





330
B385
1991:155 COPY 2

STX

The Risk-Free Rate in Heterogeneous-Agent, Incomplete-Insurance Economies--Revised

The Library of the

NOV 4 6 1991

University of Illinois
of Urbana-Champaign

Mark Huggett
Department of Economics
University of Illinois

BEBR

FACULTY WORKING PAPER NO. 91-0155

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

July 1991

The Risk-Free Rate in Heterogeneous-Agent,
Incomplete-Insurance Economies--Revised

Mark Huggett
Department of Economics

Digitized by the Internet Archive
in 2011 with funding from
University of Illinois Urbana-Champaign

<http://www.archive.org/details/riskfreerateinhe91155hugg>

ABSTRACT:

A heterogeneous-agent, incomplete-insurance economy is constructed to address why the average, real, risk-free, interest rate has been so low. The economy is calibrated and equilibria are characterized by computational methods. The risk-free interest rate generated by the calibrated economy is below those of comparable representative-agent economies.

1. Introduction

Why has the average, real, risk-free, interest rate been less than one percent?¹ The question is motivated by the work of Mehra and Prescott (1985). They argue that a class of calibrated, representative-agent models does not match the average, real return to equity and risk-free debt.² They suggest that the rate of return observations may be understood by explaining why the risk-free rate has been so low.

The conjecture that market imperfections are important for determining the risk-free rate is investigated. One approach for considering the conjecture is to describe an environment and arrangement that represent key features of actual economies. Then one can examine if the data are generated. This approach follows the modeling rules of Lucas (1987) by completely describing the game being played. Another approach, discussed by Townsend (1987), is to describe an environment and hypothesize that agents act to achieve pareto optimal allocations. Then one can check if the data result from some arrangement that achieves a pareto optimal allocation.

The first approach is explored here. One consideration motivating the choice is purely technical. Pareto optimal allocations are sometimes difficult to characterize.³ Another consideration is that the second approach is likely to be more helpful in interpreting a collection of interest rates rather than the specific one considered here. Lastly, we would like to know if features of observed arrangements are or are not important determinants of rates of return, regardless of whether we have underlying explanations for them. I view the two approaches as being largely complimentary.

This paper examines the importance of idiosyncratic shocks and incomplete insurance for determining the risk-free rate. A pure exchange economy where agents experience idiosyncratic, endowment shocks and smooth consumption by holding credit balances is constructed. Many elements that may be important determinants of the risk-free rate (eg. discrete asset levels, other assets and shocks, production, growth and government

¹Mehra and Prescott (1985) state that from 1889 to 1978 the average, real return on short term relatively riskless debt has been .8%.

²The representative-agent models in this class predict a risk-free rate that is too large and an equity premium that is too small.

³This is especially the case for environments with private information. See Green (1987), Spear and Srivastava (1987), Atkeson (1987) and Phelan and Townsend (1989) for recent advances in characterizing pareto optimal allocations in private information environments.

policy) are abstracted from to concentrate on the effects of idiosyncratic shocks and incomplete insurance on the risk-free rate. At this stage a relatively simple explanation is given for why this structure may generate a low risk-free rate. With a borrowing limit, agents must be persuaded from accumulating large credit balances so that the credit market clears. A low risk-free rate does this. In section five a more analytical explanation for the result will be provided. To examine the risk-free rate generated by this structure, the economy is calibrated and equilibria are characterized using computational methods.

There has been a considerable amount of work on heterogeneous-agent, incomplete-insurance models of asset pricing. In monetary economics, work by Bewley (1980, 1983), Lucas (1980) and Taub (1988) employ a similar structure to that used here. In other areas of economics similar structures have been used. Imrohorglu (1989) measures the potential welfare gains from eliminating aggregate fluctuations. Manuelli (1986) and Clarida (1990) study international debt markets. Diaz and Prescott (1989) study movements in the return to money and Treasury bills in response to monetary and fiscal policies. Taub (1991) analyzes the efficiency properties of money and credit in an environment with taste shocks. Aiyagari and Gertler (1989) study the effect of transaction costs on asset returns. The research above builds on the individual consumption smoothing problems analyzed by Schechtman and Escudero (1977), Mendelson and Amihud (1982), Clarida (1984) and many others. Models with a different structure that address the equity and debt observations include Mankiw (1986) and Kahn (1988).

The paper is organized in six sections. The next section, section two, describes the environment and arrangement in more detail. Section three describes the equilibrium concept and some theorems that will be useful in computing equilibria. Section four describes model calibration and computation. Section five discusses the results. Section six concludes.

2. Environment and Arrangement

This paper considers an exchange economy with a continuum of agents of total mass equal to one. Each period each agent receives an endowment $e \in E$ of the one perishable consumption good in the economy. Each agent's endowment follows a Markov process with stationary, transition probability $\pi(e' | e) = \text{Prob}(e_{t+1} = e' | e_t = e) > 0$ for $e, e' \in E$ that is independent of all other agents current and past endowments. Each agent

has preferences defined over stochastic processes for consumption given by a utility function.

$$(2.1) \quad E\left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right], \quad \text{where } \beta \in (0,1).$$

$$(2.2) \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \text{where } \sigma > 1.$$

The particular arrangement considered allows each agent to smooth consumption by holding a single asset. The asset can be interpreted as a credit balance with a central credit authority or as a one period ahead sure claim on consumption goods. I will use the credit balance interpretation. A credit balance of $a \in A$ entitles an agent to a goods this period. To obtain a credit balance of $a' \in A$ next period, an agent must pay $a'q$ goods this period, where q is the price of next period credit balances. Credit balances are restricted to never fall below a credit limit \underline{a} , $\underline{a} < 0$. An agent's decision problem will be described at a more technical level after setting down some notation.

Notation:

*An agent's position at a point in time is described by an individual state vector $x \in X$. $x = (a,e)$ indicates credit balance a and endowment e . The space of state vectors is $X = A \times E$, where $A = [\underline{a}, \infty)$, $E = \{e_1, e_2\}$ and $e_1 > e_2$.

