ROBBINS'S NEW SOLID GEOMETRY

BY

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ROBBINS'S NEW SOLID GEOMETRY.
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FOR THOSE WHOSE PRIVILEGE

IT MAY BE TO ACQUIRE A KNOWLEDGE OF

GEOMETRY

THIS VOLUME HAS BEEN WRITTEN

AND TO THE BOYS AND GIRLS WHO LEARN THE ANCIENT SCIENCE

FROM THESE PAGES, AND WHO ESTEEM THE POWER

OF CORRECT REASONING THE MORE

BECAUSE OF THE LOGIC OF

PURE GEOMETRY

THIS VOLUME IS DEDICATED

PREFACE

This "New Solid Geometry" is not only the outgrowth of the author's long experience in teaching geometry, but has profited further by suggestions from teachers who have used Robbins's "Solid Geometry" and by the recommendations of the "National Committee of Fifteen." While many new and valuable features have been added in the reconstruction, yet all the characteristics that met with widespread favor in the old book have been retained.

Among the features of the book that make it sound and teachable may be mentioned the following:

1. The book has been written for the pupil. The objects sought in the study of Geometry are (1) to train the mind to accept only those statements as truth for which convincing reasons can be provided, and (2) to cultivate a foresight that will appreciate both the purpose in making a statement and the process of reasoning by which the ultimate truth is established. Thus, the study of this formal science should develop in the pupil the ability to pursue argument coherently, and to establish geometric truths in logical order. To meet the requirements of the various degrees of intellectual capacity and maturity in every class, the reason for every statement is not printed in full but is indicated by a reference. The pupil who knows the reason need not consult the paragraph cited; while the pupil who does not know it may learn it by the reference. It is obvious that the greater progress an individual makes in assimilating the subject and in entering into its spirit, the less need there will be for the printed reference.

2. Every effort has been made to stimulate the mental activity of the pupil. To compel a young student, however, to supply his own demonstrations frequently proves unprofitable as well as

arduous, and engenders in the learner a distaste for a study in which he might otherwise take delight. This text does not aim to produce accomplished geometricians at the completion of the first book, but to aid the learner in his progress throughout the volume, wherever experience has shown that he is likely to require assistance. It is designed, under good instruction, to develop a clear conception of the geometric idea, and to produce at the end of the course a rational individual and a friend of this particular science.

3. The theorems and their demonstrations — the real subject matter of Geometry — are introduced as early in the study as possible.

4. The simple fundamental truths are explained instead of being formally demonstrated.

5. The original exercises are distinguished by their abundance, their practical bearing upon the affairs of life, their careful gradation and classification, and their independence. Every exercise can be solved or demonstrated without the use of any other exercise. Only the truths in the numbered paragraphs are necessary in working originals.

6. The exercises are introduced as near as practicable to the theorems to which they apply.

7. Emphasis is given to the discussion of original constructions.

8. The historical notes give the pupil a knowledge of the development of the science of geometry and add interest to the study.

9. The attractive open page will appeal alike to pupils and to teachers.

10. The Solid Geometry formulas are grouped together at the end of the text, as a ready means of reference.

The author desires to extend his sincere thanks to those friends and fellow teachers who, by suggestion and encouragement, have inspired him in the preparation of the text, as well as to Clement B. Davis for his original and skillful treatment of the illustrations.

EDWARD R. ROBBINS.

CONTENTS

REFERENCES TO ROBBINS'S "NEW PLANE GEOMETRY"

NOTE. Many of the theorems in the "New Solid Geometry" make reference in their proof to theorems and definitions in the "New Plane Geometry." These references are here collected for the convenience of the pupil.

16. One line is **perpendicular** to another if they meet at right angles. Either line is perpendicular to the other.

21. **Parallel lines** are straight lines that lie in the same plane and that never meet, however far they are extended in either direction.

23. A **triangle** is a portion of a plane bounded by three straight lines.

26. Two geometric figures are said to be **equal** if they have the same size or magnitude.

Two geometric figures are said to be **congruent** if, when one is superposed upon the other, they coincide in all respects.

27. **Homologous parts of congruent figures are equal.**

29. **Symbols.** The usual symbols and abbreviations employed in geometry are the following:

+	plus.	ⓢ	circles.	Ax.	axiom.
−	minus.	∠	angle.	Hyp.	hypothesis.
=	equals, is equal to.	∡	angles.	comp.	complementary.
≠	does not equal.	rt. ∠	right angle.	supp.	supplementary.
≅	congruent, *or* is congruent to.	rt. ∡	right angles.	Const.	construction.
>	is greater than.	△	triangle.	Cor.	corollary.
<	is less than.	⚠	triangles.	st.	straight.
∴	hence, therefore, consequently.	rt. ⚠	right triangles.	rt.	right.
⊥	perpendicular.	∥	parallel.	Def.	definition.
⊥s	perpendiculars.	∥s	parallels.	alt.	alternate.
⊙	circle.	▱	parallelogram.	int.	interior.
		▱s	parallelograms.	ext.	exterior.

30. An **axiom** is a statement admitted without proof to be true.

31. Axioms.

1. Magnitudes that are equal to the same thing, or to equals, are equal to each other.
2. If equals are added to, or subtracted from, equals, the results are equal.
3. If equals are multiplied by, or divided by, equals, the results are equal.

   [Doubles of equals are equal; halves of equals are equal.]
4. The whole is equal to the sum of all of its parts.
5. The whole is greater than any of its parts.
6. A magnitude may be replaced by its equal in any process.

   [Briefly called "substitution."]
7. If equals are added to, or subtracted from, unequals, the results are unequal in the same order.
8. If unequals are added to unequals in the same order, the results are unequal in that order.
9. If unequals are subtracted from equals, the results are unequal in the opposite order.
10. Doubles or halves of unequals are unequal in the same order. Also, unequals multiplied by equals are unequal in the same order.
11. If the first of three magnitudes is greater than the second, and the second is greater than the third, the first is greater than the third.
12. A straight line is the shortest line that can be drawn between two points.
13. Only one line can be drawn through a point parallel to a given line.
14. A geometrical figure may be moved from one position to another without any change in form or magnitude.

39. Only one straight line can be drawn between two points.

42. All right angles are equal.

43. Only one perpendicular to a line can be drawn from a point in the line.

44. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

45. If two adjacent angles are supplementary, their exterior sides are in the same straight line.

46. The sum of all the angles on one side of a straight line at a point equals two right angles.

47. The sum of all the angles about a point in a plane is equal to four right angles.

49. Angles that have the same supplement are equal. Or, supplements of the same angle, or of equal angles, are equal.

51. If two straight lines intersect, the vertical angles are equal.

52. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

53. Two right triangles are congruent if two legs of one are equal respectively to two legs of the other.

54. Only one perpendicular can be drawn to a line from an external point.

55. The angles opposite the equal sides of an isosceles triangle are equal.

62. Two lines in the same plane and perpendicular to the same line are parallel.

64. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.

66. If a transversal intersects two parallels, the alternate interior angles are equal.

67. If a transversal intersects two parallels, the corresponding angles are equal.

76. Two triangles are congruent if a side and the two angles adjoining it in the one are equal respectively to a side and the two angles adjoining it in the other.

77. Two right triangles are congruent if a leg and the adjoining acute angle of one are equal respectively to a leg and the adjoining acute angle of the other.

78. Two triangles are congruent, if the three sides of one are equal respectively to the three sides of the other.

80. Any point in the perpendicular bisector of a line is equally distant from the extremities of the line.

81. Any point not in the perpendicular bisector of a line is not equally distant from the extremities of the line.

82. If a point is equally distant from the extremities of a line, it is in the perpendicular bisector of the line.

83. Two points each equally distant from the extremities of a line determine the perpendicular bisector of the line.

84. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.

87. The perpendicular is the shortest line that can be drawn from a point to a straight line.

88. If from any point in a perpendicular to a line two oblique lines are drawn,

I. Oblique lines cutting off equal distances from the foot of the perpendicular are equal.

II. Equal oblique lines cut off equal distances.

III. Oblique lines cutting off unequal distances are unequal, and that one which cuts off the greater distance is the greater.

90. The method of exclusion consists in making all possible suppositions, leaving the probable one last, and then proving all these suppositions impossible, except the last, which must necessarily be true.

The method of proving the individual steps is called *reductio ad absurdum* (reduction to an absurd or impossible conclusion). This method consists in assuming as false the truth to be proved and then showing that this assumption leads to a conclusion altogether contrary to known truth or the given hypothesis.

92. If two triangles have two sides of one equal to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.

94. Every point in the bisector of an angle is equally distant from the sides of the angle.

104. The sum of the angles of any triangle is two right angles; that is, 180°.

109. Each angle of an equiangular triangle is 60°.

114. If two angles of a triangle are equal, the triangle is isosceles.

120. A **parallelogram** is a quadrilateral having its opposite sides parallel.

124. The opposite sides of a parallelogram are equal.

126. The diagonal of a parallelogram divides it into two congruent triangles.

128. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

129. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

133. Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

134. Two rectangles are congruent if the base and altitude of one are equal respectively to the base and altitude of the other.

136. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.

138. The line bisecting one leg of a trapezoid and parallel to the base bisects the other leg, is the median, and is equal to half the sum of the bases.

139. **The median of a trapezoid is parallel to the bases and equal to half their sum.**

143. **The three medians of a triangle meet in a point which is two thirds the distance from any vertex to the midpoint of the opposite side.**

145. The **number** of sides of a polygon is the same as the number of its vertices or the number of its angles.

150. Two **polygons** are **congruent** if they are mutually equiangular and their homologous sides are equal.

153. **The sum of the interior angles of an n-gon is equal to $(n-2)$ times 180°.**

155. **Each angle of an equiangular n-gon** $= \dfrac{(n-2)\,180°}{n}$.

157. **If three angles of a quadrilateral are right angles, the figure is a rectangle.**

158. **If the sides of a polygon are produced, in order, one at each vertex, the sum of the exterior angles of the polygon equals four right angles, that is, 360°.**

179. A **circle** is a plane curve all points of which are equally distant from a point in the plane, called the **center.**

180. The length of the circle is called the **circumference.**

183. **Equal circles** are circles having equal radii.

187. **All radii of the same circle are equal.**

188. **All radii of equal circles are equal.**

190. **All diameters of the same or of equal circles are equal.**

191. **The diameter of a circle bisects the circle.**

193. **In the same circle (or in equal circles) equal central angles intercept equal arcs.**

196. **In the same circle (or in equal circles) equal chords subtend equal arcs.**

197. **In the same circle (or in equal circles) equal arcs are subtended by equal chords.**

202. The line perpendicular to a radius at its extremity is tangent to the circle.

203. If a line is tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent.

208. In the same circle (or in equal circles) equal chords are equally distant from the center.

209. In the same circle (or in equal circles) chords which are equally distant from the center are equal.

210. In the same circle (or in equal circles) if two chords are unequal, the greater chord is at the less distance from the center.

211. In the same circle (or in equal circles) if two chords are unequally distant from the center, the chord at the less distance is the greater.

214. One circle and only one can be drawn through the vertices of a triangle.

219. If two circles intersect, the line joining their centers is the perpendicular bisector of their common chord.

224. To **measure a quantity** is to find the number of times it contains another quantity of the same kind, called the **unit**. This *number* is the ratio of the quantity to the unit.

225. Two quantities are called **commensurable** if there exists a common unit of measure which is contained in each a whole (integral) number of times.

Two quantities are called **incommensurable** if there does not exist a common unit of measure which is contained in each a whole number of times.

227. The **limit of a variable** is a constant, *to which* the variable cannot be equal, but *from which* the variable can be made to differ by less than any mentionable quantity.

229. If two variables are always equal and each approaches a limit, their limits are equal.

232. A central angle is measured by its intercepted arc.

233. A central right angle intercepts a quadrant of arc.

234. A right angle is measured by half a semicircle, that is, by a quadrant.

240. All angles inscribed in a semicircle are right angles.

246. The **locus** of a point is the series of positions the point must occupy in order that it may satisfy a given condition. It is the path of a point whose positions are limited or defined by a given condition, or given conditions.

280. In a proportion the product of the extremes is equal to the product of the means.

281. If the product of two quantities is equal to the product of two others, one pair may be made the extremes of a proportion and the other pair the means.

282. In any proportion the terms are also in proportion by alternation (that is, the first term is to the third as the second is to the fourth).

284. In any proportion the terms are also in proportion by composition (that is, the sum of the first two terms is to the first, or the second, as the sum of the last two terms is to the third, or the fourth).

285. In any proportion the terms are also in proportion by division (that is, the difference between the first two terms is to the first, or the second, as the difference between the last two terms is to the third, or the fourth).

287. In any proportion, like powers of the terms are in proportion, and like roots of the terms are in proportion.

288. In two or more proportions the products of the corresponding terms are in proportion.

289. A mean proportional is equal to the square root of the product of the extremes.

291. In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.

293. A line parallel to one side of a triangle divides the other sides into proportional segments.

294. If a line parallel to one side of a triangle intersects the other sides, it divides these sides proportionally.

301. Similar polygons are polygons that are mutually equiangular and whose homologous sides are proportional.

304. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

305. If a line parallel to one side of a triangle intersects the other sides, the triangle formed is similar to the original triangle.

308. If two triangles have their homologous sides proportional, they are similar.

310. If two triangles have their homologous sides perpendicular, they are similar.

313. In similar figures homologous sides are proportional.

318. If two polygons are similar, they may be decomposed into the same number of triangles similar each to each and similarly placed.

331. If in a right triangle a perpendicular is drawn from the vertex of the right angle upon the hypotenuse,

I. The triangles formed are similar to the given triangle and similar to each other.

II. The perpendicular is a mean proportional between the segments of the hypotenuse.

333. The square of a leg of a right triangle is equal to the product of the hypotenuse and the projection of this leg upon the hypotenuse.

334. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

357. The area of a rectangle is equal to the product of its base by its altitude.

359. The area of a parallelogram is equal to the product of its base by its altitude.

360. All parallelograms having equal bases and equal altitudes are equal in area.

364. The area of a triangle is equal to half the product of its base by its altitude.

366. All triangles having equal bases and equal altitudes are equal in area.

368. Two triangles having equal altitudes are to each other as their bases.

372. The area of a trapezoid is equal to half the product of the altitude by the sum of the bases.

374. If two triangles have an angle of one equal to an angle of the other, they are to each other as the products of the sides including the equal angles.

376. Two similar polygons are to each other as the squares of any two homologous sides.

395. To construct a triangle equal to a given polygon.

422. If the number of sides of an inscribed regular polygon is indefinitely increased, the apothem approaches the radius as a limit.

NOTE, page 232. It is evident that if the difference between two variables approaches zero, either

(1) one is approaching the other as a limit, or
(2) both are approaching some third quantity as their limit.

424. If the number of sides of an inscribed regular polygon and of a circumscribed regular polygon is indefinitely increased,

I. The perimeter of each polygon approaches the circumference of the circle as a limit.

II. The area of each polygon approaches the area of the circle as a limit.

428. Let C = circumference and R = radius. Then, $\boldsymbol{C = 2\pi R}$.

430. Let S = area of ⊙, C = its circumference, and R = its radius. Then, $\boldsymbol{S = \pi R^2}$.

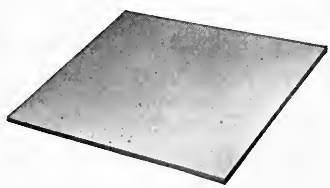
SOLID GEOMETRY

BOOK VI

LINES, PLANES, AND ANGLES IN SPACE

465. A **solid** is any limited portion of space. The boundaries of a solid are **surfaces.**

A **plane** is a surface in which, if any two points are taken, the straight line connecting them lies wholly in that surface.



Solid geometry is a science that treats of magnitudes, the parts of which are not all in the same plane.

466. The **intersection** of two surfaces is the line, or the lines, all of whose points lie in both surfaces. The intersection of a line and a surface is the point, or the points, common to both the line and the surface. The **foot of a line** intersecting a plane is their point of intersection.

467. A **straight line** is **perpendicular** to a plane if the line is perpendicular to every straight line in the plane drawn through its foot.

A **normal** is a straight line perpendicular to a plane.

468. A **straight line** is **parallel** to a plane if the line and the plane never meet, when indefinitely extended. A **straight line** is **oblique** to a plane if it is neither perpendicular nor parallel to the plane. Two **planes** are **parallel** if they never meet when indefinitely extended.

469. The **projection** of a **point** upon a plane is the foot of the perpendicular from the point to the plane.

The **projection** of a **line** upon a plane is the line formed by the projections of all the points of the given line.

470. A plane is **determined** if its position is fixed and if that position can be occupied by only one plane.

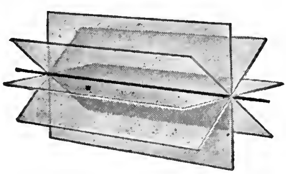
PRELIMINARY THEOREMS

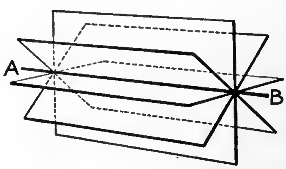
471. If two points of a straight line are in a plane, the whole line is in the plane. (Def. 465.)

472. A straight line can intersect a plane in only one point. (471.)

473. If a line is perpendicular to a plane, it is perpendicular to every line in the plane drawn through its foot. (467.)

474. Through one straight line any number of planes may be passed.





Because, if we consider a plane containing a line *AB* to revolve about *AB*, it may occupy an indefinitely great number of positions. Each of these will be a different plane containing *AB*.

475. Through a fixed straight line and an external point a plane can be passed.

Because, if we pass a plane containing this line *AB*, it may be revolved about *AB* until it contains the given point.

476. A straight line and an external point determine a plane. (See 475, 470.)

477. Three points not in a straight line determine a plane.

Because two of the points may be joined by a line; then this line and the third point determine a plane. (See 476.)

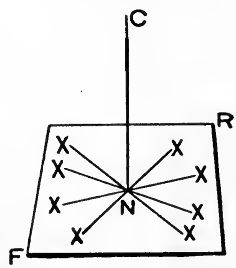
478. Two parallel lines determine a plane. (21, 476.)

479. Two intersecting straight lines determine a plane.

Because one of these lines and a point in the second line determine a plane; and this plane contains the second line. (476, 471.)

480. If two planes are parallel, no line in the one can meet any line in the other. (Def. 468.)

NOTE. A plane is represented to the eye by a quadrilateral. In some positions it appears to be a parallelogram, and in others, a trapezoid. The eye, however, must be aided by the imagination in really understanding the diagrams of solid geometry. Thus, in the adjoining figure, the line *CN* is perpendicular to the plane *FR*, and to every line in *FR* drawn through *N*. Consider several lines drawn through a point on the floor, and a cane, *CN*, occupying a vertical position, so that it is perpendicular to all these lines. Then every angle *CNX* is a right angle, though to the unskilled eye they do not all appear to be right angles in the diagram. The object of all geometrical diagrams is that the eye may assist the mind in grasping truths or in developing logical demonstrations, and the student should thoroughly examine every figure until he completely understands the relative positions of its parts. A photograph, like a geometrical diagram, represents three dimensions in a plane, and we should be as familiar with the significance of one as with the other.



When, during the process of a demonstration or elsewhere, it becomes necessary to employ a plane not already indicated, it is customary to **pass** such a plane, or to conceive it constructed.

Ex. 1. How many planes can be passed through two points? through three points in the same straight line?

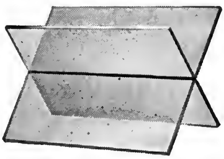
Ex. 2. Hold two pencils (representing lines) so that a plane can be passed containing both; so that no plane can be passed containing both.

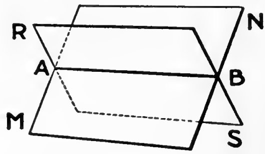
THEOREMS AND DEMONSTRATIONS

POINTS, LINES, AND PLANES

PROPOSITION I. THEOREM

481. If two planes intersect, their intersection is a straight line.





Given: Intersecting planes *MN* and *RS*.

To Prove: Their intersection is a straight line.

Proof: Suppose *A* and *B* are two points common to both planes.

Draw straight line *AB*.

Now *AB* is in plane *RS* (471).

And *AB* is in plane *MN* (?).

That is, *AB* is common to both planes.

Again, if there were a point outside of *AB* in both planes, these planes would coincide (476).

Hence *AB* contains *all points* common to the two given planes. ∴ *AB* is the intersection (466).

That is, the intersection of the two planes is a straight line. Q.E.D.

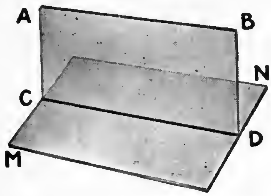
Ex. 1. Do any two planes intersect? Explain.

Ex. 2. What is meant by the statement "Two planes determine a line"? Is this universally true?

Ex. 3. What kind of lines are the folds in your letter paper? in a pamphlet? in the edges of a box or a brick? Explain.

Proposition II. Theorem

482. If two straight lines are parallel, a plane containing one, and only one, is parallel to the other line.



Given: ‖ lines *AB* and *CD*; plane *MN* containing *CD*.

To Prove: plane *MN* ‖ to line *AB*.

Proof: *AB* and *CD* are in the same plane *AD* (21).

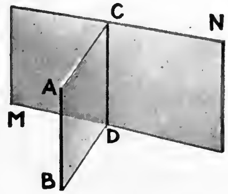
Plane *AD* intersects plane *MN* in *CD* (481).

Now as *AB* cannot meet *MN* in *CD* (hyp.) it can never meet *MN*.

∴ *AB* is ‖ to *MN* (468). Q.E.D.

Proposition III. Theorem

483. If a straight line is parallel to a plane, and another plane containing this line intersects the given plane, the intersection is parallel to the given line.



Given: *AB* ‖ to *MN*; plane *AD* containing *AB* and intersecting plane *MN* in *CD*.

To Prove: *AB* ‖ to *CD*.

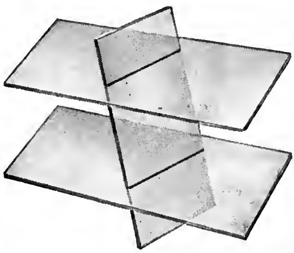
Proof: *AB* and *CD* are in the same plane *AD* (Hyp.).

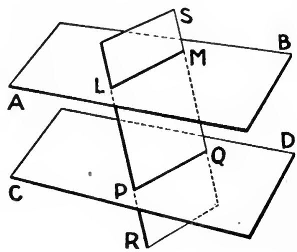
Now as *AB* cannot meet *MN* (hyp.) it can never meet *CD* in *MN*.

∴ *AB* is ‖ to *CD* (21). Q.E.D.

PROPOSITION IV. THEOREM

484. The intersections of two parallel planes by a third plane are parallel lines.





Given: ∥ planes *AB* and *CD* cut by plane *RS* in lines *LM* and *PQ*.

To Prove: *LM* ∥ to *PQ*.

Proof: *LM* and *PQ* are in the same plane *RS* (Hyp.).

Also *LM* and *PQ* can never meet (480).

∴ *LM* is ∥ to *PQ* (21).

Q.E.D.

Ex. 1. Hold a pencil parallel to the blackboard, so that its shadow falls on the blackboard. Is this shadow parallel to the pencil? Why?

Ex. 2. Can a plane intersect two planes that are not parallel so that the intersections are parallel? Illustrate your answer by passing a plane across the room so that it cuts the end and a side of the room. How must this plane be passed so that the intersections are parallel lines?

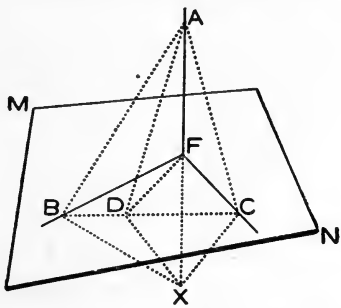
Ex. 3. Draw a line on the blackboard that will never meet the plane of the ceiling. How was it drawn?

Ex. 4. Draw a line on the blackboard and one on the floor that will meet if extended. How must these lines be drawn?

Ex. 5. When will a line on the ceiling be parallel to a line on the floor?

Proposition V. Theorem

485. A straight line perpendicular to each of two straight lines at their intersection is perpendicular to the plane of the lines.



Given: $AF \perp$ to BF and CF at F; plane MN containing BF and CF.

To Prove: $AF \perp$ to plane MN.

Proof: In plane MN draw BC; draw also DF from F to any point, D, in BC.

Prolong AF to X, making $FX =$ to AF, and draw AB, AD, AC, BX, DX, CX.

BF and CF are $\perp$ bisectors of AX (Hyp. and Const.).

In $\triangle$ ABC and BXC, $AB = BX$ and $AC = CX$ (80).

$BC = BC$ (?).

$\therefore \triangle ABC \cong \triangle BXC$ (78).

Also in $\triangle$ ABD and BXD, $\angle ABC = \angle CBX$ (27).

$BD = BD$ (?).

And $AB = BX$ (?).

$\therefore \triangle ABD \cong \triangle BXD$ (52).

$\therefore AD = DX$ (?).

Hence DF is $\perp$ to AX (83).

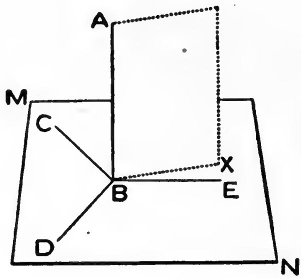
That is, AF is $\perp$ to all lines in MN through F.

$\therefore AF$ is $\perp$ to plane MN (467).

Q.E.D.

Proposition VI. Theorem

486. All straight lines perpendicular to a line at one point are in one plane, which is perpendicular to this line at this point.



Given: *AB* ⊥ to *BC*, *BD*, *BE*, etc.; plane *MN* containing *BC* and *BD*.

To Prove: *BE* is in the plane *MN* and *MN* is ⊥ to *AB* at *B*.

Proof: Pass plane *AE* containing *AB* and *BE*, and intersecting plane *MN* in line *BX*.

Now *AB* is ⊥ to plane *MN* (485).

That is, plane *MN* is ⊥ to *AB*

∴ *AB* is ⊥ to *BX* (473).

But *AB* is ⊥ to *BE* (Hyp.).

That is, *BX* and *BE* are both in plane *AX* and ⊥ to *AB* at *B*.

∴ *BX* and *BE* coincide (43).

That is, *BE* is in plane *MN*. Q.E.D.

Ex. 1. Can a line be perpendicular to two other lines if these two do not intersect? Illustrate.

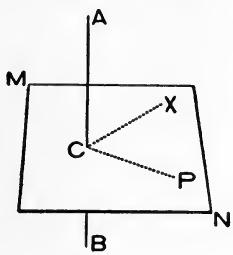
Ex. 2. If two lines are perpendicular to a third line are they necessarily in the same plane? Are the two lines necessarily parallel? Illustrate.

Ex. 3. Give a reason why the "corners" of a building are perpendicular to the horizontal plane of a level street.

487. Corollary. Through a point in a straight line one plane can be passed perpendicular to the line, and only one. (486.)

Proposition VII. Theorem

488. Through an external point one plane can be passed perpendicular to a given straight line, and only one.



Given: The line *AB* and point *P* outside of *AB*.

To Prove: Through *P*, one plane can be passed ⊥ to *AB*, and only one.

Proof: I. Draw from *P*, *PC* ⊥ to *AB*, and at *C* draw *CX*, another line ⊥ to *AB*.

PC and *CX* determine a plane *MN* (479).

Plane *MN* contains *P* and is ⊥ to *AB* (485).

II. Only one line ⊥ to *AB* can be drawn from *P* (54).

And only one plane ⊥ to *AB* can be passed at *C* (487).

That is, *MN* is the only plane ⊥ to *AB* that can be passed through *P*. Q.E.D.

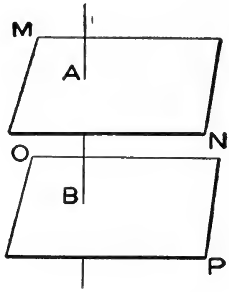
Ex. 1. Considered as lines, why are the spokes of a cart wheel perpendicular to the axle?

Ex. 2. As the hand of a clock revolves, what may it be said to describe, if it is considered of indefinite length? Why?

Ex. 3. Illustrate Proposition VI by revolving a carpenter's square, holding one straight edge against the wall or the floor.

Proposition VIII. Theorem

489. Two planes perpendicular to the same straight line are parallel.



Given: Planes *MN* and *OP* ⊥ to *AB*.

To Prove: Plane *MN* ∥ to plane *OP*.

Proof: If the planes *MN* and *OP* are not ∥, they will meet when sufficiently extended (Def. 468).

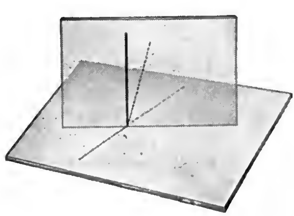
Then there would be two planes from the same point ⊥ to *AB* (⊥ by hyp.).

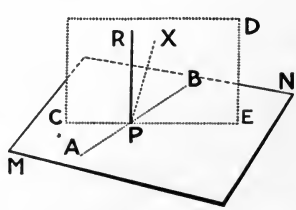
But this is impossible (488).

∴ the planes never meet and are parallel (468). Q.E.D.

Proposition IX. Theorem

490. At a given point in a plane one line can be drawn perpendicular to the given plane, and only one.





Given: Plane *MN* and point *P* within it.

To Prove: One line can be drawn ⊥ to plane *MN* at *P*, and only one.

Proof: I. In plane *MN* draw any line *AB*, through *P*.

Suppose plane *CD* is passed ⊥ to *AB* at *P*, meeting the plane *MN* in *CE*.

In plane *CD* draw *PR* ⊥ to *CE*, from *P*.

Now	*AB* is ⊥ to plane *CD*	(Const.).
	∴ *AB* is ⊥ to *PR*	(473).
	PR is ⊥ to *CE*	(Const.).
	∴ *PR* is ⊥ to plane *MN*	(485). Q.E.D.

II. Suppose another line *PX* to be ⊥ to plane *MN* at *P*.

Then	*PX* and *PR* determine a plane *CD*	(479).
And plane *CD* intersects plane *MN* in line *CE*		(481).
Then *PX* and *PR* would both be ⊥ to *CE* at *P*		(473).
But	this is impossible	(43).

That is, *PX* and *PR* coincide and *PR* is the only ⊥ to plane *MN* at *P*. Q.E.D.

Ex. 1. How many positions can a flagpole occupy without being perfectly erect? How many positions may it assume and be perfectly erect?

Ex. 2. Name all the right angles at *P*, in figure of 490, and tell why each is a right angle.

Ex. 3. What information can the mason or the surveyor obtain from a plumb bob? Does he obtain this information when the bob is swinging or when it is stationary?

Ex. 4. In transplanting a tree to a horizontal lawn a gardener may use a carpenter's square to make certain that the tree is perpendicular to the lawn. In how many different positions must he place the square against the tree to ascertain its erectness? Why?

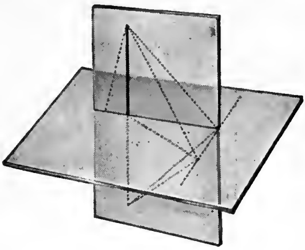
Ex. 5. In the diagram of 490, if *PX* is in plane *CD*, is it perpendicular to *CE*? Why?

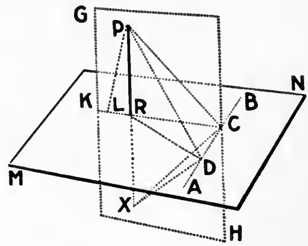
Ex. 6. In the same diagram, if *PX* is perpendicular to *CE*, is it in plane *CD*? Why?

Ex. 7. How many planes are determined by four random fixed points (that is, not all in one plane)?

Proposition X. Theorem

491. Through a given external point one line can be drawn perpendicular to a given plane, and only one.





Given: Plane MN and point P outside of it.

To Prove: One line can be drawn through $P \perp$ to plane MN, and only one.

Proof: I. In plane MN draw any line AB.

Suppose a plane GH is passed through $P \perp$ to AB, meeting plane MN in KC, and AB at C.

In plane GH draw $PR \perp$ to KC and prolong PR to X, making $RX = PR$.

Draw RD to any point in AB, except C.

Draw PC, PD, CX, DX.

Now RC is $\perp$ to PX at its midpoint (Const.).

Also AB is $\perp$ to plane GH (Const.).

Hence $\angle s$ DCP and DCX are rt. $\angle s$ (473).

In rt. $\triangle s$ DCP and DCX, $DC = DC$ (?).

$PC = CX$ (80).

$\therefore \triangle DCP \cong \triangle DCX$ (53).

$\therefore PD = XD$ (?).

$\therefore RD$ is $\perp$ to PX (83).

That is, PR is $\perp$ to RC and RD, in plane MN.

$\therefore PR$ is $\perp$ to plane MN from P (485). Q.E.D.

II. Suppose there is another line PL, $\perp$ to plane MN from P.

Then PR and PL determine a plane GH (479).

This plane intersects plane *MN* in *KC* (481).
PR and *PL* would then both be ⊥ to *KC* (473).
But this is impossible (54).

That is, *PR* and *PL* coincide, and therefore *PR* is the only line ⊥ to plane *MN* from *P*. Q.E.D.

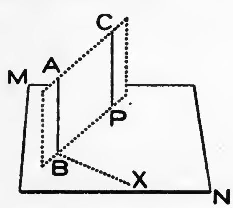
Ex. Name all the right angles at *C* in the figure of 491.

492. Corollary. **If a plane is perpendicular to a line in another plane, any line in the first plane perpendicular to the intersection of the planes is perpendicular to the second plane.**

Proof: Identical with the proof of 491, I.

Proposition XI. Theorem

493. If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other also.



Given: Plane *MN* ⊥ to line *AB*, and *AB* ∥ to *CP*.

To Prove: *CP* ⊥ to plane *MN*.

Proof: *AB* and *CP* determine a plane. (478.)

Pass this plane *BC*, intersecting plane *MN* in line *BP*.

Draw *BX* ⊥ to *BP*, in plane *MN*.

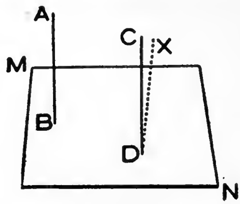
AB is ⊥ to *BX* (473).
∴ *BX* is ⊥ to plane *BC* (485).
But *AB* is ⊥ to *BP* (473).
∴ *BP* is ⊥ to *CP* (64).

That is, plane *BC* is ⊥ to *BX*, and *CP*, in plane *BC*, is ⊥ to the intersection *BP*.

∴ *CP* is ⊥ to plane *MN* (492). Q.E.D.

Proposition XII. Theorem

494. Two lines perpendicular to the same plane are parallel.



Given: Lines *AB* and *CD* ⊥ to plane *MN*.

To Prove: *AB* ‖ to *CD*.

Proof: Through *D*, the foot of *CD*, draw *DX* ‖ to *AB*.

Then *DX* is ⊥ to plane *MN* (493).

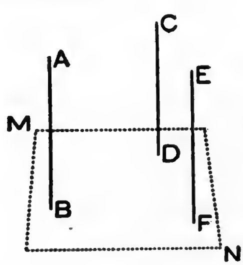
But *CD* is ⊥ to plane *MN* at *D*. (Hyp.)

∴ *DX* and *DC* coincide (490).

That is, *AB* is ‖ to *CD*. Q.E.D.

Proposition XIII. Theorem

495. Two straight lines that are parallel to a third straight line are parallel to each other.



Given: Lines *CD* and *EF* each ‖ to *AB*.

To Prove: *CD* ‖ to *EF*.

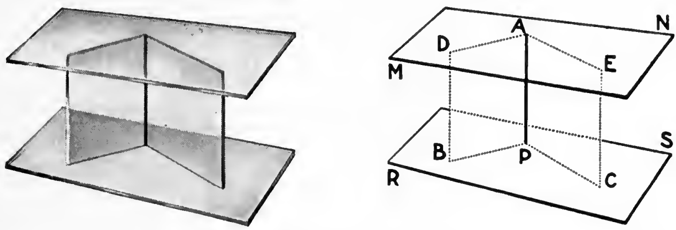
Proof: Suppose plane *MN* is passed ⊥ to *AB*.

∴ *MN* is ⊥ to *CD* and to *EF* (493).

∴ *CD* is ‖ to *EF* (494). Q.E.D.

PROPOSITION XIV. THEOREM

496. A line perpendicular to one of two parallel planes is perpendicular to the other also.



Given: Plane *MN* ∥ to plane *RS*; *AP* ⊥ to plane *RS*.

To Prove: *AP* ⊥ to plane *MN*.

Proof: Through *AP* pass any two planes, *AB* and *AC*, intersecting *MN* in *AD* and *AE*, and intersecting *RS* in *PB* and *PC*, respectively.

AD is ∥ to *PB*, and *AE* is ∥ to *PC* (484).

AP is ⊥ to *PB* and *PC* (473).

∴ *AP* is ⊥ to *AD* and *AE* (64).

∴ *AP* is ⊥ to plane *MN* (485). Q.E.D.

497. COROLLARY. If two planes are each parallel to a third plane, they are parallel to each other.

Proof: Draw a line ⊥ to the third plane.

This line is ⊥ to each of the other planes (496).

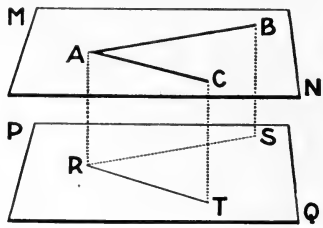
∴ The two planes are ∥ (489).

Ex. 1. Can a line be perpendicular to both of two planes if they are not parallel? Prove.

Ex. 2. Are three lines that are perpendicular to the same plane necessarily parallel?

Proposition XV. Theorem

498. If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to the given plane.



Given: Intersecting lines *AB* and *AC* in plane *MN*; each line ∥ to plane *PQ*.

To Prove: Plane *MN* ∥ to plane *PQ*.

Proof: Draw *AR* ⊥ to *MN* at *A*, meeting *PQ* at *R*.

Through *AR* and *AB* pass plane *AS*, and through *AR* and *AC* pass plane *AT*, intersecting plane *PQ* in *RS* and *RT*, respectively.

Now *AB* is ∥ to *RS*, and *AC* is ∥ to *RT* (483).

AR is ⊥ to *AB* and *AC* (473).

∴ *AR* is ⊥ to *RS* and *RT* (64).

Hence *AR* is ⊥ to plane *PQ* (485).

∴ plane *MN* is ∥ to plane *PQ* (489).

Q.E.D.

Ex. 1. Can one line be perpendicular to two other lines that do not intersect? How?

Ex. 2. Can one line be perpendicular to two other lines that do intersect? How?

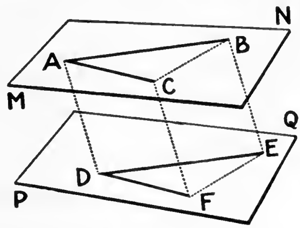
Ex. 3. Could 484 be quoted correctly as the reason that *AB* is parallel to *RS* in Proposition XV?

Ex. 4. Could 483 be quoted correctly as the reason that *AD* is parallel to *BP* in Proposition XIV?

Ex. 5. Prove Proposition XV by drawing *AR* perpendicular to *PQ* from *A*.

PROPOSITION XVI. THEOREM

499. If two angles, not in the same plane, have their sides parallel each to each, and extending in the same directions from their vertices, the angles are equal and the planes are parallel.



Given: $\angle BAC$ in plane MN and $\angle EDF$ in plane PQ; $AB \parallel$ to DE; $AC \parallel$ to DF, and extending in the same directions.

To Prove: I. $\angle BAC = \angle EDF$.

II. Plane $MN \parallel$ to plane PQ.

Proof: I. Take DE and AB equal, and DF and AC equal.

Draw AD, BE, CF, BC, EF.

The figure $ABED$ is a ▱ (129).

$\therefore AD = BE$ (124).

Also $ACFD$ is a ▱ (?).

$AD = CF$ (?).

$\therefore BE = CF$ (?).

Again, AD is $\parallel$ to BE and AD is $\parallel$ to CF (120).

$\therefore BE$ is $\parallel$ to CF (495).

$\therefore BCFE$ is a ▱ (129).

Now in $\triangle$s ABC and DEF, $AB = DE$; $AC = DF$ (Const.).

Also $BC = EF$ (124).

$\therefore \triangle ABC \cong \triangle DEF$ (78).

$\therefore \angle BAC = \angle EDF$ (27).

Q.E.D.

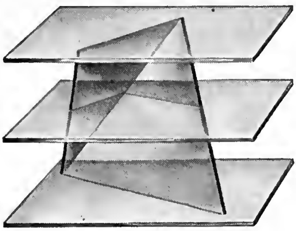
II. AB is $\parallel$ to plane PQ and AC is $\parallel$ to plane PQ (482).

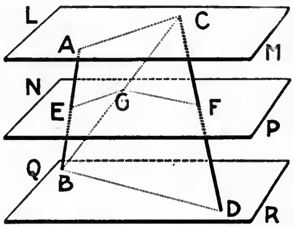
$\therefore$ plane MN is $\parallel$ to plane PQ (498).

Q.E.D.

Proposition XVII. Theorem

500. If three parallel planes intersect two straight lines, the corresponding intercepts are proportional.





Given: Parallel planes, *LM*, *NP*, *QR*, intersecting line *AB* at *A*, *E*, *B*, and *CD* at *C*, *F*, *D*, respectively.

To Prove: $AE : EB = CF : FD$.

Proof: Draw *BC*, meeting plane *NP* at *G*.

Through *AB* and *BC* pass a plane cutting *LM* in *AC* and *NP* in *EG*.

Through *BC* and *CD* pass a plane cutting *NP* in *GF* and *QR* in *BD*.

Now *EG* is ∥ to *AC* and *GF* is ∥ to *BD* (484).

$$\therefore \frac{AE}{EB} = \frac{CG}{GB} \text{ and } \frac{CG}{GB} = \frac{CF}{FD} \quad (293).$$

Consequently $AE : EB = CF : FD$ (Ax. 1).

Q.E.D.

Also $AE + EB : AE = CF + FD : CF$ (284).
Or $AE + EB : EB = CF + FD : FD$

$\therefore AB : AE = CD : CF$ (Ax. 6).
Or $AB : EB = CD : FD$

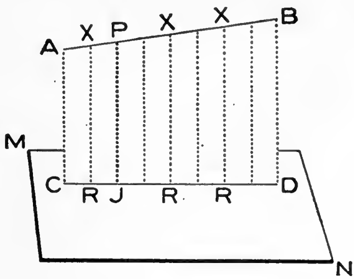
$$\therefore \frac{AB}{CD} = \frac{AE}{CF} = \frac{EB}{FD} \quad (282).$$

Ex. 1. If any number of lines which meet at a point are cut by two parallel planes, the corresponding intercepts are proportional.

Ex. 2. In the above diagram, why are not *AC* and *BD* parallel? Under what condition would *EGF* be a straight line?

