# BEBR

## FACULTY WORKING PAPER NO. 741

**Restricted Domains, Arrow Social Welfare Functions and Noncorruptible and Nonmanipulable Social Choice Correspondences: The Case of Private and Public Alternatives**

*Zvi Ritz*



College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

# BEBR

FACULTY WORKING PAPER NO. 741

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

February 1981

## Restricted Domains, Arrow Social Welfare Functions and Noncorruptible and Nonmanipulable Social Choice Correspondences: The Case of Private and Public Alternatives

Zvi Ritz, Assistant Professor
Department of Business Administration

Acknowledgment: I wish to express my deep appreciation to Ehud Kalai who introduced me to the field of social choice and whose advice and support enabled me to complete this paper. Thanks are also due to Alvin Roth for lengthy discussions and valuable suggestions. This research was supported in part by NSF grants SOC 7620953 and SOC 7907542.

## Abstract

An n-person social choice problem is considered in which the alternatives are (n+1) dimensional vectors, with the first component of such a vector being that part of the alternative affecting all the individuals together, while the (i+1) component is the part of the alternative affecting individual i alone. Assuming that individuals are selfish (individual i must be indifferent between any two alternatives with the same first and (i+1) components), that they may be indifferent among alternatives, and that each individual may choose his preferences out of a different set of permissible preferences, we prove that any set of restricted domains of preferences admits an n-person nondictatorial Arrow-type social welfare function if and only if it admits an n-person nondictatorial, nonmanipulable and noncorruptible social choice correspondence. We also characterize all the sets of restricted domains of preferences which admit two-person Arrow-type social welfare functions (and therefore also admit two-person nonmanipulable and noncorruptible social choice correspondences).

## 1. Introduction

Given the impossibility results derived by Arrow in his seminal book "Social Choice and Individual Values" on the possibility of constructing nondictatorial social welfare functions, and by Gibbard [7] and by Satterthwaite [18] on the possibility of constructing nondictatorial and nonmanipulable (strategy-proof) social choice functions, much of the recent literature on social welfare and social choice functions falls into two categories. One category includes efforts to derive possibility results by relaxing some of the criteria introduced by Arrow [1], by Gibbard [7] and by Satterthwaite [18] as desirable characteristics for social welfare and for social choice functions, while the other includes attempts to obtain possibility results by restricting the domains of preferences available to the individuals (for a comprehensive and detailed description of many of these efforts see Sen [23] and Kelly [12]).

The appeal of the "restricted domains" approach is in the observation that in most standard economic models, a considerable structure is imposed both on the individuals' preferences (e.g., assuming individuals with continuous and concave utility functions) and on the social alternatives (e.g., assuming economies with public goods only, with private goods only, etc.), thus results derived in social choice theory following this approach may be applicable to other economic models too.

The first to demonstrate the possibilities inherent in the "restricted domains" approach were Black [2] and Arrow himself ([1], page 75) who proved that majority rule is a nondictatorial social welfare function for groups of individuals with "single-peaked preferences". Arrow [1, Chap. II] also discusses the "difference between the ordering (done by an individual) of the social states according to direct consumption of the individual and the ordering when the individual adds his general standards of equity." He refers to the first type of ordering as reflecting an individual's "tastes" while the second reflects his "values." He demonstrates that this distinction can be represented as restrictions on the individuals' domains of preferences (when an individual orders by "taste" he must be indifferent between any two social alternatives which allocate to him the same (private) alternative) and he goes on to prove that this distinction by itself is not enough to guarantee the existence of a "desired" social welfare function ([1] theorem 3 and also Blau [3]).

In the model discussed here the "restricted domains" approach is used to emphasize both differences between the way individuals choose their preferences (according to "values" versus "tastes") and differences among types of alternatives (e.g., public goods as opposed to private goods). For this purpose, when dealing with the possibility of aggregating individuals' preferences into a social welfare function or a social choice function, we distinguish among three types of "restricted domains." Cases where individuals order (or choose among) the alternatives according to their "values" are cases of public alternatives only; cases where individuals order according to their "tastes" are cases of private alternatives only, and cases where individuals order the social alternatives according to their "tastes" and in addition the social alternatives are such that part of each alternative affects all the individuals together, while other parts affect each individual separately, are cases of mixed alternatives.

This distinction may be demonstrated by the following example. If a group of individuals (rational, of course) must choose a cake from among a variety of flavors, it is a case of public alternatives. If the same group must decide how to divide a given cake among its members, and each individual considers only the amount allocated to him when comparing different allocations (thus he is indifferent between any two allocations that give him the same piece of cake), it is now a case of private alternatives. On the other hand, if individuals give importance not only to the amount allocated to them but also to the amounts allocated to other members of the group, then it is again a public alternatives case. If the group must simultaneously decide on the flavor of the cake and its apportionment and each individual considers only what is allocated to him when comparing the same flavor but different divisions, then it is a case of mixed alternatives.

This work is an investigation of the possibility of constructing Arrow-type social welfare functions and Gibbard-Satterthwaite-type social choice correspondences for the case of mixed alternatives. We assume that every social alternative is a mix of public and private alternatives, individuals may differ in their sets of private alternatives as well as in their domains of preferences, and indifferences among alternatives are allowed. The principal result for this most general case is that a set of n (greater than one) restricted domains admits an n-person nondictatorial Arrow-type social welfare function if and only if it admits an n-person nondictatorial, noncorruptible Gibbard-Satterthwaite-type social choice correspondence, namely a nondictatorial, noncorruptible, nonmanipulable, rational and efficient social choice correspondence.

Intuitively a noncorruptible social choice correspondence is a choice correspondence for which no individual can, by misrepresenting his true preferences, change the social outcome without changing the value of it for himself. This concept is a generalization of the Sattherwaite-Sonnenschein [20] concept of nonbossiness. In their model they discuss strategy-proof allocation mechanisms in classical economic environments, and such a mechanism is bossy if an individual can, by misrepresenting his true preferences, change the bundles allocated to other individuals while maintaining his unchanged. Thus a mechanism can be bossy over private goods only, while a social choice correspondence may be corruptible over public and mixed alternatives as well, as is demonstrated in a later example (section 3). Although both manipulable and corruptible social choice correspondences include "strategic" behavior of individuals who misrepresent their "true" preferences, there is a conceptual difference between the two. A person manipulates a mechanism if, by behaving "strategically", he improves his position <u>directly</u>; he corrupts a social mechanism if, by changing the value of the outcome to others (without altering its value for himself), he creates the possibility of his <u>indirectly</u> improving his position by opening room for either "bribe taking" or "blackmailing". We also characterize all the sets of two or more restricted domains of preferences which admit two-person Arrow-type social welfare functions (and therefore two-person noncorruptible Gibbard-Satterthwaite-type social choice correspondences).

This work builds on, complements and generalizes a number of previous contributions. The most significant of them are results derived by Kalai-Muller [8] and Maskin [13, 14]. They independently characterized all the restricted domains of strict preferences (no indifferences among alternatives are allowed) which admit Arrow-type social welfare functions for the case of public alternatives only when all individuals have the same restricted domain of preferences. In addition they proved that such domains admit Arrow-type social welfare functions if and only if they admit rational, nonmanipulable, efficient and non-dictatorial social choice functions. (Since indifferences are not allowed, all such functions are also noncorruptible.) This further demonstrated the close reciprocity between the Arrow set and the Gibbard-Satterthwaite sets of criteria, a demonstration that started with the works of Gibbard [7] and Satterthwaite [19] combined. (Pattanaik [15] proved one direction of the equivalence for the cases in which individual preferences may be restricted and a discussion of the possibility of full equivalence for these cases appears in Blin-Satterthwaite [4].) Kalai-Ritz [9] characterized all the restricted domains of strict preferences which admit Arrow-type social welfare functions for the case of private alternatives only and individuals which are symmetric in both their alternatives and preferences set. Ritz [17] generalized the Kalai-Ritz [9] results for the case of private alternatives, nonsymmetric individuals and with indifferences among alternatives allowed. He also proved, parallel to Kalai-Muller [8] and Maskin [14], that sets of restricted domains admit Arrow-type social choice functions if and only if they admit nonmanipulable, noncorruptible, rational and non-dictatorial social choice correspondences.

As all these models are special cases of the mixed alternatives model discussed here (if all the private alternative sets contain exactly one element and all the restricted domains are the same and contain only strict preferences then the mixed alternatives model is reduced to the models discussed by Kalai-Muller [8] and Maskin [13], [14], while if the public alternatives set contains one element only, then the general model is reduced to the cases discussed by Kalai-Ritz [9] and Ritz [17]), thus our results unify and generalize some of the results derived by all the other authors. The drawback of this model is that we did not prove completely the independence between the existence of Arrow-type social welfare functions for a group and its size, and therefore we characterized only those sets of restricted domains which admit two-person Arrow-type social welfare functions, while all the previously mentioned works characterize, for their respective cases, all the restricted domains which admit Arrow-type social welfare functions for any finite (and greater than one) number of individuals. We also proved the equivalence between the Arrow type social welfare functions and the Gibbard-Satherthwaite type social choice correspondences only for Arrow social choice functions which obey positive association and therefore appears to be less general. However, since to quote Arrow ([1], page 25), "We are trying to describe social welfare and not some sort of illfare" we do not consider this as a major drawback.

The definitions and some preliminary results are introducted in section 2. In section 3 we prove the equivalence between Arrow-type social welfare functions and noncorruptible Gibbard-Satterthwaite-type social choice correspondences. In section 4 we characterize all the sets of restricted domains which admit two-person social choice functions, and in section 5 we convey some results on the dependency (or independency) between the number of individuals and the existence of Arrow-type social welfare functions.

## 2. Definition and Preliminary Results

Consider a finite set N of n individuals ($n \geq 2$). Let $A_0$ ($|A_0| \geq 1$) denote the set of all public alternatives available to all the individuals as a group. For every individual $j \in N$ and every $a \in A_0$, let $A_j(a)$ be the nonempty set of all private alternatives which are available to individual j when alternative a is available to the group. Let $(A_0;A_j) = \{(a;x) | a \in A_0 \text{ and } x \in A_j(a)\}$, then $\Sigma_j$ denotes the set of all reflexive, transitive and complete binary relations on $(A_0;A_j)$. An element of $\Sigma_j$ is called a preference relation. If for some $(a;x),(b;y) \in (A_0;A_j)$ and $r \in \Sigma_j$ it is both $(a;x)r(b;y)$ and $(b;y)r(a;x)$ then it is aib in r ((a;x) is indifferent to (b;x)) and if for some $r' \in \Sigma_j$ it is $(a;x)r'(b;y)$ and not $(b;y)r'(a;x)$ then it is $(a;x)p(b;y)$ in r' ((a;x) is preferred to (b;y)).

