MONOGRAPHS ON PHYSICS

EDITED BY

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CAVENDISH PROFESSOR OF EXPERIMENTAL PHYSICS, CAMBRIDGE

AND

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PROFESSOR OF PHYSICS IN THE UNIVERSITY OF LONDON

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RELATIVITY

AND

THE ELECTRON THEORY

BY

E. CUNNINGHAM, M.A.
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WITH DIAGRAMS

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PREFACE.

THIS monograph is an attempt to set out as clearly and simply as possible the relation of the Principle of Relativity to the generally accepted Electron Theory, showing at what points the former is the natural and necessary complement of the latter.

No attempt has been made to describe to any great extent consequences of the Principle which would be for the most part beyond the reach of experimental investigation, and the mathematical analysis has been omitted as far as possible with the hope of rendering the account useful to the general reader, especially to the experimental physicist. Those who desire to follow out the train of thought in more detail, and especially to make acquaintance with the mathematical presentation developed by Minkowski, may be referred to the author's larger book on the same subject. The author's acknowledgments are due to the Cambridge University Press for their permission to use some of the material of that work in the preparation of this.

E. C.

CAMBRIDGE, 1915.

CONTENTS.

NOTE ON VECTOR NOTATION.

VECTOR quantities are denoted throughout the book by Clarendon Type, thus, **e**, **h**, **u**.

Definitions.

(i) *Vector Product.*—The vector product of two vectors, **e**, **h**, is a vector at right angles to the directions of both, and of magnitude equal to the product of the magnitudes of the two into the sine of the angle between them. It is denoted by the symbol [**eh**].

If (e_x, e_y, e_z) (h_x, h_y, h_z) are the components of **e** and **h**, the components of [**eh**] are

$$(e_y h_z - e_z h_y, e_z h_x - e_x h_z, e_x h_y - e_y h_x).$$

(ii) *Divergence* of a vector.

$$\text{div } \mathbf{e} = \frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} + \frac{\partial e_z}{\partial z} = \underset{V \to 0}{\mathrm{L}t} \iint e_n dS$$

where dS is an element of a closed surface bounding a volume V, and e_n is the component of **e** normal to dS.

(iii) *Curl* of a vector.

$$\text{curl } \mathbf{e} = \left(\frac{\partial e_z}{\partial y} - \frac{\partial e_y}{\partial z}, \frac{\partial e_x}{\partial z} - \frac{\partial e_z}{\partial x}, \frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y}\right).$$

The component of curl **e** in a given direction $= \underset{S \to 0}{\mathrm{L}t} \int e_n ds,$

where ds is an element of the arc of a closed curve which bounds a small area S to which the given direction is normal.

CHAPTER I.

INTRODUCTORY.

1. THE Principle of Relativity is the hypothesis that it is impossible by means of physical experiments to determine the absolute velocity of a body through space. This is not a metaphysical doctrine based on the supposition that it is impossible to conceive of an absolute standard of position in space. Such an impossibility is asserted by many writers on the theory of mechanics, but it is quite irrelevant to the present subject. Physical theory has nothing to do with an absolute space. The space with which it deals is one aspect of the relations which have been observed between the phenomena with which it is concerned.

Newton in setting out his scheme of dynamics postulated an absolute standard of position in space relative to which all velocities are measured; such a beginning was suggested by the history of the astronomical problem with which he was dealing. Copernicus shifted the centre of the universe from the earth to the sun, and from this it was a very short step to the assumption that the sun was not an absolute centre any more than the earth. Hence the assumption of an unknown but definite criterion of position in space.

But when, on the background of this assumption, the science of dynamics had been developed, it was found that the laws which had been framed had a very special property. If they were universally satisfied, there was no means of determining the actual velocity of any body, though the relative velocity of two bodies was determinate. In fact, it in no way disturbs the general laws of dynamics if an arbitrary velocity is added to the velocity of every body in the system. In other words, the frame of reference assumed at the beginning is not a unique one, but may be any one of an infinite number of which any one has relative to any other a constant velocity of translation without rotation. To assert that a unique frame of reference can never be found is perhaps not absolutely proved, but neither is any scientific hypothesis. All that is done is to put the hypothesis on such a footing that it seems waste of time to endeavour to dispense with it.

This dynamical relativity is a very different thing from a statement as to what is within the power of the mind to conceive about position and motion. It is indeed doubtful whether it is possible or logical to think about either of these entirely apart from the regularities perceived in the motions of actual bodies. In any case if the doctrine of the relativity of all motion were *meta*-physical, it would have to extend beyond the scope of that indicated by Newton's laws of motion, and to deny the existence of any criterion of a fixed direction in space or of the validity of the conception of a constant velocity. It would result in an entirely agnostic position about motion which would have to be surrendered whenever it was desired to deal with actual facts of experience.

2. The limited scope of the relativity arising out of dynamical theory has always been rather unsatisfying to the mind, and so it was almost with a sense of relief that the suggestion of a universal æther was hailed as offering a possibility of a return to an absolute theory, that is to one in which a velocity can be uniquely assigned to every body, namely, its velocity relative to the æther. In fact, it seemed clear, since all astronomical observations are made by optical means, that the frame of reference in practice must be actually the medium relative to which light is propagated. Thus arose the attempt to determine the velocity of the earth relative to the æther.

Such a quest has of course no meaning apart from a definite conception of the æther. In the first instance the experiments were rather directed towards clearing up the question as to what was the best way to think of the æther, whether as another variety of matter filling all space not otherwise occupied, or as a medium of such nature that it could penetrate matter and be undisturbed by the passage of matter through it. It was the latter way of thinking of it toward which opinion gradually tended and which has been almost universally adopted. It is perhaps not wise to make a definite statement as to why this was so, but, at any rate, a very important factor was the development of the precise statement of the laws of electromagnetic phenomena as initiated by Maxwell and elaborated in the electron theory of Larmor and Lorentz.

Just as the enunciation of Newton's laws of motion gave a meaning to the notion of 'absolute direction' which before it did not possess, so the adoption of Maxwell's equations, as expressing the laws of propagation of light and electro-magnetic disturbances, gave a new meaning to the term 'position in space' by relating it to the medium whose properties the equations were thought to embody. We must remember, however, that this medium is defined only by these same properties and equations, and that to base any argument on an analogy with a hypothetical continuous medium of material and mechanical properties in the ordinary sense, properties which have not been shown to be involved in the definition of the æther, is to become liable to finding ourselves at variance with the facts. The æther is, in truth, nothing more than the aggregate of the functions which it serves, though we find it difficult to think of it except in some concrete and uniquely existent form; the exact form, however, which is adopted is largely determined by individual preference for this or that analogy.

3. The conception of the 'stagnant' æther having been adopted, it came as a great surprise that all experimental efforts to find a physical effect due to the motion of bodies relative to it proved a failure. Theory seemed to indicate several ways of observing such an effect. The failure of all these methods necessitated a reconciliation of theory and experiment, and such a reconciliation was rendered possible by the work of Larmor and Lorentz; if not complete in every detail, yet it went so far that many were led to the belief that not only some but all experiments of this nature were foredoomed to failure (e.g. Larmor, Brit. Ass., Belfast, 1902).

To admit such a possibility is to allow that the æther, whatever its nature, may for ever remain concealed. This is to weaken considerably our power of thinking of the æther as an actually existing medium, though it in no way affects the scheme of properties which it was intended to connote. Rather in the hands of Lorentz and Larmor it becomes a confirmation of the theory to find that the experiments actually do fail. But it is hardly true to say that, if we grant this hypothesis that we shall never be able to identify a unique frame of reference for the mathemetical theory, then we deny the existence of a real medium of propagation of the physical effects of light and electricity. What we do is only to question the sufficiency of the existing conceptions of the nature of this medium, and the validity of identifying it with a frame of reference which experience finds to be far from definite.

4. If the dynamics of Newton leave some ambiguity in the meaning of space relations there is no such doubt left in the case of time intervals. So definite was a time interval to him that he was able to speak of absolute time as 'flowing evenly on' as if independent of all phenomena. One of the fundamental elements in this idea of absolute time is that of the 'simultaneity of events' at different places. No ambiguity about the meaning of this appears to have arisen. But if we grant the impossibility of determining the velocity of the earth through the æther, using the term for a moment in the sense of the frame of reference for the propagation of light, this idea of 'simultaneity' becomes as vague as that of the velocity of a moving body.

If we consider that the science of astronomy and the law of gravitation are dependent for their verification on optical observations, we shall be inclined to agree that so far at any rate the criterion of simultaneity which has been used in practice has been one based on optical communication. Now we shall see later that the setting up of a standard of simultaneity by means of light signals is not possible until a definite velocity is assigned to the observer. Thus the hypothesis of relativity requires a reconsideration of the way in which we measure time.

This again reacts on the measurement of the length of a material body, the 'distance between two points' being the distance between simultaneous positions of those points. Thus it becomes necessary also to examine the way in which we measure space. It becomes impossible to consider space and time separately; the two measures are inter-related to such an extent that Minkowski felt himself constrained to say that "from henceforth time by itself and space by itself are mere shadows, that they are only two aspects of a single and indivisible manner of co-ordinating the facts of the physical world".[1]

5. In what follows, then, we have to consider first the exact way in which our fundamental notions of space, time, and motion have to be modified. This will lead us on to a statement of the status assigned to dynamical theory according to the principle of relativity, and in particular will require a discussion of the ideas of momentum and energy. It is well known that the idea of a constant mass for every body has to be modified if we take into account the effects of radiation and electrical constitution. We shall see that we can draw, from the general Principle of Relativity, some very important conclusions as to the mechanical relations of systems, without having recourse to any particular theory of the nature of an electron or of the way in which matter is built up out of them. Our growing sense of the insufficiency of the existing pictures of the constitution of matter makes this a very important consideration.

[1] Raum und Zeit, Leipzig, 1909.

CHAPTER II.

THE ORIGIN OF THE PRINCIPLE.

6. Space and Time in Newton's Dynamics.

In order to appreciate the full significance of the new point of view introduced by Einstein in 1905 and since that date associated with the technical name, "The Principle of Relativity," it is desirable to make a survey of the previous lines of thought.

Newton planted deeply in scientific thought the ideas of an absolute position in space and of an absolute time. The significance of the Copernican revolution in astronomy extends beyond the moving of the origin of reference from the observer to the sun. It makes it impossible to think even of the sun as a permanent and ultimate point of reference. We naturally go on to think of the sun as one member of a system greater than the solar system, and we can see no end to the process of shifting the point of reference. We have no conception of the actual velocity of the earth or of the sun in space. But the precise scheme of laws set forth by Newton introduced an absolute element into our conceptions of celestial motions. In the light of his laws we perceive that the reason that we are able over a large range of terrestrial phenomena to treat the earth as the point of reference and to ignore the remainder of the universe, is just that the earth's absolute velocity may be thought of as changing very slowly. Similarly, the reason that a theory of the solar system as a separate system is possible, ignoring the whole of the rest of the universe, is that we may conceive of its centre of mass as having a constant absolute velocity.

Although later we find that the 'absolute velocity' of a body is not determinable, yet the classical dynamics do give us a criterion of something which we may call 'absolute direction', though the philosopher may assert that we are not using the term in a strict sense. In the sense in which we use the term this absolute direction is not a thing that is obvious *a priori*, arising out of pure thought; it is a fact which emerges out of the perceived motions of bodies as we try to reduce them to the simplest possible descriptive scheme. The scientific conception of space is inseparable from the laws of dynamics. There is no other way of making space an object of measurement save by means of this body of law. Apart from it or from some equivalent, motion is a mere sensation, though it is doubtful whether the most primitive man is without some rudimentary notion of those regularities which have been finally summed up in the law of inertia.

7. The Relativity of Newton's Dynamics.

But now we come to the remarkable fact that the absoluteness introduced into our ideas of space by the laws of dynamics is only partial. It is a commonplace that if any frame of reference is known with respect to which the laws of motion are satisfied then any other frame which is moving relative to that known frame with a constant velocity as judged by the standards of dynamics, is equally suitable as a basis for the description of the motion by the same laws.

This partial relativity it will be convenient to speak of as 'Newtonian Relativity'. It is not a philosophic principle, it is purely empirical, and rests for its justification on the agreement of the deductions from it with the facts of observation. Thus from this point of view we have no criterion whereby we may say in any defined sense that a given point is at rest in space.

A similar position is to be taken as to the nature of time as an object of measurement. There may or may not be an *a priori* standard of the equality of two elements of duration, but we do not know what it may be. For scientific purposes of exact measurement the time of which we think is defined by the laws of motion. There is no real need to say with

Newton that "absolute time flows uniformly on," and to pretend that, given this time, we lay our phenomena of motion in thought against it and so measure them. We cannot doubt that that definition of time was invented after the system of dynamics had been worked out, and was prefixed to satisfy the demand for definitions and postulates at the beginning of any systematic treatise.

8. Time not an Absolute or Independent Concept.

Time, then, as an object of measurement is nothing more than one aspect of the relations which we are able to disentangle from the complex of phenomena of motion. But it is such a definite aspect that it seems to exist almost in its own right. The partial relativity that exists in the scheme of relations leaves no ambiguity in the term 'equality of two intervals of time' nor in the term 'equality of two distances'. These are such definite and separable elements in the construction of the scheme that it is possible to reconstruct the idea of the measurement of motion out of them. The Newtonian picture of the motions of bodies is the perfected result of a long process of analysis of perceptions and refinement of concepts. Beginning with the crude perception of change, the continual comparison of motions results in the extraction of the conceptual time and space of mathematics. In terms of these we are able to build up a mental model of the motion which is susceptible to mathematical treatment. This model is only approximate, since it necessarily ignores certain factors in the actual system. Our planetary theory, for instance, disregards the effect of the distant stars, and the neglect is justified *a posteriori* by the fact that the discrepancy between model and system is incapable of observation.

9. Mathematical Form of the Newtonian Principle of Relativity.

Before proceeding further, it will be useful to set down in mathematical form the chief aspects of 'Newtonian Relativity'.

(i) *The Space-Time Transformations.*—If (x, y, z) are a set of space coordinates and t an associated time variable which define together a Newtonian frame of reference, that is, which are such that if the motion is described by the use of these variables, the laws of motion are satisfied, then if a new set of variables is taken, defined by the equations

$$x' = x - at, \quad y' = y - \beta t, \quad z' = z - \gamma t, \quad t' = t,$$

and if the motion is now described by turning the given relations between (x, y, z), and t into relations between (x', y', z') and t', then the relations so obtained between (x', y', z') and t' will also satisfy the laws of motion; in other words, the new variables also define a Newtonian frame of reference.

In fact, a point which as measured in the first set of coordinates has the velocity (u, v, w) has in the second the velocity $(u - a, v - \beta, w - \gamma)$, and the relative velocity of two points is the same in both systems, a, β, γ being given constants.

(ii) *The Law of Addition of Velocities.*—The relations

$$u' = u - a, \quad v' = v - \beta, \quad w' = w - \gamma$$

connecting the velocities of a given point relative to two Newtonian frames of reference, of which the origin of the second has a velocity (a, β, γ) relative to the first, seem to us to be obvious relations, but it is necessary to remember that the fact that either (u, v, w) or (u', v', w') may be conveniently called the velocity of the body in the appropriate frame of reference depends upon what has been said of the possibility of maintaining unchanged the form of the dynamical laws when we change the frame of reference. We shall see later that these relations cease to be convenient when we are dealing with the extremely great velocities which we associate with the negative electrons constituting the cathode and β-rays.

(iii) Newtonian Relativity consists in the fact that either (x, y, z, t) or (x', y', z', t') are equally valid as space-time coordinates, and (u, v, w) or (u', v', w') as the corresponding measures of velocity. But there are certain quantities which are 'invariants', that is, which have the same value whichever set of coordinates we employ. These are—to mention the most simple—the 'mass' of a particle, the 'force' acting upon a particle, and the 'acceleration' of a particle. When we come to higher dynamics, we find that the 'constants of inertia' of a rigid body and other quantities have invariant values, whereas the 'energy' of a system is a *relative* quantity though the 'principle of the conservation of energy' is an *invariant relation.*

10. The Principle of Conservation of Momentum Deducible from the Principles of Conservation of Energy and of Relativity.

If we assume that the principle of the conservation of energy in the form

'*rate of increase of kinetic energy = rate of work of forces*'

is true whatever Newtonian frame of reference we use, we may deduce the principle that

'*rate of increase of momentum in any direction = sum of forces in that direction*'.

For, whatever the values of the constants α, β, γ, we must have

$$\frac{d}{dt}\Sigma\tfrac{1}{2}m\{(u-\alpha)^2+(v-\beta)^2+(w-\gamma)^2\}$$
$$=\Sigma\{X(u-\alpha)+Y(v-\beta)+Z(w-\gamma)\},$$

m being the mass of a particle of the system, and (X, Y, Z) being the components of any force applied to the system.

Hence

$$\Sigma m\left\{(u-\alpha)\frac{du}{dt}+(v-\beta)\frac{dv}{dt}+(w-\gamma)\frac{dw}{dt}\right\}$$
$$=\{X(u-\alpha)+Y(v-\beta)+Z(w-\gamma)\},$$

whatever the values of α, β, γ.

Hence the coefficients of α, β, γ separately must be equal on the two sides of the equation: or

$$\Sigma m\frac{du}{dt}=\Sigma X$$
$$\Sigma m\frac{dv}{dt}=\Sigma Y$$
$$\Sigma m\frac{dw}{dt}=\Sigma Z,$$

which equations express exactly the ordinary principle of momentum.

Exactly a similar relation between the three principles of momentum, energy, and relativity is maintained in Einstein's "Principle of Relativity," which is to be developed later.

11. The Advent of Æther Theory.

It is on the basis of time and space as described above that subsequent physical thought has rested. All motion has been tacitly referred to a Newtonian frame of reference. But the nineteenth century began a new era. Early in that century the idea of a luminiferous æther came into prominence. The question was at once asked, "*What is the velocity of the earth relative to this medium?*"

Arago, sometime before 1818,[1] devised an experiment which he thought would answer that question. On the wave theory of light the index of refraction of a piece of glass is the ratio of the velocity of the incident light to the velocity of the refracted light. If the glass were moving through the æther to meet the incident light, the relative velocity of the light at entering the prism would be greater than if the prism were at rest in the æther. The velocity of the light in the prism is however the same, so Arago thought, and thus the index of refraction is greater, and therefore the deviation of the ray of light should be greater when the prism is moving towards the ray than when it is moving away from the ray or is at rest in the æther. It was such a difference in the deviation that Arago hoped to find when he examined rays of light from stars in different directions, all of which could not make the same angle with the motion of the earth through the æther. The difference corresponding to a velocity of the earth equal to its velocity relative to the sun would have amounted to about one minute of angle, and this was well within the reach of his means of observation, so that he expected to perceive a measurable difference; but he did not. Writing to Fresnel for an opinion on his experiments, he was told that the only place at which his argument might break down was in the assumption that the light travelled through the prism at the same relative rate whether the prism was moving or at rest in the æther.

[1] See the letter from Fresnel to Arago on the subject of the experiment, "Annales de Chimie," 1818.

Fresnel's suggestion was that the prism did not communicate to the light the whole of its velocity but only a fraction of it. If u were the velocity of the light in the prism when at rest in the æther, and the prism were set in motion with velocity v relative to the æther, then Fresnel asserted that the deviation would not be altered at all, if the velocity of the light relative to the prism were now $(u - v/\mu^2)$, μ being the ordinary index of refraction; or, in other words, if the prism communicated to the ray not its whole velocity but only the fraction $(1 - 1/\mu^2)$ of it. This fraction is now commonly spoken of as 'Fresnel's convection-coefficient'.[1]

12. Matter Modified by its Motion Through the Æther.

The point of Fresnel's suggestion which is of most significance for us here is that it is a suggestion that matter, by its motion through the æther, may be modified in just such a way as to neutralize an effect which it was thought would otherwise arise. We shall see later several more instances of hypotheses that have been suggested for the purpose of explaining the non-appearance of effects to which theory seemed to point. It is one of the main objects of the Principle of Relativity to gather together these separate *ad hoc* hypotheses under a single assumption, and to deduce consequences which do not arise out of the particular and disconnected hypotheses.