PROPOSITION XVIII. THEOREM

501. The projection of a straight line upon a plane is a straight line.*



Given: Line *AB* and plane *MN*.

To Prove: The projection of *AB* on *MN* is a straight line.

Proof: Draw *PJ* ⊥ to plane *MN* from any point *P*, in *AB*.

AB and *PJ* determine a plane. (479).

Plane *AD* cuts plane *MN* in a straight line *CD*. (481).

Now in plane *AD*, draw *XR* ∥ to *PJ* from *X*, any other point in *AB*.

XR is ⊥ to plane *MN*. (493).

Now *R* is the projection of *X*. (469).

∴ *CD* is the projection of *AB*. (469).

That is, the projection of *AB* upon the plane *MN* is a straight line. Q.E.D.

Ex. 1. What is the length of the projection of a 5 ft. rod, inclined at an angle of 45°?

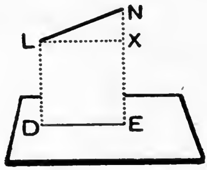
Ex. 2. A ladder 26 ft. long leans against the wall of a house, at a point 10 ft. from the ground. What is the length of the projection of the ladder on the ground?

502. COROLLARY. A straight line and its projection upon a plane are in the same plane.

* Except only if the given line is a normal to the given plane.

Proposition XIX. Theorem

503. A line not parallel to a plane is longer than its projection upon the plane.



Given: A plane and line *LN* not ∥ to the plane, and *DE* the projection of *LN* upon the plane.

To Prove: $LN > DE$.

Proof: Draw *LD* and *NE*.

Draw *LX* ⊥ to *NE* from *L*, in the plane *LE*.

LD and *NE* are ⊥ to the plane. (Def. of projection, 469).

LXED is a rectangle. (157).

Now $LN > LX$ (87).

But $LX = DE$ (124).

$\therefore LN > DE$ (Ax. 6). Q.E.D.

Ex. If a line is parallel to a plane, all points of the line are equally distant from the plane.

Proposition XX. Theorem

504. Of all lines that can be drawn to a plane from a point:

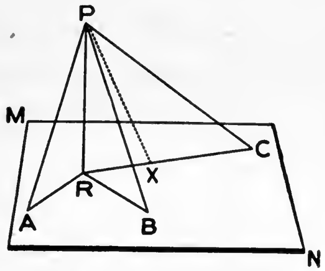
I. The perpendicular is the shortest.

II. Oblique lines having equal projections are equal.

III. Equal oblique lines have equal projections.

IV. Oblique lines having unequal projections are unequal, and the line having the greater projection is the longer.

V. Unequal oblique lines have unequal projections, and the longer line has the greater projection.



I. **Given:** Plane MN; point P; $PR \perp$ to MN; any other line from P to plane MN, as PA.

To Prove: $PR < PA$.

Proof: Draw AR.

Now PR is $\perp$ to AR (473).

And PA is not $\perp$ to AR (54).

$\therefore PR < PA$ (87). Q.E.D.

II. **Given:** Oblique lines PA and PB whose projections, AR and BR, are equal.

To Prove: $PA = PB$.

Proof: The right $\triangle$ PAR and PRB are $\cong$ (?).

III. **Given:** Equal oblique lines PA and PB.

To Prove: Their projections, AR and BR, are equal.

Proof: The right $\triangle$ PAR and PRB are $\cong$ (?).

IV. **Given:** Oblique lines PC and PA; proj. $RC >$ proj. RA.

To Prove: $PC > PA$.

Proof: In $\triangle$ PRC, take on RC, $RX = RA$, and draw PX.

Now $PC > PX$ (88, III).

But $PA = PX$ (504, II).

$\therefore PC > PA$ (Ax. 6). Q.E.D.

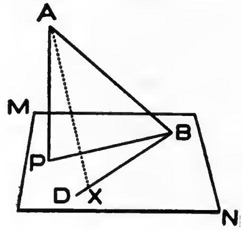
V. **Given:** Unequal oblique lines, $PC > PA$.

To Prove: Projection $RC >$ projection RA.

Proof: By method of exclusion (See 90).

Proposition XXI. Theorem

505. The acute angle that a line makes with its own projection upon a plane is the least angle that the line makes with any line of the plane.



Given: AB, any line meeting plane MN at B; BP, its projection upon MN; BD, any other line in MN, through B.

To Prove: $\angle ABP < \angle ABD$.

Proof: On BD take $BX = BP$ and draw AX.

In $\triangle$ APB and ABX,

$AB = AB$ (?).

$BP = BX$ (Const.).

But $AP < AX$ (504, I).

$\therefore \angle ABP < \angle ABD$ (92).

Q.E.D.

Ex. 1. If PR is 12 in., and AP 13 in., find the length of AR, in 504.

Ex. 2. Does the longer of two lines always have the longer projection on the same plane? Could they have equal projections? Illustrate.

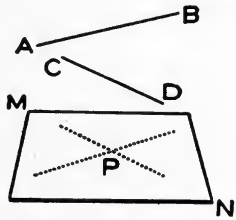
Ex. 3. With what line in a plane does a line oblique to that plane make the greatest angle?

Ex. 4. With what line in a plane does a line oblique to that plane make right angles?

Ex. 5. How do you construct the projection of a curved line upon a plane? When is this a straight line?

PROPOSITION XXII. THEOREM

506. Through a given point one plane can be passed parallel to any two given non-parallel lines in space, and only one.



Given: Point *P*; two lines, *AB* and *CD*.

To Prove: Through *P* one plane can be passed ∥ to *AB* and *CD*, and only one.

Proof: I. Through *P* draw a line ∥ to *AB* and another ∥ to *CD*.

Pass a plane *MN*, containing these lines.

MN is ∥ to both *AB* and *CD* (482).

II. Only one line can be drawn through *P* ∥ to *AB*, and only one ∥ to *CD* (Ax. 13).

∴ there is only one plane (479).

Q.E.D.

507. COROLLARY. If two lines are not in the same plane, one plane and only one can be passed through one of these lines parallel to the other.

[Through a point in one line draw a line ∥ to the other line, etc.]

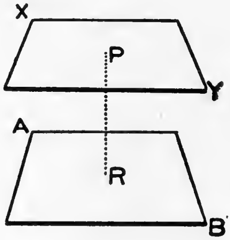
Ex. 1. Is Proposition XXII true if the given lines intersect? Is it true if they are parallel?

Ex. 2. Explain, so that a blind boy could understand, how to pass a plane through a given point and parallel to two pencils he may hold in his outstretched hands.

Ex. 3. Can two lines be parallel to a plane and not be parallel to each other? Illustrate, by means of pencils and the ceiling.

Proposition XXIII. Theorem

508. Through a given point one plane can be passed parallel to a given plane, and only one.



Given: (?). **To Prove:** (?).

Proof: I. Suppose *PR* is drawn ⊥ to plane *AB*; and plane *XY* is passed ⊥ to *PR* at *P*.

Then *XY* is ∥ to *AB* (489).

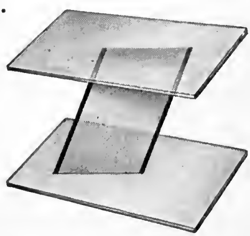
II. Only one line ⊥ to *AB* can be drawn from *P* (491).

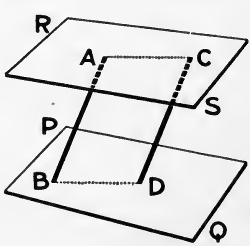
Only one plane ⊥ to *PR* can be passed at *P* (487).

∴ only one plane can contain *P* and be ∥ to *AB*. Q.E.D.

Proposition XXIV. Theorem

509. Parallel lines included between parallel planes are equal.





Given: (?). **To Prove:** (?).

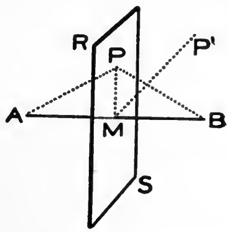
Proof: The plane determined by *AB* and *CD* intersects *RS* and *PQ* in lines *AC* and *BD*, which are ∥ (484).

∴ *ABDC* is a ▱ (Def.).

Hence $AB = CD$ (124). Q.E.D.

Proposition XXV. Theorem

510. The plane perpendicular to a line at its midpoint is the locus of points in space equally distant from the extremities of the line.



Given: Plane $RS \perp$ to AB at its midpoint, M.

To Prove: Plane RS is the locus of points in space equally distant from A and B.

Proof: (1) Take P, any point in RS.

Draw PM, PA, PB.

Now PM is $\perp$ to AB (473).

$\therefore PA = PB$ (80).

That is, any point in RS is equally distant from A and B.

(2) Take P', any point outside of RS. Draw $P'M$.

Now $P'M$ is *not* $\perp$ to AB (486).

$\therefore P'$, any point outside of plane RS, is not equally distant from A and B (81).

$\therefore$ Plane RS is the locus of points in space, equally distant from A and B (246). Q.E.D.

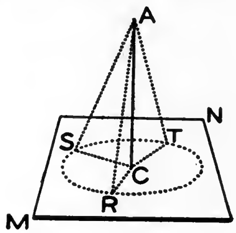
Ex. 1. How can you find a line in a plane such that each of its points is equally distant from two given points?

Ex. 2. How can you find that point in a given line which is equally distant from two given points?

Ex. 3. There are two definite lines in space. It is desired to find all points that are equally distant from the ends of one line and at the same time equally distant from the ends of the other. How can this be done?

Proposition XXVI. Theorem

511. The locus of points in space equally distant from all the points in the circumference of a circle is the line perpendicular to the plane of the circle at its center.



Given: (?). **To Prove:** (?).

Proof: I. Any point in AC is equally distant from all the points in the circumference of the circle (504, II).

II. Any point equally distant from all points of the circumference of the circle is in AC (504, III).

∴ AC is the required locus (246). Q.E.D.

Ex. 1. What is the locus of points equally distant from two given points?

Ex. 2. What is the locus of points equally distant from three given points?

Ex. 3. Draw a triangle on the blackboard and a definite straight line on the floor. Tell how to find the one point which is both equally distant from the vertices of the triangle and from the ends of the line. Is there *always* one point?

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512. The **distance** from a point to a plane is the length of the perpendicular from the point to the plane.

Thus, the word "distance," referring to the shortest line from a point to a plane, implies the perpendicular.

The **inclination** of a line to a plane is the angle between the line and its projection upon the plane.

ORIGINAL EXERCISES

1. Through one straight line a plane can be passed parallel to any other straight line in space, and only one.

Through a point of the first line draw a line ∥ to the second.

2. Two parallel planes are everywhere equally distant.

3. If a line and a plane are parallel, another line parallel to the given line and through any point in the given plane lies wholly in the given plane.

Through the given line and the point P pass a plane cutting the given plane in PX. [Use 483.]

4. A straight line parallel to the intersection of two planes, but in neither, is parallel to both planes.

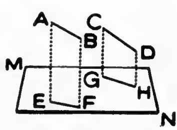
5. If two straight lines are parallel and two intersecting planes are passed, each containing one of the lines, the intersection of these planes is parallel to each of the given lines.

6. If three straight lines through a point meet the same straight line, these four lines all lie in the same plane.

7. If a straight line meets two parallel planes, its inclinations to the planes are equal.

8. Two parallel planes can be passed, each containing one of two given lines in space. Is this ever impossible?

9. If each of three straight lines intersects the other two, the three lines all lie in a plane.



10. The projections of two parallel lines on a plane are parallel.

Proof: AB is ∥ to CD (?). AE is ∥ to CG (?).
∴ planes AF and CH are ∥ (?); etc.

11. If two lines in space are equal and parallel, their projections on a plane are equal and parallel.

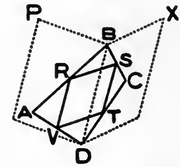
12. If a plane is parallel to one of two parallel lines, it is parallel to the other.

13. If a straight line and a plane are perpendicular to the same straight line, they are parallel.

14. Equal oblique lines drawn to a plane from one point have equal inclinations with the plane.

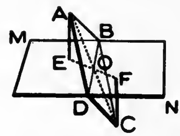
15. If a line and a plane are both parallel to the same line, they are parallel to each other.

16. Four points in space, A, B, C, D, are joined, and these four lines are bisected. Prove that the four lines joining (in order) the four midpoints of the first lines form a parallelogram.



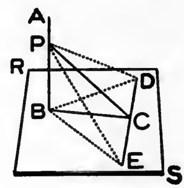
Proof: Pass plane DP through points A, D, B, and plane DX through points B, C, D, — these planes intersecting in BD. ST is $\parallel$ to BD and $= \frac{1}{2} BD$ (?); etc.

17. If a plane is passed containing a diagonal of a parallelogram and perpendiculars are drawn to the plane from the other vertices of the parallelogram, they are equal.



To Prove: $AE = CF$. **Proof:** Draw diagonal AC. Draw EO and OF in plane MN. EO, OF, and EOF are projections; etc.

18. If from the foot of a perpendicular to a plane, a line is drawn at right angles to any line in the plane, the line connecting this point of intersection with any point in the perpendicular is perpendicular to the line in the plane.



Given: $AB \perp$ to plane RS; $BC \perp$ to DE in the plane; PC drawn from C to P, in AB.

To Prove: PC is $\perp$ to DE.

Proof: Take $CD = CE$, draw PD, PE, BD, BE. BC is $\perp$ to DE at its midpoint (?). $\therefore BD = BE$ (?). $PD = PE$ (?) (504, II).
$\therefore PC$ is $\perp$ to DE (?) (83).

19. A line PB is perpendicular to a plane at B, and a line is drawn from B meeting any line DE, of the plane, at C. If PC is perpendicular to DE, BC is perpendicular to DE.

20. Are two planes that are parallel to the same straight line necessarily parallel?

21. If each of two parallel lines is parallel to a plane, is the plane of these lines also parallel to the given plane?

22. Is a three-legged stool always stable on the floor? Why? Is a four-legged chair always stable? Why?

23. What is the locus in space of points equally distant from two parallel planes? from two parallel lines?

24. What is the locus of points in space at a given distance from a given plane?

25. What is the locus of points in a plane at a given distance from an external point?

26. What is the locus of points in space equally distant from two points and equally distant from two parallel planes?

27. What is the locus of points in space equally distant from the vertices of a triangle?

28. What is the locus of all straight lines perpendicular to a given straight line at a given point?

29. What is the locus of all lines parallel to a given plane and drawn through a given point?

30. If the points in a line satisfy one condition and the points in a plane satisfy another condition, what will be true of their intersection? What will be true if they do not intersect?

31. If the points in one plane satisfy one condition and the points in another plane satisfy another condition, what is true of their intersection? What is true if the planes are parallel?

32. Construct a plane perpendicular to a given line at a given point in the line.

33. Construct a plane perpendicular to a given line through a given external point.

34. Construct a line perpendicular to a given plane through a given point in the plane; through a given external point.

35. Construct a plane parallel to a given plane through a given point.

36. Construct a number of equal oblique lines to a plane from a given external point.

37. Construct a line through a given point parallel to a given plane.

38. Construct through a given point a line parallel to each of two given intersecting planes.

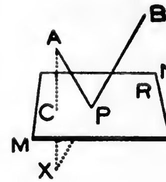
39. Construct a plane containing one given line and parallel to another.

40. Construct a plane through a given point parallel to any two given lines in space.

41. Construct a line through a given point in space which intersects two given lines not in the same plane.

When is there no such line? Is there ever more than one?

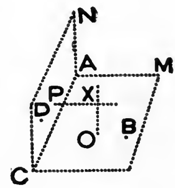
42. Find a point in a plane such that the sum of the two lines joining it to two fixed points on one side of the plane is the least possible.



Construction: Draw $AC \perp$ to plane MN and prolong it to X, making $CX = AC$. Draw BX, meeting plane MN at P. Draw AP. Take any other point R in plane MN.

Statement: $AP + PB < AR + RB$. Etc.

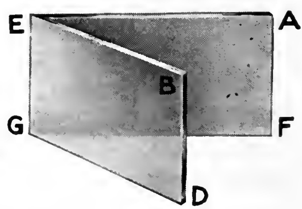
43. Find a point in a given plane equally distant from three given points. Is this ever impossible?



44. Find the one point equally distant from four given points not in the same plane.

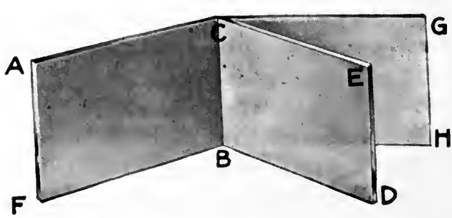
Construction: Pass plane CM, containing points A, B, C, and plane CN, containing A, D, C. Find O, the center of the ⊙ containing A, B, C. Find P, similarly. Draw the locus of points equally distant from A, B, C. (Consult 511.) Draw the locus of points equally distant from A, D, C. The plane $\perp$ to AC at its midpoint contains both these loci. (Explain.) Hence OX and PX intersect (?). ∴ X is the required point.

DIHEDRAL ANGLES



513. A **dihedral angle** is the amount of divergence of two intersecting planes. The **edge** of the dihedral angle is the line of intersection of the planes. The **faces** of the dihedral angle are the planes.

The intersecting planes AG and ED form the dihedral angle whose edge is EG, which is named $A–GE–D$; or, when there is only one dihedral angle at the edge, "the angle EG."

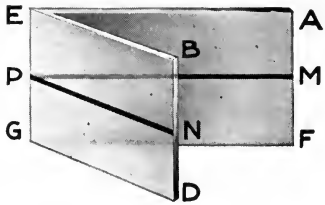


514. Adjacent dihedral angles are two dihedral angles that have the same edge and a common face between them.

Vertical dihedral angles are two dihedral angles that have the same edge, the faces of one being the extensions of the faces of the other.

515. The **plane angle** of a dihedral angle is the angle formed by two straight lines, one in each face, and perpendicular to the edge at the same point.

If *PM* is in plane *AG* and perpendicular to *EG*, and *PN* is in plane *ED* and perpendicular to *EG* at *P*, the angle *MPN* is the plane angle of the dihedral angle *EG*.



516. If one plane meets another, making the adjacent dihedral angles equal, these angles are **right dihedral angles.**

One plane is **perpendicular** to another plane if the dihedral angle formed by the two planes is a right dihedral angle.

517. Two dihedral angles are **equal** if they can be made to coincide.

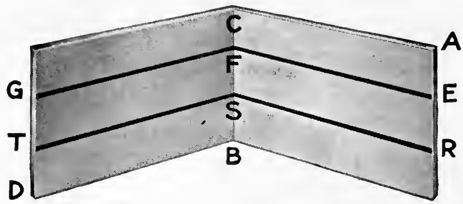
A dihedral angle is **acute, right,** or **obtuse** according as its plane angle is acute, right, or obtuse.

Dihedral angles are **complementary** or **supplementary, corresponding, alternate-interior,** etc., according as their plane angles are complementary or supplementary, corresponding, alternate-interior, etc.

NOTE. An open book often assists a student to a clear apprehension of the magnitude of dihedral angles. Thus he can see the angle increase during the act of opening the book, and observe the acute, right, and obtuse dihedrals. With the aid of two books he can understand better, perhaps, the meaning of complementary dihedrals, supplementary dihedrals, corresponding dihedrals, alternate-interior dihedrals, etc.

Proposition XXVII. Theorem

518. The plane angles of a dihedral angle are all equal.



Given: $\angle EFG$, the plane $\angle$ of dihedral $\angle BC$, at F, and $\angle RST$, the plane $\angle$ at S.

To Prove: $\angle EFG = \angle RST$.

Proof: EF is $\parallel$ to RS (62).

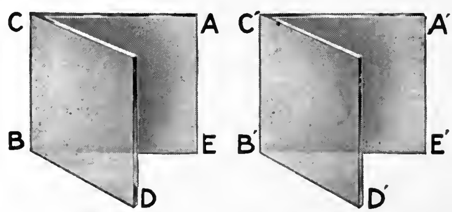
And FG is $\parallel$ to ST (?).

$\therefore \angle EFG = \angle RST$ (499). Q.E.D.

519. Corollary. The plane of the plane angle of a dihedral angle is perpendicular to the edge. (485).

Proposition XXVIII. Theorem

520. Two dihedral angles are equal if their plane angles are equal.



Given: Dihedral $\angle$s CB and $C'B'$ whose plane $\angle$s EBD and $E'B'D'$ are equal.

To Prove: Dih. $\angle CB =$ dih. $\angle C'B'$.

Proof: CB is $\perp$ to plane EBD.

And $C'B'$ is $\perp$ to plane $E'B'D'$ (519).

Apply dih. $\angle C'B'$ to dih. $\angle CB$ so that the plane $\angle E'B'D'$ coincides with its equal $\angle EBD$.

Now $C'B'$ coincides with CB (490).

$\therefore$ plane $C'D'$ coincides with plane CD and plane $C'E'$ coincides with plane CE (479).

$\therefore$ dih. $\angle CB =$ dih. $\angle C'B'$ (517).

Q.E.D.

Proposition XXIX. Theorem

521. If two dihedral angles are equal, their plane angles are equal. [Converse.]

Proof: Superpose dih. $\angle C'B'$ upon its equal dih. $\angle CB'$ making B' fall on B, and edge $B'C'$ on BC.

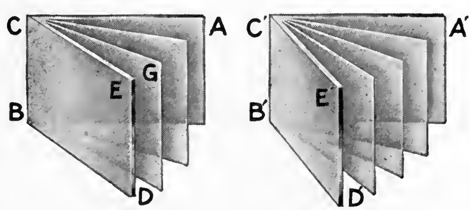
Then face $C'D'$ coincides with face CD, etc.

522. Corollary. Two vertical dihedral angles are equal. (See 520.)

523. Corollary. The plane angle of a right dihedral angle is a right angle; and if the plane angle of a dihedral angle is a right angle, the dihedral angle is right. (See 516.)

Proposition XXX. Theorem

524. Two dihedral angles have the same ratio as their plane angles.



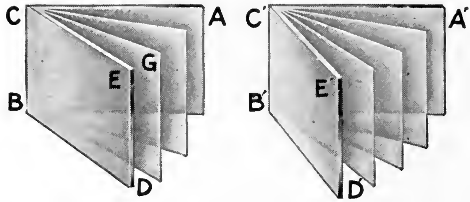
Given: Dihedral $\angle s$ $A\text{–}BC\text{–}D$ and $A'\text{–}B'C'\text{–}D'$, having plane $\angle s$ ACE and $A'C'E'$, respectively.

To Prove:

Dih. $\angle A\text{–}BC\text{–}D$: dih. $\angle A'\text{–}B'C'\text{–}D' = \angle ACE : \angle A'C'E'$.

Proof: I. If the plane angles are **commensurable.**

There exists a common unit of measure of the plane $\angle s$ ACE and $A'C'E'$ (224).



Suppose this unit when applied to these angles is contained 3 times in $\angle ACE$ and 4 times in $\angle A'C'E'$.

$$\therefore \frac{\angle ACE}{\angle A'C'E'} = \frac{3}{4} \qquad \text{(Ax. 3).}$$

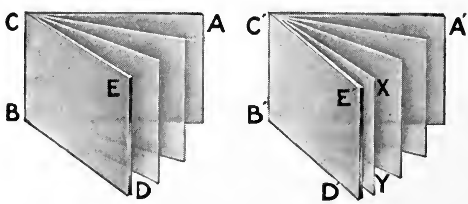
Pass planes through the edges and the several lines of division of the angles.

Dih. $\angle A\text{–}BC\text{–}D$ is divided into 3 parts; dih. $\angle A'\text{–}B'C'\text{–}D'$ is divided into 4 parts; all of these seven parts are equal (520).

$$\therefore \frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}D'} = \frac{3}{4} \qquad \text{(Ax. 3).}$$

$\therefore$ dih. $\angle A\text{–}BC\text{–}D$: dih. $\angle A'\text{–}B'C'\text{–}D' = \angle ACE : \angle A'C'E'$ (Ax. 1).

II. If the plane angles are **incommensurable.**



There does not exist a common unit. Suppose $\angle ACE$ to be divided into equal parts (any number of them).

Apply one of these as a unit of measure to $\angle A'C'E'$. There is a remainder, $XC'E'$, left over (because the $\angle s$ are incommensurable).

Pass a plane $C'Y$, determined by $B'C'$ and $C'X$.

Now $\frac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}Y} = \frac{\angle ACE}{\angle A'C'X}$ (Commensurable ∠s).

Indefinitely increase the number of subdivisions of $\angle ACE$.

Then each part, that is, our unit or divisor, is indefinitely decreased. Hence $XC'E'$, the remainder, is indefinitely decreased.

That is, $\angle XC'E'$ approaches zero as a limit.

And dih. $\angle X\text{–}B'C'\text{–}D'$ approaches zero as a limit.

$\therefore \angle A'C'X$ approaches $A'C'E'$ as a limit; and dih. $\angle A'\text{–}B'C'\text{–}Y$ approaches dih. $\angle A'\text{–}B'C'\text{–}D'$ as a limit.

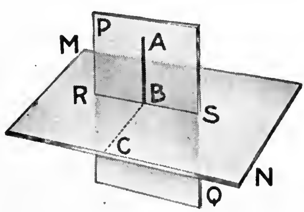
$\therefore \frac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}Y}$ approaches $\frac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}D'}$ as a limit;

and $\frac{\angle ACE}{\angle A'C'X}$ approaches $\frac{\angle ACE}{\angle A'C'E'}$ as a limit.

$$\therefore \frac{\text{dih.} \angle A\text{–}BC\text{–}D}{\text{dih.} \angle A'\text{–}B'C'\text{–}D'} = \frac{\angle ACE}{\angle A'C'E'}$$ (229). Q.E.D.

Proposition XXXI. Theorem

525. If a straight line is perpendicular to a plane, any plane containing this line is perpendicular to the given plane.



Given: Line $AB \perp$ to plane MN; plane PQ containing AB and intersecting plane MN in RS.

To Prove: Plane PQ is $\perp$ to plane MN.

Proof: In plane MN draw $BC \perp$ to RS.

Now AB is $\perp$ to RS. (473).

$\therefore \angle ABC$ is the plane $\angle$ of dih. $\angle P\text{–}SR\text{–}N$. (515).

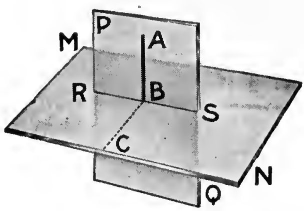
But $\angle ABC$ is a rt. $\angle$. (473).

$\therefore PQ$ is $\perp$ to MN (516). Q.E.D.

526. Corollary. If a plane is perpendicular to the edge of a dihedral angle, it is perpendicular to each face. (See 525.)

Proposition XXXII. Theorem

527. If one plane is perpendicular to another, any line in either plane, perpendicular to their intersection, is perpendicular to the other plane. [Converse of 525.]



Given: Plane *PQ* ⊥ to plane *MN*; *AB* in plane *PQ* ⊥ to the intersection, *RS*.

To Prove: *AB* ⊥ to plane *MN*.

Proof: In plane *MN* draw *BC* ⊥ to *RS*.

Now ∠ *ABC* is the plane angle of the dih. ∠ *P–SR–N*. (515).

∴ ∠ *ABC* is a rt. ∠ (523).

∴ *AB* is ⊥ to *BC* (16).

But *AB* is ⊥ to *RS* (Hyp.).

∴ *AB* is ⊥ to plane *MN* (485).

Q.E.D.

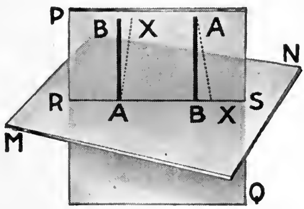
Ex. 1. In the figure of 527 prove *BC* perpendicular to the plane *PQ*. Also prove *RS* perpendicular to the plane *ABC*.

Ex. 2. In the figure of 527 prove plane *ABC* perpendicular to the planes *MN* and *PQ*.

Ex. 3. Under what condition will a line in one face of a dihedral angle meet a line in the other face?

PROPOSITION XXXIII. THEOREM

528. If one plane is perpendicular to another, a line drawn from any point in their intersection and perpendicular to one plane, lies in the other.



Given: Plane *PQ* ⊥ to plane *MN*, intersecting in *RS*; *AB* (left-hand) ⊥ to plane *MN* from *A*, in *RS*.

To Prove: *AB* is in plane *PQ*.

Proof: At *A* erect in plane *PQ*, *AX* ⊥ to *RS*.

Then *AX* is ⊥ to plane *MN* (527).

But *AB* is ⊥ to plane *MN* at *A* (Hyp.).

∴ *AB* and *AX* coincide (490).

That is, *AB* lies in plane *PQ*. Q.E.D.

PROPOSITION XXXIV. THEOREM

529. If one plane is perpendicular to another, a line drawn from any point in one plane, and perpendicular to the other, lies in the first plane.

Given: Plane *PQ* ⊥ to plane *MN*; *AB* (right-hand) ⊥ to plane *MN* from *A*, any point in plane *PQ*.

To Prove: *AB* lies in plane *PQ*.

Proof: From *A* draw in plane *PQ*, *AX* ⊥ to *RS*.

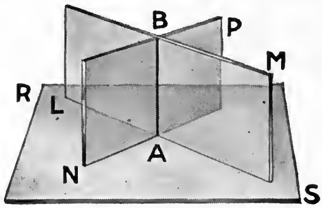
Then *AX* is ⊥ to plane *MN* (527).

∴ *AB* and *AX* coincide (491).

That is, *AB* lies in plane *PQ*. Q.E.D.

PROPOSITION XXXV. THEOREM

530. If two planes are perpendicular to a third plane, their intersection also is perpendicular to that plane.



Given: Planes *LM* and *NP*, each ⊥ to plane *RS*.

To Prove: The intersection *AB* is ⊥ to plane *RS*.

Proof: If, at *A*, a line is erected ⊥ to plane *RS*, it will lie in plane *LM* (528).

This ⊥ will lie also in plane *NP* (?).

∴ this ⊥ is the intersection *AB* (466).

That is, *AB* is ⊥ to plane *RS*. Q.E.D.

531. COROLLARY. If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection. (The same truth as 530.)

Ex. 1. If each of three planes is perpendicular to the other two, each of the three intersections is perpendicular to the remaining plane, and perpendicular to the other two intersections.

[Use figure of 530.]

Ex. 2. If two parallel planes are each perpendicular to a third plane, their intersections with that plane are parallel.

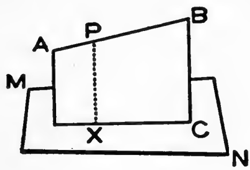
Ex. 3. Is Proposition XXXV true in the case of the intersecting walls of a building?

Ex. 4. What is the "exterior point" that, together with the plumb-bob, determines the vertical plane for the mason?

Ex. 5. If the builder keeps the edge of a building perpendicular to the plane of a level street, the two intersecting walls will also be perpendicular to the street. Why?

Proposition XXXVI. Theorem

532. Through a given line not perpendicular to a plane, one plane can be passed perpendicular to that plane, and only one.



Given: AB not ⊥ to plane MN.

To Prove: Through AB one plane can be passed ⊥ to MN, and only one.

Proof: I. From P, any point in AB, draw PX ⊥ to MN.
Through AB and PX pass plane AC.
Plane AC is ⊥ to plane MN (525).

II. Suppose another plane containing AB is ⊥ to plane MN.
Then the intersection AB, of these two planes, which are ⊥ to plane MN, will be ⊥ to plane MN (530).
But AB is not ⊥ to plane MN (Hyp.).
∴ there is only one plane containing AB that is ⊥ to plane MN. Q.E.D.

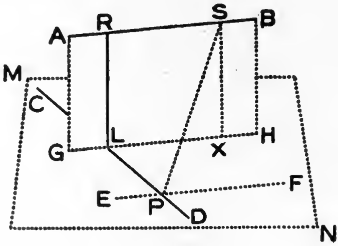
533. Corollary. The plane containing a straight line and its projection upon a plane is perpendicular to the given plane.

534. Corollary. If a line meets its projection on a plane, any line of the plane perpendicular to one of these lines at their intersection is perpendicular to the other also.

Proof: Use fig. of 505.
Plane ABP is ⊥ to plane MN (533).
A line ⊥ to plane ABP at B will lie in plane MN (528).
∴ this line is ⊥ to both AB and PB (473).

Proposition XXXVII. Theorem

535. Between any two straight lines not in the same plane, one and only one common perpendicular can be drawn, and this common perpendicular is the shortest line that can be drawn between the two lines.



Given: Lines *AB* and *CD* not in the same plane.

To Prove: I. One line can be drawn ⊥ to *AB* and *CD*.

II. Only one ⊥ can be drawn.

III. This ⊥ is the shortest line that can be drawn between *AB* and *CD*.

Proof: I. At *P*, any point in *CD*, draw *EF* ∥ to *AB*.

Pass plane *MN*, containing *CD* and *EF*.

Pass plane *AH* through *AB* and ⊥ to plane *MN*, intersecting plane *MN* in *GH*, and *CD* at *L*.

In plane *AH* draw *RL* ⊥ to *GH*.

	Plane *MN* is ∥ to *AB*	(482).
	GH is ∥ to *AB*	(483).
	RL is ⊥ to *GH*	(Const.).
	∴ *RL* is ⊥ to plane *MN*	(527).
	∴ *RL* is ⊥ to *CD*	(473).
Also	*RL* is ⊥ to *AB*	(64).
That is,	*RL* is ⊥ to both the given lines.	Q.E.D.

II. If another line can be drawn ⊥ to *AB* and *CD*, suppose *SP* is this ⊥. In plane *AH* draw *SX* ⊥ to *GH*.

Then *SX* is ⊥ to plane *MN* (527).

But if *SP* is ⊥ to *AB*, it is ⊥ to *EF* (64).

∴ *SP* is ⊥ to plane *MN* (485).

Thus there are two ⊥s from *S* to plane *MN* (*SX* and *SP*).

But this is impossible (491).

∴ there can be no second ⊥ to these two given lines.

Q.E.D.

III. Suppose *SP* is any other line between *AB* and *CD*.

Now *RL* is ∥ to *SX* (62).

∴ *RX* is a ▱ (120).

∴ $RL = SX$ (124).

But $SX < SP$ (504, I).

∴ $RL < SP$ (Ax. 6).

That is, *RL* is shorter than any other line between *AB* and *CD*. Q.E.D.

Ex. 1. In the figure of 535 prove that a plane perpendicular to *RL* at its midpoint will be parallel to *AB* and *CD*.

Ex. 2. Prove, also, that this plane will bisect *SP*.

Ex. 3. Prove that if *CD* is not perpendicular to *EF*, no plane can be passed through *AB*, perpendicular to *CD*.

Ex. 4. Tell how we can construct one plane perpendicular to another.

Ex. 5. Tell how we can construct one plane through a given point, perpendicular to any two given planes.

Ex. 6. Tell how we can construct a plane containing a given line and perpendicular to a given plane.

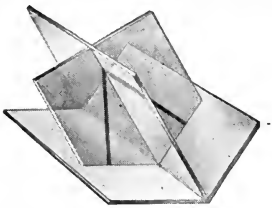
Ex. 7. Tell, so that a blind boy could understand, how to draw a line perpendicular to any two lines in space (not in the same plane).

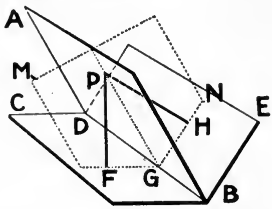
Ex. 8. If two planes are parallel, what is the form of the projection on one plane, of a circle in the other?

Ex. 9. If two planes are perpendicular, what is the form of the projection on one plane, of a circle in the other?

PROPOSITION XXXVIII. THEOREM

536. Every point in a plane bisecting a dihedral angle is equally distant from the faces of the angle.





Given: Plane *AB*, bisecting the dih. ∠ *C–BD–E*; any point *P* in plane *AB*; *PF* ⊥ to face *CB*; *PH* ⊥ to face *DE*.

To Prove: *PF* = *PH*.

Proof: Pass plane *MN*, containing *PF* and *PH*, intersecting *CB* in *FG*, *AB* in *PG*, *DE* in *HG*, *BD* at *G*.

Now plane *MN* is ⊥ to planes *CB* and *DE* (525).

∴ plane *MN* is ⊥ to *BD* (531).

∴ *BG* is ⊥ to *FG*, *PG*, and *HG* (473).

Hence ∠ *PGF* is the plane ∠ of dih. ∠ *A–BD–C* and ∠ *PGH* is the plane ∠ of dih. ∠ *A–BD–E* (515).

These dih. ∠s are = (Hyp.).

∴ ∠ *PGF* = ∠ *PGH* (521).

∠s *PFG* and *PHG* are rt. ∠s (473).

In the right △s *PFG* and *PGH*, *PG* = *PG* (?).

∠ *PGF* = ∠ *PGH*. ∴ △ *PFG* ≅ △ *PGH* (?).

∴ *PF* = *PH* (?). Q.E.D.

537. COROLLARY. Any point in a dihedral angle and equally distant from its faces is in the plane bisecting the angle.

To Prove: The plane *AB*, determined by the point *P* and the edge *BD*, bisects the dih. ∠ *C–BD–E*.

538. COROLLARY. The locus of points within a dihedral angle and equally distant from its faces is the plane bisecting that angle. **(Proof:** 536, 537.**)**

ORIGINAL EXERCISES

1. Are two planes perpendicular to the same plane necessarily parallel?

2. A straight line and a plane perpendicular to the same plane are parallel.

3. A plane perpendicular to a line in another plane is perpendicular to that plane.

4. If three planes, all perpendicular to a fourth, intersect in three lines, these lines are parallel, in pairs.

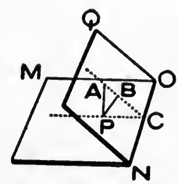
5. If the projection of any line (straight or curved) upon a plane is a straight line, the line is entirely in one plane.

6. The angle between the normals drawn to the faces of a dihedral angle from a point within the angle is the supplement of the plane angle of the dihedral angle.

7. If a line is parallel to a plane, any plane perpendicular to the line is perpendicular also to the plane.

[Construct the projection of the given line upon the given plane.]

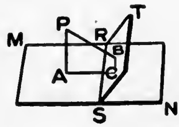
8. What is the locus of points in space equally distant from two intersecting planes?



9. If from any point in a face of a dihedral angle, a normal is drawn to each face, the plane of these normals is perpendicular to the edge of the dihedral angle.

10. If from any point in a face of a dihedral angle, a normal is drawn to each face, the angle they form is equal to the plane angle of the dihedral angle.

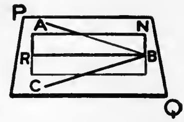
11. If a line is perpendicular to a plane, any plane parallel to the line is also perpendicular to the plane.



12. If *PA* is a normal to plane *MN*, *PB* a normal to plane *ST*, and *BC* a normal to plane *MN*, *AC* is perpendicular to *RS*, the intersection of planes *MN* and *ST*.

13. The plane perpendicular to the line that is perpendicular to two lines in space, at its middle point, bisects every straight line having its extremities in these lines.

14. The plane perpendicular to the plane of an angle and containing the bisector of the angle is the locus of points equally distant from the sides of the angle.

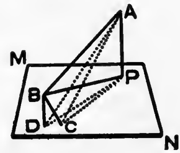


Proof: The ⊥s from any point in plane NR to AB and BC will have equal projections. (Explain by use of 94.)

∴ these perpendiculars are equal (?).

15. What is the locus of points in space equally distant from two intersecting lines?

16. If AP is a normal to plane MN and if angle PBC, in plane MN, is a right angle, angle ABC also is a right angle.

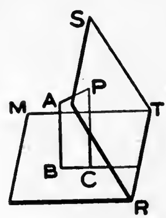


[Prove BC is ⊥ to plane APB.]

17. If AP is a normal to plane MN, and $\angle PBD$, in plane MN, is obtuse, $\angle ABD$ also is obtuse.

Proof: Take $BC = BD$; prove $PD > PC$. Then prove $AD > AC$, etc.

18. PA is perpendicular to plane RS; AB and PC are perpendicular to plane MR. Prove BC perpendicular to RT.



19. If two parallel planes are cut by a third plane, the alternate-interior dihedral angles are equal; the corresponding dihedral angles are equal; the alternate-exterior dihedral angles are equal; the adjoining interior dihedral angles are supplementary.

20. State and prove the converse theorems of those in No. 19.

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21. Construct a plane perpendicular to a given plane and containing a given line in that plane.

22. Construct a plane perpendicular to a given plane and containing a given line without that plane.

23. Construct through a given point a line which will intersect any two given lines in space.

Construction: Pass a plane through the point and one of the lines. This plane intersects the other line at a point, etc.

24. To bisect a given dihedral angle.

Construction: Pass a plane ⊥ to the edge. Bisect the plane ∠ of the given dihedral, etc.

25. Construct a line each of whose points shall be equally distant from the ends of a given line and also equally distant from the faces of a dihedral angle.

26. Find the locus of points equally distant from two points and equally distant from two intersecting planes. Discuss.

27. Find a point equally distant from three given points and equally distant from two intersecting planes. Is this problem ever impossible? Will there ever be two points? When will there be only one point?

28. Find a point equally distant from three given points and equally distant from two intersecting lines. Discuss fully.

POLYHEDRAL ANGLES

539. If three or more planes meet at a point, they form a **polyhedral angle.** The opening partially surrounded by the planes is the polyhedral angle.

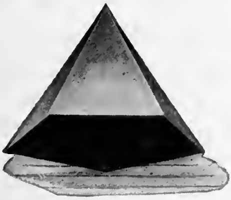
The point common to all the planes is the **vertex.**

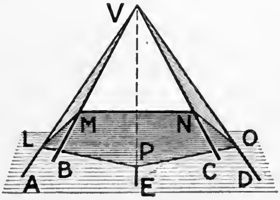
The planes are the **faces.**

The intersections of adjacent faces are the **edges.**

The angles formed at the vertex, by adjacent edges, are the **face angles.**

Thus, $V-ABCDE$ is a polyhedral angle; V is the vertex; AV, BV, etc., are edges; planes AVB, BVC, etc., are faces; $\angle s\ AVB$, BVC, etc., are face angles.

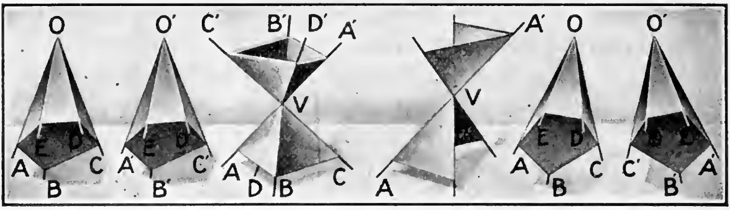




540. A **plane section** of a polyhedral angle is the plane figure bounded by the intersections of all the faces by a plane.

Polygon $LMNOP$ is a plane section of polyhedral angle $V-ABCDE$.

