Florian Cajori

RESEARCHES

ON

THE IMAGINARY ROOTS

OF

EQUATIONS,

&c. &c.

Price 3s. 6d.

J. R. Young

RESEARCHES

RESPECTING

THE IMAGINARY ROOTS

OF

NUMERICAL EQUATIONS:

BEING A CONTINUATION OF

NEWTON'S INVESTIGATIONS ON THAT SUBJECT,

AND FORMING

AN APPENDIX

TO THE

"THEORY AND SOLUTION OF EQUATIONS OF THE HIGHER ORDERS."

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PREFACE.

I HAVE called the Researches to which these few pages are devoted a continuation of those of NEWTON on the subject of the Imaginary Roots of Equations; because, as I have shown in the second chapter, a principle first announced by that illustrious analyst, in his *Arithmetica Universalis,* is competent to furnish all the more important of the conclusions to which I have arrived.

This principle was delivered by NEWTON without demonstration; and although several attempts have, at different times, been made to establish its truth, I believe that no satisfactory proof of its accuracy has till now been offered. The demonstration which I have here ventured to propose, together with the rules and criteria I have furnished for facilitating the analysis of a numerical equation, will, I hope, prove acceptable to those algebraists who take an interest in this subject,—a subject of considerable practical importance, one upon which both NEWTON and LAGRANGE expended much thought and labour, and which the more recent inquiries of Budan, Fourier, and Sturm have invested with new attractions.

Besides demonstrating and extending the above-mentioned principle of Newton, and developing its consequences, I have introduced other and independent investigations; which, in combination with that principle, have finally conducted to formulas of condition, for distinguishing imaginary roots from real, of remarkable generality and efficiency.

The tract is intended to form an Appendix to my recently published volume on the Theory of Equations in general; and will, I think, be found to supply some useful additions to that work: the nature and extent of these additions are set forth, with sufficient detail, in the accompanying Table of Contents.

J. R. YOUNG.

Belfast; Oct. 17, 1843.

CONTENTS.

ART.		PAGE
	INTRODUCTORY REMARKS	7
1.	Condition of imaginary roots deduced from the three leading coefficients of an equation	8
2.	Condition deduced from the three final coefficients	ib.
3.	General table of conditions for imaginary roots	9
4.	Rule inferred from these conditions	10
5.	Inferences from the preceding investigation	ib.
6.	The independence of the foregoing criteria of imaginary roots does not warrant the inference that there are so many imaginary pairs in the equation	11
7.	Useful applications of the criteria in the analysis of certain equations	12
8.	On determining whether the imaginary roots indicated lie in the positive or in the negative region	13
9.	The preceding criteria require no aid from common measures	14
10.	Simplified forms which the criteria take when the doubtful roots are nearly equal or imaginary	15
11.	Precepts applicable to these cases	17
12.	Another mode of arriving at the foregoing results	ib.
13.	Criteria for determining the character of a pair of roots any how related, when occupying one of the extreme intervals	19
ib.	Rule deduced from these criteria	20
14.	Investigation of a method of analysis for any intermediate interval	ib.
15.	Remarks on the preceding reasoning	22
16.	A pair of imaginary roots indicated in any interval will not necessarily have their reciprocals indicated in the reciprocal interval	23
	It is further shown in the NOTE to this article that two imaginary roots may be indicated in an interval though the real part of the imaginary pair lie wholly without that interval.	
17.	Formulas of condition to be applied to an intermediate interval	24
18.	General rule deduced from these formulas	27
19.	On the peculiar advantages of this rule	ib.
20.	On the explicitness given to the criteria at (3)	28
21.	On the simplest form of FOURIER's criteria, with a rule deduced from this form	29
22.	Examples illustrative of the criteria at page 17	ib.

ART. PAGE

23. Remarks on the practical value of these criteria 31
24. Illustrations of the value of the other criteria established in this chapter 32
25. On the examination respecting equal roots 35
26. How this examination may, in general, be dispensed with . . ib.
27. Connexion between the preceding methods of analysis and that proposed at page 307 of the *Theory of Equations* 36
28. Observations on the best practical forms for the criteria of imaginary roots 37
29. On *indicating equations* above the second degree 38
30. Method of deriving the general criteria of imaginary roots from the theorem of Sturm 39
31. On the criteria proposed by Newton; and the failure of Maclaurin's demonstration of their truth 40
32. Enunciation and examples of Newton's Rule . . . . . . ib.
33. Investigation of principles on which a demonstration of this rule is based 42
34. Determination of the general law which connects the series of limiting equations together 43
35. Proof of the identity of the criteria derived from the limiting equations with those established at page 9 45
36. Newton's Rule inferred from the preceding investigation . . . 46
37. Example of its application and extension 47
38. On the misconception of Montucla, as to the researches of Campbell and Maclaurin . . . . ib.
39. On the principles employed by Fourier in the investigation of his criterion 49
40. New and simple investigation of this criterion . . . . . . ib.
41. Observations on another method of Fourier, which, like the preceding, may be established without the aid of Lagrange's theorem . . 52

Note A. On the circumstances under which Budan's criterion fails:—The real part of an imaginary pair not necessarily situated in the interval which comprehends the indication of that pair:—The real part may lie in the positive region, and yet the pair be indicated in the negative region 54
Note B. On a paragraph in the *Theory of Equations* 56

APPENDIX

TO

THE THEORY OF EQUATIONS.

In the foregoing treatise we have proposed two methods for ascertaining the true character of those doubtful intervals which so frequently occur in the partial analysis of a numerical equation. These methods are fully developed and explained in the twelfth chapter of that work; and are there practically illustrated in connexion with the actual process of solution. Subsequent investigation has unfolded certain peculiarities respecting the methods adverted to which did not originally suggest themselves, and which will be found to confer upon them additional value in the solution of equations.

The present Appendix will be devoted to the development of these supplementary considerations, and to the discussion of certain other topics of a kindred nature; with the view of ad-

ART. PAGE

23. Remarks on the practical value of these criteria 31
24. Illustrations of the value of the other criteria established in this chapter 32
25. On the examination respecting equal roots 35
26. How this examination may, in general, be dispensed with . . ib.
27. Connexion between the preceding methods of analysis and that proposed at page 307 of the *Theory of Equations* 36
28. Observations on the best practical forms for the criteria of imaginary roots 37
29. On *indicating equations* above the second degree 38
30. Method of deriving the general criteria of imaginary roots from the theorem of STURM 39
31. On the criteria proposed by NEWTON; and the failure of MACLAURIN's demonstration of their truth 40
32. Enunciation and examples of NEWTON's Rule . . . . . . ib.
33. Investigation of principles on which a demonstration of this rule is based 42
34. Determination of the general law which connects the series of limiting equations together 43
35. Proof of the identity of the criteria derived from the limiting equations with those established at page 9 45
36. NEWTON's Rule inferred from the preceding investigation . . . 46
37. Example of its application and extension 47
38. On the misconception of MONTUCLA, as to the researches of CAMPBELL and MACLAURIN . . . . ib.
39. On the principles employed by FOURIER in the investigation of his criterion 49
40. New and simple investigation of this criterion . . . . . . ib.
41. Observations on another method of FOURIER, which, like the preceding, may be established without the aid of LAGRANGE's theorem . . 52

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NOTE B. On a paragraph in the *Theory of Equations* 56

ERRATA.

Page 22. Omit the latter part of art. (15), commencing with "It would be wrong," &c.
56, line 3, after the word "pair," insert, "in the extreme interval."

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CHAPTER I.

CRITERIA OF IMAGINARY ROOTS.

(1.) It has already been shown (page 323) that if the general equation

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots A_2 x^2 + A_1 x + A_0 = 0 \dots [1]$$

be transformed into another, by substituting $x + r$ for x, and then r be so determined that the second coefficient of the transformed equation may vanish, the third coefficient must be

$$\frac{A_{n-2}}{A_n} - \frac{n(n-1)}{2} \left\{ \frac{A_{n-1}}{nA_n} \right\}^2$$

Consequently if, when this evanescence takes place, the expression here written be positive—like the leading coefficient A_n,—the zero, then occurring between like signs in the transformed, will indicate the existence of a pair of imaginary roots in the original equation.

Hence multiplying by the positive quantity $2nA^2_n$, two imaginary roots will be indicated provided we have the condition

$$2nA_{n-2}A_n > (n-1)\, A^2_{n-1} \dots [2]$$

inasmuch as this condition secures a positive value for the third coefficient when the second is made to vanish by the suitable transformation.

(2.) If the order of the coefficients of the proposed equation be reversed, we shall have a new equation, the roots of which will be the reciprocals of the roots of the former equation; so that a pair of imaginary roots in either of these two equations necessarily implies a corresponding pair in the other. Hence, replacing the three leading coefficients of [1] by the three final ones, we shall have the new condition

$$2nA_0A_2 > (n-1)A_1^2 \;\ldots\ldots\; [3]$$

which equally with that above indicates the entrance of a pair of imaginary roots into the proposed equation.

(3.) If we were to write down, one under another, the series of limiting equations derivable from [1], we might apply the criteria [2], [3] to each in succession, and infer that when either of these criteria had place for any equation in the series, a pair of imaginary roots necessarily entered that equation, and consequently that a pair was also implied in the primitive (96.)

The application of the formula [2], however, to the leading terms of the several derived equations, would not furnish us with any new criterion, since the original form would continually recur. For, as will be readily seen, from the nature of the derivation, if A_m, A_{m-1}, A_{m-2} be the first three coefficients, either in the primitive, or in any derived equation, m being the degree of that equation, the ratio

$$\frac{2m\,A_{m-2}\,A_m}{(m-1)\,A^2_{m-1}}$$

will be constant, so that if the condition [2] has place for any equation in the series, it equally has place for all.

It is otherwise with the formula [3]. For if this be applied to the final terms of the several equations, we shall have the following group of distinct conditions, viz.

$$2nA_0A_2 > (n-1)A^2_1$$
$$3(n-1)A_1A_3 > 2(n-2)A^2_2$$
$$4(n-2)A_2A_4 > 3(n-3)A^2_3$$
$$5(n-3)A_3A_5 > 4(n-4)A^2_4$$
$$\vdots$$
$$n(n-\overline{n-2})A_{n-2}A_n > (n-1)(n-\overline{n-1})A^2_{n-1}$$

or $$2nA_{n-2}A_n > (n-1)A^2_{n-1}$$

These conditions are $n - 1$ in number: and if any of them have place we may at once infer the existence of imaginary roots in the proposed equation.

The same inference may be drawn, though $>$ be changed into $=$, except in the single case in which all the roots of the equation are equal. For it is plain that whether the third coefficient exhibited at page 323 become positive, as there supposed, or actually vanish simultaneously with the second coefficient, imaginary roots will, in either case, be indicated (68) unless indeed all the subsequent coefficients vanish likewise, implying that all the roots of the equation are severally equal to $-r$. And similar observations of course apply to each of the derived equations.