*Let $q > 0$ be the constant price of credit balances each period.

*Let $v(x;q)$ be the optimal value function for an agent who starts in state x , faces price q and behaves optimally. $v: X \times \mathbb{R}_{++} \rightarrow \mathbb{R}$.

*Let ψ be a probability measure on (S, \mathcal{S}) , where $S = [\underline{a}, \bar{a}] \times E$ and \mathcal{S} is the Borel σ -algebra. For $B \in \mathcal{S}$, $\psi(B)$ indicates the mass of agents whose individual state vector lies in B .

A functional equation that describes an agent's decision problem is then:

$$(2.3) \quad v(x;q) = \max_{(c,a')} \left[u(c) + \beta \sum_e v(a',e';q) \pi(e' | e) \right]$$

$$(c,a') \in \Gamma(x;q) = \{(c,a'): c + a'q \leq a + e; c \geq 0; a' \geq \underline{a}\}$$

If a bounded, measurable solution v to functional equation (2.3) exists, then v is the optimal value function (see Theorem 9.2 in Lucas and Stokey (1989)). If v is the optimal value function, then functions $c: X \times R_{++} \rightarrow R_+$ and $a: X \times R_{++} \rightarrow A$ are optimal decision rules provided $c(x;q)$ and $a(x;q)$ are measurable, feasible and satisfy

$$(2.4) \quad v(x;q) = u(c(x;q)) + \beta \sum_e v(a(x;q), e';q) \pi(e' | e)$$

The decision rule $a(x;q)$ and the transition probabilities define a transition function $P, P: S \times S \times R_{++} \rightarrow [0,1]$. $P(x,B;q)$ indicates the probability of being in B next period given that an agent's current state is x and the price is q . In the appendix such a transition function is constructed. The dependence of the decision rules and the transition function on q will often be suppressed for notational convenience.

3. Equilibrium

The equilibrium concept and some theorems that will be useful in computing equilibria are described. The stationary, recursive, equilibrium structure described in Lucas (1980) is employed.

Definition 1: A stationary equilibrium for this economy is $(c(x), a(x), q, \psi)$ satisfying:

1) $c(x)$ and $a(x)$ are optimal decision rules, given q .

2) Markets Clear:

$$i) \int_S c(x) d\psi = \int_S e d\psi \quad ii) \int_S a(x) d\psi = 0$$

3) ψ is a stationary probability measure

$$\psi(B) = \int_S P(x,B) d\psi \quad \text{for all } B \in S.$$

Some discussion of the equilibrium concept is in order. The first condition says that agents optimize. The second condition says that consumption and endowment averaged over the population are equal and that credit balances averaged over the population are zero. The third condition says that the distribution of agents over states is unchanging.⁴ Note that

⁴An equilibrium concept that allows for changing probability measures is not difficult to state. However, general methods for characterizing equilibria to that equilibrium concept have not been developed. Therefore, this paper considers stationary equilibria.

the measure ψ is defined over subsets of S instead of X . Subsequent arguments will show that this is legitimate.

The following theorems will be useful in computing equilibria. Theorem 1 states conditions under which for given q there exists a unique solution to (2.3) and gives a method for computing optimal decision rules. Theorem 2 lists properties of decision rules that are used in the proof of Theorem 4. Theorem 4 states conditions under which for given q there exists a unique stationary probability measure ψ on (S,S) and gives a method for computing excess demand in the credit market.

A mapping T on $C(X)$, the space of bounded, continuous, real-valued functions on X , is now defined.

$$(3.1) \quad (Tv)(x;q) = \max_{(c,a') \in \Gamma(x;q)} [u(c) + \beta \sum_{e'} v(a',e';q) \pi(e' | e)]$$

Theorem 1: For $q > 0$ and $\underline{a} + e_2 - \underline{a}q > 0$, there exists a unique solution $v(x;q) \in C(X)$ to (2.3) and $T^n v_0$ converges uniformly to v as $n \rightarrow \infty$ from any $v_0 \in C(X)$. Furthermore, $v(x;q)$ is strictly increasing, strictly concave and continuously differentiable in a .

Theorem 2: Under the conditions of theorem 1, there exist continuous, optimal decision rules $c(x;q)$ and $a(x;q)$. $a(x;q)$ is nondecreasing in a and strictly increasing in a for $(x;q)$ such that $a(x;q) > \underline{a}$.

Proofs are in the appendix. Theorems 1 and 2 are standard results in dynamic programming. Theorem 1 requires that if an agent starts out with the smallest credit balance and receives the smallest endowment shock then the agent can maintain the smallest credit balance and have strictly positive consumption. Theorem 1 is proved by an application of the contraction mapping theorem. Due to the fact that the period utility function is not bounded below, an additional argument is needed to show that T maps $C(X)$ into $C(X)$. This is handled by finding a strictly positive lower bound that consumption will not be set below.

Next, a theorem is presented for the existence of a unique, stationary, probability measure. The theorem is used to prove Theorem 4.

Assumption 1:

- i) (S, \geq) is an ordered space
- ii) S is a compact metric space
- iii) \geq is a closed order
- iv) (S, \mathcal{S}) is a measurable space and \mathcal{S} is the Borel σ -algebra.
- v) P is a transition function, $P: S \times S \rightarrow [0, 1]$.

A transition function P induces a mapping $W: P(S) \rightarrow P(S)$, where $P(S)$ is the space of probability measures on (S, \mathcal{S}) , defined by

$$(W\psi)(B) = \int_S P(s, B) d\psi \quad \text{for } B \in \mathcal{S}.$$

Theorem 3: (Hopenhayn and Prescott (1987), Theorem 2, page 17) If Assumption 1 i)-v) holds, P is increasing, S has a greatest (d) and a least (c) element in S and the following condition is satisfied:

Monotone Mixing Condition: There exists $s^* \in S$, $\varepsilon > 0$ and N such that $P^N(d, \{s: s \leq s^*\}) > \varepsilon$ and $P^N(c, \{s: s \geq s^*\}) > \varepsilon$

, then there exists a unique, stationary, probability measure ψ and, for any $\psi_0 \in P(S)$, $W^n \psi_0$ converges weakly to ψ as $n \rightarrow \infty$.