Let $\Omega_j$ be a nonempty subset of $\Sigma_j$; the elements of $\Omega_j$ are the admissible preference relations for individual j. Throughout this work we assume that any two alternatives are distinguishable, i.e., for every $j \in N$, and every $(a;x),(b;y) \in (A_0;A_j)$, if for no $r \in \Omega_j$ it is $(a;x)p(b;y)$, then there exists $r' \in \Omega_j$ such that $(b;y)p(a;x)$ in r'. Let $(A_0;A^{(n)}) = \{(x_0;x_1,\ldots,x_n) | (x_0;x_j) \in (A_0;A_j) \text{ for every } j \in N\}$. An element of $(A_0;A^{(n)})$, $X = (x_0;x_1,\ldots,x_n)$ is called an n-person social alternative (or n-person mixed alternatives allocation). Let $\Delta^{(n)}$ represent the set of all reflexive, transitive and complete binary relations on $(A_0;A^{(n)})$. An n-person social function (SF) over mixed alternatives is a function $h^n:\Omega^{(n)} \to \Delta^{(n)}$ where $\Omega^{(n)} = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$. An element of $\Omega^{(n)}$, $R = (r_1,\ldots,r_n)$ is called an n-person profile. If for a profile R and social alternatives X and Y it is both $Xh^n(R)Y$ and $Yh^n(R)X$, then it is XIY (X is indifferent to Y in $h^n(R)$), and if it is $Xh^n(R)Y$ but not $Yh^n(R)X$ then it is XPY (X is preferred to Y in $h^n(R)$). $h^n$ an n-person SF is an <u>n-person social welfare function</u> (SWF) if it obeys the following conditions:

i) <u>Unanimity</u> (U): For every $X,Y \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$ if for every $k \in N$, $(x_0;x_k)r_k(y_0;y_k)$, and there exists $j \in N$ such that $(x_0;x_j)p(y_0;y_j)$ in $r_j$ then XPY, and if for every $k \in N$, $(x_0;x_k)i(y_0;y_k)$ in $r_k$ then XIY.

ii) <u>Independence of irrelevant alternatives</u> (IIA): For every $X,Y \in (A_0;A^{(n)})$ and $R,S \in \Omega^{(n)}$ if [for every $k \in N$, $(x_0;x_k)r_k(y_0;y_k)$ iff $(x_0;x_k)s_k(y_0;y_k)$ and $(y_0;y_k)r_k(x_0;x_k)$ iff $(y_0;y_k)s_k(x_0;x_k)$] then $Xh^n(R)Y$ iff $Xh^n(S)Y$.

A SF is <u>dictatorial</u> (D) if there exists $j \in N$, s.t. for every $X,Y \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$, $(x_0;x_j)p(y_0;y_j)$ in $r_j$ implies XPY. A SF is <u>nondictatorial</u> (ND) if it is not dictatorial. An <u>n-person Arrow social welfare function</u> (ASWF) is a nondictatorial SWF.

We say that $\Omega^{(n)}$ admits an m-person ASWF ($m \leq n$) if there exists $K = \{i_1,\ldots,i_m\} \subset N$, such that $\Omega^{(K)} = \Omega_{i_1} \times \Omega_{i_2} \times \ldots \times \Omega_{i_m}$ admits an m-person ASWF. (Throughout this work, if $X = (x_0;x_1,\ldots,x_n)$, then $X_{-j} = (x_0;x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)$ and $X_{-j}|y = (x_0;x_1,\ldots,x_{j-1},y,x_{j+1},\ldots,x_n)$).

Since in the definition of an ASWF we allow for a group indifference between bundles of alternatives even if not every individual is indifferent between his respective alternatives, we first investigate the significance of this, especially the possible existence of a "trivial" ASWF, in the sense that the group is indifferent between every two bundles of alternatives over which there is no unanimous agreement among the group's members, (Kalai-Muller [8], Maskin [13] and Kalai-Ritz [9] all assumed in their respective works that a social welfare function is an antisymmetric, complete and transitive ordering of the social alternatives, thus eliminating the possibility of such an ASWF).

In lemma 1 we demonstrate that there are sets of restricted domains for which no "trivial" ASWF exists, while in lemma 2 we prove that if a set of restricted domains admits an ASWF which is not "trivial," then it also admits an ASWF which allows indifferences among social alternatives only if every individual in the group is indifferent to all of them.

For $n \geq 2$, $g^n$ is a <u>Pareto unanimity collective choice rules</u> (Pareto CCR) (Sen [21]) over mixed alternatives if for every $X,Y \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$ it is XPY in $g^n(R)$ iff for every $j \in N$ it is $(x_0;x_j)r_j(y_0;y_j)$ and for at least one individual k it is also $(x_0;x_k)p(y_0;y_k)$ in $r_k$; in any other case it is XIY in $g^n(R)$.

<u>Lemma 1</u>:

If $\Omega^{(n)}$ is such that for at least one $k \in N$, there exist $r \in \Omega_k$ and $(x_0;a),(x_0;b),(x_0;c) \in (A_0;A_k)$ such that $(x_0;a)p(x_0;b)p(x_0;c)$ in r and for at least another $\ell \in N$ $|A_\ell(x_0)| \geq 2$, then the Pareto CCR defined on $\Omega^{(n)}$, is not an ASWF.

<u>Proof</u>.

Without loss of generality (w.l.o.g.) assume $k = 1$, $\ell = 2$ and let $(x_0;x),(x_0;y) \in (A_0;A_2)$ and $r_2 \in \Omega_2$ such that $(x_0;x)p(y_0;y)$ in $r_2$. Also for $j = 3,\ldots,n$ let $(x_0;x_j) \in (A_0;A_j)$ and $r_j \in \Omega_j$. Consider the mixed bundles $X = (x_0;c,x,x_3,\ldots,x_n)$, $Y = (x_0;a,y,x_3,\ldots,x_n)$,

$Z = (x_0;b,y,x_3,\ldots,x_n)$ and the profile $R = (r,r_2,\ldots,r_n)$, then by the definition of $g^n$, the Pareto CCR, it is XIYPZIX in $g^n(R)$, hence $g^n$ is not transitive and therefore it is not an ASWF. Q.E.D.

Lemma 2:

If $\Omega^{(n)}$ admits $f^n$ an n-person ASWF which is not a Pareto CCR, then $\Omega^{(n)}$ admits $h^n$ an n-person ASWF such that for every $X,Y \in (A_0;A^{(n)})$, and $R \in \Omega^{(n)}$, XIY in $h^n(R)$ iff $(x_0;x_j)i(y_0;y_j)$ in $r_j$ for every $j \in N$.

Proof.

The ordered set of individuals $(j_1,j_2,\ldots,j_n)$ is lexicographically decisive over a set of bundles of alternatives, if for any two bundles X,Y in the set, the group prefers X to Y iff for some k, $1 \leq k \leq n$, individual $j_k$ prefers his X alternative to his Y alternative, while every individual $j_\ell$, $\ell = 1,\ldots,k-1$, is indifferent between his alternatives (we also say that X is lexicographically preferred to Y by $(j_1,\ldots,j_n)$).

Since $f^n$ is not a Pareto CCR, there exist X,Y $(A_0;A^{(n)})$ and $R \in \Omega^{(n)}$ such that w.l.o.g. $(x_0;x_1)p(y_0;y_1)$ in $r_1$, $(y_0;y_2)p(x_0;x_2)$ in $r_2$, and XPY in $f^n(R)$. Define $h^n$ as follows. For every $X,Y \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$, XIY in $h^n(R)$ iff $(x_0;x_j)i(y_0;y_j)$ in $r_j$ for every $j \in N$; XPY in $h^n(R)$ if XPY in $f^n(R)$ or if XIY in $f^n(R)$ and $(2,3,\ldots,n,1)$ lexicographically prefer X to Y. It can be easily demonstrated that $h^n$ is reflexive, complete and obeys U, IIA and ND and the proof is omitted. To complete the proof that $h^n$ is an ASWF we now prove that it is transitive. (Throughout this work, to prove that a binary relation r is transitive, we prove that for every a,b,c (i) apb and bpc imply apc and (ii) aib and bic imply aic, in r (e.g., see Sen [20] for analysis of binary relations).) By the definition of $h^n$ and the transitivity of all $r_j$, if for some $X,Y,Z \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$ XIYIZ in $h^n(R)$, then also XIZ in $h^n(R)$. Thus suppose for some $X,Y,Z \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$ XPYPZ in $h^n(R)$. This implies $Xf^n(R)Yf^n(R)Z$. If also XPZ in $f^n(R)$, then by definition XPZ in $h^n(R)$. So assume XIZ in $f^n(R)$, which implies XIYIZ in $f^n(R)$. Hence X is lexicographically preferred to Y and Y is lexicographically preferred to Z by $(2,3,\ldots,n,1)$. Let k be the first individual in $(2,3,\ldots,n,1)$ such that $(x_0;x_k)p(y_0;y_k)$ in $r_k$ and let $\ell$ be the first individual in $(2,3,\ldots,n,1)$ such that $(y_0;y_\ell)p(z_0;z_\ell)$ in $r_\ell$. Let $j = \min\{k,\ell\}$, then by the transitivity of $r_j$ $(x_0;x_j)p(z_0;z_j)$ in $r_j$ and $(x_0;x_m)i(z_0;z_m)$ in $r_m$ for $m = 2,\ldots,j-1$. Since also XIZ in $f^n(R)$ then by definition XPZ in $h^n(R)$. Q.E.D.

$h^n$—an n-person ASWF satisfies the <u>positive association</u> (PA) condition, if for every $X,Y \in (A_0;A^{(n)})$ and $R,R' \in \Omega^{(n)}$ such that for every $j \in N$ $(x_0;x_j)r_j(y_0;y_j)$ implies $(x_0;x_j)r'_j(y_0;y_j)$ and $(x_0;x_j)p(y_0;y_j)$ in $r_j$ implies $(x_0;x_j)p(y_0;y_j)$ in $r'_j$, XPY in $h^n(R)$ implies XPY in $h^n(R')$.

Observe that lemma 2 can be slightly modified as to be:

<u>Lemma 3</u>

If $\Omega^{(n)}$ admits $f^n$ an n-person ASWF which satisfies the PA condition and is not a Pareto CCR, then $\Omega^{(n)}$ admits $h^n$ an n-person ASWF which obeys the PA condition and such that for every $X,Y \in (A_0;A^{(n)})$ and $R \in \Omega^{(n)}$, XIY in $h^n(R)$ iff $(x_0;x_j)i(y_0;y_j)$ in $r_j$ for every $j \in N$.

<u>Proof</u>

Define $h^n$ as in the proof of lemma 2. Suppose for some $X,Y \in (A_0;A^{(n)})$ and $R,R' \in \Omega^{(n)}$ such that for every $j \in N$, $(x_0;x_j)r_j(y_0;y_j)$ implies $(x_0;x_j)r'_j(y_0;y_j)$ and $(x_0;x_j)p(y_0;y_j)$ in $r_j$

implies $(x_0;x_j)p(y_0;y_j)$ in $r'_j$, XPY in $h^n(R)$ and $Yh^n(R')X$. Since $f^n$ satisfies PA, it cannot be the case of XPY in $f^n(R)$, thus it must be XIY in $f^n(R)$. Let k be the first individual in (2,3,...,n,1) such that $(x_0;x_k)p(y_0;y_k)$ in $r_k$. This implies $(x_0;x_k)p(y_0;y_k)$ in $r'_k$, hence it must be both YPX in $h^n(R')$ and XIY in $f^n(R')$. Let m be the first individual in (2,...,n,1) such that $(y_0;y_m)p(x_0;x_m)$ in $r'_m$, then it must be both $(y_0;y_m)p(x_0;x_m)$ in $r_m$ and $m < k$, which contradicts XPY in $h^n(R)$.

Q.E.D.