(4.) Hence if we call any term in an equation, which lies between two terms with like signs, the *middle term*, we may embody the foregoing results in the following general principle:

If the product of the first and third of the three terms, multiplied by the exponent of the first and by n minus the exponent of the third, be *not less* than the square of the middle term multiplied by the exponent of that term and by n minus the same exponent, the equation must have imaginary roots.

This principle expressed in general symbols, m being put for the exponent of the middle term, will be as follows:

$$(m + 1)(n - \overline{m - 1})A_{m-1}A_{m+1} \text{ not less than } m(n - m)A^2_m$$

(5.) An obvious inference from the preceding investigation is that when the three leading coefficients of an equation satisfy the condition of imaginary roots, the same condition must continue to have place however we increase or diminish the roots of the equation. For if the roots of the equation [1] be increased by r', r' being either positive or negative, the resulting equation, after diminishing its roots by $r + r'$, the quantity causing the second term to vanish, would of course furnish the same third coefficient, as that obtained from diminishing the roots of [1] itself by r. But such is not necessarily the case in reference to the three final coefficients. For with respect to these, the condition implies an imaginary pair in the equation whose roots are the reciprocals of those of the proposed: increasing or diminishing the roots of the original equation by any quantity will not diminish or increase *equally* the reciprocals of those roots; so that the foregoing conclusion, which is a consequence of the equal increase or diminution of the original roots, cannot be deduced here.

An interesting corollary from this is that when the condition of imaginary roots has not place for the three leading coefficients it never can be made to have place by increasing or diminishing the roots of the equation, any more than it can be made to fail when it once exists. But with respect to the three final terms, we may possibly, by means of a transformation, arrive at the condition, though it fail at the outset. An instance of this possibility is furnished in the example at page 308 of the *Theory of Equations*.

(6.) It is further deserving of notice, that each one of the $n - 1$ inequalities, exhibited above, involves only three of the given coefficients; so that if the inequality have place for any of these sets of three, the imaginary roots thus implied, can never be converted into real roots by means of any changes among the other coefficients. The preceding conditions therefore are perfectly independent: that is, the existence of any one has no necessary connexion with the existence or failure of any other. Hence equations may be framed for which all these conditions shall have place; and since, as before observed, they are $n - 1$ in number, whilst the imaginary pairs entering an equation of the nth degree can never exceed $\frac{1}{2}n$ in number, it follows that, after the quadratic, equations may occur furnishing a greater number of these independent indications of imaginary pairs than the actual number of such pairs entering the equation can ever attain to. This consideration is sufficient to preclude the inference that there must always be as many pairs of imaginary roots as there are conditions fulfilled, and to limit our deduction from the preceding investigation to the simple fact, that when one or more of the foregoing criteria are satisfied by the coefficients of an equation, that equation must have imaginary roots. But as to the actual number of these, nothing can as yet be with certainty inferred.

In speaking, therefore, as above, of the *independence* of the criteria just established, care is to be taken that too wide a signification be not given to this term. The independence referred to is analogous to that which may be said to have place among the coefficients themselves; each of which is in a certain sense perfectly independent of the others, though all are equally dependent upon the roots of the equation, of which roots they are known functions. This is obvious from the fact that though all the coefficients but one were given, yet it would be impossible to determine that one from them alone.

Indications that are in this sense independent, that is not implied in, or deducible from one another, are not to be regarded as necessarily pointing to distinct objects: their independence does not preclude their bearing concurrent testimony to one and the same thing. It will be the business of the next chapter to investigate the principle by which the foregoing expressions are connected together; and thence to distinguish those of them which are necessarily concurrent, from those that are not; with a view to the obtaining of more enlarged information respecting the precise number of imaginary roots that may be safely inferred from them than we at present possess.

(7.) An obvious and useful application of these criteria even now offers itself, in those cases namely where by a partial analysis of the equation all the roots are ascertained to be real, in either the positive or negative region, except a single doubtful pair; since the satisfying of any one of the criteria by three consecutive coefficients of the equation will authorize the inference that the pair of roots thus left in doubt by the previous analysis must be imaginary. An example or two will furnish sufficient illustration.

1. At page 167 we have the following example, viz.

$$x^3 + 2x^2 - 3x + 2 = 0.$$

The last three coefficients are the only set of three that can satisfy the test, and it is at once seen that these succeed. Hence two of the roots must be imaginary.

2. In like manner the equation

$$x^5 - 36\,x^3 + 72\,x^2 - 37\,x + 72 = 0$$

is found, by a partial analysis, to have all its roots real except two, which are left in doubt; which doubt can be removed by FOURIER'S method only after a tedious examination. But as the last three coefficients satisfy the criterion, we immediately conclude that the doubtful roots are imaginary.

3. The equation

$$A_4 x^4 + 3\,x^3 + 2\,x^2 + 6x - A_0 = 0$$

whose coefficients satisfy the second of the preceding series of criteria, has also a pair of imaginary roots; the remaining roots being real, whatever be the numerical values of the extreme coefficients, provided only that they have opposite signs (29.)

(8). It is obvious that we can always determine, from a simple inspection of the signs of the three terms, in which region the imaginary pair indicated by those terms lies :—if the signs present a pair of variations, the imaginary roots lie in the positive region ; if a pair of permanencies, they lie in the negative region. This inference is authorized by the rule of DES CARTES, since the region occupied by the imaginary pair—or rather by the indicator of that pair—is that in which the doubtful roots would be situated if they were real. Thus in the 1st and 2d examples above, the imaginary roots occur in the positive region: in the 3d example they occur in the negative region.

4. The equation

$$x^5 + x^4 + x^3 - 2\,x^2 + 2\,x - 1 = 0$$

analysed at page 175 by the method of FOURIER, involves a good deal of numerical labour, and furnishes a striking exemplification of the practical value of the criteria here established. For from inspecting the first three coefficients, we see that a pair of imaginary roots exists in the negative region; and from inspecting the last three we see that a pair also exists in the positive region, so that the equation has four imaginary roots and one real root, and is thus completely analysed with little or no trouble.

(9.) It ought not to be overlooked that in these determinations of doubtful roots by aid of the criteria at (3), all inquiry about common measures is rendered unnecessary, since no attempts are made to separate roots thus shown to be imaginary. Imaginary roots are not however always indicated to us in this way, by the original coefficients; yet it is easy to see how we may avail ourselves of the criteria here employed in the analysis of equations in general, and thus replace the criterion of FOURIER by another of much greater efficiency. In FOURIER'S method, as taught in the preceding work, as soon as we have reduced the indices to **0 1 2**, we are to develop the root corresponding to **1**, carrying on the work through the subsequent columns, till we either separate the roots indicated by **2**, or till we arrive at coefficients, which in combination with those furnished by a superior limit to the two roots, satisfy the test of impossibility at page 165. Instead of this test of FOURIER, it will be better to appeal to the criteria here proposed, which are equally decisive, and which, as we have just seen, are likely to present themselves at a much earlier stage of the development, and are moreover independent of the results obtained by resorting to a superior limit. But we shall hereafter express in words the proper precepts for completing the analysis of an equation upon these principles, giving in a somewhat modified form the criteria to be employed for that purpose.

We have indeed already employed the criteria under the modification here adverted to, at least to a certain extent, in the preceding treatise; having applied them, to this extent, in the examples at pages 268, 277. But in the limited use there made of them there is a seeming imperfection which we shall now see does not in reality attach to them. The examples referred to are of that peculiar character in which the roots in the doubtful interval are so nearly equal as to concur in several of their leading figures : the expressions employed to develop these concurring figures are, as we shall presently see, only slight modifications of those which form the several criteria here discussed: and we have seen that they clearly enough apprize us when the critical stage of our approximation at which the concurrence ceases is reached, and where the roots, if real, must separate, inasmuch as they then present us with a discrepancy between two expressions, which up to that stage had concurred. But additional examination appeared to be required in order to determine whether the roots traced to this critical stage actually separated there or not. It will now be shown that the discrepancy adverted to is of itself fully competent to resolve this doubt without our having occasion to apply any additional tests, or to execute any new transformations or bye operations for the purpose.

(10.) From what is established at page 261, it appears that when certain roots differing but little from equality, or concurring in their leading figures, are to be developed, these common figures, after an early stage of the process, will be furnished one after another, by either of the two concurrent expressions, which, in the arrangement below, stand vertically under the function into which these roots first enter, or beneath which the index 2 appears.*

$$\begin{array}{cccccc} f_{n-2} & \cdots\cdots & f_2(x) & f_1(x) & f(x) \\[6pt] \dfrac{A_{n-1}}{nA_n} & \cdots\cdots & \dfrac{A_3}{4A_4} & \dfrac{A_2}{3A_3} & \dfrac{A_1}{2A_2} \\[10pt] \dfrac{2A_{n-2}}{(n-1)\,A_{n-1}} & \cdots\cdots & \dfrac{2A_2}{3A_3} & \dfrac{A_1}{A_2} & \dfrac{2A_0}{A_1}. \end{array}$$

* In the article referred to, the several coefficients written below are accented, to distinguish them from the coefficients of the original equation. But as no confusion can arise from omitting these accents, and as the omission contributes a little to convenience of printing, they are here suppressed accordingly: minus signs too are omitted and absolute values only regarded.

And that when there is a discrepancy between the leading figures furnished by the two expressions used, the roots, if real, are about to separate. The discrepancy here adverted to may consist either in the first expression giving a greater figure than the second, or in the second expression giving a greater figure than the first: we shall now prove, that if the former happen, the roots under examination will be *real*: if the latter, they will be *imaginary*.

For it is obvious, that if the coefficients of the quadratic, to which our approximation tends, (see pages 261-2 of the preceding treatise,) be submitted to the proper test in the series at (3), of which proper test, when applied to the leading terms, the general form is

$$2nA_{n-2}A_n > (n-1)A^2_{n-1}$$

we shall have, for $f(x)$, $n=2$; for $f_1(x)$, $n=3$; for $f_2(x)$, $n=4$; and so on: that is, the criteria corresponding to these several cases will be

$$4A_0A_2 > A^2_1, \qquad \text{or} \qquad \frac{2A_0}{A_1} > \frac{A_1}{2A_2}$$

$$6A_1A_3 > 2A^2_2 \qquad \frac{A_1}{A_2} > \frac{A_2}{3A_3}$$

$$8A_2A_4 > 3A^2_3 \qquad \frac{2A_2}{3A_3} > \frac{A_3}{4A_4}$$

$$\vdots \qquad \vdots$$

$$2nA_{n-2}A_n > (n-1)A^2_{n-1} \qquad \frac{2A_{n-2}}{(n-1)A_{n-1}} > \frac{A_{n-1}}{nA_n}.$$

Consequently, the roots under examination will be real or imaginary, according as the proper condition in the following series of conditions, and against which the derived function involving those roots is written, exists or fails, when the critical stage of the development is reached:

$$f(x) \qquad \frac{A_1}{2A_2} > \frac{2A_0}{A_1}$$

$$f_1(x) \qquad \frac{A_2}{3A_3} > \frac{A_1}{A_2}$$

$$f_2(x) \qquad \frac{A_3}{4A_4} > \frac{2A_2}{3A_3}$$

$$\vdots \qquad\qquad \vdots$$

$$f_{n-2}(x) \qquad \frac{A_{n-1}}{nA_n} > \frac{2A_{n-2}}{(n-1)A_{n-1}}.$$

(11.) Hence, in pursuing a pair of contiguous roots of $f_m(x) = 0$, we are to proceed conformably to the method employed at pages 270, 280, seeking the development of the intervening root of $f_{m+1}(x) = 0$, the successive figures of which, after a certain stage, are always furnished by the first of the expressions standing against $f_{m+1}(x)$ in the above series, carrying forward the work up to $f(x)$, and continuing the process as long as the expression referred to concurs with that which accompanies it in furnishing the same root figure; that is, till the critical stage of the development is reached. When the concurrence ceases, the roots may be pronounced *real*, if the figure given by the first expression *exceed* that given by the second; but if the contrary happen, then the roots will be imaginary. And thus their character unfolds itself spontaneously, without any appeal to external tests or to supplementary transformations.