Theorem 4: If the conditions to theorem 1 hold, $\beta < q$ and $\pi(e_1 | e_1) \geq \pi(e_1 | e_2)$, then there exists a unique, stationary, probability measure ψ (given q) on (S, \mathcal{S}) and, for any $\psi_0 \in P(S)$, $W^n \psi_0$ converges weakly to ψ as $n \rightarrow \infty$.

Theorem 4 is an application of the theorem of Hopenhayn and Prescott. The proof is done in two steps. First, it is proved that there is a set $S = [\underline{a}, \bar{a}] \times E$ that has the property that if an agent starts out in S , then the agent stays in S . This is accomplished by showing that the decision rule for credit balances has the shape shown in Figure 1. More specifically, it is shown that $a(a, e_2) < a$ for $a > \underline{a}$ and that $a(a, e_1)$ has a fixed point as a function of a . These two facts together with the fact that $a(a, e)$ is nondecreasing in its first

argument yield the desired result. Schechtman and Escudero (1977) prove a similar result for the case of independent and identically distributed shocks. As in their work, it is assumed that the period utility function has a bounded coefficient of relative risk-aversion and that the interest rate is below the time preference rate (alternatively $\beta < q$). The second step is to show that the conditions of Theorem 3 are satisfied.

4. Calibration and Computation

The economy is calibrated following the procedures described in Lucas (1981).⁵ This involves using microeconomic and macroeconomic observations to set values of the parameters $\{e_1, e_2; \pi(e_1 | e_1), \pi(e_2 | e_2); \beta; \sigma; \underline{a}\}$ and the period length. I follow Imrohorglu (1989) in interpreting e_1 and e_2 as earnings when employed and not employed. Consider the following observations:

1) Kydland (1984) calculates the standard deviation of annual hours worked for individual prime-age males from 1970-1980. He groups males by education levels. He calculates the average of group members' standard deviation as a percentage of group members' average annual hours. The statistic varies from 16% to 32% for the groups with the highest and lowest education levels.

2) The average duration of unemployment spells for men from 1948-1988 is 12.3 weeks (Handbook of Labor Statistics).

When $e_1 = 1.0$, $e_2 = 0.1$, $\pi(e_1 | e_1) = .925$, $\pi(e_2 | e_2) = .5$ and there are six model periods in one year, the standard deviation of annual earnings as a percentage of mean for an agent is 20% and the average duration of the low endowment shock is 17 weeks. The duration of the low endowment shock is higher than the data cited above. However, Clark and Summers (1979) calculate that in 1974 26% of unemployment spells for men age 20 and over ended in withdrawal from the labor force. They argue that unemployment duration understates the average time to reemployment.

The discount factor β is set to .993. As there are six model periods in one year the annual discount factor is .96. The microeconomic studies reviewed by Mehra and Prescott

⁵An earlier version of this paper used a different calibration. The calibration described here uses evidence on hours variability cited in Aiyagari and Gertler (1989). The results obtained with the previous and current calibration are similar.

(1985) estimate the risk-aversion coefficient, σ , to be about 1.5. A range of values for the credit limit are selected, $\underline{a} \in \{-2, -4, -6, -8\}$, to examine the sensitivity of the results to different credit limits. A credit limit of -5.3 is equal to one years average endowment.

The procedures used to compute equilibria to the calibrated model economies are described next. The computation method consists of four steps:

- 1) Given price q , compute $a(x;q)$ using theorem 1.
- 2) Given $a(x;q)$, iterate on $\psi_{t+1}(B) = \int_S P(x,B)d\psi_t$ from arbitrary $\psi_0 \in P(S)$ for sets B in a certain class.
- 3) Given the results from steps 1 and 2, compute $\int_S a(x;q)d\psi$.
- 4) Update q and repeat the steps until market clearing is approximately obtained.

These steps are now discussed in more detail. Step one is to iterate on on (3.1) from arbitrary bounded, concave, differentiable time 0 value function, v_0 . This is a concave programming problem. First order conditions to the time 1 problem reduce to

$$(4.1) \quad u'(a + e - a'q)q \geq \beta \sum_e v_0'(a', e) \pi(e' | e) \quad ; \text{ with equality if } a' > \underline{a}.$$

Values of $a_1(x;q)$ are given by solutions to (4.1). The first order conditions to the time 2 problem also reduce to (4.1) with $v_1'(x) = u'(a + e - a_1(x;q)q)$. This result follows from Lucas (1978) Proposition 2. Values of $a_2(x;q)$ are determined in the same manner. The iterations are repeated until convergence of decision rules is approximately obtained. To implement this procedure on a computer some changes need to be made. First, compute $u'(a + e - a'q)$ and $v_0'(a,e)$ on finite grids on $X \times A$ and X respectively. Between gridpoints let the values of the functions be given by linear interpolation. Next, solve for $a_1(a,e)$ on gridpoints using (4.1). Iterate until convergence is approximately obtained.⁶ See Figure 1 for a graph of $a(a,e)$.

Step two involves iterations on $\psi_{t+1}(B) = \int_S P(x,B)d\psi_t$ from arbitrary initial $\psi_0 \in P(S)$ for sets of the form $B = \{x \in S: x_1 \leq a, x_2 = e\}$, where $(a,e) \in S$ and $S = [\underline{a}, \bar{a}] \times E$.

⁶This computation procedure is similar to Coleman's (1988) methods for computing equilibria to representative-agent models.

To implement this procedure on a computer, define the function $F_0(a,e) = \psi_0(\{x: x_1 \leq a, x_2 = e\})$ on gridpoints. Between gridpoints let values of the function be given by linear interpolation. Then iterate on

$$(4.2) \quad F_{t+1}(a',e') = \sum_e \pi(e' | e) F_t(a^{-1}(\cdot,e)(a'), e)$$

on gridpoints (a', e') . Since $a(x)$ may not be invertible in its first argument when \underline{a} is chosen, define $a^{-1}(\cdot,e)(\underline{a})$ as the maximum a such that \underline{a} is chosen when the state is (a,e) . See Figure 2 for a graph of $F(a,e)$.