Let $\Pi^{(n)}$ be the set of all nonempty subsets of $(A_0;A^{(n)})$. An <u>n-person social choice correspondence</u> (SCC) over mixed alternatives is a correspondence $H^n: \Omega^{(n)} \times \Pi^{(n)} \to \Pi^{(n)}$. A SCC is a <u>rational social choice correspondence</u> (RSCC) if for every $B \in \Pi^{(n)}$ and $R \in \Omega^{(n)}$ it satisfies:

1. <u>Feasibility</u> (F): $H^n(R,B) \subseteq B$.
2. <u>Independence of nonoptimal alternatives</u> (INOA): For every $C \subseteq B$, if $H^n(R,B) \cap C \neq \emptyset$ then $H^n(R,C) = H^n(R,B) \cap C$.
3. <u>Unanimity</u> (P): For any $X,Y \in B$, if for every $j \in N$, $(x_0;x_j)r_j(y_0;y_j)$ and there exists $k \in N$ such that $(x_0;x_k)p(y_0;y_k)$ in $r_k$, then $Y \notin H^n(R,B)$.
4. <u>Uniqueness</u> (UQ): If $X \in H^n(R,B)$ then $Y \in H^n(R,B)$ iff $(x_0;x_j)i(y_0;y_j)$ in $r_j$ for every $j \in N$.

Since in the definition of a SCC we do not restrict the choice set to include a single social alternative only, we thus avoid the need for a tie-breaking function (needed to choose a unique social outcome among the alternatives over which the group is indifferent (see Satterthwaite [19])), however this gives rise to two different manipulability concepts.

$H^n$, an n-person SCC is point-manipulable if there exist $B \in \Pi^{(n)}$, $R,R' \in \Omega^{(n)}$ and $k \in N$ such that $r_k \neq r'_k$, for any other $j \in N$ $(j \neq k)$, $r_j = r'_j$, and there exist $X \in H^n(R',B)$ and $Y \in H^n(R,B)$, such that $(x_0;x_k)p(y_0;y_k)$ in $r_k$. $H^n$, an n-person SCC is set-manipulable if there exist $B \in \Pi^{(n)}$, $R,R' \in \Omega^{(n)}$ and $k \in N$ such that $r_k \neq r'_k$ while for any other $j \in N$ $(j \neq k)$, $r_j = r'_j$, and for every $X \in H^n(R',B)$ and for every $Y \in H^n(R,B)$ it is $(x_0;x_k)p(y_0;y_k)$ in $r_k$. If $H^n$ is not point-manipulable or not set-manipulable it is said to be point-nonmanipulable, or set-nonmanipulable respectively.

Fortunately, these two concepts are equivalent for rational social choice correspondences, as is established by the following lemma.

Lemma 4

A rational social choice correspondence is point-manipulable iff it is set-manipulable.

Proof

Part a. Suppose $H^n$, an n-person RSCC is point-manipulable by individual k. Hence there exist $B \in \Pi^{(n)}$, $R,R' \in \Omega^{(n)}$ such that $r_k \neq r'_k$, $r_j = r'_j$ for every $j \in N$ such that $j \neq k$, and for some $X \in H^n(R,B)$ and $Y \in H^n(R',B)$ it is $(y_0;y_k)p(x_0;x_k)$ in $r_k$. This implies, by UQ, that $Y \notin H^n(R,B)$ and $X \notin H^n(R',B)$. Let $C = \{X,Y\}$; since $C \subseteq B$ then by condition INOA, it is $\{X\} = H^n(R,C)$ and $\{Y\} = H^n(R',C)$, hence $H^n$ is also set-manipulable.

Part b. If $H^n$, an n-person RSCC is set-manipulable it is obviously also point-manipulable. Q.E.D.

Since in this work we are concerned with rational SCCs only, therefore from now on we use only the concept of manipulable (nonmanipulable (NM)) SCC and do not distinguish between point and set manipulability.

$H^n$, an n-person SCC is <u>corruptible</u> if there exist $B \in \Pi^{(n)}$, $R, R' \in \Omega^{(n)}$ and $k \in N$ such that $r_k \neq r'_k$, for any other $j \in N$ $(j \neq k)$ $r_j = r'_j$, for some $X \in H^n(R,B)$ and $Y \in H^n(R',B)$ it is $(x_0;x_k)i(y_0;y_k)$ in both $r_k$ and $r'_k$ and $Y \notin H^n(R,B)$. $H^n$ is <u>noncorruptible</u> (NC) if it is not corruptible. As we mentioned before, the corruptibility notion is a generalization of the bossiness concept of Satterthwaite and Sonnenschein [20]. The following example is of a corruptible but not bossy SCC.

Consider a group of three individuals with a set of three public alternatives $A = \{a,b,c\}$ and the following restricted domains of preferences: $\Omega_1 = \{r_1\}$, $\Omega_2 = \{r_2\}$, $\Omega_3 = \{r_1,r_3,r_4\}$ where apbpc in $r_1$, cpbpa in $r_2$, apbic in $r_3$ and bicpa in $r_4$. Define $\{a\} = F(R,B)$ for every $R \in \Omega_1 \times \Omega_2 \times \Omega_3$ and every $B \subseteq A$ such that $a \in B$,
$\{b\} = F((r_1,r_2,r_1),\{b,c\}) = F((r_1,r_2,r_3),\{b,c\})$ and $\{c\} = F((r_1,r_2,r_4),\{b,c\})$ and $\{b\} = F(R,\{b\})$, $\{c\} = F(R,\{c\})$ for every $R \in \Omega_1 \times \Omega_2 \times \Omega_3$. It is easy to verify that F is a nonmanipulable but corruptible (by individual 3) rational social choice correspondence. The question of whether or not there are restricted domains which admit nonmanipulable and corruptible social choice correspondences but do not admit nonmanipulable and noncorruptible social choice correspondences is still an open one.

For any $B \in \Pi^{(n)}$, let $B_j = \{(x_0;x) \in (A_0;A_j) | \text{there exists } X_{-j} | x \text{ in } B\}$. $H^n$, an n-person SCC is <u>dictatorial</u> (D) if there exists $j \in N$, such that for every $B \in \Pi^{(n)}$, $R \in \Omega^{(n)}$ and $X \in H^n(R,B)$ $(x_0;x_j)r_j(y_0;y_j)$ for every $(y_0;y_j) \in B_j$. $H^n$ is <u>nondictatorial</u> (ND) if it is not dictatorial. A SCC is <u>Gibbard-Satterthwaite type social choice correspondence</u> (GSSCC) if it is a nonmanipulable, nondictatorial and rational social choice correspondence. If it is also noncorruptible then it is <u>an extended GSSCC</u>.

We say that $\Omega^{(n)}$ admits an m-person GSSCC over mixed alternatives if there exists $K \subseteq N$, $K = \{i_1,i_2,\ldots,i_m\}$, such that $\Omega^{(K)} = \Omega_{i_1} \times \Omega_{i_2} \times \ldots \times \Omega_{i_m}$ admits an m-person GSSCC.

## 2. <u>Equivalence Between Arrow Social Welfare Functions and Noncorruptible Gibbard-Satterthwaite Type Social Choice Correspondences</u>

<u>Theorem 1</u>

Let $(A_0;A^{(n)})$ be a finite set, then for every n and m such that $n \geq m \geq 2$, $\Omega^{(n)}$ admits an m-person noncorruptible Gibbard-Satterthwaite type SCC iff it admits an m-person Arrow SWF which satisfies the positive association condition and which is not a Pareto CCR.

<u>Proof</u>

<u>Part a.</u>

Assume that $\Omega^{(n)}$ admits $H^m$ an m-person noncorruptible GSSCC, and w.l.o.g. assume that $K = \{1,2,\ldots,m\}$, i.e., $\Omega^{(K)} = \Omega_1 \times \ldots \times \Omega_m$ admits $H^m$, thus $\Omega^{(K)}$ and $\Omega^{(m)}$ will be used interchangeably. Define $h^m\colon \Omega^{(m)} \to \Delta^{(m)}$ as follows. For every $X,Y \in (A_0;A^{(m)})$ and $R \in \Omega^{(m)}$,

$Xh^m(R)Y$ iff $X \in H^m(R,\{X,Y\})$. We prove that $h^m$ is an ASWF in three steps: 1. $h^m$ is a SF, 2. $h^m$ is a SWF, and 3. $h^m$ is an ASWF.

Step 1. Since for no $B \in \Pi^{(m)}$ and $R \in \Omega^{(m)}$ is $H^m(R,B)$ empty, therefore for every $R \in \Omega^{(m)}$ and every $X,Y \in (A_0;A^{(m)})$, $X \in H^m(R,\{X\})$ and either $X \in H^m(R,\{X,Y\})$ or $Y \in H^m(R,\{X,Y\})$ (or both). Hence $h^m$ is both reflexive and complete. Suppose for some $X,Y,Z \in (A_0;A^{(m)})$ and $R \in \Omega^{(m)}$ $XIYIZ$ in $h^m(R)$. This implies both $\{X,Y\} = H^m(R,\{X,Y\})$ and $\{Y,Z\} = H^m(R,\{X,Y\})$. Then by the UQ condition it must be $(x_0;x_j)i(y_0;y_j)i(z_0;z_j)$ in $r_j$ for every $j = 1,\ldots,m$, which implies $\{X,Z\} = H^m(R,\{X,Z\})$ and therefore $XIZ$ in $h^m(R)$. Suppose for some $X,Y,Z \in (A_0;A^{(m)})$ and $R \in \Omega^{(m)}$ it is $XPYPZ$ in $h^m(R)$. Let $B = \{X,Y,Z\}$, then $XPY$ and $YPZ$ imply both $Y \notin H^m(R,\{X,Y\})$ and $Z \notin H^m(R,\{Y,Z\})$, and by INOA $Y \notin H^m(R,B)$ and $Z \notin H^m(R,B)$. Therefore it must be $\{X\} = H^m(R,B)$, by INOA $\{X\} = H^m(R,\{X,Z\})$, and by definition $XPZ$ in $h^m(R)$.

Step 2. UQ and P for $H^m$, together imply that $h^m$ satisfies unanimity. Suppose $h^m$ doesn't satisfy IIA, i.e., (w.l.o.g.) there exist $X,Y \in (A_0;A^{(m)})$, $R,S \in \Omega^{(m)}$ such that for every $j \in K$, $(x_0;x_j)r_j(y_0;y_j)$ iff $(x_0;x_j)s_j(y_0;y_j)$, $(y_0;y_j)r_j(x_0;x_j)$ iff $(y_0;y_j)s_j(x_0;x_j)$, $Xh^m(R)Y$ and $YPX$ in $h^m(S)$, which in turn implies $\{Y\} = H^m(S,\{X,Y\})$. By UQ this implies that not for every $j \in K$, $(x_0;x_j)i(y_0;y_j)$ in $s_j$ and $r_j$. This together with $X \in H^m(R,\{X,Y\})$ imply $\{X\} = h^m(R,\{X,Y\}$. Applying the method used by Schmeidler and Sonnenschein [20], let $T_0 = R$ and for $j = 1,\ldots,m$ let $T_j = (s_1,\ldots,s_j,r_{j+1},\ldots,r_m)$; w.l.o.g. assume that for

$j = 1,\ldots,m_1$ $(x_0;x_j)p(y_0;y_j)$ in $r_j$ and $s_j$, for $j = m_1+1,\ldots,m_2$ $(y_0;y_j)p(x_0;x_j)$ in $r_j$ and $s_j$ and for $j = m_2+1,\ldots,m$ $(x_0;x_j)i(y_0;y_j)$ in $r_j$ and $s_j$. Let k be such that $\{Y\} = H^m(T_k,\{X,Y\})$ and $\{X\} = H^m(T_j,\{X,Y\})$ for $j = 1,\ldots,(k-1)$. Obviously $1 < k \leq m$. If $k \leq m_2$ then $H^m$ is manipulable by individual k, a contradiction. Hence $m_2 < k \leq m$. But then $H^m$ is corruptible by individual k, again a contradiction, hence $h^m$ obeys IIA.

