A

SCHOOL ALGEBRA.

BY

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PREFACE.

This book, as the name implies, is written for High Schools and Academies, and is a thorough and practical treatment of the principles of Elementary Algebra. It covers sufficient ground for admission to any American college, and with the author's College Algebra makes as extended a course as the time allotted to this study in our best schools and colleges will allow. Great care has been taken to present the best methods, so that students in going from the lower book to the higher will have a good foundation, and have nothing to unlearn.

The problems are carefully graded. They are for the most part new; either original or selected from recent examination papers. They are sufficiently varied and interesting, and are not so difficult as to discourage the beginner. The early chapters are quite full; for even if a student is perfectly familiar with the operations of Arithmetic, he must have *time* to learn the language and the fundamental processes of Algebra.

The introductory chapter should be *read and discussed* in the recitation room. This chapter brings before the student in brief review the knowledge he has already gained from the study of Arithmetic, states and proves the general laws of numbers, sets forth clearly the advantage of using letters to represent numbers in the statement of general laws, and leads him to see at the outset that Algebra, like Arithmetic, treats of numbers. In this chapter, also, the meaning of negative quantities is explained, and the laws which regulate the combinations of different arithmetical numbers are shown to apply to algebraic numbers. It is hoped that a free discussion of these elementary principles will do much to prevent that vagueness which the beginner invariably experiences if he fails to connect the laws of Algebra with what he has learned in Arithmetic.

Answers to the problems are bound separately, in paper covers, and will be furnished free to pupils when *teachers* apply to the publishers for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

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SCHOOL ALGEBRA.

CHAPTER I.

INTRODUCTION.

1. Units. In counting separate objects the standards by which we count are called **units**; and in measuring continuous magnitudes the standards by which we measure are called **units**. Thus, in counting the boys in a school, the unit is a boy; in selling eggs by the dozen, the unit is a dozen eggs; in selling cloth by the yard, the unit is a yard of cloth; in measuring short distances, the unit is an inch, a foot, or a yard; in measuring long distances, the unit is a rod or a mile.

2. Numbers. Repetitions of the unit are expressed by numbers. If a man, in sawing logs into boards, wishes to keep a count of the logs, he makes a straight mark for every log sawed, and his record at different times will be as follows:

These representative groups are named one, two, three, four, five, six, seven, eight, nine, ten, etc., and are known collectively under the general name of **numbers**. It is obvious that these representative groups will have the same meaning, whatever the nature of the unit counted.

3. Quantities. The word "quantity" (from the Latin *quantus*, how much) implies both a unit and a number. Thus, if we inquire how much wheat a bin will hold, we mean how many bushels of wheat it will hold. If we inquire how much carpeting there is in a certain roll, we mean how many yards of carpeting. If we inquire how much wood there is on a certain wood-lot, we mean how many cords of wood.

4. Number-Symbols in Arithmetic. Instead of groups of straight marks, we use in Arithmetic the arbitrary symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, called **figures,** for the numbers one, two, three, four, five, six, seven, eight, nine.

The next number, ten, is indicated by writing the figure 1 in a different position, so that it shall signify not *one*, but *ten*. This change of position is effected by introducing a new symbol, 0, called **nought** or **zero,** and signifying *none*. Thus, in the symbol 10, the figure 1 occupying the second place from the right, signifies a collection of *ten* things, and the zero signifies that there are no single things over. The symbol 11 denotes a collection of ten things and one thing besides. All succeeding numbers up to the number consisting of 10 tens are expressed by writing the figure for the number of tens they contain in the second place from the right, and the figure for the number of units besides in the first place. The number consisting of 10 tens is called a **hundred,** and the *hundreds* of a number are written in the *third place* from the right. The number consisting of 10 hundreds is called a **thousand,** and the *thousands* are written in the *fourth place* from the right; and so on.

5. The Natural Series of Numbers. Beginning with the number *one*, each succeeding number is obtained by putting one more with the preceding number. If from a given point marked 0, we draw a straight line to the right, and beginning from this point lay off units of length, the successive repetitions of the unit will be denoted by the natural series of numbers 1, 2, 3, 4, etc. Thus,

6. The reader will notice that number symbols in Arithmetic stand for *particular numbers*, and that these symbols indicate *a* method of making up the number, but not necessarily *the* method by which the number is actually made up. Thus, if a man has 66 dollars in bank-notes, he may have, as the number 66 indicates, 6 ten-dollar bills and 6 one-dollar bills, but this is not the only way in which the 66 dollars may be made up.

7. Integral and Fractional Numbers. When the things counted are *whole* units, the numbers which count them are called **whole numbers, integral numbers,** or **integers,** where the adjective is transferred from the things counted to the numbers which count them. But if the things counted are only parts of units, the numbers which count them are called **fractional numbers,** or simply **fractions,** where again the adjective is transferred from the things counted to the numbers which count them.

To represent the parts of a given unit, two number-symbols are used, one to name the parts into which the unit is divided, and therefore called the **denominator,** and the other to denote the number of parts taken, and therefore called the **numerator.** The denominator is written below the numerator with a line between them. Thus, in the fraction $\frac{7}{9}$ the 9 shows that the unit is divided into nine equal parts, called *ninths* of the unit, and the 7 shows that seven of these equal parts are taken.

8. Principal Signs of Operations. The sign $+$, read *plus*, indicates that the number after the sign is to be *added* to the number before the sign. Thus, $5+4$ means that 4 is to be added to 5.

The sign $-$, read *minus*, indicates that the number after the sign is to be *subtracted* from the number before the sign. Thus, $8-4$ means that 4 is to be subtracted from 8.

The sign $\times$, read *times*, indicates that the number after the sign is to be multiplied by the number before the sign. Thus, 5×4 means that 4 is to be multiplied by 5.

The sign $\div$, read *divided by*, indicates that the number before the sign is to be *divided* by the number after the sign. Thus, $8 \div 4$ means that 8 is to be divided by 4.

The operation of division is also indicated by placing the dividend over the divisor with a line between them. Thus, $\frac{8}{4}$ means the same as $8 \div 4$.

9. Signs of Relation. The sign $=$, read *equals*, or *is equal to*, when placed between two numbers, indicates that they are equal. Thus, $8+4=12$ means that $8+4$ is the same as 12.

The sign $>$, read *is greater than*, indicates that the number which precedes the sign is greater than the number which follows it. Thus, $8+4>10$ means that $8+4$ is greater than 10.

The sign $<$, read *is less than*, indicates that the number which precedes the sign is less than the number which follows it. Thus, $8+4<16$ means that $8+4$ is less than 16.

10. Signs of Deduction and of Continuation. The sign $\therefore$ stands for the word "therefore" or "hence." The sign ····· or ------ stands for the words "and so on."

11. Number-Symbols in Algebra. Algebra, like Arithmetic, treats of numbers, and employs the letters of the alphabet in addition to the figures of Arithmetic to represent numbers. The letters of the alphabet are used as *general* symbols of numbers to which *any particular values* may be assigned. In any particular problem, however, a letter must be supposed to have the same particular value throughout the investigation or discussion of the problem.

These general symbols are of great advantage in investigating and stating general laws; in exhibiting the actual method in which a number is made up; and in representing unknown numbers which are to be discovered from their relations to known numbers.

The advantage of representing numbers by letters will be more clearly seen later on. For the present it will be sufficient for the beginner to understand that *every letter, and every combination of letters, and every combination of figures and letters used in Algebra, represents some number.* Thus, the number of dollars in a package of bank-notes can be represented by x; but if the package consists of ten-dollar bills, five-dollar bills, two-dollar bills, and one-dollar bills, and if we denote the number of ten-dollar bills by a, of five-dollar bills by b, of two-dollar bills by c, and of one-dollar bills by d, the whole number of dollars in the package will be represented by $10a + 5b + 2c + d$.

In this particular case x and $10a + 5b + 2c + d$ both stand for the same number.

12. Substitution. It is obvious that the same operation on each of the above expressions will produce results that agree in value, and therefore that either may be substituted for the other at pleasure. In short,

Every algebraic expression represents some number, and may be operated upon as if it were a single symbol standing for the number which it represents.

13. Factors. When a number consists of the product of two or more numbers, each of these numbers is called a **factor** of the product. If these numbers are denoted by letters, the sign $\times$ is omitted. Thus, instead of $a \times b$, we write ab.

14. Coefficients. A known factor prefixed to another factor to show how many times that factor is taken is called a **coefficient.** Thus, in $7x$ the factor 7 is the coefficient of x.

15. Powers. A product consisting of two or more **equal** factors is called a **power** of that factor.

The **index** or **exponent of a power** is a number-symbol placed at the right of a number, to show how many times the number is taken as a factor. Thus, 2^4 is written instead of $2 \times 2 \times 2 \times 2$; a^3 instead of aaa.

The second power of a number is generally called the **square** of that number; the third power of a number, the **cube** of that number.

16. Roots. The **root** of a number is one of the equal factors of that number; the *square root* of a number is one of the *two* equal factors of that number; the *cube root* of a number is one of the *three* equal factors of that number; and so on. The sign $\sqrt{\ }$, called the **radical sign,** indicates that a root is to be found. Thus, $\sqrt[2]{4}$, or $\sqrt{4}$, means that the square root of 4 is to be taken; $\sqrt[3]{8}$ means that the cube root of 8 is to be taken; and so on.

The number-symbol written above the radical sign is called the **index of the root.**

17. An **algebraic expression** is a number written with algebraic symbols; an algebraic expression consists of one symbol, or of several symbols connected by signs of operation.

A **term** is an algebraic expression the parts of which are not separated by the sign of addition or subtraction. Thus, $3ab$, $5x \times 4y$, $3ab \div 4xy$ are terms.

A **simple expression** is an expression of one term.

A **compound expression** is an expression of two or more terms.

18. Positive and Negative Terms. The terms of a compound expression preceded by the sign $+$ are called **positive terms**, and the terms preceded by the sign $-$ are called **negative terms**. The sign $+$ before the first term is omitted.

19. Parentheses. If a compound expression is to be treated as a whole it is enclosed in a parenthesis. Thus, $2\times(10+5)$ means that we are to add 5 to 10 and multiply the result by 2; if we were to omit the parenthesis and write $2\times 10+5$, the meaning would be that we were to multiply 10 by 2 and add 5 to the result.

Instead of parentheses, we use with the same meaning brackets [ ], braces { }, and a straight line called a vinculum. Thus, $(5+2)$, $[5+2]$, $\{5+2\}$, $\overline{5+2}$, $\left.\begin{matrix}5\\+2\end{matrix}\right|$, all mean that the expression $5+2$ is to be treated as the single symbol 7.

20. Rules for removing Parentheses. If a man has 10 dollars and afterwards collects 3 dollars and then 2 dollars, it makes no difference whether he adds the 3 dollars to his 10 dollars, and then the 2 dollars, or puts the 3 and 2 dollars together and adds their sum to his 10 dollars.

The first process is represented by $10+3+2$.

The second process is represented by $10+(3+2)$.

Hence $$10+(3+2)=10+3+2. \qquad (1)$$

If a man has 10 dollars and afterwards collects 3 dollars and then pays a bill of 2 dollars, it makes no difference whether he adds the 3 dollars collected to his 10 dollars and pays out of this sum his bill of 2 dollars, or pays the 2 dollars from the 3 dollars collected and adds the remainder to his 10 dollars.

The first process is represented by $10+3-2$.

The second process is represented by $10+(3-2)$.

Hence $$10+(3-2)=10+3-2. \qquad (2)$$

From (1) and (2) it follows that if a compound expression is to be *added*, the parenthesis may be removed and each term in the parenthesis retain its prefixed sign.

If a man has 10 dollars and has to pay two bills, one of 3 dollars and one of 2 dollars, it makes no difference whether he takes 3 dollars and 2 dollars in succession, or takes the 3 and 2 dollars at one time, from his 10 dollars.

The first process is represented by $10-3-2$.

The second process is represented by $10-(3+2)$.

Hence $$10-(3+2)=10-3-2. \qquad (3)$$

If a man has 10 dollars consisting of 2 five-dollar bills, and has a debt of 3 dollars to pay, he can pay his debt by giving a five-dollar bill and receiving 2 dollars.

This process is represented by $10-5+2$.

Since the debt paid is three dollars, that is, $(5-2)$ dollars, the number of dollars he has left can evidently be expressed by $$10-(5-2).$$

Hence $$10-(5-2)=10-5+2. \qquad (4)$$

From (3) and (4) it follows that if a compound expression is to be *subtracted*, the parenthesis may be removed, provided the sign before each term within the parenthesis is changed, the sign $+$ to $-$, and the sign $-$ to $+$.

Exercise 1.

Perform the operations indicated, and simplify.

1. $7+(3-2)$. **4.** $5\times(2+3)$. **7.** $(7-3)\times(5-2)$.

2. $7-(3-2)$. **5.** $(5+3)\div 2$. **8.** $(8-2)\div(5-2)$.

3. $7-(3+2)$. **6.** $5\times(3-2)$. **9.** $3\times(12-6-2)$.

21. Fundamental Laws of Numbers. We are so occupied in Arithmetic with the application of numbers to the ordinary problems of every-day life that we pay little attention to the investigation of the fundamental laws of numbers. It is, however, very important that the beginner in Algebra should have clear ideas of these laws, and of the extended meaning which it is necessary to give in Algebra to certain words and signs used in Arithmetic; and that he should see that every such extension of meaning is consistent with the meaning previously attached to the word or sign, and with the general laws of numbers. We shall, therefore, give general definitions for the fundamental operations upon numbers and then state the laws which apply to them.

22. Addition. The process of finding the result when two or more numbers are taken together is called **addition**, and the result is called the **sum**.

23. Subtraction. The process of finding the result when one number is taken from another is called **subtraction**, and the result is called the **difference** or **remainder**. The number taken away is called the **subtrahend**, and the number from which the subtrahend is taken is called the **minuend**.

In practice the difference is found by discovering the number which must be *added* to the subtrahend to give the minuend. If the subtrahend consists of two or more terms, we add these terms and then determine the number which must be added to their sum to make it equal to the minuend. Thus, if a clerk in a store sells articles for 10 cents, 15 cents, and 30 cents, and receives a dollar bill in payment, he makes change by adding these items and then adding to their sum enough change to make a dollar.

From the nature of this process it is obvious that the general laws of numbers which apply to addition apply also to subtraction, and that we may take for the general definition of subtraction

The operation of finding from two given numbers, called *minuend* and *subtrahend*, a third number, called *difference*, which *added to the subtrahend will give the minuend.*

24. Multiplication. The process of finding the result when a given number is taken as many times as there are units in another number is called **multiplication,** and the result is called the **product.**

This definition fails when the multiplier is a fraction, for we cannot take the multiplicand a *fraction of a time.* We therefore consider what extension of the meaning of multiplication can be made so as to cover the case in question. When we multiply by a fraction we divide the multiplicand into as many equal parts as there are units in the denominator and take as many of these parts as there are units in the numerator. If, for instance, we multiply 8 by $\frac{3}{4}$, we divide 8 into four equal parts and take three of these parts, getting 6 for the product. We see that $\frac{3}{4}$ is $\frac{3}{4}$ of 1, and 6 is $\frac{3}{4}$ of 8; that is, the product 6 is obtained from the multiplicand 8 precisely as the multiplier $\frac{3}{4}$ is obtained from 1. The same is true when the multiplier is an integral number.

Thus, in $6 \times 8 = 48$,

the multiplier 6 is $1 + 1 + 1 + 1 + 1 + 1$,

and the product 48 is $8 + 8 + 8 + 8 + 8 + 8$.

We may, therefore, take for the general definition of multiplication

The operation of finding from two given numbers, called *multiplicand* and *multiplier*, a third number called *product*, which is *formed from the multiplicand as the multiplier is formed from unity.*

25. Division. To divide 48 by 8 is to find the number of times it is necessary to take 8 to make 48. Here the *product* and *one factor* are given and *the other factor* is required. We may therefore take for the general definition of division

The operation by which when *the product* and *one factor* are given *the other factor* is found.

With reference to this operation the product is called the **dividend**, the given factor the **divisor**, and the required factor the **quotient.**

26. The Commutative Law. If we have a group of 3 things and another group of 4 things, we shall have a group of 7 things, whether we put the 3 things with the 4 things or the 4 things with the 3 things; that is,

$$4+3=3+4.$$

It is evident that the truth of the above statement does not depend upon the particular numbers 3 and 4, but that the statement is true for any two numbers whatever. Thus, in case of *any* two numbers we shall have

First number + second number = second number + first number.

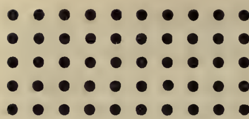
If we let a stand for the first number and b for the second number, this statement may be written in the much shorter form

$$a+b=b+a.$$

This is the commutative law of addition, and may be stated as follows:

Additions may be performed in any order.

27. Also, if we have 5 lines of dots with 10 dots in a line, the whole number of dots will be expressed by 5×10.



If we consider the dots as 10 columns with 5 dots in a column, the number will be expressed by 10×5.

That is, $$5 \times 10 = 10 \times 5.$$

Again, if we divide a given length into 6 equal parts,

one-third of the line will contain 2 of these parts, and one-half the line will contain 3 of these parts. Now one-third of one-half will be 1 of these parts, and one-half of one-third will be 1 of these parts; that is,

$$\tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{2} \times \tfrac{1}{3}.$$

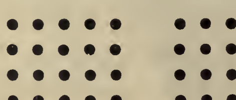
Therefore, if a and b stand for any two numbers, integral or fractional, we shall have

$$ab = ba.$$

This is the commutative law of multiplication, and may be stated as follows:

Multiplications may be performed in any order.

28. The Distributive Law. The expression $4 \times (5 + 3)$ means that we are to take the sum of the numbers 5 and 3 four times. The process can be represented by placing five dots in a line, and a little to the right three more dots in the same line, and then placing a second, third, and fourth line of dots underneath the first line and exactly similar to it.



There are $(5+3)$ dots in each line, and 4 lines. The total number of dots, therefore, is $4 \times (5+3)$.

We see that in the left-hand group there are 4×5 dots, and in the right-hand group 4×3 dots. The sum of these two numbers $(4 \times 5)+(4 \times 3)$ must be equal to the total number; that is,

$$4 \times (5+3)=(4 \times 5)+(4 \times 3).$$

Again, the expression $4 \times (8-3)$ means that 3 is to be taken from 8, and the remainder to be multiplied by 4. The process can be represented by placing eight dots in a line and crossing the last three, and then placing a second, third, and fourth line of dots underneath the first line and exactly similar to it.

The whole number of dots not crossed in each line is evidently $(8-3)$, and the whole number of lines is 4. Therefore the total number of dots not crossed is

$$4 \times (8-3).$$

The total number of dots (crossed and not crossed) is (4×8), and the total number of dots crossed is (4×3). Therefore the total number of dots not crossed is

$$(4 \times 8)-(4 \times 3);$$

that is, $$4 \times (8-3)=(4 \times 8)-(4 \times 3).$$

Hence, by the commutative law

$$(8-3) \times 4=(8 \times 4)-(3 \times 4).$$

In like manner, if a, b, c, and d stand for *any* numbers, we have $$a \times (b+c-d)=ab+ac-ad.$$

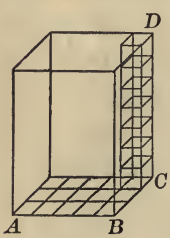
This is the distributive law, and may be stated as follows:

In multiplying a compound expression by a simple expression the result is obtained by multiplying each term of the compound expression by the simple expression, and writing down the successive products with the same signs as those of the original terms.

29. The Associative Law. The terms of an expression may be grouped in any manner. For if we have several numbers to be added, the result will evidently be the same, whether we add the numbers in succession or arrange them in groups and add the sums of these groups. Thus,

$$\begin{aligned} & a+b+c+d+e \\ &= a+(b+c)+(d+e) \\ &= (a+b)+(c+d+e). \end{aligned}$$

Likewise, if in the rectangular solid represented in the margin we suppose AB to contain 5 units of length, BC 3 units, and CD 7 units. The base may be divided into square units. There will be 3 rows of 5 square units each. Upon each square unit a cubic unit may be formed, and we shall have (3×5) cubic units. Upon these another tier of (3×5) cubic units may be formed, and then another tier of the same number, and the process continued until we have 7 tiers of (3×5) cubic units. Hence the number of cubic units in the solid will be represented by $7 \times (3 \times 5)$.



Upon the right-hand square in the back row a pile of 7 cubic units may be formed, upon the next square to the left another pile of 7 cubic units may be formed, and upon the next square another, and the process continued until we have a pile of 7 cubic units on each square in the back row. We shall then have (5×7) cubic units in the back tier, and as we can have 3 such tiers, the number of cubic units in the solid will be represented by $3 \times (5 \times 7)$.

Again, if we form a pile of 7 cubic units on the right-hand square of the back row, then another pile of 7 cubic units on the next square in front, another pile of 7 cubic units on the next square in front, we shall have a tier of (3×7) cubic units. We can have 5 such tiers, and the number of cubic units in the solid will now be represented by $5 \times (3 \times 7)$.

It follows, therefore, that the total number of cubic units in the solid may be represented by

$$7 \times (3 \times 5), \text{ or by } 3 \times (5 \times 7), \text{ or by } 5 \times (3 \times 7).$$

It is obvious that no part of this proof depends upon the particular numbers 3, 5, and 7, but the law holds for any arithmetical numbers whatever, and may be expressed by

$$c \times (a \times b) = a \times (b \times c) = b \times (a \times c).$$

This is called the **associative** law of addition and multiplication, and may be stated as follows:

The terms of an expression, or the factors of a product, may be grouped in any manner.

30. The Index Law.

Since $\quad a^2 = aa$, and $a^3 = aaa$,

$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3};$$

$$a^4 \times a = aaaa \times a = a^5 = a^{4+1}.$$

If a stands for any number, and m and n for any *integers*,

since $\quad a^m = aaa \cdots\cdots$ to m factors,

and $\quad a^n = aaa \cdots\cdots$ to n factors,

$$a^m \times a^n = (aaa \cdots\cdots \text{ to } m \text{ factors}) \times (aaa \ldots\ldots \text{ to } n \text{ factors}),$$

$$= aaa \cdots\cdots \text{ to } (m+n) \text{ factors},$$

$$= a^{m+n}.$$

Hence, the index law may be stated as follows:

The index of the product of two powers of the same number is equal to the sum of the indices of the factors.

31. These four laws, the commutative, the distributive, the associative, and the index laws, are the fundamental laws of Arithmetic, and together with the *law of signs*, which will be explained hereafter, they constitute the fundamental laws of Algebra.

32. Quantities Opposite in Kind. If a man gains 6 dollars and then loses 4 dollars, his actual gain, or, as we commonly say, his *net gain*, is 2 dollars; that is, 4 dollars' loss cancels 4 dollars of the 6 dollars' gain and leaves 2 dollars' gain. If he gained 6 dollars and then lost 6 dollars, the 6 dollars' *loss* cancels the 6 dollars' *gain*, and his *net gain* is nothing. If he gained 6 dollars and then lost 9 dollars, the 6 dollars' *gain* cancels 6 dollars of the 9 dollars' *loss*, and his *net loss* is 3 dollars. In other words, loss and gain are quantities so related that one cancels the other wholly or in part.

If the mercury in a thermometer rises 12 degrees and then falls 7 degrees, the *fall* of 7 degrees cancels 7 degrees of the *rise*, and the *net rise* is 5 degrees. If it *rises* 12 degrees and then *falls* 12 degrees, the *net rise* is nothing. If it *rises* 12 degrees and *falls* 15 degrees, there is a *net fall* of 3 degrees. In other words, *rise* and *fall* are quantities so related that one cancels the other wholly or in part.

An opposition of this kind also exists in motion *forwards* and motion *backwards;* in distances measured *east* and distances measured *west;* in distances measured *north* and distances measured *south;* in *assets* and *debts;* in time *before* and time *after* a fixed date; and so on.

33. Algebraic Numbers. If we wish to add 3 to 4, we begin at 4 in the natural series of numbers,

0 1 2 3 4 5 6 7 8 9

count 3 units *forwards*, and arrive at 7, the sum sought. If we wish to subtract 3 from 7, we begin at 7 in the natural series of numbers, count 3 units *backwards*, and arrive at 4, the difference sought. If we wish to subtract 7 from 7, we begin at 7, count 7 units backwards, and arrive at 0. If we wish to subtract 7 from 4, we cannot do it, because when we have counted backwards as far as 0 *the natural series of numbers comes to an end.*

In order to subtract a greater number from a smaller it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must proceed from zero by the repetitions of the unit, precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a **positive number**, and prefixing to it, when written, the sign $+$; and by calling every number in the left-hand series a **negative number**, and prefixing to it the sign $-$. The two series of numbers will be written thus:

$$\cdots\cdots -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \cdots\cdots$$

ALGEBRAIC SERIES OF NUMBERS.

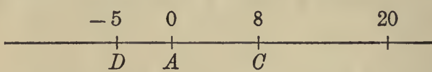
If, now, we wish to subtract 9 from 6, we begin at 6 in the positive series, count 9 units in the *negative direction* (to the left), and arrive at -3 in the negative series; that is, $6 - 9 = -3$.

The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

In general, if a and b represent any two numbers of the positive series, the expression $a - b$ will be a positive number when a is greater than b; will be zero when a is equal to b; will be a negative number when a is less than b.

In counting from left to right in the algebraic series numbers *increase* in magnitude; in counting from right to left numbers *decrease* in magnitude. Thus -3, -1, 0, $+2$, $+4$ are arranged in ascending order of magnitude.

34. We may illustrate the use of algebraic numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? *Answer:* At C, a point 8 feet to the right of A; that is, 20 feet $-$ 12 feet $=$ 8 feet; or, $20 - 12 = 8$.

Again, suppose he walks from A to the right 20 feet, and then returns 20 feet, where will he be? *Answer*: At A, the point from which he started; that is, $20 - 20 = 0$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? *Answer:* At D, a point 5 feet to the left of A; that is, $20 - 25 = -5$; and the phrase "5 feet to the left of A" is now expressed by the negative quantity, -5 feet.

35. Every algebraic number, as $+4$ or -4, consists of a *sign* $+$ or $-$ and the *absolute value* of the number. The sign shows whether the number belongs to the positive or negative series of numbers; the absolute value shows what place the number has in the positive or negative series.

When no sign stands before a number, the sign $+$ is always understood. Thus 4 means the same as $+4$, a means the same as $+a$. But *the sign $-$ is never omitted.*

36. Two algebraic numbers which have, one the sign $+$ and the other the sign $-$, are said to have *unlike signs*.

Two algebraic numbers which have the same absolute values, but unlike signs, always cancel each other when combined. Thus $+4-4=0$, $+a-a=0$.

37. Double Meanings of the Signs $+$ and $-$. The use of the signs $+$ and $-$ to indicate addition and subtraction must be carefully distinguished from the use of the signs $+$ and $-$ to indicate in which series, the positive or the negative, a given number belongs. In the first sense they are signs of *operations*, and are common to Arithmetic and Algebra; in the second sense they are signs of *opposition*, and are employed in Algebra alone.

38. Addition and Subtraction of Algebraic Numbers. An algebraic number which is to be added or subtracted is often inclosed in a parenthesis, in order that the signs $+$ and $-$, which are used to distinguish positive and negative numbers, may not be confounded with the $+$ and $-$ signs that denote the operations of addition and subtraction. Thus $+4+(-3)$ expresses the sum, and $+4-(-3)$ expresses the difference, of the numbers $+4$ and -3.

In order to add two algebraic numbers we begin at the place in the series which the first number occupies and count, *in the direction indicated by the sign of the second number*, as many units as there are in the absolute value of the second number.

Thus the sum of $+4+(+3)$ is found by counting from $+4$ three units in *the positive direction;* that is, to the right, and is, therefore, $+7$.

The sum of $+4+(-3)$ is found by counting from $+4$ three units in *the negative direction;* that is, to the left, and is, therefore, $+1$.

The sum of $-4+(+3)$ is found by counting from -4 three units in the positive direction, and is, therefore, -1.

$$\ldots\ldots -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6 \ldots\ldots$$

The sum of $-4+(-3)$ is found by counting from -4 three units in the negative direction, and is, therefore, -7.

Hence, to add two or more algebraic numbers, we have the following rules:

CASE I. When the numbers have *like* signs. *Find the sum of their absolute values, and prefix the common sign to the result.*

CASE II. When there are two numbers with *unlike* signs. *Find the difference of their absolute values, and prefix to the result the sign of the greater number.*

CASE III. When there are more than two numbers with *unlike* signs. *Combine the first two numbers and this result with the third number, and so on;* or, *find the sum of the positive numbers and the sum of the negative numbers, take the difference between the absolute values of these two sums, and prefix to the result the sign of the greater sum.*

39. The result is called the **sum.** It is often called the **algebraic sum,** to distinguish it from the *arithmetical sum,* that is, the sum of the absolute values of the numbers.

40. Subtraction. In order to subtract one algebraic number from another, we begin at the place in the series which the minuend occupies and count *in the direction opposite to that indicated by the sign of the subtrahend* as many units as there are in the absolute value of the subtrahend.

Thus, the result of subtracting $+3$ from $+4$ is found by counting from $+4$ three units in the *negative direction;* that is, in the direction *opposite to that indicated by the sign* $+$ *before* 3, and is, therefore, $+1$.

The result of subtracting -3 from $+4$ is found by counting from $+4$ three units in the positive direction; that is, in the direction *opposite to that indicated by the sign* $-$ *before* 3, and is, therefore, $+7$.

The result of subtracting $+3$ from -4 is found by counting from -4 three units in the *negative direction*, and is, therefore, -7.

The result of subtracting -3 from -4 is found by counting from -4 three units in the *positive direction*, and is, therefore, -1.

Collecting the results obtained in addition and subtraction, we have

ADDITION.	SUBTRACTION.
$+4+(-3)=+4-3=+1.$	$+4-(+3)=+4-3=+1.$
$+4+(+3)=+4+3=+7.$	$+4-(-3)=+4+3=+7.$
$-4+(-3)=-4-3=-7.$	$-4-(+3)=-4-3=-7.$
$-4+(+3)=-4+3=-1.$	$-4-(-3)=-4+3=-1.$

No part of this proof depends upon the particular numbers 4 and 3, and hence we may employ the general symbols a and b to represent the absolute values of any two algebraic numbers. We shall then have

ADDITION.	SUBTRACTION.	
$+a+(-b)=+a-b.$	$+a-(+b)=+a-b.$	(1)
$+a+(+b)=+a+b.$	$+a-(-b)=+a+b.$	(2)
$-a+(-b)=-a-b.$	$-a-(+b)=-a-b.$	(3)
$-a+(+b)=-a+b.$	$-a-(-b)=-a+b.$	(4)

From (1) and (3), it is seen that *subtracting a positive number is equivalent to adding an equal negative number.*

From (2) and (4), it is seen that *subtracting a negative number is equivalent to adding an equal positive number.*

To subtract one algebraic number from another, we have, therefore, the following rule:

Change the sign of the subtrahend, and add the subtrahend to the minuend.

This rule is consistent with the definition of subtraction given in § 23; for, if we have to subtract -4 from $+3$, we must add $+4$ to the subtrahend -4 to cancel it, and then add $+3$ to obtain the minuend; that is, we must add $+7$ to the subtrahend to get the minuend, but $+7$ is obtained by changing the sign of the subtrahend -4, making it $+4$, and adding it to $+3$, the minuend.

41. The commutative law of addition applies to algebraic numbers, for $+4+(-3)=-3+(+4)$. In the first case we begin at $+4$ in the series, count three units to the left, and arrive at $+1$; in the second case we begin at -3 in the series, count four units to the right, and arrive at $+1$.

The associative law, also, of addition is easily seen to apply to algebraic numbers.

42. Multiplication and Division of Algebraic Numbers. By the definition of multiplication, § 24.

Since $$+3=+1+1+1;$$

$$\therefore 3\times(+8)=+8+8+8$$
$$=+24,$$

and $$3\times(-8)=-8-8-8$$
$$=-24.$$

Again, since $$-3=-1-1-1;$$

$$\therefore (-3)\times 8=-8-8-8$$
$$=-24,$$

and $$(-3)\times(-8)=-(-8)-(-8)-(-8)$$
$$=+8+8+8$$
$$=+24.$$

No part of this proof depends upon the particular numbers 3 and 8. If we use a to represent the absolute value of any number, and b to represent the absolute value of any other number, we shall have

$$(+a)\times(+b)=+ab. \quad (1)$$

$$(+a)\times(-b)=-ab. \quad (2)$$

$$(-a)\times(+b)=-ab. \quad (3)$$

$$(-a)\times(-b)=+ab. \quad (4)$$

43. Law of Signs in Multiplication. From these four cases it follows that, in finding the product of two algebraic numbers,

Like signs give +, and unlike signs give –.

44. Law of Signs in Division.

Since $(+a)\times(+b)=+ab$, $\therefore +ab\div(+a)=+b$.
Since $(+a)\times(-b)=-ab$, $\therefore -ab\div(+a)=-b$.
Since $(-a)\times(+b)=-ab$, $\therefore -ab\div(-a)=+b$.
Since $(-a)\times(-b)=+ab$, $\therefore +ab\div(-a)=-b$.

That is, if the dividend and divisor have like signs, the quotient has the sign +; and if they have unlike signs, the quotient has the sign –. Hence, in division,

Like signs give +; unlike signs give –.

45. From the four cases of multiplication that we have given in § 42 it will be seen that the absolute value of each product is *independent of the signs*, and that the signs are *independent of the order of the factors.* Hence the commutative and associative laws of multiplication hold for all algebraic numbers.

46. The distributive law also holds; for, if

$$a(b+c)=ab+ac,$$

then
$$-a(b+c)=-ab-ac,$$

and
$$(b+c)(-a)=b(-a)+c(-a).$$

Therefore, for *all values* of a, b, and c,

$$a(b+c)=ab+ac.$$

From the nature of division the distributive law which applies to multiplication applies also to division.

47. We have now considered the fundamental laws of Algebra, and for convenience of reference we formulate them below:

$$\left.\begin{aligned} a+(b+c)&=a+b+c\\ a+(b-c)&=a+b-c\\ a-(b+c)&=a-b-c\\ a-(b-c)&=a-b+c \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

$$\left.\begin{aligned} (+a)\times(+b)&=+ab\\ (+a)\times(-b)&=-ab\\ (-a)\times(+b)&=-ab\\ (-a)\times(-b)&=+ab \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

The commutative law:

$$\left.\begin{aligned} &\text{Addition} && a+b=b+a\\ &\text{Multiplication} && ab=ba \end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

The associative law:

$$\left.\begin{aligned} &\text{Addition} && a+(b+c)=(a+b)+c\\ &\text{Multiplication} && a(bc)=(ab)c=abc \end{aligned}\right\} \quad \ldots \quad (4)$$

The distributive law:

$$\left.\begin{array}{ll}\text{Multiplication} & a(b+c)=ab+ac \\ \text{Division} & \dfrac{(b+c)}{a}=\dfrac{b}{a}+\dfrac{c}{a}\end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

The index law:

$$\text{Multiplication} \qquad a^m \times a^n = a^{m+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These laws are true for *all values* of the letters, but in (6) m and n are for the present restricted to *positive integral values.*

48. Value of an Algebraic Expression. Every algebraic expression stands for a number; and this number, obtained by putting for the several letters involved the numbers for which they stand, and performing the operations indicated by the signs, is called the **value** of the expression.

In finding the values of algebraic expressions, the beginner must be careful to observe what operations are actually indicated. Thus,

$4a$ means $a+a+a+a$; that is, $4 \times a$.

a^4 means $a \times a \times a \times a$.

$\sqrt{abc}$ means the square root of the product of a, b, and c.

$\sqrt{}abc$ means the product of the square root of a by bc.

NOTE. The radical sign $\sqrt{}$ before a product, without a vinculum or a parenthesis, affects only the symbol immediately following it.

$\sqrt{a}+b$ means that b is to be added to the square root of a.

$\sqrt{a+b}$ means that b is to be added to a and the square root of the sum taken.

49. In finding the value of a compound expression the operations indicated for each term must be performed *before* the operation indicated by the sign prefixed to the term.

Indicated divisions should be written in the fractional form, and the sign × omitted between a figure and a letter, or between two letters, in accordance with algebraic usage. Thus, $(b-c) \div 2 \times c + 26$ should be written

$$\frac{b-c}{2c} + 26.$$

NOTE. The line between the numerator and denominator of the fractions serves for a *vinculum*, and renders the parenthesis unnecessary.

If $b = 4$, and $c = -4$, the numerical value is

$$\frac{4-(-4)}{2 \times (-4)} + 26 = \frac{8}{-8} + 26 = -1 + 26 = 25.$$

Exercise 2.

NOTE. When there is no sign expressed between single symbols or between *compound expressions*, it must be remembered that the sign understood is the *sign of multiplication*.

If $a = 1$, $b = 2$, and $c = 3$, find the value of

1. $7a - bc.$
2. $ac + b.$
3. $4ab - c.$
4. $6ab - b - c.$
5. $2a - b + c.$
6. $ab + bc - ac.$
7. $b^2 + a^2 + c^2.$
8. $2ab - 5bc^2.$
9. $\sqrt{4abc}.$
10. $\sqrt{6abc}.$
11. $c^3 - b^3.$
12. $\sqrt[3]{c^2 - a^3}.$
13. $a - 2(b + c).$
14. $(a + b)^2 + 2(c - a)^3.$
15. $\sqrt{6bc} - (b - c).$
16. $6b - 10bc \div 12a + 2c.$
17. $5c \div (b - a) - (b + a).$
18. $\sqrt[3]{6b^2c^2}.$

If $a = 1$, $b = 2$, $c = 3$, and $d = 0$, find the value of

19. $7a - bc + 6d.$
20. $ac + b - d.$
21. $4ab - cd - d.$
22. $2a - b + c.$

23. $ab + bc - ad$.

24. $2ab \div 5bc$.

25. $\sqrt{4}\,abcd$.

26. $\sqrt{4abcd}$.

27. $b - c + d$.

28. $a - 2(b + c)$.

29. $2b(3 - 5c) \div (a - 2c)$.

30. $2(a + b)^2$.

Exercise 3.

Remove the parentheses (§ 20), and find the algebraic sum of

1. $(5 - 2) + (3 + 1)$.
2. $(6 - 2) - (2 - 3)$.
3. $(-3 + 4) - (2 + 5)$.
4. $-6 - (2 - 3 - 1)$.
5. $2 - (5 - 7 + 8)$.
6. $8 - (7 - 5 + 4)$.
7. $10 - (5 - 6 - 7)$.
8. $2 - (3 - 3 + 4)$.
9. $(5 - 10) + (3 - 2)$.
10. $-7 + (3 + 2 - 4)$.
11. $23 + (2 - 7 - 6)$.
12. $24 - (2 - 7 - 5)$.
13. $11 + (1 - 4 - 3)$.
14. $(15 - 3 - 5) - 7$.
15. $10 - (5 - 2 - 1)$.
16. $-17 - (5 - 10 - 13)$.
17. $-5 + (3 + 2 + 7)$.
18. $-(2 - 3 + 4) - 1$.
19. $-(10 - 8 - 12 + 10)$.
20. $+(8 - 2 - 3 - 1)$.

If $a = 1$, $b = 2$, and $c = -3$, find the value of

21. $a + b + c$.
22. $a - b + c$.
23. $a - b - c$.
24. $a - (-b) + c$.
25. $a - (-b) - c$.
26. $(-a) + (-b) + (-c)$.

3 + 4 = 7

4 - -1

CHAPTER II.

ADDITION AND SUBTRACTION.

INTEGRAL EXPRESSIONS.

50. If an algebraic expression contains only *integral forms*, that is, contains no *letter in the denominator of any of its terms*, it is called an **integral expression.** Thus, $x^3+7cx^2-c^3-5c^2x$, $\frac{1}{2}ax-\frac{1}{3}bcy$, are integral expressions, but $\frac{2c^2-4b-c-x}{a^2-ab+b^2}$ is a **fractional expression.**

An integral expression may have for some values of the letters a fractional value, and a fractional expression an integral value. If, for instance, a stands for $\frac{3}{4}$ and b for $\frac{1}{4}$, the integral expression $2a-5b$ stands for $\frac{6}{4}-\frac{5}{4}=\frac{1}{4}$; and the fractional expression $\frac{5a}{3b}$ stands for $\frac{15}{4}\div\frac{3}{4}=5$. Integral and fractional expressions, therefore, are so named on account of the *form of the expressions*, and with no reference whatever to the numerical value of the expressions when definite numbers are put in place of the letters.

51. A term may consist of a single symbol, as a, or may be the product of two or more factors, as $6a$, ab, $5a^2bc$. If one of the factors is an arithmetical symbol, as the factor 5 in $5a^2bc$, this factor is usually written first, and is called the **coefficient** of the term; the other factors are called **literal** factors.

NOTE. By way of distinction, a factor expressed by an arithmetical figure is called a *numerical* factor, and a factor expressed by a letter is called a *literal* factor.

52. Like Terms. Terms which have the same combination of *literal factors* are called **like** or **similar** terms; terms which do not have the same combination of literal factors are called **unlike** or **dissimilar** terms. Thus, $5a^2bc$, $-7a^2bc$, a^2bc, are like terms, but $5a^2bc$, $5ab^2c$, $5abc^2$, are unlike terms.

53. A simple expression, that is, an expression of one term, is called a **monomial.** A compound expression, that is, an expression which contains two or more terms, is called a **polynomial.** A polynomial which contains two terms is called a **binomial,** and a polynomial which contains three terms is called a **trinomial.**

54. A polynomial is said to be *arranged* according to the powers of some letter when the exponents of that letter either descend or ascend in the order of magnitude. Thus, $3ax^3 - 4bx^2 - 6ax + 8b$ is arranged according to the descending powers of x, and $8b - 6ax - 4bx^2 + 3ax^3$ is arranged according to the ascending powers of x.

55. Addition of Integral Expressions. The addition of two algebraic expressions can be represented by connecting the second expression with the first by the sign $+$. If there are no like terms in the two expressions, the operation is *algebraically complete* when the two expressions are thus connected.

If, for example, it is required to add $m + n - p$ to $a + b + c$, the result will be $a + b + c + (m + n - p)$; or, removing the parenthesis (§ 20), $a + b + c + m + n - p$.

56. If, however, there are like terms in the expressions to be added, the like terms can be *collected;* that is, every set of like terms can be replaced by a single term with a coefficient equal to the algebraic sum of the coefficients of the like terms.

If it is required to add $5a^2+4a+3$ to $2a^2-3a-4$, the result will be

$$2a^2-3a-4+(5a^2+4a+3)$$
$$=2a^2-3a-4+5a^2+4a+3 \qquad \S\ 20$$
$$=2a^2+5a^2-3a+4a-4+3 \qquad \S\ 26$$
$$=7a^2+a-1.$$

This process is more conveniently represented by arranging the terms in columns, so that like terms shall stand in the same column, as follows:

$$\begin{array}{r} 2a^2-3a-4 \\ 5a^2+4a+3 \\ \hline 7a^2+\ \ a-1 \end{array}$$

The coefficient of a^2 in the result will be $5+2$, or 7; the coefficient of a will be $-3+4$, or 1; and the last term is $-4+3$, or -1.

NOTE. When the coefficient of a term is 1, it is not written, but understood; conversely, when the coefficient of a term is not written, 1 is understood for its coefficient.

If we are to find the sum of $2a^3-3a^2b+4ab^2+b^3$, $a^3+4a^2b-7ab^2-2b^3$, $-3a^3+a^2b-3ab^2-4b^3$, and $2a^3+2a^2b+6ab^2-3b^3$, we write them in columns.

$$\begin{array}{rrrr} 2a^3 & -3a^2b & +4ab^2 & +\ b^3 \\ a^3 & +4a^2b & -7ab^2 & -2b^3 \\ -3a^3 & +\ a^2b & -3ab^2 & -4b^3 \\ 2a^3 & +2a^2b & +6ab^2 & -3b^3 \\ \hline 2a^3 & +4a^2b & & -8b^3 \end{array}$$

The coefficient of a^3 in the result will be $2+1-3+2$, or $+2$; the coefficient of a^2b will be $-3+4+1+2$, or $+4$; the coefficient of ab^2 will be $4-7-3+6$, or 0, and, therefore, the term ab^2 will not appear in the result; and the coefficient of b^3 will be $1-2-4-3$, or -8.

Exercise 4.

Add

1. $7a$, $2a$, $-3a$, and $-5a$.

2. $7xy$, $2xy$, $-4xy$, and $-5xy$.

3. $4a^2b$, $-3a^2b$, and $-5a^2b$.

4. $3xy$, $4xy$, $7ax$, and $-3ax$.

5. $a+b$ and $a-b$.

6. x^2-x and x^3-x^2.

7. $5x^2+6x-2$ and $3x^2-7x+2$.

8. $3x^2-2xy+y^2$ and $x^2-2xy+3y^2$.

9. ax^2+bx-4, $3ax^2-2bx+4$, and $-4ax^2-2bx+5$.

10. $5x+3y+z$, $3x+2y+3z$, and $x-3y-5z$.

11. $3ab-2ax^2+3a^2x$, $4ab-6a^2x+5ax^2$, $ab+a^2x-ax^2$, and $ax^2-8ab-5a^2x$.

12. $a^4-2a^3+3a^2-a+7$, $2a^4-3a^3+2a^2-a+6$, and $-a^4-2a^3+2a^2-5$.

13. $3a^2-ab+ac-3b^2+4bc-c^2$, $-5a^2-ab-ac+5bc$, $-4bc+5c^2+2ab$, and $-4a^2+b^2-5bc+2c^2$.

14. $x^4-3x^3+2x^2-4x+7$, $3x^4+2x^3+x^2-5x-6$, $-4x^4+3x^3-3x^2+9x-2$, and $2x^4-x^3+x^2-x+1$.

15. $3xy^2-4x^2y+x^3$, $5x^3-11xy^2-12xz^2$, $-7y^3+x^2y-xz^2$, and $-4xz^2+y^2-z^3$.

16. $a^4-2a^3+3a^2$, a^3+a^2+a, $4a^4+5a^3$, $2a^2+3a-2$, and $-a^2-2a-3$.

17. $x^3+2xy^2-x^2y-y^3$, $2x^3-3x^2y-4xy^2-7y^3$, and $x^3-8xy^2-7y^3$.

57. Subtraction of Integral Expressions. The subtraction of one expression from another, if none of the terms are alike, can be represented only by connecting the subtrahend with the minuend by means of the sign $-$.

If, for example, it is required to subtract $a+b+c$ from $m+n-p$, the result will be represented by

$$m+n-p-(a+b+c);$$

or, removing the parenthesis, § 20,

$$m+n-p-a-b-c.$$

If, however, some of the terms in the two expressions are alike, we can replace two like terms by a single term.

Thus, suppose it is required to subtract a^3-2a^2+2a-1 from $3a^3-2a^2+a-2$; the result may be expressed as follows:

$$3a^3-2a^2+a-2-(a^3-2a^2+2a-1);$$

or, removing the parenthesis (§ 20),

$$\begin{aligned} &3a^3-2a^2+a-2-a^3+2a^2-2a+1\\ &\quad=3a^3-a^3-2a^2+2a^2+a-2a-2+1\\ &\quad=2a^3-a-1. \end{aligned}$$

This process is more easily performed by writing the subtrahend below the minuend, mentally changing the sign of each term in the subtrahend, and adding the two expressions. Thus, the above example may be written

$$\begin{array}{rrrr} 3a^3-2a^2+ & a-2 \\ a^3-2a^2+ & 2a-1 \\ \hline 2a^3 \qquad - & a-1 \end{array}$$

The coefficient of a^3 will be $3-1$, or 2; the coefficient of a^2 will be $-2+2$, or 0, and therefore the term a^2 will not appear in the result; the coefficient of a will be $1-2$, or -1; the last term will be $-2+1$, or -1.

Again, suppose it is required to subtract $a^5 + 4a^3x^2 - 3a^2x^3 - 4ax^4$ from $a^3x^2 + 2a^2x^3 - 4ax^4$. Here terms which are alike can be written in columns, as before:

$$\begin{array}{r} a^3x^2 + 2a^2x^3 - 4ax^4 \\ a^5 + 4a^3x^2 - 3a^2x^3 - 4ax^4 \\ \hline -a^5 - 3a^3x^2 + 5a^2x^3 \end{array}$$

There is no term a^5 in the minuend, hence the coefficient of a^5 in the result is $0-1$, or -1; the coefficient of a^3x^2 will be $1-4$, or -3; the coefficient of a^2x^3 will be $2+3$, or $+5$; the coefficient of ax^4 will be $-4+4$, or 0, and therefore the term ax^4 will not appear in the result.

Exercise 5.

1. From $8a-4b-2c$ take $2a-3b-3c$.
2. From $3a-4b+3c$ take $2a-8b-c-d$.
3. From $7a^2-9x-1$ take $5a^2-6x-3$.
4. From $2x^2-2ax+a^2$ take x^2-ax-a^2.
5. From $4a-3b-3c$ take $2a-3b+4c$.
6. From $5x^2+7x+4$ take $3x^2-7x+2$.
7. From $2ax+3by+5$ take $3ax-3by-5$.
8. From $4a^2-6ab+2b^2$ take $3a^2+ab+b^2$.
9. From $4a^2b+7ab^2+9$ take $8-3ab^2$.
10. From $5a^2c+6a^2b-8a^3$ take $b^3+6a^2b-5a^2c$.
11. From a^2-b^2 take b^2.
12. From a^2-b^2 take a^2.
13. From b^2 take a^2-b^2.
14. From a^2 take a^2-b^2.
15. From $x^4+3ax^3-2bx^2+3cx-4d$ take $3x^4+ax^3-4b^2+6cx+d$.

If $A = 3a^2 - 2ab + 5b^2$, $C = 7a^2 - 8ab + 5b^2$,
$B = 9a^2 - 5ab + 3b^2$, $D = 11a^2 - 3ab - 4b^2$,
find the expression for

16. $A + C + B + D$.
17. $A - C - B + D$.
18. $C - A - B + D$.
19. $A + C - B - D$.
20. $A - C + B + D$.
21. $A + C - B + D$.

58. Parentheses. From the laws of parentheses (§ 20), we have the following equivalent expressions:

$$a + (b + c) = a + b + c, \quad \therefore a + b + c = a + (b + c);$$
$$a + (b - c) = a + b - c, \quad \therefore a + b - c = a + (b - c);$$
$$a - (b + c) = a - b - c, \quad \therefore a - b - c = a - (b + c);$$
$$a - (b - c) = a - b + c, \quad \therefore a - b + c = a - (b - c);$$

that is, if a parenthesis is preceded by the sign +, the parenthesis may be removed *without changing the sign of any term*; conversely, any number of terms may be enclosed within a parenthesis preceded by the sign +, *without changing the sign of any term.*

If a parenthesis is preceded by the sign −, the parenthesis may be removed, *provided the sign of every term within the parenthesis is changed*, namely, + to − and − to +; conversely, any number of terms may be enclosed within a parenthesis preceded by the sign −, *provided the sign of every term enclosed is changed.*

59. Expressions may occur having parentheses within parentheses. In such cases parentheses of different shapes are used, and the beginner, when he meets with one branch of a parenthesis (, or bracket [, or brace {, must look carefully for the other part, whatever may intervene; and all that is included between the two parts of each parenthesis must be treated as the sign before it directs, without regard to other parentheses. It is best to remove each parenthesis in succession, *beginning with the innermost one.* Thus,

(1)
$$\begin{aligned} & a-[b-(c-d)+e] \\ =\ & a-[b-c+d+e] \\ =\ & a-b+c-d-e. \end{aligned}$$

(2)
$$\begin{aligned} & a-\{b-[c-(d-e)+f]\} \\ =\ & a-\{b-[c-d+e+f]\} \\ =\ & a-\{b-c+d-e-f\} \\ =\ & a-b+c-d+e+f. \end{aligned}$$

Exercise 6.

Simplify the following by removing the parentheses and collecting like terms:

1. $a-b-[a-(b-c)-c]$.
2. $m-[n-(p-m)]$.
3. $2x-\{y+[4z-(y+2x)]\}$.
4. $3a-\{2b-[5c-(3a+b)]\}$.
5. $a-\{b+[c-(d-b)+a]-2b\}$.
6. $3x-[9-(2x+7)+3x]$.
7. $2x-[y-(x-2y)]$.
8. $a-[2b+(3c-2b)+a]$.
9. $(a-x+y)-(b-x-y)+(a+b-2y)$.
10. $3a-[-4b+(4a-b)-(2a-5b)]$.
11. $4c-[a-(2b-3c)+c]+[a-(2b-5c-a)]$.
12. $x+(y-z)-[(3x-2y)+z]+[x-(y-2z)]$.
13. $a-[2a+(a-2a)+2a]-5a-\{6a-[(a+2a)-a]\}$.

14. $2x-(3y+z)-\{b-(c-b)+c-[a-(c-b)]\}.$

15. $a-[b+c-a-(a+b)-c]+(2a-\overline{b+c}).$

NOTE. The sign $-$ which is written in the above problem before the first term b under the vinculum is really the sign of the vinculum, $-\overline{b+c}$ meaning the same as $-(b+c)$.

16. $10-x-\{-x-[x-(x-\overline{5-x})]\}.$

17. $2x-\{2x+(y-z)-3z+[2x-(y-\overline{z-2y})-3z]+4y\}.$

18. $a-[b-\{-c+a-(a-b)-c\}]+[2a-(b-a)].$

19. $a-\{b-[a-(c-b)+\overline{c-a}-(a-b-c)-a]+a\}.$

20. $5a-\{-3a-[3a-(2a-\overline{a-b})-a]+a\}.$

Exercise 7.

In each of the following expressions enclose the last three terms in a parenthesis preceded by the sign $-$, remembering that the sign of each term enclosed must be changed.

1. $2a-b-3c+d+3e-5f.$
2. $x-a-y-b-z-c.$
3. $a+b-c+4a-b+1.$
4. $ax+by+cz+bx-cy+cz.$
5. $3a+2b+2c-5d-3e-4f.$
6. $x-y+z-5xy-4xz+3yz.$

Considering all the factors that precede x, y, and z, respectively as the *coefficients* of these letters, we may collect in parentheses the coefficients of x, y, and z in the following expression:

$$ax-by+ay-az-cz+bx=(a+b)x+(a-b)y-(a+c)z.$$

In like manner, collect the coefficients of x, y, and z in the following expressions:

7. $ax+by+cz+bx-cy+az.$
8. $ax+2ay+4az-bx+3y-3bz-2z.$

9. $ax - 2by - 5cz - 4bx + 3cy - 7az$.

10. $ax + 3ay + 2by - bz - 11cx + 2cy - cz$.

11. $4by - 3ax - 6cz + 2bx - 7cx - 5cy - cx - cy - cz$.

12. $6az - 5by + 3cz - 2bz - 3ay + bz - ax + by$.

13. $z - by + 3az - 3cy + 2ax - 2mx - 5bz$.

14. $x + ay - az - acx + bcz - mny - y - z$.

Exercise 8.

Examples for Review.

1. Add $4x^3 - 5a^2 - 5ax^2 + 6a^2x$, $6a^3 + 3x^3 + 4ax^2 + 2a^2x$, $19ax^2 - 11x^2 - 15a^2x$, and $10x^3 + 7a^2x + 5a^3 - 18ax^2$.

2. Add $3ab + 3a + 6b$, $-ab + 2a + 4b$, $7ab - 4a - 8b$, and $6a + 12b - 2ab$.

Note. *Similar compound expressions* are added in precisely the same way as simple expressions, by finding the sum of their coefficients. Thus, $3(x - y) + 5(x - y) - 2(x - y) = 6(x - y)$.

3. Add $4(5 - x)$, $6(5 - x)$, $3(5 - x)$, and $-2(5 - x)$.

4. Add $(a + b)x^2 + (b + c)y^2 + (a + c)z^2$, $(b + c)x^2 + (a + c)y^2 + (a + b)z^2$, and $(a + c)x^2 + (a + b)y^2 + (b + c)z^2$.

5. Add $(a + b)x + (b + c)y + (c + a)z$, $(b + c)z + (c + a)x - (a + b)y$, and $(a + c)y + (a + b)z - (b + c)x$.

6. From $a^3 - x^2$ take $a^3 + 2ax + x^2$.

7. From $3a^2 + 2ax + x^2$ take $a^2 - ax - x^2$.

8. From $8x^2 - 3ax + 5$ take $5x^2 + 2ax + 5$.

9. From $a^3 + 3b^2c + ab^2 - abc$ take $ab^2 - abc + b^3$.

10. From $(a + b)x + (a + c)y$ take $(a - b)x - (a - c)y$.

11. Simplify $7a - \{3a - [4a - (5a - 2a)]\}$.

12. Simplify $3a-\{a+b-[a+b+c-(a+b+c+d)]\}$.

13. Bracket the coefficients, and arrange according to the descending powers of x
$$x^2-ax-c^2x^2-bx+bx^3-cx^2+a^2x^3-x^2-cx.$$

14. Simplify $a^2-(b^2-c^2)-[b^2-(c^2-a^2)]+[c^2-(b^2-a^2)]$.

15. If $a=1$, $b=3$, $c=5$, and $d=7$, find the value of $a-2b-\{3c-d-[3a-(5b-c-8d)]-2b\}$.

16. From $2d+11a+10b-5c$ take $2c+5a-3b$. Find the value of each of these expressions when a, b, c, and d have the values 1, 3, 5, 7, respectively, and show that the difference of these values is equal to the value of their difference.

17. If $a=1$, $b=-3$, $c=-5$, $d=0$, find the value of $a^2+2b^2+3c^2+4d^2$.

If $a=3$, $b=4$, $c=9$, and $2s=a+b+c$, find the value of

18. $s(s-a)(s-b)(s-c)$.

19. $s^2+(s-a)^2+(s-b)^2+(s-c)^2$.

20. $s^2-(s-a)(s-b)-(s-b)(s-c)-(s-c)(s-a)$.

21. If $x=a+2b-3c$, $y=b+2c-3a$, and $z=c+2a-3b$, show that $x+y+z=0$.

22. If $x=a-2b+3c$, $y=b-2c+3a$, and $z=c-2a+3b$, show that $x+y+z=2a+2b+2c$.

23. What must be added to $x^2+5y^2+3z^2$ in order that the sum may be $2y^2-z^2$?

24. What must be added to $5a^3-7a^2b+3ab^2$ in order that the sum may be $a^3-2a^2b-2ab^2+b^3$?

25. If $E=5a^3+3a^2b-2b^3$, $F=3a^3-7a^2b-b^3$,
$G=2a^2b-a^3-b^3$, $H=a^2b-2a^3-3b^3$,
find the expression for $E-[F-(G-H)]$.

CHAPTER III.

MULTIPLICATION.

INTEGRAL EXPRESSIONS.

60. The laws which govern the operation of multiplication are formulated as follows: § 47

$ab = ba$ . . . . The commutative law.

$a \times (bc) = (ab) \times c = abc$ . . The associative law.

$\left.\begin{array}{l} a(b+c) = ab + ac \\ a(b-c) = ab - ac \end{array}\right\}$ . . . . The distributive law.

$a^m \times a^n = a^{m+n}$ . . . . The index law.

$\left.\begin{array}{ll} a \times (+b) & = +ab \\ a \times (-b) & = -ab \\ (-a) \times b & = -ab \\ (-a) \times (-b) & = +ab \end{array}\right\}$ . . . The law of signs.

61. Multiplication of Monomials. When the factors are single letters, the product is represented by simply writing the letters without any sign between them. Thus, the product of a, b, and c is expressed by abc.

62. The product of $4a$, $5b$, and $3c$ is

$$4a \times 5b \times 3c = 4 \times 5 \times 3abc = 60abc.$$

NOTE. We cannot write 453 for $4 \times 5 \times 3$ because another meaning has been assigned in Arithmetic to 453, namely, $400 + 50 + 3$. Hence, between arithmetical factors the sign $\times$ must be written.

63. The product of a^2b and a^3b^2 is

$$a^2b \times a^3b^2 = a^2a^3bb^2 = a^{2+3}\,b^{1+2} = a^5b^3.$$

64. To multiply one monomial by another, therefore,

Find the product of the coefficients, and to this product annex the letters, giving to each letter in the product an index equal to the sum of its indices in the factors.

NOTE. The beginner should determine first the *sign* of the product by the law of signs, and write it down; secondly, after the sign he should write the product of the coefficients; and lastly, each letter with an index equal to the sum of its indices in the factors.

65. We may have an index affecting an expression as well as an index of a single letter. Thus, $(abc)^2$ means $abc \times abc$, which equals $aabbcc$, or $a^2b^2c^2$. In like manner, $(abc)^n = a^nb^nc^n$. That is,

The nth power of the product of several factors is equal to the product of the nth powers of the factors.

66. By the law of signs, we have

$$(-a) \times (-b) = +ab,$$

and $\quad (+ab) \times (-c) = -abc,$

that is, $\quad (-a) \times (-b) \times (-c) = -abc;$

and $\quad (-abc) \times (-d) = +abcd,$

that is, $\quad (-a) \times (-b) \times (-c) \times (-d) = +abcd.$

It is obvious, therefore, that

The product of an *even* number of negative factors will be *positive*, and the product of an *odd* number of negative factors will be *negative*.

67. Polynomials by Monomials. We have (§ 47),

$$a(b+c)=ab+ac.$$

In like manner,

$$a(b-c+d-e)=ab-ac+ad-ae.$$

To multiply a polynomial by a monomial, therefore,

Multiply each term of the polynomial by the monomial, and add the partial products.

Exercise 9.

Find the product of

1. $7c$ and $5b$.
2. $3x$ and $8y$.
3. $3a^2$ and $6a^3$.
4. $3a$ and $2a^5$.
5. $2mn$ and $3m^2n$.
6. $7ab$ and $3ac$.
7. $-2a$ and $7a^3x^2y$.
8. $-3a^2b$ and $-3ab^2$.
9. $-5mnp^2$ and $-4m^2n^2p^2$.
10. $-8a^2$, $-2b^2$, and $-3ab$.
11. $-2x^2y$, x^5y^3, and $-3x^2y$.
12. $-3x^5y$, $-2a^2b$, and $-x^2y^7a^3b^9$.
13. $5a+3b$ and $2a^2$.
14. $ab-bc$ and $5a^3bc$.
15. $ab-ac-bc$ and abc.
16. $6a^5b-7a^2b^2c$ and a^2b^2c.
17. $a^2+b^2-c^2$ and a^3bc^2.
18. $5a^2-3b^2+2c^2$ and $4ab^3c^2$.
19. $abc-3a^3bc^2$ and $-2ab^2c$.
20. $-xyz^2+x^2y^3z$ and $-x^2yz$.
21. $-2m^2np^3-mnp^2$ and $-m^2np$.

22. $x-y-z$ and $-3x^5y^7z^9$.

23. $-3x^2$ and x^2+2y^2-z.

24. $3x-2y-4$ and $5x^2$.

68. **Polynomials by Polynomials.** If we have $m+n+p$ to be multiplied by $a+b+c$, we may substitute M for the multiplicand $m+n+p$ (§ 12). Then

$$(a+b+c)M=aM+bM+cM. \qquad \text{§ 28}$$

If now we substitute for M its value $m+n+p$, we have

$$\begin{aligned}&a(m+n+p)+b(m+n+p)+c(m+n+p)\\&=am+an+ap+bm+bn+bp+cm+cn+cp.\end{aligned}$$

To find the product of two polynomials, therefore,

Multiply every term of the multiplicand by each term of the multiplier, and add the partial products.

69. In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

(1) Multiply $5a-6b$ by $3a-4b$.

$$\begin{array}{lll}5a & -\ 6b & \\ 3a & -\ 4b & \\ \hline 15a^2 & -18ab & \\ & -20ab & +24b^2 \\ \hline 15a^2 & -38ab & +24b^2\end{array}$$

We multiply $5a$, the first term of the multiplicand, by $3a$, the first term of the multiplier, and obtain $15a^2$; then $-6b$, the second term of the multiplicand, by $3a$, and obtain $-18ab$. The first line of partial products is $15a^2-18ab$. In multiplying by $-4b$, we obtain for a second line of partial products $-20ab+24b^2$, which is put one place to the right, so that the like terms $-18ab$ and

$-20ab$ may stand in the same column. We then add the coefficients of the like terms, and obtain the complete product in its simplest form.

(2) Multiply $4x + 3 + 5x^2 - 6x^3$ by $4 - 6x^2 - 5x$.

Arrange both multiplicand and multiplier according to the ascending powers of x.

$$
\begin{array}{llllll}
3 + & 4x + & 5x^2 - & 6x^3 & & \\
4 - & 5x - & 6x^2 & & & \\
\hline
12 + & 16x + & 20x^2 - & 24x^3 & & \\
 & -15x - & 20x^2 - & 25x^3 + & 30x^4 & \\
 & & -18x^2 - & 24x^3 - & 30x^4 + & 36x^5 \\
\hline
12 + & x - & 18x^2 - & 73x^3 & & +36x^5
\end{array}
$$

(3) Multiply $1 + 2x + x^4 - 3x^2$ by $x^3 - 2 - 2x$.

Arrange according to the descending powers of x.

$$
\begin{array}{llllllll}
x^4 - 3x^2 + 2x + 1 & & & & & & & \\
x^3 - 2x - 2 & & & & & & & \\
\hline
x^7 & -3x^5 & +2x^4 & +x^3 & & & & \\
 & -2x^5 & & +6x^3 & -4x^2 & -2x & & \\
 & & -2x^4 & & +6x^2 & -4x & -2 & \\
\hline
x^7 & -5x^5 & & +7x^3 & +2x^2 & -6x & -2 &
\end{array}
$$

(4) Multiply $a^2 + b^2 + c^2 - ab - bc - ac$ by $a + b + c$.

Arrange according to descending powers of a.

$$
\begin{array}{llllllllll}
a^2 - ab - ac + b^2 - bc + c^2 & & & & & & & & & \\
a + b + c & & & & & & & & & \\
\hline
a^3 & -a^2b & -a^2c & +ab^2 & -abc & +ac^2 & & & & \\
 & +a^2b & & -ab^2 & -abc & & +b^3 & -b^2c & +bc^2 & \\
 & & +a^2c & & -abc & -ac^2 & & +b^2c & -bc^2 & +c^3 \\
\hline
a^3 & & & & -3abc & & +b^3 & & & +c^3
\end{array}
$$

NOTE. The student should observe that, with a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier are arranged in the *same order.*

70. A term that is the product of three letters is said to be of three *dimensions*, or of the third *degree.* In general, a term that is the product of n letters is said to be of n *dimensions*, or of the *nth degree.* Thus, $5abc$ is of three dimensions, or of the third degree; $2a^2b^2c^2$, that is, $2aabbcc$, is of six dimensions, or of the sixth degree.

71. The *degree of a compound algebraic expression* is the degree of that term of the expression which is of *highest dimensions.*

72. When all the terms of a compound expression are of the same degree, the expression is said to be **homogeneous.** Thus, $x^3+3x^2y+3xy^2+y^3$ is a homogeneous expression, every term being of the third degree.

73. *The product of two homogeneous expressions is homogeneous.* For the different terms of the product are found by multiplying every term of the multiplicand by each term of the multiplier; and the number of dimensions of each partial product is the sum of the number of dimensions of a term of the multiplicand and of a term of the multiplier counted together. Thus, in multiplying $a^2+b^2+c^2-ab-bc-ac$ by $a+b+c$, Example (4), each term of the multiplicand is of two dimensions, and each term of the multiplier is of one dimension; we therefore have each term of the product of $2+1$, that is, three dimensions.

This fact affords an important test of the accuracy of the work of multiplication with respect to the *literal factors;* for, if any term in the product is of a degree different from the degree of the other terms, there is an error in the work of finding that term.

74. Any expression that is not homogeneous can be made so by introducing a letter, the value of which is unity. Thus, in Example (3), the expressions can be written $x^4 - 3a^2x^2 + 2a^3x + a^4$ and $x^3 - 2a^2x - 2a^3$. The product will then be $x^7 - 5a^2x^5 + 7a^4x^3 + 2a^5x^2 - 6a^6x - 2a^7$, which reduces to the product given in the example, by putting 1 for a.

75. It often happens in algebraic investigations that there is one letter in an expression of more importance than the rest, and this is therefore called the **leading letter.** In such cases the degree of the expression is generally called by the degree of the *leading letter*. Thus, a^2x^2+bx+c is of the *second degree in* x.

Exercise 10.

Find the product of

1. $x + 10$ and $x + 6$.
2. $x - 2$ and $x - 3$.
3. $x - 3$ and $x + 5$.
4. $x + 3$ and $x - 3$.
5. $x - 11$ and $x - 1$.
6. $-x + 2$ and $-x - 3$.
7. $-x - 2$ and $x - 2$.
8. $-x + 4$ and $x - 4$.
9. $-x + 7$ and $x + 7$.
10. $x - 7$ and $x + 7$.
11. $x - 3$ and $2x + 3$.
12. $2x - 3$ and $x + 3$.
13. $x - 7$ and $2x - 1$.
14. $m - n$ and $2m + 1$.
15. $m - a$ and $m + a$.
16. $3x + 7$ and $2x - 3$.
17. $5x - 2y$ and $5x + 2y$.
18. $3x - 4y$ and $2x + 3y$.
19. $x^2 + y^2$ and $x^3 - y^3$.
20. $2x^2 + 3y^2$ and $x^2 + y^2$.
21. $x + y + z$ and $x - y + z$.
22. $x + 2y - z$ and $x - y + 2z$.
23. $x^2 - xy + y^2$ and $x^2 + xy + y^2$.
24. $m^2 - mn + n^2$ and $m + n$.
25. $m^2 + mn + n^2$ and $m - n$.

26. $a^2 - 3ab + b^2$ and $a^2 - 3ab - b^2$.

27. $a^2 - 7a + 2$ and $a^2 - 2a + 3$.

28. $2x^2 - 3xy + 4y^2$ and $3x^2 + 4xy - 5y^2$.

29. $x^2 + xy + y^2$ and $x^2 - xz - z^2$.

30. $x^2 + y^2 + z^2 - xy - xz - yz$ and $x + y + z$.

31. $4a^2 - 10ab + 25a^2b^2$ and $5b + 2a$.

32. $x^2 + 4y$ and $y^2 + 4x$.

33. $x^2 + 2xy + 8$ and $y^2 + 2xy - 8$.

34. $a^2 + b^2 + 1 - ab - a - b$ and $a + 1 + b$.

35. $3x^2 - 2y^2 + 5z^2$ and $8x^2 + 2y^2 - 3z^2$.

36. $x^2 + y^2 + 2xy - 2x - 2y - 1$ and $x + y - 1$.

37. $a^m + 2a^{m-1} - 3a^{m-2} - 1$ and $a + 1$.

38. $a^n - 4a^{n-1} + 5a^{n-2} + a^{n-3}$ and $a - 1$.

39. $a^{4n+1} - 4a^{3n} + 2a^{2n-1} - a^{n-2}$ and $2a^3 - a^2 + a$.

40. $x^n - y^{n+1}$ and $x^n + y^{n+1}$.

Exercise 11.

Simplify:

1. $(a + b + c)(a + b - c) - (2ab - c^2)$.

2. $(m + n)m - [(m - n)^2 - n(n - m)]$.

3. $[ac - (a - b)(b + c)] - b[b - (a - c)]$.

4. $(x - 1)(x - 2) - 3x(x + 3) + 2[(x + 2)(x + 1) - 3]$.

5. $4(a - 3b)(a + 3b) - 2(a - 6b)^2 - 2(a^2 + 6b^2)$.

6. $(x + y + z)^2 - x(y + z - x) - y(x + z - y) - z(x + y - z)$.

7. $5[(a - b)x - cy] - 2[a(x - y) - bx]$
$- [3ax - (5c - 2a)y]$.

Exercise 12.

Examples for Review.

Multiply

1. x^2-x-19 by x^2+2x-3.
2. $1+2x+x^2$ by $1-x^3+2x^2-3x$.
3. $2x^2+2+3x$ by $2x-3x^2+2x^3$.
4. $3x^2+5-4x$ by $8+6x^2-7x$.
5. x^3+x-y by x^2-y^2+xy.
6. $3+7x^2-5x$ by $8x^2-6x-10x^3+4$
7. $b^3+6ab^2-4a^2b$ by $2a^2b-ab^2-8a^3$.
8. x^2+ax-b by x^2+5x-4.
9. x^3-mx^2+nx+r by x^2+cx+d.
10. $x^2-(a+b)x+ab$ by $x-c$.
11. $x^3+x^2y+xy^2+y^3$ by $y-x$.
12. $4x^2+9y^2-6xy$ by $4x^2+9y^2+6xy$.
13. x^4-3x^2+5 by x^2+4.
14. $x^4-x^2y^2+y^4$ by $x^4+x^2y^2+y^4$.
15. $2x^5-3x^3+4x^2-5$ by x^2-8.
16. $a^{2m}-a^my^m+y^{2m}$ by a^m+y^m.
17. $a^{3x}-a^{2x}+a^x-1$ by a^x+1.
18. $a^2+b^2+c^2+2ab-ac-bc$ by $a+b+c$.
19. Simplify $(a-2b)(b-2a)-(a-3b)(4b-a)+2ab$.
20. If $a=0$, $b=1$, and $c=-1$, find the value of
$$(a-b)(a-c)+c(3a-b-c)+2ac-(a-c)+2b.$$
21. $[(2x+y)^2+(x-2y)^2][(3x-2y)^2-(2x-3y)^2]$.
22. $a^2(b-c)-b^2(a-c)+c^2(a-b)-(a-b)(a-c)(b-c)$.
23. $(2a-b)^2+2b(a+b)-3a^2-(a-b)^2+(a+b)(a-b)$.