Step three approximates the excess demand for credit using the results from steps one and two. Theorem 4 provides the justification for this approximation.

In step four the initial value of q is selected to be the midpoint of some interval of candidate q 's. New values are increased if there is an excess demand and decreased if there is an excess supply of credit balances at the previous price. This process is stopped after approximate market clearing is obtained.

5. Results

Tables 1 and 2 present the results. In examining the Tables, note that interest rates (r) are annual rates whereas prices (q) are for model periods. Also note that a credit limit of - 5.3 is equal to one years average endowment. The experiments listed in Table 1 show that the risk-free rate is negative for sufficiently restrictive credit limits and increases as the credit limit is relaxed. For a similar result in a different context see Taub (1991). Table 2 shows the sensitivity of the results in Table 1 to changes in the coefficient of relative risk-aversion, σ . The higher value of σ reduces the risk-free rate for all credit levels examined.

Table 1
 $\sigma = 1.5$

Credit Limit	Interest Rate	Price	Excess Demand
$\underline{a} = -2$	$r = -7.4\%$	$q = 1.0129$	$-.0025$
$\underline{a} = -4$	$r = 1.2\%$	$q = .9980$	$-.0010$
$\underline{a} = -6$	$r = 3.0\%$	$q = .9951$	$-.0008$
$\underline{a} = -8$	$r = 3.5\%$	$q = .9942$	$.0044$

Table 2
 $\sigma = 3.0$

Credit Limit	Interest Rate	Price	Excess Demand
$\underline{a} = -2$	$r = -23\%$	$q = 1.0461$	$.0003$
$\underline{a} = -4$	$r = -4.5\%$	$q = 1.0077$	$-.0007$
$\underline{a} = -6$	$r = 0.5\%$	$q = .9991$	$.0004$
$\underline{a} = -8$	$r = 2.4\%$	$q = .9961$	$.0013$

The economy studied here can be compared to a similar representative-agent economy where the representative-agent receives the average endowment. In that economy the risk-free rate is equal to the annual time preference rate, approximately 4%. So, in all the experiments considered, the heterogeneous-agent, incomplete-insurance economy has a lower risk-free rate. I conjecture that there are no stationary equilibria for the economy studied here with a risk-free rate greater than the time preference rate. The basis for the conjecture is that if the interest rate were greater than the time preference rate (alternatively $\beta > q$), then agents would accumulate credit balances without bound. A more precise statement is that any agent's credit balance diverges almost surely to $+\infty$. See Mendelson and Amihud (1982) for an analysis of this point.

To close the section, two caveats are mentioned for interpreting the results. First, the results in the tables are not upper or lower bounds to true equilibrium prices and interest rates. The error involved in the computation is unknown and will be a topic of future research. Second, the issue of multiplicity of stationary equilibria has yet to be resolved.

However, for all the experiments considered, excess demand is a monotone function of the price of credit balances.

6. Conclusion

The paper investigates why the average, real, risk-free, interest rate has been so low. The main conclusion is that idiosyncratic shocks that can not be perfectly insured against can generate a risk-free rate well below that of a representative-agent model with the same aggregate fluctuations. In the future it will be interesting to look at some of the features that were abstracted from here and also consider some of the models cross sectional properties.

Adding capital to the economy would bring up a number of issues. Will the introduction of capital lead to more individual consumption smoothing or less? As the capital-output ratio for the U.S. economy is about 2.5, privately held physical capital would appear to be an important consideration in individual decision problems. It would be interesting to allow for several types of capital, say housing and non-housing capital. In the resulting economy it would be interesting to look at asset holdings for agents in different states and to look at how consumption varies with asset returns for agents in different states. On this point see Mankiw and Zeldes (1990) for an interesting look at some data.

Recall that the risk-free rate generated by the heterogeneous-agent and representative-agent models do not differ dramatically for all feasible values of the credit limit. This fact provides extra motivation for theories of endogenous credit constraints. There are several theoretical motivations for credit constraints. Limited commitment is one motivation. Kehoe and Levine (1990) provide an analysis. Private information on effort or output is another. The seminal paper of Green (1987) provides an example. In fact, the environment studied here is quite similar to Green's. The environment can be interpreted as one in which each agent has private information on the value of his own endowment. In the future it would be interesting to compare the arrangement studied here to optimal arrangements along several dimensions.

Appendix:

A transition function on the state space S is constructed.

Let (S, \mathcal{S}) be a state space and corresponding Borel σ -algebra. Let z be a random variable defined on the probability measure space $(Z, \mathcal{Z}, \lambda)$. Let g be a function mapping $S \times Z$ into S . Define a mapping $P: S \times S \rightarrow [0, 1]$ by

$$(A.1) \quad P(s, B) = \lambda(\{z: g(s, z) \in B\}) \quad \text{for } B \in \mathcal{S}$$

The following lemma gives conditions under which P is a transition function.

Lemma 5 (Hopenhayn and Prescott (1987)) If g is measurable in $S \times Z$ (with the product σ -algebra), then P described in (A.1) is a transition function for a Markov process.