Step 3. Suppose there exists individual $k \in K$ such that for every $R \in \Omega^{(m)}$ and every $X,Y \in (A_0;A^{(m)})$, $(x_0;x_k)p(y_0;y_k)$ in $r_k$ implies XPY in $h^m(R)$. Since $H^m$ is not dictatorial then there exists $B \in \Pi^{(m)}$ and $R \in \Omega^{(m)}$ such that for some $X,Y \in B$, $(y_0;y_k)p(x_0;x_k)$ in $r_k$ and $X \in H^m(R,B)$. By INOA this implies $X \in H^m(R,\{X,Y\})$ and, therefore by UQ, $Y \notin H^m(R,\{X,Y\})$, hence XPY in $h^m(R)$, a contradiction. This completes the proof that $h^m$ is indeed an Arrow SWF. Since $H^m$ satisfies UQ, therefore $h^m$ is not a Pareto CCR. Suppose $h^m$ doesn't obey PA, then there exist $X,Y \in (A_0;A^{(m)})$ and $R,R' \in \Omega^{(m)}$ such that for every $j \in K$, $(x_0;x_j)r_j(y_0;y_j)$ implies $(x_0;x_j)r'_j(y_0;y_j)$, $(x_0;x_j)p(y_0;y_j)$ in $r_j$ implies $(x_0;x_j)p(y_0;y_j)$ in $r'_j$, XPY in $h^m(R)$ but $Yh^m(R')X$. XPY in $h^m(R)$ implies that for at least one individual $k \in K$, it is $(x_0;x_k)p(y_0;y_k)$ in $r_k$ (and therefore also in $r_k$), thus $\{X,Y\} \neq H^m(R',\{X,Y\})$ and we may assume YPX in $h^m(R')$. Since $h^m$ obeys IIA there must be at least one individual $\ell \in N$ who changed his preferences between $(x_0;x_\ell)$ and $(y_0;y_\ell)$. Suppose exactly one individual $\ell$ changed his preferences, and w.l.o.g. assume $\ell = 1$, thus we may assume $r_j = r'_j$ for $j = 2,\ldots,m$.

Case 1. $(y_0;y_1)p(x_0;x_1)$ in $r_1$ and $(x_0;x_1)r_1(y_0;y_1)$. In this case $H^m$ is manipulable by individual 1 when his "true" preference is $r_1$, a contradiction.

Case 2. $(y_0;y_1)i(x_0;x_1)$ in $r_1$ and $(x_0;x_1)p(y_0;y_1)$ in $r_1'$. Then $H^m$ is manipulable by individual 1 when his "true" preference is $r_1'$, again a contradiction.

Suppose more than one individual changed his mind. W.l.o.g. assume that these individuals are $1,2,\ldots,\ell$ (notice that $\ell < m$). Using the Schmeidler-Sonnenscheim [21] method, let $T_0 = R$ and $T_j = (r_1',\ldots,r_j',r_{j+1},\ldots,r_m)$ for $j = 1,\ldots,\ell$. Since $\{X\} = H^m(T_0,\{X,Y\})$ and $\{Y\} = H^m(T_\ell,\{X,Y\})$, let k be the first integer such that $\{Y\} = H^m(T_k,\{X,Y\})$, then by cases 1 and 2 above, $H^m$ is manipulable by individual k, a contradiction. Hence $h^m$ obeys PA.

Part b.

Assume that $\Omega^{(m)}$ admits $h^m$ an m-person Arrow SWF which obeys the PA condition and which is not a Pareto CCR, and w.l.o.g. assume $K = \{1,2,\ldots,m\}$, i.e., $\Omega^{(m)}$ admits $h^m$. By lemma 4 we may assume that for every $X,Y \in (A_0;A^{(m)})$ and $R \in \Omega^{(m)}$, $XIY$ in $h^m(R)$ iff $(x_0;x_j)i(y_0;y_j)$ in $r_j$ for every $j \in K$. Define $H^m$ as follows.

For every $B \in \Pi^{(m)}$ and $R \in \Omega^{(m)}$, $X \in H^m(R,B)$ iff $X \in B$ and $Xh^m(R)Y$ for every $Y \in B$. We first prove that $H^m$ is a RSCC and then that it is also a NC GSSCC. Since $(A_0;A^{(m)})$ is a finite set, so is also every $B \in \Pi^{(m)}$ and therefore for every B there exists $X \in B$ such that $Xh^m(R)Y$ for every $Y \in B$ ($h^m$ is complete and transitive), hence $H^m$ obeys feasibility. Suppose there exist $C,B \in \Pi^{(m)}$ and $R \in \Omega^{(m)}$ such that $C \subset B$, $H^m(R,B) \cap C \neq \emptyset$ and $H^m(R,C) \neq H^m(R,B) \cap C$.

Case 1. There exists $X' \in H^m(R,C)$ such that $X' \notin H^m(R,B)$. Since $H^m(R,B) \cap C \neq \emptyset$ there exists another $Y \in C$ such that $Y \in H^m(R,B)$. Since $X' \notin H^m(R,B)$, then it must be $YPX'$ in $h^m(R)$, and since $Y \in C$, then by definition $X \notin H^m(R,C)$, a contradiction.

Case 2. There exists $X' \in H^m(R,B) \cap C$ and $X' \notin H^m(R,C)$. Since $H^m(R,C) \neq \emptyset$, there exists $Y \in C$ such that $Y \in H^m(R,C)$; this together with $X' \notin H^m(R,C)$ imply $YPX'$ in $h^m(R)$. By definition, since $Y,X' \in B$, it implies $X' \notin H^m(R,B)$, a contradiction. Hence $H^m$ obeys INOA.

Since $h^m$ obeys U, then by definition $H^m$ obeys P. The assumption that XIY in $h^m(R)$ if and only if every individual is indifferent between X and Y, guarantees that $H^m$ obeys UQ. This completes the proof that $H^m$ is a RSCC. Suppose $H^m$ is manipulable. Then there exists $R,R' \in \Omega^{(m)}$, $B \in \Pi^{(m)}$ and $X,Y \in B$ such that w.l.o.g. $r_1 \neq r_1'$, $r_j = r_j'$ for every $j = 2,\ldots,m$, $X \in H^m(R,B)$, $Y \in H^m(R',B)$ and $(y_0;y_1)p(x_0;x_1)$ in $r_1$. $(y_0;y_1)p(x_0;x_1)$ in $r_1$ implies both $Y \notin H^m(R,B)$ (because of UQ) and the existence of $k \in K$ $(k > 1)$ such that $(x_0;x_k)p(y_0;y_k)$ in $r_k$ (otherwise P is violated); since $r_k = r_k'$, then it implies $X \notin H^m(R',B)$. Together these imply XPY in $h^m(R)$ and YPX in $h^m(R')$, which either contradicts the assumption that $h^m$ obeys IIA or the assumption that it obeys PA, hence $h^m$ is nonmanipulable. Suppose $H^m$ is corruptible, then there exist $R,R' \in \Omega^{(m)}$, $B \in \Pi^{(m)}$ and $X,Y \in B$ such that w.l.o.g. $r_1 \neq r_1'$, $r_j = r_j'$ for $j = 2,\ldots,m$, $X \in H^m(R,B)$, $Y \in H^m(R',B)$, $Y \notin H^m(R,B)$ and $(x_0;x_1)i(y_0;y_1)$ in both $r_1$ and $r_1'$. This implies both XPY in $h^m(R)$ and YPX in $h^m(R')$, which contradicts the assumption that $h^m$ satisfies

IIA, hence $H^m$ is noncorruptible; which completes the proof that $h^m$ is a noncorruptible GSSCC.

Q.E.D.

## 3. Characterization of the Sets of Restricted Domains Admitting Two-Person Arrow Social Welfare Functions

We first characterize all the sets of restricted domains of preferences which admit two-person Arrow SWFs and then, in light of theorem 1, we show that we also have characterized all the sets which admit two-person noncorruptible Gibbard-Satterthwaite type SCC. Let $F = \{(X,Y) \in (A_0;A^{(2)}) \times (A_0;A^{(2)})$ there exists $r \in \Omega_1$ such that $(x_0;x_1)p(y_0;y_1)$ in $r_1\}$, F contains all the ordered pairs of social alternatives where the first individual can strictly prefer the alternative allocated to him in the first social alternative, to his second mixed alternative. We refer to F as the <u>feasible set</u>. Let $C = \{(X,Y)$ F there exists $r' \in \Omega_2$ such that $(y_0;y_2)p(x_0;x_2)$ in $r'\}$. C contains all the ordered pairs in the feasible set, over which the second individual can create conflict of interests with the first individual. C is the <u>conflict set</u>. We say that $D \subseteq (A_0;A^{(2)}) \times (A_0;A^{(2)})$ is <u>closed under decisive implications</u> (CUDI) if the following conditions hold for every $X,Y,Z \in (A_0;A^{(2)})$.

G1a. If for some $r_1 \in \Omega_1$ and $r_2 \in \Omega_2$, $(x_0;x_1)p(y_0;y_1)r_1(z_0;z_1)$, $(y_0;y_2)r_2(z_0;z_2)p(x_0;x_2)$ and $(X,Y) \in D$ then $(X,Z) \in D$.

G1b. If for some $r \in \Omega_1$, $(x_0;x_1)p(y_0;y_1)p(z_0;z_1)$ in r and $(X,Y),(Y,Z) \in D$, then $(X,Z) \in D$.

G2a. If for some $r_1 \in \Omega_1$ and $r_2 \in \Omega_2$, $(x_0;x_1)r_1(y_0;y_1)p(z_0;z_1)$, $(z_0;z_2)p(x_0;x_2)r_2(y_0;y_2)$ and $(Y,Z) \in D$ then $(X,Z) \in D$.

G2b. If for some $r \in \Omega_2$, $(x_0;x_2)p(y_0;y_2)p(z_0;z_2)$ in r and $(Z,X) \in D$, then either $(Z,Y) \in D$ or $(Y,X) \in D$.

We say that $\Omega^{(2)}$ is <u>decomposable</u> if there is a D which is closed under decisive implications. $\Omega^{(2)}$ has a <u>nontrivial decomposition</u> if it is decomposable with a D such that $(F-C) \subsetneq D \subsetneq F$.

Intuitively D contains all the ordered pairs of social alternatives over which the first individual is decisive. Thus $D \subsetneq F$ guarantees that he is not a dictator while $(F-C) \subsetneq D$ guarantees that the second individual is not a dictator. We say that $\Omega^{(n)}$ has a <u>nontrivial decomposition</u> if there exist $j,k \in N$ such that $\Omega_j \times \Omega_k$ has a nontrivial decomposition.

<u>Theorem 2</u>

$\Omega^{(n)}$ admits a two-person Arrow SWF which is not a Pareto CCR, over mixed alternatives, iff it has a nontrivial decomposition.

<u>Proof</u>:

<u>Part a</u>.