CHAPTER IV.

DIVISION.

INTEGRAL EXPRESSIONS.

76. The laws for division are expressed in symbols, as follows:

$$\left.\begin{aligned}+ab \div (+a) &= +b\\ -ab \div (+a) &= -b\\ +ab \div (-a) &= -b\\ -ab \div (-a) &= +b\end{aligned}\right\} \text{ Law of signs.} \qquad \S\, 44$$

$$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a} \quad . \quad . \quad . \quad \text{Distributive law.} \quad \S\, 47, (5)$$

77. The dividend contains all the factors of the divisor and of the quotient, and therefore the quotient contains the factors of the dividend that are not found in the divisor.

Thus, $\frac{abc}{bc} = a$; $\frac{aabx}{ab} = ax$; $\frac{124abc}{-4ab} = -31c$.

78. If we have to divide a^5 by a^2, a^6 by a^4, a^4 by a, we write them as follows:

$$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = aaa = a^3 = a^{5-2};$$

$$\frac{a^6}{a^4} = \frac{aaaaaa}{aaaa} = aa = a^2 = a^{6-4};$$

$$\frac{a^4}{a} = \frac{aaaa}{a} = aaa = a^3 = a^{4-1}.$$

79. **To divide one monomial by another,** therefore, we have the following rule:

Divide the coefficient of the dividend by the coefficient of the divisor (observing the law of signs), and subtract the index of any letter in the divisor from the index of that letter in the dividend.

Thus, $\frac{4x^2y}{2x}=2xy$; $\frac{14a^2b}{2a}=7ab$; $\frac{54x^5y^3z^2}{-6x^3y^2z}=-9x^2yz$;

$\frac{3a^{4n-1}}{3a^{4n-1}}=1$; $\frac{12x^{p-4}y^{r+3}}{3x^{p-6}y^{r-1}}=4x^2y^4$.

NOTE. Since $\frac{a^n}{a^n}=1$, and also by the rule above given, $\frac{a^n}{a^n}=a^{n-n}=a^0$, it follows that $a^0=1$. Hence, any *letter* which by the rule would appear in the quotient *with zero for an index*, may be omitted without affecting the quotient.

80. **To divide a polynomial by a monomial,** we have, by the *distributive law*, the following rule:

Divide each term of the dividend by the divisor, and add the partial quotients.

Thus, $$\frac{8ab+4ac-6ad}{2a}=\frac{8ab}{2a}+\frac{4ac}{2a}-\frac{6ad}{2a}$$
$$=4b+2c-3d.$$

$$\frac{9a^4b^2x-12a^3bx^2-3a^2x}{3a^2x}=\frac{9a^4b^2x}{3a^2x}-\frac{12a^3bx^2}{3a^2x}-\frac{3a^2x}{3a^2x}$$
$$=3a^2b^2-4abx-1.$$

$$\frac{6x^{4n+1}-4x^{3n}}{2x^{2n-1}}=\frac{6x^{4n+1}}{2x^{2n-1}}-\frac{4x^{3n}}{2x^{2n-1}}=3x^{2n+2}-2x^{n+1}.$$

NOTE. Here we have $4n+1-(2n-1)=4n+1-2n+1=2n+2$, and $3n-(2n-1)=3n-2n+1=n+1$, as indices of x in the first and last terms of the quotient respectively.

Exercise 13.

Divide

1. $3a^5$ by a^3.
2. $-42x^7$ by $6x^5$.
3. $-35z^8$ by $5z^3$.
4. $-6z^2$ by $-3z$.
5. $20x^7y^2$ by $-4xy^2$.
6. $-21x^4$ by $-7x^4$.
7. $28a^4b^3$ by $-7ab^2$.
8. $-25z^3b^2$ by $-5zb$.
9. $-24m^4n^3$ by $-4m^2n^2$.
10. $-35p^2q^2$ by $5pq^2$.
11. $-16r^2s^5$ by $-4rs^3$.
12. $28m^{10}n^{11}$ by $4m^8n^{10}$.
13. abx^2 by abx.
14. $-2a^5bx^4$ by a^3x.
15. $4abx^2$ by $-ax^2$.
16. $18b^7x^{11}$ by $-9b^4x^7$.
17. $256x^9y^8z^7$ by $8x^3y^2z$.
18. $-50a^3b^3$ by $-10a^3b^3$.
19. $84b^2x^7y^5$ by $14xy^3$.
20. $-30a^3x^2z$ by $-6xz$.
21. x^2+2xy by x.
22. a^2-2ab by a.
23. $4x^6-8x^3$ by $2x^3$.
24. $-6x^3-2x$ by $-2x$.
25. $-8a^5-16a^{10}$ by $-8a^3$.
26. $27a^3-36a^2$ by $9a^2$.
27. $-30a^7+20a^9$ by $-10a^6$.
28. $-12x^6y^4-4x^2y^2$ by $-4x^2y^2$.
29. $-3x^7z^7-6x^5z^5$ by $-3x^3z^3$.
30. $3a^3b^3c^7-9a^6b^7c^5$ by $3a^3b^3c^5$.
31. $x^2-xy-xz$ by $-x$.
32. $3a^3-6a^2b-9ab^2$ by $-3a$.
33. $x^5y^2-x^3y^3-x^2y^2$ by x^2y^2.
34. $a^3b^2c-a^2b^3c-a^2bc^3$ by abc.
35. $8a^3-4a^2b-6ab^2$ by $-2a$.
36. $5m^3n-10m^2n^2-15mn^3$ by $5mn$.

81. To divide one polynomial by another.

If the divisor (one factor) $= a+b+c$,
and the quotient (other factor) $= n+p+q$,

$$\text{then the dividend (product)} = \begin{cases} an+bn+cn \\ +ap+bp+cp \\ +aq+bq+cq. \end{cases}$$

The first term of the dividend is an; that is, the product of a, the first term of the divisor, by n, the first term of the quotient. The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n be subtracted from the dividend, the first term of the remainder ap is the product of a, the first term of the divisor, by p, the second term of the quotient; that is, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. In like manner, the third term of the quotient is obtained by dividing the first term of the new remainder by the first term of the divisor; and so on. Hence we have the following rule:

Arrange both the dividend and divisor in ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor.

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there be a remainder, consider it as a new dividend and proceed as before.

82. It is of fundamental importance to arrange the dividend and divisor *in the same order* with respect to a common letter, and *to keep this order throughout the operation.*

The beginner should study carefully the processes in the following examples:

(1) Divide $x^2 + 18x + 77$ by $x + 7$.

$$\begin{array}{l|l} x^2 + 18x + 77 & x + 7 \\ \hline x^2 + 7x & x + 11 \\ \hline \quad 11x + 77 & \\ \quad \underline{11x + 77} & \end{array}$$

NOTE. The student will notice that by this process we have in effect separated the dividend into two parts, $x^2 + 7x$ and $11x + 77$, and divided each part by $x + 7$, and that the complete quotient is the sum of the partial quotients x and 11. Thus,

$$x^2 + 18x + 77 = x^2 + 7x + 11x + 77 = (x^2 + 7x) + (11x + 77);$$

$$\therefore \frac{x^2 + 18x + 77}{x + 7} = \frac{x^2 + 7x}{x + 7} + \frac{11x + 77}{x + 7} = x + 11.$$

(2) Divide $a^2 - 2ab + b^2$ by $a - b$.

$$\begin{array}{l|l} a^2 - 2ab + b^2 & a - b \\ \hline a^2 - ab & a - b \\ \hline \quad - ab + b^2 & \\ \quad \underline{- ab + b^2} & \end{array}$$

(3) Divide $4a^4x^2 - 4a^2x^4 + x^6 - a^6$ by $x^2 - a^2$.

Arrange according to descending powers of x.

$$\begin{array}{l|l} x^6 - 4a^2x^4 + 4a^4x^2 - a^6 & x^2 - a^2 \\ \hline x^6 - a^2x^4 & x^4 - 3a^2x^2 + a^4 \\ \hline \quad - 3a^2x^4 + 4a^4x^2 - a^6 & \\ \quad \underline{- 3a^2x^4 + 3a^4x^2} & \\ \qquad\qquad a^4x^2 - a^6 & \\ \qquad\qquad \underline{a^4x^2 - a^6} & \end{array}$$

(4) Divide $22a^2b^2 + 15b^4 + 3a^4 - 10a^3b - 22ab^3$ by $a^2 + 3b^2 - 2ab$.

Arrange according to descending powers of a.

$$
\begin{array}{l|l}
3a^4 - 10a^3b + 22a^2b^2 - 22ab^3 + 15b^4 & a^2 - 2ab + 3b^2 \\
\underline{3a^4 - 6a^3b + 9a^2b^2} & 3a^2 - 4ab + 5b^2 \\
\quad - 4a^3b + 13a^2b^2 - 22ab^3 & \\
\quad \underline{- 4a^3b + 8a^2b^2 - 12ab^3} & \\
\qquad\qquad 5a^2b^2 - 10ab^3 + 15b^4 & \\
\qquad\qquad \underline{5a^2b^2 - 10ab^3 + 15b^4} &
\end{array}
$$

(5) Divide $5x^3 - x + 1 - 3x^4$ by $1 + 3x^2 - 2x$.

Arrange according to ascending powers of x.

$$
\begin{array}{l|l}
1 - x + 5x^3 - 3x^4 & 1 - 2x + 3x^2 \\
\underline{1 - 2x + 3x^2} & 1 + x - x^2 \\
\quad x - 3x^2 + 5x^3 - 3x^4 & \\
\quad \underline{x - 2x^2 + 3x^3} & \\
\qquad - x^2 + 2x^3 - 3x^4 & \\
\qquad \underline{- x^2 + 2x^3 - 3x^4} &
\end{array}
$$

(6) Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

Arrange according to descending powers of x.

$$
\begin{array}{l|l}
x^3 - 3yzx + y^3 + z^3 & x + y + z \\
\underline{x^3 + yx^2 + zx^2} & x^2 - yx - zx + y^2 - yz + z^2 \\
\quad - yx^2 - zx^2 - 3yzx + y^3 + z^3 & \\
\quad \underline{- yx^2 - y^2x - yzx} & \\
\qquad - zx^2 + y^2x - 2yzx + y^3 + z^3 & \\
\qquad \underline{- zx^2 - yzx - z^2x} & \\
\qquad\qquad y^2x - yzx + z^2x + y^3 + z^3 & \\
\qquad\qquad \underline{y^2x + y^3 + y^2z} & \\
\qquad\qquad\quad - yzx + z^2x - y^2z + z^3 & \\
\qquad\qquad\quad \underline{- yzx - y^2z - yz^2} & \\
\qquad\qquad\qquad z^2x + yz^2 + z^3 & \\
\qquad\qquad\qquad \underline{z^2x + yz^2 + z^3} &
\end{array}
$$

(7) Divide $4a^{x+1} - 30a^{x} + 19a^{x-1} + 5a^{x-2} + 9a^{x-4}$
by $a^{x-3} - 7a^{x-4} + 2a^{x-5} - 3a^{x-6}$.

$$
\begin{array}{l|l}
4a^{x+1} - 30a^{x} + 19a^{x-1} + 5a^{x-2} + 9a^{x-4} & a^{x-3} - 7a^{x-4} + 2a^{x-5} - 3a^{x-6} \\
\underline{4a^{x+1} - 28a^{x} + 8a^{x-1} - 12a^{x-2}} & \overline{4a^{4} - 2a^{3} - 3a^{2}} \\
\quad - 2a^{x} + 11a^{x-1} + 17a^{x-2} + 9a^{x-4} & \\
\quad \underline{- 2a^{x} + 14a^{x-1} - 4a^{x-2} + 6a^{x-3}} & \\
\qquad - 3a^{x-1} + 21a^{x-2} - 6a^{x-3} + 9a^{x-4} & \\
\qquad \underline{- 3a^{x-1} + 21a^{x-2} - 6a^{x-3} + 9a^{x-4}} &
\end{array}
$$

NOTE. We find the index of a in the first term of the quotient by subtracting the index of a in the first term of the divisor from the index of a in the first term of the dividend. Now $(x+1) - (x-3) = x + 1 - x + 3 = 4$. Hence 4 is the index of a in the first term of the quotient. In the same way the other indices are found.

Exercise 14.

Divide

1. $a^2 + 7a + 12$ by $a + 4$.
2. $a^2 - 5a + 6$ by $a - 3$.
3. $x^2 + 2xy + y^2$ by $x + y$.
4. $x^2 - 2xy + y^2$ by $x - y$.
5. $x^2 - y^2$ by $x - y$.
6. $4x^2 + 12x + 9$ by $2x + 3$.
7. $6x^2 - 11x + 4$ by $3x - 4$.
8. $8x^2 - 10ax - 3a^2$ by $4x + a$.
9. $3a^2 - 4a - 4$ by $2 - a$.
10. $a^3 - 8a - 3$ by $3 - a$.
11. $a^4 + 11a^2 - 12a - 5a^3 + 6$ by $3 + a^2 - 3a$.
12. $y^4 - 9y^2 + y^3 - 16y - 4$ by $y^2 + 4 + 4y$.
13. $36 + m^4 - 13m^2$ by $6 + m^2 + 5m$.
14. $1 - s - 3s^2 - s^5$ by $1 + 2s + s^2$.
15. $b^6 - 2b^3 + 1$ by $b^2 - 2b + 1$.
16. $x^4 + 2x^2y^2 + 9y^4$ by $x^2 - 2xy + 3y^2$.
17. $a^5 + b^5$ by $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.
18. $1 + 5x^3 - 6x^4$ by $1 - x + 3x^2$.

19. $8x^2y^2+9y^4+16x^4$ by $4x^2+3y^2-4xy$.

20. $x^3+y^3+z^3+3x^2y+3xy^2$ by $x+y+z$.

21. $a^3+b^3+c^3-3abc$ by $a+b+c$.

22. $x^3+8y^3+z^3-6xyz$ by $x^2+4y^2+z^2-xz-2xy-2yz$.

23. $2x^2-3y^2+xy-xz-4yz-z^2$ by $2x+3y+z$.

24. $x^2-y^2-2yz-z^2$ by $x+y+z$.

25. $x^4+x^2y^2+y^4$ by x^2+xy+y^2.

26. $x^4-9x^2+12x-4$ by x^2+3x-2.

27. $y^5-2y^4-6y^3+4y^2+13y+6$ by y^3+3y^2+3y+1.

28. $y^4-5y^2z^2+4z^4$ by $y^2-yz-2z^2$.

29. $x^2-4y^2-9z^2+12yz$ by $x+2y-3z$.

30. $x^5-41x-120$ by x^2+4x+5.

31. $x^4-3+5x+x^3-4x^2$ by $3-2x-x^2$.

32. $6-2x^4+10x^3-11x^2+x$ by $4x-3-2x^2$.

33. $1-6x^5+5x^6$ by $1-2x+x^2$.

34. $x^4+81+9x^2$ by $3x-x^2-9$.

35. x^3-y^3 by x^2+xy+y^2.

36. x^6+y^6 by x^2+y^2.

37. $x^4+a^2x^2+a^4$ by x^2-ax+a^2.

38. $a^2-2b^2+ab-3c^2+7bc+2ac$ by $3c+a-b$.

39. $ab+2a^2-3b^2-4bc-ac-c^2$ by $c+2a+3b$.

40. $15a^4+10a^3x+4a^2x^2+6ax^3-3x^4$ by $3a^2+2ax-x^2$.

41. $a^3-8b^3-1-6ab$ by $a-2b-1$.

42. $x^{3n}-3x^{2n}y^n+3x^ny^{2n}-y^{3n}$ by x^n-y^n.

43. $a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}$
by $a^m+3a^{m-1}b^n-6a^{m-2}b^{2n}$.

83. Integral expressions may have *fractional coefficients*, since an algebraic expression is integral if it has no *letter* in the denominator. The processes with fractional coefficients are precisely the same as with integral coefficients, as will be seen by the following examples worked out:

(1) Add $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$, and $\frac{5}{6}a^2 + \frac{2}{3}ab - \frac{3}{4}b^2$.

$$\begin{array}{l} \frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2 \\ \frac{5}{6}a^2 + \frac{2}{3}ab - \frac{3}{4}b^2 \\ \hline \frac{4}{3}a^2 + \frac{1}{3}ab - \frac{1}{2}b^2 \end{array}$$

(2) From $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$ take $\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{5}{6}b^2$.

$$\begin{array}{l} \frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2 \\ \frac{1}{3}a^2 - \frac{1}{2}ab + \frac{5}{6}b^2 \\ \hline \frac{1}{6}a^2 + \frac{1}{6}ab - \frac{7}{12}b^2 \end{array}$$

(3) Multiply $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$ by $\frac{1}{2}a - \frac{2}{3}b$.

$$\begin{array}{lllll} \frac{1}{2}a^2 & - \frac{1}{3}ab & + \frac{1}{4}b^2 & & \\ \frac{1}{2}a & - \frac{2}{3}b & & & \\ \hline \frac{1}{4}a^3 & - \frac{1}{6}a^2b & + \frac{1}{8}ab^2 & & \\ & - \frac{1}{3}a^2b & + \frac{2}{9}ab^2 & - \frac{1}{6}b^3 \\ \hline \frac{1}{4}a^3 & - \frac{1}{2}a^2b & + \frac{25}{72}ab^2 & - \frac{1}{6}b^3 \end{array}$$

(4) Divide $\frac{2}{3}b^3 + \frac{11}{8}bd^2 - \frac{443}{270}b^2d - \frac{5}{12}d^3$ by $\frac{3}{2}b - \frac{5}{3}d$.

$$\begin{array}{l|l} \frac{2}{3}b^3 - \frac{443}{270}b^2d + \frac{11}{8}bd^2 - \frac{5}{12}d^3 & \frac{3}{2}b - \frac{5}{3}d \\ \cline{2-2} \frac{2}{3}b^3 - \frac{20}{27}b^2d & \frac{4}{9}b^2 - \frac{3}{5}bd + \frac{1}{4}d^2 \\ \cline{1-1} \quad - \frac{9}{10}b^2d + \frac{11}{8}bd^2 - \frac{5}{12}d^3 & \\ \quad - \frac{9}{10}b^2d + bd^2 & \\ \cline{1-1} \qquad \frac{3}{8}bd^2 - \frac{5}{12}d^3 & \\ \qquad \frac{3}{8}bd^2 - \frac{5}{12}d^3 & \\ \cline{1-1} \end{array}$$

Exercise 15.

1. Add $\frac{1}{2}a^2b + \frac{1}{3}b^2c^4 + \frac{2}{15}$ and $-\frac{3}{10}a^2b - \frac{1}{5}b^2c^4 - \frac{4}{9}$.
2. From $\frac{5}{2}x^2 + 3\,ax - \frac{7}{3}a^2$ take $2x^2 - \frac{3}{2}ax - \frac{1}{2}a^2$.
3. From $\frac{1}{2}y - \frac{5}{2}a - \frac{3}{4}x + \frac{1}{3}b$ take $\frac{1}{3}y + \frac{11}{4}a - \frac{2}{3}x$.
4. Multiply $\frac{1}{3}c^2 - \frac{1}{4}c - \frac{1}{2}$ by $\frac{1}{3}c^2 - \frac{1}{4}c + \frac{1}{2}$.
5. Multiply $\frac{1}{2}x - \frac{1}{3}x^2 + \frac{1}{4}x^3$ by $\frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3$.
6. Multiply $0.5\,m^4 - 0.4\,m^3n + 1.2\,m^2n^2 + 0.8\,mn^3 - 1.4\,n^4$ by $0.4\,m^2 - 0.6\,mn - 0.8\,n^2$.
7. Divide $\frac{9}{16}a^4 - \frac{7}{8}a^3b + \frac{19}{36}a^2b^2 + \frac{1}{6}ab^3$ by $\frac{3}{2}a + \frac{1}{3}b$.
8. Divide $-\frac{1}{8}d^5 + d^2 - \frac{41}{24}d^3 + \frac{5}{6}d^4$ by $-\frac{3}{4}d^2 + 2d$.

Exercise 16.

Examples for Review.

1. Find the value of $x^3 + y^3 + z^3 - 3\,xyz$, if $x = 1$, $y = 2$, and $z = -3$.
2. Find the value of $\sqrt{2\,bc} - a$, and of $\sqrt{2\,bc - a}$, if $b = 8$, $c = 9$, and $a = 23$.
3. Add $a^2b - ab^2 + b^3$ and $a^3 - \frac{1}{2}a^2b + ab^2 - \frac{3}{4}b^3$.
4. Multiply $a^{2m} - a^mb^m + b^{2m}$ by $a^m + b^m$.
5. Multiply $4\,a^{2m+4} + 6\,a^{m+3} + 9\,a^2$ by $2\,a^{m+4} - 3\,a^3$.
6. Divide $x^3 + 8\,y^3 - 125\,z^3 + 30\,xyz$ by $x + 2\,y - 5\,z$.
7. Simplify $(x - a)^2 - (x - b)^2 - (a - b)(a + b - 3\,x)$.
8. Find the coefficient of x in the expression $x + a - 2\,[2\,a - b(c - x)]$.
9. Multiply $4\,x^{m+2n-1} - 7\,x^{2m-3n+2} + 5\,x^{2n+3m-2}$ by $5\,x^{2-m-2n}$.

10. Divide $c^2d^{3-2x} - c^4d^{4-2x} - c^{4+m}d^{3-x}$ by $c^{3-m}d^{2-2x}$.

11. Divide $m^3y^n - m^{1+x}y^{1+n} + m^{5-x}y^{n-4}$ by $m^{5-x}y^{n-4}$.

12. Divide $a^{1+3y} - a^3 + a^{2+5y}$ by a^{2-x+2y}.

13. Divide $x - x^{m-5n+4} + x^{2m}$ by x^{2-m-2n}.

14. Divide $y^p - y^{3-4m} + y^{4p+1}$ by y^{2p-m+1}.

15. Divide $2x^{3n} - 6x^{2n}y^n + 6x^ny^{2n} - 2y^{3n}$ by $x^n - y^n$.

16. Divide $x^3 - 2ax^2 + a^2x - abx - b^2x + a^2b + ab^2$ by $x^2 - ax + bx - ab$.

17. Divide $x^{4m} + x^{2m} + 1$ by $x^{2m} - x^m + 1$.

18. Divide $3x^{m+7} - 4x^{m+6} - 12x^{m+5} - 9x^{m+4}$ by $x^{m+4} - 3x^{m+3}$.

19. Divide $6x^{4m+5} - 13x^{3m+5} + 13x^{2m+5} - 13x^{m+5} - 5x^5$ by $2x^{2m} - 3x^m - 1$.

20. Divide $12a^{5n-3} - a^{4n-2} - 20a^{3n-1} + 19a^{2n} - 10a^{n+1}$ by $4a^{2n} - 3a^{n+1} + 2a^2$.

21. Arrange according to descending powers of x the following expression, and enclose the coefficient of each power in a parenthesis with a minus sign before each parenthesis except the first:

$$x^3 - 2bx - a^2x^2 - ax - ax^2 - cx - a^2x^3 - bcx.$$

22. Divide $1.2a^4x - 5.494a^3x^2 + 4.8a^2x^3 + 0.9ax^4 - x^5$ by $0.6ax - 2x^2$.

23. Multiply $\frac{1}{2}a^2 - \frac{1}{5}ab + \frac{1}{3}b^2$ by $\frac{1}{2}a + \frac{1}{5}b$.

24. Multiply $\frac{2}{3}a^2 + ab + \frac{3}{2}b^2$ by $\frac{1}{2}a - \frac{1}{3}b$.

25. Divide $\frac{1}{4}a^3 + \frac{1}{72}ab^2 + \frac{1}{12}b^3$ by $\frac{1}{2}a + \frac{1}{3}b$.

26. Subtract $\frac{1}{4}x^2 + \frac{1}{3}xy + \frac{1}{6}y^2$ from $\frac{1}{3}x^2 - \frac{1}{2}xy + \frac{1}{8}y^2$.

27. Subtract $x^2 + \frac{1}{3}xy - \frac{1}{2}y^2$ from $2x^2 - \frac{1}{2}xy + y^2$.

CHAPTER V.

SIMPLE EQUATIONS.

84. Equations. An equation is a statement in symbols that two expressions stand for the same number. Thus, the equation $3x+2=8$ states that $3x+2$ and 8 stand for the same number.

85. That part of the equation which precedes the sign of equality is called the **first member**, or **left side**, and that which follows the sign of equality is called the **second member**, or **right side**.

86. The statement of equality between two algebraic expressions, if true for all values of the letters involved, is called an **identical equation**; but if true only for certain particular values of the letters involved, it is called an **equation of condition**. Thus, $(a+b)^2=a^2+2ab+b^2$, which is true for *all values* of a and b, is an *identical equation;* and $3x+2=8$, which is true only when x stands for 2, is an *equation of condition*.

For brevity, an identical equation is called an **identity**, and an equation of condition is called simply an **equation**.

87. We often employ an equation to discover an *unknown number* from its relation to known numbers. We usually represent the unknown number by one of the *last* letters of the alphabet, as x, y, z; and, by way of distinction, we use the *first* letters, a, b, c, etc., to represent numbers that are supposed to be known, though not expressed in the number-symbols of Arithmetic. Thus, in the equation $ax + b = c$, x is supposed to represent an unknown number, and a, b, and c are supposed to represent known numbers.

88. Simple Equations. An integral equation which contains the first power of the symbol for the unknown number, x, and no higher power, is called a **simple equation**, or an **equation of the first degree.** Thus, $ax + b = c$ is a simple equation, or an equation of the first degree *in* x.

89. Solution of an Equation. To solve an equation is to find the unknown number; that is, the number which, when substituted for its symbol in the given equation, renders the equation an identity. This number is said to *satisfy* the equation, and is called the **root** of the equation.

90. Axioms. In solving an equation, we make use of the following axioms:

Ax. 1. If equal numbers be added to equal numbers, the sums will be equal.

Ax. 2. If equal numbers be subtracted from equal numbers, the remainders will be equal.

Ax. 3. If equal numbers be multiplied by equal numbers, the products will be equal.

Ax. 4. If equal numbers be divided by equal numbers, the quotients will be equal.

If, therefore, the two sides of an equation be increased by, diminished by, multiplied by, or divided by equal numbers, the results will be equal.

Thus, if $8x = 24$, then $8x + 4 = 24 + 4$, $8x - 4 = 24 - 4$, $4 \times 8x = 4 \times 24$, and $8x \div 4 = 24 \div 4$.

91. Transposition of Terms. It becomes necessary in solving an equation to bring all the terms that contain the symbol for the unknown number to one side of the equation, and all the other terms to the other side. This is called **transposing the terms.** We will illustrate by examples:

(1) Find the number for which x stands when

$$16x - 11 = 7x + 70.$$

The first object to be attained is to get all the terms which contain x on the left side of the equation, and all the other terms on the right side. This can be done by first subtracting $7x$ from both sides (Ax. 2), which gives

$$9x - 11 = 70,$$

and then adding 11 to these equals (Ax. 1), which gives

$$9x = 81.$$

If these equals be divided by 9, the coefficient of x, the quotients will be equal (Ax. 4); that is, $x = 9$.

(2) Find the number for which x stands when $x + b = a$.

The equation is $x + b = a$.

Subtract b from each side, $x + b - b = a - b$. (Ax. 2)

Since $+b$ and $-b$ in the left side cancel each other (§ 36), we have $x = a - b$.

(3) Find the number for which x stands when $x - b = a$.

The equation is $x - b = a$.

Add $+b$ to each side, $x - b + b = a + b$. (Ax. 1)

Since $-b$ and $+b$ in the left side cancel each other (§ 36), we have $x = a + b$.

(4) What number does x stand for when $ax + b = cx + d$?

This is the *general form* which every simple equation in x will assume when the like terms on each side have been collected. In this equation x represents the unknown number, and a, b, c, d represent known numbers.

If now we subtract (Ax. 2) cx and b from each side of the equation, we have

$$ax - cx = d - b;$$

or, bracketing the coefficients of x,

$$(a - c)x = d - b.$$

Whence, dividing both sides by $a - c$, the coefficient of x, we get

$$x = \frac{d - b}{a - c}.$$

92. The effect of the operation in the preceding equations, when Axioms (1) and (2) are used, is to take a term from one side and to put it on the other side with its sign changed. We can proceed in a like manner in any other case. Hence the general rule:

93. *Any term may be transposed from one side of an equation to the other provided its sign is changed.*

94. Any term, therefore, which occurs on both sides with the *same sign* may be removed from both without affecting the equality.

95. The sign of every term of an equation may be changed, for this is effected by multiplying by -1, which by Ax. 3 does not destroy the equality.

96. Verification. When the root is substituted for its symbol in the given equation, and the equation reduces to an *identity*, the root is said to be verified. We will illustrate with examples:

(1) What number added to twice itself gives 24?

Let x stand for the number;

then $2x$ will stand for twice the number,
and the number added to twice itself will be $x + 2x$.

But the number added to twice itself is 24;

$$\therefore x + 2x = 24.$$

Combining x and $2x$, $3x = 24$.

Divide by 3, the coefficient of x, $x = 8$. (Ax. 4)

The required number is 8.

VERIFICATION.

$$x + 2x = 24,$$
$$8 + 2 \times 8 = 24,$$
$$8 + 16 = 24,$$
$$24 = 24.$$

(2) If $4x - 5$ stands for 19, for what number does x stand?

We have the equation $4x - 5 = 19$.

Transpose -5 to the right side, $4x = 19 + 5$.

Combine, $4x = 24$.

Divide by 4, $x = 6$. (Ax. 4)

VERIFICATION.

$$4x - 5 = 19,$$
$$4 \times 6 - 5 = 19,$$
$$24 - 5 = 19,$$
$$19 = 19.$$

(3) If $3x - 7$ stands for the same number as $14 - 4x$, what number does x stand for?

We have the equation $3x - 7 = 14 - 4x$.

Transpose $-4x$ to the left side, and -7 to the right side, $3x + 4x = 14 + 7$.

Combine, $7x = 21$.

Divide by 7, $x = 3$.

VERIFICATION.
$$3x - 7 = 14 - 4x,$$
$$3 \times 3 - 7 = 14 - 4 \times 3,$$
$$2 = 2.$$

(4) Solve the equation $(x-3)(x-4) = x(x-1) - 30.$

We have $$(x-3)(x-4) = x(x-1) - 30.$$

Remove the parentheses,
$$x^2 - 7x + 12 = x^2 - x - 30.$$

Since x^2 on the left and x^2 on the right are precisely the same, including the sign, they may be cancelled.

Then $$-7x + 12 = -x - 30.$$

Transpose $-x$ to the left side, and $+12$ to the right side,
$$-7x + x = -30 - 12.$$

Combine,
$$-6x = -42.$$

Divide by -6, $$x = 7.$$

VERIFICATION.
$$(7-3)(7-4) = 7(7-1) - 30,$$
$$4 \times 3 = 7 \times 6 - 30,$$
$$12 = 42 - 30,$$
$$12 = 12.$$

Exercise 17.

Find what number x stands for

1. If $x - 5$ stands for 7.
2. If $x + 8$ stands for 12.
3. If $6x - 12$ stands for 18.
4. If $7x - 3$ stands for 25.
5. If $5x + 8$ stands for 43.
6. If $2x - 5 = 7 + x$.
7. If $2x - 4 = 5 - x$.
8. If $5x - 2 = 3x + 4$.
9. If $7x - 5 = 6x - 1$.
10. If $5x - 3 = 25 - 2x$.

11. If $3x$ and $x+8$ stand for the same number.

12. If $2x-5$ stands for the same number as $3x$.

Solve the equations:

13. $2x-3=8+x$.

14. $5x+4=20+x$.

15. $2x-3=7-x$.

16. $3x-4=12-x$.

17. $2x-5=7-2x$.

18. $3x+14=2-x$.

Find x

19. If $2x-5$ and $4x-11$ stand for the same number.

20. If $x(x-7)$ and x^2-70 stand for the same number.

21. If $x(3x-2)$ and $3x(x-1)+2$ stand for the same number.

22. If $3x-5=4x-10$.

23. If $2x-4=14-x$.

24. If $3x-8$ and $4x-11$ stand for the same number.

25. If $2x-5$ and $7-x$ stand for the same number.

26. If $2x^2-23$ and $(2x+1)(x-3)$ stand for the same number.

27. If $(x+3)(x-7)-(x-4)(x+1)=25$.

28. If $(2x-1)(x+3)-(x-3)(2x-3)=72$.

Solve the equations:

29. $x(x-5)=x^2-30$.

30. $x(x+3)=x^2+18$.

31. $(x-3)(x+1)=x^2-3x+1$.

32. $(x-1)(x+2)+(x+3)(x-1)=2x(x+4)-(x+1)$.

33. $x(x+8)-(x+1)(x-2)-5(x+3)+3=0$.

34. $(x-3)(x+3)-(x-4)(x+4)-x=0$.

97. Statement and Solution of Problems. The difficulties which the beginner usually meets in stating problems will be quickly overcome if he will observe the following directions:

Study the problem until you clearly understand its meaning and just what is required to be found.

Remember that x must not be put for money, length, time, weight, etc., but for the required *number* of *specified units* of money, length, time, weight, etc.

Express each statement carefully in algebraic language, and write out in full just what each expression stands for.

Do not attempt to form the equation until all the statements are made in symbols.

We will illustrate by examples:

(1) John has three times as many oranges as James, and they together have 32. How many has each?

Let x stand for the *number* of oranges James has;
then $3x$ is the number of oranges John has;
and $x + 3x$ is the number of oranges they together have.

But 32 is the number of oranges they together have;

$$\therefore x + 3x = 32;$$

or, $$4x = 32,$$

and $$x = 8.$$

Since $x = 8$, $$3x = 24.$$

Therefore James has 8 oranges, and John has 24 oranges.

NOTE. Beginners in stating the preceding problem generally write:

Let $x =$ *what* James had.

Now, we know *what* James had. He had oranges, and we are to discover simply the *number* of oranges he had.

(2) James and John together have $24, and James has $8 more than John. How many dollars has each?

Let x stand for the number of dollars John has;
then $x+8$ is the number of dollars James has;
and $x+(x+8)$ is the number of dollars they both have.

But 24 is the number of dollars they both have;

$$\therefore x+(x+8)=24.$$

Removing the parenthesis, we have

$$x+x+8=24.$$

Transposing and collecting like terms, we have

$$2x=16.$$

Dividing by 2, we get

$$x=8.$$

Since $x=8$, $\quad x+8=16.$

Therefore John has $8, and James has $16.

NOTE. The beginner must avoid the mistake of writing

Let $x=$ John's money.

We are required to find the *number* of dollars John has, and therefore x must represent this required number.

(3) The sum of two numbers is 18, and three times the greater number exceeds four times the less by 5. Find the numbers.

Let $x=$ the greater number.

Then, since 18 is the sum, and x is one of the numbers, the other number must be the sum minus x. Hence

$$18-x= \text{the smaller number.}$$

Now, three times the greater number is $3x$, and four times the less number is $4(18-x)$.

We know from the problem that $3x$ exceeds $4(18-x)$ by 5; or, in other words, we know that the excess of $3x$ over $4(18-x)$ equals 5. It only remains to determine what *sign* the word "excess" implies. If we are in doubt about it, we can apply the phrase to two arithmetical numbers. We shall have no difficulty in seeing that the excess of 50 over 40 is 10; that is, $50-40$, and hence that the sign $-$ is implied by the word "excess."

Hence, $3x - 4(18 - x) =$ the excess

But $5 =$ the excess.

$\therefore 3x - 4(18 - x) = 5,$

or $3x - 72 + 4x = 5.$

$\therefore 7x = 77,$

and $x = 11.$

Therefore the numbers are 11 and 7.

(4) Find a number whose treble exceeds 40 by as much as its double falls short of 35.

Let $x =$ the required number;

then $3x =$ its treble,

and $3x - 40 =$ the excess of its treble over 40;

also, $35 - 2x =$ the number its double lacks of 35

Hence, $3x - 40 = 35 - 2x.$

Transposing, $3x + 2x = 35 + 40,$

$\therefore 5x = 75,$

and $x = 15.$

Therefore the number required is 15.

(5) Find a number that exceeds 50 by 10 more than it falls short of 80.

Let $x =$ the required number;

then $x - 50 =$ its excess over 50,

and $80 - x =$ the number it lacks of 80.

Hence, $x - 50 - (80 - x) =$ the excess.

But $10 =$ the excess.

$\therefore x - 50 - (80 - x) = 10,$

or $x - 50 - 80 + x = 10.$

$\therefore 2x = 140,$

and $x = 70.$

Therefore the number required is 70.

Exercise 18.

1. If a number is multiplied by 7, the product is 301. Find the number.

2. The sum of two numbers is 48, and the greater is five times the less. Find the numbers.

3. The sum of two numbers is 25, and seven times the less exceeds three times the greater by 35. Find the numbers.

4. Divide 20 in two parts such that four times the greater exceeds three times the less by 17.

5. Divide 23 into two parts such that the sum of twice the greater part and three times the less part is 57.

6. Divide 19 into two parts such that the greater part exceeds twice the less part by 1 less than twice the less part.

7. A tree 84 feet high was broken so that the part broken off was five times the length of the part left standing. Required the length of each part.

8. Four times the smaller of two numbers is three times the greater, and their sum is 63. Find the numbers.

9. A farmer sold a sheep, a cow, and a horse for $216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?

10. Distribute $15 among Thomas, Richard, and Henry so that Thomas and Richard shall each have twice as much as Henry.

11. Three men, A, B, and C, pay $1000 taxes. B pays four times as much as A, and C pays as much as A and B together. How much does each pay?

12. John's age is three times the age of James, and their ages together are 16 years. What is the age of each?

13. Twice a certain number increased by 8 is 40. Find the number.

14. Three times a certain number is 46 more than the number itself. Find the number.

15. One number is four times as large as another. If I take the smaller from 12 and the greater from 21, the remainders are equal. What are the numbers?

16. The joint ages of a father and son are 70 years. If the age of the son were doubled, he would be 4 years younger than his father. What is the age of each?

17. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

18. Add $24 to a certain amount, and the sum will be as much above $80 as the amount is below $80. What is the amount?

19. Thirty yards of cloth and 40 yards of silk together cost $330; and the silk costs twice as much per yard as the cloth. How much does each cost per yard?

20. Find the number whose double diminished by 24 exceeds 80 by as much as the number itself is less than 100.

21. In a company of 180 persons composed of men, women, and children there are twice as many men as women, and three times as many women as children. How many are there of each?

22. A banker was asked to pay $56 in five-dollar and two-dollar bills in such a manner as to pay the same number of each kind of bills. How many bills of each kind must he pay?

23. How can $3,60 be paid in quarters and ten-cent pieces so as to pay twice as many ten-cent pieces as quarters?

24. I have $1,98 in ten-cent pieces and three-cent pieces, and have four times as many three-cent pieces as ten-cent pieces. How many have I of each?

NOTE. In problems involving quantities of the same kind expressed in different units, we must be careful to reduce all the quantities to the *same* unit.

25. I have $17 dollars in two-dollar bills and twenty-five-cent pieces, and have twice as many bills as coins. How many have I of each?

26. I have $6.50 in silver dollars and ten-cent pieces, and I have 20 coins in all. How many have I of each?

27. A bought 9 dozen oranges for $2.00. For a part he paid 20 cents per dozen; for the remainder he paid 25 cents a dozen. How many dozen of each kind did he buy?

28. A gentleman gave some children 10 cents apiece, and found that he had just 50 cents left. If he had had another half-dollar, he might have given each of them at first 20 cents instead of 10 cents. How many children were there?

29. A is twice as old as B and 6 years younger than C. The sum of the ages of A, B, and C is 96 years. What is the age of B?

30. Divide a line 24 inches long into two parts such that the one part shall be 6 inches longer than the other.

31. Two trains travelling, one at 25 and the other at 30 miles an hour, start at the same time from two places 220 miles apart, and move toward each other. In how many hours will the trains meet?

32. A man bought twelve yards of velvet, and if he had bought 1 yard less for the same money, each yard would have cost $1 more. What did the velvet cost a yard?

33. A and B have together $8; A and C, $10; B and C, $12. How much has each?

34. Twelve persons subscribed for a new boat, but two being unable to pay, each of the others had to pay $4 more than his share. Find the cost of the boat.

35. A man was hired for 26 days on condition that for every day he worked he was to receive $3, and for every day he was idle he was to pay $1 for his board. At the end of the time he received $58. How many days did he work?

36. A man walking 4 miles an hour starts 2 hours after another person who walks 3 miles an hour. How many miles must the first man walk to overtake the second?

37. A man swimming in a river which runs 1 mile an hour finds that it takes him three times as long to swim a mile up the river as it does to swim the same distance down. Find his rate of swimming in still water.

38. At an election there were two candidates, and 2644 votes were cast. The successful candidate had a majority of 140. How many votes were cast for each?

39. Two persons start from towns 55 miles apart and walk toward each other. One walks at the rate of 4 miles an hour, but stops 2 hours on the way; the other walks at the rate of 3 miles an hour. How many miles will each have travelled when they meet?

40. A had twice as much money as B; but if A gives B $10, B will have three times as much as A. How much has each?

41. If $2x-8$ stands for 20, for what number will $4-x$ stand?

42. A vessel containing 100 gallons was emptied in 10 minutes by two pipes running one at a time. The first pipe discharged 14 gallons a minute, and the second 9 gallons a minute. How many minutes did each pipe run?

43. A man has 8 hours for an excursion. How far can he ride out in a carriage which goes at the rate of 9 miles an hour so as to return in time, walking at the rate of 3 miles an hour?

44. If $3x - 4a = 2x - a$, find the number for which $4x - 7a$ stands.

45. If $7x - a = 9(x - a)$, find the number $\frac{2x - 3a}{x + a}$.

NOTE. When we compare the ages of two persons at a given time, and also a number of years after or before the given time, we must remember that *both* persons will be so many years older or younger. Thus, if a man is now $2x$ years old and his son x years old, 5 years ago the father was $2x - 5$ and the son $x - 5$, and 5 years hence the father will be $2x + 5$ and the son $x + 5$, years old.

46. A man is now twice as old as his son; 15 years ago he was three times as old as his son. Find the age of each.

47. A man was four times as old as his son 7 years ago, and will be only twice as old as his son 7 years hence. Find the age of each.

48. A, who is 25 years older than B, is 5 years more than twice as old as B. Find the age of each.

49. A man is 25 years older than his son; 10 years ago he was six times as old as his son. Find the age of each.

50. The difference in the squares of two consecutive numbers is 19. Find the numbers.

51. The difference in the squares of two successive odd numbers is 40. Find the numbers.

CHAPTER VI.

MULTIPLICATION AND DIVISION.

SPECIAL RULES.

98. Special Rules of Multiplication. Some results of multiplication are of so great utility in shortening algebraic work that they should be carefully noticed and remembered. The following are important:

99. Square of the Sum of Two Numbers.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

Since a and b stand for *any* two numbers, we have

RULE 1. *The square of the sum of two numbers is the sum of their squares plus twice their product.*

100. Square of the Difference of Two Numbers.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b)-b(a-b)\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2.\end{aligned}$$

Hence we have

RULE 2. *The square of the difference of two numbers is the sum of their squares minus twice their product.*

101. Product of the Sum and Difference of Two Numbers.

$$(a+b)(a-b) = a(a-b) + b(a-b)$$
$$= a^2 - ab + ab - b^2$$
$$= a^2 - b^2.$$

Hence, we have

RULE 3. *The product of the sum and difference of two numbers is the difference of their squares.*

If we put $2x$ for a and 3 for b, we have

Rule 1, $(2x+3)^2 = 4x^2 + 12x + 9.$

Rule 2, $(2x-3)^2 = 4x^2 - 12x + 9.$

Rule 3, $(2x+3)(2x-3) = 4x^2 - 9.$

Exercise 19.

Write the product of

1. $(x+y)^2.$
2. $(x-a)^2.$
3. $(x+2b)^2.$
4. $(3x-2c)^2.$
5. $(4y-5)^2.$
6. $(3a^2+4z^2)^2.$
7. $(x+y)(x-y)$
8. $(4z-3)(4z+3).$
9. $(3a^2+4b^2)(3a^2-4b^2).$
10. $(3a-c)(3a-c).$
11. $(x+7b^2)(x+7b^2).$
12. $(ax+2by)(ax-2by).$

102. If we are required to multiply $a+b+c$ by $a+b-c$, we may abridge the ordinary process as follows:

$$(a+b+c)(a+b-c) = [(a+b)+c][(a+b)-c]$$

By Rule 3, $= (a+b)^2 - c^2$

By Rule 1, $= a^2 + 2ab + b^2 - c^2.$

If we are required to multiply $a+b-c$ by $a-b+c$, we may put the expressions in the following forms, and perform the operation:

$$(a+b-c)(a-b+c)=[a+(b-c)][a-(b-c)]$$

By Rule 3, $\quad = a^2-(b-c)^2$

By Rule 2, $\quad = a^2-(b^2-2bc+c^2)$

$$= a^2-b^2+2bc-c^2.$$

Exercise 20.

Find the product of

1. $x+y+z$ and $x-y-z$.
2. $x-y+z$ and $x-y-z$.
3. $ax+by+1$ and $ax+by-1$.
4. $1+x-y$ and $1-x+y$.
5. $a+2b-3c$ and $a-2b+3c$.
6. a^2-ab+b^2 and a^2+ab+b^2.
7. m^2+mn+n^2 and m^2-mn+n^2.
8. $2+x+x^2$ and $2-x-x^2$.
9. a^2+a+1 and a^2-a+1.
10. $3x+2y-z$ and $3x-2y+z$.

103. Square of any Polynomial. If we put x for a, and $y+z$ for b, in the identity

$$(a+b)^2=a^2+2ab+b^2,$$

we shall have

$$[x+(y+z)]^2=x^2+2x(y+z)+(y+z)^2,$$

or $\quad (x+y+z)^2 = x^2+2xy+2xz+y^2+2yz+z^2$

$$= x^2+y^2+z^2+2xy+2xz+2yz.$$

It will be seen that the complete product consists of the sum of the squares of the terms of the given expression and twice the products of each term into all the terms that follow it.

Again, if we put $a-b$ for a, and $c-d$ for b, in the same identity, we shall have

$$[(a-b)+(c-d)]^2$$
$$=(a-b)^2+2(a-b)(c-d)+(c-d)^2$$
$$=(a^2-2ab+b^2)+2a(c-d)-2b(c-d)+(c^2-2cd+d^2)$$
$$=a^2-2ab+b^2+2ac-2ad-2bc+2bd+c^2-2cd+d^2$$
$$=a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd.$$

Here the same law holds as before, the sign of each double product being + or −, according as the factors composing it have *like* or *unlike* signs. The same is true for any polynomial. Hence we have the following rule:

RULE 4. *The square of a polynomial is the sum of the squares of the several terms and twice the products obtained by multiplying each term into all the terms that follow it.*

Exercise 21.

Write the square of

1. $2x-3y$.

2. $a+b+c$.

3. $x+y-z$.

4. $x-y+z$.

5. $x+y+5$.

6. $x+2y+3$.

7. $a-b+c$.

8. $3x-2y+4$.

9. $2x-3y+4z$.

10. $x^2+y^2+z^2$.

11. $2x-y-z$.

12. $a-2b-3c$.

13. $3a-b+2c$.

14. $x+2y-3z$.

15. $x+y+z+1$.

16. $4x+y+z-2$.

17. $2x-y-z-3$.

18. $x-2y-3z+4$.

See P. 92.

104. Product of Two Binomials. The product of two binomials which have the form $x+a$, $x+b$, should be carefully noticed and remembered.

(1) $$\begin{aligned}(x+5)(x+3) &= x(x+3)+5(x+3)\\ &= x^2+3x+5x+15\\ &= x^2+8x+15.\end{aligned}$$

(2) $$\begin{aligned}(x-5)(x-3) &= x(x-3)-5(x-3)\\ &= x^2-3x-5x+15\\ &= x^2-8x+15.\end{aligned}$$

(3) $$\begin{aligned}(x+5)(x-3) &= x(x-3)+5(x-3)\\ &= x^2-3x+5x-15\\ &= x^2+2x-15.\end{aligned}$$

(4) $$\begin{aligned}(x-5)(x+3) &= x(x+3)-5(x+3)\\ &= x^2+3x-5x-15\\ &= x^2-2x-15.\end{aligned}$$

Each of these results has three terms.

The first term of each result is the product of the first terms of the binomials.

The last term of each result is the product of the second terms of the binomials.

The middle term of each result has for a coefficient the *algebraic sum* of the second terms of the binomials.

The intermediate step given above may be omitted, and the products written at once by *inspection*. Thus,

(1) Multiply $x+8$ by $x+7$.

$$8+7=15, \quad 8\times 7=56.$$

$$\therefore (x+8)(x+7)=x^2+15x+56.$$

(2) Multiply $x - 8$ by $x - 7$.

$$(-8)+(-7) = -15, \quad (-8)(-7) = +56.$$
$$\therefore (x-8)(x-7) = x^2 - 15x + 56.$$

(3) Multiply $x - 7y$ by $x + 6y$.

$$-7+6 = -1, \quad (-7y)\times 6y = -42y^2.$$
$$\therefore (x-7y)(x+6y) = x^2 - xy - 42y^2.$$

(4) Multiply $x^2 + 6(a+b)$ by $x^2 - 5(a+b)$.

$$+6-5 = 1, \quad 6(a+b)\times -5(a+b) = -30(a+b)^2.$$
$$\therefore [x^2+6(a+b)][x^2-5(a+b)] = x^4+(a+b)x^2-30(a+b)^2.$$

Exercise 22.

Find by inspection the product of

1. $(x+8)(x+3)$.
2. $(x+8)(x-3)$.
3. $(x-7)(x+10)$.
4. $(x-9)(x-5)$.
5. $(x-10)(x+9)$.
6. $(a-10)(a-5)$.
7. $(x-3a)(x+2a)$.
8. $(a+2b)(a-4b)$.
9. $(a-12)(a-3)$.
10. $(a+2b)(a+4b)$.
11. $(a-3b)(a+7b)$.
12. $(a+2b)(a-9b)$.
13. $(x-3a)(x-4a)$.
14. $(x+4z)(x-2z)$.
15. $(x+6y)(x-5y)$.
16. $(x^2-9)(x^2+8)$.
17. $(x^2+2y^2)(x^2-3y^2)$.
18. $(x^2+8y^2)(x^2-4y^2)$.
19. $(ab-8)(ab+5)$.
20. $(ab-7xy)(ab+3xy)$.
21. $(x-3y)(x-3y)$.
22. $(x+6)(x+6)$.
23. $(a-3b)(a-3b)$.
24. $(x-c)(x-d)$.
25. $(x+a)(x-b)$.
26. $(x-a)(x+b)$.
27. $[(a+b)+2][(a+b)-4]$.
28. $[(x+y)-2][(x+y)+4]$.
29. $(x+y-7)(x+y+10)$.
30. $(x-y-7)(x-y-10)$.

105. In like manner the product of *any* two binomials may be written.

(1) Multiply $2a - b$ by $3a + 4b$.

$$(2a - b)(3a + 4b) = 6a^2 + 8ab - 3ab - 4b^2$$
$$= 6a^2 + 5ab - 4b^2.$$

(2) Multiply $2x + 3y$ by $3x - 2y$.

The middle term is

$$(2x) \times (-2y) + 3y \times 3x = 5xy;$$
$$\therefore (2x + 3y)(3x - 2y) = 6x^2 + 5xy - 6y^2.$$

Exercise 23.

Find the product of

1. $3x - y$ and $2x + y$.
2. $4x - 3y$ and $3x - 2y$.
3. $5x - 4y$ and $3x - 4y$.
4. $x - 7y$ and $2x - 5y$.
5. $11x - 2y$ and $7x + y$.
6. $10x - 3y$ and $10x - 7y$.
7. $3a^2 - 2b^2$ and $2a^2 + 3b^2$.
8. $a^2 + b^2$ and $a - b$.
9. $3a^2 - 2b^2$ and $2a + 3b$.
10. $a^2 - b^2$ and $a + b$.

106. Special Rules of Division. Some results in division are so important in abridging algebraic work that they should be carefully noticed and remembered.

107. Difference of Two Squares.

From § 101, $(a + b)(a - b) = a^2 - b^2$.

$$\therefore \frac{a^2 - b^2}{a + b} = a - b, \text{ and } \frac{a^2 - b^2}{a - b} = a + b. \text{ Hence}$$

RULE 1. *The difference of the squares of two numbers is divisible by the sum, and by the difference, of the numbers.*

Exercise 24.

Write by inspection the quotient of

1. $\frac{a^2-4}{a-2}$.

2. $\frac{9-x^2}{3+x}$.

3. $\frac{16-a^2}{4+a}$.

4. $\frac{x^2-25}{x-5}$.

5. $\frac{36-x^2}{6+x}$.

6. $\frac{9a^2-b^2}{3a-b}$.

7. $\frac{25x^2-36b^2}{5x+6b}$.

8. $\frac{49c^2-d^2}{7c-d}$.

9. $\frac{9a^2-1}{3a+1}$.

10. $\frac{16-4a^2}{4-2a}$.

11. $\frac{9a^4-25y^4}{3a^2+5y^2}$.

12. $\frac{4x^{16}-9y^6}{2x^8-3y^3}$.

13. $\frac{4x^{10}-a^8}{2x^5-a^4}$.

14. $\frac{a^2b^6c^8-x^{12}}{ab^3c^4+x^6}$.

15. $\frac{x^4a^8-b^{10}}{x^2a^4-b^5}$.

16. $\frac{a^2-(b+c)^2}{a-(b+c)}$.

17. $\frac{a^2-(3b-4c)^2}{a-(3b-4c)}$.

18. $\frac{1-(x-y)^2}{1+(x-y)}$.

19. $\frac{(3x-y)^2-16z^2}{(3x-y)+4z}$.

20. $\frac{(x+3a)^2-9x^2}{(x+3a)-3x}$.

21. $\frac{1-(7a-5b)^2}{1+(7a-5b)}$.

22. $\frac{(3x+2y)^2-4z^2}{(3x+2y)-2z}$.

23. $\frac{(a-b)^2-(c-y)^2}{(a-b)+(c-y)}$.

24. $\frac{x^2-(y+z)^2}{x-(y+z)}$.

25. $\frac{(x-2y)^2-25}{(x-2y)+5}$.

26. $\frac{(2x+y)^2-9z^4}{(2x+y)-3z^2}$.

108. Sum and Difference of Two Cubes. By performing the division, we find that

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2, \text{ and } \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

Hence,

RULE 2. *The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the sum of the squares of the numbers minus their product.*

RULE 3. *The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the squares of the numbers plus their product.*

Exercise 25.

Write by inspection the quotient of

1. $\frac{1-8x^3}{1-2x}$.

2. $\frac{1+8x^3}{1+2x}$.

3. $\frac{27a^3-b^3}{3a-b}$.

4. $\frac{27a^3+b^3}{3a+b}$.

5. $\frac{64x^3+27y^3}{4x+3y}$.

6. $\frac{64x^3-27y^3}{4x-3y}$.

7. $\frac{1-27z^3}{1-3z}$.

8. $\frac{1+27z^3}{1+3z}$.

9. $\frac{a^3b^3-c^3}{ab-c}$.

10. $\frac{a^3b^3+c^3}{ab+c}$.

11. $\frac{64+y^3}{4+y}$.

12. $\frac{343-8a^3}{7-2a}$.

13. $\frac{8a^3+b^6}{2a+b^2}$.

14. $\frac{x^6+729y^3}{x^2+9y}$.

15. $\frac{a^6-27b^3}{a^2-3b}$.

16. $\frac{8x^3-64y^6}{2x-4y^2}$.

109. Sum and Difference of any Two Like Powers. By performing the division, we find that

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3;$$

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3;$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4;$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$

We find by trial that

$$a^2+b^2,\ a^4+b^4,\ a^6+b^6, \text{ and so on,}$$

are *not* divisible by $a+b$ or by $a-b$. Hence,

If n is *any* positive integer,

(1) *a^n+b^n is divisible by $a+b$ if n is odd, and by neither $a+b$ nor $a-b$ if n is even.*

(2) *a^n-b^n is divisible by $a-b$ if n is odd, and by both $a+b$ and $a-b$ if n is even.*

NOTE. It is important to notice in the above examples that the terms of the quotient are all *positive* when the divisor is $a-b$, and *alternately positive and negative* when the divisor is $a+b$; also, that the quotient is homogeneous, the exponent of a decreasing and of b increasing by 1 for each successive term.

Exercise 26.

Find the quotient of

1. $\dfrac{x^6-y^6}{x-y}$.
2. $\dfrac{x^6-y^6}{x+y}$.
3. $\dfrac{x^4-1}{x-1}$.
4. $\dfrac{x^4-1}{x+1}$.
5. $\dfrac{x^4-16}{x-2}$.
6. $\dfrac{x^5-32}{x-2}$.
7. $\dfrac{x^5+32}{x+2}$.
8. $\dfrac{1-m^4}{1-m}$.
9. $\dfrac{1+m^5}{1+m}$.

CHAPTER VII.

FACTORS.

110. Rational Expressions. An expression is *rational* when none of its terms contain square or other roots.

111. Factors of Rational and Integral Expressions. By factors of a given integral number in Arithmetic we mean integral numbers that will divide the given number without remainder. Likewise by factors of a rational and integral expression in Algebra we mean rational and integral expressions that will divide the given expression without remainder.

112. Factors of Monomials. The factors of a monomial may be found by inspection. Thus, the factors of $14a^2b$ are 7, 2, a, a, and b.

113. Factors of Polynomials. The form of a polynomial that can be resolved into factors often suggests the process of finding the factors.

CASE I.

114. When all the terms have a common factor.

(1) Resolve into factors $2x^2 + 6xy$.

Since $2x$ is a factor of each term, we have

$$\frac{2x^2 + 6xy}{2x} = \frac{2x^2}{2x} + \frac{6xy}{2x} = x + 3y.$$

$$\therefore\ 2x^2 + 6xy = 2x(x + 3y).$$

Hence, the required factors are $2x$ and $x + 3y$.

(2) Resolve into factors $16a^3 + 4a^2 - 8a$.

Since $4a$ is a factor of each term, we have

$$\frac{16a^3 + 4a^2 - 8a}{4a} = \frac{16a^3}{4a} + \frac{4a^2}{4a} - \frac{8a}{4a}$$

$$= 4a^2 + a - 2.$$

$$\therefore\ 16a^3 + 4a^2 - 8a = 4a(4a^2 + a - 2).$$

Hence the required factors are $4a$ and $4a^2 + a - 2$.

Exercise 27.

Resolve into two factors:

1. $3x^2 - 6x^3$.
2. $2a^2 - 4a$.
3. $5ab - 5a^2b^3$.
4. $3a^3 - a^2 + a$.
5. $x^3 + x^2y - xy^2$.
6. $a^4 - a^3b + a^2b^2$.
7. $3a^2b - 4ab^2$.
8. $8x^3y^2 + 4x^2y^3$.
9. $3x^4 - 9x^2 - 6x^3$.
10. $8a^2x^2 - 4a^2b + 12a^2y^2$.
11. $8a^3b^2c^2 - 4a^2b^3c^3$.
12. $15a^3x - 10a^3y + 5a^3z$.

Case II.

115. When the terms can be grouped so as to show a common factor.

(1) Resolve into factors $ac + ad + bc + bd$.

$$ac + ad + bc + bd = (ac + ad) + (bc + bd) \qquad (1)$$

$$= a(c + d) + b(c + d) \qquad (2)$$

$$= (a + b)(c + d). \qquad (3)$$

Note. Since one factor is seen in (2) to be $c + d$, dividing by $c + d$ we obtain the other factor, $a + b$.

(2) Find the factors of $ac + ad - bc - bd$.

$$\begin{aligned} ac + ad - bc - bd &= (ac + ad) - (bc + bd) \\ &= a(c + d) - b(c + d) \\ &= (a - b)(c + d). \end{aligned}$$

NOTE. — Here the last two terms, $-bc - bd$, being put within a parenthesis preceded by the sign $-$, have their signs changed to $+$.

(3) Resolve into factors $3x^3 - 5x^2 - 6x + 10$.

$$\begin{aligned} 3x^3 - 5x^2 - 6x + 10 &= (3x^3 - 5x^2) - (6x - 10) \\ &= x^2(3x - 5) - 2(3x - 5) \\ &= (x^2 - 2)(3x - 5). \end{aligned}$$

(4) Resolve into factors $5x^3 - 15ax^2 - x + 3a$.

$$\begin{aligned} 5x^3 - 15ax^2 - x + 3a &= (5x^3 - 15ax^2) - (x - 3a) \\ &= 5x^2(x - 3a) - 1(x - 3a) \\ &= (5x^2 - 1)(x - 3a). \end{aligned}$$

(5) Resolve into factors $6y - 27x^2y - 10x + 45x^3$.

$$\begin{aligned} 6y - 27x^2y - 10x + 45x^3 &= (45x^3 - 27x^2y) - (10x - 6y) \\ &= 9x^2(5x - 3y) - 2(5x - 3y) \\ &= (9x^2 - 2)(5x - 3y). \end{aligned}$$

NOTE. By grouping the terms thus, $(6y - 27x^2y) - (10x - 45x^3)$, we obtain for the factors, $(2 - 9x^2)(3y - 5x)$.

But $(2 - 9x^2)(3y - 5x) = (9x^2 - 2)(5x - 3y)$, since, by the Law of Signs, *the signs of two factors, or of any even number of factors, may be changed without altering the value of the product.*

Exercise 28.

Resolve into factors:

1. $ax - bx + ay - by$.

2. $ax - bx - ay + by$.

3. $ax - cy - ay + cx$.

4. $2ab - 3ac - 2by + 3cy$.

5. $x^2 + ax - bx - ab$.

6. $x^2 + xy - ax - ay$.

7. $x^2 - xy - 6x + 6y$.

8. $2x^2 - 3xy + 4ax - 6ay$.

9. $a^2b - abx - ac + cx$.

10. $a^2bx + b^2cx - a^2cy - bc^2y$.

11. $3x^2 - 5y^2 - 6x^3 + 10xy^2$.

12. $8ax - 10bx - 12a + 15b$.

13. $2x^3 - 3x^2 - 4x + 6$.

14. $6x^4 + 8x^3 - 9x^2 - 12x$.

15. $ax^4 + bx^3 - ax - b$.

16. $3cx^4 - 2dx^3 - 9cx^2 + 6dx$.

17. $1 + 15x^4 - 5x - 3x^3$.

18. $ax^2 + a^2x + a + x$.

19. $(a+b)(c+d) - 3c(a+b)$.

20. $(x-y)^2 + 2y(x-y)$.

116. If an expression can be resolved into two equal factors, the expression is called a **perfect square**, and one of its equal factors is called its **square root.**

Thus, $16x^6y^2 = 4x^3y \times 4x^3y$. Hence, $16x^6y^2$ is a perfect square, and $4x^3y$ is its square root.

NOTE. The square root of $16x^6y^2$ may be $-4x^3y$ as well as $+4x^3y$, for $-4x^3y \times -4x^3y = 16x^6y^2$; but throughout this chapter the positive square root only will be considered.

117. The rule for extracting the square root of a perfect square, when the square is a monomial, is as follows:

Extract the square root of the coefficient, and divide the index of each letter by 2.

118. In like manner, the rule for extracting the cube root of a perfect cube, when the cube is a monomial, is,

Extract the cube root of the coefficient, and divide the index of each letter by 3.

119. By §§ 99, 100, a trinomial is a perfect square, if its first and last terms are perfect squares and positive, and its middle term is twice the product of their square roots. Thus, $16a^2 - 24ab + 9b^2$ is a perfect square.

The rule for extracting the square root of a perfect square, when the square is a trinomial, is as follows:

Extract the square roots of the first and last terms, and connect these square roots by the sign of the middle term.

Thus, if we wish to find the square root of

$$16a^2 - 24ab + 9b^2,$$

we take the square roots of $16a^2$ and $9b^2$, which are $4a$ and $3b$, respectively, and connect these square roots by the sign of the middle term, which is $-$. The square root is therefore

$$4a - 3b.$$

In like manner, the square root of

$$16a^2 + 24ab + 9b^2 \text{ is } 4a + 3b.$$

Case III.

120. When a trinomial is a perfect square.

(1) Resolve into factors $x^2 + 2xy + y^2$.

From § 119, the factors of $x^2 + 2xy + y^2$ are

$$(x + y)(x + y).$$

(2) Resolve into factors $x^4 - 2x^2y + y^2$.

From § 119, the factors of $x^4 - 2x^2y + y^2$ are

$$(x^2 - y)(x^2 - y).$$

Exercise 29.

Resolve into factors:

1. $a^2 - 6ab + 9b^2$.
2. $4a^2 + 4ab + b^2$.
3. $a^2 - 4ab + 4b^2$.
4. $x^2 + 6xy + 9y^2$.

5. $4x^2 - 12ax + 9a^2$.
6. $a^2 - 10ab + 25b^2$.
7. $4a^2 - 4a + 1$.
8. $49y^2 - 14yz + z^2$.
9. $x^2 - 16x + 64$.
10. $9x^2 + 24xy + 16y^2$.
11. $16a^2 + 8ax + x^2$.
12. $25 + 80x + 64x^2$.
13. $49x^2 - 28xy + 4y^2$.
14. $1 - 20b + 100b^2$.
15. $81a^2 + 126ab + 49b^2$.
16. $m^2n^2 - 16mna^2 + 64a^4$.
17. $4a^2 - 20ax + 25x^2$.
18. $121a^2 + 198ay + 81y^2$.
19. $a^2b^4c^6 - 2ab^2c^3x^8 + x^{16}$.
20. $49 - 140k^2 + 100k^4$.

Case IV.

121. When a binomial is the difference of two squares.

(1) Resolve into factors $x^2 - y^2$.

From § 101, $(x+y)(x-y) = x^2 - y^2$.

Hence, the difference of two squares is the product of two factors, which may be found as follows:

Take the square root of the first term and the square root of the second term.

The sum of these roots will form the first factor;

The difference of these roots will form the second factor.

Exercise 30.

Resolve into factors:

1. $a^2 - 4$.
2. $1 - x^2$.
3. $x^2 - 9y^2$.
4. $4a^2 - 49b^2$.
5. $x^2 - 4y^2$.
6. $25 - 16a^2$.
7. $16 - 25y^2$.
8. $a^2b^2 - 1$.
9. $x^2 - 100$.
10. $121a^2 - 36b^2$.
11. $81x^2 - 4y^2$.
12. $64a^4 - b^4$.
13. $m^2n^2 - 36$.
14. $x^4 - 144$.
15. $x^2 - 25$.

16. $49 - 100y^2$.
17. $1 - 49x^6$.
18. $4 - 121y^8$.
19. $1 - 169a^6$.
20. $a^2b^2 - 4c^6$.
21. $9x^8 - a^6$.
22. $4x^{16} - y^{20}$.
23. $49a^{14} - y^{12}$.
24. $64a^2 - 9b^6$.
25. $81a^4b^4 - c^4$.
26. $4a^2c - 9c^5$.
27. $20a^3b^3 - 5ab$.
28. $3a^5 - 12a^3c^2$.
29. $9a^2 - 81b^2$.
30. $25 - 64y^2$.
31. $16x^{17} - 9xy^6$.
32. $25x^{10} - 16a^8x^8$.
33. $36a^2x^2 - 49a^4$.
34. $x^2 - 16y^2$.
35. $1 - 400x^4$.
36. $4a^2c - 9c^3$.

122. If the squares are compound expressions, the same method may be employed.

(1) Resolve into factors $(x + 3y)^2 - 16a^2$.

The square root of the first term is $x + 3y$.
The square root of the second term is $4a$.
The sum of these roots is $x + 3y + 4a$.
The difference of these roots is $x + 3y - 4a$.
Therefore $(x + 3y)^2 - 16a^2 = (x + 3y + 4a)(x + 3y - 4a)$.

(2) Resolve into factors $a^2 - (3b - 5c)^2$.

The square roots of the terms are a and $(3b - 5c)$.
The sum of these roots is $a + (3b - 5c)$, or $a + 3b - 5c$.
The difference of these roots is $a - (3b - 5c)$, or $a - 3b + 5c$.
Therefore $a^2 - (3b - 5c)^2 = (a + 3b - 5c)(a - 3b + 5c)$.

123. If the factors contain like terms, these terms should be collected so as to present the results in the simplest form.

(3) Resolve into factors $(3a + 5b)^2 - (2a - 3b)^2$.

The square roots of the terms are $3a + 5b$ and $2a - 3b$.
The sum of these roots is $(3a + 5b) + (2a - 3b)$,
or $3a + 5b + 2a - 3b = 5a + 2b$.
The difference of these roots is $(3a + 5b) - (2a - 3b)$,
or $3a + 5b - 2a + 3b = a + 8b$.
Therefore $(3a + 5b)^2 - (2a - 3b)^2 = (5a + 2b)(a + 8b)$.

Exercise 31.

Resolve into factors:

1. $(x+y)^2-z^2$.
2. $(x-y)^2-z^2$.
3. $(x-2y)^2-4z^2$.
4. $(a+3b)^2-16c^2$.
5. $x^2-(y-z)^2$.
6. $a^2-(3b-2c)^2$.
7. $b^2-(2a+3c)^2$.
8. $1-(x+5b)^2$.
9. $9a^2-(x-3c)^2$
10. $16a^2-(2y-3z)^2$.
11. $(a-b)^2-(c-d)^2$.
12. $(2a+b)^2-25c^2$.
13. $(x+2y)^2-(2x-y)^2$.
14. $(x+3)^2-(3x-4)^2$.
15. $(a+b-c)^2-(a-b-c)^2$.
16. $(a-3x)^2-(3a-2x)^2$.
17. $(2a-1)^2-(3a+1)^2$.
18. $(x-5)^2-(x+y-5)^2$.
19. $(2a+b-c)^2-(a-2b+c)^2$.
20. $(a+2b-3c)^2-(a+5c)^2$.

124. By properly grouping the terms, compound expressions may often be written as the difference of two squares, and the factors readily found.

(1) Resolve into factors $a^2-2ab+b^2-9c^2$.

$$a^2-2ab+b^2-9c^2=(a^2-2ab+b^2)-9c^2$$
$$=(a-b)^2-9c^2$$
$$=(a-b+3c)(a-b-3c).$$

(2) Resolve into factors $12ab+9x^2-4a^2-9b^2$.

NOTE. Here $12ab$ shows that it is the middle term of the expression which has in its first and last terms a^2 and b^2, and the minus sign before $4a^2$ and $9b^2$ shows that these terms must be put in a parenthesis with the minus sign before it, in order that they may be made positive.

The arrangement will be

$$9x^2-(4a^2-12ab+9b^2)=9x^2-(2a-3b)^2$$
$$=(3x+2a-3b)(3x-2a+3b).$$

(3) Resolve into factors $-a^2+b^2-c^2+d^2+2ac+2bd$.

NOTE. Here $2ac$, $2bd$, and $-a^2$, $-c^2$, indicate the arrangement required.

$$
\begin{aligned}
-a^2+b^2-c^2+d^2+2ac+2bd &\\
&=(b^2+2bd+d^2)-(a^2-2ac+c^2)\\
&=(b+d)^2-(a-c)^2\\
&=(b+d+a-c)(b+d-a+c).
\end{aligned}
$$

Exercise 32.

Resolve into factors:

1. $a^2+2ab+b^2-4c^2$.
2. $x^2-2xy+y^2-9a^2$.
3. $b^2-x^2+4ax-4a^2$.
4. $4a^2+4ab+b^2-x^2$.
5. $a^2-x^2-y^2-2xy$.
6. $1-a^2-2ab-b^2$.
7. $a^2+b^2+2ab-16a^2b^2$.
8. $4a^2-9a^2+6a-1$.
9. $a^2+b^2-c^2-d^2-2ab-2cd$.
10. $x^2+y^2-2xy-2ab-a^2-b^2$.
11. $9x^2-6x+1-a^2-4ab-4b^2$.
12. $a^2+2ab-x^2-6xy-9y^2+b^2$.
13. $x^2-2x+1-b^2+2by-y^2$.
14. $9-6x+x^2-a^2-8ab-16b^2$.
15. $4-4x+x^2-4a-1-4a^2$.
16. $a^4-a^2-9+b^4+6a-2a^2b^2$.

125. A trinomial in the form of $a^4+a^2b^2+b^4$ can be written as the difference of two squares.

Since a trinomial is a perfect square when the middle term is *twice* the product of the square roots of the first and last terms, it is obvious that we must add a^2b^2 to the middle term of $a^4+a^2b^2+b^4$ to make it a perfect square.