Let $(Z, \mathcal{Z}, \lambda)$ be Lebesgue measure on the unit interval. Let $g(s, z) = (g_1(s, z), g_2(s, z))$, where $g_1(s, z) = a(s)$ and

$$g_2(s, z) = \begin{cases} e_1 & \text{if } (s_2 = e_1 \text{ and } z \in (0, \pi(e_1 | e_1))) \text{ or } (s_2 = e_2 \text{ and } z \in (0, \pi(e_1 | e_2))) \\ e_2 & \text{if } (s_2 = e_1 \text{ and } z \in (\pi(e_1 | e_1), 1)) \text{ or } (s_2 = e_2 \text{ and } z \in (\pi(e_1 | e_2), 1)) \end{cases}$$

Note that g is measurable with respect to the product σ -algebra because g_2 is measurable by construction and g_1 is measurable (S, \mathcal{S}) .⁷

Theorem 1: For $q > 0$ and $\underline{a} + e_2 - \underline{a}q > 0$, there exists a unique solution $v(x; q) \in C(X)$ to (2.3) and $T^n v_0$ converges uniformly to v as $n \rightarrow \infty$ from any $v_0 \in C(X)$. Furthermore, $v(x; q)$ is strictly increasing, strictly concave and continuously differentiable in \underline{a} .

proof: Consider the mapping T defined in (3.1). Show that $T: C(X) \rightarrow C(X)$. First, note that the maximum of the objective in the definition of T is not obtained for c too close to zero. If $|v(x; q)| < M$ for all x , then $c \in [0, c^*]$, where $c^* = u^{-1}(u(\underline{a} + e_2 - \underline{a}q) - 3\beta M)$, will never be selected. Define $H(x; q, M) = \{(c, a') \in \Gamma(x; q): c \geq c^*\}$. $H(x; q, M)$ is a continuous

⁷Conditions under which g maps $S \times Z$ into S are given in theorem 4. The function g_1 is the optimal decision rule $a(x)$. Theorem 2 states conditions under which $a(x)$ is continuous, hence $a(x)$ defined on S will be measurable (S, \mathcal{S}) .

correspondence and, for fixed x , $H(x;q,M)$ is a nonempty, compact set. Apply the Theorem of the Maximum (Lucas and Stokey (1989) p. 62) to get that $h(x;q,M)$

$$h(x;q,M) = \max_{(c,a') \in H(x;q,M)} [u(c) + \beta \sum_{e'} v(a',e';q) \pi(e' | e)]$$

is a continuous function. $h(x;q,M)$ is bounded above because the objective is bounded above. It is also bounded below because the objective on $H(x;q,M)$ is bounded below. So $h(x;q,M)$ is in $C(X)$. To show $T: C(X) \rightarrow C(X)$, note $h(x;q,M) = (Tv)(x;q)$ for any v such that $|v(x;q)| < M$ for all x .

Next note that T is a contraction because $C(X)$ with the sup norm defines a complete metric space and that Blackwell's sufficient conditions for T to be a contraction are satisfied. The contraction mapping theorem yields a unique v in $C(X)$ solving (2.3) and guarantees that $T^n v_0$ converges uniformly to v as $n \rightarrow \infty$ from any $v_0 \in C(X)$.

$v(x;q)$ is strictly increasing in a because $u(c)$ is strictly increasing and, for increases in a , it's always possible to increase c holding a' constant.

$v(x;q)$ can be shown to be strictly concave in a by standard arguments.

$v(x;q)$ can be shown to be continuously differentiable in a , $v'(x;q) = u'(c(x;q))$, by applying Proposition 2 Lucas (1978). \diamond

Theorem 2: Under the conditions of theorem 1, there exist continuous, optimal decision rules $c(x;q)$ and $a(x;q)$. $a(x;q)$ is nondecreasing in a and strictly increasing in a for (x, q) such that $a(x;q) > \underline{a}$.

proof: An application of the theorem of the maximum in theorem 1 when v is the solution to (2.3) gives an u.h.c correspondence $g: X \rightarrow A \times R_+$. The continuity of $g(x;q) = (a(x;q), c(x;q))$ follows because the program involves maximizing a strictly concave function over a convex set. To show $a(x)$ is nondecreasing in a , note that the first order conditions are

$$(A.2) \quad u'(a + e - a(a,e)q)q \geq \beta \sum_{e'} v'(a(a,e), e') \pi(e' | e) \quad ; \text{ with equality if } a(a,e) > \underline{a}.$$

For $a_1 > a_2$, assume $a(a_1, e) < a(a_2, e)$.

$$(A.3) \quad \beta \sum_{e'} v'(a(a_1, e), e') \pi(e' | e) > \beta \sum_{e'} v'(a(a_2, e), e') \pi(e' | e)$$

$$(A.4) \quad u'(a_1 + e - a(a_1, e)q)q > u'(a_2 + e - a(a_2, e)q)q$$

(A.3) holds by strict concavity of v . (A.4) holds by (A.3) and (A.2). Finally, (A.4) and the strict concavity of u implies that $(a_1 - a_2) < (a(a_1, e) - a(a_2, e))q$. Contradiction. So $a(a, e)$ is nondecreasing in a .

Now argue that $a_1 > a_2$ and $a(a_2, e) > \underline{a}$ imply that $a(a_1, e) > a(a_2, e)$. Suppose that $a(a_1, e) = a(a_2, e)$, then (A.2) implies

$$u'(a_1 + e - a(a_1, e)q) = u'(a_2 + e - a(a_2, e)q)$$

This contradicts the fact that u is strictly increasing and strictly concave. So, $a(a_1, e) > a(a_2, e)$. \diamond

Theorem 4: If the conditions to theorem 1 hold, $\beta < q$ and $\pi(e_1 | e_1) \geq \pi(e_1 | e_2)$, then there exists a unique, stationary, probability measure ψ (given q) on (S, S) and, for any $\psi_0 \in P(S)$, $W^n \psi_0$ converges weakly to ψ as $n \rightarrow \infty$.

proof: The strategy is to first justify restricting attention to a compact set $S = [\underline{a}, \bar{a}] \times E$ and then to show that the conditions to Theorem 3 hold. For the first step consider the following lemmas:

Lemma 1: Under the conditions of Theorem 4, $a(a, e_2) < a$ for $a > \underline{a}$.

proof: Define the functions v_t for $t = 0, 1, 2, \dots$ by iterating on (3.1) starting with $v_0(a, e) = 0$. Using first order conditions (4.1) and $\pi(e_1 | e_1) \geq \pi(e_1 | e_2)$, induction yields $v'_t(a, e_1) \leq v'_t(a, e_2)$ for all t . Show that $v'_t(a, e)$ converges pointwise to $v'(a, e)$. Since $v'_t(a, e) = u'(a + e - a_t(a, e)q)$, $v'(a, e) = u'(a + e - a(a, e)q)$ and u' is continuous, it is sufficient to show that $a_t(a, e)$ converges pointwise to $a(a, e)$. It is straight forward to show that the argument in Lemma 3.7 Lucas and Stokey (1989) can be applied to obtain this

result. Pointwise convergence of v'_t to v' establishes that $v'(a, e_1) \leq v'(a, e_2)$. The conclusion follows because $\beta/q < 1$ and $v'(a, e_1) \leq v'(a, e_2)$ imply that the hypothesis to Lemma 2 below holds for $e = e_2$ and any $a^* > \underline{a}$. \diamond