(12.) Although we have, for uniformity sake, arrived at these conclusions by help of the general criteria at (3), yet they might have been deduced independently, and simply from the common theory of quadratic equations, as follows.

The several approximate quadratics to which the process of development tends, according as the pair of doubtful contiguous roots belongs to $f(x) = 0$, or $f_1(x) = 0$, or $f_2(x) = 0$, &c., are exhibited in the final terms of the equations at page 261: they are

$$f(x) \qquad A_2 x^2 + \quad A_1 x + \quad A_0 = 0$$
$$f_1(x) \qquad 3\,A_3 x^2 + \quad 2\,A_2 x + \quad A_1 = 0$$
$$f_2(x) \qquad 3 \cdot 4\,A_4 x^2 + 2 \cdot 3\,A_3 x + 2\,A_2 = 0$$
&c. &c.

the general form of them being

$$f_{n-2}(x) \qquad 3 \cdot 4 \cdot 5 \ldots n A_n x^2 + 2 \cdot 3 \cdot 4 \ldots n-1\,A_{n-1} x$$
$$+ 2 . 3 \ldots n - 2\,A_{n-2} = 0\,;$$

or,

$$n(n-1)\,A_n x^2 + 2\,(n-1)\,A_{n-1}\,x + 2\,A_{n-2} = 0\,;$$

which, by the common criterion for the reality of the roots in a quadratic equation, furnishes the condition

$$2\,n A_{n-2}\,A_n > (n-1)\,A^2_{n-1}$$

as above.

These approximate quadratics might with propriety be called the *indicating quadratics* with respect to the narrow intervals under examination. They are related to the original equation, from which they are derived, in a manner analogous to that which connects the indicating curve of the second order with the surface from which it is deduced, in the general theory of curve surfaces.* It is obvious that when the indicating quadratic implies a pair of real roots, the leading figure of each, at the point of separation, may be determined from it; and not the leading figure merely, but, in general, as many leading figures as there are constant figures in the leading coefficient of the quadratic, minus one.

It must be observed, however, that it is not every pair of roots having leading figures in common that will thus continue unseparated till the indicating quadratic is reached: they may separate before the trial divisor for determining the figures of the intervening root has become fully effective; and consequently before the precepts at (11) come into operation, in which case the analysis of the interval will be accomplished independently of

* See the author's disquisition on this subject in the *Mathematical Dissertations.*

those precepts. It is only when the roots continue unseparated, after the trial divisors for the intervening root have become effective, and thus the leading figures of the first coefficient of our quadratic constant, that the foregoing tests become applicable.

(13.) When the roots occupying the doubtful interval are not in the peculiar circumstances here considered, the operation for determining their character must be conducted somewhat differently, since the general criteria at page 9 become converted into the simpler forms at page 17 only in consequence of the peculiarity alluded to.

Now in discussing the problem independently of such restrictions, it will be convenient first to dispose of the particular case in which there is known to be no doubtful interval more remote from zero, in the region under examination, than that which we seek to analyse; that is, we shall first assume, that beyond the proposed interval $[a, b]$, towards $+\infty$ or $-\infty$, according as the region occupied by it is positive or negative, no imaginary roots can exist.

For determining the character of this interval, the general expressions at page 9 furnish the following criteria of impossibility:—

$$f(x) \qquad \frac{A_1}{2A_2} < \frac{n}{n-1}\,\frac{A_0}{A_1}$$

$$f_1(x) \qquad \frac{A_2}{3A_3} < \frac{n-1}{n-2}\,\frac{A_1}{2A_2}$$

$$f_2(x) \qquad \frac{A_3}{4A_4} < \frac{n-2}{n-3}\,\frac{A_2}{3A_3}$$

$$f_{n-2}(x) \qquad \frac{A_{n-1}}{nA_n} < 2\frac{A_{n-2}}{(n-1)\,A_{n-1}}.$$

Hence two roots being indicated in either of the functions $f(x)$, $f_1(x)$, $f_2(x)$, &c., if the condition here written opposite to that function have place, anywhere within the interval $[a, b]$ comprising those roots, they may be affirmed to be imaginary. For since the fulfilment of the condition necessarily implies two imaginary roots in the interval [a, ∞], or [a, $-\infty$], according to the region in which the proposed interval is, and as no imaginary roots can exist beyond the limit b, by the supposition, it follows that the imaginary roots indicated can be no other than the pair between a and b.

It is evident that this inference is altogether independent of every condition as to the indices, preceding the index **2**, which marks the pair of roots under examination. We shall however consider this **2** to be that one of the series of indices which is preceded by **1**, but shall make no stipulation as to whether the next index in order is to be **1** or **0**. The following then is the mode of proceeding, when the character of the roots in the proposed interval is not immediately made known to us, as at (7), by an inspection of the original coefficients, but is to be discovered by actual development.

I. *To analyse a doubtful interval when imaginary roots are excluded from more advanced intervals in the same region.*

1. Disregarding the indices preceding **1**, **2** carry on the development of the root corresponding to **1**, by aid of the proper expression at page 19, suggestive of the leading figures, like as in the preceding treatise, comparing each suggested figure with that in like manner suggested by the accompanying expression, with the view of discovering whether or not the inequality at page 19 be fulfilled.

2. If a step be reached at which this inequality is fulfilled the roots may be pronounced *imaginary*. And this step necessarily will be reached, or else one at which the roots will actually separate.

(14.) In the precepts just given, it is presumed that when the indication of imaginary roots occurs, no imaginary pairs can exist beyond the proposed interval, [a, b], in the region under examination. These precepts, therefore, require no qualification when not more than a single doubtful interval occurs in each region. But when, for aught we know to the contrary, another imaginary pair may exist nearer to $+\infty$, or to $-\infty$, according as $[a, b]$ is in the positive or negative region, then an additional condition must be fulfilled, before we can affirm that the roots indicated necessarily come within the interval $[a, b]$.

For in discussing the general theory of imaginary roots it is necessary that we bear in remembrance the fact that, when the criterion of imaginary roots is satisfied by the three final terms of an equation, as here supposed, we infer the existence of such roots in the equation, simply from the circumstance that a pair is then known to exist in the reciprocal equation; inasmuch as the second coefficient of that reciprocal equation vanishes for a transforming value that renders the first and third of like signs. But without further inquiry, nothing can be positively affirmed as to the precise interval in which the imaginary roots, thus inferred, occur.

The case under consideration is this, viz.*

	A_2	A_1	A_0
(a)	+	−	+
(b)	*two var. lost.*		

Or, calling the first transformation (a), (0), it is this,

(0)	+	−	+
$(b-a)$	*two var. lost.*		

If we take the reciprocal of the equation marked (0), and transform by $\frac{1}{b-a}$, and find the results to be

	A_0	A_1	A_2			
(0)	+	−	+			
$\left(\frac{1}{b-a}\right)$	+	+	+	+	+	&c.

* There is another case:

$$- \quad + \quad -$$

two var. lost,

but as the reasoning is the same in both, it will be sufficient to confine ourselves to that in the text.

then, by the principle of BUDAN, explained in Chapter X. of the *Theory of Equations*, we know that the two roots in the interval $[a, b]$ are necessarily imaginary. Also if we find the results to be

$$\begin{array}{lcccc} (0) & + & - & + & \cdots\cdots \\ \left(\frac{1}{b-a}\right) & + & - & + & \cdots\cdots \end{array}$$

then likewise we may conclude, without completing the transformation, that the two roots indicated in the original equation in the interval $[a, b]$ are imaginary. For since more than two variations cannot possibly occur in the transformation by $\left(\frac{1}{b-a}\right)$, all the signs after the first three here exhibited must be plus; and as the roots indicated by these three signs are by hypothesis imaginary—inasmuch as the first and third terms remain plus when the middle term becomes zero—it follows that no real root can exist between $\frac{1}{b-a}$ and ∞ in the reciprocal equation, and consequently that none can exist between 0 and $b-a$ in the direct equation, so that the two roots indicated within these limits, which are the limits a, b of the original equation, must be imaginary.

(15.) It is only by these reasonings, in reference to the reciprocal equation, that we can infer anything respecting the character of a pair of roots in a given interval in the direct equation. When it is clearly ascertained that two roots necessarily exist in the direct equation between the limits a and b, and discover moreover, from consulting the reciprocal of the equation (a), or simply the three leading terms of that reciprocal, that no real roots can exist in it between the limits $\frac{1}{b-a}$ and ∞, we may then conclude with certainty that the two roots indicated are necessarily imaginary. It would be wrong to infer, because in the reciprocal equation two roots, and two only, are found to exist between the limits $\frac{1}{b-a}$ and ∞, which two roots are ascertained to be imaginary, that *therefore* two imaginary roots necessarily exist in the direct equation between the limits 0 and $b-a$: for there might not exist any roots within these limits in that equation. It is not the existence of imaginary roots in the reciprocal interval that can warrant the inference that imaginary roots also exist in the direct interval; the only legitimate principle is that the non-existence of *real* roots in the reciprocal interval, necessarily implies the non-existence of *real* roots in the direct interval; and that, consequently, if a doubtful pair have been previously found to occur in that interval they must of necessity be imaginary.

(16.) In speaking as above of the reciprocal of the equation (a) it must not be forgotten that we are supposing A_0 to be the coefficient under which the index **2** of the roots to be tested occurs. A_0 is therefore to be considered, generally, as the final coefficient of that function under which this same index appears; and it is the reciprocal of this function that is meant by the reciprocal of (a).

It is worthy of notice that when two real roots exist in the interval $[0, b-a]$, the reciprocals of them, occurring in the reciprocal equation, necessarily lie in the interval $[\frac{1}{b-a}, \infty]$. But when two imaginary roots lie in the interval $[0, b-a]$, and are indicated by the three final coefficients of the equation, then the reciprocals of them, occurring in the reciprocal equation, may lie either within or without the interval $[\frac{1}{b-a}, \infty]$. And this is obvious from what has been shown above; for when, as in the case first supposed, the signs due to $(\frac{1}{b-a})$, in the transformed reciprocal equation, are all plus, the reciprocal imaginary roots must have been overstepped, and therefore lie in the interval $[0, \frac{1}{b-a}]$; that is *without* the interval $[\frac{1}{b-a}, \infty]$: but when, as in the case next supposed, the three leading signs preserve the same variations as at first, then the imaginary roots indicated by them are still in advance, and can only be overstepped by continuing the transformations onwards towards ∞; that is, these imaginaries lie *within* the interval $[\frac{1}{b-a}, \infty]$. But when in any instance the signs due to the reciprocal transformation $(\frac{1}{b-a})$ come under neither of these cases, then the roots indicated still remain doubtful. (See NOTE A at the end.)