We must also subtract a^2b^2 to keep the value of the expression unchanged. We shall then have

$$\begin{aligned}(1)\quad a^4+a^2b^2+b^4 &= a^4+2a^2b^2+b^4-a^2b^2\\ &= (a^2+b^2)^2-a^2b^2\\ &= (a^2+b^2+ab)(a^2+b^2-ab)\\ &= (a^2+ab+b^2)(a^2-ab+b^2).\end{aligned}$$

If in the above expression we put 1 for b, we shall have

$$\begin{aligned}(2)\qquad a^4+a^2+1 &= (a^4+2a^2+1)-a^2\\ &= (a^2+1)^2-a^2\\ &= (a^2+1+a)(a^2+1-a)\\ &= (a^2+a+1)(a^2-a+1).\end{aligned}$$

(3) Resolve into factors $4x^4-37x^2y^2+9y^4$.

Twice the product of the square roots of $4x^4$ and $9y^4$ is $12x^2y^2$. We may separate the term $-37x^2y^2$ into two terms, $-12x^2y^2$ and $-25x^2y^2$, and write the expression

$$\begin{aligned}(4x^4-12x^2y^2+9y^4)-25x^2y^2&\\ &= (2x^2-3y^2)^2-25x^2y^2\\ &= (2x^2-3y^2+5xy)(2x^2-3y^2-5xy)\\ &= (2x^2+5xy-3y^2)(2x^2-5xy-3y^2).\end{aligned}$$

Exercise 33.

Resolve into factors:

1. $x^4+x^2y^2+y^4$.
2. x^4+x^2+1.
3. $9a^4-15a^2+1$.
4. $16a^4-17a^2+1$.
5. $4a^4-13a^2+1$.
6. $9a^4+26a^2b^2+25b^4$.
7. $4x^4-21x^2y^2+9y^4$.
8. $4a^4-29a^2c^2+25c^4$.
9. $4a^4+16a^2c^2+25c^4$.
10. $25x^4+31x^2y^2+16y^4$.

See P. 78.

CASE V.

126. When a trinomial has the form $x^2 + ax + b$.

From § 104 it is seen that a trinomial is often the product of two binomials. Conversely, a trinomial may, in certain cases, be resolved into two binomial factors.

127. If a trinomial of the form $x^2 + ax + b$ is such an expression that it can be resolved into two binomial factors, it is obvious that the first term of each factor will be x, and that the second terms of the factors will be two numbers whose product is b, the last term of the trinomial, and whose algebraic sum is a, the coefficient of x in the middle term of the trinomial.

(1) Resolve into factors $x^2 + 11x + 30$.

We are required to find two numbers whose product is 30 and whose sum is 11.

Two numbers whose product is 30 are 1 and 30, 2 and 15, 3 and 10, 5 and 6; and the sum of the last two numbers is 11. Hence,

$$x^2 + 11x + 30 = (x + 5)(x + 6).$$

(2) Resolve into factors $x^2 - 7x + 12$.

We are required to find two numbers whose product is 12 and whose algebraic sum is -7.

Since the product is $+12$, the two numbers are *both positive* or *both negative;* and since their sum is -7, they must both be negative.

Two negative numbers whose product is 12 are -12 and -1, -6 and -2, -4 and -3; and the sum of the last two numbers is -7. Hence,

$$x^2 - 7x + 12 = (x - 4)(x - 3).$$

(3) Resolve into factors $x^2 + 2x - 24$.

We are required to find two numbers whose product is -24 and whose algebraic sum is 2.

Since the product is -24, one of the numbers is positive and the other negative; and since their sum is $+2$, the larger number is positive.

Two numbers whose product is -24, and the larger number positive, are 24 and -1, 12 and -2, 8 and -3, 6 and -4; and the sum of the last two numbers is $+2$. Hence,

$$x^2+2x-24=(x+6)(x-4).$$

(4) Resolve into factors $x^2-3x-18$.

We are required to find two numbers whose product is -18 and whose algebraic sum is -3.

Since the product is -18, one of the numbers is positive and the other negative; and since their sum is -3, the larger number is negative.

Two numbers whose product is -18, and the larger number negative, are -18 and 1, -9 and 2, -6 and 3; and the sum of the last two numbers is -3. Hence,

$$x^2-3x-18=(x-6)(x+3).$$

(5) Resolve into factors $x^2-10xy+9y^2$.

We are required to find two expressions whose product is $9y^2$ and whose algebraic sum is $-10y$.

Since the product is $+9y^2$, and the sum $-10y$, the last two terms must both be negative.

Two negative expressions whose product is $9y^2$, are $-9y$ and $-y$, $-3y$ and $-3y$; and the sum of the first two expressions is $-10y$. Hence,

$$x^2-10xy+9y^2=(x-9y)(x-y).$$

Exercise 34.

Resolve into factors:

1. $x^2+8x+15$.
2. $x^2-8x+15$.
3. $x^2+2x-15$.
4. $x^2-3x-10$.
5. $x^2+5ax+6a^2$.
6. $x^2-5ax+6a^2$.

7. $x^2 - 2x - 15.$
8. $x^2 + 5x + 6.$
9. $x^2 - 5x + 6.$
10. $x^2 + x - 6.$
11. $x^2 - x - 6.$
12. $x^2 + 6x + 5.$
13. $x^2 - 6x + 5.$
14. $x^2 + 4x - 5.$
15. $x^2 - 4x - 5.$
16. $x^2 + 9x + 18.$
17. $x^2 - 9x + 18.$
18. $x^2 + 3x - 18.$
19. $x^2 - 3x - 18.$
20. $x^2 + 9x + 8.$
21. $x^2 - 9x + 8.$
22. $x^2 + 7x - 8.$
23. $x^2 - 7x - 8.$
24. $x^2 + 7x + 10.$
25. $x^2 - 7x + 10.$
26. $x^2 + 3x - 10.$
27. $x^2 - 5ax - 50a^2.$
28. $x^2y^2 - 3xy - 4.$
29. $a^2 - 3ax - 54x^2.$
30. $a^6 - a^3c - 2c^2.$
31. $y^2 - 28yz + 187z^2.$
32. $x^2 + ax - 6a^2.$
33. $x^2 - ax - 6a^2.$
34. $x^2 + 5xy + 4y^2.$
35. $x^2 - 3xy - 4y^2.$
36. $x^2 - 5xy + 4y^2.$
37. $x^2 + 3xy - 4y^2.$
38. $x^2 + 3xy + 2y^2.$
39. $a^2 - 7ab + 10b^2.$
40. $a^2x^2 - 3ax - 54.$
41. $x^2 - 7x - 44.$
42. $x^2 + x - 132.$
43. $x^2 - 15x + 50.$
44. $a^2 - 23a + 120.$
45. $a^2 + 17a - 390.$
46. $c^2 + 25c - 150.$
47. $c^2 - 58c + 57.$
48. $a^4 - 11a^2b^3 + 30b^6.$
49. $z^2 + 9zy + 20y^2.$
50. $x^2y^2 + 19xyz + 48z^2.$
51. $a^2b^2 - 13abc + 22c^2.$
52. $a^2 - 16ab - 36b^2.$
53. $x^2 + 17xy + 30y^2.$
54. $x^2 - 7xy - 18y^2.$
55. $c^2 + c - 20.$
56. $a^2 + 16ab - 260b^2.$

57. $y^2-5yz-84z^2$.

58. $x^2-11x-152$.

59. $18-3x-x^2=-(x^2+3x-18)$.

60. $33+8x-x^2=-(x^2-8x-33)$.

61. $78-7x-x^2$.

Case VI.

128. When a trinomial has the form ax^2+bx+c.

From § 105,

$$(3x-2)(5x+3)$$
$$=15x^2+9x-10x-6=15x^2-x-6. \quad (1)$$
$$(3x-2)(5x-3)$$
$$=15x^2-9x-10x+6=15x^2-19x+6. \quad (2)$$

Consider the resulting trinomials:

The first term in (1) and (2) is the product $3x\times 5x$.

The middle term in (1) is the algebraic sum of the products

$$3x\times 3 \text{ and } (-2)\times 5x.$$

The middle term in (2) is the algebraic sum of the products

$$3x\times(-3) \text{ and } (-2)\times 5x.$$

The last term in (1) is the product $(-2)\times 3$.

The last term in (2) is the product $(-2)\times(-3)$.

The trinomials have no monomial factor, since no one of their factors has a monomial factor. Hence,

1. If the *third* term of a given trinomial is *negative*, the *second* terms of its binomial factors have *unlike signs*.

2. If the *third* term is *positive*, the *second* terms of its binomial factors have the *same sign*, and this sign is the sign of the middle term.

3. If a trinomial has no monomial factor, neither of its binomial factors can have a monomial factor.

(1) Resolve into factors $6x^2 + 17x + 12$.

The first terms of the binomial factors must be either $6x$ and x, or $3x$ and $2x$.

The second terms of the binomial factors must be 12 and 1, or 6 and 2, or 3 and 4.

We therefore write

$$\text{I. } (6x + \quad)(x + \quad), \quad \text{or} \quad \text{II. } (3x + \quad)(2x + \quad).$$

For the second terms of these factors we must reject 1 and 12; for 12 put in the second factor of I. would make the product $6x \times 12$ too large, and put in the first factor of I., or in either factor of II., the result would show a monomial factor.

We must also reject 6 and 2; for if put in I. or II. the results would show monomial factors; and for the same reason we must reject 3 and 4 for I.

The required factors, therefore, are $(3x + 4)$ and $(2x + 3)$.

(2) Resolve into factors $14x^2 - 11x - 15$.

For a first trial we write

$$(7x \quad)(2x \quad).$$

Since the third term of the given trinomial is -15, the second terms of the binomial factors will have unlike signs, and the two products which together form the middle term will be one +, and the other −. Also, since the middle term is $-11x$, the negative product will exceed in absolute value the positive product by $11x$.

The required factors, therefore, are $(7x + 5)$ and $(2x - 3)$.

Exercise 35.

Resolve into factors:

1. $2x^2 + 5x + 3$.
2. $3x^2 - x - 2$.
3. $5x^2 - 8x + 3$.
4. $24x^2 - 2xy - 15y^2$.
5. $36x^2 - 19xy - 6y^2$.
6. $15x^2 + 19xy + 6y^2$.

7. $6x^2+7x+2$.

8. $6x^2-x-2$.

9. $15x^2+14x-8$.

10. $8x^2-10x+3$.

11. $18x^2+9x-2$.

12. $24x^2-14xy-5y^2$.

13. $24x^2-38xy+15y^2$.

14. $15x^2-26xy+8y^2$.

15. $9x^2+6xy-8y^2$.

16. $6x^2-xy-35y^2$.

17. $10x^2-21xy-10y^2$.

18. $14x^2-55xy+21y^2$.

19. $6x^2-23xy+20y^2$.

20. $6x^2+35xy-6y^2$.

Case VII.

129. When a binomial is the sum or difference of two cubes.

From § 108, $$\frac{a^3+b^3}{a+b}=a^2-ab+b^2;$$

and $$\frac{a^3-b^3}{a-b}=a^2+ab+b^2;$$

$$\therefore\ a^3+b^3=(a+b)(a^2-ab+b^2)$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2).$$

In like manner we can resolve into factors any expression which can be written as the sum or the difference of two cubes.

(1) Resolve into factors $8a^3+27b^6$.

Since by § 118, $8a^3=(2a)^3$, and $27b^6=(3b^2)^3$, we can write $8a^3+27b^6$ as $(2a)^3+(3b^2)^3$.

Since $$a^3+b^3=(a+b)(a^2-ab+b^2),$$

we have, by putting $2a$ for a and $3b^2$ for b,

$$(2a)^3+(3b^2)^3=(2a+3b^2)(4a^2-6ab^2+9b^4).$$

(2) Resolve into factors $125x^3 - 1$.

$$125x^3 - 1 = (5x)^3 - 1$$
$$= (5x - 1)(25x^2 + 5x + 1).$$

(3) Resolve into factors $x^6 + y^9$.

$$x^6 + y^9 = (x^2)^3 + (y^3)^3$$
$$= (x^2 + y^3)(x^4 - x^2y^3 + y^6).$$

130. The same method is applicable when the cubes are compound expressions.

(4) Resolve into factors $(x - y)^3 + z^3$.

Since $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,

we have, by putting $x - y$ for a and z for b,

$$(x - y)^3 + z^3 = [(x - y) + z][(x - y)^2 - (x - y)z + z^2]$$
$$= (x - y + z)(x^2 - 2xy + y^2 - xz + yz + z^2).$$

Exercise 36.

Resolve into factors:

1. $a^3 + 8b^3$.
2. $a^3 - 27a^6$.
3. $a^3 + 64$.
4. $125a^3 + 1$.
5. $27x^3y^3 - 1$.
6. $a^3 + 27b^3$.
7. $x^3y^3 - 64$.
8. $64a^6 + 125b^3$.
9. $216a^6 - b^3$.
10. $64a^3 - 27b^3$.
11. $343 - x^3$.
12. $a^3b^3 + 343$.
13. $8a^3 - b^6$.
14. $216m^3 + n^6$.
15. $(a + b)^3 - 1$.
16. $(a - b)^3 + 1$.
17. $(2x + y)^3 - (x - y)^3$.
18. $1 - (a - b)^3$.
19. $8x^3 - (x - y)^3$.
20. $8(x + y)^3 + z^3$.
21. $729y^3 - 64z^3$.
22. $(a + b)^3 - (a - b)^3$.
23. $729a^6 + 216c^6$.
24. $x^3y^3 - 512z^3$.

131. We will conclude this chapter by calling the student's attention to the following statements:

1. When a binomial has the form $x^n - y^n$, but cannot be written as the difference of two perfect squares, or of two perfect cubes, it is still possible to resolve it into two rational factors, one of which is $x-y$. Thus (§ 109),

$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4).$$

2. When a binomial has the form x^n+y^n, but cannot be written as the sum of two perfect cubes, it is still possible to resolve it into two rational factors, except when n is 2, 4, 8, 16, or some other power of 2. Thus (§ 109),

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4).$$

But a^2+b^2, a^4+b^4, a^8+b^8, cannot be resolved into *rational* factors.

3. The student must be careful to select the best method of resolving an expression into factors. Thus, a^6-b^6 can be written as the difference of two squares, or as the difference of two cubes, or be divided by $a-b$, or by $a+b$. Of all these methods, the best is to write the expression as the difference of two squares, as follows

$$\begin{aligned}(a^3)^2-(b^3)^2 &= (a^3+b^3)(a^3-b^3)\\ &= (a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2).\end{aligned}$$

4. From the last example, it will be seen that an expression can sometimes be resolved into three or more factors.

$$\begin{aligned}x^8-b^8 &= (x^4+b^4)(x^4-b^4)\\ &= (x^4+b^4)(x^2+b^2)(x^2-b^2)\\ &= (x^4+b^4)(x^2+b^2)(x+b)(x-b).\end{aligned}$$

5. When a factor occurs in every term of an expression, this factor should first be removed. Thus,

$$\begin{aligned} 8x^2-50a^2+4x-10a &= 2(4x^2-25a^2+2x-5a) \\ &= 2[(4x^2-25a^2)+(2x-5a)] \\ &= 2(2x-5a)(2x+5a+1). \end{aligned}$$

6. Sometimes an expression can be easily resolved if we replace the last term but one by two terms, one of which shall have for a coefficient an exact divisor or a multiple of the last term. Thus,

(1) $$\begin{aligned} x^3-5x^2+11x-15 &= (x^3-5x^2+6x)+(5x-15). \\ &= x(x^2-5x+6)+5(x-3) \\ &= (x-3)[x(x-2)+5] \\ &= (x-3)(x^2-2x+5). \end{aligned}$$

(2) $$\begin{aligned} x^3-9x^2+26x-24 &= (x^3-9x^2+14x)+(12x-24) \\ &= x(x^2-9x+14)+12(x-2) \\ &= x(x-7)(x-2)+12(x-2) \\ &= (x-2)(x^2-7x+12) \\ &= (x-2)(x-3)(x-4). \end{aligned}$$

(3) $$\begin{aligned} x^3-26x-5 &= (x^3-25x)-(x+5) \\ &= x(x^2-25)-(x+5) \\ &= (x+5)(x^2-5x-1). \end{aligned}$$

(4) $$\begin{aligned} x^3+3x^2-4 &= (x^3+2x^2)+(x^2-4) \\ &= x^2(x+2)+(x^2-4) \\ &= (x+2)(x^2+x-2) \\ &= (x+2)(x+2)(x-1). \end{aligned}$$

Exercise 37.

Examples for Review.

Resolve into factors:

1. $a^3 - 9a$.
2. $x^2y^2 - 4xy^4 - 3x^2y^3$.
3. $x^3 + x^2 + x + 1$.
4. $x^2 - 2y + 2x - xy$.
5. $3x^3 + 2x^2 - 9x - 6$.
6. $x^2 - 14x + 49$.
7. $36x^2 - 49y^2$.
8. $x^4 - y^4$.
9. $(x - y)^2 - b^2$.
10. $x^3 + y^3$.
11. $x^6 - y^6$.
12. $x^6 + y^6$.
13. $x^2 - (a - b)^2$.
14. $m^2 + 2mn + n^2 - 1$.
15. $a^2 - (m + n)^2$.
16. $x^2 - 11x + 18$.
17. $x^2 + 4x - 45$.
18. $x^2 + 13x + 36$.
19. $x^2 - 13x - 48$.
20. $x^2 + 9x - 36$.
21. $10x^2 + x - 21$.
22. $6x^2 - x - 12$.
23. $12x^2 - x - 1$.
24. $12x^2 - x - 20$.
25. $9a^2 + 12a + 4$.
26. $a^2 - b^2 - c^2 + 2bc$.
27. $x^4 + x^2y^2 + y^4$.
28. $z^2 - 6z - 40$.
29. $x^2 - 7x - 60$.
30. $a^2 - 19a + 84$.
31. $x^2 + 2ax + 3bx + 6ab$.
32. $x^2 + m^2 - n^2 - 2mx$.
33. $4x^4 - x^2$.
34. $x^{12} + y^{12}$.
35. $9x^4 + 21x^2y^2 + 25y^4$.
36. $x^2 - 4 + y^2 + 2xy$.
37. $2x^2 + 3xy - 2y^2$.
38. $2a^2 - 7a + 6$.
39. $x^4 - 7x^2 + 1$.
40. $1 - a^2 - b^2 - 2ab$.
41. $3x^4 - 6x^3 + 9x^2$.
42. $x^3 - 5x^2 - 2x + 10$.
43. $x^2 + ax - bx - ab$.
44. $2x^2 - 3xy + 4ax - 6ay$.

45. $ax^4 + bx^3 - ax - b$.

46. $a^3 + b^3 + a + b$.

47. $a^3 - b^3 + a - b$.

48. $(x-y)^2 - 2y(x-y)$.

49. $1 - 10xy + 25x^2y^2$.

50. $a^2 - b^2 + 2bc - c^2$.

51. $x^2 + 4y^2 - z^2 - 4xy$.

52. $a^2 - 4b^2 - 9c^2 + 12bc$.

53. $4x^2 + 9y^2 - z^2 - 12xy$.

54. $(a+b)^2 - (c-d)^2$.

55. $x^5 + y^5$.

56. $32x^5 - c^5$.

57. $a^6 + 64y^6$.

58. $729 - x^6$.

59. $x^{12} - y^{12}$.

60. $(a+b)^4 - 1$.

61. $a^2 - b^2 + a - b$.

62. $a^2 + a + 3b - 9b^2$.

63. $6a^2 - a - 77$.

64. $5c^4 - 15c^3 - 90c^2$.

65. $a^2x - c^2x + a^2y - c^2y$.

66. $16x^4 - 81$.

67. $x^4 + x^2 + 1$.

68. $27x^3 - 64a^3$.

69. $x^9 + y^9$.

70. $x^9 - y^9$.

71. $a^8 - 256$.

72. $x^4 + 16a^2x^2 + 256a^4$.

73. $1 - (x-y)^3$.

74. $(x+y)^3 + (2x-y)^3$.

75. $x^6 - 216$.

76. $3x^2 + x - 2$.

77. $2 - 3a - 2a^2$.

78. $4 - 5c - 6c^2$.

79. $2xy - x^2 - y^2 + z^2$.

80. $4a^4 - 9a^2 + 6a - 1$.

81. $a^2 - 2ab + b^2 + 12xy - 4x^2 - 9y^2$.

82. $2x^2 - 4xy + 2y^2 + 2ax - 2ay$.

83. $(a+b)^2 - 1 - ab(a+b+1)$.

84. $x^3 - x^2 + 3x + 5$. (See § 131, 6.)

85. $6x^3 - 23x^2 + 16x - 3$. (See § 131, 6.)

86. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$. (See § 103.)

87. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.

CHAPTER VIII.

COMMON FACTORS AND MULTIPLES.

132. Common Factors. A **common factor** of two or more *integral numbers* is an integral number which divides each of them without a remainder.

133. A **common factor** of two or more *integral and rational expressions* is an integral and rational expression which divides each of them without a remainder.

Thus, $5a$ is a common factor of $20a$ and $25a$; $3x^2y^2$ is a common factor of $12x^2y^2$ and $15x^3y^3$.

134. Two *numbers* are said to be **prime** to each other when they have no common factor except 1.

135. Two *expressions* are said to be **prime** to each other when they have no common factor except 1.

136. The **highest common factor** of two or more *integral numbers* is the greatest number that will divide each of them without a remainder.

137. The **highest common factor** of two or more *integral and rational expressions* is an integral and rational expression of highest degree that will divide each of them without a remainder.

Thus, $3a^2$ is the highest common factor of $3a^2$, $6a^3$, and $12a^4$; $5x^2y^2$ is the highest common factor of $10x^3y^2$ and $15x^2y^2$.

For brevity, we use H. C. F. to stand for "highest common factor."

To find the highest common factor of two algebraic expressions:

Case I.

138. When the factors can be found by inspection.

(1) Find the H. C. F. of $42a^3b^2$ and $60a^2b^4$.

$$42a^3b^2 = 2 \times 3 \times 7 \times aaa \times bb;$$
$$60a^2b^4 = 2 \times 2 \times 3 \times 5 \times aa \times bbbb.$$
$$\therefore \text{ the H. C. F.} = 2 \times 3 \times aa \times bb, \text{ or } 6a^2b^2.$$

(2) Find the H. C. F. of $2a^2x + 2ax^2$ and $3abxy + 3bx^2y$.

$$2a^2x + 2ax^2 = 2ax(a + x);$$
$$3abxy + 3bx^2y = 3bxy(a + x).$$
$$\therefore \text{ the H. C. F.} = x(a + x).$$

(3) Find the H. C. F. of $4x^2 + 4x - 48$, $6x^2 - 48x + 90$.

$$4x^2 + 4x - 48 = 4(x^2 + x - 12)$$
$$= 4(x - 3)(x + 4);$$
$$6x^2 - 48x + 90 = 6(x^2 - 8x + 15)$$
$$= 6(x - 3)(x - 5);$$
$$\therefore \text{ the H. C. F.} = 2(x - 3)$$
$$= 2x - 6.$$

Hence, to find the H. C. F. of two expressions:

Resolve each expression into its simplest factors.

Find the product of all the common factors, taking each common factor the least number of times it occurs in any of the given expressions.

Exercise 38.

Find the H.C.F. of

1. 120 and 168.
2. $36x^3$ and $27x^4$.
3. $42a^2x^3$ and $60a^3x^2$.
4. $36a^3x^2$ and $28x^3y$.
5. $48a^2b^3c$ and $60a^3c^3$.
6. $8(a+b)^2$ and $6(a+b)^3$.
7. $12a(x+y)^2$ and $4b(x+y)^3$.
8. $(x-1)^2(x+2)^2$ and $(x-3)(x+2)^3$.
9. $24a^2b^3(a+b)$ and $42a^3b(a+b)^2$.
10. $x^2(x-3)^2$ and x^2-3x.
11. x^2-16 and x^2+4x.
12. x^2-4x and x^2-6x+8.
13. $x^2-7x+12$ and x^2-16.
14. $9x^2-4y^2$ and $12x^2-xy-6y^2$.
15. x^2-7x-8 and x^2+5x+4.
16. $x^2+3xy-10y^2$ and $x^2-2xy-35y^2$.
17. $x^4-2x^3-24x^2$ and $6x^5-6x^4-180x^3$.
18. x^3-3x^2y and x^3-27y^3.
19. $1+64x^3$ and $1-4x+16x^2$.
20. x^4-81 and x^4+8x^2-9.
21. x^2+2x-3, $x^2+7x+12$.
22. x^2-6x+5, $x^2+3x-40$.
23. $3a^4+15a^3b-72a^2b^2$, $6a^3-30a^2b+36ab^2$.
24. $6x^2y-12xy^2+6y^3$, $3x^2y^2+9xy^3-12y^4$.
25. $1-16c^4$, $1+c^2-12c^4$.
26. $8x^2+2x-1$, $6x^2+7x+2$.
27. $6x^2+x-2$, $12x^2-x-6$.
28. $15x^3-19x^2y+6xy^2$, $10x^4-x^3y-3x^2y^2$.
29. $10x^3y+9x^2y^2-9xy^3$, $4xy^2+15y^3-4x^2y$.

Case II.

139. When the factors cannot be found by inspection.

The method to be employed in this case is similar to that of the corresponding case in Arithmetic. And as in Arithmetic, pairs of continually decreasing numbers are obtained, which contain as a factor the H. C. F. required, so in Algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H. C. F. required.

140. The method depends upon the following principles:

(1) *Any factor of an expression is a factor also of any multiple of that expression.*

Thus, if c is contained 3 times in A, then c is contained 9 times in $3A$, and $3m$ times in mA.

(2) *Any common factor of two expressions is a factor of their sum, their difference, and of the sum or difference of any multiples of the expressions.*

Thus, if c is contained 5 times in A, and 3 times in B, then c is contained 8 times in $A+B$, and 2 times in $A-B$.

Also, in $5A+2B$ it is contained $5\times 5+2\times 3$, or 31 times, and in $5A-2B$ it is contained $5\times 5-2\times 3$, or 19 times.

(3) *The H. C. F. of two expressions is not changed if one of the expressions is divided by a factor that is not a factor of the other expression, or if one is multiplied by a factor that is not a factor of the other expression.*

Thus, the H. C. F. of $4a^2bc^2$ and a^2c^3d is not changed if we remove the factors 4 and b from $4a^2bc^2$, and d from a^2c^3d; or if we multiply $4a^2bc^2$ by 7, and a^2c^3d by 11.

141. We will first find the greatest common factor of two arithmetical numbers, and then show that the same method is used in finding the H. C. F. of two algebraic expressions.

Find the greatest common factor of 18 and 48.

$$
\begin{array}{llll}
18)\,48\,(2 & & & \\
\quad\;\underline{36} & & & \\
\quad\;12)\,18\,(1 & & & \\
\qquad\quad\;\underline{12} & & & \\
\qquad\quad\;\;6)\,12\,(2 & & & \\
\qquad\qquad\quad\;\underline{12} & & &
\end{array}
$$

Since 6 is a factor of itself and of 12, it is, by (2), a factor of $6+12$, or 18.

Since 6 is a factor of 18, it is, by (1), a factor of 2×18, or 36; and therefore, by (2), it is a factor of $36+12$, or 48.

Hence, 6 is a common factor of 18 and 48.

Again, every common factor of 18 and 48 is, by (1), a factor of 2×18, or 36; and, by (2), a factor of $48-36$, or 12.

Every such factor, being now a common factor of 18 and 12, is, by (2), a factor of $18-12$, or 6.

Therefore, the greatest common factor of 18 and 48 is contained in 6, and cannot be greater than 6. Hence 6, which has been shown to be a common factor of 18 and 48, is the **greatest** common factor of 18 and 48.

142. It will be seen that every remainder in the course of the operation contains the greatest common factor sought; and that this is the greatest factor common to that remainder and the preceding divisor. Hence,

The greatest common factor of any divisor and the corresponding dividend is the greatest common factor sought.

143. Let A and B stand for two algebraic expressions, arranged according to the descending powers of a common letter, the degree of B being not higher than that of A.

Let A be divided by B, and let Q stand for the quotient, and R for the remainder. Then

$$\begin{array}{l} B)\ A\ (Q \\ \quad\ \underline{BQ} \\ \quad\ \ R \end{array}$$

Whence, $R = A - BQ$, and $A = BQ + R$.

Any common factor of B and R will, by (2), be a factor of $BQ + R$, that is, of A; and any common factor of A and B will, by (2), be a factor of $A - BQ$, that is, of R.

Any common factor, therefore, of A and B is likewise a common factor of B and R. That is, the common factors of A and B are *the same* as the common factors of B and R; and therefore the H.C.F. of B and R is the H.C.F. of A and B.

If, now, we take the next step in the process, and divide B by R, and denote the remainder by S, then the H.C.F. of S and R can in a similar way be shown to be the same as the H.C.F. of B and R, and therefore the H.C.F. of A and B; and so on for each successive step. Hence,

The H.C.F. of any divisor and the corresponding dividend is the H.C.F. sought.

If at any step there is no remainder, the divisor is a factor of the corresponding dividend, and is therefore the H.C.F. of itself and the corresponding dividend. Hence, the *last divisor* is the H.C.F. sought.

NOTE. From the nature of division, the successive remainders are expressions of lower and lower degrees. Hence, unless at some step the division leaves no remainder, we shall at last have a remainder that does not contain the common letter. In this case the given expressions have no common factor.

Find the H. C. F. of $2x^2+x-3$ and $4x^3+8x^2-x-6$.

$$\begin{array}{rl}
2x^2+x-3) & 4x^3+8x^2-x-6\,(2x+3 \\
 & \underline{4x^3+2x^2-6x} \\
 & \qquad 6x^2+5x-6 \\
 & \qquad \underline{6x^2+3x-9} \\
 & \qquad\qquad 2x+3)\,2x^2+x-3\,(x-1 \\
 & \qquad\qquad\qquad \underline{2x^2+3x} \\
 & \qquad\qquad\qquad -2x-3 \\
 & \qquad\qquad\qquad \underline{-2x-3}
\end{array}$$

$\therefore$ the H. C. F. $=2x+3$.

Each division is continued until the first term of the remainder is of lower degree than that of the divisor.

144. This method is of use only to determine the **compound** factor of the H. C. F. **Simple** factors of the given expressions must first be separated from them, and the H. C. F. of these must be reserved to be multiplied into the compound factor obtained.

Find the H. C. F. of

$12x^4+30x^3-72x^2$ and $32x^3+84x^2-176x$.

$$12x^4+30x^3-72x^2=6x^2(2x^2+5x-12).$$
$$32x^3+84x^2-176x=4x(8x^2+21x-44).$$

$6x^2$ and $4x$ have $2x$ *common*.

$$\begin{array}{rl}
2x^2+5x-12) & 8x^2+21x-44\,(4 \\
 & \underline{8x^2+20x-48} \\
 & \qquad x+4)\,2x^2+5x-12\,(2x-3 \\
 & \qquad\qquad \underline{2x^2+8x} \\
 & \qquad\qquad -3x-12 \\
 & \qquad\qquad \underline{-3x-12}
\end{array}$$

$\therefore$ the H. C. F. $=2x(x+4)$.

145. Modifications of this method are sometimes needed.

(1) Find the H.C.F. of $4x^2-8x-5$ and $12x^2-4x-65$.

$$
\begin{array}{r}
4x^2-8x-5)\,12x^2-4x-65\,(3 \\
\underline{12x^2-24x-15} \\
20x-50
\end{array}
$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x-50$ is made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

The H.C.F. sought *is contained in the remainder* $20x-50$, and is a *compound factor.* Hence if the *simple factor* 10 is removed, the H.C.F. must still be contained in $2x-5$, and therefore the process may be continued with $2x-5$ for a divisor.

$$
\begin{array}{r}
2x-5)\,4x^2-8x-5\,(2x+1 \\
\underline{4x^2-10x} \\
2x-5 \\
\underline{2x-5}
\end{array}
$$

$$\therefore \text{ the H.C.F.} = 2x-5.$$

(2) Find the H.C.F. of

$$21x^3-4x^2-15x-2 \text{ and } 21x^3-32x^2-54x-7.$$

$$
\begin{array}{r}
21x^3-4x^2-15x-2)\,21x^3-32x^2-54x-7\,(1 \\
\underline{21x^3-4x^2-15x-2} \\
-28x^2-39x-5
\end{array}
$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, the expression $21x^3-4x^2-15x-2$ must be *multiplied* by the simple factor 4 to make its first term exactly divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H.C.F. sought, for 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$
\begin{array}{lr}
28x^2+39x+5) & 84x^3-16x^2-60x-8\,(3x \\
 & \underline{84x^3+117x^2+15x} \\
 & -133x^2-75x-8 \\
\text{Multiply by } -4, & -4 \\
 & 532x^2+300x+32\,(19 \\
 & \underline{532x^2+741x+95} \\
\text{Divide by } -63, & -63)\,\underline{-441x-63} \\
 & 7x+1
\end{array}
$$

$$
\begin{array}{lr}
7x+1) & 28x^2+39x+5\,(4x+5 \\
 & \underline{28x^2+4x} \\
 & 35x+5 \\
\therefore \text{ the H.C.F.} = 7x+1. & \underline{35x+5}
\end{array}
$$

(3) Find the H.C.F. of

$$8x^2+2x-3 \text{ and } 6x^3+5x^2-2.$$

$$
\begin{array}{lr}
 & 6x^3+5x^2-2 \\
 & \underline{4} \\
8x^2+2x-3) & 24x^3+20x^2-8\,(3x+7 \\
 & \underline{24x^3+6x^2-9x} \\
 & 14x^2+9x-8 \\
\text{Multiply by } 4, & \underline{4} \\
 & 56x^2+36x-32 \\
 & \underline{56x^2+14x-21} \\
\text{Divide by } 11, & 11)\,\underline{22x-11}
\end{array}
$$

$$
\begin{array}{lr}
2x-1) & 8x^2+2x-3\,(4x+3 \\
 & \underline{8x^2-4x} \\
 & 6x-3 \\
\therefore \text{ the H.C.F.} = 2x-1. & \underline{6x-3}
\end{array}
$$

The following arrangement of the work will be found most convenient:

$$
\begin{array}{r|r|l}
8x^2+2x-3 & 6x^3+5x^2-2 & \\
\underline{8x^2-4x} & \underline{4} & \\
6x-3 & 24x^3+20x^2-8 & 3x \\
\underline{6x-3} & \underline{24x^3+6x^2-9x} & \\
 & 14x^2+9x-8 & \\
 & \underline{4} & \\
 & 56x^2+36x-32 & +7 \\
 & \underline{56x^2+14x-21} & \\
 & 11\overline{)22x-11} & \\
 & 2x-1 & 4x+3
\end{array}
$$

146. From the foregoing examples it will be seen that, in the algebraic process of finding the H.C.F., the following steps, in the order here given, must be carefully observed:

I. Simple factors of the given expressions are to be removed from them, and the H.C.F. of these is to be reserved as a factor of the H.C.F. sought.

II. The resulting compound expressions are to be arranged according to the *descending* powers of a common letter; and that expression which is of the lower degree is to be taken for the divisor; or, if both are of the same degree, that whose first term has the smaller coefficient.

III. Each division is to be continued until the remainder is of lower degree than the divisor.

IV. If the final remainder of any division is found to contain a factor that is not a *common* factor of the given expressions, *this factor is to be removed;* and the resulting expression is to be used as the next divisor.

V. A dividend whose first term is not exactly divisible by the first term of the divisor, is to be *multiplied* by such a number as will make it thus divisible.

Exercise 39.

Find by division the H. C. F. of

1. $4x^2+3x-10,\ 4x^3+7x^2-3x-15.$
2. $2x^3-6x^2+5x-2,\ 8x^3-23x^2+17x-6.$
3. $20x^3+2x^2-18x+48,\ 20x^4-17x^2+48x-3.$
4. $4x^3-2x^2-16x-91,\ 12x^3-28x^2-37x-42.$
5. $12x^3+4x^2+17x-3,\ 24x^3-52x^2+14x-1.$
6. $2x^3+5x^2-2x+3,\ 3x^3+2x^2-17x+12.$
7. $8x^4-6x^3-x^2+15x-25,\ 4x^3+7x^2-3x-15.$
8. $4x^3-4x^2-5x+3,\ 10x^2-19x+6.$
9. $6x^4-13x^3+3x^2+2x,\ 6x^4-10x^3+4x^2-6x+4.$
10. $2x^4-3x^3+2x^2-2x-3,\ 4x^4+3x^2+4x-3.$
11. $3x^4-x^3-2x^2+2x-8,\ 6x^3+13x^2+3x+20.$
12. $3x^5+2x^4+x^2,\ 3x^4+2x^3-3x^2+2x-1.$
13. $6x^5-9x^4+11x^3+6x^2-10x,\ 4x^5+10x^4+10x^3+4x^2+60x.$
14. $2x^5-11x^2-9,\ 4x^5+11x^4+81.$
15. $x^4-4x^3+10x^2-12x+9,\ x^4+2x^2+9.$
16. $2x^3-3x^2-16x+24,\ 4x^5+2x^4-28x^3-16x^2-32x.$
17. $x^4-x^3-14x^2+x+1,\ x^5-4x^4-x^3-2x^2+8x+2.$
18. $6x^3-14ax^2+6a^2x-4a^3,\ x^4-ax^3-a^2x^2-a^3x-2a^4.$
19. $2a^4-2a^3-3a^2-2a,\ 3a^4-a^3-2a^2-16a.$
20. $2x^3+7ax^2+4a^2x-3a^3,\ 4x^3+9ax^2-2a^2x-a^3.$
21. $2x^3-9ax^2+9a^2x-7a^3,\ 4x^3-20ax^2+20a^2x-16a^3.$
22. $2x^4+9x^3+14x+3,\ 3x^4+14x^3+9x+2.$
23. $9x^5-7x^3+8x^2+2x-4,\ 6x^4-7x^3-10x^2+5x+2.$

147. The H. C. F. of three expressions may be obtained by resolving them into their prime factors; or by finding the H. C. F. of two of them, and then of that and the third expression.

For, if A, B, and C are three expressions,
and D the highest common factor of A and B,
and E the highest common factor of D and C,
Then D contains every factor common to A and B,
and E contains every factor common to D and C.
$\therefore$ E contains every factor common to A, B, and C.

Exercise 40.

Find the H. C. F. of

1. $x^2+3x+2,\ x^2+4x+3,\ x^2+6x+5.$
2. $x^2-9x-10,\ x^2-7x-30,\ x^2-11x+10.$
3. $x^3-1,\ x^3-2x^2+1;\ x^3-2x+1.$
4. $6x^2+x-2,\ 2x^2+7x-4,\ 2x^2-7x+3.$
5. $a^2+2ab+b^2,\ a^2-b^2,\ a^3+2a^2b+2ab^2+b^3.$
6. $x^2-5ax+4a^2,\ x^2-3ax+2a^2,\ 3x^2-10ax+7a^2.$
7. $x^2+x-6,\ x^3-2x^2-x+2,\ x^3+3x^2-6x-8.$
8. $x^3+7x^2+5x-1,\ x^2+3x-3x^3-1,\ 3x^3+5x^2+x-1.$
9. $x^3-6x^2+11x-6,\ x^3-8x^2+19x-12,\ x^3-9x^2+26x-24.$

148. Common Multiples. A common multiple of two or more *numbers* is a number which is exactly divisible by each of the numbers.

A common multiple of two or more *expressions* is an expression which is exactly divisible by each of the expressions. Thus, 48 is a common multiple of 4, 6, and 8; $48(x^2-y^2)$ is a common multiple of $3(x-y)$ and $8(x+y)$.

149. The **lowest common multiple** of two or more *numbers* is the least number that is exactly divisible by each of the given numbers.

The **lowest common multiple** of two or more *expressions* is the expression of lowest degree that is exactly divisible by each of the given expressions. Thus, $24(x^2-y^2)$ is the lowest common multiple of $3(x-y)$ and $8(x+y)$.

We use L. C. M. to stand for "lowest common multiple."

To find the L. C. M. of two or more algebraic expressions:

Case I.

150. When the factors of the expressions can be found by inspection.

(1) Find the L. C. M. of $42a^3b^2$ and $60a^2b^4$.

$$42a^3b^2 = 2 \times 3 \times 7 \times a^3 \times b^2;$$
$$60a^2b^4 = 2 \times 2 \times 3 \times 5 \times a^2 \times b^4.$$

The L. C. M. must evidently contain each factor the greatest number of times that it occurs in either expression.

$$\therefore \text{L. C. M.} = 2 \times 2 \times 3 \times 7 \times 5 \times a^3 \times b^4,$$
$$= 420a^3b^4.$$

(2) Find the L. C. M. of

$$4x^2+4x-48,\ 6x^2-48x+90,\ 4x^2-10x-6.$$

$$4x^2+4x-48=4(x^2+x-12) = 2 \times 2(x-3)(x+4);$$
$$6x^2-48x+90=6(x^2-8x+15) = 2 \times 3(x-3)(x-5);$$
$$4x^2-10x-6=2(2x^2-5x-3) = 2(x-3)(2x+1).$$
$$\therefore \text{L. C. M.} = 2 \times 2 \times 3 \times (x-3)(x+4)(x-5)(2x+1).$$

Hence, to find the L. C. M. of two or more expressions:

Resolve each expression into its simplest factors.

Find the product of all the different factors, taking each factor the greatest number of times it occurs in any of the given expressions.

Exercise 41.

Find the L. C. M. of

1. 24, 32, and 60.
2. $24a^2x^3$, $60a^3x^2$, and $32a^2x^2$.
3. $x^2-2xy+y^2$ and x^2-y^2.
4. x^2-4x+4, x^2+4x+4, and x^2-4.
5. x^3+a^3 and x^2-a^2.
6. x^2+ax+a^2, x^2-a^2, and x^3-a^3.
7. $x^2(x-3)^2$ and x^2-5x+6.
8. $x^2+7x+12$ and x^4-9x^2.
9. $x^2-7x+10$, x^2-4x-5, x^2-x-2.
10. $1-3x-4x^2$, $1-4x-5x^2$, $1-9x-10x^2$.
11. $6x^2+7xy-3y^2$, $3x^2+11xy-4y^2$, $2x^2+11xy+12y^2$.
12. $8-14a+6a^2$, $4a+4a^2-3a^3$, $4a^2+2a^3-6a^4$.
13. $6x^3+7x^2-3x$, $3x^2+14x-5$, $6x^2+39x+45$.
14. $6ax+9bx-2ay-3by$, $6x^2+3ax-2xy-ay$.
15. $12ax-9ay-8xy+6y^2$, $6ax+3ay-4xy-2y^2$.
16. $27x^3-a^3$, $6x^2+ax-a^2$, $15x^2-5ax+3bx-ab$.
17. x^3-1, $2x^2-x-1$, $3x^2-x-2$.

Case II.

151. When the factors of the expressions cannot be found by inspection.

In this case the factors of the given expressions may be found by finding their H. C. F. and dividing each expression by this H. C. F.

Find the L. C. M. of

$$6x^3 - 11x^2y + 2y^3 \text{ and } 9x^3 - 22xy^2 - 8y^3.$$

$$\begin{array}{l|l|l}
6x^3 - 11x^2y \qquad + 2y^3 & 9x^3 - 22xy^2 - 8y^3 & 3 \\
\underline{6x^3 - 8x^2y - 4xy^2} & \underline{2} & \\
-3x^2y + 4xy^2 + 2y^3 & 18x^3 - 44xy^2 - 16y^3 & \\
\underline{-3x^2y + 4xy^2 + 2y^3} & \underline{18x^3 - 33x^2y + 6y^3} & \\
 & 11y)\underline{33x^2y - 44xy^2 - 22y^3} & \\
 & \quad 3x^2 - 4xy - 2y^2 & 2x - y
\end{array}$$

$\therefore$ the H. C. F. $= 3x^2 - 4xy - 2y^2$.

Hence, $6x^3 - 11x^2y + 2y^3 = (2x - y)(3x^2 - 4xy - 2y^2)$,

and $9x^3 - 22xy^2 - 8y^3 = (3x + 4y)(3x^2 - 4xy - 2y^2)$.

$\therefore$ the L. C. M. $= (2x - y)(3x + 4y)(3x^2 - 4xy - 2y^2)$.

Exercise 42.

Find the L. C. M. of

1. $6x^3 - 7ax^2 - 20a^2x, \; 3x^2 + ax - 4a^2$.
2. $3x^3 - 13x^2 + 23x - 21, \; 6x^3 + x^2 - 44x + 21$.
3. $3x^3 - 3x^2y + xy^2 - y^3, \; 4x^3 - x^2y - 3xy^2$.
4. $c^4 - 2c^3 + c, \; 2c^4 - 2c^3 - 2c - 2$.
5. $x^3 - 8x + 3, \; x^6 - 3x^5 + 21x - 8$
6. $a^3 - 6a^2x + 12ax^2 - 8x^3, \; 2a^2 - 8ax + 8x^2$.
7. $2x^3 + x^2 - 12x + 9, \; 2x^3 - 7x^2 + 12x - 9$.
8. $7x^3 - 2x^2 - 5, \; 7x^3 + 12x^2 + 10x + 5$.
9. $x^4 - 13x^2 + 36, \; x^4 - x^3 - 7x^2 + x + 6$.
10. $2x^3 + 3x^2 - 7x - 10, \; 4x^3 - 4x^2 - 9x + 5$.
11. $12x^3 - x^2 - 30x - 16, \; 6x^3 - 2x^2 - 13x - 6$.
12. $6x^3 + x^2 - 5x - 2, \; 6x^3 + 5x^2 - 3x - 2$.
13. $x^3 - 9x^2 + 26x - 24, \; x^3 - 12x^2 + 47x - 60$.

CHAPTER IX.

FRACTIONS.

152. An **algebraic fraction** is the indicated quotient of two expressions, written in the form $\frac{a}{b}$.

The dividend a is called the **numerator**, and the divisor b is called the **denominator.**

The numerator and denominator are called the **terms** of the fraction.

153. The introduction of the same factor into the dividend and divisor does not alter the value of the quotient, and the rejection of the same factor from the dividend and divisor does not alter the value of the quotient. Thus $\frac{12}{4} = 3$, $\frac{2 \times 12}{2 \times 4} = 3$, $\frac{12 \div 2}{4 \div 2} = 3$. It follows, therefore, that

The value of a fraction is not altered if the numerator and denominator are both multiplied, or both divided, by the same factor.

Reduction of Fractions.

154. To reduce a fraction is to change its *form* without altering its *value.*

Case I.

155. To reduce a fraction to its lowest terms.

A fraction is in its *lowest terms* when the numerator and denominator have no common factor. We have, therefore, the following rule :

Resolve the numerator and denominator into their prime factors, and cancel all the common factors; or, *divide the numerator and denominator by their highest common factor.*

Reduce the following fractions to their lowest terms:

(1) $\dfrac{38a^2b^3c^4}{57a^3bc^2} = \dfrac{2 \times 19a^2b^3c^4}{3 \times 19a^3bc^2} = \dfrac{2b^2c^2}{3a}.$

(2) $\dfrac{a^3 - x^3}{a^2 - x^2} = \dfrac{(a-x)(a^2+ax+x^2)}{(a-x)(a+x)} = \dfrac{a^2+ax+x^2}{a+x}.$

(3) $\dfrac{a^2+7a+10}{a^2+5a+6} = \dfrac{(a+5)(a+2)}{(a+3)(a+2)} = \dfrac{a+5}{a+3}.$

(4) $\dfrac{6x^2-5x-6}{8x^2-2x-15} = \dfrac{(2x-3)(3x+2)}{(2x-3)(4x+5)} = \dfrac{3x+2}{4x+5}.$

(5) $\dfrac{x^3-4x^2+4x-1}{x^3-2x^2+4x-3}.$

We find by the method of division the H.C.F. of the numerator and denominator to be $x - 1$.

The numerator divided by $x-1$ gives $x^2 - 3x + 1$.

The denominator divided by $x-1$ gives $x^2 - x + 3$.

$\therefore \dfrac{x^3-4x^2+4x-1}{x^3-2x^2+4x-3} = \dfrac{x^2-3x+1}{x^2-x+3}.$

Exercise 43.

Reduce to lowest terms:

1. $\dfrac{6ab^2}{9a^2b}.$
2. $\dfrac{3ab^2c}{15a^2b^2c^2}.$
3. $\dfrac{26x^2y^3}{39xy^5}.$
4. $\dfrac{42m^2b}{49mn^2}.$
5. $\dfrac{30xy^3z^4}{18x^2y^2z^2}.$
6. $\dfrac{21m^2n^2}{28m^2p}.$
7. $\dfrac{34ax^3y^2}{51a^2xy^7}.$
8. $\dfrac{35a^6b^4c^2}{5a^3b^3c}.$
9. $\dfrac{58ab^2c^3}{87a^4b^3c^2}.$

10. $\frac{9xy - 12y^2}{12x^2 - 16xy}$.

11. $\frac{4a^2 - 9c^2}{4a^2 + 6ac}$.

12. $\frac{3a^2 + 6a}{a^2 + 4a + 4}$.

13. $\frac{b^2 - 5b}{b^2 - 4b - 5}$.

14. $\frac{20(a^3 - c^3)}{4(a^2 + ac + c^2)}$.

15. $\frac{x^3 + y^3}{x^2 + 2xy + y^2}$.

16. $\frac{x^3 - 27}{x^2 + 2x - 15}$.

17. $\frac{x^2 - 8x + 15}{2x^2 - 13x + 21}$.

18. $\frac{x^2 - x - 20}{2x^2 - 7x - 15}$.

19. $\frac{4x^2 + 12ax + 9a^2}{8x^3 + 27a^3}$.

20. $\frac{x^2 - y^2 - 2yz - z^2}{x^2 + 2xy + y^2 - z^2}$.

21. $\frac{x^4 + x^2y^2 + y^4}{x^3 - y^3}$.

22. $\frac{2a^2 + 17a + 21}{3a^2 + 26a + 35}$.

23. $\frac{(a + b)^2 - c^2}{(a + b + c)^2}$.

24. $\frac{x^2 - y^2}{y - x}$.

25. $\frac{(x + a)^2 - b^2}{(x + b)^2 - a^2}$.

26. $\frac{(a + b)^2 - (c + d)^2}{(a + c)^2 - (b + d)^2}$.

27. $\frac{(a + c)^2 - b^2}{4a^2c^2 - (a^2 + c^2 - b^2)^2}$.

Reduce by finding the H. C. F. of the terms:

28. $\frac{x^3 - 6x - 4}{3x^3 - 8x + 8}$.

29. $\frac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}$.

30. $\frac{3x^3 - x^2 - x - 1}{3x^3 - 4x^2 - x + 2}$.

31. $\frac{x^4 - 13x^2 + 36}{x^4 - x^3 - 7x^2 + x + 6}$.

32. $\frac{3x^3 + 17x^2 + 22x + 8}{6x^3 + 25x^2 + 23x + 6}$.

33. $\frac{x^3 - 3x^2 - 15x + 25}{x^3 + 7x^2 + 5x - 25}$.

34. $\frac{2x^3 + x^2 - 8x + 3}{3x^3 + 8x^2 + x - 2}$.

35. $\frac{x^3 + 4x^2 - 8x + 24}{x^4 - x^3 + 8x - 8}$.

Case II.

156. **To reduce a fraction to an integral or mixed expression.**

(1) Reduce $\frac{x^3-1}{x-1}$ to an integral expression.

$$\frac{x^3-1}{x-1}=x^2+x+1. \quad (\S\, 108)$$

(2) Reduce $\frac{x^3-1}{x+1}$ to a mixed expression.

$$\begin{array}{l|l} x^3-1 & x+1 \\ \hline x^3+x^2 & x^2-x+1 \\ \hline -x^2-1 & \\ -x^2-x & \\ \hline \quad x-1 & \\ \quad x+1 & \\ \hline \quad -2 & \end{array}$$

$$\therefore \frac{x^3-1}{x+1}=x^2-x+1-\frac{2}{x+1}.$$

Note. By the Law of Signs for division,

$$\frac{-2}{x+1} \text{ and } \frac{2}{-(x+1)}=-\frac{2}{x+1}.$$

The last form is the form usually written.

157. If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to a mixed or integral expression by the following rule:

Divide the numerator by the denominator.

Note. If there is a remainder, this remainder must be written as the numerator of a fraction of which the divisor is the denominator, and this fraction with its proper sign must be annexed to the integral part of the quotient.

Exercise 44.

Reduce to integral or mixed expressions:

1. $\frac{4x^2+12x+3}{4x}$.

2. $\frac{3x^2-9x-2}{3x}$.

3. $\frac{x^2+y^2}{x+y}$.

4. $\frac{x^2+y^2}{x-y}$.

5. $\frac{x^3+y^3}{x-y}$.

6. $\frac{x^3-y^3}{x+y}$.

7. $\frac{x^4+81}{x+3}$.

8. $\frac{2a^2-ab-b^2}{a+b}$.

9. $\frac{3x^2+2x+1}{x+4}$.

10. $\frac{a^3+2x^3}{a+2x}$.

11. $\frac{4x^2-3x-54}{x-4}$.

12. $\frac{3x^3-x^2-2}{x^2+x-3}$.

13. $\frac{4x^2+6ax+9a^2}{2x-3a}$.

14. $\frac{a^3+4a^2-5}{a^2+a-2}$.

Case III.

158. To reduce a mixed expression to a fraction.

The process is precisely the same as in Arithmetic. Hence,

Multiply the integral expression by the denominator, to the product add the numerator, and under the result write the denominator.

(1) Reduce to a fraction $\frac{x-3}{x-4}+x-5$.

$$\frac{x-3}{x-4}+x-5=\frac{x-3+(x-4)(x-5)}{x-4}$$

$$=\frac{x-3+x^2-9x+20}{x-4}$$

$$=\frac{x^2-8x+17}{x-4}.$$

(2) Reduce to a fraction $a-b-\frac{a^2-ab-b^2}{a+b}$.

$$a-b-\frac{a^2-ab-b^2}{a+b}=\frac{(a-b)(a+b)-(a^2-ab-b^2)}{a+b}$$

$$=\frac{a^2-b^2-a^2+ab+b^2}{a+b}$$

$$=\frac{ab}{a+b}.$$

NOTE. The dividing line between the terms of a fraction has the force of a vinculum affecting the numerator. If, therefore, a *minus sign* precedes the dividing line, as in Example (2), and this line is removed, the numerator of the given fraction must be enclosed in a parenthesis preceded by the minus sign, or the sign of every term of the numerator must be changed.

Exercise 45.

Reduce to a fraction:

1. $x+1+\frac{3x-4}{2x}$.
2. $x+1-\frac{2x-3}{2x}$.
3. $a+b-\frac{2ab}{a+b}$.
4. $x-1-\frac{2}{x-2}$.
5. $1-\frac{a-b}{a+b}$.
6. $5a-\frac{1+5a^2}{a}$.
7. $a+x-\frac{a^2+x^2}{a+x}$.
8. $x+4-\frac{x-12}{x-3}$.
9. $a^2-ax+x^2-\frac{x^3}{a+x}$.
10. $a^2+ax+x^2-\frac{a^3}{a-x}$.
11. $\frac{a-3x}{4}-a+2x$.
12. $3a-2b-\frac{3a^2-2b^2}{a+b}$.
13. $2x-7-\frac{21-13x}{x-3}$.
14. $5x-3+\frac{3x+21}{x+7}$.

Case IV.

159. To reduce fractions to their lowest common denominator.

Since the value of a fraction is not altered by multiplying its numerator and denominator by the same factor (§ 153), any number of fractions can be reduced to equivalent fractions having the same denominator.

The process is the same as in Arithmetic. Hence we have the following rule:

Find the lowest common multiple of the denominators; this will be the required denominator. Divide this denominator by the denominator of each fraction.

Multiply the first numerator by the first quotient, the second numerator by the second quotient, and so on.

The products will be the respective numerators of the equivalent fractions.

Note. Every fraction should be in its lowest terms before the common denominator is found.

(1) Reduce $\frac{3x}{4a^2}$, $\frac{2y}{3a}$, and $\frac{5}{6a^3}$ to equivalent fractions having the lowest common denominator.

The L. C. M. of $4a^2$, $3a$, and $6a^3 = 12a^3$.
The respective quotients are $3a$, $4a^2$, and 2.
The products are $9ax$, $8a^2y$, and 10.
Hence, the required fractions are

$$\frac{9ax}{12a^3},\ \frac{8a^2y}{12a^3},\ \text{and}\ \frac{10}{12a^3}.$$

(2) Reduce $\frac{1}{x^2+5x+6}$, $\frac{2}{x^2+4x+3}$, $\frac{3}{x^2+2x+1}$ to equivalent fractions having the lowest common denominator.

$$\frac{1}{x^2+5x+6},\ \frac{2}{x^2+4x+3},\ \frac{3}{x^2+2x+1}$$

$$=\frac{1}{(x+3)(x+2)},\ \frac{2}{(x+3)(x+1)},\ \frac{3}{(x+1)(x+1)}$$

∴ the lowest common denominator (L. C. D.) is

$$(x+3)(x+2)(x+1)(x+1).$$

The respective quotients are

$$(x+1)(x+1),\ (x+2)(x+1),\ \text{and}\ (x+3)(x+2).$$

The respective products are

$$1(x+1)(x+1),\ 2(x+2)(x+1),\ \text{and}\ 3(x+3)(x+2).$$

Hence the required fractions are

$$\frac{(x+1)(x+1)}{(x+3)(x+2)(x+1)^2},\ \frac{2(x+2)(x+1)}{(x+3)(x+2)(x+1)^2},$$

$$\frac{3(x+3)(x+2)}{(x+3)(x+2)(x+1)^2}$$

Exercise 46.

Express with lowest common denominator:

1. $\frac{a-2x}{3a},\ \frac{3x^2-2ax}{9ax}.$
2. $\frac{1}{x+2},\ \frac{2}{x+3}.$
3. $\frac{a}{x-a},\ \frac{a^2}{x^2-a^2}.$
4. $\frac{4a^2+c^2}{4a^2-c^2},\ \frac{2a+c}{2a-c}.$
5. $\frac{x^2+y^2}{25x^2-4y^2},\ \frac{1}{5x+2y}.$
6. $\frac{x+2}{x-2},\ \frac{x-2}{x+2}.$
7. $\frac{1}{x+y},\ \frac{1}{x-y},\ \frac{2}{x^2-y^2}.$
8. $\frac{1}{1+2x},\ \frac{1}{1-4x^2},\ \frac{3}{1-2x}.$
9. $\frac{5}{9-x^2},\ \frac{7}{3+x},\ \frac{3}{3-x}.$
10. $\frac{1}{x^2-9x+18},\ \frac{1}{x^2-10x+24}.$

Addition and Subtraction of Fractions.

160. The algebraic sum of two or more fractions which have the same denominator, is a fraction whose numerator is the algebraic sum of the numerators of the given fractions, and whose denominator is the common denominator of the given fractions. This follows from the distributive law of division.

If the fractions to be added have not the same denominator, they must first be reduced to equivalent fractions having the same denominator. (§ 159.)

To add fractions, therefore,

Reduce the fractions to equivalent fractions having the same denominator; and write the sum of the numerators of these fractions over the common denominator.

161. When the denominators are simple expressions.

(1) Simplify $\frac{3a-4b}{4}-\frac{2a-b+c}{3}+\frac{a-4c}{12}$.

The L. C. D. = 12.

The multipliers, that is, the quotients obtained by dividing 12 by 4, 3, and 12, are 3, 4, and 1.

Hence the sum of the fractions equals

$$\frac{9a-12b}{12}-\frac{8a-4b+4c}{12}+\frac{a-4c}{12}$$

$$=\frac{9a-12b-(8a-4b+4c)+a-4c}{12}$$

$$=\frac{9a-12b-8a+4b-4c+a-4c}{12}$$

$$=\frac{2a-8b-8c}{12}$$

$$=\frac{a-4b-4c}{6}.$$

The above work may be arranged as follows:

The L. C. D. = 12.

The multipliers are 3, 4, and 1, respectively.

$$\begin{array}{rcll} 3(3a-4b) & = & 9a-12b & = \text{1st numerator.} \\ -4(2a-b+c) & = & -8a+4b-4c & = \text{2d numerator.} \\ 1(a-4c) & = & a\qquad\quad -4c & = \text{3d numerator.} \\ \hline & & 2a-8b-8c & \\ & \text{or} & 2(a-4b-4c) & = \text{the sum of the numerators.} \end{array}$$

$$\therefore \text{ sum of fractions} = \frac{2(a-4b-4c)}{12} = \frac{a-4b-4c}{6}.$$

Exercise 47.

Simplify:

1. $\frac{x-1}{2}-\frac{x-3}{5}-\frac{x-7}{10}+\frac{x-2}{5}.$

2. $\frac{2x-1}{3}-\frac{x+7}{6}+\frac{x-4}{4}-\frac{x-3}{2}.$

3. $\frac{7x-5}{8}-\frac{3x+2}{3}+\frac{x+1}{4}-\frac{5x-10}{12}.$

4. $\frac{2x+3}{9}+\frac{x-2}{6}-\frac{5x+4}{12}-\frac{2x-4}{3}.$

5. $\frac{2x+3}{2x}+\frac{x+3}{4x}-\frac{18x+5}{8x^2}-\frac{x-3}{x}.$

6. $\frac{x}{2}-\frac{2x-11}{3}-\frac{x+3}{4}+\frac{x-7}{6}-\frac{x-1}{12}.$

7. $\frac{4a^2}{b^2}-\frac{a+b}{ab}+\frac{4b^2}{a^2}+\frac{a^2b+ab^2-4a^4}{a^2b^2}.$

8. $\frac{5x-11}{4}-\frac{x-1}{10}+\frac{11x-1}{12}-\frac{12x-5}{3}.$

9. $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{4}-\frac{6x+7}{8}.$

10. $\dfrac{2x-6}{5x}-\dfrac{8x-4}{15x}+\dfrac{56x-48}{45x}$.

11. $\dfrac{11xy+2}{x^2y^2}-\dfrac{5y^2-3}{xy^3}-\dfrac{6x^2-5}{x^3y}$.

12. $\dfrac{3}{2x^2y}+\dfrac{1}{6y^2z}-\dfrac{1}{2xz^2}+\dfrac{2x-z}{4x^2z^2}+\dfrac{y-6z}{4x^2yz}$.

162. When the denominators have compound expressions.

(1) Simplify $\dfrac{2a+b}{a-b}-\dfrac{2a-b}{a+b}-\dfrac{6ab}{a^2-b^2}$.

The L. C. D. is $(a-b)(a+b)$.
The multipliers are $a+b$, $a-b$, and 1, respectively.

$$\begin{array}{rcll}
(a+b)(2a+b) & = & 2a^2+3ab+b^2 & = \text{1st numerator.}\\
-(a-b)(2a-b) & = & -2a^2+3ab-b^2 & = \text{2d numerator.}\\
-1(6ab) & = & -6ab & = \text{3d numerator.}\\
\hline
 & & 0 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ sum of fractions $=0$.

(2) Simplify $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$.

The L. C. D. is $(x-2)(x-3)(x-4)$.

$$\begin{array}{rcll}
(x-1)(x-3)(x-4) & = & x^3-8x^2+19x-12 & = \text{1st numerator.}\\
(x-2)(x-2)(x-4) & = & x^3-8x^2+20x-16 & = \text{2d numerator.}\\
(x-2)(x-3)(x-3) & = & x^3-8x^2+21x-18 & = \text{3d numerator.}\\
\hline
 & & 3x^3-24x^2+60x-46 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ sum of fractions $=\dfrac{3x^3-24x^2+60x-46}{(x-2)(x-3)(x-4)}$.

Exercise 48.

Simplify:

1. $\dfrac{1}{x+6}+\dfrac{1}{x-5}$.

2. $\dfrac{1}{1+x}-\dfrac{1}{1-x}$.

3. $\dfrac{1}{1+x}-\dfrac{2}{1-x^2}$.

4. $\dfrac{x+y}{x-y}-\dfrac{x^2-y^2}{(x-y)^2}$.

5. $\frac{x-y}{x+y}-\frac{(x-y)^2}{(x+y)^2}$.

6. $\frac{1}{2a(a+b)}+\frac{1}{2a(a-b)}$.

7. $\frac{1+x}{1+x+x^2}-\frac{1-x}{1-x+x^2}$.

8. $\frac{1}{a-3c}-\frac{(a-3c)^2}{a^3-27c^3}$.

9. $\frac{x+y}{x-y}-\frac{x-y}{x+y}-\frac{4xy}{x^2-y^2}$.

10. $\frac{x}{a-x}+\frac{x}{a+x}+\frac{2x^2}{a^2+x^2}$.

11. $\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}$.

12. $\frac{6-2a}{9-a^2}-\frac{2}{3+a}-\frac{1}{3-a}$.

13. $\frac{1}{x+2y}+\frac{1}{x-2y}-\frac{x}{x^2-4y^2}$.

14. $\frac{1}{y-1}-\frac{2}{y}+\frac{1}{y+1}-\frac{1}{y^2-1}$.

15. $\frac{x}{x-1}-1-\frac{1}{x^2-x}+\frac{1}{x}$.

16. $\frac{b}{a+b}-\frac{ab}{(a+b)^2}-\frac{ab^2}{(a+b)^3}$.

17. $\frac{x+y}{y}-\frac{2x}{x+y}-\frac{x^2(x-y)}{y(x^2-y^2)}$.

HINT. Reduce the last fraction to lowest terms.

18. $\frac{3}{x-a}+\frac{4a}{(x-a)^2}-\frac{5a^2}{(x-a)^3}$.

19. $\frac{1}{x-2a}+\frac{a^2}{x^3-8a^3}-\frac{x+a}{x^2+2ax+4a^2}$.

20. $\frac{x^2-2x+3}{x^3+1}+\frac{x-2}{x^2-x+1}-\frac{1}{x+1}$.

21. $\frac{1}{x-3}+\frac{x-1}{x^2+3x+9}+\frac{x^2+x-3}{x^3-27}$.

22. $\frac{x^2+8x+15}{x^2+7x+10}-\frac{x-1}{x-2}$.

HINT. Reduce the first fraction to lowest terms.

23. $\dfrac{x^2-5ax+6a^2}{x^2-8ax+15a^2}-\dfrac{x-7a}{x-5a}$.

24. $\dfrac{1}{x-2}+\dfrac{1}{x^2-3x+2}-\dfrac{2}{x^2-4x+3}$.

Hint. Express the denominators of the last two fractions in prime factors.

25. $\dfrac{1}{a^2-7a+12}+\dfrac{2}{a^2-4a+3}-\dfrac{3}{a^2-5a+4}$.

26. $\dfrac{3}{10a^2+a-3}-\dfrac{4}{2a^2+7a-4}$.

27. $\dfrac{3}{2-x-6x^2}-\dfrac{1}{1-x-2x^2}$.

163. Since $\dfrac{ab}{b}=a$, and $\dfrac{-ab}{-b}=a$, it follows that

The value of a fraction is not altered if the signs of the numerator and denominator are both changed.

It follows, also, by the Law of Signs, that

The value of a fraction is not altered if the signs of *any even number of factors* in the numerator and denominator of a fraction are changed.

164. Since changing the sign before a fraction is equivalent to changing the sign before the numerator or the denominator, it follows that

The sign before the denominator may be changed, provided the sign before the fraction is changed.

Note. If the denominator is a compound expression, the beginner must remember that the sign of the denominator is changed by changing the sign of every term of the denominator. Thus,

$$\frac{x}{a-x}=-\frac{x}{x-a}.$$

These principles enable us to change the signs of fractions, if necessary, so that their denominators shall be *arranged in the same order.*

(1) Simplify $\frac{2}{x}-\frac{3}{2x-1}+\frac{2x-3}{1-4x^2}$.

Changing the signs before the terms of the denominator of the third fraction and the sign before the fraction, we have

$$\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}.$$

The L.C.D. $= x(2x-1)(2x+1)$.

$$\begin{aligned}
2(2x-1)(2x+1) &= 8x^2-2 &&= \text{1st numerator.}\\
-3x(2x+1) &= -6x^2-3x &&= \text{2d numerator.}\\
-x(2x-3) &= -2x^2+3x &&= \text{3d numerator.}\\
\hline
& \ -2 &&= \text{sum of numerators.}
\end{aligned}$$

$\therefore$ sum of the fractions $= -\frac{2}{x(2x-1)(2x+1)}$.

(2) Simplify

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$$

NOTE. Change the sign of the factor $(b-a)$ in the denominator of the second fraction, and change the sign before the fraction.

Change the signs of the two factors $(c-a)$ and $(c-b)$ in the denominator of the third fraction. We now have

$$\frac{1}{a(a-b)(a-c)}-\frac{1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)}.$$

The L.C.D. $= abc(a-b)(a-c)(b-c)$.

$$\begin{aligned}
bc(b-c) &= b^2c-bc^2 &&= \text{1st numerator.}\\
-ac(a-c) &= -a^2c+ac^2 &&= \text{2d numerator.}\\
ab(a-b) &= a^2b-ab^2 &&= \text{3d numerator.}\\
\hline
a^2b-a^2c-ab^2&+ac^2+b^2c-bc^2 &&= \text{sum of numerators.}
\end{aligned}$$

$$\begin{aligned}
&= a^2(b-c)-a(b^2-c^2)+bc(b-c),\\
&= [a^2-a(b+c)+bc][b-c],\\
&= [a^2-ab-ac+bc][b-c],\\
&= [(a^2-ac)-(ab-bc)][b-c],\\
&= [a(a-c)-b(a-c)][b-c],\\
&= (a-b)(a-c)(b-c).
\end{aligned}$$

$\therefore$ sum of the fractions $= \frac{(a-b)(a-c)(b-c)}{abc(a-b)(a-c)(b-c)} = \frac{1}{abc}$.

Exercise 49.

Simplify:

1. $\frac{x^2}{x^2-1}+\frac{x}{x+1}-\frac{x}{1-x}.$

2. $\frac{a}{a-x}+\frac{3a}{a+x}+\frac{2ax}{x^2-a^2}.$

3. $\frac{3}{2a-3}-\frac{2}{3+2a}+\frac{15}{9-4a^2}.$

4. $\frac{a-b}{b}+\frac{2a}{a-b}+\frac{a^3+a^2b}{b^3-a^2b}.$

5. $\frac{3}{x}+\frac{5}{1-2x}-\frac{2x-7}{4x^2-1}.$

6. $\frac{1}{(x+a)^2}+\frac{1}{(a-x)^2}+\frac{1}{x^2-a^2}.$

7. $\frac{1}{x-y}+\frac{x-y}{x^2+xy+y^2}-\frac{xy-2x^2}{y^3-x^3}.$

8. $\frac{1}{(x-2)(x-3)}+\frac{2}{(x-1)(3-x)}+\frac{1}{(x-1)(x-2)}.$

9. $\frac{bc}{(c-a)(a-b)}+\frac{ac}{(a-b)(b-c)}+\frac{ab}{(b-c)(c-a)}.$

10. $\frac{b+c}{(a-b)(a-c)}+\frac{a+c}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$

11. $\frac{3}{(a-b)(b-c)}-\frac{4}{(b-a)(c-a)}-\frac{6}{(a-c)(c-b)}.$

12. $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}-\frac{1}{xyz}.$

13. $\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2+ac}{(b-a)(b+c)}+\frac{c^2+ab}{(c-a)(c+b)}.$

MULTIPLICATION AND DIVISION OF FRACTIONS.

165. Multiplication of Fractions.

Find the product of $\frac{a}{b} \times \frac{c}{d}$.

Let $\frac{a}{b} = x$, and $\frac{c}{d} = y$.

Then $a = bx$, and $c = dy$.

The product of these two equations is

$$ac = bdxy.$$

Divide by bd, $\frac{ac}{bd} = xy.$

But $\frac{a}{b} \times \frac{c}{d} = xy.$

Therefore, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$

To find the product of two fractions, therefore,

Find the product of the numerators for the required numerator, and the product of the denominators for the required denominator.

166. In like manner,

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf};$$

and so on for any number of fractions.

Again, $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}.$

In like manner,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

167. Division of Fractions. If the product of two numbers is equal to 1, each of the numbers is called the **reciprocal** of the other.

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$,

for $$\frac{b}{a}\times\frac{a}{b}=\frac{ba}{ab}=1.$$

The reciprocal of a fraction, therefore, is the fraction inverted.

Since $$\frac{a}{b}\div\frac{a}{b}=1,$$

and $$\frac{b}{a}\times\frac{a}{b}=1,$$ it follows that

To divide by a fraction is the same as to multiply by its reciprocal.

To divide by a fraction, therefore,

Invert the divisor and multiply.

NOTE. Every mixed expression should first be reduced to a fraction, and every integral expression should be written as a fraction having 1 for the denominator. If a factor is common to a numerator and a denominator, it should be cancelled, as the cancelling of a common factor *before* the multiplication is evidently equivalent to cancelling it *after* the multiplication.

(1) Find the product of $\frac{2a^2b}{3cd^2}\times\frac{6c^2d}{5ab}\times\frac{5ab^2c}{8a^2c^2d^2}$.

$$\frac{2a^2b}{3cd^2}\times\frac{6c^2d}{5ab}\times\frac{5ab^2c}{8a^2c^2d^2}=\frac{2\times6\times5a^3b^3c^3d}{3\times5\times8a^3bc^3d^4}=\frac{b^2}{2d^3}.$$

(2) Find the product of

$$\frac{x^2-y^2}{x^2-3xy+2y^2}\times\frac{xy-2y^2}{x^2+xy}\times\frac{x^2-xy}{(x-y)^2}.$$

$$\frac{x^2-y^2}{x^2-3xy+2y^2}\times\frac{xy-2y^2}{x^2+xy}\times\frac{x^2-xy}{(x-y)^2}$$
$$=\frac{(x-y)(x+y)}{(x-y)(x-2y)}\times\frac{y(x-2y)}{x(x+y)}\times\frac{x(x-y)}{(x-y)(x-y)}$$
$$=\frac{y}{x-y}.$$

NOTE. The common factors cancelled are $(x-y)$, $(x+y)$, $(x-2y)$, x, and $x-y$.