13. Fizeau's Verification of Fresnel's Hypothesis.

In 1851 Fizeau set out to verify directly Fresnel's suggestion of a convection-coefficient by actually observing the effect of the motion of a stream of water on the velocity of light through it. He devised an experiment in which a beam of light is divided into two parts which traverse two parallel tubes filled with water which can be set in motion with a measurable velocity. A beam of light from the source *a* (Fig. 1) rendered parallel by the lens *b* is divided into two parts by the half-silvered plate *c*. The two parts are reflected by the

[1] Larmor, "Aether and Matter," 1900, p. 38, shows that Fresnel's hypothesis is not only sufficient but is necessary if Arago's experimental evidence is universally confirmed.

mirrors *d* into the two tubes AB, CD which contain the water, and are sent back through the opposite tube by the totally reflecting prism *f*; they are then reunited by the mirrors *d* and the plate *c* and enter the telescope *g*, through which interference fringes are observed.

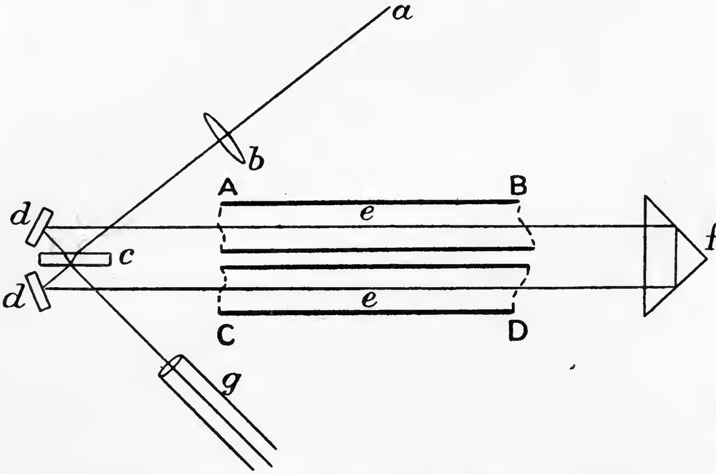


FIG. 1.—Fizeau's apparatus as improved by Michelson and Morley.

Explanation of Diagram.—*a*, source of light; *b*, condensing lens; *c*, half-silvered plate of glass to divide the beam of light; *d*, *d*, mirrors; *e*, *e*, tubes carrying stream of water in opposite directions; *f*, totally reflecting prism; *g*, observing telescope.

The water in the tubes is then caused to circulate, moving in opposite directions along AB and CD, so that it is moving with one part of the beam of light and against the other. A shifting of the interference fringes is at once observed, which is proportional to the velocity of the water.

Supposing that the velocity of light through the moving water is $c' + kv$, where c' is the velocity when the velocity v is zero, the times taken by the two parts of the beam to traverse the total path l in the water are respectively

$$\frac{l}{c' + kv} \text{ and } \frac{l}{c' - kv},$$

the velocity v being in opposite senses for the two parts. Thus the retardation of the one beam relative to the other is

$$\frac{l}{c' - kv} - \frac{l}{c' + kv} = \frac{2lkv}{c'^2},$$

neglecting $(v/c')^2$.

On measuring the displacement of the interference fringes, Fizeau found that Fresnel's value for the convection-coefficient k, viz. $1 - \mu^{-2}$, fitted his results quite well. The conclusion was confirmed later by Michelson and Morley, who repeated the experiment with modern refinements. The detailed discussion of Michelson and Morley's results will be referred to later (p. 41, §§ 36-8). It will be seen that they are entirely in agreement with the predictions of the principle of relativity.

14. The Significance of Fresnel's Convection-Coefficient, and of Fizeau's Verification of It.

The remarkable experimental verification of Fresnel's brilliant guess considerably strengthened the tendency towards the stagnant æther theory. It might have seemed that the hypothesis was a very artificial one, but we must admit that Fresnel's preference for this conception was not without reason. To quote his own words:—

"If we admitted that our globe communicates its velocity to the æther which surrounds it, it would be easy to see why the same prism refracts light in the same manner from whatever direction it comes. But it seems impossible to explain the aberration of the stars on this hypothesis; so far I have not been able to think clearly of this phenomenon except by supposing that the æther passes freely through the earth, and that the velocity given to this subtle fluid is only a very small part of that of the earth; not more than one hundredth part, for example."

The outstanding fact of aberration must, indeed, not be forgotten in considering the various experiments and hypotheses that have been devised to meet the various questions that arise.

Sir George Stokes, a great student of the motion of fluids, tried hard to evolve a theory of the æther which would allow of it being dragged along with the earth, and at the same time be consistent with the law of aberration. But his theory never received a large amount of support, and was practically abandoned when the establishment of the identity of light and electric disturbances required that the æther should be conceived as much in relation to electro-magnetic phenomena as to optical. But the immediate significance of Fizeau's experiment was that it was thereby definitely shown that the velocity of light relative to a material medium is not a constant depending on the nature of the medium alone, but is also dependent on the velocity of the medium through space, or rather through the æther, which was conceived by Fresnel to be at rest in space.

The effect being a first order one, that is, being proportional to the first power of the velocity, the experiment throws no light on the velocity of the earth itself, since the part of the effect due to this velocity remains practically constant, the observed effect being proportional to the change in the velocity of the water in the tubes relative to the earth. But henceforth it was clear that in respect of its optical properties a material medium must be admitted to be in some sense modified by its motion.

When at a later date light was identified as an electro-magnetic disturbance this became one deciding factor as between Hertz' theory of electro-magnetic phenomena in moving bodies, and that of Maxwell and Lorentz. Hertz' idea of a moving body was just an extension of the Newtonian idea of a rigid body in motion. The properties of the body were conceived by him to be entirely unchanged when it was set in motion, just as the dynamical mass of a body was unaltered; and among other implications was the complete convection of light, that is, the invariance of the velocity of light relative to the body. In view of Fizeau's experiment it was not possible to maintain this theory, and other experiments soon showed it to be lacking in other respects.

Fresnel's suggestion of the interpenetration of æther and matter therefore assumed greater prominence, and became the basis of the later theory developed by Lorentz and by Larmor.

15. The Michelson-Morley Experiment.

The question was, however, by no means regarded as settled by Fizeau's results. In 1881 we find A. A. Michelson returning to it with an attempt at a direct investigation of the difference in the velocity of light relative to the earth in different directions by a method which did not involve propagation through material bodies. It was hoped thus to prevent the possibility of an explanation of the results by means of hypotheses about the modification of matter by its motion through the æther such as Fresnel had advanced in the case of Arago's experiment. The result of this experiment on subsequent thought was so far-reaching that it will be well to describe it in detail.

The experiment was first carried out by Michelson in 1881, and was repeated with greater refinement with the assistance of E. W. Morley[1] in 1887, and again with even greater care by Morley and D. B. Miller in 1905.[2]

Fizeau's experiment, as has been pointed out, being a first order experiment, would only reveal the influence of velocity relative to the earth. In the Michelson-Morley experiment it was proposed to seek for an effect depending on the square of the velocity, and to avoid any question concerning the internal constitution of moving matter. The arrangement was as follows, the figure showing the path of the light relative to the moving apparatus :—

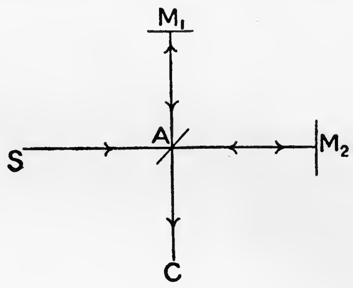


FIG. 2.

A beam of light from a source S was divided by partial reflection at a plate of glass A into two portions travelling along the paths AM_1, AM_2. These two portions were reflected back by mirrors M_1 and M_2, and, on striking the plate again, a portion of the beam from M_1 is transmitted and brought to interference with a portion of the beam from M_2 reflected along AC. The whole apparatus could be rotated into any position desired.

[1] "Phil. Mag.," 1887. [2] *Ibid.*, 1905.

16. The Theory of the Experiment.

Suppose first that AM_2 is in the direction of the earth's velocity relative to the æther, which will be called v. Then since $(c - v)$ is the velocity of light relative to the apparatus on the forward journey and $(c + v)$ is the relative velocity on the return journey, the time taken by the light to travel from A to M_2 and back is

$$\frac{l_2}{c - v} + \frac{l_2}{c + v} = \frac{2l_2c}{c^2 - v^2},$$

where l_2 is the distance AM_2.

In considering the time taken to go to M_1 and back we have to take a ray whose velocity relative to the apparatus is along AM_1, which is assumed perpendicular to AM_2. The relative velocity along this line will thus be $(c^2 - v^2)^{\frac{1}{2}}$, and the time taken is therefore

$$\frac{2l_1}{(c^2 - v^2)^{\frac{1}{2}}},$$

where l_1 is the distance AM_1.

Thus the retardation in time of the former beam relative to the latter when they are brought together again is

$$2\left\{\frac{l_2c}{c^2 - v^2} - \frac{l_1}{(c^2 - v^2)^{\frac{1}{2}}}\right\}.$$

If the apparatus is now rotated so that AM_1 comes into the direction of the light, l_1 and l_2 change places, and the retardation is altered to

$$2\left\{\frac{l_2}{(c^2 - v^2)^{\frac{1}{2}}} - \frac{l_1c}{c^2 - v^2}\right\}.$$

Thus the change in the retardation is

$$2(l_2 + l_1)\left\{\frac{c}{c^2 - v^2} - \frac{1}{(c^2 - v^2)^{\frac{1}{2}}}\right\}.$$

17. Alternative Explanation.

The following explanation, alternative to that given above, together with fig. 3 which illustrates the absolute path of the rays which interfere, may perhaps make the theory of the experiment a little clearer.

At a certain moment t_3 let A_3 be the position of the moving plate and let us consider the paths of the two parts of the beam which are reunited at that moment. Let A_1 be the position of the plate at the instant t_1 at which that element of disturbance leaves A_1 which, travelling by the path $A_1M_1A_3$, arrives at A_3 at time t_3. Then the time t_1 is found thus :—

$$\begin{aligned} A_1A_3 &= v(t_3 - t_1), \\ c(t_3 - t_1) &= A_1M_1 + M_1A_3 \\ &= 2(l_1^2 + \tfrac{1}{4}A_1A_3^2)^{\frac{1}{2}} \\ &= \{4l_1^2 + v^2(t_3 - t_1)^2\}^{\frac{1}{2}}, \end{aligned}$$

giving $$(c^2 - v^2)(t_3 - t_1)^2 = 4l_1^2,$$

or $$t_3 - t_1 = \frac{2l_1}{(c^2 - v^2)^{\frac{1}{2}}}.$$

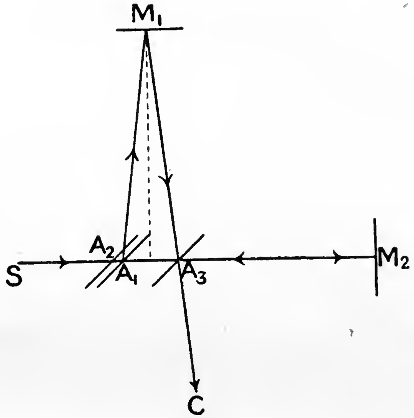


FIG. 3.

In the same way let A_2 be the position of the plate at the moment t_2 of the emission of the element of disturbance which, travelling by the path $A_2M_2A_3$, arrives at A_3 at time t_3.

Then $$\begin{aligned} A_3A_2 &= v(t_3 - t_2), \\ A_2M_2 &= c(\tau - t_2), \\ M_2A_3 &= c(t_3 - \tau), \end{aligned}$$

where τ is the moment of reflection.

Also l_2 is the distance of M_2 from the position of the plate at time τ.

Thus :— $$\begin{aligned} l_2 &= A_2M_2 - v(\tau - t_2) \\ &= (c - v)(\tau - t_2), \end{aligned}$$

and similarly $$l_2 = (c + v)(t_3 - \tau),$$

giving $$\frac{l_2}{c - v} + \frac{l_2}{c + v} = t_3 - t_2,$$

or $$t_3 - t_2 = \frac{2l_2c}{c^2 - v^2}.$$

Thus the disturbances which are united at A_3 at time t_3 do not leave the plate A simultaneously, but at instants separated by an interval

$$t_2 - t_1 = \frac{2l_1}{(c^2 - v^2)^{\frac{1}{2}}} - \frac{2l_2c}{c^2 - v^2},$$

and therefore will differ in phase by this amount, exactly as obtained above. The question to be answered by the experiment is whether this difference of phase will be altered when the apparatus is turned round.

Looking at the theory from this point of view it becomes necessary to consider whether the directions of the reunited parts of the beam will necessarily be the same. This involves the question of the reflection of a ray of light at a moving mirror. Let us consider this by means of Huygens' principle.

Let AB be a reflector placed at any angle α with SS′, and moving with velocity v in the direction SS′. Let AX be a plane wave-front incident on the mirror at A at a certain instant; at a small time τ later, the unreflected portion of this will have advanced a distance $c\tau$, being now part of the line X′N. In the same time the reflector has advanced a distance AA′ $= v\tau$; so that, drawing A′B′ parallel to AB to meet X′N in C, C is now the point of incidence of the wave-front on the reflector. Thus the reflection is exactly that which would take place at a fixed mirror in the position AC.

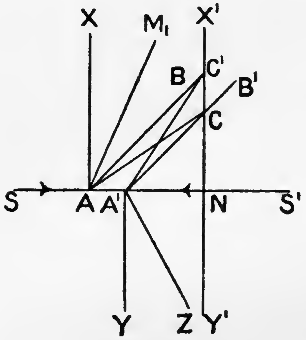


FIG. 4.

In the same way, considering a ray incident on the other side of AB, moving in the opposite direction, we find that A′C′ is the direction of the equivalent fixed mirror.

Thus if the mirror is set at an angle of 45° with AS′ the direction of the reflected part of the beam at the first incidence will be AM_1 where the angle

$$XAM_1 = 2C'AC,$$

and the direction of the transmitted ray after its final reflection will be A′Z where

$$YA'Z = 2C'A'C.$$

The condition that the reunited rays shall be parallel is that

$$XAM_1 = YA'Z,$$

that is that

$$CAC' = C'A'C,$$

and this is satisfied since the right-angled triangle ANC′ is isosceles.

18. The Result of the Experiment.

Interpreted according to this calculation, if the velocity of the earth relative to the æther had been as much as a quarter of the velocity of the earth in its orbit, there would have been a displacement of the fringes produced by the interference of the two beams of such magnitude that it could not have escaped observation with the apparatus used. But no trace of such a displacement was found.

Morley and Miller, repeating the experiment in 1905, with still greater refinement, also came to the conclusion that there was no displacement, though they could have observed one due to one-tenth of the velocity of the earth.

19. The FitzGerald-Lorentz Contraction.

If, as we seem bound to, we accept the above results as showing the theory which has been given to be inadequate, the hope of determining v by this means vanishes, and the difficulty remains of reconciling the null result with the hypothesis of an æther which is not convected along with the optical system by the earth.

FitzGerald[1] threw out a suggestion that if the æther can percolate through matter, it may affect the apparatus, and change its dimensions when it is rotated. Such a suggestion had become feasible in view of the adoption of the electromagnetic theory of light, and the growing knowledge of the electrical relations of matter.[2]

If such an effect is to nullify the change in the retardation, we must have, if l'_1, l'_2 are the changed lengths of AM_1, AM_2,

$$\frac{l_1}{(c^2 - v^2)^{\frac{1}{2}}} - \frac{l_2 c}{c^2 - v^2} = \frac{l'_1 c}{c^2 - v^2} - \frac{l'_2}{(c^2 - v^2)^{\frac{1}{2}}},$$

and this is satisfied if

$$\frac{l_2}{l'_2} = \frac{l'_1}{l_1} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}};$$

that is, if either arm contracts in the ratio $1 : (1 - v^2/c^2)$ when turned into the direction of motion, as compared with its length when at right angles to this direction.

[1] See O. Lodge, "Aberration Problems," "Phil. Trans.," 184A (1893), p. 727; also Presidential Address to the British Association, 1913.

[2] E.g. Maxwell's law connecting the index of refraction with the dielectric constant.

FitzGerald's suggestion was not carried further at the time, and it was left to Lorentz to make it independently,[1] and to show at the same time some plausible reason why a contraction of exactly this amount might be expected.

The theory to which Lorentz was led in his investigations into the optical behaviour of moving bodies is so fundamental to the present subject, and is, if not always in its original form, so generally accepted, that some account of it must now be given.

20. The Null Result Independent of the Material Constituting the Apparatus.

Before this is done, however, we may note a fact of great importance for the general significance of the Michelson-Morley experiment. As first carried out, the whole apparatus was mounted on a sandstone block which floated in mercury. In the repetition by Morley and Miller, the distance between the mirrors was intentionally maintained by wooden rods.

Thus if the null effect is to be explained by FitzGerald's suggestion, the contraction of the right amount must automatically take place in two such different materials as sandstone and pine. It is difficult to think of this as a mere coincidence, so that we are naturally led to think that if we accept the contraction hypothesis, whatever explanation of it we may adopt, it must be one that is universal, and inherent in the constitution of matter, even down to the structure of the chemical atom.

21. The Influence of the Michelson-Morley Experiment on the Development of Electrical Theory.

The contraction hypothesis of FitzGerald is a second instance of an *ad hoc* addition to existing theory for the express purpose of making it fit the facts, of which the first instance, cited above, was Fresnel's hypothesis of a convection-coefficient. But it was not possible for the matter to rest at this point. Why should matter contract to exactly the required

[1] "Versuch einer Theorie der Elektrischen und optischen Erscheinungen in bewegten Körpern," Leiden, 1895.

extent, and that, too, in the case of two such different materials as stone and wood? The necessity of answering this question supplied a direct and powerful stimulus to the construction of a theory of matter in which the suggested change in the length of a body is no longer an arbitrary hypothesis, but from which it is a necessary deduction.

The result of these efforts was the electron theory of Lorentz and Larmor. In this theory an ideal picture of matter in general is constructed. The special properties characteristic of the particular chemical constitutions of bodies are eliminated from the consideration at the outset; the general laws which govern them are supposed to be common to all kinds of matter. All matter is conceived to be composed of electrons, that is, of small nuclei which are indivisible, and, though this is not absolutely necessary, of identical nature for all kinds of matter. As far as possible the exact nature of these electrons, their size and shape, and the way in which the electric charge is distributed on them, is left out of consideration. These electrons are in motion within the body which they constitute, and set up an electro-magnetic field according to the laws which were first completely formulated by Maxwell.

This field in its turn is supposed to determine the way in which the electrons move. But the general laws which determine the field when the motions of the electrons are given in advance are not sufficient in themselves to do this.[1] To supplement them some further assumption has to be made as to the nature of the reaction of the æther on the electron.

We may see at once how the desire to explain the automatic contraction of a body when set in motion through the æther regulated the choice between the various hypotheses that might present themselves to the mind. Lorentz showed that all that was necessary beyond the equations of Maxwell to determine the motion of an electron was to assume (i) that its exact size and shape, or, more precisely, the exact distribution of the electric charge, should be given, and (ii) that the whole inertia or kinetic reaction of the electron was due to

[1] For a fuller discussion, see Chapter IV, pp. 44 ff.

the electric field which by its motion it was setting up in the æther. This second assumption is known as the '**hypothesis of purely electro-magnetic inertia**'; it was shown to be a reasonable one by the results of the experiments of Kaufmann in 1901, on the apparent inertia of the free electrons which are supposed to constitute the cathode rays.

Now what was the assumption which Lorentz found it desirable to make about the configuration of the electron? Starting with a desire to prove the reasonableness of the contraction-hypothesis as a universal characteristic of all matter, he was led to assign the same property to the ultimate common element. That is, assuming an electron at rest to have a spherical configuration, an electron moving with velocity v is supposed to have a spheroidal shape, the whole electron being squeezed together along the direction of the velocity in the ratio $(1 - v^2/c^2)^{\frac{1}{2}} : 1$.