Lemma 2: If $v'(a, e) > (\beta/q) E[v'(a, e') | e]$ for $a \geq a^* > \underline{a}$, then $a(a, e) < a$ for $a \geq a^*$.

proof: An agent's first order condition is

$$(A.5) \quad u'(a + e - a(a, e)q)q \geq \beta \sum_{e'} v'(a(a, e), e') \pi(e' | e) \quad ; \text{ with equality if } a(a, e) > \underline{a}.$$

For $a \geq a^*$, either $a(a, e) = \underline{a}$ or $a(a, e) > \underline{a}$. If the first occurs then $a(a, e) < a$. If the second occurs, then (A.5), the hypothesis, $v'(a, e) = u'(a + e - a(a, e)q)$ and v' decreasing in a imply that $a(a, e) < a$. v concave and differentiable implies that v' is decreasing in a . \diamond

Lemma 3: Under the conditions of Theorem 4, there exists a such that $a(a, e_1) = a$.

proof: Suppose not. Then $a(a, e_1) > a$ for all a . Lemma 1 then implies that $a(a, e_1) \geq a(a, e_2)$ for all a . Three inequalities follow:

$$a + e_2 - a(a, e_1)q \leq a + e_2 - a(a, e_2)q$$

$$c(a, e_1) - (e_1 - e_2) \leq c(a, e_2)$$

$$c(a, e_2)/c(a, e_1) \geq 1 - (e_1 - e_2)/c(a, e_1)$$

Note that v bounded, increasing, v' decreasing and $v'(a, e) = u'(c(a, e))$ imply that $c(a, e_1) \rightarrow \infty$ as $a \rightarrow \infty$. So for all sufficiently large a ,

$$v'(a, e_1)/v'(a, e_2) = (c(a, e_2)/c(a, e_1))^\sigma \geq (1 - (e_1 - e_2)/c(a, e_1))^\sigma$$

Since $\beta/q < 1$, there is an a^* such that $v'(a, e_1)/v'(a, e_2) > \beta/q$ for $a \geq a^*$. This fact and $v'(a, e_1) \leq v'(a, e_2)$ from Lemma 1 imply that the hypothesis of Lemma 2 holds for $e = e_1$. Contradiction. \diamond

The previous Lemmas imply that there is $S = [\underline{a}, \bar{a}] \times E$ such that if an agent starts with state x in S then the agent stays in S . Choose \bar{a} to be the smallest solution to $a(a, e_1) = a$. Now show that the conditions to Theorem 3 hold. First, define an order \geq on S . For $x, x' \in S$, where $x = (x_1, x_2)$

$$x \geq x' \text{ iff } [(x_1 \geq x_1' \text{ and } x_2 = x_2') \text{ or } (x' = c = (\underline{a}, e_2)) \text{ or } (x = d = (\bar{a}, e_1))]$$

This is a closed order with minimum (c) and maximum (d) elements.

Next, define P as described in the appendix. To show that P is increasing, Hopenhayn and Prescott (1987) prove that it is sufficient to show

$$x, x' \in S \quad x \geq x' \text{ imply } \int_S f P(x, dx) \geq \int_S f P(x', dx)$$

$$\text{where } f = \chi_B, B = \{y \in S: y \geq x \text{ for some } x \text{ in } B\} \in S.$$

Let $B_x = \{z \in Z: g(x, z) \in B\}$ and $B_{x'} = \{z \in Z: g(x', z) \in B\}$, where g was defined earlier in the construction of the transition function. Show $B_{x'} \subseteq B_x$. This is obvious if $g(x, z)$ is monotone in x for fixed z . $g(x, z)$ can be shown to be monotone by considering all possible cases. Therefore, $P(x, B) \geq P(x', B)$ as was to be shown.

Lastly, show that the mixing condition holds. Choose $s^* = (a(\underline{a}, e_1) + \bar{a})/2, e_1$. Define a sequence $x_1 = \underline{a}, x_2 = a(x_1, e_1), x_3 = a(x_2, e_1), \dots$ and a sequence $y_1 = \bar{a}, y_2 = a(y_1, e_2), y_3 = a(y_2, e_2), \dots$. Note that $\{x_n\} \rightarrow \bar{a}$ monotonically and $\{y_n\} \rightarrow \underline{a}$ monotonically. Therefore, there is an N_1 such that an agent goes from c to $\{x \in S: x \geq s^*\}$ with positive probability in N_1 or greater steps and there is an N_2 such that an agent goes from d to $\{x \in S: x \leq s^*\}$ with positive probability in N_2 or greater steps. Choose $N = \max\{N_1, N_2\}$ in the mixing condition. The conclusion to Theorem 4 follows by Theorem 3. \diamond

References:

Aiyagari, R. and Gertler, M., 1989. "Asset Returns with Transaction Costs and Uninsured Risk: A Stage III Exercise", December, mimeo

Atkeson, A., 1987. "International Lending with Moral Hazard and Risk of Repudiation", Stanford University, Graduate School of Business, mimeo.

Bewley, T. 1980. "The Optimum Quantity of Money", in Models of Monetary Economies, ed. Kareken, J. and Wallace, N., Federal Reserve Bank of Minneapolis.

Clarida, R., 1984. "On the Stochastic Steady-State Behavior of Optimal Asset Accumulation in the Presence of Random Wage Fluctuations and Incomplete Markets", Cowles Foundation Discussion Paper No. 701.

Clarida, R., 1990. "International Lending and Borrowing in a Stochastic, Stationary Equilibrium", *International Economic Review*, August, pp. 543-58.