(3) Find the quotient of $\dfrac{ax}{(a-x)^2} \div \dfrac{ab}{a^2-x^2}$.

$$\frac{ax}{(a-x)^2} \div \frac{ab}{a^2-x^2} = \frac{ax}{(a-x)(a-x)} \times \frac{(a-x)(a+x)}{ab}$$
$$= \frac{x(a+x)}{b(a-x)}.$$

The common factors cancelled are a and $a-x$.

Exercise 50.

Simplify:

1. $\dfrac{12x^2}{7y^2} \times \dfrac{14xy}{9x^2z^3}$.
2. $\dfrac{3a^2b^2c^3}{4amn} \times \dfrac{20m^2n^3}{21a^4c^5}$.
3. $\dfrac{6a^2b^2c^5}{7mxy} \times \dfrac{5m^2x^3}{3a^3c^6}$.
4. $\dfrac{9m^2n^2}{8x^3y^3} \times \dfrac{4x^2y^2}{15mn}$.
5. $\dfrac{16a^4b^2}{21x^3y^2} \div \dfrac{4a^3b}{3x^2y}$.
6. $\dfrac{7xy}{12z^2} \div \dfrac{35xz}{36yz^2}$.
7. $\dfrac{x^2-y^2}{x^2+y^2} \times \dfrac{4x}{x+y}$.
8. $\dfrac{3x^2-x}{a} \times \dfrac{2a}{2x^2-4x}$.
9. $\dfrac{3x^2}{5x-10} \times \dfrac{3x-6}{4x^3}$.
10. $\dfrac{8a^3}{a^3-b^3} \div \dfrac{4a^2}{a^2+ab+b^2}$.
11. $\dfrac{x^2+y^2}{x^2-y^2} \div \dfrac{3x^2+3y^2}{x+y}$.
12. $\dfrac{ab-b^2}{a(a+b)} \div \dfrac{b^2}{a(a^2-b^2)}$.
13. $\dfrac{a^2-4x^2}{a^2+4ax} \div \dfrac{a^2-2ax}{ax+4x^2}$.
14. $\dfrac{x^4-y^4}{(x-y)^2} \div \dfrac{x^2+xy}{x-y}$.
15. $\dfrac{x^3+a^3}{x^2-9a^2} \times \dfrac{x+3a}{x+a}$.
16. $\dfrac{a^3-b^3}{a^3+b^3} \div \dfrac{a-b}{a^2-ab+b^2}$.
17. $\dfrac{x^2-1}{x^2-4x-5} \times \dfrac{x^2-25}{x^2+2x-3}$.
18. $\left(1-\dfrac{y^4}{x^4}\right) \div \left(\dfrac{x^2+y^2}{xy}\right)$.

19. $\left(\frac{4x^2}{y^2}-1\right)\left(\frac{2x}{2x-y}-1\right).$

20. $\left(\frac{8x^3}{y^3}-1\right)\left(\frac{4x^2+2xy}{4x^2+2xy+y^2}-1\right).$

21. $\left(x-\frac{xy-y^2}{x+y}\right)\left(x-\frac{xy^2-y^3}{x^2+y^2}\right)\div\left(1-\frac{xy-y^2}{x^2}\right).$

22. $\frac{8a^2b}{c}\times\frac{c^2d}{8a^3}\times\frac{4ab}{cd}\times\frac{bcd-cd^2}{4(b^2-bd)}.$

23. $\frac{y(x^3-y^3)}{x(x+y)}\times\frac{(x^2-y^2)^2}{x^2+xy+y^2}\div\frac{(x-y)^3}{(x+y)^2}.$

24. $\left[\frac{(a+b)^2-c^2}{a^2+ab-ac}\div\frac{(a+c)^2-b^2}{a}\right]\times\frac{(a-b)^2-c^2}{ab-b^2-bc}.$

25. $\frac{(a+b)^2-c^2}{a^2-(b-c)^2}\times\frac{c^2-(a-b)^2}{c^2-(a+b)^2}\times\frac{c-a-b}{ac-a^2+ab}.$

26. $\frac{(x-a)^2-b^2}{(x-b)^2-a^2}\times\frac{x^2-(b-a)^2}{x^2-(a-b)^2}\times\frac{ax+a^2-ab}{bx-ab+b^2}.$

27. $\frac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\times\frac{a+b-c}{a-b+c}.$

28. $\left[\frac{x^2+(x+1)^2}{x(x+1)}+\frac{(x+1)^2-x^2}{x^2+x}\right]\times\frac{2x+1}{x}.$

29. $\frac{2ax^3+2a^3x}{(x-a)^2(x+a)^2}\times\frac{x^2-a^2}{2(x^2+a^2)}\times\frac{x+a}{ax}.$

30. $\frac{a^2-b^2-c^2-2bc}{a^2-ab-ac}\times\left(1-\frac{2c}{a+b+c}\right).$

31. $\frac{a^4+a^2b^2+b^4}{a^6-b^6}\times\frac{a+b}{a^3+b^3}\times\frac{a^2-b^2}{a}.$

32. $\frac{x^2+7xy+12y^2}{x^2+5xy+6y^2}\times\frac{x^2+xy-2y^2}{x^2+3xy-4y^2}.$

168. Complex Fractions. A complex fraction is one that has a fraction in the numerator, or in the denominator, or in both.

(1) Simplify $\dfrac{3x}{x-\frac{1}{4}}$.

$$\frac{3x}{x-\frac{1}{4}}=\frac{3x}{\frac{4x-1}{4}}=\frac{3x}{1}\div\frac{4x-1}{4}=\frac{3x}{1}\times\frac{4}{4x-1}$$

$$=\frac{12x}{4x-1}.$$

NOTE. Generally, the shortest way to simplify a complex fraction is to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator. Thus, in (1), if we multiply both terms by 4, we have at once $\dfrac{12x}{4x-1}$.

(2) Simplify $\dfrac{\frac{a+x}{a-x}-\frac{a-x}{a+x}}{\frac{a+x}{a-x}+\frac{a-x}{a+x}}$.

The L. C. D. of the fractions in the numerator and denominator is

$$(a-x)(a+x).$$

Multiply by $(a-x)(a+x)$, and the result is

$$\frac{(a+x)^2-(a-x)^2}{(a+x)^2+(a-x)^2}$$

$$=\frac{(a^2+2ax+x^2)-(a^2-2ax+x^2)}{(a^2+2ax+x^2)+(a^2-2ax+x^2)}$$

$$=\frac{a^2+2ax+x^2-a^2+2ax-x^2}{a^2+2ax+x^2+a^2-2ax+x^2}$$

$$=\frac{4ax}{2a^2+2x^2}$$

$$=\frac{2ax}{a^2+x^2}.$$

(3) Simplify $\dfrac{x}{1-\left(\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}\right)}$.

$$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}=\frac{x}{1-\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}}$$

$$=\frac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}$$

$$=\frac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}.$$

$$=\frac{x+x^2+x^4}{1+x^2}.$$

NOTE. In a fraction of this kind, called a *continued fraction*, we begin at the *bottom*, and reduce step by step. Thus, in the last example, we take out the fraction $\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}$, and multiply the numerator and denominator by $1-x+x^2$, getting for the result, $\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}$, which simplified is $\dfrac{x-x^2+x^3}{1+x+x^3}$.

Putting this fraction in the given complex fraction for

$$\frac{x}{1+x+\dfrac{x}{1-x+x^2}},$$

we have

$$\frac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}.$$

Multiplying both terms by $1+x+x^3$, we get

$$\frac{x(1+x+x^3)}{1+x+x^3-x+x^2-x^3}$$

$$=\frac{x+x^2+x^4}{1+x^2}.$$

Exercise 51.

Simplify:

1. $$\frac{\frac{x}{m}+\frac{y}{m}}{\frac{z}{m}}.$$

2. $$\frac{x+\frac{y}{z}}{z-\frac{y}{x}}.$$

3. $$\frac{\frac{ab}{c}-3d}{3c-\frac{ab}{d}}.$$

4. $$\frac{1+\frac{1}{x-1}}{1-\frac{1}{x+1}}.$$

5. $$\frac{1+\frac{y}{x-y}}{1-\frac{y}{x+y}}.$$

6. $$\frac{m+\frac{mn}{m-n}}{m-\frac{mn}{m+n}}.$$

7. $$\frac{3}{a+1}-\frac{2a-1}{a^2+\frac{a}{2}-\frac{1}{2}}.$$

8. $$\frac{\frac{2m+n}{m+n}-1}{1-\frac{n}{m+n}}.$$

9. $$\frac{\frac{x^3+y^3}{x^2-y^2}}{\frac{x^2-xy+y^2}{x-y}}.$$

10. $$\frac{\frac{1}{a-b}-\frac{a}{a^2-b^2}}{\frac{a}{ab+b^2}-\frac{b}{a^2+ab}}.$$

11. $$\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}}{\frac{a^2-(b+c)^2}{ab}}.$$

12. $$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}.$$

13. $$\frac{\frac{1}{ab}-\frac{1}{ac}-\frac{1}{bc}}{\frac{a^2-(b-c)^2}{a}}.$$

14. $$1+\frac{x}{1+x+\frac{2x^2}{1-x}}.$$

15. $\dfrac{1}{a+\dfrac{1}{1+\dfrac{a+1}{3-a}}}$.

16. $\dfrac{x+y}{x+y+\dfrac{1}{x-y+\dfrac{1}{x+y}}}$.

17. $\dfrac{x+\dfrac{1}{y}}{x+\dfrac{1}{y+\dfrac{1}{z}}}-\dfrac{1}{y(xyz+x+z)}$.

18. $\dfrac{3abc}{bc+ac+ab}-\dfrac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}$.

19. $\left(\dfrac{m+n}{m-n}+\dfrac{m^2+n^2}{m^2-n^2}\right)\div\left(\dfrac{m-n}{m+n}-\dfrac{m^2+n^2}{m^2-n^2}\right)$.

20. $\left(\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}\right)\times\left(\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}\right)$.

21. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left(1+\dfrac{b^2+c^2-a^2}{2bc}\right)$.

Exercise 52.

Examples for Review.

1. Find the value of $\sqrt{a^3+b^3+c^3}-(a-b-c)^2$, when $a=2$, $b=-2$, and $c=4$.

2. Reduce to lowest terms $\dfrac{3x^3+10x^2+7x-2}{3x^3+13x^2+17x+6}$.

Simplify:

3. $\dfrac{1}{(x-3)(x-2)}-\dfrac{x-4}{(x-1)(x-3)}+\dfrac{x-3}{(x-1)(x-2)}$.

4. $\dfrac{x^2+yz}{(x-y)(x-z)}+\dfrac{y^2+xz}{(y-z)(y-x)}+\dfrac{z^2+xy}{(z-x)(z-y)}.$

5. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right).$

6. $\dfrac{1}{c}\left(\dfrac{1}{x-c}+\dfrac{1}{x+2c}\right)-\dfrac{3}{x^2+cx-2c^2}.$

7. $\dfrac{x^4-y^4}{x^2y^2}\left(\dfrac{x^2}{x^2-y^2}-1+\dfrac{y^2}{x^2+y^2}\right).$

8. $\dfrac{3(x^2+x-2)}{x^2-x-2}-\dfrac{3(x^2-x-2)}{x^2+x-2}-\dfrac{8x}{x^2-4}.$

9. $\left(\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right)\div\left(\dfrac{x}{y}+2-\dfrac{x}{x+y}\right).$

10. $\left(1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right)\left(1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right).$

11. $\dfrac{x^2-xy+y^2}{x^2+xy+y^2}\times\dfrac{x^3-y^3}{x^3+y^3}\div\dfrac{(y-x)^2}{(x+y)^2}.$

12. $\dfrac{2a-b-c}{(a-b)(a-c)}+\dfrac{2b-c-a}{(b-c)(b-a)}+\dfrac{2c-a-b}{(c-a)(c-b)}.$

13. $\dfrac{\dfrac{1}{x+1}-\dfrac{2}{(x+2)(x+1)}}{\dfrac{1}{x+2}-\dfrac{1}{(x+1)(x+2)}}.$

14. $\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}\div\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}.$

15. $\left(1-\dfrac{1-x}{1+x}+\dfrac{1+2x^2}{1-x^2}\right)\left(\dfrac{x+1}{2x+1}\right).$

16. $\dfrac{x^3-8y^3}{x(x-y)}\times\dfrac{x^2-xy+y^2}{x^2+2xy+4y^2}\times\dfrac{x(x^2-y^2)}{x^3+y^3}.$

17. $\dfrac{2}{a}+\dfrac{2}{b}+\dfrac{2}{c}-\dfrac{b+c-a}{bc}-\dfrac{c+a-b}{ac}-\dfrac{a+b-c}{ab}.$

CHAPTER X.

FRACTIONAL EQUATIONS.

169. To reduce Equations containing Fractions.

(1) Solve $\frac{x}{3} - \frac{x-1}{11} = x - 9.$

Multiply by 33, the L. C. M. of the denominators.

Then,
$$11x - 3x + 3 = 33x - 297,$$
$$11x - 3x - 33x = -297 - 3,$$
$$-25x = -300.$$
$$\therefore x = 12.$$

NOTE. Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x, the first term of the numerator, is changed to −, and the − before 1, the second term of the numerator, is changed to +.

Therefore, to clear an equation of fractions,

Multiply each term by the L. C. M. *of the denominators.*

If a fraction is preceded by a minus sign, *the sign of every term of the numerator must be changed when the denominator is removed.*

(2) Solve $\frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{x-7}{x-8} - \frac{x-8}{x-9}.$

NOTE. The solution of this and similar problems will be much easier by combining the fractions on the left side and the fractions on the right side than by the rule given above.

$$\frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)} = \frac{(x-7)(x-9)-(x-8)^2}{(x-8)(x-9)}.$$

By simplifying the numerators, we have

$$\frac{-1}{(x-5)(x-6)}=\frac{-1}{(x-8)(x-9)}.$$

Since the numerators are equal, the denominators are equal.

Hence, $(x-5)(x-6)=(x-8)(x-9)$.

Solving, we have $x=7$.

Exercise 53.

Solve:

1. $\frac{3x-1}{4}-\frac{2x+1}{3}-\frac{4x-5}{5}=4.$

2. $2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}.$

3. $\frac{5x+1}{3}+\frac{19x+7}{9}-\frac{3x-1}{2}=\frac{7x-1}{6}.$

4. $4+\frac{x}{7}=\frac{3x-2}{2}-\frac{11x+2}{14}-\frac{2-7x}{3}.$

5. $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x.$

6. $\frac{6x+7}{9}+\frac{7x-13}{27}=\frac{2x+4}{3}.$

7. $\frac{9x+5}{14}+\frac{8x-7}{14}=\frac{36x+15}{56}+\frac{41}{56}.$

8. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}.$

9. $11-\left(\frac{3x-1}{4}+\frac{2x+1}{3}\right)=10-\left(\frac{2x-5}{3}+\frac{7x-1}{8}\right).$

10. $\frac{7x-4}{9}+\frac{3x-1}{5}-\frac{5(x-1)}{6}=\frac{3(3x-1)}{20}+\frac{x}{7}.$

11. $6x - \frac{27x-1}{4} - \frac{2(4x-1)}{5} = \frac{9x-5}{4} - \frac{11x-2}{3} + 22.$

12. $\frac{10x+11}{6} - \frac{12x-13}{3} - 4 = \frac{7-6x}{4}.$

13. $\frac{3}{y-4} + \frac{5}{2(y-4)} + \frac{9}{2(y-4)} = \frac{1}{2}.$

14. $\frac{2}{x-1} - \frac{5}{2(x-1)} = \frac{8}{3(x-1)} - \frac{x}{x-1} + \frac{5}{18}.$

15. $\frac{2x-3}{2x-4} - 6 = \frac{x+5}{3x-6} - \frac{11}{2}.$

16. $\frac{10-7x}{6-7x} = \frac{5x-4}{5x}.$

17. $\frac{5+8x}{3+2x} = \frac{45-8x}{13-2x}.$

18. $\frac{5x-1}{2x+3} = \frac{5x-3}{2x-3}.$

19. $\frac{x}{3} - \frac{x^2-5x}{3x-7} = \frac{2}{3}.$

20. $\frac{2x+1}{2x-1} - \frac{8}{4x^2-1} = \frac{2x-1}{2x+1}.$

21. $\frac{5-2x}{x-1} - \frac{2-7x}{x+1} = \frac{5x^2+4}{x^2-1}.$

22. $\frac{6}{x+2} - \frac{x+2}{x-2} + \frac{x^2}{x^2-4} = 0.$

23. $\frac{4}{1+x} + \frac{x+1}{1-x} - \frac{x^2-3}{1-x^2} = 0.$

24. $\frac{2x+1}{3x-3} = \frac{7x-1}{6x+6} - \frac{2x^2-3x-45}{4x^2-4}.$

25. $\frac{x^2-x+1}{x-1} + \frac{x^2+x+1}{x+1} = 2x.$

26. $\frac{9x+5}{6(x-1)} + \frac{3x^2-51x-71}{18(x^2-1)} = \frac{15x-7}{9(x+1)}.$

27. $\frac{4}{x+2} + \frac{7}{x+3} - \frac{37}{x^2+5x+6} = 0.$

28. $\frac{1}{x-1} - \frac{1}{x-2} = \frac{1}{x-3} - \frac{1}{x-4}.$

170. If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

(1) Solve $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}$.

Multiply both sides by 14.

Then, $$8x+5+\frac{7(7x-3)}{3x+1}=8x+12.$$

Transpose and combine, $$\frac{7(7x-3)}{3x+1}=7.$$

Divide by 7 and multiply by $3x+1$,

$$7x-3=3x+1.$$
$$\therefore x=1.$$

(2) Solve $\dfrac{3-\dfrac{4x}{9}}{4}=\dfrac{1}{4}-\dfrac{\dfrac{7x}{9}-3}{10}$.

Simplify the complex fractions by multiplying both terms of each fraction by 9.

Then, $$\frac{27-4x}{36}=\frac{1}{4}-\frac{7x-27}{90}.$$

Multiply both sides by 180.

$$135-20x=45-14x+54,$$
$$-6x=-36.$$
$$\therefore x=6.$$

Exercise 54.

Solve:

1. $\frac{4x+3}{10}-\frac{2x-5}{5x-1}=\frac{2x-1}{5}$.

2. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$.

3. $\frac{10x+17}{18} - \frac{12x+2}{13x-16} = \frac{5x-4}{9}.$

4. $\frac{6x+7}{9} + \frac{7x-13}{6x+3} = \frac{2x+4}{3}.$

5. $\frac{6x+7}{15} - \frac{2x-2}{7x-6} = \frac{2x+1}{5}.$

6. $\frac{6x+1}{15} - \frac{2x-4}{7x-16} = \frac{2x-1}{5}.$

7. $\frac{11x-13}{14} - \frac{22x-75}{28} = \frac{13x+7}{2(3x+7)}.$

8. $\frac{2x+8\frac{1}{2}}{9} - \frac{13x-2}{17x-32} + \frac{x}{3} = \frac{7x}{12} - \frac{x+16}{36}.$

9. $\frac{6-5x}{15} - \frac{7-2x^2}{14(x-1)} = \frac{1+3x}{21} - \frac{2x-2\frac{1}{5}}{6} + \frac{1}{105}.$

10. $\frac{2x-5}{5} + \frac{x-3}{2x-15} = \frac{4x-3}{10} - 1\frac{1}{10}.$

171. Literal Equations. Literal equations are equations in which some or all of the given numbers are represented by letters; the numbers regarded as known numbers are usually represented by the *first* letters of the alphabet.

(1) $(a-x)(a+x) = 2a^2 + 2ax - x^2.$

Then,
$$a^2 - x^2 = 2a^2 + 2ax - x^2,$$
$$-2ax = a^2.$$
$$\therefore x = -\frac{a}{2}.$$

Exercise 55.

Solve:

1. $ax + 2b = 3bx + 4a.$
2. $x^2 + b^2 = (a-x)^2.$
3. $(a+x)(b+x) = x(x-c).$
4. $(x-a)(x+b) = (x-b)(x-c).$

5. $\dfrac{x}{a+ax}=\dfrac{b}{c+cx}.$

6. $\dfrac{c+d}{ab+bx}=\dfrac{m-x}{an+nx}.$

7. $\dfrac{x+2}{x-2}=\dfrac{m+n}{m-n}.$

8. $\dfrac{m+n}{2+x}=\dfrac{m-n}{2-x}.$

9. $\dfrac{a+bx}{a+b}=\dfrac{c+dx}{c+d}.$

10. $\dfrac{6x+a}{4x+b}=\dfrac{3x-b}{2x-a}.$

11. $\dfrac{x}{a}+\dfrac{x}{b}+\dfrac{x}{c}=d.$

12. $\dfrac{ax-b}{c}-\dfrac{bx+c}{a}=abc.$

13. $\dfrac{5ax}{a-b}-3a=8x.$

14. $\dfrac{x}{a-b}-\dfrac{5a}{a+b}=\dfrac{2bx}{a^2-b^2}.$

15. $\dfrac{1}{n}+\dfrac{n}{x+n}=\dfrac{x+n}{nx}.$

16. $\dfrac{x}{a}+\dfrac{x}{b-a}=\dfrac{a}{b+a}.$

17. $\dfrac{x-a}{x-b}=\left(\dfrac{2x-a}{2x-b}\right)^2.$

18. $\dfrac{x-a}{2}=\dfrac{(x-b)^2}{2x-a}.$

19. $\dfrac{\dfrac{m+n}{x}}{\dfrac{1}{m}}=\dfrac{a}{b}.$

20. $\dfrac{\dfrac{ax-b}{x}}{\dfrac{a}{b}}=\dfrac{\dfrac{ax+b}{x}}{\dfrac{b}{a}}.$

21. $\dfrac{\dfrac{cx+d}{a}}{\dfrac{cx}{d}}=\dfrac{2d}{a}.$

22. $\dfrac{\dfrac{a+x-3}{3}}{\dfrac{a-x+3}{3}}=\dfrac{b}{c}.$

23. $\dfrac{a-b}{bx+c}+\dfrac{a+b}{ax-c}=0.$

24. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}.$

25. $\dfrac{x+a}{b}-x=b-\dfrac{x-b}{c}+\dfrac{c-bx}{b}.$

26. $\dfrac{20a-bx}{5a}+\dfrac{9c-ax}{3c}+\dfrac{6d-cx}{2d}=10.$

27. $\dfrac{ax}{b}-\dfrac{b-x}{2c}+\dfrac{a(b-x)}{3d}=a.$

172. Problems involving Fractional Equations.

Ex. The sum of the third and fourth parts of a certain number exceeds 3 times the difference of the fifth and sixth parts by 29. Find the number.

Let $x =$ the number.

Then $\frac{x}{3}+\frac{x}{4} =$ the sum of its third and fourth parts,

$\frac{x}{5}-\frac{x}{6} =$ the difference of its fifth and sixth parts,

$3\left(\frac{x}{5}-\frac{x}{6}\right) =$ 3 times the difference of its fifth and sixth parts,

$\frac{x}{3}+\frac{x}{4}-3\left(\frac{x}{5}-\frac{x}{6}\right) =$ the given excess.

But $29 =$ the given excess.

$$\therefore \frac{x}{3}+\frac{x}{4}-3\left(\frac{x}{5}-\frac{x}{6}\right)=29.$$

Multiply by 60 the L. C. D. of the fractions.

$$20x+15x-36x+30x=60\times 29.$$

Combining, $29x = 60 \times 29.$

$$\therefore x = 60.$$

Exercise 56.

1. The sum of the sixth and seventh parts of a number is 13. Find the number.

2. The sum of the third, fourth, and sixth parts of a number is 18. Find the number.

3. The difference between the third and fifth parts of a number is 4. Find the number.

4. The sum of the third, fourth, and fifth parts of a number exceeds the half of the number by 17. Find the number.

5. There are two consecutive numbers, x and $x+1$, such that one-fourth of the smaller exceeds one-ninth of the larger by 11. Find the numbers.

6. Find three consecutive numbers such that if they are divided by 7, 10, and 17, respectively, the sum of the quotients will be 15.

7. Find a number such that the sum of its sixth and ninth parts shall exceed the difference of its ninth and twelfth parts by 9.

8. The sum of two numbers is 91, and if the greater is divided by the less the quotient is 2, and the remainder is 1. Find the numbers.

HINT. $\frac{\text{Dividend} - \text{Remainder}}{\text{Divisor}} = \text{Quotient}.$

9. The difference of two numbers is 40, and if the greater is divided by the less the quotient is 4, and the remainder 4. Find the numbers.

10. Divide the number 240 into two parts such that the smaller part is contained in the larger part 5 times, with a remainder of 6.

11. In a mixture of alcohol and water the alcohol was 24 gallons more than half the mixture, and the water was 4 gallons less than one-fourth the mixture. How many gallons were there of each?

12. The width of a room is three-fourths of its length. If the width was 4 feet more and the length 4 feet less, the room would be square. Find its dimensions.

Ex. Eight years ago a boy was one-fourth as old as he will be one year hence. How old is he now?

Let $x =$ the number of years old he is now.
Then $x - 8 =$ the number of years old he was eight years ago,
and $x + 1 =$ the number of years old he will be one year hence.

$$\therefore x - 8 = \tfrac{1}{4}(x + 1).$$

Solving, $x = 11.$

13. A is 60 years old, and B is three-fourths as old. How many years since B was just one-half as old as A?

14. A father is 50 years old, and his son is half that age. How many years ago was the father two and one-fourth times as old as his son?

15. A father is 40 years old, and his son is one-third that age. In how many years will the son be half as old as his father?

Ex. A can do a piece of work in 6 days, and B can do it in 7 days. How long will it take both together to do the work?

Let x = the number of days it will take both together.

Then $\frac{1}{x}$ = the part both together can do in one day,

$\frac{1}{6}$ = the part A can do in one day,

$\frac{1}{7}$ = the part B can do in one day,

and $\frac{1}{6}+\frac{1}{7}$ = the part both together can do in one day.

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$$\therefore \frac{1}{6}+\frac{1}{7}=\frac{1}{x}.$$

$$7x+6x=42.$$

$$13x=42.$$

$$x=3\tfrac{3}{13}.$$

Therefore they together can do the work in $3\frac{3}{13}$ days.

16. A can do a piece of work in 3 days, B in 4 days, and C in 6 days. How long will it take them to do it working together?

17. A can do a piece of work in $2\frac{1}{3}$ days, B in $3\frac{1}{2}$ days, and C in $4\frac{2}{3}$ days. How long will it take them to do it working together?

18. A and B can separately do a piece of work in 12 days and 20 days, and with the help of C they can do it in 6 days. How long would it take C to do the work?

19. A and B together can do a piece of work in 10 days, A and C in 12 days, and A by himself in 18 days. How many days will it take B and C together to do the work? How many days will it take A, B, and C together?

20. A and B can do a piece of work in 10 days, A and C in 12 days, B and C in 15 days. How long will it take them to do the work if they all work together?

21. A cistern can be filled by three pipes in 15, 20, and 30 minutes respectively. In what time will it be filled if the pipes are all running together?

22. A cistern can be filled by two pipes in 25 minutes and 30 minutes, respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three pipes are running together?

23. A tank can be filled by three pipes in 1 hour and 20 minutes, 2 hours and 20 minutes, and 3 hours and 20 minutes, respectively. In how many minutes can it be filled by all three together?

Ex. A courier who travels 6 miles an hour is followed, after 5 hours, by a second courier who travels $7\frac{1}{2}$ miles an hour. In how many hours will the second courier overtake the first?

Let $x =$ the number of hours the first travels.

Then $x - 5 =$ the number of hours the second travels,

$6x =$ the number of miles the first travels,

and $(x-5)7\frac{1}{2} =$ the number of miles the second travels.

They both travel the same distance.

$$\therefore \quad 6x = (x-5)7\tfrac{1}{2},$$

or

$$12x = 15x - 75.$$

$$\therefore \quad x = 25.$$

Therefore the second courier will overtake the first in 20 hours.

24. A sets out and travels at the rate of 9 miles in 2 hours. Seven hours afterwards B sets out from the same place and travels in the same direction at the rate of 5 miles an hour. In how many hours will B overtake A?

25. A man walks to the top of a mountain at the rate of $2\frac{1}{2}$ miles an hour, and down the same way at the rate of 4 miles an hour, and is out 5 hours. How far is it to the top of the mountain?

26. In going from Boston to Portland, a passenger train, at 27 miles an hour, occupies 2 hours less time than a freight train at 18 miles an hour. Find the distance from Boston to Portland.

27. A person has 6 hours at his disposal. How far may he ride at 6 miles an hour so as to return in time, walking back at the rate of 3 miles an hour?

28. A boy starts from Exeter and walks towards Andover at the rate of 3 miles an hour, and 2 hours later another boys starts from Andover and walks towards Exeter at the rate of $2\frac{1}{2}$ miles an hour. The distance from Exeter to Andover is 28 miles. How far from Exeter will they meet?

Ex. A hare takes 5 leaps while a greyhound takes 3, but 1 of the greyhound's leaps is equal to 2 of the hare's. The hare has a start of 50 of her own leaps. How many leaps must the greyhound take to catch her?

Let $3x =$ the number of leaps the greyhound takes.
Then $5x =$ the number of leaps the hare takes in the same time.
Also, let $a =$ the number of feet in one leap of the hare.
Then $2a =$ the number of feet in one leap of the hound.
Hence $3x \times 2a$ or $6ax =$ the whole distance,
and $(5x + 50)a$ or $5ax + 50a =$ the whole distance.

$$\therefore 6ax = 5ax + 50a.$$

$$\therefore \quad x = 50,$$

and $$3x = 150.$$

Therefore the greyhound must take 150 leaps.

29. A hare takes 7 leaps while a dog takes 5, and 5 of the dog's leaps are equal to 8 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

30. A dog makes 4 leaps while a hare makes 5, but 3 of the dog's leaps are equal to 4 of the hare's. The hare has a start of 60 of the dog's leaps. How many leaps will each take before the hare is caught?

NOTE. If the number of units in the breadth and length of a rectangle are represented by x and $x + a$, respectively, then $x(x + a)$ will represent the number of units of area in the rectangle.

31. A rectangle whose length is $2\frac{1}{2}$ times its breadth would have its area increased by 60 square feet if its length and breadth were each 5 feet more. Find its dimensions.

32. A rectangle has its length 4 feet longer and its width 3 feet shorter than the side of the equivalent square. Find its area.

33. The width of a rectangle is an inch more than half its length, and if a strip 5 inches wide is taken off from the four sides, the area of the strip is 510 square inches. Find the dimensions of the rectangle.

NOTE. If x pounds of metal lose 1 pound when weighed in water, 1 pound of metal will lose $\frac{1}{x}$ of a pound.

34. If 1 pound of tin loses $\frac{5}{37}$ of a pound, and 1 pound of lead loses $\frac{2}{23}$ of a pound, when weighed in water, how many pounds of tin and of lead in a mass of 60 pounds that loses 7 pounds when weighed in water?

35. If 19 ounces of gold lose 1 ounce, and 10 ounces of silver lose 1 ounce, when weighed in water, how many ounces of gold and of silver in a mass of gold and silver weighing 530 ounces in air and 495 ounces in water?

Ex. Find the time between 2 and 3 o'clock when the hands of a clock are together.

At 2 o'clock the hour-hand is 10 minute-spaces ahead of the minute-hand.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 10 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves 12 times as fast as the hour-hand,

$12(x - 10) =$ the number of spaces the minute-hand moves over.

$$\therefore x = 12(x - 10),$$

and
$$11x = 120.$$
$$\therefore x = 10\tfrac{10}{11}.$$

Therefore the time is $10\frac{10}{11}$ minutes past 2 o'clock.

36. At what time between 2 and 3 o'clock are the hands of a watch at right angles?

37. At what time between 3 and 4 o'clock are the hands of a watch pointing in opposite directions?

38. At what time between 7 and 8 o'clock are the hands of a watch together?

Ex. A merchant adds yearly to his capital one-third of it, but takes from it, at the end of each year, $5000 for expenses. At the end of the third year, after deducting the last $5000, he has twice his original capital. How much had he at first?

Let $x =$ number of dollars he had at first.

Then $\frac{4x}{3} - 5000$, or $\frac{4x - 15000}{3}$,

will stand for the number of dollars at the end of first year,

and $\frac{4}{3}\left(\frac{4x - 15000}{3}\right) - 5000$, or $\frac{16x - 105000}{9}$,

will stand for the number of dollars at the end of second year,

and $\frac{4}{3}\left(\frac{16x - 105000}{9}\right) - 5000$, or $\frac{64x - 555000}{27}$,

will stand for the number of dollars at the end of third year.

But $2x$ stands for the number of dollars at the end of third year.

$$\therefore \frac{64x - 555000}{27} = 2x.$$

Whence $x = 55{,}500.$

39. A trader adds yearly to his capital one-fourth of it, but takes from it, at the end of each year, \$800 for expenses. At the end of the third year, after deducting the last \$800, he has $1\frac{7}{8}$ times his original capital. How much had he at first?

40. A trader adds yearly to his capital one-fifth of it, but takes from it, at the end of each year, \$2500 for expenses. At the end of the third year, after deducting the last \$2500, he has $1\frac{7}{10}$ times his original capital. Find his original capital.

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41. A's age now is two-fifths of B's. Eight years ago A's age was two-ninths of B's. Find their ages.

42. A had five times as much money as B. He gave B 5 dollars, and then had only twice as much as B. How much had each at first?

43. At what time between 12 and 1 o'clock are the hour and minute hands pointing in opposite directions?

44. Eleven-sixteenths of a certain principal was at interest at 5 per cent, and the balance at 4 per cent. The entire income was \$1500. Find the principal.

45. A train which travels 36 miles an hour is $\frac{3}{4}$ of an hour in advance of a second train which travels 42 miles an hour. In how long a time will the last train overtake the first?

46. An express train which travels 40 miles an hour starts from a certain place 50 minutes after a freight train, and overtakes the freight train in 2 hours and 5 minutes. Find the rate per hour of the freight train.

47. A messenger starts to carry a despatch, and 5 hours after a second messenger sets out to overtake the first in 8 hours. In order to do this, he is obliged to travel $2\frac{1}{2}$ miles an hour more than the first. How many miles an hour does the first travel?

48. The fore and hind wheels of a carriage are respectively $9\frac{1}{2}$ feet and $11\frac{2}{5}$ feet in circumference. What distance will the carriage have made when one of the fore wheels has made 160 revolutions more than one of the hind wheels?

49. When a certain brigade of troops is formed in a solid square there is found to be 100 men over; but when formed in column with 5 men more in front and 3 men less in depth than before, the column needs 5 men to complete it. Find the number of troops.

50. An officer can form his men in a hollow square 14 deep. The whole number of men is 3136. Find the number of men in the front of the hollow square.

51. A trader increases his capital each year by one-fourth of it, and at the end of each year takes out $2400 for expenses. At the end of 3 years, after deducting the last $2400, he finds his capital to be $10,000. Find his original capital.

52. A and B together can do a piece of work in $1\frac{1}{2}$ days, A and C together in $1\frac{3}{4}$ days, and B and C together in $1\frac{7}{8}$ days. How many days will it take each alone to do the work?

53. A fox pursued by a hound has a start of 100 of her leaps. The fox makes 3 leaps while the hound makes 2; but 3 leaps of the hound are equivalent to 5 of the fox. How many leaps will each take before the hound catches the fox?

173. Formulas and Rules. When the given numbers of a problem are represented by letters, the result obtained from solving the problem is a general expression which includes all problems of that class. Such an expression is called a **formula**, and the translation of this formula into words is called a **rule.**

We will illustrate by examples:

(1) The sum of two numbers is s, and their difference d; find the numbers.

Let $x =$ the smaller number;

then $x + d =$ the larger number.

Hence $x + x + d = s,$

or $2x = s - d.$

$$\therefore\ x = \frac{s-d}{2},$$

and $$x + d = \frac{s - d + 2d}{2},$$

$$= \frac{s+d}{2}.$$

Therefore the numbers are $\frac{s+d}{2}$ and $\frac{s-d}{2}$.

As these formulas hold true whatever numbers s and d stand for, we have the general rule for finding two numbers when their sum and difference are given:

Add the difference to the sum and take half the result for the greater number.

Subtract the difference from the sum and take half the result for the smaller number.

(2) If A can do a piece of work in a days, and B can do the same work in b days, in how many days can both together do it?

Let $x =$ the required number of days.

Then, $\frac{1}{x} =$ the part both together can do in one day.

Now $\frac{1}{a} =$ the part A can do in one day,

and $\frac{1}{b} =$ the part B can do in one day;

therefore $\frac{1}{a} + \frac{1}{b} =$ the part both together can do in one day.

$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$$

Whence $$x = \frac{ab}{a+b}.$$

The translation of this formula gives the following rule for finding the time required by two agents together to produce a given result, when the time required by each agent separately is known:

Divide the product of the numbers which express the units of time required by each to do the work by the sum of these numbers; the quotient is the time required by both together.

174. Interest Formulas. The elements involved in computation of interest are the *principal*, *rate*, *time*, *interest*, and *amount*.

Let $p =$ the principal,
$r =$ the interest of \$1 for 1 year, at the given rate,
$t =$ the time expressed in years,
$i =$ the interest for the given time and rate,
$a =$ the amount (sum of principal and interest).

175. Given the Principal, Rate, and Time; to find the Interest.

Since r is the interest of \$1 for 1 year, pr is the interest of \$$p$ for 1 year, and prt is the interest of \$$p$ for t years.

$$\therefore \mathbf{i = prt}. \qquad \text{(Formula 1.)}$$

RULE. *Find the product of the principal, rate, and time.*

176. Given the Interest, Rate, and Time; to find the Principal.

By Formula 1, $prt = i.$

Divide by rt, $\mathrm{p} = \dfrac{\mathrm{i}}{\mathrm{rt}}$ (Formula 2.)

177. Given the Amount, Rate, and Time; to find the Principal.

From formula 2, $p + prt = a,$

or $p(1 + rt) = a.$

Divide by $1 + rt$, $\mathrm{p} = \dfrac{\mathrm{a}}{1 + \mathrm{rt}}.$ (Formula 3.)

178. Given the Amount, Principal, and Rate; to find the Time.

From formula 2, $p + prt = a.$

Transpose p, $prt = a - p.$

Divide by pr, $\mathrm{t} = \dfrac{\mathrm{a} - \mathrm{p}}{\mathrm{pr}}.$ (Formula 4.)

179. Given the Amount, Principal, and Time; to find the Rate.

From formula 2, $p + prt = a.$

Transpose p, $prt = a - p.$

Divide by pt, $\mathrm{r} = \dfrac{\mathrm{a} - \mathrm{p}}{\mathrm{pt}}.$ (Formula 5.)

Exercise 57.

Solve by the preceding formulas:

1. The sum of two numbers is 40, and their difference is 10. Find the numbers.

2. The sum of two angles is 100°, and their difference is 21° 30′. Find the angles.

3. The sum of two angles is 116° 24′ 30″, and their difference is 56° 21′ 44″. Find the angles.

4. A can do a piece of work in 6 days, and B in 5 days. How long will it take both together to do it?

5. Find the interest of $2750 for 3 years at $4\frac{1}{2}$ per cent.

6. Find the interest of $950 for 2 years 6 months at 5 per cent.

7. Find the amount of $2000 for 7 years 4 months at 6 per cent.

8. Find the rate if the interest on $680 for 7 months is $35.70.

9. Find the rate if the amount of $750 for 4 years is $900.

10. Find the rate if a sum of money doubles in 16 years and 8 months.

11. Find the time required for the interest on $2130 to be $436.65 at 6 per cent.

12. Find the time required for the interest on a sum of money to be equal to the principal at 5 per cent.

13. Find the principal that will produce $161.25 interest in 3 years 9 months at 8 per cent.

14. Find the principal that will amount to $1500 in 3 years 4 months at 6 per cent.

15. How much money is required to yield $2000 interest annually if the money is invested at 5 per cent?

16. Find the time in which $640 will amount to $1000 at 6 per cent.

17. Find the principal that will produce $100 per month, at 6 per cent.

18. Find the rate if the interest on $700 for 10 months is $25.

CHAPTER XI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

180. If we have two unknown numbers and but one relation between them, we can find an unlimited number of pairs of values for which the given relation will hold true. Thus, if x and y are unknown, and we have given only the one relation $x+y=10$, we can *assume* any value for x, and then from the relation $x+y=10$ find the corresponding value of y. For from $x+y=10$ we find $y=10-x$. If x stands for 1, y stands for 9; if x stands for 2, y stands for 8; if x stands for -2, y stands for 12; and so on without end.

181. We may, however, have two equations that express *different* relations between the two unknowns. Such equations are called **independent equations.** Thus, $x+y=10$ and $x-y=2$ are independent equations, for they evidently express *different* relations between x and y.

182. Independent equations involving the *same* unknowns are called **simultaneous equations.**

If we have two unknowns, and have given two independent equations involving them, there is but *one* pair of values which will hold true for both equations. Thus, if in § 181, besides the relation $x+y=10$, we have also the relation $x-y=2$, the only pair of values for which both equations will hold true is the pair $x=6$, $y=4$.

Observe that in this problem x stands for the same number in *both* equations; so also does y.

183. Simultaneous equations are solved by combining the equations so as to obtain a single equation with one unknown number; this process is called **elimination.**

There are three methods of elimination in general use:

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

184. Elimination by Addition or Subtraction.

(1) Solve:

$$\left.\begin{aligned} 5x - 3y &= 20 \\ 2x + 5y &= 39 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 5, and (2) by 3,

$$25x - 15y = 100 \qquad (3)$$
$$6x + 15y = 117 \qquad (4)$$

Add (3) and (4),

$$31x \quad = 217$$
$$\therefore\ x = 7.$$

Substitute the value of x in (2),

$$14 + 5y = 39.$$
$$\therefore\ y = 5.$$

In this solution y is eliminated by *addition.*

(2) Solve:

$$\left.\begin{aligned} 6x + 35y &= 177 \\ 8x - 21y &= 33 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 4, and (2) by 3,

$$24x + 140y = 708 \qquad (3)$$
$$24x - 63y = 99 \qquad (4)$$

Subtract,

$$203y = 609$$
$$\therefore y = 3.$$

Substitute the value of y in (2),

$$8x - 63 = 33.$$
$$\therefore\ x = 12.$$

In this solution x is eliminated by *subtraction.*

185. To eliminate by addition or subtraction, therefore,

Multiply the equations by such numbers as will make the coefficients of one of the unknown numbers equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal coefficients have unlike or like signs.

NOTE. It is generally best to select the letter to be eliminated which requires the smallest multipliers to make its coefficients equal; and the smallest multiplier for each equation is found by dividing the L. C. M. of the coefficients of this letter by the given coefficient in that equation. Thus, in example (2), the L. C. M. of 6 and 8 (the coefficients of x) is 24, and hence the smallest multipliers of the two equations are 4 and 3 respectively.

Sometimes the solution is simplified by first adding the given equations, or by subtracting one from the other.

(3)
$$x + 49y = 51 \quad (1)$$
$$49x + y = 99 \quad (2)$$

Add (1) and (2), $50x + 50y = 150$ (3)

Divide (3) by 50, $x + y = 3.$ (4)

Subtract (4) from (1), $48y = 48.$

$\therefore y = 1.$

Subtract (4) from (2), $48x = 96.$

$\therefore x = 2.$

Exercise 58.

Solve by addition or subtraction:

1. $\left.\begin{aligned} 5x + 2y &= 39 \\ 2x - y &= 3 \end{aligned}\right\}$

2. $\left.\begin{aligned} x + 3y &= 22 \\ 2x - 4y &= 4 \end{aligned}\right\}$

3. $\left.\begin{aligned} 7x - 2y &= 11 \\ x + 5y &= 28 \end{aligned}\right\}$

4. $\left.\begin{aligned} 4x - 5y &= 26 \\ 3x - 6y &= 15 \end{aligned}\right\}$

5. $\left.\begin{aligned} x + 2y &= 35 \\ 3x - 2y &= 17 \end{aligned}\right\}$

6. $\left.\begin{aligned} x + 4y &= 35 \\ 2x - 3y &= 26 \end{aligned}\right\}$

7. $\left.\begin{array}{r}3x+5y=50\\ x-7y=8\end{array}\right\}$

8. $\left.\begin{array}{r}5x+2y=36\\ 2x+3y=43\end{array}\right\}$

9. $\left.\begin{array}{r}3x+7y=50\\ 5x-2y=15\end{array}\right\}$

10. $\left.\begin{array}{r}2x+y=3\\ 7x+5y=21\end{array}\right\}$

11. $\left.\begin{array}{r}x+2y=9\\ 3x-3y=90\end{array}\right\}$

12. $\left.\begin{array}{r}4x-3y=39\\ 3x-4y=17\end{array}\right\}$

13. $\left.\begin{array}{r}7x-2y=69\\ x-10y=39\end{array}\right\}$

14. $\left.\begin{array}{r}3x+7y=16\\ 2x+5y=13\end{array}\right\}$

186. Elimination by Substitution.

(1) Solve: $\left.\begin{array}{r}5x+4y=32\\ 4x+3y=25\end{array}\right\}$

$$5x+4y=32. \qquad (1)$$

$$4x+3y=25. \qquad (2)$$

Transpose $4y$ in (1), $5x=32-4y.$ (3)

Divide by coefficient of x, $x=\frac{32-4y}{5}.$ (4)

Substitute the value of x in (2),

$$4\left(\frac{32-4y}{5}\right)+3y=25,$$

$$\frac{128-16y}{5}+3y=25,$$

$$128-16y+15y=125,$$

$$-y=-3.$$

$$\therefore y=3.$$

Substitute the value of y in (2),

$$4x+9=25.$$

$$\therefore x=4.$$

To eliminate by substitution, therefore,

From one of the equations obtain the value of one of the unknown numbers in terms of the other.

Substitute for this unknown number its value in the other equation, and reduce the resulting equation.

Exercise 59.

Solve by substitution:

1. $\left.\begin{aligned} 2x - 7y &= 0 \\ 3x - 5y &= 11 \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x - 5y &= 4 \\ 3x - 2y &= 10 \end{aligned}\right\}$

3. $\left.\begin{aligned} 2x - 3y &= 1 \\ 3x - 2y &= 29 \end{aligned}\right\}$

4. $\left.\begin{aligned} x + y &= 19 \\ 2x + 7y &= 88 \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x - y &= 5 \\ x + 2y &= 25 \end{aligned}\right\}$

6. $\left.\begin{aligned} 19x - 15y &= 23 \\ 13x - 5y &= 21 \end{aligned}\right\}$

7. $\left.\begin{aligned} x + 10y &= 73 \\ 7x - 2y &= 7 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x - 2y &= 28 \\ 2x + 5y &= 63 \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x - 3y &= 23 \\ 5x + 2y &= 29 \end{aligned}\right\}$

10. $\left.\begin{aligned} 6x - 7y &= 11 \\ 5x - 6y &= 8 \end{aligned}\right\}$

11. $\left.\begin{aligned} 7x + 6y &= 20 \\ 2x + 5y &= 32 \end{aligned}\right\}$

12. $\left.\begin{aligned} x + 5y &= 37 \\ 3x + 2y &= 46 \end{aligned}\right\}$

13. $\left.\begin{aligned} 3x - 7y &= 40 \\ 4x - 3y &= 9 \end{aligned}\right\}$

14. $\left.\begin{aligned} 5x + 9y &= -17 \\ 3x + 11y &= 1 \end{aligned}\right\}$

187. Elimination by Comparison.

Solve: $\left.\begin{aligned} 2x - 5y &= 66 \\ 3x + 2y &= 23 \end{aligned}\right\}$

$$2x - 5y = 66. \quad (1)$$
$$3x + 2y = 23. \quad (2)$$

Transpose $5y$ in (1), and $2y$ in (2),

$$2x = 66 + 5y, \quad (3)$$
$$3x = 23 - 2y. \quad (4)$$

Divide (3) by 2, $x = \frac{66 + 5y}{2}.$ (5)

Divide (4) by 3, $x = \frac{23 - 2y}{3}.$ (6)

Equate the values of x, $\frac{66 + 5y}{2} = \frac{23 - 2y}{3}.$ (7)

Reduce (7),
$$198 + 15y = 46 - 4y,$$
$$19y = -152.$$
$$\therefore\ y = -8.$$

Substitute the value of y in (1),
$$2x + 40 = 66.$$
$$\therefore\ x = 13.$$

188. To eliminate by comparison, therefore,

From each equation obtain the value of one of the unknown numbers in terms of the other.

Form an equation from these equal values and reduce the equation.

Exercise 60.

Solve by comparison:

1. $\left.\begin{array}{r} x + y = 30 \\ 3x - 2y = 25 \end{array}\right\}$

2. $\left.\begin{array}{r} 7x + 3y = 70 \\ 5x - 4y = 7 \end{array}\right\}$

3. $\left.\begin{array}{r} 9x + 4y = 54 \\ 4x + 9y = 89 \end{array}\right\}$

4. $\left.\begin{array}{r} 7x + 2y = 63 \\ 8x - y = 3 \end{array}\right\}$

5. $\left.\begin{array}{r} 2x - 33y = 29 \\ 3x - 47y = 46 \end{array}\right\}$

6. $\left.\begin{array}{r} 2x - y = 9 \\ 5x - 3y = 14 \end{array}\right\}$

7. $\left.\begin{array}{r} 11x - 7y = 6 \\ 9x - 5y = 10 \end{array}\right\}$

8. $\left.\begin{array}{r} 5x + 9y = 188 \\ 13x - 2y = 57 \end{array}\right\}$

9. $\left.\begin{array}{r} 2x - 3y = 1 \\ 5x + 2y = 126 \end{array}\right\}$

10. $\left.\begin{array}{r} 50x - 9y = 1 \\ 7x - y = 3 \end{array}\right\}$

11. $\left.\begin{array}{r} x + 21y = 2 \\ 27y + 2x = 19 \end{array}\right\}$

12. $\left.\begin{array}{r} 10x + 3y = 174 \\ 3x + 10y = 125 \end{array}\right\}$

13. $\left.\begin{array}{r} 6x - 13y = 2 \\ 5x - 12y = 4 \end{array}\right\}$

14. $\left.\begin{array}{r} 2x + y = 108 \\ 10x + 2y = 60 \end{array}\right\}$

15. $\left.\begin{array}{r} 3x - 5y = 5 \\ 7x + y = 265 \end{array}\right\}$

16. $\left.\begin{array}{r} 12x + 7y = 176 \\ 3y - 19x = 3 \end{array}\right\}$

189. Each equation must be simplified, if necessary, before the elimination.

Solve:
$$\left.\begin{aligned}\tfrac{3}{4}x - \tfrac{1}{2}(y+1) &= 1\\ \tfrac{1}{3}(x+1) + \tfrac{3}{4}(y-1) &= 9\end{aligned}\right\}$$

$$\tfrac{3}{4}x - \tfrac{1}{2}(y+1) = 1. \qquad (1)$$
$$\tfrac{1}{3}(x+1) + \tfrac{3}{4}(y-1) = 9. \qquad (2)$$

Multiply (1) by 4, and (2) by 12,

$$3x - 2y - 2 = 4, \qquad (3)$$
$$4x + 4 + 9y - 9 = 108. \qquad (4)$$

From (3), $3x - 2y = 6.$ (5)

From (4), $4x + 9y = 113.$ (6)

Multiply (5) by 4, and (6) by 3,

$$\begin{aligned}12x - 8y &= 24\\ 12x + 27y &= 339\\ \hline 35y &= 315\\ \therefore y &= 9\end{aligned}$$

Substitute value of y in (1), $x = 8.$

Exercise 61.

Solve:

1. $\left.\begin{aligned}\frac{x}{3}+\frac{y}{2}&=\frac{4}{3}\\ \frac{x}{2}+\frac{y}{3}&=\frac{7}{6}\end{aligned}\right\}$

2. $\left.\begin{aligned}\frac{x+y}{3}+\frac{y-x}{2}&=9\\ \frac{x}{2}+\frac{x+y}{9}&=5\end{aligned}\right\}$

3. $\left.\begin{aligned}\frac{4x+5y}{40}&=x-y\\ \frac{2x-y}{3}+2y&=\frac{1}{2}\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{x-1}{8}+\frac{y-2}{5}&=2\\ 2x+\frac{2y-5}{3}&=21\end{aligned}\right\}$

5. $\left.\begin{aligned}\frac{3x-5y}{2}+3&=\frac{2x+y}{5}\\ 8-\frac{x-2y}{4}&=\frac{x}{2}+\frac{y}{3}\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{x+3y}{x-y}&=8\\ \frac{7x-13}{3y-5}&=4\end{aligned}\right\}$

7. $$\left.\begin{array}{l}\frac{x+1}{3}-\frac{y+2}{4}=\frac{2(x-y)}{5}\\ \frac{x+6}{4}-\frac{y+1}{3}=\frac{2x-y}{2}\end{array}\right\}$$

8. $$\left.\begin{array}{l}\frac{x+2y+1}{2x-y+1}=2\\ \frac{3x-y+1}{x-y+3}=5\end{array}\right\}$$

9. $$\left.\begin{array}{l}\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}\\ \frac{2y+4}{3}=\frac{4x+y+26}{8}\end{array}\right\}$$

10. $$\left.\begin{array}{l}\frac{2x-y+3}{3}=\frac{x-2y+19}{4}\\ \frac{3x-4y+3}{4}=\frac{2y-4x+21}{3}\end{array}\right\}$$

11. $$\left.\begin{array}{l}\frac{7+x}{5}-\frac{2x-y}{4}=3y-5\\ \frac{5y-7}{2}+\frac{4x-3}{6}=18-5x\end{array}\right\}$$

12. $$\left.\begin{array}{l}\frac{26+5x-6y}{13}=4y-3x\\ 12+\frac{5x-6y}{6}=\frac{3(x+2y)}{4}\end{array}\right\}$$

13. $$\left.\begin{array}{l}\frac{x+8}{2}=2-\frac{3y-x}{6}\\ \frac{2x+y}{2}-\frac{9x-7}{8}=\frac{7y-4x+36}{16}\end{array}\right\}$$

14. $$\left.\begin{array}{l}\frac{x+2y+3}{13}=\frac{5y-4x-6}{3}\\ \frac{6x-5y+4}{3}=\frac{3x+2y+1}{19}\end{array}\right\}$$

15. $$\left.\begin{aligned} \frac{x+y}{y-x} &= \frac{5}{3} \\ 3x - \frac{3y+44}{7} &= 13 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} \frac{y-4}{4} &= \frac{x+1}{10} \\ \frac{x}{6} + \frac{y+2}{5} &= 3\tfrac{1}{2} \end{aligned}\right\}$$

17. $$\left.\begin{aligned} \frac{5x-6y}{11} + 2x &= 3(y-1) \\ \frac{5x+6y}{10} - \frac{4x-3y}{3} &= y-2 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} \frac{4x-3y-7}{5} &= \frac{9x-4y-25}{30} \\ \frac{y-1}{3} + \frac{10x-3y-20}{20} &= \frac{3x+2y+3}{30} \end{aligned}\right\}$$

NOTE. In solving the following problems proceed as in § 170.

19. $$\left.\begin{aligned} \frac{6y+5}{8} - \frac{4x-5y+3}{4x-2y} &= \frac{9y-4}{12} \\ \frac{8x+3}{4} + \frac{x-3y}{7-x} &= \frac{6x-1}{3} \end{aligned}\right\}$$

20. $$\left.\begin{aligned} x - \frac{2y-x}{23-x} &= x - 9\tfrac{1}{2} \\ y + \frac{y-3}{x-18} &= y - 5\tfrac{2}{3} \end{aligned}\right\}$$

21. $$\left.\begin{aligned} \frac{4x+7}{3} + \frac{5x-4y}{2x+8} &= \frac{17+8x}{6} \\ \frac{5x-12}{4} - \frac{4x-6y-13}{2x-3y} &= \frac{10x-53}{8} \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{7+8x}{10} - \frac{3(x-2y)}{2(x-4)} &= \frac{11+4x}{5} \\ \frac{3(2y+3)}{4} &= \frac{6y+21}{4} - \frac{3y+5x}{2(2y-3)} \end{aligned}\right\}$$

190. Literal Simultaneous Equations.

Solve:
$$\left.\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$$

NOTE. The letters a', b' are read *a prime, b prime*. In like manner, a'', a''' are read *a second, a third*, and a_1, a_2, a_3, are read *a sub one, a sub two, a sub three*. It is sometimes convenient to represent different numbers that have a common property by the same letter marked by *accents* or *suffixes*. Here a and a' have a common property as coefficients of x.

$$ax + by = c. \qquad (1)$$
$$a'x + b'y = c'. \qquad (2)$$

To find the value of y, multiply (1) by a', and (2) by a,

$$aa'x + a'by = a'c$$
$$aa'x + ab'y = ac'$$
$$a'by - ab'y = a'c - ac'$$
$$y = \frac{a'c - ac'}{a'b - ab'}.$$

To find the value of x, multiply (1) by b', and (2) by b,

$$ab'x + bb'y = b'c$$
$$a'bx + bb'y = bc'$$
$$ab'x - a'bx = b'c - bc'$$
$$x = \frac{b'c - bc'}{ab' - a'b}$$

Exercise 62.

Solve:

1. $\left.\begin{aligned} x + y &= s \\ x - y &= d \end{aligned}\right\}$

2. $\left.\begin{aligned} mx + ny &= r \\ m'x + n'y &= r' \end{aligned}\right\}$

3. $\left.\begin{aligned} ax - by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$

4. $\left.\begin{aligned} x - y &= mn \\ cx + aby &= ms \end{aligned}\right\}$

5. $\left.\begin{aligned} bx + ay &= abc \\ x &= dy \end{aligned}\right\}$

6. $\left.\begin{aligned} bx + ay &= 1 \\ b'x - a'y &= 1 \end{aligned}\right\}$

7. $\left.\begin{aligned} 3bx + 2ay &= 3ab \\ 4bx - 3ay &= \tfrac{7}{6}ab \end{aligned}\right\}$

8. $\left.\begin{aligned} 2x - 3y &= a - b \\ 3x - 2y &= a + b \end{aligned}\right\}$

9. $$\left.\begin{aligned}\frac{bx}{a^2-b^2}+\frac{cy}{b^2-a^2}&=\frac{1}{a+b}\\ bx+cy&=a+b\end{aligned}\right\}$$

10. $$\left.\begin{aligned}\frac{x+m}{y-n}&=\frac{a}{b}\\ bx+ay&=c\end{aligned}\right\}$$

11. $$\left.\begin{aligned}\frac{x}{a}+\frac{2y}{b}&=1\\ \frac{2x}{a}-\frac{y}{b}&=\frac{1}{3}\end{aligned}\right\}$$

12. $$\left.\begin{aligned}\frac{x}{a+b}+\frac{y}{a-b}&=2\\ x+y&=2a\end{aligned}\right\}$$

13. $$\left.\begin{aligned}\frac{2x}{3a}+\frac{4y}{3b}&=\frac{3y}{b}-\frac{x}{a}\\ x-y&=a-b\end{aligned}\right\}$$

14. $$\left.\begin{aligned}ax+by&=c\\ bx+ay&=c\end{aligned}\right\}$$

15. $$\left.\begin{aligned}3a^2+ax&=b^2+by\\ ax+2by&=d\end{aligned}\right\}$$

16. $$\left.\begin{aligned}\frac{x}{a+b}-\frac{y}{a-b}&=\frac{1}{a+b}\\ \frac{x}{a+b}+\frac{y}{a-b}&=\frac{1}{a-b}\end{aligned}\right\}$$

17. $$\left.\begin{aligned}\frac{x}{m-a}+\frac{y}{m-b}&=1\\ \frac{x}{n-a}+\frac{y}{n-b}&=1\end{aligned}\right\}$$

191. Fractional simultaneous equations, of which the denominators are simple expressions and contain the unknown numbers, may be solved as follows:

(1) Solve:
$$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{c}{x}+\frac{d}{y}&=n\end{aligned}\right\}$$

We have $$\frac{a}{x}+\frac{b}{y}=m, \qquad (1)$$

and $$\frac{c}{x}+\frac{d}{y}=n. \qquad (2)$$

To find the value of y.

Multiply (1) by c, $$\frac{ac}{x}+\frac{bc}{y}=cm. \qquad (3)$$

Multiply (2) by a, $\frac{ac}{x} + \frac{ad}{y} = an.$ (4)

Subtract (4) from (3), $\frac{bc - ad}{y} = cm - an.$

Multiply both sides by y, $bc - ad = (cm - an)y.$

$$\therefore\ y = \frac{bc - ad}{cm - an}.$$

To find the value of x.

Multiply (1) by d, $\frac{ad}{x} + \frac{bd}{y} = dm.$ (5)

Multiply (2) by b, $\frac{bc}{x} + \frac{bd}{y} = bn.$ (6)

Subtract (6) from (5), $\frac{ad - bc}{x} = dm - bn.$

Multiply both sides by x, $ad - bc = (dm - bn)x.$

$$\therefore\ x = \frac{ad - bc}{dm - bn}.$$

(2) Solve:

$$\left.\begin{aligned}\frac{5}{3x} + \frac{2}{5y} &= 7\\ \frac{7}{6x} - \frac{1}{10y} &= 3\end{aligned}\right\}$$

We have $\frac{5}{3x} + \frac{2}{5y} = 7.$ (1)

and $\frac{7}{6x} - \frac{1}{10y} = 3.$ (2)

Multiply (1) by 15, the L.C.M. of 3 and 5, and (2) by 30,

$$\frac{25}{x} + \frac{6}{y} = 105. \quad (3)$$

$$\frac{35}{x} - \frac{3}{y} = 90. \quad (4)$$

Multiply (4) by 2, and add the result to (3),

$$\frac{95}{x} = 285.$$

$$\therefore\ x = \frac{1}{3}.$$

Substitute the value of x in (1), and we get

$$y = \frac{1}{5}.$$

Exercise 63.

Solve:

1. $\left.\begin{array}{l}\frac{5}{x}+\frac{6}{y}=2\\ \frac{15}{x}-\frac{3}{y}=2\frac{1}{2}\end{array}\right\}$

2. $\left.\begin{array}{l}\frac{5}{x}+\frac{13}{y}=49\\ \frac{7}{x}+\frac{3}{y}=23\end{array}\right\}$

3. $\left.\begin{array}{l}\frac{3}{x}+\frac{8}{y}=3\\ \frac{15}{x}-\frac{4}{y}=4\end{array}\right\}$

4. $\left.\begin{array}{l}\frac{2}{x}+\frac{5}{y}=19\\ \frac{8}{x}-\frac{3}{y}=7\end{array}\right\}$

5. $\left.\begin{array}{l}\frac{4}{5x}+\frac{5}{6y}=5\frac{11}{15}\\ \frac{5}{4x}-\frac{4}{5y}=\frac{11}{20}\end{array}\right\}$

6. $\left.\begin{array}{l}\frac{1}{2x}+\frac{2}{3y}=3\\ \frac{3}{4x}+\frac{4}{5y}=3.9\end{array}\right\}$

7. $\left.\begin{array}{l}\frac{1}{x}+\frac{1}{y}=m\\ \frac{1}{x}-\frac{1}{y}=n\end{array}\right\}$

8. $\left.\begin{array}{l}\frac{m}{x}+\frac{1}{y}=b\\ \frac{n}{x}+\frac{1}{y}=c\end{array}\right\}$

9. $\left.\begin{array}{l}\frac{m}{x}+\frac{n}{y}=a\\ \frac{r}{x}+\frac{s}{y}=b\end{array}\right\}$

10. $\left.\begin{array}{l}\frac{a}{x}-\frac{b}{y}=\frac{ac}{b}\\ \frac{b}{x}-\frac{a}{y}=\frac{bc}{a}\end{array}\right\}$

11. $\left.\begin{array}{l}\frac{3}{ax}-\frac{2}{by}=5\\ \frac{2}{ax}-\frac{3}{by}=2\end{array}\right\}$

12. $\left.\begin{array}{l}\frac{a}{bx}+\frac{b}{ay}=a+b\\ \frac{b}{x}+\frac{a}{y}=a^2+b^2\end{array}\right\}$

192. If three simultaneous equations are given, involving three unknown numbers, one of the unknowns must be eliminated between *two pairs* of the equations; then a second unknown between the two resulting equations.

Likewise, if four or more equations are given, involving four or more unknown numbers, one of the unknowns must be eliminated between three or more pairs of the equations; then a second between the pairs that can be formed of the resulting equations; and so on.

NOTE. The pairs chosen to eliminate from must be independent pairs, so that *each of the given equations* shall be used in the process of the eliminations.

Solve:

$$\left.\begin{aligned} 2x - 3y + 4z &= 4 \quad (1)\\ 3x + 5y - 7z &= 12 \quad (2)\\ 5x - y - 8z &= 5 \quad (3)\end{aligned}\right\}$$

Eliminate z between the equations (1) and (3).

Multiply (1) by 2, $4x - 6y + 8z = 8$ (4)

(3) is $5x - y - 8z = 5$

Add, $9x - 7y = 13$ (5)

Eliminate z between the equations (1) and (2).

Multiply (1) by 7, $14x - 21y + 28z = 28$

Multiply (2) by 4, $12x + 20y - 28z = 48$

Add, $26x - y = 76$ (6)

We now have two equations (5) and (6) involving two unknowns, x and y.

Multiply (6) by 7, $182x - 7y = 532$ (7)

(5) is $9x - 7y = 13$

Subtract (5) from (7), $173x = 519$

$$\therefore x = 3.$$

Substitute the value of x in (6), $78 - y = 76$.

$$\therefore y = 2.$$

Substitute the values of x and y in (1),

$$6 - 6 + 4z = 4.$$

$$\therefore z = 1.$$

Exercise 64.

1. $$\left.\begin{aligned} x+y-8&=0\\ y+z-28&=0\\ x+z-14&=0 \end{aligned}\right\}$$

2. $$\left.\begin{aligned} 4x+3y+2z&=25\\ 3x-2y+5z&=20\\ 10x-5y+3z&=17 \end{aligned}\right\}$$

3. $$\left.\begin{aligned} 5x-2y-2z&=12\\ x+y+z&=8\\ 7x+3y+4z&=42 \end{aligned}\right\}$$

4. $$\left.\begin{aligned} x-y+z&=11\\ 3x+3y-2z&=60\\ 10x-5y-3z&=0 \end{aligned}\right\}$$

5. $$\left.\begin{aligned} 10x-y+3z&=42\\ 7x+2y+z&=51\\ 3x+3y-z&=24 \end{aligned}\right\}$$

6. $$\left.\begin{aligned} 5x+2y-3z&=160\\ 3x+9y+8z&=115\\ 2x-3y-5z&=45 \end{aligned}\right\}$$

7. $$\left.\begin{aligned} 6x-2y+5z&=53\\ 5x+3y+7z&=33\\ x+y+z&=5 \end{aligned}\right\}$$

8. $$\left.\begin{aligned} 3x-3y+4z&=20\\ 6x+2y-7z&=5\\ 2x-y+8z&=45 \end{aligned}\right\}$$

9. $$\left.\begin{aligned} 2x+7y+10z&=25\\ x+y-z&=9\\ 7x-7y-11z&=73 \end{aligned}\right\}$$

10. $$\left.\begin{aligned} 5x+2y-20z&=20\\ 3x-6y+7z&=51\\ 4x+8y-9z&=53 \end{aligned}\right\}$$

11. $$\left.\begin{aligned} x+2y+10z&=44\\ 3x+3y+7z&=384\\ 2x+y+z&=256 \end{aligned}\right\}$$

12. $$\left.\begin{aligned} 10x&=y+4z+56\\ 3y&=2x+3z-98\\ 2z&=x-3y-18 \end{aligned}\right\}$$

13. $$\left.\begin{aligned} 3x-5y-2z&=14\\ 5x-8y-z&=12\\ x-3y-3z&=1 \end{aligned}\right\}$$

14. $$\left.\begin{aligned} 2x+3y+z&=31\\ x-y+3z&=13\\ 10y+5x-2z&=48 \end{aligned}\right\}$$

15. $$\left.\begin{aligned} 2x+3y-4z&=1\\ 10x-6y+12z&=6\\ x+12y+2z&=5 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} 3x+6y+2z&=3\\ 12y+4z-6x&=2\\ 9x+18y-4z&=4 \end{aligned}\right\}$$

17. $$\left.\begin{aligned} 2x+y+2z&=3\\ 5y-4x-4z&=1\\ 3x+9y+z&=9 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} 3x+2y+z&=20\tfrac{1}{2}\\ 2x-y+3z&=26\tfrac{1}{2}\\ x+y+10z&=55 \end{aligned}\right\}$$

19. $\left.\begin{aligned} \frac{1}{x}-\frac{2}{y}+4&=0 \\ \frac{1}{y}-\frac{1}{z}+1&=0 \\ \frac{2}{z}+\frac{3}{x}-14&=0 \end{aligned}\right\}$

23. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{a}&=0 \\ \frac{1}{x}+\frac{1}{z}-\frac{1}{b}&=0 \\ \frac{1}{y}+\frac{1}{z}-\frac{1}{c}&=0 \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=36 \\ \frac{1}{x}+\frac{3}{y}-\frac{1}{z}&=28 \\ \frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}&=20 \end{aligned}\right\}$

24. $\left.\begin{aligned} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}&=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}&=38 \end{aligned}\right\}$

21. $\left.\begin{aligned} \frac{1}{x}+\frac{2}{y}-\frac{3}{z}&=1 \\ \frac{5}{x}+\frac{4}{y}+\frac{6}{z}&=24 \\ \frac{7}{x}-\frac{8}{y}+\frac{9}{z}&=14 \end{aligned}\right\}$

25. $\left.\begin{aligned} \frac{2}{x}+\frac{1}{y}-\frac{3}{z}&=0 \\ \frac{3}{z}-\frac{2}{y}-2&=0 \\ \frac{1}{x}+\frac{1}{z}-\frac{4}{3}&=0 \end{aligned}\right\}$

22. $\left.\begin{aligned} \frac{4}{x}-\frac{3}{y}&=\frac{1}{20} \\ \frac{2}{z}-\frac{3}{x}&=\frac{1}{15} \\ \frac{4}{z}-\frac{5}{y}&=\frac{1}{12} \end{aligned}\right\}$

26. $\left.\begin{aligned} \frac{15}{x}-\frac{4}{y}+\frac{5}{z}&=38 \\ \frac{2}{x}+\frac{3}{y}+\frac{12}{z}&=61 \\ \frac{8}{x}-\frac{5}{y}+\frac{40}{z}&=161 \end{aligned}\right\}$

CHAPTER XII.

PROBLEMS INVOLVING TWO OR MORE UNKNOWN NUMBERS.

193. It is often necessary in the solution of problems to employ two or more letters to represent the numbers to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown numbers employed.

194. If there are *more* equations than unknown numbers, some of them are superfluous or inconsistent; if there are *fewer* equations than unknown numbers, the problem is indeterminate.

(1) If A gives B $10, B will have three times as much money as A. If B gives A $10, A will have twice as much money as B. How much has each?

Let $x =$ number of dollars A has,
and $y =$ number of dollars B has.

Then, after A gives B $10,

$x - 10 =$ the number of dollars A has,
$y + 10 =$ the number of dollars B has.

$$\therefore y + 10 = 3(x - 10). \qquad (1)$$

If B gives A $10,

$x + 10 =$ the number of dollars A has,
$y - 10 =$ the number of dollars B has.

$$\therefore x + 10 = 2(y - 10). \qquad (2)$$

From the solution of equations (1) and (2), $x = 22$, and $y = 26$.

Therefore A has $22, and B has $26.

(2) If the smaller of two numbers is divided by the greater, the quotient is 0.21, and the remainder 0.0057; but if the greater is divided by the smaller, the quotient is 4 and the remainder 0.742. Find the numbers.

Let $x =$ the greater number,
and $y =$ the smaller number.

Then $$\frac{y - 0.0057}{x} = 0.21, \quad (1)$$

and $$\frac{x - 0.742}{y} = 4. \quad (2)$$

$$\therefore\ y - 0.21x = 0.0057, \quad (3)$$

$$x - 4y = 0.742. \quad (4)$$

Multiply (3) by 4, $$4y - 0.84x = 0.0228 \quad (5)$$

(4) is $$-4y + x = 0.742$$

By adding, $$0.16x = 0.7648$$

$$\therefore\ x = 4.78.$$

Substituting the value of x in (4),

$$-4y = -4.038,$$

$$\therefore\ y = 1.0095.$$

Exercise 65.

1. If A gives B $100, A will then have half as much money as B; but if B gives A $100, B will have one-third as much as A. How much has each?

2. If the greater of two numbers is divided by the smaller, the quotient is 4 and the remainder 0.37; but if the smaller is divided by the greater, the quotient is 0.23 and the remainder 0.0149. Find the numbers.

