$$

$$=\frac{-1}{(ab+1)[c(ab+1)+a]}.$$

To prove the property in a general manner, let

$$\frac{P}{P'}, \frac{Q}{Q'}, \frac{R}{R'}$$

be three consecutive converging fractions, corresponding to the three integral fractions $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$. Then

$$\frac{P}{P'}-\frac{Q}{Q'}=\frac{PQ'-P'Q}{P'Q'},$$

and $$\frac{Q}{Q'}-\frac{R}{R'}=\frac{R'Q-RQ'}{R'Q'};$$

but $R=Qc+P$, and $R'=Q'c+P'$, (Art. 350.)

Substituting these values in the last equation, we find

$$\frac{Q}{Q'}-\frac{R}{R'}=\frac{(Q'c+P')Q-(Qc+P)Q'}{R'Q'}=\frac{P'Q-PQ'}{R'Q'}.$$

But the numerator of this result $P'Q-PQ'$ is the same with a contrary sign as the numerator of $\frac{P}{P'}-\frac{Q}{Q'}$, which we have before shown is $+1$. Hence, the difference between the numerators of any two consecutive approximating fractions, when reduced to a common denominator, is the same with a contrary sign, as that which exists between the last numerator and the numerator of the fraction immediately following.

But it has been already shown that the difference of the numerators of the 1st and 2nd fractions is $+1$; the difference of the numerators of the 2nd and 3rd fractions is -1; therefore, the difference of the numerators of the 3rd and 4th is $+1$, and so on. And since (Art. 348) any converging fraction of an *even* order is *less* than the true value, and of an *odd* order *greater* than the true value; therefore, if a converging fraction of an even order be subtracted from the consecutive converging fraction of an odd order, the numerator of the difference will be $+1$; and, conversely, if a converging fraction of an odd order be subtracted from the consecutive converging fraction of an even order, the numerator of the difference will be -1.

ART. 352. *To show that every converging fraction is in its lowest terms; and to find the limit of error in taking any convergent for the true fraction* $\frac{a}{b}$.

If $\frac{A}{B}$ and $\frac{C}{D}$ be any two consecutive converging fractions, by Art. 351 $\frac{A}{B}-\frac{C}{D}=+\frac{1}{BD}$, or $-\frac{1}{BD}$; that is, AD—BC=+1, or —1. Now if A and B have a common divisor *greater than* 1, it will divide AD and BC, and consequently their difference ±1; that is, a quantity greater than 1 is a divisor of 1, which is impossible; hence, $\frac{A}{B}$ is in its lowest terms.

Again, if $\frac{A}{B}$ and $\frac{C}{D}$ be any two consecutive convergents, as has just been shown, AD—BC=±1; and of $\frac{A}{B}$ and $\frac{C}{D}$ we know that $\frac{a}{b}<$ one and $>$ the other (Art. 348); therefore, the difference between $\frac{a}{b}$, and either of them, is less than the difference between $\frac{A}{B}$ and $\frac{C}{D}$; that is, $<\frac{1}{BD}$, since $\frac{A}{B}\sim\frac{C}{D}$, or $\frac{AD\sim BC}{BD}=\frac{1}{BD}$.

But, since D is greater than B, $\frac{1}{BD}$ is greater $\frac{1}{D^2}$; hence, since the result is true to within $\frac{1}{BD}$, it is certainly true to within $\frac{1}{D^2}$; that is, *the approximate result which is obtained, is true to within unity, divided by the square of the denominator of the last converging fraction.*

Thus, in the example, (Art. 348,) $\frac{1}{5}$ differs from $\frac{30}{157}$ by a quantity less than $\frac{1}{5^2}$; $\frac{4}{21}$ differs from $\frac{30}{157}$ by less than $\frac{1}{21^2}=\frac{1}{441}$, and so on.

ART. **353.** *To express* $\sqrt{N}$, *when* $N=a^2+1$, *in the form of a continued fraction.*

$$\sqrt{a^2+1}=a+\sqrt{a^2+1}-a=a+\frac{1}{\sqrt{a^2+1}+a}\ (*),$$

$$=a+\cfrac{1}{a+a+\sqrt{a^2+1}-a},$$

$$=a+\cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a+}}}, \text{ \&c.}$$

$$(*)\quad \sqrt{a^2+1}-a=\frac{1}{\sqrt{a^2+1}+a},$$

because $(\sqrt{a^2+1}-a)(\sqrt{a^2+1}+a)=1.$

Ex. $\sqrt{17}=\sqrt{4^2+1}=4+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+}}}$, &c.;

the converging fractions to be added to 4, are $\frac{1}{8}, \frac{8}{65}, \frac{65}{528}$, &c.

ART. **354.** To convert $\sqrt{N}$, where $N=a^2+b$, into a continued fraction.

$$\sqrt{N}=a+\sqrt{N}-a=a+\frac{b}{\sqrt{N}+a}=a+\frac{1}{\frac{1}{b}(\sqrt{N}+a)};$$

let r_1 be the nearest integer to $\frac{1}{b}(\sqrt{N}+a)$;

$$\therefore \frac{1}{b}(\sqrt{N}+a)=r_1+\frac{1}{b}(\sqrt{N}+a-r_1b)=r_1+\frac{1}{b}(\sqrt{N}-a_1)$$

$$=r_1+\frac{b_1}{\sqrt{N}+a_1}=r_1+\frac{1}{\frac{1}{b_1}(\sqrt{N}+a_1)};$$

by making $a_1=r_1b-a$; $b_1=\frac{1}{b}(N-a_1^2)$.

Similarly, $\frac{1}{b_1}(\sqrt{N}+a_1)=r_2+\frac{1}{\frac{1}{b_2}(\sqrt{N}+a_2)}$,

and we must proceed till we get a quotient $2a$, after which the quotients will recur in the same order

Thus, $\sqrt{19}=4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+, \&c.}}}$

The quotients are 4, 2, 1, 3, 1, 2, 8;

$\therefore$ fractions are $\frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{22}{5}, \frac{79}{18}, \frac{101}{23}, \frac{281}{64}$.

Art. 355. *To find the value of a continued fraction, when the denominators* q, r, s, *&c., of the integral fractions recur ad infinitum in a certain order.*

Ex. 1. Let $\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{q+\cfrac{1}{r}+, \&c., \text{ ad infinitum.}}}}$

then $\cfrac{1}{q+\cfrac{1}{r+x}}=x$, or $\frac{r+x}{qr+qx+1}=x$;

hence, $r+x=qrx+qx^2+x$, and $x^2+rx-\frac{r}{q}=0$.

From the solution of this equation, the value of x is easily found.

Art. 356. *To find in the form of a continued fraction, the value of* x, *which satisfies the equation* $a^x=b$.

Substitute for x the numbers 0, 1, 2, 3, &c., until two consecutive numbers are found n, and $n+1$, such that

$$a^n<b, \text{ and } a^{n+1}>b;$$

then it is evident that $x<n+1$, and $>n$.

Let $x=n+\frac{1}{y}$, where $y>1$.

then $a^{n+\frac{1}{y}}=b$, or $a^n.a^{\frac{1}{y}}=b$;

hence, $a^{\frac{1}{y}}=\frac{b}{a^n}$, or $\left(\frac{b}{a^n}\right)^y=a$.

Again, since $\frac{b}{a^n}>1$, and $<a$, by substituting the numbers between 1 and a for y in the last equation, two consecutive numbers, p, and $p+1$, will be found, such that $y>p$ and $<p+1$, so

that $y=p+\frac{1}{z}$; and $\therefore\ x=n+\cfrac{1}{p+\cfrac{1}{z}}$; and by continuing the process in the same manner, the fraction expressing the value of x may be continued.

Ex. Required the value of x in the equation $10^x=2$.

By substituting 0 and 1 for x, it appears that $x>0$ and <1;

$$\text{let}\qquad x=\frac{1}{y},\ \text{then}\ 10^{\frac{1}{y}}=2,\ \text{or}\ 2^y=10.$$

Since $2^3=8$, and $2^4=16$, one of which is less and the other greater than 10, therefore, $y>3$, and <4; let $y=3+\frac{1}{z}$;

$$\text{then}\qquad 2^{3+\frac{1}{z}}=10,$$

$$\text{or}\quad 2^3.2^{\frac{1}{z}}=10,\ \text{or}\ 2^{\frac{1}{z}}=\tfrac{10}{8}=1.25;$$

$$\therefore\ (1.25)^z=2.$$

Again, it appears that $z>3$, and <4; let $z=3+\frac{1}{u}$, then

$$(1.25)^{3+\frac{1}{u}}=(1.25)^3(1.25)^{\frac{1}{u}}=2\ \therefore\ (1.25)^{\frac{1}{u}}=\frac{2}{(1.25)^3}=1.024;$$

$$\therefore\ (1.024)^u=1.25,\ \text{and by trial}\ u>9\ \text{and}\ <10.$$

$$\text{Hence,}\ x=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+,\ \&c.}}}$$

This gives $x=\frac{1}{3}-,\ \frac{3}{10}+,\ \frac{28}{93}-=.30107$ nearly, &c.

EXAMPLES FOR PRACTICE.

Reduce each of the following fractions to a continued fraction, and find the successive integral and converging fractions.

1. $\frac{130}{421}$. *Ans.* Integral fractions $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$.
   Converging fractions $\frac{1}{3}, \frac{4}{13}, \frac{21}{68}, \frac{130}{421}$.

2. $\frac{130}{291}$. *Ans.* Integral fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$.
   Converging fractions $\frac{1}{2}, \frac{4}{9}, \frac{21}{47}, \frac{130}{291}$.

3. $\frac{157}{972}$. *Ans.* Integral fractions $\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.
   Converging fractions $\frac{1}{6}, \frac{5}{31}, \frac{21}{130}, \frac{68}{421}, \frac{157}{972}$.

4. The hight of Mt. Etna is 10963 feet, and of Vesuvius 3900 feet; required the approximate ratio of the hight of the former to that of the latter.

Ans. $\frac{1}{2}, \frac{1}{3}, \frac{5}{14}, \frac{16}{45}, \frac{37}{104}, \frac{90}{253}, \frac{127}{357}, \frac{3900}{10963}$.

5. The hight of Mt. Perdu, the highest of the Pyrenees, is 11283 feet; that of Mt. Hecla is 4900 feet; required the approximate ratio of the hight of the former to that of the latter.

Ans. $\frac{1}{2}, \frac{3}{7}, \frac{10}{23}, \frac{33}{76}, \frac{76}{175}$, &c.

6. When the diameter of a circle is 1, the circumference is found to be greater than 3.1415926, and less than 3.1415927; required the series of fractions converging to the ratio of the circumference to the diameter. *Ans.* $\frac{1}{3}, \frac{7}{22}, \frac{106}{333}$, and $\frac{113}{355}$.

Show that this last ratio, $\frac{113}{355}$, is true to within less than three ten millionths of the circumference.

SUGGESTION.— In examples of this kind the integral fractions, corresponding to both fractions, should be found, and then the converging fractions calculated from those integral fractions that are the same in both series.

7. Express approximately the ratio of 24 hours to 5 hours, 48 minutes, 49 seconds, the excess of the solar year above 365 days.

Ans. $\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{31}{128}, \frac{39}{161}, \frac{655}{2704}, \frac{694}{2865}, \frac{1349}{5569}, \frac{20929}{86400}$.

Hence, after every 4 years, we must have had 1 intercalary day, as in leap year; after every 29 years, we ought to have had 7 intercalary days; after every 33 years we ought to have had 8 intercalary days. This last was the correction used by the Persian astronomers, who had seven regular leap years, and then deferred the eighth until the fifth year, instead of having it on the fourth.

8. Find the least fraction with only two figures in each term, approximating to $\frac{1947}{3359}$. *Ans.* $\frac{11}{19}$.

9. The lunar month, calculated on an average of 100 years, is 27.321661 days. Find a series of common fractions approximating nearer and nearer to this quantity.

Ans. $\frac{27}{1}, \frac{82}{3}, \frac{765}{28}, \frac{3907}{143}$, &c.

10. Find a series of fractions converging to $\sqrt{2}$.

Ans. $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, &c.

11. Show that $\sqrt{5}$ is greater than $\frac{682}{305}$ and less than $\frac{2889}{1292}$.

12. If $8^x=32$, find x. *Ans.* $\frac{5}{3}$.

13. If $3^x=15$, find x. *Ans.* 2.465.

LOGARITHMS.

ART. **357.** In a system of logarithms, all numbers are considered as the powers of some one number, arbitrarily assumed, which is called the *base* of the system; and *the exponent of that power of the base, which is equal to any given number, is called the* LOGARITHM *of that number.*

Thus, if a is the base of a system of logarithms, N any number, and x such that

$$a^x = N$$

then x is called the logarithm of N, in the system whose base is a.

For particular examples suppose we have the equations $a^2 = N$, and $a^3 = N'$, then 2 is the logarithm of N, and 3 is the logarithm of N′.

The base of the common system of logarithms (called from their inventor "Brigg's Logarithms") is the number 10. If we designate the logarithm of any number in this system by *l.* or log., we shall have

$(10)^0 = 1$	; hence,	0	is the	log. of 1;
$(10)^1 = 10$	; "	1	" "	log. of 10;
$(10)^2 = 100$	; "	2	" "	log. of 100;
$(10)^3 = 1000$	; "	3	" "	log. of 1000;
$(10)^4 = 10000$;	"	4	" "	log. of 10000;
&c.,				&c.

From this it appears that, in the common system, the logarithm of every number between 1 and 10 is some number between 0 and 1; that is, a proper fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2; that is, 1 plus a fraction. The logarithm of every number between 100 and 1000 is some number between 2 and 3; that is, 2 plus a fraction; and so on.

ART. **358.** The integral part of a logarithm is called the *index* or *characteristic* of the logarithm.

Since the logarithm of 1 is 0, of 10 is 1, of 100 is 2, of 1000 is 3, and so on; therefore,

The characteristic of the logarithm of any number greater than unity, is one less than the number of integral figures in the given number.

Thus, the logarithm of 123 is 2 plus a fraction; the logarithm of 1234 is 3 plus a fraction, and so on.

ART. **359.** The computation of the logarithms of numbers in the common system, consists in finding the values of x in the equation

$$10^x=N, \text{ when N is successively } 1, 2, 3, \&c.$$

One method of finding an approximate value of x has been explained in Art. 356, but other methods more expeditious will be given hereafter.

The following table contains the logarithms of numbers from 1 to 100 in the common system:

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

In the common tables, only the fractional part of the logarithm is given. Thus, in searching for the logarithm of such a number as 3530 we find in the table opposite to 3530 the number 547775; but since 3530 is expressed by four figures the characteristic of the logarithm is 3; hence,

$$\text{log. } 3530=3.547775.$$

GENERAL PROPERTIES OF LOGARITHMS.

ART. **360.** Let N and N′ be any two numbers, x and x' their respective logarithms, and a the base of the system. Then, by the definition of logarithms (Art. 357),

$$a^x = N \quad . \ . \ . \ . \ (1),$$

$$a^{x'} = N' \quad . \ . \ . \ . \ (2).$$

Multiplying equations (1) and (2) together, we find

$$a^x \times a^{x'} = a^{x+x'} = NN'.$$

But, by the definition of logarithms, $x+x'$, the exponent of a, is the logarithm of NN′; hence, we have

PROPERTY I.— *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

It may be shown similarly that the sum of the logarithms of three or more factors, is equal to the logarithm of their product. Hence, *to multiply two or more numbers together, add their logarithms together, and the product will be the number corresponding to this sum.*

ART. **361.** Taking the same equations, (Art. 360), we have

$$a^x = N \quad . \ . \ . \ . \ (1),$$

$$a^{x'} = N' \quad . \ . \ . \ . \ (2).$$

Dividing equation (1) by equation (2), we find

$$\frac{a^x}{a^{x'}} = a^{x-x'} = \frac{N}{N'}.$$

But, by the definition of logarithms, $x-x'$, the exponent of a is the logarithm of $\frac{N}{N'}$; hence,

PROPERTY II.— *The logarithm of the dividend, minus the logarithm of the divisor, is equal to the logarithm of the quotient.*

The same principle may be expressed otherwise thus, *the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

From this article, and the preceding, we see that by means of logarithms, the operation of *Multiplication* is performed by *Addition*, and of *Division* by *Subtraction*.

Ex. 1. Find the product of 9 and 6 by means of logarithms.

By the table (page 316) the log. of 9 is 0.954243
" " the log. of 6 is 0.778151

The sum of these logarithms is 1.732394
and the number corresponding in the table is 54.

2. Find the quotient of 63, divided by 9, by means of logarithms.

The log. of 63 is 1.799341
" log. of 9 is 0.954243

The difference is 0.845098
and the number corresponding to this log. is 7.

By means of logarithms

3. Find the product of 7 and 8.

4. Find the continued product of 2, 3, and 7.

5. Find the quotient of 85 divided by 17.

6. Find the quotient of 91 divided by 13.

Art. **362.** Resuming equation (1), (Art. 360), we have

$$a^x = N.$$

Raising both sides to the m^{th} power, we find

$$a^{mx} = N^m.$$

But, by the definition (Art. 357), mx is the logarithm of N^m; that is, m times log. $N =$ log. N^m. Hence,

Property III.—*If we multiply the logarithm of a number by any exponent, the product will be the logarithm of that power of the given number.*

Art. **363.** Taking the same equation

$$a^x = N,$$

and extracting the n^{th} root of both sides, we have

$$a^{\frac{x}{n}} = N^{\frac{1}{n}}$$

But, by the definition, (Art. 357), $\frac{x}{n}$ is the logarithm of $N^{\frac{1}{n}}$;

that is, $\frac{1}{n}$ of log. $N =$ log. $N^{\frac{1}{n}}$. Hence,

Property IV.— *If we divide the logarithm of a number by any index, the quotient will be the logarithm of that root of the given number.*

From this article and the preceding, we see that by means of logarithms, the operation of raising a number to any given power is performed by a *simple multiplication*, and the extraction of any root, by a *simple division*.

Ex. 1. Find the third power of 4 by means of logarithms.

The logarithm of 4 is	0.602060
Multiply by the exponent 3.	3
The product is	1.806180

which is the logarithm of 64.

2. Extract the fifth root of 32 by means of logarithms.

The logarithm of 32 is	1.505150
Dividing by the index 5, the quotient is	0.301030

which is the logarithm of 2, the required root.

Solve the following examples by means of logarithms:

3. Find the square of 7.

4. Find the fourth power of 3.

5. Extract the cube root of 27.

6. Extract the sixth root of 64.

The preceding properties and examples will suffice to show the great utility of logarithms in mathematical calculations. It is, however, rather the province of algebra, to explain the principles of logarithms, than their use in actual calculations, as the latter requires a set of logarithmic tables, which are usually inserted in works on Trigonometry, Surveying, &c.

ART. **364.** By means of *negative* exponents, we can also express the logarithm of fractions less than 1. Thus, in the common system, since

$(10)^{-1}=\frac{1}{10}=.1$, therefore -1 is the log. of .1;

$(10)^{-2}=\frac{1}{100}=.01$, " -2 " log. " .01;

$(10)^{-3}=\frac{1}{1000}=.001$, " -3 " log. " .001;

$(10)^{-4}=\frac{1}{10000}=.0001$ " -4 " log. " .0001;

&c., &c.

The logarithm of any fraction between one and one-tenth, for example, seven-tenths, may be expressed thus,

$$\log.\ (\tfrac{7}{10})=\log.\ (\tfrac{1}{10}\times 7)=\log.\ \tfrac{1}{10}+\log.\ 7=-1+\log.\ 7.$$

In like manner the logarithm of any fraction between one-tenth and one-hundredth, may be expressed thus,

$$\log.\ (\tfrac{3}{100})=\log.\ (\tfrac{1}{100}\times 3)=\log.\ \tfrac{1}{100}+\log.\ 3=-2+\log.\ 3.$$

Similarly, for fractions between $\frac{1}{100}$ and $\frac{1}{1000}$, thus,

$$\text{log. } (\tfrac{4}{1000}) = \text{log. } \tfrac{1}{1000} + \text{log. } 4 = -3 + \text{log. } 4.$$

It is customary not to perform the subtraction thus indicated but to unite the logarithm of the numerator of the decimal considered as a whole number, to the negative characteristic. Thus,

$$\text{log. } 0.7 \quad = -1 + \text{log. } 7 = -1.845098$$
$$\text{log. } 0.03 \; = -2 + \text{log. } 3 = -2.477121,$$
$$\text{log. } 0.004 = -3 + \text{log. } 4 = -3.602060.$$

Since the logarithm of .1 is -1, of .01 is -2, of .001 is -3, and so on; therefore,

The characteristic of the logarithm of a decimal fraction is a negative number, and is one more than the number of zeros immediately following the decimal point.

Art. **365.** *To explain the principle generally, by means of which the logarithms of decimals are represented.*

Let a represent a decimal fraction containing m zeros immediately following the decimal point, and n other places of figures; then the number of zeros in the denominator will be $m+n$, and by the nature of decimals the fraction will be represented by

$$\frac{a}{(10)^{m+n}}.$$

$$\text{log. } \left\{\frac{a}{(10)^{m+n}}\right\} = \text{log. } a-(m+n) \text{ log. } 10 = \text{log. } a-(m+n),$$

since log. $10=1$.

But, by supposition, a contains n figures; hence, the characteristic of its logarithm (Art. 358,) is $n-1$; and if d represent the decimal part of the log., the entire log. of a will be $n-1+d$. Substituting this instead of log. a, we have

$$\text{log. } \left\{\frac{a}{(10)^{m+n}}\right\} = n-1+d-(m+n) = -(m+1)+d.$$

Hence, *to find the logarithm of any decimal fraction, find the decimal part of the logarithm from the tables, as if the fraction were a whole number, and unite to it a negative characteristic, greater by unity than the number of zeros immediately following the decimal point.*

Art. **366.** It is of the highest importance to the student to make himself familiar with the application of the properties of logarithms (Arts. 360 to 364) to algebraic calculations. The following examples will afford a useful exercise :

1. $\log.\ (a.b.c.d. \ .\ ) = \log.\ a + \log.\ b + \log.\ c + \log.\ d\ .\ .$

2. $\log.\ \left(\frac{abc}{de}\right) = \log.\ a + \log.\ b + \log.\ c - \log.\ d - \log.\ e.$

3. $\log.\ (a^m.b^n.c^p.) = m \log.\ a + n \log.\ b + p \log.\ c.$

4. $\log.\ \left(\frac{a^m.b^n}{c^p}\right) = m \log.\ a + n \log.\ b - p.\ \log.\ c.$

5. $\log.\ (a^2-x^2) = \log.\ [(a+x)(a-x)] = \text{l}.\ (a+x) + \text{l}\ (a-x).$

6. $\log.\ \sqrt{a^2-x^2} = \frac{1}{2} \log.\ (a+x) + \frac{1}{2} \log.\ (a-x).$

7. $\log.\ (a^3 \times \sqrt[4]{a^3}) = 3\frac{3}{4} \log.\ a.$

8. $\log.\ \frac{\sqrt{a^2-x^2}}{(a+x)^2} = \frac{1}{2}\{\log.\ (a-x) - 3 \log.\ (a+x)\}.$

ART. **367.** Let us resume the equation

$$a^x = \text{N},$$

in which x is the logarithm of N.