The reason for this contraction of the electron is not considered; neither are the agencies which hold the electron together; these are eliminated by means of the purely kinematic assumption. But granting it to exist, Lorentz was able to show[1] considerable reason for anticipating that a body constituted according to his scheme would automatically undergo the required contraction when set in motion.

This, put rather bluntly, is the general result of the work of Lorentz. Of course, there was actually more support for this conception of the electron than the mere fact that it furnished an explanation of the hypothetical contraction. Such, for instance, was the fact that it also gave an explanation of the failure of the experiments of Rayleigh and Brace which sought to find an evidence of double refraction in an isotropic transparent body when set in motion through the æther.[2] But this does not alter the fact that the suggestion of the contracting electron is in reality only throwing the mystery a stage further back. If we accept the suggestion, it

[1] Lorentz, "Proc. Roy. Soc.," Amst., 1904; also, Larmor, "Aether and Matter," 1900. Larmor, on the ground of the theory developed here, demurred to Lord Rayleigh's expectation of any effect (Brit. Ass., Belf., 1902).

[2] See below, p. 54.

will not be long before we go on to ask the further question, 'Why is this so?' This can only be answered by an analysis of the construction of the electron, in terms of its parts, so that we should at once be carried beyond the stage at which the electron is an ultimate element in our thought. At this point, therefore, the assumption must be left until something arises which may throw further light on it, or furnish us with an alternative.

22. Lorentz's Argument Anticipates the Principle of Relativity.

At the present moment it is desired to call attention to the fact that the hypothesis of the contracting electron is nothing more than throwing back on the elementary conceptual constituent of matter the very property which it desired to establish of the objects of experience.

This is in reality an application of what is now known as the Principle of Relativity: that is, it is a result of the assumption that the failure of experiments to discover the motion of the earth relative to the æther is no accident, but is the result of universal relations inherent in the constitution of matter. If Lorentz's or Larmor's theory be true, then the velocity of bodies relative to the æther must for ever remain unknown.[1] This is the point from which the principle of relativity starts. But the deduction of consequences from this point is independent of any constitutive theory of the nature of matter. Lorentz's theory, as far as it goes, implies the relativity of all its consequences, but many of those results can be obtained as an immediate deduction from the hypothesis of relativity apart from the details of the theory. Not only so, but in regions where the pure electron theory becomes inapplicable, we may say what laws are consistent with the relativity of all phenomena and what laws are not.

The law of gravitation, for instance, in the commonly accepted form is not consistent with the relativity implied by the electron theory. If matter were completely constituted according to that theory, the law of gravitation would come within the

[1] See Chapter IV, p. 48.

scope of the theory, and we should therefore expect that, even if we could use gravitational phenomena in our investigations, we should still be unable to find any evidence of the earth's motion relative to the æther.

It seems at first sight that there is an obvious impossibility of including the fact of gravitation within the scope of electrical theory, inasmuch as this would probably imply that the effect is propagated with the velocity of light, whereas it has commonly been supposed that the velocity of gravitation is, at any rate, enormously greater than that. It should not be forgotten, however, that modifications of the law of gravitation have been proposed—by Gerber,[1] for instance—in which the influence does travel with the velocity of light and which have been confidently asserted to be consistent with astronomical observations.

Two questions are thus suggested :—

(1) Are the observations of astronomy consistent with the hypothesis of relativity in the sense in which we are now thinking of it? That is: Do existing observations give us a means, if only the proper way can be devised, of determining the velocity of the earth through the æther?

(2) If we give up hope of this, and grant that in some way the law of gravitation must be included in the scope of the principle of relativity, is this because, contrary to the usual opinion, gravitation is really a product of the electrical constitution of matter, or is the relativity of phenomena in the special sense of which we are thinking, something of wider range and more deep-seated than the electron theory itself?

This fact of gravitation and our incapacity to coordinate it with the other physical qualities of material bodies constitutes perhaps the most important region in which the theory of relativity can go beyond the theories of the constitution of matter.[2] It is just where these theories become insufficient to give a complete account of the mechanism on which the properties of matter depend that the Principle of Relativity becomes of importance as a supplementary hypothesis.

[1] See "Encyk. der Math. Wiss.," Vol. V, p. 49.

[2] See Chapter VII, p. 81.

CHAPTER III.

THE RELATIVITY OF SPACE AND TIME.

23. In order to arrive at the essential point which Einstein added to the argument of Lorentz, we will now begin to examine some positive results of the fundamental assumption that it is impossible to determine the actual velocity of the earth relative to the æther, conceived after the manner of Lorentz as a unique medium which is capable of use as a frame of reference.

It is, of course, assumed that the fundamental property of the æther, that of propagating effects with a definite velocity in all directions is one of the phenomena with which we are concerned. This implies that we shall assume that the result of the experiment of Michelson and Morley is general; that is, we assume that we are ever to remain in ignorance of any difference between the velocities of light in different directions relative to the earth or, more generally, to any given observer.

Now this means that for all practical purposes we may at any instant suppose the observer to have any velocity we choose. But we may not, of course, attribute to him any arbitrary acceleration; because, as was said at the outset, it is not assumed that the relativity extends to the concealment of all motion, but only of a motion which is uniform. In practically all that follows, when we speak of an observer we mean an observer who is moving uniformly to the æther, so that we may choose a frame of reference in which he is permanently at rest.

24. On the Idea of Simultaneity.

As a result of the fact that an unambiguous conception of time emerges out of the system of dynamics, the idea of 'simultaneous events at different places' has come to be considered as one about which there is no lack of definition.

If we examine the development of the idea, however, we see that it arose gradually in some such way as this. In the first instance, events which were seen by the eye at the same moment were considered to be simultaneous; but in reality the simultaneity was not in the events, but in the observer. This is, in fact, the only kind of absolute coincidence about the meaning of which there is no doubt. For as soon as the finite velocity of propagation of light is recognised, it has to be allowed for in estimating the moments of occurrence of events at a distance from the observer. Not only this, but the velocity of the observer must be allowed for if he is thought to be moving. In doing this the actual observations made are necessarily of what may be called 'coincidences'. Measurement only becomes exact when mental judgment of distances and intervals is eliminated. It is the chief aim of the experimenter to do this as far as possible.

If we seek to date some celestial phenomenon, what we observe is a coincidence, that of a light impression falling upon some terrestrial object, as the eye. In allowing for the time of transmission of the light, we enter the realm of theory, and postulate some law such as that of the constancy of the velocity of light. If this is all we postulate, and if we have no other means than that of light signals for setting up a scale of 'simultaneity' for events on the earth, and at distant celestial points, then it will be seen that we are not able to date the event uniquely.

If the conception of a definite and unique æther be adopted, then for an observer at rest in it a criterion of simultaneity can be set up as follows. Let A, B be two points, each of which is supposed to be at permanent rest in the æther, and let a ray of light be emitted from A at an instant t_1, in the direction AB. Let this ray be reflected back on its arrival at B, the moment of reflection being called t_2. Let the reflected ray arrive at A again at an instant which we will call t_3. Then on the ordinary view, A and B being at rest in the æther, t_2

must be considered as simultaneous with the instant midway between t_1 and t_3. We may write

$$t_2 = \tfrac{1}{2}(t_1 + t_3) \quad . \quad . \quad . \quad (1)$$

On the other hand, if A and B have a common velocity v in the direction AB, and the same process of transmission of a ray of light from A to B and back again is carried out, this will not be so. For if l represents the constant distance AB and the instants corresponding to t_1, t_2, t_3 are denoted by t'_1, t'_2, t'_3, then we have

$$t'_2 - t'_1 = l/(c - v),$$
$$t'_3 - t'_2 = l/(c + v).$$

These equations lead to

$$t_2 = \frac{t'_1 + t'_3}{2} + \frac{v(t'_3 - t'_1)}{2c} \quad . \quad . \quad (2)$$

Thus in this case the instant of reflection is not to be considered as simultaneous with the instant midway between the instants of the start and return of the signal.

Now as long as we have any hope of knowing the velocity of a given point relative to the æther, this fact is in no sense confusing. But if we grant the hypothesis of relativity, and reconcile ourselves to the thought that we shall never know the actual velocity of the points A and B, then we must also face the consequence that we can never know what instant at B can be called, more than any other, 'simultaneous' with a given instant at A. Or we may say that *the phrase 'simultaneous events at different points' has no meaning until the velocity of those points is stated.*

This is one of the main distinctions between the outlook here suggested and that of the earlier thought. We cannot, granting the fundamental assumption, say that there exists on physical grounds a unique means of ordering phenomena in time regardless of their position. If on *meta*physical grounds we desire to think that there is really a unique and absolute meaning for the time which we measure, yet we must admit that we have no means of knowing whether we are using it or not. It is exactly the same position to which we are brought in respect of the æther. We may assert its existence, but we have no means of identifying it, and the definition of it in terms of its properties determines it only as one of an infinite number, all of which have identical properties.

25. On the Measurement of Length and the Conception of Rigidity.

If it be granted that we must leave the question of the simultaneity of events an unanswered one, we are also faced with a difficulty about the notion of the distance between two given points of a material body. In the ordinary way of measuring lengths on a moving body, the length of a given line in the body must be defined as the distance between the two points of space which are occupied by the ends of the line at simultaneous instants, the conception of distances between the points of space being supposed, like the conception of absolute intervals of time, to be an *a priori* one.

But if we grant that we cannot speak unambiguously of 'simultaneous instants' until we have chosen an arbitrary body to be the one which is at rest at a particular moment, then it is clear that we cannot, without further definition, speak of the length of a given line in a body until we know what velocity is assigned to it.

It is taken for granted, in the classical treatment of the dynamics of rigid bodies, that the distances between the various parts of a body are invariable in the motion. This conception of an ideal rigid body is fundamental, and space is conceived to be so graduated that a given body occupies a region of exactly the same size and shape at every instant, and that this region is independent of the velocity of the body.

We see now that this conception is breaking down at various points. First we have the suggestion of FitzGerald that every body contracts on being set in motion; then we have the difficulty of defining what we mean by the region occupied by the body at a given instant on account of the difficulty about what we mean by simultaneous positions of different points of the body. Not only this, but we have a further practical difficulty in the fact that if our conception of metrical space is based on that of an ideal rigid body this seems to react on the measurement of time, since we have the apparent truth that equal distances are traversed by a ray of light in equal intervals of time.

In order to deal with the apparently confused situation so created it is necessary to make an entirely new beginning and to build up afresh our statement of what we mean by the system of measurement of space and time that we employ.

26. The Construction of All Possible Space and Time Systems.

The one definite fact upon which we are going to build is the *assumption* that we are unable to determine a difference in the velocity of light in different directions. If we did not make some assumption of this nature to replace the graduation of space by means of a rigid body and of time by the rotation of the earth, we should have no definiteness in our system of time and space at all. We can order the events which happen in the universe in an infinite number of ways. Suppose, for instance, that we associate with any point of space the quantities (x, y, z), and with any instant of time at that point the quantity t. If we take four quantities (x', y', z', t'), defined as any arbitrary functions of (x, y, z, t), then with any occurrence is associated a set of quantities (x, y, z, t), and consequently a set of quantities (x', y', z', t'). Events may be described, as far as their order is concerned, by relations between these quantities. For example, the motion of a moving point may be described by saying in what way (x, y, z) vary as t varies, or in what way (x', y', z') vary with t'.

It is only when we begin to speak of the form of the relations which hold between the motions of different points that the choice of the variables which are used to describe the motions is in any way limited. The Newtonian form of the laws of motion limits us, not to a single set of time and space coordinates, but to a group of an infinite number, the relation between any one set (x_1, y_1, z_1) and any other (x_2, y_2, z_2) being of the form

$$x_2 = x_1 - ut_1,\ y_2 = y_1 - vt_1,\ z_2 = z_1 - wt_1,\ t_2 = t_1.$$

A fixed point in the second set is a point which in the first set satisfies the relations

$$x_1 - ut_1 = \text{const.},\ y_1 - vt_1 = \text{const.},\ z_1 - wt_1 = \text{const.};$$

that is, it is a point moving with velocity (u, v, w). Here (u, v, w) are any arbitrary velocities whatever. (See p. 9.)

27. THE HYPOTHESIS OF THE CONSTANCY OF THE VELOCITY OF LIGHT.

What now is the corresponding limitation imposed by our attempt to maintain, instead of the Newtonian laws, the fact of the constant and universal velocity of light? This limitation is simply that between any two possible sets of space-time coordinates (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) there must be such relations that, if a point is moving with velocity c in the first set, it is also moving with velocity c in the second set.

This may be set down in mathematical form thus:—

If the moving point changes its space coordinates by amounts (ξ_1, η_1, ζ_1) in an interval of time τ_1, and the corresponding quantities in the second system of reference are (ξ_2, η_2, ζ_2) and τ_2; then as a consequence of the equation

$$\xi_1^2 + \eta_1^2 + \zeta_1^2 = c^2\tau_1^2$$

we must have

$$\xi_2^2 + \eta_2^2 + \zeta_2^2 = c^2\tau_2^2.$$

Now, it is a purely mathematical problem to find out for what kind of relation between the two sets of coordinates this consequence can and must arise; and it is a problem which is capable of complete solution. The complete solution gives us, of course, certain obvious changes of coordinates which we need not discuss; namely (i) the addition of mere constants to x, y, z, and t, and (ii) a uniform change in the scale of all four coordinates, which amounts to nothing more than a multiplication of the units of space and time by the same quantity, a process which clearly will not alter the measure of the velocity of a moving point. There is also the obvious possibility of turning the space frame of reference round into any other angular position.

But apart from these there is still an infinite number of ways of changing the coordinates. Of these there is an infinite number which are linear transformations,[1] that is, in which each of the new set of coordinates can be derived from those of the old set by taking sums of constant multiples of the four coordinates of that set. We may show by a simple algebraical calculation that these can all be obtained from a change of variables of the following simple type :—

$$x_2 = \beta(x_1 - vt_1),\ y_2 = y_1,\ z_2 = z_1,\ t_2 = \beta(t_1 - vx_1/c^2) \quad . \quad \text{(A)}$$

where $\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$

Suppose now that we consider a point which is at rest in the system of coordinates (x_2, y_2, z_2, t_2), that is a point for which the space coordinates (x_2, y_2, z_2) have fixed values. Then applying the relation (A) we find that for this point

$$(x_1 - vt_1),\ y_1 \text{ and } z_1$$

have constant values, that is, the point has a velocity v parallel to the axis of x_1, if $(x_1, y_1, z_1, t_1,)$ are considered to be its space-time coordinates. This velocity v is of arbitrary magnitude and direction, since the axis of x may be taken in any direction we please; so that we may write down a transformation of this type which will ascribe to a point at rest in one system of coordinates any velocity that we choose to assign.

28. If we put

$$x_2 = \beta\, x_1 + a\, ct_1,\ y_2 = y_1,\ z_2 = z_1,\ ct_2 = \gamma\, x_1 + \delta\, ct_1,$$

this gives identically

$$x_2^2 + y_2^2 + z_2^2 - c^2t_2^2 \equiv x_1^2 + y_1^2 + z_1^2 - c^2t_1^2,$$

provided that $\beta^2 - \gamma^2 = \delta^2 - a^2 = 1$

and $a\beta - \gamma\delta = 0.$

These conditions lead to

$$a^2 = \beta^2 - 1,\ \gamma^2 = \beta^2 - 1,\ \delta^2 = \beta^2,$$

so that, if we put $a/\beta = -v/c$, we have

$$\beta^2(1 - v^2/c^2) = 1,$$

so that the formulæ for the change of coordinates are as stated.

29. The Lorentz-Transformation.

The transformations of type (A) lie at the heart of all the history and developments of the Principle of Relativity. They are commonly known as Lorentz-Transformations.

[1] It is possible to generalize all that follows to include the transformations which are not linear.

They were not first arrived at in the way which has been given above. Lorentz was the first to see their significance for the theory of the optical and electrical effects in moving bodies, though he neglected the square of v/c, and so took $\beta = 1$. His results only professed to be accurate as far as the first order in v/c. The approximation was carried a stage further by Larmor,[1] who showed by the use of the above variables how the second order effects of motion through the æther would be concealed. Einstein was the first to use the transformations in the significance which we have seen them to have.

What we have to do now is to see whether the facts of physical observation other than the special phenomenon of the propagation of light with a definite velocity—the only phenomenon so far taken into account—are in agreement with the suggestion that any two systems of measurement of space and time which are related to one another by a Lorentz-transformation are equally valid. We have seen that it is consistent with the facts as known of the propagation of light through free space. It remains to be seen whether the known facts of electro-magnetism and dynamics (including observational astronomy) can be fitted into this framework of space and time without contradiction; these being taken as representing the exact branches of physical science. If this can be done, the result will be that all the facts at our disposal are unable to distinguish between these sets of coordinates as valid or convenient measures of space and time.

30. On the Dimensions of a Moving Body.

We are now able to express definitely what is to be assumed of the dimensions of a moving body under the fundamental hypothesis that we have made.

The 'length of any line in a body' is, as was said above, defined to be the distance between two points of space which are occupied simultaneously by the ends of the line. Let us suppose that we have a body which is at rest in the system of coordinates (x, y, z, t). Let the time-space coordinates

[1] "Aether and Matter," Cambridge, 1900, pp. 173-7.

for two particular points of the body be (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2), and let the corresponding quantities in another system of reference be denoted by the addition of an accent.

Then applying the equation (A) twice and subtracting we have

$$x_2' - x_1' = \beta\{x_2 - x_1 - v(t_2 - t_1)\}, \quad . \quad (3)$$

$$y_2' - y_1' = y - y_1, \quad z_2' - z_1' = z_2 - z_1,$$

$$t_2' - t_1' = \beta\{t_2 - t_1 - v(x_2 - x_1)/c^2\} \quad . \quad (4)$$

The last equation shows that if we take simultaneous positions of the two points in the second system by putting t_2' equal to t_1', the corresponding times in the first system differ by an amount given by

$$0 = t_2 - t_1 - v(x_2 - x_1) c^2.$$

But since the body is at rest in the first system, x_1 and x_2 are independent of t_1 and t_2.

Putting $(t_2' - t_1')$ equal to the value given above (4), we obtain from (3) for the difference between the simultaneous values of x_2' and x_1' the value

$$x_2' - x_1' = \beta(x_2 - x_1)(1 - v^2/c^2) = (x_2 - x_1)(1 - v^2/c^2)^{\frac{1}{2}},$$

that is the apparent length of a body in the direction of the velocity which it is conceived to have, is less than the apparent length when it is conceived to be at rest, in the ratio $1 : (1 - v^2/c^2)^{\frac{1}{2}}$, the dimensions at right angles to the direction of the velocity being the same.

31. Is the FitzGerald Contraction a Real One?

The difficulty which naturally arises in the mind about this result is as to whether the apparent contraction when the system of reference is changed is in any sense a real one. To meet this question one or two suggestions may be given. Lorentz, working always with the conception of a unique æther, insists that the contraction is a real one, and not, as it seems to him, the position above outlined implies, a purely subjective change in the system of measurement.

But the question which must be asked is, "What is meant by a real contraction?" In what sense is it possible to speak of a contraction at all? It must be a contraction relative to something, something which is conceived to have a permanent configuration. This something may be taken to be the conceptual frame of reference, and so long as the frame of reference is a unique one the question as to the existence is one with a meaning. But until the frame is defined this is not so.

The position taken then in regard to the contraction of the apparatus of Michelson's experiment when rotated relative to the earth is this. Let us assume that when we take a frame of reference in which the apparatus is at rest the configuration of the apparatus is unchanged when it is turned round so slowly that the effect of centrifugal forces in straining the body may be neglected, as well as the effect due to the altered strain arising from the change in position relative to the gravitational field of the earth. That is, we assume, in accordance with the Principle of Relativity, that an observer on the earth will observe no effect on the apparatus as a consequence of the motion of the earth. This being so, the conclusion drawn from the fundamental hypothesis of the principle by the analysis given above is that if we refer everything to a frame of reference relative to which the earth is moving then there will actually be a contraction of the apparatus as measured by this standard.