In these circumstances the mode of proceeding most likely to suggest itself is that of BUDAN, discussed at length in the *Theory of Equations.* This method, by a series of transformations, reciprocal and direct, aims at bringing about the indications of imaginary roots considered in the first case above; that is, by continually diminishing the direct interval, it seeks at length to inclose the reciprocal imaginary pair *within* the limits 0 and $\frac{1}{b-a}$; or which is the same thing, to exclude them from the interval $[\frac{1}{b-a}, \infty]$; an object which it is easy to see is always attainable, since the interval $[0, \frac{1}{b-a}]$ widens as $b-a$ contracts; and in the pursuit of this object the roots if real must separate.

In prosecuting this method of BUDAN we lose the advantage of the other indications of imaginary roots considered in the second case noticed above, and should go on with our transformations and contractions even after these latter indications had shown the roots to be imaginary: it would be of importance therefore to combine these latter indications with those of BUDAN, in prosecuting his method of analysis.

But this method, though sometimes affording very early information as to the character of the interval under analysis, is often exceedingly tedious and operose: we have already discussed its merits and peculiarities in the preceding volume. The general rule, about to be given, will comprehend its principal advantages without involving the objections here adverted to.

(17.) The particular rule at page 20 is, as we have already seen, fully effective when applied to the analysis of one of the extreme intervals; and, in so applying it, if the roots indicated prove to be real, we may employ it for the analysis of the doubtful interval next in order; and so on: so that in many cases, even where several doubtful intervals occur, the rule above will prove sufficient. We shall now show that by prefixing to it a preliminary step, of very easy execution, we may render it perfectly general for all cases that can arise.

In order to this it will only be necessary to transform our intermediate interval into an extreme one, and then to submit this changed interval to the analysis prescribed by the rule. The change is effected simply by reversing the coefficients—thus converting the equation into its reciprocal—and then transforming by $\frac{1}{b-a}$ exactly, thus far, as in the method of BUDAN. This transformed equation we already know cannot have more than two roots between 0 and ∞, and to the development of these, or the discovery of their impossibility, the rule adverted to is of course fully adequate. If they prove to be real, then calling them r and r', the roots in the proposed interval will be the reciprocals of $\frac{1}{b-a} + r$ and $\frac{1}{b-a} + r'$ increased by a. If the roots are imaginary, the preliminary transformation adverted to will often disclose to us the circumstance, as in BUDAN'S method; and thus, as remarked above, the most valuable peculiarity of that method is secured to this: the character of the roots in the case supposed may also unfold itself without actually executing any transformation at all: for if the condition of imaginary roots hold for the pair under examination, and if moreover we can foresee that in the reciprocal of the equation to which this pair immediately refers the transformation by $(\frac{1}{b-a})$ would furnish two variations in the first three terms, then as explained at page 22, we might at once infer the impossibility of the roots in question. There is no difficulty in foreseeing whether or not this can happen: for we have only to ascertain whether the transforming value for which the second coefficient of the reciprocal equation referred to becomes zero, be greater or less than $\frac{1}{b-a}$. The transforming value for which a second coefficient becomes zero is obtained by dividing the second coefficient by m times the first, m being the degree of the equation, (*Theory of Equa.* p. 86;) so that if this value exceed $\frac{1}{b-a}$, then in transforming by the latter, the three variations will remain undisturbed, and hence, as explained at page 22, two imaginary roots will be indicated in the interval

$[a, b]$. If therefore we call, in general, the final coefficient of the derived function under which the index **2** occurs A_p, the preceding coefficient being A_{p+1}, and m the degree of that function, the condition

$$\frac{A_{p+1}}{mA_p} > \frac{1}{b-a} \text{ or } b-a > \frac{mA_p}{A_{p+1}}$$

when found to exist in conjunction with the ordinary criterion of imaginary roots at page 9, will authorize the conclusion that the two roots indicated in the interval $[a, b]$ are necessarily imaginary. When therefore the criterion is satisfied we ought to try whether or not this second condition holds before applying the preparatory tranformation of Budan to the reciprocal equation in our attempts to analyse a doubtful intermediate interval; as much subsequent labour may thereby be spared. We shall here write down in order the proper pairs of criteria corresponding to each derived function. They are as follow:

$$f(x)\left\{\begin{array}{l} b-a > n\dfrac{A_0}{A_1} \\[2ex] \dfrac{A_1}{2A_2} < \dfrac{n}{n-1}\dfrac{A_0}{A_1} \end{array}\right.$$

$$f_1(x)\left\{\begin{array}{l} b-a > (n-1)\dfrac{A_1}{2A_2} \\[2ex] \dfrac{A_2}{3A_3} < \dfrac{n-1}{n-2}\dfrac{A_1}{2A_2} \end{array}\right.$$

$$f_2(x)\left\{\begin{array}{l} b-a > (n-2)\dfrac{A_2}{3A_3} \\[2ex] \dfrac{A_3}{4A_4} < \dfrac{n-2}{n-3}\dfrac{A_2}{3A_3} \end{array}\right.$$

$$\vdots$$

$$f_{n-2}(x)\left\{\begin{array}{l} b-a > 2\dfrac{A_{n-2}}{(n-1)A_{n-1}} \\[2ex] \dfrac{A_{n-1}}{nA_n} < 2\dfrac{A_{n-2}}{(n-1)A_{n-1}} \end{array}\right.$$

And when either of these pairs of criteria hold for the corresponding function, the two roots indicated in that function, and occupying the interval $[a, b]$, will be imaginary, whatever be the indices preceding **2**.

When the coefficients A_0, A_1, &c. are small, the second of each pair of criteria may be more readily computed by the original forms at page 9; and the same is the case with respect to the series of criteria before given. But after a few steps of the development the coefficients usually become large, when the forms above are to be preferred.

(18.) We are now prepared to furnish the general rule adverted to above.

II. *To analyse any doubtful intermediate interval.*

1. Two roots being indicated between a and b, see whether the proper pair of conditions at page 26 corresponding to the function to which the index **2** of these roots belongs, have place: if so, the roots are imaginary.

2. If these conditions have not place, reverse the coefficients of (a), and transform the resulting reciprocal by $\left(\frac{1}{b-a}\right)$: if no variations are left the roots are also imaginary.

3. But if two variations appear, apply the rule at page 20 to the roots indicated by them: if these prove to be imaginary, so are those in the proposed interval: if they prove to be real, then, calling them r, r', the reciprocals of $\frac{1}{b-a} + r$ and $\frac{1}{b-a} + r'$, each increased by a, will be the roots indicated between a and b.

(19.) This general rule combines several advantages: the precepts 1 and 2 apply, whatever be the indices preceding the index **2** referred to; and they will frequently enable us to discover the existence of imaginary roots in the proposed interval without any development. The second of these precepts will, indeed, often detect other pairs of imaginary roots besides the pair to which our index **2** applies, provided such pairs exist in the same interval, as shown in the example at page 241 of the *Theory of Equations*. When, however, actual development becomes necessary to determine the character of the doubtful pair, then, by applying precept 3, or proceeding according to the rule at page 20, we are furnished at the outset with a criterion whereby to test the character of the roots, whatever index precedes **1**, **2**; and are not obliged to wait till our development has at length reduced the indices to **0**, **1**, **2**, as in FOURIER's method. If the character of the roots be not detected till this state of the indices be reached, then we shall have at our disposal a choice of criteria:—we may either continue to employ the criterion of the rule, or may apply that of FOURIER, whichever appears to be the more eligible. Should the determination of the nature of the roots be delayed till the trial divisors have become permanently effective, then the simpler conditions at page 17 may be brought into operation.

(20.) The simple and easily applied relations which, at page 26, accompany the ordinary criteria of imaginary roots, give a desirable extension and explicitness to those criteria; as they afford us an insight, to a certain extent, into the intervals occupied by the roots which those criteria indicate. For if two roots be found to exist between a and b, at the same time that the proper pair of conditions at page 19 has place, we become apprized of the interval within which the imaginary roots indicated must be situated; and we have only to enlarge the interval a, b till the relation accompanying the criterion is attained, in order to obtain the limits within which the imaginary roots indicated must necessarily occur, whatever other roots may at the same time be comprehended within this enlarged interval. In this way, whenever the criterion of imaginary roots has place, may we always determine an interval, beyond the limits of which the imaginary roots indicated cannot exist.

The preceding criteria may of course be varied in form, and may indeed be made to assume a simpler appearance: that above has been adopted for the purpose of keeping the trial expressions for the successive root figures distinct; and for preserving uniformity with the group of criteria before given.

(21.) When the indices preceding the **2** are reduced to **0**, **1**, much simpler formulas may be obtained. FOURIER's is

$$\left\{\frac{f(a)}{f_1(a)} + \frac{f(b)}{f_1(b)}\right\} > b - a$$

But it is desirable to avoid the trouble of repeated transformations by a superior limit; disregarding therefore the second fraction we have

$$\frac{f(a)}{f_1(a)} > b - a, \text{ or } \frac{A_0}{A_1} > b - a$$

And this, when $b - a$ is taken equal to unit, furnishes perhaps the simplest criterion for the case supposed, that can be given. We deduce from it the following rule:

III. *To analyse an interval when the indices are* **0**, **1**, **2**.

1. Continue to develope the root **1** of the middle function $f_{m+1}(x) = 0$ till $\frac{A_m}{(m+1)A_{m+1}}$ exceeds *unit* in the place of the *last found* figure, unless the roots separate before.

2. Having reached this condition, transform by an additional unit; that is, increase the last found figure by 1. If the roots do not separate for this transformation they must be imaginary.

It is of course supposed here, as in Rules I and II, that development is actually necessary to the analysis of the interval; otherwise the doubt becomes immediately removed by the application of the formula to the original coefficients.

(22.) We shall now select an example or two, from among those discussed in the *Theory of Equations*, in illustration of the principles established in the preceding articles.

1. The equation

$$x^5 + 173\,x^4 + 2356\,x^3 + 10468\,x^2 - 14101\,x + 4183 = 0$$

is partially analysed at page 303 by the method of BUDAN, and a pair of roots is found to lie between 0 and 1. The development of the intervening root in the immediately inferior derived equation is carried on at page 271 till the expressions

$$\frac{A_1}{2A_2}, \quad \frac{2A_0}{A_1}$$

which furnish the concurring root figures, differ; which they are found to do at the sixth step, the expressions then becoming

$$\frac{1314}{2 \times 147\ldots} = 4 \text{ and } \frac{2 \times 139}{131} = 2;$$

and as the first exceeds the second, we immediately infer that the roots indicated in the specified interval are *real.*

In the same equation another doubtful interval occurs, in the negative region, viz., between -5 and -6 (page 303). The roots in this interval are at once seen to be *imaginary,* because the second criterion at (3) is fulfilled.