1st. If we make $x=1$, we have

$$a^1 = \text{N} = a, \text{ hence } \log.\ a = 1;$$

that is, *whatever be the base of the system, its logarithm in that system is* 1.

2nd. If we make $x=0$, in the equation $a^x=\text{N}$, we have

$$a^0 = \text{N} = 1, \text{ hence } \log.\ 1 = 0;$$

that is, *in any system the logarithm of* 1 *is* 0.

ART. **368.** In the equation $a^x=\text{N}$, consider $a>1$, as in the common system, and suppose x negative, we then have

$$a^{-x} = \frac{1}{a^x} = \text{N}.$$

As x increases the value of the fraction $\frac{1}{a^x}$ will diminish ; and when x is infinite, the value of the fraction becomes 0; that is,

$$\frac{1}{a^\infty} = a^{-\infty} = 0; \text{ or, } \log.\ 0 = -\infty.$$

Hence, *the logarithm of* 0 *in a system whose base is greater than* 1 *is an infinite number and negative.*

In the Naperian, as well as the common system of logarithms, the base is *greater* than 1; but it may be shown that in a system whose base is *less* than 1, the logarithm of 0 is *infinite* and *positive*.

ART. **369.** In the equation $a^x=N$, every *positive* value of x gives a corresponding *positive* value of N.

If x is *negative*, we have $a^{-x}=\frac{1}{a^x}=N$. Hence, for every *negative* value of x the corresponding value of N is also *positive*. Therefore, whether x is positive or negative, the corresponding value of N is *positive*; hence, *Negative numbers have no real logarithms.*

COMPUTATION OF LOGARITHMS.

ART. **370.** Before proceeding to explain the methods of computing logarithms, we may observe that *it is only necessary to compute the logarithms of the prime numbers.*

This is obvious when we consider that every *composite* number is the product of two or more prime numbers, and that the logarithm of any product is equal to the sum of the logarithms of its factors. (Art. 360.)

For example, if we have the logarithms of 1, 2, 3, 5, 7, we can find the logarithms of all composite numbers produced by the multiplication of two or more of these numbers together. Thus,

$4=2^2$; hence, log. 4=2 log. 2, (Art. 362);
$6=2\times3$; " log. 6= log. 2+ log. 3;
$8=2^3$; " log. 8=3 log. 2;
$9=3^2$; " log. 9=2 log. 3;
$10=2\times5$; " log. 10= log. 2+ log. 5;
$12=3\times4$; " log. 12= log. 3+ log. 4;

We can proceed in a similar manner to find the logarithms of 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, and so on.

Exercise 1. Suppose the logarithms of the numbers 2, 3, 5 and 7 to be known; show how the logarithms of the numbers just named may be found.

2. Of what numbers between 30 and 100, may the logarithms be found from those of 2, 3, 5, and 7; and why?

Ans. Of 23 different numbers, from 32 to 98.

ART. **371.** In the common system the equation $a^x=N$ (Art 357) becomes $10^x=N$.

If we multiply both sides by 10, we have

$$10^x\times10=10^{x+1}=10N;$$

also, $10^x\times100=10^x\times10^2=10^{x+2}=100N.$

Hence, in the common system, the logarithm of any number will become the logarithm of 10 times that number, by increasing the characteristic by 1; of 100 times by increasing the characteristic by 2, and so on.

Thus, the log. of	3	is.	0.477121,
" "	30	".	1.477121,
" "	300	".	2.477121.
Also, the log. of	.2583	is	—1.412124,
"	2.583	"	0.412124,
"	25.83	"	1.412124.

Art. 372. If we compare the different powers of 10 with their logarithms in the common system, we have

numbers	1,	10,	100,	1000,	10000,
logarithms	0,	1,	2,	3,	4, and so on.

Hence, in the common system, while the numbers are in *geometrical progression,* their logarithms are in *arithmetical progression.* Therefore, if we take a geometrical mean between two numbers, and an arithmetical mean between their logarithms, the latter number will be the logarithm of the former. Thus, the geometrical mean between 10 and 1000 is $\sqrt{10\times1000}=100$, and the arithmetical mean between their logarithms, 1 and 3, is $(1+3)\div2=2$.

In general, if N and N′ are two numbers, and x and x' their logarithms in the common system, then the

$$\text{log. of } \sqrt{\text{NN}'} \text{ is } \frac{x+x'}{2}.$$

By means of this principle, the common, or Briggean, system of logarithms was originally calculated. To exemplify the method of operation, let it be required to calculate the logarithm of 5.

First.— The proposed number lies between 1 and 10; hence, its logarithm will lie between 0 and 1.

The geometrical mean between 1 and 10 is $\sqrt{(1\times10)}=3.162277$; the arithmetical mean between 0 and 1 is $(0+1)\div2=0.5$.

Hence, the log. of 3.162277 is 0.5.

Secondly.— Take the numbers 3.162277 and 10, and their logarithms .5 and 1, we find

the geometrical mean is $\sqrt{(3.162277 \times 10)} = 5.623413$;
the arithmetical mean is $(.5+1) \div 2 = 0.75$.

Hence, the log. of 5.623413 is 0.75.

Thirdly.— Take the numbers 3.162277 and 5.623413, and their logarithms 0.5 and 0.75, we find
the geometrical mean is $\sqrt{(3.162277 \times 5.623413)} = 4.216964$;
the arithmetical mean is $(.5+.75) \div 2 = 0.625$.

Hence, the logarithm of 4.216964 is 0.625.

Fourthly.— Take the numbers 4.216964 and 5.623413, and their logarithms 0.625 and 0.75, we find
the geometrical mean is $\sqrt{(4.216964 \times 5.623413)} = 4.869674$;
the arithmetical mean is $(.625+.75) \div 2 = 0.6875$.

Hence, the logarithm of 4.869674 is 0.6875.

By continuing this process, observing always to take the two numbers nearest to 5, one of which is *less* and the other *greater*, and finding *their geometrical mean*, and the corresponding *arithmetical mean of their logarithms*, at each step we shall obtain a number nearer to 5 than either of the preceding, with its corresponding logarithm. And after twenty-two operations we obtain the number 5.000000+, and its corresponding logarithm 0.698970+.

Having the logarithm of 5 we readily find that of 2,

since $2 = \frac{10}{5}$, and log. 2 = log. 10 — log. 5 = 1 — 0.698970 = 0.301030.

We might now proceed to find the logarithm of 3 by taking the numbers 2 and 3.162277, and their logarithms 0.301030, and 0.5, and pursuing a process similar to that used in finding the logarithm of 5. But the method of series is much shorter, and is the one now generally used.

Art. 373. Logarithmic Series.— The most convenient method of computing logarithms is by means of *Series*, which we shall now proceed to explain.

Let x be a number whose logarithm is to be expressed in a series, and let us apply the method of Indeterminate Coëfficients (Art. 314). If we assume

$$\log. x = A + Bx + Cx^2 + Dx^3 +, \&c.,$$

and make $x=0$, we have

$$\log. 0 = A. \quad \text{But } \log. 0 = \infty \text{ (Art. 368); hence,}$$

$$\infty = A, \text{ which is absurd.}$$

If we assume $\log. x = Ax + Bx^2 + Cx^3 +$, &c., and make $x = 0$,

we have $\log. 0 = 0$; that is, (Art. 368),

$\infty = 0$, which is also absurd.

Hence, it is impossible to develop the logarithm of a number in powers of that number.

But if we assume

$$\log. (1+x) = Ax + Bx^2 + Cx^3 + Dx^4 +, \&c. \;.\;.\;. (1)$$

and make $x = 0$, we have

$\log. 1 = 0$, which is correct, (Art. 367).

In like manner, also assume

$$\log. (1+z) = Az + Bz^2 + Cz^3 + Dz^4 +, \&c. \;.\;.\;. (2)$$

Subtracting equation (2) from (1) we get

$$\log. (1+x) - \log. (1+z) = A(x-z) + B(x^2-z^2)$$
$$+ C(x^3-z^3) +, \&c. \;.\;.\;. (3).$$

The second member of this equation is divisible by $x-z$ (Art. 83); let us reduce the first member to a form in which it shall also be divisible by the same factor.

Since the logarithm of a fraction is equal to the logarithm of the numerator, minus the logarithm of the denominator (Art. 361), therefore,

$$\log. (1+x) - \log. (1+z) = \log. \left(\frac{1+x}{1+z}\right).$$

But, by division, we find $\frac{1+x}{1+z} = 1 + \frac{x-z}{1+z}$; therefore,

$$\log. \left(\frac{1+x}{1+z}\right) = \log. \left(1 + \frac{x-z}{1+z}\right).$$

Now regarding $\frac{x-z}{1+z}$ as a single quantity, we may assume

$$\log. \left(1+\frac{x-z}{1+z}\right) = A.\frac{x-z}{1+z} + B\left(\frac{x-z}{1+z}\right)^2 + C\left(\frac{x-z}{1+z}\right)^3 +, \&c.$$

Substituting this development in the place of $\log. (1+x) - \log. (1+z)$, in equation (3), and dividing both sides by $x-z$, we obtain

$$A.\frac{1}{1+z} + B.\frac{x-z}{(1+z)^2} + C.\frac{(x-z)^2}{(1+z)^3} +, \&c.,$$
$$= A + B(x+z) + C(x^2+xz+z^2) +, \&c.,$$

Since this equation, like the preceding, is true for all values of x and z, it must be true when $x=z$. Making this supposition, we have

$$A.\frac{1}{1+x}=A+2Bx+3Cx^2+4Dx^3+5Ex^4+, \&c.;$$

or, performing the division of 1 by $1+x$, we have

$$A(1-x+x^2-x^3+x^4-\ .\ .\ .\ )=A+2Bx+3Cx^2+4Dx^3+\ .\ .\ .$$

Equating the coëfficients of the like powers of x (Art. 314), we obtain

$$A=A,\ -A=2B,\ A=3C,\ -A=4D\ .\ .\ .\ .$$

whence,

$$A=A,\ B=-\frac{A}{2},\ C=\frac{A}{3},\ D=-\frac{A}{4}\ .\ .\ .\ .$$

The law of this series is obvious, the coëfficient of the n^{th} term being $\pm\frac{A}{n}$, according as n is *odd* or *even*.

Hence, $\log.(1+x)=Ax-\frac{A}{2}x^2+\frac{A}{3}x^3-\frac{A}{4}x^4+\ .\ .\ .\ .$

$$=A(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\ .\ .\ .\ .\ )\quad (4)$$

There still remains one quantity, A, undetermined. This is as it should be, for the question to find the logarithm of a given number is indeterminate, unless the base of the system be given. The value of the quantity A may be considered as dependent on the base of the system, so that when A is given the base may be determined; or, when the base is known, A may be determined.

If we denote the series in the parenthesis in equation (4) by x', we may write

$$\log.(1+x)=Ax'.$$

Hence, the logarithm of a number consists of two factors, one of which depends on the number itself, and the other on the base of the system in which the logarithm is taken. *That factor which depends on the base is called the* MODULUS *of the system of logarithms.*

Lord Napier, the inventor of logarithms, assumed the modulus equal to unity, and the system resulting from such a modulus, is called the *Naperian system.*

Designating the logarithms in this system by log'., we have

$$\log'.(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+, \&c.\quad (5)$$

By making $x=0, 1, 2, 3$, &c., we may obtain from this equation the Naperian logarithms of all numbers.

Thus, if $x=0$, we find $\log'. 1=0$, as in Art. 367.

If we make $x=1$, we have

$$\log'. 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-, \text{\&c.}$$

ART. **374.** The preceding series converges so slowly that it would be necessary to take a great number of terms to obtain a near approximation. But we may obtain a more converging series in the following manner:

Resuming equation (5),

$$\log'. (1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-, \text{\&c.} \quad . \; . \; . \quad (5).$$

Substituting $-x$ for x, in this equation, we obtain

$$\log'. (1-x)=-\frac{x}{1}-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-, \text{\&c.} \quad . \; . \quad (6).$$

Subtracting equation (6) from (5), and observing that

$$\log'. (1+x)-\log'. (1-x)=\log'. \left(\frac{1+x}{1-x}\right), \text{ we have}$$

$$\log'. \frac{1+x}{1-x}=2\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+. \; . \; . \right).$$

Since $\dfrac{1+x}{1-x}=1+\dfrac{2x}{1-x}$, let $\dfrac{1+x}{1-x}=1+\dfrac{1}{z}$, $\therefore \; x=\dfrac{1}{2z+1}$,

$$\text{and } \log'. \frac{1+x}{1-x}=\log'. \left(1+\frac{1}{z}\right)=\log'. \left(\frac{z+1}{z}\right)$$

$$=\log'. (z+1)-\log'. z.$$

By substitution, the preceding series becomes

$$\log'. (z+1)-\log'. z=2\left\{\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+. \; . \right\};$$

or, log.'

$$(z+1)=\log'. z+2\left\{\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+. \; . \right\} \quad (7).$$

ART. **375.** By means of this series the Naperian logarithm of any number may be computed, when the logarithm of the *preceding number* is known. But the log'. of 1 is 0, (Art. 367); therefore, making $z=1, 2, 4, 6$, &c., we obtain the following

NAPERIAN, OR HYPERBOLIC LOGARITHMS.

$$\log'. 2 = \log'. 1 + 2\left\{\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \ldots\right\} = 0.693147$$

$$\log'. 3 = \log'. 2 + 2\left\{\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \frac{1}{7\cdot 5^7} + \ldots\right\} = 1.098612$$

$$\log'. 4 = 2. \log. 2 \ldots\ldots = 1.386294$$

$$\log'. 5 = \log'. 4 + 2\left\{\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \ldots\right\} = 1.609438$$

$$\log'. 6 = \log'. 2 + \log'. 3 \ldots\ldots = 1.791759$$

$$\log'. 7 = \log'. 6 + 2\left\{\frac{1}{13} + \frac{1}{3\cdot 13^3} + \frac{1}{5\cdot 13^5} + \ldots\right\} = 1.945910$$

$$\log'. 8 = 3 \log'. 2, \text{ or } \log'. 2 + \log'. 4 \ldots\ldots = 2.079442$$

$$\log'. 9 = 2 \log'. 3 \ldots\ldots = 2.197225$$

$$\log'. 10 = \log'. 2 + \log'. 5 \ldots\ldots = 2.302585$$

In this manner the Naperian logarithms of all numbers may be computed.

When the numbers are large their logarithms are computed more easily than in the case of small numbers. Thus, in calculating the logarithm of 101, the first term of the series gives the result true to seven places of decimals.

ART. **376.** *To explain the method of computing common logarithms from Naperian logarithms.*

We have already found (Art. 373, Equation 4),

$$\log. (1+x) = A\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\right).$$

Denoting the Naperian logarithm by an accent, we have

$$\log'. (1+x) = A'\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\right).$$

Since the series in the second members are the same, we have

$$\log. (1+x) : \log'. (1+x) : : A : A'.$$

Therefore, *the logarithms of the same number, in two different systems, are to each other as the moduli of those systems.*

But in Napier's system the modulus $A'=1$. Therefore,

$$\log. (1+x) = A \log'. (1+x).$$

Hence, *to find the common logarithm of any number, multiply the Naperian logarithm of the number by the modulus of the common system.*

It now remains to find the Modulus of the common system.

From the equation, log. $(1+x)=$A. log'. $(1+x)$,

$$\text{we find } A=\frac{\log.\ (1+x)}{\log'.\ (1+x)}.$$

Hence, *the modulus of the common system is equal to the common logarithm of any number divided by the Naperian logarithm of the same number.*

But the common logarithm of 10 is 1, and we have calculated the Naperian logarithm of 10, (Art. 375); therefore,

$$A=\frac{\log.\ 10}{\log.'\ 10}=\frac{1}{2.302585}=.4342944,$$

which is the modulus of the common system.

Hence, if N is any number, we have

$$\text{com. log. } N=.4342944\times \text{ Nap. log. } N.$$

On account of the importance of the number A, its value has been calculated with great exactness. It is

$$A=.4342944819032518276 5.$$

Art. **377.** *To calculate the common logarithms of numbers directly.*

Having found the modulus of the common system, if we multiply both members of equation (7), Art. 374, by A, and recollect that A$\times$ Nap. log. N= com. log. N, the series becomes

$$\log.\ (z+1)=\log.\ z+2A\left\{\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+..\right\}.$$

Or, by changing z into P, for the sake of distinction, and putting B, C, D, &c., to represent the terms immediately preceding those in which they are used, we have

$$\log.\ (P+1)=\log.\ P+\frac{2A}{2P+1}+\frac{B}{3(2P+1)^2}+\frac{3C}{5(2P+1)^2}$$

$$+\frac{5D}{7(2P+1)^2}+\frac{7E}{9(2P+1)^2}+\frac{9F}{11(2P+1)^2}+,\ \&c.$$

We shall now exemplify its use in finding the logarithm of 2.

Here P=1, and 2P+1=3.

$$\log. P = \log. 1 \ldots\ldots\ldots = .00000000;$$

$$\frac{2A}{2P+1} = \frac{.86858896}{3} \ldots\ldots = .28952965; \quad (B.)$$

$$\frac{B}{3(2P+1)^2} = \frac{.28952965}{3\times 3^2} \ldots\ldots = .01072332; \quad (C.)$$

$$\frac{3C}{5(2P+1)^2} = \frac{3\times .01072332}{5\times 3^2} \ldots = .00071489; \quad (D.)$$

$$\frac{5D}{7(2P+1)^2} = \frac{5\times .00071489}{7\times 3^2} \ldots = .00005674; \quad (E.)$$

$$\frac{7E}{9(2P+1)^2} = \frac{7\times .00005674}{9\times 3^2} \ldots = .00000490; \quad (F.)$$

$$\frac{9F}{11(2P+1)^2} = \frac{9\times .00000490}{11\times 3^2} \ldots = .00000045; \quad (G.)$$

$$\frac{11G}{13(2P+1)^2} = \frac{11\times .00000045}{13\times 3^2} \ldots = .00000004; \quad (H.)$$

$$\therefore \text{ common logarithm of } 2 \qquad = .30102999.$$

Exercise. In a similar manner let the pupil calculate the common logarithms of 3, 5, 7, and 11.

For the results to 6 places of decimals, see the Table, page 316.

ART. **378.** *To find the base of the Naperian system of logarithms.*

If we designate the base by e, we have, (Art. 376),

$$\log. e : \log'. e :: A : A'.$$

But $A=.4342944$, $A'=1$, and $\log'. e=1$, (Art. 367);

$$\text{hence,} \quad \log. e : 1 :: .4342944 : 1,$$

$$\text{whence } \log. e = .4342944,$$

But since we have explained the method of calculating common logarithms, they are supposed to be known, and we may use them to obtain the number of which the logarithm is .4342944, which we shall find to be

$$e=2.71828128.$$

We thus see that in both the common and the Naperian systems of logarithms, the base is *greater* than unity.

Brigg's logarithms are used in the ordinary operations of multiplication, division, &c., and hence are called *common* logarithms. Napier's logarithms are used in the applications of the Calculus. These are the only systems much used.

Art. 379. The student may prove the following theorems:

1. No system of logarithms can have a negative base, or have unity for its base.

2. The logarithms of the same numbers in two different systems have the same ratio to each other.

3. The difference of the logarithms of two consecutive numbers is less as the numbers themselves are greater.

SINGLE AND DOUBLE POSITION.

Note.— On account of the use made of Double Position in the solution of exponential and other equations, it becomes necessary to explain the principles on which it is founded. We shall also explain Single Position.

Art. 380. Single Position.— The Rule of Single Position is applied to the solution of those questions in which there is a result which is increased or diminished in the *same ratio* with some unknown quantity which it is required to find. Of this class are all questions which give rise to an equation of the form

$$ax = m \quad (1).$$

If we assume x' to be the value of x, and denote by m' the result of the substitution of x' for x, we have

$$ax' = m' \quad (2).$$

Comparing equations (1) and (2), we have

$$m' : m :: ax' : ax : x' : x;$$

that is, *As the result of the supposition is to the result in the question, so is the supposed number to the number required.*

Art. 381. Double Position.— The Rule of Double Position is applied to those questions in which the result, although it is dependent on the unknown quantity, does not increase or diminish in the same ratio with it. The class of questions to which it is particularly applicable, gives rise to an equation of the form

$$ax + b = m \quad (1).$$

If we suppose x' and x'' to be near values of x, and e' and e'' to be the errors, or the differences between the true result and the results obtained by substituting x' and x'' for x, we have

$$ax' + b = m + e' \quad (2),$$
$$ax'' + b = m + e'' \quad (3).$$

If we subtract equation (1) from (2), and (3) from (2), we have

$$a(x' - x) = e' \quad (4).$$
$$a(x' - x'') = e' - e'' \quad (5).$$

From these equations, we easily obtain

$$\frac{x'-x''}{e'-e''}=\frac{x'-x}{e'} \quad (6).$$

By subtracting equation (1) from (3) we also find

$$a(x''-x)=e'', \text{ and thence,}$$

$$\frac{x'-x''}{e'-e''}=\frac{x''-x}{e''} \quad (7).$$

Hence (Art. 263), *The difference of the errors is to the difference of the two assumed numbers, as the error of either result is to the difference between the true result and the corresponding assumed number.*

When the question gives rise to an equation of the form $ax+b=m$, this rule gives a result absolutely correct; but when the equation is of a less simple form, as in exponential equations (Art. 383), the result obtained is only approximately true.

Cor. The value of x, found either from equation (6) or (7), is $x=\frac{e'x''-e''x'}{e'-e''}$. This, expressed in ordinary language, furnishes the common arithmetical rule.