But, of course, this real contraction will not be capable of detection; for in order to measure it physically it is necessary to lay alongside of the apparatus another comparison body, and to allow this body to remain alongside and to share the displacement of the apparatus. It will therefore share also in the universal contraction, and the measurements will remain as before.

32. Criticisms of the Use of Light Signals.

The objection is often raised against the argument of the preceding sections that the introduction of this fundamental assumption of the constant velocity of light for the purpose of controlling the time standards at different places is in reality a very artificial proceeding. This objection should be considered carefully before going any further.

The first question that may be asked is, "Have we any right to suppose that there is no means of communicating between distant points other than, and independent of, the transmission of light, which might be used to set up a standard of simultaneity which is different from that given by the equation (2), p. 29".

It is often suggested, for instance, that gravitation is possibly propagated instantaneously, or, at any rate, with a velocity enormously greater than that of light. It has been already remarked, however, that this statement is very questionable, and it will be shown later that, starting from the method of light signals for the purpose of defining simultaneity, we may develop an empirical modification of the law of gravitation which is strictly consonant with the known facts of planetary motion.

A second form of the same objection is that in practice our measure of time is controlled by the dynamical motion of the earth. This amounts, as a matter of fact, to saying that the time which we use in practice is the uniquely defined time of the Newtonian dynamics. But, as we shall see later, the classical dynamics cannot be considered any longer as absolute in its accuracy. This is obvious merely from the fact that we know that moving charged bodies do not obey the Newtonian laws of motion; the deviation has been actually observed in the case of the cathode rays and the β-rays. We shall hardly expect therefore, that with the modern view that matter is largely, if not wholly, of electrical constitution, the dynamics of material bodies will conform exactly to the simple scheme propounded by Newton. We are almost prepared, in fact, to believe that the exact dynamics of matter will be a very complicated matter. Abraham, for example, dealing with the mechanics of the electron, obtains a formula[1] for the apparent mass of a single electron which makes it appear almost hopeless to think of further progress in the direction of evaluating the mass of a much more complicated system.

Now, as we shall see later[2] just at this point the Principle of Relativity is able to step in and indicate a general modification of the usual formulæ of dynamics which renders them

[1] "Theorie der Elektrizität," tome ii, 2nd edition, Leipzig, 1908, p. 180.
[2] Chapter VII.

consistent with the standard of time and space set up in the preceding sections. The modifications required are everywhere of the order of the square or some higher power of the ratio of the velocity of the body to that of light, and if these quantities are neglected the formulæ reduce to the ordinary Newtonian form.

These modifications are found to be in strict accordance with the direct evidence available from experiments on the motion of the free negative electrons in the cathode and β-rays,[1] which at present are the only moving systems in which the velocities are sufficiently great to render the deviation from Newtonian mechanics appreciable.

Further, there is strong indirect support in the null results of the experiments of Rayleigh and Brace, Trouton and Noble, which cannot be explained by a combination of existing electrical theory with Newtonian dynamics.[2]

The objections raised are therefore such as can only be maintained by showing either that there are some experimental facts which are not reconcilable with the predictions of the Principle of Relativity, or else by showing that an unnecessary complication is introduced into our thought by accepting the principle.

33. The Velocity Addition Formula.

If we consider a moving point which is displaced in time δt through distances δx, δy, δz parallel to the three coordinate axes, the corresponding displacements when measured in the system (x', y', z', t') can be obtained by applying the equations (A) twice for the positions at the beginning and end of the interval δt. On subtraction we get

$$\delta x' = \beta(\delta x - v\delta t),\ \delta y' = \delta y,\ \delta z' = \delta z,\ \delta t' = \beta(\delta t - v\delta x/c^2) \quad (5)$$

From these, dividing $\delta x'$, $\delta y'$, $\delta z'$ by $\delta t'$, we get expressions for the velocities as estimated in the new frame of reference :—

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2},\quad u'_y = \frac{\beta u_y}{1 - vu_x/c^2},\quad u_z = \frac{\beta u_z}{1 - vu_x/c^2} \quad (a)$$

If we put u equal to zero in these equations, u becomes equal to $(-v)$ in the direction of the axis of x.

[1] *Vide* § 54, p. 65. [2] Pp. 54 ff.

Conversely, since as we have seen the formulæ (A) can be reversed by simply changing the sign of v, if u is zero, u is equal to v in the positive sense.

Thus a point at rest in the first system of coordinates and a point in the second system have a velocity v relative to one another.

34. Relative Velocity.

In the ordinary way of speaking the relative velocity of two points is defined as the difference of their velocities, this being the same for all frames of reference. But it is clear that it will not be so as a consequence of the equations ( ). Thus this definition would not give us a determinate value for the relative velocity. We therefore substitute the following definition: *the velocity of a point* P *relative to a point* Q *is the velocity of P in a system of coordinates in which* Q *is at rest.*

Such a system can always be chosen; for if we take an arbitrary system, and the velocity of Q in this system is v then taking the axis of x in the direction of this velocity, and applying the transformation (a), the velocity of Q in the new system is, as we have seen, zero.

Thus, for example, if we have three points A, B, C, of which B has a velocity u relative to A, and C has a velocity v relative to B in the same direction as u, then the velocity of C relative to A is not $(u + v)$, but $(u + v)/(1 + uv/c^2)$.

35. The Velocity of Light as a Critical Velocity.

From the relations above (5), p. 58, we have identically

$$\delta x'^2 + \delta y'^2 + \delta z'^2 - c^2\delta t'^2 = \delta x^2 + \delta y^2 + \delta z^2 - c^2\delta t,^2$$

or

$$\delta t'^2(u'^2 - c^2) = \delta t^2(u^2 - c^2); \quad . \quad . \quad (6)$$

hence if u is less than c, so is u'; that is, if the velocity of a moving point is less than the velocity of light in any system of reference, it is also less in any other, provided always that the velocity v is also less than c.

An example of this may be given in reference to an objection that has often been raised to the point of view of the Principle of Relativity. It is suggested[1] that whereas the analysis of

[1] *Vide* Soddy, "Nature," November, 1913.

the principle indicates that relative velocities greater than that of light are physically impossible, yet there is an actual possibility of observing in nature a relative velocity considerably greater. We may imagine, for instance, two β-particles shot out by a radio-active body in opposite directions with velocities, each equal to nine-tenths of that of light; and here is an actual difference of velocity equal to $1 \cdot 8c$. This is quite true, and the principle of relative has nothing to say against it. The principle maintains that a velocity greater than c *relative to the observer* cannot be observed; this is in accordance with the facts in the case in question; further, the relative velocity of two bodies, as defined above, cannot be greater than c; this also is in agreement with the example chosen; for if in the expression $(u + v)/(1 + uv/c^2)$ we put both u and v equal to $\cdot 9c$, we get, not $1 \cdot 8c$, but $1 \cdot 8/1 \cdot 81c$, which is certainly less than c.

The position is not that velocities greater than c are not conceivable, but that real bodies become illusory in observation if they are conceived to be moving faster than light. We shall also see later that the electrical constitution of matter seems to indicate that a body would suffer dissolution if it were accelerated so that its velocity were made greater than c. Not only so, but the energy of a body tends to an infinite value as its velocity is increased towards c, so that consistently with our ordinary conceptions of energy, it would not be possible to set a body in motion from rest with this velocity. Of course, there is nothing to hinder us if we please from thinking of bodies which have a real configuration for a velocity which is greater than c, only such bodies would suffer dissolution if they were caused to move with a velocity less than c, and an infinite amount of energy would be used in the retardation.

36. The Fizeau Experiment and Fresnel's Convection-coefficient.

We may now consider in what way the Principle of Relativity gives an account of the result of Fizeau's experiment. We suppose that a light disturbance travels through a refracting substance with a relative velocity c/μ; that is, this is the velocity in a frame of reference in which the refracting material is at rest. Suppose now that we change to a frame of reference in which the material has a velocity v in the direction of propagation of the light.

Then, applying the addition equation, the velocity of the light disturbance in this frame of reference is

$$\frac{\frac{c}{\mu} + v}{1 + \frac{c}{\mu}\frac{v}{c^2}} = \frac{c}{\mu} + v\left(1 - \frac{1}{\mu^2}\right) + v^2 \ldots$$

Thus, neglecting v^2, we arrive at once at the convection-coefficient $(1 - \mu^{-2})$.

In the repetition of Fizeau's experiment by Michelson and Morley the figure arrived at for this coefficient was ·442 ± ·02.[1] Taking the known value of μ for sodium light the theoretical value is ·438. Thus the agreement is satisfactory enough.

37. The Doppler Effect.

The Principle of Relativity gives also a simple account of the Doppler effect.

If light, of period τ to a given observer, is travelling along the axis of x, any component of the disturbance may be taken as proportional to a harmonic function

$$\sin\left\{\frac{2\pi}{\tau}(x - ct) + a\right\}$$

where x and t are space-time coordinates in which the observer is at rest.

If now we change to a system in which the observer has velocity $(-v)$ by putting

$$x = \beta(x' + vt'),\ t = \beta(t' + vx'/c^2)$$

we have

[1] "Amer. Journ. of Sci.," 31, 1885. The figure given in this paper is ·434, but there appears to be an arithmetical error, the data given leading to ·442.

$$\frac{2\pi}{\tau}(x - ct) = \frac{2\pi\beta(1 - v/c)}{\tau}(x' - ct')$$

$$= \frac{2\pi}{\tau'}(x' - ct')$$

where
$$\frac{\tau}{\tau'} = \beta(1 - v/c)$$

$$= 1 - v/c \text{ neglecting } (v/c)^2.$$

Here τ' is the period as seen by an observer at rest in the new frame of reference. The result indicates that the effect of a motion with velocity v opposite to that of the light shortens the apparent period in the ratio $(1 - v/c) : 1$.

38. The Convection-coefficient in a Dispersive Medium.

In treating of the convection-coefficient in § 36, μ was defined by the fact that c/μ was the velocity of the disturbance when the medium was treated as stationary. In a dispersive medium μ is a function of the period of the light. The value to be taken in the formula obtained is clearly that corresponding to the period τ as seen by the observer to whom the medium is stationary. But the period of the light as observed in Fizeau's experiment is τ', that which is apparent to the observer relative to whom the medium is moving with velocity v.

By the same method as in the last section we obtain

$$\tau' = \tau\left(1 - \frac{v}{c/\mu}\right).$$

Thus, if μ' is the index corresponding to τ'

$$\mu' - \mu = (\tau' - \tau)\frac{\partial\mu}{\partial\tau}$$

$$= -\frac{v\mu\tau}{c}\frac{\partial\mu}{\partial\tau}.$$

Thus
$$\frac{c}{\mu} = \frac{c}{\mu'\left(1 + \frac{v\tau}{c}\frac{\partial\mu}{\partial\tau}\right)}$$

$$= \frac{c}{\mu'} - \frac{v\tau}{\mu'}\frac{\partial\mu}{\partial\tau}, \text{ neglecting } (v/c)^2,$$

and the velocity of the light becomes to the first order in v

$$\frac{c}{\mu'} + v\left(1 - \frac{1}{\mu^2} - \frac{\tau}{\mu}\frac{\partial\mu}{\partial\tau}\right),$$

where in the last term we neglect the difference between μ and μ'.

The additional term in the coefficient of v increases the theoretical value of the convection-coefficient in the case examined by Michelson and Morley to ·451, but this is still well within the limits indicated by their experiments, namely, ·442 ± ·02. Zeeman[1] has recently performed some further very careful experiments with light of different colours with a view to confirming the formula last given. He has succeeded in showing a dispersive effect, but the detailed results of his experiments are not yet published.

39. The Contraction in Volume of a Moving Body.

It follows at once from the formula obtained above that if a body, all of whose points have the same velocity, has a volume V_0 when it is considered to be at rest, and volumes V, V′ when its velocities are u and u' respectively, then

$$V = V_0(1 - u^2/c^2)^{\frac{1}{2}}, \; V' = V_0(1 - u'^2/c^2)^{\frac{1}{2}}.$$

Now if u' is the velocity obtained from the velocity u by the equations (A), we find on substitution from the addition formula that

$$1 - u'^2/c'^2 = (1 - vu_x/c^2)^{-2}$$

which leads to $$\frac{(1 - u^2/c^2)^{\frac{1}{2}}}{(1 - u'^2/c'^2)^{\frac{1}{2}}} = \beta\,(1 - vu_x/c^2),$$

so that $$V/V' = \beta\,(1 - vu_x/c^2) \quad . \quad . \quad . \quad (7)$$

This equation gives us the contraction in a volume which is not at rest in either frame of reference. It will be of importance when we come to consider later whether the measure of an electric charge is an absolute or relative magnitude.

[1] "Amst. Proc." XVII. (1914), p. 445.

CHAPTER IV.

THE RELATIVITY OF THE ELECTRO-MAGNETIC VECTORS.

40. The Definitions of Electric and Magnetic Intensities.

In the ordinary treatment of electrical theory the definitions given of the electric and magnetic intensities at a point are somewhat as follows:—

The 'electric intensity' is the force which would be exerted per unit charge on a small charged body placed at rest at the point.

The 'magnetic intensity' is definable from the law that the force per unit charge on a small moving charged body is equal to the sum of the electric intensity and the vector product[1] of the velocity and the magnetic intensity.

An alternative definition of the magnetic intensity may be given as the force per unit pole on a magnetic pole placed at the point. As this introduces the non-existent isolated magnetic pole, it will be as well to confine the discussion to the former definition.

We see that, apart from the fact that the measurement introduces a reference to the mechanical category of 'force', the definition involves the knowledge of the velocity of the small test body. But we have already learned to look upon the velocity of a body as a quantity which, physically, is so far indeterminate. To assign a definite value to it we have to decide which of all possible frames of reference we will adopt. Thus the definitions of the electric and magnetic intensities are relative ones; their values are 'relative' in the same sense as the measures of space and time.

[1] See note on vectors, p. 1.

We need therefore to add to the relation (A) (p. 33), a relation between the values assigned to the electric and magnetic vectors in two different systems of reference. We cannot obtain this from the definitions given just now, because they involve the conception of 'force', and we have not yet seen whether this is to be looked upon also as a relative or absolute quantity. We shall see as a matter of fact when we come to the consideration of dynamical quantities that we are bound to treat it as relative.

In order therefore to obtain the required relations we must adopt the same method as that used above in respect of space and time, and consider the measures of these quantities in the light of the laws in which they occur, the laws which express the uniformities which we believe on experimental grounds to hold between electrical phenomena.

These laws are the equations of the electro-magnetic field. We will take them in the form adopted by Lorentz,[1] namely,

$$\frac{1}{c}\left(\frac{\partial \mathbf{e}}{\partial t} + \rho \mathbf{u}\right) = \operatorname{curl} \mathbf{h} \quad . \quad . \quad . \quad \text{I}$$

$$0 = \operatorname{div} \mathbf{h} \quad . \quad . \quad \text{II}$$

$$-\frac{1}{c}\frac{\partial \mathbf{h}}{\partial t} = \operatorname{curl} \mathbf{e} \quad . \quad . \quad . \quad \text{III}$$

$$\rho = \operatorname{div} \mathbf{e} \quad . \quad . \quad \text{IV}$$

where **e** and **h** are the electric and magnetic intensities, and ρ is the density of charge.

The fifth equation of Lorentz's theory,

$$\mathbf{F} = \mathbf{e} + [\mathbf{uh}]/c$$

where **F** is the 'mechanical force' per unit of charge moving with velocity **u**, will supply us with a means of connecting up with mechanical theory, but gives us no criterion about the purely electrical magnitudes, the mechanical categories now occupying a derivative place.

41. The Invariant Form of the Fundamental Equations.

The first thing that may be noticed about the form of the equations I-IV, is that if we change the scale on which we

[1] The units here used are those of Lorentz. See "Theory of Electrons,' Leipzig, 1909, p. 5.

measure **e** and **h**, we alter the scale on which we measure ρ in the same ratio. Thus it becomes possible to limit the consideration to such changes of the variables as leave the charge carried by any particular element of volume unaltered.

The fundamental hypothesis of electrical theory which takes the place of the Newtonian conception of 'conservation of mass' is that of '*conservation of electric charge*'. We proceed to see, therefore, whether it is possible to make such changes in the variables **e** and **h**, when we change the space-time coordinates according to the transformation (A), that *the form of the equations I-IV may be unaltered* and at the same time *the charge carried by any particular element of volume may be unaltered.* It was the discovery that it is possible to satisfy these requirements that gave rise to the Principle of Relativity.

The following are the equations which give us the necessary changes in the vectors **e** and **h** :—

$$\left.\begin{aligned} e_x' &= e_x,\ e_y' = \beta(e_y - vh_z/c),\ e_z' = \beta(e_z + vh_y/c) \\ h_x' &= h_x,\ h_y' = \beta(h_y + ve_z/c),\ h_z' = \beta(h_z - ve_y/c) \end{aligned}\right\} \quad (B)$$

If we take the equations of Lorentz and make the changes of variables given by (A) and (B), we arrive directly at the equations

$$\frac{1}{c}\left(\frac{\partial \mathbf{e}'}{\partial t'} + \rho'\mathbf{u}'\right) = \text{curl}'\,\mathbf{h}' \quad \text{I}'$$

$$0 = \text{div}'\,\mathbf{h}' \quad \text{II}'$$

$$-\frac{1}{c}\frac{\partial \mathbf{h}'}{\partial t'} = \text{curl}'\,\mathbf{e}' \quad \text{III}'$$

$$\rho' = \text{div}'\,\mathbf{e}' \quad \text{IV}'$$

where $\mathbf{u}'$ is a velocity whose components are

$$\left.\begin{aligned} u'_x &= \frac{u_x - v}{1 - vu_x/c^2} \\ u'_y &= \frac{\beta u_y}{1 - vu_x/c^2} \\ u'_z &= \frac{\beta u_z}{1 - vu_x/c^2} \end{aligned}\right\} (a)$$

and ρ' is defined by the equation

$$\rho' = \beta\rho(1 - vu_x/c^2) \quad (C)$$

In order to see the significance of these quantities $\mathbf{u}'$ and ρ', we must recall one or two of the immediate consequences of the change in the manner in which space is measured from the coordinates (x, y, z, t) to (x', y', z', t').

We have seen, (p. 38) that the equations (a) are exactly those which connect the estimates of the velocities of a moving point as measured in the two systems respectively.

We have also seen, (7), p. 43, that if δV is the volume of a small region of a body moving with velocity $\mathbf{u}$ as measured in (x, y, z, t), and $\delta V'$, $\mathbf{u}'$ are the corresponding volume and velocity as measured in (x', y', z', t') then

$$\delta V' = \delta V/\beta(1 - vu_x/c^2).$$

If we put this equation and (C) together we obtain

$$\rho'\delta V' = \rho\delta V.$$

Thus the equation (C) is exactly that which is required to express that the *charges in the corresponding elements of volume are the same.*

The equations (A), (B), (C) then effect an exact transformation of the fundamental equations I-IV into equations of the same form, and are consistent with an unique value being assigned to any element of charge. We say that '*the charge is an invariant*'.

42. The Consequence of this Transformation.

The result of the exactness of this transformation is that, so long as we are dealing with phenomena of which the sole laws are the field equations in the form which we have taken, we shall not be able to discriminate between one set of space-time coordinates and another, as that one relative to which the laws have the required form. In other words, remembering that as we know it, the æther is simply the frame of reference relative to which these equations hold, as long as we are dealing with phenomena which are entirely governed by these laws, we can never expect to identify the æther. Unless we can light on some measurable phenomena which are not subject to these field equations alone, the æther remains an unknown frame of reference, exactly as the frame of reference for the laws of dynamics is indeterminate.