A **convex polyhedral angle** is one whose plane sections are all convex.



EQUAL POLYHEDRAL ANGLES

VERTICAL POLYHEDRAL ANGLES

VERTICAL DIHEDRAL ANGLES

SYMMETRICAL POLYHEDRAL ANGLES

541. Two polyhedral angles are **equal** if they can be made to coincide in all particulars. That is, if two polyhedral angles are equal, their homologous dihedral angles are equal; their homologous face angles are equal, and they are arranged in the same order. The length of the edges or the extent of the faces does not affect the size of the angle.

Two polyhedral angles are **vertical** if the edges of one are the prolongations of the edges of the other.

Two polyhedral angles are **symmetrical** if all the parts of one are equal to the corresponding parts of the other, but arranged in opposite order.

NOTE. It is apparent from the definitions that equal polyhedral angles are mutually equiangular as to the face angles and as to the dihedral angles.

Vertical polyhedral angles are mutually equiangular as to their face angles and as to their dihedral angles, but the order is reversed.

Symmetrical polyhedral angles are also mutually equiangular as to their face angles and as to their dihedral angles, but the order is reversed. Thus, if one follows around the polygon $A'D'$ in alphabetical order, he is moving as the hands of a clock, if the eye is at the vertex O'; but if he follows around AD alphabetically, he is moving in a direction opposite to the motion of the hands of a clock, if the eye is at the vertex O. Hence, it is apparent that, in general, symmetrical polyhedral angles cannot be made to coincide.

542. A **trihedral angle** is a polyhedral angle having three and only three faces.

A trihedral angle is **rectangular** if it contains a right dihedral angle ; **birectangular** if it contains two right dihedral angles; **trirectangular** if it contains three right dihedral angles.

A trihedral angle is **isosceles** if two of its face angles are equal.

PRELIMINARY THEOREMS

543. Two vertical polyhedral angles are symmetrical.

Proof : Their homologous face angles are equal and arranged in reverse order, and their homologous dihedral angles are equal and arranged in reverse order.

$\therefore$ they are symmetrical (Def. 541).

544. If two polyhedral angles are symmetrical, the vertical polyhedral angle of the one is equal to the other.

Because the corresponding parts are equal and they are arranged in the same order.

545. Provided two trihedral angles have their parts arranged in the same order, they are equal :

I. If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

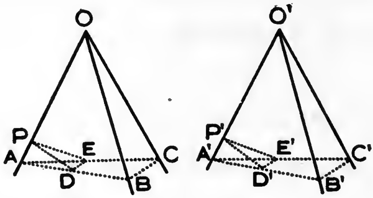
II. If a face angle and the two dihedral angles adjoining it of the one are equal respectively to a face angle and the two dihedral angles adjoining it, of the other.

Proof : By method of superposition, as in plane $\triangle$s.

Ex. Illustrate a birectangular trihedral angle by means of an open book standing on a table. Similarly illustrate a trirectangular trihedral angle. Similarly use two closed books to illustrate a rectangular trihedral angle.

Proposition XXXIX. Theorem

546. Provided two trihedral angles have their parts arranged in the same order, they are equal, if the three face angles of one are equal respectively to the three face angles of the other.



Given: Trih. ⦣s O and O';

$$\angle AOB = \angle A'O'B';$$
$$\angle BOC = \angle B'O'C';$$
$$\angle COA = \angle C'O'A'.$$

To Prove: Trih. $\angle O =$ trih. $\angle O'$, that is, dih. $\angle OA =$ dih. $\angle O'A'$, etc.

Proof: Take $OA = OB = OC = O'A' = O'B' = O'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

Take, on edges AO and $A'O'$, $AP = A'P'$ and in face AOB draw $PD \perp$ to AO.

$\angle OAB$ is acute ($\triangle AOB$ is isosceles). $\therefore PD$ will meet AB.

In face AOC draw $PE \perp$ to AO, meeting AC at E.

Draw DE. Similarly draw $P'D'$, $P'E'$, $D'E'$.

Now ⦣s EPD and $E'P'D'$ are the plane ⦣s of the dihedral ⦣s AO and $A'O'$ (515).

To prove that these ⦣s are equal requires the proof that eight pairs of ▵s are equal.

(1) $\triangle OAB = \triangle O'A'B'$ (Explain).
(2) $\triangle OBC = \triangle O'B'C'$ (Explain).
(3) $\triangle OAC = \triangle O'A'C'$ (Explain).

$\therefore AB = A'B'$, $BC = B'C'$, $AC = A'C'$ (?).

$\angle OAB = \angle O'A'B'$,
$\angle OAC = \angle O'A'C'$ (?).

(4) $\triangle APD = \triangle A'P'D'$ (Explain).
(5) $\triangle APE = \triangle A'P'E'$ (Explain).
$\left.\begin{array}{l} \therefore AD = A'D' \\ AE = A'E' \\ PD = P'D', \text{etc.} \end{array}\right\}$ (?).

(6) $\triangle ABC = \triangle A'B'C'$ (Explain). $\therefore \angle CAB = \angle C'A'B'$ (?).
(7) $\triangle AED = \triangle A'E'D'$ (Explain). $\therefore ED = E'D'$ (?).
(8) $\triangle PED = \triangle P'E'D'$ (Explain). $\therefore \angle EPD = \angle E'P'D'$ (?).
Hence dih. $\angle AO =$ dih. $\angle A'O'$ (520).

Similarly, one may prove the other pairs of homologous dihedral angles equal.

$\therefore$ trihedral $\angle O =$ trihedral $\angle O'$ (541).

Q.E.D.

Proposition XL. Theorem

547. Provided two trihedral angles have their parts arranged in reverse order, they are symmetrical:

I. If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

II. If a face angle and the two dihedral angles adjoining it of the one are equal respectively to a face angle and the two dihedral angles adjoining it of the other.

III. If the three face angles of one are equal respectively to the three face angles of the other.

Proof: In each case construct a third trihedral $\angle$ symmetrical to the first. This third figure will have its parts = to the parts of the second, and arranged in the same order (Def. 541).

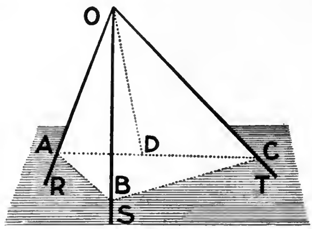
$\therefore$ the third = the second (545 and 546).
$\therefore$ the first is symmetrical to the second. (Ax. 6.)

Q.E.D.

Ex. The three planes bisecting the three dihedral angles of a trihedral angle intersect in a straight line.

Proposition XLI. Theorem

548. The sum of any two face angles of a trihedral angle is greater than the third face angle.



Given: Trih. $\angle O\text{-}RST$ in which face angle ROT is the greatest.

To Prove: $\angle ROS + \angle SOT > \angle ROT$.

Proof: Construct, in face ROT, $\angle ROD = \angle ROS$.

Take $OD = OB$; draw ADC, meeting OT at C. Draw AB and BC.

$\triangle AOD \cong \triangle AOB$ (Explain).

$\therefore AB = AD$ (?).

Now $AB + BC > AD + DC$ (Ax. 12).

But $AB = AD$ (?).

Subtracting, $BC > DC$ (Ax. 7).

Now $OB = OD$ (?), $OC = OC$ (?) and $BC > DC$ (Just proved).

$\therefore \angle BOC > \angle DOC$ (92).

But $\angle AOB = \angle AOD$ (?).

Adding, $\angle AOB + \angle BOC > \angle AOC$ (Ax. 7).

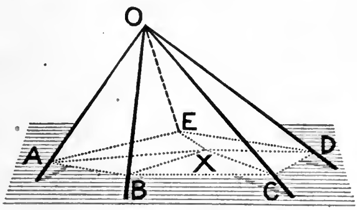
That is, $\angle ROS + \angle SOT > \angle ROT$ (Ax. 6).

Ex. 1. Prove theorem of 548 for the case of an isosceles trihedral angle.

Ex. 2. The three planes containing the three bisectors of the three face angles of a trihedral angle and perpendicular to those faces intersect in a straight line.

Proposition XLII. Theorem

549. The sum of the face angles of any polyhedral angle is less than four right angles or 360°.



Given: Polyhedral ∠ O, having n faces.

To Prove: The sum of the face ∠s at $O <$ 4 rt. ∠s or 360°.

Proof: Pass a plane AD, intersecting all the faces, and the edges at A, B, C, etc.

In this section take any point X and join X to all the vertices of the polygon.

(1) There are n face △s having their vertices at O (Hyp.).

(2) There are n base △s having their vertices at X (Const.).

(3) The sum of the ∠s of the face △s = $2n$ rt. ∠s (104).

(4) The sum of the ∠s of the base △s = $2n$ rt. ∠s (104).

(5) ∴ the sum of the ∠s of the face △s = the sum of the ∠s of the base △s (Ax. 1).

Now $\angle OAE + \angle OAB > \angle EAB$ (548).

And $\angle OBA + \angle OBC > \angle ABC$ (?), etc., etc.

Adding, the sum of the base ∠s of the face △s > the sum of the base ∠s of the base △s (Ax. 8).

Subtracting this inequality from equation (5) above, the sum of the face ∠s at $O <$ the sum of the ∠s at X (Ax. 9).

But the sum of all the ∠s at X = 4 rt. ∠s (47).

∴ the sum of the face ∠s at $O <$ 4 rt. ∠s, or 360° (Ax. 6).

Q.E.D.

ORIGINAL EXERCISES

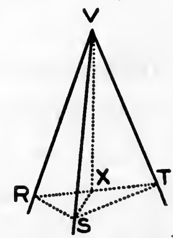
1. In the figure of 549, as the vertex O approaches the base, does the sum of the face angles at O increase or decrease? What limit does this sum approach? Does the sum ever become equal to this limit?

2. Can a polyhedral angle have for its faces three equilateral triangles? four? five? six? seven?

3. Can a polyhedral angle have for its faces four squares? five? three?

4. What other regular polygons can be used for the faces of a polyhedral angle?

5. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.



Given: $\angle RVS = \angle SVT$.

To Prove: Dih. $\angle RV =$ dih. $\angle TV$.

Proof: Pass plane SVX bisecting dih. $\angle SV$. Prove trih. ∠s $V\text{–}RSX$ and $V\text{–}TSX$ are sym. by (547, 1).

6. An isosceles trihedral angle and its symmetrical trihedral angle are equal.

7. Find the locus of points equally distant from the three faces of a trihedral angle.

8. Find the locus of points equally distant from the three edges of a trihedral angle.

9. If the three face angles of a trihedral angle are equal, the three dihedral angles also are equal.

10. If the three face angles of a trihedral angle are right angles, the three dihedral angles also are right angles.

11. In any trihedral angle the greatest dihedral angle has the greatest face opposite it.

12. If the edges of one trihedral angle are perpendicular to the faces of a second trihedral angle, then the edges of the second are perpendicular to the faces of the first.

13. Construct, through a given point, a plane which shall make, with the faces of a polyhedral angle having four faces, a section that is a parallelogram.

Construction: Extend one pair of opp. faces to obtain their line of intersection. Similarly extend the other pair. Any plane section ∥ to these lines will be a ▱. [Explain.]

BOOK VII

POLYHEDRONS

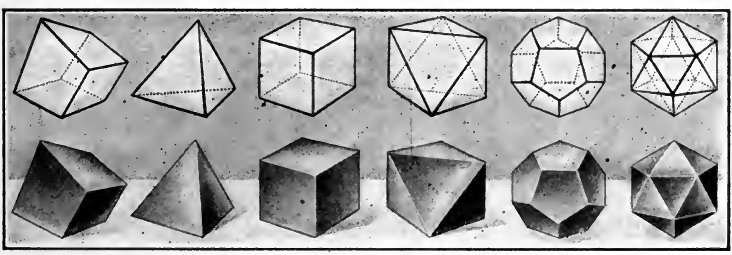
550. A **polyhedron** is a solid bounded by planes.

The **edges** of a polyhedron are the intersections of the bounding planes.

The **faces** are the portions of the bounding planes included by the edges.

The **vertices** are the intersections of the edges.

The **diagonal** of a polyhedron is a straight line joining two vertices not in the same face.



POLY-HEDRON TETRA-HEDRON HEXAHEDRON CUBE OCTA-HEDRON DODECA-HEDRON ICOSA-HEDRON

551. A **tetrahedron** is a polyhedron having four faces.

A **hexahedron** is a polyhedron having six faces.

An **octahedron** is a polyhedron having eight faces.

A **dodecahedron** is a polyhedron having twelve faces.

An **icosahedron** is a polyhedron having twenty faces.

552. A polyhedron is **convex** if the section made by every plane is a convex polygon.

Only convex polyhedrons are considered in this book.

PRISMS

553. A **prism** is a polyhedron two of whose opposite faces are congruent polygons in parallel planes, and whose other faces are all parallelograms.

The **bases** of a prism are the congruent parallel polygons.

The **lateral faces** of a prism are the parallelograms.

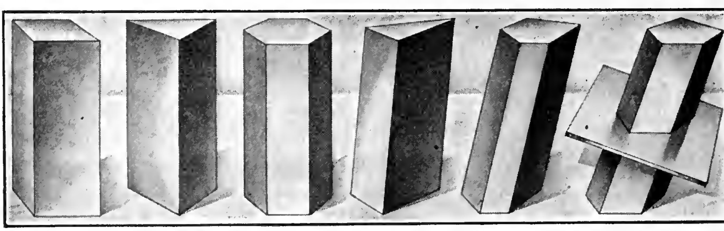
The **lateral edges** of a prism are the intersections of the lateral faces.

The **lateral area** of a prism is the sum of the areas of the lateral faces.

The **total area** of a prism is the sum of the lateral area and the areas of the bases.

The **altitude** of a prism is the perpendicular distance between the planes of the bases.

A **triangular prism** is a prism whose bases are triangles.



PRISM TRIANGULAR PRISM REGULAR PRISM OBLIQUE PRISMS RIGHT SECTION TRUNCATED PRISMS

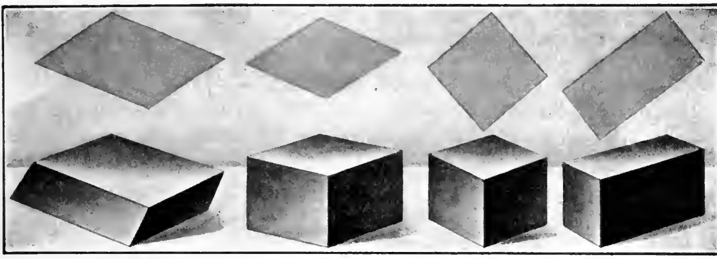
554. A **right prism** is a prism whose lateral edges are perpendicular to the planes of the bases.

A **regular prism** is a right prism whose bases are regular polygons.

An **oblique prism** is a prism whose lateral edges are not perpendicular to the planes of the bases.

A **truncated prism** is the portion of a prism included between the base and a plane not parallel to the base.

A **right section** of a prism is the section made by a plane perpendicular to the lateral edges of the prism.



PARALLELEPIPED — RIGHT PARALLELEPIPED — CUBE — RECTANGULAR PARALLELEPIPED

555. A **parallelepiped** is a prism whose bases are parallelograms.

A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to the planes of the bases.

A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

An **oblique parallelepiped** is a parallelepiped whose lateral edges are not perpendicular to the planes of the bases.

A **cube** is a rectangular parallelepiped whose six faces are squares.

556. The **unit of volume** is a cube whose edges are each a unit of length. The **volume of a solid** is the number of units of volume it contains. The volume of a solid is the ratio of that solid to the unit of volume.

The three edges of a rectangular parallelepiped meeting at any vertex are the **dimensions** of the parallelepiped.

Equal solids are solids that have equal volumes.

Congruent solids are solids that can be made to coincide.

Ex. What is the base of a rectangular parallelepiped? of a right parallelepiped? of an oblique parallelepiped? Illustrate, by removing the cover and bottom of an ordinary cardboard box and distorting the shape of the frame that remains, the three kinds of parallelepipeds.

Note. The space that is bounded by the surfaces of a solid, independent of the solid, is called a **geometrical solid.**

That is, if a material or physical body should occupy a certain position and then be removed elsewhere, there is a definite portion of space that is the same shape and size as the solid, and can be conceived as bounded by exactly the same surfaces as bounded the solid when in that original position. In order that we may pass planes and draw lines through solids, and superpose one solid upon another, it is convenient in studying the properties of solids to consider them usually as geometric solids, — the material body being removed for the time.

PRELIMINARY THEOREMS

557. Theorem. The lateral edges of a prism are equal. (?.)

558. Theorem. Any two lateral edges of a prism are parallel. (495.)

559. Theorem. Any lateral edge of a right prism equals the altitude. (509.)

560. Theorem. The lateral faces of a right prism are perpendicular to the bases. (525.)

561. Theorem. The lateral faces of a right prism are rectangles. (Def. 554.)

562. Theorem. The faces and bases of a rectangular parallelepiped are rectangles. (Def. 555.)

563. Theorem. All the faces of any parallelepiped are parallelograms. (?)

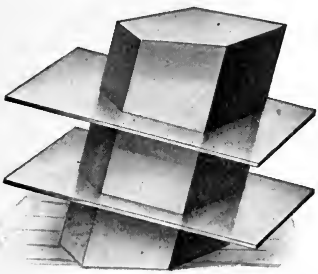
564. Axiom. A polyhedron cannot have fewer than four faces.

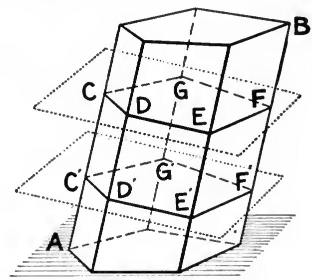
565. Axiom. A polyhedron cannot have fewer than three faces at each vertex.

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

566. The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.





Given: Prism AB; $\parallel$ sections CF and $C'F'$.

To Prove: Polygon $CF \cong$ polygon $C'F'$.

Proof: CD is $\parallel$ to $C'D'$, DE is $\parallel$ to $D'E'$, etc. (484).

$\therefore CD'$, DE', EF', etc. are ▱s (?).

$\therefore CD = C'D'$, $DE = D'E'$, $EF = E'F'$, etc. (124).

$\angle GCD = \angle G'C'D'$, $\angle CDE = \angle C'D'E'$, etc. (499).

$\therefore$ Polygon $CF \cong$ polygon $C'F'$ (150).

Q.E.D.

Ex. 1. All right sections of a prism are equal.

Ex. 2. Any section of a parallelepiped made by a plane cutting two pairs of opposite faces is a parallelogram.

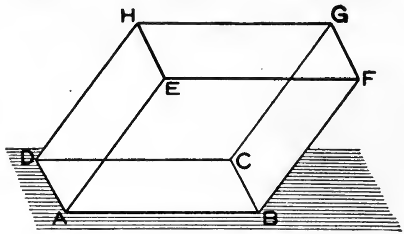
Ex. 3. How many edges has a cube? how many vertices? how many dihedral angles? how many trihedral angles?

Ex. 4. In the figure of 566, if DE is the altitude of one of the lateral faces, how could one express the area of that face?

Ex. 5. In 566, is CF' a prism? Why? Is CB a prism? Why? Is AF a prism? Why? What name is given CB?

Proposition II. Theorem

567. The opposite faces of a parallelepiped are congruent and parallel.



Given: (?).

To Prove: Face $AF \cong$ and $\parallel$ to face DG.

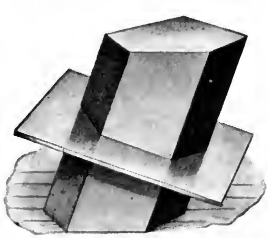
Proof:

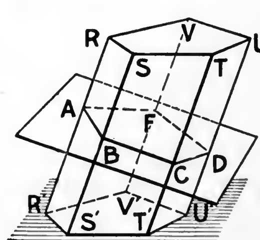
Faces AF and DG are ▱s	(?).
$AB = DC$, $AE = DH$	(124).
AB is $\parallel$ to DC and AE is $\parallel$ to DH	(?).
$\therefore \angle EAB = \angle HDC$	(499).
$\therefore$ face $AF \cong$ face DG	(133).
Also face AF is $\parallel$ to face DG	(499).

Q.E.D.

Proposition III. Theorem

568. The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.





Given: Prism RU'; edge $= E$; right section AD.

To Prove: Lateral area of $RU' = E \times$ perimeter of AD.

Proof: AB is $\perp$ to RR', BC is $\perp$ to SS', etc. (473).

Area $\square\ RS' = E \cdot AB$ (359).

Area $\square\ ST' = E \cdot BC$ (?).

Area $\square\ TU' = E \cdot CD$ (?).

etc. etc. Adding,

The lateral area $= E \cdot (AB + BC + CD + \text{etc.})$ (Ax. 2).

$= E \cdot$ perimeter of rt. sect. (Ax. 6).

Q.E.D.

569. Corollary. The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.

$$L = H \cdot Pr.$$

(Where L = lateral area, H = altitude, and Pr = perimeter of base, of a right prism.)

570. Corollary. $T = L + 2B.$

(Where T = total area, and B = area of base.)

Ex. 1. Any section of a parallelepiped made by a plane parallel to any edge is a parallelogram.

Ex. 2. The sum of the face angles at all the vertices of any parallelepiped is equal to 24 right angles.

Ex. 3. The sum of the plane angles of all the dihedral angles of any parallelepiped is equal to 12 right angles.

Proof: Pass three planes $\perp$ to three intersecting edges. Prove these sections ▱ whose ∠s are the plane angles of the dihedral angles, etc.

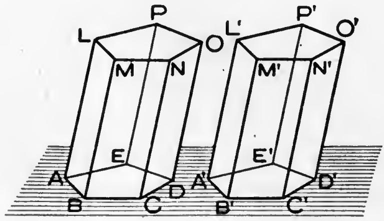
Ex. 4. Find the lateral area of a right prism whose altitude is 8 ft. and each side of whose triangular base is 5 ft.

Ex. 5. Find the total area of a regular prism whose base is a regular hexagon, 10 in. on a side, if the altitude of the prism is 15 in.

Ex. 6. Find the lateral area of a prism whose edge is 12 in. and whose right section is a pentagon, the sides of which are 3 in., 5 in., 6 in., 9 in., and 11 in.

Proposition IV. Theorem

571. Two prisms are congruent if three faces including a trihedral angle of one are congruent, respectively, to three faces including a trihedral angle of the other, and similarly placed.



Given: Prisms AO and $A'O'$; face $AM \cong$ face $A'M'$; face $AP \cong$ face $A'P'$; face $AD \cong$ face $A'D'$.

To Prove: Prism $AO \cong$ prism $A'O'$.

Proof: The three face $\angle$s at A are respectively = to the three face $\angle$s at A' (27).

$\therefore$ trih. $\angle A =$ trih. $\angle A'$ (546).

Superpose prism AO upon prism $A'O'$, making the equal trihedral $\angle$s A and A' coincide.

Face AD coincides with face $A'D'$, face AM with $A'M'$, face AP with $A'P'$ (They are $\cong$ by hyp.).

That is, point L falls on L'; M on M'; and P on P'.

$\therefore$ the plane LO falls upon the plane $L'O'$ (477).

Polygon $LO \cong$ polygon $L'O'$ (Ax. 1).

$\therefore$ these bases coincide (?).

Similarly face BN coincides with $B'N'$, CO with $C'O'$, etc.

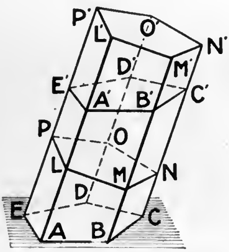
$\therefore$ the prisms are $\cong$ (Def. 556). Q.E.D.

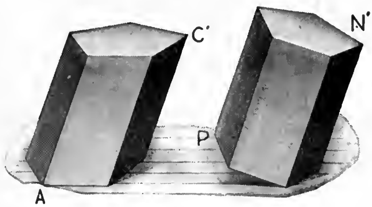
572. Corollary. **Two right prisms are congruent if they have congruent bases and equal altitudes.** (Explain.)

573. Corollary. **Two truncated prisms are congruent if three faces including a trihedral angle of one are congruent, respectively, to three faces including a trihedral angle of the other, and are similarly placed.** (Explain.)

Proposition V. Theorem

574. An oblique prism is equal to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to the lateral edge of the oblique prism.





Given: Oblique prism AC'; right prism PN' whose base is PN, a right section of AC', and whose altitude $PP' =$ edge EE'.

To Prove: Oblique prism $AC' =$ right prism PN'.

Proof: Edge $EE' = PP'$ (Hyp.).

Subtract PE' from each, and $EP = E'P'$ (Ax. 2).

Likewise $AL = A'L'$, $BM = B'M'$, $CN = C'N'$, etc.

(1) Face $AC \cong$ face $A'C'$ (553).

(2) In faces AP and $A'P'$, $EP = E'P'$, $AL = A'L'$ (Ax. 2).

Also $AE = A'E'$, $PL = P'L'$ (124).

That is, face AP and face $A'P'$ are mutually equilateral.

Also $\angle EAL = \angle E'A'L'$, $\angle PEA = \angle P'E'A'$, $\angle ALP = \angle A'L'P'$, $\angle EPL = \angle E'P'L'$ (67).

That is, faces AP and $A'P'$ are mutually equiangular.

$\therefore$ face $AP \cong$ face $A'P'$ (150).

(3) Similarly, face $AM \cong$ face $A'M'$.

$\therefore$ truncated prism $AN \cong$ truncated prism $A'N'$ (573).

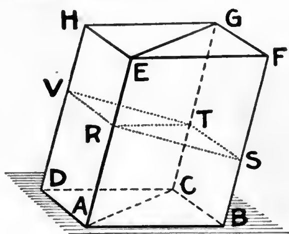
Now, add, solid $PC' =$ solid PC' (Iden.).

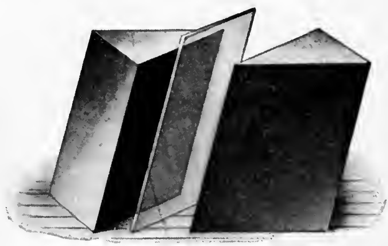
Oblique prism $AC' =$ right prism PN' (Ax. 2). Q.E.D.

Ex. Prove, in the figure of 574, that truncated prism AN is congruent to truncated prism $A'N'$, by the method of superposition.

PROPOSITION VI. THEOREM

575. The plane containing two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equal triangular prisms.





Given: Parallelepiped *BH* and plane *AG* containing the opposite edges *AE* and *CG*.

To Prove: Prism *ABC* – *F* = prism *ADC* – *H*.

Proof: Pass a right section *RSTV* intersecting the given plane in *RT*.

Face *AF* is ∥ to face *DG* (?).

∴ *RS* is ∥ to *VT* (484).

Also *RV* is ∥ to *ST* (?).

∴ *RSTV* is a ▱ (?).

∴ △ *RST* ≅ △ *RVT* (126).

Prism *ABC–F* = a right prism whose base is *RST* and whose altitude = *EA* (574).

Prism *ADC–H* = a right prism whose base is *RVT* and whose altitude = *EA* (?).

But these imaginary right prisms are congruent (572).

∴ prism *ABC–F* = prism *ADC–H* (Ax. 1).

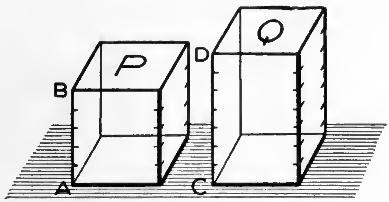
Q.E.D.

Ex. 1. The section of a parallelepiped made by a plane containing two diagonally opposite edges, is a parallelogram.

Ex. 2. Are the two triangular prisms in the diagram of 575 congruent? Why?

Proposition VII. Theorem

576. Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.



Given: Rectangular parallelepipeds *P* and *Q*, having ≅ bases, and their altitudes *AB* and *CD*, respectively.

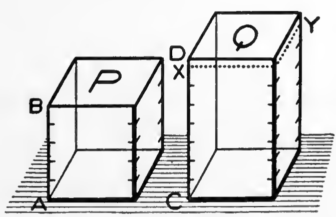
To Prove: $P : Q = AB : CD$.

Proof: I. If the altitudes are **commensurable.**

(Consult 524.)

II. If the altitudes are **incommensurable.**

There does not exist a common unit (225).



Suppose *AB* divided into equal parts. Apply one of these as a unit of measure to *CD*. There will be a remainder, *DX* (?).

Pass a plane *XY*, through *X* and ∥ to the base.

Now $P : CY = AB : CX$ (?).

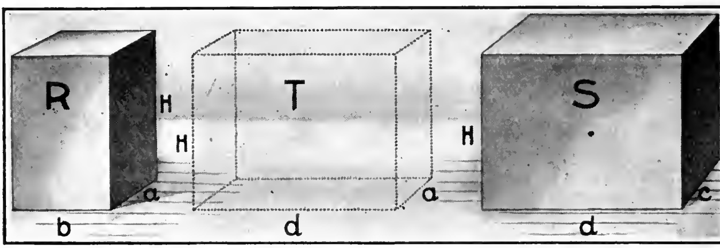
Indefinitely increase, etc. — as in 524.

577. Corollary. Two rectangular parallelepipeds having two dimensions of the one equal respectively to two dimensions of the other, are to each other as their third dimension.

The faces having the sides of one equal to the sides of the other, respectively, may be considered the bases and the third dimensions the altitudes. Thus this statement is the same as 576.

Proposition VIII. Theorem

578. Two rectangular parallelepipeds having equal altitudes are to each other as their bases.



Given: Rectangular parallelepipeds R and S, having the same altitude H; and other dimensions a, b, and c, d, respectively.

To Prove: $\frac{R}{S}=\frac{a \cdot b}{c \cdot d}$.

Proof: Construct a third rectangular parallelepiped, having altitude = to H, another dimension = to a, a third = to d.

$$\frac{R}{T}=\frac{b}{d} \text{ and } \frac{T}{S}=\frac{a}{c} \qquad (577).$$

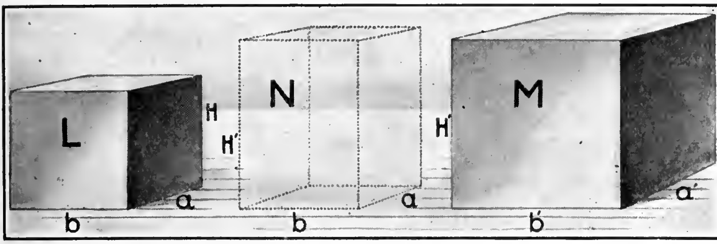
Multiplying, $$\frac{R}{S}=\frac{a \cdot b}{c \cdot d} \qquad (\text{Ax. } 3).$$

Q.E.D.

579. Corollary. Two rectangular parallelepipeds having one dimension in common, are to each other as the products of the other dimensions.

PROPOSITION IX. THEOREM

580. THEOREM. Any two rectangular parallelepipeds are to each other as the products of their three dimensions.



Given: Rectangular parallelepipeds L and M, whose dimensions are a, b, H, and a', b', H', respectively.

To Prove: $\frac{L}{M} = \frac{a \cdot b \cdot H}{a' \cdot b' \cdot H'}$.

Proof: Construct N, whose dimensions are a, b, H'.

Then $$\frac{L}{N} = \frac{H}{H'} \qquad (577).$$

And $$\frac{N}{M} = \frac{a \cdot b}{a' \cdot b'} \qquad (579).$$

Multiplying, $$\frac{L}{M} = \frac{a \cdot b \cdot H}{a' \cdot b' \cdot H'} \qquad (\text{Ax. } 3).$$

Q.E.D.

Ex. 1. In the above diagram, if $a = 6$ in., $b = 6$ in., and $H = 7$ in., find the length of the diagonal of L.

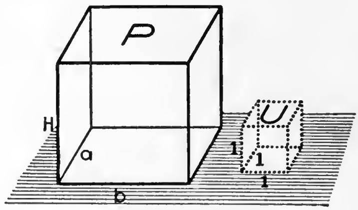
Ex. 2. Find the length of the diagonal of a room a ft. long, b ft. wide, and c ft. high.

Ex. 3. Show that the four diagonals of a rectangular parallelepiped are all equal.

Ex. 4. Find the diagonal of a cube whose edge is 4 in. Find the edge of a cube whose diagonal is 10 in.

Proposition X. Theorem

581. The volume of a rectangular parallelepiped is equal to the product of its three dimensions.



Given: (?). **To Prove:** (?).

Proof: Let U be a unit of vol.

$$\frac{P}{U} = \frac{a \cdot b \cdot H}{1 \cdot 1 \cdot 1} = a \cdot b \cdot H \qquad (580).$$

But $$\frac{P}{U} = \text{vol. of } P \qquad (556).$$

$$\therefore \text{ vol. of } P = a \cdot b \cdot H \qquad (\text{Ax. } 1).$$

Q.E.D.

582. Corollary. **The volume of a rectangular parallelepiped is equal to the product of its base by its altitude.**

$$V = B \cdot H. \qquad (\text{See } 581.)$$

583. Corollary. **The volume of a cube is equal to the cube of its edge.**

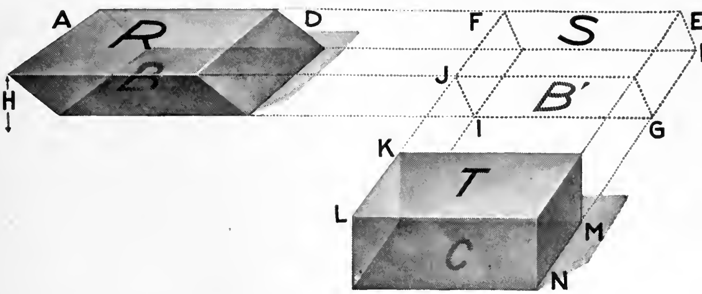
Ex. 1. A rectangular tank is 56 in. long, 44 in. wide, and 60 in. deep (inside). How many gallons will it hold? [231 cu. in. = 1 gal.]

Ex. 2. Three persons measured the above tank inaccurately. A found it to be 57 in. × 44 in. × 60 in.; B, 56 in. × 45 in. × 60 in.; and C, 56 in. × 44 in. × 61 in., — each recording two dimensions correctly, and one, 1 in. too great. Whose error was most serious, judging by the capacity of the tank? Which dimensions should be most carefully measured?

Ex. 3. What is the volume of a cube 6 in. on each edge? What is the edge of a cube having double the volume? What happens to the volume of a cube if we double each edge? if we halve each edge?

Proposition XI. Theorem

584. The volume of any parallelepiped is equal to the product of its base by its altitude.



Given: Parallelepiped R, whose base $= B$ and alt. $= H$.

To Prove: Volume of $R = B \cdot H$.

Proof: Prolong the edge AD and all edges ∥ to AD.

On the prolongation of AD, take $EF =$ to AD.

Through E and F pass planes EG and FI, ⊥ to EF, forming the right parallelepiped S.

Again, prolong FJ and all the edges ∥ to FJ.

On the prolongation of FJ, take $KL =$ to FJ.

Through K and L pass planes KM and LN, ⊥ to KL, forming the rectangular parallelepiped T.

Consider FI the base of S, and EF its altitude.

Then $R = S$ (574).

Also $B = B'$ (360).

Consider FP the base of S, KM the base of T, and KL its altitude.

Then $S = T$; also $B' \cong C$ (574, 134).

Hence $R = T$ and $B = C$ (Ax. 1).

And the altitude of T is H (509).

But the volume of $T = C \cdot H$ (582).

$\therefore$ **V of $R = \mathbf{B \cdot H}$** (Ax. 6). Q.E.D.

585. Corollary. **Two parallelepipeds having equal altitudes and equal bases are equal.** (Ax. 1.)

586. Corollary. **Two parallelepipeds having equal altitudes are to each other as their bases.**

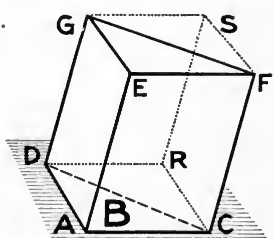
Proof: $Q = B \cdot H$ and $R = B' \cdot H$. $\therefore \frac{Q}{R} = \frac{B}{B'}$ (584, Ax. 3).

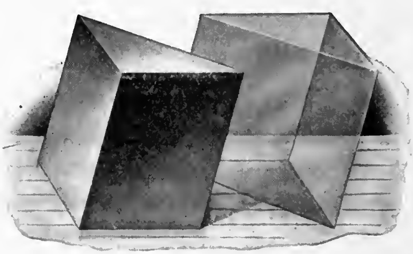
587. Corollary. **Two parallelepipeds having equal bases are to each other as their altitudes.** (?.)

588. Corollary. **Any two parallelepipeds are to each other as the products of their bases by their altitudes.** (?.)

Proposition XII. Theorem

589. The volume of a triangular prism is equal to the product of its base by its altitude.





Given: Triangular prism $ACD–F$; base $= B$; alt. $= H$.

To Prove: Volume of $ACD–F = B \cdot H$.

Proof: Construct parallelepiped AS having as three of its lateral edges AE, CF, DG.

Vol. $AS = ACRD \cdot H$ (584).

Hence $\frac{1}{2}$ volume of $AS = \frac{1}{2} ACRD \cdot H$ (Ax. 3).

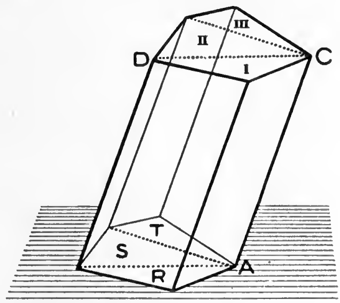
But $\frac{1}{2}$ volume of $AS =$ volume of prism $ACD–F$ (575).

And $\frac{1}{2} ACRD = B$ (126).

$\therefore$ volume of $ACD–F = B \cdot H$ (Ax. 6). Q.E.D.

Proposition XIII. Theorem

590. The volume of any prism is equal to the product of its base by its altitude.



Given: Prism AD; base = to B; altitude = to H.

To Prove: Vol. of $AD = B \cdot H$.

Proof: Through any lateral edge, AC, and other lateral edges not adjoining AC, pass planes cutting the prism into triangular prisms I, II, III, having bases R, S, T, respectively.

$$\left.\begin{array}{l}\text{Vol. of prism I} = R \cdot H \\ \text{Vol. of prism II} = S \cdot H \\ \text{Vol. of prism III} = T \cdot H\end{array}\right\} \text{ (589).}$$

Adding,

Vol. of prism $AD = (R + S + T)H = B \cdot H$ (Ax. 2). Q.E.D.

591. Corollary. Two prisms having equal altitudes and equal bases are equal.

592. Corollary. Two prisms having equal altitudes are to each other as their bases.

593. Corollary. Two prisms having equal bases are to each other as their altitudes.

594. Corollary. Any two prisms are to each other as the products of their bases by their altitudes.

ORIGINAL EXERCISES

1. Which rectangular parallelepiped contains the greater volume, one whose edges are 5 in., 7 in., 9 in., or one whose edges are 4 in., 6 in., 13 in.?

2. The base of a prism is a right triangle whose legs are 8 m. and 12 m. and the altitude of the prism is 20 m. Find its volume.

3. During a rain, half an inch of water fell. How many gallons fell on a level ten-acre park, allowing $7\frac{1}{2}$ gal. to the cubic foot?

4. Counting 38 cu. ft. of coal to a ton, how many tons will a coal bin 18 ft. long, 6 ft. wide, and $9\frac{1}{2}$ ft. deep contain, when even full?

5. How many faces has a parallelepiped? edges? vertices? How many faces has a hexagonal prism? edges? vertices?

6. Every lateral face of a prism is parallel to the lateral edges not in that face.

7. Every lateral edge of a prism is parallel to the faces that do not contain it.

8. Every plane containing one and only one lateral edge of a prism is parallel to all the other lateral edges.

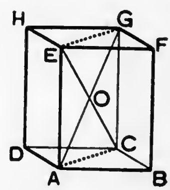
9. Any lateral face of a prism is less than the sum of the other lateral faces.

10. The diagonals of a rectangular parallelepiped are equal.

Proof: Pass the plane $ACGE$. This is a rectangle (?), etc.

11. The four diagonals of a parallelepiped bisect each other.

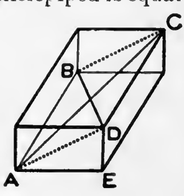
[First prove that one pair bisect each other; thus prove that any pair bisect each other, etc.]



12. Two triangular prisms are equal if their lateral faces are equal each to each.

13. Any prism is equal to the parallelepiped having the same altitude and an equal base.

14. The square of the diagonal of a rectangular parallelepiped is equal to the sum of the squares of its three dimensions.

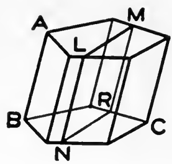


To Prove: $\overline{AC}^2 = \overline{AE}^2 + \overline{ED}^2 + \overline{DC}^2$.

Proof: AD is the hypotenuse of rt. $\triangle AED$, and AC, of rt. $\triangle ACD$.

15. The diagonal of a cube is equal to the edge multiplied by $\sqrt{3}$.

16. The volume of a triangular prism is equal to half the product of the area of any lateral face by the perpendicular drawn to that face from any point in the opposite edge.



17. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.

To Prove: *LMRN* a ▱. **Proof**: *LM* is ∥ to *NR* (?). *LN* and *MR* are each ∥ to any edge. (Explain.)

18. Every polyhedron has an even number of face angles.

Proof: Consider the faces as separate polygons. The number of sides of these polygons = double the number of edges of the polyhedron. (Explain.) But the number of sides of these polygons = the number of their angles, that is, the number of face angles. ∴ the number of face angles = double the number of edges = an even number (?).

19. There is no polyhedron having fewer than 6 edges.

20. Find the contents and total area of a room 7 m. × 5 m. × 3 m.

21. Find the volume, lateral area, and total area of an 8-inch cube.

22. A right prism whose height is 12 ft. has for its base a right triangle whose legs are 6 ft. and 8 ft. Find the volume, lateral area, and total area of the prism.

23. Find the altitude of a rectangular parallelepiped whose base is 21 in. × 30 in., equivalent to a rectangular parallelepiped whose dimensions are 27 in. × 28 in. × 35 in.

24. A cube and a rectangular parallelepiped whose edges are 6 in., 16 in., and 18 in., have the same volumes. Find the edge of the cube.

25. Fiud the volume of a rectangular parallelepiped whose total area is 620 sq. in. and whose base is 14 in. × 9 in.

26. How many bricks each 8 in. × $2\frac{3}{4}$ in. × 2 in. will be required to build a wall 22 ft. × 3 ft. × 2 ft. (not allowing for mortar)?

27. If a triangular prism is 20 in. high and each side of its base is 8 in., how many cubic inches does it contain?

28. Find the lateral area, total area, and volume of a regular hexagonal prism each side of whose base is 10 in. and whose altitude is 15 in.