EXPONENTIAL EQUATIONS.

Art. **382.** An *exponential equation* is an equation in which the unknown quantity appears in the form of an exponent or index, as

$$a^x=b,\ x^x=a,\ a^{b^x}=c, \ \&c.$$

Such equations are most easily solved by means of logarithms. Thus, in the equation

$$a^x=b$$

if we take the logarithms of both members,

$$\text{we have} \quad x \log. a=\log. b,$$

$$\text{or,} \quad x=\frac{\log. b}{\log. a}.$$

Ex. 1. What is the value of x in the equation $2^x=64$?

$$\text{Here} \quad x \log. 2=\log. 64.$$

$$\text{whence, } x=\frac{\log. 64}{\log. 2}=\frac{1.806180}{.301030}=6. \quad \textit{Ans.}$$

Art. **383.** If the equation is of the form $x^x=a$, the value of x may be found by Double Position as follows:

Find by *trial* two numbers nearly equal to the value of x; substitute them for x in the given equation, and note the results. Then,

As the difference of the errors;

Is to the difference of the two assumed numbers;

So is the error of either result;

To the correction to be applied to the corresponding assumed number.

Ex. 1. Given $x^x=100$, to find the value of x.

The value of x is evidently between 3 and 4, since $3^3=27$, and $4^4=256$; hence, taking the logarithms of both sides of the equation, we have

$$x \log. x = \log. 100 = 2.$$

By trial, we readily find that x is greater than 3.5, and less than 3.6; then let us assume 3.5 and 3.6 for the two numbers.

First Supposition.		*Second Supposition.*	
$x=3.5$; log. x	$=.544068$	$x=3.6$; log. x	$=.556303$
multiply by 3.5 we find		multiply by 3.6 we find	
$x.$ log. x	$=1.904238$	$x.$ log. x	$=2.002690$
true no.	$=2.000000$	true no.	$=2.000000$
error	$=-.095762$	error	$+.002690$

Diff. results : Diff. assumed nos. : : Error 2nd result : Its cor.
.098452 : 0.1 : : .002690 : .00273

Hence, $x=3.6-.00273=3.59727$ nearly.

By trial we find that 3.5972 is less, and 3.5973 greater than the true value; and by repeating the operation with these numbers we would find $x=3.5972849$ nearly.

EXAMPLES FOR PRACTICE.

2. Given $20^x=100$, to find x. *Ans.* $x=1.53724$.

3. Given $100^x=250$, to find x. *Ans.* $x=1.19897$.

4. Given $x^x=5$, to find x. *Ans.* $x=2.129372$.

5. Given $x^x=42.8454$, to find x. *Ans.* $x=3.2164$.

6. How many places of figures will there be in the number expressing the 64^{th} power of 2? *Ans.* 20.

7. Given $a^{bx+d}=c$, to find x. *Ans.* $x=\frac{\log. c-d. \log. a}{b. \log. a}$.

8. Given $a^{mx}.b^{nx}=c$, to find x

$$Ans.\ x=\frac{\log. c}{m.\log. a+n.\log. b}.$$

9. Given $c^{mx}=a.b^{nx-1}$, to find x.

$$Ans.\ x=\frac{\log. a-\log. b}{m.\log. c-n.\log. b}.$$

10. Given $x+y=a$, and $m^{x-y}=n$, to find x and y.

$Ans.$ $x=\frac{1}{2}(a+\log. n\div\log. m)$, $y=\frac{1}{2}(a-\log. n\div\log. m)$.

11. Given $a^x.b^y=c$, and $my=nx$, to find x and y.

$$Ans.\ x=\frac{m.\log. c.}{m.\log. a+n.\log. b},\ y=\frac{n.\log. c}{m.\log. a+n.\log. b}.$$

12. Given $2^x.3^z=2000$, and $3z=5x$, to find the values of x and z.

$$Ans.\ x=\frac{3(3+\log. 2)}{3\log. 2+5\log. 3},\ z=\frac{5(3+\log. 2)}{3\log. 2+5\log. 3}.$$

13. Given $a^{2x}-2a^x=8$, to find x. $Ans.$ $x=\frac{2\log. 2}{\log. a}$.

SUGGESTION.— This is a quadratic form, therefore let $a^x=y$ and complete the square.

14. Given $2^{2x}+2^x=12$, to find x. $Ans.$ $x=1.58496$.

15. Given $2a^{4x}+a^{2x}=a^{6x}$, to find x.

$$Ans.\ x=\frac{\log. (\sqrt{2}+1)}{2\log. a}.$$

16. Given $a^x+\frac{1}{a^x}=b$, to find x.

$$Ans.\ x=\frac{\log. \frac{1}{2}(b\pm\sqrt{b^2-4})}{\log. a}.$$

17. Given $x^y=y^x$, and $x^3=y^2$, to find x and y.

$Ans.$ $x=2\frac{1}{4}$, $y=3\frac{3}{8}$.

18. Given $(a^2-b^2)^{2(x-1)}=(a-b)^{2x}$, to find x.

$$Ans.\ x=1+\frac{\log. (a-b)}{\log. (a+b)}.$$

19. Given $(a^4-2a^2b^2+b^4)^{x-1}=(a-b)^{2x}(a+b)^{-2}$, to find x.

$$Ans.\ x=\frac{\log. (a-b)}{\log. (a+b)}.$$

20. Given $x^y=y^x$, and $x^p=y^q$, to find x and y.

$$Ans.\ x=\left(\frac{p}{q}\right)^{\frac{q}{p-q}},\ y=\left(\frac{p}{q}\right)^{\frac{p}{p-q}}.$$

21. Given $3^{(x^2-4x+5)}=1200$, to find x.

$Ans.$ $x=4.33$, or -0.33.

INTEREST AND ANNUITIES.

Art. **384.** The solution of all questions connected with interest and annuities, may be simplified, and also generalized, by means of algebraical formulæ.

We shall employ the following notation :

Let $P=$ the principal, or sum at interest in dollars.

$r=$ the interest of 1$ for *one* year.

$t=$ the time in years that P draws interest.

$A=$ the amount of principal and interest, at the end of t years.

Note.— It must be recollected that r is not the *rate per cent.*, but only the hundredth part of it. Thus, at 5 per cent., $r=.05$$, at 6 per cent. $r=.06$$; and so on.

Art. **385.** Simple Interest.— Since the interest of the same sum for 2 years, is *twice* the interest for 1 year ; for 3 years, *three* times the interest for 1 year, and so on ; therefore, if

$r=$ the interest of 1$ for *one* year,

$tr=$ the interest of 1$ for t years,

$Ptr=$ the interest of P$ for t years,

$$\therefore\ A=P+Ptr=P(1+tr) \quad \ldots\ldots\ldots (1).$$

From this equation, any three of the quantities P, r, t, A, being given, the fourth may be found. Thus,

$$P=\frac{A}{1+tr},\quad t=\frac{A-P}{Pr},\quad r=\frac{A-P}{Pt}.$$

Examples may be taken from any treatise on arithmetic to illustrate these formulæ.

Art. **386.** Compound Interest.— Let $R=1+r$, the amount of 1$ for one year ; then at the end of the first year, R may be considered as the principal or sum due, and since the amount is proportional to the principal, that is, the amount of R$ for 1 year is R times the amount of 1$ for the same time ; therefore,

$1 : R :: R : R^2$, the amount of 1$ in 2 years.

$1 : R :: R^2 : R^3$, the amount of 1$ in 3 years.

And in like manner R^t is the amount of 1$ in t years.

Then, since for the same time the amount is proportional to the principal, the amount of P$ will be P times the amount of 1$. Hence,

$$A=P.R^t=P(1+r)^t \text{; whence,}$$

$$\log. A=\log. P+t. \log. (1+r) \qquad (1).$$

$$\log. P=\log. A-t. \log. (1+r) \qquad (2).$$

$$t=\frac{\log. A-\log. P}{\log. (1+r)} \qquad (3).$$

$$\log. (1+r)=\frac{\log. A-\log. P}{t} \qquad (4).$$

Cor. 1.— The interest $=A-P=PR^t-P=P(R^t-1)$.

Cor. 2.— If the interest is paid *half-yearly*, then $2t$ will be the number of payments, and $\frac{r}{2}$ the rate of interest; hence, in this case we have

$$A=P\left(1+\frac{r}{2}\right)^{2t} \ldots\ldots (5).$$

If paid *quarterly*, $A=P\left(1+\frac{r}{4}\right)^{4t} \ldots\ldots (6).$

Cor. 3.— From the equation $A=P.R^t$, we can readily find the time in which any sum at compound interest, will amount to *twice*, *thrice*, or m times itself.

Thus, if $A=2P$; then $2P=PR^t$ ∴ $R^t=2$, and $t=\frac{\log. 2}{\log. R}$.

if $A=3P$; then $R^t=3$, and $t=\log. 3 \div \log. R$;

if $A=mP$; then $R^t=m$, and $t=\log. m \div \log. R$.

Ex. Let it be required to find the time in which any sum will double itself at 10 per cent. compound interest.

Here $r=.10$, $R=1+r=1+.10=1.10$;

$$\text{hence, } t=\frac{\log. 2}{\log. R}=\frac{.301030}{.041393}=7.272 \text{ yrs. } Ans.$$

Art. 387. The increase of the population of a country may be computed on the same principles as compound interest. Thus if we know the population at two different periods, we may find the rate of increase; or, if we know the population at any given period, with the rate of increase, we may determine the population at any future period.

Ex. The population of the United States in 1790 was 3900000, and in 1840, 17000000. Required the average rate of increase for each 10 years.

Here there are 5 periods of 10 years each. Hence, by comparing the quantities given, with those in equation (4), Art. 386, we have $A=17000000$, $P=3900000$, and $t=5$.

log. A, (see table, page 316), ...	7.230449
log. P	6.591065
Divide by 5	5)0.639384
log. $(1+r)$ 1.342.	0.127877

Hence $r=1.342-1=.342=34\frac{1}{5}$ per cent. *Ans.*

Art. **388.** Compound Discount.—The present value of a sum P, due t years hence, reckoning compound interest, is easily obtained from Art. 386.

Let $P'=$ the present worth, then in t years, P' at compound interest, will amount to P, ∴

$$P=P'(1+r)^t, \therefore\ P'=\frac{P}{(1+r)^t}. \qquad (1).$$

Let $D=$ Comp. Discount, then $D=P-P'=P-\frac{P}{(1+r)^t}$. (2).

From equation (1), $\log. P'=\log. P-t\log.(1+r)$ (3).

Art. **389.** Annuities Certain.—An *Annuity* is a sum of money which is payable at equal intervals of time.

When the annuity has already commenced, it is said to be *in possession;* but should it not begin until some particular event has happened, or a certain number of years has elapsed, it is then called a *deferred* annuity, or an annuity in *reversion.*

An *annuity certain* is one which is limited to a certain number of years; a *life annuity* is one which terminates with the life of any person, and a *perpetuity,* or *perpetual* annuity, is one which is entirely unlimited in its duration.

All the computations relating to annuities are made according to compound interest.

Art. **390.** *To find the amount of an annuity in any number of years, at compound interest.*

Let a denote the annuity, p the present value, m the amount; and r, R, t, the same as in the preceding articles.

The first annuity a, becomes due at the end of the year, and thus, in $t-1$ years, will amount to aR^{t-1} (Art. 386). The second annuity becomes due at the end of two years, and in $t-2$ years it will amount to aR^{t-2}. In like manner, the third annuity will amount to aR^{t-3}, and so on to the last annuity, which is simply a. Hence, the entire amount is the sum of a geometrical series, whose first term $=aR^{t-1}$, common ratio $=R$, and last term $=a$; therefore, by reversing the order of the terms, we have

$$m=a+aR+aR^2+aR^3+\ldots\ldots+aR^{t-2}+aR^{t-1}.$$

$$\therefore \text{(Art. 297)},\ m=a\frac{R^t-1}{R-1}=a\frac{(1+r)^t-1}{r}.$$

If the annuity is to be received in *half-yearly* installments,

then we have $m=\dfrac{a}{2}\cdot\dfrac{(1+\frac{1}{2}r)^{2t}-1}{\frac{1}{2}r}=a.\dfrac{(1+\frac{1}{2}r)^{2t}-1}{r}$.

If *quarterly*, $m=\dfrac{a}{4}\cdot\dfrac{(1+\frac{1}{4}r)^{4t}-1}{\frac{1}{4}r}=a.\dfrac{(1+\frac{1}{4}r)^{4t}-1}{r}$.

Cor. If d dollars are placed out annually for n successive years, and the whole be allowed to accumulate at compound interest, then will the amount $A=dR+dR^2+dR^3+\ldots\ldots+dR^n$.

$$A=dR(1+R+R^2+\ldots\ldots+R^{n-1})=dR.\frac{R^n-1}{R-1}.$$

ART. 391. *To find the present value of an annuity to be paid* t *years, at compound interest.*

Let p denote the present value of the annuity a; then the amount of p\$ in t years $=pR^t$ (Art. 386), and the amount of the annuity a in the same time is (Art. 390) $a.\dfrac{R^t-1}{R-1}$; but these two amounts must be equal to each other; hence, we get

$$pR^t=a.\frac{R^t-1}{R-1},\ \text{and}\ p=a.\frac{R^t-1}{R^t(R-1)}=\frac{a}{R-1}\left(1-\frac{1}{R^t}\right).$$

Cor. If the annuity is to continue forever, t is infinite, and therefore R^t is infinitely great, and $\dfrac{1}{R^t}$ vanishes;

$$\text{hence, } p=\frac{a}{R-1}=\frac{a}{r}.$$

ART. 392. *To find the present value of an annuity in reversion; that is, an annuity which is to commence at the end of* n *years, and to continue* t *years.*

By Art. 391, the present value of the annuity for $n+t$ years, is $\frac{a}{R-1}\left(1-\frac{1}{R^{n+t}}\right)$, and the present value of the annuity for n years is $\frac{a}{R-1}\left(1-\frac{1}{R^n}\right)$; and the difference of these two sums is obviously the value in reversion,

$$\therefore p=\frac{a}{R-1}\left(\frac{1}{R^n}-\frac{1}{R^{n+t}}\right)=\frac{a}{rR^n}\left(1-\frac{1}{R^t}\right).$$

If the annuity is payable *forever* after the expiration of n years, then the value of the *reversion of the perpetuity* is (since t is infinite),

$$p=\frac{a}{rR^n}.$$

EXAMPLES IN INTEREST AND ANNUITIES.

1. What is the amount of 1$ for 100 years, at 6 per cent. per annum, compound interest? *Ans.* $339.30.

2. How many figures will it require to express the amount of 1$ for 1000 years, at 6 per cent. per annum, compound interest? *Ans.* 26.

3. How many years will it require for any sum of money to double itself at compound interest, at the rates of 5, 6, 7, and 8 per cent. per annum respectively?
Ans. 14.2066, 11.8956, 10.2447, and 9.0064 yrs.

4. Find in what time, at compound interest, reckoning 5 per cent. per annum, $10 will amount to $100. *Ans.* 47.14 yrs.

5. If P$, at compound interest, amount to M$ in t years, what sum must be paid down to receive P$ at the end of t years?
Ans. $\frac{\$P^2}{M}$.

6. Three children, A, B, C, who come of age at the end of a, b, c, years, are to have a sum of money $P divided among them, so that their shares being placed at compound interest, each shall receive at coming of age the same sum. Find the share of A, the youngest. *Ans.* $\frac{P}{1+R^{a-b}+R^{a-c}}$.

7. To what sum will an annuity of $120 for 20 years amount to at 6 per cent. per annum? *Ans.* $4414.27.

8. What is the present worth of an annuity of \$250, payable yearly for 30 years, at 5 per cent. per annum?

Ans. \$3843.1135.

9. What is the present value of an annuity of \$112.50, to commence at the end of 10 years, and to continue 20 years, at 4 per cent.? *Ans.* \$1032.877.

10. A debt of a\$, accumulating at compound interest, is discharged in n years, by equal annual payments of b\$; find the value of n. *Ans.* $n=\frac{\log. b-\log. (b-ra)}{\log. (1+r)}$.

CHAPTER XII.

GENERAL THEORY OF EQUATIONS.

ART. **393.** An equation is the statement of equality between two algebraic expressions. Equations are of different degrees.

From what has been already shown (Art. 113), it is obvious that

$ax+b=0$, is an equation of the 1st degree.

$x^2+bx+c=0$, is an equation of the 2nd degree.

$x^3+bx^2+cx+d=0$, is an equation of the 3rd degree; and in general,

$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+Tx+V=0$,

is an equation of the n^{th} degree. The coëfficients, A, B, C, &c., may be positive or negative, integral or fractional; and either of them may be equal to zero. The coëfficient of the highest power of x is represented by unity, because if it is not unity, the equation may be reduced to this form by dividing by such coëfficient.

ART. **394.** The root of an equation is such a number, or quantity, that being substituted for the unknown quantity, the equation will be verified. Thus, in the cubic equation $x^3+2x^2-14x-3=0$, the root is 3, because when this number is substituted for x, the first member becomes equal to the second.

Every equation must have at least one root, for if there is *no quantity whatever* that will satisfy the equation when substituted for the unknown quantity, then is the equation itself not true.

A *function* of a quantity is any expression dependent on that quantity. Thus, $2x+3$ is a function of x,

$5x^2$, is a function of x,

$7x-3y^2$, is a function of x and y.

In a series, when the signs of two successive terms are *alike*, they constitute a *permanence*, when they are *unlike*, a *variation*. Thus, in the polynomial,

$$-r-s+t+u,$$

the signs of the first and second terms constitute a permanence, of the second and third a variation, and of the third and fourth a permanence.

ART. **395.** PROP. I.—*If a is a root of any equation,*

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+Tx+V=0, \ (n),$$

then will the equation be divisible by $x-a$.

For if a is one value of x, the equation will be verified when a is substituted for x. This gives

$$a^n+Aa^{n-1}+Ba^{n-2}+Ca^{n-3}+\ldots\ldots+Ta+V=0;$$

$$\text{or, } V=-a^n-Aa^{n-1}-Ba^{n-2}-Ca^{n-3}-\ldots\ldots-Ta.$$

Substituting this value of V in the given equation, and arranging the terms according to the same powers of x and a, we have

$$(x^n-a^n)+A(x^{n-1}-a^{n-1})+B(x^{n-2}-a^{n-2})+\ldots\ldots+T(x-a)=0.$$

Now, (Art. 83), each of the expressions (x^n-a^n), $(x^{n-1}-a^{n-1})$, &c., is divisible by $x-a$, therefore the given equation is divisible by $x-a$.

Cor. Conversely, if the equation

$x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+Tx+V=0$, (n) is divisible by $x-a$, then a is a root of the equation.

For if the equation (n) is divisible by $x-a$, if we call the quotient Q, we have $\quad (x-a)Q=0 \ (n),$

which may be satisfied by making $x-a=0$, whence $x=a$.

D'ALEMBERT'S PROOF OF PROP. I.—If said division leave a remainder, let it be called R, and the quotient Q; then the equation (n) becomes

$$(x-a)Q+R=0.$$

But $x-a=0$, $\therefore$ R$=0$; that is, there is no remainder on dividing equation (n) by $x-a$.

ILLUSTRATION 1. In the equation $x^3-9x^2+26x-24=0$, the roots are 2, 3, and 4; and the equation is divisible by $x-2$ $x-3$, and $x-4$.

2. In the equation $x^3+x^2-14x-24=0$, the roots are -2, -3, and 4; and the equation is divisible by $x+2$, $x+3$, and $x-4$.

Art. 396. Prop. II.—*Every equation containing but one unknown quantity, has as many roots as there are units in the number denoting its degree; that is, an equation of the* n*th* *degree has* n *roots.*

Let a be a root of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+Tx+V=0 \quad (n)$$

By Art. 395 this equation is divisible by $x-a$. If we perform the division, and denote by A_1, B_1, &c., the coëfficients of the powers of x in the quotient after the highest, equation (n) becomes

$$(x-a)(x^{n-1}+A_1x^{n-2}+B_1x^{n-3}+\ldots\ldots+T_1x+V_1)=0.$$

This equation will be satisfied by making

$$x^{n-1}+A_1x^{n-2}+B_1x^{n-3}+\ldots\ldots+T_1x+V_1=0.$$

Now this equation must also have a root, which may be denoted by b; it is therefore (Art. 395) divisible by $x-b$, and may be placed under the form

$$(x-b)(x^{n-2}+A_2x^{n-3}+B_2x^{n-4}+\ldots\ldots+T_2x+V_2)=0.$$

This equation will be satisfied by placing the second member equal to zero, which gives another equation of a degree still lower by a unit, and as x must here also have some value, as c, this equation must be divisible by $x-c$; and if the division be performed we shall have an equation of a degree still lower by a unit.

It is evident that if this operation be continued, the exponent n will be exhausted, and the last quotient will be unity; hence, calling the last root l, we shall have

$(x-a)(x-b)(x-c)(x-d),\ldots\ldots(x-l)=0$, which is satisfied by making $x=a, b, c, d,\ldots\ldots$ or l; that is, there are n quantities, either of which, when substituted for x, will satisfy the conditions of the equation; or, in other words, the equation has n roots, a, b, c, d, &c.

Cor. 1. From this theorem it follows that if we know one root of an equation we may, by dividing (Art. 395), find the equation containing the remaining roots. Hence, when all the roots of an equation but two are known, it may be reduced to a quadratic by division, and the remaining roots found by methods already given.

Thus, one root of the equation $x^3-12x^2+47x-60=0$, is 5, and by dividing it by $x-5$, the quotient is $x^2-7x+12=0$, of which the roots are found to be $+3$ and $+4$.

Cor. 2. From the preceding, it is obvious that when any equation, whose right hand member is zero, can be separated into factors, the roots of the equation may be found by placing each of the factors equal to zero. Thus, in the equation $x^2-1=0$, by factoring we have $(x+1)(x-1)=0$, $\therefore$ $x+1=0$, and $x-1=0$, whence $x=-1$, and $x=+1$.

Again if $x^2+4x=0$, we have $x(x+4)=0$, whence $x=0$, and $x=-4$. (See Art. 254.)

EXAMPLES FOR PRACTICE.

1. One root of the equation $x^3-11x^2+23x+35=0$ is -1; find the equation containing the remaining roots.

Ans. $x^2-12x+35=0$.