2. The equation

$$x^6 + 378\,x^5 + 38189\,x^4 + 492368\,x^3 - 572554\,x^2$$

$$+ 213720\,x - 26352 = 0$$

is found at page 305, to have two of its roots indicated between 0 and 1. The expressions which concur in furnishing the successive figures of the intervening root of the derived equation $f_2(x) = 0$, are

$$\frac{A_2}{3A_3}, \quad \frac{A_1}{A_2};$$

and as shown at page 280 this concurrence has place as far as the fourth decimal in the root: the expressions then differ, the first giving 4 for the leading figure, and the second giving 3. We infer, therefore, that the roots are *real,* and that they separate at the fifth decimal.

3. The equation

$$12x^3 - 120x^2 + 326x - 127 = 0,$$

analysed at page 305, has a pair of roots indicated between 4 and 5. By developing the intervening root of $f_1(x)=0$, as at page 308, we find the trial divisor to become effective after the first decimal of the root is reached. The expression for the next root figure, viz., $\frac{A_1}{2A_2}$, does not concur with $\frac{2A_0}{A_1}$ in furnishing the same value for that figure: the former giving 6 and the latter, 8. Consequently the roots of $f(x)=0$, in the proposed interval, are *imaginary*, by (12).

(23.) These examples, which are of more than the ordinary degree of difficulty, sufficiently prove the practical value of the criteria proposed at p. 17 when employed in connexion with the process of development recommended in the preceding treatise. They evidently furnish the precise information wanted as to the true character of the doubtful roots, as soon as our development has conducted us to that critical stage of the process at which a decision becomes practicable. In the *Theory of Equations* we have arrived at this decision by aid of external considerations, and supplementary transformations. What is here done, however, saves us this trouble, and gives the desired completion to this more delicate part of the analysis.

Sometimes, as in the examples at (7), the impossibility of a pair of roots in a specified interval may be indicated at once, without any development at all, by the three coefficients referring to these roots, satisfying the proper condition at (3); or, which is the same thing, the corresponding condition at (13). Or it may happen, as in the fourth of the examples referred to, that the entire constitution of the equation may in this way be made known. In other cases too, not included in those examples, the application of the same tests to the original coefficients will be equally effective in discovering the presence of imaginary roots. But to determine in general the greatest number of imaginary roots of which the character is impressed upon the coefficients of the equation, by aid of the tests hitherto employed, requires the investigation of a rule first announced by NEWTON in the *Universal Arithmetic,* but which, for want of a demonstration of its truth, has very properly been accepted with hesitancy; and, notwithstanding its importance, has of late ceased to occupy a place among the established principles of analysis. We shall furnish a demonstration of this rule in the next chapter.

(24.) It remains for us now to give a practical illustration of the other criteria established in this chapter: we shall select for this purpose the example analysed with so much difficulty by BUDAN's method at pages 201, 202 of the *Theory of Equations,* viz.

$$4x^7 - 6x^6 - 7x^5 + 8x^4 + 7x^3 - 23x^2 - 22x - 5 = 0$$

The precepts 1, 2, at page 27, furnish the same results as those arrived at by the method of BUDAN as exhibited at the place referred to, and they are obtained by those precepts in the same manner. The first step of precept 3 leads also to the result

$$5x^7 + 48x^6 + 179x^5 + 317x^4 + 248x^3 + 33x^2 - 38x + 4 = 0$$

as in the same method of BUDAN, this being the transformed equation at which we arrive by diminishing the roots of the reciprocal of the proposed equation by 2. It is to this transformed reciprocal that the development recommended in precept 3, or in the rule at page 20, is to be applied;* and the operation is as follows:—

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* In applying the rule at page 20, it will perhaps be advisable, even when the interval is an extreme one, always to employ, as a preliminary, BUDAN's reciprocal transformation, before proceeding with the actual development; or at least to execute as much of this transformation as may be necessary to enable us to foresee whether or not the resulting signs will all be plus. If the entire transformation be effected, we may then apply our development either to this or to the original—if the same number of variations occur in both—whichever appears to offer the greater facility. As respects the equation above, we may remark that had BUDAN's method been applied to it as at page 202 of the *Equations,* as far as the transformation (6)′, and then our criterion of imaginary roots, the analysis would have been completed at that step.

<table>
<tr><td>5 + 48</td><td>+ 179</td><td>+ 317</td><td>+ 248</td><td>+ 33</td><td>— 38</td><td>+ 4(·166</td></tr>
<tr><td>·5</td><td>4·85</td><td>18·385</td><td>33·5385</td><td>28·15385</td><td>6·115385</td><td>— 3·1884615</td></tr>
<tr><td>48·5</td><td>183·85</td><td>335·385</td><td>281·5385</td><td>61·15385</td><td>— 31·884615</td><td>·8115385 [1]</td></tr>
<tr><td>·5</td><td>4·9</td><td>18·875</td><td>35·4260</td><td>31·69645</td><td>9·285030 [1]</td><td>— 8035931</td></tr>
<tr><td>49</td><td>188·75</td><td>354·260</td><td>316·9645</td><td>92·85030</td><td>— 22·599585</td><td>·0079454 [2]</td></tr>
<tr><td>·5</td><td>4·95</td><td>19·370</td><td>37·3630</td><td>35·43275 [1]</td><td>9·206366</td><td>— 78787</td></tr>
<tr><td>49·5</td><td>193·70</td><td>373·630</td><td>354·3275</td><td>128·28305</td><td>— 13·39321|9</td><td>·0000667 [3]</td></tr>
<tr><td>·5</td><td>5</td><td>19·87</td><td>39·35 [1]</td><td>25·15639</td><td>10·81068 [2]</td><td></td></tr>
<tr><td>50</td><td>198·7</td><td>393·50</td><td>393·6775</td><td>153·4394|4</td><td>— 2·58253</td><td></td></tr>
<tr><td>·5</td><td>5·05</td><td>20·375 [1]</td><td>25·5956</td><td>26·7386</td><td>1·26941</td><td></td></tr>
<tr><td>50·5</td><td>203·75</td><td>413·875</td><td>419·273|1</td><td>180·178|0</td><td>— 1·3131|2</td><td></td></tr>
<tr><td>·5</td><td>5·1 [1]</td><td>12·718</td><td>26·370</td><td>28·368 [2]</td><td>1·2877 [3]</td><td></td></tr>
<tr><td>51</td><td>208·85</td><td>426·59|3</td><td>445·64|3</td><td>208·546</td><td>— ·0254</td><td></td></tr>
<tr><td>·5 [1]</td><td>3·11</td><td>12·91</td><td>27·16</td><td>3·022</td><td></td><td></td></tr>
<tr><td>51·5</td><td>211·9|6</td><td>439·5|0</td><td>472·8|0</td><td>211·56|8</td><td></td><td></td></tr>
<tr><td>·3</td><td>3·1</td><td>13·1</td><td>28 [2]</td><td>3·04</td><td></td><td></td></tr>
<tr><td>51·|8</td><td>215|·1</td><td>452·|6</td><td>500·8</td><td>214·6|1</td><td></td><td></td></tr>
<tr><td>|5|2</td><td>3</td><td>13</td><td>2·9</td><td>3·1 [3]</td><td></td><td></td></tr>
<tr><td></td><td>21|8</td><td>46|6</td><td>503·|7</td><td>217·7</td><td></td><td></td></tr>
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<tr><td></td><td></td><td>|4|8</td><td>50|6</td><td></td><td></td><td></td></tr>
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Without completing the transformation **3** it is obvious that the condition of imaginary roots is satisfied by the three final coefficients; and thus the roots in the proposed interval are imaginary. The imaginary roots of the equation

$$5x^7 - 22x^6 + 23x^5 + 7x^4 - 8x^3 - 7x^2 + 6x + 4 = 0$$

are therefore

$$2{\cdot}166 \ldots \pm \beta\sqrt{-1}$$

and consequently those of the original equation, indicated within the prescribed limits, are

$$\frac{-1}{2{\cdot}166 \ldots \pm \beta\sqrt{-1}}.$$

Hence between those limits the equation has two *imperfect real roots*, each equal to $\frac{-1}{2{\cdot}166}$, the correction of A_7 being less than $-{\cdot}0000667$. (See *Theory of Equations*, p. 311.)

The operation here exhibited in full is very considerably shorter than that by which the corresponding result was obtained by the method of Budan at pages 201, 202 of the preceding treatise. In the work above, only three transformations are employed, and these are blended together in one continuous series of steps; in the operation adverted to, nine transformations are necessary, all separate and distinct, and unsusceptible of abbreviation. It is true that these transformations are each by unit only, except the last, but this is a circumstance of comparatively but trifling importance.

It is obvious that the rule at page 29 might have been appealed to in the foregoing example, at the transformation **3**; and it would have decided the doubt, after transformation by an additional unit.

As an illustration of the simple formula at (21), the example already considered, viz.,

$$f(x) = 12x^3 - 120x^2 + 326x - 127 = 0$$

may be taken.

After the third decimal 7, of the root of $f_1(x) = 0$, is determined, as at page 308 (*Theory of Equations*), we find

$$\frac{A_0}{A_1} = \frac{\cdot 0475}{\cdot 0056} > 8 > b - a.$$

Consequently, the roots are discovered to be imaginary at this step of the approximation; or, by Rule III, at the preceding step.

(25.) It is scarcely necessary to remind the reader of the preceding treatise, that when the character of the doubtful roots is not immediately indicated by the coefficients of the proposed equation, but is to be discovered only by entering upon the operation of actual development, it is supposed to have been ascertained, by a previous examination, that the doubtful roots are not equal. If by applying the simple tests, substituted at page 185 for the process of the common measure, they should prove to be equal, then we should have further to inquire whether these equal roots reduce to zero the advanced functions taken in order up to $f(x)$; that is, whether these functions are all divisible by the quadratic function involving the equal roots. If they are, the number and values of the equal roots of the given equation, situated within the proposed interval, will thus have been determined. But if they are not, then the equal roots, discovered in the intermediate function, or derived equation, will at once imply a pair of imaginary roots in the original equation; since when equal roots occur in any equation, the first member of it, and of that immediately derived from it, vanish simultaneously; and the occurrence of consecutive zeros, in the series of derived polynomials, always indicates imaginary roots in the primitive equation (68). In this case, therefore, we should proceed in exactly the same manner as if the two equal roots had turned out to be two imaginary roots, and which mode of proceeding is fully explained in Chapter X of the preceding work.