3. A certain number of persons paid a bill. If there had been 10 persons more, each would have paid $2 less; but if there had been 5 persons less, each would have paid $2.50 more. Find the number of persons and the amount of the bill.

4. A train proceeded a certain distance at a uniform rate. If the speed had been 6 miles an hour more, the time occupied would have been 5 hours less; but if the speed had been 6 miles an hour less, the time occupied would have been $7\frac{1}{2}$ hours more. Find the distance.

HINT. If $x =$ the number of hours the train travels, and y the number of miles per hour, then $xy =$ the distance.

5. A man bought 10 cows and 50 sheep for $750. He sold the cows at a profit of 10 per cent, and the sheep at a profit of 30 per cent, and received in all $875. Find the average cost of a cow and of a sheep.

6. It is 40 miles from Dover to Portland. A sets out from Dover, and B from Portland, at 7 o'clock A.M., to meet each other. A walks at the rate of $3\frac{1}{2}$ miles an hour, but stops 1 hour on the way; B walks at the rate of $2\frac{1}{2}$ miles an hour. At what time of day and how far from Portland will they meet?

7. The sum of two numbers is 35, and their difference exceeds one-fifth of the smaller number by 2. Find the numbers.

8. If the greater of two numbers is divided by the smaller, the quotient is 7 and the remainder 4; but if three times the greater number is divided by twice the smaller, the quotient is 11 and the remainder 4. Find the numbers.

9. If 3 yards of velvet and 12 yards of silk cost $60, and 4 yards of velvet and 5 yards of silk cost $58, what is the price of a yard of velvet and of a yard of silk?

10. If 5 bushels of wheat, 4 of rye, and 3 of oats are sold for $9; 3 bushels of wheat, 5 of rye, and 6 of oats for $8.75; and 2 bushels of wheat, 3 of rye, and 9 of oats for $7.25; what is the price per bushel of each kind of grain?

NOTE I. A fraction the terms of which are unknown may be represented by $\frac{x}{y}$.

Ex. A certain fraction becomes equal to $\frac{1}{2}$ if 2 is added to its numerator, and equal to $\frac{1}{3}$ if 3 is added to its denominator. Find the fraction.

Let $\frac{x}{y}$ = the required fraction.

Then $\frac{x+2}{y} = \frac{1}{2}$,

and $\frac{x}{y+3} = \frac{1}{3}$.

The solution of these equations gives 7 for x, and 18 for y.

Therefore the required fraction is $\frac{7}{18}$.

11. A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator and 1 to its denominator, and equal to $\frac{1}{4}$ if 3 is subtracted from its numerator and from its denominator. Find the fraction.

12. A certain fraction becomes equal to $\frac{9}{11}$ if 1 is added to double its numerator, and equal to $\frac{1}{3}$ if 3 is subtracted from its numerator and from its denominator. Find the fraction.

13. Find two fractions with numerators 11 and 5 respectively, such that their sum is $1\frac{4}{9}$, and if their denominators are interchanged their sum is $2\frac{1}{9}$.

14. There are two fractions with denominators 20 and 16 respectively. The fraction formed by taking for a numerator the sum of the numerators, and for a denominator the sum of the denominators, of the given fractions, is equal to $\frac{1}{3}$; and the fraction formed by taking for a numerator the difference of the numerators, and for a denominator the difference of the denominators of the given fractions, is equal to $\frac{1}{2}$. Find the fractions.

Note II. A number consisting of *two* digits which are unknown may be represented by $10x + y$, in which x and y represent the digits of the number. Likewise, a number consisting of *three* digits which are unknown may be represented by $100x + 10y + z$, in which x, y, and z represent the digits of the number. For example, the expression 364 means $300 + 60 + 4$; or, 100 *times* 3 + 10 *times* 6 + 4.

Ex. The sum of the two digits of a number is 10, and if 18 is added to the number, the digits will be reversed. Find the number.

Let $x =$ tens' digit,

and $y =$ units' digit.

Then $10x + y =$ the number.

Hence $x + y = 10$, (1)

and $10x + y + 18 = 10y + x$. (2)

From (2), $9x - 9y = -18$,

or $x - y = -2$. (3)

Add (1) and (3), $2x = 8$,

and therefore $x = 4$.

Subtract (3) from (1), $2y = 12$,

and therefore $y = 6$.

Therefore the number is 46.

15. The sum of the two digits of a number is 9, and if 27 is subtracted from the number, the digits will be reversed. Find the number.

16. The sum of the two digits of a number is 9, and if the number is divided by the sum of the digits, the quotient is 5. Find the number.

17. A certain number is expressed by two digits. The sum of the digits is 11. If the digits are reversed, the new number exceeds the given number by 27. Find the number.

18. A certain number is expressed by three digits. The sum of the digits is 21. The sum of the first and last digits is twice the middle digit. If the hundreds' and tens' digits are interchanged, the number is diminished by 90. Find the number.

19. A certain number is expressed by three digits, the units' digit being zero. If the hundreds' and tens' digits are interchanged, the number is diminished by 180. If the hundreds' digit is halved, and the tens' and units' digits are interchanged, the number is diminished by 336. Find the number.

20. A number is expressed by three digits. If the digits are reversed, the new number exceeds the given number by 99. If the number is divided by nine times the sum of its digits, the quotient is 3. The sum of the hundreds' and units' digits exceeds the tens' digit by 1. Find the number.

NOTE III. If a boat moves at the rate of x miles an hour in still water, and if it is on a stream that runs at the rate of y miles an hour, then

$x + y$ represents its rate *down* the stream,
$x - y$ represents its rate *up* the stream.

21. A boatman rows 20 miles down a river and back in 8 hours. He finds that he can row 5 miles down the river in the same time that he rows 3 miles up the river. Find the time he was rowing down and up respectively.

22. A boat's crew which can pull down a river at the rate of 10 miles an hour finds that it takes twice as long to row a mile up the river as to row a mile down. Find the rate of their rowing in still water and the rate of the stream.

23. A boatman rows down a stream, which runs at the rate of $2\frac{1}{2}$ miles an hour, for a certain distance in 1 hour and 30 minutes; it takes him 4 hours and 30 minutes to return. Find the distance he pulled down the stream and his rate of rowing in still water.

NOTE IV. It is to be remembered that if a certain work can be done in x units of time (days, hours, etc.), the part of the work done in *one* unit of time will be represented by $\frac{1}{x}$.

Ex. A cistern has three pipes, A, B, and C. A and B will fill the cistern in 1 hour and 10 minutes, A and C in 1 hour and 24 minutes, B and C in 2 hours and 20 minutes. How long will it take each pipe alone to fill it?

1 hour and 10 minutes = 70 minutes.
1 hour and 24 minutes = 84 minutes.
2 hours and 20 minutes = 140 minutes.

Let x = number of minutes it takes A to fill it,
y = number of minutes it takes B to fill it,
and z = number of minutes it takes C to fill it.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ = the parts A, B, and C can fill in one minute respectively,

and $\frac{1}{x}+\frac{1}{y}$ = the part A and B together can fill in one minute.

But $\frac{1}{70}$ = the part A and B together can fill in one minute.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{70}. \quad (1)$$

In like manner,
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{84}, \quad (2)$$

and
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{140}. \quad (3)$$

Add, and divide by 2,
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{60}. \quad (4)$$

Subtract (1) from (4), $\frac{1}{z}=\frac{1}{420}$.

Subtract (2) from (4), $\frac{1}{y}=\frac{1}{210}$.

Subtract (3) from (4), $\frac{1}{x}=\frac{1}{105}$.

Therefore x, y, z = 105, 210, 420, respectively.

Hence A can fill it in 1 hour and 45 minutes, B in 3 hours and 30 minutes, and C in 7 hours.

24. A and B can do a piece of work together in 3 days, A and C in 4 days, B and C in $4\frac{1}{2}$ days. How long will it take each alone to do the work?

25. A and B can do a piece of work in $2\frac{1}{2}$ days, A and C in $3\frac{1}{3}$ days, B and C in 4 days. How long will it take each alone to do the work?

26. A and B can do a piece of work in a days, A and C in b days, B and C in c days. How long will it take each alone to do the work?

NOTE V. If x represents the number of linear units in the length, and y in the width, of a rectangle, xy will represent the number of its units of surface; the surface unit having the same name as the linear unit of its side.

27. If the length of a rectangular field were increased by 5 yards and its breadth increased by 10 yards, its area would be increased by 450 square yards; but if its length were increased by 5 yards and its breadth diminished by 10 yards, its area would be diminished by 350 square yards. Find its dimensions.

28. If the floor of a certain hall had been 2 feet longer and 4 feet wider, it would have contained 528 square feet more; but if the length and width were each 2 feet less, it would contain 316 square feet less. Find its dimensions.

29. If the length of a rectangle was 4 feet less and the width 3 feet more, the figure would be a square of the same area as the given rectangle. Find the dimensions of the rectangle.

NOTE VI. In considering *the rate of increase or decrease* in quantities, it is usual to take 100 as a *common standard of reference*, so that the increase or decrease is calculated for every 100, and therefore called *per cent.*

It is to be observed that the representative of the number resulting after an increase has taken place is 100 + increase per cent; and after a decrease, 100 − decrease per cent.

Interest depends upon the *time* for which the money is lent, as well as upon the *rate per cent* charged; the rate per cent charged being the rate per cent on the principal for *one year*. Hence,

$$\text{Simple interest} = \frac{\text{Principal} \times \text{Rate per cent} \times \text{Time}}{100},$$

where Time means *number of years or fraction of a year.*

$$\text{Amount} = \text{Principal} + \text{Interest}.$$

In questions relating to stocks, 100 is taken as the representative of the *stock*, the *price* represents its market value, and the *per cent* represents the *interest* which the *stock* bears. Thus, if six per cent stocks are quoted at 108, the meaning is, that the price of $100 of the stock is $108, and that the interest derived from $100 of the *stock* will be $\frac{6}{100}$ of $100, that is, $6 a year. The rate of interest on the *money invested* will be $\frac{100}{108}$ of 6 per cent.

30. A man has $10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 6 per cent; the income from his 5 per cent investment is $60 more than from his 6 per cent. How much has he in each investment?

31. A sum of money, at simple interest, amounted in 4 years to $29,000, and in 5 years to $30,000. Find the sum and the rate of interest.

32. A sum of money, at simple interest, amounted in 10 months to $2100, and in 18 months to $2180. Find the sum and the rate of interest.

33. A person has a certain capital invested at a certain rate per cent. Another person has $2000 more capital, and his capital invested at one per cent better than the first, and he receives an income of $150 greater. A third person has $3000 more capital, and his capital invested at two per cent better than the first, and he receives an income of $280 greater. Find the capital of each and the rate at which it is invested.

34. A sum of money, at simple interest, amounted in m years to c dollars, and in n years to d dollars. Find the sum and the rate of interest.

35. A sum of money, at simple interest, amounted in m months to a dollars, and in n months to b dollars. Find the sum and the rate of interest.

36. A person has $18,375 to invest. He can buy 3 per cent bonds at 75, and 5 per cent bonds at 120. How much of his money must he invest in each kind of bonds in order to have the same income from each investment?

HINT. Notice that the 3 per cent bonds at 75 pay 4 per cent on the money invested, and 5 per cent bonds at 120 pay $4\frac{1}{6}$ per cent.

37. A man makes an investment at 4 per cent, and a second investment at $4\frac{1}{2}$ per cent. His income from the two investments is $715. If the first investment had been at $4\frac{1}{2}$ per cent and the second at 4 per cent, his income would have been $730. Find the amount of each investment.

(1) In a mile race A gives B a start of 20 yards and beats him by 30 seconds. At the second trial A gives B a start of 32 seconds and beats him by $9\frac{5}{11}$ yards. Find the number of yards each runs a second.

Let x = number of yards A runs a second,
and y = number of yards B runs a second.

Since there are 1760 yards in a mile,

$$\frac{1760}{x} = \text{number of seconds it takes A to run a mile.}$$

Since B has a start of 20 yards, he runs 1740 yards the first trial; and as he was 30 seconds longer than A,

$$\frac{1760}{x} + 30 = \text{the number of seconds B was running.}$$

But $\frac{1740}{y}$ = the number of seconds B was running.

$$\therefore \frac{1740}{y} = \frac{1760}{x} + 30. \qquad (1)$$

In the second trial B runs $1760 - 9\frac{5}{11} = 1750\frac{6}{11}$ yards.

$$\therefore \frac{1750\frac{6}{11}}{y} = \frac{1760}{x} + 32. \qquad (2)$$

From the solution of equations (1) and (2), $x = 5\frac{1}{1}\frac{3}{5}$, and $y = 5\frac{3}{11}$.

Therefore A runs $5\frac{1}{1}\frac{3}{5}$ yards a second, and B runs $5\frac{3}{11}$ yards a second.

(2) A train, after travelling an hour from A towards B, meets with an accident which detains it half an hour; after which it proceeds at four-fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have been only an hour late. Find the usual rate of the train.

Since the train was detained $\frac{1}{2}$ an hour and arrived $1\frac{1}{4}$ hours late, the *running time* was $\frac{3}{4}$ of an hour more than usual.

Let y = number of miles from A to B,

and $5x$ = number of miles the train travels per hour.

Then $y - 5x$ = number of miles the train has to go after the accident.

Hence $\frac{y-5x}{5x}$ = number of hours required usually,

and $\frac{y-5x}{4x}$ = number of hours actually required.

$\therefore \frac{y-5x}{4x} - \frac{y-5x}{5x}$ = loss in hours of running time.

But $\frac{3}{4}$ = loss in hours of running time.

$$\therefore \frac{y-5x}{4x} - \frac{y-5x}{5x} = \frac{3}{4}. \qquad (1)$$

If the accident had happened 30 miles farther on, the remainder of the journey would have been $y-(5x+30)$, and the loss in running time would have been $\frac{1}{2}$ an hour.

$$\therefore \frac{y-(5x+30)}{4x} - \frac{y-(5x+30)}{5x} = \frac{1}{2}. \qquad (2)$$

From the solution of equations (1) and (2), $x = 6$, and $5x = 30$.

Therefore the usual rate of the train is 30 miles an hour.

38. Two men, A and B, run a mile, and A wins by 2 seconds. In the second trial B has a start of $18\frac{1}{3}$ yards, and wins by 1 second. Find the number of yards each runs a second, and the number of miles each would run in an hour.

39. In a mile race A gives B a start of 3 seconds, and is beaten by $12\frac{4}{7}$ yards. In the second trial A gives B a start of 10 yards, and the race is a tie. Find the number of yards each runs a second. At this rate, how many miles could each run in an hour?

40. In a mile race A gives B a start of 44 yards, and is beaten by 1 second. In a second trial A gives B a start of 6 seconds, and beats him by $9\frac{7}{9}$ yards. Find the number of yards each runs a second.

41. An express train, after travelling an hour from A towards B, meets with an accident which delays it 15 minutes. It afterwards proceeds at two-thirds its usual rate, and arrives 24 minutes late. If the accident had happened 5 miles farther on, the train would have been only 21 minutes late. Find the usual rate of the train.

42. A train, after running 2 hours from A towards B, meets with an accident which delays it 20 minutes. It afterwards proceeds at four-fifths its usual rate, and arrives 1 hour and 40 minutes late. If the accident had happened 40 miles nearer A, the train would have been 2 hours late. Find the usual rate of the train.

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43. A and B can do a piece of work in $2\frac{1}{2}$ days, A and C in $3\frac{1}{3}$ days, B and C in $3\frac{3}{4}$ days. In what time can all three together, and each one separately, do the work?

44. A sum of money, at interest, amounts in 8 months to \$1488, and in 15 months to \$1530. Find the principal and the rate of interest.

45. A number is expressed by two digits, the units' digit being the larger. If the number is divided by the sum of its digits, the quotient is 4. If the digits are reversed and the resulting number is divided by 2 more than the difference of the digits, the quotient is 14. Find the number.

46. A and B together can dig a well in 10 days. They work 4 days, and B finishes the work in 16 days. How long would it take each alone to dig the well?

47. The denominator of the greater of two fractions is 20, and this is the greater of the two denominators. The fraction formed by taking for a numerator the sum of the numerators of the two fractions, and for a denominator the sum of the denominators, is equal to $\frac{2}{3}$. The fraction similarly formed with the difference of the numerators, and of the denominators, is equal to $\frac{1}{2}$. The sum of the numerators is twice the difference of the denominators. Find the fractions.

48. A cistern can be filled in 5 hours by two pipes, A and B, together. Both are left open for 3 hours and 45 minutes, and then A is shut, and B takes 3 hours and 45 minutes longer to fill the cistern. How long would it take each pipe alone to fill the cistern?

49. A man put at interest $20,000 in three sums, the first at 5 per cent, the second at $4\frac{1}{2}$ per cent, and the third at 4 per cent, receiving an income of $905 a year. The sum at $4\frac{1}{2}$ per cent is one-third as much as the other two sums together. Find the three sums.

50. An income of $335 a year is obtained from two investments, one in $4\frac{1}{2}$ per cent stock and the other in 5 per cent stock. If the $4\frac{1}{2}$ per cent stock should be sold at 110, and the 5 per cent at 125, the sum realized from both stocks together would be $8300. How much of each stock is there?

51. A boy bought some apples at 3 for 5 cents, and some at 4 for 5 cents, paying for the whole $1. He sold them at 2 cents apiece, and cleared 40 cents. How many of each kind did he buy?

52. Find the area of a rectangular floor, such that if 3 feet were taken from the length and 3 feet added to the breadth, its area would be increased by 6 square feet, but if 5 feet were taken from the breadth and 3 feet added to the length, its area would be diminished by 90 square feet.

53. A courier was sent from A to B, a distance of 147 miles. After 7 hours, a second courier was sent from A, who overtook the first just as he was entering B. The time required by the first to travel 17 miles added to the time required by the second to travel 76 miles is 9 hours and 40 minutes. How many miles did each travel per hour?

54. A box contains a mixture of 6 quarts of oats and 9 of corn, and another box contains a mixture of 6 quarts of oats and 2 of corn. How many quarts must be taken from each box in order to have a mixture of 7 quarts, half oats and half corn?

55. A train travelling 30 miles an hour takes 21 minutes longer to go from A to B than a train which travels 36 miles an hour. Find the distance from A to B.

56. A man buys 570 oranges, some at 16 for 25 cents, and the rest at 18 for 25 cents. He sells them all at the rate of 15 for 25 cents, and gains 75 cents. How many of each kind does he buy?

57. A and B run a mile race. In the first heat B receives 12 seconds start, and is beaten by 44 yards. In the second heat B receives 165 yards start, and arrives at the winning post 10 seconds before A. Find the time in which each can run a mile.

INDETERMINATE PROBLEMS.

195. If a *single* equation is given which contains *two* unknown numbers, and no other condition is imposed, the number of its solutions is *unlimited;* for, if *any* value be assigned to one of the unknowns, a *corresponding* value may be found for the other. Such an equation is said to be **indeterminate.**

196. The values of the unknown numbers in an indeterminate equation are *dependent upon each other;* so that, though they are unlimited in number, they are confined to a *particular range.*

This range may be still further limited by requiring these values to satisfy some given condition; as, for instance, that they shall be *positive integers.* With such restrictions the equation may admit of a definite number of solutions.

Ex. A number is expressed by two digits. If the number is divided by the sum of its digits diminished by 4, the quotient is 6. Find the number.

The single statement is

$$\frac{10x+y}{x+y-4}=6.$$

Whence $$4x=5y-24,$$

and $$x=y+\frac{y}{4}-6$$

$$=y-6+\frac{y}{4}.$$

We see from $\frac{y}{4}$ that the values of y which will be *integral* are 4, 8, 12, 16, or some other multiple of 4, and from the relation $x=y-6+\frac{y}{4}$ that the least positive integral value of y which will give to x a *positive* integral value, is 8. If we put 8 for y in (1), we find $x=4$. Hence the number required is 48.

Exercise 66.

1. A number is expressed by two digits. If the number is divided by the last digit, the quotient is 15. Find the number.

2. A number is expressed by three digits. The sum of the digits is 20. If 16 is subtracted from the number and the remainder divided by 2, the digits will be reversed. Find the number.

Here $$x + y + z = 20,$$

and $$\frac{100x + 10y + z - 16}{2} = 100z + 10y + x.$$

Eliminate y and reduce, and we have

$$4x = 7z + 8.$$

3. A man spends $114 in buying calves at $5 apiece, and pigs at $3 apiece. How many did he buy of each?

4. In how many ways can a man pay a debt of $87 with five-dollar bills and two-dollar bills?

5. Find the smallest number which when divided by 5 or by 7 gives 4 for a remainder.

Let $n =$ the number, then $\frac{n-4}{5} = x$, and $\frac{n-4}{7} = y$.

6. A farmer sells 15 calves, 14 lambs, and 13 pigs for $200. Some days after, at the same price, he sells 7 calves, 11 lambs, and 16 pigs, for which he receives $141. What was the price of each?

Discussion of Problems.

197. The *discussion* of a problem consists in making various suppositions as to the relative values of the given numbers, and explaining the results. We will illustrate by an example:

Two couriers were travelling along the same road, and in the same direction, from C towards D. A travels at the rate of m miles an hour, and B at the rate of n miles an hour. At 12 o'clock B was d miles in advance of A. When will the couriers be together?

Suppose they will be together x hours *after* 12. Then A has travelled mx miles, and B has travelled nx miles, and as A has travelled d miles more than B

$$mx = nx + d,$$

or

$$mx - nx = d.$$

$$\therefore x = \frac{d}{m-n}.$$

Discussion of the Problem. 1. If m is greater than n, the value of x, namely, $\frac{d}{m-n}$, is positive, and it is evident that A will overtake B *after* 12 o'clock.

2. If m is less than n, then $\frac{d}{m-n}$ will be negative. In this case B travels faster than A, and as he is d miles ahead of A at 12 o'clock, it is evident that A cannot overtake B *after 12 o'clock*, but that B passed A *before* 12 o'clock by $\frac{d}{n-m}$ hours. The supposition, therefore, that the couriers were together *after* 12 *o'clock* was incorrect, and the *negative* value of x points to an error in the supposition.

3. If m equals n, then the value of x, that is, $\frac{d}{m-n}$, assumes the form $\frac{d}{0}$. Now if the couriers were d miles apart at 12 o'clock, and if they had been travelling at the same rates, and continue to travel at the same rates, it is obvious that they never had been together, and that they never will be together, so that the symbol $\frac{d}{0}$ may be regarded as the **symbol of impossibility.**

4. If m equals n and d is 0, then $\frac{d}{m-n}$ becomes $\frac{0}{0}$. Now if the couriers were together at 12 o'clock, and if they had been travelling at the same rates, and continue to travel at the same rates, it is obvious that they had been together all the time, and that they will continue to be together all the time, so that the symbol $\frac{0}{0}$ may be regarded as the **symbol of indetermination.**

Exercise 67.

1. A train travelling b miles per hour is m hours in advance of a second train which travels a miles per hour. In how many hours will the second train overtake the first?

Ans. $\frac{bm}{a-b}$.

Discuss the result (1) when $a > b$; (2) when $a = b$; (3) when $a < b$.

2. A man setting out on a journey drove at the rate of a miles an hour to the nearest railway station, distant b miles from his house. On arriving at the station he found that the train for his place of destination left c hours before. At what rate should he have driven in order to reach the station just in time for the train?

Ans. $\frac{ab}{b-ac}$.

Discuss the result (1) when $c = 0$; (2) when $c = \frac{b}{a}$; (3) when $c = -\frac{b}{a}$. In case (2), how many hours did the man have to drive from his house to the station? In case (3), what is the meaning of the *negative* value of c?

3. A wine merchant has two kinds of wine which he sells, one at a dollars, and the other at b dollars per gallon. He wishes to make a mixture of l gallons, which shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Ans. $\frac{(m-b)l}{a-b}$ of the first; $\frac{(a-m)l}{a-b}$ of the second.

Discuss the question (1) when $a = b$; (2) when a or $b = m$; (3) when $a = b = m$; (4) when $a > b$ and $< m$; (5) when $a > b$ and $b > m$.

CHAPTER XIII.

INEQUALITIES.

198. An inequality consists of two unequal numbers connected by the sign of inequality. Thus, $12 > 4$ and $4 < 12$ are inequalities.

199. Two inequalities are said to be of the **same direction** if the first members are both greater or both less than the second members; that is, if the signs of inequality point in the same direction.

200. Two inequalities are said to be the **reverse** of each other if the signs point in opposite directions.

201. If equal numbers are added to, or subtracted from, the members of an inequality, the inequality remains in the same direction. Thus, if $a > b$, then $a + c > b + c$, and $a - c > b - c$. Hence,

A term can be transposed from one member of an inequality to the other without altering the inequality, provided its sign is changed.

202. If unequals are taken from equals, the result is an inequality which is the reverse of the given inequality. Thus, if $x = y$, and $a > b$, then $x - a < y - b$.

203. If the signs of the terms of an inequality are changed, the inequality is reversed. Thus, if $a > b$, then $-a < -b$. (See § 33.)

204. Hence, if the members of an inequality are multiplied or divided by the same *positive* number, the inequality remains in the same direction, by the same *negative* number, the inequality is reversed.

(1) Simplify $4x - 3 > \frac{3x}{2} - \frac{3}{5}$.

We have $$4x - 3 > \frac{3x}{2} - \frac{3}{5}.$$

Multiply by 10, $$40x - 30 > 15x - 6.$$

Transpose, $$25x > 24.$$

Divide by 25, $$x > \tfrac{24}{25}.$$

Therefore the value of x is greater than $\frac{24}{25}$.

(2) Find the limits of x, given

$$x - 4 > 2 - 3x,$$
$$3x - 2 < x + 3.$$

We have $$x - 4 > 2 - 3x, \quad (1)$$

and $$3x - 2 < x + 3. \quad (2)$$

Transpose in (1), $$4x > 6.$$
$$\therefore x > 1\tfrac{1}{2}.$$

Transpose in (2), $$2x < 5.$$
$$\therefore x < 2\tfrac{1}{2}.$$

Therefore, the value of x lies between $1\frac{1}{2}$ and $2\frac{1}{2}$.

(3) If a and b stand for unequal and positive numbers, then $a^2 + b^2 > 2ab$.

Since $(a-b)^2$ is positive, whatever the values of a and b,

$$(a-b)^2 > 0.$$
$$a^2 - 2ab + b^2 > 0.$$
$$\therefore a^2 + b^2 > 2ab.$$

Exercise 68.

1. Simplify $(x+1)^2 < x^2 + 3x - 5$.

2. Simplify $\frac{4x-2}{3} > \frac{3-5x}{7}$.

3. Simplify $x + 2b > 7x$.

4. Simplify $3x - 2 < \frac{x}{2} + 7\frac{1}{3}$.

Find the limiting values of x, given

5. $4x - 6 < 2x + 4$,
$2x + 4 > 16 - 2x$.

6. $\frac{ax}{5} + bx - ab > \frac{a^2}{5}$,
$\frac{bx}{7} - ax + ab < \frac{b^2}{7}$.

Find the integral value of x, given

7. $\frac{1}{4}(x+2) + \frac{1}{3}x < \frac{1}{2}(x-4) + 3$,
$\frac{1}{4}(x+2) + \frac{1}{3}x > \frac{1}{2}(x+1) + \frac{1}{3}$.

8. Twice a certain integral number increased by 7 is not greater than 19; and three times the number diminished by 5 is not less than 13. Find the number.

If the letters stand for unequal and positive numbers, show that

9. $a^2 + 3b^2 > 2b(a+b)$.

10. $a^3 + b^3 > a^2b + ab^2$.

11. $a^2 + b^2 + c^2 > ab + ac + bc$.

12. $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$.

13. $\frac{a}{b} + \frac{b}{a} > 2$.

14. $\frac{a+b}{2} > \frac{2ab}{a+b}$.

CHAPTER XIV.

INVOLUTION AND EVOLUTION.

205. The operation of raising an expression to any required *power* is called **Involution.**

Every case of involution is merely an example of *multiplication*, in which the factors are *equal.*

206. Index Law. If m is a positive integer, by definition

$$a^m = a \times a \times a \cdots\cdots \text{ to } m \text{ factors.}$$

Consequently, if m and n are both positive integers,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots\cdots \text{ to } m \text{ factors} \\ &= (a \times a \cdots\cdots \text{ to } n \text{ factors})(a \times a \cdots\cdots \text{ to } n \text{ factors}) \\ &\qquad \cdots\cdots \text{ taken } m \text{ times} \\ &= a \times a \times a \cdots\cdots \text{ to } mn \text{ factors.} \\ &= a^{mn}.\end{aligned}$$

This is the **index law** for involution.

207. Also, $(a^m)^n = a^{mn} = (a^n)^m$.

And $$\begin{aligned}(ab)^n &= ab \times ab \cdots\cdots \text{ to } n \text{ factors} \\ &= (a \times a \cdots\cdots \text{ to } n \text{ factors})(b \times b \cdots\cdots \text{ to } n \text{ factors}) \\ &= a^n b^n.\end{aligned}$$

208. If the exponent of the required power is a composite number, the exponent may be resolved into prime factors, the power denoted by one of these factors found, and the result raised to a power denoted by a second factor of the exponent; and so on. Thus, the fourth power may be obtained by taking the second power of the second power;

the sixth by taking the second power of the third power; and so on.

209. From the *Law of Signs* in multiplication it is evident that all *even* powers of a number are *positive;* all *odd* powers of a number have the *same sign* as the number itself.

Hence, no *even* power of *any* number can be *negative;* and the even powers of two compound expressions which have the same terms with opposite signs are identical.

Thus, $(b-a)^2 = \{-(a-b)\}^2 = (a-b)^2.$

210. Binomials. By actual multiplication we obtain,

$$(a+b)^2 = a^2 + 2ab + b^2;$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In these results it will be observed that:

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. b appears in the second term with 1 for an exponent, and its exponent increases by 1 in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying the coefficient of that term by the exponent of a, and dividing the product by a number greater by one than the exponent of b.

If b is negative, the terms in which the *odd* powers of b occur are negative. Thus,

(1) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

(2) $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

By the above rules any power of a binomial of the form $a \pm b$ may be written at once.

NOTE. The double sign $\pm$ is read *plus or minus;* and $a \pm b$ means $a+b$ or $a-b$.

211. The same method may be employed when the terms of a binomial have *coefficients* or *exponents.*

Since $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$

putting $5x^2$ for a, and $2y^3$ for b, we have

$$\begin{aligned}&(5x^2 - 2y^3)^3,\\ &= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3,\\ &= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9.\end{aligned}$$

Since $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4,$

putting x^2 for a, and $\frac{1}{2}y$ for b, we have

$$\begin{aligned}&(x^2 - \tfrac{1}{2}y)^4,\\ &= (x^2)^4 - 4(x^2)^3(\tfrac{1}{2}y) + 6(x^2)^2(\tfrac{1}{2}y)^2 - 4x^2(\tfrac{1}{2}y)^3 + (\tfrac{1}{2}y)^4,\\ &= x^8 - 2x^6y + \tfrac{3}{2}x^4y^2 - \tfrac{1}{2}x^2y^3 + \tfrac{1}{16}y^4.\end{aligned}$$

212. In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its terms in parentheses, so as to give the expression the form of a binomial. Thus,

(1) $$\begin{aligned}(a+b+c)^3 &= [a + (b+c)]^3,\\ &= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3,\\ &= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc\\ &\quad + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.\end{aligned}$$

(2) $(x^3 - 2x^2 + 3x + 4)^2$,

$= [(x^3 - 2x^2) + (3x + 4)]^2$,

$= (x^3 - 2x^2)^2 + 2(x^3 - 2x^2)(3x + 4) + (3x + 4)^2$,

$= x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16$,

$= x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16$.

Exercise 69.

Raise to the required power:

1. $(a^4)^3$.
2. $(a^2b^3)^5$.
3. $\left(\frac{2xy^2}{3ab^3}\right)^4$.
4. $(-5ab^2c^3)^4$.
5. $(-7x^2yz^3)^3$.
6. $\left(-\frac{3a^2b^3c^4}{2x^2y^5}\right)^5$.
7. $(-2x^2y^4)^6$.
8. $(-3a^2b^3x^4)^5$.
9. $\left(-\frac{3x^2y^3}{4z^3}\right)^4$.
10. $(x+2)^5$.
11. $(x^2-2)^4$.
12. $(x+3)^5$.
13. $(2x+1)^6$.
14. $(2m^2-1)^3$.
15. $(2x+3y)^5$.
16. $(2x-y)^6$.
17. $(xy-2)^7$.
18. $(1-x+x^2)^2$.
19. $(1-2x+3x^2)^2$.
20. $(1-a+a^2)^3$.
21. $(3-4x+5x^2)^2$.

Evolution.

213. The *n*th **root** of a number is one of the *n* equal factors of that number.

The operation of finding any required root of an expression is called **Evolution.**

Every case of evolution is merely an example of *factoring*, in which the required factors are all *equal*. Thus, the square, cube, fourth, ... roots of an expression are found by taking one of its *two*, *three*, *four* ..... *equal factors*.

The symbol which denotes that a square root is to be extracted is $\sqrt{\ }$; and for other roots the same symbol is used, but with a figure written above to indicate the root; thus, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., signifies the *third* root, *fourth* root, etc.

214. Index Law. If m and n are positive integers, we have, (§ 206),

$$(a^m)^n = a^{mn}.$$

Consequently, $$\sqrt[n]{a^{mn}} = a^m.$$

Thus, the *cube root* of a^6 is a^2; the *fourth root* of $81a^{12}$ is $3a^3$; and so on.

This is the **index law** for evolution.

215. Also, since $$(ab)^n = a^n b^n,$$

conversely, $$\sqrt[n]{a^n b^n} = ab = \sqrt[n]{a^n} \times \sqrt[n]{b^n},$$

and $$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$$

Hence, to find the root of a simple expression:

Divide the exponent of each factor by the index of the root, and take the product of the resulting factors.

216. From the Law of Signs it is evident that

I. Any *even* root of a *positive* number will have the double sign, $\pm$.

II. There can be no *even* root of a *negative* number.

For $\sqrt{-x^2}$ is neither $+x$ nor $-x$; since the square of $+x = +x^2$, and the square of $-x = +x^2$.

The indicated even root of a negative number is called an **imaginary number.**

III. Any *odd* root of a number will have the same sign as the number.

Thus, $\sqrt{\dfrac{16x^2}{81y^2}} = \pm\dfrac{4x}{9y}$; $\sqrt[3]{-27m^3n^6} = -3mn^2$;

$$\sqrt[4]{\frac{16x^8y^{12}}{81a^{16}}} = \pm\frac{2x^2y^3}{3a^4}.$$

217. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. Thus, to find the square root of 3,415,104:

2^3	3415104
2^3	426888
3^2	53361
7	5929
7	847
11	121
	11

$$\therefore \quad 3{,}415{,}104 = 2^6 \times 3^2 \times 7^2 \times 11^2.$$

$$\therefore \sqrt{3{,}415{,}104} = 2^3 \times 3 \times 7 \times 11 = 1848.$$

Exercise 70.

Simplify:

1. $\sqrt{4x^2y^4}$.
2. $\sqrt[3]{64x^9}$.
3. $\sqrt[4]{16x^8y^{12}}$.
4. $\sqrt[5]{-32a^{10}}$.
5. $\sqrt[3]{-27x^3}$.
6. $\sqrt{25a^4}$.
7. $\sqrt[3]{-8a^3b^6}$.
8. $\sqrt[6]{64x^{12}}$.
9. $\sqrt[3]{-216a^{12}}$.
10. $\sqrt[6]{729x^{18}}$.
11. $\sqrt[5]{243y^5z^{10}}$.
12. $\sqrt[3]{-1728d^3}$.
13. $\sqrt[3]{-343a^6}$.
14. $\sqrt[4]{81a^{24}}$.
15. $\sqrt[3]{512a^{12}b^{15}}$.
16. $\sqrt[3]{x^{3n}y^{12m}}$.
17. $\sqrt{\dfrac{9a^2b^6}{16x^4y^2}}$.
18. $\sqrt[3]{-\dfrac{8x^3y^6}{27z^9}}$.
19. $\sqrt[5]{-\dfrac{32a^{10}}{243x^{15}}}$.
20. $\sqrt[4]{\dfrac{16x^4}{81a^8b^{12}}}$.
21. $\sqrt[3]{\dfrac{125x^{21}}{216a^{24}}}$.

SQUARE ROOTS OF COMPOUND EXPRESSIONS.

218. Since the square of $a + b$ is $a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to find a method of extracting the root $a + b$ when $a^2 + 2ab + b^2$ is given:

Ex. The first term, a, of the root is obviously the square root of the first term, a^2, in the expression.

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \;\underline{|\, a + b} \\ & a^2 \\ \hline 2a + b & 2ab + b^2 \\ & 2ab + b^2 \end{array}$$

If the a^2 is subtracted from the given expression, the remainder is $2ab + b^2$. Therefore the second term, b, of the root is obtained when the first term of this remainder is divided by $2a$, that is, by *double the part of the root already found.* Also, since

$$2ab + b^2 = (2a + b)b,$$

the divisor *is completed by adding to the trial-divisor the new term of the root.*

(1) Find the square root of $25x^2 - 20x^3y + 4x^4y^2$.

$$\begin{array}{r|l} & 25x^2 - 20x^3y + 4x^4y^2 \;\underline{|\, 5x - 2x^2y} \\ & 25x^2 \\ \hline 10x - 2x^2y & -20x^3y + 4x^4y^2 \\ & -20x^3y + 4x^4y^2 \end{array}$$

The expression is *arranged* according to the ascending powers of x.

The square root of the first term is $5x$, and $5x$ is placed at the right of the given expression, for the first term of the root.

The second term of the root, $-2x^2y$, is obtained by dividing $-20x^3y$ by $10x$, and this new term of the root is also annexed to the divisor, $10x$, to complete the divisor.

219. The same method will apply to longer expressions, if care be taken to obtain the *trial-divisor* at each stage of the process, by *doubling the part of the root already found,* and to obtain the *complete divisor by annexing the new term of the root to the trial-divisor.*

Ex. Find the square root of

$$1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x.$$

$$\begin{array}{r|l}
 & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 \;\underline{|\,4x^3 - 3x^2 + 2x - 1} \\
 & 16x^6 \\
8x^3 - 3x^2 & -24x^5 + 25x^4 \\
 & -24x^5 + 9x^4 \\
8x^3 - 6x^2 + 2x & 16x^4 - 20x^3 + 10x^2 \\
 & 16x^4 - 12x^3 + 4x^2 \\
8x^3 - 6x^2 + 4x - 1 & -8x^3 + 6x^2 - 4x + 1 \\
 & -8x^3 + 6x^2 - 4x + 1
\end{array}$$

The expression is arranged according to the descending powers of x.

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its *last term doubled.*

Exercise 71.

Find the square root of

1. $x^4 - 8x^3 + 18x^2 - 8x + 1.$
2. $9a^4 - 6a^3 + 13a^2 - 4a + 4.$
3. $4x^4 - 12x^3y + 29x^2y^2 - 30xy^3 + 25y^4.$
4. $1 + 4x + 10x^2 + 12x^3 + 9x^4.$
5. $16 - 96x + 216x^2 - 216x^3 + 81x^4.$
6. $x^4 - 22x^3 + 95x^2 + 286x + 169.$
7. $4x^4 - 11x^2 + 25 - 12x^3 + 30x.$
8. $9x^4 + 49 - 12x^3 - 28x + 46x^2.$
9. $49x^4 + 126x^3 + 121 - 73x^2 - 198x.$
10. $16x^4 - 30x - 31x^2 + 24x^3 + 25.$
11. $\dfrac{x^4}{a^4} - \dfrac{2x^3}{a^3} + \dfrac{3x^2}{a^2} - \dfrac{2x}{a} + 1.$

12. $4x^4 + 4x^3 - \frac{1}{2}x + \frac{1}{16}.$

13. $\frac{4a^2}{b^2} + 8 + \frac{4b^2}{a^2}.$

14. $a^4 - 2a^3 + \frac{3a^2}{2} - \frac{a}{2} + \frac{1}{16}.$

15. $x^4 + \frac{2x^3}{3} + \frac{10x^2}{9} + \frac{x}{3} + \frac{1}{4}.$

16. $\frac{4x^2}{y^2} + \frac{3x}{y} + \frac{41}{16} + \frac{3y}{4x} + \frac{y^2}{4x^2}.$

17. $\frac{a^2}{4} - ax + \frac{3a}{2} + x^2 - 3x + \frac{9}{4}.$

18. $16x^4 + \frac{16}{3}x^2y + 8x^2 + \frac{4}{9}y^2 + \frac{4}{3}y + 1.$

19. $\frac{9x^4}{4} - \frac{3x^3}{2} + \frac{43x^2}{4} - \frac{7x}{2} + \frac{49}{4}.$

20. $4a^2 + \frac{9}{a^2} - \frac{6}{a} - 11 + 4a.$

Find to three terms the square root of

21. $a^2 + b.$
22. $x^2 + \frac{1}{2}y.$
23. $1 + 2a.$
24. $1 + a.$
25. $1 - 2a.$
26. $4a^2 + 2b.$
27. $4x^2 + 3.$
28. $4 - 3a.$
29. $4a^2 - 1.$

220. Arithmetical Square Roots. In the general method of extracting the square root of a number expressed by figures, the first step is to mark off the figures in *groups*.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any number expressed by *one* or *two* figures is a number of *one* figure; the square root of any number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral square number is divided into groups of two figures each, from the right to the left, the number of figures in the root will be equal to the number of groups of figures. The last group to the left may have only one figure.

Ex. Find the square root of 3249.

$$\begin{array}{r|l} & 32\,49(57 \\ & \underline{25} \\ 107) & 749 \\ & \underline{749} \end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is, a^2, and the complete divisor $2a + b$ is $2 \times 50 + 7 = 107$.

221. The same method will apply to numbers of more than two groups of figures by considering a in the typical form to represent at each step *the part of the root already found.*

It must be observed that *a represents so many tens with respect to the next figure of the root.*

Ex. Find the square root of 5,322,249.

$$\begin{array}{r|l} & 5\,32\,22\,49(2307 \\ & \underline{4} \\ 43) & 132 \\ & \underline{129} \\ 4607) & 32249 \\ & \underline{32249} \end{array}$$

222. If the square root of a number has decimal places, the number itself will have *twice* as many. Thus, if 0.21 is the square root of some number, this number will be $(0.21)^2 = 0.21 \times 0.21 = 0.0441$; and if 0.111 be the root, the number will be $(0.111)^2 = 0.111 \times 0.111 = 0.012321$.

Therefore, the number of *decimal* places in every square decimal will be *even*, and the number of decimal places in the root will be *half* as many as in the given number itself.

Hence, if a given number contains a decimal, we divide it into groups of two figures each, by beginning at the decimal point and marking toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *two* figures, annexing a cipher when necessary.

Ex. Find the square roots of 41.2164 and 965.9664.

```
       41.21 64(6.42              9 65.96 64(31.08
       36                         9
   124)521                     61)65
       496                        61
  1282)2564                  6208)49664
       2564                       49664
```

223. If a number contains an *odd* number of decimal places, or if any number gives a *remainder* when as many figures in the root have been obtained as the given number has groups, then its exact square root cannot be found. We may, however, approximate to its exact root as near as we please by annexing ciphers and continuing the operation.

The square root of a common fraction whose denominator is not a perfect square can be found approximately by reducing the fraction to a decimal and then extracting the root; or by reducing the fraction to an equivalent fraction whose denominator is a perfect square, and extracting the square root of both terms of the fraction. Thus,

$$\sqrt{\frac{5}{8}} = \sqrt{0.625} = 0.79057;$$

or

$$\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{\sqrt{10}}{4} = \frac{3.16227}{4} = 0.79057.$$

Ex. Find the square roots of 3 and 357.357.

3.(1.732.....	3 57.35 70 (18.903.....
1	1
27) 200	28) 257
189	224
343) 1100	369) 3335
1029	3321
3462) 7100	37803) 147000
6924	113409

Exercise 72.

Find the square root of

1. 289.	6. 150.0625.	11. 640.343025.
2. 1225.	7. 118.1569.	12. 100.240144.
3. 12544.	8. 172.3969.	13. 316.021729.
4. 253009.	9. 5200.140544.	14. 454.585041.
5. 529984.	10. 1303.282201.	15. 5127.276025.

Find to four decimal places the square root of

16. 10.	19. 0.5.	22. 0.607.	25. $\frac{2}{3}$.	28. $\frac{6}{7}$.
17. 3.	20. 0.7.	23. 0.521.	26. $\frac{3}{4}$.	29. $\frac{5}{8}$.
18. 5.	21. 0.9.	24. 0.687.	27. $\frac{4}{5}$.	30. $\frac{9}{11}$.

224. Cube Roots of Compound Expressions. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the cube root of

$$a^3+3a^2b+3ab^2+b^3 \text{ is } a+b.$$

It is required to devise a method for extracting the cube root $a+b$ when $a^3+3a^2b+3ab^2+b^3$ is given:

(1) Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{l|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & \underline{a+b} \\
3a^2 & a^3 & \\
\quad +3ab + b^2 & 3a^2b + 3ab^2 + b^3 & \\
\hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 &
\end{array}$$

The first term a of the root is obviously the cube root of the first term a^3 of the given expression.

If a^3 is subtracted, the remainder is $3a^2b + 3ab^2 + b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of a*.

Also, since $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$, the *complete divisor* is obtained by adding $3ab + b^2$ to the *trial-divisor* $3a^2$.

(2) Find the cube root of $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

$$\begin{array}{r|l|l}
 & 8x^3 + 36x^2y + 54xy^2 + 27y^3 & \underline{2x+3y} \\
12x^2 & 8x^3 & \\
(6x+3y)3y = \quad 18xy + 9y^2 & 36x^2y + 54xy^2 + 27y^3 & \\
\hline
12x^2 + 18xy + 9y^2 & 36x^2y + 54xy^2 + 27y^3 &
\end{array}$$

The cube root of the first term is $2x$, and this is therefore the first term of the root. $8x^3$, the cube of $2x$, is subtracted.

The second term of the root, $3y$, is obtained by dividing $36x^2y$ by $3(2x)^2 = 12x^2$, which corresponds to $3a^2$ in the typical form, and the divisor is completed by annexing to $12x^2$ the expression

$$\{3(2x) + 3y\}3y = 18xy + 9y^2,$$

which corresponds to $3ab + b^2$ in the typical form.

225. The same method may be applied to longer expressions by considering a in the typical form $3a^2 + 3ab + b^2$ to represent at each stage of the process *the part of the root already found.* Thus, if the part of the root already found is $x + y$, then $3a^2$ of the typical form will be represented by $3(x+y)^2$; and if the third term of the root is $+z$, the $3ab + b^2$ will be represented by $3(x+y)z + z^2$. So that the complete divisor, $3a^2 + 3ab + b^2$, will be represented by $3(x+y)^2 + 3(x+y)z + z^2$.

Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$
\begin{array}{rl|l}
 & & x^2 - x - 1 \\
 & & x^6 - 3x^5 + 5x^3 - 3x - 1 \\
 & 3x^4 & x^6 \\
(3x^2 - x)(-x) = & -3x^3 + x^2 & -3x^5 + 5x^3 \\
 & 3x^4 - 3x^3 + x^2 & -3x^5 + 3x^4 - x^3 \\
 & & -3x^4 + 6x^3 - 3x - 1 \\
3(x^2 - x)^2 = & 3x^4 - 6x^3 + 3x^2 & \\
(3x^2 - 3x - 1)(-1) = & -3x^2 + 3x + 1 & \\
 & 3x^4 - 6x^3 + 3x + 1 & -3x^4 + 6x^3 - 3x - 1
\end{array}
$$

The root is placed above the given expression for convenience of arrangement.

The first term of the root, x^2, is obtained by taking the cube root of the first term of the given expression; and the first trial-divisor, $3x^4$, is obtained by taking three times the square of this term.

The first complete divisor is found by annexing to the trial-divisor $(3x^2 - x)(-x)$, which expression corresponds to $(3a + b)b$ in the typical form.

The part of the root already found (a) is now represented by $x^2 - x$; therefore $3a^2$ is represented by $3(x^2 - x)^2 = 3x^4 - 6x^3 + 3x^2$, the second trial-divisor; and $(3a + b)b$ by $(3x^2 - 3x - 1)(-1)$; therefore, in the second complete divisor, $3a^2 + (3a + b)b$ is represented by

$$(3x^4 - 6x^3 + 3x^2) + (3x^2 - 3x - 1) \times (-1) = 3x^4 - 6x^3 + 3x + 1.$$

Exercise 73.

Find the cube root of

1. $a^3 + 3a^2x + 3ax^2 + x^3$.
2. $8 + 12x + 6x^2 + x^3$.
3. $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.
4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.
5. $1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6$.
6. $x^6 + 1 - 6x - 6x^5 + 15x^2 + 15x^4 - 20x^3$.

7. $64x^6 - 144x^5 + 8 - 36x + 102x^2 - 171x^3 + 204x^4.$

8. $27a^6 - 27a^5 - 18a^4 + 17a^3 + 6a^2 - 3a - 1.$

9. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1.$

10. $27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6.$

11. $a^3 - a^2b + \frac{ab^2}{3} - \frac{b^3}{27}.$

12. $\frac{x^9}{y^{15}} + \frac{3x^8}{y^{14}} - \frac{5x^6}{y^{12}} + \frac{3x^4}{y^{10}} - \frac{x^3}{y^9}.$

226. Arithmetical Cube Roots. In extracting the cube root of a number expressed by figures, the first step is to separate it into groups.

Since $1 = 1^3$, $1000 = 10^3$, $1{,}000{,}000 = 100^3$, and so on, it follows that the cube root of any number between 1 and 1000, that is, of any number which has *one*, *two*, or *three* figures, is a number of *one* figure; and that the cube root of any number between 1000 and 1,000,000, that is, of any number which has *four*, *five*, or *six* figures, is a number of *two* figures; and so on.

If, therefore, an integral cube number be divided into groups of three figures each, from right to left, the number of figures in the root will be equal to the number of groups. The last group to the left may consist of one, two, or three figures.

227. If the cube root of a number have decimal places, the number itself will have *three times* as many. Thus, if 0.11 be the cube root of a number, the number is $0.11 \times 0.11 \times 0.11 = 0.001331$. Hence, if a given number contains a decimal, we divide the figures of the number into groups of three figures each, by beginning at the decimal point and marking toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *three* figures, annexing ciphers when necessary.

228. Notice that if a denotes the first term, and b the second term of the root, the *first complete divisor* is

$$3a^2 + 3ab + b^2,$$

and the *second trial-divisor* is $3(a+b)^2$, that is,

$$3a^2 + 6ab + 3b^2,$$

which may be obtained by adding to the preceding complete divisor *its second term and twice its third term.*

Ex. Extract the cube root of 5 to five places of decimals.

	5.000(1.70997
	1
$3 \times 10^2 = 300$	4000
$3(10 \times 7) = 210$	
$7^2 = 49$	
559	3913
259	87000000
$3 \times 1700^2 = 8670000$	
$3(1700 \times 9) = 45900$	
$9^2 = 81$	
8715981	78443829
45981	85561710
$3 \times 1709^2 = 8762043$	78858387
	67033230
	61334301

After the first two figures of the root are found, the next trial-divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor being three times the square of the part of the root already found.

Exercise 74.

Find the cube root of

1. 4913. **3.** 1404928. **5.** 385828.352.

2. 42875. **4.** 127263527. **6.** 1838.265625.

Find to four decimal places the cube root of

7. 87. **9.** 3.02. **11.** 0.05. **13.** $\frac{2}{3}$. **15.** $\frac{9}{11}$.

8. 10. **10.** 2.05. **12.** 0.677. **14.** $\frac{3}{4}$. **16.** $\frac{1}{13}$.

229. Since the fourth power is the square of the square, and the sixth power the square of the cube, the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root.* In like manner, the eighth, ninth, twelfth, ... roots may be found.

Exercise 75.

Find the fourth root of

1. $81x^4 + 108x^3 + 54x^2 + 12x + 1$.

2. $16x^4 - 32ax^3 + 24a^2x^2 - 8a^3x + a^4$.

3. $1 + 4x + x^8 + 4x^7 + 10x^6 + 16x^3 + 10x^2 + 19x^4 + 16x^5$.

Find the sixth root of

4. $1 + 6d + d^6 + 6d^5 + 15d^4 + 20d^3 + 15d^2$.

5. $729 - 1458x + 1215x^2 - 540x^3 + 135x^4 - 18x^5 + x^6$.

6. $1 - 18y + 135y^2 - 540y^3 + 1215y^4 - 1458y^5 + 729y^6$.

CHAPTER XV.

THEORY OF EXPONENTS.

230. If n is a positive integer, we have defined a^n to mean the product obtained by taking a as a factor n times. Thus a^3 stands for $a \times a \times a$; b^4 stands for $b \times b \times b \times b$.

231. From this definition we have obtained the following laws for positive and integral exponents:

$$\text{I.} \quad a^m \times a^n = a^{m+n}.$$

$$\text{II.} \quad (a^m)^n = a^{mn}.$$

$$\text{III.} \quad \frac{a^m}{a^n} = a^{m-n}, \text{ if } m > n.$$

$$\text{IV.} \quad \sqrt[n]{a^{mn}} = a^m.$$

$$\text{V.} \quad (ab)^n = a^n b^n.$$

232. Since by the definition of a^n the exponent n denotes simply *repetitions* of a as a factor, such expressions as $a^{\frac{2}{3}}$ and a^{-3} have no meaning whatever. It is found convenient, however, to extend the meaning of a^n so as to include fractional and negative values of n.

233. If we do not define the meaning of a^n when n is a fraction or negative, but require that the meaning of a^n must in all cases be such that the fundamental index law shall always hold true, namely,

$$a^m \times a^n = a^{m+n},$$

we shall find that this condition alone will be sufficient to define the meaning of a^n for all cases.

234. To find the Meaning of a Fractional Exponent.

Assuming the index law to hold true for fractional exponents, we have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}} = a,$$
$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^{\frac{3}{3}} = a,$$
$$a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3,$$
$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \cdots\cdots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n} \cdots\cdots \text{ to } n \text{ terms}} = a^{\frac{n}{n}} = a,$$
$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \cdots\cdots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n} \cdots\cdots \text{ to } n \text{ terms}} = a^{\frac{mn}{n}} = a^m.$$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a,

$a^{\frac{1}{3}}$ is one of the three equal factors of a,

$a^{\frac{3}{4}}$ is one of the four equal factors of a^3,

$a^{\frac{1}{n}}$ is one of the n equal factors of a,

$a^{\frac{m}{n}}$ is one of the n equal factors of a^m.

Hence $a^{\frac{1}{2}} = \sqrt{a}$; $a^{\frac{1}{3}} = \sqrt[3]{a}$;

$a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. (§ 213)

Also, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \cdots\cdots$ to m factors.

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n} \cdots\cdots \text{ to } m \text{ terms}} = a^{\frac{m}{n}}.$$
$$\therefore\ a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

The meaning, therefore, of $a^{\frac{m}{n}}$, where m and n are positive integers, is, the nth root of the mth power of a, or the mth power of the nth root of a.

Hence the numerator of a fractional exponent indicates a power, and the denominator a root; and the result is the same when we first extract the root and raise this root to the required power, as when we first find the power and extract the required root of this power.

235. To find the Meaning of a⁰.

By the index law,

$$a^0 \times a^m = a^{0+m} = a^m.$$

$$\therefore\ a^0 = a^m \div a^m.$$

$$\therefore\ a^0 = 1, \text{ whatever the value of } a \text{ is.}$$

236. To find the Meaning of a Negative Exponent.

If n stands for a positive integer, or a positive fraction, we have by the index law,

$$a^n \times a^{-n} = a^{n-n} = a^0.$$

But $$a^0 = 1.$$

$$\therefore\ a^n \times a^{-n} = 1.$$

That is, a^n and a^{-n} are *reciprocals* of each other (§ 167), so that $a^{-n} = \frac{1}{a^n}$, and $a^n = \frac{1}{a^{-n}}$.

237. Hence, we can change any *factor* from the numerator of a fraction to the denominator, or from the denominator to the numerator, *provided we change the sign of its exponent.*

Thus $\frac{ab^2}{c^3d^3}$ may be written $ab^2c^{-3}d^{-3}$, or $\frac{1}{a^{-1}b^{-2}c^3d^3}$.

238. We have now assigned definite meanings to fractional and negative exponents, meanings obtained by subjecting them to the fundamental index law of positive integral exponents; and we will now show that Law II., namely, $(a^m)^n = a^{mn}$, which has been established for positive integral exponents, holds true for fractional and negative exponents.

(1) If n is a positive integer, whatever the value of m,

We have $$(a^m)^n = a^m \times a^m \times a^m \text{ ..... to } n \text{ factors,}$$

$$= a^{m+m+m \text{ ..... to } n \text{ terms}},$$

$$= a^{mn}.$$

(2) If n is a positive fraction $\frac{p}{q}$, where p and q are positive integers, we have

$$(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} \qquad \S\ 234$$

$$= \sqrt[q]{a^{mp}} \qquad (1)$$

$$= a^{\frac{mp}{q}} \qquad \S\ 234$$

$$= a^{m \times \frac{p}{q}}$$

$$= a^{mn}.$$

(3) If n is a negative integer, and equal to $-p$, we have

$$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} \qquad \S\ 236$$

$$= \frac{1}{a^{mp}} \qquad (1)$$

$$= a^{-mp} \qquad \S\ 236$$

$$= a^{m(-p)}$$

$$= a^{mn}.$$

(4) If n is negative and equal to the fraction $-\frac{p}{q}$, where p and q are positive integers, we have

$$(a^m)^n = (a^m)^{-\frac{p}{q}} = \frac{1}{(a^m)^{\frac{p}{q}}} \qquad \S\ 236$$

$$= \frac{1}{\sqrt[q]{a^{mp}}} \qquad \S\ 234$$

$$= \frac{1}{a^{\frac{mp}{q}}} \qquad \S\ 234$$

$$= \frac{1}{a^{m \times \frac{p}{q}}}$$

$$= \frac{1}{a^{m(-n)}}$$

$$= \frac{1}{a^{-mn}} \qquad (1)$$

$$= a^{mn}. \qquad \S\ 236$$

Hence, $(a^m)^n = a^{mn}$, for *all values* of m and n.

239. In like manner it may be shown that all the index laws of positive integral exponents apply also to fractional, and negative, exponents. We will now give some examples.

(1) $27^{-\frac{1}{3}} = \frac{1}{27^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}.$

(2) $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = \pm 2^3 = \pm 8.$

(3) $a^{\frac{3}{4}} \times a^{-\frac{2}{3}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{1}{12}} = \sqrt[12]{a}.$

(4) $a^{\frac{5}{6}} \times a^{\frac{1}{2}} \times a^{-\frac{4}{3}} = a^{\frac{5}{6}+\frac{1}{2}-\frac{4}{3}} = a^0 = 1.$

(5) $(a^{-\frac{1}{3}})^6 = a^{6(-\frac{1}{3})} = a^{-2} = \frac{1}{a^2}.$

(6) $\frac{a^{-3}}{a^{-5}} = a^{-3-(-5)} = a^{-3+5} = a^2.$

(7) $\sqrt[3]{a^2b^{-3}c^{-4}d} = a^{\frac{2}{3}}b^{-1}c^{-\frac{4}{3}}d^{\frac{1}{3}}.$

(8) $(4a^{-\frac{2}{3}})^{-\frac{3}{2}} = \frac{1}{(4a^{-\frac{2}{3}})^{\frac{3}{2}}} = \frac{1}{4^{\frac{3}{2}}a^{-\frac{2}{3}\times\frac{3}{2}}} = \frac{1}{8a^{-1}} = \frac{a}{8}.$

(9) $\left(\frac{16a^{-4}}{81b^3}\right)^{-\frac{3}{4}} = \left(\frac{81b^3}{16a^{-4}}\right)^{\frac{3}{4}} = \frac{27b^{\frac{9}{4}}}{8a^{-3}} = \frac{27a^3b^{\frac{9}{4}}}{8}.$

(10) $(3^{\frac{3}{8}}a^{-3})^{-\frac{2}{3}} = \frac{1}{(3^{\frac{3}{8}}a^{-3})^{\frac{2}{3}}} = \frac{1}{3^{\frac{2}{3}\times\frac{3}{8}}a^{-2}} = \frac{a^2}{3^{\frac{1}{4}}} = \frac{a^2}{\sqrt[4]{3}}.$

Exercise 76.

Express with fractional exponents:

1. $\sqrt[3]{a^2}$. 3. $\sqrt[5]{a^7}$. 5. $\sqrt[4]{a^3}$. 7. $\sqrt{a}+\sqrt[3]{x}+\sqrt[4]{16b^2}$.

2. $\sqrt{a^3}$. 4. $\sqrt[3]{-8}$. 6. $\sqrt[3]{a^5}$. 8. $\sqrt[5]{a^2x^5}+\sqrt[3]{a^3c}$.

Express with root-signs:

9. $a^{\frac{3}{4}}$. 11. $a^{\frac{1}{2}}b^{\frac{1}{3}}$. 13. $x^{\frac{1}{3}}y^{-\frac{1}{3}}$. 15. $a^{\frac{1}{2}}-x^{\frac{1}{3}}c^{\frac{1}{3}}$.

10. $c^{\frac{2}{3}}$. 12. $a^{\frac{1}{3}}b^{\frac{2}{3}}$. 14. $3x^{\frac{2}{3}}y^{-\frac{3}{4}}$. 16. $a^{\frac{2}{3}}+x^{\frac{2}{3}}c^{\frac{2}{3}}$.

Express with positive exponents:

17. a^{-3}. 19. $3x^{-2}y^{3}$. 21. $4x^{-3}y^{-2}$. 23. $\dfrac{2a^{-1}x^{2}}{3^{-2}b^{2}y^{-5}}$.

18. $a^{-\frac{2}{3}}$. 20. $4xy^{-7}$. 22. $3a^{-4}b^{\frac{3}{2}}$.

Write without denominators:

24. $\dfrac{3y^{2}z^{3}}{x^{-5}}$. 26. $\dfrac{abc}{a^{-2}bc^{-2}d}$. 28. $\dfrac{x^{-2}y^{\frac{2}{3}}}{a^{-2}b^{-\frac{6}{5}}}$.

25. $\dfrac{x^{2}z}{x^{-3}z^{-2}}$. 27. $\dfrac{a^{-1}b^{-2}c^{-3}}{a^{-2}b^{-2}c^{-4}}$. 29. $\dfrac{a^{-4}b^{-5}c^{-6}}{a^{-7}b^{-5}c^{-3}}$.

Find the value of

30. $8^{\frac{5}{3}}$. 32. $27^{\frac{2}{3}}$. 34. $36^{\frac{3}{2}}$. 36. $(-27)^{\frac{1}{3}}\times 25^{\frac{5}{2}}$.

31. $16^{-\frac{5}{4}}$. 33. $(-8)^{-\frac{2}{3}}$. 35. $(-27)^{\frac{5}{3}}$. 37. $81^{-\frac{3}{2}}\times 16^{\frac{3}{4}}$.

Simplify:

38. $8^{\frac{2}{3}}\times 4^{-\frac{1}{2}}$. 40. $(\frac{1}{125})^{-\frac{2}{3}}\times(\frac{1}{36})^{\frac{1}{2}}$. 42. $(a^{-\frac{2}{3}}b^{3})^{-\frac{3}{2}}$.

39. $(\frac{1}{25})^{\frac{1}{2}}\times 16^{-\frac{3}{4}}$. 41. $a^{-\frac{3}{4}}b^{\frac{2}{3}}\times a^{\frac{1}{4}}b^{\frac{1}{3}}$. 43. $(a^{-\frac{1}{2}}b^{-1})^{-2}$.

If $a=4$, $b=2$, $c=1$, find the value of

44. $a^{\frac{1}{2}}b^{-1}$. 46. $a^{-\frac{1}{2}}b^{2}$. 48. $3(ab)^{\frac{1}{3}}$. 50. $(ab^{2}c)^{\frac{1}{4}}$.

45. ab^{-2}. 47. $a^{-\frac{1}{2}}c^{-\frac{2}{3}}$. 49. $2(ab)^{-\frac{1}{3}}$. 51. $(ab^{2}c)^{-\frac{1}{4}}$.

240. Compound expressions are multiplied and divided as follows:

(1) Multiply $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

$$\begin{array}{llllll}
x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}} & & & & & \\
x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}} & & & & & \\
\hline
x & +x^{\frac{3}{4}}y^{\frac{1}{4}} & +x^{\frac{1}{2}}y^{\frac{1}{2}} & & & \\
 & -x^{\frac{3}{4}}y^{\frac{1}{4}} & -x^{\frac{1}{2}}y^{\frac{1}{2}} & -x^{\frac{1}{4}}y^{\frac{3}{4}} & & \\
 & & +x^{\frac{1}{2}}y^{\frac{1}{2}} & +x^{\frac{1}{4}}y^{\frac{3}{4}} & +y & \\
\hline
x & & +x^{\frac{1}{2}}y^{\frac{1}{2}} & & +y &
\end{array}$$

(2) Divide $\sqrt[3]{x^2}+\sqrt[3]{x}-12$ by $\sqrt[3]{x}-3$.

$$\begin{array}{l|l}
x^{\frac{2}{3}}+\ x^{\frac{1}{3}}-12 & x^{\frac{1}{3}}-3 \\
\cline{2-2}
x^{\frac{2}{3}}-3x^{\frac{1}{3}} & x^{\frac{1}{3}}+4 \\
\cline{1-1}
\quad +4x^{\frac{1}{3}}-12 & \\
\quad +4x^{\frac{1}{3}}-12 & \\
\cline{1-1}
\end{array}$$

Exercise 77.

Multiply:

1. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

2. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

3. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

4. $x^{\frac{3}{2}}+2x$ by $x^{\frac{3}{2}}-2x$.

5. $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-2}-x^{-1}y^{-1}+y^{-2}$.

6. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

7. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

8. $1+b^{-1}+b^{-2}$ by $1-b^{-1}+b^{-2}$.

9. $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

10. $a^{\frac{2}{3}}b^{-\frac{1}{2}}+2a^{\frac{1}{3}}-3b^{\frac{1}{2}}$ by $2b^{-\frac{1}{2}}-4a^{-\frac{1}{3}}-6a^{-\frac{2}{3}}b^{\frac{1}{2}}$.

Divide:

11. $a-b$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

12. $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

13. $a-b$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$.

14. $a+b$ by $a^{\frac{1}{5}}+b^{\frac{1}{5}}$.

15. $2x^{-2}+6x^{-1}y^{-1}-16x^2y^{-4}$ by $2x+2x^2y^{-1}+4x^3y^{-2}$.

16. $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

17. $x-3x^{\frac{2}{3}}+3x^{\frac{1}{3}}-1$ by $x^{\frac{1}{3}}-1$.

18. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

19. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}+1+6x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}}-2$.

20. $9x-12x^{\frac{1}{2}}-2+4x^{-\frac{1}{2}}+x^{-1}$ by $3x^{\frac{1}{2}}-2-x^{-\frac{1}{2}}$.

Find the square root of

21. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}+1$.

22. $4a^{\frac{2}{3}}-4a^{\frac{1}{3}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

23. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}+4$.

24. $4a^{-2}+4a^{-1}+1$.

25. $9a-12a^{\frac{1}{2}}+10-4a^{-\frac{1}{2}}+a^{-1}$.

26. $49x^{\frac{4}{3}}-28x+18x^{\frac{2}{3}}-4x^{\frac{1}{3}}+1$.

27. $m^2+2m-1-2m^{-1}+m^{-2}$.

28. $1+4y^{-\frac{1}{3}}-2y^{-\frac{2}{3}}-4y^{-1}+25y^{-\frac{4}{3}}-24y^{-\frac{5}{3}}+16y^{-2}$.

Expand:

29. $(x-x^{-1})^3$.

30. $(\sqrt[3]{x}-\frac{1}{2}x)^4$.

31. $(2a^{\frac{1}{2}}-a^{-\frac{1}{2}})^3$.

32. $(2x^{-2}+x^{\frac{1}{2}})^5$.

33. $(\frac{1}{2}\sqrt{x}-\frac{1}{3}\sqrt[3]{x})^4$.

34. $(\frac{1}{3}\sqrt{x^{-3}}+\frac{1}{4}x^{-1})^3$.

CHAPTER XVI.

RADICAL EXPRESSIONS.

241. A **radical expression** is an expression affected with the radical sign; as, $\sqrt{a}$, $\sqrt[6]{9}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a+b}$, $\sqrt[5]{32}$.

242. An indicated root that cannot be exactly obtained is called a **surd.** An indicated root that can be exactly obtained is said to have the *form* of a surd.

The required root shows the **order** of a surd; and surds are named quadratic, cubic, biquadratic . . . according as the second, third, fourth . . . roots are required.

The product of a rational factor and a surd factor is called a **mixed surd**; as, $3\sqrt{2}$, $b\sqrt{a}$. The rational factor of a mixed surd is called the **coefficient** of the radical.

When there is no rational factor outside of the radical sign, that is, when the coefficient is 1, the surd is said to be **entire**; as, $\sqrt{2}$, $\sqrt{a}$.

243. A surd is in its *simplest form* when the expression under the radical sign is *integral and as small as possible.*

Surds which, when reduced to the simplest form, have the same surd factor, are said to be **similar.**

NOTE. In operations with surds, arithmetical numbers contained in the surds should be expressed in their prime factors.

REDUCTION OF RADICALS.

244. To reduce a radical is to change its *form* without changing its *value.*

Case I.

245. **When the radical is a perfect power and has for an exponent a factor of the index of the root.**

(1) $\sqrt[4]{a^2} = a^{\frac{2}{4}} = a^{\frac{1}{2}} = \sqrt{a}$;

(2) $\sqrt[4]{36\,a^2b^2} = \sqrt[4]{(6\,ab)^2} = (6\,ab)^{\frac{2}{4}} = (6\,ab)^{\frac{1}{2}} = \sqrt{6\,ab}$;

(3) $\sqrt[6]{25\,a^4b^2c^8} = \sqrt[6]{(5\,a^2bc^4)^2} = (5\,a^2bc^4)^{\frac{2}{6}} = (5\,a^2bc^4)^{\frac{1}{3}}$
$= \sqrt[3]{5\,a^2bc^4}$.

We have, therefore, the following rule:

Divide the exponent of the power by the index of the root.

Exercise 78.

Simplify:

1. $\sqrt[4]{25}$.
2. $\sqrt[8]{16}$.
3. $\sqrt[6]{27}$.
4. $\sqrt[4]{49}$.
5. $\sqrt[6]{64}$.
6. $\sqrt[6]{a^2b^2}$.
7. $\sqrt[4]{a^2b^2}$.
8. $\sqrt[8]{a^4b^4}$.
9. $\sqrt[6]{27\,a^3b^6}$.
10. $\sqrt[8]{16\,a^4b^4}$.
11. $\sqrt[6]{\dfrac{25\,a^2}{64\,b^2}}$.
12. $\sqrt[4]{\dfrac{16\,x^2}{(x-3)^2}}$.
13. $\sqrt[6]{\dfrac{a^3b^6}{8\,x^3y^3}}$.

Case II.

246. **When the radical is the product of two factors, one of which is a perfect power of the same degree as the radical.**

Since $\sqrt[n]{a^nb} = \sqrt[n]{a^n} \times \sqrt[n]{b} = a\sqrt[n]{b}$ (§ 215), we have

(1) $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$;

(2) $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = 3\sqrt[3]{4}$;

(3) $4\sqrt{72a^2b^3} = 4\sqrt{36a^2b^2 \times 2b} = 4\sqrt{36a^2b^2} \times \sqrt{2b}$

$= 4 \times 6ab\sqrt{2b} = 24ab\sqrt{2b}$;

(4) $2\sqrt[3]{54a^4b} = 2\sqrt[3]{27a^3 \times 2ab} = 2\sqrt[3]{27a^3} \times \sqrt[3]{2ab}$

$= 2 \times 3a\sqrt[3]{2ab} = 6a\sqrt[3]{2ab}$.

We have, therefore, the following rule:

Resolve the radical into two factors, one of which is the greatest perfect power of the same degree as the radical.

Remove this factor from under the radical sign, extract the required root, and multiply the coefficient by the root obtained.

Exercise 79.

Simplify:

1. $\sqrt{28}$.
2. $\sqrt{72}$.
3. $\sqrt[3]{72}$.
4. $\sqrt[3]{500}$.
5. $\sqrt[3]{432}$.
6. $\sqrt[3]{192}$.
7. $\sqrt[5]{128}$.
8. $\sqrt[4]{243}$.
9. $\sqrt[4]{176}$.
10. $\sqrt{405}$.
11. $2\sqrt[4]{112}$.
12. $3\sqrt[3]{864}$.
13. $7\sqrt[4]{144}$.
14. $8\sqrt{m^2n}$.
15. $3\sqrt[4]{b^8a^3}$.
16. $2\sqrt[5]{a^{18}c^4}$.
17. $11\sqrt[6]{a^{12}b^{15}}$.
18. $7\sqrt[3]{8a^3b}$.
19. $6\sqrt[3]{27m^2n^3}$.
20. $4\sqrt[4]{x^7y^8}$.
21. $\sqrt[3]{1029}$.
22. $\sqrt[3]{-2187}$.
23. $\sqrt[4]{1250}$.
24. $4\sqrt[3]{648}$.
25. $\sqrt[3]{\dfrac{64x^6y}{27m^3n^3}}$.
26. $\sqrt{\dfrac{4a^3b}{9}}$.
27. $\sqrt[3]{\dfrac{125x^3}{216y^3}}$.
28. $2\sqrt[5]{\dfrac{m^6n}{243}}$.
29. $\sqrt[4]{\dfrac{x^5y^7}{1296}}$.
30. $\sqrt[3]{\dfrac{(x-y)^3z^3}{512}}$.
31. $\dfrac{3ab}{2c}\sqrt{\dfrac{20c^2}{9a^2b^2}}$.