2. One root of the equation $x^3-9x^2+26x-24=0$ is 3; find the remaining roots. *Ans*. 2 and 4.

3. One root of the equation $x^3-7x+6=0$ is 2; find the remaining roots. *Ans*. 1 and -3.

4. Two roots of the equation $x^4+2x^3-41x^2-42x+360=0$, are 3 and -4; required the remaining roots. *Ans*. 5 and -6.

5. Two roots of the equation $x^4-3x^3-5x^2+9x-2=0$, are $+1$, and -2; find the remaining roots.

Ans. $2+\sqrt{3}$, and $2-\sqrt{3}$.

REMARKS. 1. When it is stated, for example, that $x=4$ and $x=3$, in the same equation, it is not to be understood that x is equal to 4 and 3 at the same time, but that x is equal either to 4 or 3.

2. This proposition proves that an equation of the nth degree is composed of n binomial factors, but these are not *necessarily unequal*. Two or more of them may be equal to each other. Thus, the equation $x^3-6x^2+12x-8=0$, is the same as $(x-2)(x-2)(x-2)=0$, or $(x-2)^3=0$, from which, by placing each factor equal to zero, we find the three roots to be $x=2$, $x=2$, and $x=2$.

ART. **397.** PROP. III.— *No equation can have a greater number of roots than there are units in the number denoting its degree, that is an equation of the* n*th degree can have only* n *roots.*

If it be possible let the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ .\ .\ .\ .\ +Tx+V=0.$$

Besides the n roots a, b, c, d, &c., have another root, r, not identical with either of the roots a, b, c, d, &c.; then since r is a root of the equation it must be divisible by $x-r$ (Art. 395); this gives

$$x^n+Ax^{n-1}+Bx^{n-2}+, \&c., =(x-r)(x^{n-1}+A'x^{n-2}+, \&c.,)$$

or $(x-a)(x-b)(x-c) \ldots (x-l)=(x-r)(x^{n-1}+A'x^{n-2}+, \&c.)$

But since r is a value of x, we have, by substitution,

$$(r-a)(r-b)(r-c) \ldots (r-l)=(r-r)(x^{n-1}+A'x^{n-2}+, \&c.)$$

Now the second member of this equation is $=0$, because $(r-r)=0$; but the other side cannot be 0, since r is not equal to any of the quantities a, b, c, &c.; hence the supposition is absurd that x can have any value other than $a, b, c, d, \ldots l$.

Art. 398. Prop. IV.— *To discover the relations between the coëfficients of an equation, and its roots.*

$$\text{Let } \left.\begin{array}{l} x=a, \\ x=b, \\ x=c, \\ x=d, \text{ \&c.} \end{array}\right\} \quad \text{Then, } \left\{\begin{array}{l} x-a=0, \\ x-b=0, \\ x-c=0, \\ x-d=0, \text{ \&c.} \end{array}\right.$$

By multiplying together the corresponding terms of the last set of equations, we have $(x-a)(x-b)(x-c)(x-d)=0\times0\times0\times0=0$.

If we perform the actual multiplication of the factors, we find

$$\left.\begin{array}{l|l|l|l} x^4-a & x^3+ab & x^2-abc & x+abcd \\ \quad\;\, -b & \quad\;\, +ac & \quad\;\, -abd & \\ \quad\;\, -c & \quad\;\, +ad & \quad\;\, -acd & \\ \quad\;\, -d & \quad\;\, +bc & \quad\;\, -bcd & \\ & \quad\;\, +bd & & \\ & \quad\;\, +cd & & \end{array}\right\}=0.$$

Similarly, in the equation of the n^{th} degree,

$$x^n+Ax^{n-1}+Bx^{n-2}+, \&c., =(x-a)(x-b)(x-c) \ldots (x-l)=0.$$

If we perform the multiplication of the n factors, we shall have

$$\begin{array}{c} -a-b-c \ldots -k-l=A; \\ ab+ac+ \ldots +kl=B; \\ -abc-abd \ldots akl=C; \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \pm abcd \ldots kl=V. \end{array}$$

The double sign is placed before the last term, because the product $-a\times-b\times-c \ldots \times-l$, will be *plus* or *minus*, according as the degree of the equation is *even* or *odd*. Hence,

1. *The coëfficient of the second term of any equation, is equal to the sum of all the roots, with their signs changed.*

2. *The coëfficient of the third term is equal to the sum of the products of all the roots taken two and two.*

3. *The coëfficient of the fourth term is equal to the sum of the products of all the roots taken three and three, with their signs changed.*

And so on, and

4. *The last, or absolute term, is equal to the product of all the roots.*

Cor. 1. *If any term of an equation is wanting it is because its coëfficient is* 0.

2. *If the* 2nd *term of any equation is wanting, the sum of the roots is equal to* 0.

3. *If the* 3rd *term of any equation is wanting, the sum of the products of the roots, taken two and two in a product, is equal to* 0.

4. *If the absolute term is wanting, the product of the roots must be* 0, *and hence one of the roots must be* 0.

5. *Since the last term is the product of all the roots, therefore it must be divisible by each of them; that is, every rational root of an equation is a divisor of the last term.*

EXAMPLES ILLUSTRATING THE PRECEDING PRINCIPLES.

1. Form the equation whose roots are 3, 4, and -5.

The equations $x=3$, $x=4$, and $x=-5$, give $x-3=0$, $x-4=0$, and $x+5=0$;

hence, $(x-3)(x-4)(x+5)=x^3-2x^2-23x+60=0$.

Here $3+4-5=+2$, the coëfficient of the 2nd term with a contrary sign.

$3\times4+3\times-5+4\times-5=-23$, the coëfficient of the 3rd term.

$3\times4\times-5=-60$, the last term, with the minus sign, because the degree of the equation is odd.

2. What is the equation whose roots are 2, 3, and -5? (See Cor. 2.) *Ans.* $x^3-19x+30=0$.

3. Find the equation whose roots are 3, -2, and 7. *Ans.* $x^3-8x^2+x+42=0$.

4. Form the equation with roots 0, -1, 2, and -5. *Ans.* $x^4+4x^3-7x^2-10x=0$.

5. Form the equation whose roots are -2, $+4$, and $+4$ (See Cor. 3.) *Ans.* $x^3-6x^2+32=0$.

6. Find the equation whose roots are $1+\sqrt{3}$, and $1-\sqrt{3}$.
Ans. $x^2-2x-2=0$.

7. Find the equation whose roots are $1\pm\sqrt{2}$ and $2\pm\sqrt{3}$.
Ans. $x^4-6x^3+8x^2+2x-1=0$.

8. What is the 4^{th} term of the equation whose roots are -2, $-1, 1, 3, 4$? *Ans.* $29x^2$.

9. Find the middle term of the equation whose roots are $5, 3, 1, -1, -2, -4$. *Ans.* $28x^3$.

10. Form an equation of the 4^{th} degree, when two of the roots are $-\sqrt{2}$, and $+\sqrt{-3}$. *Ans.* $x^4+x^2-6=0$.

ART. **399**. PROP. V.— *No equation having unity for the coëfficient of the first term, and all the other coëfficients integers, can have a root equal to a rational fraction.*

Take the general equation of the n^{th} degree, and suppose all its coëfficients integers,

$$x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+Tx+V=0.$$

If possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of this equation; then by substituting it for x we have

$$\frac{a^n}{b^n}+A\frac{a^{n-1}}{b^{n-1}}+B\frac{a^{n-2}}{b^{n-2}}+\ldots\ldots+\frac{Ta}{b}+V=0.$$

Reducing all the terms to a common denominator,

$$\frac{a^n}{b^n}+\frac{Aa^{n-1}b}{b^n}+B\frac{a^{n-2}b^2}{b^n}+\ldots\ldots+\frac{Tab^{n-1}}{b^n}+\frac{Vb^n}{b^n}=0.$$

Transposing all the terms to the second member, except the first, and omitting the common denominator,

$$a^n=-Aa^{n-1}b-Ba^{n-2}b^2-\ldots\ldots-Tab^{n-1}-Vb^n.$$

Dividing both members by b,

$$\frac{a^n}{b}=-Aa^{n-1}-Ba^{n-2}b-\ldots\ldots-Tab^{n-2}-Vb^{n-1}.$$

But, by hypothesis, a and b contain no common factor, therefore $\frac{a^n}{b}$ is an irreducible fraction, and the right member is a series of integral quantities; therefore, an irreducible fraction is equal to a series of integers, which is *absurd*. Hence, the supposition which leads to this conclusion is absurd, namely that the equation has a fractional root.

REMARK.— This proposition only proves that in an equation of the form described, the real roots must be integers, otherwise they cannot be *exactly* expressed in numbers. It often happens that the roots of an equation can be expressed *approximately* by fractions. Thus, in the equation $x^3-3x^2-5x+10=0$, one of the roots is -2, and the other two are expressed *nearly* by 1.382, and 3.618.

When a real root cannot be expressed *exactly* in numbers it is termed *incommensurable.*

ART. **400.** PROP. VI.—*If the signs of the alternate terms of an equation be changed, the signs of all the roots will be changed.*

Let a be a root of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+V=0, \qquad (1)$$

then $$a^n+Aa^{n-1}+Ba^{n-2}+Ca^{n-3}+\ldots\ldots+V=0, \qquad (2)$$

By changing the signs of the alternate terms of equation (1) it becomes

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+\ldots\ldots\pm V=0. \qquad (3)$$

By substituting $-a$ for x in this equation, we have

$$a^n-Aa^{n-1}+Ba^{n-2}-Ca^{n-3} \qquad \pm V=0. \qquad (4)$$

Now if n be *even*, the 2nd, 4th, &c., terms will contain odd powers of a, which will be negative (Art. 193), and the signs of the terms being negative, the results of each term will be positive; hence, the whole result will be the same as that produced by the substitution of a for x in equation (1).

But if n be *odd*, the odd powers of a will be negative, and the even powers positive; and the signs of the same terms being negative, these terms will be negative, which will render all the terms of (4) negative.

But this result is the same as that which would be produced by multiplying all the terms of (2) by -1. Hence, if a is a root of equation (1), $-a$ is a root of (3), whether n be odd or even.

REMARK.— If the signs of *all* the terms be changed, the signs of the roots will remain unchanged, because this is the same as multiplying both members by -1. (Art. 148.)

Ex. 1. The roots of the equation $x^2+2x-24=0$, are 4 and -6; what are the roots of the equation $x^2-2x-24=0$?

Ans. -4 and 6.

2. The roots of the equation $x^3-3x^2-10x+24=0$, are 2, -3, and 4; what are the roots of the equation $x^3+3x^2-10x-24=0$.

Art. 401. Prop. VII.— *When the coëfficients of an equation are real, if it contains imaginary roots, the number of these roots must be even.*

If $a+b\sqrt{-1}$ be a root of the equation

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+, \&c., =0;$$

then $a-b\sqrt{-1}$ is also a root.

In the equation substitute $a+b\sqrt{-1}$ for x, and the result will consist of two parts: 1st, possible quantities which involve the odd and even powers of a, and the *even* powers of $b\sqrt{-1}$; and 2nd, impossible quantities which involve the *odd* powers of $b\sqrt{-1}$; call the sum of the possible quantities P, and of the impossible $Q\sqrt{-1}$, then $P+Q\sqrt{-1}$ is the whole result; hence,

$$P+Q\sqrt{-1}=0.$$

But the first quantity being real, and the second imaginary, in order to satisfy the equation, each of the quantities must be 0; this gives $P=0$, and $Q\sqrt{-1}=0$.

Again, let $a-b\sqrt{-1}$ be substituted for x, and the 1st part of the result will be the same as before, and the 2nd part, which arises from the odd powers of $b\sqrt{-1}$, will differ from the former imaginary part only in its sign; therefore, the result will be $P-Q\sqrt{-1}$; but since $P=0$, and $Q\sqrt{-1}=0$, we must have

$$P-Q\sqrt{-1}=0.$$

Hence $a-b\sqrt{-1}$ is a root of the equation, since its substitution for x gives a result equal to 0.

Cor 1.— If for $b\sqrt{-1}$ we put $\sqrt{b}$, it is evident that in the result we must put $\sqrt{Q}$ instead of $Q\sqrt{-1}$, so that $P+\sqrt{Q}=0$ and therefore $P-\sqrt{Q}=0$; hence, surd roots of the form $a\pm\sqrt{b}$, enter an equation by pairs.

Cor. 2.— In the same manner it may be proved that roots of the form $\pm b\sqrt{-1}$, or $\pm\sqrt{b}$ enter equations by pairs, for in both cases we have only to make $a=0$.

Cor. 3.— Since irrational and imaginary roots always occur in pairs where the coëfficients are real, it follows that every equation of an odd degree must have at least *one* real root.

Cor. 4.— Corresponding to any pair of imaginary roots $a\pm b\sqrt{-1}$, we shall have in the equation, the quadratic factor

$$\{x-(a+b\sqrt{-1})\}\{x+(a-b\sqrt{-1})\}=(x-a)^2+b^2;$$

therefore every equation of an *even* order, with real coëfficients, is composed of *real* factors of the second degree.

Ex. 1. One root of the equation $x^3-26x+60=0$ is -6; required the other roots. *Ans.* $3\pm\sqrt{-1}$.

Ex. 2. One root of the equation $x^3-15x+4=0$ is -4; required the other roots. *Ans.* $2\pm\sqrt{3}$.

Ex. 3. Two of the roots of the equation $x^4-4x^3-7x^2+26x-14=0$ are $3+\sqrt{2}$, and $3-\sqrt{2}$; required the remaining roots. *Ans.* $-1\pm\sqrt{3}$.

Ex. 4. One root of $x^3-7x^2+13x-3=0$, is $2-\sqrt{3}$; find the other roots. *Ans.* $2+\sqrt{3}$ and 3.

Ex. 5. One root of $x^4-3x^2-42x-40=0$ is $-\frac{1}{2}(3+\sqrt{-31})$; find the other roots. *Ans.* $-\frac{1}{2}(3-\sqrt{-31})$, 4, and -1.

Ex. 6. Two roots of $x^5-10x^4+29x^3-10x^2-62x+60=0$ are 3 and $\sqrt{2}$; find the other roots. *Ans.* $-\sqrt{2}$, 2, and 5.

ART. **402.** PROP. VIII.— DESCARTES' RULE OF THE SIGNS.— *No equation can have a greater number of* POSITIVE *roots than there are* VARIATIONS *of sign; nor a greater number of* NEGATIVE *roots than there are* PERMANENCES *of sign.*

In the equation $x-a=0$, where the value of x is $+a$, there is *one variation*, and *one positive root.*

In the equation $x+a=0$, where the value of x is $-a$, there is *one permanence*, and *one negative root.*

In the equation $x^2-(a+b)x+ab=0$, where the values of x are $+a$ and $+b$, there are *two variations* and *two positive* roots.

In the equation $x^2+(a+b)x+ab=0$, where the values of x are $-a$, and $-b$, there are *two permanences*, and *two negative* roots.

In the equation $x^2-x-12=0$, where $x=+4$, and -3, there is *one variation*, and *one positive* root, and *one permanence*, and *one negative* root.

If we form an equation of the third degree, (Art. 397), whose roots are $+2, +3, +4$, we shall have $x^3-9x^2+26x-24=0$, where there are *three variations*, and *three positive* roots.

But if we form an equation whose roots are $-2, -3, +4$, we shall have $x^3+x^2-14x-24=0$, where there is *one variation*, and *one positive* root, and *two permanences*, and *two negative* roots.

To prove the proposition generally, let the signs of the terms in their order, in any *complete* equation, be

$+ + - + - + + +$, and let a new factor $x-a=0$, corresponding to a new positive root be introduced, the signs in the partial and final products will be

$$\begin{array}{l} + + - + - + + + \\ + - \\ \hline + + - + - + + + \\ \quad - - + - + - - - \\ \hline + \pm - + - + \pm \pm - . \end{array}$$

Now in this product, it is obvious, that *each permanence is changed into an ambiguity ;* hence, the permanences, take the ambiguous sign as you will, are not *increased* in the final product by the introduction of the positive root $+a$; but the number of signs is increased by *one*, and therefore the number of variations must be increased by *one.* Hence, the introduction of any positive root introduces, at least, one additional variation of sign.

Now the equation $x-a=0$, contains one positive root, and has one variation of sign. Therefore, since every additional positive root introduces, at least, one additional variation of sign, *the number of positive roots can never exceed the number of variations of sign.*

Again, if we change the signs of the alternate terms, the roots will be changed from positive to negative, and conversely (Art. 400). Hence, the permanences in the proposed equation will be replaced by variations in the changed equation, and the variations in the former by permanences in the latter ; and since the changed equation cannot have a greater number of positive roots than there are variations of sign, *the proposed equation cannot have a greater number of negative roots than there are permanences of sign.*

Cor. 1. Since the whole number of variations and permanences is evidently equal to the degree of the equation, (the equation if not complete being rendered so by the introduction of ciphers) Therefore, if the roots of an equation be *all real,* the number of *positive* roots must be *equal* to the number of *variations,* and the number of *negative* roots to the number of *permanences.* (See examples, pages 343, 345.)

2. By means of this theorem we can often determine whether there are imaginary roots in an equation.

For example, the equation

$$x^2+16=0,$$

may be written $$x^2\pm 0x+16=0.$$

Now, if we take the upper sign there are no variations, hence there is no positive root; and if we take the lower sign there are no permanences, hence there is no negative root. But since the equation has two roots (Art. 396), they must, therefore, both be imaginary.

In like manner the cubic equation

$$x^3+Bx+C=0,$$

may be written $$x^3\pm 0x^2+Bx+C=0.$$

Now if we take the upper sign there are no variations, and consequently no positive root. But if we take the lower sign, there is one permanence, hence there can be but one negative root. Therefore, the other two roots must be imaginary.

ART. **403**. PROP. IX.— *If two numbers, when substituted for the unknown quantity in an equation, give results affected with different signs, one root at least of this equation lies between these numbers.*

Let the equation, for example, be

$$x^3-x^2+x-8=0.$$

If we substitute 2 for x in this equation, the result is -2; and if we substitute 3 for x, the result is $+13$. These results have different signs, and it is required to show that there must be one real root, at least, between 2 and 3.

The equation may evidently be written thus,

$$(x^3+x)-(x^2+8)=0.$$

Now in substituting 2 for x, $x^3+x=10$, and $x^2+8=12$,

$$\therefore\ x^3+x<x^2+8;$$

also, in substituting 3 for x, $x^3+x=30$, and $x^2+8=17$,

$$\therefore\ x^3+x>x^2+8.$$

Now both these quantities increase while x increases, but the first increases more rapidly than the second, since when $x=2$, it is *less* than the second, but when $x=3$ it is *greater*. Consequently, for some value of x between 2 and 3, we must have $x^3+x=x^2+8$, and this value of x is, therefore, a real root of the equation.

In general, suppose we have an equation X=0, where X represents a polynomial involving x, and that two numbers, p and q, when substituted for x, give results with contrary signs. Let P be the sum of the positive, and N the sum of the negative terms;

also suppose that when $x=p$, P—N is negative, or $P<N$, and that when $x=q$, P—N is positive, or $P>N$.

Suppose x to change by imperceptible degrees from p to q, then P and N must also change by imperceptible degrees, and both increase, but P must increase faster than N, otherwise from having been less it could never become greater; there must, therefore, be some value of x between p and q, which renders $P=N$, or satisfies the equation $X=0$, and this value of x is, therefore, a real root of the equation.

Cor. If the difference of the two numbers, p and q, which give results with contrary signs, is equal to *unity*, it is evident that we have found the *integral* part of one of the roots.

Ex. 1. Find the integral part of one value of x in the equation

$$x^4-4x^3+3x^2+x-5=0.$$

If $x=3$, the value of the equation is -2, but if $x=4$, the value is 47. Hence, a root lies between 3 and 4; that is, 3 is the first figure of one of the roots.

2. Required the first figure of one of the roots of the equation $x^3-5x^2-x+1=0$. *Ans.* 5.

TRANSFORMATION OF EQUATIONS.

ART. **404.** The transformation of an equation is the changing it into another of the same degree, whose roots shall have a specified relation to the roots of the given equation.

Thus, in the general equation of the n^{th} degree

$$x^n+Ax^{n-1}+Bx^{n-2}. \quad . \quad . \quad . \quad +Tx+V=0; \qquad (1)$$

if $-y$ be substituted for x, the equation will be transformed into another whose roots are the same as those in (1), but with contrary signs, for $y=-x$.

If $\frac{1}{y}$ be substituted for x, the roots of the new equation in y will be the reciprocals of those of equation (1), for $y=\frac{1}{x}$.

ART. **405.** PROP. I.— *To transform an equation into one whose roots are the roots of the given equation multiplied or divided by any given quantity.*

Let a, b, c, &c., be the roots of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}. \quad . \quad . \quad . \quad +Tx+V=0;$$

assume $y=kx$, or $x=\frac{y}{k}$; then substitute this value for x, and the proposed equation becomes

$$\frac{y^n}{k^n}+A\frac{y^{n-1}}{k^{n-1}}+B\frac{y^{n-2}}{k^{n-2}}\cdot\cdot\cdot\cdot+\frac{Ty}{k}+V=0;$$

then, multiplying by k^n, we have

$$y^n+Aky^{n-1}+Bk^2y^{n-2}.\ .\ .\ .\ +Tk^{n-1}y+k^nV=0.$$

Since $y=kx$, the roots of this equation are ka, kb, kc, &c.

It is evident this equation may be derived from the proposed equation, by multiplying the successive terms by $1, k, k^2, k^3$, &c., and changing x into y.

In the case of division, assume $y=\frac{x}{k}$, or $x=ky$, and substitute.

Cor. By this transformation an equation may be cleared of fractions, or if the first term be affected with a coëfficient, that coëfficient may be removed.

Ex. Let it be required to transform the equation

$$x^3+\tfrac{1}{2}px^2+\tfrac{1}{3}qx+r=0,$$

into one which is clear of fractions, and which has unity for the coëfficient of the highest term.

By multiplying by 6, we have

$$6x^3+3px^2+2qx+6r=0.$$

Let $y=6x$, or $x=\frac{1}{6}y$, and the equation becomes

$$6\frac{y^3}{6^3}+3p\frac{y^2}{6^2}+2q\frac{y}{6}+6r=0;$$

and multiplying by 6^2, we have

$$y^3+3py^2+12qy+216r=0,$$

an equation of the required form.