(26.) But without entering upon these preliminary examinations, except in so far as may be necessary to assure us that the proposed equation has itself no equal roots in the specified interval, we may if we please proceed at once with our approximation, with the certainty that if the roots we are aiming to separate are in reality equal, the imaginary roots indicated by them, and which must belong to a higher equation in the series, must eventually betray themselves, by the results furnished by the column of work connected with that higher equation diverging from zero, or at most, converging towards a fixed constant, as already explained at page 299. And whenever this happens we shall at least learn that two imaginary roots are indicated in the contracted interval to which, by our approximations, the original interval may have been reduced. It will thus happen that if, disregarding the preliminary inquiries adverted to in last article, we should ever proceed in search of the indicating quadratic of contiguous roots when the actual equality of those roots renders it unattainable the information which we ultimately seek, and which by the path pursued could never be reached, will nevertheless spontaneously offer itself from a different quarter. And it is deserving of especial notice that when no greater number of roots of the proposed equation $f(x)=0$ than three occupy the interval under examination, all inquiry about equal roots in the subordinate equations may be dispensed with: we shall merely have to satisfy ourselves as to whether the equation $f(x)=0$ itself have equal roots in that interval; and thus the method of analysis explained at page 308 will always prove valuable by whatever criterion we propose to ourselves at the outset to test the character of the doubtful roots as the approximation proceeds. We have shown that that method accomplishes the object in view without applying any criterion to its several steps, and now that these other advantages are seen to attach to it, additional reasons are furnished to justify the terms in which we have spoken of it at page 312. It is probable that, in the estimation of some, the peculiarities here noticed, viewed in connexion with the simplicity and obviousness of the process, may invest this method with claims to a preference over that involving the criteria discussed in the present chapter.

(27.) The method just referred to, and that which has been more especially dwelt upon in this Appendix are, however, more intimately connected together than might at first be supposed. In the method here discussed it has been seen that if the expression which furnishes the successive figures of the developed root continue greater than the collateral expression to be compared with it, the doubtful roots must separate; but if, on the contrary, the latter expression always exceed the former, the roots must be imaginary. In this latter case, therefore, the root figure actually put in the quotient, and which is furnished by the less of the two expressions, is too small a quotient figure, in reference to the divisor, or denominator of the other expression, for the demands of the dividend, or numerator of the same expression. And as this continues to be the case, however far the operation be carried, it follows that the said dividend can never be exhausted, nor even diminished below a certain limit: and thus is the method adverted to above deduced from that delivered at (10) in this Appendix; so that we might by this route arrive at the inference that the diminution of the dividend or absolute number, in the case referred to, down to zero, can be effected only by the aid of an imaginary increment connected with the real part of the development; as otherwise shown at page 163 of the *Theory of Equations*.

(28.) It may be here remarked that the series of criteria at (13), although virtually the same as those at (3), are in a better form for use in approximations than the others. The left hand members of the inequalities at (13) are those actually in request at every step of the development of the root, and are thus necessary to facilitate that development, being in fact no other than the trial expressions for the root figure itself. But, independently of this important consideration, these forms are to be preferred to those originally deduced on other grounds, namely, that in general they involve less computation than the original ones. For when the coefficients composing these forms are large numbers, a disagreement in the leading figures of the two *quotients* at (13) is, of course, much more readily discovered than the corresponding disagreement between the two *products* at (3). It may indeed be taken as a general principle, that whenever we have to compute two such products, simply for the purpose of determining which exceeds the other, it will be better to convert the products into quotients; as the disagreement in the results may then always be detected at an earlier stage, each quotient being developed, figure by figure, alternately. In this way the common test for the reality of the roots of a quadratic equation,

$$A_2 x^2 + A_1 x + A_0 = 0,$$

which is always exhibited in the form

$$4A_0 A_2 < A_1^2,$$

ought, with a view to practical convenience in the case of large numbers, to be expressed in the form

$$\frac{4A_0}{A_1} < \frac{A_1}{A_2} \text{ or } \frac{2A_0}{A_1} < \frac{A_1}{2A_2},$$

as at page 17.

In like manner, in applying the ordinary test for the reality of the roots of the cubic equation

$$x^3 - px + q = 0,$$

viz.

$$\left(\frac{p}{3}\right)^3 > \left(\frac{q}{2}\right)^2$$

If the computation must be actually entered upon, and the coefficients be large numbers, the form had better be changed into

$$\sqrt{\frac{p}{3}} > \frac{3q}{2p}$$

and the computation of the two members carried on simultaneously, or rather a figure deduced from one, then the corresponding figure from the other, and so on alternately.

(29.) In reference to the preceding researches we have only further to add, that when there are three contiguous roots in a given interval, that is, three roots concurring in their leading figures, like the three roots developed at page 280 of the foregoing work, we shall, by proceeding as there exhibited, arrive at an indicating cubic, as soon as a disagreement occurs among the three expressions at page 281, supposing the first of these to have previously become effective for the true root-figure of $f_2(x) = 0$. The discussion of this cubic will make known to us whether the three roots are real, or involve an imaginary pair: if they prove to be real, the same cubic will furnish the initial values of the three portions of the roots still undeveloped. And similar observations apply to indicating equations of higher orders, when a greater number of roots all concur in their leading figures. But it is useless to dwell upon these indicators of the higher orders; since, like as in the analysis of a curve surface, they may always be ultimately reduced to indicators of the second degree.

(30.) It may be interesting to notice in conclusion, that the fundamental expressions [2], [3], at page 8 of this Appendix, and upon which the preceding investigations are based, are easily deducible from STURM's function X_2, the general expression for which is found at page 226 of the *Theory of Equations*, to be

$$X_2 = \{ (n-1) A^2_{n-1} - 2n A_n A_{n-2} \} x^{n-2} + \&c.$$

For we know from STURM's theorem, that if the leading coefficient in this function, that is the expression within the braces, be negative, the proposed equation must have a pair of imaginary roots. Hence, as at p. 8, we have the criterion

$$2nA_n A_{n-2} > (n-1) A^2_{n-1}$$

And if the order of the coefficients be reversed, we shall, in like manner, have the new criterion,

$$2nA_0 A_2 > (n-1) A^2_1$$

from which the series of criteria at (3) may be deduced as before.

CHAPTER II.

DEMONSTRATION OF NEWTON'S CRITERIA OF IMAGINARY ROOTS.

(31.) We have already adverted (23) to a rule proposed by Newton in the *Universal Arithmetic*, by aid of which important information respecting the number of imaginary roots entering an equation may often be obtained from an inspection of the coefficients. An investigation of this rule, delivered by Newton without demonstration, was entered upon by Maclaurin, in No. 394 of the *Philosophical Transactions;* and, after the lapse of three years, was resumed and concluded in No. 408 of the same work. From an examination of these elaborate investigations, it will appear that though Maclaurin entered upon the inquiry under the impression that he would be led to a general and satisfactory proof of the rule in question, yet in the second of the papers referred to he appears to have abandoned this expectation, admitting that the results to which he is conducted only go the length of showing that "*some* imaginary roots exist in an equation," whenever any of Newton's criteria have place; and do not embrace the more general affirmation of the rule, that there are always as many pairs of such roots as there are distinct criteria fulfilled. The rule itself is as follows:

Newton's Rule.

(32.) Form a series of fractions, whose denominators are the numbers 1, 2, 3, 4, 5, &c., in succession, going on to the number which marks the degree of the equation, and whose numerators are the same numbers taken in a contrary order. Divide each fraction, commencing with the second, by that which immediately precedes it. Place the results over the middle terms of the equation: and under any of the middle terms, if its square multiplied by the fraction written over it be greater than the product of the two adjacent terms, place the sign $+$; but if it be less,* the sign $-$. And under the first and last terms place the sign $+$. There will be as many impossible roots in the equation, as there are changes in this series of underwritten signs, from $+$ to $-$, and from $-$ to $+$.

For example, if the equation be

$$x^7 - 2x^6 + 3x^5 - 2x^4 + x^3 + 0x^2 + 0x - 3 = 0;$$

then the fractions to be written in order, over the terms intermediate between the first and last, are to be deduced from the series

$$\frac{7}{1}, \quad \frac{6}{2}, \quad \frac{5}{3}, \quad \frac{4}{4}, \quad \frac{3}{5}, \quad \frac{2}{6}, \quad \frac{1}{7}$$

by dividing the second by the first, the third by the second, and so on.

We thus have, agreeably to the rule

$$x^7 - \overset{\frac{3}{7}}{2x^6} + \overset{\frac{5}{9}}{3x^5} - \overset{\frac{3}{5}}{2x^4} + \overset{\frac{3}{5}}{x^3} + \overset{\frac{5}{9}}{0x^2} + \overset{\frac{3}{7}}{0x} - 3 = 0$$

$$+ \quad - \quad + \quad - \quad + \quad - \quad + \quad +$$

And as the underwritten signs have six changes, we infer that the equation has six imaginary roots.

Again, let the equation be

$$x^5 - 4x^4 + 4x^3 - 2x^2 - 5x - 4 = 0.$$

In this case the series of fractions, out of which those to be written over the terms are formed, are

$$\frac{5}{1}, \quad \frac{4}{2}, \quad \frac{3}{3}, \quad \frac{2}{4}, \quad \frac{1}{5};$$

* Or not greater.

and therefore the equation, with the proper fractions over its terms, and the proper signs, deduced from them as in the rule, underneath, will be as follows:

$$\begin{array}{cccccc} & \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & \\ x^5 & - 4x^4 & + 4x^3 & - 2x^2 & - 5x & - 4 = 0 \\ + & + & - & + & + & + \end{array}$$

As the underwritten signs have here only two changes, but two imaginary roots are indicated. These occur in the positive region, since the signs of $x^5 - 4x^4 + 4x^3$, the terms, to which the two changes are due, having two variations, imply two positive roots. And as from the entire number of variations, the roots in the positive region are three in number, it follows that the equation has but one real positive root, two imaginary roots in the positive region, and two roots, of what character we know not, in the negative region.

These examples are taken from NEWTON: we shall now proceed to demonstrate the rule to which they refer.

(33.) From the general equation,

$$A_n x^n + \ldots\ldots A_3 x^3 + A_2 x^2 + A_1 x + A_0 = 0,$$

we deduce the following series of derived equations, viz.:

$$nA_n x^{n-1} + \ldots . 4A_4 x^3 + 3A_3 x^2 + 2A_2 x + A_1 = 0,$$

$$n(n-1)A_n x^{n-2} + \ldots . 4 \cdot 5A_5 x^3 + 3 \cdot 4A_4 x^2 + 2 \cdot 3A_3 x + 2A_2 = 0,$$

$$n(n-1)(n-2)A_n x^{n-3} + \ldots . 4 \cdot 5 \cdot 6A_6 x^3 + 3 \cdot 4 \cdot 5A_5 x^2 + 2 \cdot 3 \cdot 4A_4 x + 2 \cdot 3A_3 = 0,$$

&c. &c.

If any of these equations have imaginary roots, then imaginary roots also enter the primitive equation. Also if the reciprocal equations deduced from these, or limiting equations derived from such reciprocals, have imaginary roots, then likewise imaginary roots must also enter the primitive.