Lorentz remarks that it is really a matter of personal predisposition whether this leads a physicist to deny the existence of the æther or to the more agnostic position which merely says that "the æther is real but will always be hidden from our knowledge". This may be so, but few to whom the means of the propagation of electrical effects, and, indeed, the whole structure of matter is a question of the first interest, will feel content to take either of these positions. If the æther, in the ordinary sense of a stationary medium, does not exist, what is it that conveys the waves of light through space? If it does exist in the ordinary sense, is it a pure accident that its laws are of such a form that it must for ever remain concealed; or is it that we have naturally lighted first on those aspects of its working which, as being independent of motion through it, would be most readily found by an observer who did not know his velocity through it?

In any case, the impression made by the situation is that there is a break in our knowledge. The Principle of Relativity represents an attempt to see what are the conclusions to be drawn from an admission of our inability to identify the æther, in the hope that it may throw light on the question where we are to look for the way out of this difficulty. In order to see at what point the Principle is an independent hypothesis and something which is not contained in a purely mathematical result arising out of the particular form of a set of equations, we must go a little further, and consider the way in which the electron theory professes to give an account of the structure of matter.

43. The Method of Coincidences in Experimental Observation.

The fact that we cannot have a unique measure of the electric and magnetic intensities may perhaps be a little more readily accepted if we consider that in practice we never really profess to measure them directly. When we, for example, measure an electric potential by means of an electrometer, the only actual observation is one of a configuration of relative equilibrium. We observe the *coincidence* of a pointer or of a spot of light with a certain mark on a scale. Only by eliminating the uncertain factor of human judgment of an interval of space and time is it possible to make accurate measurements. We make our instruments do all the observing for us. Then if we want to interpret the indications of the instrument in terms of the values of the strength of an electric field, or of an electric current or the like, we have to do so by means of exactly those theoretical equations which we have seen to have a purely relative significance.

Now it is a fundamental property of the systems of coordinates that we have been considering, that events which are simultaneous and coincident in space at a given instant in a given set of coordinates are also simultaneous and coincident in any other of the sets of coordinates. To a given set of values of (x, y, z, t) there is only one value of each of the coordinates (x', y', z', t'), and these values do not depend upon any particular circumstances associated with the place or instant. Thus *if we read an electrometer, the coincidence of pointer and a certain mark on the scale does not depend on the frame of reference that we are using.* But the significance that we attach to the observation does do so. The quantitative description of the state of an electrical system depends on the frame of reference used; the nature of this dependence is limited by the fact that the laws of the field are to be independent of the choice of the frame of reference, and this means that the estimates of electric and magnetic force for two different frames of reference will be related in exactly the way represented by the equations (B) which are necessary for the preservation of the form of the laws.

44. The Incompleteness of Purely Electrical Schemes of the Constitution of Matter.

The foundation of the electron theory of matter is the scheme of equations which has been considered in the preceding sections. But there are certain other hypotheses that have to be added to those general laws.

The first is that of '*the atomic nature of electricity*'. The negative '*electrons*' are supposed to be small discrete nuclei, of identical structure and charge. It is not necessary here to go into the evidence for this. The magnitude of these nuclei is not definitely known; it is only possible for us to construct tentative theories as to their exact structure. For certain purposes we may be able to treat them as points, for others we may need to adopt some provisional conception of their extent and constitution. For the present so little is known about the properties of positive electricity that we must be content to admit that it is by no means certain that a similar statement of the identity of the elementary quanta of positive electricity is possible.

The picture which we have of the constitution of a portion of matter is one in which these electrons are grouped and held together in a permanent configuration; not in a static state, but in a state of motion; the motion of each electron is governed by certain laws of which the theoretical equations that have been already referred to are a *partial* expression.

As they stand it is impossible for them to be the complete scheme of laws. For it is easily seen that they are not mathematically sufficient to determine from a knowledge of the whole field at any given instant, the way in which the field and the distribution of charge will vary through all time. The equations II and IV (p. 45) are not independent of I and III,[1] so that we really have only two vector equations

[1] From III, since div curl $\mathbf{e} = 0$ (see note on Vector Analysis, p. 1), we have $\frac{\partial}{\partial t}\text{div}\,\mathbf{h} = 0$, showing that at a given point div $\mathbf{h}$ has a constant value. If at any preceding time there was no field, this constant value must be zero. Thus Eqn. II is deduced from III. From I and IV we obtain

$$\frac{\partial}{\partial t}(\text{div}\,\mathbf{e}) + \text{div}\,(\rho\mathbf{u}) = 0,$$

or

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho\mathbf{u}) = 0.$$

This is what is commonly known as the equation of continuity or of conservation of electricity. Integrated through a closed volume it gives—

connecting the three vectors **e**, **h**, and **u**, ρ being connected with **u** by the condition of 'the conservation of electricity,' which requires that the change in the distribution of electricity takes place entirely by convection.

We cannot therefore pretend that our scheme of the structure of matter is even theoretically complete until we have added to the general equations some further condition. It is just at this point that the structure of the electron has to be taken into account. If we choose to assume that we may neglect its dimensions, and treat it as a point charge, this is a sufficient condition.[1] The quantity ρ is then everywhere zero, and the solutions of the equations which represent the field are such as are restricted to possess singularities of a certain definite type. But as a matter of fact this is not a possible way of building up a practical theory because it introduces mathematical difficulties in the very simplification of ignoring the dimensions of the electron. The specification of the exact way in which the electric and magnetic forces become infinite in the neighbourhood of a point-electron is a matter of considerable difficulty, and the only way of avoiding this is to assume the electron to have a structure of finite size, and to assume some definite relation between the distribution of charge through its volume, and the quantities **e**, **h**, and **u**. Such a hypothesis will be purely empirical; it must be chosen according to the indications which we have of facts that are not explained by the general equations.

So the question arises: "What light is thrown by experiment on the structure of the electron?" This question has been rather obscured recently by the more tangible one of the structure of the atom as a group of electrons; but this has only become possible because, on any theory of the constitution of the electron, its extreme minuteness does allow of it being treated for many purposes as a material particle of de-

$$-\frac{\partial}{\partial t}\iiint \rho dV = \iint \rho \mathbf{u}_n dS,$$

that is, the rate of loss of electricity included in volume is exactly accounted for by the outward flow over the surface; there is no creation or destruction of electricity.

[1] As in Larmor's "Aether and Matter".

finite mass and charge, so that it is amenable to ordinary mechanical treatment.

It would seem that if the structure of the electron can be ignored in problems of the atom, it would *a fortiori* be so in the discussion of the grouping of atoms to form matter. But in the particular problem which we are considering, namely, the failure of experiment to determine uniquely the frame of reference which we call the æther, this is not so. For the preceding analysis of the fundamental equations has shown us that this failure arises partly because the equations of the electric field have a form which is invariant under a certain transformation. We are thus able to form any number of pictures of a given field, each using a different frame of reference, and each subject to the same fundamental equations. But if we wish to extend our picture so as to take in not only the æthereal field but the whole of the matter with which it is in interaction, it is necessary that any further hypotheses, introduced to supplement the equations of the field to obtain a complete scheme for the determination of the configuration and motion of the various parts of the matter, must also be of such form that they too are independent of the particular frame of reference that is adopted.

In order to illustrate this point we may consider a little more in detail the explanations offered by Lorentz of the failure of specific experiments, as representing the point of view of one who thinks in terms of the actual existence of a unique stationary æther. (Cf. Chapter II.)

45. The Experiment of Michelson and Morley.

We have already seen that the failure of this experiment to discover any trace of a motion relative to the æther can be completely accounted for by the hypothesis of a contraction of the apparatus. The magnitude of this contraction is exactly that which is apparent when, the body being supposed unaltered, the frame of reference is changed according to the transformation of Einstein. What reasons are given by Lorentz's account for supposing that the body will actually undergo this contraction when its velocity relative to a fixed frame of reference is changed?

His argument, relieved of the mathematical details, is as follows: When a body is at rest in the æther each electron within it has a specific motion determined according to certain laws. If we know these motions completely we can by means of the space-time correlation build up a picture of another system which is moving as a whole with velocity v relative to the same æther; in this the motion of any particular electron is obtained from that of the corresponding electron in the original body by merely turning the relations between (x, y, z) and t that express the way in which it moves into relations connecting (x', y', z') with t'. If this is done it is clear that the configuration of the body as a whole in the new picture is exactly that of the original body contracted in the ratio $1 : (1 - v^2/c^2)^{\frac{1}{2}}$, as on page 35. But, the argument goes on, the electro-magnetic field of the electrons in the new body satisfies the same fundamental laws as that of the original body, and is therefore equally capable of existing. Thus if these were the only laws in question we should be fairly justified in concluding that the original body would automatically change into the one of which we have so formed a picture when it is set in motion with the velocity v.[1]

46. Incompleteness of the Argument for the FitzGerald Contraction.

It is clear that there are two gaps in this argument. First, there is the obvious restriction that it must be assumed that, provided that the acceleration of the body is sufficiently slow, so as not to produce mechanical distortion, change of temperature or other disturbances, there is a *unique* configuration for a given grouping of electrons moving as a whole with a given velocity relative to the æther. This does not seem to offer any serious difficulty, though it must be remembered that the external configuration of a material body must be thought of as a statistical one; that is, in view of a probable kinetic constitution, to a fixed external configuration will correspond a multitude of rapidly changing distributions of electrons in the

[1] Cf. also "Aether and Matter," p. 176.

interior; but this will not be considered here. The more important point for discussion is that the correlation which has been shown to exist in the case of the 'field equations,' must be *assumed* to hold for all the other relations which play a part in determining the motions of the electrons.

It has been remarked that it is enough for the purpose in hand to supplement the field equations by an empirical relation between the distribution of the charge that constitutes the electron and its velocity. If this is done, it must be a relation which maintains its form in the correlated moving system that we have built up. Since it is a purely geometrical relation it must be a relation which is invariant under, or is a consequence of, the fundamental transformation (A). Hence Lorentz's assumption that the electron, which is naturally thought of as symmetrical round a centre when it is at rest, is, when in motion, of spheroidal shape, being obtained from the spherical stationary electron by the simple application to it of the FitzGerald contraction. If this is not assumed, the correlation between the stationary and the moving system is not perfect, and we have no reason, therefore, for thinking that the desired contraction of the whole body will take place.

Thus the whole explanation reduces to the formation of a conception of the electron, which really embodies the fact that we are trying to explain. The origin of the contraction is left in obscurity. The conception is only arrived at by assuming the result of the experiment to be, not an accident arising out of the particular circumstances of the case, but inherent in the constitution of matter down to its most minute elements. In other words, *the formation of this conception of the electron is itself a direct application of the Principle of Relativity.*

47. THE EXPERIMENTS OF RAYLEIGH AND BRACE.

If we adopt the conception of the electron that has been described there is no need to go any further into the details of any phenomena which are explicable in terms of the motions of electrons within a body in order to be sure that we shall not be able to obtain a positive effect due to the motion of bodies through the æther. For as we have already remarked more than once, the adoption of a definite relation between the velocity and the configuration of the electron, so that the latter is determined uniquely by the former, makes the scheme of the laws of the motions of the electrons complete, and the whole scheme is now a relative one; the æther is completely concealed.

It may, however, help to make the consequences of the conception clearer if we consider specially some of those results of it; especially some which directly affect certain experiments which were devised to detect effects which, it was thought, would follow from the contraction hypothesis.

It occurred to Lord Rayleigh that a shrinkage of a transparent body in one direction would cause it to cease to be isotropic, and that therefore there might arise a double refraction of a beam of light traversing it in a direction across the line of the motion. This was demurred to by Sir Joseph Larmor when the paper was read in 1902 at the British Association meeting on the ground of an argument similar to that given above. But the experiment was carried out again with extreme refinement by Brace in 1906, the only result being a complete fulfilment of the prediction of its failure.

The mechanism of refraction as ordinarily conceived is that the light waves travelling through the body set in vibration the electrons which go to make up the atoms, and their inertia, number, and distribution determine the amount of the refraction. If these are not isotropically disposed the body will produce a different refraction for different directions of the components of the electric force in the wave, and so a double refraction will arise. Now, Lorentz[1] was able to show that, starting from the conception of the electron which he had developed, the effective inertia of an electron is not the same for all directions. On allowing for this it appears that the inequality in the inertia is exactly of that amount which is required to neutralize the inequality in distribution produced by

[1] "Theory of Electrons," p. 217.

the contraction of the body. This, of course, in the light of what we have said above, is exactly what we should expect.

48. The Experiment of Trouton and Rankine.

In spite of the results of earlier experiments and of the theory of Lorentz and Larmor it still seemed to some that there might be need of putting the matter to a further test. Again reasoning from the FitzGerald contraction as producing a lack of isotropy in a body, Professor Trouton[1] suggested that there might be a difference of electrical conductivity in different directions in a conducting body carried along with the earth through the æther.

The conductivity of a body like its refracting power is interpreted in terms of the inertia and distribution of electrons which move within the body; and again, if the difference in apparent inertia for different directions is taken into account, the effect of the contraction on the distribution is annulled. The expectation of a positive effect is therefore unjustified, as we might predict on general grounds without any calculation of the inertia of the electron at all. The result of the experiment again verified this conclusion.

49. The Experiment of Trouton and Noble.[2]

In this experiment, which arose out of a suggestion made by FitzGerald, a parallel plate condenser was suspended with the plates in a vertical plane and capable of turning about a vertical axis. An attempt was made to detect a tendency for the condenser to set itself in a definite direction when charged. This was expected to happen for the following reason: If a charged condenser is in motion through the æther, the electric force between the plates being E, then a magnetic force is set up

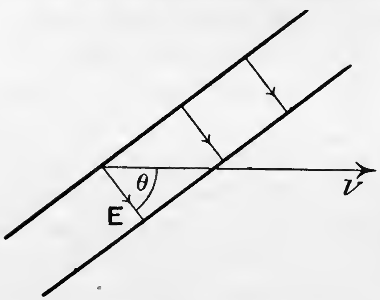


Fig. 5.

[1] "Proc. Roy. Soc." (A), 8 (1908), p. 428.
[2] "Proc. Roy. Soc.," 72 (1903), p. 132.

in the region between the plates equal to $Ev\sin\theta/c$, to the first order in (v/c), where v is the velocity of the condenser and θ is the angle between the normal and the direction of motion. This magnetic field makes a contribution to the energy of the moving system proportional to $E^2v^2\sin^2\theta/c^2$. But in addition to this the electric force between the plates is altered by a quantity of the order E^2v^2/c^2, and so there is a further change in the energy which is also of the order E^2v^2/c^2. A calculation gives for the total change of energy the quantity

$$W = 4\pi CV^2(\sin^2\theta - \tfrac{1}{4})v^2/c^2$$

where C is the capacity of the condenser and V the potential difference. If now the condenser is rotated so as to alter the angle θ without altering the charge, work will have to be done in changing the energy; this indicates that there is a couple acting on the condenser of magnitude $-\dfrac{dW}{d\theta}$ that is, $-8\pi CV^2\sin\theta\cos\theta\, v^2/c^2$. This couple vanishes if θ is equal either to 0 or to $\frac{1}{2}\pi$; that is if the condenser is either normal or parallel to the direction of motion, the stable position being that in which the energy is a minimum, that is, when the condenser is normal to the velocity.

The method of observation was to reverse the charge of the condenser at regular intervals of time equal to half the time of the natural period of swing of the condenser. By this means it was hoped that the cumulative effect of the alternating couple would gradually set up a measurable swing in the system. The experiment was tried at different times of day so as to ensure that by no accident the actual velocity of the earth relative to the æther was normal to the plane of the condenser; but no effect could be detected in any case.

If the reasoning whereby we were led to anticipate this result is correct, the expression for the energy, which is obtained after allowing for the FitzGerald contraction, leads to a disagreement between the result of experiment and the conclusions drawn from applying the ordinary mechanical principles. This brings us to the point that was overlooked by the designers of the experiment. In experiments which take account of the second order in v/c, the Newtonian mechanics require correction. This is clear from the case of the electron that has already been referred to. The apparent inertia being different in different directions, the force on the electron is not in the direction of its acceleration. It will clearly be a matter of some difficulty to derive from a constitutive theory of matter as composed of electrons of Lorentz's type the general way in which the dynamics of a rigid body are to be modified. The only feasible way of finding the kind of correction required is to start from the hypothesis of relativity in the new sense, not in the old dynamical sense, and to see what is the simplest modification that can be made in accordance with it, just as Lorentz's conception of the electron is the simplest type of nucleus that can be devised to accord with it. Some indication of the solution of this problem will be given in a later chapter, where a fuller discussion of the experiment will be found. (See p. 83.)

CHAPTER V.

MECHANICS AND THE PRINCIPLE OF RELATIVITY.

50. The Transition from Electrical to Mechanical Theory.

In making the transition from the purely electrical theory that we have been chiefly considering in the last two chapters to mechanical theory, it is necessary first to state clearly what is the connecting-link.

We might follow the method of the critical school which treats of force as a mere concept, the formation of which is possible on account of the physical laws of conservation of mass and momentum. This school of thought begins with the hypothesis, based on experience, that the accelerations of two particles which act on one another are in a constant ratio. This is, in effect, Newton's law of equal and opposite action and reaction, combined with the conception of a mass-ratio for two particles which is constant and independent of their velocities and of any other circumstances. In order to adapt this hypothesis to the requirements of the Principle of Relativity, we should have to modify it in such a way that the kinematic law which we put in its place has a form which remains the same when we change by a Lorentz transformation from one set of space-time coordinates to another. Having done this, we might define the force acting on a particle in a convenient way, and hence obtain the manner in which we conceive the measure or direction of a force to change when we change the frame of reference. Then subsequently we might bring the force, as so defined, into relation with the electrical field by seeking for a quantity in terms of the electro-magnetic vectors which is subject to the same transformation as that which has been suggested on kinetic grounds. For any equation which embodies a relation actually subsisting in physical phenomena must, according to the Principle of Relativity, have a form which is independent of the particular frame of reference employed.

It may be as well to point out by an illustration that some modification is necessary.

The addition equation of velocities which is fundamental in the relativity kinematics renders the Newtonian form of the equation of the conservation of momentum no longer possible. If we consider two particles moving in a straight line under no forces save their own mutual action, we have the equation

$$m_1 u_1 + m_2 u_2 = \text{constant.}$$

If we change the frame of reference, by the use of the equation (*a*), p. 38, we have in place of this

$$m_1 \frac{u_1' + v}{1 + u_1' v/c^2} + m_2 \frac{u_2' + v}{1 + u_2' v/c^2} = \text{constant,}$$

which is of an entirely different form.

It is not easy, without further analytical assistance, to see in what way the original equation is to be modified in order to obtain a form which is the same in both frames of reference. Minkowski devised an analytical method of great power and elegance by which it can be done very quickly. But it would be beyond the scope of this work to give any satisfactory account of it. We turn, therefore, to a means of bridging the gap between electrical and mechanical theory, which is nearer to that which has been usual in electro-dynamics, and which is followed by Lorentz, who makes the transition by introducing the expression for the 'mechanical force per unit charge,' $\mathbf{e} + [\mathbf{uh}]/c$.

51. Application of the Lorentz Transformation; Motion of a Charged Particle.

We adopt this definition and proceed to consider the motion of a charged particle. In order to maintain the relativity of the dynamical equation which determines its motion, we seek to equate the 'force' to a vector quantity depending on the velocity and acceleration of the particle which is subject to the same transformation as the expression for the 'force,' when we change the frame of reference in which the quantities **u**, **e**, and **h** are measured.

In order to do this it is sufficient to consider a transformation between two systems, in one of which the particle is instantaneously at rest. We shall assume that in this system the acceleration of the particle is given by

$$q\,\mathbf{e} = m\,\mathbf{f},$$

q being the charge on the particle, **f** the acceleration, and m being a definite constant. That is, we assume that for a particle whose velocity may be neglected, the motion is given by an equation of the ordinary Newtonian form, m being the 'apparent mass' when the velocity is zero.