29. A box is 12 in. × 9 in. × 8 in. What is the length of its diagonal?

30. Each edge of a cube is 8 in. Find its diagonal.

31. The diagonal of a cube is $10\sqrt{3}$ in. Find its edge, volume, and total area.

32. A trench is 180 ft. long and 12 ft. deep, 7 ft. wide at the top and 4 ft. at the bottom. How many cubic yards of earth have been removed?

33. A metallic tank, open at the top, is made of iron 2 in. thick; the internal dimensions of the tank are, 4 ft. 8 in. long, 3 ft. 6 in. wide, 4 ft. 4 in. deep. Find the weight of the tank if empty; if full of water.

[Water weighs $62\frac{1}{2}$ lb. to the cubic foot and iron is 7.2 times as heavy as water.]

34. The base of a right parallelepiped is a rhombus whose sides are each 25 in., and the shorter diagonal is 14 in. The height of the parallelepiped is 40 in. Find its volume and total surface.

35. If the diagonal of a cube is 12 ft., find its surface.

36. If the total surface of a cube is 54 sq. ft., find its volume.

37. A right prism whose altitude is 25 in. has for its base a triangle whose sides are 11 in., 13 in., 20 in. Find its lateral area, total area, and volume.

PYRAMIDS

595. A **pyramid** is a polyhedron, one of whose faces is a polygon and whose other faces are all triangles having a common vertex.

The **lateral faces** of a pyramid are the triangles.

The **lateral edges** of a pyramid are the intersections of the lateral faces. The **vertex** of a pyramid is the common vertex of all the lateral faces.

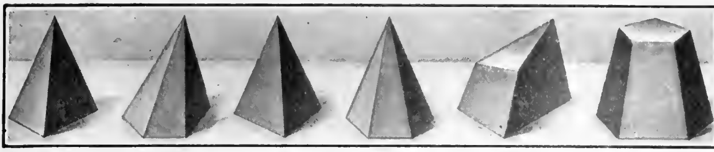
The **base** of a pyramid is the face opposite the vertex.

The **lateral area** of a pyramid is the sum of the areas of the lateral faces. The **total area** of a pyramid is the sum of the lateral area and the area of the base.

The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

A **triangular pyramid** is a pyramid whose base is a triangle. It is called also a tetrahedron. (See 551.)

596. A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude, from the vertex, meets the base at its center.



PYRAMIDS REGULAR PYRAMIDS TRUNCATED PYRAMID FRUSTUM OF A PYRAMID

The **slant height** of a regular pyramid is the line drawn in a lateral face, from the vertex perpendicular to the base of the triangular face. It is the altitude of any lateral face.

597. The **frustum** of a pyramid is the part of a pyramid included between the base and a plane parallel to the base.

The **altitude** of a frustum of a pyramid is the perpendicular distance between the planes of its bases.

The **slant height** of the frustum of a regular pyramid is the perpendicular distance, in a face, between the bases of that face.

A **truncated pyramid** is the part of a pyramid included between the base and a plane cutting all the lateral edges.

PRELIMINARY THEOREMS

598. Theorem. The lateral edges of a regular pyramid are all equal. (504, II.)

599. Theorem. The lateral faces of a regular pyramid are congruent isosceles triangles. (598, 78.)

600. Theorem. The lateral edges of the frustum of a regular pyramid are all equal. (Ax. 2.)

601. Theorem. The lateral faces of the frustum of a regular pyramid are congruent isosceles trapezoids. (484.)

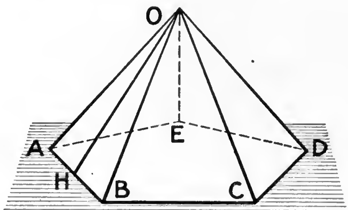
602. Theorem. The lateral faces of the frustum of any pyramid are trapezoids. (?.)

603. Theorem. The slant height of a regular pyramid is the same length in all the lateral faces.

THEOREMS AND DEMONSTRATIONS

Proposition XIV. Theorem

604. The lateral area of a regular pyramid is equal to half the product of the perimeter of the base by the slant height.



Given: Regular pyramid $O\text{–}ABCDE$; lateral area = to L; perimeter of base = to P; slant height OH = to s.

To Prove: $L = \frac{1}{2} P \cdot s$.

Proof:

$$\left.\begin{array}{l}\text{Area } \triangle AOB = \frac{1}{2} AB \cdot s \\ \text{Area } \triangle BOC = \frac{1}{2} BC \cdot s\end{array}\right\} \quad (364).$$

etc. etc.

Adding, Lateral area $= \frac{1}{2} AB \cdot s + \frac{1}{2} BC \cdot s +$ etc. (Ax. 2).

That is, $L = \frac{1}{2}(AB + BC + \text{etc.}) \cdot s$, or,

Lateral area, $L = \frac{1}{2} P \cdot s$ (Ax. 6). Q.E.D.

Ex. 1. Prove that the bases of any frustum of a pyramid are mutually equiangular.

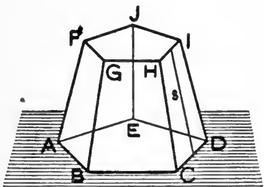
Ex. 2. The foot of the altitude of a regular pyramid drawn from the vertex, coincides with the center of the circles inscribed in, and circumscribed about, the base.

Ex. 3. The sum of the medians of the lateral faces of the frustum of a pyramid is equal to half the sum of the perimeters of the bases.

Ex. 4. To what rectangle is the lateral area of a regular pyramid equal? the total area?

Proposition XV. Theorem

605. The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height.



Given: (?).

To Prove: $L = \frac{1}{2}(P + p) \cdot s$.

Proof: Area trapezoid $CI = \frac{1}{2}(CD + HI) \cdot s$ (?).

Area trapezoid $BH = \frac{1}{2}(BC + GH) \cdot s$ (?).

Area trapezoid $AG = \frac{1}{2}(AB + FG) \cdot s$ (?) etc.

Adding, **Lateral area,** $L = \frac{1}{2}(P + p) \cdot s$ Explain.

Q.E.D.

Ex. 1. Using L for lateral area and B for area of base express the formula for the total area, T, of a regular pyramid.

Ex. 2. The slant height of a regular pyramid whose base is a square, of which each side is 8 ft., is 15 ft. Find the lateral area; the total area.

Ex. 3. A regular pyramid stands on a hexagonal base 16 in. on a side, and the slant height is 2 ft. Find the lateral and total areas.

Ex. 4. The slant height of the frustum of a regular pyramid is 12 in., and the bases are squares, 10 in. and 6 in. on a side, respectively. Find the lateral area and the total area.

Ex. 5. How many square feet of tin will be required to line a vat in the form of the frustum of a regular pyramid, having inside measurements as follows: the slant height is 14 ft.; the bases are regular hexagons whose sides are 7 ft. and 6 ft. respectively.

Ex. 6. The lateral edge of a pyramidal church spire is 61 ft. Each side of its octagonal base is 22 ft. What will be the cost of painting the spire at $2\frac{1}{2}$¢ a square foot?

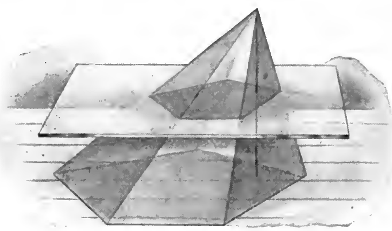
Proposition XVI. Theorem

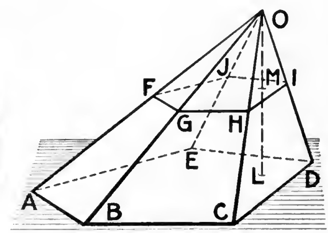
606. If a pyramid is cut by a plane parallel to the base:

I. The lateral edges and altitude are divided proportionally.

II. The section is a polygon similar to the base.

III. The area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.





Given: Pyr. $O\text{–}ABCDE$; plane $FI \parallel$ to the base; altitude $= OL$.

To Prove: I. $\frac{OF}{OA} = \frac{OG}{OB} = \frac{OH}{OC} = \cdots = \frac{OM}{OL}$.

II. Section FI is similar to the base AD.

III. $\frac{\text{section } FI}{\text{base } AD} = \frac{\overline{OM}^2}{\overline{OL}^2}$.

Proof: I. Imagine a plane through $O \parallel$ to plane AD.

This plane is $\perp$ to OL (496).

And $\parallel$ to plane FI (489).

$\therefore \frac{OF}{OA} = \frac{OG}{OB} = \frac{OH}{OC} = \cdots = \frac{OM}{OL}$ (500). Q.E.D.

II. FG is $\parallel$ to AB, GH is $\parallel$ to BC, etc. (484).

$\therefore \angle FGH = \angle ABC$; $\angle GHI = \angle BCD$; etc. (499).

That is, the polygons are mutually equiangular.

$\triangle OFG$ is similar to $\triangle OAB$; $\triangle OGH$, to $\triangle OBC$; etc. (305).

$$\therefore \frac{FG}{AB} = \left(\frac{OG}{OB}\right) = \frac{GH}{BC} = \left(\frac{OH}{OC}\right) = \frac{HI}{CD} = \text{etc.} \qquad (313).$$

$\therefore$ section FI is similar to base AD (301). Q.E.D.

III. Section FI is similar to base AD (II, above).

$$\frac{\text{section } FI}{\text{base } AD} = \frac{\overline{FG}^2}{\overline{AB}^2}. \qquad (376).$$

Now $\triangle OFG$ is similar to $\triangle OAB$ (305).

$$\therefore \frac{FG}{AB} = \frac{OF}{OA} \qquad (313).$$

But $$\frac{OF}{OA} = \frac{OM}{OL} \qquad \text{(I, above).}$$

$$\therefore \frac{FG}{AB} = \frac{OM}{OL} \qquad \text{(Ax. 1).}$$

And $$\frac{\overline{FG}^2}{\overline{AB}^2} = \frac{\overline{OM}^2}{\overline{OL}^2} \qquad (287).$$

Hence $$\frac{\text{section } FI}{\text{base } AD} = \frac{\overline{OM}^2}{\overline{OL}^2} \qquad \text{(Ax. 1).} \quad \text{Q.E.D.}$$

Ex. 1. The bases of the frustum of a regular pyramid are equilateral triangles whose sides are 12 in. and 20 in., respectively. The slant height is 40 in. Find the lateral area; the total area.

Ex. 2. The bases of the frustum of a regular pyramid are regular hexagons whose sides are 8 in. and 18 in., respectively. The slant height is 25 in. Find the lateral area and total area.

Ex. 3. A pyramid whose altitude is 10 in. and whose base contains 80 sq. in. is cut by a plane bisecting the altitude and parallel to the base. Find the area of the section of the pyramid made by this plane.

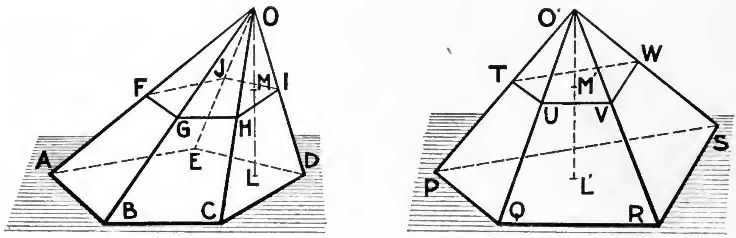
Ex. 4. Cutting a pyramid whose altitude is 16 ft. is a plane parallel to the base and 6 ft. from it. The area of the base is 192 sq. ft. What is the area of the section?

Ex. 5. If a plane parallel to the base of a pyramid bisects the altitude, how does the area of the section compare with the area of the base?

Ex. 6. Two planes, parallel to the base of a pyramid, trisect the altitude. How do the areas of the sections compare with the area of the base?

Proposition XVII. Theorem

607. If two pyramids have equal altitudes and equal bases, sections made by planes parallel to the bases and at equal distances from the vertices are equal.



Given: Pyramids $O\text{–}ABCDE$ and $O'\text{–}PQRS$; alt. $OL = O'L'$; base $AD =$ base PR; planes of the sections FI and $TV \parallel$ to the bases; $OM = O'M'$.

To Prove: Section $FI =$ section TV.

Proof: $\dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\overline{OM}^2}{\overline{OL}^2}$ and $\dfrac{\text{section } TV}{\text{base } PR} = \dfrac{\overline{O'M'}^2}{\overline{O'L'}^2}$ (606, III).

But $OM = O'M'$ and $OL = O'L'$ (Hyp.).

$\therefore \dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\text{section } TV}{\text{base } PR}$ (Ax. 1).

Now base $AD =$ base PR (Hyp.).

Multiplying, section $FI =$ section TV (Ax. 3).

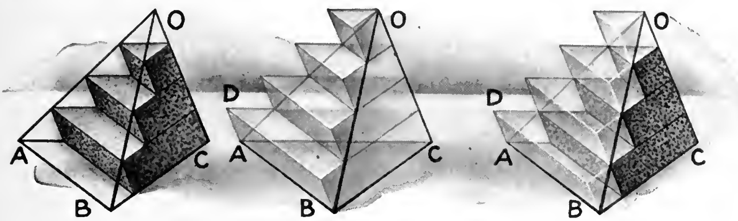
Q.E.D.

608. If a plane is passed parallel to the base of a pyramid, intersecting all the lateral edges, and upon this section, as a base, a prism is constructed wholly inside the pyramid, but having one lateral edge in a lateral edge of the pyramid, this prism is called an **inscribed prism**.

If upon this section as a base a prism is constructed partly outside the pyramid, having one lateral edge in one of the lateral edges of the pyramid, this prism is called a **circumscribed prism**.

Proposition XVIII. Theorem

609. The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms, having equal altitudes, if the number of prisms is indefinitely increased.



Given: Triangular pyramid *O–ABC*, having a series of prisms inscribed in it, and another series circumscribed about it, all the prisms having equal altitudes.

To Prove: *O–ABC* is the limit of the sum of each series of prisms as their number is indefinitely increased.

Proof: Denote the volume of the pyramid by V, the sum of the volumes of the series of inscribed prisms by S_i, and the sum of the volumes of the series of circumscribed prisms by S_c.

The uppermost circumscribed prism = the uppermost inscribed prism (591).

The second pair of prisms also are equal (?).

And so on, until the last circumscribed prism, *D–ABC*, remains, for which there is no equivalent inscribed prism.

Hence it is evident that $S_c - S_i = D\text{–}ABC$, the lowest circumscribed prism.

Now by indefinitely increasing the number of the prisms, the altitude of *D–ABC* becomes indefinitely small, and hence the volume of *D–ABC* approaches zero as a limit.

The altitude can never actually equal zero, nor can the volume equal zero. Hence $S_c - S_i$ can be made less than any mentionable quantity, but cannot equal zero.

Now $S_c = S_c$
and $V > S_i$ (Ax. 5).
$\therefore S_c - V < S_c - S_i$ (Ax. 9).

$V < S_c$ (Ax. 5).
and $S_i = S_i$.
$\therefore V - S_i < S_c - S_i$ (Ax. 7).

That is, $S_c - V$ and $V - S_i$ are each less than $S_c - S_i$, which itself approaches zero.

Hence $S_c - V$ approaches zero and $V - S_i$ approaches zero.

$\therefore S_c$ approaches V as a limit, and S_i approaches V as a limit (227). (See note on p. xviii.)

Q.E.D.

Ex. 1. If a plane is passed parallel to the base of a pyramid, cutting the lateral edges, the section is to the base as the square of the lateral edge of the pyramid cut away by this plane is to the square of the lateral edge of the original pyramid. [See proof of 606, III.]

Ex. 2. If two pyramids have equal altitudes and are cut by planes parallel to the bases and at equal distances from the vertices, the sections formed will be to each other as the bases of the pyramids.

Ex. 3. In the figure of 609 prove that the planes of the faces of the prisms that are opposite *OC* are parallel to *OC* and *AB*.

Ex. 4. State the theorems leading up to the theorem of 575.

Ex. 5. State the theorems leading up to the theorem of 584.

Ex. 6. State the theorems leading up to the theorem of 590.

Ex. 7. The base of a pyramid is 180 sq. in. and its altitude is 15 in. What is the area of the section made by a plane parallel to the base, and 5 in. from the vertex?

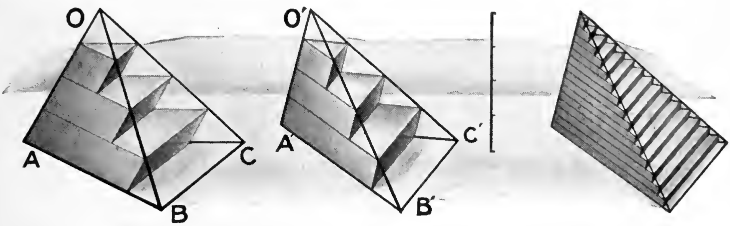
Ex. 8. The base of a pyramid is 200 sq. in., and its altitude is 12 in. At what distance from the vertex must a plane be passed so that the section shall contain half the area of the base?

Ex. 9. The base of a pyramid is B sq. in., and the altitude is h in. How far from the vertex must a plane parallel to the base be passed so that the area of the section shall contain $\frac{1}{3}B$ sq. in.? $\frac{1}{4}B$ sq. in.? Find the area of the section if the plane is passed $\frac{1}{3}h$ in. from the vertex; if it is passed $\frac{1}{4}h$ in. from the vertex.

Ex. 10. A pyramid having an altitude of 2 ft. and a base which is an equilateral triangle of 8 in. on a side, is cut by a plane parallel to the base and 18 in. from it. Find the area of the section.

PROPOSITION XIX. THEOREM

610. Two triangular pyramids having equal altitudes and equal bases are equal.



Given: Triangular pyramids $O\text{-}ABC$ and $O'\text{-}A'B'C'$ having equal altitudes and base ABC = base $A'B'C'$.

To Prove: $O\text{-}ABC = O'\text{-}A'B'C'$.

Proof: Divide the altitude of each pyramid into any number of equal parts.

Through these points of division pass planes ∥ to the bases, forming triangular sections.

Upon these sections as bases construct inscribed prisms.

Denote the volumes of the pyramids by V and V', and the sums of the volumes of these series of prisms by S and S'.

The corresponding sections are equal (607).

∴ the corresponding prisms are equal (591).

Hence $S = S'$ (Ax. 2).

By indefinitely increasing the number of equal parts into which the altitudes are divided, the number of prisms becomes indefinitely great. ∴ S approaches V as a limit (609).

And S' approaches V' as a limit (?).

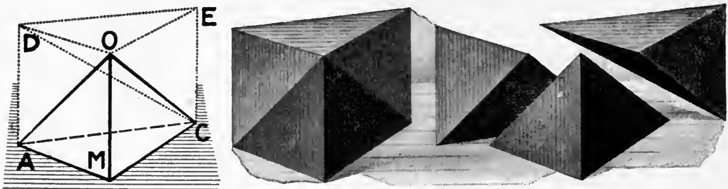
∴ $V = V'$ (229).

That is, $O\text{-}ABC = O'\text{-}A'B'C'$. Q.E.D.

NOTE. As in plane geometry, $\triangle ABC$ is the same as $\triangle BAC$, so in solid geometry the pyramid $O\text{-}ABC$ is the same as the pyramid $A\text{-}BCO$ or $B\text{-}ACO$ or $C\text{-}ABO$.

Proposition XX. Theorem

611. The volume of a triangular pyramid is equal to one third the product of its base by its altitude.



Given: Triangular pyramid $O\text{–}AMC$, whose base $= B$ and altitude $= H$.

To Prove: Volume $O\text{–}AMC = \frac{1}{3} B \cdot H$.

Proof: Construct a prism $AMC\text{–}DOE$, having AMC as its base, and OM as one of its lateral edges. Pass a plane through DO and OC, cutting the face AE in line CD. The prism is now divided into three triangular pyramids.

In pyramids $O\text{–}AMC$ and $C\text{–}ODE$, the altitudes are = (509).

The bases AMC and ODE are ≅ (553).

∴ pyramid $O\text{–}AMC$ = pyramid $C\text{–}ODE$ (610).

In pyramids $C\text{–}AMO$ and $C\text{–}AOD$, the altitudes are the same line, a ⊥ from C to plane DM (491).

The bases AMO and AOD are ≅ (126).

∴ pyramid $C\text{–}AMO$ = pyramid $C\text{–}AOD$ (610).

Hence $O\text{–}AMC = C\text{–}ODE = C\text{–}AOD$ (Ax. 1).

That is, $O\text{–}AMC = \frac{1}{3}$ the prism.

But the volume of the prism $= B \cdot H$ (?).

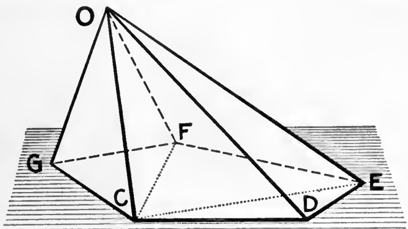
∴ volume of pyramid $O\text{–}AMC = \frac{1}{3} B \cdot H$ (Ax. 6). Q.E.D.

Ex. 1. In the figure of 611 prove pyramid $O\text{–}ACD = O\text{–}CDE$.

Ex. 2. The area of the base of a triangular pyramid is 30 sq. in., and its altitude is 20 in. Find the volume. Find the volume of the prism having the same base and altitude.

Proposition XXI. Theorem

612. The volume of any pyramid is equal to one third the product of its base by its altitude.



Given: Pyramid $O\text{–}CDEFG$, whose base $= B$ and altitude $= H$.

To Prove: Volume of $O\text{–}CDEFG = \frac{1}{3} B \cdot H$.

Proof: Through any lateral edge, OC, and lateral edges not adjoining OC, pass planes dividing the pyramid into triangular pyramids.

$$\left.\begin{aligned} \text{Vol. of } O\text{–}CFG &= \tfrac{1}{3}\, CFG \cdot H \\ \text{Vol. of } O\text{–}CDE &= \tfrac{1}{3}\, CDE \cdot H \\ \text{Vol. of } O\text{–}CEF &= \tfrac{1}{3}\, CEF \cdot H \end{aligned}\right\} \quad (611). \quad \text{Adding,}$$

$$\text{Vol. of } O\text{–}CDEFG = \tfrac{1}{3} B \cdot H \quad \text{(Ax. 2, 4).}$$

Q.E.D.

613. Corollary. **Any two pyramids having equal altitudes and equal bases are equal.** (Ax. 1.)

614. Corollary. **Two pyramids having equal altitudes are to each other as their bases.** (Prove.)

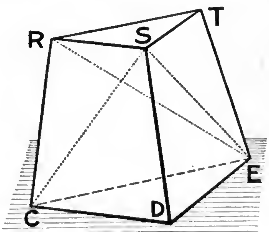
615. Corollary. **Two pyramids having equal bases are to each other as their altitudes.** (Prove.)

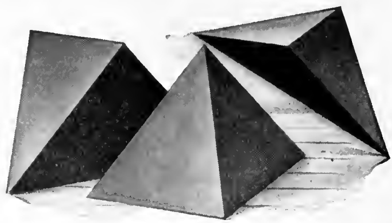
616. Corollary. **Any two pyramids are to each other as the products of their bases by their altitudes.** (Prove.)

Ex. The altitude of a pyramid is 15 ft., and each side of its square base is 8 ft. Find the volume. What is the volume of a prism having the same base and altitude?

PROPOSITION XXII. THEOREM

617. The volume of the frustum of a triangular pyramid is equal to one third the altitude multiplied by the sum of the lower base, the upper base and a mean proportional between the bases of the frustum.





Given: The frustum RD of a triangular pyramid whose lower base $= B$; upper base $= b$; altitude $= H$.

To Prove: Volume of $RD = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Proof: Pass a plane through edge CE and vertex S, and another through edge RS and vertex E, dividing the frustum into three triangular pyramids, $S\text{–}CDE$, $E\text{–}RST$, $E\text{–}CRS$.

I. $S\text{–}CDE = \frac{1}{3} H \cdot B$ (611).

II. $E\text{–}RST = \frac{1}{3} H \cdot b$ (?).

III. We shall now prove

$$E\text{–}CRS = \frac{1}{3} H \cdot \sqrt{B \cdot b}.$$

$$\frac{E\text{–}CSD}{E\text{–}CRS} = \frac{\triangle CSD}{\triangle CRS} \qquad (614).$$

$$\frac{\triangle CSD}{\triangle CRS} = \frac{CD}{RS} \qquad (368).$$

$$\therefore \frac{E\text{–}CSD}{E\text{–}CRS} = \frac{CD}{RS} \qquad (\text{Ax. } 1).$$

Likewise $$\frac{S\text{–}CER}{S\text{–}ERT} = \frac{\triangle CER}{\triangle ERT} \qquad (?).$$

And $$\frac{\triangle CER}{\triangle ERT} = \frac{CE}{RT} \quad (?).$$

$$\therefore \frac{S\text{–}CER}{S\text{–}ERT} = \frac{CE}{RT} \quad (?).$$

But $\triangle$ *CDE* and *RST* are similar (606, II).

$$\therefore \frac{CD}{RS} = \frac{CE}{RT} \quad (?).$$

Hence $$\frac{E\text{–}CSD}{E\text{–}CRS} = \frac{S\text{–}CER \text{ or } E\text{–}CRS}{S\text{–}ERT} \quad (\text{Ax. } 1).$$

That is, $$\frac{\frac{1}{3}H \cdot B}{E\text{–}CRS} = \frac{E\text{–}CRS}{\frac{1}{3}H \cdot b}$$

(Substituting from I and II).

$$\therefore E\text{–}CRS = \sqrt{\tfrac{1}{3}H \cdot B \cdot \tfrac{1}{3}H \cdot b} = \tfrac{1}{3}H \cdot \sqrt{B \cdot b} \quad (289).$$

$\therefore$ **Volume of the frustum** $= \frac{1}{3}H[B + b + \sqrt{B \cdot b}]$ (Ax. 2).

Q.E.D.

NOTE. The theorem of 617 is sometimes stated thus:

The frustum of a triangular pyramid is equal to the sum of three pyramids whose altitudes are the same as the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

Ex. 1. Find the volume of the pyramid whose altitude is 18 in. and whose base is 10 in. square.

Ex. 2. Find the volume of the frustum of a triangular pyramid whose altitude is 20 in. and the areas of whose bases are 18 sq. in. and 32 sq. in.

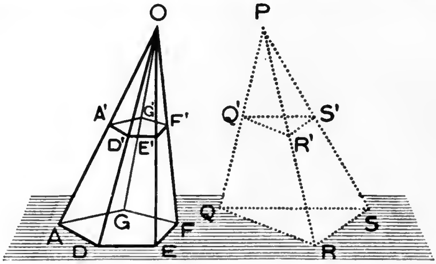
Ex. 3. If, in the formula for the volume of the frustum of a triangular pyramid, $b = B$, show that the formula becomes the correct formula for the volume of a prism. And, again, if $b = 0$, show that the formula of 617 becomes the correct formula for the volume of a pyramid.

Ex. 4. What is the locus of the vertices of all pyramids upon the same base and having the same volume?

Historical Note. The Greeks, as early as the fourth century B.C., knew that the pyramid is the third part of the prism having the same base and altitude. Eudoxus, an Athenian mathematician, is given credit for the discovery and proof of this truth.

Proposition XXIII. Theorem

618. The volume of the frustum of any pyramid is equal to one third the altitude multiplied by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.



Given: Pyr. $O\text{–}ADEFG$; frustum $A'F$, whose lower base $= B$, upper base $= b$, altitude $= H$.

To Prove: Volume of frustum $= \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Proof: Construct a $\triangle QRS =$ to polygon AF (395).

Upon $\triangle QRS$ as a base, construct a pyramid whose altitude $=$ the altitude of $O\text{–}ADEFG$.

Pass a plane $Q'R'S' \parallel$ to QRS and at a distance from $QRS =$ to H.

Vol. of $Q'R = \frac{1}{3} H[\triangle QRS + \triangle Q'R'S' + \sqrt{\triangle QRS \cdot \triangle Q'R'S'}]$ (617).

The alt. of $P\text{–}Q'R'S' =$ alt. of $O\text{–}A'D'E'F'G'$ (Ax. 2).

Also $QRS = B$ (Const.); and $Q'R'S' = b$ (607).

$\therefore$ vol. of $O\text{–}ADEFG =$ vol. of $P\text{–}QRS$ (613).

And vol. of $O\text{–}A'D'E'F'G' =$ vol. of $P\text{–}Q'R'S'$ (?).

Subtracting, vol. of frustum $A'F =$ vol. of frustum $Q'R$ (Ax. 2).

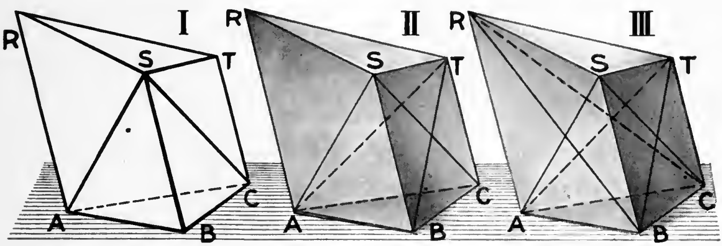
Vol. of frustum $A'F = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$ (Ax. 6).

Q.E.D.

Ex. The bases of the frustum of a pyramid are regular hexagons whose sides are 10 in. and 6 in., respectively. The altitude of the frustum is 2 ft. Find its volume.

PROPOSITION XXIV. THEOREM

619. A truncated triangular prism is equal to three triangular pyramids whose bases are the base of the prism and whose vertices are the three vertices of the face opposite the base (the inclined section).



Given: The truncated triangular prism *ABC–RST*, whose base is *ABC* and whose opposite vertices are *R*, *S*, *T*. Let it be divided by the planes *ACS*, *ABT*, *BCR*.

To Prove: *ABC–RST* = *R–ABC* + *S–ABC* + *T–ABC* (III).

Proof: In Fig. I, *S–ABC* is obviously one of these pyramids.

In Fig. II, *A–CST* = *A–BCT* (613).

That is, *A–CST* = *T–ABC*.

In Fig. III, *T–ARS* = *T–ABR* (613).

T–ABR = *C–ABR* (613).

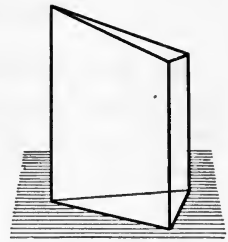
∴ *T–ARS* = *R–ABC* (Ax. 1).

Now *ABC–RST* = *T–ARS* + *S–ABC* + *A–CST* (Ax. 4).

Hence *ABC–RST* = *R–ABC* + *S–ABC* + *T–ABC* (Ax. 6).

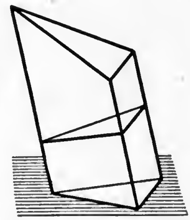
Q.E.D.

Ex. There are approximately $1\frac{1}{4}$ cu. ft. in a bushel. Find the capacity, in bushels, of a grain elevator, 30 ft. high, in the shape of the frustum of a square pyramid, and having bases 24 ft. square and 16 ft. square.



620. Corollary. The volume of a truncated triangular prism is equal to the product of the base by one third the sum of the three altitudes drawn to the base from the three vertices opposite the base.

621. Corollary. The volume of a truncated right triangular prism is equal to the product of the base by one third the sum of its lateral edges.



622. Corollary. The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.

Proof: The right section divides the solid into two truncated right prisms.

Hence the volume = the right section × $\frac{1}{3}$ the sum of the lateral edges. (621.) Q.E.D.

Ex. 1. A pyramid whose volume is V and whose altitude is h, is bisected by a plane parallel to the base. Find the distance of this plane from the vertex.

Ex. 2. The altitude of a square pyramid each side of whose base is 6 ft., is 10 ft. Parallel to the base and 2 ft. from the vertex a plane is passed. Find the area of the section. Find the volumes of the two pyramids concerned, and hence find the volume of the frustum.

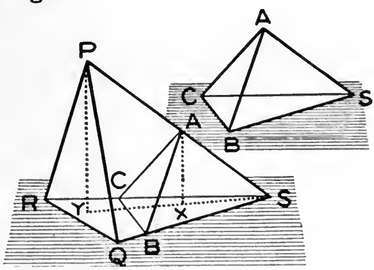
Ex. 3. The base of a pyramid is a rhombus whose diagonals are 7 m. and 10 m. Find the volume if the altitude is 15 m.

Ex. 4. The areas of the bases of the frustum of a pyramid are 3 sq. ft. and 27 sq. ft. The volume is 104 cu. ft. Find the altitude.

Ex. 5. State what distances must be known in order to find the volume of a truncated right triangular prism, and of a truncated oblique triangular prism. Now explain how to use these lengths to find the volume.

Proposition XXV. Theorem

623. Two triangular pyramids (tetrahedrons) having a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges including the equal trihedral angles.



Given: Triangular pyramids $S\text{–}ABC$, $S\text{–}PQR$; having the trih. ∡s at S equal; and their volumes V and V'.

To Prove: $\dfrac{V}{V'} = \dfrac{SA \cdot SB \cdot SC}{SP \cdot SQ \cdot SR}$.

Proof: Place the pyramids so that the equal trihedral ∡s coincide. Draw the altitudes AX and PY and the projection SXY, of the edge PS, in plane SQR.

$$\triangle SAX \text{ is similar to } \triangle SPY \quad (304).$$

Now $$\frac{V}{V'} = \frac{\triangle SBC \cdot AX}{\triangle SQR \cdot PY} = \frac{\triangle SBC}{\triangle SQR} \cdot \frac{AX}{PY} \quad (616).$$

But $\dfrac{\triangle SBC}{\triangle SQR} = \dfrac{SB \cdot SC}{SQ \cdot SR}$ (374) and $\dfrac{AX}{PY} = \dfrac{SA}{SP}$ (313).

Hence, by substituting in the first equation above,

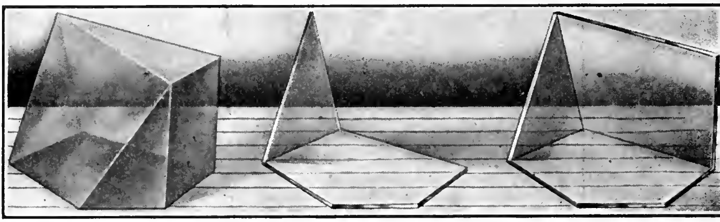
$$\frac{V}{V'} = \frac{SB \cdot SC}{SQ \cdot SR} \cdot \frac{SA}{SP} = \frac{SA \cdot SB \cdot SC}{SP \cdot SQ \cdot SR} \quad (\text{Ax. } 6). \quad \text{Q.E.D.}$$

Ex. 1. To what is the ratio of $\triangle SAB$ to $\triangle SPQ$ equal?

Ex. 2. Reduce the formula of 623 if $SA = SB = SC$ and $SP = SQ = SR$.

Proposition XXVI. Theorem

624. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.



Given: A polyhedron; E = number of edges; F = number of faces; V = number of vertices.

To Prove: $E + 2 = V + F$.

Proof: Suppose the surface of the polyhedron is put together, face by face.

For one face, $E = V$ (145). (Begin with the base.)

By attaching an adjoining face, the number of edges is one greater than the number of vertices.

That is, for two faces, $E = V + 1$.

For three faces, $E = V + 2$.

For four faces, $E = V + 3$.

For five faces, $E = V + 4$.

. . . .

For n faces, $E = V + (n - 1)$.

For $F - 1$ faces, $E = V + (F - 2)$.

By attaching the last face, neither the number of edges nor the number of vertices is increased.

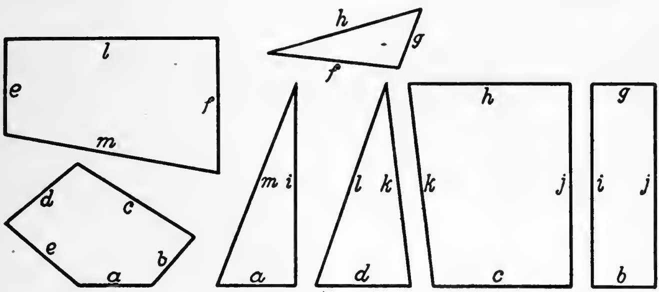
That is, for F faces, $E = V + F - 2$.

$\therefore$ for the complete solid, $E + 2 = V + F$ (Ax. 2). Q.E.D.

625. Corollary. In any polyhedron the difference between the number of edges and the number of faces is two less than the number of vertices, that is, $E - F = V - 2$.

PROPOSITION XXVII. THEOREM

626. The sum of all the face angles of any polyhedron is equal to 4 right angles multiplied by two less than the number of vertices, that is, $S \measuredangle s = (V - 2)\,4$ rt. $\measuredangle s = (V - 2)\,360°$.



Given: A polyhedron; E = number of edges; F = number of faces; V = number of vertices.

To Prove: Sum of all the face $\measuredangle s = (V - 2)\,4$ rt. $\measuredangle s$.

Or, $S \measuredangle s = (V - 2)\,360°$.

Proof: If the faces are considered as separate polygons, it is obvious that each edge is a side of two polygons, that is, the number of sides of the several faces = $2E$.

∴ the number of vertices of all the polygons = $2E$ (145).

Suppose an exterior ∠ formed at each of these $2E$ vertices.

Then the sum of the exterior $\measuredangle s$ of each face = 4 rt. $\measuredangle s$ (?).

Sum of int. and ext. $\measuredangle s$ at each vertex = 2 rt. $\measuredangle s$ (?).

∴ the int. $\measuredangle s$ + ext. $\measuredangle s$ at all the $2E$ vertices = $4E$ rt. $\measuredangle s$

But the ext. $\measuredangle s$ of all the F faces = $4F$ rt. $\measuredangle s$ (?).

∴ the int. $\measuredangle s$ of these polygons = $4E$ rt. $\measuredangle s - 4F$ rt. $\measuredangle s$ (?).

$= (E - F) \cdot 4$ rt. $\measuredangle s$.

But $E - F = V - 2$ (625).

∴ $S \measuredangle s = (V - 2)\,4$ rt. $\measuredangle s = (V - 2)\,360°$ (Ax. 6). Q.E.D.

627. Prismatoid. A **prismatoid** is a polyhedron two of whose faces, called **bases**, are polygons in parallel planes, and whose other faces are triangles, trapezoids, or parallelograms.

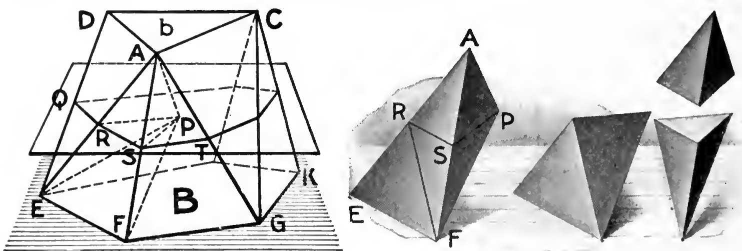
The **altitude** of a prismatoid is the distance between the planes of the bases.

The **mid-section** of a prismatoid is the section made by a plane parallel to the bases and bisecting the altitude.

A **prismoid** is a prismatoid whose lateral faces are either trapezoids or parallelograms.

Proposition XXVIII. Theorem

628. The volume of a prismatoid is equal to the product of one sixth of its altitude by the combined sum of the two bases and four times the mid-section.



Given: The prismatoid $ACD\text{–}EFGKT$, with bases b and B, mid-section M, altitude H, and volume V.

To Prove: $V = \frac{1}{6} H [b + B + 4M]$.

Proof: Through any point, P, in the mid-section, pass planes, each containing an edge of the solid.

These planes will divide the prismatoid into pyramids:

I. The pyramid $P\text{–}ACD$, whose vertex is P, base b, altitude $\frac{1}{2} H$.

Of this pyramid, $v_1 = \frac{1}{6} b \cdot H$ (612).

II. The pyramid $P\text{–}EFGKT$, whose vertex is P, base B, altitude $\frac{1}{2} H$.

Of this pyramid, $v_2 = \frac{1}{6} BH$.

III. Several pyramids, like $P\text{–}AEF$, whose combined volume $= \frac{4}{6} M \cdot H$, as we shall now prove. $AE = 2\,AR$ (?).

$$\therefore \ \triangle AEF = 4 \triangle ARS \qquad (376).$$

$$\therefore \text{ pyramid } P\text{–}AEF = 4 \cdot \text{pyramid } P\text{–}ARS \qquad (614).$$

But pyramid $P\text{–}ARS = \frac{1}{3} \triangle PRS \cdot \frac{1}{2} H = \frac{1}{6} \triangle PRS \cdot H$ (611).

That is, pyramid $P\text{–}AEF = \frac{4}{6} \triangle PRS \cdot H$.

Hence, for the sum of all such pyramids in the prismatoid,

$$v_3 = \tfrac{4}{6} M \cdot H \qquad \text{(Ax. 4).}$$

By addition, $V = \frac{1}{6} bH + \frac{1}{6} BH + \frac{4}{6} MH$ (Ax. 2).

Or, $\boldsymbol{V = \frac{1}{6} H [b + B + 4\,M]}$. Q.E.D.

Ex. 1. A prismatoid has an upper base 5 sq. in., a lower base 11 sq. in., a mid-section 8 sq. in., and an altitude 9 in. Find the volume.

Ex. 2. How does the mid-section of a prism compare with the base? the mid-section of a pyramid? of a cube?

Ex. 3. Will the formula for the volume of a prismatoid give the volume of a cube? Will it give the formula for the volume of a prism?

Ex. 4. Reduce the prismatoid formula to a formula for the volume of a parallelepiped.

Ex. 5. Derive the formula for the volume of a pyramid from the prismatoid formula.

Ex. 6. Is a prism a prismatoid? is a pyramid? is a truncated prism? is the frustum of a pyramid? is a parallelepiped?

Historical Note. The prismatoid formula was discovered by a German, E. F. August, in the latter part of the nineteenth century. Its importance in the mensuration of polyhedrons was recognized at once by mathematicians throughout the world.

REGULAR AND SIMILAR POLYHEDRONS

629. A **regular polyhedron** is a polyhedron whose faces are equal regular polygons and whose polyhedral angles are all equal.

Similar polyhedrons are polyhedrons which have the same number of faces similar each to each and similarly placed, and which have their homologous polyhedral angles equal.

PROPOSITION XXIX. THEOREM

630. There can exist no more than five kinds of regular polyhedrons.

Proof: The faces must be equilateral △s, squares, regular pentagons, or some other regular polygons. (629.)

There must be at least three faces at each vertex. (565.)

The sum of the face ∠s at each vertex is $< 360°$. (549.)

I. Each ∠ of an equilateral $\triangle = 60°$ (?). Hence we may form a polyhedral ∠ by placing 3, 4, or 5 equilateral △s at a vertex, but not 6 (?). That is, only three regular polyhedrons can be formed having equilateral triangles for faces.

II. Each ∠ of a square $= 90°$ (?). Hence we may form a polyhedral ∠ by placing 3 squares at a vertex; but not 4. That is, only one regular polyhedron can be formed having squares for faces.