Ex. 1. Find the equation whose roots are those of the equation $x^4+7x^2-4x+3=0$ multiplied by 3.

Ans. $y^4+63y^2-108y+243=0$.

2. Find the equation whose roots are each 5 times those of the equation $x^4+2x^3-7x-1=0$.

Ans. $y^4+10y^3-875y-625=0$.

3. What is the equation whose roots are each $\frac{1}{2}$ of those of $x^3-3x^2+4x+10=0$? *Ans.* $4y^3-6y^2+4y+5=0$.

4. What is the equation whose roots are each $\frac{1}{3}$ of those of $x^3+18x^2+99x+81=0$ taken negatively?

Ans. $y^3-6y^2+11y-3=0$.

5. Transform the equation $x^3-2x^2+\frac{1}{3}x-10=0$, into one having integral coëfficients. *Ans.* $y^3-6y^2+3y-270=0$.

ART. **406.** PROP. II.—*To transform an equation into one whose roots are greater or less by any given quantity than the corresponding roots of the proposed equation.*

Let the proposed equation be

$$x^n+Ax^{n-1}+Bx^{n-2}. \quad . \quad . \quad . +Tx+V=0,$$

whose roots are a, b, c, &c.

The relation between x and y will be expressed by the equation $y=x\pm r$. As the principle is the same, whether x is increased or diminished, we will consider the case where $y=x-r$. This gives $x=y+r$, and by substituting this for x in the proposed equation, we have

$$(y+r)^n+A(y+r)^{n-1}+B(y+r)^{n-2}. \quad . \quad . \quad . +T(y+r)+V=0.$$

Developing the different powers of $y+r$ by the Binomial theorem, and arranging the terms according to the powers of y, we have

$$\left.\begin{array}{l} y^n+\left.\begin{array}{r} nr \\ +A \end{array}\right| y^{n-1}+\left.\begin{array}{r} \frac{n(n-1)}{1\cdot 2}r^2 \\ +(n-1)Ar \\ +B \end{array}\right| y^{n-2} \quad . \quad . \quad . \quad . \quad . \quad \begin{array}{l} +r^n \\ +Ar^{n-1} \\ +Br^{n-2} \\ \vdots \\ +Tr \\ +V \end{array} \end{array}\right\}=0.$$

Now since $y=x-r$, the values of y in this equation are $a-r$, $b-r$, $c-r$, &c.

ART. **407.** *Cor.*—By means of the preceding transformation we may remove any intermediate term of an equation. Thus, to transform an equation into one which shall want the second term, r must be assumed, so that $nr+A=0$, or $r=-\frac{A}{n}$. To take away the third term, $\frac{1}{2}n(n-1)r^2+(n-1)Ar+B$ must be put $=0$, and the value of r derived from the solution of this equation.

The pupil may solve the following examples by the preceding principles:

1. Transform the equation $x^3-7x+7=0$ into another whose roots shall be less by 1 than the corresponding roots of this equation. *Ans.* $y^3+3y^2-4y+1=0$.

2. Find the equation whose roots are less by 3 than those of the equation $x^4-3x^3-15x^2+49x-12=0$.

Ans. $y^4+9y^3+12y^2-14y=0$.

3. Transform the equation $x^3-6x^2+8x-2=0$ into another whose second term shall be absent.

Here $A=-6$, $n=3$, $\therefore$ $r=2$; hence, $x=y+2$.

Ans. $y^3-4y-2=0$.

4. Transform the equation $x^2+2px-q=0$ into another wanting the second term. *Ans.* $y^2-p^2-q=0$.

ART. **408.** There is a more easy and elegant method of performing the operation of transformation, so as to increase or diminish the roots of an equation, than by direct substitution, which we will now proceed to explain.

Let the proposed equation be

$$Ax^4+Bx^3+Cx^2+Dx+E=0, \qquad (1)$$

and let it be required to transform it into another, whose roots shall be less by r; then $y=x-r$ and $x=y+r$.

By substituting $y+r$, instead of x, we have

$$A(y+r)^4+B(y+r)^3+C(y+r)^2+D(y+r)+E=0.$$

By developing the powers of $y+r$, and arranging the terms according to the powers of y, as in Art. 406, the transformed equation will take the form

$$Ay^4+B_1y^3+C_1y^2+D_1y+E_1=0. \qquad (2)$$

where the coëfficient A must evidently be the same as in equation (1), while the coëfficients B_1, C_1, D_1, and E_1, are unknown quantities, whose values are now to be determined. For y, substitute its value $x-r$, and equation (2) becomes

$$A(x-r)^4+B_1(x-r)^3+C_1(x-r)^2+D_1(x-r)+E_1=0 \quad (3)$$

Now since the values of x are the same in equations (1) and (3), it is evident these equations are identical. Hence, whatever operation is performed on equation (1), the result will be the same as if this process had been applied to equation (3). Now as the object is to obtain the values of the coëfficients B_1, C_1, &c., let equation (3) or (1) be divided by $x-r$, and it is evident that the quotient will be

$$A(x-r)^3+B_1(x-r)^2+C_1(x-r)+D_1,$$

and the remainder will be the last coëfficient E_1; hence, E_1 is determined.

Again, divide this quotient by $x-r$, and the next quotient will be

$$A(x-r)^2+B_1(x-r)+C_1 \text{ with a remainder } D_1;$$

hence, D_1 is determined. Dividing again by $x-r$ we get the remainder C_1; and lastly, by another division, we obtain the remainder B_1; and thus we find all the coëfficients of equation (2).

To illustrate this method we will now proceed to solve Ex. 1, Art. 407; that is, to find the equation whose roots are less by 1, than those of the equation $x^3-7x+7=0$.

Here $y=x-1$, and we proceed to divide the proposed equation and the successive quotients, by $x-1$. The successive remainders will be the coëfficients of y in the transformed equation, except that of the highest power, which will have the same coëfficient as the highest power of x in the proposed equation.

$$\begin{array}{l} x-1)x^3-7x+7(x^2+x-6 \quad \text{1st. quot.} \\ \quad x^3-x^2 \\ \hline \quad +x^2-7x \\ \quad x^2-x \\ \hline \quad -6x+7 \\ \quad -6x+6 \\ \hline \text{1st. rem.} =+1. \end{array} \qquad \begin{array}{l} x-1)x^2+x-6(x+2 \quad \text{2nd. quot.} \\ \quad x^2-x \\ \hline \quad +2x-6 \\ \quad 2x-2 \\ \hline \text{2nd. rem.} =-4 \\ \\ x-1)x+2(1, \quad \text{3rd quot.} \\ \quad x-1 \\ \hline \text{3rd. rem.} =+3 \end{array}$$

Here the last quotient is 1, and the successive remainders are $+3, -4$, and $+1$. Comparing these with the general equation, we have $A=1$, $B_1=+3$, $C_1=-4$, and $D_1=+1$. Placing these as coëfficients to the respective powers of y, the transformed equation is $y^3+3y^2-4y+1=0$.

This method of transforming an equation, would not, in general, be shorter than direct substitution, were it not that the successive divisions may be greatly shortened by a process, called from its discoverer, HORNER'S SYNTHETIC METHOD OF DIVISION, which we shall now proceed to explain.

ART. **409.** SYNTHETIC DIVISION.—This may be considered as an abridgment of the method of division by Detached Coëfficients (Art. 77). To explain the process, we shall first divide $5x^4-12x^3+3x^2+4x-5$ by $x-2$, by detached coëfficients.

$$\begin{array}{l}
\text{Divisor } 1-2)5-12+3+4-5(5-2-1+2 \text{ Quotient} \\
\qquad\quad 5-10 \qquad\qquad \text{or } 5x^3-2x^2-x+2. \\
\qquad\quad \overline{-2+3} \\
\qquad\quad -2+4 \\
\qquad\qquad \overline{-1+4} \\
\qquad\qquad -1+2 \\
\qquad\qquad\quad \overline{+2-5} \\
\qquad\qquad\quad 2-4 \\
\qquad\qquad\qquad \overline{-1} \text{ Rem.}
\end{array}$$

By changing the signs of the terms of the divisor, except that of the first term, which must not be changed, as by means of that we determine the signs of the respective terms of the quotient, and *adding* each partial product instead of subtracting it, except the *first* term, which being always the same as the first term of each dividend, may be *omitted*, the operation may be represented thus:

$$\begin{array}{l}
1+2)5-12+3+4-5(5-2-1+2 \\
\qquad\quad *+10 \\
\qquad\quad \overline{-2+3} \\
\qquad\qquad\quad *-4 \\
\qquad\qquad \overline{-1+4} \\
\qquad\qquad\qquad *-2 \\
\qquad\qquad\quad \overline{+2-5} \\
\qquad\qquad\qquad\quad *+4 \\
\qquad\qquad\qquad\quad \overline{-1}
\end{array}$$

Let it be observed that the figures over the stars are the coëfficients of the several terms of the quotient. It will also be seen that it is unnecessary to bring down the several terms of the dividend. Hence, the last operation may be represented as follows:

$$\begin{array}{r}
+2)5-12+3+4-5 \\
+10-4-2+4 \\
\hline
-\;2-1+2-1
\end{array}$$

In this operation 5 is the first term of the quotient, +10 is the product of +2, the divisor, by 5; the sum of +10 and —12 gives —2, the second term of the quotient, —4 is the product of +2, the divisor, by —2, the second term of the quotient, and the sum of —4 and +3 gives —1, the third term of the quotient, and so on. The last term, —1, is the remainder.

Supplying the powers of x, the quotient is $5x^3-2x^2-x+2$, with a remainder —1.

A similar method may be used when the divisor contains three terms; and if the coëfficient of the first term of the divisor is not unity, it may be made unity by dividing both dividend and divisor by the coëfficient of the first term of the divisor. The method, however, is rarely used except when the divisor is a binomial, the coëfficient of whose first term is 1.

In the application of Synthetic division, when any term of an equation is absent, its place must be supplied with a zero.

ART. **410.** We shall now illustrate the use of Synthetic division in the transformation of equations, by the method described in Art. 408.

1. Let it be required to find the equation whose roots are less by 1 than those of the equation x^3-7x+7.

To effect this transformation, it is required to find the successive remainders which arise from dividing x^3-7x+7, and the successive quotients, by $x-1$.

Since the second term is wanting, its place must be supplied with 0. Also, in arranging the operation, it is customary to place the second term of the changed divisor on the right, as in division.

OPERATION BY SYNTHETIC DIVISION.

$$\begin{array}{llllll}
1 & \pm 0 & -7 & +7 & (+1 & \\
 & +1 & +1 & -6 & & \\
\hline
 & +1 & -6 & +1 & \therefore\ +1= & 1^{st}\text{ R.} \\
 & +1 & +2 & & & \\
\hline
 & +2 & -4 & \therefore\ -4= & 2^{nd}\text{ R.} & \\
 & +1 & & & & \\
\hline
 & +3 & \therefore\ +3= & 3^{rd}\text{ R.} & &
\end{array}$$

Hence, the required coëfficients are 1, +3, —4, and +1.

$\therefore\ y^3+3y^2-4y+1=0$ is the transformed equation required.

2. Transform the equation $5x^4+28x^3+51x^2+32x-1=0$, into another having its roots greater by 2 than those of the given equation.

Here, $y=x+2$; hence, to find the coëfficients of the transformed equation, we must find the successive remainders arising from dividing the proposed equation and the successive quotients, by $x+2$. Changing the sign of $+2$, the operation is as follows:

$$\begin{array}{llllll}
5 & +28 & +51 & +32 & -1 & (-2 \\
 & -10 & -36 & -30 & -4 & \\ \cline{2-5}
 & +18 & +15 & +2 & -5 & \therefore\ -5 = 1^{st}\text{ R.} \\
 & -10 & -16 & +2 & & \\ \cline{2-4}
 & +8 & -1 & +4 & \multicolumn{2}{l}{\therefore\ +4 = 2^{nd}\text{ R.}} \\
 & -10 & +4 & & & \\ \cline{2-3}
 & -2 & +3 & \multicolumn{3}{l}{\therefore\ +3 = 3^{rd}\text{ R.}} \\
 & -10 & & & & \\ \cline{2-2}
 & -12 & \multicolumn{4}{l}{\therefore\ -12 = 4^{th}\text{ R.}}
\end{array}$$

Comparing this with the general equation (Art. 408,) we find $A=5$, $B_1=-12$, $C_1=+3$, $D_1=+4$, and $E_1=-5$.

$\therefore\ 5y^4-12y^3+3y^2+4y-5=0$ is the transformed equation required.

3. Find the equation whose roots are less by 1.7 than those of the equation $x^3-2x^2+3x-4=0$.

If we transform this equation into another whose roots are less by 1, the resulting equation is $y^3+y^2+2y-2=0$. We may then transform this into another whose roots are less by .7, and the result will be the equation required, or, the whole operation may be performed at once as follows:

$$\begin{array}{lllll}
1 & -2 & +3 & -4 & (+1.7 \\
 & +1.7 & -.51 & +4.233 & \\ \cline{2-4}
 & -.3 & +2.49 & +.233 & \therefore\ +.233 = 1^{st}\text{ R.} \\
 & +1.7 & +2.38 & & \\ \cline{2-3}
 & +1.4 & +4.87 & \multicolumn{2}{l}{\therefore\ +4.87 = 2^{nd}\text{ R.}} \\
 & +1.7 & & & \\ \cline{2-2}
 & +3.1 & \multicolumn{3}{l}{\therefore\ 3.1 = 3^{rd}\text{ R.}}
\end{array}$$

Hence, the required equation is $y^3+3.1y^2+4.87y+.233=0$.

It is generally preferable to perform the operation by successive transformations, using only one figure at a time, as there is less liability to error. It is not necessary, however, after each transformation to arrange the coëfficients in a horizontal line.

Thus, the two transformations necessary in the preceding example may be combined as follows:

1	−2	+3	−4	(1.7
	+1	−1	+2	
	−1	+2	−2.*	
	+1	0	+2.233	
	0	+2.*	.233*	
	+1	+1.19		
	+1*	3.19		
	.7	1.68		
	+1.7	4.87*		
	.7			
	+2.4			
	.7			
	3.1*			

The stars indicate the coëfficients after each transformation

EXAMPLES FOR PRACTICE.

4. Find the equation whose roots are each less by 3 than the roots of $x^3-27x-36=0$. *Ans.* $y^3+9y^2-90=0$.

5. Find the equation whose roots are each less by 3 than the roots of $x^4-27x^2-14x+120=0$.
Ans. $y^4+12y^3+27y^2-68y-84=0$.

6. Required the equation whose roots are less by 5 than those of the equation $x^4-18x^3-32x^2+17x+9=0$.
Ans. $y^4+2y^3-152y^2-1153y-2331=0$.

7. Required the equation whose roots are less by 1.2 than those of the equation $x^5-6x^4+7.4x^3+7.92x^2-17.872x-.79232=0$. *Ans.* $y^5-7y^3+2y-8=0$.

Transform the following equations into others wanting the 2nd term. (See Art. 407.)

8. $x^3-6x^2+7x-2=0$. *Ans.* $y^3-5y-4=0$.

9. $x^3-6x^2+5=0$. *Ans.* $y^3-12y-11=0$.

10. $x^3-6x^2+12x+19=0$. *Ans.* $y^3+27=0$.

11. $3x^3+15x^2+25x-3=0$. *Ans.* $27y^3-152=0$.

Transform the following equations into others wanting the 3rd term.

12. $x^3-6x^2+9x-20=0$.
Ans. $y^3+3y^2-20=0$, or $y^3-3y^2-16=0$.

13. $x^3-4x^2+5x-2=0$. *Ans.* $y^3-y^2=0$, or $y^3+y^2-\frac{4}{27}=0$.

ART. **411.** PROP. III.— *To determine the law of Derived Polynomials.*

Let X represent the general equation of the n^{th} degree ; that is,

$$X=x^n+Ax^{n-1}+Bx^{n-2}. \quad . \quad . \quad . \quad +Tx+V=0.$$

If we substitute $x+h$ for x, and put X_1 to represent the new value of X, we have

$$X_1=(x+h)^n+A(x+h)^{n-1}+B(x+h)^{n-2}+, \text{ \&c.},$$

and if we expand the different powers of $x+h$ by the Binomial theorem, we have $X_1=$

$$\begin{array}{ll|l|l} x^n & +nx^{n-1} & h+\quad n(n-1)x^{n-2} & \dfrac{h^2}{1\cdot 2}+, \text{ \&c.} \\ +Ax^{n-1} & +(n-1)Ax^{n-2} & +(n-1)(n-2)\,Ax^{n-3} & \\ +Bx^{n-2} & +(n-2)\,Bx^{n-3} & +(n-2)(n-3)\,Bx^{n-4} & \\ +, \text{ \&c.} & & & \end{array}$$

But the first vertical column is the same as the original equation, and if we put X′, X″, X‴, &c., to represent the succeeding columns, we have

$$X\ =x^n+Ax^{n-1}+Bx^{n-2}+, \text{ \&c.},$$

$$X'=nx^{n-1}+(n-1)Ax^{n-2}+(n-2)Bx^{n-3}+, \text{ \&c.},$$

$$X''=n(n-1)x^{n-2}+(n-1)(n-2)Ax^{n-3}+, \text{ \&c.},$$

$$\text{\&c.,} \qquad\qquad \text{\&c.}$$

By substituting these in the development of X_1, we have

$$X_1=X+X'h+\frac{X''}{1\cdot 2}h^2+\frac{X'''}{1\cdot 2\cdot 3}h^3+, \text{ \&c.}$$

The expressions X′, X″, X‴, &c., are called *derived polynomials* of X, or *derived functions* of X. X′ is called the *first derived polynomial* of X, or *first derived function* of X ; X″ is called the *second*, X‴ the *third*, and so on.

It is easily seen that X′ may be derived from X, *by multiplying each term by the exponent of x in that term, and diminishing the exponent by unity.* And each succeeding polynomial may be derived from that which precedes it by the same law.

ART. **412.** *Cor.* If we transpose X we have $X_1-X=X'h+\frac{X''}{1\cdot 2}h^2+$, &c. Now it is evident that h may be taken so small that *the sign of the sum* $X'h+\frac{X''}{1\cdot 2}h^2+$, &c., *will be the same as the sign of the first term* $X'h$. For, since

$$X'h+\tfrac{1}{2}X''h^2+,\ \&c.,=h(X'+\tfrac{1}{2}X'\,h+,\ \&c.),$$

if h be taken so small, that $\frac{1}{2}X''h+\frac{1}{6}X'''h^2+$, &c., becomes less than X' (their magnitudes alone being considered), the sign of the sum of these two expressions must be the same as the sign of the greater X'.

ART. **413**. By comparing the transformed equation in Art. 406, with the development of X_1 in Art. 411, it is easily seen that X_1 may be considered the transformed equation, y corresponding to x, and r to h. Hence, the transformed equation may be obtained by substituting the values of X, X', &c., in the development of X_1. As an example, let it be required to find the equation whose roots are less by 1 than those of the equation $x^3-7x+7=0$.

$$\text{Here}\quad \begin{aligned} X &= x^3-7x+7,\\ X' &= 3x^2-7,\\ X'' &= 6x,\\ X''' &= 6,\\ X^{iv} &= 0. \end{aligned}$$

Observing that $h=1$, and substituting these values in the equation $X_1=X+X'h+\frac{X''}{1\cdot 2}h^2+\frac{X'''}{1\cdot 2\cdot 3}h^3+$, &c., we have

$$X_1=(x^3-7x+7)+(3x^2-7)1+(6x)\frac{1}{1\cdot 2}+\frac{6}{1\cdot 2\cdot 3}.$$

$=x^3+3x^2-4x+1$, in which the value of x is equal to that of x in the given equation diminished by 1.

By this method the learner may solve the examples in Art. 410.

EQUAL ROOTS.

ART. **414**. *To determine the equal roots of an equation.*

We have already seen (Art. 396, Rem. 2,) that an equation may have two or more of its roots equal to each other. Thus the equation $x^3-6x^2+12x-8=0$, or $(x-2)(x-2)(x-2)=(x-2)^3=0$, has three roots, each of which is 2. We now propose to determine when an equation has equal roots, and how to find them.

If we take the equation $(x-2)^3=0$ (1)

Its first derived polynomial is $3(x-2)^2=0$.

Hence, we see that if any equation contains the same factor taken *three* times, its first derived polynomial will contain the same factor taken *twice;* this last factor is, therefore, *a common divisor* of the given equation, and its first derived polynomial.

In general, if we have an equation $X=0$, containing the factors $(x-a)^m(x-b)^n$, its first derived polynomial will contain the factors $m(x-a)^{m-1}n(x-b)^{n-1}$; that is, the *greatest common divisor* of the given equation, and its first derived polynomial will be $(x-a)^{m-1}(x-b)^{n-1}$, and the given equation will have m roots, each equal to a, and n roots, each equal to b.

Therefore, to determine whether an equation has equal roots, *find the greatest common divisor between the equation and its first derived polynomial. If there is no common divisor the equation has no equal roots.*

If the greatest common divisor contains a factor of the form $x-a$, then it has *two* roots equal to a; if it contains a factor of the form $(x-a)^2$ it has *three* roots equal to a, and so on.

If it has a factor of the form $(x-a)(x-b)$ it has two roots equal to a, and two equal to b; and so on.

Ex. 1. Given the equation $x^3-x^2-8x+12=0$, to determine whether it has equal roots, and if so, to find them.

We have for the first derived polynomial (Art. 411),

$$3x^2-2x-8.$$

The greatest common divisor of this and the given equation (Art. 108) is $x-2$.

Hence $x-2=0$, and $x=+2$.

Therefore, the equation has two roots equal to 2.

Now since the equation has *two* roots equal to 2, it must be divisible by $(x-2)(x-2)$, or $(x-2)^2$. (Art. 395.)

$$\text{Whence, } x^3-x^2-8x+12=(x-2)^2(x+3)=0,$$

$$\text{and } x+3=0, \text{ or } x=-3.$$

It is evident that when an equation contains other roots besides the equal roots, that these may be found, and the degree of the equation depressed by division (Art. 395), after which the unequal roots may be found by other methods.

The following equations have equal roots; find all the roots.

2. $x^3-2x^2-15x+36=0$. *Ans.* 3, 3, —4.

3. $x^4-9x^2+4x+12=0$. *Ans.* 2, 2, —1, —3.

4. $x^4-6x^3+12x^2-10x+3=0$. *Ans.* 1, 1, 1, 3.