The limiting cubics derived from these reciprocals are:

$$4 \cdot 5 \ldots n A_0 x^3 + 3 \cdot 4 \ldots (n-1) A_1 x^2 + 2 \cdot 3 \ldots (n-2) A_2 x + 2 \cdot 3 \ldots (n-3) A_3 = 0,$$

$$4 \cdot 5 \ldots (n-1) A_1 x^3 + 3 \cdot 4 \ldots (n-2) 2 A_2 x^2 + 2 \cdot 3 \ldots (n-3) 3 A_3 x + 2 \cdot 3 \ldots (n-4) 4 A_4 = 0,$$

$$4 \cdot 5 \ldots (n-2) 2 A_2 x^3 + 3 \cdot 4 \ldots (n-3) 2 \cdot 3 A_3 x^2 + 2 \cdot 3 \ldots (n-4) 3 \cdot 4 A_4 x + 2 \cdot 3 \ldots (n-5) 4 \cdot 5 A_5 = 0,$$

&c. &c.

or expunging common numerical factors they are,

$$n(n-1)(n-2) A_0 x^3 + 3(n-1)(n-2) A_1 x^2 + 2 \cdot 3 (n-2) A_2 x + 2 \cdot 3 A_3 = 0,$$

$$(n-1)(n-2)(n-3) A_1 x^3 + 3(n-2)(n-3) 2 A_2 x^2 + 2 \cdot 3 (n-3) 3 A_3 x + 2 \cdot 3 \cdot 4 A_4 = 0,$$

$$(n-2)(n-3)(n-4) 2 A_2 x^3 + 3(n-3)(n-4) 2 \cdot 3 A_3 x^2 + 2 \cdot 3 (n-4) 3 \cdot 4 A_4 x + 2 \cdot 3 \cdot 4 \cdot 5 A_5 = 0,$$

&c. &c.

Now if any of these limiting cubics indicate imaginary roots, when submitted to the criteria at (3), such indications will imply imaginary roots in the proposed equation. But several indications, apparently distinct, may offer themselves in these equations, which upon closer examination may be found to be necessarily dependent, or concurrent. Distinct imaginary pairs can of course be inferred only from independent and non-concurring conditions. We have therefore to inquire how these are to be discovered in the above series of equations.

(34.) And first we may remark, that since only one imaginary pair can enter into a cubic equation, it follows that whether the criterion of imaginary roots is satisfied by the three leading terms of any of the above cubics, or by the three final terms, or simultaneously by both sets of three, one imaginary pair, and one only is implied. Hence, when both sets of three terms furnished by any cubic fulfil the proposed conditions, these conditions, though really independent, that is not necessarily implied one in the other, nevertheless necessarily concur in indicating the same thing. Thus only a single imaginary pair can be inferred from any one of the limiting cubics, whether the criterion is satisfied for one set of three terms, or for the two consecutive sets.

Again: a glance at the final set in one cubic, and the leading set in that next in order, is sufficient to show that if the criterion is fulfilled for the former set, it must of necessity be also fulfilled for the latter, and vice versâ. In this case, therefore, the conditions are necessarily dependent; the existence of one implying, of necessity, the existence of the other; so that, as before, though from a dissimilar cause, the fulfilment of the conditions by two consecutive sets of three terms, implies but a single imaginary pair.

We thus discover the general law which connects the above series of limiting equations together, as respects the indications of imaginary roots which they severally furnish, and are thence enabled to distinguish those indications which are really independent and non-concurrent, and which therefore point to distinct imaginary pairs, from those which, in virtue of this connexion, unite in testifying to one and the same thing.

If the first set of three, that is the leading terms in the first cubic, satisfy the criterion, we can immediately infer the existence of one imaginary pair. If the next set, the final terms of the same cubic, also satisfy it, the preceding condition merely recurs, and supplies no additional information. In this case the following set of three, the leading terms of the next cubic, must of necessity furnish the same concurring condition, by the second principle stated above; and so on, till we arrive at a set of three terms for which the condition *fails,* thus putting a stop to the series of concurring indications, and preparing the way for new and distinct conditions altogether unconnected with the former. As soon as the criterion again holds the condition, being thus entirely independent of, and unconnected with the former, must imply another and distinct imaginary pair. And so on to the end of the series.

(35.) Now the criteria which we have here supposed to be applied to the terms, taken three at a time, of the successive limiting cubics under examination, supply one after another the very expressions exhibited at page 9; the three final terms of one cubic always furnishing the same one of these expressions as the three leading terms of the next, agreeably to what is shown above; so that in deducing from our cubics the expressions alluded to, these repetitions may be omitted. Attending to this, and applying the criterion of imaginary roots for a cubic equation, to each of the foregoing in succession, we have the following inequalities, viz.

1st. $2^2 \cdot 3^2 n(n-1)(n-2)^2 A_0 A_2 > 2 \cdot 3^2 (n-1)^2 (n-2)^2 A_1^2$

or suppressing the common factors,

$$2nA_0 A_2 > (n-1) A_1^2 \quad . \; . \; . \; . \quad [1],$$

2d. $2^2 \cdot 3^3 (n-1)(n-2) A_1 A_3 > 2^3 \cdot 3^2 (n-2)^2 A_2^2$;

or suppressing the common factors,

$$3(n-1) A_1 A_3 > 2(n-2) A_2^2 \quad . \; . \; . \; . \quad [2],$$

3d. $2^3 \cdot 3^3 \cdot 4 (n-2)(n-3) A_2 A_4 > 2^3 \cdot 3^4 (n-3)^2 A_3^2$;

or suppressing the common factors,

$$4(n-2) A_2 A_4 > 3(n-3) A_3^2 \quad . \; . \; . \; . \quad [3],$$

4th. $2^3 \cdot 3^4 \cdot 4 \cdot 5 (n-3)(n-4) A_3 A_5 > 2^3 \cdot 3^4 \cdot 4^2 (n-4)^2 A^2{}_4$;

or suppressing the common factors,

$$5(n-3) A_3 A_5 > 4(n-4) A_4^2 \quad . \; . \; . \; . \quad [4],$$

&c. &c.

And thus, as stated above, are we led to the series of criteria already established at page 9; and which we now know to be so connected together that, if when proceeding from one set of three terms in an equation to the three next in order, the consecutive criteria both have place, the recurrence is to be regarded merely as a second indication of the same thing—the existence of a single imaginary pair: and that as soon as the condition fails, preparation is made for a new and independent indication; and so on till all the sets of three have been examined.

(36.) Hence the indications that are really non-concurrent, and consequently the number of imaginary pairs inferrible from them, may be thus noted:

Under the first and last terms of the proposed equation, write the sign *plus*. Then taking each of the intermediate terms in succession for a middle term, write under it the sign *minus* when the criterion holds, and *plus* when it fails. The alternations of signs, thus furnished, will denote the number of imaginary roots, which must of necessity enter the equation: there may in certain circumstances be more, but there can never be fewer.

This rule is virtually the same as that of NEWTON. For it is obvious that NEWTON'S over-written fractions are no other than the quantities

$$\frac{n-1}{2n}, \quad \frac{2(n-2)}{3(n-1)}, \quad \frac{3(n-3)}{4(n-2)}, \quad \frac{4(n-4)}{5(n-3)}, \text{ \&c.}$$

But without encumbering the terms with these over-written fractions, or with the under-written signs, we may apply the criteria at once, in order, as they stand at page 9, reckoning a single imaginary pair as soon as one of them is fulfilled, which single pair is all that is to be counted upon till a failure takes place, preparatory to a second fulfilment of the condition, and if this happen, a second pair is to be counted, and so on.

(37.) This rule of NEWTON, now established, is obviously a valuable adjunct in the modern theory of numerical equations. It is plain that most of the conclusions of the last chapter are deducible from it, and may therefore be regarded as legitimate inferences from NEWTON'S principle: but it was thought better to obtain those conclusions from independent considerations. In fact, these were all arrived at before the preceding investigation suggested itself: and would therefore have remained undisturbed, though NEWTON'S rule had proved inaccurate.

Examples and illustrations of this rule may of course be framed at pleasure: we shall adduce but one.

Let the following equation be proposed for analysis, viz.

$$5x^8 - 2x^7 + 3x^6 - 24x^5 - 16x^4 + x^3 - 4x^2 - 2x - 60 = 0.$$

Taking $2x^7$ for the middle term the condition is fulfilled: hence there is a pair of imaginary roots in the positive region. Taking the next term $3x^6$, the condition in like manner holds; so that no new pair is indicated. Taking the next term $24x^5$, the condition fails; the signs of the adjacent terms are unlike. Taking the next term $16x^4$, the condition again fails. For the next term x^3, it holds: hence there is a second imaginary pair in the positive region. For the next term $4x^2$, the condition again fails; but for the term following it succeeds: hence there is a third pair of imaginary roots: this last pair being in the negative region. We conclude therefore that the equation has six imaginary roots: and since the last term is minus, we know that the two remaining roots must be real; one positive and the other negative.

It is obvious that the criteria established at page 26, in reference to the limits of the imaginary pairs, give a useful extension to the foregoing rule, as already explained at page 28.

(38.) Before closing this chapter, it may be proper to notice that some authors have referred to the foregoing rule of NEWTON as having been demonstrated by MACLAURIN.

Thus, MONTUCLA speaks of it as "une règle assez simple, mais encore assez imparfaite. Elle n'etoit d'ailleurs pas démontrée, ce qui a engagé MM. Maclaurin et Campbell à s'en occuper, et ils sont parvenus non-seulement à démontrer, mais encore à perfectionner la règle de Newton."* But any one who examines with care the papers of CAMPBELL and MACLAURIN, will find that the rules given in those papers do not enable us to detect the existence of more than *two* imaginary roots in an equation. MACLAURIN was fully aware of this; and hence terminates his investigations as follows:—

"I might show, in the next place, how the rules deduced from the 11th and 12th propositions may be extended so as to discover when more than two roots of an equation are imaginary, and in general to determine the number of imaginary roots in any equation; but as it would require a long discussion, and some lemmata, to demonstrate this strictly, I shall only observe that these 11th and 12th propositions will be found to be still the most useful of all those we have given for that purpose."†

* Histoire des Mathématiques. Tome iii. p. 31.
† Phil. Trans. No. 408, p. 77.

CHAPTER III.

DEMONSTRATION OF THE CRITERION OF FOURIER.

(39.) EVERYTHING relating to the analysis and solution of numerical equations has at length been brought under the dominion of common algebra, with the single exception of the rule which FOURIER has proposed for discovering the character of a pair of roots indicated in a given interval. Of this rule FOURIER has given two investigations;—one founded upon the analytical theory of curves, and the other involving a principle of the higher calculus known by the name of LAGRANGE's theorem on the limits of TAYLOR's series. The former of these investigations, as being the simpler of the two, is that which we have adopted at page 164 of the preceding treatise. But it is desirable that the reasoning, by which this rule is established, should be stripped of its transcendental form; and thus be reduced to a level with the other general principles that now constitute the doctrine of numerical equations. It is the intention of the following investigation to accomplish this object.