Clark, K.B. and Summers, L.H., 1979. "Labor Market Dynamics and Unemployment: A Reconsideration", *Brookings Papers on Economic Activity* pp. 16-60.

Coleman, W., 1988. "Money, Interest and Capital in a Cash-in-Advance Economy", Federal Reserve Board, *International Finance Discussion Papers*, No. 323.

Diaz-Gimenez, J. and Prescott, E.C., 1989. "Computable General Equilibrium Heterogeneous Agent Monetary Economies", Federal Reserve Bank of Minneapolis.

Green, E., 1987. "Lending and the Smoothing of Uninsurable Income", in Contractual Arrangements for Intertemporal Trade, Prescott, E.C. and Wallace, N. editors, University of Minnesota Press.

Hopenhayn, H. and Prescott, E.C., 1987. "Invariant Distributions for Monotone Markov Processes", University of Minnesota, Center for Economic Research, Discussion Paper No. 242.

- Imrohoroglu, A., 1988. "The Welfare Costs of Inflation", University of Southern California, Department of Finance and Business Economics, mimeo.
- Imrohoroglu, A., 1989. "Cost of Business Cycles with Indivisibilities and Liquidity Constraints", *Journal of Political Economy*, December, 1364-83.
- Kahn, J.A., 1988. "Moral Hazard, Imperfect Risk-Sharing and the Behavior of Asset Returns", Rochester Working Paper No. 152.
- Kehoe, T.J. and Levine, D.K., 1990. "Debt Constrained Asset Markets", Federal Reserve Bank of Minneapolis, mimeo.
- Kydland, F., 1984. "Labor-Force Heterogeneity and the Business Cycle" in *Carnegie-Rochester Conference Series on Public Policy*, 21, 173-208.
- Lucas, R.E., 1978. "Asset Prices in an Exchange Economy", *Econometrica*, 46, 1429-1445.
- Lucas, R.E., 1980. "Equilibria in a Pure Currency Economy" in Models of Monetary Economies, Kareken, J. and Wallace, N. editors, Federal Reserve Bank of Minneapolis.
- Lucas, R.E., 1981. "Methods and Problems in Business Cycle Theory", in Studies in Business Cycle Theory, MIT Press.
- Lucas, R.E. and Stokey, N.L., 1989. Recursive Methods in Economic Dynamics, Harvard University Press.
- Mankiw, G., 1986. "The Equity Premium and the Concentration of Aggregate Shocks", NBER Working Paper No. 1788.
- Mankiw, G. and Zeldes, S., 1990. "The Consumption of Stockholders and Non-stockholders", NBER Working Paper No. 3402.
- Manuelli, R.E., 1986. *Topics in Intertemporal Economics*, Ph.D. Thesis, University of Minnesota.

Mehra, R. and Prescott, E.C., 1985. "The Equity Premium: A Puzzle", *Journal of Monetary Economics* 15, 145-162.

Mendelson, H. and Amihud, Y., 1982. "Optimal Consumption Policy Under Uncertain Income", *Management Science*, Vol. 28 No. 6.

Phelan, C. and Townsend, R.M., 1989. "Computing Multiperiod Information-Constrained Optima", University of Chicago, mimeo.

Schechtman, J. and Escudero, V., 1977. "Some Results on "An Income Fluctuation Problem"", *Journal of Economic Theory* 16, 151-166.

Spear, S. and Srivastava, S., 1987. "On Repeated Moral Hazard with Discounting", *Review of Economic Studies*, 180, 599-618.

Taub, B., 1988. "Efficiency in a Pure Currency Economy with Inflation", *Economic Inquiry*, 26, 567-83.

Taub, B., 1991. "Currency and Credit are Equivalent Mechanisms", University of Illinois, mimeo.

Townsend, R.M., 1987. "Arrow-Debreu Programs as Microfoundations of Macroeconomics", in Advances in Economic Theory: Fifth World Congress, ed. Bewley, T., Cambridge University Press.