III. Each ∠ of a regular pentagon $= 108°$ (155). Hence we may form a polyhedral ∠ by placing 3, but not 4 regular pentagons at a vertex. That is, only one regular polyhedron can be formed having regular pentagons for faces.

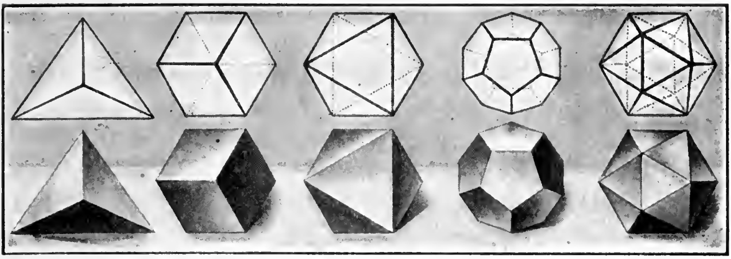
IV. Each ∠ of a regular hexagon $= 120°$ (?).

∴ no polyhedral ∠ can be formed by hexagons (?).

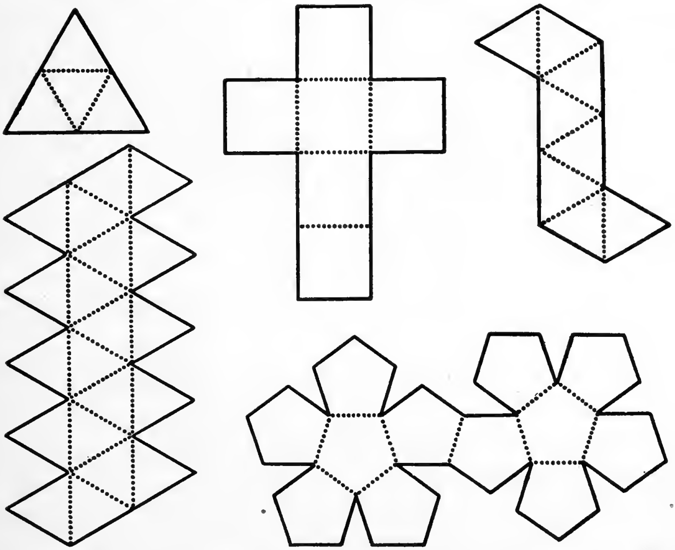
Consequently there can be no more than five kinds of regular polyhedrons, — three kinds bounded by triangles, one kind by squares, and one by pentagons. Q.E.D.

631. The **names** of the regular polyhedrons.

NAMES	TOTAL NUMBER OF FACES	NUMBER OF FACES AT EACH VERTEX	KINDS OF FACES
Regular tetrahedron	4	3	Equilateral triangles
Regular hexahedron (cube)	6	3	Squares
Regular octahedron	8	4	Equilateral triangles
Regular dodecahedron	12	3	Regular pentagons
Regular icosahedron	20	5	Equilateral triangles



REGULAR TETRAHEDRON — CUBE — REGULAR OCTAHEDRON — REGULAR DODECAHEDRON — REGULAR ICOSAHEDRON



DIRECTIONS FOR CONSTRUCTION. — Mark on cardboard larger figures similar to the drawings. Cut the dotted lines half through and the solid lines entirely through. Fold along the dotted lines, closing the solids up and forming the figures. Paste strips of paper along the edges.

Historical Note. Pythagoras knew about the existence of all the regular polyhedrons except the dodecahedron. This was discovered in 470 B.C. by Hippasus, who having boasted of his discovery was drowned by the other Pythagoreans. The regular polyhedrons were supposed to have certain magical properties and their study was greatly emphasized.

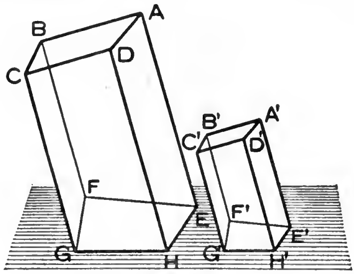
Proposition XXX. Theorem

632. **In two similar polyhedrons:**

I. Homologous edges are proportional.

II. Homologous faces are to each other as the squares of any two homologous edges.

III. Total surfaces are to each other as the squares of any two homologous edges.



Proof: I. Homologous faces are similar (629).

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DH}{D'H'} = \frac{AE}{A'E'} = \frac{BF}{B'F'} = \text{etc.} \quad (313).$$

II. $$\frac{\text{Face } DG}{\text{Face } D'G'} = \frac{\overline{DH}^2}{\overline{D'H'}^2} = \frac{\text{face } AH}{\text{face } A'H'^2} = \frac{\overline{AE}^2}{\overline{A'E'}^2} = \text{etc.} \quad (376).$$

III. $$\frac{\text{Face } DG}{\text{Face } D'G'} = \frac{\text{face } AH}{\text{face } A'H'} = \frac{\text{face } GE}{\text{face } G'E'} = \text{etc.} \quad (\text{Ax. } 1).$$

$$\frac{\text{Total surface of } AG}{\text{Total surface of } A'G'} = \frac{\text{face } DG}{\text{face } D'G'} = \frac{\overline{DH}^2}{\overline{D'H'}^2} = \frac{\overline{AE}^2}{\overline{A'E'}^2} = \text{etc.} \quad (291).$$

Or, $T : T' = e^2 : e'^2$. Q.E.D.

633. Corollary. **If a pyramid is cut by a plane parallel to the base, the pyramid cut away is similar to the original pyramid.** (629.)

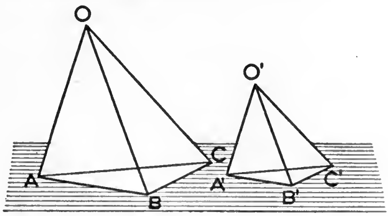
Ex. 1. Show that the theorem of 624 holds true in the case of a cube.

Ex. 2. Show that the theorem of 626 holds true in the case of a cube.

Ex. 3. Show that 624 and 626 are true of a regular octahedron.

Proposition XXXI. Theorem

634. Two similar tetrahedrons are to each other as the cubes of any two homologous edges.



Given: Similar tetrahedrons $O\text{–}ABC$ and $O'\text{–}A'B'C'$; whose volumes $= V$ and V'.

To Prove: $V : V' = \overline{AB}^3 : \overline{A'B'}^3 = \text{etc.}$

Proof: Trihedral $\angle A =$ trihedral $\angle A'$ (629).

$$\therefore \frac{V}{V'} = \frac{AB \cdot AC \cdot AO}{A'B' \cdot A'C' \cdot A'O'} \quad (623).$$

$$= \frac{AB}{A'B'} \cdot \frac{AC}{A'C'} \cdot \frac{AO}{A'O'}.$$

But $$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{AO}{A'O'} \quad (632, \text{I}).$$

$$\therefore \frac{V}{V'} = \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} \quad (\text{Ax. } 6).$$

That is, $V : V' = \overline{AB}^3 : \overline{A'B'}^3 = \overline{AC}^3 : \overline{A'C'}^3 = \text{etc.}$ Q.E.D.

Or, $S : V' = e^3 : e'^3$.

Ex. 1. If, in the figure of 634, $AO = 4$ in. and $A'O' = 1$ in., what is the ratio of the total surfaces of the tetrahedrons? of their volumes?

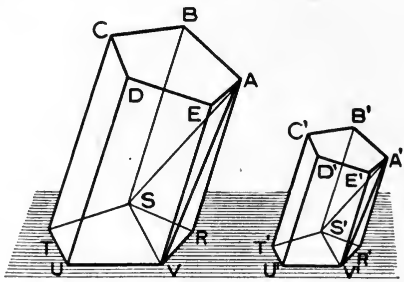
Ex. 2. Two homologous edges of two similar polyhedrons are 2 in. and 3 in. What is the ratio of their total surfaces?

Ex. 3. Two homologous edges of two similar tetrahedrons are 2 in. and 5 in. The total surface of the less is 28 sq. in., and its volume is 40 cu. in. Find the area of the total surface and the volume of the other.

Ex. 4. Show that the theorems of 624 and 626 are true in the cases of regular dodecahedrons and regular icosahedrons.

PROPOSITION XXXII. THEOREM

635. Two similar polyhedrons can be decomposed into the same number of tetrahedrons similar each to each and similarly placed.



Given: (?).

To Prove: (?).

Proof: Suppose diagonals drawn in every face of AT, except the faces containing vertex A, dividing the faces into $\triangle$. (The figure shows only SV.)

Suppose lines drawn from A to the several vertices of these $\triangle$. (The figure shows only AS, AV.)

Obviously, this process divides the solid (by planes) into tetrahedrons, each of which has a vertex at A.

Then construct homologous lines in solid $A'T'$. There will evidently be as many lines in $A'T'$ as in AT and as many tetrahedrons, and these will be similarly placed.

Now, in the tetrahedrons $A\text{–}SVR$ and $A'\text{–}S'V'R'$,

$\triangle AVR$ is similar to $\triangle A'V'R'$;

$\triangle ASR$ is similar to $\triangle A'S'R'$;

$\triangle SVR$ is similar to $\triangle S'V'R'$ (318).

Also $$\frac{AV}{A'V'} = \frac{VR}{V'R'} = \frac{VS}{V'S'} = \frac{RS}{R'S'} = \frac{AS}{A'S'} \quad (313).$$

$$\therefore \frac{AV}{A'V'} = \frac{VS}{V'S'} = \frac{AS}{A'S'} \quad (\text{Ax. } 1).$$

Hence $\triangle ASV$ is similar to $\triangle A'S'V'$ (308).
Also the trihedral $\angle s$ R and R' are equal; S and S' are equal; V and V' are equal, etc. (546).

$\therefore$ the two tetrahedrons are similar (Def. 629).

Furthermore, after removing these tetrahedrons, the remaining polyhedrons are similar. (Def. 629.)

By the same process other pairs of tetrahedrons may be removed and proved similar, and the process may be continued until the polyhedrons are completely decomposed into tetrahedrons similar each to each and similarly placed. Q.E.D.

Proposition XXXIII. Theorem

636. The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.

Given: Similar polyhedrons AT and $A'T'$; volumes V and V'; AR and $A'R'$, any two homologous edges. (See figure of 635.)

To Prove: (?).

Proof: These solids may be decomposed, etc. (635).

Denote the volumes of tetrahedrons of AT by w, x, y, z, etc.; of $A'T'$ by w', x', y', z', etc.

Then $$\frac{w}{w'}=\frac{\overline{AR}^3}{\overline{A'R'}^3};\ \frac{x}{x'}=\frac{\overline{AR}^3}{\overline{A'R'}^3};\ \frac{y}{y'}=\frac{\overline{AR}^3}{\overline{A'R'}^3};\ \text{etc.} \qquad (634).$$

Hence $$\frac{w}{w'}=\frac{x}{x'}=\frac{y}{y'}=\text{etc.} \qquad (\text{Ax. } 1).$$

Therefore $$\frac{w+x+y+\text{etc.}}{w'+x'+y'+\text{etc.}}=\frac{w}{w'} \qquad (291).$$

That is, $$\frac{V}{V'}=\frac{\overline{AR}^3}{\overline{A'R'}^3} \qquad (\text{Ax. } 6).$$ Q.E.D.

637. Corollary. The volumes of two similar pyramids are to each other as the cubes of their altitudes. (Explain.)

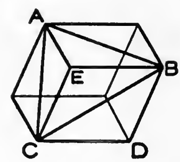
ORIGINAL EXERCISES

1. What plane through the vertex of a given tetrahedron will divide it into two equal parts? Prove.

2. The area of the base of any pyramid is less than the sum of the lateral faces.

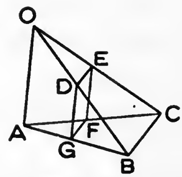
[Draw the altitudes of the lateral faces and the projections of the altitudes upon the base.]

3. Three of the edges of a parallelepiped that meet in a point are also the lateral edges of a pyramid. What part of the parallelepiped is this pyramid?



4. A plane is passed containing one vertex of a parallelepiped and a diagonal of a face not containing that vertex. What part of the volume of the parallelepiped is the pyramid thus cut off?

5. Any section of a tetrahedron made by a plane parallel to two opposite edges, is a parallelogram.



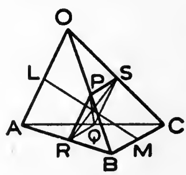
Given: Section $DEFG \parallel$ to OA and BC.

To Prove: $DEFG$ is a ▱.

Proof: EF is $\parallel$ to OA, DG is $\parallel$ to OA (?). Also DE is $\parallel$ to BC and GF is $\parallel$ to BC (?), etc.

6. The three lines that join the midpoints of the opposite edges of a tetrahedron meet in a point and bisect one another.

Given: LM, PQ, RS, three lines, etc. **To Prove**: (?).



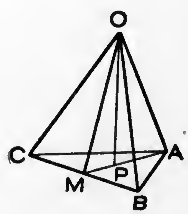
Proof: Join PS, SQ, QR, PR. PS is $\parallel$ to and $= \frac{1}{2} BC$; RQ is $\parallel$ to and $= \frac{1}{2} BC$. (Explain.)

$\therefore$ fig. $PSQR$ is a ▱ (?).

Similarly, discuss LM and SR.

7. A pyramid having one of the faces of a cube for its base and the center of the cube for its vertex, contains one sixth of the volume of the cube.

8. A plane containing an edge of a regular tetrahedron and the midpoint of the opposite edge:



(a) contains the medians of two faces;

(b) is perpendicular to the opposite edge;

(c) is perpendicular to these two faces;

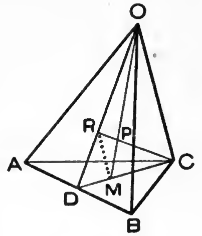
(d) contains two altitudes of the tetrahedron.

9. The altitude of a regular tetrahedron meets the base at the point of intersection of the medians of the base.

10. The altitude of a regular tetrahedron $= \frac{1}{3}\sqrt{6}$ times the edge.

11. The altitudes of a regular tetrahedron meet at a point.

12. The lines joining the vertices of *any* tetrahedron to the point of intersection of the medians of the opposite face meet in a point that divides each line into segments in the ratio 3 : 1.



Given: OM, CR, two such lines.

To Prove: The four such lines meet, etc.

Proof: OM and CR lie in the plane determined by OC and point D, the midpoint of AB.

$\therefore$ OM and CR intersect. Draw RM.

$$\left.\begin{aligned} DR &= \tfrac{1}{2} RO \\ DM &= \tfrac{1}{2} MC \end{aligned}\right\} \text{(?)}. \quad \therefore \frac{DR}{DM} = \frac{RO}{MC} \qquad \text{(?)}.$$

$\therefore$ RM is $\parallel$ to OC (?).

$\therefore$ $\triangle DMR$ and DCO are similar (?); and $DR : DO = RM : OC$. (?)

Thus $RM = \frac{1}{3} OC$. (Explain.)

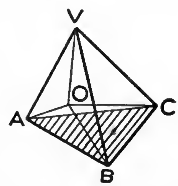
Also $\triangle PRM$ and OPC are similar (?).

Hence $OP : PM = OC : RM = 3 : 1.$
And $CP : PR = OC : RM = 3 : 1.$ (Explain.)

Q.E.D.

NOTE. This point P is called the **center of gravity** of the tetrahedron.

13. There can be no polyhedron having seven edges and only seven.



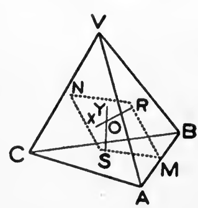
14. The planes bisecting the dihedral angles of any tetrahedron meet in a point that is equally distant from the faces.

15. The lines perpendicular to the faces of any tetrahedron, at the centers of the circles circumscribed about the faces, meet in a point that is equally distant from the vertices.

Proof: RX and SY are loci of points, etc. (511).

Plane MN, $\perp$ to AB at M, the midpoint of AB, is the locus of points, etc.

$\therefore$ RX and SY lie in MN and intersect at O, etc.

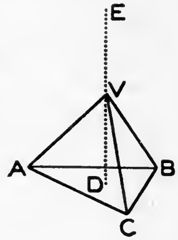


16. If a plane is passed through the midpoints of the three edges of a parallelepiped that meet at a vertex, what part of the whole solid is the pyramid thus cut off?

17. The plane bisecting a dihedral angle of a tetrahedron divides the opposite edge into two segments proportional to the areas of the faces that form the dihedral angle.

18. Two tetrahedrons are similar if a dihedral angle of one equals a dihedral angle of the other and the faces forming these dihedral angles are respectively similar.

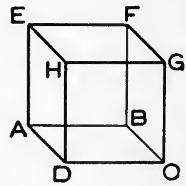
19. If from any point within a regular tetrahedron perpendiculars to the four faces are drawn, their sum is constant and equal to the altitude of the tetrahedron.



20. Construct a regular tetrahedron upon a given edge.

Construction: Upon AB, construct an equilateral $\triangle ABC$. Erect $ED \perp$ to plane of $\triangle ABC$, at D, the center of circumscribed ⊙. Take V on ED such that $AV = BV = CV = AB$, etc.

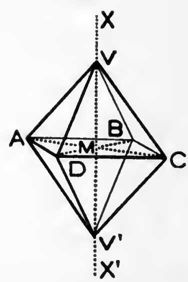
21. Construct a regular hexahedron upon a given edge.



Construction: Upon AB construct a square $ABCD$. At the vertices erect ⊥s = to AB and join the extremities, etc.

22. Construct a regular octahedron upon a given edge.

Construction: Upon AB construct a square $ABCD$. At M, the center of the square, erect XX' ⊥ to plane of $ABCD$. On XX' take $MV = MV' = MD$. Draw the edges from V and V'.



Statement : VV' is a regular octahedron.

Proof: The right ▲s DMV, DMC, DMV' are equal. (Explain.)

Thus the 12 edges are equal and the 8 faces are equal. (Explain.)

Figures $AVCV'$, $DVBV'$, $ABCD$ are equal squares. (Explain.)

Then, pyramids V–$ABCD$, D–$AVCV'$, etc., are equal and the 6 polyhedral angles are equal. (Explain.) ∴ etc.

23. Pass a plane through a cube so that the section will be a regular hexagon.

24. Pass planes through three given lines in space, no two of which are parallel, which shall inclose a parallelepiped.

25 Find the lateral area and the total area of a regular pyramid whose slant height is 20 in. and whose base is a square, 1 ft. on a side.

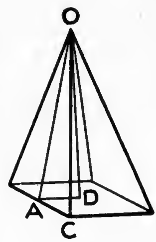
26. Find the volume of a pyramid whose altitude is 18 in. and whose base is an equilateral triangle each side of which is 8 in.

27. A regular hexagonal pyramid has an altitude of 9 ft. and each edge of the base is 6 ft. Find the volume.

28. The base of a pyramid is an isosceles triangle whose sides in inches are 14, 25, 25, and the altitude of the pyramid is 12 in. Find its volume.

29. The altitude of the frustum of a pyramid is 25 in., and the bases are squares whose sides are 4 in. and 10 in., respectively. Find the volume of the frustum.

30. The frustum of a regular pyramid has hexagons for bases whose sides are 5 in. and 9 in., respectively. The slant height of the frustum is 14 in. Find its lateral area. Find its total area.

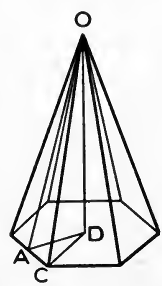


31. The altitude of a regular pyramid is 15 in., and each side of its square base is 16 in. Find the slant height, the lateral edge, the total area, and the volume.

$\overline{OA}^2 = \overline{OD}^2 + \overline{DA}^2 = (15)^2 + (8)^2 = 289.$

$\therefore AO = 17. \quad OC^2 = OA^2 + AC^2 = 289 + 64 = 353.$

$\therefore OC = \sqrt{353} = 18.78^+.$



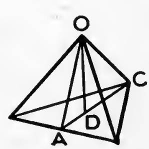
32. The slant height of a regular pyramid is 39 ft., the altitude is 36 ft., and the base is a square. Find the lateral area and the volume.

33. The lateral edge of a regular pyramid is 37 in. and each side of the hexagonal base is 12 in. Find the slant height, the lateral area, the total area, and the volume.

In rt. $\triangle ACD$, $CD = 12$, $AC = 6$, $\therefore AD = 6\sqrt{3}$.

In rt. $\triangle ACO$, $CO = 37$, $AC = 6$, $\therefore AO = \sqrt{1333}$.

In rt. $\triangle CDO$, $CO = 37$, $CD = 12$, $\therefore OD = 35$, etc.



34. Find the total area and volume of a regular tetrahedron whose edge is 6 in.

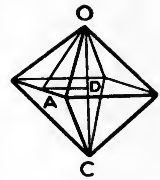
The four faces are equal equilateral $\triangle$s. $\therefore AO = AC = 3\sqrt{3}$ in.; $\therefore AD = \sqrt{3}$ in. and $CD = 2\sqrt{3}$ in.

Hence $OD = 2\sqrt{6}$ in. Area of any face $= 9\sqrt{3}$ sq. in., etc.

35. Find the total area and the volume of a regular tetrahedron whose edge is 10 in.

36. Find the total area and the volume of a regular hexahedron whose edge is 8 in.

37. Find the total area and the volume of a regular octahedron whose edge is 16 in.



The 8 faces are equal equilateral △. $AO = 8\sqrt{3}$. In $\triangle ADO$, one finds $OD = 8\sqrt{2}$. The volume of the octahedron = the volume of two pyramids, etc.

38. Find the total area and the volume of a regular octahedron whose edge is 18 in.

39. The altitude of a regular pyramid is 16 in. and each side of the square base is 24 in. Find the lateral area and the volume.

40. The slant height of a regular pyramid is 16 in. and each side is an equilateral triangle whose side is $20\sqrt{3}$ in. Find the total area and the volume.

41. The altitude of a regular pyramid is 29 in. and its base is a regular hexagon whose side is 10 in. Find the total area and the volume.

42. Find the total area and the volume of a regular tetrahedron whose edge is 18 in.

43. Find the total area and the volume of a regular octahedron whose edge is 20 in.

44. If the edge of a regular tetrahedron is e in., show that the total area is $e^2\sqrt{3}$ in. and the volume is $\frac{1}{12}e^3\sqrt{2}$ cu. in.

45. If the edge of a regular octahedron is e in., show that the total area is $2e^2\sqrt{3}$ sq. in. and the volume is $\frac{1}{3}e^3\sqrt{2}$ cu. in.

46. A pyramid whose base is a square 9 in. on a side, contains 360 cu. in. Find its height.

47. A pyramid has for its base a hexagon whose side is $7\frac{1}{2}$ units and the pyramid contains 675 cubic units. Find the altitude.

48. The volume of a regular tetrahedron is $144\sqrt{2}$ cu. in. Find its edge.

49. The volume of a regular octahedron is $243\sqrt{2}$ cu. in. Find its edge.

50. The volume of a square pyramid is 676 cu. in. and the altitude is 1 ft. Find the side of the base. Find the lateral area.

51. The altitude of the Great Pyramid is 480 ft. and its base is 764 ft. square. It is said to have cost $10 a cubic yard and $3 more for each square yard of lateral surface (considered as planes). What was the cost?

52. The total surface of a regular tetrahedron is $324\sqrt{3}$ sq. in. Find its volume.

53. The base of a pyramid is an isosceles right triangle whose hypotenuse is 8 in. The altitude of the pyramid is 15 in. Find the volume.

54. Find the area of the section of a triangular pyramid, each side of whose base is 8 in. and whose altitude is 18 in., made by a plane parallel to the base and 1 ft. from the vertex.

55. The altitude of a frustum of a pyramid is 6 in., and the areas of the bases are 20 sq. in. and 45 sq. in. Find the altitude of the complete pyramid. Find the volume of this frustum by two distinct methods.

56. A granite monument in the form of a frustum of a pyramid, having rectangular bases one of which is 8 ft. wide and 12 ft. long, and the other 6 ft. wide, is 30 ft. high. It is surmounted by a granite pyramid having the same base as the less base of the frustum, and 10 ft. in height. Find the entire volume and the weight. [1 cu. ft. of water weighs $62\frac{1}{2}$ lb. and granite is 3 times as heavy as water.]

57. If a square pyramid contains 40 cu. in. and its altitude is 15 in., find the side of its base.

58. A church spire in the form of a regular hexagonal pyramid whose base edge is 8 ft. and whose altitude is 75 ft. is to be painted at the rate of 18 ¢ per square yard. Find the cost.

59. Find the edge of a cube whose volume is equal to the volumes of two cubes whose edges are 4 in. and 6 in.

60. The base of a certain pyramid is an isosceles trapezoid whose parallel sides are 20 ft. and 30 ft. and the equal sides each 13 ft. Find the volume of the pyramid if its altitude is 12 yd.

61. The lateral edge of the frustum of a regular square pyramid is 53 in. and the sides of the bases are 10 in. and 66 in. Find the altitude, the slant height, the lateral area, and the volume.

62. The sides of the base of a triangular pyramid in inches are 33, 34, 65, and the altitude of the pyramid is 80. Find its volume.

63. The sides of the base of a tetrahedron in inches are 17, 25, 26, and its altitude is 90. Find its volume.

64. If there are $1\frac{1}{4}$ cu. ft. in a bushel, what is the capacity (in bushels) of a hopper in the shape of an inverted pyramid, 12 ft. deep and 8 ft. square at the top?

65. In the corner of a cellar is a pyramidal heap of coal. The base of the heap is an isosceles right triangle whose hypotenuse is 20 ft. and the altitude of the heap is 7 ft. If there are 35 cu. ft. in a ton of coal, how many tons are there in this heap?

66. How many cubic yards of earth must be removed in digging an artificial lake 15 ft. deep, whose base is a rectangle 180 ft. $\times$ 20 ft. and whose top is a rectangle 216 ft. $\times$ 24 ft.? [The frustum of a pyramid.]

67. One pair of homologous edges of two similar tetrahedrons are 3 ft. and 5 ft. Find the ratio of their surfaces; of their volumes.

68. A pair of homologous edges of two similar polyhedrons are 5 in. and 7 in. Find the ratio of their surfaces; of their volumes.

69. The edge of a cube is 3 in. What is the edge of a cube twice as large? four times as large? half as large?

70. An edge of a tetrahedron is 6 ft. What is the edge of a similar tetrahedron three times as large? eight times as large? nine times as large? one third as large?

71. An edge of a regular icosahedron is 3 in. What is the edge of a similar solid five times as large? ten times as large? fifty times as large? a thousand times as large?

72. The edges of a trunk are 2 ft., 3 ft., 5 ft. Another trunk is twice as long (the other edges 2 ft. $\times$ 3 ft.). How do their volumes compare? A third trunk has each dimension double those of the first. How does its volume compare with the first? How do their surfaces compare?

73. If the altitude of a certain regular pyramid is doubled, but the base remains unchanged, how is the volume affected? If each edge of the base is doubled, but the altitude is unchanged, how is the volume affected? If the altitude and each edge of the base are all doubled, how is the volume affected?

74. A contractor agrees to build a dam 60 ft. long, 15 ft. high, 11 ft. wide at the bottom and 7 ft. wide at the top for $8.25 a cubic yard. Find his profit if it costs him only $2000.

75. A pyramid is cut by a plane parallel to the base and bisecting the altitude. What part of the entire pyramid is the less pyramid cut away by this plane?

76. The volume of a certain pyramid, one of whose edges is 7 in., is 686 cu. in. Find the volume of a similar pyramid whose homologous edge is 8 in.

77. A certain polyhedron whose shortest edge is 2 in. weighs 40 lb. What is the weight of a similar polyhedron whose shortest edge is 5 in.?

78. An edge of a polyhedron is 5 in. and the homologous edge of a similar polyhedron is 7 in. The entire surface of the first is 250 sq. in. and its volume is 375 cu. in. Find the entire surface and volume of the second.

79. A berry box, sold to contain a quart of berries, is in the form of the frustum of a pyramid 5 in. square at the top, $4\frac{1}{2}$ in. square at the bottom, and $2\frac{7}{8}$ in. deep. If a U.S. dry quart contains 67.2 cu. in., does this box contain more or less than a quart?

BOOK VIII

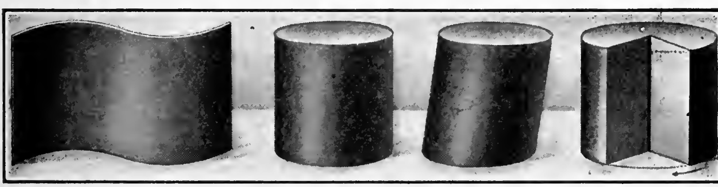
CYLINDERS AND CONES

CYLINDERS

638. A **cylindrical surface** is a surface generated by a moving straight line which continually intersects a given curved line in a plane, and which is always parallel to a given straight line not in the plane of the curve.

The generating line is the **generatrix.** The directing curve is the **directrix.**

An **element** of a cylindrical surface is the generating line in any position.



CYLINDRICAL SURFACE — RIGHT CIRCULAR CYLINDER — OBLIQUE CYLINDER — CYLINDER OF REVOLUTION

639. A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes.

The **bases** of a cylinder are the parallel plane sections.

The **lateral area** of a cylinder is the area of the cylindrical surface included between the planes of the bases.

The **total area** of a cylinder is the sum of the lateral area and the areas of the bases.

The **altitude** of a cylinder is the perpendicular distance between the planes of the bases.

640. A **right cylinder** is a cylinder whose elements are perpendicular to the planes of the bases.

A **circular cylinder** is a cylinder whose base is a circle.

An **oblique cylinder** is a cylinder whose elements are not perpendicular to the planes of the bases.

A **right circular cylinder** is a right cylinder whose base is a circle.

A **cylinder of revolution** is a cylinder generated by the revolution of a rectangle about one of its sides as an axis.

Similar cylinders of revolution are cylinders generated by similar rectangles revolving on homologous sides.

641. A **right section** of a cylinder is a section made by a plane perpendicular to all the elements.

A **plane** is **tangent** to a cylinder if it contains one element of the cylindrical surface and only one, however far it may be extended.

A **prism** is **inscribed** in a cylinder if its lateral edges are elements of the cylinder and the bases of the prism are inscribed in the bases of the cylinder.

A **prism** is **circumscribed** about a cylinder if its lateral faces are tangent to the cylinder and the bases of the prism are circumscribed about the bases of the cylinder.

PRELIMINARY THEOREMS

642. THEOREM. **Any two elements of a cylinder are parallel and equal.** (495 and 509.)

643. THEOREM. **A line drawn through any point in a cylindrical surface, parallel to an element, is itself an element.** (Ax. 13.)

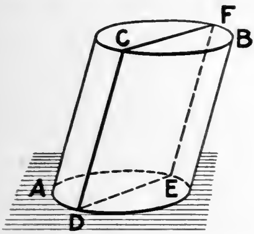
644. THEOREM. **A right circular cylinder is a cylinder of revolution.**

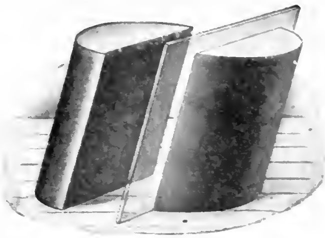
Ex. If a plane is defined as a surface generated by a moving straight line, what is the directrix?

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

645. Every section of a cylinder made by a plane containing an element is a parallelogram.





Given: Cylinder *AB*; plane *CE* containing element *CD*.

To Prove: *CE* is a ▱.

Proof: At *E* draw *EF* ∥ to *CD* in plane *CE*. Also, *EF* is an element of the cylinder. (643.)

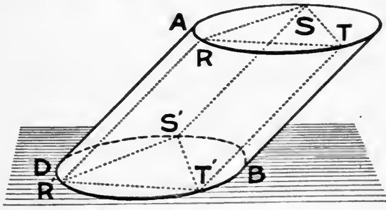
∴ *EF* is the intersection of the plane and the cylindrical surface. (466.)

Also *CF* is ∥ to *DE* (484).

∴ *CDEF* is a ▱ (120).

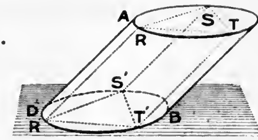
Proposition II. Theorem

646. The bases of a cylinder are congruent.



Given: (?). **To Prove:** (?).

Proof: Suppose *R*, *S*, *T* three points in the perim. of base *AC*.



Draw elements RR', SS', TT'.
Also draw RS, ST, RT, $R'S'$, $S'T'$, $R'T'$.

RR' = and is ∥ to SS'
RR' = and is ∥ to TT' (642).
SS' = and is ∥ to TT'

∴ RS' is a ▱, RT' is a ▱, ST' is a ▱ (129).
∴ $RS = R'S'$; $ST = S'T'$; $RT = R'T'$ (124).
$\triangle RST \cong \triangle R'S'T'$ (?).

∴ base AC may be placed upon base BD so that R, S, and T coincide with R', S', and T', respectively. But S is *any point* on the boundary; hence *every point* on the boundary of AC will coincide with a corresponding point on the boundary of BD.

∴ base $AC \cong$ base BD (Def. 26).

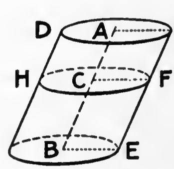
Q.E.D.

647. Corollary. Parallel plane sections of a cylinder (cutting all the elements) are congruent.

648. Corollary. Every section of a right cylinder made by a plane containing an element is a rectangle.

649. Corollary. The line joining the centers of the bases of a circular cylinder is equal and parallel to an element.

650. Corollary. All sections of a circular cylinder parallel to its bases are equal circles, and the straight line joining the centers of the bases passes through the centers of all the parallel sections.



To Prove: HF = any other ∥ section and C is its center.

Proof: Pass a plane AE cutting the three planes in AG, CF, and BE.

DF is a cylinder (Def. 639).

∴ $HF \cong DG$ (646).

That is, HF is a ⊙ and equal to any other section ∥ to DG.

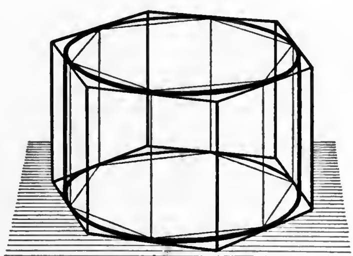
Now AG is ∥ to CF (484).
And AB is ∥ to GE (649).
∴ AF is a ▱ (120).
And $AG = CF$ (124).

But HF is a ⊙ and F is any point on it. Hence C is equally distant from all points on HF, and is, therefore, the center. (179.)

Q.E.D.

Proposition III. Theorem

651. Theorem. If a regular prism is inscribed in, or circumscribed about, a right circular cylinder and the number of sides of the base is indefinitely increased, the lateral area of the cylinder is the limit of the lateral area of the prism.



Given: A regular prism inscribed in and a regular prism circumscribed about a right circular cylinder; the lateral area of the cylinder $= L$, and of the prisms, L_i and L_c, respectively.

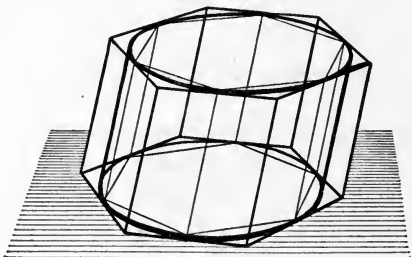
To Prove: That as the number of sides of the bases of the prisms is indefinitely increased, L is the limit of both L_i and L_c.

Proof: If the number of sides of the bases of the prisms is indefinitely increased, their perimeters will approach the circumference of the base of the cylinder as a limit. (?.)

Hence, it is obvious that the lateral area of the cylinder is the limit of the lateral area of either prism. Q.E.D.

Proposition IV. Theorem

652. If a prism having a regular polygon for a base is inscribed in, or circumscribed about, any circular cylinder and the number of the sides of the base of the prism is indefinitely increased, the volume of the cylinder is the limit of the volume of the prism.



Given: (?). **To Prove**: (?).

Proof: If the number of sides of the base of either prism is indefinitely increased, the area of the base of the prism approaches the area of the base of the cylinder. (424, II.)

∴ it is obvious that the volume of the cylinder is the limit of the volume of either prism. Q.E.D.

Ex. 1. If the cylindrical surface of a cylinder is cut along an element, and this surface is placed in coincidence with a plane, what plane geometrical figure will it become?

Ex. 2. What two lines determine the size of a right circular cylinder?

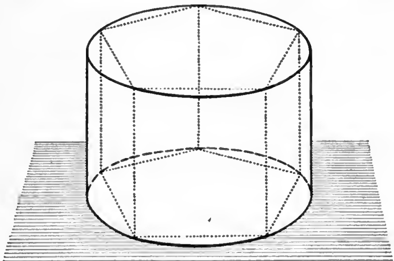
Ex. 3. What is the locus of all points 2 in. from a circular cylindrical surface? Is the answer to this question affected by the radius of the given surface? If so, explain.

Ex. 4. From a log 36 in. in diameter at its less end, is to be cut the largest prismatic piece of timber possible, having square ends. Find the side of this square.

Ex. 5. A lead pencil whose ends are regular hexagons was cut from a cylindrical piece of wood, with the least waste of wood. If the original piece was 8 in. long and $\frac{1}{2}$ in. in diameter, find the volume of the pencil.

Proposition V. Theorem

653. The lateral area of a right circular cylinder is equal to the product of the circumference of the base by an element.



Given: A right circular cylinder, the circumference of whose base = C, and whose element = E.

To Prove: Lateral area $L = C \cdot E$.

Proof: Inscribe in the cylinder a regular prism, the perimeter of whose base is P, whose lateral edge is E, and whose lateral area is L'.

Then $L' = P \cdot E$ (?).

If the number of sides of the base of the prism is indefinitely increased, L' approaches L as a limit (?).

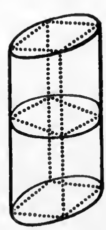
P approaches C as a limit (?).

$\therefore L = C \cdot E$ (?). Q.E.D.

654. Corollary. Area of a right circular cylinder:

$$L = 2\pi RH. \qquad T = 2\pi RH + 2\pi R^2 = 2\pi R(H + R).$$

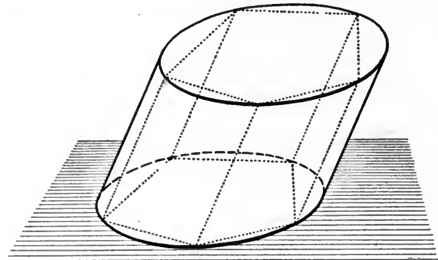
(Where L = lateral area, H = altitude, R = radius of base, and T = total area.)



Note. The lateral area of an oblique circular cylinder equals the product of the perimeter of a right section of the cylinder by an element. The right section of an oblique circular cylinder is not a circle. The right section of an inscribed prism, having a regular polygon for a base, is not a regular polygon. Hence the proof of this theorem is omitted.

PROPOSITION VI. THEOREM

655. The volume of a circular cylinder is equal to the product of its base by its altitude.



Given: A circular cylinder whose base $= B$, altitude $= H$, and volume $= V$.

To Prove: $V = B \cdot H$.

Proof: Inscribe a prism having a regular polygon for its base, whose base $= B'$ and volume $= V'$.

Then $V' = B' \cdot H$ (?).

If the number of sides of the base of the prism is indefinitely increased, V' approaches V as a limit (?).

B' approaches B as a limit (?).

$B' \cdot H$ approaches $B \cdot H$ as a limit.

$\therefore V = B \cdot H$ (229). Q.E.D.

656. COROLLARY. **Volume of a circular cylinder:**

$$V = \pi R^2 H.$$

(Where V = volume, H = altitude, and R = radius of base.)

Ex. 1. Find the lateral area and the total area of a cylinder whose altitude is 15 in. and radius 14 in.

Ex. 2. How many square inches of tin are required to make a cylindrical pipe $10\frac{1}{2}$ in. in diameter and 8 ft. long?

Ex. 3. What is the capacity of a cylindrical pail 1 ft. high and 9 in. in diameter?

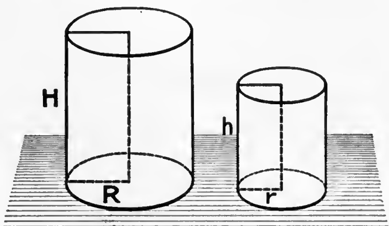
PROPOSITION VII. THEOREM

657. Of two similar cylinders of revolution:

I. The lateral areas are to each other as the squares of their altitudes or as the squares of the radii of their bases.

II. The total areas are to each other as the squares of their altitudes or as the squares of the radii of their bases.

III. The volumes are to each other as the cubes of their altitudes or as the cubes of the radii of their bases.



Given: Two similar cylinders of revolution whose lateral areas $= L$ and l; total areas $= T$ and t; volumes $= V$ and v; altitudes $= H$ and h, and radii are R and r.

To Prove: I. $L : l = H^2 : h^2 = R^2 : r^2$.

II. $T : t = H^2 : h^2 = R^2 : r^2$.

III. $V : v = H^3 : h^3 = R^3 : r^3$.

Proof: The generating rectangles are similar. (Def. 640.)

$\therefore H : h = R : r$ (?).

Hence $\frac{H+R}{h+r} = \frac{H}{h} = \frac{R}{r}$ (291).

I. $\frac{L}{l} = \frac{2\pi RH}{2\pi rh} = \frac{RH}{rh} = \frac{R}{r} \cdot \frac{H}{h} = \frac{H}{h} \cdot \frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$ (Ax. 6).

II. $\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R}{r} \cdot \frac{H+R}{h+r} = \frac{H}{h} \cdot \frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$ (Ax. 6).

III. $\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2 H}{r^2 h} = \frac{R^2}{r^2} \cdot \frac{H}{h} = \frac{H^2}{h^2} \cdot \frac{H}{h} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$ (Ax. 6).

Q.E.D.

ORIGINAL EXERCISES (NUMERICAL)

$\pi = 3\frac{1}{7}$. 1 bu. = 2150.42 cu. in. 1 gal. = 231 cu. in.

In a cylinder of revolution:

1. If $R = 5$ in., $H = 14$ in., find L; T; V.
2. If $R = 7$ m., $H = 10$ m., find L; T; V.
3. If $R = 4\frac{2}{3}$ ft., $H = 18$ ft., find L; T; V.
4. If $R = 6$ in., $L = 792$ sq. in., find H; T; V.
5. If $R = 4$ ft., $T = 352$ sq. ft., find H; L; V.
6. If $R = 2$ in., $V = 22$ cu. in., find H; L; T.
7. If $H = 5.6$ in., $L = 352$ sq. in., find R; T; V.
8. If $H = 9$ in., $T = 440$ sq. in., find R; L; V.
9. If $H = 9\frac{1}{3}$ in., $V = 66$ cu. in., find R; L; T.

10. How many square inches of tin will be required to make a cylindrical pail 10 in. in diameter and 1 ft. in height, without any lid? How many gallons will it contain?

11. The diameter of a cylindrical well is $5\frac{1}{2}$ ft. and the water is 14 ft. deep. How many gallons of water does the well hold?