Assume that $\Omega^{(n)}$ has a nontrivial decomposition. W.l.o.g. assume that $j = 1$, $k = 2$ are such that $\Omega^{(2)} = \Omega_1 \times \Omega_2$ has the nontrivial decomposition. Define $h^2$ as follows. For every $X,Y \in (A_0;A^{(2)})$ and $R \in \Omega^{(2)}$,

a. $XIY$ in $h^2(R)$ iff $(x_0;x_1)i(y_0;y_1)$ and $(x_0;x_2)i(y_0;y_2)$ in $r_1$ and $r_2$ respectively;

b. $XPY$ in $h^2(R)$ if

(i) $(x_0;x_1)r_1(y_0;y_1)$, $(x_0;x_2)r_2(y_0;y_2)$ and either $(x_0;x_1)p(y_0;y_1)$ or $(x_0;x_2)p(y_0;y_2)$ (or both), or

(ii) $(x_0;x_1)p(y_0;y_1)$ in $r_1$ $((y_0;y_2)p(x_0;x_2)$ in $r_2)$ and $(X,Y) \in D$, or

(iii) $(x_0;x_2)p(y_0;y_2)$ in $r_2$ $((y_0;y_1)p(x_0;x_1)$ in $r_1)$ and $(Y,X) \notin D$.

To prove that $h^2$ is a two person Arrow SWF, first observe that part (a) of the definition implies $Xh^2(R)X$ for any $X \in (A_0;A^{(2)})$ and every $R \in \Omega^{(2)}$, thus $h^2$ is reflexive. Then notice that if $h^2$ is not a complete order, then there exist $X,Y \in (A_0;A^{(2)})$ and $R \in \Omega^{(2)}$ such that neither $Xh^2(R)Y$ nor $Yh^2(R)X$. Since it is not $XIY$ in $h^2(R)$, then it must either be the case of $(x_0;x_1)p(y_0;y_1)$ and $(y_0;y_2)p(x_0;x_2)$, or be the case of $(y_0;y_1)p(x_0;x_1)$ and $(x_0;x_2)p(y_0;y_2)$ in $r_1$ and $r_2$ respectively. W.l.o.g. assume $(x_0;x_1)p(y_0;y_1)$ in $r_1$ and $(y_0;y_2)p(x_0;x_2)$ in $r_2$ (since these two cases are symmetric to each other). This together with not $Xh^2(R)Y$ imply $(X,Y) \notin D$, but by part (b.(iii)) in the definition, it then must be $YPX$ in $h^2(R)$, a contradiction. Parts (a) and (b.(i)) imply that $h^2$ obeys the unanimity condition, and since $h^2$ was defined for pairs of social alternatives only, it also obeys the IIA condition. Thus to demonstrate that $h^2$ is at least a social welfare function, it is only left to prove that it obeys transitivity. Suppose it is not a transitive relation, and there exist $X,Y,Z \in (A_0;A^{(2)})$ and $R \in \Omega^{(2)}$ such $XPYPZh^2(R)X$. We distinguish among the following three cases.

Case 1. XPY because of unanimity $((x_0;x_1)r_1(y_0;y_1), (x_0;x_2)r_2(y_0;y_2))$. If also YPZ because of unanimity, then $(x_0;x_1)r_1(z_0;z_1)$, $(x_0;x_2)r_2(z_0;z_2)$ and either 1 or 2 (or both) strictly prefers X to Z. By definition this implies XPZ in $h^2(R)$, a contradiction. If $(y_0;y_1)p(z_0;z_1)$ and $(z_0;z_2)p(y_0;y_2)$ in $r_1$ and $r_2$ respectively, then YPZ implies $(Y,Z) \in D$; $(x_0;x_1)r_1(y_0;y_1)p(z_0;z_1)$ and $Zh^2(R)X$ imply $(z_0;z_2)p(x_0;x_2)$ in $r_2$, and together with condition (G2a), this implies $(X,Z) \in D$. Then by definition XPZ in $h^2(R)$, a contradiction. If $(z_0;z_1)p(y_0;y_1)$ and $(y_0;y_2)p(z_0;z_2)$ in $r_1$ and $r_2$ respectively, then YPZ implies both $(Z,Y) \notin D$ and $(x_0;x_2)r_2(y_0;y_2)p(z_0;z_2)$. $Zh^2(R)X$ implies both $(z_0;z_1)p(x_0;x_1)r_1(y_0;y_1)$ and $(Z,X) \in D$, and by condition (G1a), this implies $(Z,Y) \in D$, a contradiction.

Case 2. XPY in $h^2(R)$ because of $(x_0;x_1)p(y_0;y_1)$ in $r_1$ and $(X,Y) \in D$. If YPZ because of unanimity, then $Zh^2(R)X$ and $(x_0;x_1)p(y_0;y_1)r_1(z_0;z_1)$ imply $(y_0;y_2)r_2(z_0;z_2)p(x_0;x_2)$ and $(X,Z) \notin D$. But $(X,Y) \in D$ and the above $r_1$ and $r_2$ imply, by condition (G1a), $(X,Z) \in D$, a contradiction. If $(y_0;y_1)p(z_0;z_1)$ and $(z_0;z_2)p(y_0;y_2)$ in $r_1$ and $r_2$ respectively, then $(Y,Z) \in D$, and by condition (G1b), $(X,Z) \in D$. Then, by definition, $(x_0;x_1)p(z_0;z_1)$ implies XPZ in $h^2(R)$, a contradiction. Thus it must be $(z_0;z_1)p(y_0;y_1)$ and $(y_0;y_2)p(z_0;z_2)$ in $r_1$ and $r_2$ respectively, in which case YPZ implies $(Z,Y) \notin D$. If also $(x_0;x_1)p(z_0;z_1)$, then $Zh^2(R)X$ implies $(y_0;y_2)p(z_0;z_2)p(x_0;x_2)$ in $r_2$ and $(X,Z) \notin D$. But then $r_2$, $(X,Y) \in D$, $(X,Z) \notin D$ and $(Z,Y) \notin D$ together contradict condition (G2b). So suppose $(z_0;z_1)r_1(x_0;x_1)p(y_0;y_1)$. If $(y_0;y_2)p(z_0;z_2)r_2(x_0;x_2)$ then by condition (G2a), $(X,Y) \in D$ implies $(Z,Y) \in D$, a contradiction. This leaves as the only possible case, $(y_0;y_2)p(x_0;x_2)p(z_0;z_2)$ in $r_2$. Then $Zh^2(R)X$ implies both $(z_0;z_1)p(x_0;x_1)$ and $(Z,X) \in D$. Then, by condition (G1b), it is $(Z,Y) \in D$, again a contradiction.

Case 3. XPY because of $(y_0;y_1)p(x_0;x_1)$, $(x_0;x_2)p(y_0;y_2)$ in $r_1$ and $r_2$ respectively and $(Y,X) \notin D$. If $Zh^2(R)X$ because of unanimity then $(z_0;z_2)r_2(x_0;x_2)p(y_0;y_2)$ implies both $(y_0;y_1)p(z_0;z_1)r_1(x_0;x_1)$ and $(Y,Z)$ D, and by condition (G1a), together they imply $(Y,X)$ D, a contradiction. If $Zh^2(R)X$ because of $(x_0;x_1)p(z_0;z_1)$ in $r_1$, $(z_0;z_2)p(x_0;x_2)$ in $r_2$ and $(X,Z) \notin D$, then YPZ implies $(Y,Z) \in D$, and together with $r_2$, $(X,Z) \notin D$, and condition (G2b) imply $(Y,X) \in D$, a contradiction. This leaves $Zh^2(R)X$ because of $(z_0;z_1)p(x_0;x_1)$ in $r_1$, $(x_0;x_2)p(z_0;z_2)$ in $r_2$ and $(Z,X) \in D$. Then, either YPZ because of unanimity which together with condition (G2a) implies, $(Y,X) \in D$, YPZ because of $(y_0;y_1)p(z_0;z_1)$ in $r_1$, $(z_0;z_2)p(y_0;y_2)$ in $r_2$ and $(Y,Z) \in D$ which together with condition (G1b), again implies $(Y,X) \in D$, or YPZ because of $(z_0;z_1)p(y_0;y_1)$, $(y_0;y_2)p(z_0;z_2)$ and $(Z,Y) \notin D$, and again $(Y,X) \in D$ (by condition (G2b)). In all these three cases we reach a contradiction. Hence $h^2$ is transitive and therefore a social welfare function. To complete the proof that $h^2$ is an Arrow SWF, notice that since $\Omega^{(2)}$ has a nontrivial decomposition, there exist $X',Y' \in (A_0;A^{(2)})$ such that $(X',Y') \in C$ and $(X',Y') \notin D$ and $X'',Y'' \in (A_0;A^{(2)})$ such that $(X'',Y'') \in C \cap D$. These two cases imply that $h^2$ is a ND social welfare function. $h^2$ is not a Pareto CCR because of parts (b.ii) and (b.iii) of the definition and the fact that $\Omega^{(2)}$ has a nontrivial decomposition.

Part b.

Suppose $\Omega^{(n)}$ admits a two-person Arrow SWF which is not a Pareto CCR. W.l.o.g. assume $\Omega^{(2)} = \Omega_1 \times \Omega_2$ admits that 2-person ASWF. Define $D = \{(X,Y) \in F | (x_0;x_1)p(y_0;y_1)$ in $r_1$ implies XPY in $h^2(R)\}$. Since $h^2$ is not a Pareto CCR (and by lemma 2), then there exist $X',Y' \in (A_0;A^{(2)})$ and $R' \in \Omega^{(2)}$ such that w.l.o.g. $(x_0';x_1')p(y_0';y_1')$ in $r_1'$, $(y_0';y_2')p(x_0';x_2')$ in $r_2'$ and X'PY' in $h^2(R')$. Then either by IIA or by U $(x_0';x_1')p(y_0';y_1')$ in $r_1$ will always imply X'PY' in $h^2(R)$, thus $(X',Y') \in D$ and $(F-C) \subsetneq D$. $h^2$ is also ND, then there exist $R'' \in \Omega^{(2)}$ and $X'',Y'' \in (A_0;A^{(2)})$ such that $(y_0'';y_1'')p(x_0'';x_1'')$ in $r_1''$, $(x_0'';x_2'')p(y_0'';y_2'')$ in $r_2''$, and $X''h^2(R'')Y''$. Hence $(Y'',X'') \in C$ and $(Y'',X'') \notin D$, therefore $(F-C) \subsetneq D \subsetneq F$. To complete the proof that $\Omega^{(2)}$ has a nontrivial decomposition, we show that D is closed under decisive implications. Suppose there exist $r_1' \in \Omega_1$, $r_2' \in \Omega_2$ and $X,Y,Z \in (A_0;A^{(2)})$ such that $(x_0;x_1)p(y_0;y_1)r_1'(z_0;z_1)$, $(y_0;y_2)r_2'(z_0;z_2)p(x_0;x_2)$ and $(X,Y) \in D$. This implies both XPY in $h^2(R')$ and $Yh^2(R')Z$. Then by transitivity XPZ in $h^2(R')$ and by either IIA or U, XPZ in $h^2(R)$ for any R such that $(x_0;x_1)p(z_0;z_1)$ in $r_1$, hence $(X,Z) \in D$ which proves that D obeys condition (G1a). By using similar arguments it is straightforward to show that D also obeys (G1b) and (G2a), thus to complete the proof we show that D also obeys (G2b). Suppose there exist $X,Y,Z \in (A_0;A^{(2)})$ and $r_2' \in \Omega_2$ such that $(x_0;x_2)p(y_0;y_2)p(z_0;z_2)$ in $r_2'$, $(Z,X) \in D$ and $(Y,X) \notin D$. $(Y,X) \notin D$ implies XPY in $h^2(R)$ for every $R \in \Omega^{(2)}$ such that $(x_0;x_2)p(y_0;y_2)$ in $r_2$. $(Z,X) \in D$ implies ZPX in $h^2(R)$ for every $R \in \Omega^{(2)}$ such that $(z_0;z_1)p(x_0;x_1)$ in $r_1$ (and since $(Z,X) \in D \subseteq F$, there exists such $r_1$). Choose $R \in \Omega^{(2)}$ such that $r_2$ is $r_2'$ and $(z_0;z_1)p(x_0;x_1)$ in $r_1$ and then ZPXPY in $h^2(R)$. By transitivity

ZPY in $h^2(R)$, and since $(y_0;y_2)p(z_0;z_2)$ in $r_2$, then it must be $(Z,Y) \in D$.