CASE III.

247. When the radical expression is a fraction, the denominator of which is not a perfect power of the same degree as the radical.

$$\sqrt{\frac{5}{8}}=\sqrt{\frac{10}{16}}=\sqrt{10\times\frac{1}{16}}=\tfrac{1}{4}\sqrt{10}.$$

$$\sqrt{\frac{7}{12}}=\sqrt{\frac{7}{4\times 3}}=\sqrt{\frac{7\times 3}{4\times 9}}=\sqrt{21\times\frac{1}{36}}=\tfrac{1}{6}\sqrt{21};$$

$$\sqrt[3]{\frac{5}{18}}=\sqrt[3]{\frac{5}{9\times 2}}=\sqrt[3]{\frac{5\times 3\times 4}{27\times 8}}=\sqrt[3]{60\times\frac{1}{27\times 8}}$$

$$=\frac{1}{3\times 2}\sqrt[3]{60}=\tfrac{1}{6}\sqrt[3]{60}.$$

We have, therefore, the following rule:

Multiply both terms of the fraction by such a number as will make the denominator a perfect power of the same degree as the radical; and then proceed as in Case II.

Exercise 80.

Simplify:

1. $2\sqrt{\tfrac{1}{2}}$.
2. $\tfrac{3}{4}\sqrt{\tfrac{2}{3}}$.
3. $\tfrac{1}{4}\sqrt{\tfrac{1}{5}}$.
4. $7\sqrt{\tfrac{4}{5}}$.
5. $\sqrt[4]{\tfrac{25}{16}}$.
6. $3\sqrt{\tfrac{9}{80}}$.
7. $\sqrt[3]{\tfrac{8}{3}}$.
8. $\sqrt[3]{\tfrac{9}{8}}$.
9. $\sqrt[3]{\tfrac{24}{343}}$.
10. $2\sqrt[3]{\tfrac{5}{3}}$.
11. $3\sqrt[5]{\tfrac{2}{81}}$.
12. $2\sqrt[5]{\tfrac{3}{16}}$.
13. $\sqrt{\dfrac{a^4c}{b^3}}$.
14. $\sqrt[4]{\dfrac{b^4}{a^3}}$.
15. $\sqrt[3]{\dfrac{ax^4}{b^2}}$.
16. $\sqrt[3]{\dfrac{7a}{125x}}$.
17. $\sqrt{\dfrac{a^2cy^2}{bd^2}}$.
18. $2\sqrt[3]{\dfrac{2a^3b^2c}{3x^2yz^3}}$.

CASE IV.

248. To reduce a mixed surd to an entire surd.

Since $a\sqrt[n]{b} = \sqrt[n]{a^n} \times \sqrt[n]{b} = \sqrt[n]{a^n b}$, we have

(1) $3\sqrt{5} = \sqrt{3^2 \times 5} = \sqrt{9 \times 5} = \sqrt{45}$;

(2) $a^2b\sqrt{bc} = \sqrt{(a^2b)^2 \times bc} = \sqrt{a^4b^2 \times bc} = \sqrt{a^4b^3c}$;

(3) $2x\sqrt[3]{xy} = \sqrt[3]{(2x)^3 \times xy} = \sqrt[3]{8x^3 \times xy} = \sqrt[3]{8x^4y}$;

(4) $3y^2\sqrt[4]{x^3} = \sqrt[4]{(3y^2)^4 \times x^3} = \sqrt[4]{81y^8x^3}$.

We have, therefore, the following rule:

Raise the coefficient to a power of the same degree as the radical, multiply this power by the given surd factor, and indicate the required root of the product.

Exercise 81.

Express as entire surds:

1. $5\sqrt{5}$.	5. $2\sqrt[4]{3}$.	9. $-2\sqrt[3]{y}$.	13. $\frac{1}{2}\sqrt{a}$.
2. $3\sqrt{11}$.	6. $3\sqrt[5]{2}$.	10. $-3\sqrt{y^3}$.	14. $-\frac{2}{3}\sqrt[3]{a^2}$.
3. $3\sqrt[3]{3}$.	7. $2\sqrt[6]{2}$.	11. $-m\sqrt[6]{10}$.	15. $\frac{3}{4}\sqrt{m^3}$.
4. $2\sqrt[3]{4}$.	8. $2\sqrt[4]{4}$.	12. $-2\sqrt[4]{x}$.	16. $-\frac{1}{2}\sqrt[4]{m^7}$.

CASE V.

249. To reduce radicals to a common index.

(1) Reduce $\sqrt{2}$ and $\sqrt[3]{3}$ to a common index.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}.$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

Hence,

Write the radicals with fractional exponents, and change these fractional exponents to equivalent exponents having the least common denominator. Raise each radical to the power denoted by the numerator, and indicate the root denoted by the common denominator.

Exercise 82.

Reduce to surds of the same order:

1. $\sqrt[4]{3}$ and $\sqrt[6]{5}$.
2. $\sqrt[3]{14}$ and $\sqrt{6}$.
3. $\sqrt{2}$ and $\sqrt[3]{4}$.
4. $\sqrt{a}$ and $\sqrt[3]{b^2}$.
5. $\sqrt{5}$ and $\sqrt[6]{75}$.
6. $2^{\frac{1}{2}}$, $2^{\frac{2}{3}}$, and $2^{\frac{3}{4}}$.
7. $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$.
8. $\sqrt[6]{a^2}$, $\sqrt[3]{b}$, and $\sqrt{c}$.
9. $\sqrt[5]{a^4}$, $\sqrt[10]{c^3}$, and $\sqrt{x^3}$.
10. $\sqrt[4]{x^2y}$, $\sqrt[3]{abc}$, and $\sqrt[6]{2z}$.
11. $\sqrt[6]{x-y}$ and $\sqrt[4]{x+y}$.
12. $\sqrt[3]{a+b}$ and $\sqrt{a-b}$.

NOTE. Surds of different orders may be reduced to surds of the same order and then compared in respect to magnitude.

Arrange in order of magnitude:

13. $\sqrt[3]{15}$ and $\sqrt{6}$.
14. $\sqrt[3]{4}$ and $\sqrt{3}$.
15. $\sqrt[6]{80}$, $\sqrt[3]{9}$, and $\sqrt{8}$.
16. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{7}$.

Addition and Subtraction of Radicals.

250. In the addition of surds, each surd must be reduced to its simplest form; and, if the resulting surds are similar,

Find the algebraic sum of the coefficients, and to this sum annex the common surd factor.

If the resulting surds are not similar,

Connect them with their proper signs.

(1) Simplify $\sqrt{27}+\sqrt{48}+\sqrt{147}$.

$$\sqrt{27}=(3^2\times 3)^{\frac{1}{2}}=3\times 3^{\frac{1}{2}}=3\sqrt{3};$$

$$\sqrt{48}=(2^4\times 3)^{\frac{1}{2}}=2^2\times 3^{\frac{1}{2}}=4\times 3^{\frac{1}{2}}=4\sqrt{3};$$

$$\sqrt{147}=(7^2\times 3)^{\frac{1}{2}}=7\times 3^{\frac{1}{2}}=7\sqrt{3}.$$

$$\therefore\ \sqrt{27}+\sqrt{48}+\sqrt{147}=(3+4+7)\sqrt{3}=14\sqrt{3}.\ \textit{Ans.}$$

(2) Simplify $2\sqrt[3]{320}-3\sqrt[3]{40}$.

$$2\sqrt[3]{320}=2(2^6\times 5)^{\frac{1}{3}}=2\times 2^2\times 5^{\frac{1}{3}}=8\sqrt[3]{5};$$

$$3\sqrt[3]{40}=3(2^3\times 5)^{\frac{1}{3}}=3\times 2\times 5^{\frac{1}{3}}=6\sqrt[3]{5}.$$

$$\therefore\ 2\sqrt[3]{320}-3\sqrt[3]{40}=(8-6)\sqrt[3]{5}=2\sqrt[3]{5}.\quad \textit{Ans.}$$

(3) Simplify $2\sqrt{\frac{5}{3}}-3\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{15}}$.

$$2\sqrt{\tfrac{5}{3}}=2\sqrt{\tfrac{15}{9}}=2\sqrt{15\times\tfrac{1}{9}}=\tfrac{2}{3}\sqrt{15};$$

$$3\sqrt{\tfrac{3}{5}}=3\sqrt{\tfrac{15}{25}}=3\sqrt{15\times\tfrac{1}{25}}=\tfrac{3}{5}\sqrt{15};$$

$$\sqrt{\tfrac{4}{15}}=\sqrt{\frac{4\times 15}{15^2}}=\sqrt{15\times\frac{4}{15^2}}=\tfrac{2}{15}\sqrt{15}.$$

$$\therefore\ 2\sqrt{\tfrac{5}{3}}-3\sqrt{\tfrac{3}{5}}+\sqrt{\tfrac{4}{15}}=(\tfrac{2}{3}-\tfrac{3}{5}+\tfrac{2}{15})\sqrt{15}=\tfrac{1}{5}\sqrt{15}.\ \textit{Ans.}$$

Exercise 83.

Simplify:

1. $4\sqrt{11}+3\sqrt{11}-5\sqrt{11}$.

2. $2\sqrt{3}-5\sqrt{3}+9\sqrt{3}$.

3. $5\sqrt[3]{4}+2\sqrt[3]{32}-\sqrt[3]{108}$.

4. $3\sqrt[5]{2}+4\sqrt[5]{2}-\sqrt[5]{64}$.

5. $\frac{1}{2}\sqrt[3]{5}+2\frac{1}{2}\sqrt[3]{5}+\frac{1}{4}\sqrt[3]{40}$.

6. $3\sqrt[4]{3}-5\sqrt[4]{48}+\sqrt[4]{243}$.

7. $\sqrt{27}+\sqrt{48}+\sqrt{75}$.

8. $4\sqrt{147}+3\sqrt{75}+\sqrt{192}$.

9. $\sqrt{a}+\frac{1}{2}\sqrt{a}+\frac{3}{2}\sqrt{a}$.

10. $\sqrt[3]{a^2}+\frac{1}{2}\sqrt[3]{a^2}-3\sqrt[3]{27a^2}$.

11. $\sqrt{a^3}+b\sqrt{a}-3\sqrt{a}$.

12. $\sqrt{25\,b}+2\sqrt{9\,b}-3\sqrt{4\,b}$.

13. $2\sqrt{175}-3\sqrt{63}+5\sqrt{28}$.

14. $\sqrt{2}+3\sqrt{32}+\frac{1}{2}\sqrt{128}-6\sqrt{18}$.

15. $\sqrt{75}+\sqrt{48}-\sqrt{147}+\sqrt{300}$.

16. $20\sqrt{245}-\sqrt{5}+\sqrt{125}-2\frac{1}{2}\sqrt{180}$.

17. $2\sqrt{20}+\frac{1}{2}\sqrt{12}-2\sqrt{27}+5\sqrt{45}-9\sqrt{12}$.

18. $7\sqrt{25}+4\sqrt{45}-\sqrt{9}-2\sqrt{80}+\sqrt{20}-4\sqrt{64}$.

19. $\sqrt[3]{54}+\sqrt{\frac{1}{2}}-\sqrt[3]{250}-\frac{3}{4}\sqrt{\frac{2}{9}}$.

20. $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{15}}$.

21. $\sqrt[3]{27\,c^4}-\sqrt[3]{8\,c^4}+\sqrt[3]{125\,c}$.

22. $\sqrt[5]{a^5b}-\sqrt[5]{b^6}+\sqrt[5]{32\,b}$.

23. $\sqrt{a^4x}+\sqrt{b^4x}-\sqrt{4\,a^2b^2x}$.

24. $\sqrt{4\,x^2y^2z}+\sqrt{y^4z}+\sqrt{x^4z}$.

25. $\sqrt{a^2b^2c}-a\sqrt{4\,c}+b\sqrt{a^2c}$.

26. $\sqrt[4]{81\,a^5}-\sqrt[4]{16\,a}+\sqrt[4]{256\,a^5}$.

27. $\sqrt[3]{27\,m^4}-\sqrt[3]{125\,m}+\sqrt[3]{216\,m}$.

28. $\sqrt{8\,a}-\sqrt{50\,a^3}-3\sqrt{18\,a}$.

29. $6\,a\sqrt{63\,ab^3}-3\sqrt{112\,a^3b^3}+2\,ab\sqrt{343\,ab}$.

30. $3\sqrt{125\,m^3n^2}+n\sqrt{20\,m^3}-\sqrt{500\,m^3n^2}$.

31. $\sqrt{32\,a^4b^5}+6\sqrt{72\,b}+3\sqrt{128\,a^2b^3}$.

32. $2\sqrt[3]{a^6b}-3\,a^2\sqrt[3]{64\,b}+5\,a\sqrt[3]{a^3b}+2\,a^2\sqrt[3]{125\,b}$.

MULTIPLICATION OF RADICALS.

251. Since $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$, we have

(1) $3\sqrt{8} \times 5\sqrt{2} = 3 \times 5 \times \sqrt{8} \times \sqrt{2} = 15\sqrt{16} = 60$;

(2) $3\sqrt{2} \times 4\sqrt[3]{3} = 3\sqrt[6]{8} \times 4\sqrt[6]{9} = 12\sqrt[6]{72}$.

We have, therefore, the following rule:

Express the radicals with a common index. Find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor.

Reduce the result to its simplest form.

Exercise 84.

1. $\sqrt{3} \times \sqrt{27}$.
2. $\sqrt{5} \times \sqrt{20}$.
3. $\sqrt{2} \times \sqrt{18}$.
4. $\sqrt[3]{3} \times \sqrt[3]{9}$.
5. $\sqrt[3]{2} \times \sqrt[3]{32}$.
6. $\sqrt[4]{27} \times \sqrt[4]{3}$.
7. $\sqrt[5]{4} \times \sqrt[5]{8}$.
8. $\sqrt[5]{27} \times \sqrt[5]{9}$.
9. $\sqrt{2} \times \sqrt{12}$.
10. $\sqrt{3} \times \sqrt{6}$.
11. $\sqrt[3]{3} \times \sqrt[3]{18}$.
12. $\sqrt[4]{6} \times \sqrt[4]{8}$.
13. $\sqrt[5]{54} \times \sqrt[5]{9}$.
14. $2\sqrt{8} \times \sqrt{2}$.
15. $\sqrt[5]{8} \times \sqrt[5]{-4}$.
16. $\sqrt[3]{7} \times \sqrt[3]{-49}$.
17. $\sqrt[3]{81} \times \sqrt[3]{-45}$.
18. $\frac{2}{3}\sqrt[3]{18} \times \frac{3}{4}\sqrt[3]{3}$.
19. $(\sqrt{18} + 2\sqrt{72} - 3\sqrt{8}) \times \sqrt{2}$.
20. $(\sqrt[3]{32} - \frac{1}{2}\sqrt[3]{864} + 3\sqrt[3]{4}) \times \sqrt[3]{2}$.
21. $(\frac{1}{3}\sqrt{27} - \frac{1}{4}\sqrt{2187} + \frac{1}{8}\sqrt{432}) \times \sqrt{3}$.
22. $\sqrt{5} \times \sqrt[3]{4}$.
23. $\sqrt[3]{16} \times \sqrt[6]{250}$.
24. $\sqrt[4]{64} \times \sqrt[8]{16}$.
25. $\sqrt{3} \times \sqrt[3]{72}$.
26. $\sqrt{\frac{2}{9}} \times \sqrt[3]{\frac{1}{2}}$.
27. $\sqrt{\frac{7}{12}} \times \sqrt[3]{\frac{3}{7}}$.
28. $\sqrt[3]{81} \times \sqrt{3}$.
29. $\sqrt[3]{\frac{5}{6}} \times \sqrt{\frac{3}{5}}$.
30. $\sqrt[4]{\frac{2}{3}} \times \sqrt[3]{\frac{1}{2}}$.
31. $\sqrt[3]{2x} \times \sqrt[4]{x^3}$.
32. $\sqrt[5]{y^4} \times \sqrt[7]{2y^3}$.
33. $\sqrt[3]{7} \times \sqrt{5}$.

252. Compound radicals are multiplied as follows:

Ex. Multiply $2\sqrt{3}+3\sqrt{x}$ by $3\sqrt{3}-4\sqrt{x}$.

$$\begin{array}{l} 2\sqrt{3}+3\sqrt{x} \\ 3\sqrt{3}-4\sqrt{x} \\ \hline 18+9\sqrt{3x} \\ \qquad -8\sqrt{3x}-12x \\ \hline 18+\sqrt{3x}-12x \end{array}$$

Exercise 85.

Multiply:

1. $\sqrt{5+\sqrt{4}}$ by $\sqrt{5-\sqrt{4}}$.
2. $\sqrt{9-\sqrt{17}}$ by $\sqrt{9+\sqrt{17}}$.
3. $3+2\sqrt{5}$ by $2-\sqrt{5}$.
4. $8+3\sqrt{2}$ by $2-\sqrt{2}$.
5. $5+2\sqrt{3}$ by $3-5\sqrt{3}$.
6. $3-\sqrt{6}$ by $6-3\sqrt{6}$.
7. $2\sqrt{6}-3\sqrt{5}$ by $\sqrt{3}+2\sqrt{2}$.
8. $7-\sqrt{3}$ by $\sqrt{2}+\sqrt{5}$.
9. $\sqrt[3]{9}-2\sqrt[3]{4}$ by $4\sqrt[3]{3}+\sqrt[3]{2}$.
10. $2\sqrt{30}-3\sqrt{5}+5\sqrt{3}$ by $\sqrt{8}+\sqrt{3}-\sqrt{5}$.
11. $3\sqrt{5}-2\sqrt{3}+4\sqrt{7}$ by $3\sqrt{7}-4\sqrt{5}-5\sqrt{3}$.
12. $4\sqrt{8}+\frac{1}{2}\sqrt{12}-\frac{1}{4}\sqrt{32}$ by $8\sqrt{32}-4\sqrt{50}-2\sqrt{2}$.
13. $\sqrt[3]{6}-\sqrt[3]{3}+\sqrt[3]{16}$ by $\sqrt[3]{36}+\sqrt[3]{9}-\sqrt[3]{4}$.
14. $2\sqrt{\frac{2}{3}}-8\sqrt{\frac{3}{8}}+3\sqrt{\frac{3}{2}}$ by $3\sqrt{\frac{2}{3}}-\sqrt{12}-\sqrt{6}$.
15. $2\sqrt{\frac{5}{6}}-4\sqrt{\frac{3}{2}}-7\sqrt{\frac{6}{5}}$ by $3\sqrt{\frac{5}{6}}-5\sqrt{30}-2\sqrt{\frac{15}{2}}$.
16. $2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{3}}$ by $2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{3}}$.

Division of Radicals.

253. Since $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{a}} = \dfrac{\sqrt[n]{a} \times \sqrt[n]{b}}{\sqrt[n]{a}} = \sqrt[n]{b}$, we have

$$(1)\quad \frac{4\sqrt{8}}{2\sqrt{2}} = 2\sqrt{4} = 4;$$

$$(2)\quad \frac{4\sqrt[3]{3}}{2\sqrt{2}} = \frac{4\sqrt[6]{3^2}}{2\sqrt[6]{2^3}} = \frac{4\sqrt[6]{3^2 \times 2^3}}{2\sqrt[6]{2^6}} = \sqrt[6]{72}.$$

We have, therefore, the following rule:

Express the radicals with a common index. Find the quotient of the coefficients for the required coefficient, and the quotient of the surd factors for the required surd factor.

Reduce the result to its simplest form.

Exercise 86.

Divide:

1. $\sqrt{243}$ by $\sqrt{3}$.
2. $\sqrt[3]{81}$ by $\sqrt[3]{3}$.
3. $\sqrt{3a^7}$ by $\sqrt{a^3}$.
4. $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{2}{9}}$.
5. $\sqrt{\frac{5}{6}}$ by $\sqrt{\frac{3}{5}}$.
6. $\sqrt{\frac{7}{12}}$ by $\sqrt{\frac{3}{7}}$.
7. $\sqrt{\frac{14}{15}}$ by $\sqrt{\frac{24}{35}}$.
8. $\sqrt{\frac{22}{27}}$ by $\sqrt{\frac{6}{11}}$.
9. $\sqrt{\frac{52}{85}}$ by $\sqrt{\frac{17}{65}}$.
10. $3\sqrt{6} + 45\sqrt{2}$ by $3\sqrt{3}$.
11. $42\sqrt{5} - 30\sqrt{3}$ by $2\sqrt{15}$.
12. $84\sqrt{15} + 168\sqrt{6}$ by $3\sqrt{21}$.
13. $30\sqrt[3]{4} - 36\sqrt[3]{10} + 30\sqrt[3]{90}$ by $3\sqrt[3]{20}$.
14. $50\sqrt[3]{18} + 18\sqrt[3]{20} - 48\sqrt[3]{5}$ by $2\sqrt[3]{30}$.
15. $\sqrt{54}$ by $\sqrt[4]{36}$.
16. $\sqrt[3]{49}$ by $\sqrt{7}$.
17. $\sqrt[3]{12}$ by $\sqrt{6}$.
18. $\sqrt{\frac{8}{45}}$ by $\sqrt[4]{6\frac{1}{4}}$.
19. $\sqrt{\frac{3}{5}}$ by $\sqrt[6]{3\frac{3}{8}}$.
20. $\sqrt[3]{2x}$ by $\sqrt{x^3}$.
21. $\sqrt[6]{0.064}$ by $\sqrt{10}$.
22. $\sqrt{x^2 - y^2}$ by $x + y$.

254. The quotient of one surd by another may be found by *rationalizing the divisor;* that is, by multiplying the dividend and divisor by a factor which will free the divisor of surds.

255. This method is of great utility when we wish to find the approximate numerical value of the quotient of two simple surds, and is the method required when the divisor is a compound surd.

(1) Divide $3\sqrt{8}$ by $\sqrt{6}$.

$$\frac{3\sqrt{8}}{\sqrt{6}}=\frac{6\sqrt{2}}{\sqrt{6}}=\frac{6\sqrt{2}\times\sqrt{6}}{\sqrt{6}\times\sqrt{6}}=\frac{6\sqrt{12}}{6}=\sqrt{12}=2\sqrt{3}.$$

(2) Divide $3\sqrt{5}-4\sqrt{2}$ by $2\sqrt{5}+3\sqrt{2}$.

$$\frac{3\sqrt{5}-4\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}=\frac{(3\sqrt{5}-4\sqrt{2})(2\sqrt{5}-3\sqrt{2})}{(2\sqrt{5}+3\sqrt{2})(2\sqrt{5}-3\sqrt{2})}=\frac{54-17\sqrt{10}}{20-18}$$

$$=\frac{54-17\sqrt{10}}{2}=27-8\tfrac{1}{2}\sqrt{10}.$$

(3) Given $\sqrt{2}=1.41421$, find the value of $\frac{5}{\sqrt{2}}$.

$$\frac{5}{\sqrt{2}}=\frac{5\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{5\sqrt{2}}{2}=\frac{7.07105}{2}=3.53553.$$

Exercise 87.

Divide:

1. $\sqrt{a}+\sqrt{b}$ by $\sqrt{ab}$.
2. $\sqrt{125}$ by $5\sqrt{65}$.
3. 3 by $11+3\sqrt{7}$.
4. $3\sqrt{2}-1$ by $3\sqrt{2}+1$.
5. 17 by $3\sqrt{7}+2\sqrt{3}$.
6. 1 by $\sqrt{2}+\sqrt{3}$.
7. $3+5\sqrt{7}$ by $3-5\sqrt{7}$.
8. $21\sqrt{3}$ by $4\sqrt{3}-3\sqrt{2}$.
9. $75\sqrt{14}$ by $8\sqrt{2}+2\sqrt{7}$.
10. $\sqrt{5}-\sqrt{3}$ by $\sqrt{5}+\sqrt{3}$.
11. $\sqrt{8}+\sqrt{7}$ by $\sqrt{7}-\sqrt{2}$.
12. $7-3\sqrt{10}$ by $5+4\sqrt{5}$.

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$; find to four places of decimals the value of

13. $\frac{10}{\sqrt{2}}$.	16. $\frac{1}{\sqrt{500}}$.	19. $\frac{1}{3\sqrt{2}}$.	22. $\frac{7-3\sqrt{5}}{5+4\sqrt{5}}$.
14. $\frac{8}{\sqrt{3}}$.	17. $\frac{1}{\sqrt{243}}$.	20. $\frac{1}{\sqrt{125}}$.	23. $\frac{3+\sqrt{5}}{\sqrt{5}-2}$.
15. $\frac{12}{\sqrt{5}}$.	18. $\frac{1}{2\sqrt{3}}$.	21. $\frac{1}{4\sqrt{5}}$.	24. $\frac{3\sqrt{2}-1}{3\sqrt{2}+1}$.

Involution and Evolution of Radicals.

256. Any power or any root of a radical is easily found by using fractional exponents.

(1) Find the square of $2\sqrt[3]{a}$.

$$(2\sqrt[3]{a})^2=(2a^{\frac{1}{3}})^2=2^2a^{\frac{2}{3}}=4a^{\frac{2}{3}}=4\sqrt[3]{a^2}.$$

(2) Find the cube of $2\sqrt{a}$.

$$(2\sqrt{a})^3=(2a^{\frac{1}{2}})^3=2^3a^{\frac{3}{2}}=8a^{\frac{3}{2}}=8a\sqrt{a}.$$

(3) Find the square root of $4x\sqrt{a^3b^3}$.

$$(4x\sqrt{a^3b^3})^{\frac{1}{2}}=(4xa^{\frac{3}{2}}b^{\frac{3}{2}})^{\frac{1}{2}}=4^{\frac{1}{2}}x^{\frac{1}{2}}a^{\frac{3}{4}}b^{\frac{3}{4}}=4^{\frac{1}{2}}x^{\frac{2}{4}}a^{\frac{3}{4}}b^{\frac{3}{4}}=2\sqrt[4]{a^3b^3x^2}.$$

(4) Find the cube root of $4x\sqrt{a^3b^3}$.

$$(4x\sqrt{a^3b^3})^{\frac{1}{3}}=(4xa^{\frac{3}{2}}b^{\frac{3}{2}})^{\frac{1}{3}}=4^{\frac{1}{3}}x^{\frac{1}{3}}a^{\frac{1}{2}}b^{\frac{1}{2}}=4^{\frac{2}{6}}x^{\frac{2}{6}}a^{\frac{3}{6}}b^{\frac{3}{6}}=\sqrt[6]{16a^3b^3x^2}.$$

Exercise 88.

Perform the operations indicated:

1. $(\sqrt[3]{m^2})^3$.	3. $(\sqrt[5]{x^4})^{15}$.	5. $\sqrt[4]{\sqrt{(x-y)^8}}$.
2. $(\sqrt{m^3})^5$.	4. $(\sqrt[3]{y^5})^{12}$.	6. $\sqrt[3]{\sqrt[6]{(a-b)^{36}}}$.

7. $(\sqrt{2a^3b})^4$.
8. $(\sqrt[3]{x^2-y^2})^6$.
9. $(\sqrt[8]{x})^4$.
10. $\sqrt{\sqrt[3]{a^2}}$.
11. $\sqrt[4]{\sqrt[3]{729}}$.
12. $\sqrt[3]{\sqrt{125}}$.
13. $\sqrt[7]{\sqrt[3]{(3a-2b)^{14}}}$.
14. $\sqrt[3]{\sqrt[5]{32a^{45}}}$.
15. $\sqrt[7]{128\sqrt[5]{243a^{70}}}$.

PROPERTIES OF QUADRATIC SURDS.

257. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.* Thus,

$$\sqrt{ab}\times\sqrt{abc}=ab\sqrt{c};$$
$$\sqrt{abc}\div\sqrt{ab}=\sqrt{c}.$$

For every quadratic surd, when simplified, will have under the radical sign one or more factors raised only to the first power; and two surds which are *dissimilar* cannot have *all* these factors alike.

Hence, their product or quotient will have *at least one factor* raised only to the *first* power, and will therefore be a surd.

258. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For if $\sqrt{a}\pm\sqrt{b}$ could equal a rational number c, we should have, by squaring,

$$a\pm 2\sqrt{ab}+b=c^2;$$

that is, $$\pm 2\sqrt{ab}=c^2-a-b.$$

Now, as the right side of this equation is rational, the left side would be rational; but, by § 257, $\sqrt{ab}$ cannot be rational. Therefore, $\sqrt{a}\pm\sqrt{b}$ cannot be rational.

In like manner, it may be shown that $\sqrt{a}\pm\sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

259. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For if $\sqrt{a}$ could equal $c+\sqrt{b}$, we should have, by squaring,

$$a=c^2+2c\sqrt{b}+b,$$

and, by transposing,

$$2c\sqrt{b}=a-b-c^2.$$

That is, a surd equal to a rational number, which is impossible.

260. *If* $a+\sqrt{b}=x+\sqrt{y}$, *then* a *will equal* x, *and* b *will equal* y.

For, by transposing, $\sqrt{b}-\sqrt{y}=x-a$; and if b were not equal to y, the difference of two unequal surds would be rational, which by § 258 is impossible.

$$\therefore\ b=y, \text{ and } a=x.$$

In like manner, if $a-\sqrt{b}=x-\sqrt{y}$, a will equal x, and b will equal y.

261. To extract the square root of a binomial surd.

Ex. Extract the square root of $a+\sqrt{b}$.

Suppose $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$. (1)

By squaring, $a+\sqrt{b}=x+2\sqrt{xy}+y$. (2)

$\therefore\ a=x+y$ and $\sqrt{b}=2\sqrt{xy}$. § 260

Therefore, $a-\sqrt{b}=x-2\sqrt{xy}+y$, (3)

and $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$. (4)

Multiplying (1) by (4),

$$\sqrt{a^2-b}=x-y.$$

But $a=x+y$.

Adding, and dividing by 2, $x = \frac{a + \sqrt{a^2 - b}}{2}$.

Subtracting, and dividing by 2,

$$y = \frac{a - \sqrt{a^2 - b}}{2}.$$

$$\therefore \sqrt{a + \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

From these two values of x and y, it is evident that this method is practicable only when $a^2 - b$ is a perfect square.

(1) Extract the square root of $7 + 4\sqrt{3}$.

Let $\sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}}$.

Then $\sqrt{x} - \sqrt{y} = \sqrt{7 - 4\sqrt{3}}$.

Multiplying, $x - y = \sqrt{49 - 48}$.

$\therefore x - y = 1$.

But $x + y = 7$.

$\therefore x = 4$, and $y = 3$.

$\therefore \sqrt{x} + \sqrt{y} = 2 + \sqrt{3}$.

$\therefore \sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$.

A root may often be obtained by inspection. For this purpose, write the given expression in the form $a + 2\sqrt{b}$, and determine what two numbers have their sum equal to a, and their product equal to b.

(2) Find by inspection the square root of $75 - 12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore, $75 - 12\sqrt{21}$ must be put in the form

$$75 - 2\sqrt{756}.$$

The two numbers whose sum is 75 and whose product is 756 are 63 and 12.

Then $75 - 2\sqrt{756} = 63 + 12 - 2\sqrt{63 \times 12}$,

$= (\sqrt{63} - \sqrt{12})^2$.

That is, $\sqrt{63} - \sqrt{12} =$ square root of $75 - 12\sqrt{21}$;

or, $3\sqrt{7} - 2\sqrt{3} =$ square root of $75 - 12\sqrt{21}$.

Exercise 89.

Find the square root of

1. $7 - 4\sqrt{3}$.
2. $11 + \sqrt{72}$.
3. $7 + 2\sqrt{10}$.
4. $18 + 8\sqrt{5}$.
5. $8 + 2\sqrt{15}$.
6. $15 - 4\sqrt{14}$.
7. $16 + 5\sqrt{7}$.
8. $75 + 12\sqrt{21}$.
9. $19 + 8\sqrt{3}$.
10. $8\sqrt{6} + 20$.
11. $28 - 16\sqrt{3}$.
12. $51 + 36\sqrt{2}$.
13. $94 + 42\sqrt{5}$.
14. $11 - 2\sqrt{30}$.
15. $47 - 4\sqrt{33}$.
16. $29 + 6\sqrt{22}$.
17. $83 + 12\sqrt{35}$.
18. $55 - 12\sqrt{21}$.

Equations containing Radicals.

262. An equation containing a *single* radical may be solved by arranging the terms so as to have the radical alone on one side, and then raising both sides to a power corresponding to the order of the radical.

Ex. $\sqrt{x^2 - 9} + x = 9$.

$$\sqrt{x^2 - 9} = 9 - x.$$

By squaring,

$$x^2 - 9 = 81 - 18x + x^2.$$
$$18x = 90.$$
$$\therefore x = 5.$$

263. If *two* radicals are involved, two steps may be necessary.

Ex. $\sqrt{x + 15} + \sqrt{x} = 15$.

$$\sqrt{x + 15} + \sqrt{x} = 15.$$

Squaring and simplifying, we have

$$\sqrt{x^2 + 15x} = 105 - x.$$

Squaring, we have

$$x^2 + 15x = 11025 - 210x + x^2.$$
$$225x = 11025.$$
$$\therefore x = 49.$$

Exercise 90.

Solve:

1. $2\sqrt{x+5}=\sqrt{28}.$
2. $3\sqrt{4x-8}=\sqrt{13x-3}.$
3. $\sqrt{x+9}=5\sqrt{x-3}.$
4. $4=2\sqrt{x}-3.$
5. $5-\sqrt{3y}=4.$
6. $7+2\sqrt[3]{3x}=5.$
7. $\sqrt[3]{2x-3}=-3.$
8. $\sqrt[3]{3x+7}=3.$
9. $14+\sqrt[3]{4x-40}=10.$
10. $\sqrt[3]{10y-4}=\sqrt[3]{7y+11}.$
11. $2\sqrt{x-2}=\sqrt[4]{32(x-2)^3}.$
12. $\sqrt{\frac{15}{4}+x}=\frac{3}{2}+\sqrt{x}.$
13. $\sqrt{32+x}=16-\sqrt{x}.$
14. $\sqrt{x}-\sqrt{x-5}=\sqrt{5}.$
15. $\sqrt{x+20}-\sqrt{x-1}-3=0.$
16. $\sqrt{x+15}-7=7-\sqrt{x-13}.$
17. $x=7-\sqrt{x^2-7}.$
18. $\sqrt{x-7}=\sqrt{x+1}-2.$
19. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}-2}.$
20. $\dfrac{1}{2}-\dfrac{3}{x}=\sqrt{\dfrac{1}{4}-\dfrac{1}{x}\sqrt{9-\dfrac{36}{x}}}.$
21. $\dfrac{1+(1-x)^{\frac{1}{2}}}{1-(1-x)^{\frac{1}{2}}}=3.$
22. $(x-3)^{\frac{1}{2}}+x^{\frac{1}{2}}=\dfrac{3}{(x-3)^{\frac{1}{2}}}.$
23. $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}.$
24. $\sqrt{ax}-1=4+\frac{1}{2}\sqrt{ax}-\frac{1}{2}.$
25. $3\sqrt{x}-3\sqrt{a}=\sqrt{x}-\sqrt{a}+2\sqrt{a}.$
26. $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}.$

CHAPTER XVII.

IMAGINARY EXPRESSIONS.

264. An **imaginary expression** is any expression which involves the indicated even root of a negative number.

It will be shown hereafter that *any* indicated even root of a negative number may be made to assume a form which involves only an indicated *square root* of a negative number. In considering imaginary expressions, we accordingly need consider only expressions which involve the indicated square roots of negative numbers.

Imaginary expressions are also called **imaginary numbers** and **complex numbers.** In distinction from imaginary numbers, all other numbers are called **real numbers.**

265. Imaginary Square Roots. If a and b are both positive, we have

$$\text{I. } \sqrt{ab} = \sqrt{a} \times \sqrt{b}. \qquad \text{II. } (\sqrt{a})^2 = a.$$

If one of the two numbers a and b is positive and the other negative, Law I. is *assumed* still to apply; we have, accordingly:

$$\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4} \times \sqrt{-1} = 2\sqrt{-1};$$
$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{5} \times \sqrt{-1} = 5^{\frac{1}{2}}\sqrt{-1};$$
$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a} \times \sqrt{-1} = a^{\frac{1}{2}}\sqrt{-1};$$

and so on.

It appears, then, that every imaginary square root can be made to assume the form $a\sqrt{-1}$, where a is a real number.

266. The symbol $\sqrt{-1}$ is called the **imaginary unit**, and may be defined as an expression the square of which is -1.

Hence, $\sqrt{-1}\times\sqrt{-1}=(\sqrt{-1})^2=-1;$

$$\begin{aligned}\sqrt{-a}\times\sqrt{-b}&=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}\\&=\sqrt{a}\times\sqrt{b}\times(\sqrt{-1})^2\\&=\sqrt{ab}\times(-1)\\&=-\sqrt{ab}.\end{aligned}$$

267. It will be useful to form the successive powers of the imaginary unit.

$$\begin{aligned}&(\sqrt{-1})\ \ldots\ldots\ldots\ldots &&=+\sqrt{-1};\\&(\sqrt{-1})^2\ \ldots\ldots\ldots\ldots &&=-1;\\&(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1} &&=(-1)\sqrt{-1}=-\sqrt{-1};\\&(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2 &&=(-1)(-1)=+1;\\&(\sqrt{-1})^5=(\sqrt{-1})^4\sqrt{-1} &&=(+1)\sqrt{-1}=+\sqrt{-1};\end{aligned}$$

and so on. We have, therefore,

$$\begin{aligned}(\sqrt{-1})^{4n+1}&=+\sqrt{-1};\\(\sqrt{-1})^{4n+2}&=-1;\\(\sqrt{-1})^{4n+3}&=-\sqrt{-1};\\(\sqrt{-1})^{4n+4}&=+1.\end{aligned}$$

268. Every imaginary expression may be made to assume the form $a+b\sqrt{-1}$, where a and b are real numbers, and may be integers, fractions, or surds.

If $b=0$, the expression consists of only the real part a, and is therefore real.

If $a=0$, the expression consists of only the imaginary part $b\sqrt{-1}$, and is called a **pure imaginary.**

269. The form $a+b\sqrt{-1}$ is the **typical form** of imaginary expressions.

Reduce to the typical form $6+\sqrt{-8}$.

This may be written $6+\sqrt{8}\times\sqrt{-1}$, or $6+2\sqrt{2}\times\sqrt{-1}$; here $a=6$, and $b=2\sqrt{2}$.

270. Two expressions of the form $a+b\sqrt{-1}$, $a-b\sqrt{-1}$, are called **conjugate imaginaries.**

To find the sum and product of two conjugate imaginaries,

$$\begin{array}{l} a+b\sqrt{-1} \\ a-b\sqrt{-1} \\ \hline 2a \end{array}$$

The sum is $2a$

$$\begin{array}{lll} a & +b\sqrt{-1} & \\ a & -b\sqrt{-1} & \\ \hline a^2 & +ab\sqrt{-1} & \\ & -ab\sqrt{-1} & +b^2 \\ \hline a^2 & & +b^2 \end{array}$$

The product is a^2+b^2

From the above it appears that the *sum* and *product* of two conjugate imaginaries are both *real.*

271. *An imaginary expression cannot be equal to a real number.*

For, if possible, let

$$a+b\sqrt{-1}=c.$$

Then transposing a, $b\sqrt{-1}=c-a$,

and squaring, $-b^2=(c-a)^2$.

Since b^2 and $(c-a)^2$ are both positive, we have a negative number equal to a positive number, which is impossible.

272. *If two imaginary expressions are equal, the real parts are equal and the imaginary parts are equal.*

For, let $$a+b\sqrt{-1}=c+d\sqrt{-1}.$$
Then $$(b-d)\sqrt{-1}=c-a;$$
squaring, $$-(b-d)^2=(c-a)^2,$$
which is impossible unless $b=d$ and $a=c$.

273. *If x and y are real and $x+y\sqrt{-1}=0$, then $x=0$ and $y=0$.*

For, $$y\sqrt{-1}=-x,$$
$$-y^2=x^2,$$
$$x^2+y^2=0,$$
which is true only when $x=0$ and $y=0$.

274. Operations with Imaginaries.

(1) Add $5+7\sqrt{-1}$ and $8-9\sqrt{-1}$.

The sum is $$5+8+7\sqrt{-1}-9\sqrt{-1},$$
or $$13-2\sqrt{-1}.$$

(2) Multiply $3+2\sqrt{-1}$ by $5-4\sqrt{-1}$.

$$(3+2\sqrt{-1})(5-4\sqrt{-1})$$
$$=15-12\sqrt{-1}+10\sqrt{-1}-8(-1)$$
$$=23-2\sqrt{-1}.$$

(3) Divide $14+5\sqrt{-1}$ by $2-3\sqrt{-1}$.

$$\frac{14+5\sqrt{-1}}{2-3\sqrt{-1}}=\frac{(14+5\sqrt{-1})(2+3\sqrt{-1})}{(2-3\sqrt{-1})(2+3\sqrt{-1})}$$
$$=\frac{13+52\sqrt{-1}}{4-(-9)}$$
$$=\frac{13+52\sqrt{-1}}{13}$$
$$=1+4\sqrt{-1}.$$

Exercise 91.

Reduce to the form $b\sqrt{-1}$:

1. $\sqrt{-9}$.
2. $\sqrt{-16}$.
3. $\sqrt{-25}$.
4. $\sqrt{-144}$.
5. $\sqrt{-169}$.
6. $\sqrt{-x^2}$.
7. $\sqrt{-81}$.
8. $\sqrt{-256}$.
9. $\sqrt{-625}$.
10. $\sqrt{-36}$.
11. $\sqrt[6]{-64}$.
12. $\sqrt[6]{-729}$.
13. $\sqrt{-289}$.
14. $\sqrt[10]{-1024}$.
15. $\sqrt{-x^8}$.
16. $\sqrt{-x^9}$.
17. $\sqrt[6]{-x^{18}}$.
18. $\sqrt{-\frac{1}{4}}$.
19. $\sqrt{-a^4b^{-2}}$.
20. $\sqrt{-9x^4}$.
21. $\sqrt[6]{-(2x-3y)^{12}}$.
22. $\sqrt[10]{-(x-2y)^{20}}$.
23. $\sqrt{-(x^2+y^2)}$.
24. $\sqrt{-(x^2-y^2)}$.

Add:

25. $\sqrt{-25}+\sqrt{-49}-\sqrt{-121}$.
26. $\sqrt{-64}+\sqrt{-1}-\sqrt{-36}$.
27. $\sqrt{-a^4}+\sqrt{-4a^4}+\sqrt{-16a^4}$.
28. $\sqrt{-a^2}+\sqrt{-81a^2}-\sqrt{-a^2}$.
29. $a-b\sqrt{-1}+a+b\sqrt{-1}$.
30. $2+3\sqrt{-1}-2+3\sqrt{-1}$.
31. $a+b\sqrt{-1}+c-d\sqrt{-1}$.
32. $3a\sqrt{-1}-(2a-b)\sqrt{-1}$.

Multiply:

33. $\sqrt{-3}$ by $\sqrt{-5}$.
34. $-\sqrt{-5}$ by $\sqrt{-5}$.
35. $\sqrt{-16}$ by $\sqrt{-9}$.
36. $\sqrt{-x^2}$ by $\sqrt{-x}$.
37. $\sqrt{-x^2}$ by $\sqrt{-y^2}$.
38. $\sqrt{-8}$ by $\sqrt{-16}$.

39. $\sqrt{-25}$ by $\sqrt{-64}$.

40. $\sqrt{-(a+b)}$ by $\sqrt{-(a-b)}$.

41. $3\sqrt{-3}$ by $2\sqrt{-2}$.

42. $-5\sqrt{-2}$ by $2\sqrt{-5}$.

43. $\sqrt{-2}+\sqrt{-3}$ by $\sqrt{-4}-\sqrt{-5}$.

44. $x-\dfrac{1+\sqrt{-3}}{2}$ by $x-\dfrac{1-\sqrt{-3}}{2}$.

45. $a\sqrt{-a}+b\sqrt{-b}$ by $a\sqrt{-a}-b\sqrt{-b}$.

46. $2\sqrt{-2}+3\sqrt{-3}$ by $3\sqrt{-4}-2\sqrt{-5}$.

47. $\sqrt{3}+2\sqrt{-3}$ by $\sqrt{3}-2\sqrt{-3}$.

48. $m-3\sqrt{-b}$ by $n+4\sqrt{-c}$.

Perform the divisions indicated:

49. $\dfrac{a}{\sqrt{-1}}$.

50. $\dfrac{b}{\sqrt{-b^2}}$.

51. $\dfrac{c}{\sqrt{-4}}$.

52. $\dfrac{\sqrt{-9}}{\sqrt{-81}}$.

53. $\dfrac{\sqrt{-a}}{\sqrt{-b}}$.

54. $\dfrac{\sqrt{-ax}}{\sqrt{-x}}$.

55. $\dfrac{x}{\sqrt{-x}}$.

56. $\dfrac{x}{\sqrt{-x^2}}$.

57. $\dfrac{\sqrt{-x^2}}{-\sqrt{-x}}$.

58. $\dfrac{\sqrt{-8}}{\sqrt{-2}}$.

59. $\dfrac{\sqrt{-10x^3}}{\sqrt{-5x}}$.

60. $\dfrac{8\sqrt{-x^2}}{2\sqrt{x}}$.

61. $\dfrac{1}{3-\sqrt{-2}}$.

62. $\dfrac{2+\sqrt{-2}}{1-\sqrt{-1}}$.

63. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}$.

64. $\dfrac{\sqrt{5}+\sqrt{-6}}{\sqrt{6}-\sqrt{-8}}$.

65. $\dfrac{2a+3b\sqrt{-1}}{2a-3b\sqrt{-1}}$.

66. $\dfrac{\frac{1}{2}a-4b\sqrt{-1}}{4a-\frac{1}{2}b\sqrt{-1}}$.

CHAPTER XVIII.

QUADRATIC EQUATIONS.

275. We have already considered equations of the first degree in one or more unknowns. We pass now to the treatment of equations containing one or more unknowns to a degree not exceeding the second. An equation which contains the *square* of the unknown, but no higher power, is called a **quadratic equation.**

276. A quadratic equation which involves but one unknown number can contain only:

(1) Terms involving the square of the unknown number.

(2) Terms involving the first power of the unknown number.

(3) Terms which do not involve the unknown number.

Collecting similar terms, every quadratic equation can be made to assume the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are known numbers, and x the unknown number.

If a, b, c are numbers expressed by figures, the equation is a **numerical quadratic.** If a, b, c are numbers represented wholly or in part by letters, the equation is a **literal quadratic.**

277. In the equation $ax^2 + bx + c = 0$, a, b, and c are called the **coefficients** of the equation. The third term c is called the **constant term.**

If the first power of x is wanting, the equation is a **pure quadratic**; in this case $b = 0$.

If the first power of x is present, the equation is an **affected** or **complete quadratic.**

Pure Quadratic Equations.

278. Examples.

(1) Solve the equation $5x^2 - 48 = 2x^2$.

We have $5x^2 - 48 = 2x^2$.

Collect the terms, $3x^2 = 48$.

Divide by 3, $x^2 = 16$.

Extract the square root, $x = \pm 4$.

It will be observed that there are *two* roots, and that these are numerically equal, but of opposite signs. There can be only two roots, since any number has only two square roots.

It may seem as though we ought to write the sign $\pm$ before the x as well as before the 4. If we do this, we have $+x = +4$, $-x = -4$, $+x = -4$, $-x = +4$.

From the first and second equations, $x = 4$; from the third and fourth, $x = -4$; these values of x are both given by the equation $x = \pm 4$. Hence it is *unnecessary* to write the $\pm$ sign on *both* sides of the reduced equation.

(2) Solve the equation $3x^2 - 15 = 0$.

We have $3x^2 = 15$,

or $x^2 = 5$.

Extract the square root, $x = \pm \sqrt{5}$.

The roots cannot be found exactly, since the square root of 5 cannot be found exactly; it can, however, be determined approximately to any required degree of accuracy; for example, the roots lie between 2.23606 and 2.23607; and between -2.23606 and -2.23607.

(3) Solve the equation $3x^2 + 15 = 0$.

We have $3x^2 = -15$,

or $x^2 = -5$.

Extract the square root, $x = \pm\sqrt{-5}$.

There is no square root of a negative number, since the square of any number, positive or negative, is necessarily positive.

The square root of -5 differs from the square root of $+5$ in that the latter can be found as accurately as we please, while the former cannot be found at all.

279. A root which can be found exactly is called an **exact** or **rational** root. Such roots are either whole numbers or fractions.

A root which is indicated but can be found only approximately is called a **surd**. Such roots involve the roots of imperfect powers.

Rational and surd roots are together called **real** roots.

A root which is indicated but cannot be found, either exactly or approximately, is called an **imaginary** root. Such roots involve the even roots of negative numbers.

Exercise 92.

Solve:

1. $3x^2 - 2 = x^2 + 6.$
2. $5x^2 + 10 = 6x^2 + 1.$
3. $7x^2 - 50 = 4x^2 + 25.$
4. $6x^2 - \frac{1}{6} = 4x^2 + \frac{11}{9}.$
5. $\dfrac{x^2+1}{5} = 10.$
6. $\dfrac{3x^2-8}{10} = 4.$
7. $\dfrac{x^2-9}{4} = \dfrac{x^2+1}{5}.$
8. $\dfrac{2x^2-4}{7} + \dfrac{x^2+4}{5} = 8.$
9. $\dfrac{3-x^2}{11} + \dfrac{x^2+5}{6} = 3.$
10. $\dfrac{5x^2+3}{8} - \dfrac{17-x^2}{4} = 4.$
11. $\dfrac{3}{4x^2} - \dfrac{1}{6x^2} = \dfrac{7}{3}.$
12. $\dfrac{5}{3x^2} - \dfrac{3}{5x^2} = \dfrac{4}{15}.$
13. $\dfrac{1}{x-1} - \dfrac{1}{x+1} = \dfrac{1}{4}.$
14. $\dfrac{15}{8-x} + \dfrac{7}{2-3x} = 2.$

15. $3x^2 + 11x = 10x + 8 + x^2 + x.$

16. $(x+4)(x+5) = 3(x+1)(x+2) - 4.$

17. $3(x-2)(x+3) = (x+1)(x+2) + x^2 + 5.$

18. $(2x+1)(3x-2) + (1-x)(3+4x) = 3x^2 - 15.$

19. $\frac{x^2+9}{17} - \frac{2x^2-5}{9} + \frac{3x^2+10}{5} = 14.$

20. $\frac{3x^2-5}{7} + \frac{2x^2+4}{9} - \frac{x^2-3}{2} = 5.$

21. $\frac{10x^2+7}{18} - \frac{12x^2+2}{11x^2-8} = \frac{5x^2-9}{9}.$

22. $\frac{x-1}{x+1} + \frac{x+1}{x-1} = \frac{5}{2}.$

23. $ax^2 + b = c.$

24. $ax^2 + b = bx^2 + a.$

25. $x^2+2bx+c = b(2x+1).$

26. $\frac{a}{x} + \frac{x}{a} = \frac{9a^2 - x^2}{ax}.$

27. $\frac{x+a}{x-a} + \frac{x-a}{x+a} = \frac{5}{2}.$

28. $\frac{2b}{x-b} + \frac{5x+2b}{3x} = -1$

29. $2\{(x+a)(x+b) + (x-a)(x-b)\} = a^2 + 4b^2.$

30. $2\{(x-a)(x+b) + (x+a)(x-b)\} = 9a^2 + 2ab + b^2.$

Affected Quadratic Equations.

280. Since $(x \pm b)^2 = x^2 \pm 2bx + b^2$, it is evident that the expression $x^2 \pm 2bx$ lacks only the *third term*, b^2, of being a perfect square.

This third term is the square of half the coefficient of x.

Every affected quadratic may be made to assume the form $x^2 \pm 2bx = c$, by dividing the equation through by the coefficient of x^2.

To solve such an equation:

The first step is to add to both members *the square of half the coefficient of x*. This is called completing the square.

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

(1) Solve the equation $x^2 - 8x = 20$.

We have $x^2 - 8x = 20.$

Complete the square, $x^2 - 8x + 16 = 36.$

Extract the square root, $x - 4 = \pm 6.$

Reduce, $x = 4 + 6 = 10,$

or $x = 4 - 6 = -2.$

The roots are 10 and -2.

Verify by putting these numbers for x in the given equation:

$x = 10,$	$x = -2,$
$10^2 - 8(10) = 20,$	$(-2)^2 - 8(-2) = 20,$
$100 - 80 = 20.$	$4 + 16 = 20.$

(2) Solve the equation $\frac{x+1}{x-1} = \frac{4x-3}{x+9}$.

Free from fractions, $(x+1)(x+9) = (x-1)(4x-3).$

Simplify, $3x^2 - 17x = 6.$

We can reduce the equation to the form $x^2 - 2bx$ by dividing by 3.

Divide by 3, $x^2 - \frac{17}{3}x = 2.$

Half the coefficient of x is $\frac{1}{2}$ of $-\frac{17}{3} = -\frac{17}{6}$, and the square of $-\frac{17}{6}$ is $\frac{289}{36}$. Add the square of $-\frac{17}{6}$ to both sides, and we have

$$x^2 - \frac{17x}{3} + \left(\frac{17}{6}\right)^2 = 2 + \frac{289}{36},$$

or

$$x^2 - \frac{17}{3}x + \left(\frac{17}{6}\right)^2 = \frac{361}{36}.$$

Extract the root, $x - \frac{17}{6} = \pm\frac{19}{6}.$

Reduce, $$x-\frac{17}{6}=\pm\frac{19}{6}.$$

$$\therefore\ x=\frac{17}{6}+\frac{19}{6}=\frac{36}{6}=6,$$

or $$x=\frac{17}{6}-\frac{19}{6}=-\frac{2}{6}=-\frac{1}{3}.$$

The roots are 6 and $-\frac{1}{3}$.

Verify by putting these numbers for x in the original equation:

$$x=6.$$

$$\frac{6+1}{6-1}=\frac{24-3}{6+9},$$

$$\frac{7}{5}=\frac{21}{15}.$$

$$x=-\frac{1}{3}.$$

$$\frac{-\frac{1}{3}+1}{-\frac{1}{3}-1}=\frac{-\frac{4}{3}-3}{-\frac{1}{3}+9},$$

$$-\frac{2}{4}=-\frac{13}{26}.$$

Exercise 93.

Solve:

1. $x^2+2x=8.$
2. $x^2-6x=7.$
3. $x^2-4x=12.$
4. $x^2+4x=5.$
5. $x^2+5x=14.$
6. $x^2-3x=28.$
7. $2x^2+x=15.$
8. $5x^2+3x=2.$
9. $x^2+\frac{2}{3}x=40.$
10. $3x^2-4x=4.$
11. $6x^2+x=1.$
12. $6x^2-x=2.$
13. $12x^2-11x+2=0.$
14. $15x^2-2x-1=0.$
15. $\dfrac{(x+1)(x+2)}{5}-\dfrac{(x-1)(x-2)}{2}=3.$
16. $\dfrac{(2x-3)x}{4}-\dfrac{(x+4)(x-1)}{6}=1.$

17. $\frac{3x+5}{x+4}+\frac{2x-5}{x-2}=3.$

18. $\frac{x-6}{x-2}+\frac{x+5}{2x+1}=1.$

19. $\frac{4-3x}{2+x}-\frac{1+2x}{1-x}=\frac{9}{2}.$

20. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}.$

21. $5x^2-4x=1.$

22. $\frac{x+2}{x-1}-\frac{4-x}{2x}=\frac{7}{3}.$

23. $\frac{x+3}{x-2}=\frac{5x+8}{x+4}.$

24. $\frac{2x-1}{3}+\frac{3}{2x-1}=2.$

25. $\frac{x+1}{x+2}+\frac{x+2}{x+1}=\frac{13}{6}.$

26. $7x^2-8x=-1.$

ANOTHER METHOD OF COMPLETING THE SQUARE.

281. When the coefficient of x^2 is not unity, we may proceed as in the preceding section, or we may complete the square by another method.

Since $(ax \pm b)^2 = a^2x^2 \pm 2abx + b^2$, it is evident that the expression $a^2x^2 \pm 2abx$ lacks only the *third term*, b^2, of being a complete square.

It will be seen that this third term is *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

282. Every affected quadratic may be made to assume the form of $a^2x^2 \pm 2abx = c$.

To solve such an equation:

The first step is to *complete the square;* that is, to add to each side *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

The second step is to *extract the square root* of each side of the resulting equation.

The third and last step is to *reduce* the two resulting simple equations.

283. Examples.

(1) Solve the equation $16x^2+5x-3=7x^2-x+45$.

We have $\quad 16x^2+5x-3=7x^2-x+45.$
Simplify, $\quad 9x^2+6x=48.$

The square root of $9x^2$ is $3x$, and twice $3x$ is $6x$. The second term divided by $6x$ is 1. Square 1 and add it to both sides.

$$9x^2+6x+1=49.$$

Extract the square root, $3x+1=\pm 7.$
Reduce, $\quad 3x=-1+7 \text{ or } -1-7,$
$$\therefore x=2 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 2 for x in the equation:

$$16x^2+5x-3=7x^2-x+45,$$
$$16(2)^2+5(2)-3=7(2)^2-(2)+45,$$
$$64+10-3=28-2+45,$$
$$71=71.$$

Verify by substituting $-2\frac{2}{3}$ for x in the equation:

$$16x^2+5x-3=7x^2-x+45,$$
$$16\left(-\frac{8}{3}\right)^2+5\left(-\frac{8}{3}\right)-3=7\left(-\frac{8}{3}\right)^2-\left(-\frac{8}{3}\right)+45,$$
$$\frac{1024}{9}-\frac{40}{3}-3=\frac{448}{9}+\frac{8}{3}+45,$$
$$1024-120-27=448+24+405,$$
$$877=877.$$

(2) Solve the equation $3x^2-4x=32$.

Since the exact root of 3, the coefficient of x^2, cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number*.

Multiply by 3, $\quad 9x^2-12x=96.$
Complete the square,
$$9x^2-12x+4=100.$$
Extract the square root, $3x-2=\pm 10.$
Reduce, $\quad 3x=2+10 \text{ or } 2-10;$
$$\therefore x=4 \text{ or } -2\tfrac{2}{3}.$$

Or, divide by 3, $x^2 - \frac{4x}{3} = \frac{32}{3}$.

Complete the square, $x^2 - \frac{4x}{3} + \frac{4}{9} = \frac{32}{3} + \frac{4}{9} = \frac{100}{9}$.

Extract the square root, $x - \frac{2}{3} = \pm \frac{10}{3}$.

$$\therefore x = \frac{2 \pm 10}{3},$$

$$= 4 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 4 for x in the original equation:

$$48 - 16 = 32,$$

$$32 = 32.$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation:

$$21\tfrac{1}{3} - (-10\tfrac{2}{3}) = 32,$$

$$32 = 32.$$

(3) Solve the equation $-3x^2 + 5x = -2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is

$$3x^2 - 5x = 2.$$

Multiply by 3, $9x^2 - 15x = 6$.

Complete the square,

$$9x^2 - 15x + \frac{25}{4} = \frac{49}{4}.$$

Extract the square root, $3x - \frac{5}{2} = \pm \frac{7}{2}$.

Reduce, $3x = \frac{5 \pm 7}{2}$.

$$3x = 6 \text{ or } -1.$$

$$\therefore x = 2 \text{ or } -\frac{1}{3}.$$

Or, divide by 3, $x^2 - \frac{5x}{3} = \frac{2}{3}$.

Complete the square,

$$x^2 - \frac{5x}{3} + \frac{25}{36} = \frac{49}{36}.$$

Extract the square root, $x - \frac{5}{6} = \pm \frac{7}{6}$.

$$\therefore x = \frac{5 \pm 7}{6},$$

$$= 2 \text{ or } -\frac{1}{3}.$$

284. If the equation $3x^2 - 5x = 2$ be multiplied by *four times the coefficient of* x^2, fractions will be avoided.

We have $36x^2 - 60x = 24$.

Complete the square,

$$36x^2 - 60x + 25 = 49.$$

Extract the square root, $6x - 5 = \pm 7$.

$$6x = 5 \pm 7.$$

$$6x = 12 \text{ or } -2.$$

$$\therefore x = 2 \text{ or } -\frac{1}{3}.$$

It will be observed that the number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2 - 5x = 2$.

NOTE. If the coefficient of x is an *even* number, we may multiply by the *coefficient* of x^2, and add to each member the square of *half* the coefficient of x in the given equation.

(1) Solve the equation $4x^2 - 23x = -30$.

Multiply by four times the coefficient of x^2, and add to each side the square of the coefficient of x,

$$64x^2 - (\ ) + (23)^2 = 529 - 480 = 49.$$

Extract the square root, $8x - 23 = \pm 7$.

Reduce, $8x = 23 \pm 7$;

$$8x = 30 \text{ or } 16.$$

$$\therefore x = 3\tfrac{3}{4} \text{ or } 2.$$

NOTE. If a trinomial is a perfect square, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(2) Solve the equation $72x^2 - 30x = -7$,

Since $72 = 2^3 \times 3^2$, if the equation is multiplied by 2, the coefficient of x^2 in the resulting equation, $144x^2 - 60x = -14$, will be a square number, and the term required to complete the square will be $\left(\frac{60}{24}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$. Hence, if the original equation is multiplied by 4×2, the coefficient of x^2 in the result will be a square number, and fractions will be avoided in the work. We shall then have

$$576x^2 - 240x = -56.$$

$$\therefore 576x^2 - (\ ) + 25 = -31.$$

Extract the root, $$24x - 5 = \pm\sqrt{-31}.$$

$$x = \tfrac{1}{24}(5 \pm \sqrt{-31}).$$

Exercise 94.

Solve:

1. $3x^2 - 2x = 8.$
2. $5x^2 - 6x = 27.$
3. $2x^2 + 3x = 5.$
4. $2x^2 - 5x = 7.$
5. $3x^2 + 7x = 6.$
6. $5x^2 - 7x = 24.$
7. $8x^2 + 3x = 26.$
8. $7x^2 + 5x = 150.$
9. $6x^2 + 5x = 14.$
10. $7x^2 - 2x = \frac{3}{4}.$
11. $8x^2 + 7x = 51.$
12. $7x^2 - 20x = 75.$
13. $11x^2 - 10x = 24.$
14. $3x^2 + \dfrac{2x}{3} = 25.$
15. $x^2 - \dfrac{3x}{4} = 3x + 1.$
16. $\dfrac{x^2}{2} - \dfrac{x}{3} = 2(x - 2).$
17. $\dfrac{2x^2}{3} + \dfrac{3x}{2} = 15.$
18. $\dfrac{3x}{4} - 2x^2 = \dfrac{1}{16}.$
19. $\frac{3}{5}x^2 + \frac{5}{8}x = \dfrac{20}{3}.$
20. $2x - 3 = \dfrac{2}{x}.$
21. $\dfrac{7x}{5} - \dfrac{5}{3x} = \dfrac{20}{3}.$
22. $\dfrac{2x}{3} + \dfrac{3}{2x} = \dfrac{10}{3}.$

23. $(x+2)(2x+1)+(x-1)(3x+2)=57.$

24. $3x(2x+5)-(x+3)(3x-1)=1.$

25. $\frac{(2x+5)(x-3)}{3}+\frac{x(3x+4)}{5}=5.$

26. $\frac{2}{3}(5x^2-8x-6)-\frac{1}{2}(x^2-3)=2x+1.$

27. $\frac{2}{x+3}+\frac{5}{x}=2.$

28. $\frac{5}{x-1}-\frac{3}{2x+1}=\frac{4}{3}.$

29. $\frac{7}{3x-2}+\frac{4}{2x-5}=5.$

30. $\frac{2}{x-1}=\frac{3}{x-2}+\frac{2}{x-4}.$

31. $\frac{x+2}{x-4}+\frac{x+3}{x-2}=-5.$

32. $\frac{2x-3}{x-4}+\frac{5-3x}{x+2}=1.$

33. $\frac{11-3x}{1-x}+\frac{2(7-4x)}{1-2x}=1.$

34. $\frac{x+1}{x^2-4}+\frac{1-x}{x+2}=\frac{2}{5(x-2)}.$

35. $\frac{2x+7}{2x-3}+\frac{3x-2}{x+1}=5.$

36. $\frac{2x+3}{2(2x-1)}-\frac{7-x}{2(x+1)}=\frac{7-3x}{4-3x}.$

37. $\frac{2x-1}{3}-\frac{3}{x-8}=\frac{x-2}{x-8}+5.$

38. $\frac{3x+2}{2x-1}+\frac{7-x}{2x+1}=\frac{7x-1}{4x^2-1}+5.$

39. $\frac{x-5}{x+3}+\frac{x-8}{x-3}=\frac{80}{x^2-9}+\frac{1}{2}.$

40. $\frac{2x+1}{7-x}+\frac{4x+1}{7+x}=\frac{45}{49-x^2}+1.$

LITERAL QUADRATICS.

285. Examples.

(1) Solve the equation $ax^2 + bx + c = 0$.

Transpose c, $ax^2 + bx = -c$.

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2 + (\ ) + b^2 = b^2 - 4ac.$$

Extract the root, $2ax + b = \pm\sqrt{b^2 - 4ac}$.

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(2) Solve the equation $(a^2 + 1)x = ax^2 + a$.

Transpose ax^2 and change the sign,

$$ax^2 - (a^2 + 1)x = -a.$$

Multiply by $4a$, and complete the square,

$$4a^2x^2 - (\ ) + (a^2 + 1)^2 = -4a^2 + a^4 + 2a^2 + 1$$
$$= a^4 - 2a^2 + 1.$$

Extract the root, $2ax - (a^2 + 1) = \pm(a^2 - 1)$.

Reduce, $2ax = (a^2 + 1) \pm (a^2 - 1)$,

$$= 2a^2 \text{ or } 2.$$

$$\therefore\ x = a \text{ or } \frac{1}{a}.$$

(3) Solve the equation $adx - acx^2 = bcx - bd$.

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)x = bd.$$

Multiply by $4ac$, and complete the square,

$$4a^2c^2x^2 + (\ ) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root,

$$2acx + (bc - ad) = \pm(bc + ad).$$

Reduce, $2acx = -(bc - ad) \pm (bc + ad)$

$$= 2ad \text{ or } -2bc.$$

$$\therefore\ x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

(4) Solve the equation $px^2 - px + qx^2 + qx = \frac{pq}{p+q}$.

Express the left member in *two terms*,

$$(p+q)x^2 - (p-q)x = \frac{pq}{p+q}.$$

Multiply by four times the coefficient of x^2,

$$4(p+q)^2x^2 - 4(p^2-q^2)x = 4pq.$$

Complete the square,

$$4(p+q)^2x^2 - (\ ) + (p-q)^2 = p^2 + 2pq + q^2.$$

Extract the root,

$$2(p+q)x - (p-q) = \pm(p+q).$$

Reduce,
$$2(p+q)x = (p-q) \pm (p+q),$$
$$= 2p \text{ or } -2q.$$
$$\therefore x = \frac{p}{p+q} \text{ or } -\frac{q}{p+q}.$$

NOTE. The left-hand member of the equation when simplified must be expressed in *two terms, simple or compound*, one term containing x^2, and the other term containing x.

Exercise 95.

Solve:

1. $x^2 + 2ax = 3a^2$.
2. $x^2 - 4ax = 12a^2$.
3. $x^2 + 8bx = 9b^2$.
4. $x^2 + 3bx = 10b^2$.
5. $x^2 + 5ax = 14a^2$.
6. $3x^2 + 4cx = 4c^2$.
7. $5ax - 2x^2 = 2a^2$.
8. $6x^2 - ax - a^2 = 0$.
9. $2a^2x^2 + ax - 1 = 0$.
10. $12b^2x^2 - 5bx = 3$.
11. $\frac{2x^2}{3} + \frac{ax}{4} = 11a(x - 3a)$.
12. $\frac{3x^2}{4} + \frac{x}{2a} = \frac{3}{2a^2}$.
13. $x^2 - \frac{x}{a} = \frac{3}{4a^2}$.
14. $\frac{3ax^2}{4} + \frac{2x}{3} = \frac{13}{3a}$.
15. $\frac{x^2 + ax + a^2}{3} + \frac{a^2 - x^2}{4} = ax$.

16. $2x^2 - x + a = 2a^2.$

17. $\dfrac{x(a-x)}{a+x} + \dfrac{x}{3} = 5a.$

18. $\dfrac{x(3x-a)}{4x+a} = \dfrac{a}{12}.$

19. $x^2 + \dfrac{m^2-n^2}{mn}x = 1.$

20. $\dfrac{x+a}{b-a} + \dfrac{b-a}{x+a} = 2.$

21. $\dfrac{a+2b}{3b-x} + \dfrac{a+b}{x+2a} = 2.$

22. $x^2 + (a-b)x = ab.$

23. $\dfrac{2a+x}{2a-x} + \dfrac{a-2x}{a+2x} = \dfrac{8}{3}.$

24. $\dfrac{2x-3a}{x+4a} + \dfrac{3x+2a}{4x-a} = \dfrac{10}{7}.$

25. $\dfrac{x^2}{6b^2} - \dfrac{5x}{6ab} + \dfrac{1}{a^2} = 0.$

26. $\dfrac{a-b+x}{a+b+x} + \dfrac{a+b}{x+b} = 2.$

27. $\dfrac{1}{a+b+x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}.$

28. $\dfrac{(3x-a)(2x+a)}{2} + \dfrac{a^2-4x^2}{3} = 2a^2.$

29. $9x^2 - 3(a+2b)x + 2ab = 0.$

30. $(2a+1)x^2 + 3a^2x + a^3 - a^2 = 0.$

31. $(1-a^2)x^2 - 2(1+a^2)x + 1 - a^2 = 0.$

32. $(a+b)^2x^2 - (a^2-b^2)x = ab.$

33. $a^2x^2 + 2bcx + a^2 = c^2x^2 + 2a^2x + b^2.$

34. $(a+b)x^2 - (2a+b)x + a = 0.$

35. $\dfrac{2x-3b}{x-2b} - \dfrac{3x}{x+2a} = \dfrac{a}{2(a-b)}.$

36. $\dfrac{3x-2b}{x-b} - \dfrac{4x+2b}{x+a} = \dfrac{a-b}{a+b}.$

37. $(3a^2+b^2)(x^2-x+1) = (a^2+3b^2)(x^2+x+1).$

38. $x^2 - (a+b)x + ac + bc - c^2 = 0.$

39. $x^2 - px = (p+q+r)(q+r).$

40. $(a^2+ab)(x^2-1) - (a^2+b^2)x + (a+b)(a-2b) = 0.$

SOLUTIONS BY FACTORING.

286. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved *by inspection* into factors.

In this case, the roots are seen at once without completing the square.

(1) Solve $x^2 + 7x - 60 = 0$.

Since $x^2 + 7x - 60 = (x + 12)(x - 5)$,
the equation $x^2 + 7x - 60 = 0$
may be written $(x + 12)(x - 5) = 0$.

If *either* of the factors $x + 12$ or $x - 5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x + 12 = 0$, or $x - 5 = 0$.

$\therefore x = -12$, or $x = 5$.

(2) Solve $2x^3 - x^2 - 6x = 0$.

The equation $2x^3 - x^2 - 6x = 0$
becomes $x(2x^2 - x - 6) = 0$,
and is satisfied if $x = 0$, or if $2x^2 - x - 6 = 0$.

By solving $2x^2 - x - 6 = 0$, the two roots 2 and $-\frac{3}{2}$ are found.

Hence the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(3) Solve $x^3 + x^2 - 4x - 4 = 0$.

The equation $x^3 + x^2 - 4x - 4 = 0$
becomes $x^2(x + 1) - 4(x + 1) = 0$,
$(x^2 - 4)(x + 1) = 0$.

Hence the roots of the equation are -1, 2, -2.

(4) Solve $x^3 - 2x^2 - 11x + 12 = 0$.

By trial we find that 1 satisfies the equation, and is therefore a root (§ 89).

Divide by $x - 1$; the given equation may be written

$$(x - 1)(x^2 - x - 12) = 0,$$

and is satisfied if $x - 1 = 0$, or if $x^2 - x - 12 = 0$.

The roots are found to be 1, 4, -3.

(5) Solve the equation $x(x^2-9)=a(a^2-9)$.

If we put a for x, the equation is satisfied; therefore a is a root (§ 89).

Transpose all the terms to the left-hand member and divide by $x-a$.

The given equation may be written

$$(x-a)(x^2+ax+a^2-9)=0,$$

and is satisfied if $x-a=0$,

or if $x^2+ax+a^2-9=0$.