Now, if we take the addition equations

$$u_x' = \frac{u_x + v}{1 + u_x v/c^2}, \qquad u_y' = \frac{u_y}{\beta(1 + u_x v/c^2)}, \qquad u_z' = \frac{u_z}{\beta(1 + u_x v/c^2)},$$

and assume that **u** is very small, these give us to the first order in **u**

$$u_x' = v + u_x(1 - v^2/c^2), \quad u_y' = u_y/\beta, \quad u_z' = u_z/\beta.$$

Now, let us suppose that the velocity **u** is the velocity communicated to the particle from rest in a short time δt; the corresponding interval of time in the other frame of reference is to be obtained from the equation

$$\delta t' = \beta(\delta t + v\ \delta x/c^2)$$

in which we have to put δx equal to zero since the velocity of the particle is momentarily zero. Thus

$$\delta t' = \beta\ \delta t.$$

Hence, dividing the increments of velocity by the corresponding intervals of time, we have for the relation between the expression for the accelerations in the two systems

$$f_x' = f_x/\beta^3, \quad f_y' = f_y/\beta^2, \quad f_z' = f_z/\beta^2.$$

But we have also from the transformation of the electric and magnetic intensities,

$$e_x' = e_x, \quad \beta\left(e_y' + \frac{v}{c}h_z'\right) = e_y, \quad \beta\left(e_z' - \frac{v}{c}h_y'\right) = e_z.$$

Thus the equations of motion become

$$qe_x' = \beta^3 mf_x',$$

$$q\beta\left(e_y' + \frac{v}{c}h_z'\right) = \beta^2 mf_z,$$

$$q\beta\left(e_z' - \frac{v}{c}h_y'\right) = \beta^2 mf_z';$$

or, introducing the 'force' vector $\mathbf{F}' = (\mathbf{e}' + [\mathbf{u}'\mathbf{h}]/c)$, since $\mathbf{u}' = (v, 0, 0)$ we may write these equations

$$qF_x' = \beta^3 mf_x', \quad qF_y' = \beta mf_y', \quad qF_z' = \beta mf_z'.$$

These are the equations which are satisfied by the motion of the particle relative to the system in which the particle has a velocity v parallel to the axis of x. They are commonly interpreted by saying that the particle has a 'longitudinal mass' equal to $\beta^3 m$, and a 'transverse mass' equal to βm; the word 'mass' being understood as the 'force per unit of acceleration produced,' and the 'force per unit charge' being defined to be $(\mathbf{e} + [\mathbf{uh}]/c)$.

As far as the dynamics of the particle are concerned, the above equations rest on the assumptions, (i) that observations of the acceleration of a charged particle by an electro-magnetic field will not furnish us with a means of distinguishing between the different systems of coordinates for which the laws of purely electrical phenomena preserve their form; and (ii) that if the velocity of a particle relative to the frame of reference is sufficiently small, the law of its motion reduces to the Newtonian form, the form which has been adopted as the result of the observation of the motions of bodies whose velocities relative to the earth are all very small compared with the velocity of light.

52. The Experiments on the Deviation of the β-Rays by an Electro-Magnetic Field.

We may now proceed to consider what light is thrown upon the equations and assumptions that have been written down above by the well-known experiments of Kaufmann and others on the inertia of the negative electrons. In the first form of these experiments a stream of the β-rays from a small piece of a radio-active substance passes through a fine hole in a screen and impinges on a photographic plate. Electric and magnetic fields are then set up in a common direction at right angles to the stream; it is then found that, instead of meeting the screen in a single small spot, the stream produces on the plate a curved trace.

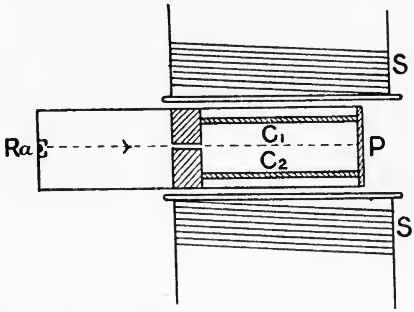


FIG. 6.—C_1, C_2, condenser; P, photographic plate; S, S, electro-magnet; *Ra*, radium fluoride.

The interpretation of this is as follows. Supposing the rays to consist of a stream of small charged particles, they will be deflected by the electric and magnetic fields. Let us take the axis of x to be in the direction of motion, and the axis of y to be in the common direction of the fields; let the accelerations in the directions of the axes of y and z be f_y and f_z, the velocity along the axis of x being v; then if l is the distance traversed through the field, and t is the time of transit,

$$l = vt;$$

and the deviations in the directions of the axes of y and z are

$$y = \tfrac{1}{2} f_y t^2 = \tfrac{1}{2} f_y l^2/v^2,$$
$$z = \tfrac{1}{2} f_z t^2 = \tfrac{1}{2} f_z l^2/v^2.$$

The measurement of the deviations y and z on the photographic plate thus gives the values of f_y/v^2 and f_z/v^2.

Let E, H be the respective intensities of the electric and magnetic fields. Then, if the equations obtained above are a correct representation of the facts, we have, since with the given arrangement the components of **F** are (0, E, vH/c),

$$q\text{E} = \beta^2 m f_y, \quad qv\text{H}/c = \beta^2 m f_z,$$

from which we have

$$\frac{v\text{H}}{c\text{E}} = \frac{f_y}{f_z} = \frac{y}{z}.$$

Thus, corresponding to any particular point on the trace of the ray on the plate, we can obtain an estimate of the velocity of the particles which strike the plate at that point, the strengths of the fields being supposed to be known. We are then able to test the correctness of the assumptions that led to the above equations by means of either equation. The first gives us

$$\frac{q}{m} = \frac{\beta^2 f_y}{E},$$

and each factor on the right-hand side is now a quantity of which we have an estimate. If it be true that all the particles forming the stream are identical in nature, then the values of q/m obtained from all the different points of the trace of the rays on the plate should be the same.

53. The Method of the Crossed Fields.

An improvement on this method was introduced by Bestelmeyer. It is now known as the method of 'crossed fields'.

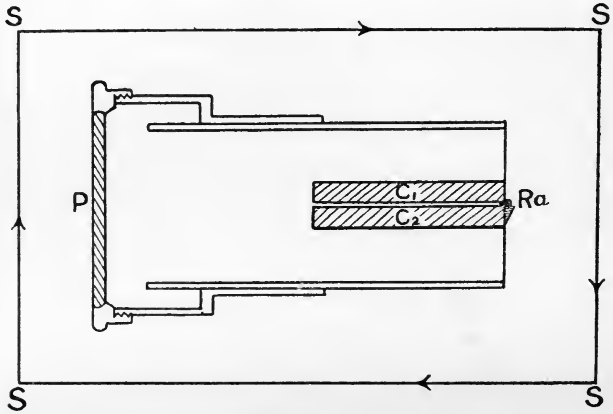


Fig. 7.

The figure shows the essential parts of the apparatus as used by Bucherer and Wolz. C_1 and C_2 are the plates of a condenser, consisting of two parallel optically-plane plates of glass silvered on the inner side, and kept at a fixed very small distance (about ·25 mm.) by pieces of a thin sheet of quartz. R*a* is a grain of radium fluoride from which the β-rays can pass between the plates of the condenser to strike the photographic plate P, which is at a measurable and adjustable distance from the condenser. The whole of this part of the apparatus is placed transversely inside a long solenoid, of which the rectangular section SSSS is shown in the figure. The electric field between the plates of the condenser and the magnetic field due to the solenoid are now at right angles, and the β-particles move across them both. As they are emitted in all directions by the radium fluoride, they cross the magnetic field at different angles with its direction. If we consider one which moves with velocity v at an angle α with the magnetic field H, it is subject to a force $qv\mathrm{H}\sin\alpha/c$ normally to the plates of the condenser, while, at the same time, it is subject to a force $q\mathrm{E}$ due to the electric field E. These two forces will balance one another if $v\mathrm{H}\sin\alpha = c\mathrm{E}$, and, if this is so, the electron will move in a straight line and so be able to pass right through the narrow gap between the plates of the condenser. (The gap between the condenser plates being so narrow compared with their length, it may be taken that the electrons emerging at a given angle α have a very definite velocity given by this equation.) On emerging they will be subject to the magnetic field only, and will therefore be deflected from a straight path.

According to the above theory there should be an acceleration perpendicular to v and to H of magnitude $qv\mathrm{H}\sin\alpha/mc\beta^2$, and the distance to be traversed is $d/\sin\alpha$, d being the distance from the condenser to the plate.

Thus to a first approximation the deflection z will be given by

$$z = \tfrac{1}{2}(qv\mathrm{H}\sin\alpha/mc\beta^2)\,(d/v\sin\alpha)^2$$
$$= \tfrac{1}{2}\,\frac{qd^2\mathrm{H}^2}{m\beta^2c^2\mathrm{E}}\,.$$

In this expression for z every factor is independent of α except β^{-2} which is equal $(1 - \mathrm{E}^2/\mathrm{H}^2\sin^2\alpha)$.

Thus the electrons will meet the plate in a curved line with a maximum deflection for the rays incident normally, that is, for $\alpha = \tfrac{1}{2}\pi$.

The measurements in the third part of the table below were made by the application of this method, which has proved capable of much more accuracy than the original method of Kaufmann.

54. Table of Results.

For the purpose of comparison of results obtained by different methods and for electrons of different velocities, it is convenient to put together the values of the ratio q/m, m being, as above, the apparent mass for small velocities, the equations of motion being assumed to be those obtained above by the application of the Principle of Relativity. It will be seen that there is a remarkable agreement between the values obtained.

The first four lines in the table give the ratio q/m for comparatively slow rays, as the cathode rays, and the rays emitted by a glowing body. For these there is no question as to the dependence of inertia upon velocity.

Next is given the value which is predicted by the best measurements of the Zeeman effect, interpreted on the hypothesis that the spectral lines are due to the vibrations of an electron.

The last part of the table gives the values obtained from β-rays of velocity comparable with that of light.

	v/c	$q/m \times 10^{-7}$
J. Classen, "Verh. d. Deut. Phys. Cres.," 10 (1908), p. 700	*	1·773
J. Malassez, "Ann. d. Chim. et de Phys.," 23 (1911), p. 231	*	1·769
A. Bestelmeyer, "Ann. der Phys.," 35 (1911), p. 909	*	1·766
E. Alberti, "Ann. der Phys.," 39 (1912), p. 1133	*	1·766
Weiss and Cotton, "J. de Phys.," 6 (1907), p. 429	†	1·767
P. Gmelin, "Ann. der Phys.," 28 (1909), p. 1079		1·771
A. Bucherer, "Ann. der Phys." 28 (1908), p. 513	·31	1·752[1]
A. Bucherer, "Ann. der Phys." 29 (1909), p. 1063	·69	1·767
R. Wolz, "Ann. der Phys.," 30 (1909), p. 273	·5	1·7676[2]
	·7	1·7672[2]
C. Schaefer (G. Neumann), "Phys. Zeit.," xiv (1913), p. 1117	·4 to ·8	1·7676[3]

In addition to the experiments of Wolz, Bucherer, and Neumann, reference should be made to those of Hupka.[4]

* Negligible.

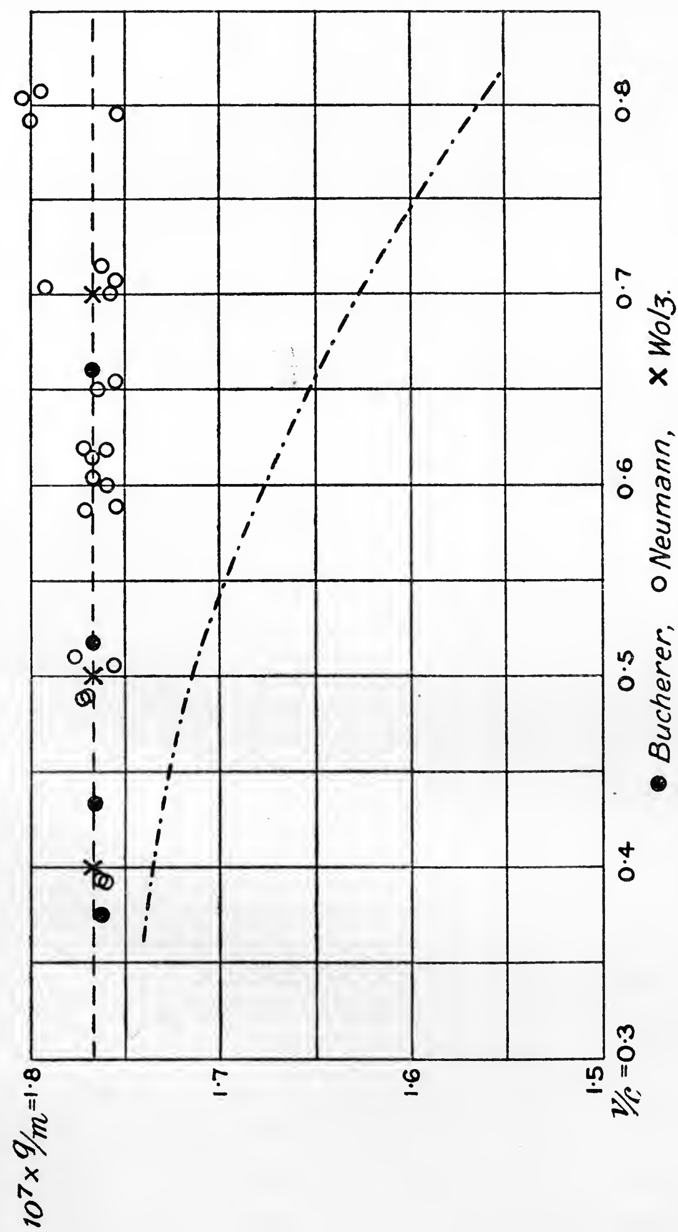
† Assumed to be negligible; experiments on Zeeman effect.

[1] Bucherer, in a later paper, points out a correction which should be made for the edge effect of the condenser. This would probably raise this result to the same value as the others. The following experiments of Wolz took particular note of this correction.

[2] $\pm$ ·0025.

[3] Of 26 results obtained, all lie between 1·75 and 1·8; all but four between 1·75 and 1·78. Possible error, $\pm$ ·0025. See Figure (8) below.

[4] "Ann. d. Phys.," 31 (1910), p. 169.



The curve –·–·– shews values of q/m calculated by the formula of Abraham

Fig. 8.

This author does not give an absolute value of q/m. He shows, however, that over a considerable range of velocities the formula of Lorentz gives a very constant value for q/m, e.g. in one set of Hupka's experiments in which the values of v/c varied between ·346 and ·501 the calculated values of q/m did not vary by more than ·03 per cent.

55. The Significance of this Verification.

The experiments which have just been discussed were, in the first instance, undertaken by Kaufmann merely with the intention of examining how far the inertia of an electron was due to the charge which it carried. It had been suggested by Sir J. J. Thomson as early as 1881 that the inertia of a body was increased by the presence of an electric charge on it. The result of Kaufmann's first experiments on the cathode rays was to show that, within the range of velocities there present, the whole inertia of the particles varied exactly in the same way as the electro-magnetic part of it should do, as calculated by Abraham from the hypothesis that the electron was a spherical distribution of charge.

Three years later, on the publication of the suggestion of the contracting electron to explain the results of the Michelson and Morley experiment, Kaufmann repeated his experiments with the β-rays in which higher velocities were accessible, with the object of distinguishing between the two theories. It cannot be said that the results obtained were entirely in favour of Lorentz's hypothesis; but more recent experiments, as the figures reproduced in the above tables show, were distinctly so. We may now say with certainty that the contraction hypothesis is not contradicted by experiment.

But to say that the results confirm the hypothesis 'that the whole of the inertia is attributable to purely electro-magnetic origin' is to forget the nature of the assumptions that are made in making the calculations.

The argument, as given by Lorentz, rests essentially on the hypothesis of the contracting electron. It was remarked above that to make this hypothesis is in reality only to shift the difficulty of explaining the facts back from the FitzGerald hypothesis, which refers to concrete matter, to the same hypothesis about a purely conjectural element in the constitution of matter, and there it must be left. If we accept the crude picture of the electron as a spherical shell carrying a surface layer of charge, we must not ask how this shell is held together against the mutual repulsions of the parts of the charge which is carried; in fact, we hardly dare think of the charge on the electron as divisible, much less may we ask why the electron contracts, and whether there are 'forces' which cause it to do so. All that we can do is merely to assume that it is exactly so constituted as to fall in with the experimental facts.

There is then something over and above purely electro-magnetic theory required for the complete explanation of the facts observed. The simple comparison of the results of Abraham and Lorentz, which give entirely different values for the apparent inertia of an electron whose velocity is comparable with that of light, shows that the choice of this additional element to supplement the electro-magnetic laws is one which may seriously affect the results predicted by those laws. This is, for example, clearly shown by the lower curve in fig. 8, which exhibits side by side the values of q/m calculated from the experimental results of Neumann (i) by Lorentz's theory, (ii) by Abraham's theory. The latter theory differs only from that of Lorentz in the assumption that the electron is always a sphere, whatever velocity it is supposed to have.

Thus the agreement of experiment with the predictions of Lorentz's theory confirms the general correctness of his hypothesis. But it cannot be said to confirm the particular hypothesis of the contracting electron, which is only one of many that are consistent with the Principle of Relativity. In any case, it cannot be said that Lorentz's theory is a *purely* electro-magnetic one.

The results recorded above are therefore to be considered as confirming the general hypothesis of relativity as a principle that goes beyond the mere invariant form of the electro-magnetic equations.

CHAPTER VI.

MINKOWSKI'S FOUR-DIMENSION VECTORS.

56. INTRODUCTORY.

THE position at which our discussion has now arrived is this. We saw in Chapter III. that, confining our attention to the facts of the propagation of light in free space, the possible methods of partitioning time and space were not independent of one another, that there was no unique sense to be attached to the phrase "simultaneity of events". This has been followed by a consideration of physical phenomena of a varied nature, with the result that we have arrived at the conclusion that the ambiguity in, and the interdependence of, the modes of measurement of time and space has, up to the present, not been removed.

Minkowski says of the situation so created: "*From henceforth, space by itself, and time by itself, are mere shadows, and only a kind of blend of the two exists in its own right*".[1]

All future discussion of the exact relations of the constitution of matter must be influenced by the general evidence for the validity of the hypothesis of relativity. If the principle is not universal it will sooner or later prove its own destruction by predicting results which are at variance with experiments. We must remember, however, that the principle is on a different plane from such "physical" laws as those embodied in Maxwell's equations or the law of gravitation. It is a general mode of thought. We are not concerned with any particular quantitative law, but with the question as to whether all the

[1] "Raum und Zeit," 1908. Reprinted in "Das Relativitäts prinzip," Leipzig, 1913.

quantitative laws of physics conform to a certain common criterion.

There is one such common criterion which is commonly accepted. We may call it the "vectorial" criterion, or the criterion that a mere change of orientation of a physical system in space will not alter its intrinsic properties. Space is said to be isotropic as far as it is merely a mode of ordering physical properties. The result is that, although we often employ a particular set of coordinate axes as a frame of reference, we recognize that the form of the equations which embody our physical laws must be one which is unaltered if we turn our axes into any other angular position. What is known as "vector analysis" is a notation and calculus which recognizes this principle from the beginning, and which is adapted specially and only to the expression of such laws as conform to it.

The criterion of relativity is very analogous to this, and was, in fact, shown by Minkowski to be expressible as a generalization of it. An attempt will now be made to present the main idea introduced by Minkowski.

57. The Lorentz Transformation as a Rotation of Axes.

The change in the measures of space and time, which we have called a Lorentz transformation, may be written thus—

$$x' = x\cos\theta - u\sin\theta,\quad y' = y,$$
$$u' = x\sin\theta + u\cos\theta,\quad z' = z,$$

where, i standing as usual for $\sqrt{(-1)}$,

$$u = ict,\ u' = ict',\ \cos\theta = \frac{c}{(c^2 - v^2)^{\frac{1}{2}}},\ \sin\theta = \frac{iv}{(c^2 - v^2)^{\frac{1}{2}}},$$

the last two quantities satisfying the equation

$$\cos^2\theta + \sin^2\theta = 1.$$

The analogous equations in three variables,

$$x' = x\cos\theta - z\sin\theta,\quad y' = y,$$
$$z' = x\sin\theta + z\cos\theta,$$

are the equations which correspond to a simple rotation of the axes of coordinates through an angle θ about the axis of y.