Figure 1

DECISION RULE

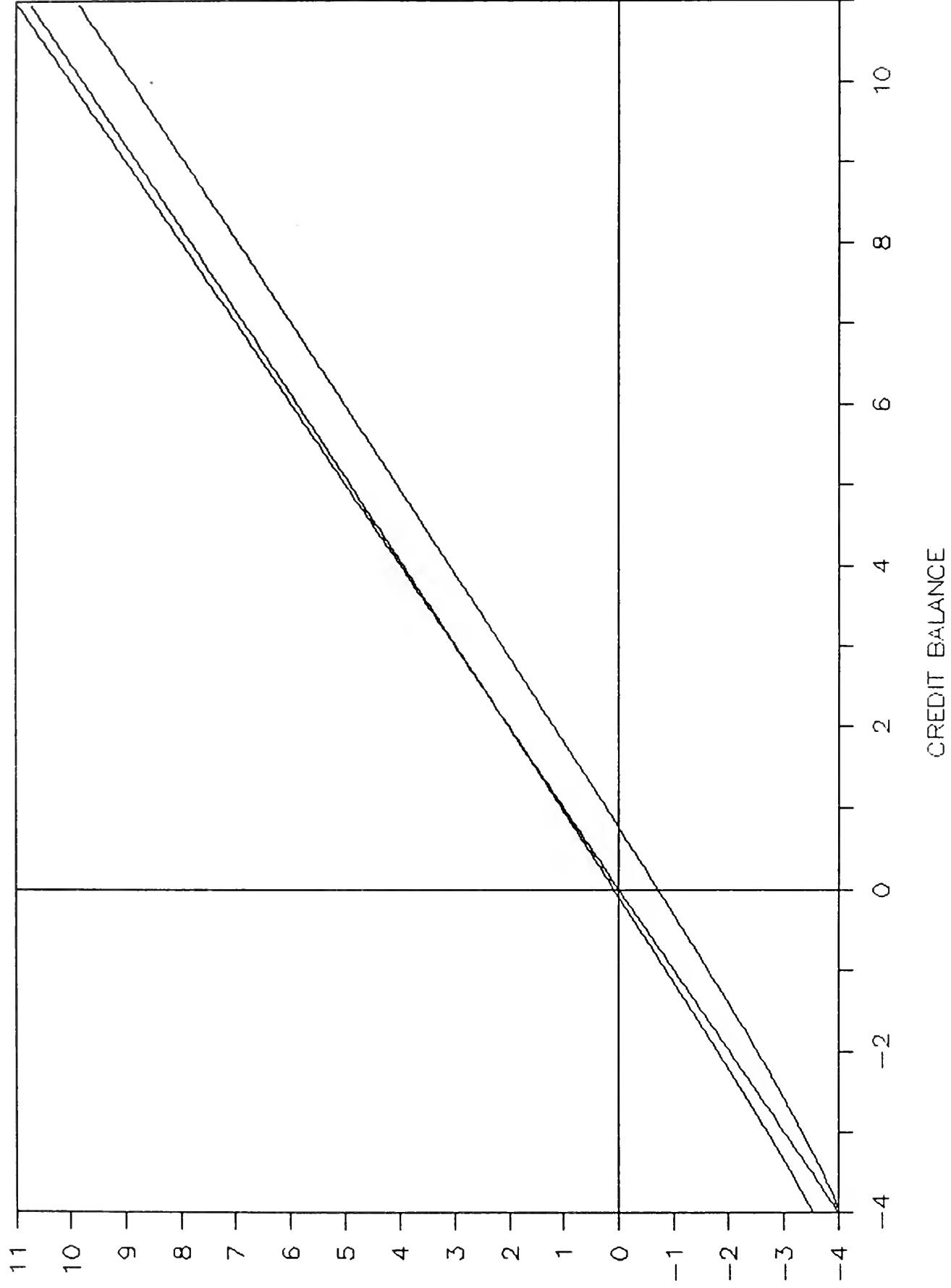
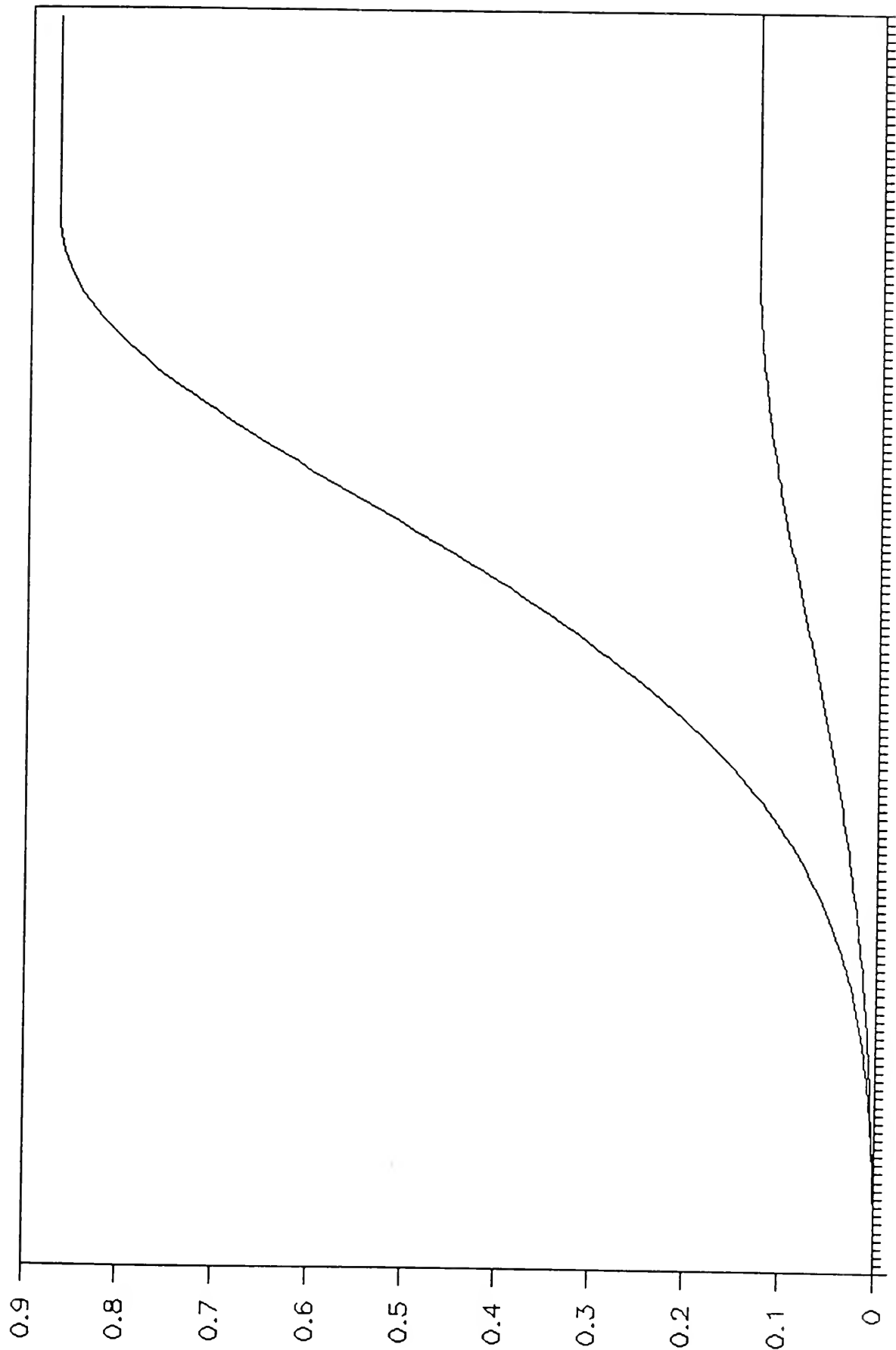


Figure 2

STATIONARY DISTRIBUTION



CREDIT BALANCE

HECKMAN
BINDERY INC.



JUN 95

Bound-To-Pleas[®] N. MANCHESTER,
INDIANA 46962

UNIVERSITY OF ILLINOIS-URBANA



3 0112 060295893