The roots are found to be

$$a,\ \frac{-a+\sqrt{36-3a^2}}{2},\ \frac{-a-\sqrt{36-3a^2}}{2}.$$

Exercise 96.

Find all the roots of

1. $x^2-5x+4=0$.
2. $6x^2-5x-6=0$.
3. $2x^2-x-3=0$.
4. $10x^2+x-3=0$.
5. $x^3+x^2-6x=0$.
6. $x^3-8=0$.
7. $x^3+8=0$.
8. $x^4-16=0$.
9. $(x-1)(x-3)(x^2+5x+6)=0$.
10. $(2x-1)(x-2)(3x^2-5x-2)=0$.
11. $(x^2+x-2)(2x^2+3x-5)=0$.
12. $x^3+x^2-4(x+1)=0$.
13. $3x^3+2x^2-(3x+2)=0$.
14. $x^3-27-13x+39=0$.
15. $x^3+8+3(x^2-4)=0$.
16. $x(x^2-1)-6(x-1)=0$.
17. $2x^3-2x^2-(x^2-1)=0$.
18. $x^3-3x-2=0$.
19. $2x^3+2x^2+(x^2-5x-6)=0$.
20. $x^4-4x^3-x^2+16x-12=0$.

Solutions by a Formula.

287. Every quadratic equation can be made to assume the form $ax^2 + bx + c = 0$.

Solving this equation (§ 285, Ex. 1), we obtain for its two roots

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are two roots, and but two roots, since there are two, and but two, square roots of the expression $b^2 - 4ac$.

By this formula, the values of x in an equation of the form $ax^2 + bx + c = 0$ may be written at once.

Ex. Find the roots of the equation $3x^2 - 5x + 2 = 0$.

Here $a = 3,\ b = -5,\ c = 2.$

Putting these values for the letters in the above formulas, we have

$$x = \frac{5 + \sqrt{25 - 24}}{6}, \text{ or } \frac{5 - \sqrt{25 - 24}}{6},$$
$$= \tfrac{6}{6} \text{ or } \tfrac{4}{6}$$
$$= 1 \text{ or } \tfrac{2}{3}.$$

Exercise 97.

Solve by the above formulas:

1. $2x^2 + 3x = 14.$
2. $3x^2 - 5x = 12.$
3. $x^2 - 7x = 18.$
4. $5x^2 - x = 42.$
5. $6x^2 - 7x = 10.$
6. $3x^2 - 11x = -6.$
7. $5x^2 - 7x = -2.$
8. $4x^2 - 9x = 28.$
9. $5x^2 + 7x = 12.$
10. $11x^2 - 9x = -\frac{10}{9}.$
11. $7x^2 + 5x = 38.$
12. $5x^2 - 7x = 6.$

Equations in the Quadratic Form.

288. An equation is in the *quadratic form* if it contains but two powers of the unknown number, and the exponent of one power is exactly twice that of the other power.

289. Equations not of the second degree, but of the quadratic form, may be solved by completing the square.

(1) Solve: $8x^6 + 63x^3 = 8.$

This equation is in the *quadratic form* if we regard x^3 as the unknown number.

We have $8x^6 + 63x^3 = 8.$

Multiply by 32 and complete the square,

$$256x^6 + (\ ) + (63)^2 = 4225.$$

Extract the square root, $16x^3 + 63 = \pm 65.$

Hence, $x^3 = \frac{1}{8}$ or $-8.$

Extracting the cube root, two values of x are $\frac{1}{2}$ and -2. There are four other values of x which we do not find at present.

(2) Solve: $\sqrt{x^3} - 3\sqrt[4]{x^3} = 40.$

Using fractional exponents, we have

$$x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 40.$$

This equation is in the quadratic form if we regard $x^{\frac{3}{4}}$ as the unknown number.

Complete the square, $4x^{\frac{3}{2}} - 12x^{\frac{3}{4}} + 9 = 169.$

Extract the root, $2x^{\frac{3}{4}} - 3 = \pm 13.$

$$\therefore 2x^{\frac{3}{4}} = 16 \text{ or } -10,$$

$$x^{\frac{3}{4}} = 8 \text{ or } -5,$$

$$x = 16 \text{ or } 5\sqrt[3]{5}.$$

There are other values of x which we do not find at present.

(3) Solve completely the equation $x^3 = 1$.

We have $x^3 - 1 = 0$.
Factoring, $(x-1)(x^2+x+1) = 0$.
Therefore, either $x - 1 = 0$
or $x^2 + x + 1 = 0$.

$x - 1 = 0$.	$x^2 + x + 1 = 0$.
$\therefore x = 1$.	Solving, $x = \dfrac{-1 \pm \sqrt{-3}}{2}$.

The three values, 1, $\dfrac{-1+\sqrt{-3}}{2}$, $\dfrac{-1-\sqrt{-3}}{2}$, are the *three cube roots of* 1.

(4) Solve: $(2x-3)^2 - (2x-3) = 6$.

Put y for $2x-3$, and therefore y^2 for $(2x-3)^2$.

We have $y^2 - y = 6$.
Solving, $y = 3$ or -2.
Putting now $2x - 3$ for y,

$2x - 3 = 3$,	$2x - 3 = -2$,
$x = 3$.	$x = \frac{1}{2}$.

Exercise 98.

1. $x^4 - 5x^2 + 4 = 0$.
2. $x^4 - 13x^2 + 36 = 0$.
3. $x^4 - 21x^2 = 100$.
4. $4x^6 - 3x^3 = 27$.
5. $2x^4 + 5x^2 = 21\frac{3}{8}$.
6. $10x^4 - 21 = x^2$.
7. $\sqrt[3]{x^2} + 3\sqrt[3]{x} = 1\frac{3}{4}$.
8. $3\sqrt[4]{x} - 2\sqrt{x} = -20$.
9. $5x^{2n} + 3x^n = 6\frac{3}{4}$.
10. $(8x+3)^2 + (8x+3) = 30$.
11. $2(x^2 - x + 1) - \sqrt{x^2 - x + 1} = 1$.
12. $x^6 - 9x^3 + 8 = 0$.
13. $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = \frac{13}{6}$.
14. $(x+1) + \sqrt{x+1} = 6$.
15. $x^4 - 13x^2 = -36$.
16. $2x^2 + 4x + 9 + 3\sqrt{2x^2 + 4x + 9} = 40$.
17. $\dfrac{12x^3 - 11x^2 + 10x - 78}{8x^2 - 7x + 6} = 1\frac{1}{2}x - \dfrac{1}{2}$.

RADICAL EQUATIONS.

290. If an equation involves radical expressions, we first clear of radicals as follows:

Solve $\sqrt{x+4}+\sqrt{2x+6}=\sqrt{7x+14}$.

Square both sides,

$$x+4+2\sqrt{(x+4)(2x+6)}+2x+6=7x+14.$$

Transpose and combine, $2\sqrt{(x+4)(2x+6)}=4x+4.$

Divide by 2 and square, $(x+4)(2x+6)=(2x+2)^2.$

Reduce, $x^2-3x=10.$

Hence, $x=5$ or $-2.$

Of these two values, only 5 will satisfy the original equation.

The value -2 will satisfy the equation

$$\sqrt{x+4}-\sqrt{2x+6}=\sqrt{7x+14}.$$

In fact, squaring both members of the original equation is equivalent to transposing $\sqrt{7x+14}$ to the left member, and then multiplying by the rationalizing factor $\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14}$, so that the equation stands

$$(\sqrt{x+4}+\sqrt{2x+6}-\sqrt{7x+14})(\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14})=0,$$

which reduces to $\sqrt{(x+4)(2x+6)}-(2x+2)=0.$

Transposing and squaring again is equivalent to multiplying by

$$(\sqrt{x+4}-\sqrt{2x+6}+\sqrt{7x+14})(\sqrt{x+4}-\sqrt{2x+6}-\sqrt{7x+14}).$$

Multiplying out and reducing, we have

$$x^2-3x-10=0.$$

Therefore, the equation $x^2-3x-10=0$ is really obtained from

$$(\sqrt{x+4}+\sqrt{2x+6}-\sqrt{7x+14})$$
$$\times(\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14})$$
$$\times(\sqrt{x+4}-\sqrt{2x+6}-\sqrt{7x+14})$$
$$\times(\sqrt{x+4}-\sqrt{2x+6}+\sqrt{7x+14})=0.$$

This equation is satisfied by any value that will satisfy any one of the *four* factors of its left member. The first factor is satisfied by 5, and the last factor by -2, while no values can be found to satisfy the second or third factor.

Hence, if a radical equation of this form is proposed for solution, if there is a value of x that will satisfy the particular equation given, that value must be retained, and any value that does not satisfy the equation given must be rejected. (See Wentworth, McLellan and Glashan's Algebraic Analysis, pp. 278–281.)

291. Some radical equations may be solved as follows:

Solve $7x^2 - 5x + 8\sqrt{7x^2 - 5x + 1} = -8$.

Add 1 to both sides,

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Put $\sqrt{7x^2 - 5x + 1} = y$; the equation becomes

$$y^2 + 8y = -7.$$

Hence,

$$y = -1 \text{ or } -7,$$

$$y^2 = 1 \text{ or } 49.$$

We now have $7x^2 - 5x + 1 = 1$, or $7x^2 - 5x + 1 = 49$.

Solving these, we find for the values of x,

$$0, \frac{5}{7} \;\Big|\; 3, \frac{16}{7}.$$

These values all satisfy the given equation when we take the *negative* value of the square root of the expression $7x^2 - 5x + 1$; they are in fact the four roots of the biquadratic obtained by clearing the given equation of radicals.

Exercise 99.

Solve:

1. $\sqrt{9x + 40} - 2\sqrt{x + 7} = \sqrt{x}$.

2. $\sqrt{a + x} + \sqrt{a - x} = \sqrt{b}$.

3. $\dfrac{3x + \sqrt{4x - x^2}}{3x - \sqrt{4x - x^2}} = 2$.

4. $\sqrt{x - 3} - \sqrt{x - 14} = \sqrt{4x - 155}$.

5. $\sqrt{x + 4} - \sqrt{x} = \sqrt{x + \frac{3}{2}}$.

6. $\dfrac{3\sqrt{x}-4}{2+\sqrt{x}}-\dfrac{15+3\sqrt{x}}{40+\sqrt{x}}=0.$

7. $\sqrt{14x+9}+2\sqrt{x+1}+\sqrt{3x+1}=0.$

8. $\sqrt{5x+1}-2-\sqrt{x+1}=0.$

9. $\sqrt{x-2}+\sqrt{x+3}-\sqrt{4x+1}=0.$

10. $\sqrt{7-x}+\sqrt{3x+10}+\sqrt{x+3}=0.$

11. $3\sqrt{x^3+17}+\sqrt{x^3+1}+2\sqrt{5x^3+41}=0.$

12. $2x-\sqrt{2x-1}=x+2.$

13. $\sqrt{x+2}-\sqrt{x-2}-\sqrt{2x}=0.$

14. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}=12.$

15. $\sqrt{3x}+\sqrt{3x+13}=\dfrac{91}{\sqrt{3x+13}}.$

16. $x-\sqrt[3]{x^3-2x^2}-2=0.$

17. $\dfrac{3x-\sqrt{x^2-8}}{x-\sqrt{x^2-8}}=x+\sqrt{x^2-8}.$

18. $2x^2+3x-5\sqrt{2x^2+3x+9}=-3.$

19. $3x^2+15x-2\sqrt{x^2+5x+1}=2.$

20. $x^2-\frac{3}{2}x+3\sqrt{2x^2-3x+2}=7.$

21. $2x^2-\sqrt{x^2-2x-3}=4x+9.$

22. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

23. $3x^2-7+3\sqrt{3x^2-16x+21}=16x.$

292. Problems involving Quadratics. Problems which involve quadratic equations apparently have two solutions, since a quadratic equation has two roots. When both roots of the quadratic equation are positive integers, they will, generally, both be admissible solutions.

Fractional and negative roots will in some problems give admissible solutions; in other problems they will not give admissible solutions.

No difficulty will be found in selecting the result which belongs to the particular problem we are solving. Sometimes, by a change in the statement of the problem, we may form a new problem which corresponds to the result that was inapplicable to the original problem.

Imaginary roots indicate that the problem is impossible.

Here as in simple equations x stands for an unknown *number*.

(1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,
and $x + 1 =$ the other.
Then $x^2 + (x + 1)^2 = 481,$
or $2x^2 + 2x + 1 = 481.$

The solution of which gives $x = 15$ or -16.

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4 .....

(2) A pedler bought a number of knives for \$2.40. Had he bought 4 more for the same money, he would have paid 3 cents less for each. How many knives did he buy, and what did he pay for each?

Let $x =$ number of knives he bought.
Then $\frac{240}{x} =$ number of cents he paid for each.

But if $x + 4 =$ number of knives he bought,

$\frac{240}{x+4} =$ number of cents he paid for each,

$\frac{240}{x} - \frac{240}{x+4} =$ the difference in price.

But $3 =$ the difference in price.

$$\therefore \frac{240}{x} - \frac{240}{x+4} = 3.$$

Solving, $x = 16$ or -20.

He bought 16 knives, therefore, and paid $\frac{240}{16}$, or 15 cents for each.

If the problem is changed so as to read: A pedler bought a number of knives for \$2.40, and if he had bought 4 *less* for the same money, he would have paid 3 cents *more* for each, the equation will be

$$\frac{240}{x-4} - \frac{240}{x} = 3.$$

Solving, $x = 20$ or -16.

This second problem is therefore the one which the negative answer of the first problem suggests.

(3) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ number of eggs for a shilling.

Then $\frac{1}{x} =$ cost of 1 egg in shillings,

and $\frac{12}{x} =$ cost of 1 dozen in shillings.

But if $x + 2 =$ number of eggs for a shilling,

$\frac{12}{x+2} =$ cost of 1 dozen in shillings.

$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12}$ (1 penny being $\frac{1}{12}$ of a shilling).

The solution of which gives $x = 16$, or -18.

And, if 16 eggs cost a shilling, 1 dozen will cost 9 pence.

Therefore, the price of the eggs is 9 pence per dozen.

If the problem is changed so as to read: What is the price of eggs per dozen when two *less* in a shilling's worth *raises* the price 1 penny per dozen? the equation will be

$$\frac{12}{x-2}-\frac{12}{x}=\frac{1}{12}.$$

The solution of which gives $x=18$, or -16.

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result.

Exercise 100.

1. The sum of the squares of two consecutive integers is 761. Find the numbers.

2. The sum of the squares of two consecutive numbers exceeds the product of the numbers by 13. Find the numbers.

3. The square of the sum of two consecutive even numbers exceeds the sum of their squares by 336. Find the numbers.

4. Twice the product of two consecutive numbers exceeds the sum of the numbers by 49. Find the numbers.

5. The sum of the squares of three consecutive numbers is 110. Find the numbers.

6. The difference of the cubes of two successive odd numbers is 602. Find the numbers.

7. The length of a rectangular field exceeds its breadth by 2 rods. If the length and breadth of the field were each increased by 4 rods, the area would be 80 square rods. Find the dimensions of the field.

8. The area of a square may be doubled by increasing its length by 10 feet and its breadth by 3 feet. Determine its side.

9. A grass plot 12 yards long and 9 yards wide has a path around it. The area of the path is $\frac{2}{3}$ of the area of the plot. Find the width of the path.

10. The perimeter of a rectangular field is 60 rods. Its area is 200 square rods. Find its dimensions.

11. The length of a rectangular plot is 10 rods more than twice its width, and the length of a diagonal of the plot is 25 rods. What are the dimensions of the field?

12. The denominator of a certain fraction exceeds the numerator by 3. If both numerator and denominator be increased by 4, the fraction will be increased by $\frac{1}{8}$. Determine the fraction.

13. The numerator of a fraction exceeds twice the denominator by 1. If the numerator be decreased by 3, and the denominator increased by 3, the resulting fraction will be the reciprocal of the given fraction. Find the fraction.

14. A farmer sold a number of sheep for $120. If he had sold 5 less for the same money, he would have received $2 more per sheep. How much did he receive per sheep?

State the problem to which the negative solution applies.

15. A merchant sold a certain number of yards of silk for $40.50. If he had sold 9 yards more for the same money, he would have received 75 cents less per yard. How many yards did he sell?

16. A man bought a number of geese for $27. He sold all but 2 for $25, thus gaining 25 cents on each goose sold. How many geese did he buy?

17. A man agrees to do a piece of work for $48. It takes him 4 days longer than he expected, and he finds that he has earned $1 less per day than he expected. In how many days did he expect to do the work?

18. Find the price of eggs per dozen when 10 more in one dollar's worth lowers the price 4 cents a dozen.

19. A man sold a horse for \$171, and gained as many per cent on the sale as the horse cost dollars. How much did the horse cost?

20. A drover bought a certain number of sheep for \$160. He kept 4, and sold the remainder for \$10.60 per head, and made on his investment $\frac{3}{4}$ as many per cent as he paid dollars for each sheep bought. How many sheep did he buy?

21. Two pipes running together can fill a cistern in $5\frac{5}{6}$ hours. The larger pipe will fill the cistern in 4 hours less time than the smaller. How long will it take each pipe running alone to fill the cistern?

22. A and B can do a piece of work together in 18 days, and it takes B 15 days longer to do it alone than it does A. In how many days can each do it alone?

23. A boat's crew row 4 miles down a river and back again in 1 hour and 30 minutes. Their rate in still water is 2 miles an hour faster than twice the rate of the current. Find the rate of the crew and the rate of the current.

24. A number is formed by two digits. The units' digit is 2 more than the square of half the tens' digit, and if 18 be added to the number, the order of the digits will be reversed. Find the number.

25. A circular grass plot is surrounded by a path of a uniform width of 3 feet. The area of the path is $\frac{1}{3}$ the area of the plot. Find the radius of the plot.

26. If a carriage wheel 11 feet round took $\frac{1}{4}$ of a second less to revolve, the rate of the carriage would be 5 miles more per hour. At what rate is the carriage travelling?

CHAPTER XIX.

SIMULTANEOUS QUADRATIC EQUATIONS.

293. Quadratic equations involving two unknown numbers require different methods for their solution, according to the form of the equations.

CASE I.

294. When from one of the equations the value of one of the unknown numbers can be found in terms of the other, and this value substituted in the other equation.

Ex. Solve:
$$\left.\begin{aligned} 3x^2 - 2xy &= 5 \\ x - y &= 2 \end{aligned}\right\} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Transpose x in (2), $\qquad y = x - 2.$

In (1) put $x - 2$ for y,

$$3x^2 - 2x(x - 2) = 5,$$

The solution of which gives $x = 1$, or $x = -5$.

If $\qquad x = 1,$

$$y = 1 - 2 = -1;$$

and if $\qquad x = -5,$

$$y = -5 - 2 = -7.$$

We have therefore the *pairs* of values,

$$\left.\begin{aligned} x &= 1 \\ y &= -1 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= -5 \\ y &= -7 \end{aligned}\right\}.$$

The original equations are both satisfied by either pair of values. But the values $x = 1$, $y = -7$, will not satisfy the equations; nor will the values $x = -5$, $y = -1$.

The student must be careful to join to each value of x *the corresponding value of* y.

Case II.

295. **When the left side of each of the two equations is homogeneous and of the second degree.**

Solve:
$$\left.\begin{aligned} 2y^2 - 4xy + 3x^2 &= 17 \\ y^2 - x^2 &= 16 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Let $y = vx$, and substitute vx for y in both equations.

From (1), $$2v^2x^2 - 4vx^2 + 3x^2 = 17.$$

$$\therefore x^2 = \frac{17}{2v^2 - 4v + 3}.$$

From (2), $$v^2x^2 - x^2 = 16.$$

$$\therefore x^2 = \frac{16}{v^2 - 1}.$$

Equate the values of x^2, $$\frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1},$$

$$32v^2 - 64v + 48 = 17v^2 - 17,$$

$$15v^2 - 64v = -65,$$

$$225v^2 - 960v = -975,$$

$$225v^2 - (\ ) + (32)^2 = 49,$$

$$15v - 32 = \pm 7.$$

$$\therefore v = \frac{5}{3} \text{ or } \frac{13}{5}.$$

If $$v = \frac{5}{3},$$

$$y = vx = \frac{5x}{3}.$$

Substitute in (2),

$$\frac{25x^2}{9} - x^2 = 16,$$

$$x^2 = 9,$$

$$x = \pm 3,$$

$$y = \frac{5x}{3} = \pm 5.$$

If $$v = \frac{13}{5},$$

$$y = vx = \frac{13x}{5}.$$

Substitute in (2),

$$\frac{169x^2}{25} - x^2 = 16,$$

$$x^2 = \frac{25}{9},$$

$$x = \pm\frac{5}{3},$$

$$y = \frac{13x}{5} = \pm\frac{13}{3}.$$

CASE III.

296. When the two equations are symmetrical with respect to x and y; that is, when x and y are similarly involved.

Thus, the expressions

$2x^3+3x^2y^2+2y^3$, $2xy-3x-3y+1$, $x^4-3x^2y-3xy^2+y^4$,

are symmetrical expressions. In this case the general rule is to combine the equations in such a manner as to remove the highest powers of x and y.

Solve:
$$\left.\begin{aligned} x^4+y^4&=337 \\ x+y&=7 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

To remove x^4 and y^4, raise (2) to the fourth power,

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=2401$$

Add (1),
$$x^4 \qquad\qquad\qquad\qquad +y^4=337$$

$$2x^4+4x^3y+6x^2y^2+4xy^3+2y^4=2738$$

Divide by 2, $x^4+2x^3y+3x^2y^2+2xy^3+y^4=1369.$

Extract the square root, $x^2+xy+y^2=\pm 37.$ (3)

Subtract (3) from $(2)^2$, $xy=12$ or 86.

We now have to solve the two pairs of equations,

$$\left.\begin{aligned} x+y&=7 \\ xy&=12 \end{aligned}\right\};\quad \left.\begin{aligned} x+y&=7 \\ xy&=86 \end{aligned}\right\}.$$

From the first,
$$\left.\begin{aligned} x&=4 \\ y&=3 \end{aligned}\right\};\quad \text{or} \quad \left.\begin{aligned} x&=3 \\ y&=4 \end{aligned}\right\}.$$

From the second,
$$\left.\begin{aligned} x&=\frac{7\pm\sqrt{-295}}{2} \\ y&=\frac{7\mp\sqrt{-295}}{2} \end{aligned}\right\}.$$

297. The preceding cases are *general methods* for the solution of equations which belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may be found more easily than by the general method.

(1) Solve:

$$\left.\begin{aligned} x + y &= 40 \\ xy &= 300 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Square (1), $x^2 + 2xy + y^2 = 1600.$ (3)

Multiply (2) by 4, $4xy = 1200.$ (4)

Subtract (4) from (3),

$$x^2 - 2xy + y^2 = 400. \qquad (5)$$

Extract root of each side, $x - y = \pm 20.$ (6)

From (1) and (6),

$$\left.\begin{aligned} x &= 30 \\ y &= 10 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= 10 \\ y &= 30 \end{aligned}\right\}.$$

(2) Solve:

$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{9}{20} \\ \frac{1}{x^2} + \frac{1}{y^2} &= \frac{41}{400} \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Square (1), $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{81}{400}.$ (3)

Subtract (2) from (3), $\frac{2}{xy} = \frac{40}{400}.$ (4)

Subtract (4) from (2),

$$\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{400}.$$

Extract the root, $\frac{1}{x} - \frac{1}{y} = \pm \frac{1}{20}.$ (5)

From (1) and (5),

$$\left.\begin{aligned} x &= 4 \\ y &= 5 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= 5 \\ y &= 4 \end{aligned}\right\}.$$

(3) Solve: $\left.\begin{aligned} x - y &= 4 \\ x^2 + y^2 &= 40 \end{aligned}\right\}$ (1) (2)

Square (1), $x^2 - 2xy + y^2 = 16.$ (3)

Subtract (2) from (3), $-2xy = -24.$ (4)

Subtract (4) from (2),

$x^2 + 2xy + y^2 = 64.$

Extract the root, $x + y = \pm 8.$ (5)

From (1) and (5), $\left.\begin{aligned} x &= 6 \\ y &= 2 \end{aligned}\right\}$; or $\left.\begin{aligned} x &= -2 \\ y &= -6 \end{aligned}\right\}$.

(4) Solve: $\left.\begin{aligned} x^3 + y^3 &= 91 \\ x + y &= 7 \end{aligned}\right\}$ (1) (2)

Divide (1) by (2), $x^2 - xy + y^2 = 13.$ (3)

Square (2), $x^2 + 2xy + y^2 = 49.$ (4)

Subtract (3) from (4), $3xy = 36.$

Divide by -3, $-xy = -12.$ (5)

Add (5) and (3), $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1.$ (6)

From (2) and (6), $\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}$; or $\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}$.

(5) Solve: $\left.\begin{aligned} x^3 + y^3 &= 18xy \\ x + y &= 12 \end{aligned}\right\}$ (1) (2)

Divide (1) by (2), $x^2 - xy + y^2 = \frac{3xy}{2}.$ (3)

Square (2), $x^2 + 2xy + y^2 = 144.$ (4)

Subtract (4) from (3), $-3xy = \frac{3xy}{2} - 144,$

which gives $xy = 32.$

We now have, $\left.\begin{aligned} x + y &= 12 \\ xy &= 32 \end{aligned}\right\}$.

Solving, we find, $\left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}$; or $\left.\begin{aligned} x &= 4 \\ y &= 8 \end{aligned}\right\}$.

Exercise 101.

1. $\left.\begin{aligned} x+y&=7 \\ xy&=10 \end{aligned}\right\}$

2. $\left.\begin{aligned} x+y&=12 \\ xy&=27 \end{aligned}\right\}$

3. $\left.\begin{aligned} x-y&=6 \\ xy&=-8 \end{aligned}\right\}$

4. $\left.\begin{aligned} x-y&=10 \\ xy&=11 \end{aligned}\right\}$

5. $\left.\begin{aligned} x+y&=12 \\ x^2+y^2&=80 \end{aligned}\right\}$

6. $\left.\begin{aligned} x+y&=3 \\ x^2+y^2&=29 \end{aligned}\right\}$

7. $\left.\begin{aligned} x-y&=9 \\ x^2+y^2&=45 \end{aligned}\right\}$

8. $\left.\begin{aligned} x+2y&=7 \\ x^2+y^2&=10 \end{aligned}\right\}$

9. $\left.\begin{aligned} 3x-y&=12 \\ x^2-y^2&=16 \end{aligned}\right\}$

10. $\left.\begin{aligned} y&=3x+1 \\ x^2+xy&=33 \end{aligned}\right\}$

11. $\left.\begin{aligned} 5x-4y&=10 \\ 3x^2-4y^2&=8 \end{aligned}\right\}$

12. $\left.\begin{aligned} x+7y&=23 \\ xy&=6 \end{aligned}\right\}$

13. $\left.\begin{aligned} 2x-3y&=2 \\ x^2-2xy&=-7 \end{aligned}\right\}$

14. $\left.\begin{aligned} 2x-3y&=1 \\ 3x^2-4xy&=32 \end{aligned}\right\}$

15. $\left.\begin{aligned} x^2-xy+y^2&=21 \\ x+y&=9 \end{aligned}\right\}$

16. $\left.\begin{aligned} x^2-3xy+2y^2&=0 \\ 2x+3y&=7 \end{aligned}\right\}$

17. $\left.\begin{aligned} x^2-y^2&=9 \\ x-y&=1 \end{aligned}\right\}$

18. $\left.\begin{aligned} x^2+3y+17&=0 \\ 3x-y&=3 \end{aligned}\right\}$

19. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=5 \\ \frac{1}{x^2}-\frac{1}{y^2}&=5 \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{5}{6} \\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{13}{36} \end{aligned}\right\}$

21. $\left.\begin{aligned} \frac{2}{x}+\frac{3}{y}&=2 \\ xy&=6 \end{aligned}\right\}$

22. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=11 \\ \frac{1}{x^2}+\frac{1}{y^2}&=61 \end{aligned}\right\}$

23. $\left.\begin{aligned} \frac{6}{x}+\frac{8}{y}&=4 \\ xy&=16 \end{aligned}\right\}$

24. $\left.\begin{aligned} x^3+y^3&=35 \\ x+y&=5 \end{aligned}\right\}$

25. $$\left.\begin{array}{l} x^3 - y^3 = 61 \\ x - y = 1 \end{array}\right\}$$

26. $$\left.\begin{array}{l} x^3 + y^3 = 65 \\ x + y = 5 \end{array}\right\}$$

27. $$\left.\begin{array}{l} x^2y + xy^2 = 120 \\ x + y = 8 \end{array}\right\}$$

28. $$\left.\begin{array}{l} x^3 - y^3 = \frac{7}{64} \\ x - y = \frac{1}{4} \end{array}\right\}$$

29. $$\left.\begin{array}{l} x^3 + y^3 = 126 \\ x^2 - xy + y^2 = 21 \end{array}\right\}$$

30. $$\left.\begin{array}{l} x^3 - y^3 = 56 \\ x^2 + xy + y^2 = 28 \end{array}\right\}$$

31. $$\left.\begin{array}{l} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{35}{3} \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{12} \end{array}\right\}$$

32. $$\left.\begin{array}{l} \dfrac{x^2}{y} - \dfrac{y^2}{x} = \dfrac{19}{2} \\ \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{18} \end{array}\right\}$$

33. $$\left.\begin{array}{l} x^2 + xy = 24 \\ xy + y^2 = 40 \end{array}\right\}$$

34. $$\left.\begin{array}{l} x^2 - xy = 8 \\ xy - y^2 = 7 \end{array}\right\}$$

35. $$\left.\begin{array}{l} x^2 + 2xy = 24 \\ 2xy + 4y^2 = 120 \end{array}\right\}$$

36. $$\left.\begin{array}{l} 4x^2 + 5xy = 14 \\ 7xy + 9y^2 = 50 \end{array}\right\}$$

37. $$\left.\begin{array}{l} x^2 + xy + y^2 = 39 \\ 2x^2 + 3xy + y^2 = 63 \end{array}\right\}$$

38. $$\left.\begin{array}{l} x^2 + 3y^2 = 52 \\ xy + 2y^2 = 40 \end{array}\right\}$$

39. $$\left.\begin{array}{l} 2x^2 - y^2 = 46 \\ xy + y^2 = 14 \end{array}\right\}$$

40. $$\left.\begin{array}{l} x^2 + xy + 2y^2 = 44 \\ 2x^2 - 3xy + 2y^2 = 16 \end{array}\right\}$$

41. $$\left.\begin{array}{l} x^2 + 3y^2 = 31 \\ 4xy + y^2 = 33 \end{array}\right\}$$

42. $$\left.\begin{array}{l} 3x^2 + 7xy = 82 \\ x^2 + 5xy + 9y^2 = 279 \end{array}\right\}$$

43. $$\left.\begin{array}{l} x^4 + y^4 = 97 \\ x + y = 5 \end{array}\right\}$$

44. $$\left.\begin{array}{l} x^4 + y^4 = 17 \\ x + y = 3 \end{array}\right\}$$

45. $$\left.\begin{array}{l} x^4 + y^4 = 881 \\ x - y = 1 \end{array}\right\}$$

46. $$\left.\begin{array}{l} x^5 + y^5 = 211 \\ x + y = 1 \end{array}\right\}$$

47. $$\left.\begin{array}{l} x^5 - y^5 = 242 \\ x - y = 2 \end{array}\right\}$$

48. $$\left.\begin{array}{l} x^2 + y^2 = xy + 7 \\ x + y = xy - 1 \end{array}\right\}$$

49. $$\left.\begin{array}{l} x^3 - y^3 = 7xy \\ x - y = 2 \end{array}\right\}$$

50. $\left.\begin{array}{l} x^3+y^3=36xy \\ x+y=24 \end{array}\right\}$

51. $\left.\begin{array}{l} x^3+3xy^2=62 \\ 3x^2y+y^3=63 \end{array}\right\}$

52. $\left.\begin{array}{l} x^2+xy+y^2=61 \\ x^4+x^2y^2+y^4=1281 \end{array}\right\}$

53. $\left.\begin{array}{l} x^2-xy+y^2=3 \\ x^4+x^2y^2+y^4=21 \end{array}\right\}$

54. $\left.\begin{array}{l} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3} \\ x^2+y^2=20 \end{array}\right\}$

55. $\left.\begin{array}{l} \dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}=\dfrac{24}{5} \\ 3x+4y=36 \end{array}\right\}$

56. $\left.\begin{array}{l} x^2+y^2+x+y=32 \\ xy+16=0 \end{array}\right\}$

57. $\left.\begin{array}{l} x-y-3=0 \\ 2(x^2-y^2)=3xy \end{array}\right\}$

58. $\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}=7 \\ \dfrac{1}{x+1}+\dfrac{1}{y+1}=\dfrac{31}{20} \end{array}\right\}$

59. $\left.\begin{array}{l} x^4+y^4=272 \\ x^2+y^2=3xy-4 \end{array}\right\}$

60. $\left.\begin{array}{l} x^2+y^2=x^2y^2+1 \\ x+y=2xy-1 \end{array}\right\}$

61. $\left.\begin{array}{l} x^3-y^3=a^3 \\ x-y=a \end{array}\right\}$

62. $\left.\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=1 \\ \dfrac{a}{x}+\dfrac{b}{y}=4 \end{array}\right\}$

63. $\left.\begin{array}{l} x^2=ax+by \\ y^2=bx+ay \end{array}\right\}$

64. $\left.\begin{array}{l} x^2+y^2=2(a^2+b^2) \\ xy=a^2-b^2 \end{array}\right\}$

65. $\left.\begin{array}{l} x^2+y^2=\dfrac{a^4+b^4}{a^2b^2} \\ xy=1 \end{array}\right\}$

66. $\left.\begin{array}{l} x^2-y^2=\dfrac{a-b}{a+b} \\ xy=\dfrac{ab}{(a+b)^2} \end{array}\right\}$

67. $\left.\begin{array}{l} x^2-xy=2ab+2b^2 \\ xy-y^2=2ab-2b^2 \end{array}\right\}$

68. $\left.\begin{array}{l} x^2-y^2=a^2 \\ xy=b^2 \end{array}\right\}$

69. $\left.\begin{array}{l} x^2-y^2=8ab \\ xy=a^2-4b^2 \end{array}\right\}$

70. $\left.\begin{array}{l} x^3+y^3=a^3+b^3 \\ x+y=a+b \end{array}\right\}$

Exercise 102.

1. The area of a rectangle is 60 square feet, and its perimeter is 34 feet. Find the length and breadth of the rectangle.

2. The area of a rectangle is 108 square feet. If the length and breadth of the rectangle are each increased by 3 feet, the area will be 180 square feet. Find the length and breadth of the rectangle.

3. If the length and breadth of a rectangular plot are each increased by 10 feet, the area will be increased by 400 square feet. But if the length and breadth are each diminished by 5 feet, the area will be 75 square feet. Find the length and breadth of the plot.

4. The area of a rectangle is 168 square feet, and the length of its diagonal is 25 feet. Find the length and breadth of the rectangle.

5. The diagonal of a rectangle is 25 inches. If the rectangle were 4 inches shorter and 8 inches wider, the diagonal would still be 25 inches. Find the area of the rectangle.

6. A rectangular field, containing 180 square rods, is surrounded by a road 1 rod wide. The area of the road is 58 square rods. Find the dimensions of the field.

7. Two square gardens have a total surface of 2137 square yards. A rectangular piece of land whose dimensions are respectively equal to the sides of the two squares, will have 1093 square yards less than the two gardens united. What are the sides of the two squares?

8. The sum of two numbers is 22, and the difference of their squares is 44. Find the numbers.

9. The difference of two numbers is 6, and their product exceeds their sum by 39. Find the numbers.

10. The sum of two numbers is equal to the difference of their squares, and the product of the numbers exceeds twice their sum by 2. Find the numbers.

11. The sum of two numbers is 20, and the sum of their cubes is 2060. Find the numbers.

12. The difference of two numbers is 5, and the difference of their cubes exceeds the difference of their squares by 1290. Find the numbers.

13. A number is formed of two digits. The sum of the squares of the digits is 58. If twelve times the units' digit be subtracted from the number, the order of the digits will be reversed. Find the number.

14. A number is formed of three digits, the third digit being twice the sum of the other two. The first digit plus the product of the other two digits is 25. If 180 be added to the number, the order of the first and second digits will be reversed. Find the number.

15. There are two numbers formed of the same two digits in reverse orders. The sum of the numbers is 33 times the difference of the two digits, and the difference of the squares of the numbers is 4752. Find the numbers.

16. The sum of the numerator and denominator of a certain fraction is 5; and if the numerator and denominator be each increased by 3, the value of the fraction will be increased by $\frac{1}{6}$. Find the fraction.

17. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.

CHAPTER XX.

PROPERTIES OF QUADRATICS.

298. Every affected quadratic can be reduced to the form $ax^2 + bx + c = 0$, of which the two roots are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}. \quad \S\, 285.$$

CHARACTER OF THE ROOTS.

299. As regards the character of the two roots, there are three cases to be distinguished.

I. **If $b^2 - 4ac$ is positive and not zero.** In this case the roots are *real* and *unequal*. The roots are real, since the square root of a positive number can be found exactly or approximately. If $b^2 - 4ac$ is a perfect square, the roots are rational; if $b^2 - 4ac$ is not a perfect square, the roots are surds.

The roots are unequal, since $\sqrt{b^2 - 4ac}$ is not zero.

II. **If $b^2 - 4ac$ is zero.** In this case the two roots are *real* and *equal*, since they both become $-\frac{b}{2a}$.

III. **If $b^2 - 4ac$ is negative.** In this case the roots are *imaginary*, since they both involve the square root of a negative number.

The two imaginary roots of a quadratic cannot be equal, since $b^2 - 4ac$ is not zero. They have, however, the same real part, $-\frac{b}{2a}$, and the same imaginary parts, but with opposite signs; such expressions are called **conjugate imaginaries.** The expression b^2-4ac is called the **discriminant** of the expression ax^2+bx+c.

300. The above cases may also be distinguished as follows:

CASE I. $b^2-4ac>0$, roots real and unequal.
CASE II. $b^2-4ac=0$, roots real and equal.
CASE III. $b^2-4ac<0$, roots imaginary.

301. By calculating the value of b^2-4ac we can determine the character of the roots of a given equation without solving the equation.

(1) $x^2-5x+6=0$.

Here $a=1,\ b=-5,\ c=6.$
$b^2-4ac=25-24=1.$

The roots are real and unequal, and rational.

(2) $3x^2+7x-1=0$.

Here $a=3,\ b=7,\ c=-1.$
$b^2-4ac=49+12=61.$

The roots are real and unequal, and are both surds.

(3) $4x^2-12x+9=0$.

Here $a=4,\ b=-12,\ c=9.$
$b^2-4ac=144-144=0.$

The roots are real and equal.

(4) $2x^2-3x+4=0$.

Here $a=2,\ b=-3,\ c=4.$
$b^2-4ac=9-32=-23.$

The roots are both imaginary.

(5) Find the values of m for which the following equation has its two roots equal:

$$2mx^2 + (5m+2)x + (4m+1) = 0.$$

Here $a = 2m,\ b = 5m+2,\ c = 4m+1.$

If the roots are to be equal, we must have

$$b^2 - 4ac = 0, \text{ or } (5m+2)^2 - 8m(4m+1) = 0.$$

This gives $m = 2$, or $-\frac{2}{7}$.

For these values of m the equation becomes

$$4x^2 + 12x + 9 = 0, \text{ and } 4x^2 - 4x + 1 = 0,$$

each of which has its roots equal.

Exercise 103.

Determine without solving the character of the roots of each of the following equations:

1. $x^2 + 5x + 6 = 0.$
2. $x^2 + 2x - 15 = 0.$
3. $x^2 + 2x + 3 = 0.$
4. $3x^2 + 7x + 2 = 0.$
5. $9x^2 + 6x + 1 = 0.$
6. $6x^2 - 7x - 3 = 0.$
7. $5x^2 - 5x - 3 = 0.$
8. $2x^2 - x + 5 = 0.$
9. $6x^2 + x - 77 = 0.$
10. $5x^2 + 8x + \frac{16}{5} = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

11. $(m+1)x^2 + (m-1)x + m + 1 = 0.$
12. $(2m-3)x^2 + mx + m - 1 = 0.$
13. $2mx^2 + x^2 + 4x + 2mx + 2m - 4 = 0.$
14. $2mx^2 + 3mx - 6 = 3x - 2m - x^2.$
15. $mx^2 + 9x - 10 = 3mx - 2x^2 + 2m.$

Relations of Roots and Coefficients.

302. Consider the equation $x^2 - 10x + 24 = 0$. Resolve into factors, $(x-6)(x-4) = 0$. The two values of x are 6 and 4; their sum is 10, the coefficient of x with its sign changed; their product is 24, the third term.

303. In general, representing the roots of the quadratic equation $ax^2 + bx + c = 0$ by r_1 and r_2, we have (§ 285),

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Adding, $\quad r_1 + r_2 = -\frac{b}{a};$

multiplying, $\quad r_1 r_2 = \frac{c}{a}.$

If we divide the equation $ax^2 + bx + c = 0$ through by a, we have the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; this may be written $x^2 + px + q = 0$ where $p = \frac{b}{a}$, $q = \frac{c}{a}$.

It appears, then, that if any quadratic equation be made to assume the form $x^2 + px + q = 0$, the following relations hold between the coefficients and roots of the equation:

(1) The **sum** of the two roots is equal to the coefficient of x with its sign changed.

(2) The **product** of the two roots is equal to the constant term.

Thus, the sum of the two roots of the equation

$$x^2 - 7x + 8 = 0$$

is 7, and the product of the roots 8.

304. Resolution into Factors. By § 303, if r_1 and r_2 are the roots of the equation $x^2+px+q=0$, the equation may be written

$$x^2-(r_1+r_2)x+r_1r_2=0.$$

The left member is the product of $x-r_1$ and $x-r_2$, so that the equation may be also written

$$(x-r_1)(x-r_2)=0.$$

It appears, then, that the factors of the *quadratic expression* x^2+px+q are $x-r_1$ and $x-r_2$, where r_1 and r_2 are the roots of the *quadratic equation* $x^2+px+q=0$.

The factors are real and different, real and alike, or imaginary, according as r_1 and r_2 are real and unequal, real and equal, or imaginary.

If $r_2=r_1$, the equation becomes $(x-r_1)(x-r_1)=0$, or $(x-r_1)^2=0$; if, then, the two roots of a quadratic equation are equal, the left member, when all the terms are transposed to that member, will be a perfect square as regards x.

305. If the equation is in the form $ax^2+bx+c=0$, the left member may be written

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right),$$

$$a(x-r_1)(x-r_2).$$

306. If the roots of a quadratic equation are given, we can readily form the equation.

Ex. Form the equation of which the roots are 3 and $-\frac{5}{2}$.

The equation is $\quad (x-3)\left(x+\frac{5}{2}\right)=0,$

or $\quad (x-3)(2x+5)=0,$

or $\quad 2x^2-x-15=0.$

307. Any quadratic expression may be resolved into factors by putting the expression equal to zero, and solving the equation thus formed.

(1) Resolve into two factors $x^2 - 5x + 3$.

Write the equation

$$x^2 - 5x + 3 = 0.$$

Solve this equation, and the roots are found to be

$$\frac{5 + \sqrt{13}}{2} \text{ and } \frac{5 - \sqrt{13}}{2}.$$

Therefore, the factors of $x^2 - 5x + 3$ are

$$x - \frac{5 + \sqrt{13}}{2} \text{ and } x - \frac{5 - \sqrt{13}}{2}.$$

(2) Resolve into factors $3x^2 - 4x + 5$.

Write the equation

$$3x^2 - 4x + 5 = 0.$$

Solve this equation, and the roots are found to be

$$\frac{2 + \sqrt{-11}}{3} \text{ and } \frac{2 - \sqrt{-11}}{3}.$$

Therefore, the expression $3x^2 - 4x + 5$ may be written (§ 305),

$$3\left(x - \frac{2 + \sqrt{-11}}{3}\right)\left(x - \frac{2 - \sqrt{-11}}{3}\right).$$

Exercise 104.

Form the equations of which the roots are

1. $7, 6$.
2. $5, -3$.
3. $1\frac{1}{2}, -2$.
4. $4, 2\frac{1}{3}$.
5. $1\frac{1}{2}, -1\frac{1}{3}$.
6. $-1\frac{1}{6}, -1\frac{2}{3}$.
7. $13, -4\frac{1}{3}$.
8. $\frac{3}{11}, \frac{2}{11}$.
9. $3 + \sqrt{2}, 3 - \sqrt{2}$.
10. $1 + \sqrt{-1}, 1 - \sqrt{-1}$
11. $a, a - b$.
12. $a + b, a - b$.

Resolve into factors, real or imaginary:

13. $12x^2 + x - 1$.
14. $3x^2 - 14x - 24$.
15. $x^2 - 2x - 2$.
16. $x^2 - 2x + 3$.
17. $x^2 + x + 1$.
18. $x^2 - 2x + 9$.

CHAPTER XXI.

RATIO, PROPORTION, AND VARIATION.

308. Ratio of Numbers. The relative magnitude of two numbers is called their **ratio** when expressed by the indicated quotient of the first by the second. Thus the ratio of a to b is $\frac{a}{b}$, or $a \div b$, or $a:b$.

The first term of a ratio is called the **antecedent**, and the second term the **consequent**. When the antecedent is equal to the consequent, the ratio is called a ratio of **equality**; when the antecedent is greater than the consequent, the ratio is called a ratio of **greater inequality**; when less, a ratio of **less inequality**.

When the antecedent and consequent are interchanged, the resulting ratio is called the **inverse** of the given ratio. Thus, the ratio $3:6$ is the *inverse* of the ratio $6:3$.

309. A ratio will not be altered if both its terms are multiplied by the same number. For the ratio $a:b$ is represented by $\frac{a}{b}$, the ratio $ma:mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, we have $ma:mb = a:b$.

310. A ratio will be altered if different multipliers of its terms are taken; and will be increased or diminished according as the multiplier of the antecedent is greater than or less than that of the consequent. Thus,

If	$m > n$,	If	$m < n$,
then	$ma > na$,	then	$ma < na$,
and	$\frac{ma}{nb} > \frac{na}{nb}$;	and	$\frac{ma}{nb} < \frac{na}{nb}$;
but	$\frac{na}{nb} = \frac{a}{b}$.	but	$\frac{na}{nb} = \frac{a}{b}$.
$\therefore$	$\frac{ma}{nb} > \frac{a}{b}$,	$\therefore$	$\frac{ma}{nb} < \frac{a}{b}$,
or	$ma : nb > a : b$.	or	$ma : nb < a : b$.

311. Ratios are *compounded* by taking the product of the fractions that represent them. Thus, the ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

The ratio compounded of $a : b$ and $a : b$ is the *duplicate* ratio $a^2 : b^2$; the ratio compounded of $a : b$, $a : b$, and $a : b$ is the *triplicate* ratio $a^3 : b^3$; and so on.

312. Ratios are *compared* by comparing the fractions that represent them.

Thus, $a : b >$ or $< c : d$,

according as $\frac{a}{b} >$ or $< \frac{c}{d}$,

as $\frac{ad}{bd} >$ or $< \frac{bc}{bd}$,

as $ad >$ or $< bc$.

313. Proportion of Numbers. Four numbers, a, b, c, d, are said to be in **proportion** when the ratio $a : b$ is equal to the ratio $c : d$.

We then write $a : b = c : d$, and read this, the ratio of a to b equals the ratio of c to d, or a is to b as c is to d.

A proportion is also written $a : b :: c : d$.

The four numbers, a, b, c, d, are called **proportionals**; a and d are called the **extremes**, b and c the **means**.

314. When four numbers are in proportion, the product of the extremes is equal to the product of the means.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying by bd, $$ad = bc.$$

The equation $ad = bc$ gives $a = \frac{bc}{d}$, $b = \frac{ad}{c}$; so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean. If three terms of a proportion are given, it appears from the above that the fourth term has one value, and but one value.

315. If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.

For, if $$ad = bc,$$

then, dividing by bd, $$\frac{ad}{bd} = \frac{bc}{bd},$$

or $$\frac{a}{b} = \frac{c}{d}.$$

$$\therefore\ a : b = c : d.$$

316. Transformations of a Proportion. If four numbers, a, b, c, d, are in proportion, they will be in proportion by:

I. Inversion; that is, b will be to a as d is to c.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$1 \div \frac{a}{b} = 1 \div \frac{c}{d}.$$

or $$\frac{b}{a} = \frac{d}{c}.$$

$$\therefore\ b : a = d : c.$$

II. **Composition**; that is, $a+b$ will be to b as $c+d$ is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore\ a+b : b = c+d : d.$$

III. **Division**; that is, $a-b$ will be to b as $c-d$ is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

$$\therefore\ a-b : b = c-d : d.$$

IV. **Composition and Division**; that is, $a+b$ will be to $a-b$ as $c+d$ is to $c-d$.

For, from II., $$\frac{a+b}{b} = \frac{c+d}{d},$$

and from III., $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Dividing, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore\ a+b : a-b = c+d : c-d.$$

V. Alternation; that is, a will be to c as b is to d.

For, if $$a:b=c:d,$$

then $$\frac{a}{b}=\frac{c}{d}.$$

Multiplying by $\frac{b}{c}$, $$\frac{ab}{bc}=\frac{bc}{cd},$$

or $$\frac{a}{c}=\frac{b}{d}.$$

$$\therefore\ a:c=b:d.$$

317. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

For, if $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$

r may be put for each of these ratios.

Then $$\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{e}{f}=r,\ \frac{g}{h}=r.$$

$$\therefore\ a=br,\ c=dr,\ e=fr,\ g=hr.$$

$$\therefore\ a+c+e+g=(b+d+f+h)r.$$

$$\therefore\ \frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}.$$

$$\therefore\ a+c+e+g:b+d+f+h=a:b.$$

In like manner it may be shown that

$$ma+nc+pe+qg:mb+nd+pf+qh=a:b.$$

318. Continued Proportion. Numbers are said to be in **continued proportion** when the first is to the second as the second is to the third, and so on. Thus, a, b, c, d, are in continued proportion when $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$.

319. If a, b, c are proportionals, so that $a:b=b:c$, then b is called a **mean proportional** between a and c, and c is called a **third proportional** to a and b.

If $$a:b=b:c, \text{ then } b=\sqrt{ac}.$$

For, if $$a:b=b:c,$$

then $$\frac{a}{b}=\frac{b}{c},$$

and $$b^2=ac.$$

$$\therefore\ b=\sqrt{ac}.$$

320. The products of the corresponding terms of two or more proportions are in proportion.

For, if $$a:b=c:d,$$
$$e:f=g:h,$$
$$k:l=m:n,$$

then $$\frac{a}{b}=\frac{c}{d},\ \frac{e}{f}=\frac{g}{h},\ \frac{k}{l}=\frac{m}{n}.$$

Taking the product of the left members, and also of the right members of these equations,

$$\frac{aek}{bfl}=\frac{cgm}{dhn}.$$

$$\therefore\ aek:bfl=cgm:dhn.$$

321. Like powers, or like roots, of the terms of a proportion are in proportion.

For, if $$a:b=c:d,$$

then $$\frac{a}{b}=\frac{c}{d}.$$

Raising both sides to the nth power,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$

$$\therefore\ a^n:b^n=c^n:d^n.$$

Extracting the nth root, $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \dfrac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$.

$$\therefore \ a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

322. The laws that have been established for ratios should be remembered when ratios are expressed in fractional form.

(1) Solve: $\dfrac{x^2 + x + 1}{x^2 - x - 1} = \dfrac{x^2 - x + 2}{x^2 + x - 2}$.

By composition and division,

$$\frac{2x^2}{2(x+1)} = \frac{2x^2}{-2(x-2)}.$$

This equation is satisfied when $x = 0$. For any other value of x, we may divide by x^2.

We then have $\dfrac{1}{x+1} = \dfrac{1}{2-x}$,

and therefore $x = \frac{1}{2}$.

(2) If $a : b = c : d$, show that

$$a^2 + ab : b^2 - ab = c^2 + cd : d^2 - cd.$$

If $\dfrac{a}{b} = \dfrac{c}{d}$,

then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$,

and $\dfrac{a}{-b} = \dfrac{c}{-d}$.

$$\therefore \ \frac{a}{-b} \times \frac{a+b}{a-b} = \frac{c}{-d} \times \frac{c+d}{c-d};$$

that is, $\dfrac{a^2 + ab}{b^2 - ab} = \dfrac{c^2 + cd}{d^2 - cd}$,

or $a^2 + ab : b^2 - ab = c^2 + cd : d^2 - cd$.

(3) If $a:b=c:d$, and a is the *greatest term*, show that $a+d$ is greater than $b+c$.

Since $\frac{a}{b}=\frac{c}{d}$, and $a>c$, (1)

the denominator $b>d$.

From (1), by division, $\frac{a-b}{b}=\frac{c-d}{d}$. (2)

Since $b>d$,

from (2), $a-b>c-d$.

Now, $b+d=b+d$,

Adding, $a+d>b+c$.

323. Ratio of Quantities. To *measure* a quantity of any kind is to find out how many times it contains another *known* quantity of the *same kind*, called the **unit of measure.** Thus, if a line contains 5 times the linear unit of measure, one yard, the length of the line is 5 yards.

324. Commensurable Quantities. If two quantities of the *same kind* are so related that a unit of measure can be found which is contained in each of the quantities an integral number of times, this unit of measure is a *common measure* of the two quantities, and the two quantities are said to be **commensurable.**

If two commensurable quantities are measured by the same unit, their **ratio** is simply the ratio of the two numbers by which the quantities are expressed. Thus, $\frac{1}{6}$ of a foot is a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet, being contained in the first 15 times and in the second 22 times.

The ratio of $2\frac{1}{2}$ feet to $3\frac{2}{3}$ feet is therefore the ratio of 15 : 22.

Evidently two quantities *different in kind* can have no ratio.

325. Incommensurable Quantities. The ratio of two quantities of the same kind cannot always be expressed by the ratio of two whole numbers. Thus, the side and diagonal of a square have no common measure; for, if the side is a inches long, the diagonal will be $a\sqrt{2}$ inches long, and no measure can be found which will be contained in each an integral number of times.

Again, the diameter and circumference of a circle have no common measure, and are therefore incommensurable.

In this case, as there is no common measure of the two quantities, we cannot find their ratio by the method of § 324. We therefore proceed as follows:

Suppose a and b to be two incommensurable quantities of the *same kind*. Divide b into any integral number (n) of equal parts, and suppose one of these parts is contained in a more than m times and less than $m+1$ times. Then the ratio $\frac{a}{b} > \frac{m}{n}$, but $< \frac{m+1}{n}$; that is, the value of $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is less than $\frac{1}{n}$. But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

Hence, the ratio of two incommensurable quantities cannot be expressed *exactly* in figures, but it may be expressed *approximately* to any desired degree of accuracy.

Thus, if b represent the side of a square, and a the diagonal,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356.....$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b is taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth*, a *trillionth*, *or any other assigned value whatever.*

326. The ratio of two incommensurable quantities is an **incommensurable ratio**, and is *a fixed value* toward which its successive approximate values constantly tend as the error is made less and less.

327. Proportion of Quantities. In order for four quantities, A, B, C, D, to be in proportion, A and B must be of the *same kind*, and C and D of the same kind (but C and D need not necessarily be of the same kind as A and B), and in addition the ratio of A to B must be equal to the ratio of C to D.

If this is true, we have the proportion

$$A : B = C : D.$$

When four quantities are in proportion, the numbers by which they are expressed are four abstract numbers in proportion.

328. The laws of § 316, which apply to proportion of numbers, apply also to proportion of quantities, except that alternation will apply only when the four quantities in proportion are *all* of the same kind.

Exercise 105.

1. Find the duplicate of the ratio $3:4$.

2. Find the ratio compounded of the ratios $2:3$, $3:4$, $6:7$, $14:8$.

3. Find a third proportional to 21 and 28.

4. Find a mean proportional between 6 and 24.

5. Find a fourth proportional to 3, 5, and 42.

6. Find x if $5+x:11-x=3:5$.

7. Find the number which must be added to both the terms of the ratio $3:5$ in order that the resulting ratio may be equal to the ratio $15:16$.

If $a:b=c:d$, show that

8. $ac:bd=c^2:d^2$.

9. $ab:cd=a^2:c^2$.

10. $a^2-b^2:c^2-d^2=a^2:c^2$.

11. $2a+b:2c+d=b:d$.

12. $5a-b:5c-d=a:c$.

13. $a-3b:a+3b=c-3d:c+3d$.

14. $a^2+ab+b^2:a^2-ab+b^2=c^2+cd+d^2:c^2-cd+d^2$.

Find x in the proportion

15. $45:68=90:x$.

16. $6:3=x:7$.

17. $x:1\frac{1}{2}=1\frac{5}{7}:1\frac{4}{5}$.

18. $3:x=7:42$.

19. Find two numbers in the ratio $2:3$, the sum of whose squares is 325.

20. Find two numbers in the ratio $5:3$, the difference of whose squares is 400.

21. Find three numbers which are to each other as $2:3:5$, such that half the sum of the greatest and least exceeds the other by 25.

22. Find x if $6x - a : 4x - b = 3x + b : 2x + a$.

23. Find x and y from the proportions

$$x : y = x + y : 42; \quad x : y = x - y : 6.$$

24. Find x and y from the proportions

$$2x + y : y = 3y : 2y - x;$$
$$2x + 1 : 2x + 6 = y : y + 2.$$

25. If $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$, show that $\frac{a}{b} = \frac{c}{d}$.

Variation.

329. A quantity which in any particular problem has a fixed value is called a **constant quantity**, or simply a **constant**; a quantity which may change its value is called a **variable quantity**, or simply a **variable.**

Variable numbers, like unknown numbers, are generally represented by x, y, z, etc.; constant numbers, like known numbers, by a, b, c, etc.

330. Two variables may be so related that when a value of one is given, the corresponding value of the other can be found. In this case one variable is said to be a *function* of the other; that is, one variable depends upon the other for its value. Thus, if the rate at which a man walks is known, the distance he walks can be found when the time is given; the distance is in this case a *function* of the time.

331. There is an unlimited number of ways in which two variables may be related. We shall consider in this chapter only a few of these ways.

332. When x and y are so related that their ratio is constant, y is said to vary as x; this is abbreviated thus:

$y \propto x$. The sign $\propto$, called the **sign of variation**, is read "varies as." Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude is changed in any ratio, the area will be changed in the same ratio.

In this case, if we represent the constant ratio by m,

$$y : x = m, \text{ or } \frac{y}{x} = m; \ \therefore y = mx.$$

Again, if y', x' and y'', x'' be two sets of corresponding values of y and x, then

$$y' : x' = y'' : x'',$$

or $$y' : y'' = x' : x''. \qquad \text{§ 316, V.}$$

333. When x and y are so related that the ratio of y to $\frac{1}{x}$ is constant, y is said to vary *inversely* as x; this is written $y \propto \frac{1}{x}$. Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen be doubled, halved, or changed in any other ratio, the time required will be halved, doubled, or changed in the inverse ratio.

In this case, $y : \frac{1}{x} = m$; $\therefore y = \frac{m}{x}$, and $xy = m$; that is, the product xy is constant.

As before, $$y' : \frac{1}{x'} = y'' : \frac{1}{x''},$$

$$x'y' = x''y'',$$

or, $$y' : y'' = x'' : x'. \qquad \text{§ 315}$$

334. If the ratio of $y : xz$ is constant, then y is said to *vary jointly* as x and z.

In this case, $y = mxz$,

and $$y' : y'' = x'z' : x''z''.$$

335. If the ratio $y:\frac{x}{z}$ is constant, then y varies *directly* as x and *inversely* as z.

In this case, $$y=\frac{mx}{z},$$

and $$y':y''=\frac{mx'}{z'}:\frac{mx''}{z''}=\frac{x'}{z'}:\frac{x''}{z''}.$$

336. Theorems.

I. If $y \propto x$, and $x \propto z$, then $y \propto z$.

For $$y=mx \text{ and } x=nz.$$
$$\therefore\ y=mnz;$$
$$\therefore\ y \text{ varies as } z.$$

II. If $y \propto x$, and $z \propto x$, then $(y \pm z) \propto x$.

For $$y=mx \text{ and } z=nx.$$
$$\therefore\ y \pm z=(m \pm n)x;$$
$$\therefore\ y \pm z \text{ varies as } x.$$

III. If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \propto xz$ when x and z are both variable.

Let x', y', z', and x'', y'', z'' be two sets of corresponding values of the variables.

Let x change from x' to x'', z remaining constant, and let the corresponding value of y be Y.

Then $$y':Y=x':x''. \tag{1}$$

Now let z change from z' to z'', x remaining constant.

Then $$Y:y''=z':z''. \tag{2}$$

From (1) and (2),

$$y'Y:y''Y=x'z':x''z'', \qquad \S\ 320$$

or $$y':y''=x'z':x''z'',$$

or $$y':x'z'=y'':x''z''. \qquad \S\ 316,\ \text{V.}$$

$\therefore$ the ratio $\frac{y}{xz}$ is constant, and y varies as xz.

In like manner it may be shown that if y varies as each of any number of quantities x, z, u, etc., when the rest are unchanged, then when they all change, $y \propto xzu$, etc. Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

337. Examples.

(1) If y varies inversely as x, and when $y = 2$ the corresponding value of x is 36, find the corresponding value of x when $y = 9$.

Here $$y = \frac{m}{x}, \text{ or } m = xy.$$

$$\therefore\ m = 2 \times 36 = 72.$$

If 9 and 72 be substituted for y and m respectively in

$$y = \frac{m}{x},$$

the result is $$9 = \frac{72}{x}, \text{ or } 9x = 72.$$

$$\therefore\ x = 8. \ \textit{Ans.}$$

(2) The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weighs 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
D represent the diameter.

Then $W \propto V$ and $V \propto D^3$.

$\therefore\ W \propto D^3$. § 336, I.

Put $W = mD^3$;

then, since 20 and 4 are corresponding values of W and D,

$$20 = m \times 64.$$

$$\therefore\ m = \frac{20}{64} = \frac{5}{16}.$$

$$\therefore\ W = \tfrac{5}{16} D^3.$$

$\therefore$ when $D = 5$, $W = \frac{5}{16}$ of $125 = 39\frac{1}{16}$.

Exercise 106.

1. If $x \propto y$, and if $y = 3$ when $x = 5$, find x when y is 5.

2. If W varies inversely as P, and W is 4 when P is 15, find W when P is 12.

3. If $x \propto y$ and $y \propto z$, show that $xz \propto y^2$.

4. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

5. If x varies inversely as $y^2 - 1$, and is equal to 24 when $y = 10$, find x when $y = 5$.

6. If x varies as $\frac{1}{y} + \frac{1}{z}$, and is equal to 3 when $y = 1$ and $z = 2$, show that $xyz = 2(y + z)$.

7. If $x - y$ varies inversely as $z + \frac{1}{z}$, and $x + y$ varies inversely as $z - \frac{1}{z}$, find the relation between x and z if $x = 1$, $y = 3$, when $z = \frac{1}{2}$.

8. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 1 foot is 3.1416 square feet. Find the area of a circle whose radius is 20 feet.

9. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4.188 cubic feet. Find the volume of a sphere whose radius is 2 feet.

10. If a sphere of given material 3 inches in diameter weighs 24 lbs., how much will a sphere of the same material weigh if its diameter is 5 inches?

11. The *velocity* of a falling body varies as the time during which it has fallen from rest. If the velocity of a falling body at the end of 2 seconds is 64 feet, what is its velocity at the end of 8 seconds?

12. The *distance* a body falls from rest varies as the square of the time it is falling. If a body falls through 144 feet in 3 seconds, how far will it fall in 5 seconds?

The volume of a right circular cone varies jointly as its height and the square of the radius of its base.

If the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet:

13. Compare the volume of two cones, one of which is twice as high as the other, but with one half its diameter.

14. Find the volume of a cone 9 feet high with a base whose radius is 3 feet.

15. Find the volume of a cone 7 feet high with a base whose radius is 4 feet.

16. Find the volume of a cone 9 feet high with a base whose radius is 4 feet.

17. The volume of a sphere varies as the cube of its radius. If the volume is $179\frac{2}{3}$ cubic feet when the radius is $3\frac{1}{2}$ feet, find the volume when the radius is 1 foot 6 inches.

18. Find the radius of a sphere whose volume is the sum of the volumes of two spheres with radii $3\frac{1}{2}$ feet and 6 feet respectively.

19. The distance of the offing at sea varies as the square root of the height of the eye above the sea-level, and the distance is 3 miles when the height is 6 feet. Find the distance when the height is 24 feet.

CHAPTER XXII.

PROGRESSIONS.

338. A succession of numbers that proceed according to some fixed law is called a **series**; the successive numbers are called the **terms** of the series.

A series that ends at some particular term is a **finite series**; a series that continues without end is an **infinite series.**

339. The number of different forms of series is unlimited; in this chapter we shall consider only Arithmetical Series, Geometrical Series, and Harmonical Series.

Arithmetical Progression.

340. A series is called an **arithmetical series** or an **arithmetical progression** when each succeeding term is obtained by adding to the preceding term a *constant difference.*

The general representative of such a series will be

$$a,\ a+d,\ a+2d,\ a+3d\ \ldots\ldots,$$

in which a is the first term and d the common difference; the series will be *increasing* or *decreasing* according as d is positive or negative.

341. The nth Term. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d will always be one less than the number of the term, so that the nth term is $a+(n-1)d$.

If the nth term is represented by l, we have

$$l=a+(n-1)d. \qquad \text{I.}$$

342. Sum of the Series. If l denotes the nth term, a the first term, n the number of terms, d the common difference, and s the sum of n terms, it is evident that

$$s = a + (a+d) + (a+2d) + \cdots + (l-d) + l, \text{ or}$$
$$s = l + (l-d) + (l-2d) + \cdots + (a+d) + a,$$
$$\therefore 2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)$$
$$= n(a+l).$$
$$\therefore \quad s = \frac{n}{2}(a+l). \qquad \text{II.}$$

343. From the equations

$$l = a + (n-1)d, \qquad \text{I.}$$
$$s = \frac{n}{2}(a+l), \qquad \text{II.}$$

any *two* of the five numbers a, d, l, n, s, may be found when the other *three* are given.

(1) Find the sum of ten terms of the series 2, 5, 8, 11 .....

Here $a = 2,\ d = 3,\ n = 10.$

From I., $l = 2 + 27 = 29.$

Substituting in II., $s = \frac{10}{2}(2 + 29) = 155.$ *Ans.*

(2) The first term of an arithmetical series is 3, the last term 31, and the sum of the series 136. Find the series.

From I., $31 = 3 + (n-1)d.$ (1)

From II., $136 = \frac{n}{2}(3 + 31).$ (2)

From (2), $n = 8.$

Substituting in (1), $d = 4.$

The series is 3, 7, 11, 15, 19, 23, 27, 31.

(3) How many terms of the series 5, 9, 13, ..., must be taken in order that their sum may be 275?

From I., $$l = 5 + (n-1)4.$$
$$\therefore\ l = 4n + 1. \qquad (1)$$

From II., $$275 = \frac{n}{2}(5 + l). \qquad (2)$$

Substituting in (2) the value of l found in (1),

$$275 = \frac{n}{2}(4n + 6),$$

or $$2n^2 + 3n = 275.$$

This is a quadratic with n for the unknown number.

Complete the square,

$$16n^2 + (\ ) + 9 = 2209.$$

Extract the root, $$4n + 3 = \pm 47.$$

Therefore, $n = 11$, or $-12\frac{1}{2}$.

We use only the positive result.

(4) Find n when d, l, s are given.

From I., $$a = l - (n-1)d.$$

From II., $$a = \frac{2s - ln}{n}.$$

Therefore, $$l - (n-1)d = \frac{2s - ln}{n}.$$

$$\therefore\ ln - dn^2 + dn = 2s - ln,$$

$$\therefore\ dn^2 - (2l + d)n = -2s.$$

This is a quadratic with n for the unknown number.

Complete the square,

$$4d^2n^2 - (\ ) + (2l + d)^2 = (2l + d)^2 - 8ds.$$

Extract the root,

$$2dn - (2l + d) = \pm\sqrt{(2l + d)^2 - 8ds}.$$

Therefore, $$n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$$

(5) Find the series in which the 15th term is -25 and 41st term -41.

If a is the first term and d the common difference, the 15th term is $a + 14d$, and the 41st term is $a + 40d$.

Therefore, $a + 14d = -25$, (1)

and $a + 40d = -41$. (2)

Subtracting, $-26d = 16$.

$$\therefore\ d = -\frac{8}{13}.$$

Substituting in (1), $a = -16\frac{5}{13}$.

Hence, the series is $-16\frac{5}{13}, -17, -17\frac{8}{13}, \ldots\ldots$

344. The **arithmetical mean** between two numbers is the number which stands between them, and makes with them an arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, then a, A, b are in arithmetical progression, and by the definition of an arithmetical series, § 340,

$$A - a = d,$$

and

$$b - A = d.$$

$$\therefore\ A - a = b - A.$$

$$\therefore\ A = \frac{a + b}{2}.$$

345. Sometimes it is required to insert several arithmetical means between two numbers.

Ex. Insert six arithmetical means between 3 and 17.

Here the whole number of terms is eight; 3 is the first term and 17 the eighth.

By I., $17 = 3 + 7d$.

$$d = 2.$$

The series is 3, [5, 7, 9, 11, 13, 15,] 17, the terms in brackets being the means required.

346. When the sum of a number of terms in arithmetical progression is given, it is convenient to represent the terms as follows:

Three terms by $x-y,\ x,\ x+y$;

four terms by $x-3y,\ x-y,\ x+y,\ x+3y$;

and so on.

Ex. The sum of three numbers in arithmetical progression is 36, and the square of the mean exceeds the product of the two extremes by 49. Find the numbers.

Let $x-y,\ x,\ x+y$ represent the numbers.

Then, adding, $3x=36.$

$\therefore x=12.$

Putting for x its value, the numbers are

$12-y,\ 12,\ 12+y.$

Then $(12)^2-(12-y)(12+y)=$ the excess.

But $49=$ the excess.

Therefore, $144-144+y^2=49.$

$\therefore y=\pm 7.$

The numbers are 5, 12, 19; or 19, 12, 5.

Exercise 107.

$$l=a+(n-1)d;\quad s=\frac{n}{2}(a+l)=\frac{n}{2}[2a+(n-1)d].$$

Find l and s, if

1. $a=7,\ d=4,\ n=13.$
2. $a=5,\ d=3,\ n=12.$
3. $a=\frac{1}{2},\ d=6,\ n=30.$
4. $a=63,\ d=-5,\ n=8.$
5. $a=\frac{4}{7},\ d=\frac{5}{7},\ n=15.$
6. $a=3n,\ d=2n,\ n=36.$

Find d and s, if

7. $a = 2,\ l = 134,\ n = 13.$

8. $a = 0,\ l = 200,\ n = 51.$

9. $a = 169,\ l = 8,\ n = 24.$

10. $a = 5a^3,\ l = 145a^3,\ n = 21.$

Insert eight arithmetical means between

11. 13 and 76.

12. 1 and 2.

13. $\frac{1}{2}$ and $\frac{3}{4}$.

14. 47 and 2.

Find a and s, if

15. $d = 7,\ l = 149,\ n = 22.$

16. $d = 21,\ l = 242,\ n = 12.$

Find n and s, if

17. $a = 17,\ l = 350,\ d = 9.$

18. $a = 34,\ l = 10,\ d = -2.$

Find d and n, if

19. $a = 1\frac{1}{2},\ l = 54,\ s = 999.$

20. $a = 2,\ l = 87,\ s = 801.$

Find d and l, if

21. $a = 10,\ n = 14,\ s = 1050.$

22. $a = 1,\ n = 20,\ s = 305.$

Find a and d, if

23. $l = 21,\ n = 7,\ s = 105.$

24. $l = 105,\ n = 16,\ s = 840.$

Find a and l, if

25. $n = 21,\ d = 4,\ s = 1197.$

26. $n = 25,\ s = -75,\ d = \frac{1}{2}.$

Find n and l, if

27. $s = 636,\ a = 9,\ d = 8.$

28. $s = 798,\ a = 18,\ d = 6.$

Find a and n, if

29. $s = 623,\ d = 5,\ l = 77.$

30. $s = 1008,\ d = 4,\ l = 88.$

Find the arithmetical series in which

31. The 15th term is 25, and the 29th term 46.

How many terms must be taken of

32. The series $-16, -15, -14, \ldots\ldots$ to make -100?

33. The series $20, 18\frac{3}{4}, 17\frac{1}{2}, \ldots\ldots$ to make $162\frac{1}{2}$?

34. The sum of three numbers in arithmetical progression is 9, and the sum of their squares is 29. Find the numbers.

35. The sum of three numbers in arithmetical progression is 12, and their product is 60. Find the numbers.

Geometrical Progression.

347. A series is called a **geometrical series** or a **geometrical progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier*.

The general representative of such a series will be

$$a, \; ar, \; ar^2, \; ar^3, \; ar^4 \ldots\ldots,$$

in which a is the first term and r the constant multiplier or ratio.

The terms increase or decrease in numerical magnitude according as r is numerically greater than or numerically less than unity.