(40.) Let a, b represent the numbers which bound the doubtful interval comprehending the roots to be examined. We may consider these numbers to be positive, giving rise to the following variations of signs in the three final functions:—

	$f_2(x)$	$f_1(x)$	$f(x)$
(a) - - -	$+$	$-$	$+$
	0	1	2
(b) - - -	$+$	$+$	$+$

Let $a + h$ be one of the intervening roots of $f(x) = 0$, the least, if the roots be real, and let $b - k$ be the other root. We shall proceed on the assumption that the roots in question are real, and consequently that h and k are real positive quantities.

Now from common algebraical principles, we have

$$f_2(a + h) = f_2(a) + f_3(a)\,h + f_4(a)\frac{h^2}{2} + f_5(a)\frac{h^3}{2 \cdot 3} + \ldots\ldots$$
$$+ f_n(a)\frac{h^{n-2}}{2 \cdot 3 \ldots (n-2)}.$$

And the right hand member of this is the second limiting polynomial derived from

$$f_2(a)\frac{h^2}{2} + f_3(a)\frac{h^3}{2 \cdot 3} + \ldots\ldots + f_n(a)\frac{h^n}{2 \cdot 3 \ldots n-1} \ldots [1],$$

these limiting polynomials being

$$f_2(a)h + f_3(a)\frac{h^2}{2} + f_4(a)\frac{h^3}{2 \cdot 3} + \ldots\ldots + f_n(a)\frac{h^{n-1}}{2 \cdot 3 \ldots (n-1)} \ldots [2],$$

$$f_2(a) + f_3(a)h + f_4(a)\frac{h^2}{2} + \ldots\ldots + f_n(a)\frac{h^{n-2}}{2 \cdot 3 \ldots (n-2)} \ldots [3].$$

The positive roots of the equations

$$[1] = 0, \qquad [2] = 0, \qquad [3] = 0,$$

which are the only values of h in which we are interested, when written in ascending order, are known to arrange themselves as follows:—

$$\begin{array}{ccccccccc} 0 & & 0 & & a_1 & & a_2 & \ldots \\ & 0 & & b_1 & & b_2 & \ldots & \\ & & c_1 & & c_2 & & \ldots & \end{array}$$

Consequently [1] cannot undergo any change of sign during the progress of h from $h = 0$, up to $h = c_1$, the least positive root of

$[3] = 0$. And, therefore, since $f_2(a)$ is positive by hypothesis, and that the power of h connected with it is positive, [1] continues positive from $h = 0$, up to $h = c_1$ (*Theory of Equations*, p. 14); and, consequently throughout the interval $[a, b]$, since the root c_1 of $[3] = 0$, is not reached in this interval.

A similar conclusion of course has place when in [1], [2], [3],—k is substituted for h, and b for a.

Now, by hypothesis,

$$f(a+h) = f(a) + f_1(a)\,h + f_2(a)\,\frac{h^2}{2} + \ldots\ldots = 0 \ldots\ldots [4],$$

$$f(b-k) = f(b) - f_1(b)\,k + f_2(b)\,\frac{k^2}{2} - \ldots\ldots = 0 \ldots\ldots [5].$$

And by the conclusions just established

$$f_2(a)\,\frac{h^2}{2} + \ldots\ldots$$

$$\text{and} \quad f_2(b)\,\frac{k^2}{2} - \ldots\ldots$$

are both *positive* quantities. Also by hypothesis $f_1(a)$ is negative, and $f_1(b)$ positive: consequently in the equation

$$-\frac{f(a)}{f_1(a)} = h + \frac{f_2(a)}{f_1(a)}\,\frac{h^2}{2} + \ldots\ldots$$

immediately deduced from [4], the terms after h are, in the aggregate, *negative:* and in the equation

$$-\frac{f(b)}{f_1(b)} = -k + \frac{f_2(b)}{f_1(b)}\,\frac{k^2}{2} - \ldots\ldots$$

deduced in like manner from [5], the terms after $-k$ are, in the aggregate, *positive.* Hence, by subtracting the latter from the former, we have

$$-\frac{f(a)}{f_1(a)} + \frac{f(b)}{f_1(b)} = h + k + \text{a negative quantity.}$$

The first of these expressions $-\frac{f(a)}{f_1(a)}$, on account of the sign of $f_1(a)$, is of course positive, like the second; hence, omitting the minus sign, and regarding only absolute numerical values, we have

$$\frac{f(a)}{f_1(a)} + \frac{f(b)}{f_1(b)} < h + k.$$

But $b - a$ is necessarily not less than $h + k$: consequently

$$\frac{f(a)}{f_1(a)} + \frac{f(b)}{f_1(b)} < b - a,$$

the condition which must always be fulfilled whenever, as assumed above, the doubtful roots are real. And this is the criterion of FOURIER.

So long as this condition has place, we are to continue narrowing the interval $[a, b]$ carrying on the approximation to the intervening root of $f_1(x) = 0$ either till the roots actually separate, or till the process is stopped by the non-fulfilment of the condition; when we may conclude that the roots are necessarily imaginary.

(41.) The above is the criterion to which FOURIER has given the most prominent place in his researches into the analysis of equations, and it appears to be the most convenient of all that he has proposed. But on account of its involving the superior limit (b) of the doubtful pair of roots, it will in general be found far less eligible than the criteria established in the former chapters of this Appendix. Of FOURIER's other methods, here alluded to, there is one at the close of his work, deduced like that just discussed, from the theory of curves, combined with Lagrange's theorem before adverted to.* This method bears some analogy to that developed in the first chapter, page 15. But it is of inferior efficacy, and involves, like that above, the superior limit of the roots. The methods delivered in this Appendix are deduced from

* This method may however be investigated upon the same principles as those employed above; and thus LAGRANGE's theorem dispensed with.

principles purely algebraical; and although, as before remarked, (*Theory of Equations*, p. 153,) geometrical considerations sometimes conduct to views and methods but obscurely indicated by our algebraical symbols, yet it often happens, on the contrary, that purely analytical reasonings terminate in results of a much higher practical value than those furnished by geometry. And this is more especially the case when, as in inquiries connected with the numerical solution of equations, these results furnish approximations only. The geometrical investigations of FOURIER, just alluded to, and which assimilate the neighbouring roots of numerical equations to the approximations of parabolic curves to one another, may be adduced as an instance of this.

(42). We shall merely remark, in conclusion, that the method at page 15, as explained in the precepts of article (11), might suffice for a general rule for the analysis of a doubtful interval. For by developing, as there directed, the roots, if real, and not related to one another in the peculiar manner there supposed, would eventually separate whilst we were seeking to reach the indicating quadratic, and would thus put a stop to the search: and if imaginary, the roots would then be under the circumstances implied in that method, inasmuch as they could never separate, and consequently could not interfere with the certain attainment of the indicating quadratic. But the methods subsequently delivered dispense with the necessity of waiting for this quadratic, when, at the same time, the character of the roots is impressed upon the coefficients at an earlier stage of the development. The object of those methods being to detect the earliest indications that can occur.

Note A, page 24.

When two imaginary roots of an equation are indicated between a and b, it is the object of Budan's method, as already stated, to exclude the reciprocals of them from the reciprocal interval $[\frac{1}{b-a}, \infty]$; and we have already shown the practicability of effecting this object, either by directly narrowing the interval $[a, b]$, or, by means of a succession of transformations reciprocal and direct, virtually accomplishing the same end, as in the tenth chapter of the *Equations*. It may here be well to show, as indeed Budan himself has also done, under what circumstances the exclusion adverted to fails to be effected at the first reciprocal transformation: that is to say, what peculiarities must distinguish the imaginary pair in question, in order that the first reciprocal transformation may fail to have its signs all plus.

Let the pair of imaginary roots in the interval $[a, b]$ be $x = \alpha \pm \beta\sqrt{-1}$, the reciprocals of them, entering the reciprocal equation, will be

$$\frac{1}{x} = \frac{1}{\alpha \pm \beta\sqrt{-1}} = \frac{\alpha \mp \beta\sqrt{-1}}{\alpha^2 + \beta^2}$$

and consequently, the first reciprocal transformation will reduce them to

$$\frac{1}{x} - 1 = \left(\frac{\alpha}{\alpha^2 + \beta^2} - 1\right) \mp \frac{\beta}{\alpha^2 + \beta^2}\sqrt{-1}$$

Now if the signs of this transformation are not all plus, it will be a proof that the real part of the imaginary expression just written still remains positive, requiring further diminution to render it negative. (See *Theory of Equations*, page 131.) But this real part cannot be positive unless the denominator $\alpha^2 + \beta^2$

be less than α, and this is not possible unless α and β are both fractions. Moreover β cannot exceed a certain limit: for since

$$\alpha^2+\beta^2 < \alpha \therefore \beta^2 < \alpha(1-\alpha)$$

and as the greatest product that can arise from multiplying a fraction by its defect from unity is $\frac{1}{4}$, it follows that, under the circumstances supposed, β must be less than $\frac{1}{2}$. Hence, the first reciprocal transformation can fail to present an uninterrupted series of permanencies—the roots being imaginary—only when α and β are both fractions, and β less than $\frac{1}{2}$.* It is in such cases as these that our second condition, respecting the limits of the imaginary roots, becomes useful.

In the preceding reasoning the interval [a, b), comprehending the indication of the imaginary roots, is supposed to be unit: but if this interval be different from unit—whether less or greater—it may in a similar manner be shown, that for the criterion of Budan to fail, the imaginary pair must be such that

$$\alpha < (b-a) \text{ and } \beta < \frac{b-a}{2}.$$

Contracting the interval affects α only, and not β; which remains constant, however the roots be increased or diminished by real quantities: hence the diminution of the interval must at length preclude the fulfilment of the second of the above conditions, and thus cause the criterion to hold—as otherwise inferred in the text.

It may not be superfluous here to caution the reader against supposing that when an imaginary pair is indicated between a and b, there must always exist the condition $\alpha < (b-a)$, or that the real part of that pair necessarily lies between the proposed limits: it may on the contrary be wholly excluded from those limits. The transformed equation 4, at page 308 of the *Theory of Equations*, has a pair of imaginary roots indicated between ·00005 and ·00006: but, as appears from page 311, the real part of this pair is ·00011... And when in the text, we

* For an examination of the circumstances under which the second, third, &c, reciprocal equation fails, in prosecuting Budan's method, see a paper by Mr. James R. Christie, in the *Philosophical Magazine* for August, 1842.

speak for brevity of the passage over an imaginary pair, we merely mean the passage over the indicator of that pair. It is clear that a passage over the real part of an imaginary pair can never precede the passage over the indicator: for the two variations are not lost till the indicator is passed; yet these must be lost by the passage of the real part of the imaginary pair indicated. (See *Theory of Equations,* page 131.)

Note B.

(*Page* 220 *of the Theory of Equations.*)

At page 220 of the Theory of Equations, in reference to the remark terminating in the word "explained," at line 13, the following should have been introduced as a foot-note:—

In this determination, X_p is to be treated as a constant; since the roots of $\frac{X_1}{X_p} = 0$ are interposed between those of $\frac{X}{X_p} = 0$, exactly as the same roots of $X_1 = 0$ are interposed between those of $X = 0$.

Also to the last line of the text, at page 445, should be added, "provided $a^2 - b$ be one."

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