12. In a cylinder of revolution generated by a rectangle 30 in. × 14 in. revolving about its shorter side as an axis, find L; T; V.

13. In a cylinder of revolution generated by the rectangle of No. 12, revolving about its longer side as an axis, find L; T; V.

14. A cylindrical vessel 9 in. high, closed at one end, required $361\frac{3}{7}$ sq. in. of tin in its construction. Find its radius.

15. A cylindrical pail 12 in. high holds exactly 2 gal. Find R.

16. How many cubic feet of metal are there in a hollow cylindrical tube 42 ft. long, whose outer and inner diameters are 10 in. and 6 in.?

17. A tunnel whose cross section is a semicircle 18 ft. high is 1 mi. long. How many cubic yards of material were removed in the excavation?

18. An irregular stone is placed in a cylindrical vessel a in. in diameter and partly full of water. The water rises b in. Find volume of stone.

19. A rod of copper 18 ft. long and 2 in. square at the end is melted and formed into a wire $\frac{1}{8}$ in. in diameter. Find the length of the wire.

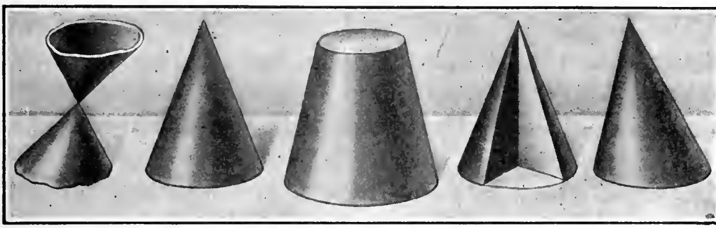
20. How many miles of platinum wire $\frac{1}{32}$ in. in diameter can be made from a cubic foot of platinum?

21. If a cubic foot of copper weighs 550 lb., what is the weight of a copper wire $\frac{1}{8}$ in. in diameter and 5 mi. long?

CONES

658. A **conical surface** is a surface generated by a moving straight line that continually intersects a given curve in a plane, and passes through a fixed point not in this plane.

The generating line is the **generatrix.** The directing curve is the **directrix.** The fixed point is the **vertex** of the conical surface. An **element** of a conical surface is the generating line in any position.



CONICAL SURFACE — RIGHT CIRCULAR CONE — FRUSTUM OF A CONE — CONE OF REVOLUTION — OBLIQUE CONE

659. A **cone** is a solid bounded by a conical surface and a plane cutting all the elements.

The **base** of a cone is its plane surface.

The **lateral area** of a cone is the area of the conical surface.

The **total area** of a cone is the sum of the lateral area and the area of the base.

The **altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.

660. A **circular cone** is a cone whose base is a circle.

The **axis** of a circular cone is the line drawn from the vertex to the center of the base.

A **right circular cone** is a circular cone whose axis is perpendicular to the plane of the base.

An **oblique circular cone** is one whose axis is oblique to the plane of the base.

A **cone of revolution** is a cone generated by the revolution of a right triangle about one of the legs as an axis.

Similar cones of revolution are cones generated by the revolution of similar right triangles revolving about homologous sides.

The **slant height** of a cone of revolution is any one of its elements.

661. A **frustum** of a cone is the portion of a cone between the base and a plane parallel to the base.

The **altitude** of a frustum of a cone is the perpendicular distance between the planes of its bases. The **slant height** of a frustum of a cone is the portion of an element included between the bases. The **lateral area** of a frustum is the area of its curved surface. The **total area** of a frustum is the sum of the lateral area and the area of the bases. The **mid-section** of a frustum is the section made by a plane parallel to the bases and bisecting the altitude and the slant height.

662. A **plane** is **tangent** to a cone if it contains one element of the conical surface and only one, however far it may be extended.

A **pyramid** is **inscribed** in a cone if its base is inscribed in the base of the cone, and its vertex is the vertex of the cone.

A **pyramid** is **circumscribed** about a cone if its base is circumscribed about the base of the cone, and its vertex is the vertex of the cone.

The **frustum** of a pyramid is **inscribed** in, or **circumscribed** about, the frustum of a cone if the bases of the pyramid are inscribed in, or circumscribed about, the bases of the cone.

Ex. 1. Find the slant height of a right circular cone whose altitude is 8 in. and whose radius is 6 in.

Ex. 2. What is the locus of all points 3 in. from a conical surface?

Ex. 3. What is the locus of all lines forming a given angle with a given line at a given point in the line?

PRELIMINARY THEOREMS

663. Theorem. **The elements of a right circular cone are all equal.** (504, II.)

664. Theorem. **A right circular cone is a cone of revolution.**

665. Theorem. **The altitude of a cone of revolution is the axis of the cone.**

666. Theorem. **A straight line drawn from the vertex of a cone to any point in the perimeter of the base is an element.** (39.)

667. Theorem. **The lateral edges of a pyramid inscribed in a cone are elements of the cone.**

668. Theorem. **The lateral faces of a pyramid circumscribed about a cone are tangent to the conical surface.**

669. Theorem. **The slant height of a regular pyramid circumscribed about a right circular cone is the same as the slant height of the cone.**

670. Theorem. **The slant height of the frustum of a regular pyramid circumscribed about the frustum of a right circular cone is the same as the slant height of the frustum of the cone.**

671. Theorem. **The radius of the mid-section of a frustum of a right circular cone is equal to half the sum of the radii of the bases.**

Proof: The radius of the mid-section is the median of a trapezoid whose bases are the radii of the bases of the frustum. That is, $m = \frac{1}{2}(R + r)$. (139.)

Q.E.D.

Ex. If a conical surface is cut along an element and the surface there placed in coincidence with a plane, what geometrical figure does the surface become?

672. Theorem. **If a regular pyramid is inscribed in, or circumscribed about, a right circular cone and the number of sides of the base is indefinitely increased, the lateral area of the cone is the limit of the lateral area of the pyramid.** (See Fig. A.)

Demonstration is similar to that of 651.

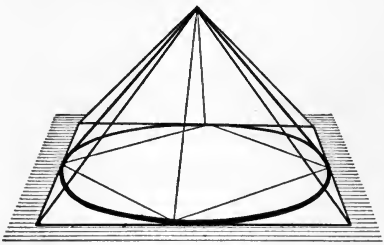


Fig. A

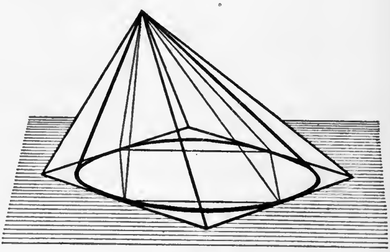


Fig. B

673. Theorem. **If a pyramid having a regular polygon for a base is inscribed in, or circumscribed about, any circular cone and the number of sides of its base is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid.** (See Fig. B.)

Demonstration is similar to that of 652.

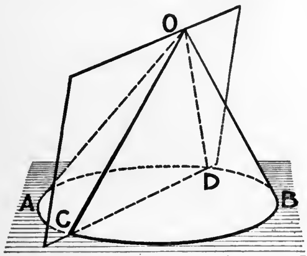
674. Theorem. **If a frustum of a regular pyramid is inscribed in, or circumscribed about, the frustum of a right circular cone and the number of sides of the bases is indefinitely increased, the lateral area of the frustum of the cone is the limit of the lateral area of the frustum of the pyramid.**

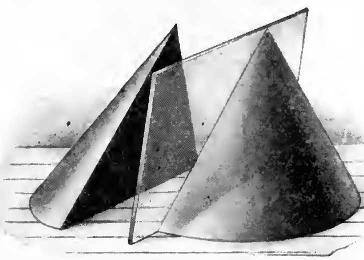
675. Theorem. **If the frustum of a pyramid having regular polygons for its bases is inscribed in, or circumscribed about, a frustum of any circular cone and the number of sides of the bases of the frustum is indefinitely increased, the volume of the frustum of the cone is the limit of the volume of the frustum of the pyramid.**

THEOREMS AND DEMONSTRATIONS

Proposition VIII. Theorem

676. Any section of a cone made by a plane passing through the vertex is a triangle.





Given: Cone *O–AB*; plane *OCD*.

To Prove: Section *OCD* is a △.

Proof: Draw straight lines *OC*, *OD*, in plane *OCD*.

They are elements (666).

∴ *OC* and *OD* compose the intersection of the plane and the conical surface. (466.)

Also *CD* is a straight line (?).

∴ *OCD* is a △ (23).

Q.E.D.

Ex. 1. Can the plane section of a cone, containing the vertex, ever be a right triangle? Explain. Can it be an isosceles triangle? Explain. What kind of triangle is this section, in general?

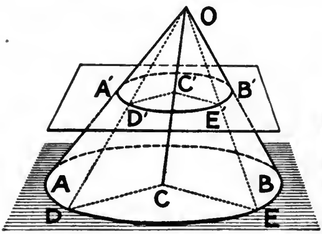
Ex. 2. What is the locus of all straight lines making a given angle with a given plane at a given point in the plane?

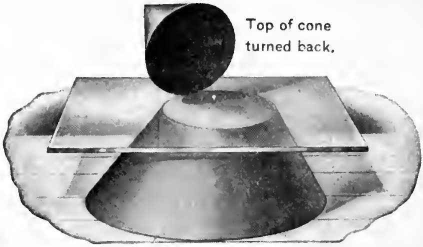
Ex. 3. Does every cone have a slant height, or only certain kinds of cones?

Ex. 4. If a circular disk is held between a source of light and a wall, and parallel to the wall, what is the shape of the shadow on the wall? What is the shape of the shadow region between the light and the wall? (Consider the source of light a point.)

Proposition IX. Theorem

677. Any section of a circular cone made by a plane parallel to the base is a circle, whose center is the intersection of the plane with the axis.





Given: Cone $O\text{–}AB$; circle C its base; section $A'B' \parallel$ to base, and axis OC.

To Prove: $A'B'$ also a ⊙, whose center is C'.

Proof: Pass planes OCD, OCE intersecting the base in CD, CE respectively, and the section in $C'D'$, $C'E'$.

In $\triangle$s OCD and OCE, $D'C'$ is $\parallel$ to DC;

$C'E'$ is $\parallel$ to CE (484).

$\therefore \triangle OC'D'$ is similar to $\triangle OCD$;

$\triangle OC'E'$ is similar to $\triangle OCE$ (305).

$$\therefore \frac{OC'}{OC} = \frac{C'D'}{CD} \text{ and } \frac{OC'}{OC} = \frac{C'E'}{CE} \quad (313).$$

$$\therefore \frac{C'D'}{CD} = \frac{C'E'}{CE} \quad (\text{Ax. } 1).$$

But $CD = CE$ (187).

$\therefore$ by multiplying, $C'D' = C'E'$ (Ax. 3).

That is, all points on the boundary of $A'B'$ are equally distant from C'.

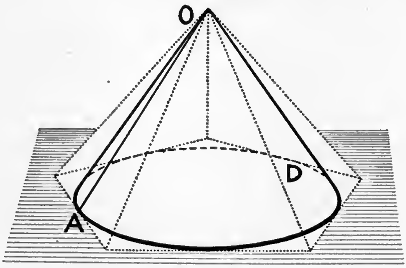
$\therefore A'B'$ is a ⊙ whose center is C (179). Q.E.D.

Ex. Any section of a circular cone parallel to the base is to the base as the square of its distance from the vertex is to the altitude of the cone.

Proof: $A'B' : AB = \overline{C'D'}^2 : \overline{CD}^2 = \overline{OC'}^2 : \overline{OC}^2$. (Explain.)

Proposition X. Theorem

678. The lateral area of a right circular cone is equal to half the product of the circumference of the base by the slant height.



Given: Right circular cone $O\text{–}AD$, the circumference of whose base $= C$ and whose slant height $= s$.

To Prove: Lateral area $= \frac{1}{2} C \cdot s$.

Proof: Circumscribe a regular pyramid and denote the lateral area by L' and the perimeter of the base by P.

Slant height $OA = s$. (663.)

Then $L' = \frac{1}{2} P \cdot s$ (598).

Now indefinitely increase the number of sides of the base of the pyramid and L' approaches L as a limit (672),

P approaches C as a limit (424, I),

$\frac{1}{2} P \cdot s$ approaches $\frac{1}{2} C \cdot s$ as a limit (?).

Hence $L = \frac{1}{2} C \cdot s$ (229). Q.E.D.

679. Corollary. Area of a right circular cone:

$$L = \frac{1}{2}(2\pi R)s = \pi R s. \qquad T = \pi R s + \pi R^2 = \pi R(s + R).$$

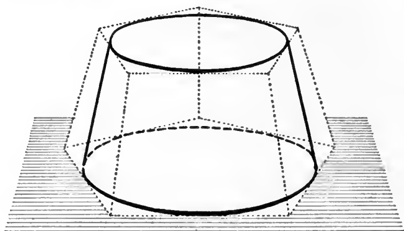
(Where $L =$ lateral area, $T =$ total area, $s =$ slant height, and $R =$ radius of base.)

Ex. 1. Of a right circular cone whose slant height is 15 in. and radius is 9 in., find the lateral area, the total area, and the altitude.

Ex. 2. If the radius of a right circular cone is 8 in. and the altitude is 15 in., find the slant height, the lateral area, and the total area.

PROPOSITION XI. THEOREM

680. The lateral area of the frustum of a right circular cone is equal to half the sum of the circumferences of the bases multiplied by the slant height.



Given: Frustum of right circular cone, whose lateral area is L; whose slant height is s; and the circumferences of whose bases are C and c.

To Prove: $L = \frac{1}{2}(C + c) \cdot s$.

Proof: Circumscribe a frustum of a regular pyramid and denote its lateral area by L', the perimeters of its bases by P and p. The slant height of frustum of pyramid $= s$ (670).

Now $L' = \frac{1}{2}(P + p) \cdot s$ (605).

Indefinitely increase the number of the sides of the bases of the frustum of the pyramid and L' approaches L as a limit, P approaches C, and p approaches c (?).

$\frac{1}{2}(P + p) \cdot s$ approaches $\frac{1}{2}(C + c) \cdot s$ as a limit.

Hence $L = \frac{1}{2}(C + c) \cdot s$ (229). Q.E.D.

681. COROLLARY. **Area of the frustum of a right circular cone:**

$$L = \frac{1}{2}(2\pi R + 2\pi r)s = \pi(R + r)s.$$

$$L = \pi(2m)s = 2\pi ms.$$

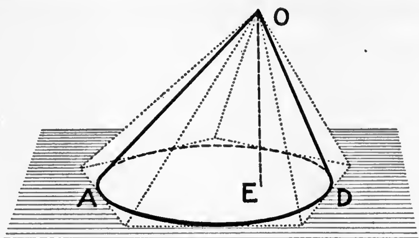
$$T = \pi(R + r)s + \pi R^2 + \pi r^2.$$

$$\therefore T = \pi[(R + r)s + R^2 + r^2].$$

(Where L, T, R, r and s denote magnitudes as before, and m = the radius of the midsection.)

Proposition XII. Theorem

682. The volume of a circular cone is equal to one third the product of the area of the base by the altitude.



Given: Circular cone $O\text{–}AD$, whose volume $= V$; area of whose base $= B$; altitude $= OE = H$.

To Prove: $V = \frac{1}{3} B \cdot H$.

Proof: Circumscribe (or inscribe) a pyramid having a regular polygon for its base. Denote the volume of the pyramid by V', its base by B'. Its altitude $= OE = H$ (491).

$\therefore\ V' = \frac{1}{3} B' \cdot H$ (?).

Indefinitely increase the number of the sides, etc.

Then V' approaches V as a limit (?).

B' approaches B as a limit (424, II).

$\frac{1}{3} B' \cdot H$ approaches $\frac{1}{3} B \cdot H$ as a limit.

Hence $V = \frac{1}{3} B \cdot H$ (229). Q.E.D.

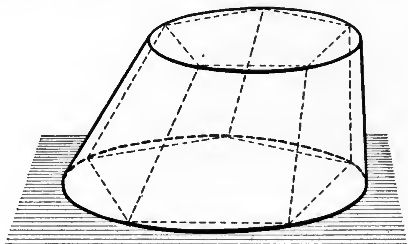
683. Corollary. Volume of a circular cone: $V = \frac{1}{3}\pi R^2 H$. (Where $V =$ volume, $H =$ altitude, and $R =$ radius of base.)

Ex. 1. Find the volume of a circular cone whose altitude is 14 in. and the radius of the base, 6 in.

Ex. 2. A right triangle, whose legs are 15 in. and 20 in., is revolved about the lesser leg as an axis, forming a cone of revolution. Find the lateral area, the total area, and the volume. Find the lateral area, the total area, and the volume of the cone formed by revolving this triangle about the greater leg as an axis.

Proposition XIII. Theorem

684. The volume of the frustum of a circular cone is equal to one third the product of the altitude by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.



Given: A frustum of any circular cone, whose volume $= V$, whose bases are B and b, whose altitude is H.

To Prove: $V = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Proof: Inscribe (or circumscribe) a frustum of a pyramid having regular polygons for bases.

Denote its volume by V', bases by B' and b', and altitude by H.

Then $V' = \frac{1}{3} H [B' + b' + \sqrt{B' \cdot b'}]$ (618).

Indefinitely increase, etc. V' approaches V as a limit (?),

B' and b' approach B and b respectively as limits (?),

$\sqrt{B' \cdot b'}$ approaches $\sqrt{B \cdot b}$ as a limit.

$\frac{1}{3} H [B' + b' + \sqrt{B' \cdot b'}]$ approaches $\frac{1}{3} H [B + b + \sqrt{B \cdot b}]$.

Hence $V = \frac{1}{3} H [B + b + \sqrt{B \cdot b}]$ (229). Q.E.D.

685. Corollary. **Volume of the frustum of a circular cone,**

$$V = \frac{1}{3} H[\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \cdot \pi r^2}]$$

$$\therefore \boldsymbol{V = \frac{1}{3} \pi H [R^2 + r^2 + R \cdot r]}.$$

(Where $V =$ volume, etc.)

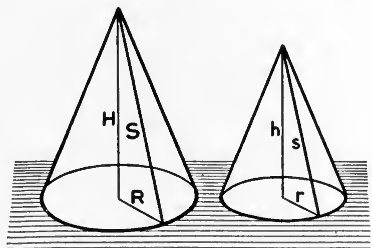
Proposition XIV. Theorem

686. Of two similar cones of revolution:

I. The lateral areas are to each other as the squares of their altitudes, or as the squares of their radii, or as the squares of their slant heights.

II. The total areas are to each other as the squares of their altitudes, or as the squares of their radii, or as the squares of their slant heights.

III. The volumes are to each other as the cubes of their altitudes, or as the cubes of their radii, or as the cubes of their slant heights.



Given: Two similar cones of revolution, whose respective lateral areas are L and l, total areas are T and t, volumes are V and v, altitudes are H and h, radii are R and r, slant heights are S and s.

To Prove: I. $\frac{L}{l}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$. II. $\frac{T}{t}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$.

III. $\frac{V}{v}=\frac{H^3}{h^3}=\frac{R^3}{r^3}=\frac{S^3}{s^3}$.

Proof: Generating $\triangle$s are similar and $\frac{H}{h}=\frac{R}{r}=\frac{S}{s}$ (?).

Hence $\frac{R+S}{r+s}=\frac{R}{r}=\frac{S}{s}=\frac{H}{h}$ (291).

I. $\frac{L}{l}=\frac{\pi RS}{\pi rs}=\frac{R}{r}\cdot\frac{S}{s}=\frac{H}{h}\cdot\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$ (Ax. 6).

II. $\frac{T}{t}=\frac{\pi R(R+S)}{\pi r(r+s)}=\frac{R}{r}\cdot\frac{R+S}{r+s}=\frac{H}{h}\cdot\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$ (Ax. 6).

III. $\frac{V}{v}=\frac{\frac{1}{3}\pi R^2H}{\frac{1}{3}\pi r^2h}=\frac{R^2}{r^2}\cdot\frac{H}{h}=\frac{H^2}{h^2}\cdot\frac{H}{h}=\frac{H^3}{h^3}=\frac{R^3}{r^3}=\frac{S^3}{s^3}$ (?). Q.E.D.

ORIGINAL EXERCISES

In a cone of revolution:

1. If $H = 12$ in., $s = 13$ in., find R.

2. If $H = 15$ ft., $R = 8$ ft., find S.

3. If $R = 18$ cm., $s = 30$ cm., find H.

4. If $H = 6$ in., $s = 10$ in., find R; L; T; V.

5. If $H = 20$ ft., $R = 21$ ft., find s; L; T; V.

6. If $R = 7$ m., $s = 25$ m., find H; L; T; V.

7. If $L = 4070$ sq. in., $s = 37$ in., find R; H; T; V.

8. If $L = 46.64$ sq. in., $R = 2.8$ in., find s; H; T; V.

9. If $L = 400$ sq. ft., $T = 500$ sq. ft., find s; H; R; V.

10. If $T = 80\,\pi$ sq. in., $R = 5$ in., find s; H; L; V.

11. If $T = 10\,\pi$ sq. ft., $s = 3$ ft., find R; H; L; V.

12. If $V = 462$ cu. in., $R = 21$ in., find H; s; L; T.

13. If $V = 8\frac{8}{175}$ cu. ft., $H = 3$ ft., find R; s; L; T.

14. What would be the cost at 10¢ a square foot of painting a conical church steeple, 112 ft. high and 30 ft. in diameter at the base?

15. The sides of an equilateral triangle are each 12 in. Find the lateral surface, the total surface, and the volume of the solid generated by revolving this triangle about an altitude as an axis.

16. An isosceles right triangle whose legs are each 8 in. is revolved about the hypotenuse as an axis. Find the total surface and the volume of the solid generated.

17. The sides of an equilateral triangle are each 10 in. Find the total surface and the volume of the solid generated by revolving this triangle about one of its sides as an axis.

18. Find the volume of a cone of revolution whose slant height is 16 in. and whose lateral area is $192\,\pi$ sq. in.

19. Find the lateral area of a cone of revolution whose altitude is 20 in. and whose volume is $240\,\pi$ cu. in.

20. How many bushels are there in a conical heap of grain whose base is a circle 35 ft. in diameter, and whose height is 25 ft.?

21. A regular hexagon whose side is 6 in. revolves about one of the longer diagonals. Find the surface and the volume of solid generated.

22. Find the volumes of the right circular cones inscribed in and circumscribed about a regular tetrahedron whose edge is a m.

23. A right triangle whose legs are 15 in. and 20 in. is revolved about the hypotenuse as an axis. Find the surface and the volume of the solid generated.

In the frustum of a right circular cone:

24. If $H = 8$ in., $R = 10$ in., $r = 4$ in., find s; L; T; V.

25. If $H = 30$ cm., $s = 34$ cm., $r = 5$ cm., find R; L; T; V.

26. If $s = 19\frac{1}{2}$ ft., $R = 10\frac{1}{2}$ ft., $r = 3$ ft., find H; L; T; V.

27. How many square feet of tin are required to make a funnel 2 ft. long, if the diameters of the ends are 20 in. and 56 in., respectively?

28. A chimney 150 ft. high has a cylindrical flue 3 ft. in diameter. The bases of the chimney are circles whose diameters are 28 ft. and 7 ft. Find the number of cubic yards of masonry in the chimney.

29. A plane is passed parallel to the base of a right circular cone and $\frac{2}{5}$ the distance from the vertex to the base. Find the ratio of the smaller cone thus formed to the original cone. Compare the volume of the less cone with the frustum formed.

$$\frac{\text{Original cone}}{\text{Less cone}} = \frac{5^3}{2^3}\ (?) = \text{etc.}$$

Hence $$\frac{\text{Original cone} - \text{Less cone}}{\text{Less cone}} = \frac{125 - 8}{8}\ (?), \text{ etc.}$$

30. The altitude of a cone of revolution is 12 in. What is the altitude of the frustum of this cone that shall contain one fourth the volume of the whole cone?

31. The altitudes of two similar cylinders of revolution are 3 in. and 5 in. What is the ratio of their lateral areas? of their total areas? of their volumes?

32. The altitudes of two similar cylinders of revolution are 5 in. and 6 in., and the lateral area of the first is 200 sq. in. Find the lateral area of the second. If the volume of the first is 500 cu. in., what is the volume of the second?

33. The total areas of two similar cones of revolution are 24π sq. in. and 216π sq. in. and the radius of the first is 3 in. Find the radius of the second. The slant height of the first is 5 in. Find the lateral area of the second. Find the altitude of the first and the volume of the second.

34. The volumes of two similar cones of revolution are 27π cu. in. and 343π cu. in. The altitude of the first is 9 in. Find the altitude of the second. Find the radius of the base of each.

35. A cone of revolution whose radius is 10 in. and altitude 20 in., has the same volume as a cylinder of revolution whose radius is 15 in. Find the altitude of the cylinder.

36. A cylinder of revolution whose radius is 8 in. and altitude 30 in., is formed into a cone of revolution whose altitude is 40 in. Find the radius of its base.

37. The heights of two equivalent cylinders of revolution are in the ratio of 4 : 9. If the diameter of the first is 92 ft., what is the diameter of the second?

38. A cylinder of revolution 8 ft. in diameter is equivalent to a cone of revolution 7 ft. in diameter. If the height of the cone is 16 ft., what is the height of the cylinder?

39. Two circular cylinders having equal altitudes are to each other as their bases.

40. Two circular cylinders having equal bases are to each other as their altitudes.

41. Two circular cylinders having equal bases and equal altitudes are equal.

42. Two circular cones having equal altitudes are to each other as their bases.

43. Two circular cones having equal bases are to each other as their altitudes.

44. Two circular cones having equal bases and equal altitudes are equal.

45. If the altitude of a right circular cylinder is equal to the radius of the base, the lateral area is half the total area.

46. If the altitude of a right circular cylinder is half the radius of the base, the lateral area is equal to the area of the base.

47. If the slant height of a right circular cone is equal to the diameter of the base, the lateral area is double the area of the base.

48. The lateral area of a cone of revolution is equal to the area of a circle whose radius is a mean proportional between the slant height and the radius of the base.

49. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

50. What relation does the section of a circular cone made by a plane parallel to the base have to the base? Prove.

51. At what distance from the vertex of a right circular cone whose altitude is h must a plane parallel to the base be passed, so as to bisect the lateral area? At what distance must it be passed so as to bisect the volume?

52. What does the volume V of a right circular cone become, if the altitude is doubled and the base undisturbed? Prove. What does the volume V become if the radius of the base is doubled but the altitude undisturbed? Prove. What does the volume become if both radius of base and altitude are doubled? Prove.

53. The intersection of two planes tangent to a cylinder is a line parallel to an element.

54. The intersection of two planes tangent to a cone is a line through the vertex.

55. One straight line can be drawn upon a cylindrical surface through a given point, and only one.

56. If two cylinders of revolution have equivalent lateral areas, their volumes are to each other as their radii.

57. If a rectangle is revolved about its unequal sides as axes, the volumes of the two solids generated are inversely proportional to the axes, and directly proportional to the radii of the bases.

58. Show that the formula for the volume of a circular cone can be derived from the formula for the volume of a frustum of a circular cone if one base of the frustum becomes a point.

59. Reduce the formula for the volume of a frustum of a circular cone if the radius of one base is double the radius of the other.

60. Could you prove the theorem of 680 by inscribing a frustum of a pyramid? Could you prove the theorem of 684 by circumscribing a frustum of a pyramid? Give reasons for your answer.

61. Could you prove the theorem of 678 by inscribing a pyramid? Could you prove the theorem of 682 by inscribing a pyramid? Give reasons.

62. Pass a plane tangent to a circular cylinder and containing a given element.

Construction: Draw a line in plane of base tangent to the base at the end of the given element, etc.

63. Pass a plane tangent to a circular cone and containing a given element.

64. Divide the lateral surface of a cone of revolution into two equivalent parts by a plane parallel to the base.

65. Pass a plane tangent to a circular cylinder and through a given point without it.

Construction: From the point draw a line ∥ to an element, meeting the plane of the base. From this point of intersection draw a line tangent to the base of the cylinder. Through the point of contact draw an element, etc.

66. Pass a plane tangent to a circular cone through a given point without it.

Construction: Connect this point with the vertex of the cone and prolong this line to meet the plane of the base, etc.

67. Find the locus of points at a given distance from a given straight line.

68. Find the locus of points equally distant from two given points and at a given distance from a straight line. Discuss.

69. Find the locus of points at a given distance from a given plane and at a given distance from a given line. Discuss.

70. Find the locus of points at a given distance from a given cylindrical surface whose generatrix is a circle, and whose elements are perpendicular to the plane of the circle.

71. Find the locus of a point at a given distance from a given line and equally distant from two given planes. Discuss.

72. Find a point X, at a given distance from a given line and equally distant from three given points. Discuss.

73. A grain elevator in the form of a frustum of a right circular cone is 30 ft. high, and the radii of its bases are 12 ft. and 8 ft. respectively. If a bushel contains approximately $1\frac{1}{4}$ cu. ft., how many bushels of wheat will this elevator hold?

74. A certain coffee pot is 7 in. deep, $3\frac{1}{2}$ in. in diameter at the top, and 5 in. at the bottom. If there are 5 cups in a quart, how many cups of coffee will this coffee pot hold? (Disregard fractional parts of a cup.)

75. Find the radius of a circle having the same area as the lateral area of a cone of revolution whose radius is 4 and slant height is 9.

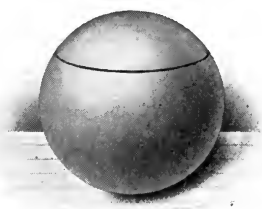
76. The frustum of a circular cone is 15 in. high and the bases are circles whose radii are 3 in. and 5 in. Find the edge of an equivalent cube.

77. The frustum of a right circular cone has a slant height of 9 ft. and the radii are 5 ft. and 7 ft. Find the lateral area and the total area. What is the length of the altitude of this frustum? Find the altitude of the cone that was removed to leave this frustum. Now find in two ways the volume of the frustum.

BOOK IX

THE SPHERE

687. A **sphere** is a solid bounded by a surface, all points of which are equally distant from a point within, called the **center.** The surface of a sphere is called a **spherical surface.**



A **radius** of a sphere is a straight line drawn from the center to any point of the surface.

A **diameter** of a sphere is a straight line that contains the center and has its extremities in the surface.

Ex. 1. What is the locus of all points 2 in. from the surface of a sphere whose radius is 10 in.?

Ex. 2. Consider the center of a sphere at one of the vertices of a rectangular box or room. What part of the sphere is within the box or room?

Ex. 3. Name several familiar objects that are usually regarded as spheres.

Ex. 4. What is the shape of the celestial bodies?

Ex. 5. Why do you believe the earth to be spherical?

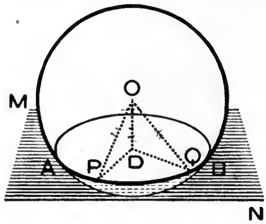
Ex. 6. Why do you believe the moon to be spherical?

Ex. 7. Describe fully the locus of points 3 in. from a line 8 in. long.

Historical Note. Archimedes was the discoverer of the formulas for the surface and volume of the sphere. Menelaus (100 A.D.) gave us the properties of spherical triangles.

Proposition I. Theorem

688. Every plane section of a sphere is a circle.



Given: Sphere whose center is O; plane MN intersecting sphere in AB.

To Prove: The figure AB is a ⊙.

Proof: Draw $OD \perp$ to plane MN, meeting the plane at D.

Take P and Q, any two points on the perimeter of the section, and draw DP, DQ, OP, OQ.

△ ODP and ODQ are rt. △ (?).

In right △ ODP and ODQ,

$OD = OD$ (Iden.).

$OP = OQ$ (687).

∴ the right △ are congruent (?).

Hence $DP = DQ$ (?).

That is, all points of the boundary of AB are equally distant from D.

Hence the section AB is a circle (179). Q.E.D.

689. A **great circle** of a sphere is a section of the sphere made by a plane containing the center of the sphere.

A **small circle** of a sphere is a section of the sphere made by a plane that does not contain the center of the sphere.

The **axis of a circle** of a sphere is the diameter of the sphere perpendicular to the plane of the circle.

The **poles of a circle** of a sphere are the ends of its axis.

A **quadrant** (in spherical geometry) is one fourth of a great circle.

Equal spheres are spheres having equal radii.

690. A **plane** is **tangent** to a sphere if it touches the sphere in one and only one point. Two **spheres** are **tangent** to each other if they are tangent to the same plane at the same point. They may be tangent internally or externally.

A **line** is tangent to a sphere if it touches the sphere in one and only one point and does not intersect it.

A **line** is tangent to the circle of a sphere if it lies in the plane of the circle and touches the circle in one and only one point. In all cases this common point is the **point of contact** or **point of tangency.**

691. A **sphere** is **inscribed** in a polyhedron if all the faces of the polyhedron are tangent to the sphere.

A **sphere** is **circumscribed** about a polyhedron if all the vertices of the polyhedron are in the spherical surface.

692. The **distance** between two points on the surface of a sphere is the less arc of a great circle passing through them.

The distance between a point on a circle of a sphere and the nearer pole of the circle is the **polar distance** of the point.

693. The **angle between two** intersecting **curves** is the angle formed by their tangents at the point of intersection.

A **spherical angle** is the angle between two great circles of a sphere.

PRELIMINARY THEOREMS

694. Theorem. **All radii of a sphere are equal.** (687.)

695. Theorem. **All radii of equal spheres are equal.** (689.)

696. Theorem. **All diameters of the same sphere or of equal spheres are equal.**

697. Theorem. **All great circles of the same sphere or of equal spheres are equal.**

698. Theorem. **In the same sphere or in equal spheres:**

I. Equal plane sections are equally distant from the center.

II. Plane sections equally distant from the center are equal. [Converse.]

III. Of two unequal plane sections, the greater is at the less distance from the center.

IV. Of two plane sections unequally distant from the center, the section at the greater distance is the less. [Converse.]

In each case the diameters of the sections are chords of great circles. These theorems follow from 208–211.

699. Theorem. **Two great circles of a sphere bisect each other.** (Because they have a common diameter.)

700. Theorem. **Every great circle of a sphere bisects the sphere and the spherical surface.** (?).

701. Theorem. **A sphere may be generated by the revolution of a semicircle about the diameter as an axis.** (?).

702. Theorem. **One and only one great circle can be drawn through two points, not the ends of a diameter, on the surface of a sphere.** (477.)

703. Theorem. **One and only one circle can be drawn through three points on the surface of a sphere.** (477.)

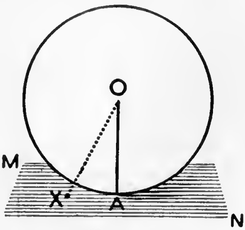
704. Theorem. **A point is without a sphere if its distance from the center is greater than the radius, and conversely if a point is without a sphere, its distance from the center is greater than the radius.**

705. Theorem. **The axis of a circle of a sphere passes through its center.** (This is proved in the proof of 688.)

THEOREMS AND DEMONSTRATIONS

Proposition II. Theorem

706. A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.



Given: Radius OA of sphere O, and plane $MN \perp$ to OA at A.

To Prove: MN tangent to the sphere.

Proof: Take *any* point X in MN (except A) and draw OX.

$OX > OA$ (504, I).

$\therefore X$ lies without the sphere (704).

Hence every point of plane MN, except A, is without the sphere; that is, plane MN is tangent to the sphere. (690.)

Q.E.D.

Proposition III. Theorem

707. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact. [Converse.]

Given: Plane MN tangent to sphere O at A; radius OA.

To Prove: OA is $\perp$ to plane MN.

Proof: Every point in MN, except A, is without the sphere (690).

Take any point X in MN and draw OX.

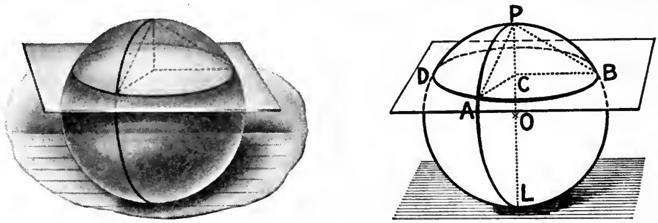
Now OX is $> OA$ (704).

That is, OA is the shortest line from O to MN.

$\therefore OA$ is $\perp$ to MN (504, I). Q.E.D.

Proposition IV. Theorem

708. All points in the circumference of a circle of a sphere are equally distant from either pole; that is, the polar distances of all points in the circumference of a circle are equal.



Given: P and L, the poles of ⊙ C on sphere O, and great Ⓢ PAL, PBL.

To Prove: arc PA = arc PB;
arc AL = arc BL.

Proof: Draw the axis PL meeting plane of ⊙ C at C. Draw AC, AP, BC, BP.

PC is ⊥ to plane DAB (Def. of axis, 689).

∴ chord PA = chord PB (?).

Hence arc PA = arc PB (?).

Likewise arc AL = arc BL. Q.E.D.

709. Corollary. The polar distance of a great circle is a quadrant.

Ex. 1. If the radius of a sphere is 26 in. and a plane is passed, 10 in. from the center, find the radius of the circular section.

Ex. 2. What geographical circles on the earth's surface are great circles? Which are small circles?

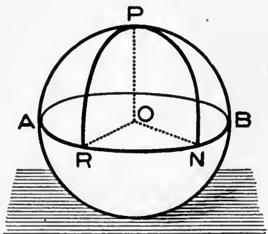
Ex. 3. Can two circles on the surface of a sphere intersect in more than two points? Why?

Ex. 4. The area of a section of a sphere 45 in. from the center is 784π sq. in. Find the radius of the sphere.

Ex. 5. The area of a section of a sphere 7 in. from the center is 576π sq. in. Find the area of a section 6 in. from the center.

PROPOSITION V. THEOREM

710. If a point on the surface of a sphere is at the distance of a quadrant from two other points on the surface, not the ends of a diameter, it is the pole of the great circle containing these two points.



Given: *P*, a point, and *R* and *N*, two other points (not the ends of a diameter),—all on the surface of sphere *O*; arcs *PR* and *PN*, quadrants; great circle *ARNB*.

To Prove: *P* is the pole of ⊙ *ARNB*.

Proof: Draw the radii *OP*, *OR*, *ON*.

$\angle$s *PON* and *POR* are rt. $\angle$s (232).

∴ *PO* is ⊥ to plane *AB* (485).

∴ *PO* is the axis of ⊙ *ARNB* (Def. 689).

∴ *P* is the pole of ⊙ *ARNB* (Def. 689).

Q.E.D.

Ex. 1. Considering the earth as a true sphere, what is the pole of the equator? If a city has a latitude of 43°, what is its polar distance? What is the polar distance of a place upon the equator?

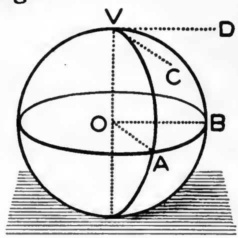
Ex. 2. In the diagram above, *O* is a trihedral angle. What arcs are the measures of its face angles? What is an isosceles trihedral angle? Explain by this diagram that the name is consistent with the etymology of the word.

Ex. 3. Prove that two lines tangent to a sphere at a point determine a plane tangent to the sphere at that point.

Ex. 4. Find the volume of a cube circumscribed about a sphere whose radius is 6 in. Find the volume of the cube inscribed in this sphere.

Proposition VI. Theorem

711. A spherical angle is measured by the arc of a great circle having the vertex of the angle as a pole and intersected by the sides of the angle.



Given: Spherical $\angle AVB$; arc AB of great $\odot$ whose pole is V, on sphere O.

To Prove: $\angle AVB$ is measured by arc AB.

Proof: Draw radii OA, OB, OV, and at V draw VC tangent to $\odot$ VA, and VD tangent to $\odot$ VB. VB is a quadrant. (709.)

OV is $\perp$ to VD (203),

OV is $\perp$ to OB (232);

OV is $\perp$ to VC and to OA (?).

$\therefore$ VD is $\parallel$ to OB, and VC is $\parallel$ to OA (62).

$\therefore$ $\angle CVD = \angle AOB$ (499).

$\angle CVD$ is the spherical $\angle AVB$ (693).

$\angle AOB$ is measured by arc AB (232).

$\therefore$ $\angle CVD$ is measured by arc AB (Ax. 6).

That is, $\angle AVB$ is measured by arc AB (Ax. 6).

Q.E.D.

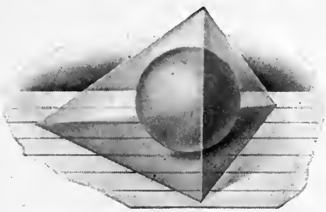
712. Corollary. All arcs of great circles containing the pole of a great circle are perpendicular to the great circle.

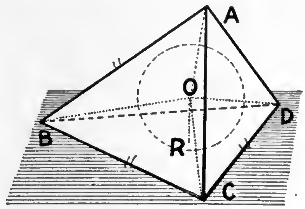
713. Corollary. A spherical angle is equal to the plane angle of the dihedral angle formed by the planes of its sides.

714. Corollary. If two great circles are perpendicular to each other, each contains the pole of the other. (528; 689.)

Proposition VII. Theorem

715. A sphere may be inscribed in any tetrahedron.





Given: Tetrahedron *A–BCD*. **To Prove:** (?).

Proof: Pass plane *OAB* bisecting dih. ∠ *AB*, and plane *OBC* bisecting dih. ∠ *BC*, and plane *OCD* bisecting dih. ∠ *CD*, the three planes meeting at point *O*.

Point *O*, in plane *OAB*, is equally distant from faces *ABC* and *ABD*. (?.)

Point *O*, in plane *OBC*, is equally distant from faces *ABC* and *BCD*. (?.)

Point *O*, in plane *OCD*, is equally distant from faces *BCD* and *ACD*. (?.)

∴ *O* is equally distant from all four faces (Ax. 1).

Hence the sphere constructed with *O* as a center and *OR* as a radius, is tangent to each of the four faces.

∴ that sphere is inscribed in the tetrahedron (691). Q.E.D.

716. Corollary. The six planes bisecting the six dihedral angles of any tetrahedron meet in a point.

Ex. 1. The volume of any polyhedron circumscribed about a sphere, is equal to one third the product of the surface of the polyhedron and the radius of the sphere.

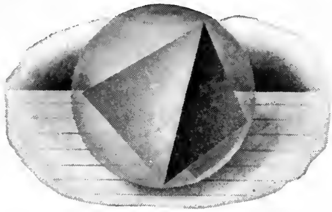
Proof: Pass planes each containing the center of the sphere and two vertices of the polyhedron. These form pyramids whose altitude . . . etc.

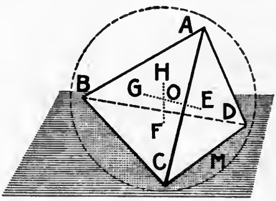
Ex. 2. What is true of the point at which the six planes meet in 716?