348. The nth Term. Since the exponent of r increases by one for each succeeding term after the first, the exponent will always be one less than the number of the term, so that the nth term is ar^{n-1}.

If the nth term is represented by l, we have

$$l = ar^{n-1}. \qquad \text{I.}$$

349. Sum of the Series. If l represent the nth term, a the first term, n the number of terms, r the common ratio, and s the sum of n terms, then

$$s = a + ar + ar^2 + \cdots\cdots ar^{n-1}. \tag{1}$$

Multiply by r,

$$rs = ar + ar^2 + ar^3 + \cdots\cdots ar^{n-1} + ar^n. \tag{2}$$

Subtracting the first equation from the second,

$$rs - s = ar^n - a,$$

or

$$(r-1)s = a(r^n - 1).$$

$$\therefore\ s = \frac{a(r^n - 1)}{r - 1}. \qquad \text{II.}$$

Since $l = ar^{n-1}$, $rl = ar^n$, and II. may be written

$$s = \frac{rl - a}{r - 1}. \qquad \text{III.}$$

350. From the two equations I. and II., or the two equations I. and III., any *two* of the five numbers a, r, l, n, s, may be found when the other *three* are given.

(1) The first term of a geometrical series is 3, the last term 192, and the sum of the series 381. Find the number of terms and the ratio.

From I., $\quad 192 = 3\,r^{n-1}, \quad$ (1)

From III., $\quad 381 = \dfrac{192\,r - 3}{r - 1}, \quad$ (2)

From (2), $\quad r = 2.$

Substituting in (1), $\quad 2^{n-1} = 64.$

$\therefore\ n = 7.$

The series is 3, 6, 12, 24, 48, 96, 192.

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(2) Find l when r, n, s are given.

From I., $$a = \frac{l}{r^{n-1}}.$$

Substituting in III., $$s = \frac{rl - \dfrac{l}{r^{n-1}}}{r-1},$$

$$(r-1)s = \frac{(r^n - 1)}{r^{n-1}}l.$$

$$\therefore\ l = \frac{(r-1)r^{n-1}s}{r^n - 1}.$$

351. The **geometrical mean** between two numbers is the number which stands between them, and makes with them a geometrical series.

If a and b denote two numbers, and G their geometrical mean, then a, G, b are in geometrical progression, and by the definition of a geometrical series, § 347,

$$\frac{G}{a} = r, \text{ and } \frac{b}{G} = r.$$

$$\therefore\ \frac{G}{a} = \frac{b}{G}.$$

$$\therefore\ G = \sqrt{ab}.$$

352. Sometimes it is required to insert several geometrical means between two numbers.

Ex. Insert three geometrical means between 3 and 48.

Here the whole number of terms is five; 3 is the first term and 48 the fifth.

By I., $$48 = 3r^4,$$

$$r^4 = 16.$$

$$r = \pm 2.$$

The series is one of the following:

3, [6, 12, 24], 48;

3, [−6, 12, −24,] 48.

The terms in brackets are the means required.

353. Infinite Geometrical Series. When r is less than 1, the successive terms become numerically smaller and smaller; by taking n large enough we can make the nth term, ar^{n-1}, as small as we please, although we cannot make it absolutely zero.

The sum of n terms, $\frac{rl-a}{r-1}$, may be written $\frac{a}{1-r}-\frac{rl}{1-r}$; this sum differs from $\frac{a}{1-r}$ by the fraction $\frac{rl}{1-r}$; by taking enough terms we can make l, and consequently the fraction $\frac{rl}{1-r}$, as small as we please; the greater the number of terms taken the nearer does their sum approach $\frac{a}{1-r}$. Hence $\frac{a}{1-r}$ is called the sum of an infinite number of terms of the series.

(1) Find the sum of the infinite series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots\cdots$

Here, $a=1$, $r=-\frac{1}{2}$.

The sum of the series is $\frac{1}{1+\frac{1}{2}}$ or $\frac{2}{3}$. *Ans.*

We find for the sum of n terms $\frac{2}{3}+\frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}$; this sum evidently approaches $\frac{2}{3}$ as n increases.

(2) Find the value of the recurring decimal

$$0.12135135\ldots\ldots$$

Consider first the part that recurs; this may be written $\frac{135}{100000}+\frac{135}{100000000}+\cdots\cdots$, and the sum of this series is $\frac{\frac{135}{100000}}{1-\frac{1}{1000}}$, which reduces to $\frac{1}{740}$. Adding 0.12, the part that does not recur, we obtain for the value of the decimal $\frac{12}{100}+\frac{1}{740}$, or $\frac{449}{3700}$. *Ans.*

Exercise 108.

$$l = ar^{n-1}; \; s = \frac{a(r^n - 1)}{r - 1} = \frac{rl - a}{r - 1}.$$

Find l and s, if

1. $a = 4$, $r = 2$, $n = 7$.

2. $a = 9$, $r = \frac{1}{3}$, $n = 11$.

Find r and s, if

3. $a = 1$, $n = 4$, $l = 64$.

4. $a = 7$, $n = 8$, $l = 896$.

5. Insert 1 geometrical mean between 14 and 686.

6. Insert 3 geometrical means between 31 and 496.

7. Find a and s, if $l = 128$, $r = 2$, $n = 7$.

8. Find s and n, if $a = 9$, $l = 2304$, $r = 2$.

9. Find r and n, if $a = 2$, $l = 1458$, $s = 2186$.

10. If the 5th term is $\frac{8}{9}$ and the 7th term $\frac{32}{81}$, find the series.

11. Find three numbers in geometrical progression whose sum is 14, and the sum of whose squares is 84.

Sum to infinity:

12. $1, \frac{1}{2}, \frac{1}{4}, \dots\dots$

13. $8, 2, \frac{1}{2}, \dots\dots$

14. $2, \frac{2}{7}, \frac{2}{49}, \dots\dots$

15. $5, 3, \frac{9}{5}, \dots\dots$

16. $1, \frac{1}{4}, \frac{1}{16}, \dots\dots$

17. $3, -2, \frac{4}{3}, \dots\dots$

Find the value of

18. $0.1\dot{6}$.

19. $0.3\dot{7}\dot{8}$.

20. $0.\dot{8}\dot{6}$.

21. $0.5\dot{4}$.

22. $0.7\dot{3}\dot{6}$.

23. $0.\dot{3}6\dot{3}$.

HARMONICAL PROGRESSION.

354. A series is called a **harmonical series**, or a **harmonical progression**, when the reciprocals of its terms form an arithmetical series.

The general representative of such a series will be

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots\ldots,\ \frac{1}{a+(n-1)d}.$$

Questions relating to harmonical series are best solved by writing the reciprocals of its terms, and thus forming an arithmetical series.

355. If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore\ H=\frac{2ab}{a+b}.$$

356. Sometimes it is required to insert several harmonical means between two numbers.

Ex. Insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{19}{72}, \frac{14}{72}, \frac{9}{72}$; therefore, the harmonical means are $\frac{72}{19}, \frac{72}{14}, \frac{72}{9}$; or $3\frac{15}{19}$, $5\frac{1}{7}$, 8.

357. Since $A=\frac{a+b}{2}$, and $G=\sqrt{ab}$, and $H=\frac{2ab}{a+b}$,

$$H=\frac{G^2}{A},\ \text{and}\ \ G=\sqrt{AH}.$$

That is, the geometrical mean between two numbers is also the geometrical mean between the arithmetical and harmonical means of the numbers.

Exercise 109.

1. If a, b, c are in harmonical progression, show that $a-b:b-c=a:c$.

2. Show that if the terms of a harmonical series are all multiplied by the same number, the products will form a harmonical progression.

3. The second term of a harmonical series is 2, and the fourth term 6. Find the series.

4. Insert the harmonical mean between 2 and 3.

5. Insert 2 harmonical means between 1 and $\frac{1}{4}$.

6. Insert 5 harmonical means between 1 and $\frac{1}{7}$.

7. The first term of a harmonical progression is 1, and the third term $\frac{1}{3}$. Find the 8th term.

8. The first term of a harmonical progression is 1, and the sum of the first three terms is $1\frac{5}{6}$. Find the series.

9. If a is the arithmetical mean between b and c, and b the geometrical mean between a and c, show that c is the harmonical mean between a and b.

10. The arithmetical mean between two numbers exceeds the harmonical mean by 1, and twice the square of the arithmetical mean exceeds the sum of the squares of the harmonical and geometrical means by 11. Find the numbers.

CHAPTER XXIII.

PROPERTIES OF SERIES.

358. Convergent and Divergent Series. By performing the indicated division, we obtain from the fraction $\frac{1}{1-x}$ the infinite series $1+x+x^2+x^3+\cdots\cdots$. This series, however, is not equal to the fraction for all values of x.

359. If x is numerically less than 1, the series is equal to the fraction. In this case we can obtain an approximate value for the sum of the series by taking the sum of a number of terms; the greater the number of terms taken, the nearer will this approximate sum approach the value of the fraction. The approximate sum will never be exactly equal to the fraction, however great the number of terms taken; but by taking enough terms, it can be made to differ from the fraction as little as we please.

Thus, if $x=\frac{1}{2}$, the value of the fraction is 2, and the series is

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots$$

The sum of four terms of this series is $1\frac{7}{8}$; the sum of five terms, $1\frac{15}{16}$; the sum of six terms, $1\frac{31}{32}$; and so on. The successive approximate sums approach, but never reach, the finite value 2.

360. An infinite series is said to be **convergent** when the sum of the terms, as the number of terms is indefinitely increased, *approaches some fixed finite value;* this finite value is called the **sum** of the series.

361. In the series $1 + x + x^2 + x^3 + \cdots$ suppose x numerically greater than 1. In this case, the greater the number of terms taken, the greater will their sum be; by taking enough terms, we can make their sum as large as we please. The fraction, on the other hand, has a definite value. Hence, when x is numerically greater than 1, the series is *not* equal to the fraction.

Thus, if $x = 2$, the value of the fraction is -1, and the series is

$$1 + 2 + 4 + 8 + \cdots$$

The greater the number of terms taken, the larger the sum. Evidently the fraction and the series are not equal.

362. In the same series suppose $x = 1$. In this case the fraction is $\frac{1}{1-1} = \frac{1}{0}$, and the series $1 + 1 + 1 + 1 + \cdots$ The more terms we take, the greater will the sum of the series be, and the sum of the series does *not* approach a *fixed finite value.*

If x, however, is not exactly 1, but is a little less than 1, the value of the fraction $\frac{1}{1-x}$ will be very great, and the fraction will be equal to the series.

Suppose $x = -1$. In this case the fraction is $\frac{1}{1+1} = \frac{1}{2}$, and the series $1 - 1 + 1 - 1 + \cdots$ If we take an even number of terms, their sum is 0; if an odd number, their sum is 1. Hence the fraction is *not* equal to the series.

363. A series is said to be **divergent** when the sum of the terms, as the number of terms is indefinitely increased, either increases without end, or oscillates in value *without approaching any fixed finite value.*

No reasoning can be based on a divergent series; hence, in using an infinite series it is necessary to make such restrictions as will cause the series to be convergent. Thus, we can use the infinite series $1 + x + x^2 + x^3 + \cdots\cdots$ when, and only when, x lies between $+1$ and -1.

364. Identical Series. *If two series, arranged by powers of x, are equal for all values of x that make both series convergent, the corresponding coefficients are equal each to each.*

For, if $A + Bx + Cx^2 + \cdots\cdots = A' + B'x + C'x^2 + \cdots\cdots,$
by transposition,

$$A - A' = (B' - B)x + (C' - C)x^2 + \cdots\cdots$$

Now, by taking x sufficiently small, the right side of this equation can be made *less* than any assigned value whatever, and therefore less than $A - A'$, if $A - A'$ have any value whatever. Hence, $A - A'$ cannot have any value.

Therefore,

$$A - A' = 0 \text{ or } A = A'.$$

Hence, $Bx + Cx^2 + Dx^3 + \cdots\cdots = B'x + C'x^2 + D'x^3 + \cdots\cdots$

or $\qquad (B - B')x = (C' - C)x^2 + (D' - D)x^3 + \cdots\cdots;$

by dividing by x,

$$B - B' = (C' - C)x + (D' - D)x^2 + \cdots\cdots;$$

and, by the same proof as for $A - A'$,

$$B - B' = 0 \text{ or } B = B'.$$

In like manner,

$$C = C', \quad D = D', \text{ and so on.}$$

Hence, the equation

$$A + Bx + Cx^2 + \cdots\cdots = A' + B'x + C'x^2 + \cdots\cdots,$$

if true for all finite values of x, is an identical equation; that is, *the coefficients of like powers of x are the same.*

365. Indeterminate Coefficients.

Ex. Expand $\dfrac{2+3x}{1+x+x^2}$ in ascending powers of x.

Assume $$\frac{2+3x}{1+x+x^2}=A+Bx+Cx^2+Dx^3 \cdots\cdots;$$

then, by clearing of fractions,

$$\begin{aligned} 2+3x = A + Bx + Cx^2 + Dx^3 + \cdots\cdots \\ + Ax + Bx^2 + Cx^3 + \cdots\cdots \\ + Ax^2 + Bx^3 + \cdots\cdots \end{aligned}$$

$$\therefore\ 2+3x=A+(B+A)x+(C+B+A)x^2+(D+C+B)x^3+\cdots\cdots$$

By § 364, $A=2,\ B+A=3,\ C+B+A=0,\ D+C+B=0;$

whence $B=1,\ C=-3,\ D=2;$ and so on.

$$\therefore\ \frac{2+3x}{1+x+x^2}=2+x-3x^2+2x^3+\cdots\cdots$$

The series is of course equal to the fraction for only such values of x as make the series convergent.

NOTE. In employing the method of Indeterminate Coefficients, the form of the given expression must determine what powers of the variable x must be assumed. It is necessary and sufficient that the assumed equation, when simplified, shall have in the right member all the powers of x that are found in the left member.

If any powers of x occur in the *right* member that are not in the *left* member, the coefficients of these powers in the right member will vanish, so that in this case the method still applies; but if any powers of x occur in the *left* member that are not in the *right* member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x; and this leads to absurd results. Thus, if it were assumed that

$$\frac{2+3x}{1+x+x^2}=Ax+Bx^2+Cx^3+\cdots\cdots,$$

there would be in the simplified equation no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x, 2, which is $2x^0$, would have to be put equal to $0x^0$; that is, $2=0$, an absurdity.

Exercise 110.

Expand to four terms:

1. $\dfrac{1}{1+2x}$.

2. $\dfrac{1}{2-3x}$.

3. $\dfrac{1+x}{2+3x}$.

4. $\dfrac{1-x}{1+x+x^2}$.

5. $\dfrac{5-2x}{1+x-x^2}$.

6. $\dfrac{2-3x}{1-2x+3x^2}$.

7. $\dfrac{3+x}{1-x-x^2}$.

8. $\dfrac{1+x}{1+x+x^2}$.

9. $\dfrac{1-8x}{1-x-6x^2}$.

Expand to five terms:

10. $\dfrac{4}{2+x}$.

11. $\dfrac{2-x}{3+x}$.

12. $\dfrac{5-2x}{1+3x-x^2}$.

13. $\dfrac{x^2-x+1}{x(x-2)}$.

14. $\dfrac{3x-2}{x(x-1)^2}$.

15. $\dfrac{x^2-x+1}{(x-1)(x^2+1)}$.

366. Partial Fractions. To resolve a fraction into *partial fractions* is to express it as the sum of a number of fractions of which the respective denominators are the factors of the denominator of the given fraction. This process is the reverse of the process of *adding* fractions which have different denominators.

Resolution into partial fractions may be easily accomplished by the use of **indeterminate coefficients** and the theorem of § 364.

In decomposing a given fraction into its simplest partial fractions, it is important to determine what form the assumed fractions must have.

Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator; moreover, all the factors of the given denominator must be taken as denominators of the assumed fractions.

Since the required partial fractions are to be in their simplest form incapable of further decomposition, the numerator of each required fraction must be assumed with reference to this condition. Thus, if the denominator is x^n or $(x \pm a)^n$, the assumed fraction must be of the form $\frac{A}{x^n}$ or $\frac{A}{(x \pm a)^n}$; for if it had the form $\frac{Ax+B}{x^n}$ or $\frac{Ax+B}{(x \pm a)^n}$, it could be decomposed into two fractions, and the partial fractions would not be in the simplest form possible.

When all the monomial factors, and all the binomial factors, of the form $x \pm a$, have been removed from the denominator of the given expression, there may remain quadratic factors which cannot be further resolved; and the numerators corresponding to these quadratic factors may each contain the first power of x, so that the assumed fractions must have either the form $\frac{Ax+B}{x^2 \pm ax + b}$, or the form $\frac{Ax+B}{x^2+b}$.

(1) Resolve $\frac{3}{x^3+1}$ into partial fractions.

Since $x^3+1=(x+1)(x^2-x+1)$, the denominators will be $x+1$ and x^2-x+1.

Assume $$\frac{3}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1};$$

then $$3=A(x^2-x+1)+(Bx+C)(x+1)$$
$$=(A+B)x^2+(B+C-A)x+(A+C);$$

whence, $$3=A+C,\ B+C-A=0,\ A+B=0,$$

and $$A=1,\ B=-1,\ C=2.$$

Therefore, $$\frac{3}{x^3+1}=\frac{1}{x+1}-\frac{x-2}{x^2-x+1}.$$

(2) Resolve $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2}$ into partial fractions.

The denominators may be x, x^2, $x+1$, $(x+1)^2$.

Assume $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$.

$$\therefore\ 4x^3 - x^2 - 3x - 2 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$
$$= (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B;$$

whence,
$$A + C = 4,$$
$$2A + B + C + D = -1,$$
$$A + 2B = -3,$$
$$B = -2;$$

and
$$\therefore\ B = -2,\quad A = 1,\quad C = 3,\quad D = -4.$$

Therefore, $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2} = \dfrac{1}{x} - \dfrac{2}{x^2} + \dfrac{3}{x+1} - \dfrac{4}{(x+1)^2}$.

Exercise 111.

Resolve into partial fractions:

1. $\dfrac{7x+1}{(x+4)(x-5)}$.

2. $\dfrac{7x-1}{(1-2x)(1-3x)}$.

3. $\dfrac{5x-1}{(2x-1)(x-5)}$.

4. $\dfrac{x-2}{(x-5)(x+2)}$.

5. $\dfrac{3}{x^3-1}$.

6. $\dfrac{x^2-x-3}{x(x^2-4)}$.

7. $\dfrac{3x^2-4}{x^2(x+5)}$.

8. $\dfrac{7x^2-x}{(x-1)^2(x+2)}$.

9. $\dfrac{2x^2-7x+1}{x^3-1}$.

10. $\dfrac{7x-1}{(6x+1)(x-1)}$.

11. $\dfrac{x^2-3}{(x^2+1)(x+2)}$.

12. $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$.

CHAPTER XXIV.

BINOMIAL THEOREM.

367. Binomial Theorem, Positive Integral Exponent. By successive multiplication we obtain the following identities:

$(a+b)^2 = a^2 + 2ab + b^2$;

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$;

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

The expressions on the right may be written in a form better adapted to show the law of their formation:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2;$$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3;$$

$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4.$$

NOTE. The dot between the Arabic figures means the same as the sign $\times$.

368. Let n represent the exponent of $(a+b)$ in any one of these identities; then, in the expressions on the right, we observe that the following laws hold true:

I. The number of terms is $n+1$.

II. The first term is a^n, and the exponent of a is one less in each succeeding term.

The first power of b occurs in the second term, the second power in the third term, and the exponent of b is one greater in each succeeding term.

The sum of the exponents of a and b in any term is n.

III. The coefficient of the first term is 1; of the second term, n; of the third term, $\frac{n(n-1)}{1\cdot 2}$; and so on.

369. Consider the coefficient of any term; the number of factors in the numerator is the same as the number of factors in the denominator, and the number of factors in each is the same as the exponent of b in that term; this exponent is one less than the number of the term.

370. Proof of the Theorem. That the laws of § 368 hold true when the exponent is *any* positive integer, is shown as follows:

We know that the laws hold for the fourth power; suppose, for the moment, that they hold for the kth power.

We shall then have

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1\cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3 + \cdots\cdots \quad (1)$$

Multiply both members of (1) by $a+b$; the result is

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots\cdots \quad (2)$$

In (1) put $k+1$ for k; this gives

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1} + (k+1)a^kb + \frac{(k+1)(k+1-1)}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)(k+1-1)(k+1-2)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots\cdots \\ &= a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots\cdots \quad (3)\end{aligned}$$

Equation (3) is seen to be the same as equation (2).

Hence (1) holds when we put $k+1$ for k; that is, if the laws of § 368 hold for the kth power, they must hold for the $(k+1)$th power.

But the laws hold for the fourth power; therefore they must hold for the fifth power.

Holding for the fifth power, they must hold for the sixth power; and so on for any positive integral power.

Therefore they must hold for the nth power, if n is a positive integer; and we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots\cdots \quad \text{(A)}$$

NOTE. The above proof is an example of a proof by *mathematical induction.*

371. This formula is known as the **binomial theorem.**

The expression on the right is known as the **expansion** of $(a+b)^n$; this expansion is a *finite series* when n is a positive integer. That the series is finite may be seen as follows:

In writing out the successive coefficients we shall finally arrive at a coefficient which contains the factor $n-n$; the corresponding term will vanish. The coefficients of all the succeeding terms likewise contain the factor $n-n$, and therefore all these terms will vanish.

372. If a and b are interchanged, the identity (**A**) may be written

$$(a+b)^n = (b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{1 \cdot 2} b^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} b^{n-3}a^3 + \cdots\cdots$$

This last expansion is the expansion of (A) written in reverse order. Comparing the two expansions, we see that: the coefficient of the last term is the same as the coefficient of the first term; the coefficient of the last term but one is the same as the coefficient of the first term but one; and so on.

In general, the coefficient of the rth term from the end is the same as the coefficient of the rth term from the beginning. In writing out an expansion by the binomial theorem, after arriving at the middle term, we can shorten the work by observing that the remaining coefficients are those already found, taken in reverse order.

373. If b is negative, the terms which involve even powers of b will be positive, and those which involve odd powers of b negative. Hence,

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots \qquad \text{(B)}$$

Also, putting 1 for a and x for b, in (A) and (B),

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \qquad \text{(C)}$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \qquad \text{(D)}$$

374. Examples:

(1) Expand $(1+2x)^5$.

In (C) put $2x$ for x and 5 for n. The result is

$$(1+2x)^5 = 1 + 5(2x) + \frac{5\cdot 4}{1\cdot 2}4x^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}8x^3$$
$$+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}16x^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}32x^5$$
$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

(2) Expand to three terms $\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6$.

Put a for $\frac{1}{x}$, and b for $\frac{2x^2}{3}$; then, by (B),

$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - \cdots\cdots$$

Replacing a and b by their values,

$$\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6 = \left(\frac{1}{x}\right)^6 - 6\left(\frac{1}{x}\right)^5\left(\frac{2x^2}{3}\right) + 15\left(\frac{1}{x}\right)^4\left(\frac{2x^2}{3}\right)^2 - \cdots\cdots$$
$$= \frac{1}{x^6} - \frac{4}{x^3} + \frac{20}{3} - \cdots\cdots$$

375. Any Required Term. From (A) it is evident (§ 372) that the $(r+1)$th term in the expansion of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots \text{ to } r \text{ factors}}{1\times 2\times 3\ldots\ldots r}a^{n-r}b^r.$$

NOTE. In finding the coefficient of the $(r+1)$th term, write the series of factors $1\times 2\times 3 \ldots\ldots r$ for the denominator of the coefficient, then write over this series the factors $n(n-1)$, $(n-2)$, etc., writing just as many factors in the numerator as there are in the denominator.

The $(r+1)$th term in the expansion of $(a-b)^n$ is the same as the above if r is even, and the negative of the above if r is odd.

Ex. Find the eighth term of $\left(4-\frac{x^2}{2}\right)^{10}$.

Here $\qquad a=4,\ b=\frac{x^2}{2},\ n=10,\ r=7.$

The term required is $\frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(4)^3\left(-\frac{x^2}{2}\right)^7$,

which reduces to $\qquad -60x^{14}$.

376. A trinomial may be expanded by the binomial theorem as follows:

Expand $(1+2x-x^2)^3$.

Put $\qquad 2x-x^2=z;$

then $\qquad (1+z)^3=1+3z+3z^2+z^3.$

Replace z with $2x-x^2$.

$$\therefore (1+2x-x^2)^3=1+3(2x-x^2)+3(2x-x^2)^2+(2x-x^2)^3$$
$$=1+6x+9x^2-4x^3-9x^4+6x^5-x^6.$$

Exercise 112.

Expand:

1. $(a+b)^7$.
2. $(x-2)^5$.
3. $(3x-2y)^4$.
4. $\left(\frac{x}{y}-\frac{y}{x}\right)^6$.
5. $(4+3y)^4$.
6. $(a^3+b)^7$.
7. $(m^2+n^3)^8$.
8. $(a-b^3)^7$.
9. $(a^{\frac{1}{2}}+b^{\frac{2}{3}})^5$.
10. $(a^{-1}+b^{-2})^3$.
11. $(m^{-\frac{1}{2}}+n^2)^4$.
12. $(x^{-2}+z^{\frac{2}{3}})^6$.
13. $(2x^2+y^{\frac{1}{2}})^5$.
14. $(a^{\frac{1}{2}}-c^{\frac{2}{3}})^4$.
15. $(2a^2-\frac{1}{2}\sqrt{a})^5$.
16. $\left(\frac{a^2}{b}-\frac{\sqrt{b}}{2a}\right)^7$.
17. $\left(\frac{2a}{b^2}+\frac{1}{2}b\sqrt{a}\right)^3$.
18. $(2x^2y^{-1}-y\sqrt{y})^4$.
19. $\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}+\frac{\sqrt[3]{b}}{a}\right)^7$.
20. $\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}-\frac{1}{2}\sqrt{\frac{b}{a}}\right)^5$.
21. $(2ab^{-2}-ba^{\frac{1}{2}})^7$.

22. $\left(\frac{a}{b}\sqrt{\frac{c}{d}}-\frac{c}{d}\right)^6.$ 23. $\left(\frac{a}{b}\sqrt{\frac{b}{a}}-\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^5.$

24. Find the fourth term of $(2x-3y)^7$.

25. Find the ninety-seventh term of $(2a-b)^{100}$.

NOTE. As the expansion has 101 terms, the ninety-seventh term from the beginning is the fifth term from the end.

26. Find the eighth term of $(3x-y)^{11}$.

27. Find the tenth term of $(2a^2-\frac{1}{2}a)^{20}$.

28. Find the fifth term of $(a-2\sqrt{b})^{25}$.

29. Find the eleventh term of $(2-a)^{16}$.

30. Find the fifteenth term of $(x+y)^{20}$.

31. Find the fourth term of $(3-2x^2)^9$.

32. Find the twelfth term of $(a^2-a\sqrt{x})^{17}$.

33. Find the seventh term of $(y^2-1)^{38}$.

34. Find the fifth term of $(\frac{1}{2}a-b\sqrt{b})^{21}$.

35. Find the fourth term of $(\sqrt{a}-\sqrt[3]{b^2})^{20}$.

36. Find the third term of $(\sqrt{a}-\sqrt{-b})^7$.

37. Find the sixth term of $(\sqrt[3]{a^2}-\sqrt{-1})^9$.

38. Find the eighth term of $(\sqrt{\frac{2}{3}a}+\sqrt{\frac{3}{4}x})^{20}$.

39. Find the ninth term of $(x\sqrt{-1}+y\sqrt{-1})^{16}$.

40. Find the fifth term of $\left(a^3b-\frac{3b^{-2}}{\sqrt{a^5}}\right)^{31}$.

41. Find the seventh term of $(x+x^{-1})^{2n}$.

377. Binomial Theorem, Any Exponent. We have seen (§ 370) that when n is a positive integer we have the identity

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

We proceed to the case of fractional and negative exponents.

I. Suppose n is a positive fraction, $\frac{p}{q}$. We may assume that

$$(1+x)^p = (A + Bx + Cx^2 + Dx^3 + \cdots\cdots)^q, \qquad (1)$$

provided x be so taken that the series

$$A + Bx + Cx^2 + Dx^3 + \cdots\cdots$$

is convergent (§ 360).

That this assumption is allowable may be seen as follows:

Expand both members of (1). We obtain

$$1 + px + \frac{p(p-1)}{1\cdot 2}x^2 + \frac{p(p-1)(p-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots,$$

and $$A^q + qA^{q-1}Bx + \left[\frac{q(q-1)}{1\cdot 2}A^{q-2}B^2 + qA^{q-1}C\right]x^2 + \cdots\cdots$$

In the first k coefficients of the second series there enter only the first k of the coefficients $A, B, C, D, \ldots\ldots$ If, then, we equate the coefficients of corresponding terms in the two series (§ 364) as far as the kth term, *we shall have just k equations to find k unknown numbers $A, B, C, D, \ldots\ldots$* Hence the assumption made in (1) is allowable.

Comparing the two first terms and the two second terms, we obtain

$$A^q = 1, \quad \therefore\ A = 1;$$

$$qA^{q-1}B = p, \text{ or } qB = p, \quad \therefore\ B = \frac{p}{q}.$$

Extracting the qth root of both members of (1), we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + Cx^2 + Dx^3 + \cdots\cdots, \qquad (2)$$

where x is to be so taken that the series on the right is convergent.

II. Suppose n is a negative number, integral or fractional. Let $n = -m$, so that m is positive; then

$$(1+x)^n = (1+x)^{-m} = \frac{1}{(1+x)^m}.$$

From (2), whether m is integral or fractional, we may assume

$$\frac{1}{(1+x)^m} = \frac{1}{1 + mx + cx^2 + dx^3 + \cdots\cdots}.$$

By actual division this gives an equation in the form

$$(1+x)^{-m} = 1 - mx + Cx^2 + Dx^3 + \cdots\cdots \qquad (3)$$

378. It appears from (2) and (3) § 377 that whether n be integral or fractional, positive or negative, we may assume

$$(1+x)^n = 1 + nx + Cx^2 + Dx^3 + \cdots\cdots,$$

provided the series on the right is convergent.

Squaring both members,

$$\begin{aligned}(1+2x+x^2)^n = 1 + 2nx &+ 2Cx^2 + 2Dx^3 + \cdots\cdots \qquad (1)\\ &+ n^2x^2 \;\; + 2nCx^3 + \cdots\cdots\end{aligned}$$

Also, since

$$(1+y)^n = 1 + ny + Cy^2 + Dy^3 + \cdots\cdots,$$

we have, putting $2x + x^2$ for y,

$$\begin{aligned}(1+2x+x^2)^n &= 1 + n(2x+x^2) + C(2x+x^2)^2 \\ &\qquad + D(2x+x^2)^3 \cdots\cdots \\ &= 1 + 2nx + nx^2 \;\; + 4Cx^3 + \cdots\cdots \\ &\qquad\qquad + 4Cx^2 + 8Dx^3 + \cdots\cdots \qquad (2)\end{aligned}$$

Comparing corresponding coefficients in (1) and (2),

$$n+4C=2C+n^2,$$
$$4C+8D=2D+2nC.$$

$$\therefore\ 2C=n^2-n, \text{ and } \quad C=\frac{n(n-1)}{1\cdot 2},$$

$$3D=(n-2)C, \text{ and } D=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3};$$

and so on.

Hence, whether n be integral or fractional, positive or negative, we have

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots\cdots,$$

provided, always, x be so taken that the series on the right is convergent.

The series obtained will be an infinite series unless n is a positive integer (§ 371).

379. If x is negative,

$$(1-x)^n=1-nx+\frac{n(n-1)}{1\cdot 2}x^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots\cdots$$

Also, if $x<a$,

$$(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$$
$$=a^n\left[1+n\frac{x}{a}+\frac{n(n-1)}{1\cdot 2}\frac{x^2}{a^2}+\cdots\cdots\right]$$
$$=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\cdots\cdots;$$

if $x>a$,

$$(a+x)^n=(x+a)^n=x^n\left(1+\frac{a}{x}\right)^n$$
$$=x^n\left[1+n\frac{a}{x}+\frac{n(n-1)}{1\cdot 2}\frac{a^2}{x^2}+\cdots\cdots\right]$$
$$=x^n+nax^{n-1}+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}+\cdots\cdots$$

380. Examples.

(1) Expand $(1+x)^{\frac{1}{3}}$.

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

$$= 1 + \tfrac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots\cdots$$

The above equation is only true for those values of x which make the series convergent.

(2) Expand $\dfrac{1}{\sqrt[4]{1-x}}$.

$$\frac{1}{\sqrt[4]{1-x}} = (1-x)^{-\frac{1}{4}}$$

$$= 1 - (-\tfrac{1}{4})x + \frac{-\frac{1}{4}\cdot -\frac{5}{4}}{1\cdot 2}x^2 - \frac{-\frac{1}{4}\cdot -\frac{5}{4}\cdot -\frac{9}{4}}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

$$= 1 + \tfrac{1}{4}x + \frac{1\cdot 5}{4\cdot 8}x^2 + \frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}x^3 + \cdots\cdots$$

if x is so taken that the series is convergent.

A root may often be extracted by means of an expansion.

(3) Extract the cube root of 344 to six decimal places.

$$344 = 343\left(1+\frac{1}{343}\right) = 7^3\left(1+\frac{1}{343}\right).$$

$$\therefore\ \sqrt[3]{344} = 7\left(1+\frac{1}{343}\right)^{\frac{1}{3}},$$

$$= 7\left[1 + \frac{1}{3}\left(\frac{1}{343}\right) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}\left(\frac{1}{343}\right)^2 + \cdots\cdots\right]$$

$$= 7(1 + 0.000971815 - 0.000000944 + \cdots\cdots),$$

$$= 7.006796.$$

(4) Find the eighth term of $\left(x - \dfrac{3}{4\sqrt{x}}\right)^{-\frac{1}{2}}$.

Here $a = x,\quad b = \dfrac{3}{4\sqrt{x}} = \dfrac{3}{4x^{\frac{1}{2}}},\quad n = -\dfrac{1}{2},\quad r = 7.$

The term is $\dfrac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}\cdot -\frac{9}{2}\cdot -\frac{11}{2}\cdot -\frac{13}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^{-\frac{15}{2}}\left(-\dfrac{3}{4x^{\frac{1}{2}}}\right)^7,$

or $\dfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 3^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 4^7\cdot x^{11}}.$

Exercise 113.

Expand to four terms.

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{-2}$.
3. $(1+x)^{-\frac{1}{2}}$.
4. $(1-x)^{\frac{2}{3}}$.
5. $(1-x)^{-\frac{2}{3}}$.
6. $(1+x)^{\frac{5}{3}}$.
7. $(1+x)^{-\frac{5}{3}}$.
8. $(1+x)^{-3}$.
9. $(1+5x)^{-5}$.
10. $(1+5x)^{\frac{5}{3}}$.
11. $(2x+3y)^{\frac{3}{4}}$.
12. $(2x+3y)^{-\frac{3}{4}}$.
13. $\dfrac{1}{\sqrt[4]{a^2-x}}$.
14. $\dfrac{1}{\sqrt[5]{(a-x)^2}}$.
15. Find the fourth term of $\left(a-\dfrac{3}{2\sqrt{x}}\right)^{\frac{1}{2}}$.
16. Find the fifth term of $\dfrac{1}{\sqrt[3]{(a-2x)^2}}$.
17. Find the third term of $(4-7x)^{\frac{2}{7}}$.
18. Find the sixth term of $(a^2-2ax)^{\frac{5}{3}}$.
19. Find the fifth term of $(1-2x)^{-\frac{1}{2}}$.
20. Find the fifth term of $(1-x)^{-3}$.
21. Find the seventh term of $(1-x)^{\frac{1}{2}}$.
22. Find the third term of $(1+x)^{-\frac{1}{2n}}$.
23. Find the fourth term of $(1+x)^{-\frac{5}{3}}$.
24. Find the sixth term of $\left(2-\dfrac{1}{x}\right)^{-\frac{2}{7}}$.
25. Find the fifth term of $(2x-3y)^{-\frac{3}{4}}$.
26. Find the fourth term of $(1-5x)^{-\frac{5}{3}}$.

CHAPTER XXV.

LOGARITHMS.

381. If the natural numbers are regarded as powers of ten, the exponents of the powers are the **Common or Briggs Logarithms** of the numbers. If A and B denote natural numbers, a and b their logarithms, then $10^a = A$, $10^b = B$; or, written in logarithmic form, $\log A = a$, $\log B = b$.

382. The logarithm of a product is found by adding the logarithms of its factors.

For $$A \times B = 10^a \times 10^b = 10^{a+b}.$$

Therefore, $$\log (A \times B) = a + b = \log A + \log B.$$

383. The logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend.

For $$\frac{A}{B} = \frac{10^a}{10^b} = 10^{a-b}.$$

Therefore, $$\log \frac{A}{B} = a - b = \log A - \log B.$$

384. The logarithm of a power of a number is found by multiplying the logarithm of the number by the exponent of the power.

For $$A^n = (10^a)^n = 10^{an}.$$

Therefore, $$\log A^n = an = n \log A.$$

385. The logarithm of the root of a number is found by dividing the logarithm of the number by the index of the root.

For $$\sqrt[n]{A} = \sqrt[n]{10^a} = 10^{\frac{a}{n}}.$$

Therefore, $$\log \sqrt[n]{A} = \frac{a}{n} = \frac{\log A}{n}.$$

386. The logarithms of 1, 10, 100, etc., and of 0.1, 0.01, 0.001, etc., are integral numbers. The logarithms of all other numbers are fractions.

Since $10^0 = 1$, $\quad 10^{-1} (= \frac{1}{10}) = 0.1$,
$10^1 = 10$, $\quad 10^{-2} (= \frac{1}{100}) = 0.01$,
$10^2 = 100$, $\quad 10^{-3} (= \frac{1}{1000}) = 0.001$,
therefore $\log 1 = 0$, $\quad \log 0.1 = -1$,
$\log 10 = 1$, $\quad \log 0.01 = -2$,
$\log 100 = 2$, $\quad \log 0.001 = -3$.

Also, it is evident that the common logarithms of all numbers between

1 and 10 will be $0 +$ a fraction,
10 and 100 will be $1 +$ a fraction,
100 and 1000 will be $2 +$ a fraction,
1 and 0.1 will be $-1 +$ a fraction,
0.1 and 0.01 will be $-2 +$ a fraction,
0.01 and 0.001 will be $-3 +$ a fraction.

387. If the number is less than 1, the logarithm is negative (§ 386), but is written in such a form that the *fractional part* is always *positive.*

388. Every logarithm, therefore, consists of two parts: a positive or negative integral number, which is called the **characteristic,** and a *positive* proper fraction, which is called the **mantissa.** Thus, in the logarithm 3.5218, the integral number 3 is the characteristic, and the fraction .5218 the mantissa. In the logarithm $0.7825 - 2$, which is sometimes written $\overline{2}.7825$, the integral number -2 is the characteristic, and the fraction .7825 is the mantissa.

389. If the logarithm has a negative characteristic, it is customary to change its form by adding 10, or a multiple of 10, to the characteristic, and then indicating the subtraction of the same number from the result. Thus, the logarithm $\overline{2}.7825$ is changed to $8.7825-10$ by adding 10 to the characteristic and writing -10 after the result. The logarithm $\overline{13}.9273$ is changed to $7.9273-20$ by adding 20 to the characteristic and writing -20 after the result.

390. The following rules are derived from § 386:

Rule 1. If the number is *greater than* 1, make the *characteristic* of the logarithm *one unit less* than the number of figures on the left of the decimal point.

Rule 2. If the number is *less than* 1, make the characteristic of the logarithm *negative*, and *one unit more* than the number of zeros between the decimal point and the first significant figure of the given number.

Rule 3. If the characteristic of a given logarithm is *positive*, make the number of figures in the integral part of the corresponding number *one more* than the number of units in the characteristic.

Rule 4. If the characteristic is *negative*, make the number of zeros between the decimal point and the first significant figure of the corresponding number *one less* than the number of units in the characteristic.

Thus, the characteristic of log 7849.27 is 3; the characteristic of log 0.037 is $-2=8.0000-10$. If the characteristic is 4, the corresponding number has five figures in its integral part. If the characteristic is -3, that is, $7.0000-10$, the corresponding fraction has two zeros between the decimal point and the first significant figure.

391. The *mantissa* of the common logarithm of any integral number, or decimal fraction, depends only upon the digits of the number, and is unchanged so long as the *sequence of the digits* remains the same.

For changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its common logarithm, therefore, will be increased or diminished by the *exponent* of that power of 10; and since this exponent is *integral*, the *mantissa*, or decimal part of the logarithm, will be unaffected.

$$\text{Thus,}\quad 27196 = 10^{4.4345}, \qquad 2.7196 = 10^{0.4345},$$
$$2719.6 = 10^{3.4345}, \qquad 0.27196 = 10^{9.4345-10},$$
$$27.196 = 10^{1.4345}, \qquad 0.0027196 = 10^{7.4345-10}.$$

One advantage of using the number *ten* as the base of a system of logarithms consists in the fact that the *mantissa* depends only on the *sequence of digits*, and the *characteristic* on the *position of the decimal point.*

392. In simplifying the logarithm of a root the equal positive and negative numbers to be added to the logarithm should be such that the resulting negative number, when divided by the index of the root, gives a quotient of -10.

Thus, if the $\log 0.002^{\frac{1}{3}} = \frac{1}{3}$ of $(7.3010-10)$, the expression $\frac{1}{3}$ of $(7.3010-10)$ may be put in the form $\frac{1}{3}$ of $(27.3010-30)$, which is $9.1003-10$, since the addition of 20 to the 7, and of -20 to the -10, produces no change in the *value* of the logarithm.

Exercise 114.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the common logarithms of the following numbers by resolving the numbers into factors, and taking the sum of the logarithms of the factors:

1. log 6.	**5.** log 25.	**9.** log 0.021.	**13.** log 2.1.
2. log 15.	**6.** log 30.	**10.** log 0.35.	**14.** log 16.
3. log 21.	**7.** log 42.	**11.** log 0.0035.	**15.** log 0.056.
4. log 14.	**8.** log 420.	**12.** log 0.004.	**16.** log 0.63.

Find the common logarithms of the following:

17. 2^3.	**20.** 5^5.	**23.** $5^{\frac{1}{5}}$.	**26.** $7^{\frac{2}{7}}$.	**29.** $5^{\frac{7}{2}}$.
18. 5^2.	**21.** $2^{\frac{1}{3}}$.	**24.** $7^{\frac{1}{11}}$.	**27.** $5^{\frac{5}{3}}$.	**30.** $2^{\frac{11}{7}}$.
19. 7^4.	**22.** $5^{\frac{1}{2}}$.	**25.** $2^{\frac{3}{4}}$.	**28.** $3^{\frac{9}{11}}$.	**31.** $5^{\frac{3}{4}}$.

393. The logarithm of the reciprocal of a number is called the **cologarithm** of the number.

If A denote any number, then

$$\text{colog}\, A = \log\frac{1}{A} = \log 1 - \log A \ (\S\ 383) = -\log A \ (\S\ 386).$$

Hence, the cologarithm of a number is equal to the logarithm of the number with the minus sign prefixed, which sign affects the entire logarithm, both characteristic and mantissa.

In order to avoid a negative mantissa in the cologarithm, it is customary to substitute for $-\log A$ its equivalent $(10 - \log A) - 10$.

Hence, the cologarithm of a number is found by subtracting the logarithm of the number from 10, and then annexing -10 to the remainder.

The best way to perform the subtraction is to begin on the left and subtract each figure of $\log A$ from 9 until we reach the last significant figure, which must be subtracted from 10.

If $\log A$ is greater in absolute value than 10 and less than 20, then in order to avoid a negative mantissa, it is necessary to write $-\log A$ in the form $(20 - \log A) - 20$. So that, in this case, $\text{colog}\, A$ is found by subtracting $\log A$ from 20, and then annexing -20 to the remainder.

(1) Find the cologarithm of 4007.

$$\begin{array}{llr} & & 10 \qquad -10 \\ \text{Given:} & \log 4007 = & 3.6028 \qquad\quad \\ \text{Therefore,} & \text{colog}\, 4007 = & 6.3972 - 10 \end{array}$$

(2) Find the cologarithm of 103992000000.

$$\begin{array}{llr} & & 20 \qquad -20 \\ \text{Given:} & \log 103992000000 = & 11.0170 \qquad\quad \\ \text{Therefore,} & \text{colog}\, 103992000000 = & 8.9830 - 20 \end{array}$$

If the characteristic of $\log A$ is negative, then the subtrahend, -10 or -20, will vanish in finding the value of $\text{colog}\, A$.

(3) Find the cologarithm of 0.004007.

$$\begin{array}{llr} & & 10 \qquad -10 \\ \text{Given:} & \log 0.004007 = & 7.6028 - 10 \\ \text{Therefore,} & \text{colog}\, 0.004007 = & 2.3972 \qquad\quad \end{array}$$

By using cologarithms the inconvenience of subtracting the logarithm of a divisor is avoided. For dividing by a number is equivalent to multiplying by its reciprocal. Hence, instead of subtracting the logarithm of a divisor, its cologarithm may be added.

(4) Find the logarithm of $\frac{5}{0.002}$.

$$\log \frac{5}{0.002} = \log 5 + \text{colog}\ 0.002.$$

$$\begin{aligned} \log 5 &= 0.6990 \\ \text{colog}\ 0.002 &= \underline{2.6990} \\ \log \text{quotient} &= 3.3980 \end{aligned}$$

(5) Find the logarithm of $\frac{0.07}{2^3}$.

$$\log \frac{0.07}{2^3} = \log 0.07 + \text{colog}\ 2^3.$$

$$\begin{aligned} \log 0.07 &= 8.8451 - 10 \\ \text{colog}\ 2^3 = (10 - 3 \log 2) - 10 &= \underline{9.0970 - 10} \\ \log \text{quotient} &= 7.9421 - 10 \end{aligned}$$

Exercise 115.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$; $\log 11 = 1.0414$.

Find the logarithms of the following quotients:

1. $\frac{2}{7}$.	8. $\frac{5}{11}$.	15. $\frac{0.03}{44}$.	22. $\frac{8^5}{3^7}$.
2. $\frac{4}{5}$.	9. $\frac{7}{11}$.	16. $\frac{44}{0.05}$.	23. $\frac{55^3}{28^2}$.
3. $\frac{5}{6}$.	10. $\frac{3}{8}$.	17. $\frac{35}{0.11}$.	24. $\frac{77^2}{15^7}$.
4. $\frac{7}{5}$.	11. $\frac{8}{11}$.	18. $\frac{24}{55}$.	25. $\frac{33^2}{0.005}$.
5. $\frac{7}{11}$.	12. $\frac{5}{22}$.	19. $\frac{11^3}{12^2}$.	26. $\frac{0.007}{15^3}$.
6. $\frac{11}{7}$.	13. $\frac{0.7}{11}$.	20. $\frac{5^5}{6^4}$.	27. $\frac{0.011}{0.05^4}$.
7. $\frac{5}{7}$.	14. $\frac{6.05}{4}$.	21. $\frac{7^3}{2^5}$.	28. $\frac{0.05^5}{0.011^3}$.

394. Tables. A table of *four-place* common logarithms is given at the end of this chapter, which contains the common logarithms of all numbers under 1000, *the decimal point and characteristic being omitted.* The logarithms of single digits, 1, 8, etc., will be found at 10, 80, etc.

Tables containing logarithms of more places can be procured, but this table will serve for many practical uses, and will enable the student to use tables of five-place, seven-place, and ten-place logarithms, in work that requires greater accuracy.

In working with a four-place table, the numbers corresponding to the logarithms, that is, the **antilogarithms**, as they are called, may be carried to *four significant digits.*

395. To Find the Logarithm of a Number in this Table.

(1) Suppose it is required to find the logarithm of 65.7. In the column headed "N" look for the first two significant figures, and at the top of the table for the third significant figure. In the line with 65, and in the column headed 7, is seen 8176. To this number prefix the characteristic and insert the decimal point. Thus,

$$\log 65.7 = 1.8176.$$

(2) Suppose it is required to find the logarithm of 20347. In the line with 20, and in the column headed 3, is seen 3075; also in the line with 20, and in the 4 column, is seen 3096, and the difference between these two is 21. The difference between 20300 and 20400 is 100, and the difference between 20300 and 20347 is 47. Hence, $\frac{47}{100}$ of $21 = 10$, nearly, must be added to 3075; that is,

$$\log 20347 = 4.3085.$$

(3) Suppose it is required to find the logarithm of 0.0005076. In the line with 50, and in the 7 column, is seen 7050; in the 8 column, 7059: the difference is 9. The

difference between 5070 and 5080 is 10, and the difference between 5070 and 5076 is 6. Hence, $\frac{6}{10}$ of $9 = 5$ must be added to 7050; that is,

$$\log 0.0005076 = 6.7055 - 10.$$

396. To Find a Number when its Logarithm is Given.

(1) Suppose it is required to find the number of which the logarithm is 1.9736.

Look for 9736 in the table. In the column headed "N," and in the line with 9736, is seen 94, and at the head of the column in which 9736 stands is seen 1. Therefore, write 941, and insert the decimal point as the characteristic directs; that is, the number required is 94.1.

(2) Suppose it is required to find the number of which the logarithm is 3.7936.

Look for 7936 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 7931 and 7938; their difference is 7, and the difference between 7931 and 7936 is 5. Therefore, $\frac{5}{7}$ of the difference between the numbers corresponding to the mantissas, 7931 and 7938, must be added to the number corresponding to the mantissa 7931.

The number corresponding to the mantissa 7938 is 6220.

The number corresponding to the mantissa 7931 is 6210.

The difference between these numbers is 10,

and $$6210 + \tfrac{5}{7} \text{ of } 10 = 6217.$$

Therefore, the number required is 6217.

(3) Suppose it is required to find the number of which the logarithm is $7.3882 - 10$.

Look for 3882 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 3874 and 3892; the difference between the two mantissas is 18, and the difference between 3874 and the given mantissa 3882 is 8.

The number corresponding to the mantissa 3892 is 2450.
The number corresponding to the mantissa 3874 is 2440.
The difference between these numbers is 10,

and $$2440 + \tfrac{8}{18} \text{ of } 10 = 2444.$$

Therefore, the number required is 0.002444.

Exercise 116.

Find, from the table, the common logarithms of:

1. 50.	**4.** 7803.	**7.** 7063.	**10.** 0.5234.
2. 201.	**5.** 4325.	**8.** 1202.	**11.** 0.01423.
3. 888.	**6.** 8109.	**9.** 0.00789.	**12.** 0.1987.

Find antilogarithms to the following common logarithms:

13. 4.1432.	**15.** 2.3177.	**17.** 9.0380 − 10.
14. 3.5317.	**16.** 1.3709.	**18.** 9.9204 − 10.

397. Examples.

(1) Find the product of $908.4 \times 0.05392 \times 2.117$.

$$\begin{aligned} \log\ 908.4 &= 2.9583 \\ \log\ 0.05392 &= 8.7318 - 10 \\ \log\ 2.117 &= 0.3257 \\ \hline &\ 2.0158 = \log 103.7. \ \textit{Ans.} \end{aligned}$$

When any of the factors are *negative*, find their logarithms without regard to the signs; write − after the logarithm that corresponds to a negative number. If the number of logarithms so marked is *odd*, the product is *negative*; if *even*, the product is *positive*.

(2) Find the quotient of $\dfrac{-8.3709 \times 834.637}{7308.946}$.

$$\begin{aligned} \log\ 8.3709 &= 0.9227 \quad - \\ \log\ 834.637 &= 2.9215 \quad + \\ \text{colog}\ 7308.946 &= 6.1362 - 10 \ + \\ \hline &\ 9.9804 - 10 = \log -0.9558. \ \textit{Ans.} \end{aligned}$$

(3) Find the cube of 0.0497.

$$\begin{array}{lr} & \log 0.0497 = 8.6964 - 10 \\ \text{Multiply by 3,} & 3 \\ \hline & 6.0892 - 10 = \log 0.0001228. \quad \textit{Ans.} \end{array}$$

(4) Find the fourth root of 0.00862.

$$\begin{array}{lr} & \log\ 0.00862 = 7.9355 - 10 \\ \text{Add } 30 - 30, & 30. \quad - 30 \\ \text{Divide by 4,} & 4\overline{)37.9355 - 40} \\ & 9.4839 - 10 = \log 0.3047. \quad \textit{Ans.} \end{array}$$

(5) Find the value of $\sqrt[5]{\dfrac{3.1416 \times 4771.21 \times 2.7183^{\frac{1}{2}}}{30.103^4 \times 0.4343^{\frac{1}{2}} \times 69.897^4}}$.

$$\begin{array}{rrll} \log & 3.1416 = & 0.4971 & = 0.4971 \\ \log & 4771.21 = & 3.6786 & = 3.6786 \\ \frac{1}{2}\log & 2.7183 = & 0.4343 \div 2 & = 0.2172 \\ 4\ \text{colog} & 30.103 = & 4(8.5214 - 10) & = 4.0856 - 10 \\ \frac{1}{2}\ \text{colog} & 0.4343 = & 0.3622 \div 2 & = 0.1811 \\ 4\ \text{colog} & 69.897 = & 4(8.1555 - 10) & = 2.6220 - 10 \\ \hline & & & 11.2816 - 20 \\ & & & 30 \quad - 30 \\ & & & 5\overline{)41.2816 - 50} \\ & & & 8.2563 - 10 \\ & & & = \log 0.01804. \quad \textit{Ans.} \end{array}$$

398. An **exponential equation,** that is, an equation in which the exponent involves the unknown number, is easily solved by Logarithms.

Ex. Find the value of x in $81^x = 10$.

$$81^x = 10.$$

$$\therefore \log(81^x) = \log 10,$$

$$x \log 81 = \log 10,$$

$$x = \frac{\log 10}{\log 81} = \frac{1.0000}{1.9085} = 0.524. \quad \textit{Ans.}$$

Exercise 117.

Find by logarithms the following products:

1. 948.7×0.04387.
2. 3.409×0.008763.
3. 830.7×0.0003769.
4. 8.439×0.9827.
5. $7564 \times (-0.003764)$.
6. $3.764 \times (-0.08349)$.
7. $-5.845 \times (-0.00178)$.
8. -8945.7×73.84.

Find by logarithms:

9. $\dfrac{7065}{5401}$.
10. $\dfrac{7.652}{-0.06875}$.
11. $\dfrac{0.07654}{83.94 \times 0.8395}$.
12. $\dfrac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.
13. 0.1768^5.
14. 1.211^{10}.
15. $11^{\frac{1}{5}}$.
16. $(\frac{73}{61})^{11}$.
17. $(\frac{14}{51})^7$.
18. $906.8^{\frac{1}{4}}$.
19. $(\frac{951}{823})^6$.
20. $(7\frac{6}{11})^{0.38}$.
21. $2.563^{\frac{3}{11}}$.
22. $(8\frac{3}{4})^{2.8}$.
23. $(5\frac{31}{37})^{0.375}$.
24. $(9\frac{20}{43})^{\frac{1}{5}}$.
25. $\sqrt[3]{\dfrac{0.0075^2 \times 78.34 \times 172.4^{\frac{1}{3}} \times 0.00052}{4285^{\frac{1}{3}} \times 54.27^4 \times 0.001 \times 86.79^{\frac{1}{2}}}}$.
26. $\sqrt[4]{\dfrac{0.03271^2 \times 3.429 \times 0.7752^3}{32.79 \times 0.00371^4}}$.
27. $\sqrt[3]{\dfrac{7.126 \times \sqrt{0.1327} \times 0.05738}{\sqrt{0.4346} \times 17.38 \times \sqrt{0.006372}}}$.

Find x from the equations:

28. $5^x = 10$.
29. $4^x = 20$.
30. $7^x = 40$.
31. $(1.3)^x = 4.2$.
32. $(0.4)^{-x} = 3$.
33. $(0.9)^{-x} = 2$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

CHAPTER XXVI.

GENERAL REVIEW EXERCISE.

If $a = 6$, $b = 5$, $c = -4$, $d = -3$, find the value of

1. $\sqrt{b^2 + ac} + \sqrt{c^2 - 2ac}$.

2. $\dfrac{a^2 - \sqrt{b^2 + ac}}{2a - \sqrt{b^2 - ac}}$.

3. $\sqrt{b^2 + ac + \sqrt{c^2 - 2ac}}$.

4. $\dfrac{c + \sqrt{d^2 + c^2}}{c^3 + 2d(d^2 - c^2)}$.

Find the value of

5. $\dfrac{x}{a} + \dfrac{x}{b}$, when $x = \dfrac{abc}{a+b}$.

6. $\dfrac{1}{a}(b-x) + \dfrac{1}{b}(c-x) + \dfrac{1}{a}(x-c)$, when $x = \dfrac{ab - b^2 + bc}{a}$.

7. $\dfrac{x}{a} + \dfrac{x}{b-a}$, when $x = \dfrac{a^2(b-a)}{b(b+a)}$.

8. $(a+x)(b+x) - a(b+c) + x^2$, when $x = \dfrac{ac}{b}$.

9. $\dfrac{a(1+b)+bx}{a(1+b)-bx} - \dfrac{a}{a-2bx}$, when $x = -a$.

10. Add $(a-b)x^2 + (b-c)y^2 + (c-a)z^2$, $(b-c)x^2 + (c-a)y^2 + (a-b)z^2$, and $(c-a)x^2 + (a-b)y^2 + (b-c)z^2$.

11. Add $(a+b)x + (b+c)y - (c+a)z$, $(b+c)z + (c+a)x - (a+b)y$, and $(a+c)y + (a+b)z - (b+c)x$.

12. Show that $x^3 + y^3 + z^3 - 3xyz = 0$, if $x + y + z = 0$.

13. Show that $x^3 - 8y^3 - 27z^3 - 18xyz = 0$, if $x = 2y + 3z$.

Simplify by removing parentheses and collecting terms:

4. $3a-2(b-c)-[2(a-b)-3(c+a)]-[9c-4(c-a)]$.

15. $7(2a+b)-\{19b-[13(c-a)+12(b-c)]\}$.

16. $x-\{4y+[3(z-x)-(x+2y)]-(2y+z-2x)\}$.

17. $1+2\{x+4-3[x+5-4(x+1)]\}$.

18. $10x-\{4[5x-3(x-1)]-3[4x-3(x+1)]\}$.

19. $3x^2-\{2x^2-(3x-7)-[2x^2-(3x-x^2)]-[5-(2x^2-4x)]\}$.

20. $(x-2)(x-3)-(x-7)(x-1)+(x-1)(x-2)$.

21. $(x+y)^2-2x(3x+2y)-(y-x)(-x+y)$.

22. $3a-[2a-(3a-b)^2]+3a\left(2b-3a-\frac{b^2}{3a}\right)$.

Resolve into lowest factors:

23. $(x+y)^2-4z^2$.

24. $(x^2+y^2)^2-4x^2y^2$.

25. $a^2-b^2-c^2+2bc$.

26. $(x^2-y^2-z^2)^2-4y^2z^2$.

27. $9x^8-\frac{3}{2}x^4y^2+\frac{1}{16}y^4$.

28. $\left(\frac{a}{b}\right)^{2m}+\left(\frac{b}{a}\right)^{2m}-2$.

29. $81a^4-1$.

30. $a^{12}-b^{12}$.

31. $(a^2-b^2+c^2-d^2)^2-(2ac-2bd)^2$.

32. $x^2-19x+84$.

33. $\frac{1}{4}x^2+2\frac{1}{2}x-36$.

34. $x^2-8x+15$.

35. $9x^2-150x+600$.

36. $56x^2+3xy-20y^2$.

37. $12x^2+37x+21$.

38. $33-14x-40x^2$.

39. $6x^2+5x-4$.

40. x^3-y^6.

41. $8+a^3x^3$.

42. x^5-a^{10}.

43. $27a^3-64$.

44. $x^{10}-32y^5$.

45. $a^{12}-b^8$.

46. $x^{15}+1024y^{10}$.

47. $a^3-(b+c)^3$.

48. $8x^3-6xy(2x+3y)+27y^3$.

Find the H. C. F. of

49. $6x^4-2x^3+9x^2+9x-4$, and $9x^4+80x^2-9$.

50. $3x^5-5x^3+2$, and $2x^5-5x^2+3$.

51. $x^3-93x-308$, and $x^3-21x^2+131x-231$.

52. $x^4-2x^3+4x^2-6x+3$, and $x^4-2x^3-2x^2+6x-3$.

53. $x^5-4x^3-x^2+2x+2$, and x^3-x^2-2x+2.

54. $3x^3+10x^2+7x-2$, and $3x^3+13x^2+17x+6$.

55. $4x^4-9x^2+6x-1$, and $6x^3-7x^2+1$.

56. $x^5+11x-12$, and x^5+11x^3+54.

Find the L. C. M. of

57. $4x^2+4x-3$, and $4x^2+2x-6$.

58. x^2-4y^2, and $x^2+xy-6y^2$.

59. $7a^2x(a-x)$, $21ax(a^2-x^2)$, and $12ax^2(a+x)$.

60. $9x^3-x-2$, and $3x^3-10x^2-7x-4$.

61. x^2-5x+6, x^2-4x+3, and x^2-3x+2.

62. $2x^3+5x^2y-5xy^2+y^3$, and $2x^3-7x^2y+5xy^2-y^3$.

Simplify:

63. $\dfrac{x+1}{x(x-2)}-\dfrac{3x+2}{x(x+1)}+\dfrac{2x-1}{x^2-x-2}$.

64. $\dfrac{1+x}{1-x}+\dfrac{1-x}{1+x}-\dfrac{1+x^2}{1-x^2}-\dfrac{1-x^2}{1+x^2}$.

65. $\dfrac{x-1}{(x+2)(x+5)}-\dfrac{2(x+2)}{(x+5)(x-1)}+\dfrac{x+5}{(x-1)(x+2)}$.

66. $\dfrac{1}{ax-a^2}+\dfrac{1}{ax+2a^2}-\dfrac{3}{x^2+ax-2a^2}$.

67. $\dfrac{x}{(x+3)(x-1)}+\dfrac{x-1}{(x+3)(2-x)}-\dfrac{x-3}{(2-x)(1-x)}$.

68. $\left(1+\frac{4}{x-1}+\frac{12}{x-3}\right)\left(1+\frac{4}{x+1}-\frac{12}{x+3}\right).$

69. $\left(\frac{x+2y}{x+y}+\frac{x}{y}\right)\div\left(\frac{x}{x+y}-\frac{x+2y}{y}\right).$

70. $\frac{a^3-b^3}{a^4-b^4}-\frac{a-b}{2(a^2-b^2)}-\frac{1}{2}\left(\frac{a+b}{a^2+b^2}-\frac{1}{a+b}\right).$

71. $\dfrac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}+\frac{2}{ab}}.$

72. $\dfrac{\frac{1}{1-a}-\frac{1}{1+a}}{\frac{a}{1-a}+\frac{1}{1+a}}.$

73. $\dfrac{1-\frac{1}{2}[1-\frac{1}{3}(1-x)]}{1-\frac{1}{3}[1-\frac{1}{2}(1-x)]}.$

74. $\dfrac{\frac{1}{a}+\frac{1}{ab^3}}{b-1+\frac{1}{b}}.$

75. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}.$

76. $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1+a}}}.$

Solve:

77. $\frac{6x+13}{15}-\frac{9x+15}{5x-25}+3=\frac{2x+15}{5}.$

78. $\frac{2x+a}{3(x-a)}+\frac{3x-a}{2(x+a)}=2\frac{1}{6}.$

79. $\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7}.$

80. $\left.\begin{aligned}\frac{x}{3}+\frac{5}{y}&=4\tfrac{1}{3}\\ \frac{x}{6}+\frac{10}{y}&=2\tfrac{2}{3}\end{aligned}\right\}$

81. $\left.\begin{aligned}\frac{5x}{0.7}+\frac{0.3}{y}&=6\\ \frac{10x}{7}+\frac{9}{y}&=31\end{aligned}\right\}$

82. $\left.\begin{aligned}\frac{2x}{a}+\frac{3y}{b}-\frac{4z}{c}&=1\\ \frac{4x}{a}+\frac{2y}{b}-\frac{3z}{c}&=3\\ \frac{3x}{a}+\frac{4y}{b}-\frac{2z}{c}&=5\end{aligned}\right\}$

83. $\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}+\frac{c}{z}&=3\\ \frac{a}{x}+\frac{b}{y}-\frac{c}{z}&=1\\ \frac{2a}{x}-\frac{b}{y}-\frac{c}{z}&=0\end{aligned}\right\}$

Find the arithmetical value of

84. $36^{\frac{1}{2}}$; $27^{\frac{1}{3}}$; $16^{\frac{1}{4}}$; $32^{\frac{1}{5}}$; $4^{\frac{5}{2}}$; $8^{\frac{2}{3}}$; $27^{\frac{5}{3}}$; $64^{\frac{2}{3}}$.

85. $32^{\frac{3}{5}}$; $64^{\frac{5}{6}}$; $81^{\frac{3}{4}}$; $(3\frac{3}{8})^{\frac{1}{3}}$; $(5\frac{1}{16})^{\frac{1}{4}}$; $(1\frac{9}{16})^{\frac{3}{2}}$.

86. $(0.25)^{\frac{1}{2}}$; $(0.027)^{\frac{2}{3}}$; $49^{0.5}$; $32^{0.2}$; $81^{0.75}$.

87. $36^{-\frac{1}{2}}$; $27^{-\frac{1}{3}}$; $(\frac{9}{16})^{-\frac{3}{2}}$; $(0.16)^{-\frac{3}{2}}$; $(0.0016)^{-\frac{3}{4}}$.

88. Interpret a^{-2}; a^0; $a^{\frac{1}{2}}$; $a^{-\frac{1}{3}}$; $(a^{-\frac{1}{2}})^{-\frac{1}{3}}$.

Simplify:

89. $a^{\frac{1}{2}}\times a^{\frac{1}{3}}$; $c^{\frac{1}{3}}\times c^{\frac{1}{6}}$; $m^{\frac{1}{2}}\times m^{-\frac{1}{6}}$; $n^{\frac{3}{4}}\times n^{-\frac{1}{12}}$.

90. $a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{1}{6}}\times a^{-\frac{2}{3}}b^{-\frac{1}{3}}c^{-\frac{1}{6}}$; $a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{4}}\times a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{\frac{1}{2}}d$.

91. $(2ab+2bc+2ac-a^2-b^2-c^2)\div(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}})$.

92. $\sqrt{12}$; $\sqrt{8}$; $\sqrt{50}$; $\sqrt[3]{16}$; $4\sqrt[3]{250}$; $\sqrt{\frac{1}{2}}$; $\sqrt[3]{\frac{1}{4}}$; $\sqrt{\frac{8}{27}}$.

93. $5\sqrt[3]{-320}$; $\sqrt{a^3b^7}$; $\sqrt[3]{a^5}$; $\sqrt{a^2x+a^3}$; $3\frac{1}{3}\sqrt[3]{54x^9}$.

94. $2\sqrt{18}-3\sqrt{8}+2\sqrt{50}$; $\sqrt[3]{81}+\sqrt[3]{24}-\sqrt[3]{192}$.

95. $\frac{3}{2}\sqrt{\frac{5}{9}}+\sqrt{80}-\frac{1}{4}\sqrt{20}$; $8\sqrt{\frac{5}{16}}+10\sqrt{\frac{20}{25}}-2\sqrt{\frac{5}{4}}$.

Rationalize the divisor, and find the value of

96. $\dfrac{1}{2-\sqrt{3}}$.

97. $\dfrac{3}{3+\sqrt{6}}$.

98. $\dfrac{5}{\sqrt{2}+\sqrt{7}}$.

99. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

100. $\dfrac{7\sqrt{5}}{\sqrt{5}+\sqrt{7}}$.

101. $\dfrac{2\sqrt{6}}{\sqrt{3}-\sqrt{5}}$.

Solve:

102. $\dfrac{5}{x-2}-\dfrac{4}{x}=\dfrac{3}{x+6}.$

103. $\dfrac{x+3}{2x-7}-\dfrac{2x-1}{x-3}=0.$

104. $\dfrac{x+4}{x-4}+\dfrac{x-2}{x-3}=6\frac{1}{3}.$

105. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a^2+b^2}{ab}.$

106. $\dfrac{ax+b}{a+bx}=\dfrac{cx+d}{c+dx}.$

107. $\dfrac{c}{x-d}-\dfrac{d}{x+c}=\dfrac{c-d}{x}.$

108. $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b.$

109. $\dfrac{1}{a+x}+\dfrac{1}{b+x}=\dfrac{a+b}{ab}.$

110. $ax^2-\dfrac{6c^2}{a+b}=cx-bx^2.$

111. $\dfrac{x+a}{x+b}+\dfrac{x+b}{x+c}=2.$

112. $\left.\begin{array}{l}3xy-5y^2=1\\5xy+3x^2=22\end{array}\right\}$

113. $\left.\begin{array}{l}x^2+10xy=11\\5xy-3y^2=2\end{array}\right\}$

114. $\left.\begin{array}{l}\sqrt{x+y}=\sqrt{y}+2\\x-y=7\end{array}\right\}$

115. $\left.\begin{array}{l}x^2+xy+y^2=52\\xy-x^2=8\end{array}\right\}$

Form the equations of which the roots are:

116. $a-b,\ a+b.$

117. $a-2b,\ a+3b.$

118. $a+2b,\ 2a+b.$

119. $1+\sqrt{3},\ 1-\sqrt{3}.$

120. $-1+\sqrt{3},\ -1-\sqrt{3}.$

121. $1+\sqrt{-3},\ 1-\sqrt{-3}.$

Solve:

122. $x^4-5x^2+4=0.$

123. $x^6-9x^3+8=0.$

124. $9x^4-13x^2+4=0.$

125. $4x^4-17x^2+4=0.$

126. $2x^4-5x^2+2=0.$

127. $2x^6-19x^3+24=0.$

128. $x^4-1=0.$

129. $x^6-1=0.$

130. $x^{\frac{2}{3}}+8x^{\frac{1}{3}}-9=0.$

131. $16x^{\frac{4}{5}}-17x^{\frac{2}{5}}+1=0.$

132. $2x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 1 = 0.$

133. $3\sqrt[3]{x} - 5\sqrt[6]{x} + 2 = 0.$

134. $6\sqrt{x} - 3\sqrt[4]{x} - 45 = 0.$

135. $21\sqrt[3]{x^2} - 5\sqrt[3]{x} - 74 = 0.$

136. $3\sqrt{x^5} + 4\sqrt[4]{x^5} - 20 = 0.$

137. $x^{-\frac{1}{2}} + 5x^{-\frac{1}{4}} - 14 = 0.$

138. $4x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} - 27 = 0.$

139. $x^{2n} + 3x^n - 4 = 0.$

140. $3x^{\frac{1}{3}} - 2ax^{\frac{1}{6}} - a^2 = 0.$

141. $\sqrt{2x} - \sqrt[4]{2x} - 2 = 0.$

142. $3\sqrt[3]{9x^2} + 4\sqrt[3]{3x} - 39 = 0.$

143. $\sqrt{3ax} + a\sqrt[4]{3ax} - 2a^2 = 0.$

144. $3\sqrt[3]{2bx} - 5b\sqrt[6]{2bx} - 2b^2 = 0.$

145. $\sqrt{x+4} + \sqrt{3x+1} = \sqrt{9x+4}.$

146. $\sqrt{5x+1} + 2\sqrt{4x-3} = 10\sqrt{x-2}.$

147. $2\sqrt{x+2} - 3\sqrt{3x-5} + \sqrt{5x+1} = 0.$

148. $\sqrt{11-x} + \sqrt{8-2x} - \sqrt{21+2x} = 0.$

Expand:

149. $(a^{\frac{1}{2}} - x^{\frac{2}{3}})^9$; $(3a^{\frac{1}{3}} - 2b^2)^5$; $(a^{\frac{2}{3}} - \frac{1}{2}b^{\frac{1}{2}})^7$; $\left(2 - \frac{x^2}{3}\right)^6$; $\left(3 - \frac{x^{\frac{1}{2}}}{2}\right)^7$.

150. $\left(x^2 - \frac{y^3}{x}\right)^5$; $(2x^2 + \sqrt{3x})^4$; $(\sqrt{a^2+1} + a^3)^5$; $(1 + 2x - x^2 - x^3)^3$.

151. Expand to four terms

$(1-3x)^{-\frac{1}{3}}$; $(1-4x^2)^{-\frac{2}{3}}$; $(1-\frac{3}{2}x^{\frac{1}{2}})^{\frac{2}{3}}$; $(a - 2x^{-\frac{1}{3}})^{-5}$.

152. Find the eighty-seventh term of $(2x-y)^{90}$.

153. Resolve into partial fractions

$$\frac{3-2x}{1-3x+2x^2};\quad \frac{3-2x}{(1-x)(1-3x)};\quad \frac{1}{1-x^3}.$$

154. Expand to five terms $\frac{3-2x}{1-3x+2x^2}$.

1 Early School
2 State Attitude
3 Philanthropic Move —
4 Chap IV

$7(a+b)^6 a+$

$(a+b)^7$ $\frac{1}{8}$

$4\frac{7}{8}$

Log 4

$\sqrt{4^3}$

$4^{3/2}$