5. $x^4-7x^3+9x^2+27x-54=0$. *Ans.* x=3, 3, 3, —2.

6. $x^4+2x^3-3x^2-4x+4=0$. *Ans.* —2, —2, +1, +1.

7. $x^4-12x^3+50x^2-84x+49=0$. *Ans.* $3\pm\sqrt{2}$, $3\pm\sqrt{2}$.

8. $x^5-2x^4+3x^3-7x^2+8x-3=0$.

Ans. 1, 1, 1, $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-11}$.

9. $x^6+3x^5-6x^4-6x^3+9x^2+3x-4=0$.

Ans. 1, 1, 1, —1, —1, —4.

SUGGESTION.— When the greatest common divisor of the given equation and its first derived polynomial, contains a factor of the form $(x-a)^2$, or of any higher degree than the first, it is evident that the first and second derived polynomials will also contain a common divisor, of which the first or some higher power of $x-a$ is also a factor. This principle may be sometimes used, as in the last example, to simplify the solution.

LIMITS OF THE ROOTS OF EQUATIONS.

ART. **415.** Limits to a root of an equation are any two numbers between which that root lies. A *superior limit* to the positive roots is a number numerically greater than the greatest positive root; and an *inferior limit* to the negative roots, is a number greater without regard to its sign, than the greatest negative root.

The characteristic of a superior limit is, that when it or any number greater than it, is substituted for x in the equation, the result is *positive.*

The characteristic of an inferior limit is, that its substitution for x produces a *negative* result, as likewise do all negative numbers numerically greater, provided the equation is of an odd degree.

The object of ascertaining the limits of the roots is to diminish the labor necessary in finding them.

ART. **416.** PROP. I.— *The greatest negative coëfficient, increased by unity, is greater than the greatest root of the equation.*

Take the general equation

$$x^n+Ax^{n-1}+Bx^{n-2}. \ . \ . \ . +Tx+V=0,$$

and let us suppose A to be the greatest negative coëfficient.

The reasoning will not be affected if we suppose all the coëfficients to be negative, and each equal to A.

It is required to find what number substituted for x, will make

$$x^n > A(x^{n-1}+x^{n-2}+x^{n-3}. \ . \ . \ . \ +x+1).$$

By Art. 297, the sum of the series in the parenthesis is $\frac{x^n-1}{x-1}$; hence, we must have

$$x^n > A\left(\frac{x^n-1}{x-1}\right), \text{ or } x^n > \frac{Ax^n}{x-1}-\frac{A}{x-1}.$$

But if $x^n=\frac{Ax^n}{x-1}$, we find $x=A+1$; therefore, $A+1$ substituted for x will render $x^n=\frac{Ax^n}{x-1}$, and consequently

$$x^n > \frac{Ax^n}{x-1}-\frac{A}{x-1}.$$

It is evident that by considering all the coëfficients after the first negative, we have taken the most unfavorable case; if either of them, as B, were positive, the sum of the terms in the parenthesis would be *less* than $\frac{x^n-1}{x-1}$.

ART. 417. PROP. II.—*If we increase by unity that root of the greatest negative coëfficient, whose index is equal to the number of terms preceding the first negative term, the result will be greater than the greatest positive root of the equation.*

Let Cx^{n-r} be the first negative term, C being the greatest negative coëfficient, then any value of x which makes

$$x^n > C(x^{n-r}+x^{n-r-1}. \ . \ . \ . \ +x+1) \qquad (1)$$

will evidently render the first member of the equation >0, or positive; because this supposes all the coëfficients after C negative, and each equal to the greatest, which is evidently the most unfavorable case.

By Art. 297, the series in the parenthesis is equal to $\frac{x^{n-r+1}-1}{x-1}$; hence, by substitution, the inequality (1) becomes

$$x^n > C\left(\frac{x^{n-r+1}-1}{x-1}\right), \text{ or } x^n > \frac{Cx^{n-r+1}}{x-1}-\frac{C}{x-1}.$$

But this inequality will be true if

$$x^n=\frac{Cx^{n-r+1}}{x-1}, \text{ or } > \frac{Cx^{n-r+1}}{x-1};$$

or, by multiplying both members by $x-1$, and dividing by x^{n-r+1}, when $(x-1)x^{r-r}=C$, or $>C$ (2).

But $x-1$ is $<x$, and $\therefore$ $(x-1)^{r-1}<x^{r-1}$;

therefore, (2) will be true, if we have

$$(x-1)(x-1)^{r-1}, \text{ or } (x-1)^r=C, \text{ or } >C;$$

$$\text{or } x-1=\sqrt[r]{C}, \text{ or } >\sqrt[r]{C};$$

$$\text{or } x=1+\sqrt[r]{C}, \text{ or } >1+\sqrt[r]{C}.$$

Find superior limits of the roots of the following equations:

1. $x^4-5x^3+37x^2-3x+39=0$.

Here $C=5$, and $r=1$ $\therefore$ $1+\sqrt[r]{C}=1+5^{\frac{1}{1}}=6$. *Ans.*

2. $x^5+7x^4-12x^3-49x^2+52x-13=0$.

Here $1+\sqrt[r]{C}=1+\sqrt[2]{49}=1+7=8$. *Ans.*

3. $x^4+11x^2-25x^2-67=0$.

By supposing the second term $+0x^3$, we have $r=3$; hence, the limit is $1+\sqrt[3]{67}$, or 6.

4. $3x^3-2x^2-11x+4=0$.

Dividing by 3, $x^3-\frac{2}{3}x^2-\frac{11}{3}x+\frac{4}{3}=0$.

Here the limit is $1+\frac{11}{3}$, or 5.

ART. **418.** To determine the inferior limit to the negative roots, change the signs of the alternate terms; this will change the signs of the roots (Art. 400); then the *superior* limit of the roots of this equation, by changing its sign, will be the *inferior* limit of the roots of the proposed equation.

ART. **419.** PROP. III.—*If the real roots of an equation, taken in the order of their magnitudes, be* a, b, c, d, &c., a *being greater than* b, b *greater than* c, *and so on; then if a series of numbers,* a′, b′, c′, d′, &c., *in which* a′ *is greater than* a, b′ *a number between* a *and* b, c′ *a number between* b *and* c, *and so on, be substituted for* x *in the proposed equation, the results will be alternately positive and negative.*

The first member of the proposed equation is equivalent to

$$(x-a)(x-b)(x-c)(x-d)\ldots\ldots=0.$$

Substituting for x the proposed series of numbers a', b', c', &c., we obtain the following results:

$(a'-a)(a'-b)(a'-c)(a'-d)$, &c. . $=+$ product, since *all* the factors are $+$.

$(b'-a)(b'-b)(b'-c)(b'-d)$, &c. . $=-$ product, since only **one** factor is $-$.

$(c'-a)(c'-b)(c'-c)(c'-d)$, &c. . $=+$ product, since two factors are $-$, and the rest $+$.

$(d'-a)(d'-b)(d'-c)(d'-d)$, &c. . $=-$ product, since an odd number of factors is $-$, and so on.

Cor. 1. If two numbers be successively substituted for x, in any equation, and give results with *contrary* signs, then between these numbers there must be *one, three, five,* or some *odd* number of roots.

Cor. 2. If two numbers, when substituted successively for x, give results affected with the *same* sign, then between these numbers there must be *two, four,* or some *even* number of real roots, or *no* roots at all.

Cor. 3. If a quantity q, and every quantity greater than q, render the results continually positive, q is greater than the greatest root of the equation.

Cor. 4. Hence, if the signs of the alternate terms be changed, and if p, and every quantity greater than p, renders the result positive, then $-p$ is less than the least root of the equation.

Illustration.—If we form the equation whose roots are 5, 2, and -3, the result is $x^3-4x^2-11x+30=0$. Now if we substitute any number whatever for x, greater than 5, the result is *positive.* When we substitute 5 for x, the result is zero, as it should be.

If we substitute for x, any number less than 5, and greater than 2, the result is *negative.* When we substitute 2 the result is zero.

When we substitute for x, any number less than 2, and greater than -3, the result is *positive.* When we substitute -3, the result is zero.

If we substitute for x, any number less than -3, the result is negative.

By means of Corollaries 3 and 4, it is easy to find when we have *passed all the real roots,* either in the ascending or descending scale.

STURM'S THEOREM.

ART. 420. *To find the number of real and imaginary roots of an equation.*

In 1834 M. Sturm gained the mathematical prize of the French Academy of Sciences, by the discovery of a beautiful theorem, by means of which the *number* and *situation* of all the real roots of an equation can, with certainty, be determined. This theorem we shall now proceed to explain.

Let $X = x^n + Ax^{n-1} + Bx^{n-2} \ldots + Tx + V = 0$,

be any equation of the n^{th} degree, which we may suppose contains no equal roots; for if the given equation contains equal roots these may be found (Art. 414), and the degree of the equation diminished by division.

Let the first derived function of X (Art. 411) be denoted by X_1.

Divide X by X_1 until the remainder is of a lower degree with respect to X than the divisor, and call this remainder $-X_2$; that is, let the remainder *with its sign changed*, be denoted by X_2. Divide X_1 by X_2 in the same manner, and so on, as in the margin, denoting the successive remainders with their *signs changed* by X_3, X_4, &c., until we arrive at a remainder which does not contain x, which must always happen, since the equation having no equal roots, there can be no factor containing x, common to the equation and its first derived function. Let this remainder, having its sign changed, be called X_{r+1}.

$$X_1)X \quad (Q_1$$
$$\underline{X_1Q_1}$$
$$X - X_1Q_1 = -X_2$$

$$X_2)X_1 \quad (Q_2$$
$$\underline{X_2Q_2}$$
$$X_1 - X_2Q_2 = -X_3$$

$$X_3)X_2 \quad (Q_3$$
$$\underline{X_3Q_3}$$
$$X_2 - X_3Q_3 = -X_4$$

In making these successive divisions, we may either multiply or divide the dividends and divisors by any *positive* number, for the purpose of avoiding fractions, as this will not affect the *signs* of the functions X, X_1, X_2, &c.

By this operation we obtain the series of quantities

$$X, X_1, X_2, X_3 \ldots X_{r+1} \qquad (1),$$

which, for the convenience of reference, we shall call series (1). Each member of this series is of a lower degree with respect to x than the preceding, and the last does not contain x. We shall also call X the *primitive* function, and X_1, X_2, &c., *auxiliary* functions.

The following, then, is Sturm's Theorem :

If p *and* q *be any two numbers, of which* p *is less than* q, *and if these numbers be substituted for* x *in the functions* X, X_1, X_2, *&c., in the series* (1), *we shall have two series of signs, the one resulting from the substitution of* p *for* x, *giving* k *variations of signs, and the other resulting from the substitution of* q *for* x, *giving* k′ *variations of signs; then the exact number of real roots of the given equation between the limits* p *and* q *will be* k—k′.

To simplify the demonstration of this theorem we shall employ the following Lemmas:

ART. **421.** LEMMA I.—*Two consecutive functions,* X_1, X_2, *for example, cannot both vanish for the same value of x.*

From the process by which X_1, X_2, &c., are obtained, we have the following equations:

$$X = X_1Q_1 - X_2 \quad . \; . \; . \; . \; . \; . \quad (1)$$
$$X_1 = X_2Q_2 - X_3 \quad . \; . \; . \; . \; . \; . \quad (2)$$
$$X_2 = X_3Q_3 - X_4 \quad . \; . \; . \; . \; . \; . \quad (3)$$
$$. \; . \; . \; . \; . \; . \; . \; . \; . \; .$$
$$X_{r-1} = X_rQ_r - X_{r+1} \quad . \; . \; . \; . \quad (r-1).$$

If possible let $X_1=0$, and $X_2=0$, then by eq. (2) we have $X_3=0$; hence, since $X_2=0$, and $X_3=0$, then by eq. (3) we have $X_4=0$; and proceeding in the same way we shall find $X_5=0$, $X_6=0$, and finally $X_{r+1}=0$. But this is impossible, since X_{r+1} does not contain *x*, and therefore cannot vanish for any value of *x*.

ART. **422**. LEMMA II.—*If one of the auxiliary functions vanishes for any particular value of* x, *the two adjacent functions must have contrary signs for the same value of* x.

Let us suppose that $X_3=0$, when $x=a$; then because $X_2=X_3Q_3-X_4$, and $X_3=0$, therefore $X_2=-X_4$; that is, X_2 and X_4 have contrary signs.

ART. **423.** LEMMA III.—*If any of the auxiliary functions vanishes when* x=a, *and* h *be taken so small that no root of any of the other functions in series* (1) *lies between* a—h, *and* a+h, *then will the number of variations and permanences when* a—h *and* a+h *are substituted for* x *in this series, be precisely the same.*

Suppose, for example, the substitution of *a* for *x* causes the function X_3 to vanish, then by Art. 421 neither of the functions

X_2 or X_4 can vanish for the same value of x; and since when X_3 vanishes, X_2 and X_4 have contrary signs, (Art. 422), therefore, the substitution of a for x in the three functions X_2, X_3, X_4 must give

$$\begin{array}{ccccccc} X_2, & X_3, & X_4, & \text{or} & X_2, & X_3, & X_4. \\ + & 0 & -, & & - & 0 & + \end{array}$$

And since h is taken so small that no root either of $X_2=0$, or $X_4=0$, lies between $a-h$ and $a+h$, the signs of these functions will continue the same whether we substitute $a-h$, or $a+h$ for x (Art. 419). Hence, whether we suppose X_3 to be $+$ or $-$ by the substitution of $a-h$ and $a+h$ for x, there will be *one* variation and *one* permanence. Thus we shall have either

$$\begin{array}{ccccccc} X_2, & X_3, & X_4, & \text{or} & X_2, & X_3, & X. \\ + & \pm & - & & - & \pm & + \end{array}$$

So that no alteration in the number of variations and permanences can be made in passing from $a-h$ to $a+h$.

ART. 424. LEMMA IV.—*If* a *is a root of the equation* $X=0$, *then the series of functions* X, X_1, X_2, *&c., will lose one variation of signs in passing from* $a-h$ *to* $a+h$; h *being taken so small that no root of the function* $X_1=0$, *lies between* $a-h$ *and* $a+h$.

For x substitute $a+h$ in the equation $X=0$, and denote the result by H. Also put A, A', A'' for the values of X and its derived functions, when $a+h$ is substituted for x; we shall then have (Art. 411)

$$H=A+A'h+\tfrac{1}{2}A''h^2+, \&c.$$

But since a is a root of the equation $X=0$, we shall have $A=0$, while A' cannot be 0, since the equation $X=0$ has no equal roots. Therefore,

$$H=A'h+\tfrac{1}{2}A''h^2+, \&c., =h(A'+\tfrac{1}{2}A''h+, \&c).$$

Now h may be taken so small that the quantity within the parenthesis shall have the same sign as its first term A', (since A' expresses the first derived function of X, corresponding to X. in Art. 412), therefore, the sign of X, when $x=a+h$, will be the *same* as the sign of X_1.

If we substitute $a-h$ for x in the equation $X=0$, and denote the result by H', we then have, by changing h into $-h$, in the expression for H,

$$H'=-h(A'-\tfrac{1}{2}A''h+, \&c).$$

Now it is evident that for very small values of h, the sign of H′ will depend upon the first term $-A'h$, and consequently will be *contrary* to that of A′. Hence, when $x=a-h$, there is a variation of signs in the first two terms of the series X, X_1; and when $x=a+h$, there is a continuation of the same sign. Therefore, *one variation of signs is lost in passing from* $x=a-h$ *to* $a+h$.

If any of the auxiliary functions should vanish at the same time by making $x=a$, the number of variations will not be affected on this account (Art. 423), and therefore, one variation of signs will still be lost in passing from $a-h$ to $a+h$.

ART. **425.** STURM'S THEOREM.—*If any two numbers,* p *and* q, (p *being less than* q) *be substitutud for* x *in the series of functions* X, X_1, X_2, *&c., the substitution of* p *for* x *giving* k *variations, and that of* q *for* x, *giving* k′ *variations, then* k—k′ *will be the exact number of real roots of the equation* X=0, *which lies between* p *and* q.

Let us suppose that $-\infty$ is substituted for x, (by which sign is meant any quantity so great that the signs of the different functions X, X_1, X_2, &c., depend on the sign of the first term only), and suppose that x continually increases and passes through all degrees of magnitude till it becomes 0, and after this let it continually increase till it becomes $+\infty$.

Now it is evident, that so long as x, with its minus sign, is less than any of the roots of $X=0, X_1=0$, &c., no alteration will take place in the signs of any of these functions (Art. 419); but when x becomes equal to the least root (with its sign) of any of the auxiliary functions, although a change may occur in the sign of this function, yet we have seen (Art. 423) that it is the *order* only, and not the *number* of variations which is affected. But when x becomes equal to any of the roots of the primitive function, then one variation of signs is always lost. Since, then, a variation is always lost whenever the value of x passes through a root of the primitive function $X=0$, and since a variation cannot be lost in any other way, nor can one be ever introduced, it follows that the excess of the number of variations given by $x=p$, above that given by $x=q$, $(p<q)$ is exactly equal to the number of real roots of $X=0$, which lie between p and q.

Cor. If the equation is of the n^{th} degree, and m represents the number of real roots, then (Art. 396) the number of imaginary roots will be $n-m$.

ART. **426.** To determine merely the *number* of real roots, we may substitute $-\infty$ and $+\infty$ for x in the several functions. In this case the sign of each function will be that of its first term.

If we substitute 0 for x, the number of variations lost from $-\infty$ to 0, will be the number of *negative* roots; and from 0 to $+\infty$, the number of *positive* roots.

ART. **427.** *To determine the situation of each real root; that is, the figures between which it lies.*

Substitute the numbers $0, -1, -2, -3$, &c., for x, in series (1), till we find a number which produces as many variations as $x=-\infty$ produced. This number will be the limit of the greatest negative root.

We then substitute the numbers 1, 2, 3, &c., till we find a positive number which gives the same number of variations that $x=+\infty$ does. This will be the limit of the greatest positive root. By observing where one or more variations is lost, we find the situation of the roots. If two or more variations are lost between two of the substitutions, we must substitute smaller numbers, until only *one* variation is lost between two substitutions. This operation is termed the *separation of the roots.*

Ex. 1. Find the number and situation of the real roots of the equation $4x^3-12x^2+11x-3=0$.

Here we have $X = 4x^3-12x^2+11x-3$,
and (Art. 411) $X_1=12x^2-24x+11$.

Multiplying X by 3, to render the first term divisible by the first term of X_1, and proceeding as in the method of finding the greatest common divisor (Art. 108), we have for a remainder $-2x+2$. Canceling the factor $+2$, and changing the signs (Art. 420) we have $X_2=x-1$. Dividing X_1 by X_2 we have for a remainder -1; hence, $X_3=+1$. Therefore, the series of functions are

$$X = 4x^3-12x^2+11x-3.$$
$$X_1=12x^2-24x+11.$$
$$X_2= x-1.$$
$$X_3=+1.$$

Put $-\infty$ and $+\infty$ for x in the leading terms of these functions, and the signs of the results are

For $x = -\infty$, $- + - +$ three variations, $\therefore$ k$=3$
$x = +\infty$, $+ + + +$ no variation, $\therefore$ k$'=0$
$\therefore$ k$-$k$'=3-0=3$, the number of real roots.

Next, to find the situation of the roots, we must employ narrower limits than $-\infty$ and $+\infty$. But if we substitute 0 in each of the functions, we find three variations, the same number as for $-\infty$; hence, there is no real root between 0 and $-\infty$. This we might also have learned from Art. 402, Cor. 1, since there is no permanence in the proposed equation.

In practice it is customary to substitute integral numbers first, and afterward fractional; particularly where two or more roots lie between two whole numbers. In the present example, however, for the sake of illustration, we shall at once substitute fractions.

	X	X_1	X_2	X_3	
For $x=-\infty$ the signs are	$-$	$+$	$-$	$+$	giving 3 var.
$x=\ 0$	$-$	$+$	$-$	$+$	" 3 "
$x=+\frac{1}{4}$	$-$	$+$	$-$	$+$	" 3 "
$x=+\frac{1}{2}$	0	$+$	$-$	$+$	
$x=+\frac{3}{4}$	$+$	$-$	$-$	$+$	" 2 "
$x=+1$	0	$-$	0	$+$	
$x=+1\frac{1}{4}$.	$-$	$-$	$+$	$+$	" 1 "
$x=+1\frac{1}{2}$.	0	$+$	$+$	$+$	
$x=+1\frac{3}{4}$.	$+$	$+$	$+$	$+$	" 0 "
$x=+\infty$.	$+$	$+$	$+$	$+$	" 0 "

Here we see that the roots are $\frac{1}{2}$, 1, and $1\frac{1}{2}$, but if these numbers had not been substituted, we would have noticed that *one* variation was lost in passing from $\frac{1}{4}$ to $\frac{3}{4}$; *one* in passing from $\frac{3}{4}$ to $1\frac{1}{4}$, and lastly, *one* in passing from $1\frac{1}{4}$ to $1\frac{3}{4}$, which would have given the *situation* of the roots.

A careful study of this example will serve to illustrate the theorem. Thus we see that there are *three* changes of sign of the primitive function, *two* of the first auxiliary function, and *one* of the second. We observe, however, that no variation is lost by the change of sign of either of the auxiliary functions, while each change of sign of the primitive function occasions a loss of one variation.

2. How many real roots has the equation

$$x^3-3x^2+x-3=0.$$

Here,
$$X = x^3-3x^2+x-3$$
$$X_1=3x^2-6x+1$$
$$X_2= x+2$$
$$X_3=-25.$$

For $x=-\infty$ the signs are $-+--$, 2 variations, $\therefore$ k$=$2

$x=+\infty$ the signs are $+++-$, 1 variation, $\therefore$ k$'$=1

$\therefore$ k$-$k$'$=2$-$1=1, the number of real roots.

The root is +3, and by substitution it will be found that one variation is lost in passing from 2 to 4.