Ex. 3. Explain, so that a blind boy could understand it, the process of inscribing a sphere in a tetrahedron.

PROPOSITION VIII. THEOREM

717. A sphere may be circumscribed about any tetrahedron.





Given: (?). **To Prove**: (?).

Proof: Take *E* and *F*, the centers of circles circumscribed about the faces *ACD* and *BCD*, respectively. Erect *EG* and *FH* ⊥ to these faces. Find *M*, the midpoint of edge *CD*.

EG is the locus of all points equally distant from points *A*, *D*, and *C*. (511.)

FH is the locus of all points equally distant from points *B*, *C*, and *D*. (?.)

That is, all points in *EG* and *FH* are equally distant from *C* and *D*. (Ax. 1.)

But all points equally distant from *C* and *D* are in a plane ⊥ to *CD* at *M*. (510.)

∴ *EG* and *FH* are in this plane and are not parallel. (Not ⊥ to the same plane.)

That is, *EG* and *FH* must intersect at *O*.

Hence *O* is equally distant from *A*, *B*, *C*, and *D*. (Ax. 1.)

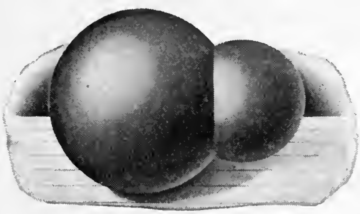
That is, using *O* as a center and *OA*, or *OB*, or *OC*, or *OD*, as a radius, a sphere may be circumscribed about the tetrahedron *A–BCD*. (691.) Q.E.D.

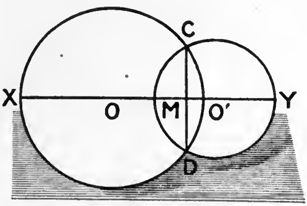
718. COROLLARY. **Through any four points not in the same plane a sphere may be described.**

719. COROLLARY. **The six planes perpendicular to the edges of any tetrahedron at their midpoints meet in a point.**

Proposition IX. Theorem

720. The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is in that line.





Given: Two intersecting circles O and O'; common chord CD; line of centers XY, intersecting CD at M.

To Prove: The spherical surfaces generated by the revolution of these ⓢ intersect in a circle.

Proof: If these ⓢ are revolved upon XY as an axis, they will generate spheres. (701.)

$CM = MD$ (219).

Point C, common to both ⓢ, will generate the intersection of the spherical surfaces. (466.)

CM is always $\perp$ to XY (219).

∴ the curved line generated by C is in one plane (486).

∴ the intersection is a circle (179).

Also the plane of this ⊙ is $\perp$ to OO' (486).

And the center of the ⊙ is M, in OO' (179).

Q.E.D.

Ex. 1. What can be said about the point mentioned in 719?

Ex. 2. Explain, so that a blind boy could understand, how to circumscribe a sphere about a tetrahedron.

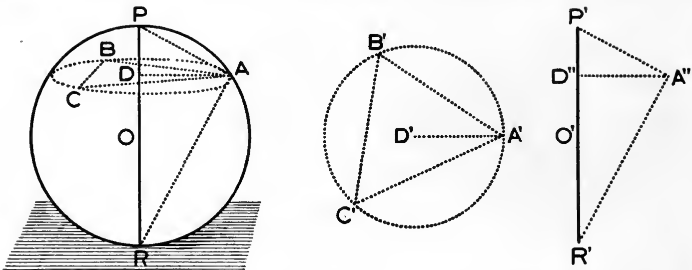
Ex. 3. Find a point equally distant from four points in space, not all in the same plane.

Ex. 4. Practical illustrations of the truth of 720 can be given by use of soap bubbles, or by two eggs.

CONSTRUCTION PROBLEMS

Proposition X. Problem

721. To find the radius of a material sphere.



Given: A material sphere. **Required**: To find its radius.

Construction: *First*, place one point of the compasses at P, and using any opening of the compasses, as AP, with the other point draw a circumference on the surface of the sphere.

Upon this circumference take three points, A and B and C, and by means of the compasses measure the straight lines AB, AC, BC.

Second, construct a $\triangle A'B'C'$, whose sides equal AB, AC, BC. Circumscribe a circle about this $\triangle$, and draw the radius $A'D'$.

Third, construct a right $\triangle P'A''D''$, whose hypotenuse equals the known line PA and whose leg equals the known radius $A'D'$. At A'' erect $A''R' \perp$ to $P'A''$ meeting $P'D''$, produced, at R'. Bisect $P'R'$ at O'.

Statement: $O'P'$ = the required radius. Q.E.F.

Proof: P is equally distant from A, B, and C (Const.).
D is equally distant from A, B, and C (179).
O is equally distant from A, B, and C (694).

$\therefore$ if the diameter $PDOR$ could be drawn it would be $\perp$ to the plane of ABC. (511.)

$\therefore \angle PDA$ is a rt. $\angle$ (473).

Now $\triangle ABC \cong \triangle A'B'C'$ (78).

$\therefore DA = D'A'$ (188).

Also $\triangle PDA \cong \triangle P'D''A''$ (84).

$\therefore \angle P = \angle P'$ (?).

Now $\angle PAR$ is a rt. $\angle$ (240).

Also $\triangle PRA \cong \triangle PR'A''$ (77).

$\therefore PR = P'R'$ (?).

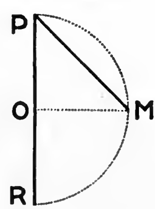
$\therefore OP = O'P'$ (Ax. 3).

That is, $OP =$ the radius of the sphere. Q.E.D.

NOTE. The most common construction on the surface of a sphere is the drawing of a great circle. For this construction, the compasses must open so that the distance between their points is equal to the chord of a quadrant of a great circle. To find this length, as in 722, we must find the radius of the solid sphere, as in 721.

PROPOSITION XI. PROBLEM

722. To find the chord of a quadrant of a material sphere.



Given: (?).

Required: (?).

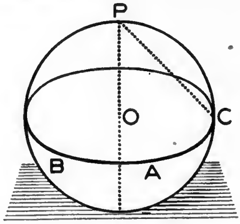
Construction: Find the radius of the sphere (by 721). Using this radius OP and any center O, describe a semicircle PMR. Erect radius $OM \perp$ to the diameter PR, and draw PM.

Statement: Arc PM is a quadrant of a great circle of the given sphere and chord PM is the required chord. Q.E.F.

Proof: Arc PM is a quadrant (?). Q.E.D.

Proposition XII. Problem

723. To describe a great circle through two given points on the surface of a sphere.



Given: The points *A* and *B* on the surface of the sphere *O*.

Required: To describe a great circle through *A* and *B*.

Construction: Find the chord of a quadrant of the given sphere (by 722).

Place one point of the compasses at *A*, and using the chord just found as an opening, describe an arc on the surface of the sphere.

Place one point of the compasses at *B*, and using the same opening, describe an arc, meeting the former arc at *P*.

Now place one point of the compasses at *P* and describe the circle *BAC*, using the same opening as before.

Statement: Circle *BAC* is the required great circle.

Proof: Points *A* and *B* are each at the distance of a quadrant from *P*. (Construction.)

∴ ⊙ *BAC* is a great circle whose pole is *P* (710). Q.E.D.

Ex. 1. A line tangent to a sphere lies in the plane tangent to the sphere at the same point.

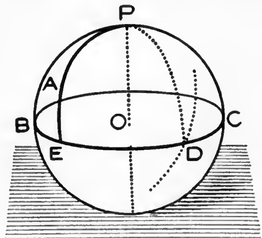
Proof: The line is ⊥ to the radius drawn to the point of contact (?). The plane also is ⊥ to the radius (?).

∴ the line is in the plane (486).

Ex. 2. At a point on the surface of a sphere there can be only one tangent plane.

PROPOSITION XIII. PROBLEM

724. To draw an arc of a great circle through a given point on the surface of a sphere and perpendicular to a given great circle.



Given: Point *A* on sphere *O*, and great circle *BC*, whose pole is *P*.

Required: To draw through *A* an arc of a great circle ⊥ to the great circle *BC*.

Construction: Place one point of the compasses at *A* and, with an opening equal to the chord of a quadrant of the given sphere, describe an arc of a great circle intersecting the given great circle at *D*.

Now place one point of the compasses at *D* and similarly draw arc of great circle *PAE*. Draw *PD*, the arc of a great circle (by 723).

Statement: Arc *PAE* is ⊥ to circle *BEDC*. Q.E.F.

Proof:	*ED*, *EP*, and *PD* are quadrants	(709).
	∴ *E* is the pole of arc *PD*	(710).
Hence	∠ *PED* is measured by quadrant *PD*	(711).
	∴ ∠ *PED* is a right angle	(232).
That is,	arc *PAE* is ⊥ to circle *BEDC*.	Q.E.D.

Ex. 1. Construct a plane tangent to a sphere at a given point on the surface.

Ex. 2. Construct a plane tangent to a sphere from a given point without the sphere. How many such planes are there?

SPHERICAL TRIANGLES

725. A **spherical triangle** is a portion of the surface of a sphere bounded by three arcs of great circles.

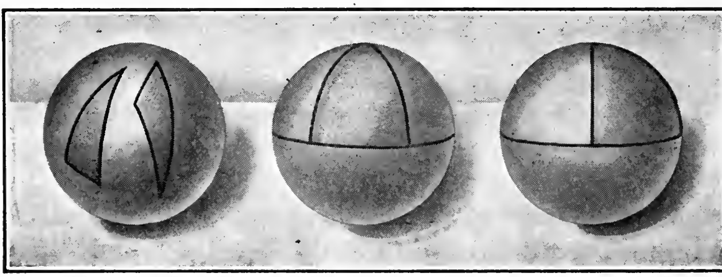
The bounding arcs are the **sides** of the triangle.

The intersections of the sides are the **vertices** of the triangle.

The angles formed by the sides are the **angles** of the triangle.

Spherical triangles are equilateral, equiangular, isosceles, scalene, acute, right, obtuse, under the same conditions as in plane triangles.

726. A **birectangular spherical triangle** is a spherical triangle two of whose angles are right angles.

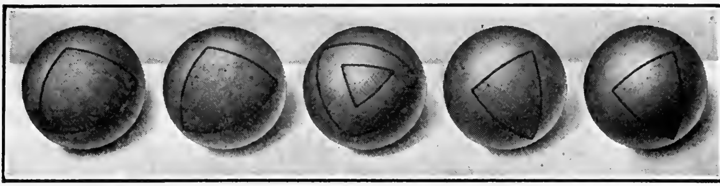


A **trirectangular spherical triangle** is a spherical triangle all of whose angles are right angles.

The **unit** usually employed in measuring the sides of a spherical triangle is the degree.

It is obvious that three great circles (not meeting at a point) divide the surface of a sphere into eight spherical triangles.

727. Two spherical triangles are **mutually equilateral** if the sides of the triangles are equal each to each; and they are **mutually equiangular** if their angles are equal each to each.



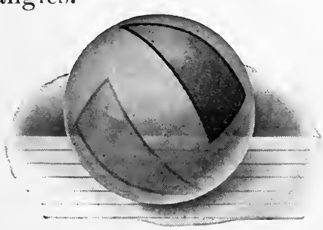
MUTUALLY EQUILATERAL SPHERICAL TRIANGLES — POLAR TRIANGLES — MUTUALLY EQUIANGULAR SPHERICAL TRIANGLES

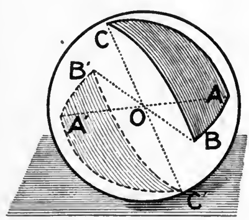
728. If three great circles are described, having as their poles the vertices of a spherical triangle, one of the eight triangles thus formed is the **polar triangle** of the first.

The **polar triangle** is the one whose vertices are nearest the vertices of the original triangle.

729. **Symmetrical spherical triangles** are triangles that have their parts equal but arranged in reverse order. They correspond to symmetrical trihedral angles.

Vertical spherical triangles correspond to vertical trihedral angles.





If the diameters of a sphere are drawn to the vertices of a spherical triangle, the original triangle and the triangle whose vertices are the opposite ends of these diameters are **vertical spherical triangles**.

730. A **spherical polygon** is a portion of the sphere bounded by three or more arcs of great circles.

Two spherical polygons are **congruent** if they can be made to coincide. The **diagonal** of a spherical polygon is the arc of a great circle connecting two vertices not in the same side.

Only *convex* spherical polygons are considered in this book.

PRELIMINARY THEOREMS

731. Theorem. The planes of the sides of a spherical triangle form a trihedral angle:

I. Whose vertex is the center of the sphere.

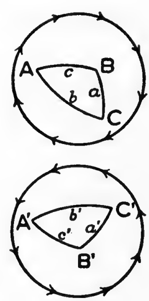
II. Each of whose face angles is measured by the intercepted side of the triangle. (232.)

III. Each of whose dihedral angles is equal to the corresponding angle of the triangle. (713.)

732. Theorem. Two symmetrical spherical triangles are mutually equilateral and mutually equiangular.

(51, 193, 522, 713.)

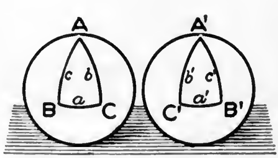
733. Theorem. The homologous parts of two symmetrical spherical triangles are arranged in reverse order.



Proof: If the eye is at the center of the sphere, the order of the vertices A, B, C is the same in direction as the motion of the hands of a clock. But the order of A', B', C' is in the opposite direction. (See 541, Note.) Hence the parts are arranged in reverse order. Q.E.D.

734. Theorem. The homologous parts of two symmetrical spherical triangles are equal. (732.)

735. Theorem. Two symmetrical isosceles spherical triangles are congruent.



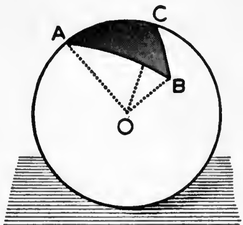
Proof: The method of superposition, as in the case of plane triangles.

Historical Note. It was not until the seventeenth century that polar triangles were invented by Gerard of Holland. It was he also who found the formulas for the area of a spherical triangle and of a spherical polygon.

THEOREMS AND DEMONSTRATIONS

PROPOSITION XIV. THEOREM

736. One side of a spherical triangle is less than the sum of the other two.



Given: (?). **To Prove**: $AB < AC + BC$.

Proof: Draw radii OA, OB, OC.

In the trihedral $\angle O$,

$\angle AOB < \angle AOC + \angle BOC$ (548).

$\angle AOB$ is measured by arc AB, etc. (?).

$\therefore$ arc $AB <$ arc $AC +$ arc BC (Ax. 6).

Q.E.D.

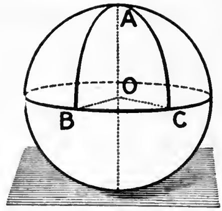
PROPOSITION XV. THEOREM

737. In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the third angle is measured by the third side.

Given: Birectangular $\triangle ABC$; $\angle B$ and $\angle C$, right $\angle$s.

To Prove: I. AB and AC quadrants.

II. $\angle A$ is measured by arc BC.



Proof: I. Draw radii *OA*, *OB*, *OC*.
Arc *AB* is ⊥ to arc *BC* and
arc *AC* is ⊥ to arc *BC* (Hyp.).
∴ *A* is the pole of arc *BC* (714).
∴ *AB* and *AC* are quadrants (709).
II. ∠ *A* is measured by arc *BC*. (711).
Q.E.D.

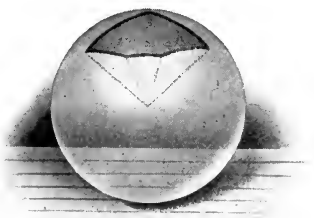
738. Corollary. The three sides of a trirectangular spherical triangle are quadrants.

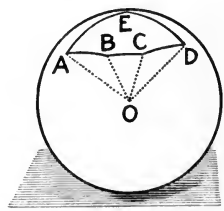
Ex. 1. If two sides of a spherical triangle are quadrants, the triangle is birectangular. (710, 712.)

Ex. 2. If all sides of a spherical triangle are quadrants, the triangle is trirectangular.

Proposition XVI. Theorem

739. The sum of the sides of any spherical polygon is less than 360°.





Given: (?). **To Prove**: (?).

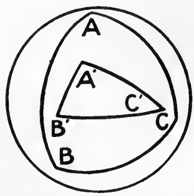
Proof: Draw radii to the several vertices of the polygon, forming the polyhedral ∠ *O*.

Then ∠ *AOB* + ∠ *BOC* + ∠ *COD* + ∠ *AOD* < 360° (549).
But ∠ *AOB* is measured by arc *AB*, etc. (?).
∴ arcs *AB* + *BC* + *CD* + *AD* < 360° (Ax. 6). Q.E.D.

740. Corollary. **The sum of the sides of any spherical polygon is less than the circumference of a great circle.**

Proposition XVII. Theorem

741. If one spherical triangle is the polar of a second triangle, then the second is the polar of the first.



Given: Spherical $\triangle ABC$ and its polar $\triangle A'B'C'$.

To Prove: $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.

Proof: A is the pole of arc $B'C'$ (Hyp.).

$\therefore$ B' is the distance of a quadrant from A (709).

C is the pole of arc $A'B'$ (?).

$\therefore$ B' is the distance of a quadrant from C (?).

Hence B' is the pole of arc AC (710).

Also A' is the pole of BC, and C' is the pole of AB.

$\therefore$ ABC is the polar $\triangle$ of $\triangle A'B'C'$ (728).

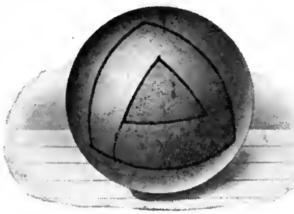
Q.E.D.

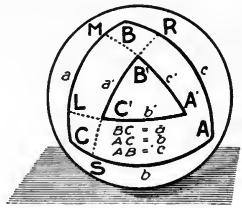
Note. Many properties of a trihedral angle are common to the corresponding spherical triangle. The polyhedral angle is similarly related to the spherical polygon. Sometimes it is advantageous to employ one, sometimes the other. The spherical triangle is perhaps simpler and more suggestive of properties than the corresponding trihedral angle, when the plane angles of its dihedral angles appear in the same diagram.

It is a most instructive and helpful exercise for the student to draw spherical triangles and their polars, etc., on a material sphere, such as a slate globe, a large apple, a ball, or other spherical object. A great many geometrical truths can be fixed in the mind by an orange and three long needles.

PROPOSITION XVIII. THEOREM

742. In two polar spherical triangles each angle of one and the opposite side of the other are supplementary.





Given: Polar $\triangle$ ABC and $A'B'C'$.

To Prove: $\angle A + a' = 180°$; $\angle A' + a = 180°$;
$\angle B + b' = 180°$; $\angle B' + b = 180°$;
$\angle C + c' = 180°$; $\angle C' + c = 180°$.

Proof: Prolong arc $B'C'$ to meet arc AB at R and arc AC at S.

$B'S = 90°$ and $C'R = 90°$ (709).

$\therefore B'S + C'R = 180°$ (Ax. 2).

That is, $C'S + B'C' + C'R$ or $RS + B'C' = 180°$ (Ax. 4).

Now RS is the measure of $\angle A$ (711).

Also $B'C' = a'$.

$\therefore \angle A + a' = 180°$ (Ax. 6).

Similarly, $\angle B + b' = 180°$; $\angle C + c' = 180°$.

Again, prolong arcs $A'B'$ and $A'C'$ to meet arc BC at M and L.

$BL = 90°$ and $CM = 90°$ (709).

$\therefore BL + CM = 180°$ (Ax. 2).

That is, $LM + MB + CM = 180°$, or $LM + BC = 180°$ (Ax. 4).

Now LM is the measure of $\angle A'$ (?).

Also $BC = a$.

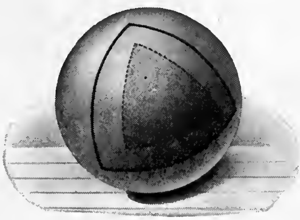
$\therefore \angle A' + a = 180°$ (Ax. 6).

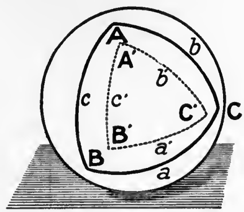
Similarly, $\angle B' + b = 180°$; $\angle C' + c = 180°$. Q.E.D.

743. Corollary. **In two polar spherical triangles each angle of one is measured by the supplement of the opposite side of the other.**

Proposition XIX. Theorem

744. **The sum of the angles of a spherical triangle is greater than 180° and less than 540°.**





Given: A spherical $\triangle ABC$.

To Prove: I. $\angle A + \angle B + \angle C > 180°$;

II. $\angle A + \angle B + \angle C < 540°$.

Proof: I. Construct $\triangle A'B'C'$, the polar $\triangle$ of $\triangle ABC$.

$\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$ (742).

Adding, $\angle A + \angle B + \angle C + a' + b' + c' = 540°$ (Ax. 2).

But $a' + b' + c' < 360°$ (739).

Subtracting, $\angle A + \angle B + \angle C > 180°$ (Ax. 9).

Q.E.D.

II. Again $\angle A + \angle B + \angle C + a' + b' + c' = 540°$ (Ax. 2).

But $a' + b' + c' > 0°$ (725).

Subtracting, $\angle A + \angle B + \angle C < 540°$ (Ax. 9).

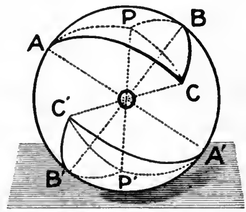
Q.E.D.

745. Corollary. **The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.**

746. Corollary. **A spherical triangle may have one, two, or three obtuse angles.**

PROPOSITION XX. THEOREM

747. Two symmetrical spherical triangles are equal.



Given: Two symmetrical spherical △ ABC and $A'B'C'$.

To Prove: $\triangle ABC = \triangle A'B'C'$.

Proof: Suppose P is the pole of the ⊙ containing A, B, C. Draw the diameters AA', BB', CC', PP', and the arcs of great ⓢ, PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

$$\angle POA = \angle P'OA' \qquad (?).$$

$$\therefore \text{arc } PA = \text{arc } P'A' \qquad (193).$$

Also arc $PB =$ arc $P'B'$ and arc $PC =$ arc $P'C'$.

But $$PA = PB = PC \qquad (708).$$

$$\therefore P'A' = P'B' = P'C' \qquad (\text{Ax. } 1).$$

Hence
$$\left.\begin{aligned} \triangle APB &\cong \triangle A'P'B' \\ \triangle ACP &\cong \triangle A'C'P' \\ \triangle BPC &\cong \triangle B'P'C' \end{aligned}\right\} \qquad (735).$$

Adding, $$\triangle ABC = \triangle A'B'C' \qquad (\text{Ax. } 2).$$

Q.E.D.

NOTE. If the pole P should be without the $\triangle ABC$, one of the pairs of equal △ would be without the original △ and would be subtracted from the sum of the others to obtain △ ABC and $A'B'C'$.

748. COROLLARY. Vertical spherical triangles are symmetrical and equal.

Ex. 1. Are symmetrical spherical triangles ever congruent?

Ex. 2. What unit is used in measuring the sides and angles of a spherical triangle?

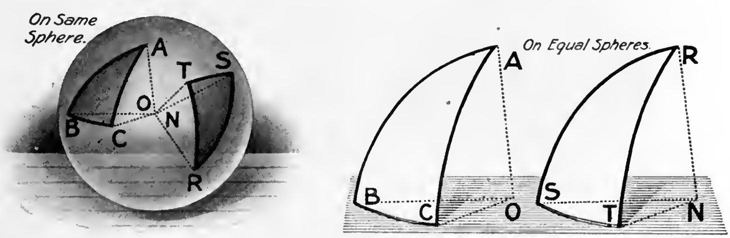
Proposition XXI. Theorem

749. Provided two spherical triangles on the same sphere (or on equal spheres) have their parts arranged in the same order, they are congruent:

I. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

II. If a side and the two angles adjoining it of one are equal respectively to a side and the two angles adjoining it in the other.

III. If three sides of the one are equal respectively to three sides of the other; that is, if they are mutually equilateral.



Given: (?).

To Prove: $\triangle ABC \cong \triangle RST$.

Proof: I and II. Superposition as in plane ⩓.

III. Draw radii of the sphere to all the vertices of the ⩓.

The face ∠s of the trih. $\angle O =$ the face ∠s of the trih. $\angle N$, respectively. (?.)

Hence trih. $\angle O =$ trih. $\angle N$ (546).

∴ dih. $\angle OA =$ dih. $\angle NR$; dih. $\angle OB =$ dih. $\angle NS$; etc.

∴ the ⩓ are mutually equiangular (731, III).

Hence the ⩓ can be made to coincide.

∴ $\triangle ABC \cong \triangle RST$ (26).

Q.E.D.

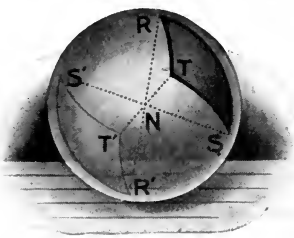
Proposition XXII. Theorem

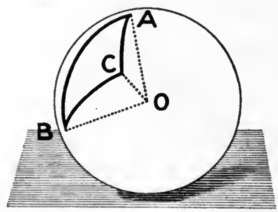
750. Provided two spherical triangles on the same sphere (or on equal spheres) have their parts arranged in reverse order, they are symmetrical:

I. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

II. If a side and the two angles adjoining it of one are equal respectively to a side and the two angles adjoining it of the other.

III. If three sides of one are equal respectively to three sides of the other; that is, if they are mutually equilateral.





Given: (?). **To Prove:** (?).

Proof: In each of these cases construct a third spherical △ $R'S'T'$, symmetrical to the △ RST.

Then △ $R'S'T'$ will have its parts equal to the parts of △ ABC and arranged in the same order.

∴ △ $R'S'T' \cong$ △ ABC (749).

Hence △ RST is symmetrical to △ ABC (Ax. 6).

Q.E.D.

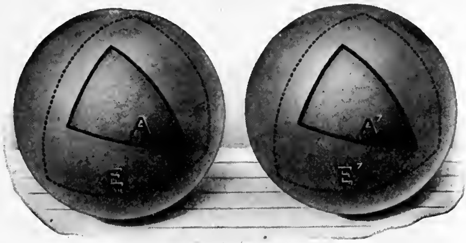
751. Corollary. Two mutually equilateral spherical triangles are mutually equiangular and are congruent or symmetrical.

When are they *congruent?* When are they *symmetrical?*

Ex. Is the corresponding theorem about plane triangles true?

PROPOSITION XXIII. THEOREM

752. Two mutually equiangular spherical triangles on the same sphere (or on equal spheres) are mutually equilateral, and are congruent or symmetrical.



Given: △ A and A', mutually equiangular.

To Prove: △ A and A' mutually equilateral, and congruent or symmetrical.

Proof: Construct △ E and E', the polar △ of A and A'.

The sides of E are supplements of the ∠s of A.
The sides of E' are supplements of the ∠s of A'. } (742).

But the ∠s of A are = respectively to the ∠s of A' (Hyp.).

∴ △ E and E' are mutually equilateral (49).

Hence △ E and E' are mutually equiangular (751).

Again △ A and A' are the polar △ of E and E' (741).

∴ the sides of A are supplements of the ∠s of E
the sides of A' are supplements of the ∠s of E' } (?).

Hence △ A and A' are mutually equilateral (?).

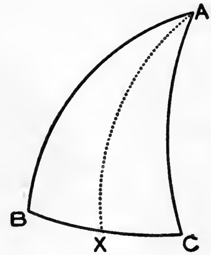
∴ they are congruent (when ?);
or symmetrical (when ?).

Q.E.D.

Ex. Is the corresponding theorem about plane triangles true? Is there any theorem concerning the congruence of plane triangles that is not true of spherical triangles? If the parts of two plane triangles were arranged in reverse order and they were kept in a plane, could they be made to coincide?

Proposition XXIV. Theorem

753. The angles opposite the equal sides of an isosceles spherical triangle are equal.



Given: (?).

To Prove: $\angle B = \angle C$.

Proof: Suppose X the midpoint of BC.

Draw AX, the arc of a great ⊙. Now the two spherical △ ABX and AXC are mutually equilateral. (Explain.)

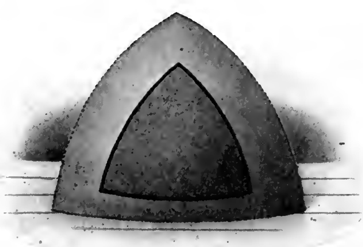
∴ they are mutually equiangular and symmetrical (751).

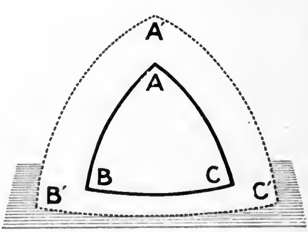
$\therefore \angle B = \angle C$ (734). Q.E.D.

754. Corollary. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the midpoint of the base bisects the vertex angle and is perpendicular to the base.

Proposition XXV. Theorem

755. If two angles of a spherical triangle are equal, the sides opposite are equal.





Given: (?). **To Prove:** (?).

Proof: Construct $\triangle A'B'C'$, the polar $\triangle$ of $\triangle ABC$.

Then $A'B'$ is the supplement of $\angle C$.
And $A'C'$ is the supplement of $\angle B$. (?.)

$\therefore A'B' = A'C'$ (49).

$\therefore \angle B' = \angle C'$ (753).

Again $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$ (741).

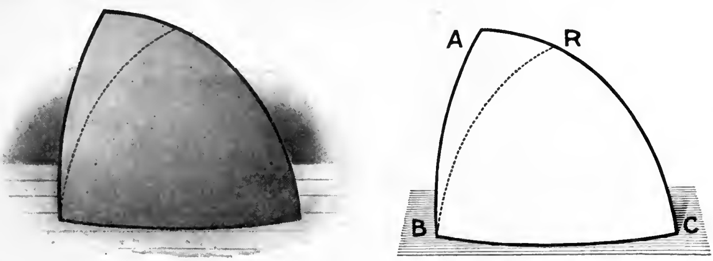
AB is the supplement of $\angle C'$, and AC of $\angle B'$ (?).

$\therefore AB = AC$ (49).

Q.E.D.

Proposition XXVI. Theorem

756. If two angles of a spherical triangle are unequal, the sides opposite are unequal and the greater side is opposite the greater angle.



Given: $\triangle ABC$; $\angle ABC > \angle C$.

To Prove: $AC > AB$.

Proof: Suppose BR drawn, the arc of a great $\odot$, making $\angle CBR = \angle C$ and meeting AC at R.

Now $AR + BR > AB$ (736).

But $BR = CR$ (755).

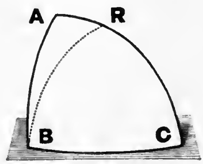
$\therefore AR + CR > AB$ (Ax. 6).

That is, $AC > AB$ Q.E.D.

Ex. Compare Propositions XXIV, XXV, XXVI with the corresponding theorems about plane triangles.

PROPOSITION XXVII. THEOREM

757. If two sides of a spherical triangle are unequal, the angles opposite are unequal and the greater angle is opposite the greater side. [Converse.]



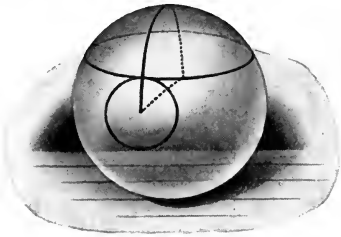
Given: (?). **To Prove:** $\angle ABC > \angle C$.

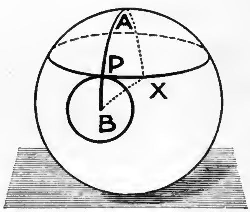
Proof: $\angle ABC$ is either $< \angle C$ or $= \angle C$ or $> \angle C$.

Continue by method of exclusion (90).

PROPOSITION XXVIII. THEOREM

758. If two circles on a sphere contain a point on the arc of a great circle that joins their poles, they have no other point in common.





Given: Point P on the arc AB of a great ⊙ of a sphere, and P common to two circles whose poles are A and B.

To Prove: P is the only point common to these Ⓢ.

Proof: Suppose X is another common point.

Draw arcs of great Ⓢ AX and BX.

Then $AX + BX > AP + BP$ (736).

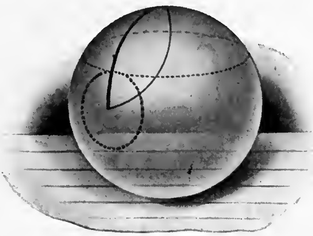
But $AX = AP$ (708).

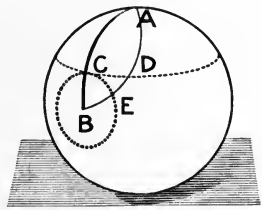
Subtracting, $BX > BP$ (Ax. 7).

That is, X is without the ⊙ B and cannot be in both the circles. Q.E.D.

PROPOSITION XXIX. THEOREM

759. The shortest line that can be drawn on the surface of a sphere, between two points on the surface, is the less arc of the great circle containing the two points.





Given: Points *A* and *B*, and *AB* the arc of a great ⊙ joining them; line *ADEB*, any other line on the surface of the sphere, between *A* and *B*.

To Prove: Arc *AB* < line *ADEB*.

Proof: Take on arc *AB* any point *C*, and describe two circles through *C*, having *A* and *B* as their poles, and intersecting *ADEB* at *D* and *E*. Point *C* is the only point common to these two ⓢ. (758.)

No matter what kind of line *AD* is, a line of equal length can be drawn from *A* to *C*, on the surface; and a line can be drawn from *B* to *C* equal in length to *BE*.

[Imagine *AD* revolved on the surface of the sphere, using *A* as a pivot, and *D* will move along the ⊙ to point *C*. Similarly with *BE*.]

There is now a line from *A* to *B*, through *C*, < *ADEB*.

That is, whatever the nature of *ADEB*, there is a shorter line from *A* to *B*, which contains *C*, *any* point of arc *AB*.

Thus the shortest line contains all the points of *AB* and therefore *is the line AB*. Q.E.D.

NOTE. This theorem justifies the definition of the "distance" between two points, etc., in 692.

ORIGINAL EXERCISES

1. Vertical spherical angles are equal.

2. If two spherical triangles, on the same or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.

3. The polar triangle of an isosceles spherical triangle is isosceles.

4. The polar triangle of a birectangular spherical triangle is birectangular.

5. If two dihedral angles of a trihedral angle are equal, the opposite face angles also are equal.

Proof: Construct a sphere having the vertex as center, etc.

6. If two face angles of a trihedral angle are equal, the opposite dihedral angles also are equal.

7. A trirectangular spherical triangle is its own polar triangle.

8. Two symmetrical spherical polygons are equal.

9. Any side of a spherical polygon is less than the sum of the other sides. [Draw diagonals from a vertex.]

10. If the three face angles of a trihedral angle are equal, the three dihedral angles also are equal.

11. State and prove the converse of No. 10.

12. A straight line cannot meet a spherical surface in more than two points.

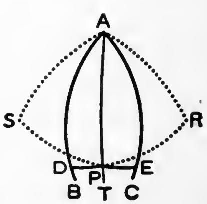
13. If two dihedral angles of a trihedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater dihedral angle.

14. State and prove the converse of No. 13.

15. All the tangent lines drawn to a sphere from an external point are equal.

16. The volume of any tetrahedron is equal to one third the product of its total surface by the radius of the inscribed sphere.

17. Every point of a great circle that is perpendicular to an arc at its midpoint is equally distant from the ends of the arc.



18. The points of contact of all lines tangent to a sphere from an external point lie in the circumference of a circle.

19. Any point in the arc of a great circle that bisects a spherical angle is equally distant from the sides of the angle.

20. If the opposite sides of a spherical quadrilateral are equal, the opposite angles are equal.

21. If the opposite sides of a spherical quadrilateral are equal, the diagonals bisect each other.

22. If the diagonals of a spherical quadrilateral bisect each other, the opposite sides are equal.

23. The exterior angle of a spherical triangle is less than the sum of the opposite interior angles.

24. The sum of the angles of a spherical quadrilateral is more than four right angles.

25. If two spheres are tangent to each other, the straight line joining their centers passes through the point of contact.

26. The sum of the angles of a spherical polygon is more than $2n - 4$ right angles and less than $2n$ right angles.

27. The arcs of great circles bisecting the angles of a spherical triangle meet in a point.

28. If a tangent line and a secant are drawn to a sphere from an external point, the tangent is a mean proportional between the whole secant and the external segment.

29. The product of any secant that can be drawn to a sphere from an external point, by its external segment, is constant for all secants drawn through the same point.

30. If two spherical surfaces intersect and a plane is passed containing their intersection, tangents from any point in this plane to the two spherical surfaces are equal.

31. Find the distance from the center of a sphere whose radius is 15 in. to the plane of a small circle whose radius is 8 in.

32. The polar distance of a small circle is 60° and the radius of the sphere is 12 in. Find the radius of the circle.

33. The total surface of a tetrahedron is 90 sq. m., and the radius of the inscribed sphere is 4 m. Find the volume of the tetrahedron.

34. Find the radius of the sphere inscribed in a tetrahedron whose volume is 250 cu. in. and total surface is 150 sq. in.

35. Find the total surface of a tetrahedron whose volume is 320 cu. in., if the radius of the inscribed sphere is 8 in.

36. Find the radius of the sphere inscribed in a regular tetrahedron whose edges are each 10 in.

37. Find the radius of the sphere circumscribed about a regular tetrahedron whose edges are each 18 in.

38. Find the radii of the spheres inscribed in and circumscribed about a cube whose edges are each 10 in.

39. The sides of a spherical triangle are 60°, 80°, 110°. Find the angles of its polar triangle.

40. The angles of a spherical triangle are 74°, 119°, 87°. Find the sides of its polar triangle.

41. The chord of the polar distance of the circle of a sphere is 12 m., and the radius of the sphere is 9 m. Find the radius of the circle.

42. The polar distance of a circle is 60° and the diameter of the circle is 8 ft. Find the diameter of the sphere.

[Denote by R, each side of an equilateral triangle whose altitude is 4 ft.]

43. The radii of two spherical surfaces are 11 in. and 13 in., and their centers are 20 in. apart. Find the radius of the circle of their intersection. Find also the distances from the centers of the spheres to the center of this circle.

44. The radii of two spherical surfaces are 20 m. and 37 m., and the distance between their centers is 19 m. What is the length of the diameter of their intersection?

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45. Bisect an arc of a great circle.

46. Draw an arc of a great circle perpendicular to a given arc of a great circle through a given point in the arc.

47. Bisect a spherical angle.

48. Bisect an arc of a small circle.

49. Circumscribe a circle about a given spherical triangle.

50. Construct a spherical angle equal to a given spherical angle at a given point on the same sphere.

51. Construct a spherical triangle having the three sides given.

52. Construct a spherical triangle having the three angles given.

53. Construct a plane tangent to a sphere at a given point on the surface.

54. Construct a spherical surface having the radius given and containing three given points.

55. Construct a spherical surface that shall have a given radius, touch a given plane, and contain two given points.

56. Construct a spherical surface that shall have a given radius, shall be tangent to a given sphere, and contain two given points.

57. Construct a spherical surface that shall contain four given points.

58. Construct a plane that shall contain a given line and be tangent to a given sphere.

59. Construct a plane tangent to a given sphere and parallel to a given plane.

60. What is the locus of points on the surface of a sphere :

(*a*) Equally distant from two given points on the surface?

(*b*) Equally distant from two given points not on the surface?

61. What is the locus of the centers of those spherical surfaces that pass through two given points?

62. What is the locus of the centers of the spherical surfaces of given radius that contain two given points?

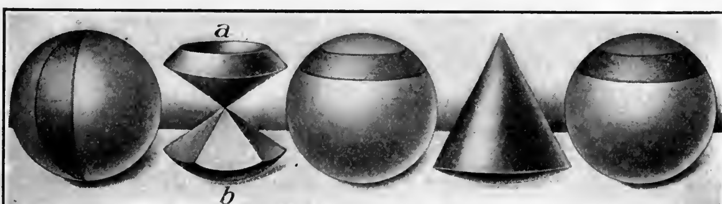
63. What is the locus of the centers of the spherical surfaces that pass through three given points?

SPHERICAL AREAS AND VOLUMES

760. A **lune** is a portion of the surface of a sphere bounded by two great semicircles.

The points of intersection of the sides of a lune are the **vertices** of the lune.

The angles made at the vertices by the sides are the **angles** of the lune.



LUNE (*a*) SPHERICAL SECTOR (*b*) SPHERICAL PYRAMID ZONE SPHERICAL CONE SPHERICAL SEGMENT

761. A **zone** is a portion of the surface of a sphere bounded by two circles whose planes are parallel.

The **bases** of a zone are the circles bounding it.

The **altitude** of a zone is the perpendicular distance between the planes of its bases.

If one of the planes is tangent to the sphere, the zone is a **zone of one base.**

762. A **spherical degree** is $\frac{1}{720}$ of the surface of a sphere. If the surface of a sphere is divided into 720 equal parts, each part is a spherical degree.

The size of a spherical degree depends on the size of the sphere.

It may be easily conceived to be half a lune whose angle is 1 degree, that is, a birectangular spherical triangle whose third angle is 1°.

How many spherical degrees are there in a trirectangular spherical triangle?

763. The **spherical excess** of a spherical triangle is the sum of its angles less 180°. That is, $E = A + B + C - 180°$.

764. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of its sides.

The **vertex** of a spherical pyramid is the center of the sphere.

The **base** of a spherical pyramid is the spherical polygon.

765. A **spherical sector** is the solid generated by the revolution of the sector of a circle about any diameter of the circle as an axis.

The **base** of the spherical sector is the zone generated by the arc of the circular sector.

A **spherical cone** is a spherical sector whose base is a zone of one base.

766. A **spherical segment** is a portion of a sphere included between two parallel planes that intersect the sphere.

The **bases** of a spherical segment are the circular sections made by the parallel planes.

The **altitude** of a spherical segment is the perpendicular distance between the bases.

A **spherical segment of one base** is a segment one of whose bounding planes is tangent to the sphere.

A **hemisphere** is a spherical segment of one base, which base is a great circle.

A **spherical wedge** is a portion of a sphere bounded by a lune and the planes of its sides.

Ex. 1. What is the spherical excess of a spherical triangle whose angles are 60°, 70°, and 100°?

Ex. 2. Distinguish between a zone and a spherical segment.

Ex. 3. Find the area of a spherical degree on a sphere whose surface is 3600 sq.in.

Ex. 4. Find the area of a spherical triangle containing 80 spherical degrees, on a sphere whose surface is 450 sq. ft.

Ex. 5. Find the area of a spherical polygon containing 152 spherical degrees on a sphere whose surface is 630 sq. yd.

Ex. 6. A spherical triangle containing 128 spherical degrees has an area of 72 sq. in. What is the area of the spherical surface?