We may then think of the Lorentz change of variables as a rotation of the imaginary set of axes through the imaginary angle θ in the four-dimension space in which the coordinates are (x, y, z, ict). This is, of course, only a way of visualizing what are in effect algebraic processes involving only real quantities.

58. Extension of the Idea of a "Vector".

Again, in three dimensions we speak of 'vector quantities,' as directed quantities which combine according to the parallelogram law.

If a certain vector is represented by components (X, Y, Z) along the axes of (x, y, z), and by the components (X′, Y′, Z′) when the axes of (x', y', z') are used, the relations connecting (X′, Y′, Z′) with (X, Y, Z) are exactly the same as those connecting (x', y', z') with (x, y, z). A vector quantity can be exactly represented by a line.

So Minkowski speaks of '**4-vectors**' as quantities with four components which when the frame of reference is changed are subject to exactly the same equations as are the space-time coordinates (x, y, z, u). As a particular case, for the Lorentz transformation the 4-vector (X, Y, Z, U) is changed to (X′, Y′, Z′, U′), according to the equations

$$X' = \beta(X - ivU),\ Y' = Y,\ Z' = Z,\ U' = \beta(U + ivX/c^2),$$

where the velocity v is arbitrary, and the direction of the axis of x is also an arbitrary direction in space.

These equations are equivalent to

$$X' = \beta(X + vT),\ Y' = Y,\ Z' = Z,\ T' = \beta(T + vX/c^2).$$

59. Examples of 4-Vectors.

Let two point-instants be in a certain frame of reference (x_1, y_1, z_1, t_1) (x_2, y_2, z_2, t_2). Each of these sets of quantities is a 4-vector, and clearly so is their difference

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1, t_2 - t_1).$$

If we write for the differences $(\delta x, \delta y, \delta z, \delta t)$, remembering the fundamental property that

$$\delta x^2 + \delta y^2 + \delta z^2 - c^2\delta t^2$$

is an invariant (p. 39), it follows that

$$(\delta x, \delta y, \delta z, \delta t)/\kappa\delta t$$

is a 4-vector; that is, for a moving point

$$\mathfrak{u} = \kappa(u_x, u_y, u_z, 1) \text{ is a 4-vector}$$

where $$\kappa = \{1 - (u_x^2 + u_y^2 + u_z^2)/c^2\}^{-}.$$

More generally,

if $\mathfrak{X} = (X, Y, Z, U)$ is any 4-vector

then $\dot{\mathfrak{X}} = \kappa \frac{\partial}{\partial t}(X, Y, Z, U)$ is a 4-vector,

where X, Y, Z, U are supposed to be any functions of the space-time coordinates (x, y, z, t) of a certain moving point.

The following may be stated to be 4-vectors without proof:—

$$\mathfrak{e} = \{\mathbf{e} + [\mathbf{uh}]/c,\ i(\mathbf{ue})/c\}$$
$$\mathfrak{h} = \{\mathbf{h} - [\mathbf{ue}]/c,\ i(\mathbf{uh})/c\},$$

where the symbols have the same meanings as in the fundamental equations, and the first three components of the 4-vector are the components of the ordinary vectors specified.

60. Space and Time as Two Aspects of a Unity.

We now see what Minkowski had in mind when he spoke of "only a blend of space and time existing in its own right". The 4-vector is thought of as a single entity, just as we think of a force as a single entity. The *components* of a force do not exist by themselves. They are only convenient means of specifying the *force* which is one and definite. So space and time coordinates are to Minkowski only particular, complementary, and inseparable means of specifying a single fact or occurrence.

Analytically Minkowski transports himself to a space of four dimensions in which the distinction between space and time vanishes. In this four-dimensional region, the whole of space and time is portrayed in one construct. The motion of a moving point through all time is represented by a single curve, the points on the curve being ordered to correspond with the succession of events in time, but the interpretation of the curve as representing an ordinary motion is not unique; it depends upon the choice of the direction in the four-dimensional region which is chosen to be the time axis.

Thus three-dimensional kinematics becomes four-dimensional geometry. This relation extends further, it reaches into the domain of mechanical quantities. Three-dimensional dynamics can be interpreted as a four-dimensional statics, an exact generalization of three-dimensional statics. An example of this will be given shortly.

61. A Second Kind of Four-Dimensional Vector.

Before passing on to this, however, it will be well in passing to indicate the way in which Minkowski, thinking in terms of this four-dimension space, pictures the significance of the relativity of the electro-magnetic vectors **e** and **h.**

In ordinary three-dimension space an element of a straight line—i.e. a one-dimensional region—and an element of area—i.e. a two-dimensional region—may be equally well represented as vectors. But in four dimensions, of the elements of regions of one, two, and three dimensions, only the first and third are capable of representation as what we have called 4-vectors. A two-dimensional element will have to be represented by six-components, corresponding to the number of ways of choosing two out of the four coordinates of a point.

Now Minkowski was able to show that the transformations (B, p. 46), affecting $(E_x, E_y, E_z,)$ (iH_x, iH_y, iH_z) are exactly of the same form as those which affect the six-components of a two-dimensional region in four-dimension space when the fundamental geometrical transformation of (x, y, z, t) is carried out. Hence he introduces a new conception; he defines a '**6-vector**' to be a quantity with six-components which transforms in exactly this way. Then the validity of the transformation (B) is contained in the phrase

$$\mathfrak{F} = (\mathbf{e}, i\mathbf{h}) \textit{ is a } \mathbf{6\text{-}vector.}$$

One property which follows from this, and which may be deduced immediately from (B), is that

$$\mathfrak{F}^2 = \mathbf{e}^2 - \mathbf{h}^2 \textit{ is invariant.}$$

We might here extend the remark of Minkowski quoted above (p. 70), and say that "*from this point the electric intensity and the magnetic intensity apart from each other have no significance, but that only a single quantity compounded of the two exists in its own right*".

62. The Principle of Least Action a Generalization of that of Least Potential Energy.

It is now possible to give a striking example of the way in which these conceptions give to dynamical equations a symmetrical form which is an exact generalization of the ordinary vectorial form of statical equations.

In electrostatics the 'total energy' of a field may be written

$$\tfrac{1}{2}\int \mathbf{e}^2 dV,$$

where dV is an element of volume [1]; and there is a well-known theorem that *the electro-static field is such as to render this total energy a minimum subject only to the restrictions imposed on it by a given distribution of charge.* The generalization of this is:—

If $d\mathfrak{V}$ represent an element of the four-dimensional space, and $\int \mathfrak{F}^2 d\mathfrak{V}$ be called the 'total action,' then the electro-dynamic field is such as to make this total action a minimum, subject only to the restrictions implied by a given distribution of moving charges.

This is only the expression in the language of four vectors of the known theorem that the equations of the electro-magnetic field are such as to make the time integral of the difference of the magnetic and electric energies a minimum. In ordinary notation this integral is

$$\tfrac{1}{2}\iiiint (\mathbf{e}^2 - \mathbf{h}^2)\, dx\, dy\, dz\, dt.$$ [2]

In comparing these theorems we may note that whereas in the former the 'total energy' is a quantity which is independent of the frame of reference, so in the second the 'total action' is a quantity that does not change when we alter the space-time system of measurement.[3]

63. The Fundamental Equations in Symmetrical Form.

It may be worth while to set down the equations of the field as Minkowski arranges them. For fuller details see the author's larger work, Chapter IX.

Writing

$$(F_{yz}, F_{zx}, F_{xy}, F_{xu}, F_{yu}, F_{zu}) = (-H_x, -H_y, -H_z, iE_x, iE_y, iE_z),$$

and $$F_{yz} = -F_{zy}, \text{ etc.};$$

also $$(S_x, S_y, S_z, S_u) = \rho\,(u_x, u_y, u_z, ic)/c,$$

and $$u = ict,$$

[1] Remembering that we are using Lorentz's units.

[2] See Larmor, "Aether and Matter," pp. 82 ff.

[3] It has been pointed out above that $\mathfrak{F}^2$ is an invariant. In changing from the variables (x, y, z, t) to (x', y', z', t') we substitute for $d\mathfrak{V}$ the quantity $\dfrac{\partial(x, y, z, t)}{\partial(x', y', z', t')} d\mathfrak{V}'$, and it is easily seen that $\dfrac{\partial(x, y, z, t)}{\partial(x', y', z', t')}$ is unity.

the equations of the field become

$$\left.\begin{array}{l} \qquad\quad\ \frac{\partial F_{yx}}{\partial y}+\frac{\partial F_{zx}}{\partial z}+\frac{\partial F_{ux}}{\partial u}=S_x \\ \frac{\partial F_{xy}}{\partial x}\qquad\quad\ +\frac{\partial F_{zy}}{\partial z}+\frac{\partial F_{uy}}{\partial u}=S_y \\ \frac{\partial F_{xz}}{\partial x}+\frac{\partial F_{yz}}{\partial y}\qquad\quad\ +\frac{\partial F_{uz}}{\partial u}=S_z \\ \frac{\partial F_{xu}}{\partial x}+\frac{\partial F_{yu}}{\partial y}+\frac{\partial F_{zu}}{\partial u}\qquad\quad\ =S_u \end{array}\right\}$$

$$\left.\begin{array}{l} \qquad\quad\ \frac{\partial F_{uz}}{\partial y}+\frac{\partial F_{yu}}{\partial z}+\frac{\partial F_{zy}}{\partial u}=0 \\ \frac{\partial F_{zu}}{\partial x}\qquad\quad\ +\frac{\partial F_{ux}}{\partial z}+\frac{\partial F_{xz}}{\partial u}=0 \\ \frac{\partial F_{uy}}{\partial x}+\frac{\partial F_{xu}}{\partial y}\qquad\quad\ +\frac{\partial F_{yx}}{\partial u}=0 \\ \frac{\partial F_{yz}}{\partial x}+\frac{\partial F_{zx}}{\partial y}+\frac{\partial F_{xy}}{\partial z}\qquad\quad\ =0 \end{array}\right\}$$

The symmetrical form of the equations is now obvious. They may be easily shown to have an invariant form, and may be looked upon as the generalization of the vectorial equations of the electro-static field

$$\text{div}\ \mathbf{e} = \rho$$
$$\text{curl}\ \mathbf{e} = 0.$$

(S_x, S_y, S_z, S_u) is easily seen to be a 4-vector by means of the equations (*a*) and (C), p. 46.

64. Summary.

To sum up what has been briefly sketched in this chapter, it may be said that the relativity of physical phenomena, in the sense in which we are using the term, is equivalent to the possibility of expressing all the relations between them as vectorial relations in this four-dimensional sense. Once the general principle is admitted, no equations would be considered as representing actual physical phenomena unless they fulfil this condition.

Thus, for example, if we are dealing with the dynamics of a single actual particle, it is necessary that its velocity shall conform to Einstein's addition equation, which, as we have seen, may be put into the condition that $\mathfrak{u}$ (§ 59) is a 4-vector.

This last remark bears upon the question as to whether we are allowed to think of an actual "objective æther" once we admit the principle of relativity. Clearly it is not if we identify the æther with a frame of reference which is not unique. But if we allow that the æther may be a moving medium—something after the manner, for instance, of Sir J. J. Thomson's moving tubes of force—there will be no *a priori* reason against it. For if the velocity of the æther at a point is defined in terms of the electric and magnetic intensities at the point, the actual value of it will depend upon the frame of reference used; and it is quite possible that the velocities so obtained for two different systems of reference may be related in the manner required by our principle. This will be discussed at greater length in Chapter VIII.

CHAPTER VII.

THE NEW MECHANICS.

65. 'Force' and 'Work' as Two Aspects of a Single Concept.

In the classical mechanics there have always been two strains of thought. The two aspects of 'force' as 'the time rate of change of momentum,' and as 'the space rate of change on energy,' have with different writers been given different degrees of prominence. Galileo developed the former, Huyghens the latter. In the light of four-dimensional vectors the two ideas become unified, and differ only as partial aspects of a greater concept, just as space and time are unified and become partial aspects of the whole extension of the universe.

The equations of motion for a body moving as a whole with velocity $\mathbf{u}$ without rotation

$$\frac{d\mathbf{g}}{dt} = \mathbf{k}$$

and

$$\frac{d\mathbf{w}}{dt} = (\mathbf{ku})$$

can be brought into a single equation thus.

We have seen that, $(\delta x, \delta y, \delta z, ic\delta t)$ being a 4-vector, $\delta t/\kappa$ is an invariant.

Hence if $\mathfrak{f}$ is any 4-vector, and $\delta\mathfrak{f}$ is a small change in it,

$$\kappa\frac{\delta\mathfrak{f}}{dt} \text{ or, in the limit, } \kappa\frac{d\mathfrak{f}}{dt}$$

is another 4-vector.

Now let $(\mathfrak{k}) = (\mathbf{k}, i(\mathbf{ku})/c)$ be a 4-vector. If we were to assume that $(\mathbf{g}, iw/c)$ is a 4-vector $\mathfrak{g}$, the equation

$$\mathfrak{k} = \kappa\frac{d\mathfrak{g}}{dt}$$

would be an equation of invariant form, that is, would express relations which are in accord with the Principle of Relativity.

The first three components of this equation give

$$\mathbf{k} = \kappa \frac{d\mathbf{g}}{dt}$$

which if $(v/c)^2$ be neglected reduces to the Newtonian form,

$$\mathbf{k} = \frac{d\mathbf{g}}{dt};$$

while the fourth component is, to the same order,

$$(\mathbf{ku}) = \frac{dw}{dt}$$

which is the usual equation of energy.

Thus in order to bring the mechanics of a particle into line with the Principle of Relativity, we have only to assume

(i) *that if* $\mathbf{g}$ *is the momentum and* w *the energy then* $(\mathbf{g}, iw/c)$ *is a 4-vector;*

(ii) *that if* $\mathbf{k}$ *is the force acting on the particle and* $\mathbf{u}$ *its velocity then*

$(\mathbf{k}, i(\mathbf{ku})/c)$ *is a 4-vector;*

(iii) *that the ordinary equations of Newton are only an approximation to the more exact equations*

$$\mathfrak{k} = \kappa \frac{d\mathfrak{g}}{dt}.$$

The distinction between the schools of Galileo and Huyghens, between 'force' and 'work,' is here lost, and we see them each as partial aspects of a greater whole.

66. Relation Between Momentum and Energy.

The stipulation that $\mathfrak{g}$ shall be a 4-vector requires us to modify the usual relations which connect the momentum and energy of a moving particle with its velocity. Let us make the assumption that *for a particle at rest the momentum is zero.* Then for that case

$$\mathfrak{g}_0 = (0, 0, 0, iw_0/c)$$

where w_0 is the energy of the stationary particle.

Now, for any particle whatever we may choose a frame of reference of which the origin has momentarily the same velocity as the particle. The extended velocity vector $\mathfrak{u} = \kappa(\mathbf{u}, ic)$ then becomes

$$\mathfrak{u}_o = (o, o, o, ic),$$

since in this frame of reference the particle is at rest.

The two vectors $\mathfrak{g}_o$, $\mathfrak{u}_o$ are thus in the same direction, and we may write

$$\mathfrak{g}_o = \frac{w_o}{c^2}\mathfrak{u}_o.$$

Now the equality of two vectors is an invariant relation; so that, whatever the frame of reference, we must have for the same particle

$$\mathfrak{g} = \frac{w_o}{c^2}\mathfrak{u}.$$

The last equation resolves into

$$\mathbf{g} = \kappa w_o \mathbf{u}/c^2 = \frac{w_o \mathbf{u}}{c^2(1 - \mathbf{u}^2/c^2)^{\frac{1}{2}}},$$

and

$$w = \kappa w_o = \frac{w_o}{(1 - \mathbf{u}^2/c^2)^{\frac{1}{2}}},$$

$\mathbf{g}$ and w being connected by the relation

$$\mathbf{g} = w\mathbf{u}/c^2.$$

Of these results we may note the following consequences if we neglect higher powers of $\mathbf{u}/c$ than the second:—

(i) $$w = w_o(1 + \tfrac{1}{2}\mathbf{u}^2/c^2) = w_o + \tfrac{1}{2}m\mathbf{u}^2,$$

(ii) $$\mathbf{g} = w_o\mathbf{u}/c^2 = m\mathbf{u}, \text{ where } m = w_o/c^2.$$

The quantity m here enters in exactly the same way as the Newtonian mass; thus we are led to contemplate a relation between the mass of a particle and its energy. That it is a possible relation appears from the following figures. If the particle were to give out energy, as in the case of radioactive bodies, the apparent mass should, according to this relation, diminish. But if we take the actual rate of loss of energy by a gram-atom (225 grms.) of radium in its disintegration, which is about 10^{12} ergs per hour, and divide by c^2, the rate of diminution of the mass would be only about 10^{-5} gr. per year, which is quite inappreciable by ordinary methods.

In the case of large velocities the concept of 'mass' becomes meaningless, but we still have the relation

$$\mathbf{g} = w\mathbf{u}/c^2$$

which indicates that the convection of energy by a moving body implies an amount of momentum equal to the product of energy and velocity divided by c^2.

Thus if a body moving uniformly is radiating energy, it is also losing momentum. The equation

$$\mathbf{k} = \beta \frac{d\mathbf{g}}{dt}$$

then indicates that there is a force acting on the body even though its velocity remains constant.

It has sometimes been suggested as a paradox, that whereas ordinary electro-magnetic theory clearly indicates a resistance to the motion of a radiating body through the æther,[1] yet the Principle of Relativity must allow that a body radiating equally in all directions must be capable of remaining in uniform motion through the æther if subject to no external force.

The above analysis resolves the paradox; it admits the resistance, but finds it to be equal to the rate at which momentum is lost owing to the radiation of energy, even though the velocity be unchanged.

This proposition of the momentum or inertia of energy is one which, in the development of mechanics subject to the Principle of Relativity, is universal. A familiar instance is to be found in the mechanics of the free æther as ordinarily stated. Poynting, in an analysis of electro-magnetic energy, suggested that, at any point of free space, there is a flux of energy represented by the vector c[EH]; while later, Abraham showed that we may maintain the proposition of the conservation of momentum as between æther and electric charge if we allow of a distribution of momentum in the æther of intensity [EH]/c. Here again the flux of energy is equal to the momentum multiplied by c^2, in exact agreement with the corresponding result for a material particle.

67. Gravitation and the Principle of Relativity.

The statement of Laplace that if gravitation is propagated through space with a finite velocity, that velocity must be

[1] See Larmor, "International Congress of Mathematicians," Cambridge, 1912, p. 216.

vastly greater than that of light is well known, and upon it is based a criticism of the Principle of Relativity that has already been referred to. But it does not appear to be equally well known that there are certain deviations of the planets from perfect elliptic motion which have not yet been explained on the Newtonian law of gravitation, notably a secular change in the apse line of the orbits, and that several attempts have been made to explain this particular irregularity by the introduction of a finite velocity of propagation of the same order as the velocity of light.

Laplace's statement would remain true if no other modification were made in the planetary dynamics than the introduction of the finite velocity of propagation. But we have seen that the Principle of Relativity requires the introduction of other ideas which are foreign to the classical dynamics. It necessitates, for instance, the giving up of the idea of a constant mass, and of a force which is a vector in the ordinary three-dimension sense and is independent of the velocity of the point upon which it acts.