Part c.

Suppose the only ASWFs $\Omega^{(n)}$ admits, are Pareto CCRs, then there are no $j,k \in N$, such that $\Omega_j \times \Omega_k$ has a nontrivial decomposition, since otherwise it is possible, as was done in part (a) to define a two-person ASWF which is not a Pareto CCR. Q.E.D.

Corollary 1

$\Omega^{(n)}$ admits a two-person noncorruptible Gibbard-Satterthwaite type SCC iff it is has a nontrivial decomposition.

Proof

Since any two-person Arrow SWF also obeys the PA condition, then by theorem 1 and theorem 2 together, the proof is immediate. Q.E.D.

The following corollary demonstrates that some of the previous characterizations of domains admitting 2-person ASWFs, are special cases of the characterization introduced in this paper.

Corollary 2

(i) If $|A_0| = 1$, then conditions (G1a) to (G2b) are reduced to conditions (D1a) to (D2c) of Ritz [17];

(ii) if $|A_0| = 1$, $A_j = A$, $\Omega_j = \Omega$ for every $j \in N$ and no indifferences among alternatives are allowed, then conditions (G1a)to (G2b) are reduced to conditions (DI1) and (DI2) of Kalai-Ritz [9];

(iii) if for every $j \in N$ and every $a \in A_0$, $A_j(a) = A$, $\Omega_j = \Omega$, $|A| = 1$ and no indifferences among alternatives are allowed, then conditions (G1a) to (G2b) are reduced to conditions (DI1a) to (DI2b) of Kalai-Muller [8].

Proof

(i) Since $|A_0| = 1$ (say $A_0 = \{a\}$), then instead of $(a;x_1)$ we use $x_1$ and instead of $X = (a;x_1,\ldots,x_n)$ we use $X = (x_1,\ldots,x_n)$. Conditions (D1a) to (D2c) are as follows.

D1a. If for some $r \in \Omega_1$, $x_1py_1rz_1$ and $(X,Y) \in D$ then $(X,(z_1,y_2)) \in D$.

D1b. If for some $r \in \Omega_1$, $x_1ry_1pz_1$ and $(Y,Z) \in D$ then $((x_1,y_2),Z) \in D$.

D1c. If for some $r \in \Omega_1$, $x_1py_1pz_1$ and $(X,Y),(Y,Z) \in D$, then $(X,Z) \in D$.

D2a. If for some $r \in \Omega_2$, $x_2py_2rz_2$ and $(Z,X) \in D$ then $((z_1,y_2),X) \in D$.

D2b. If for some $r \in \Omega_2$, $x_2ry_2pz_2$ and $(Z,X) \in D$ then $(Z,(x_1,y_2)) \in D$.

D2c. If for some $r \in \Omega_2$, $x_2py_2pz_2$ and $(Z,X) \in D$ then either $(Z,Y) \in D$ or $(Y,X) \in D$ (or both).

If $(X,Y) \in D$ and $(X,Y) \notin C$ then $x_1py_1rz_1$ implies $(X,(z_1,y_2)) \in D$; if $(X,Y) \in C$ then condition (G1a) implies $(X,(z_1,y_2)) \in D$. In both cases condition (D1a) is derived. Similarly (G2a) implies (D1b), condition (G1b) implies (D1c), condition (G2a) implies (D2a), condition (G1a) implies (D2b) and condition (G2b) implies condition (D2c).

(ii) Conditions (DI1) and (DI2) are as follows.

DI1. If for some $p \in \Omega$, $x_1py_1pz_1$, $(X,Y) \in D$ and $(Y,Z) \in D$, then $(X,Z) \in D$.

DI2. If for some $p \in \Omega$, $x_2py_2pz_2$ and $(Z,X) \in D$, then $(Z,Y) \in D$ or $(Y,X) \in D$.

Condition (G1b) implies condition (DI1) while condition (G2b) implies (DI2).

(iii) Since for every $j \in N$ and every $a \in A_0$, $A_j(a) = A$ and $|A| = 1$ say $A = \{x\}$, then $(A_0;A_j) = (A_0;x)$ for every $j \in N$. Thus instead of $X = (x_0;x,\ldots,x)$ we just use $x_0$ etc.

Conditions (DI1a) to (DI2b).

DI1. If there are $p_1,p_2 \in \Omega$ such that $xpypz$ and $ypzpx$ then

DI1a. $(x,y) \in D$ implies that $(x,z) \in D$, and

DI1b. $(z,x) \in D$ implies that $(y,x) \in D$.

DI2. If there is a $p \in \Omega$ such that $xpypz$ then

DI2a. $(x,y) \in D$ and $(y,z) \in D$ imply $(x,z) \in D$, and

DI2b. $(z,x) \in D$ implies that either $(y,x) \in D$ or $(z,y) \in R$ (or both).

Condition (G1a) implies (DI1a), condition (G2a) implies condition (DI1b) (when $X = y$, $Y = z$ and $Z = x$). Condition (G1b) implies condition (DI2a) and (G2b) implies (DI2b). Q.E.D.

To show the usefulness of theorem 2, we discuss two examples.

<u>Example 1.</u> <u>Restricted domains containing inseparable pairs of alternatives</u>

For $j \in N$, $(x_0;x),(y_0;y) \in (A_0;A_j)$ is an <u>inseparable pair of alternatives in $\Omega_j$</u> (denoted by $((x_0;x),(y_0;y))$ if there exists $r \in \Omega_j$ such that $(x_0;x)p(y_0;y)$ in $r$, and for no $(z_0;z) \in (A_0;A_j)$ and no $r' \in \Omega_j$ it is $(x_0;x)p(z_0;z)p(y_0;y)$ in $r'$. (See Kalai-Ritz [9] and Ritz [17] for discussion of inseparable pairs of private alternatives and Kalai-Ritz

[10] for the case of inseparable pairs of public alternatives. Kim-Roush [11] proved for the case of public alternatives, no indifferences allowed and all individuals have the same restricted domain of preferences that the domains containing exactly one inseparable pair are the largest restricted domains that still admit Arrow SWFs).

Assume that for every $j \in N$ $(n = 2)$ and every $a \in A_0$, $A_j(a) = A_1$, $\Omega_j = \Omega$ and $\Omega$ contains only antisymmetric, complete and transitive binary relations. Suppose $((\bar{x}_0;\bar{x}),(\bar{y}_0;\bar{y}))$ is an inseparable pair in $\Omega$.

Claim 1

If $|A_0 \times A_1| > 2$ or if $|A_0 \times A_1| = 2$ and $|\Omega| = 2$, then $\Omega\times\Omega$ has a non-trivial decomposition.

Proof

Define $D = \{(X,Y) \in F$ such that either $(y_0;y_2) \neq (\bar{x}_0;\bar{x})$ or $(x_0;x_2) \neq (\bar{y}_0;\bar{y})\}$.

Obviously $D \subseteq F$. If $|A_0 \times A_1| = 2$ and $|\Omega| = 2$ then actually $\{A_0 \times A_1\} = \{(\bar{x}_0;\bar{x}),(\bar{y}_0;\bar{y})\}$, $\Omega = \{p_1,p_2\}$ s.t. $(\bar{x}_0;\bar{x})p_1(\bar{y}_0;\bar{y})$ and $(\bar{y}_0;\bar{y})p_2(\bar{x}_0;\bar{x})$. Hence $(\bar{Y},\bar{X}) \notin D$ and $(\bar{X},\bar{Y}) \in D$ where $\bar{X} = (x_0;\bar{x},\bar{x})$ and $\bar{Y} = (\bar{y}_0;\bar{y},\bar{y})$, thus $F-C \subsetneq D \subsetneq F$. If $|A_0 \times A_1| > 2$ then there exists $(z_0;z) \in A_0 \times A_1$ such that $(\bar{x}_0;\bar{x}) \neq (z_0;z) \neq (\bar{y}_0,\bar{y})$ and again it is easy to demonstrate that $D \neq \emptyset$, $D \subsetneq F$ and $F-C \subsetneq D$. To prove that D is closed under decisive implications is straightforward and the proof is omitted.

Q.E.D.

Example 2. Decentralization.

Let $|A_0| \geq 2$. Assume that for every $j \in N$ ($n = 2$), the only restrictions on $\Omega_j$ are that for every $a,b \in A_0$ and every $r \in \Omega_j$, if $(a;\bar{x})p(b;\bar{y})$ in r for some $\bar{x} \in A_j(a)$ and $\bar{y} \in A_j(b)$, then $(a;\bar{x})p(b;\bar{y})$ in r for every $x \in A_j(a)$ and $y \in A_j(b)$. We titled this example "decentralization" since such restrictions allow separation of issues where different individuals are endowed with the responsibility for different areas and a coordunator is responsible for decisions pertaining to matters involving more than one area.

Claim 2

The above $\Omega_1 \times \Omega_2$ has a nontrivial decomposition.

Proof

Define $D = \{(X,Y) \in F \mid x_0 \neq y_0\}$. The proof that $F-C \subsetneq D \subsetneq F$ and that D is closed under decisive implications is straightforward and is omitted. Q.E.D.

## 4. The Group's Size and the Existence of Arrow Social Welfare Functions

Kalai-Muller [8], Maskin [13], [14], Kalai-Ritz [9] and Ritz [17] all proved for their respective cases the independence of the existence of an Arrow SWF from the group's size. That is to say that if a set of restricted domain of preferences admits a k-person Arrow SWF for a finite k greater than one, then it admits an n-person Arrow SWF for any given finite n greater than one (though these ASWFs need not be the same). These results enabled the characterization of the domains admitting Arrow SWFs in general, rather than characterizing only the domains admitting two-person Arrow SWF. Unfortunately, we did not manage to prove a parallel result for the general case analyzed here and the issue of complete independence is still an open question, nevertheless the following two theorems do contain some interesting results about the relationship between the group's size and the possible existence of Arrow SWFs. Theorem 4 also displays some of the difficulties which arise when dealing with a mix of private and public alternatives.

Theorem 3

For $n \geq 2$, if $\Omega^{(n)}$ admits a k-person Arrow SWF then it admits an m-person Arrow SWF for any $2 \leq k \leq m \leq n$.

Proof

Suppose $\Omega^{(n)}$ admits $h^k$ ($k \geq 2$), a k-person Arrow SWF. W.l.o.g. let $K = \{1,2,\ldots,k\} \subseteq N$ be such that $\Omega^{(K)}$ admits $h^k$. Define $h^{k+1}$ on $\Omega_1 \times \ldots \times \Omega_k \times \Omega_{k+1}$ as follows.

For any $X,Y \in (A_0;A^{(k+1)})$ and $R \in \Omega^{(k+1)}$

(i) $XIY$ in $h^{k+1}(R)$ iff $(x_0;x_j)i(y_0;y_j)$ in $r_j$ for every $j = 1,\ldots,k+1$,

(ii) $XPY$ in $h^{k+1}(R)$ if

(1) $X_{-(k+1)}PY_{-(k+1)}$ in $h^k(R_{-(k+1)})$, or

(2) $X_{-(k+1)}IY_{-(k+1)}$ in $h^k(R_{-(k+1)})$ and $(x_0;x_{k+1})p(y_0;y_{k+1})$ in $r_k$.