PRELIMINARY THEOREMS

767. Theorem. Either angle of a lune is measured by the arc of a great circle described with the vertex of the lune as a pole, and included between the sides of the lune. (711.)

768. Theorem. The angles of a lune are equal.

769. Theorem. Every great circle of a sphere divides the sphere into two equal hemispheres, and the surface into two equal zones.

770. Theorem. The spherical excess of a spherical *n*-gon is equal to the sum of its angles less $(n - 2)$ 180°.

Proof: (?).

771. Theorem. If a regular polygon having an even number of sides is inscribed in, or circumscribed about, a circle, and the figure is made to revolve about one of the longest diagonals of the polygon, the surface generated by the perimeter of the polygon approaches the surface of the sphere generated by the circle, as a limit, if the number of sides of the polygon is indefinitely increased.

772. Theorem. If a polyhedron is circumscribed about a sphere and the number of its faces is indefinitely increased, the surface of the polyhedron approaches the surface of the sphere as a limit, and the volume of the polyhedron approaches the volume of the sphere as a limit.

Note. If a regular polygon having an even number of sides is inscribed in, or circumscribed about, a circle, and the figure is made to revolve about one of the longest diagonals of the polygon, the surface generated by the polygon is composed of the surfaces of cones, a cylinder, and frustums, and the surface generated by the circle is a spherical surface.

Ex. 1. Find the spherical excess of a polygon whose angles are 80°, 110°, 140°, 130°, 160°.

Ex. 2. The spherical excess of a spherical polygon is the difference between the sum of its angles and the sum of the angles of a plane polygon having the same number of sides.

Ex. 3. The sum of the angles of a spherical quadrilateral is less than eight right angles.

Ex. 4. Find the spherical excess of a spherical hexagon if each of its angles equals 128°. If each angle equals 155°, find the excess.

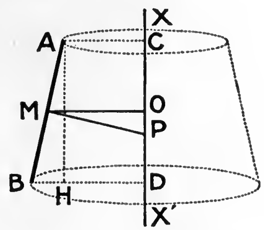
Ex. 5. If the opposite angles of a spherical quadrilateral are equal, the opposite sides are also equal.

Proof: Prolong one pair of opposite sides in both directions until they meet. Now prove two triangles congruent.

THEOREMS AND DEMONSTRATIONS

Proposition XXX. Theorem

773. The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line upon the axis by the circumference of a circle whose radius is the line perpendicular to the revolving line at its midpoint, and terminating in the axis.



Given: Line AB revolving about axis XX'; CD = projection of AB on XX'; $MP = a = \perp$ erected at midpoint of AB and terminating in XX'; MO = radius of mid-section.

To Prove: Surface generated by $AB = CD \cdot 2\pi a$.

Proof: I. The surface generated by AB is the surface of the frustum of a right circular cone whose bases are generated by AC and BD, and the mid-section, by MO.

Area of surface $= 2\pi MO \cdot AB$ (681).

Now $\triangle ABH$ and MPO are similar (310).

$\therefore MO : AH = MP : AB$ (?).

Hence $MO \cdot AB = AH \cdot MP = CD \cdot a$ (?).

$\therefore$ area of surface $= 2\pi CD \cdot a = CD \cdot 2\pi a$ (Ax. 6).

II. If AB is $\parallel$ to XX', the surface is cylindrical and equals $CD \cdot 2\pi a$ (654).

III. If AB meets XX' at C, the entire surface is conical and equals $\pi BD \cdot AB$ (680).

Now $BD = 2\,MO$ (136).

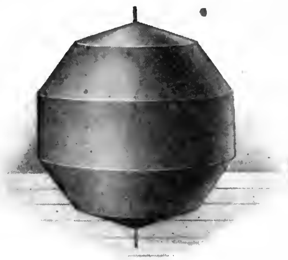
And $MO \cdot AB = CD \cdot a.$ (?.)

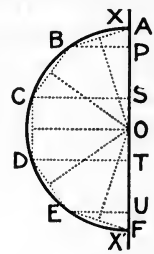
$\therefore \pi BD \cdot AB = \pi \cdot 2\,MO \cdot AB = \pi \cdot 2 \cdot CD \cdot a = CD \cdot 2\pi a$ (Ax. 6).

$\therefore$ the area of the surface $= CD \cdot 2\pi a$ (Ax. 6). Q.E.D.

PROPOSITION XXXI. THEOREM

774. The surface of a sphere is equal in area to four great circles; that is, to $4\pi R^2$.





Given: Semicircle ACF; diameter AF; S = surface of sphere generated by revolving the semicircle about AF as an axis; R = radius of this sphere.

To Prove: $S = 4\pi R^2$.

Proof: Inscribe in this semicircle half of a regular polygon having an even number of sides. Draw the apothems, a. Draw the projections of the sides of the polygon on the diameter. Now, if the figure revolves on AF as an axis,

$$\left.\begin{aligned}&\text{the surface } AB = AP \cdot 2\pi a\\ &\text{the surface } BC = PS \cdot 2\pi a\\ &\text{the surface } CD = ST \cdot 2\pi a\end{aligned}\right\} \quad (773).$$

etc. Adding,

the entire surface $= (AP + PS + ST + \text{etc.}) \cdot 2\pi a$ (Ax. 2).

$= AF \cdot 2\pi a$ (Ax. 6).

Now, if the number of sides of the polygon is indefinitely increased, the entire surface generated by the polygon approaches S as a limit. (771.)

a approaches R as a limit (422).

Also $AF \cdot 2\pi a$ approaches $AF \cdot 2\pi R$.

$\therefore S = AF \cdot 2\pi R$ (229).

But $AF = 2R$ (?).

$\therefore S = 4\pi R^2$ (Ax. 6). Q.E.D.

775. Corollary. **The area of a spherical degree equals $\frac{4\pi R^2}{720} = \frac{\pi R^2}{180}$ sq. units.**

776. Corollary. **The areas of the surfaces of two spheres are to each other as the squares of their radii and as the squares of their diameters.**

Proof: $\frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2}D)^2}{(\frac{1}{2}D')^2} = \frac{D^2}{D'^2}$.

Proposition XXXII. Theorem

777. The area of a zone is equal to the product of its altitude by the circumference of a great circle.

Given: (The same as in 774).

To Prove: The area of the zone generated by the arc $BC = PS \times 2\pi R$.

Proof: The area generated by chord $BC = PS \cdot 2\pi a$ (773).

If the number of sides of the inscribed polygon is indefinitely increased, the length of chord BC approaches arc BC and the surface generated by chord BC approaches the area of a zone.

Also $PS \cdot 2\pi a$ will approach $PS \cdot 2\pi R$.

Hence **Area of zone $BC = PS \cdot 2\pi R$** (229). Q.E.D.

778. Corollary. **Area of a zone,**

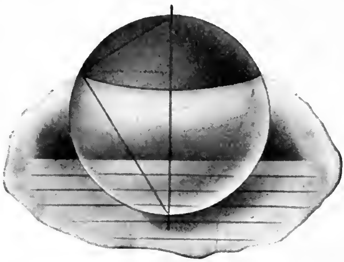
$$Z = 2\pi RH.$$

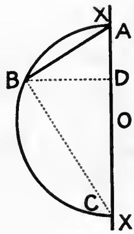
(Where Z = area of the zone, H = its altitude, and R = radius of sphere.)

Ex. 1. On a sphere whose radius is 6 in., find the area of a zone $2\frac{1}{2}$ in. in height.

Ex. 2. What does the formula for the area of a zone become when the altitude is the diameter? when the altitude is half the radius?

779. Corollary. The area of a zone of one base is equal to the area of a circle whose radius is the chord of the generating arc.





Given: Arc AB of semicircle ABC; diameter AC; chord AB.

To Prove: Area of zone generated by arc $AB = \pi\overline{AB}^2$.

Proof: Area of zone $AB = AD \cdot 2\pi R$ (777).

That is, area of zone $AB = \pi \cdot AD \cdot 2R$.

Draw chord BC. $\triangle ABC$ is a rt. $\triangle$ (?).

$\therefore AD \cdot AC = \overline{AB}^2$ (333).

That is, $AD \cdot 2R = \overline{AB}^2$ (Ax. 6).

Hence area of zone $AB = \pi\overline{AB}^2$ (Ax. 6).

That is, **area of zone of one base = π (chord)^2.** Q.E.D.

Ex. 1. What is the area of a zone of one base whose chord is 7 in. in length? of one whose chord is 14 in. in length?

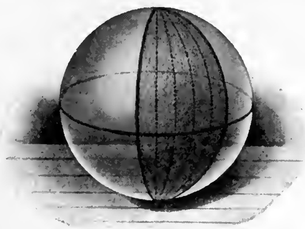
Ex. 2. What does the formula for the area of a zone of one base become when the generating arc is a semicircle? when the generating arc is a quadrant?

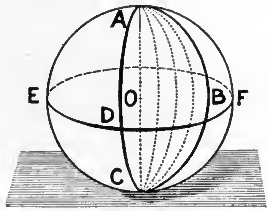
Ex. 3. If the radius of the earth is approximately 4000 mi. and the altitude of the north temperate zone is 2080 mi., what is the area of the north temperate zone?

Ex. 4. Prove that on the same or equal spheres, zones having equal altitudes have equal areas.

Proposition XXXIII. Theorem

780. The area of a lune is to the area of the surface of its sphere as the angle of the lune is to 360°.





Given: Lune $ABCDA$ on sphere O; L = area of lune; S = area of sphere; great ⊙ EB whose pole is A.

To Prove: $L : S = \angle A : 360$.

Proof: I. If arc BD and the circumference of ⊙ EB are **commensurable.** There exists a common unit of measure. Suppose this unit contained 5 times in BD; 32 times in the circumference. $\therefore$ arc BD : circumference $= 5 : 32$ (?).

Arc BD measures $\angle A$ (711).

$\therefore \angle A : 360 = 5 : 32$ (Ax. 6).

Pass great ⓢ through the several points of division of circumference EB and vertex A, dividing the surface of the sphere into 32 equal lunes. Then $L : S = 5 : 32$ (Ax. 3).

Hence $L : S = \angle A : 360$ (Ax. 1). Q.E.D.

II. If the arc and circumference are **incommensurable.**

The proof is similar to that found in 293, 524.

781. Corollary. The number of spherical degrees in the area of a lune is double the number of degrees in its angle.

Proof: Let $L°$ denote the area of the lune, expressed in spherical degrees.

Then $L° : 720° = \angle A : 360°$ (Subst. in 780).

$\therefore L° = 2 \angle A$. Q.E.D.

782. Corollary. **The area of a lune expressed in square units is $\frac{\angle A \cdot \pi R^2}{90}$.**

Proof: Substituting in 780, $L : 4\pi R^2 = \angle A : 360$.

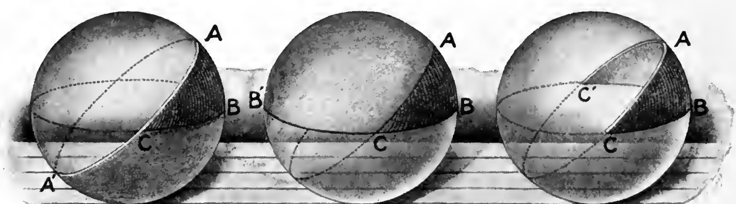
$$\therefore L = \frac{\angle A \cdot \pi R^2}{90}.$$ Q.E.D.

783. Corollary. **Two lunes on the same or equal spheres are to each other as their angles.**

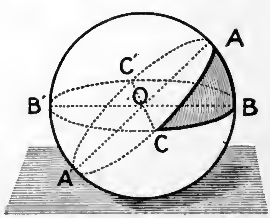
Proof: $L : S = \angle A : 360$, and $L' : S = \angle A' : 360$ (780).
Dividing, $L : L' = \angle A : \angle A'$ (Ax. 3). Q.E.D.

Proposition XXXIV. Theorem

784. The number of spherical degrees in a spherical triangle is equal to the spherical excess of the triangle.



Lune, ABA′CA Lune, BAB′CB Lune, CAC′BC



Given: Spherical $\triangle ABC$ on sphere O; spherical excess of the $\triangle = E$.

To Prove: Number of spherical degrees in $\triangle ABC = E$.

Proof: Continue the sides of the $\triangle ABC$ to form the lunes $ABA'CA$, $BAB'CB$, $CAC'BC$; draw diameters AA', BB', CC'.

$$\triangle ABC' = \triangle A'B'C$$ (747).

Lune $CAC'BC = \triangle ABC + \triangle AC'B = \triangle ABC + \triangle A'B'C$ (Ax. 6).

Now $\triangle ABC + \triangle A'B'C = \text{lune } CAC'BC.$
And $\triangle ABC + \triangle A'BC = \text{lune } ABA'CA.$ (Ax. 4.)
And $\triangle ABC + \triangle AB'C = \text{lune } BAB'CB.$ Adding,

$$2 \triangle ABC + \triangle ABC + \triangle A'B'C + \triangle A'BC + \triangle AB'C$$
$$= \text{lune } A + \text{lune } B + \text{lune } C \quad \text{(Ax. 2)}.$$

Now, *first*, 4 of these $\triangle$s compose a hemisphere and = 360 spherical degrees;

second, the 3 lunes $= 2\angle A + 2\angle B + 2\angle C$ (781).

By substituting in the long equation above,

$$2 \triangle ABC + 360 = 2\angle A + 2\angle B + 2\angle C \quad \text{(Ax. 6)}.$$

$$\therefore \triangle ABC = \angle A + \angle B + \angle C - 180 \quad \text{(Ax. 3)}.$$

That is, $\triangle ABC = E$ (763). Q.E.D.

785. Corollary. The area of a spherical triangle expressed in square units is $\frac{E \cdot \pi R^2}{180}$.

Proof: 1 spherical degree $= \frac{\pi R^2}{180}$ sq. units (775).

$\therefore$ **Area of a spherical $\triangle = \frac{E \cdot \pi R^2}{180}$ sq. units** (Ax. 3).

Q.E.D.

Ex. 1. How many spherical degrees are there in a lune whose angle equals 25°?

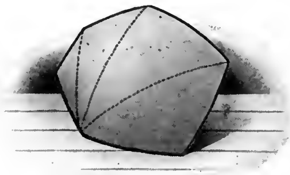
Ex. 2. On a sphere whose radius is 10 in., is a lune whose angle is 10°. Find the area of the lune in square inches.

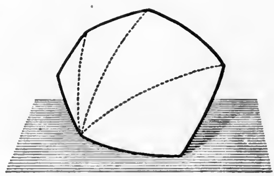
Ex. 3. Reduce the formula for the area of a lune in square units, if the angle is 90°; if the angle is 180°.

Ex. 4. On a sphere whose radius is 9 ft. is a spherical triangle whose angles are 70°, 145°, and 60°. Find the area of the triangle.

PROPOSITION XXXV. THEOREM

786. The number of spherical degrees in a spherical polygon is equal to its spherical excess.





Given: A spherical n-gon.

To Prove: The number of spherical degrees in this n-gon = the excess of the polygon.

Proof: From any vertex draw diagonals, dividing the polygon into $(n-2)$ △; let the sums of the ∠ of these △ be denoted by s, s_1, s_2, ⋯ etc.

Now the number of spherical degrees in one △ = $s - 180°$ (784).

Number of spherical degrees in another △ = $s_1 - 180°$ (?).

Etc., for $(n-2)$ △.

Adding, the number of spherical degrees in the n-gon = the sum of its ∠ − $(n-2)$ 180° (Ax. 2).

The excess of n-gon = sum of its ∠ − $(n-2)$ 180° (770).

∴ **the number of spherical degrees in a spherical polygon = the excess of the polygon** (Ax. 1). Q.E.D.

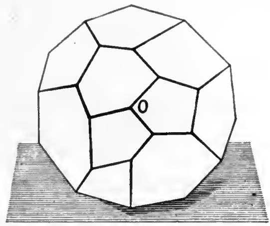
Ex. 1. Find the area of a spherical triangle whose angles are 80°, 125°, and 95°, on a sphere whose radius is 6.3 in.

Ex. 2. Find the area of a spherical polygon whose angles are 135°, 105°, 85°, 155°, 120°, on a sphere whose radius is 15 ft.

Ex. 3. Find the area of a spherical triangle whose angles are 72°, 97°, and 101°, on a sphere whose radius is $3\frac{1}{2}$ in.

PROPOSITION XXXVI. THEOREM

787. The volume of a sphere $= \frac{4\pi R^3}{3}$.



Given: Sphere O; radius $= R$; surface $= S$; volume $= V$.

To Prove: $V = \frac{4\pi R^3}{3}$.

Proof: Suppose a polyhedron circumscribed about the sphere, its surface denoted by S', and its volume by V'.

Suppose planes are passed through the edges of the polyhedron and the center of the sphere, thus dividing the polyhedron into pyramids whose vertices are all at the center, and whose common altitude is R.

The volume of one such pyramid $= \frac{1}{3} R \cdot$ its base (612).

$\therefore$ volume of *all* pyramids $= \frac{1}{3} R \cdot$ the sum of all the bases (Ax. 2).

That is, $V' = \frac{1}{3} R \cdot S'$.

Indefinitely increase the number of faces of the polyhedron, thus indefinitely decreasing each face,

and V' approaches V as a limit
and S' approaches S as a limit } (772).

Hence $\frac{1}{3} R \cdot S'$ approaches $\frac{1}{3} R \cdot S$ as a limit.

$\therefore V = \frac{1}{3} R \cdot S$ (229).

But $S = 4\pi R^2$ (?).

$\therefore V = \frac{4\pi R^3}{3}$. (Ax. 6). Q.E.D.

788. Corollary. **The volumes of two spheres are to each other as the cubes of their radii or as the cubes of their diameters.**

Proof:

$$\frac{V}{V'}=\frac{4\,\pi R^3}{3}\div\frac{4\,\pi R'^3}{3}=\frac{R^3}{R'^3}=\frac{(\frac{1}{2}\,D)^3}{(\frac{1}{2}\,D')^3}=\frac{D^3}{D'^3} \quad \text{(Ax. 6).}$$

Q.E.D.

789. Corollary. **The volume of a spherical pyramid is equal to one third the product of the polygon that is its base, by the radius of the sphere.**

$$V=\frac{1}{3}(\text{area of base})R.$$

Proof: Similar to the proof of 787.

790. Corollary. **The volume of a spherical wedge is to the volume of the sphere as the angle of its base is to 360°.**

Proof: Similar to the proof of 780.

791. Corollary. **Volume of a spherical wedge,**

$$V=\frac{\angle A\cdot\pi R^3}{270}.$$

(Where A = the $\angle$ of the lune, and R = the radius of sphere.)

Proof: $V:\frac{4}{3}\,\pi R^3=\angle A:360°$ (790).

$$\therefore V=\frac{\angle A\cdot\pi R^3}{270}.$$

792. Corollary. **The volume of a spherical sector is equal to one third the product of the zone that is its base by the radius of the sphere.**

Proof: Similar to the proof of 787.

793. Corollary. **Volume of a spherical sector or a spherical cone,**

$$V=\frac{1}{3}\,Z\cdot R=\frac{2}{3}\,\pi R^2H.$$

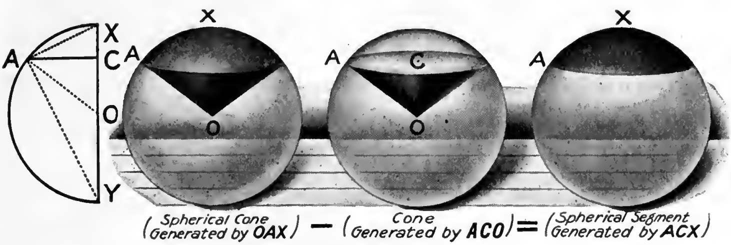
(Where V = the volume of the spherical sector or cone, H = the altitude of its base, R = the radius of the sphere.)

PROPOSITION XXXVII. PROBLEM

794. To derive a formula for the volume of a spherical segment.

There are Three Cases

1. Spherical segment of one base.



Given: Spherical segment generated by the figure ACX; semicircle XAY; $AC = r$; radius of sphere $= R$; altitude $= CX = H$.

Required: To find the volume of the spherical segment.

Computation: Draw chords AX, AY, and radius AO.

The right $\triangle ACO$ will generate a cone of revolution (Def. 660).

The volume of spherical segment $ACX =$ the volume of spherical cone OAX **minus** the volume of cone ACO.

Volume of spherical cone $OAX = \frac{2}{3}\pi R^2 \cdot H$ (793).

Volume of cone $ACO = \frac{1}{3}\pi r^2 \cdot CO$ (683).

Now $r^2 = CX \cdot CY = H(2R - H)$ (331, II).

Also $CO = R - H.$

$\therefore$ vol. $ACO = \frac{1}{3}\pi H(2R - H)(R - H)$ (Ax. 6).

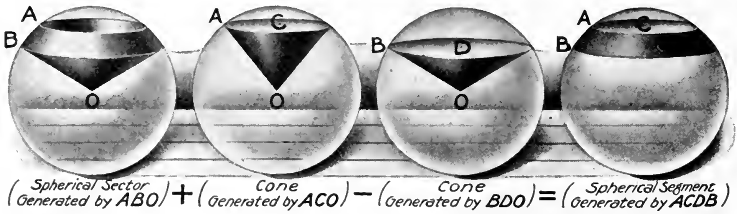
Hence volume of spherical segment

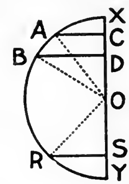
$ACX = \frac{2}{3}\pi R^2 H - (\frac{2}{3}\pi R^2 H - \pi R H^2 + \frac{1}{3}\pi H^3)$ (Ax. 6).

$\therefore$ **Volume of spherical segment of one base** $= \frac{1}{3}\pi H^2(3R - H)$.

Q.E.F.

2. Spherical segment not including the center.





Given: Spherical segment generated by figure $ACDB$; semicircle $XABY$; $AC = r$; $BD = r'$; radius of sphere $= R$; altitude $= CD = H$.

Required: To find the volume, V, of the spherical segment.

Computation: The $\triangle$ ACO and BDO generate cones of revolution (Def. 660).

Denote OD by d.

The volume of spherical segment $ACDB$
= the volume of spherical sector ABO
plus the volume of cone ACO
minus the volume of cone BDO.

Now the volume of spherical sector $ABO = \frac{2}{3}\,\pi R^2 H$ (792).

And the volume of cone $ACO = \frac{1}{3}\,\pi r^2(d + H)$ (683).

And the volume of cone $BDO = \frac{1}{3}\,\pi r'^2 d$ (683).

$$\therefore\ V = \tfrac{2}{3}\,\pi R^2 H + \tfrac{1}{3}\,\pi r^2(d+H) - \tfrac{1}{3}\,\pi r'^2 d$$

$$= \frac{\pi}{3}[2\,R^2H + r^2H + d(r^2 - r'^2)] \quad . \quad . \quad (1)$$

But in rt. $\triangle$ ACO, $R^2 = r^2 + (d+H)^2$ . . (2)
and in rt. $\triangle$ BDO, $R^2 = r'^2 + d^2$ . . . . . (3)
} (334).

Subtracting and solving,

$$d = \frac{r'^2 - r^2 - H^2}{2\,H}. \quad . \quad . \quad (4)$$

Substituting in (3),

$$R^2 = \frac{r^4 + r'^4 + H^4 - 2\,r^2r'^2 + 2\,r^2H^2 + 2\,r'^2H^2}{4\,H^2} \quad (5)$$

Substituting (4) and (5) in (1), and simplifying,

$$V = \frac{\pi}{3}\left[\frac{3\,r^2H + 3\,r'^2H + H^3}{2}\right].$$

$$\therefore V = \tfrac{1}{2}\pi H(r^2 + r'^2) + \tfrac{1}{6}\pi H^3. \qquad \text{Q.E.F.}$$

3. Spherical segment including the center.

Given: Spherical segment generated by figure *BDSR*; etc.

Required: To find the volume, V, of the spherical segment.

Computation: V = the volume of spherical sector *BOR*
plus the volume of cone *BDO*
plus the volume of cone *RSO*.

(Computation similar to that in 2, with same final formula.)

ORIGINAL EXERCISES

1. Prove that the area of the surface of a sphere is equal to the square of the diameter multiplied by π; that is, $S = \pi D^2$.

2. Prove that the volume of a sphere is equal to one sixth the cube of the diameter multiplied by π; that is, $V = \frac{1}{6}\pi D^3$.

3. The surface of a sphere is equal to the cylindrical surface of the circumscribed cylinder.

4. The total surface of a hemisphere is three fourths the surface of the sphere.

5. The volume of a sphere is two thirds the volume of the circumscribed cylinder.

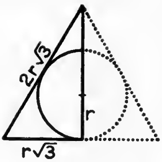
6. Upon the same circle as a base are constructed a hemisphere, a cylinder of revolution, and a cone of revolution, all having the same altitude. Prove that their total areas are $3\,\pi R^2$, $4\,\pi R^2$, $\pi R^2(1 + \sqrt{2})$, respectively, and their volumes are $\frac{2}{3}\pi R^3$, πR^3, $\frac{1}{3}\pi R^3$, respectively.

7. Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.

8. The area of the surface of a sphere is equal to the area of the circle whose radius is the diameter of the sphere.

9. Show that the formula for the volume of a spherical segment of one base reduces to the correct formula for the volume of a hemisphere when the base of the segment is a great circle; and to the correct formula for the volume of a sphere when the planes are both tangent.

10. In an equilateral triangle is inscribed a circle, and the figure is revolved about an altitude of the triangle as an axis. Prove:



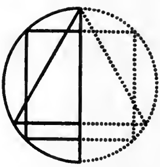
(*a*) That the surface generated by the circumference is two thirds the lateral surface generated by the triangle.

(*b*) That the volume generated by the circle is four ninths the volume generated by the triangle.

11. Derive a formula for the surface of a sphere, containing only V and π.

12. Derive a formula for the volume of a sphere, containing only S and π.

13. In a circle whose radius is R, there are inscribed a square and an equilateral triangle having their bases parallel; the whole figure is then revolved about the diameter perpendicular to the base of the triangle. Find, in terms of R:



(*a*) The total areas of the three surfaces generated.

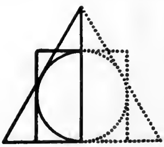
(*b*) The volumes of the three solids generated.

14. If a cylinder of revolution having its altitude equal to the diameter of its base, and a cone of revolution having its slant height equal to the diameter of its base, are both inscribed in a sphere:

(*a*) The total area of the cylinder is a mean proportional between the area of the surface of the sphere and the total area of the cone.

(*b*) The volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

15. About a circle whose radius is a there are circumscribed a square and an equilateral triangle having their bases in the same straight line. The whole figure is then revolved about an altitude of the triangle. Find, in terms of a:



(*a*) The total areas of the three surfaces generated.

(*b*) The volumes of the three surfaces generated.

16. If a cylinder of revolution having its altitude equal to the diameter of its base, and a cone of revolution having its slant height equal to the diameter of its base, is circumscribed about a sphere:

(*a*) The total area of the cylinder is a mean proportional between the area of the surface of the sphere and the total area of the cone.

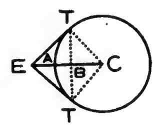
(*b*) The volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

17. The line joining the centers of two intersecting spherical surfaces is perpendicular to the plane of the intersection at the center of the intersection.

18. A cube and a sphere have equal surfaces. Show that the sphere has the greater volume.

19. Prove that the parallel of latitude through a point having 30° north latitude bisects the surface of the northern hemisphere.

20. Prove that in order that the eye may observe one sixth of the surface of a sphere it must be at a distance from the center of the sphere equal to $\frac{3}{2}$ of the radius.



Proof: Zone $TT = \frac{1}{3}$ surface of sphere (Hyp.).

$\therefore AB = \frac{1}{6}$ diam. $= \frac{1}{3} R$. Hence $BC = \frac{2}{3} R$.

In rt. $\triangle ETC$, $\overline{TC}^2 = EC \cdot BC$ (?); $\therefore R^2 = EC \cdot \frac{2}{3} R$, or $EC = \frac{3}{2} R$. (Explain.) Q.E.D.

21. How many miles above the surface of the earth (diameter of earth $= 7960$ mi.) must a person be in order that he may see one sixth of the earth's surface?

22. If the area of a zone of one base is a mean proportional between the area of the remaining zone of the sphere and the area of the entire sphere, the altitude of the zone is $R(\sqrt{5} - 1)$.

23. The area of a lune is to the area of a trirectangular spherical triangle as the angle of the lune is to 45°.

24. A cone, a sphere, and a cylinder have the same diameters and altitudes. Prove that their volumes are in arithmetical progression.

25. The surface of a sphere bears the same ratio to the total surface of the circumscribed cylinder of revolution as the volume of the sphere bears to the volume of the cylinder.

26. The smallest circle upon a sphere whose plane passes through a given point within the sphere, is the circle whose plane is perpendicular to the diameter through the given point.

27. What part of the surface of the earth could one see if he were at the distance of a diameter above the surface?

28. Prove that if any number of lines in space are drawn through a point, and from any other point perpendiculars to these lines are drawn, the feet of all of these perpendiculars lie on the surface of a sphere.

29. The volume of a sphere is to the volume of the circumscribed cube as $\pi : 6$. The volume of a sphere is to the volume of the inscribed cube as $\pi : \frac{2}{3}\sqrt{3}$.

30. There are five spheres that touch the four planes of the faces of a tetrahedron.

31. If two angles of a spherical triangle are supplementary, the sides of the polar triangle, opposite these angles, are supplementary.

32. A square, whose side is a, is revolved about a diagonal, and also about an axis bisecting two opposite sides. Which of these figures contains the greater volume? Which has the greater surface?

33. Find the area of the surface and the volume of a sphere whose radius is 6 in.

34. Find the area of a zone whose altitude is 4 in. on a sphere whose radius is 14 in.

35. Find the area of a lune whose angle is 30° on a sphere whose radius is 8 in.

36. Find the area of a spherical triangle whose angles are 110°, 41°, 92° on a sphere whose radius is 10 in.

37. Find the volume of a sphere whose radius is 5 in.

38. Find the volume of a spherical pyramid whose base is 35 sq. in. on a sphere whose radius is 12 in.

39. Find the area of a spherical polygon whose angles are 87°, 108°, 121°, 128° on a sphere whose radius is 25 cm.

40. What is the radius of a sphere whose surface is 1386 sq. yd.?

41. What is the radius of a sphere whose volume is $\frac{500\pi}{3}$ cu. in.?

42. What is the area of the surface of a sphere whose volume is 288π cu. ft.?

43. What is the volume of a sphere the area of whose surface is 2464 sq. in.?

44. Find the area of a zone whose altitude is $3\frac{1}{2}$ in. if the radius of the sphere is $7\frac{1}{2}$ in.

45. Find the volume of a spherical sector the altitude of whose base is $5\frac{1}{4}$ in. if the radius of the sphere is 6 in.

46. Find the diameter, the circumference of a great circle, and the volume of a sphere the area of whose surface is 25π sq. ft.

47. By how many cubic inches is a 9-in. cube greater than a 9-in. sphere?

48. The radius of a sphere is 15 in., and the angles of the base of a spherical pyramid are 160°, 127°, 96°, 145°, and 117°. Find the volume of the pyramid.

49. A cylindrical vessel 10 in. in diameter contains a liquid. A metal ball is immersed in the liquid and the surface rises $\frac{5}{6}$ in. What is the diameter of the ball?

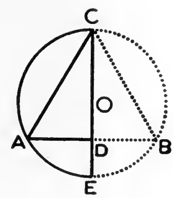
50. If a sphere 3 ft. in diameter weighs 99 lb., how much will a sphere of the same material 4 ft. in diameter weigh?

51. The radii of the bases of a frustum of a cone of revolution are 5 in. and 6 in., and the altitude of the frustum is $19\frac{1}{2}$ in. What is the diameter of an equal sphere?

52. What is the radius of a sphere whose surface is equal to the total surface of a right circular cylinder having an altitude equal to 21 in. and radius of the base equal to 6 in.?

53. Find the volume generated by the revolution of an equilateral triangle inscribed in a circle whose radius is 8 in. about an altitude of the triangle as an axis. (See Fig. of Ex. 55.)

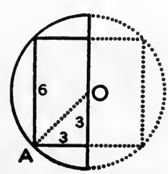
54. In the figure of Ex. 55, find the volume of the segment generated by the figure AED revolving about CD as an axis.



55. Find the area of the surface and the volume of the sphere generated by a circle that is circumscribed about an equilateral triangle whose side is 10 in.

56. Circumscribing a sphere whose radius is 18 m. is a cylinder of revolution. Compare their total areas; their volumes.

57. Circumscribing a cylinder of revolution whose altitude and diameter are each 6 in. is a sphere. Find the volume and area of the surface of the sphere.



58. Circumscribing a cylinder whose altitude is 4 in. and diameter is 3 in. is a sphere. Find the radius and volume of the sphere.

59. Each edge of a cube is 8 in. What is the area of the surface and the volume of the circumscribed sphere?

60. Find the volume of one of the segments cut from a 10 in. sphere by the plane of one of the faces of the inscribed cube.

61. The volume of a certain sphere is $179\frac{2}{3}$ cu. ft. Find the radius of a sphere 8 times as large. Find the radius of a sphere 3 times as large.

62. The radius of a certain sphere is 5 in. What is the radius of a sphere twice as great? half as great? two thirds as great?

63. A hollow sphere has an outer diameter of 20 in. and an inner diameter of 16 in. Find the volume of the metal in the shell.

64. Find the diameter of that sphere whose volume is, numerically, equal to the area of its surface.

65. A projectile consists of a right circular cylinder having a hemisphere at each end. If the cylinder is 9 in. long and 7 in. in diameter, what is the volume of one projectile?

66. Inscribed in a regular tetrahedron whose edge is 4 in., and circumscribed about it, are two spheres. Find their radii.

67. Find the radii of the spheres inscribed in and circumscribed about a regular hexahedron whose edge is 8 m.

68. Find the radii of the spheres inscribed in and circumscribed about a regular octahedron whose edge is 12 in.

69. How many spherical bullets $\frac{1}{2}$ in. in diameter can be made from a cube of lead 5 in. on each edge?

70. The area of a spherical triangle whose angles are 158°, 77°, 95° is $288\frac{3}{4}$ sq. ft. Find the radius of the sphere.

71. The area of a spherical triangle whose excess is 75° is 135π sq. in. Find the radius of the sphere.

72. If the radius of a sphere is 2.5 in., and the sides of a triangle on it are 104°, 115°, 101°, find the area of the polar triangle.

73. In a trihedral angle the plane angles of the dihedral angles are 75°, 85°, 110°. Find the number of degrees of surface of a sphere whose center is the vertex of the trihedral angle inclosed by the faces of this trihedral angle.

74. What is the area of a spherical hexagon, each of whose angles is 145°, on a sphere whose radius is 15 m.?

75. How many miles above the earth must a person be in order that he may see a third of its surface? one eighth of its surface?

76. Find the altitude of the zone whose area is equal to the area of a great circle of a sphere.

77. If the radius of a sphere is doubled, how is the amount of surface affected? the volume? the weight?

78. At a distance ($= d$) from the center of a sphere whose radius is r is an illuminating point. What is the altitude of the zone illuminated?

79. On a sphere having a radius of 5 in. is an equiangular spherical triangle whose area is 5π sq. in. Find the angles of the triangle.

80. Find the area of the surface of a sphere whose volume is 1 cu. yd.

81. Find the volume of a sphere whose surface is 1 sq. yd.

82. If a circumference is described on the surface of a sphere by a pair of compasses whose points are $2\frac{4}{5}$ in. apart, what is the area of the zone bounded by this circumference?

83. On a sphere the area of whose surface is 288 sq. ft. is a birectangular spherical triangle whose vertex angle is 100°. Find the area of this triangle.

84. Five inches from the center of a sphere whose diameter is two feet a plane is passed. Find the areas of the two zones formed. Find the chords of their generating arcs.

85. The diameter of the moon is about 2000 mi.; that of the earth, about 8000 mi. How do their surfaces compare? their volumes?

86. The radii of two concentric spheres are 12 in. and 13 in. A plane is tangent to the inner sphere. Find area of section of outer sphere.

87. If a solid sphere 4 ft. in diameter weighs 500 lb., what is the weight of a spherical shell, whose external diameter is 10 ft., if it is made of the same material and a foot thick?

88. The sun's diameter is about 109 times the diameter of the earth. How do the areas of their surfaces compare? their volumes?

89. How many quarter-inch spherical bullets can be made from a sphere of lead a foot in diameter?

90. A 12-inch cube of lead is melted and cast in the form of a spherical cannon ball. What is the radius of the cannon ball?

91. Find the angles of an equiangular spherical triangle equal to the sum of three equiangular spherical triangles (upon the same sphere) whose angles are each 75°.

92. What is the radius of a sphere equal to the sum of two spheres whose radii are 3 in. and 4 in. respectively?

93. What is the radius of a sphere equal to the difference of two spheres whose radii are 5 in. and 4 in. respectively?

94. The area of an equiangular spherical triangle is π sq. in., and the radius of the sphere is 4 in. Find the angles of the triangle.

95. The volumes of two spheres are to each other as 64 : 343. What is the ratio of their surfaces?

96. Find the volumes of the segments of a sphere whose radius is 12 in. formed by a plane whose distance from the center is 9 in.

97. If the radius of a sphere is 20 in., find:

(*a*) The area of its surface.
(*b*) The area of a zone whose altitude is 2 in.
(*c*) The edge of a cube inscribed in the sphere.
(*d*) The area of a lune whose angle is 80°.
(*e*) The area of a spherical triangle whose angles are 75°, 53°, 72°.
(*f*) The area of a spherical polygon whose angles are 68°, 119°, 128°, 147°, 150°.
(*g*) The area of a birectangular spherical triangle whose vertex-angle is 54°.
(*h*) The area of a zone of one base whose altitude is 5 in.
(*i*) The radius of a sphere whose surface is four times as large.
(*j*) The volume of the sphere.
(*k*) The volume of a wedge whose angle is 36°.
(*l*) The volume of a spherical pyramid whose base is the triangle of exercise (*e*).
(*m*) The volume of the spherical sector whose base is the zone of exercise (*b*).
(*n*) The volume of the spherical cone whose base is the zone of exercise (*h*).
(*o*) The volume of a spherical segment of one base, whose altitude is 6 in.
(*p*) The volume of a spherical segment whose altitude is 4 in. and the radii of whose bases are 12 in. and 16 in.
(*q*) The radius of a sphere whose volume is 64 times as large.

98. The angles of a spherical triangle are 80°, 90°, 100°. Find the angle of an equal lune.

99. In a sphere whose radius is 26 in. two parallel planes are passed 34 in. apart. The radii of the two sections are 10 in. and 24 in. Find the volume of the spherical segment included between the planes.

100. In a certain refrigerating plant is a large tank of ice water. From this tank to a faucet is a pipe $\frac{3}{4}$ in. inside diameter, and 42 ft. long. The faucet is opened and 1 qt. of water runs out every 4 seconds. In what length of time will the cold water from the tank appear at the faucet?

101. A sphere 2 ft. in diameter is trisected by two concentric spherical surfaces. Find the radii of these surfaces, in inches.

102. If a cylindrical leaden bar, *a* ft. long and *b* in. in diameter, is melted and made into bullets, $\frac{1}{8}$ in. in diameter, explain the successive steps necessary to be taken to ascertain the number of bullets there will be.

SUMMARY OF FORMULAS OF SOLID GEOMETRY

B = area of base.
b = area of upper base.
E = number of edges.
e, e' = homologous edges.
F = number of faces.
H = altitude.
L = lateral area.
M = mid-section.
m = radius of mid-section.
P = perimeter of base.
P_r = perimeter of right section.
p = perimeter of upper base.
R, r = radius of base.
s = slant height.
T = total area.
V = volume; number of vertices.

PRISMS AND PYRAMIDS

Parallelepiped	$V = BH$	(584).
Prism	$L = H \cdot P_r$	(569).
	$T = L + 2B$	(570).
	$V = B \cdot H$	(590).
Prismatoid	$V = \frac{1}{6}H(b + B + 4M)$	(628).
Regular Pyramid	$L = \frac{1}{2}P \cdot s$	(604).
	$T = L + B$	(595).
Pyramid	$V = \frac{1}{3}B \cdot H$	(612).
Frustum of pyramid	$L = \frac{1}{2}(P + p)s$	(605).
	$V = \frac{1}{3}H(B + b + \sqrt{B \cdot b})$	(618).
Polyhedron.	$E + 2 = V + F$	(624).
	Sum of face $\measuredangle$s $= (V - 2)360°$	(626).
Similar polyhedrons . .	$T : T' = e^2 : e'^2$	(632).
	$V : V' = e^3 : e'^3$	(634).

CYLINDERS AND CONES

Right circular cylinder . . .	$L = 2\pi RH$	(654).
	$T = 2\pi R(H + R)$	(654).
Circular cylinder	$V = \pi R^2 H$	(656).
Right circular cone	$L = \pi Rs$	(679).
	$T = \pi R(s + R)$	(679).
Circular cone	$V = \frac{1}{3}\pi R^2 H$	(683).
Frustum of right circular cone	$L = \pi(R + r)s$	(681),
	$= 2\pi ms$	(681).
	$T = \pi[(R + r)s + R^2 + r^2]$	(681).
	$V = \frac{1}{3}\pi H[R^2 + r^2 + R \cdot r]$	(685).

A = angle of lune.

E = spherical excess.
H = altitude.
L = area of lune.
R = radius of sphere.

r, r' = radii of bases of spherical segment.
S = area of spherical surface.
V = volume.
Z = area of zone.
Δ = area of triangle.

The Sphere

Spherical surface	$S = 4\pi R^2$	(774).
Zone	$Z = 2\pi RH$	(778).
Zone of one base	$Z = \pi\ (\text{chord})^2$	(779).
Lune	$L = 2A$ spherical degrees	(781).
	or $L = \dfrac{A \cdot \pi R^2}{90}$ square units	(782).
Spherical triangle	$\Delta = E$ spherical degrees	(784).
	or $\Delta = \dfrac{E \cdot \pi R^2}{180}$ square units	(785).
Spherical polygon . .	Polygon = E spherical degrees	(786).
Sphere	$V = \dfrac{4\pi R^3}{3}$	(787).
Spherical pyramid . . .	$V = \frac{1}{3}\ (\text{base})\ R$	(789).
Spherical wedge	$V = \dfrac{\angle A \cdot \pi R^3}{270}$	(791).
Spherical sector	$V = \frac{2}{3}\pi R^2 H$	(793).
Spherical cone	$V = \frac{2}{3}\pi R^2 H$	(793).
Spherical segment, one base,	$V = \frac{1}{3}\pi H^2(3R - H)$	(794).
Any spherical segment . .	$V = \frac{1}{2}\pi H(r^2 + r'^2) + \frac{1}{6}\pi H^3$	(794).

INDEX OF DEFINITIONS

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