Find the number and situation of the real roots in each of the following equations:

3. $x^3-2x^2-x+2=0$. *Ans.* Three, -1, $+1$, $+2$.
4. $8x^3-36x^2+46x-15=0$. *Ans.* Three. One between 0 and 1, one between 1 and 2, and one between 2 and 3.
5. $x^3-3x^2-4x+11=0$. *Ans.* Three, one between -2 and -1, one between 1 and 2, and one between 3 and 4.
6. $x^3-2x-5=0$. *Ans.* One between 2 and 3.
7. $x^3-15x-22=0$. *Ans.* Three. One root is -2, one between $-2\frac{1}{4}$ and $-2\frac{1}{2}$, and one between 4 and 5.
8. $x^4+x^3-x^2-2x+4=0$. *Ans.* No real roots.
9. $x^4-4x^3-3x+23=0$. *Ans.* Two. One between 2 and 3 and one between 3 and 4.
10. $x^4-2x^3-7x^2+10x+10=0$. *Ans.* Four. The limits are $(-3, -2)$; $(0, -1)$; $(2, 3)$; $(2, 3)$.
11. $x^5-10x^3+6x+1=0$. *Ans.* Five. The limits are $(-4, -3)$; $(-1, 0)$; $(-1, 0)$; $(0, 1)$; $(3, 4)$.

CHAPTER XIII.

RESOLUTION OF NUMERICAL EQUATIONS

ART. **428.** In the preceding Articles we have demonstrated the most important propositions in the theory of equations, and in some cases have shown how to find their roots. The general solution of an equation higher than the fourth degree has never yet been effected, but the class of equations which most frequently occurs in philosophical investigations is *numerical;* that is, those that have numerical coëfficients. When the roots of these are real, we can find them either exactly, or approximately, as near as we please. The way for doing this has been prepared in the preceding articles, by finding the limits of the roots, and separating them from each other.

RATIONAL ROOTS.

Art. 429. Prop. I.— *To determine the integral roots of an equation.*

If a be an integral root of the equation

$$Ax^4+Bx^3+Cx^2+Dx+E=0,$$

we shall have $Aa^4+Ba^3+Ca^2+Da+E=0;$

$$\therefore \frac{E}{a}=-Aa^3-Ba^2-Ca-D.$$

Now since the second member of the last equation is evidently a whole number, E is divisible by a. Put $\frac{E}{a}=E'$; transpose D to the first member, and divide by a; this gives

$$\frac{E'+D}{a}=-Aa^2-Ba-C;$$

therefore, a is also a divisor of $E'+D$.

Put $E'+D=D'$, transpose C, and divide by a; this gives

$$\frac{D'+C}{a}=-Aa-B;\ \therefore\ a \text{ is a divisor of } D'+C.$$

Again put $\frac{D'+C}{a}=C'$, transpose B, and divide by a, we find

$$\frac{C'+B}{a}=-A.$$

Lastly, making $C'+B=B'$, and transposing A, we have

$$B'+A=0.$$

If then, *all* these conditions are satisfied, a is a root of the proposed equation; but if any one of them fails, a is not a root. Hence, we have the following

Rule for finding the integral roots of an equation.— *Divide the last term of the equation by any of its divisors* a, *and add to the quotient the coëfficient of the term containing* x.

Divide this sum by a, *and add to the quotient the coëfficient of* x^2.

Proceed in this manner unto the first term, and if a *be a root of the equation, all these quotients will be whole numbers, and the result will be* 0.

Cor. 1. It will be more easy to substitute the divisors $+1$ and -1, at once in the given equation, and therefore they may be omitted in the operation. Also, by ascertaining the limits to the positive and negative roots (Art. 417), we shall frequently find that several of the divisors fall beyond the limits; and therefore, these may be omitted.

Cor. 2. If the coëfficient of the first term be not unity, the equation may have a fractional root. To determine if this be the case, transform the equation into one in which the coëfficient of the first term shall be unity (Art. 405, Cor. 1), and then all the rational roots will be integers (Art. 399).

Cor. 3. When all the roots except two are integral, the integral roots may be found by the rule, and then the proposed equation reduced to one of the second degree by division (Art. 396, Cor. 1), and solved as a quadratic.

Ex. 1. Find the rational roots of the equation

$$x^3+3x^2-4x-12=0.$$

Here, by Art. 417, no positive root can exceed $1+\sqrt[3]{12}$, or 4, and the limit of the negative roots is $1+3=4$.

It is also found, by trial, that $+1$, and -1 are not roots.

We then proceed to arrange the divisors of -12, among which it is *possible* to find the roots, and proceed with the operation as follows:

Last term -12

Divisors.	$+2$,	$+3$,	$+4$,	-2,	-3,	-4
Quotients	-6,	-4,	-3,	$+6$,	$+4$,	$+3$
Add -4	-10,	-8,	-7,	$+2$,	-0,	-1
Quotients	-5,	*,	*,	-1,	0	*
Add $+3$	-2,			$+2$,	$+3$,	
Quotients	-1			-1,	-1,	
Add $+1$	0,			0,	0.	

Since -8 is not divisible by $+3$, we proceed no further with this divisor, as it is evident that it is not a root of the equation; in like manner $+4$ and -4 cannot be roots. But we find that $+2$, -2, and -3 are roots of the proposed equation.

Find the roots of the following equations:

2. $x^3-7x^2+36=0$. *Ans.* 3, 6, and -2.

SUGGESTION.—When any term is wanting, as the third term in this example, its place must be supplied with 0.

3. $x^3-6x^2+11x-6=0$. *Ans.* 1, 2, 3.

4. $x^3+x^2-4x-4=0$. *Ans.* 2, -1, -2.

5. $x^3-3x^2-46x-72=0$. *Ans.* 9, -2, -4.

6. $x^3-5x^2-18x+72=0$. *Ans.* 3, 6, -4.

7. $x^4-10x^3+35x^2-50x+24=0$. *Ans.* 1, 2, 3, 4.

8. $x^4+4x^3-x^2-16x-12=0$. *Ans.* 2, -1, -2, -3.

9. $x^4-4x^3-19x^2+46x+120=0$. *Ans.* 4, 5, -2, -3.

10. $x^4-27x^2+14x+120=0$. *Ans.* 3, 4, -2, -5

11. $x^4+x^3-29x^2-9x+180=0$. *Ans.* 3, 4, -3, -5.

12. $x^3-2x^2-4x+8=0$. *Ans.* 2, 2, -2.

13. $x^3+3x^2-8x+10=0$. (See Cor. 3.) *Ans.* -5, $1\pm\sqrt{-1}$.

14. $x^4-9x^3+17x^2+27x-60=0$. *Ans.* 4, 5, $\pm\sqrt{3}$.

15. $2x^3-3x^2+2x-3=0$. (See Cor. 2.) *Ans.* $\frac{3}{2}$, $\pm\sqrt{-1}$.

16. $3x^3-2x^2-6x+4=0$. *Ans.* $\frac{2}{3}$, $\pm\sqrt{2}$.

17. $8x^3-26x^2+11x+10=0$. *Ans.* $\frac{5}{2}$, $\frac{1}{8}(3\pm\sqrt{41})$.

18. $6x^4-25x^3+26x^2+4x-8=0$. *Ans.* 2, 2, $\frac{2}{3}$, $-\frac{1}{2}$.

19. $x^4-9x^3+\frac{45}{4}x^2+\frac{27}{2}x-\frac{81}{4}=0$. *Ans.* $\frac{3}{2}$, $\frac{3}{2}$, $3\pm3\sqrt{2}$.

IRRATIONAL ROOTS — METHODS OF APPROXIMATION.

After we have found all the integral roots of an equation, we must have recourse to the methods of approximation, the best of which is Horner's, by which we can always obtain the numerical values of the real roots, to any required degree of accuracy.

ART. 430. HORNER'S METHOD OF APPROXIMATION.

The principle of this method depends on the successive transformation of the given equation, so as to diminish its roots at each step, and the operation is performed by Synthetic Division, as explained in Art. 410.

Let the equation, one of whose roots is to be found, be

$$Px^n+Qx^{n-1}\ldots\ldots+Tx+V=0.$$

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Suppose a to be the integral part of the root required, and r, s, t. . the decimal digits taken in order, so that $x=a+r+s+t$. . . Let a be found by trial, or by Sturm's theorem (Art. 427), and transform the equation into one whose roots shall be diminished by a, by the method explained in Art. 410.

Let $Py^n+Q'y^{n-1}. . . . +T'y+V'=0$ be the transformed equation, then the value of y is the decimal $r+s+t$. . . ; and since this root is contained between 0 and 1, we may easily find its first digit r. Again, let the roots of this equation be diminished by r, and let the transformed equation be

$$Pz^n+Q''z^{n-1}. . . . +T''z+V''=0.$$

Now the value of z in this equation is $s+t$. . . , and the value of s lies between .00 and .1; that is, it is either .00, .01, .02, . . . or .09. But since the figure s is in the second place of decimals, the terms containing z^2, z^3. . . will be small, and we may generally find s, the next figure of the root, from the equation $T''z+V''=0$; that is, s is nearly equal to the quotient of $-V''$ divided by T''.

Having found s, we next proceed to diminish the roots of the last equation by s, and then from the last two terms, $T'''z'+V'''$, of the resulting equation, find t the next decimal figure, and so on.

ART. **431.** The absolute number, or last term of the equation, is sometimes called the *dividend*, and the coëfficient of the first power of the unknown quantity,(for example T'') the *incomplete* or *trial divisor*.

The correctness of the values of the figures s, t, &c., obtained by means of the trial divisor, will always be verified in the next operation. For when we multiply by s, in the operation of transformation, to obtain the product to be subtracted from V'', the number multiplied by s (sometimes called the *complete divisor*) ought to be contained in V'' only s times. But if it should be contained a *greater* or *less* number of times, then s must be *increased* or *diminished*.

In some cases, where it is small, and when the equation does not exceed the third degree, r, the first decimal figure of the root, may be found by dividing V' by T'. The accuracy with which each succeeding decimal figure may be found, increases as the value of the figure decreases. In general, after threeor four decimal figures have been found, the next three or four figures may be obtained accurately by division, as in the method of finding each previous figure.

ART. **432**. To find the negative roots of the proposed equation, change the signs of the alternate terms (Art. 400) and find the *positive* roots of the resulting equation; these will be the *negative* roots of the proposed equation.

REMARK.— Instead of finding the first decimal figure of the root by trial, or by division, it may be found from the transformed equation by Sturm's theorem; but in general it can be obtained more easily by trial.

ART. **433.** To illustrate this method, let it be required to find the positive root of the equation $x^2-4x-10.768649=0$.

We readily find that x must be greater than 5 and less than 6 therefore $a=5$. We then proceed to transform this equation into another whose roots shall be less by 5. (See Art. 410.)

$$\begin{array}{lll} 1-4 & -10.768649 & (\overset{a}{5} \\ \underline{+5} & \underline{+\ 5} & \\ +1 & -\ 5.768649 & \\ \underline{+5} & & \\ \underline{+6} & & \end{array}$$

1st Trans. eq. $y^2+6y \quad -5.768649=0$.

Here we may find the value of y nearly, by dividing 5.7 by 6, which gives .9; but this is too great because we neglected the square of y. If we assume $y=.8$, and deduct $y^2=.64$ from 5.7, and then divide by 6, we see that y must be .8. Let us now transform the equation into another whose roots shall be less by .8.

$$\begin{array}{llll} 1 & +6 & -5.768649 & (\overset{s}{.8} \\ & \underline{.8} & \underline{+5.44} & \\ & +6.8 & -\ .328649 & \\ & \underline{.8} & & \\ & \underline{7.6} & & \end{array}$$

2nd Trans. eq. $z^2.+7.6z-.328649=0$.

The approximate value of z in this equation is the second decimal figure of the root. This is readily found by dividing the absolute term by the coëfficient of z, the first term, z^2, being now so small that it may be neglected. Thus, $.328\div7.6=.04=s$.

We next proceed to diminish the roots of the last equation by .04.

			s
1	$+7.6$	$-.328649$	(.04
	$.04$	$.3056$	
	$+7.64$	$.023049$	
	$.04$		
	$+7.68$		

3rd Trans. eq. $z'^2+7.68z' \quad -.023049=0.$

Here z' is nearly $.023 \div 7.68 = .003 = t$.

By diminishing the roots of the last equation by .003 we have

			t
1	$+7.68$	$-.023049$	(.003
	$.003$	$.023049$	
	$+7.683$	$.0$	

The remainder being zero, shows that we have obtained the exact root, which is 5.843.

By changing the sign of the second term of the proposed equation, we have $x^2+4x-10.768649=0$. The root of this equation may be found in a similar manner; it is 1.843. Hence, the two roots are $+5.843$ and -1.843.

Ex. 2. To illustrate this method further, let us form the equation whose roots are $3, +\sqrt{2}, -\sqrt{2}$, which gives $x^3-3x^2-2x+6=0$. Let it now be required to find, by Horner's method, the root which lies between 1 and 2; that is, $\sqrt{2}$.

By examination, we readily see that one root lies between 1 and 2; hence, $a=1$, and the first step is to transform the equation so as to diminish its roots by 1.

				a
1	-3	-2	$+6$	(1
	$+1$	-2	-4	
	-2	-4	$+2$	
	$+1$	-1		
	-1	-5		
	$+1$			
	0			

$$r=\frac{V'}{T'}=\frac{2}{5}=.4$$

Hence, $y^2\pm0y^2-5y+2=0$, is the first transformed equation. By dividing the absolute term 2, by 5, the trial divisor or coëfficient of y, the second figure, r, of the root is readily found $=.4$, and we proceed to transform the equation so as to diminish its roots by .4.

$$\begin{array}{lrrrl}
 & & & & r \\
1 & \pm 0 & -5 & +2 & (.4 \\
 & .4 & .16 & -1.936 & \\ \hline
 & .4 & -4.84 & +\ .064 & \\
 & .4 & +\ .32 & & \\ \hline
 & .8 & -4.52 & & \\
 & .4 & & & \\ \hline
 & 1.2 & & &
\end{array}
\qquad s=\frac{V''}{T''}=\frac{.064}{4.52}=.01$$

This gives $z^3+1.2z^2-4.52z+.064=0$, for the 2nd transformed equation; and for s the next figure of the root .01. The next step is to transform this equation so as to diminish its roots by .01.

$$\begin{array}{lrrrl}
 & & & & s \\
1 & +1.2 & -4.52 & +.064 & (.01 \\
 & .01 & .0121 & -.045079 & \\ \hline
 & 1.21 & -4.5079 & +.018921 & \\
 & .01 & .0122 & & \\ \hline
 & 1.22 & -4.4957 & & \\
 & .01 & & & \\ \hline
 & 1.23 & & &
\end{array}
\qquad t=\frac{V'''}{T'''}=\frac{.0189}{4.495}=.004$$

This gives $z'^3+1.23z'^2-4.4957z'+.018921=0$, for the 3rd transformed equation; and for the next figure of the root $t=.004$. The next step is to transform this equation so as to diminish its roots by .004.

$$\begin{array}{lrrrl}
 & & & & t \\
1 & +1.23 & -4.4957 & +.018921 & (.004 \\
 & .004 & +\ .004936 & -.017963056 & \\ \hline
 & 1.234 & -4.490764 & .000957944 & \\
 & .004 & +\ .004952 & & \\ \hline
 & 1.238 & -4.485812 & & \\
 & .004 & & & \\ \hline
 & 1.242 & & &
\end{array}$$

Having obtained three decimal places in the root, we may obtain several of the succeeding figures accurately by division; thus, $.000957944 \div 4.485812 = .0002135$, which is true to the last decimal place, as will be found by extracting the square root of 2. Hence, $x=1.4142135$.

To illustrate the process fully, the preceding operation has been presented in the most extended form. In practice it is customary to make some abridgments. Thus, by marking with a * the coëfficients of the unknown quantity in each transformed equation, it is not necessary to rewrite it. Also, when the root is required only to five or six places of decimals we need not use more than this number in the operation.

We shall now give the solution of an equation of the 4th degree, presenting the operation in a concise form.

3. Given $x^4-8x^3+14x^2+4x-8=0$, to find a value of x.

OPERATION.

1	—8	+14	+4	—8 (5.236068
	5	—15	—5	—5
	—3	— 1	—1	*—13
	5	10	45	10.6576
	2	9	*44	*— 2.3424
	5	35	9.288	1.93880241
	7	*44	53.288	*— .40359759
	5	2.44	9.784	.39905490
1	*12	46.44	*63.072	*— .00454269
	.2	2.48	1.554747	.00400954
	12.2	48.92	64.626747	*—.00053315
	.2	2.52	1.566321	
	12.4	*51.44	*66.193068	
	.2	.3849	.31608	
	12.6	51.8249	66.50915	
	.2	3858	.31656	
1	*12.8	52.2107	*66.82571	
	.03	3867		
	12.83	*52.5974		
	03	08		
	12.86	52.68		
	.03	.08		
	12.89	52.76		
	03			
1	*12.92			

As the root is found only to six decimal places, it is not necessary to carry the true divisor for the third figure (6) to more than five decimal places; this divisor is 66.50915, which multiplied by .006, gives eight decimal places, and the dividend ought to be carried to seven or eight decimal places, in order that the figure in the sixth decimal place of the root may be correct. So the divisor, 66.825, for the fifth figure of the root, requires to be carried only to three decimal places, for the product of this number by .00006 gives eight decimal places as it ought to do. So the divisor for the last figure (8) of the root would require to be carried only to two decimal places. The numbers in the vertical columns preceding the divisors, require to be carried to still fewer places, as the pupil will readily perceive.

After obtaining the third figure of the root, the next three may be obtained merely by division; thus, $.00454269 \div 66.82571 = .000068$ nearly.

The pupil must observe that where decimals are omitted, we always take the figure next to the omitted places, *to the nearest unit.* Thus, .07752 is nearer .08 than .07; therefore, the former is taken.

Art. **434.** The process illustrated in the preceding examples may be extended to equations of any degree, and is justly regarded as the most elegant method of approximating to the roots of equations yet discovered. It may be briefly expressed by the following

Rule.—1. *Find by trial, or by Sturm's theorem, the integral part of the required root.*

2. *Transform the equation* (Art. 410) *into another whose roots shall be those of the proposed equation, diminished by the part of the root already found.*

3. *With the absolute term in the first transformed equation for a dividend, and the coëfficient of* x *for a divisor, find the first decimal figure of the root.*

4. *Transform the last equation into another whose roots shall be diminished by the part of the root already found, and from the first two terms of this equation, find the second figure of the root.*

5. *Continue this process, till the root is found to the required degree of accuracy.*

6. *To find the negative roots, change the signs of the alternate terms, and proceed as for a positive root.*

Remarks.—1. If any figure, found by trial, is either too great or too small, it will be made manifest in the next transformation. (See Art. 431.)

2. In general, after three figures of the root have been found accurately, the next three may be obtained by dividing the absolute term by the coëfficient of x.

EXAMPLES FOR PRACTICE.

Let the pupil find at least one value of x in each of the following equations:

1. $x^2+5x-12.24=0$. *Ans.* $x=1.8$.
2. $x^2+12x-35.4025=0$. *Ans.* $x=2.45$.
3. $4x^2-28x-61.25=0$. *Ans.* $x=8.75$
4. $8x^2-120x+394.875=0$. *Ans.* $x=10.125$
5. $5x^2-7.4x-16.08=0$. *Ans.* $x=2.68$.
6. $x^2+x-1=0$. *Ans.* $x=.618034$.
7. $x^2-6x+6=0$. *Ans.* $x=4.73205$.
8. $x^3+4x^2-9x-57.623625=0$. *Ans.* $x=3.45$.
9. $2x^3-50x+32.994306=0$. *Ans.* $x=4.63$.
10. $x^3+x^2+x-1=0$. *Ans.* $x=.543689$.
11. $x^3+4x^2-5x-20=0$. *Ans.* $x=2.23608$.
12. $x^3-2x-5=0$. *Ans.* $x=2.0945515$.
13. $x^3+10x^2-24x-240=0$. *Ans.* $x=4.8989795$.
14. $x^3+12x^2-18x=216$. *Ans.* $x=4.2426407$.
15. $x^4-8x^3+20x^2-15x+.5=0$. *Ans.* $x=1.284724$.
16. $x^4+x^2-8x-15=0$. *Ans.* $x=2.302775$.
17. $x^4-59x^2+840=0$. *Ans.* $x=4.8989795$.
18. $2x^4+5x^3+4x^2+3x=8002$. *Ans.* $x=7.335554$.
19. $x^5+4x^4-3x^3+10x^2-2x=962$. *Ans.* $x=3.385777$.
20. $x^5+2x^4+3x^3+4x^2+5x=54321$. *Ans.* $x=8.414455$.

ART. **435.** *To extract the roots of numbers by Horner's Method.*

The extraction of any root of a number is only a particular case of the solution of an equation of the same degree; for if we call the number N, the root x, and the index of the root n, we shall have $x^n=N$, or $x^n-N=0$; an equation of the n^{th} degree in which all the terms are wanting except the first and last.

In performing the operation we may find the successive integral figures in the same manner as the successive decimal places were found in the preceding article. It is only necessary to bear in mind that any two figures in consecutive integer places, have the same relation to each other as if they were in consecutive decimal places. In extracting any root, the cube root for example, it is necessary to point off the given number into periods, as in the operation by the common rule. We shall now illustrate the method of operation by finding the cube root of 12977875; that is, by finding one root of the equation $x^3-12977875=0$.

1	0	0	12977875 (235
	2	4	8
	2	4	4977
	2	8	4167
	4	*12	810875
	2	189	810875
1	*6	1389	
	3	198	
	63	*1587	
	3	3475	
	66	162175	
	3		
1	*69		
	5		
	695		

Should the learner not readily understand the reason for the manner in which the figures are placed in the successive columns, let him perform the operation, using the numbers 200, and 30, instead of 2 and 3, and all the difficulties will vanish.

By the same method find

2. The cube root of 34012224. *Ans.* 324.
3. The cube root of 9. *Ans.* 2.080084.
4. The cube root of 30. *Ans.* 3.107233.
5. The fifth root of 68641485507. *Ans.* 147.

APPROXIMATION BY DOUBLE POSITION.

Art. 436. Double Position furnishes one of the most useful methods of approximating to the roots of equations. It has the advantage of being applicable, whether the equation is *fractional*, *radical*, or *exponential*, or to any other form of function.

Let X=0, represent any equation; and suppose that a and b, when substituted for x, give results, the one too *small*, and the other too *great*, so that one root of this equation lies between a and b. (Art. 403.)