When allowance is made for these facts, Laplace's conclusion is invalidated. This was perceived to be the case as soon as the conception of an electro-magnetic constitution of matter was foreshadowed. Weber, Riemann, Levy, Lorentz, and Gerber each suggested modifications of the law of gravitation for moving bodies, which reduce to the Newtonian law when the velocities of the attracting bodies are neglected. The hypotheses of Levy and Gerber actually show that it is not only possible to make a modified law of gravitation which is equally satisfactory with that of Newton, but that it is even possible so to explain the outstanding difficulty of the secular motion of the perihelion of the planet Mercury.[1] Each of these writers assumes a velocity of propagation equal to that of light.

The laws which are developed by the various authors mentioned are based upon various arbitrary hypotheses as to the nature of matter. The Principle of Relativity proceeds, on

[1] For a general summary of the various hypotheses and their consequences see J. Zenneck, article on "Gravitation," "Encyk. der Math. Wiss.," V. ii., pp. 46-51.

the other hand, from a general hypothesis which, as far as we can tell, is experimentally valid, without any reference to a hypothetical detailed structure. Poincaré[1] was the first to show how modifications of the law of gravitation can be devised which are consistent with the general principle. Minkowski also showed, by the use of his four-dimensional calculus, a simple way of effecting the same process. The result is that the Principle of Relativity indicates that there is an infinite number of laws which fulfil the required condition of reducing to the Newtonian law for bodies at rest. Of these laws two have been thoroughly tested by de Sitter.[2] For the first of them the deviations in the planetary motions from those predicted by the Newtonian law are in every particular too small to be observed. In the second case, the only appreciable effect is a motion of 7″ per century in the perihelion of Mercury; an effect of the same kind, though much smaller, as that of 40″ per century, which is known to exist, but which still awaits explanation.

Thus we conclude that in the only region of dynamical phenomena where we have any experimental evidence, that of planetary motion, there is no contradiction between the facts and the Principle of Relativity.[3]

68. The Trouton-Noble Experiment.[4]

There is an apparent contradiction, somewhat similar to that referred to in § 6 in the case of a radiating body in uniform motion, which arises in respect of the moment of the forces acting on a system in uniform motion. We shall see in the next chapter that, in order to reconcile the mechanical principles of conservation of momentum and conservation of energy in the æther with the known laws of propagation of electro-magnetic disturbances, it is necessary to assign to the æther a velocity which is not in general in the direction of the momentum. In the same way it is possible so to modify the mechanical equations of any deformable continuous system,

[1] "Rend. del Circ. Mat. di Pal.," 21 (1906), p. 166.
[2] Monthly notices of Roy. Astr. Soc., March, 1911, p. 388.
[3] Further details are given in the author's larger work, pp. 171-180.
[4] See p. 56.

using the same methods that have been indicated above for modifying the dynamics of a single particle, that they may be in accord with the Principle of Relativity.

In doing this two important results emerge. The first is that the relation referred to above between the flux of energy[1] and the momentum must be made universal. The second is that, in general, *the momentum per unit volume at any point of the medium need not be in the direction of the velocity of the medium at that point.*

In a case where this difference of direction arises it is easy to see that, for an element of volume moving with uniform velocity and with constant linear momentum, *the moment of momentum about a fixed point is not constant* as it is when the directions are the same. For in the figure let P, P′ be the positions of the element at two consecutive instants, so that $PP' = v\delta t$, and let the linear momentum be in the direction NP and of constant magnitude g.

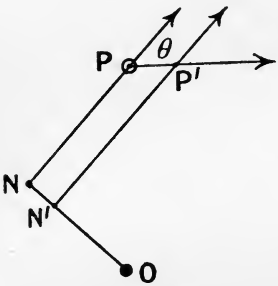


Fig. 9.

Then it is clear from the figure that in the time δt the moment of the momentum about O diminishes by $g \times NN'$, that is, by $gv \sin \theta\delta t$, a quantity which only vanishes if g and v are in the same straight line.

Thus in the mechanics of the Principle of Relativity we shall not expect that the moment of momentum of a body moving uniformly relative to a given frame of reference will always remain constant. Exactly such a case arises in the conditions of the experiment of Trouton and Noble (see p. 56). Here a charged plane condenser is supposed to be placed obliquely to its direction of motion relative to a given frame of reference. The classical theory shows clearly that there must be a couple acting on it in a sense which would, according to ordinary dynamics, tend to turn it so as to be normal to the direction of motion. But if the dynamics of the Principle of Relativity are applied to this case we find that the couple is exactly equal to the rate of

[1] *Flux of Energy = Momentum* $\times c^2$.

change of the moment of momentum which arises as above from the fact that the linear momentum, though constant, is in a different direction from the velocity. Thus the mechanical conditions are completely fulfilled without any rotation of the condenser being set up.[1]

69. Extension of Analogy Between Dynamics and Statics.

In § 62 we have shown that the relation of the laws of the electro-dynamic field to the principle of least action can be looked upon as the extension to Minkowski's four-dimensional world of the relation of the laws of the electro-static field to the principle of least potential energy. In that statement, however, no account was taken of the mechanical constraints imposed upon the charge or current; the distribution of these was assumed to be given, and the variation of energy or current was subject to the given distribution.

In the electro-static case, if the charges or conductors carrying them are supposed to be moved, we know that the increment in the electro-static energy is equal to the virtual work of the non-electrical forces which maintain the equilibrium of the conductors against the electrical attractions and repulsions. Or again, if the non-electrical forces are conservative so that we may speak of a corresponding potential energy, the variation of the total potential energy in a small displacement from the equilibrium position is zero. We note, of course, that the total energy is a scalar quantity, and is independent of the choice of any particular system of reference.

In the dynamical case we have the corresponding theorem that if A_1 represents the electro-dynamic action as defined in § 62, and if there be a non-electrical action A_2, arising from non-electrical inertia and forces, then if $A = A_1 + A_2$ is called the 'total action,' then, for any slight variation of the currents and motion of the carriers of charge and current from the actual motion which takes place, the variation in A is zero.

This very general proposition must, as in the simpler case, be independent of the choice of a frame of reference; that is,

[1] For a fuller discussion see the author's larger work, p. 170.

if we maintain the relativity of all the phenomena, it must have an invariant form for any change of axes in Minkowski's four-dimensional space. Now we know, as in § 62, that A_1, the electro-dynamic part of the action, is an invariant. Hence the Principle of Relativity requires that if there is a non-electrical action A_2, it too must be an invariant.

This hypothesis would be a sufficient basis for the development of a system of mechanics which is consonant with the Principle, but it is beyond the scope of this work to develop it in any detail.

70. One simple development may be set down as an indication of the application of the above result.

If $\mathbf{k} = (k_x, k_y, k_z)$ is a typical force acting at a point (x, y, z) at time t, and we consider a slightly varied motion of the system in which the coordinates of the same point are $(x + \delta x, y + \delta y, z + \delta z)$ at time $(t + \delta t)$, then the variation in the action arising from $\mathbf{k}$ is known to be

$$k_x \delta x + k_y \delta y + k_z \delta z - (\mathbf{ku})\delta t,$$

where $\mathbf{u}$ is the velocity of the point in question (see e.g. Routh, "Rigid Dynamics," Vol. II, p. 280).

In Minkowski's notation this can be written $\mathfrak{k}\delta\mathfrak{r}$ where $\mathfrak{k} = (\mathbf{k}, i(\mathbf{ku})/c)$ and $\delta\mathfrak{r} = (\delta x, \delta y, \delta z, icdt)$. The invariance of this quantity, $\delta\mathfrak{r}$ being an arbitrary 4-vector, requires that $\mathfrak{k}$ shall also be a 4-vector.

CHAPTER VIII.

RELATIVITY AND AN OBJECTIVE ÆTHER.

71. A Necessary Criterion for an Objective Æther.

A point upon which much emphasis was laid in the earlier part of this account of the Principle of Relativity was that the stationary æther commonly spoken of in electrical theory is a conceptual medium whose only properties are represented by the vectors which are known as the electric and magnetic intensities, and by the distribution and motion of the electric charges. Since the equations which describe the sequences of electrical phenomena, as far as we know them, do not involve any assumed velocity of the æther, it has become natural to think of it as stationary, and, in fact, to identify it with the frame of reference.

Hence arose largely the controversial attitude which has been adopted by many in regard to the Principle of Relativity. On the one hand, it is maintained that the frame of reference is not unique, and that therefore, since there cannot be an infinite number of æthers, there is no justification in speaking of one; while, on the other hand, it is urged that this is to do away with the æther, and to disregard the necessity for a real means of propagation of electrical effects with a definite velocity.[1]

Now there is another alternative position which has not been sufficiently considered. Those who admit that the frame of reference is not unique are certainly justified in saying that a real objective æther must not be identified with the frame of reference; but no objection can be taken to the conception of an æther which moves relative to the frame of reference,

[1] Cf. Larmor, "Int. Cong. of Math.," Cambridge, 1912, p. 214, footnote.

provided that when the frame of reference is changed by means of the Einstein transformation, the velocity attributed to any element of the æther is subject to the same kind of change as that which is applicable to the velocity of any other identifiable point; as for example, the velocity of a material particle. In other words, the velocities of the æther in two different frames of reference must be related according to the Einstein addition equation. If this were so, then the Principle of Relativity, instead of discarding the æther as fictitious, would naturally recognize it as objectively[1] existent; and the objections of those who demand a *real* æther to carry *real* effects will be thus met.

72. Statement of the Problem.

The following problem may therefore be propounded:—

Given the electric intensity $\mathbf{E}$ *and the magnetic intensity* $\mathbf{H}$ *at a point in a given frame of reference, is it possible to determine in terms of* $\mathbf{E}$ *and* $\mathbf{H}$ *a velocity* $\mathbf{u} = \mathbf{f}(\mathbf{E}, \mathbf{H})$, *the vector function* $\mathbf{f}$ *being of such a nature that if* $\mathbf{u}' = \mathbf{f}(\mathbf{E}', \mathbf{H}')$, $\mathbf{E}'$, $\mathbf{H}'$ *being the electric and magnetic intensities at the corresponding point in a second frame of reference, the relation between* $\mathbf{u}$ *and* $\mathbf{u}'$ *is exactly the Einstein relation for the transformation of a velocity from the first frame of reference to the second.*

73. The Mechanics of the Æther.

The problem of thus assigning a velocity to the æther is intimately associated with that of the mechanical properties of the æther. In fact, the conception of the æther as the means of propagating electrical effects is probably never completely separated from the conception of a mechanical medium of some kind in which a state of stress and motion is associated with a transfer either of momentum or of energy.[2]

[1] For our purposes here the definition of an 'objectively existent' entity is one of which a definite state of motion may be specified.

[2] In the scheme proposed by Lorentz we have seen that the mechanical categories of "force" and "momentum" are introduced in order to give a complete set of equations for the determination of the motion of the electrons. In the form proposed by Larmor the usual equations are shown to be deducible from the mechanical principle of least action used; the variation of the time integral of the difference of magnetic and potential energy is made to vanish,

Now, the usual analysis of the stress in the æther is incomplete just at this point. The *transfer of energy* represented by what is known as the Poynting Vector is not identified with *the rate of work of the stress* in the æther. This, in fact, can only be done if the medium is supposed to be in motion; for a stress on a stationary element of area does not transmit any energy across that element. One way therefore of seeking for a suitable state of motion is to look for a distribution of velocity which will be consistent with the ordinary principles of momentum and energy.

74. The Customary Analysis of Æther-stress and Momentum.

In showing that this can be done it will be best to begin with a statement of the analysis as far as it has hitherto been carried. In the electro-static field it was shown by Faraday that the forces on charged bodies could be accounted for by supposing that there existed a tension along the lines of force of intensity equal to $\frac{1}{2}\mathbf{E}^2$ (in ordinary electro-static units, $\mathbf{E}^2/8\pi$), together with a uniform pressure in all directions at right angles to this of the same intensity.

In a static magnetic field a similar result may be obtained, the lines of electric force being replaced by those of magnetic force. But in a general electro-dynamic field, if the two states of stress so defined are superposed, the total stress over any closed surface drawn in the æther is not self-equilibrating, and to maintain the mechanical principle of momentum in its entirety a distribution of momentum has therefore to be assigned to the æther. The combined stress over the surface of any element of volume is exactly what is needed to produce the actual rate of change of momentum within the element, if the momentum is represented by the vector $[\mathbf{EH}]/c$ per unit volume.

But this statement is in reality nothing more than a certain analytical relation obtained from the relations connecting $\mathbf{E}$ and $\mathbf{H}$, the momentum being *defined* as the above quantity, and the stress being understood to represent the rate of transfer of this momentum across *stationary* elements of area. Now, in ordinary mechanics the stress in a moving body measures the transfer of momentum across elements of area *moving with the body.* Thus, if we attribute velocity to the æther, the 'true stress' would be defined as differing from what we may call the 'Faraday-Maxwell stress' by an amount representing the rate at which momentum is *convected* by the æther across stationary elements of area.

It is this 'true stress' which, if we are to retain our ordinary mechanical notions, does work by acting on moving elements, and transfers energy through the æther.

75. The Poynting Vector, Flow of Energy, Determination of a Velocity for the Æther.

We may now turn to the relation first stated by Poynting. Taking $\frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)$ as the energy per unit volume, he showed from the equations of the field that the changes in the energy may be accounted for by supposing a flow of energy to exist at all points the intensity of which is intensity represented by the vector $c\,[\mathbf{EH}]$ across stationary elements of area. Across an element moving with velocity $\mathbf{v}$ the flow of energy will, of course, be less by the vector $W\mathbf{v}$ where W is the density of energy. In order to maintain the mechanical principle of energy $\mathbf{v}$ must be such that this modified *flow of energy across elements of area moving with the æther is equal to the rate at which the 'true stress' does work on those elements.*

Now it is found on carrying out the analysis[1] that the condition that this shall be so determines the value of that component velocity of the æther which is in the direction of the vector $[\mathbf{EH}]$, that is, in the direction of the Poynting energy-flux.

If we call v_g this component we find, in fact, that

$$(c^2 + v_g^2)g = 2Wv_g,$$

where g is the magnitude of the momentum $[\mathbf{EH}]/c$ and W is, as above, the density of energy. This being a quadratic equation, there are two possible values of v_g; their product is

[1] The proofs of all the following statements are given in the author's larger work, Chapter XV.

c^2, so that one must be greater than, and the other less than c. They are always real values.

As to the remaining component of the velocity of the æther, we find that it lies in a definite direction in the plane of **E** and **H**, but that its magnitude is arbitrary as far as the above mechanical conditions are concerned. In order to make the specification of the velocity complete, we may make the further limitation, suggested by the fact that we wish to make the complete specification consonant with the Principle of Relativity, that *the total velocity at any point shall have the magnitude c.* Since c is an invariant velocity we know that this condition will not introduce any discrepancy with the Principle; and, at the same time, it determines which of the values of v_g is to be taken, namely, that which is less than c.

We now find that *the velocity so determined is actually subject to the transformation of Einstein*, so that a moving æther thus defined may be taken as a unique objective medium, as was explained at the beginning of this chapter.

76. The True Stresses.

When the velocity has been thus defined, the specification of the 'true stress' in the medium may be completed; and it is very remarkable that whereas the composite Faraday-Maxwell stress was obtained by superposing two separate distributions, each of which consisted of a tension in one direction with equal pressures all round that direction, the 'true stress' reduces to a single distribution of the same type. There is a *tension* P in a defined direction in the plane of **E** and **H**, namely, the direction of the component velocity of the æther in that plane, and *an equal pressure* P in all directions perpendicular to this. P is given in terms of the energy and momentum by the equation

$$P^2 = W^2 - c^2 g^2.$$

A further remarkable fact is that this principal stress P *is invariant* under the transformation of the electric and magnetic intensity from one frame of reference to another. That is, whereas the 'electric intensity,' 'magnetic intensity,' and many other quantities have only relative values, there is no ambiguity about the value assignable to the 'true stress'. We may compare this with the corresponding fact in Newtonian mechanics, that whereas, 'momentum' and 'energy' have only relative values, 'force' and 'stress' are measured by the same quantities in all possible systems of reference.

77. Summary of Specification of Velocity, Stress, and Momentum.

The mechanical specification of the æther is thus completely given in any particular frame of reference in terms of

(i) The magnitude of the principal tension.

(ii) The direction of the total velocity c of the æther.

(iii) The direction of the momentum **g**.

The magnitude of the momentum is then given by the equation

$$\mathbf{g} = \frac{2v_g}{c^2 - v_g^2}\,\mathrm{P},$$

and of the energy by

$$\mathrm{W} = \frac{c^2 + v_g^2}{c^2 - v_g^2}\,\mathrm{P},$$

while the rates of change of the momentum and energy are expressible in terms of the stress by the ordinary mechanical principles.

The mechanical equations of the æther are, however, not complete. For if we imagine the complete specification to be given at an instant t, we are able, knowing the stress, to calculate the direction and magnitude of the momentum, and the intensity of the energy after an interval of time δt; hence we can find the magnitude of P after time δt from the equation $\mathrm{W}^2 - c^2g^2 = \mathrm{P}$; but the change in the direction of the velocity c is not completely determined. All that is known is the new value of the component v_g, the direction of the other component being undetermined.

Thus it cannot be said that a complete mechanical model of the æther has thus been constructed. All that has been done is to reconcile with one another the conception of an objective æther of which velocity may be predicated, the Principle of

Relativity, and the mechanical principles of momentum and energy.

78. Examples of the Specification of Velocity and Stress in the Æther.

(i) *A train of plane waves of light.*

Here the electric and magnetic intensities at any point are equal to one another and both are perpendicular to the direction of propagation.

Thus $cg = EH = E^2 = \frac{1}{2}(E^2 + H^2) = W.$

Hence $c^2 + v_g{}^2 = 2cv_g$

giving $v_g = c.$

Further $P^2 = W^2 - c^2g^2 = 0.$

Thus in this case v_g is the whole velocity of the æther; that is, the æther must be supposed to be moving as a whole in the direction of propagation of the waves, while the stress vanishes. We may therefore think of this case as that of a purely convected system of momentum and energy without any stress.

(ii) *The field of a moving point charge.*

It is known that in this case, if **E** is the electric intensity at any point P at time t, and $\mathbf{r}_1$ is a unit-vector in the direction OP, where O is that point which was occupied by the point charge at the time t_1 such that $OP = c(t - t_1)$,

then $\mathbf{H} = [\mathbf{r}_1\mathbf{E}]$.[1]

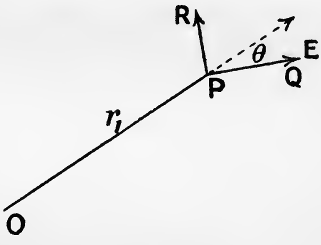


Fig. 10.

Let PQ be the direction of **E**; **H** is then perpendicular to the plane OPQ, and the direction PR of the momentum vector is in the plane OPQ and at right angles to PQ.

We have

$$H = E \sin\theta,$$

$$g = E^2 \sin\theta / c,$$

so that

$$\frac{2v_g c}{c^2 + v_g{}^2} = \frac{2E^2 \sin\theta}{E^2(1 + \sin^2\theta)},$$

giving

$$\frac{v_g}{c} = \sin\theta.$$

[1] See Abraham, "Theorie der Elektrizität," p. 96 (2nd edition, 1908).

Thus the component velocity of the æther in the direction PR is $c \sin \theta$.

On examining the direction of the remaining component of the velocity of the æther, we find that it is along PQ. Hence it follows at once that the total velocity c is along the line OP.

Hence it follows that the path of the element of the æther which is at P is a straight line starting from the position O of the electron at time t_1 and traversed uniformly with velocity c. In other words, the æther moves as if constantly emitted from the electron with velocity c, every element travelling uniformly in a straight line after emission.

In this case we find that the principal tension is along the line PQ, and is of magnitude $\frac{1}{2}E^2 \cos^2 \theta$.

The above examples are suggestive of a new form of emission theory of electrical action; but it should be borne in mind that no substantially new facts have been introduced; everything depends upon the fundamental and commonly accepted equations of the electro-magnetic field in free spaces. Nor can we return from the mechanical relations to the electro-magnetic. The mechanical specification of the æther is not sufficient to determine the ordinary electro-magnetic specification.

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