The proof that $h^{k+1}$ is an Arrow SWF is straightforward and is omitted. By repeating the above step as many times as needed we can prove that $\Omega^{(n)}$ admits an m-person Arrow SWF for any m such that $k \leq m \leq n$.

Q.E.D.

We now consider a special case of the general mixed alternatives case, where the public and private alternatives are independent of each other, every individual in the group has access to the same set of private alternatives, indifferences among alternatives are not allowed and all individuals have the same restricted domain of preferences.

Assume that for every $j \in N$ and every $a \in A_0$, $A_j(a) = A_1$, $\Omega_j = \Omega'$, and $\Omega'$ contains only antisymmetric, transitive and complete binary relations. We say that $\Omega'$ admits an n-person ASWF, if $\Omega'^{(n)}$ admits an n-person ASWF. Also assume that no group indifferences are allowed.

Theorem 4

For $n > 3$, $\Omega'$ admits an n-person ASWF iff it admits a 3-person ASWF.

Proof

Part a.

Assume that $\Omega'$ admits a 3-person ASWF, then by theorem 3, $\Omega'$ also admits an n-person ASWF.

Part b.

Assume that $\Omega'$ admits $h^n$ an n-person ASWF, for $n > 3$. If $\Omega'$ contains an inseparable pair of alternatives and $|A_0 \times A_1| > 2$ or $|\Omega'| \geq 2$, then by claim 1 of example 1, $\Omega'$ admits $h^2$ a 2-person ASWF, and then by theorem 3 it also admits $h^3$ a 3-person ASWF, so let's assume that $\Omega'$ doesn't contain any ISP (notice that if $|A_0 \times A_1| = 2$ and $|\Omega'| = 1$ then it is either the case of $|A_0| = 2$, $|A_1| = 1$ and then for no $n \geq 2$ is there

$h^n$ an n-person ASWF, or it is the case of $|A_0| = 1$, $|A_1| = 2$ and then $\Omega'$ admits $h^3$ a 3-person ASWF as was proved in Kalai-Ritz ([10], theorem 3, (ii))).

Individuals $i,j \in N$ are identical over $X,Y \in A_0 \times A_1^n$ (notice that now $(A_0;A^{(n)}) = (A_0 \times A_1^n)$, denoted by $(i,j)$, if $r_i = r_j$, $(x_0;x_i) = (x_0;x_j)$ and $(y_0;y_i) = (y_0;y_j)$. Individual k is dictator by (i,j), denoted $k = d(i,j)$, if whenever $(i,j)$ are identical, k is a dictator, i.e., for every $X,Y \in A_0 \times A_1^n$ and every $R \in \Omega'^n$ such that $r_i = r_j$, $x_i = x_j$ and $y_i = y_j$, then $(x_0;x_k)r_k(y_0;y_k)$ implies $Xh^n(R)Y$ (notice that both $r_k$ and $h^n$ are strict orders).

Lemma 5

For $n \geq 4$, if $h^n$ a SWF is such that whenever two individuals are identical there exists a dictator by them, then this dictator is unique.

Proof

Suppose the claim is false. Case 1. For every $(i,j)$, $d(i,j) \in \{i,j\}$. Since $|A_1| \geq 2$, let $(x_0;x),(x_0;y) \in A_0 \times A_1$ and $r \in \Omega'$ be such that $(x_0;x)r(x_0;y)$. Let $X = (x_0;x,x,y,y,\ldots,y)$, $Y = (x_0;y,y,x,x,y,\ldots,y)$ and $R = (r,\ldots,r)$, then $Xh^n(R)Y$ because $d(1,2) \in \{1,2\}$ and also $Yh^n(R)X$ because $d(3,4) \in \{3,4\}$, a contradiction.

Case 2. There exists at least one identical pair $(i,j)$ s.t. $d(i,j) \notin \{i,j\}$. W.l.o.g. assume that $1 = d(2,3)$. Suppose there exists $(i,j)$ s.t. $d(i,j) \neq 1$.

Subcase 1. $1 \notin \{i,j\}$. Then if $k = d(i,j)$, let $X = (x_0;x,y,\ldots,y)$, $Y = (x_0;y,x,\ldots,x)$ and $R = (r,\ldots,r)$. Then $Xh^n(R)Y$ because of $1 = d(2,3)$ and $Yh^n(R)X$ because $d(i,j) = k \neq 1$, a contradiction.

Subcase 2. $1 \in \{i,j\}$. Since $d(1,j) \neq 1$, then also $d(1,j) \neq j$ or $d(1,j) \notin \{1,j\}$, and a contradiction can be easily constracted.

Q.E.D.

The following lemmas are proved for $n = 4$. This is done only in order to simplify the notations, the proofs for the cases of $n > 4$ are identical.

Individual i is <u>b-dictator</u> if for every $R \in \Omega'^n$ and $X,Y \in A_0 \times A_1^n$ s.t. $x_0 = y_0 = b$, $(b;x_i)r_i(b;y_i)$ implies $Xh^n(R)Y$.

Lemma 6

Let $h^4$ be a SWF. If for every two identical individuals there exists a dictator, and if for some $b \in A_0$ there is a b-dictator then these dictators are the same.

Proof

W.l.o.g. assume $1 = d(i,j)$ for every $i,j \in \{1,2,3,4\}$ $(i \neq j)$. Suppose for some $b \in A_0$, $\ell \neq 1$ is the b-dictator. Then let $(b;x),(b;y) \in A_0 \times A_1$ and $r \in \Omega'$ s.t. $(b;x)r(b;y)$ and let $X = (b;x,y,y,y)$, $Y = (b;y,x,x,x)$ and $R = (r,r,r,r)$. Then by the above assumptions it is $Xh^4(R)Yh^4(R)X$, a contradiction. Q.E.D.

j is a <u>public-dictator</u> if for every $R \in \Omega'^n$ and $X,Y \in A_0 \times A_1^n$ such that $x_1 = x_2 = \dots = x_n$ and $y_1 = y_2 = \dots = y_n$, then $(x_0;x)r_i(y_0;y)$ implies $Xh^n(R)Y$.

Lemma 7

Let $h^4$ be a SWF. If for every two identical individuals there exists a dictator k, and there is also j a public-dictator then $k = j$.

Proof

Suppose $k \neq j$. W.l.o.g. assume $k = 1$ and $j = 2$. Let $(a;x),(b;y) \in A_0 \times A_1$ and $r,r' \in \Omega'$ be such that $(a;x)r(b;y)$ and $(b;y)r'(a;x)$ (if there are no such $r$ and $r'$, then $|\Omega'| = 1$ and $h^2$ a 2-person ASWF can be defined directly). Let $X = (a;x,x,x,x)$, $Y = (b;y,y,y,y)$ and $R = (r,r',r',r')$ then $Xh^4(R)Y$ because of $1 = k(2,3)$ and $Yh^4(R)X$ because of $j = 2$, a contradiction.

Q.E.D.

k is an <u>induced dictator</u> if he is the dictator of lemmas 6 and 7 combined.

Preferences $(r,r') \in \Omega'^2$ are <u>$((a;x_1,x_2),(b;y_1,y_2))$ connected</u> if there exists $r'' \in \Omega'$ which agrees with $r$ over $(a;x_1),(b;y_1)$ and with $r'$ over $(a;x_2),(b;y_2)$.

Lemma 8

Let $h^4$ be an ASWF for which there exists k, an induced dictator. If there exist $X,Y \in A_0 \times A_1{}^4$ and $R \in \Omega'^4$ such that $(y_0;y_k)r_k(x_0;x_k)$ and $Xh^4(R)Y$ then there are no $i,j \in N$ $(i \neq j)$ such that $(r_i,r_j)$ are $((x_0;x_i,x_j),(y_0;y_i,y_j))$ connected.

Proof

W.l.o.g. assume that $k = 1$, $(r_2,r_3)$ are $((\bar{x}_0;\bar{x}_2,\bar{x}_3),(\bar{y}_0;\bar{y}_2,\bar{y}_3))$ connected, $(\bar{y}_0;\bar{y}_1)r_1(\bar{x}_0;\bar{x}_1)$ and $\bar{X}h^4(R)\bar{Y}$. Since $(r_2,r_3)$ are connected there exists $r' \in \Omega'$ which agrees with $r_2$ over $(\bar{x}_0;\bar{x}_1),(\bar{y}_0;\bar{y}_1)$, and with $r_2$ over $(\bar{x}_0;\bar{x}_2),(\bar{y}_0;\bar{y}_2)$ w.l.o.g. assume $(\bar{x}_0;\bar{x}_2)r'(\bar{x}_0;\bar{x}_3)$ and $(\bar{y}_0;\bar{y}_3)r'(\bar{y}_0;\bar{y}_2)$. Let $R' = (r_1,r',r',r_4)$. By IIA it is $\bar{X}h^4(R')\bar{Y}$.

Let $Y = (\overline{y}_0;\overline{y}_1,\overline{y}_2,\overline{y}_2,\overline{y}_4)$ and $X = (\overline{x}_0;\overline{x}_1,\overline{x}_2,\overline{x}_2,\overline{x}_4)$. Then it is $\overline{Y}h^4(R')Y$ by unanimity, $Yh^4(R')X$ since 1 is an induced dictator, and $Xh^4(R')\overline{X}$ by unanimity which leads to $\overline{Y}h^4(R')\overline{X}$, a contradiction. Q.E.D.

Individual k is <u>x-essential</u> if there are $X,Y \in A_0 \times A_1^n$ and $R \in \Omega'^n$ such that $x_0 \neq y_0$, $x_k = y_k = x$, $(x_0;x)r_k(y_0;x)$, $(y_0;y_j)r_j(x_0;x_j)$ for every $j \in N$, $j \neq k$ and $Xh^n(R)Y$.

<u>Lemma 9</u>

If $h^4$ is an ASWF for which there exists k an induced dictator, then for no $x \in A_1$ is there an individual $j \neq k$ who is x-essential.

<u>Proof</u>

Suppose the claim is false. W.l.o.g. assume $k = 1$ and $j = 2$ is x-essential, i.e., there exist $X,Y \in A_0 \times A_1^4$ and $R \in \Omega'^4$ such that $X = (x_0;x_1,x,x_3,x_4)$, $Y = (y_0;y_1,x,x_3,x_4)$, $(x_0;x)r_2(y_0;x)$, $(y_0;y_j)r_j(x_0;x_j)$ for $j = 1,3,4$ and $Xh^4(R)Y$. Since 1 is an induced dictator, then by lemma 8, no two preferences of $r_2$, $r_3$ and $r_4$ can be connected. Therefore $(x_0;x)r_2(y_0;x)$ implies $(y_0;x)r_3(x_0;x)$ and $(y_0;x)r_4(x_0;x)$, $(y_0;y_3)r_3(x_0;x_3)$ implies $(x_0;x_3)r_2(y_0;y_3)$ and $(x_0;x_3)r_4(y_0;y_3)$ and $(y_0;y_4)r_4(x_0;x_4)$ implies $(x_0;x_4)r_2(y_0;y_4)$ and $(x_0;x_4)r_3(y_0;y_4)$.