Let A and B be the results arising from the substitution of a and b for x, in the equation $X=0$. Let $x=a+h$, and $b=a+k$. then if we substitute $a+h$, and $a+k$ for x, in the equation $X=0$, we shall have

$$X=A+A'h+\tfrac{1}{2}A''h^2+, \&c.$$

$$B=A+A'k+\tfrac{1}{2}A''k^2+, \&c.$$

Here A', A'', &c., are the derived functions of A (Art. 411). Now if h and k be so small that their second and higher powers may be neglected without much error, we shall have

$$X-A=A'h \text{ nearly};$$

$$B-A=A'k \quad \text{"} \quad .$$

Whence, $B-A : X-A :: A'k : Ah : k : h$;

or $B-A : k :: X-A : h$, (Art. 270);

or $B-A : b-a :: X-A : h$, since $k=b-a$.

Hence we have the following

RULE.— Find by trial, two numbers which substituted for x in the proposed equation, give one a result too small, and the other too great. Then say,

As the difference of the results;

Is to the difference of the suppositions;

So is the difference between the true result and either of the former results;

To the correction of the corresponding supposition.

This correction is to be added to the corresponding supposition when it is too little, and subtracted when it is too great, and the result will be the first approximation.

Substitute this root for the unknown quantity, and the result will show whether the supposition is too small or too great; then take another number such that the true root may lie between it and the last supposition, and proceed, as before, to obtain a second approximate value of the required root; and so on.

It is generally best to begin with two integers which differ from each other by unity, and to carry the first approximation only to *one* place of decimals. In the next operation the difference of the suppositions may be 0.1, and the second quotient may be carried to *two* places, and so on, doubling the number of places of decimals at each approximation.

Ex 1. Given $x^3+x^2+x=100$, to find x.

It is easily found that x lies between 4 and 5. We then substitute these two numbers for x in the given equation, and the result is as follows:

4		x		5
64		x^3		125
16		x^2		25
4		x		5
84		results		155
155		5		100
84		4		84
71	:	1	::	16 : 0.22;

therefore, $x=4.2$, the first approximation.

Again, substituting 4.2 and 4.3 for x in the given equation, and proceeding as before, we get for a second approximation $x=4.264$. By assuming $x=4.264$, and $x=4.265$, and repeating the operation, we obtain for a third approximation $x=4.2644299$ nearly.

Find one root of each of the following equations:

2. $x^3+30x=420$. *Ans.* $x=6.170103$.
3. $144x^3-973x=319$. *Ans.* $x=2.75$.
4. $x^3+10x^2+5x=2600$. *Ans.* $x=11.00679$.
5. $2x^3+3x^2-4x=10$. *Ans.* $x=1.62482$.
6. $x^4-x^3+2x^2+x=4$. *Ans.* $x=1.14699$.
7. $x^4+x^3+2x^2-x=4$. *Ans.* $x=1.09059$.
8. $x^4-12x+7=0$. *Ans.* $x=2.04727$.
9. $2x^4-13x^2+10x-19=0$. *Ans.* $x=2.4573$.
10. $\sqrt[3]{7x^3+4x^2}+\sqrt{10x(2x-1)}=28$. *Ans.* $x=4.51066$.

ART. **437.** NEWTON'S METHOD OF APPROXIMATION.— This method of approximation is but little used, yet it is so often referred to, that it is desirable the learner should be acquainted with the principle on which it is founded.

Find by trial, two numbers which, substituted for the unknown quantity, give results with different signs; then (Art. 403) one real root, at least, lies between these two numbers. Now by increasing one of the limits, and diminishing the other, an approximation may be made to the root. When the quantity a thus found is within 0.1 of the value of the root, we may substitute $a+y$ for x in the given equation, and it will be of this form

$$A+A'y+\tfrac{1}{2}A''y^2+\tfrac{1}{6}A'''y^3+, \&c., =0 \quad \text{(Art. 411)},$$

where A, A′, A″, &c., are known quantities dependent on a.

From this equation, by transposing and dividing, we find

$$y=-\frac{A}{A'}-\tfrac{1}{2}\frac{A''}{A'}y^2-\tfrac{1}{6}\frac{A'''}{A'}y^3-, \&c.;$$

and since y is <0.1, y^2 will be <0.01, $y^3<0.001$, and so on. Therefore, if the sum of the terms containing y^2, y^3, &c., be less than .01, we shall, in neglecting them, obtain a value of y within .01 of the truth. Let, then, $y=-\frac{A}{A'}$, which gives for x the value $a-\frac{A}{A'}$. This will differ from the true value of x by less than .01. It is not necessary, however, to carry on the division of A by A beyond the second place of decimals, as the accuracy of the figure in the third place, could not be relied on.

Now, put b for this approximate value of x, and let $x=b+z$; we have then as before

$$B+B'z+\tfrac{1}{2}B''z^2+\tfrac{1}{6}B'''z^3+, \&c., =0;$$

and as z is supposed to be less than .01, z^2 will be $<.0001$. If then we neglect the terms containing z^2, z^3, &c., we shall obtain a probable value of z within .0001; and so on.

Since A is what the proposed equation becomes when $x=a$, and A′ what the first derived function becomes when $x=a$, therefore the corrections $-\frac{A}{A'}$, $\frac{B}{B'}$, &c., are easily found.

Newton gave but a single example, viz.: to find the value of x in the equation $x^3-2x-5=0$. *Ans.* $x=2.09455149$.

The pupil desirous of additional exercises may solve the examples in the preceding article by this method.

CARDAN'S RULE FOR SOLVING CUBIC EQUATIONS.

ART. **438.** In its most general form, a cubic equation may be represented by

$$x^3+px^2+qx+r=0;$$

but as we can always take away the second term by the method described in Art. 407, we will suppose, in order to avoid fractions, that it is reduced to the form

$$x^3+3qx+2r=0.$$

Assume $x=y+z$, and the equation becomes

$$y^3+z^3+3yz(y+z)+3q(y+z)+2r=0.$$

Now since we have two unknown quantities in this equation, y and z, and have made only one supposition respecting them, namely, that $y+z=x$, we are at liberty to make another. Let, therefore, $yz=-q$; then, by substituting this in the equation, it becomes $y^3+z^3+2r=0$; but since $yz=-q$, we have $z^3=-\frac{q^3}{y^3}$;

$$\text{hence } y^3-\frac{q^3}{y^3}+2r=0,$$

$$\text{or } \quad y^6+2ry^3=q^3.$$

$$\text{whence } \quad y^3=-r+\sqrt{r^2+q^3}=A^3,$$

and similarly $\quad z^3=-r-\sqrt{r^2+q^3}=B^3$;

the radical quantity being taken positive in one of these expressions, and negative in the other, to render them different. And since $x=y+z$, we have

$$x=\sqrt[3]{(-r+\sqrt{r^2+q^3})}+\sqrt[3]{(-r-\sqrt{r^2+q^3})}.$$

This formula would appear to give but one of the roots. But since the values of y and z are found by extracting the cube roots of A^3 and B^3, it will now be shown that each of them must have three values.

Since $y^3=A^3$, we have $y^3-A^3=0$, or, by factoring (Art. 83), $(y-A)(y^2+Ay+A^2)=0$; putting each of these factors equal to 0, and solving the resulting equations, we have

$$y=A,\ y=\frac{-1+\sqrt{-3}}{2}A, \text{ and } y=\frac{-1-\sqrt{-3}}{2}A.$$

Similarly, from the equation $z^3=B^3$, we find

$$z=B,\ z=\frac{-1+\sqrt{-3}}{2}B, \text{ and } z=\frac{-1-\sqrt{-3}}{2}B.$$

By combining each value of y with the three values of z, it might appear that x had 9 values; that is, that an equation of the third degree has nine roots, which is impossible (Art. 397).

That this is impossible from the solution is thus shown:

We supposed that $y\,z=-q$.

But six of the products of the values of yz give imaginary values, and since $yz=-q$, a real negative quantity, therefore, the combinations giving imaginary products must be rejected, and the three values of x are

1st $\quad A+B,$

2nd $\quad \frac{-1+\sqrt{-3}}{2}A+\frac{-1-\sqrt{-3}}{2}B,$

3rd $\frac{-1-\sqrt{-3}}{2}A+\frac{-1+\sqrt{-3}}{2}B.$

ART. **439.** If r^2+q^3 be negative, that is, if $r^2+q^3<0$, the values of x become apparently imaginary when they are actually real, and we shall now show that

Cardan's Method of Solution does not extend to those cases in which the equation has three real and unequal roots.

Since every cubic equation has at least one real root (Art. 401, Cor. 3), we may suppose this to be a; and the other two roots arising from the solution of a quadratic, may be represented by $b+\sqrt{3c}$, and $b-\sqrt{3c}$, in which, if $3c$ be positive, the roots are real, and if $3c$ be negative they are imaginary; and because the second term of the equation is 0, we have (Art. 395, Corollaries)

$$0=a+(b+\sqrt{3c})+(b-\sqrt{3c})=a+2b;$$
$$3q=a\times 2b+b^2-3c=-3b^2-3c;$$
$$2r=-a(b^2-3c)=2b^3-6bc.$$

Hence we have

$$r^2+q^3=(b^3-3bc)^2-(b^2+c)^3$$
$$=-9b^4c+6b^2c^2-c^3=-c(3b^2-c)^2;$$
$$\therefore\ \sqrt{r^2+q^3}=(3b^2-c)\sqrt{-c}.$$

Now this expression is real when c is negative, and imaginary when c is positive, or when the equation has three real roots.

If we suppose c to become zero, the value of $\sqrt{r^2+q^3}$ ceases to be imaginary. But this supposition reduces the roots to a, b, and b; hence, Cardan's Rule is applicable to equations of the third degree containing two equal roots.

ART. **440.** From the last article we see that when the roots of the quadratic equation in Art. 438 are imaginary, the roots of the cubic equation are all real. As this appears paradoxical, we will show by the direct solution of a particular example, that the value of x, in this case, is a real quantity.

Ex. To find the three roots of the equation $x^3-15x-4=0$.

By substituting $y+z$ for x, we have

$$y^3+z^3+3yz(y+z)-15(y+z)-4=0;$$

and, therefore, as in Art. 438,

$$3yz=15,\ y^3+z^3-4=0.$$

From the solution of these equations we obtain $y^3=2+11\sqrt{-1}$; and it may be proved, by actual multiplication, that $y=2+\sqrt{-1}$; likewise, $z^3=2-11\sqrt{-1}$, and $z=2-\sqrt{-1}$.

Hence, $x=y+z=(2+\sqrt{-1})+(2-\sqrt{-1})=4$.

By dividing the given equation by $x-4$, and placing the result equal to 0, we find the other two roots are $x=-2+\sqrt{3}$, and $-2-\sqrt{3}$.

As no means have yet been discovered for reducing the imaginary forms to real values, Cardan's rule fails when all the roots are real.

This is termed the *irreducible case* of cubic equations, and has been a subject of great perplexity to mathematicians.

ART. **441.** We shall present some examples for solution in the case to which Cardan's rule applies; that is, when the proposed equation contains one real and two imaginary roots. When the proposed equation contains the second term it must first be removed (See Art. 407), and the equation reduced to the form $x^3+3qx+2r=0$.

Then $x=\sqrt[3]{(-r+\sqrt{r^2+q^3})}+\sqrt[3]{(-r-\sqrt{r^2+q^3})}$, will be the real root of the proposed equation.

Having the real root, the imaginary roots may be found from the formula in Art. 438, or by reducing the proposed equation to a quadratic.

Ex. 1. Solve the equation $v^3+3v^2+9v-13=0$.

If we transform this equation into another which shall want its second term, by substituting $x-1$ for v (Art. 407), we have

$$x^3+6x-20=0.$$

Comparing this with the equation $x^3+3qx+2r=0$, we find $q=2$, $r=-10$; hence,

$$x=\sqrt[3]{(10+\sqrt{108})}+\sqrt[3]{(10-\sqrt{108})}=2.732-.732=2.$$

Whence $v=x-1=2-1=1$.

The other two roots are easily found to be $-1\pm3\sqrt{-1}$.

Solve the following equations by Cardan's Rule:

2. $x^3-9x+28=0$. *Ans.* $x=-4,\ 2\pm\sqrt{-3}$.

3. $x^3+6x-2=0$. *Ans.* $x=\sqrt[3]{4}-\sqrt[3]{2}=.32748$.

4. $x^3-6x^2+13x-10=0$. *Ans.* $x=2,\ 2\pm\sqrt{-1}$.

5. $x^3+6x^2-32=0$. *Ans.* $x=2,\ -4,\ -4$.

6. $x^3+6x^2+27x-26=0$. *Ans.* $x=.801245$.

7. $x^3-9x^2+6x-2=0$. *Ans.* $x=8.306674$.

REMARK.—It is not deemed necessary to introduce the Rules of Ferrari, Euler, Descartes, or Simpson, for the Solution of Equations of the fourth degree, since they are applicable only to special cases, and, together with Cardan's Rule for the solution of cubic equations, are regarded, since the discovery of Horner's method, and Sturm's theorem, as little more than analytical curiosities.

RECIPROCAL OR RECURRING EQUATIONS.

ART. **442.** A *recurring* or *reciprocal equation* is one such that if a be one of its roots, the reciprocal of a, that is $\frac{1}{a}$, will be another.

Prop. I. *In a recurring equation the coëfficients, when taken in a direct and in an inverse order, are the same.*

Let $x^n+Ax^{n-1}+Bx^{n-2}\ldots\ldots+Sx^2+Tx+V=0$,
be a recurring equation; that is, one that is satisfied by the substitution of $\frac{1}{x}$ for x; this gives

$$\frac{1}{x^n}+\frac{A}{x^{n-1}}+\frac{B}{x^{n-2}}\ldots\ldots+\frac{S}{x^2}+\frac{T}{x}+V=0,$$

and multiplying by x^n.

$$1+Ax+Cx^2\ldots\ldots+Sx^{n-2}+Tx^{n-1}+Vx^n=0,$$

which proves the proposition.

NOTE.—Equations of this kind are called *Recurring equations* from the forms of their *coëfficients*, and *Reciprocal equations* from the forms of their *roots*.

PROP. II. *A recurring equation of an odd degree has one of its roots equal to* $+1$, *when the signs of the like coëfficients are different, but equal to* -1, *when their signs are alike.*

Since every power of $+1$ is positive; when the signs of the like coëfficients are *different*, if we substitute $+1$ for x, the corresponding terms will be equal, but of *different* signs; hence, they will destroy each other. But when the signs of the like coëfficients are the *same*, then since one of them will belong to an *odd* power, and the other to an *even* power, if we substitute -1 for x, the corresponding terms will be equal, but of *different* signs, and, therefore, they will destroy each other. Hence, in either case the equation will be satisfied, and may be reduced one degree lower by dividing by $x-1$, or $x+1$.

PROP. III. *A recurring equation of an even degree, in which the like coëfficients have opposite signs, and whose middle term is wanting, is divisible by* x^2-1, *and therefore, two of its roots are* $+1$, *and* -1.

Let $x^{2n}+Ax^{2n-1}+Bx^{2n-2} \ldots -Bx^2-Ax-1=0$,

be an equation of the kind specified. It may evidently be arranged thus

$$(x^{2n}-1)+Ax(x^{2n-2}-1)+Bx^2(x^{2n-4}-1)+, \text{\&c.} \ldots =0,$$

which is divisible by x^2-1 (Art. 83).

Cor. An equation of this form may therefore be reduced two degrees lower by division.

The most convenient method of reduction, either in this case or the preceding, is by means of Synthetic Division (Art. 409).

PROP. IV. *Every recurring equation of an even degree above the second, may be reduced to an equation of half that degree.*

For, $x^{2n}-Ax^{2n-1}+Bx^{2n-2}-$, &c. . . $+Bx^2-Ax+1=0$, by dividing by x^n, and collecting the pairs of terms equi-distant from the extremes, becomes of the form

$$\left(x^n+\frac{1}{x^n}\right)-A\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+B\left(x^{n-2}+\frac{1}{x^{n-2}}\right)-, \text{\&c.}, =0.$$

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, by squaring; also,

$$\left(x^3+\frac{1}{x^3}\right)=\left(x^2+\frac{1}{x^2}\right)z-\left(x+\frac{1}{x}\right)=(z^2-2)z-z;$$

and generally $\left(x^n+\frac{1}{x^n}\right)=\left(x^{n-1}+\frac{1}{x^{n-1}}\right)z-\left(x^{n-2}+\frac{1}{x^{n-2}}\right)$.

Hence, each of the binomials may be expressed in terms of z, and the resulting equation will be of the n^{th} degree.

If the signs of the terms from the beginning and end be different, let $z=x-\frac{1}{x}$, and a similar result will be obtained.

Ex. 1. Given $x^4-5x^3+6x^2-5x+1=0$, to find x.

Here $x^2-5x+6-\frac{5}{x}+\frac{1}{x^2}=0$, or $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0$

Let $x+\frac{1}{x}=z$, then $z^2-5z+4=0$, and $z=4$ or 1;

$$\text{also } x+\frac{1}{x}=4, \text{ gives } x=2\pm\sqrt{3};$$

$$\text{and } x+\frac{1}{x}=1, \text{ gives } x=\tfrac{1}{2}(1\pm\sqrt{-3}).$$

$$\text{Hence, } x=2+\sqrt{3} \quad 2-\sqrt{3}, \quad \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}.$$

The learner may not see readily that the second of these values is the reciprocal of the first, and the fourth of the third, we will, therefore, explain.

$$\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}.$$

In like manner $\frac{2}{1+\sqrt{-3}}=\frac{1-\sqrt{-3}}{2}$.

EXAMPLES IN RECURRING EQUATIONS.

1. $x^4-10x^3+26x^2-10x+1=0$

Ans. $x=3\pm2\sqrt{2}, 2\pm\sqrt{3}$.

2. $x^4+5x^3+2x^2+5x+1=0$.

Ans. $x=\frac{1}{2}(-5\pm\sqrt{21})$; $\pm\sqrt{-1}$.

3. $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$.

Ans. $x=2, \frac{1}{2}$; $\pm\sqrt{-1}$.

4. $x^4-3x^3+3x-1=0$. *Ans.* $x=\pm1, \frac{1}{2}(3\pm\sqrt{5})$.

5. $x^5-11x^4+17x^3+17x^2-11x+1=0$.

Ans. $x=-1, \frac{9+\sqrt{77}}{2}, \frac{9-\sqrt{77}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$.

6. $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$

Ans. $x=2, \frac{1}{2}, 2, \frac{1}{2}, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$.

BINOMIAL EQUATIONS.

ART. **443.** Binomial equations are those of the form

$$y^n\pm A=0.$$

Let $\sqrt[n]{A}=a$; that is, $A=a^n$.

Then $y^n\pm a^n=0$.

Let $y=ax$, then $a^nx^n\pm a^n=0$,

or $x^n\pm 1=0$,

which is a recurring equation.

ART. **444.** I.— The roots of the equation $x^n\pm 1=0$, are all unequal; for the first derived polynomial nx^{n-1}, evidently has no divisor in common with $x^n\pm 1$, and therefore there are no equal roots (Art. 414).

II.— If n be even the equation $x^n-1=0$, or $x^n=1$, has two real roots, $+1$ and -1, and no more. That it has these two roots is evident from Art. 442, Prop. III; and that it has no other real root is evident because no other number can by its involution produce 1.

By dividing $x^n-1=0$ by $(x+1)(x-1)=x^2-1$, we have

$$x^{n-2}+x^{n-4}+\ldots\ldots+x^4+x^2+1=0,$$

a recurring equation in which all the $n-2$ roots must be imaginary.

For example, the equation $x^6=1$, or $x^6-1=0$ divided by x^2-1 gives $x^4+x^2+1=0$;

$$\text{whence } x=\pm\sqrt{\left\{\frac{-1\pm\sqrt{-3}}{2}\right\}}.$$

This gives for the six roots of 1

$$+1, \qquad -1,$$

$$+\sqrt{\frac{-1+\sqrt{-3}}{2}},\quad -\sqrt{\frac{-1+\sqrt{-3}}{2}},$$

$$+\sqrt{\frac{-1-\sqrt{-3}}{2}},\quad -\sqrt{\frac{-1-\sqrt{-3}}{2}}.$$

III.— If n be odd, the equation $x^n-1=0$ has only one real root, viz.: $+1$; for $+1$ is the only real number of which the odd powers are $+1$. Dividing $x^n-1=0$ by $x-1$, we have the recurring equation

$$x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots+x^2+x+1=0,$$

of which the $n-1$ roots are imaginary.

For example, the equation $x^3=1$, or $x^3-1=0$, divided by $x-1$ gives $x^2+x+1=0$;

$$\text{whence } x=\frac{-1\pm\sqrt{-3}}{2}.$$

Hence the three third roots of 1 are

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}.$$

IV.— If n be even, the equation $x^n+1=0$, or $x^n=-1$, has no real root, since $\sqrt[n]{-1}$ is then impossible. Hence, all the roots of this equation are imaginary. For example, the four roots of the equation $x^4+1=0$, as determined by the method explained in Art. 442, are

$$\frac{-1+\sqrt{-1}}{\sqrt{2}}, \frac{-1-\sqrt{-1}}{\sqrt{2}}, \frac{1+\sqrt{-1}}{\sqrt{2}}, \frac{1-\sqrt{-1}}{\sqrt{2}}.$$

V.— If n be odd, the equation $x^n+1=0$, or $x^n=-1$, has one real root, viz.: -1, and no more, because this is the only real number of which an odd power is -1. For example, the equation

$x^3+1=0$, divided by $x+1$ gives
$x^2-x+1=0$,

whence $x=\frac{1\pm\sqrt{-3}}{2}$ $\therefore$ the three third roots of -1, are -1, $\frac{1+\sqrt{-3}}{2}$, and $\frac{1-\sqrt{-3}}{2}$.

Binomial equations have other properties, but some of them cannot be discussed without a knowledge of Analytical Trigonometry.

For exercises the pupil may find

1. The four fourth roots of unity.

Ans. $+1, -1, +\sqrt{-1}, -\sqrt{-1}$.

2. The five fifth roots of unity.

Ans. 1,

$\frac{1}{4}\{\sqrt{5}-1+\sqrt{(-10-2\sqrt{5})}\}$,

$\frac{1}{4}\{\sqrt{5}-1-\sqrt{(-10-2\sqrt{5})}\}$,

$-\frac{1}{4}\{\sqrt{5}+1+\sqrt{(-10+2\sqrt{5})}\}$,

$-\frac{1}{4}\{\sqrt{5}+1-\sqrt{(-10+2\sqrt{5})}\}$.

THE END.