<u>Case 1.</u> Suppose $(y_0;y_4)r_3(y_0;y_3)$. If also $(y_0;y_3)r_4(y_0;y_4)$, then it implies $(x_0;x_4)r_3(y_0;y_3)$, $(x_0;x_3)r_4(y_0;y_4)$, and therefore $(x_0;x_1,x,x_3,x_3)h^4(R)(x_0;x_1,x,x_3,x_4)h^4(R)(y_0;y_1,x,y_3,y_4)$ and $(x_0;x_1,x,x_4,x_4)h^4(R)(y_0;y_1,x,y_3,y_4)$. If $(x_0;x_3)r_2(y_0;y_4)$ then $r_2$ and $r_4$ are connected over $((x_0;x,x_3),(y_0;x,y_4)$, a contradiction, and if $(y_0;y_4)r_2(x_0;x_3)$ then $(x_0;x_4)r_2(y_0;y_3)$ and $r_2,r_3$ are

$((x_0;x,x_4),(y_0;x,y_3))$-connected, again a contradiction. If $(y_0;y_4)r_4(y_0;y_3)$, then by unanimity $(x_0;x_1,x,x_4,x_4)h^4(R)(y_0;y_1,y,y_3,y_3)$, but then either $r_2,r_3$ or $r_2,r_4$ are $((x_0;x,x_4),(y_0;x,y_3))$-connected, a contradiction.

Case 2. Suppose $(y_0;y_3)r_3(y_0;y_4)$. If also $(x_0;x_4)r_3(x_0;x_3)$ then by unanimity $(x_0;x_1,x,x_4,x_4)h^4(R)(y_0;y_1,y,y_4,y_4)$ and $r_2$ is connected either with $r_3$ or with $r_4$, a contradiction. If $(x_0;x_3)r_3(x_0;x_4)$ then $(x_0;x_3)r_3(y_0;y_4)$ and $(x_0;x_1;x,x_3,x_4)h^4(R)(y_0;y_1,x,y_4,y_4)$; if also $(y_0;y_3)r_4(y_0;y_4)$ then it must be $(x_0;x_3)r_4(y_0;y_4)r_4(x_0;x_4)$, $(x_0;x_1,x,x_3,x_3)h^4(R)(y_0;y_1;x,y_4;y_4)$ and $r_3,r_4$ are connected, a contradiction; but if $(y_0;y_4)r_4(y_0;y_3)$ then it must be $(x_0;x_4)r_4(y_0;y_4)$ and $r_2$ is connected either with $r_4$ or with $r_3$ respectively, a contradiction.

Q.E.D.

We say that k is (x,r)-dictatorial by j, denoted $k = d_j(x,r)$, if for every $X,Y \in A_0 \times A_1^4$ and every $R \in \Omega'^4$ s.t. $x_j = y_j = x$ and $r_j = r$, then $(x_0;x_k)r_k(y_0;y_k)$ implies $Xh^4(R)Y$.

Lemma 10

Let $h^4$ be an ASWF for which there is k, an induced dictator. If there are $x \in A_1$, $j \in N$ and $r \in \Omega'$ such that $d_j(x,r)$ exists then $k = d_j(x,r)$.

Proof

W.l.o.g. assume $k = 1$, $j \in N$, $x' \in A_1$, and $r' \in \Omega'$ s.t. $d_j(x',r') = 2 \neq 1$. Also assume $(x_0;x)r'(y_0;x)$. Let $r \in \Omega'$ be such that $(y_0;y)r(x_0;z)$ for some $y,z \in A_1$ (notice that there must be such r). Suppose $j \neq 1$.

Then since $2 = d_j(x,p)$ it must be $(x_0;z,x,x,x)h^4((r,r',r',r'))(y_0;y,x,x,x)$, which contradicts the assumption that 1 is an induced dictator. If $j = 1$, then since $2 = d_1(x,p)$ it is $(y_0;x,y,y,y)h^4((r',r,r,r))(x_0;x,z,z,z)$ which again contradicts the assumption that 1 is an induced dictator.

Q.E.D.

Lemma 11

If $h^4$ is an ASWF and there exists k an induced dictator, then there exist at least one individual $j \in N$, alternative $x \in A_1$ and $r \in \Omega'$ for which $d_j(x,r)$ does not exist.

Proof

Suppose this is not the case. Assume w.l.o.g. that $k = 1$. Then by lemma 10, $1 = d_j(x,r)$ for every $j \in N$, $x \in A_1$ and $r \in \Omega'$. Since $h^4$ is an ASWF, there exist $X,Y \in A_0 \times A_1{}^4$ and $R \in \Omega'^4$ s.t. $(y_0;y_1)r_1(x_0;x_1)$ and $Xh^4(R)Y$. This implies that there exists $j \in N$, say $j = 2$, s.t. $(x_0;x_2)r_2(y_0;y_2)$. Since $((y_0;y_1),(x_0;x_1))$ is not an ISP, then there exist $(z_0;z_1) \in A_0 \times A_1$ and $r \in \Omega'$ such that $(y_0;y_1)r(z_0;z_1)r(x_0;x_1)$. Let $R' = (r,r_2,r_3,r_4)$ then by IIA also $Xh^4(R')Y$. Since $1 = d_2(y_2,r_2)$ then $(y_0;y_1,y_2,y_3,y_4)h^4(R')(z_0;z_1,y_2,x_3,x_4)$, and since also $1 = d_3(x_3,r_3)$ then $(z_0;z_1,y_2,x_3,x_4)h^4(R')(x_0;x_1,x_2,x_3,x_4)$, a contradiction.

Q.E.D.

Let us now complete the proof of the theorem. If $h^n$ $(n \geq 4)$ is such that there exists an induced dictator for it, then by lemma 11 there exist $x \in A_1$, $j \in N$ and $r \in \Omega'$ for which there is no $d_j(x,r)$. W.l.o.g. assume $j = n$ and define $h^{n-1}$ $(n \geq 4)$ as follows. For every $X,Y \in A_0 \times A_1^{n-1}$

and $R \in \Omega'^{n-1}$, $Xh^{n-1}(R)Y$ iff $(x_0;x_1,\ldots,x_{n-1},x)h^n((r_1,\ldots,r_{n-1},r))$ $(y_0;y_1,\ldots,y_{n-1},x)$. It is straightforward to show that $h^{n-1}$ is an ASWF. If there is no induced dictator for $h^n$, then: <u>Case 1</u>. There exist $i,j \in N$ such that there is no $k \in N$ which is dictator by $(i,j)$. W.l.o.g. assume $(i,j)$ to be $(n-1,n)$. Define $h^{n-1}$ as follows. For every $X,Y \in A_0 \times A_1^{n-1}$ and $R \in \Omega'^{n-1}$, $Xh^{n-1}(R)Y$ iff $(x_0;x_1,\ldots,x_{n-1},x_{n-1})h^n((r_1,\ldots,r_{n-1},r_{n-1}))(y_0;y_1,\ldots,y_{n-1},y_{n-1})$. The proof that $h^{n-1}$ is an ASWF is straightforward. <u>Case 2</u>. For every $(i,j)$ there exists $k$ a dictator by $(i,j)$. Since there is no induced dictator, there must exist $b$ $A_0$ such that there is no $b$-dictator for $h^n$, i.e., there exist $r \in \Omega'$, $R' \in \Omega'^n$, $\overline{X} = (b;\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x}_n)$ and $\overline{Y} = (b;\overline{y}_1,\ldots,\overline{y}_{n-1},\overline{y}_n)$ such that $(b;\overline{y}_1)r_1(b;\overline{x}_1)$, $\overline{X}h^n(R')\overline{Y}$ when $r_n = r$. Define $g\colon A_0 \times A_1^{n-1} \to A_1$ as:

$$g(X) = \overline{y}_n \text{ if } X = (b;\overline{y}_1,\ldots,\overline{y}_n)$$

$$= \overline{x}_n \text{ otherwise.}$$

Define $h^{n-1}$ as follows: For every $X,Y \in A_0 \times A_1^{n-1}$ and $R \in \Omega'^{n-1}$, $Xh^{n-1}(R)Y$ iff $(X,g(X))h^n((r_1,\ldots,r_{n-1},r))(Y,g(Y))$. Again, the proof that $h^{n-1}$ is an ASWF is straightforward. This completes the proof of theorem 4. Q.E.D.

## References

1. Arrow, K. J., Social Choice and Individual Values, 2nd ed. New York: Wiley, 1963.

2. Black, D., "The Decision of a Committee Using a Special Majority," Econometrica, 16 (1948).

3. Blau, J. H., "The Existence of Social Welfare Functions," Econometrica, Vol. 25.

4. Blin, J. M. and M. A. Satterthwaite, "Individual Decisions and Group Decisions," Journal of Public Economics.

5. Dutta, B., "Existence of Stable Situations, Restricted Preferences and Strategic Manipulation under Democratic Group Decision Rules," Journal of Economic Theory, 15, 99-111 (1977).

6. Fishburn, P. C., The Theory of Social Choice. Princeton, N.J.: Princeton University Press, 1973.

7. Gibbard, A., "Manipulation of Voting Schemes: A General Result," Econometrica, 41, 587-601 (1973).

8. Kalai, E. and E. Muller, "Characterizations of Domains Admitting Nondictatorial Social Welfare Functions and Nonmanipulable Voting Procedures," Journal of Economic Theory, 16 (1977).

9. Kalai, E. and Z. Ritz, "Characterization of the Private Alternatives Domains Admitting Arrow Social Welfare Functions," Journal of Economic Theory, 22 (1980).

10. Kalai, E. and Z. Ritz, "An Extended Single Peak Condition in Social Choice," Discussion Paper No. 325, The Center for Mathematical Studies in Economics and Management Science.

11. Kim, K. H. and F. W. Roush, "On Domains Admitting Nondictatorial Social Welfare Functions," mimeo, Alabama State University.

12. Kelly, J. S., Arrow Impossibility Theorems. Academic Press, 1978.

13. Maskin, E., "Social Welfare Functions on Restricted Domains," Harvard University and Darwin College, Cambridge, Mass., 1976.

14. Maskin, E., "On Strategy Proofness and Social Welfare Functions When Preferences are Restricted," Harvard University and Darwin College, Cambridge, Mass., 1976.

15. Pattanaik, P. K., "Collective Rationality and Strategy-Proofness of Group Decision Rules," Theory and Decisions, 7, 191-204.

16. Pattanaik, P. K., Strategy and Group Choice. Amsterdam: North-Holland (1978).

17. Ritz, Z., "Restricted Domains, Arrow Social Welfare Functions and Noncorruptible and Nonmanipulable Social Choice Correspondences: The Case of Private Alternatives," Faculty Working Paper No. 748, Bureau of Economic and Business Research, University of Illinois.

18. Satterthwaite, M. A., "Existence of a Strategy Proof Voting Procedure: A Topic in Social Choice Theory," (unpublished Ph.D. dissertation, University of Wisconsin, Madison, 1973).

19. Satterthwaite, M. A., "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," Journal of Economic Theory, 10 (1975).

20. Satterthwaite, M. A. and H. Sonnenschein, "Strategy-Proof Allocation Mechanisms," Discussion Paper No. 395, The Center for Mathematical Studies in Economics and Management Science.

21. Schmeidler, D. and H. Sonnenschein, "The Possibility of a Cheat Proof Social Choice Function: A Theorem of A. Gibbard and M. Satterthwaite," Discussion Paper No. 89 (May 1974), The Center for Mathematical Studies in Economics and Management Science, Northwestern University.

22. Sen, A. K., Collective Choice and Social Welfare. San Francisco: Holden-Day, 1970.

23. Sen, A., "Social Choice Theory," February 1979. (A paper prepared for the Handbook of Mathematical Economics, edited by K. J. Arrow and M. Intriligator.)

24. Sengupta, M. and B. Dutta, "A Condition for Nash Stability under Binary and Democratic Group Decision Functions," Theory and Decisions, 10, 293-310 (1979).